

# Analysis 2 Practicals

Notes, University (of Technology) Graz  
based on the lecture by Wolfgang Ring

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## 1 Practicals

- Florian Kruse
- Analysis 2 practicals, every Thu, 15:00–16:30
- Sprechstunde: Tue, 14–15

## 2 Sheet 1, Exercise 1

**Exercise 1.** The Euclidean norm of  $v = (v^1, v^2, \dots, v^n)^T \in \mathbb{R}^n$  is defined as

$$\|v\|_2 := \sqrt{(v^1)^2 + (v^2)^2 + \dots + (v^n)^2}$$

Show: A sequence  $(x_k) \subset \mathbb{R}^n$  converges in regards of the Euclidean norm to  $x \in \mathbb{R}^n$  iff they converge componentwise to  $x$

$$\lim_{k \rightarrow \infty} \|x_k - x\|_2 = 0 \iff \forall j \in \{1, \dots, n\} : \lim_{k \rightarrow \infty} x_k^j = x^j$$

Direction  $\Rightarrow$ .

Let  $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$ .

Consider:  $|x_{jk} - x_j|$  for arbitrary  $j \in \{1, \dots, n\}$ .

It holds that

$$0 \leq |x_{jk} - x_j| = \sqrt{(x_{jk} - x_j)^2} \leq \sqrt{(x_{1k} - x_1)^2 + \dots + (x_{nk} - x_n)^2} = \|x_k - x\| \rightarrow 0$$

$$\implies \lim_{k \rightarrow \infty} |x_{jk} - x_j| = 0 \forall j$$

Direction  $\Leftarrow$ .

Let  $\lim_{k \rightarrow \infty} x_{jk} = x_j \forall j \in \{1, \dots, n\}$ .

The square root function is continuous.

$$\lim_{k \rightarrow \infty} \|x_k - x\| = \sqrt{(x_{1k} - x_1)^2 + \dots + (x_{nk} - x_n)^2}$$

$$\sqrt{(\lim_{k \rightarrow \infty} x_{1k})^2 - 2(\lim_{k \rightarrow \infty} x_{1k})x_1 + x_1^2 + \dots + (\lim_{k \rightarrow \infty} x_{nk})^2 - 2(\lim_{k \rightarrow \infty} x_{nk})x_n + x_n^2}$$

$$= \sqrt{\underbrace{x_1^2 - 2x_1^2 + x_1^2}_{=0} + \cdots + \underbrace{x_n^2 - 2x_n^2 + x_n^2}_{=0}} = 0$$

**Remark:** In  $\mathbb{R}^n$ , all norms are equivalent. This exercise showed this property. So if you pick two numbers in  $\mathbb{R}^n$  and they get “closer”, they get “closer” in every norm.

### 3 Sheet 1, Exercise 2

**Exercise 2.** In the lecture, we discussed the SCNF.  $d_{\text{SCNF}} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ . For some fixed  $p \in \mathbb{R}^2$  it is defined as

$$d_{\text{SCNF}} := \begin{cases} \|x - y\|_2 & \text{if } \exists \lambda > 0 : y = p + \lambda(x - p) \\ \|x - p\|_2 + \|y - p\|_2 & \text{else} \end{cases}$$

For  $p := (0, 0)^T$  and  $x := (1, 1)^T$ , sketch the set  $B_R(x)$  for  $R = 1$  and  $R = 2$ .

$$B_R(x) := \{y \in \mathbb{R}^2 \mid d_{\text{SCNF}} < R\}$$

### 4 Sheet 1, Exercise 3

**Exercise 3.** Let  $(M, d)$  be a metric space and  $x \in M$ . Furthermore let  $(x_k) \subset M$  be a sequence with property that every subsequence of  $(x_k)$  contains a subsequence converging to  $x$ . Prove by contradiction, that  $(x_k)$  converges to  $x$ .

$x_0 \not\rightarrow x$ .

There exists  $\varepsilon_0 > 0$  for infinitely many  $n \in \mathbb{N} : d(x_n, x) \geq \varepsilon_0$ . Choose a subsequence  $(x_{n_j})_{j \in \mathbb{N}}$  with  $d(x_{n_j}, x) \geq \varepsilon_0 \forall j \in \mathbb{N}$ . Then there does not exist a subsequence of  $(x_{n_j})$  with limit  $x$ .

### 5 Sheet 1, Exercise 4

**Exercise 4.** Let  $(M, d)$  be a metric space and complete space. The diameter of a nonempty set  $A \subset M$  is given by

$$\text{diam}(A) := \sup \{d(x, y) \mid x, y \in A\}$$

Let  $(A_j)_{j \in \mathbb{N}}$  be a sequence of nonempty, closed sets in  $M$  with  $A_{j+1} \subset A_j$  for all  $j \in \mathbb{N}$ . Furthermore it holds that  $\text{diam}(A_j) \rightarrow 0$  for  $j \rightarrow \infty$ . Prove that  $x \in M$  exists with  $\bigcap_{j=1}^{\infty} A_j = \{x\}$  and that  $x$  is unique.

$A_j \subseteq M$ , because its a complete, metric space.

$$\implies \bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$$

Assume  $\exists y_0 \in M : y_0 \neq x_0 \implies d(y_0, x_0) \geq \varepsilon > 0$

$$\forall j \in \mathbb{N} : \text{diam}(A_j) \geq \varepsilon$$

This is a contradiction. However, this is not the equality, we are looking for. Assume  $\bigcap_{j=1}^{\infty} A_j = \{x_0\} = \{y_0\} \implies x_0 = y_0$ . This is the equality, that was meant to be proven.

### 5.1 Prove $\bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$

**Hint:** If the assignment mentions that completeness must be proven, usually you have to construct a Cauchy sequence.

Construct  $(x_j)_{j \in \mathbb{N}}$ . Choose for  $x_j$  some element of  $A_j$ . Choose  $x_j \in A_j$  for  $j \in \mathbb{N}$ . This defines a Cauchy sequence  $(x_j)_{j \in \mathbb{N}}$ . Let  $j \in \mathbb{N}$ .  $x_i \in A_j \supset A_{j+1}$  and  $x_{j+1} \in A_{j+1} \forall i \in \mathbb{N}$ .

$$\implies d(x_j, x_{j+i}) \leq \text{diam}(A_j) \forall i \in \mathbb{N}$$

where  $\text{diam}(A_j) \rightarrow 0$  with  $j \rightarrow \infty$ .

$$\implies \exists x \in M : \lim_{j \rightarrow \infty} (x_j) = x$$

Because  $(x_j)_{j \geq J} \subseteq A_j$  and  $\lim_{j \rightarrow \infty} (x_j)_{j \geq J} = x$ , it follows that  $x \in A_j$  and then it follows that  $x \in \bigcap_{j=1}^{\infty} A_j$ .

*This lecture took place on 2018/03/22.*

## 6 Sheet 2, Exercise 1

### 6.1 Blackboard solution

Let  $B$  be bounded.

$$\text{diam}(B) < \infty \quad \text{diam}(B) = \sup(\{d(x, y) \mid x, y \in B\})$$

$$d(B_k, B_{k+1}) = \inf(\{d(x, y) \mid x \in B_k, y \in B_{k+1}\})$$

Exercise (a).

Prove:

$$\sum_{k=1}^{\infty} \text{diam}(B_k) < \infty \wedge \sum_{k=1}^{\infty} d(B_k, B_{k+1}) \implies \text{diam}\left(\bigcup_{k=1}^{\infty} B_k\right) < \infty$$

$$\text{diam}(B_k \cup B_{k+1}) \leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1})$$

We distinguish 3 cases:

1.  $x \in B_k, y \in B_k : d(x, y) \leq \text{diam}(B_k) \leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1})$
2.  $x \in B_{k+1}, y \in B_{k+1}, d(x, y) \leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1})$
3.  $\forall x \in B_k \forall y \in B_{k+1}$

Choose  $x_0$  and  $y_0$  on the border of sets  $B_k$  and  $B_{k+1}$  respectively. But  $x_0, y_0$  do not necessarily exist if compactness is not given. But let  $\varepsilon > 0$ . Find  $x_0, y_0$  with  $d(x_0, y_0) \leq d(B_k, B_{k+1}) + \varepsilon$ .

$$d(x, y) \leq \underbrace{d(x, x_0)}_{\leq \text{diam}(B_k)} + \underbrace{d(x_0, y_0)}_{\leq d(B_k, B_{k+1}) + \varepsilon} + \underbrace{d(y_0, y)}_{\leq \text{diam}(B_{k+1})} \leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1}) + \varepsilon$$

Laurent Pfeiffer continued the following solution (until Exercise 2):

$$\text{diam}((B_k \cup B_{k+1}) \cup B_{k+2}) \leq \text{diam}(B_k \cup B_{k+1}) + \underbrace{d((B_k \cup B_{k+1}), B_{k+2})}_{\leq d(B_{k+1}, B_{k+2})} + \text{diam}(B_{k+2})$$

$$\leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1}) + d((B_k \cup B_{k+1}), B_{k+2}) + \text{diam}(B_{k+2})$$

By induction it follows that

$$\text{diam}(B_k \cup B_{k+1} \cup \dots \cup B_n) \leq \text{diam}(B_k) + d(B_k, B_{k+1}) + \text{diam}(B_{k+1}) + d(B_{k+1}, B_{k+2}) + \dots + d(B_{n-1}, B_n) + \text{diam}(B_n)$$

$$\text{diam}(B_k \cup \dots \cup B_n) \leq \underbrace{\sum_{i=1}^n \text{diam}(B_i) + d(B_i, B_{i+1})}_D$$

Choose  $x, y \in \bigcup_{i=1}^{\infty} B_i$ . Then there exists some  $k \in \mathbb{N}$  such that  $x \in B_k$ . There exists  $n$  such that  $y \in B_n$ .

$$d(x, y) \leq \text{diam}(B_k) + \dots + \text{diam}(B_n) \leq D$$

Exercise (b).

Let  $x \in M$ . We define:  $B_{k+1} = B_{k+2} = \dots = \{x\}$ . For all  $i \geq k$  it holds that

$$\text{diam}(B_i) = 0$$



$$d(B_i, B_{i+1}) = 0$$

Therefore,

$$\sum_{i=1}^{\infty} \text{diam}(B_i) = \sum_{i=1}^k \underbrace{\text{diam}(B_i)}_{< +\infty} < +\infty$$

What about the distances?

$$\int_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^k d(B_i, B_{i+1}) < +\infty$$

By (a), it follows that

$$\left( \bigcup_{i=1}^{\infty} B_i \right) \text{ is bounded} \implies \left( \bigcup_{i=1}^k B_i \right) \subseteq \left( \bigcup_{i=1}^{\infty} B_i \right) \text{ is also bounded}$$

Exercise (c).

We define

$$B_i = \left[ \sum_{j=1}^i \frac{1}{j}, \sum_{j=1}^{i+1} \frac{1}{j} \right]$$

Then it holds that

$$\text{diam}(B_i) = \frac{1}{i+1} \xrightarrow{i \rightarrow \infty} 0$$

$$\sum_{i=1}^{\infty} \text{diam}(B_i) = \infty$$

$$B_i \cap B_{i+1} = \left\{ \sum_{j=1}^{i+1} \frac{1}{j} \right\} \implies d(B_i, B_{i+1}) = 0$$

$$B_1 \cup \dots \cup B_i = \left[ 1, \underbrace{\sum_{j=1}^{i+1} \frac{1}{j}}_{\rightarrow \infty} \right] \implies \underbrace{\bigcup_{i=1}^{\infty} B_i}_{\text{not bounded}} = [1, \infty)$$

We define  $B_i = \left\{ \sum_{j=1}^i \frac{1}{j} \right\}$ . For all  $i$ :

- $\text{diam}(B_i) = 0 \implies \sum_{i=1}^{\infty} \text{diam}(B_i) = 0$
- 

$$d(B_i, B_{i+1}) = \left( \sum_{j=1}^{i+1} \frac{1}{j} \right) - \left( \sum_{j=1}^i \frac{1}{j} \right) = \frac{1}{i+1} \xrightarrow{i \rightarrow \infty} 0$$

$$\sum_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty$$

The union is *not* bounded, because  $\sum_{j=1}^i \frac{1}{j} \in \bigcup_{j=1}^{\infty} B_j$ .

## 7 Sheet 2, Exercise 2

**Exercise 5.** Let  $(X, d)$  be a sequentially compact, metric space. Show:

- a.  $X$  is bounded.
- b.

### 7.1 Blackboard solution

Exercise (a).

Let  $X$  be unbounded. Hence, there exists a tuple  $(x_N, y_N) \in X \times X$  for every  $N \in \mathbb{N}$  with  $d(x_N, y_N) > N$ . Because  $(X, d)$  is sequentially compact, there exists a convergent subsequence  $(x_{N_{k_i}}, y_{N_{k_i}})$  we can choose such that

$$\begin{aligned} \lim_{k \rightarrow \infty} x_{N_k} = \infty \quad \lim_{i \rightarrow \infty} y_{N_{k_i}} = y_0 \quad \lim_{i \rightarrow \infty} (x_{N_{k_i}}) = x_0 \\ \implies \underbrace{N_{k_i}}_{\xrightarrow{i \rightarrow \infty} \infty} < d(x_{N_{k_i}}, y_{N_{k_i}}) \xrightarrow{i \rightarrow \infty} d(x_0, y_0) \end{aligned}$$

By this contradiction, it follows that  $X$  is bounded.

Exercise (b).

Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $X$ . Let  $X$  be sequence compact  $\implies$  there exists a convergent subsequence  $x_{n_k} \xrightarrow{k \rightarrow \infty} x \in X$ . Show that  $x_n \xrightarrow{n \rightarrow \infty} x$ .

Let  $\varepsilon > 0$  be arbitrary. Choose  $N \in \mathbb{N}$  such that  $\forall n, m \geq N : d(x_n, x_m) < \frac{\varepsilon}{2}$ . Choose  $k \in \mathbb{N}$  such that  $n_k \geq N$  and  $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ .

$$\forall n \geq n_k : d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) < \varepsilon$$

Exercise (c).

Show that  $A \subset X$  is sequentially compact iff  $A$  is closed.

$\implies$  Let  $(x_n)_{n \in \mathbb{N}}$  be a convergent sequence,  $(x_n)_{n \in \mathbb{N}} \subset A$ ,  $\lim_{n \rightarrow \infty} x_n = x_0 \in X$ . Show that  $x_0 \in A$ .

Set  $A$  is sequentially compact. Choose subsequence  $(x_{n_k})_{k \in \mathbb{N}} \subset A$ ,  $\lim_{k \rightarrow \infty} x_{n_k} = x_0 \in A \implies A$  is closed.

$\Leftarrow$   $A$  is closed. Show that  $A$  is sequentially compact.

Let  $(x_n)_{n \in \mathbb{N}} \subset A$  and there exists subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} x_{n_k} = x_0 \in X$ , because  $X$  is sequentially compact.  $(x_{n_k})_{k \in \mathbb{N}} \subset A \implies A$  is sequentially compact.

## 8 Sheet 2, Exercise 2

**Exercise 6.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sqrt{1+x^2}$ .

1. Show that  $|f(x) - f(y)| < |x - y| \forall x, y \in \mathbb{R}$  with  $x \neq y$
2. Investigate which conditions of Banach's Fixed Point Theorem are [not] met.
3. Is Banach's Fixed Point Theorem applicable? Does  $f$  have a fixed point?

Exercise (a).

$$\begin{aligned}
 |f(x) - f(y)| &< |x - y| \quad x, y \in \mathbb{R}, x \neq y \\
 \left| \sqrt{1+x^2} - \sqrt{1+y^2} \right| &< |x - y| \\
 1 + x^2 + 1 + y^2 - 2\sqrt{(1+x^2)(1+y^2)} &< x^2 + y^2 - 2xy \\
 2 - 2\sqrt{(1+x^2)(1+y^2)} &< -2xy \\
 1 + xy &< \sqrt{(1+x^2)(1+y^2)}
 \end{aligned}$$

We need to distinguish 2 cases here ( $x$  and  $y$  have same signum,  $x$  and  $y$  have different signum). This is trivial.

$$\begin{aligned}
 1 + 2xy + x^2y^2 &< 1 + x^2 + y^2 + x^2y^2 \\
 0 &< x^2 + y^2 - 2xy \\
 0 &< (x - y)^2
 \end{aligned}$$

Exercise (b and c).

Let  $x \in \mathbb{R}$ .

$$\begin{aligned}
 f(x) &= x \\
 \sqrt{1+x^2} &= x \\
 1 + x^2 &= x^2 \\
 1 &= 0
 \end{aligned}$$

This lecture took place on 2018/04/12.

## 9 Sheet 3, Exercise 4

**Exercise 7.** Let  $(X, d)$  be a metric space and  $x_0 \in X$ . A function  $f : X \rightarrow \mathbb{R}$  is called half-continuous from below in  $x_0$ , if for every  $\varepsilon > 0$  some  $\delta > 0$  exists, such that  $d(x, x_0) < \delta$  implies  $f(x_0) - f(x) < \varepsilon$ . If  $f$  is half-continuous from below in every  $x_0 \in X$ , then  $f$  is called half-continuous from below.

Obviously, continuity implies half-continuity.

### 9.1 Sheet 3, Exercise 4a

**Exercise 8.** Give some half-continuous from below  $f : [-1, 1] \rightarrow \mathbb{R}$  such that  $f$  is non-continuous.

Let  $f : [-1, 1] \rightarrow \mathbb{R}$ .

$$x \mapsto \begin{cases} -1 & x = -1 \\ -x & x \neq -1 \end{cases}$$
$$\underbrace{f(-1)}_{=-1} - \underbrace{f(x)}_{\geq -1} \leq 0 < \varepsilon$$

### 9.2 Sheet 3, Exercise 4b

**Exercise 9.** Give some half-continuous from below  $f : [-1, 1] \rightarrow \mathbb{R}$ , but does not have a maximum.

Same  $f$  can be chosen.

### 9.3 Sheet 3, Exercise 4c

**Exercise 10.** Give some half-continuous from below  $f : [-1, 1] \rightarrow \mathbb{R}$ , but does not have a minimum.

$f$  as  $f|_{[-1,1]}$  can be chosen.

### 9.4 Sheet 3, Exercise 4d

**Exercise 11.** Prove that every half-continuous from below function in a compact set has a minimum.

**Hint:** It is assumed that cover-compactness seems to be more cumbersome than sequential compactness.

**Remark:** This is a generalization of the theorem, that every continuous, compact function has a minimum and maximum.

Let  $K \subseteq X$  be compact.  $f : K \rightarrow \mathbb{R}$  is half-continuous from below.

Show that  $f^k = \inf(f(K)) \in f(K)$ .

$$\exists (x_n)_{n \in \mathbb{N}} \subseteq K \text{ with } f(x_n) - f^k < \frac{1}{n}$$

$K$  is compact. Hence, there exists  $(x_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} x_{n_k} := x^* \in K$ . Let  $\varepsilon > 0$  be arbitrary. By half-continuity from below, it follows that  $\exists \delta > 0 : d(x^*, x) < \delta \implies f(x^*) - f(x) < \varepsilon$ .

$$\begin{aligned} \exists K \in \mathbb{N} \forall k \geq K : d(x^k, x_{n_k}) < \delta &\implies f(x^k) - f(x_{n_k}) < \varepsilon \iff f(x^*) < f(x_{n_k}) + \varepsilon \\ &\implies f(x^*) \leq \lim_{k \rightarrow \infty} f(x_{n_k}) \implies f(x^*) \leq \lim_{n \rightarrow \infty} f(x_n) = f^* \\ &\implies f(x^*) = f^* \implies f^* \text{ is minimum of } f(X) \end{aligned}$$

## 10 Sheet 3, Exercise 3

**Exercise 12.** Let  $(X, d)$  and  $(Y, e)$  be metric spaces, where  $d : X \rightarrow \mathbb{R}$  is a discrete metric, hence

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

### 10.1 Sheet 3, Exercise 3a

**Exercise 13.** Every map  $f : X \rightarrow Y$  is continuous.

Let  $f : X \rightarrow Y$  be arbitrary. Let  $x_0 \in X$  and  $\varepsilon > 0$  be arbitrary. Show that

$$\exists \delta > 0 : d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$$

$$K_{\frac{1}{2}}(x_0) = \{x_0\}$$

### 10.2 Sheet 3, Exercise 3b

**Exercise 14.** A map  $f : X \rightarrow Y$  is not necessarily bounded.

$M \geq 0$  arbitrary.  $\exists x, y \in f(X) : e(x, y) > M$ .

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{Z} & x &\mapsto x \\ f(x) &= \mathbb{Z} & x = 0 & \quad y = M + 1 \end{aligned}$$

$e = |\cdot|$ .

### 10.3 Sheet 3, Exercise 3c

**Exercise 15.** Every map  $g : Y \rightarrow X$  is bounded.

Let  $g : Y \rightarrow X$  be arbitrary. Show that  $\exists M \geq 0 \forall x, y \in g(Y) : d(x, y) \leq M$ . Choose  $M = 2$ .  $\forall x, y \in X : d(x, y) \leq 1 \leq 2$ .

### 10.4 Sheet 3, Exercise 3d

**Exercise 16.** In case  $(Y, e) = (\mathbb{R}, |\cdot|)$ , every non-constant map  $g : Y \rightarrow X$  is non-continuous.

We show: continuity implies constant.

Let  $g : \mathbb{R} \rightarrow X$  continuous. Let  $x_0 \in \mathbb{R}$  be arbitrary and  $\varepsilon = \frac{1}{2}$ .  $\exists \delta_0 > 0 : |x_0 - x| < \delta \implies d(g(x_0), g(x)) < \frac{1}{2}$  for  $x_0 \in \mathbb{R}$  there exists  $\delta_0$  such that  $\forall x \in (x_0 - \delta, x_0 + \delta) : g(x) = g(x_0)$ .

$$\sup \{s \in [x_0, \infty) \mid g(x) = g(x_0) \forall x \in [x_0, s)\}$$

## 11 Sheet 3, Exercise 2

**Exercise 17.** Let  $V$  be the vector space of bounded, complex sequences, hence

$$V := \{(a_k)_{k \in \mathbb{N}} \subset \mathbb{C} \mid \exists M \in \mathbb{R} \text{ with } |a_k| \leq M \forall k \in \mathbb{N}\}$$

additionally with norm

$$\|(a_k)_{k \in \mathbb{N}}\|_\infty := \sup \{|a_k| \mid k \in \mathbb{N}\}$$

This solution was done by Mr. Kruse himself.

### 11.1 Sheet 3, Exercise 2b

**Exercise 18.** The unit sphere in  $(V, \|\cdot\|_\infty)$ ,

$$B_1(0) = \{a \in V \mid \|a\|_\infty \leq 1\}$$

is closed and bounded, but not sequentially compact.

We need to prove boundedness.

Let  $C, D \in B_1(0)$ .

$$\begin{aligned} \Rightarrow \left\| \underbrace{C}_{=(c_k)} - \underbrace{D}_{=(d_k)} \right\|_{\infty} &\leq 2 \\ \sup \left\{ \left| \underbrace{c_k - d_k}_{\substack{\leq |c_k| + |d_k| \\ \leq 1\forall k \quad \leq 1\forall k}} \right| : k \in \mathbb{N} \right\} &\leq 2 \end{aligned}$$

We need to prove closedness.

$$(A^n)_{n \in \mathbb{N}} \subset B_1(0) \text{ with } \lim_{n \rightarrow \infty} A^n = A$$

Show that  $A \in B_1(0)$ .

$$\text{For every } A^n := (a_k^n)_{k \in \mathbb{N}} \text{ it holds that } \left\| \underbrace{(a_k^n)_{k \in \mathbb{N}}}_{= \sup\{|a_k^n| : k \in \mathbb{N}\} \leq 1} \right\|_{\infty} \leq 1$$

$$(A^n)_{n \in \mathbb{N}} \subset B_1(0) \text{ with } \lim_{n \rightarrow \infty} A^n = A$$

$$\iff \lim_{n \rightarrow \infty} \|A^n - A\|_{\infty} = 0$$

$|a_k^n|$  in

$$\sup \{|a_k^n| : k \in \mathbb{N}\}$$

converges to  $|a_k| \leq 1$  for  $n \rightarrow \infty$ .

We need to prove sequentially non-compact of  $B_1(0)$ . So we only need to find some sequence that does not have some converging subsequence.

We define

$$A^n := (a_k^n)_{k \in \mathbb{N}} := \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

for every  $n \in \mathbb{N}$ . As such we get a sequence

$$\implies (A^n)_{n \in \mathbb{N}} \subset B_1(0)$$

but it holds that  $\|A^n - A^m\|_{\infty} = 1 \forall n \neq m$ . This is also not a Cauchy sequence.

## 12 Sheet 3, Exercise 1

**Exercise 19.** Let  $(X, d)$  be a metric space. A set  $K \subset X$  is called *cover-compact*, if for every family of open sets  $(U_i)_{i \in I} \subset X$  with  $K \subset \bigcup_{i \in I} U_i$  it holds that: There exists a finite set  $J \subset I$  with  $K \subset \bigcup_{i \in J} U_i$ . Let  $K \subset X$  be cover-compact.

### 12.1 Sheet 3, Exercise 1a

**Exercise 20.** Show that  $K$  is totally bounded, hence for every  $r > 0$ , there exists  $x_1, \dots, x_n$  in  $K$  with  $K \subset \bigcup_{i=1}^n B_r(x_i)$ .

Construct a family of open spheres  $((B_r(x))_{x \in K} \subset K$  covering  $K$ ). By cover-compactness it follows there exists some finite  $J \subset K$  with  $K \subset \bigcup_{x \in J} B_r(x)$ .

### 12.2 Sheet 3, Exercise 1b

**Exercise 21.** Prove that  $K$  is sequentially compact.

Proof by contradiction: Assume  $K$  is not sequentially compact.

Then there exists a sequence  $(x_n)_{n \in \mathbb{N}} \in K$  which has a subsequence  $(x_{n_k})_{k \in \mathbb{N}} \rightarrow c \notin K$ .

$$\forall x \in K : \exists r_x > 0 : B_{r_x}(x) \text{ contains finitely many sequence elements}$$

Because  $\bigcup_{x \in K} B_{r_x}(x) \supset K$  it holds: there exists  $J \subset K$  finite  $\bigcup_{x \in J} B_{r_x}(x) \supset K$ . This contradicts with  $(x_n)_{n \in \mathbb{N}} \subset K$ .

### 12.3 Sheet 4, Exercise 1

**Exercise 22.** Let  $(M, d)$  be a complete metric space and  $(A_k)_{k \in \mathbb{N}} \subset M$  is a sequence of closed sets. Use Cantor's Theorem to prove:  $\bigcup_{k \in \mathbb{N}} A_k$  contains an open set if at least one  $A_k$  contains an open set. Illustrate this statement for  $(M, d) = (\mathbb{R}, |\cdot|)$ .

First we illustrate it in  $\mathbb{R}$ .

$$(A_k) = \{a_k\}$$

where  $a_k \in \mathbb{R}$ .

Consider some



## 13 Sheet 4, Exercise 2

**Exercise 23.** Let  $f : [-1, 1] \rightarrow \mathbb{C}$  be continuous and  $O \subset \mathbb{C}$  is an open set. In the lecture we have seen that  $f^{-1}(O)$  is open. Review the result and prove for  $O = \mathbb{C}$ .

1. The set  $O$  is open.
2. It holds that  $f^{-1}(O) = [-1, 1]$
3. The set  $[-1, 1] \subset \mathbb{R}$  is not open.
4. The statement of the lecture about  $f^{-1}(O)$  is still correct.

### 13.1 Sheet 4, Exercise 2a

Show that  $\mathbb{C}$  is open.

Let  $z \in \mathbb{C}$ .  $\exists \varepsilon > 0$ ,

$$B(z, \varepsilon) \subseteq \mathbb{C}$$

### 13.2 Sheet 4, Exercise 2b

Follows from the definition of a function.

### 13.3 Sheet 4, Exercise 2c

If it is an open set, there must be a neighborhood of arbitrary  $\varepsilon$  such that this neighborhood is completely in the set.

Let  $\varepsilon > 0$ . Choose  $x \in B(1, \varepsilon)$  with  $x = 1 + \frac{\varepsilon}{2}$ .

$$\implies x \in B(1, \varepsilon) \wedge x \notin [-1, 1]$$

### 13.4 Sheet 4, Exercise 2d

Let  $(X, d)$  and  $(Y, e)$  be metric spaces and  $f : X \rightarrow Y$  continuous then  $f^{-1}(O)$  is open  $\forall O \subseteq Y$  open.

Show:

$$\forall x \in [-1, 1] \exists \varepsilon > 0 : \underbrace{B(x, \varepsilon)}_{=\{z \in [-1, 1] \mid d(x, z) < \varepsilon\}} \subseteq [-1, 1]$$

So the difference is the domain of  $z$  ( $[-1, 1]$  unlike exercise c, where we used  $\mathbb{R}$ ).

The point was to illustrate how to read the theorem properly.

## 14 Sheet 4, Exercise 3

**Exercise 24.** Let  $\Omega$  be a non-empty set and  $B(\Omega)$  the vector space of real-valued bounded functions on  $\Omega$ . Hence,

$$B(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \exists M \in \mathbb{R} \text{ with } |f(x)| \leq M \forall x \in \Omega\}$$

with norm

$$\|f\|_{\infty} := \sup \{|f(x)| \mid x \in \Omega\}$$

Prove the following statements:

1.  $(B(\Omega), \|\cdot\|_{\infty})$  is a complete normed vector space.
2. The unit circle  $U$  in  $B(\Omega)$  is closed and bounded.

$$U = \{f \in B(\Omega) \mid \|f\|_{\infty} \leq 1\}$$

3. The unit circle is sequentially compact if and only if  $\Omega$  is finite.

### 14.1 Sheet 4, Exercise 3a

Given  $\Omega \neq \emptyset$ .

$$B(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \exists M \in \mathbb{R} : |f(x)| \leq M \quad \forall x \in \Omega\}$$

First, we show that  $\|\cdot\|_{\infty}$  is indeed a norm. We just show absolute homogeneity for illustrative purposes:

$$\begin{aligned} \|\lambda f\|_{\infty} &= \sup \{|\lambda \cdot f(x)| \mid x \in \Omega\} \\ &= \sup \{|\lambda| \cdot |f(x)| \mid x \in \Omega\} \\ &= |\lambda| \cdot \sup \{|f(x)| \mid x \in \Omega\} \\ &= |\lambda| \cdot \|f\| \end{aligned}$$

We show completeness of  $(B(\Omega), \|\cdot\|_{\infty})$ . Equivalently, all Cauchy sequences in  $B(\Omega)$  are convergent. Equivalently, for all Cauchy sequences  $(f_n)_{n \in \mathbb{N}} : \exists f \in B(\Omega) : \|f_n - f\|_{\infty} \rightarrow 0$  for  $n \rightarrow \infty$ .

Let  $(f_n)_{n \in \mathbb{N}}$  be an arbitrary Cauchy sequence. Hence,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m > N \implies \|f_n - f_m\|_{\infty} = \sup \{(f_n - f_m)(x) \mid x \in \Omega\} < \varepsilon$$

$$\forall \varepsilon > 0 : n, m > N$$

$$\forall x \in \Omega : |(f_n - f_m)(x)| < \varepsilon$$

$$\implies \forall x \in \Omega : (f_n(x))_{n \in \mathbb{N}} \subseteq R$$

is a Cauchy sequence in  $\mathbb{R}$ .

$$\iff \forall x \in \Omega : (f_n(x))_{n \in \mathbb{N}} \text{ converges}$$

$$\forall x \in \Omega : (f_n(x))_{n \in \mathbb{N}} \rightarrow f(x) \forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \implies |f_n(x) - f(x)| < \varepsilon$$

$$\exists N \in \mathbb{N} \forall n > N : \|f_n - f\|_\infty < 1$$

$$\|f\|_\infty = \|f - f_N + f_N\|_\infty \leq \underbrace{\|f - f_N\|_\infty}_{<1} + \underbrace{\|f_N\|_\infty}_{\leq M} < 1 + M$$

## 14.2 Sheet 4, Exercise 3b

Let  $K_1 := \{f \in B(\Omega) \mid \|f\|_\infty \leq 1\}$ . Show  $K_1$  is bounded and closed.

### 14.2.1 $K_1$ is bounded

Let  $f, g \in K_1$  be arbitrary.

$$\|f - g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 1 + 1 = 2$$

2 is a boundary and therefore  $K_1$  is bounded.

### 14.2.2 $K_1$ is closed

Let  $(f_n)_{n \in \mathbb{N}}$  be a convergent sequence in  $K_1$  with  $\lim_{n \rightarrow \infty} f_n = f \iff \lim_{n \rightarrow \infty} \|f_n - f\| = 0$ .

Show  $f \in K_1$ .

$$\begin{aligned} & \forall f_n \in K_1 : \|f_n\|_\infty \leq 1 \\ \|f\|_\infty &= \|f - f_n\|_\infty \leq \underbrace{\|f - f_n\|_\infty}_{\xrightarrow{n \rightarrow \infty} 0} + \underbrace{\|f_n\|_\infty}_{\leq 1} \leq 1 \\ & \implies \|f\|_\infty \leq 1 \implies f \in K_1 \end{aligned}$$

### 14.3 Sheet 4, Exercise c

$f$  is sequentially compact if and only if  $\Omega$  is finite? Equivalently, every sequence  $(f_n)_{n \in \mathbb{N}} \subseteq K_1$  has a convergent subsequence with limit in  $K_1$ .

Direction  $\implies$ .

Let  $\Omega$  be infinite. Then  $\exists$  a sequence  $(f_n)_{n \in \mathbb{N}}$  without convergent subsequence. We build a sequence  $(f_n)_{n \in \mathbb{N}}$  in  $K_1$ .

Let  $(x_i)_{i \in \mathbb{N}}$  be an arbitrary sequence in  $\Omega$  with  $x_i \neq x_j \forall i \neq j$ .

$$f_n(x) := \begin{cases} 1 & \text{if } x = x_n \\ 0 & \text{else} \end{cases}$$

Then it holds that  $\forall n \neq m$ ,

$$\|f_n - f_m\|_\infty = 1$$

Assume there exists a convergent subsequence in  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  with limit  $f$ .

$$\implies \exists M > 0 : k > M : \|f_{n_k} - f\|_\infty < \frac{1}{2}$$

Let  $k, l > M$  with  $k \neq l$

$$\implies \|f_{n_k} - f_{n_l}\|_\infty \leq \|f_{n_k} - f\|_\infty + \|f_{n_l} - f\|_\infty < \frac{1}{2} + \frac{1}{2} = 1$$

This is a contradiction to  $\|f_n - f_m\|_\infty = 1$ .

Direction  $\impliedby$ .

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $K_1$  without limit. Let  $n \in \mathbb{N}$ .

$$\Omega = \{x_1, \dots, x_n\} \implies |\{f_n(x_1), \dots, f_n(x_n)\}| < \infty$$

Let  $f_n \in K_1 \implies |f_n(x_i)| \leq 1 \forall i \in \{1, \dots, m\} \forall n \in \mathbb{N}$ .

Consider  $x_1 \in \Omega$ .

$$(f_n(x_1)) = y_n^1 \in [-1, 1]$$

$[-1, 1]$  compact  $\implies (y_n^1)_{n \in \mathbb{N}}$  has convergent subsequence  $(y_{n_k}^1)_{k \in \mathbb{N}} \rightarrow \tilde{y}^1$

$$(f_{n_k}(x_1))_{k \in \mathbb{N}} = (y_{n_k}^1)_{k \in \mathbb{N}} \rightarrow \tilde{y}^1 := f(x_1)$$

and this goes on up to

$$(f_n(x_m))_{n \in \mathbb{N}} \rightarrow f(x_m)$$

For every  $\varepsilon > 0$

$$\exists N_1 : \forall n \in N_1 : \left| f_n(x_1) - f(x_1) \right| < \varepsilon$$

$$\exists N_m : \forall n \in N_m : \left| \begin{matrix} \vdots \\ f_n(x_m) - f(x_m) \\ \vdots \end{matrix} \right| < \varepsilon$$

Choose  $N := \max N_1, \dots, N_m$ . For all  $n \geq N$ ,

$$\Rightarrow \left\| \begin{matrix} f_n \\ \vdots \end{matrix} \right\|_{\infty} < \varepsilon$$

## 15 Sheet 4, Exercise 4

**Exercise 25.** Let  $k \in \mathbb{N}$ . Show:  $\exists \phi_k : \sqrt{k\pi} \leq \xi_k \leq \sqrt{(k+1)\pi}$  such that

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2) dx = \frac{(-1)^k}{\xi_k}$$

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2) dx = \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \frac{x \cdot \sin(x^2)}{x} dx = \frac{1}{\xi_k} \cdot \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} x \cdot \sin(x^2) dx$$

But this IVT is unconventional.

$$= \frac{1}{\xi_k} \cdot \left( -\frac{1}{2} \cdot \cos(x^2) \right) \Big|_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}$$

If  $k$  is even:

$$\frac{1}{\xi_k} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{\xi_k}$$

If  $k$  is odd:

$$\frac{1}{\xi_k} \left( -\frac{1}{2} - \frac{1}{2} \right) = -\frac{1}{\xi_k}$$

This implies a boundary of

$$\frac{(-1)^k}{\xi_k}$$

This lecture took place on 2018/04/26.

## 16 Sheet 5, Exercise 1

**Exercise 26.** Let  $\mathcal{R}[a, b]$  be the vector space of real-valued regulated functions on  $[a, b] \subseteq \mathbb{R}$ , hence

$$\mathcal{R}[a, b] := \left\{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is a regulated function} \right\}$$

annotated with a norm  $\|\cdot\|_\infty$  of Sheet 4 Exercise 3. Prove that  $(\mathcal{R}[a, b], \|\cdot\|_\infty)$  is a complete normed vector space with a sequentially non-compact unit sphere.

## 17 Sheet 5, Exercise 2

**Exercise 27.** Let  $f, g \in \mathcal{R}[a, b]$  with

$$f_+(x) = g_+(x) \quad \forall x \in [a, b)$$

$$f_-(x) = g_-(x) \quad \forall x \in (a, b]$$

1. For  $\alpha, \beta \in [a, b]$  :  $\int_\alpha^\beta f(x) dx = \int_\alpha^\beta g(x) dx$  holds.
2. For every antiderivative  $F : [a, b] \rightarrow \mathbb{R}$  of  $f$  there exists an antiderivative  $G : [a, b] \rightarrow \mathbb{R}$  of  $g$  with  $F(x) = G(x)$  for all  $x \in [a, b]$ .

### 17.1 Sheet 5, Exercise 2a

Let  $f, g \in \mathcal{R}[a, b]$ .

$$F'_+(x) := f_+(x) = g_+(x)$$

$$F'_-(x) := f_-(x) = g_-(x)$$

Show:  $\int_\alpha^\beta f(x) dx = \int_\alpha^\beta g(x) dx$ .

In general  $f_+(x) \neq f(x) \neq f_-(x)$ .

$$F := \int f(x) dx$$

$$G := \int g(x) dx$$

$$\int_\alpha^\beta f(x) dx = F|_\alpha^\beta \stackrel{(b)}{=} \underbrace{F(\beta) + K}_{G(\beta)} - \underbrace{(F(\alpha) - K)}_{G(\alpha)} = \int_\alpha^\beta g(x) dx$$

## 17.2 Sheet 5, Exercise 2b

$F$  is an antiderivative of  $f$  if and only if

$$F = \int f(x) dx$$

$$F'_+(x) = f_+(x) = g_+(x) = g_+(x) \quad \forall x \in [a, b)$$

$$F'_-(x) = f_-(x) = g_-(x) = g_-(x) \quad \forall x \in (a, b]$$

## 18 Sheet 5, Exercise 3

**Exercise 28.** 1. Let  $f : [a, b] \rightarrow \mathbb{R}$  continuously differentiable with  $f(x) \neq 0 \forall x \in [a, b]$ . Show that

$$\int_a^b \frac{f'(x)}{f(x)} dx = \ln |f(b)| - \ln |f(a)|$$

2. Determine the value of  $I$  using  $\cos(x) = \frac{1}{2}(\sin x + \cos x + \cos x - \sin x)$

$$I := \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

3. Determine  $I$  using the substitution  $y(x) = \frac{\pi}{2} - x$ .

### 18.1 Sheet 5, Exercise 3a

$$\begin{aligned} \int_a^b \frac{f'(x)}{f(x)} dx &= \left| \frac{t = f(x)}{dt = f'(x) dx} \right| = \int_{f(a)}^{f(b)} \frac{1}{t} dt \\ &= [\ln |t|]_{f(a)}^{f(b)} = \ln |f(b)| - \ln |f(a)| \end{aligned}$$

### 18.2 Sheet 5, Exercise 3b

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x) + \cos(x)} &= \underbrace{\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin(x) + \cos(x)}{\sin(x) + \cos(x)} dx}_{\frac{\pi}{4}} + \underbrace{\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} dx}_{f(x)} \\ &= \frac{\pi}{4} + \ln \left| \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) \right| - \ln |\cos(0) + \sin(0)| \\ &= \frac{\pi}{4} + 0 \end{aligned}$$

### 18.3 Sheet 5, Exercise 3c

$$u(x) = \frac{\pi}{2} - x$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x) + \cos(x)} dx &= \int_{\frac{\pi}{2}}^0 -\frac{\cos(\frac{\pi}{2} - u)}{\sin(\frac{\pi}{2} - u) + \cos(\frac{\pi}{2} - u)} du \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2} - u)}{\sin(\frac{\pi}{2} - u) + \cos(\frac{\pi}{2} - u)} du \\ &= \int_0^{\frac{\pi}{2}} \frac{\sin(u)}{\sin(u) + \cos(u)} du \\ \Rightarrow 2I &= \int_0^{\frac{\pi}{2}} \frac{\sin(u)}{\sin(u) + \cos(u)} du + \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\sin(u) + \cos(u)} du \\ 2I &= \int_0^{\frac{\pi}{2}} \frac{\sin(u) + \cos(u)}{\sin(u) + \cos(u)} du \\ 2I &= \frac{\pi}{2} \iff I = \frac{\pi}{4} \end{aligned}$$

### 19 Sheet 5, Exercise 4

**Exercise 29.** 1. Evaluate using integration by parts:  $\int_0^{\pi} (\sin x)^2 dx$

2. Determine (for  $n \in \mathbb{N}$ ) by integration by parts:  $\int_0^{\frac{\pi}{2}} (\cos x)^{2n} dx$

3. Determine by integration by parts followed by substitution:  $\int_0^1 \log(x+1) dx$

#### 19.1 Sheet 5, Exercise 4a

Let  $u := \sin(x)$ ,  $u' = \cos(x)$ ,  $v' := \sin(x)$  and  $v = -\cos(x)$ .

$$\begin{aligned} \int_0^{\pi} (\sin(x))^2 dx &= [-\sin(x) \cos(x)]_0^{\pi} - \int_0^{\pi} -\cos(x) \cos(x) dx \\ &= \int_0^{\pi} 1 - \sin(x)^2 dx \\ \iff \int_0^{\pi} 2 \cdot \sin(x)^2 dx &= \int_0^{\pi} 1 = \pi \\ &= \frac{\pi}{2} \end{aligned}$$



## 19.2 Sheet 5, Exercise 4b

Let  $n \in \mathbb{N} \setminus \{0\}$ .

$$\int_0^{\frac{\pi}{2}} (\cos(x))^{2n} dx$$

We prove by complete induction: Consider  $n = 0$ .

$$\int_0^{\frac{\pi}{2}} (\cos(x))^{2n} dx = \frac{\pi}{2}$$

Consider  $n - 1 \rightarrow n$ .

$$\int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} dx = \int_0^{\frac{\pi}{2}} \underbrace{\cos(x)^{2n+1}}_u \underbrace{\cos(x)}_{v'} dx$$

$$\int_0^{\frac{\pi}{2}} (\cos(x))^2 = \frac{\pi}{4}$$

$$\begin{aligned} \text{By induction hypothesis } \int_0^{\frac{\pi}{2}} \cos(x)^{2n} dx &= \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos(x)^{2(n-1)} \\ &= \left| \begin{array}{ll} u' &= -(2n+1) \sin(x) \cos(x)^{2n} \\ v &= \sin(x) \end{array} \right| \end{aligned}$$

$$\begin{aligned} [\cos(x)^{2n+1} \cdot \sin(x)]_0^{\frac{\pi}{2}} + (2n+1) \cdot \int_0^{\frac{\pi}{2}} \cos(x)^{2n} \cdot \sin(x)^2 dx &= (2n+1) \cdot \int_0^{\frac{\pi}{2}} \cos(x)^{2n} dx - (2n+1) \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} dx \\ \Rightarrow (2n+2) \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} dx &= (2n+1) \int_0^{\frac{\pi}{2}} \cos(x)^{2n} dx \\ \Rightarrow \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} dx &= \frac{(2n+1)}{2n+2} \int_0^{\frac{\pi}{2}} \cos(x)^{2n} dx \\ &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \dots \cdot \frac{1}{2} \cdot \frac{\pi}{2} \end{aligned}$$

## 19.3 Sheet 5, Exercise 4c

$$\begin{aligned} \int_0^1 x \cdot \log(x+1) dx &= \left| \begin{array}{ll} u' = x & u = \frac{x^2}{2} \\ v = \log(x+1) & v' = \frac{1}{1+x} \end{array} \right| \\ \left[ \frac{x^2}{2} \log(x+1) \right]_0^1 - \int_0^1 \left( \frac{x^2}{2} \cdot \frac{1}{1+x} \right) dx & \quad u(x) = 1+x \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{x^2}{2} \log(x+1) \right]_0^1 - \frac{1}{2} \underbrace{\int_1^2 (u-1)^2 \cdot \frac{1}{u} du}_{\int_1^2 \left( \frac{u^2+1-2u}{u} \right) du = \int_1^2 u + \frac{1}{u} - 2 du} \\
&\quad \frac{\log(2)}{2} - \frac{1}{2} \left[ \frac{u^2}{2} + \log(u) - 2u \right]_1^2 = \frac{1}{4}
\end{aligned}$$

It is valid to assume that  $\log = \ln$  in this exercise, because it is not specified otherwise. But you can also consider a factor  $a$ , which normalizes it to  $\ln$ .

## 20 Sheet 6, Exercise 1

**Exercise 30.** Let  $\mathcal{R}[a, b]$  be the set of regulated functions,  $C[a, b]$  be the set of continuous functions and  $\mathcal{M}[a, b]$  be the set of monotonic functions on  $[a, b] \subset \mathbb{R}$ . Show:

1.  $f \in C[a, b] \implies f \in \mathcal{R}[a, b]$
2.  $f \in \mathcal{M}[a, b] \implies f \in \mathcal{R}[a, b]$
3.  $f \in C[a, b], g \in \mathcal{R}[a, b] \wedge g([a, b]) \subset [a, b] \implies f \circ g \in \mathcal{R}[a, b]$

### 20.1 Sheet 6, Exercise 1a

Assume  $f \in C[a, b]$ . For all  $x \in [a, b]$ ,  $f$  has one-sided limits.

### 20.2 Sheet 6, Exercise 1b

Let  $x \in [a, b]$ . Consider  $x_{n \in \mathbb{N}} \nearrow x$ . Show that  $\lim_{n \rightarrow \infty} f(x_n)$  exists. We consider a monotonic subsequence

$$\begin{aligned}
f(x_{n_k}) &\geq f(x_{n_{k+1}}) \forall k \in \mathbb{N} \\
f(x) &\leq f(x_{n_k}) \forall k \in \mathbb{N}
\end{aligned}$$

### 20.3 Sheet 6, Exercise 1c

$(x_n)_{n \in \mathbb{N}} \nearrow x$ .

$$\begin{aligned}
&\lim_{n \rightarrow \infty} f(g(x_n)) \text{ exists} \\
&\lim_{n \rightarrow \infty} \underbrace{g(x_n)}_{=: y_n} = y \in \mathbb{R} \\
&\lim_{n \rightarrow \infty} f(y_n) = f(\lim_{n \rightarrow \infty}) = f(\lim_{n \rightarrow \infty} y_n) \text{ TODO}
\end{aligned}$$

$g : [a, b] \rightarrow [a, b]$ .  $f \in \mathcal{R}[a, b]$ ,  $g \in C([a, b])$ ,  $g([a, b]) \subset [a, b]$ .

## 21 Sheet 6, Exercise 2

**Exercise 31.** Determine all antiderivatives:

$$\int \frac{1}{x(\ln x)^3} dx \quad (x > 0) \quad (1)$$

$$\int \sin^3(x) \cos^4(x) dx \quad (2)$$

$$\int \operatorname{arsinh}(x) dx \quad (3)$$

### 21.1 Sheet 6, Exercise 2a

We apply integration by substitution:

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b f(g(u)) \cdot g'(u) du$$

We consider:

$$f(x) = \left(\frac{1}{x^3}\right) = \frac{1}{x^3}$$

$$g(x) = \ln(x) \quad g'(x) = \frac{1}{x}$$

$$\int \frac{1}{x(\ln x)^3} dx = \int \left(\frac{1}{u^3}\right) du = \int u^{-3} du = \frac{u^{-2}}{-2} + c = \frac{1}{-2 \cdot u^2} + c = \frac{1}{-2 \cdot \ln(x)^2} + c$$

**Hint.** Because we apply Backsubstitution, we do what we usually do by computing the integral over some specified limits. Therefore the improper integral is exact as well.

## 21.2 Sheet 6, Exercise 2b

$$\begin{aligned}
 \int \sin(x)^2 \cdot \sin(x) \cdot \cos(x)^4 dx &= \int (1 - \cos(x)^2) \cdot \cos(x)^4 \cdot \sin(x) dx \\
 &= \int (\cos(x)^4 - \cos(x)^6) \cdot \sin(x) dx \\
 &\quad \left| \begin{array}{l} u = \cos(x) \\ u' = -\sin(x) \\ du = dx \cdot u' \end{array} \right| \\
 &= \int (u^4 - u^6) \cdot (-1) du = \int (-u^4 + u^6) du \\
 &= \frac{u^7}{7} - \frac{u^5}{5} + c = \frac{\cos(x)^7}{7} - \frac{\cos(x)^5}{5} + c
 \end{aligned}$$

## 21.3 Sheet 6, Exercise 2c

$$\begin{aligned}
 \int \operatorname{arsinh}(x) dx &= \int \ln(x + \sqrt{x^2 + 1}) dx \\
 &\quad \left| \begin{array}{l} u = \ln(x + \sqrt{x^2 + 1}) \\ v' = 1 \\ v = x \\ u' = \frac{1}{\sqrt{x^2 + 1}} \end{array} \right| \\
 &= \ln(x + \sqrt{x^2 + 1})x - \int \frac{1}{\sqrt{x^2 + 1}} x dx \\
 &\quad \left| \begin{array}{l} u = x^2 + 1 \\ u' = 2x \\ du = 2x dx \end{array} \right| \\
 &= \operatorname{arsinh}(x) \cdot x - \int \frac{1}{\sqrt{u}} \frac{1}{2} du \\
 &= \operatorname{arsinh}(x) \cdot x - \sqrt{u} + c \\
 &= \operatorname{arsinh}(x) \cdot x - \sqrt{x^2 + 1} + c
 \end{aligned}$$

## 22 Sheet 6, Exercise 3

**Exercise 32.** For  $a = 0$  and  $a > 0$ , determine all antiderivatives:

$$\int \frac{\ln(x)}{\sqrt{a+x}} dx \quad (x > 0)$$

Case  $a = 0$ :

$$\begin{aligned}
 \int \frac{\ln(x)}{\sqrt{x}} \left| \begin{array}{l} u' = \frac{1}{\sqrt{x}} \quad u = 2\sqrt{x} \\ v = \ln(x) \quad v' = \frac{1}{x} \end{array} \right| \\
 = \ln(x) \cdot 2\sqrt{x} \dots \\
 = \ln(x) \cdot \sqrt{x} - 4\sqrt{x} + c
 \end{aligned}$$

Case  $a > 0$ :

$$\begin{aligned}
 \int \frac{\ln(x)}{\sqrt{x+a}} &= \int \frac{\ln(x)}{\sqrt{x+a}} \cdot 2\sqrt{x+a} du \\
 \left| \begin{array}{l} u = \sqrt{x+a} \\ \frac{du}{dx} = \frac{1}{2\sqrt{x+a}} \implies dx = 2\sqrt{x+a} du \\ u = \sqrt{x+a} \implies x = u^2 - a \end{array} \right| \\
 &= 2 \int \ln(x) du \\
 &= 2 \ln(u^2 - a) du \\
 &= 2 \int \ln(u + \sqrt{a}) + \ln(u - \sqrt{a}) du \\
 &= 2 \left( \int (u + \sqrt{a}) du + \int \ln(u - \sqrt{a}) du \right)
 \end{aligned}$$

We compute separately:

$$\begin{aligned}
 \int \ln(x+c) dx &= \int 1 \cdot \ln(x+c) dx \\
 \left| \begin{array}{l} u' = 1 \implies u = x \\ v = \ln(x+c) \implies v' = \frac{1}{x+c} \end{array} \right| \\
 &= x \ln(x+c) - \int \frac{x+c-c}{x+c} \\
 &= x \ln(x+c) - x + c \ln(x+c) \\
 &= (x+c) \ln(x+c) - x + c
 \end{aligned}$$

with

$$\int \frac{x+c}{x+c} - \frac{c}{x+c} = \int 1 - \frac{c}{x+c} = x - c \ln(x+c) + c$$

We continue:

$$\begin{aligned}
&= 2((u + \sqrt{a}) \ln(u + \sqrt{a}) - (u + \sqrt{a}) + (u - \sqrt{a}) \ln(u - \sqrt{a}) - (u - \sqrt{a})) + c \\
&= 2(u \ln(u^2 - a) + \sqrt{a} \ln\left(\frac{u + \sqrt{a}}{u - \sqrt{a}}\right) - 2u) + c \\
&= 2\sqrt{x+a} \ln(x) + \sqrt{a} \ln\left(\frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} - \sqrt{a}}\right) - 4\sqrt{x+a} + c
\end{aligned}$$

## 23 Sheet 6, Exercise 4

**Exercise 33.** Let  $k \in \mathbb{Z}$ ,  $I_k := ((2k-1)\pi, (2k+1)\pi)$  and

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) := \frac{1}{3 \cos(x) + 5}$$

1. Prove for all  $x \in I_k$  the identity

$$\cos(x) = \frac{1 - \tan(x/2)^2}{1 + \tan(x/2)^2}$$

2. Determine all antiderivatives:

$$\int f(x) dx, x \in I_k$$

Begin by integration by substitution with  $u(x) = \tan(\frac{x}{2})$ .

3. Construct a continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$ , that is an antiderivative of  $f$  on every compact interval.

### 23.1 Sheet 6, Exercise 4a

$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos(x)}$$

Proof: Let  $u = \frac{x}{2}$  and  $x = 2u$ .

$$\tan(u) = \frac{\sin 2u}{1 + \cos(2u)} = \frac{2 \sin(u) \cos(u)}{1 + \cos^2(u) - \sin^2(u)} = \frac{2 \sin(u) \cos(u)}{2 \cos^2(u)} = \frac{\sin(u)}{\cos(u)} = \tan(u)$$

Then,

$$\begin{aligned}
\frac{1 - \tan(x/2)^2}{1 + \tan(x/2)^2} &= \frac{1 - \frac{\sin^2(x)}{1 + \cos(x)}}{1 + \frac{\sin^2(x)}{(1 + \cos(x))^2}} \\
&= \frac{(1 + \cos(x))^2 - \sin^2(x)}{(1 + \cos(x))^2 + \sin^2(x)} \\
&= \frac{1 + 2\cos(x) + \cos(x)^2 - \sin(x)}{1 + 2\cos(x) + \underbrace{\cos(x)^2 + \sin^2(x)}_{=1}} \\
&= \frac{2\cos(x)(1 + \cos(x))}{2(1 + \cos(x))} \\
&= \cos(x)
\end{aligned}$$

## 23.2 Sheet 6, Exercise 4b

Let  $x \in I_k$ .

$$\begin{aligned}
\int f(x) dx &= \int \frac{1}{3\cos(x) + 5} dx \\
&= \int \frac{1}{3\left(\frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}\right) + 5} dx \\
&\quad \left| \begin{array}{l} u = \tan(x/2) \\ du = \frac{1}{2\cos^2(x/2)} dx \end{array} \right| \\
&= \int \frac{1}{3\left(\frac{1 - u^2}{1 + u^2} + 5\right)} 2\cos^2(x/2) du \\
&= 2 \int \frac{1}{\left(3\left(\frac{1 - u^2}{1 + u^2} + 5\right) + 5\right)(1 + u^2)} du
\end{aligned}$$

$$\cos(x) = \frac{1}{1 + \tan^2(x)}$$

We compute separately:

$$\begin{aligned}
&\left(\frac{3(1 - u^2) + 5}{1 + u^2} + 5\right)(1 + u^2) = \frac{3(1 - u^2)}{1 + u^2}(1 + u^2) + 5(1 + u^2) = 2(4 + u^2) \\
&= 2 \int \frac{1}{2} \frac{1}{4 + u^2} du = \int \frac{1}{4 + u^2} du = \left| \frac{t = \frac{u}{2}}{dt = \frac{1}{2} du} \right| = 2 \int \frac{1}{4 + 4t^2} dt = \frac{2}{4} \int \frac{1}{1 + t^2} dt \\
&\frac{1}{2} \arctan(t) + c = \frac{1}{2} \arctan\left(\frac{u}{2}\right) + c = \frac{1}{2} \arctan\left(\frac{\tan(x/2)}{2}\right) + c
\end{aligned}$$

Is expected to be continuously differentiable.

## 24 Sheet 7, Exercise 1

**Exercise 34.** Use the direct comparison criterion to determine the convergence of these integrals:

$$(a) \int_1^{\infty} \frac{1}{x^2 + 5x + 1} dx \quad (b) \int_0^{\infty} \frac{1}{x^s + x^{\frac{1}{s}}} dx \quad s \in \mathbb{R} \setminus \{0\}$$

### 24.1 Sheet 7, Exercise 1a

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 5x + 1} dx &\leq \int_1^{\infty} \frac{1}{x^2} dx \\ \int_1^{\infty} \frac{1}{x^p} < \infty &\iff p > 1 \end{aligned}$$

### 24.2 Sheet 7, Exercise 1b

**Case  $s = 1$**

$$\begin{aligned} \int_0^{\infty} \frac{1}{x + x} dx &= \frac{1}{2} \int_0^{\infty} \frac{1}{x} dx = \frac{1}{2} \left( \int_0^1 \frac{1}{x} dx + \int_1^{\infty} \frac{1}{x} dx \right) \\ &= \frac{1}{2} \lim_{t \rightarrow \infty} \int_0^t \frac{1}{x} dx \\ &= \frac{1}{2} \left( \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx + \lim_{t \rightarrow \infty} \int_t^1 \frac{1}{x} dx \right) \end{aligned}$$

**Case  $s < 0$**

$$\int_0^{\infty} \frac{1}{x^s + x^{\frac{1}{s}}} dx$$

Because  $s < 0$ ,  $x^s + x^{\frac{1}{s}}$  is monotonically decreasing and positive.

$$\frac{1}{x^s + x^{\frac{1}{s}}}$$

is monotonically increasing. More specifically:

$$\begin{aligned} \int_0^1 \underbrace{\frac{1}{x^s + x^{\frac{1}{s}}}}_{\geq 0} + \int_1^{\infty} \underbrace{\frac{1}{x^s + x^{\frac{1}{s}}}}_{\geq 1} dx \\ \int_1^{\infty} 1 dx = \infty \end{aligned}$$



## 25 Sheet 7, Exercise 2

**Exercise 35.** Prove the following statements:

1.  $\forall k \in \mathbb{N} \cup \{0\} : \int_{k\pi}^{(k+1)\pi} |\text{sinc}(x)| \, dx \geq \frac{2}{(k+1)\pi}$ .
2. The improper integral  $\int_0^\infty |\text{sinc}(x)| \, dx$  does not exist.

### 25.1 Sheet 7, Exercise 2a

We apply the Mean Value Theorem:

$$\begin{aligned} \exists \xi \in [k\pi, (k+1)\pi] : I &= \frac{1}{\xi} \int_{k\pi}^{(k+1)\pi} |\sin(x)| \, dx \\ \int_{k\pi}^{(k+1)\pi} |\sin(x)| \, dx &= \left| \int_{k\pi}^{(k+1)\pi} \sin(x) \, dx \right| = \left| -\cos(x) \Big|_{k\pi}^{(k+1)\pi} \right| = 2 \\ \implies I &= \frac{1}{\xi} 2 \geq \frac{2}{(k+1)\pi} \quad \forall n \in \mathbb{N} \\ \underbrace{\text{sinc}(0)}_{=1} &\geq \overbrace{\frac{\sin(0)}{\pi}}^{=0} \end{aligned}$$

Let  $k = 0$ :

$$\int_0^\infty \text{sinc}(x) \, dx \xRightarrow{\text{for } x \neq 0} \text{sinc}(x) = \frac{\sin(x)}{x} \geq \frac{\sin(x)}{\pi} \quad \forall x \in (0, \pi]$$

We can exclude the case  $x = 0$ , because individual finitely many values don't matter.

$$\begin{aligned} &\geq \int_0^\pi \frac{\sin(x)}{\pi} \, dx = \frac{2}{\pi} = \frac{2}{(k+1)\pi} \\ \implies \text{sinc}(x) &\geq \frac{\sin(x)}{\pi} \quad \forall x \in [0, \pi] \end{aligned}$$

### 25.2 Sheet 7, Exercise 2b

Sketch of the proof (but it lacks details acc. to the tutor)

$$\int_0^\infty |\sin(x)| \, dx = \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} |\text{sinc}(x)| \, dx$$

$$\geq \lim_{n \rightarrow \infty} \sum_{k=0}^N \underbrace{\frac{2}{(k+1)\pi}}_{\rightarrow \infty} = \sum_{k=0}^{\infty} \frac{2}{k\pi + \pi}$$

$$\lim_{N \rightarrow \infty} \frac{2}{\pi} \sum_{k=1}^N \frac{1}{k}$$

$$\int_{k\pi}^{(k+1)\pi} |\sin(x)| \geq \frac{2}{(k+1)\pi}$$

We add some details:

$$\lim_{N \rightarrow \infty} \sum_{k=0}^N T_n \cdot \Delta x =: \int f$$

$$\int_a^b f dx + \int_b^c f dx = \int_a^c f dx$$

$$\lim_{R \rightarrow \infty} \int_0^{R\pi} |\sin(x)| dx \geq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} |\sin(x)| dx$$

## 26 Sheet 7, Exercise 3

Exercise 36.

## 27 Sheet 7, Exercise 4

**Exercise 37.** Let  $n \in \mathbb{N}$ . For  $k \in \{0, 1, \dots, n\}$ , we define  $x_k := \frac{k}{n}$  and the step function

$$T_n : [0, 1] \rightarrow \mathbb{R} \quad T_n(x) := \begin{cases} x_k^2 & \text{if } x \in [x_{k-1}, x_k) \\ 1 & \text{if } x = 1 \end{cases}$$

1. Show that: For every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $\|T_n(x) - x^2\|_\infty < \varepsilon$  for all  $n \geq N$ .
2. Determine  $\int_0^1 x^2 dx$  using sequence  $(T_n)_{n \in \mathbb{N}}$ .

### 27.1 Sheet 7, Exercise 4a

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n \geq N : \|T_n(x) - x^2\|_\infty < \varepsilon$$

$$\|T_n(x) - x^2\|_\infty = 1 - x_{n-1}^2$$

$$\|T_n(x) - x^2\|_\infty = 1 - x_{n-1}^2 = 1 - \left(\frac{n-1}{n}\right)^2 = \frac{2n-1}{n^2}$$

$$1. \forall x \in [x_{k-1}, x_k] : |T_n(x) - x^2| \leq x_k^2 - x_{k-1}^2 = \left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2 = \frac{2k-1}{n^2}.$$

Remark: Also  $\frac{2k-1}{n^2} \leq \frac{2n-1}{n^2} \rightarrow 0$ .

Remark:  $x_k^2 - x_{k-1}^2 = (x_k - x_{k-1})(x_k + x_{k-1}) = \frac{1}{n}\delta$  with  $0 \leq \delta \leq 2$ .

$$2. \forall k \in \{0, 1, \dots, n-2\} : x_{k+1}^2 - x_k^2 < x_{k+2}^2 - x_{k+1}^2$$

$$\begin{aligned} \left(\frac{k+1}{n}\right)^2 - \left(\frac{k}{n}\right)^2 &= \frac{k^2 + 2k + 1 - k^2}{n^2} < \frac{2k+3}{n^2} \\ &= \frac{k^2 + 4k + 4 - (k^2 + 2k + 1)}{n^2} = \left(\frac{k+2}{n}\right)^2 - \left(\frac{k+1}{n}\right)^2 \end{aligned}$$

## 27.2 Sheet 7, Exercise 4b

$$\int_0^1 x^2 dx$$

By exercise (4a), it follows that  $\lim_{n \rightarrow \infty} \|T_n - x^2\|_\infty = 0$ .

$$\int_0^1 x^2 dx = \lim_{n \rightarrow \infty} \int_0^1 T_n(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n x_k^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{1}{3}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{6} \cdot \left[1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)\right]$$

The integral is independent of the particular chosen approximating sequence (see lecture notes).

## 27.3 Remark on integrals

You are allowed to change a regulated function in countable infinite many points. Its limit won't change.

$$\begin{aligned} &\int_a^b f dx \\ \tilde{f} &:= \begin{cases} f(x) & x \in (a, b] \\ 0 & x = a \end{cases} \end{aligned}$$

Then  $\int_a^b f dx = \int \tilde{f} dx$ .

This lecture took place on 2018/05/24.

## 28 Sheet 8, Exercise 1

**Exercise 38.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) := \cosh(2x)$ .

1. Determine  $f^{(n)}(x)$  and  $T_f^n(x; 0)$  for  $n \geq 0$  and furthermore  $T_f(x; 0)$
2. Show that for all  $x \in \mathbb{R}$  it holds that  $R_f^{n+1}(x; 0) \rightarrow 0$  for  $n \rightarrow \infty$ . You can use the Lagrange representation of the Taylor remainder  $R_f^{n+1}$ .

### 28.1 Sheet 8, Exercise 1a

$$\begin{aligned} T_f^n(x; 0) &= \sum_{k=0}^n \frac{1}{k!} f^{(k)}(0) x^k = \sum_{\substack{k=0 \\ k \text{ even}}}^n \frac{1}{k!} \underbrace{2^k \cosh(0)}_{=1} x^k + \sum_{\substack{k=0 \\ k \text{ odd}}}^n \frac{1}{k!} \underbrace{2^k \sinh(0)}_{=0} x^k \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^n \frac{2^k}{k!} x^{-k} \\ T_f(x; 0) &= \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k} \end{aligned}$$

### 28.2 Sheet 8, Exercise 1b

$$\begin{aligned} R_p^{n+1}(x; 0) &= \frac{1}{(n+1)!} f^{n+1}(\xi) x^{n+1} \\ \xi &\in (x, 0) \cup (0, x) \\ \left| \frac{1}{(n+1)!} f^{n+1}(\xi) x^{n+1} \right| &\leq \frac{x^{n+1}}{(n+1)!} 2^{n+1} |\cosh(2\xi)| \\ &\leq \frac{|x^{n+1}|}{(n+1)!} \underbrace{2^{n+1} \cosh(2x)}_{\text{constant}} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

### 28.3 Sheet 8, Exercise 1c

$$T_f(x; 0) = \sum_{k=0}^{\infty} \frac{2^{(2k)}}{(2k)!} x^{2k}$$

$$\underbrace{\lim_{n \rightarrow \infty} R_f^{n+1}(x; 0)}_0 = \lim_{n \rightarrow \infty} (f(x) - T_f^n(x; 0)) = f(x) - T_f(x; 0)$$

$$0 = f(x) - \lim_{n \rightarrow \infty} T_f^n(x; 0)$$

with  $\lim_{n \rightarrow \infty} (f(x) - T_f^n(x; 0)) = \lim_{n \rightarrow \infty} T_f^n(x; 0)$ . As  $\lim_{n \rightarrow \infty} (f(x) - T_f^n(x; 0))$  converges, it holds that  $\lim_{n \rightarrow \infty} (f(x) - T_f^n(x; 0)) = \lim_{n \rightarrow \infty} f(x) - \lim_{n \rightarrow \infty} T_f^n(x; 0)$ . So we do not need to show convergence of  $\lim_{n \rightarrow \infty} T_f^n(x; 0)$ .

## 28.4 Sheet 8, Exercise 1d

Show that

$$\left| f(x) - T_f^8(x; 0) \right| < \frac{|x|^9}{700} |\sinh(2x)| < \frac{|x|^9}{1400} e^{2|x|}$$

$$\begin{aligned} \left| R_f^9(x; 0) \right| &= \frac{1}{9!} \left| f^{(9)}(\xi) x^9 \right| \quad \xi \in (0, x) \vee (x, 0) \\ &= \frac{|x|^9}{9!} 2^9 |\sinh(2\xi)| < \frac{|x|^9}{700} |\sinh(2\xi)| \end{aligned}$$

Show:

1.

$$\begin{aligned} \frac{2^9}{9!} &\stackrel{!}{<} \frac{1}{700} \\ \frac{2 \cdot 2^2 \cdot 2^3 \cdot 2^3 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} &= \frac{4}{81 \cdot 35} < \frac{4}{80 \cdot 35} = \frac{1}{700} \end{aligned}$$

2.

$$\begin{aligned} |\sinh(2\xi)| &< |\sinh(2x)| \\ x > 0 &\implies \xi > 0 \end{aligned}$$

because of monotonicity.

$$x < 0 \implies x < \xi < 0$$

## 28.5 Sheet 8, Exercise 1e

$$\begin{aligned} \sinh(2x) &= \frac{1}{2} \left( \underbrace{e^{2x}}_{>0} - \underbrace{e^{-2x}}_{>0} \right) < \frac{1}{2} e^{2x} \\ \frac{1}{2} |e^{2x}| &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned}
x < 0 : |\sinh(2x)| &< +\frac{e^{2|x|}}{2} \quad \forall x \in [-1, 1] \\
\left| f(x) - T_f^8(x; 0) \right| &< \frac{6}{1000} \quad x = 0 \\
\left| f(x) - T_f^8(x; 0) \right| &< \frac{|x|^9}{1400} e^{2|x|} \\
\frac{|x|^9}{1400} e^{2|x|} &\leq \frac{|x|^9}{1400} e^2 < \frac{2.8^2}{1400} = \frac{28 \cdot 26}{140000} = \frac{7}{1250} < \frac{6}{1000}
\end{aligned}$$

## 29 Sheet 8, Exercise 2

**Exercise 39.** Let  $n \in \mathbb{N} \cup \{0\}$ ,  $a > 0$ ,  $I := [-a, a]$  and  $f : I \rightarrow \mathbb{R}$   $n$ -times differentiable.

1. Show: If  $f$  is even, i.e.  $f(x) = f(-x) \forall x \in I$ ,  $T_f^n(x; 0)$  is even
2. Show: If  $f$  is odd, i.e.  $f(x) = -f(-x) \forall x \in I$ ,  $T_f^n(x; 0)$  is odd
3. Prove that the inverse statements of (a) and (b) are wrong. Use  $g : I \rightarrow \mathbb{R}$ ,  $g(x) := x^{n+1}$  for  $x > 0$ ,  $g(x) = 0$ .
4. Prove that a and b also hold for  $T_f(x; 0)$  instead of  $T_f^n(x; 0)$  if  $f$  is arbitrary often differentiable.
5. Show that the inverse of statements (a) and (b) are also wrong for  $T_f(x; 0)$  instead of  $T_f^n(x; 0)$ , if  $f$  is arbitrarily often differentiable.

### 29.1 Sheet 8, Exercise 2a

$$T_f^n = \ln(x_0) + \sum_{k=1}^n \underbrace{\frac{(-1)^{k+1}}{x_0^k \cdot k}}_{a_k} (x - x_0)^k$$

### 29.2 Sheet 8, Exercise 2b

$$\text{Cauchy-Hadamard} \implies \rho = \left( \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} \right)^{-1}$$

Area of convergence:  $(0, 2x)$

Outside the area of convergence, the series diverges.

$$\left( \limsup_{k \rightarrow \infty} \frac{1}{|x_0| \cdot \sqrt[k]{k}} \right)^{-1} = \left( \frac{1}{|x_0|} \right)^{-1} = x_0$$

Consider  $x = 2x_0$ :

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{x_0^k \cdot k} \cdot x_k^k \implies \text{converges}$$

Consider  $x = 0$ :

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}(-1)}{x_0^k \cdot k} x_0^k \implies \text{diverges}$$

Thus, the actual area of converge is  $(0, 2x_0]$ .

### 29.3 Sheet 8, Exercise 2c

Show that:

$$\lim_{n \rightarrow \infty} R_f^{n+1}(x; x_0) = 0$$

$$\begin{aligned} \left| R_f^{n+1}(x; x_0) \right| &= \left| \frac{1}{n!} \int_{x_0}^x (x-t)^n \cdot f^{(n+1)}(t) dt \right| \\ &= \left| \frac{1}{n!} \int_{x_0}^x (x-t)^n \frac{(-1)^n n!}{t^{n+1}} dt \right| \\ &= \left| \int_{x_0}^x \frac{(x-t)^n}{t^{n+1}} dt \right| \\ &= \left| \int_{x_0}^x \frac{1}{t} \cdot \underbrace{\left( \frac{x}{t} - 1 \right)}_{=: q}^n dt \right| \end{aligned}$$

$$\begin{aligned} &\sup \left\{ \left| \frac{x}{t} - 1 \right|_* \right\} \\ &t \in [x_0, x] \\ &x \in [x_0, 2x_0) \\ &= \underbrace{\frac{x}{x_0}}_{<2} - 1 < 1 \end{aligned}$$

Whence, consider  $x = x_0$ ,

$$\left| \int_{x_0}^x \frac{1}{t} \cdot (q)^n dt \right| \leq |\tilde{q}^n| \cdot |\ln(x) - \ln(x_0)| \xrightarrow{n \rightarrow \infty} 0$$

The identity in the assignment implies that  $T_f(x; x_0)$  converges.

$T_f(x; x_0)$  does not converge at  $x = 0$ .

### 30 Sheet 8, Exercise 3

**Exercise 40.** Let  $n \in \mathbb{N} \cup \{0\}$ ,  $a > 0$ ,  $I := [-a, a]$  and  $f : I \rightarrow \mathbb{R}$   $n$ -times differentiable.

1. Show: If  $f$  is even, i.e.  $f(x) = f(-x) \forall x \in I$ ,  $T_f^n(x; 0)$  is even.
2. Show: If  $f$  is odd, i.e.  $f(x) = -f(-x) \forall x \in I$ ,  $T_f^n(x; 0)$  is odd.

#### 30.1 Sheet 8, Exercise 3a

$f(x)$  is even, then  $f'(x)$  is odd  $\iff f'(x) = -f'(-x)$ . How?

$$f(x) = f(-x) \iff f(x) = f((-1) \cdot (x)) \implies f'(x) = -f'(-x)$$

$$\begin{aligned} f(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ T_f^n(x; 0) &= \sum_{k=0}^n \frac{1}{k!} f^{(k)}(0) \cdot x^k \\ &= \sum_{\substack{k=0 \\ k \bmod 2=0}}^n \frac{1}{k!} f^{(k)}(0) \cdot x^k + \sum_{\substack{k=1 \\ k \bmod 2=1}}^n \overbrace{\frac{1}{k!} f^{(k)}(0) \cdot x^k}^{=0} \\ &= \sum_{\substack{k=0 \\ k \bmod 2=0}}^n \frac{1}{k!} f^{(k)}(0) \cdot (x)^k = T_f^n(-x, 0) \end{aligned}$$

#### 30.2 Sheet 8, Exercise 3b

Analogous to Exercise 3a.



### 30.3 Sheet 8, Exercise 3c

$$\begin{aligned} g : I &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} x^{n+1} & x > 0 \\ 0 & x \leq 0 \end{cases} \\ \sum_{k=0}^n \frac{1}{k!} g^{(k)}(0) \cdot x^k \\ g^{(0)} &= 0 \quad g^{(k)}(0) = 0 \forall k \leq n \end{aligned}$$

Do not skip to show that  $x = 0$  in all derivatives is zero.

### 30.4 Sheet 8, Exercise 3d

$$\begin{aligned} f(x) = f(-x) &\stackrel{!}{\implies} T_f(x_0) = T_f(-x, 0) \\ T_f(x, 0) &= \lim_{n \rightarrow \infty} T_f^n(x, 0) = \lim_{n \rightarrow \infty} (T_f^n(-x, 0)) \end{aligned}$$

This implies that the Taylor series converges.

### 30.5 Sheet 8, Exercise 3e

Find a function that is differentiable infinitely often, is even and odd and  $f(0) = 0$ .

$$k(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

## 31 Sheet 9, Exercise 3

### 31.1 Sheet 9, Exercise 3a

$$n \in \mathbb{N}, t \in \mathbb{R} : \frac{1}{1+t^2} = \frac{(-t^2)^{n+1}}{1+t^2} + \sum_{k=0}^n (-t^2)^k$$

Let  $z := -t^2$  and we are done (the domains of  $-t^2$  and  $z$  also match).

### 31.2 Sheet 9, Exercise 3b

We already know:

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

By the fundamental theorem of differential and integration calculus, variant 1:

$$\Rightarrow \int_0^x \left( \frac{d}{dt} \arctan(x) \right) dt = \arctan(x)$$

$$\begin{aligned} \arctan(x) &= \int_0^x \left[ \frac{(-t^2)^{n+1}}{1+t^2} + \sum_{k=0}^n (-t^2)^k \right] \\ &= \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt + \sum_{k=0}^n \int_0^x (-t^2)^k dt \\ &= \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt + \sum_{k=0}^n (-1)^k \frac{x^{2k+1}}{2k+1} \end{aligned}$$

### 31.3 Sheet 9, Exercise 3c

$$\forall x \in [-1, 1] : \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

Show that  $\lim_{n \rightarrow \infty} \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt = 0$ .

$$\begin{aligned} \left| \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} dt \right| &\leq \left| \int_0^x (-t^2)^{n+1} dt \right| = \left| \int_0^x |-t^2|^{n+1} dt \right| \\ &= \left| \int_0^x t^{2n+2} dt \right| = \left| \frac{t^{2n+3}}{2n+3} \Big|_0^x \right| = \frac{|x|^{2n+3}}{2n+3} \leq \frac{1}{2n+3} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

### 31.4 Sheet 9, Exercise 3d

$$1 - \frac{1}{3} + \frac{1}{5} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 1^{2k+1}}{2k+1} = \arctan(1)$$

## 32 Sheet 9, Exercise 4

### 32.1 Sheet 9, Exercise 4a

$$P(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad Q(z) = \sum_{k=0}^{\infty} a_k (f(z) - z_0)^k$$

$$f : \mathbb{C} \rightarrow \mathbb{C} \quad \hat{z} \mapsto \bar{z}$$

Show that  $P(z)$  converges  $\iff Q(\hat{z})$  converges.

$$\begin{aligned} P(z) &= \sum_{k=0}^{\infty} a_k (z - z_0)^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k (\bar{z} - z_0)^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k (f(\hat{z}) - z_0)^k \\ &= \sum_{k=0}^{\infty} a_k (f(\hat{z}) - z_0)^k = Q(\hat{z}) \end{aligned}$$

### 32.2 Sheet 9, Exercise 4b

$$P(z) = \sum_{k=0}^{\infty} z^k \quad Q(z) = \sum_{k=0}^{\infty} (-1)^k (z^2)^k = \sum_{k=0}^{\infty} (-z^2)^k$$

with  $P(z) = \frac{1}{1-z}$ .

$$f(z) = -z^2$$

$$|z| < 1 \iff |z^2| < 1 \iff |-z^2| < 1$$

### 32.3 Sheet 9, Exercise 4c

Determine the root function of  $Q(z)$  with  $z \in \mathbb{R}$ .

$$\int Q(z) dz = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} + C$$

These are all root functions. But are these all root functions? Yes. There is some  $C$  such that this integral becomes  $\arctan$ , specifically  $C = 0$ .

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan(z) \quad \forall z \in (-1, 1)$$

## 33 Sheet 9, Exercise 2

### 33.1 Sheet 9, Exercise 2a

$$\begin{aligned}f &: \mathbb{R} \setminus \{-1, 2\} \rightarrow \mathbb{R} \\f(x) &= \frac{x+3}{x^2-x-2} \\&= \frac{-\frac{2}{3}}{x+1} + \frac{\frac{5}{3}}{x-2} \\f^{(n)}(x) &= \frac{\frac{2}{3}n! \cdot (-1)^n}{(x+1)^{n+1}} + \frac{\frac{5}{3}n!(-1)}{(x-2)^{n+1}} \\T_f^n(x; 0) &= \sum_{k=0}^n \frac{-\frac{2}{3}k!(-1)^k + \frac{5}{3}k!(-1)^k}{(-2)^{k+1}} k! \cdot x^k\end{aligned}$$

### 33.2 Sheet 9, Exercise 2b

$$T_f^2(x; 0) = -\frac{3}{2} + \frac{x}{4} - \frac{7}{8}x^2$$

$\xi \in (0, x)$ .

$$R_3 = \left| \frac{f^3(\xi)}{3!} x^3 \right| = \left| \frac{\frac{4}{(\xi+1)^4} - \frac{10}{(\xi-2)^4}}{6} x^3 \right| = \frac{\frac{5}{(\xi-2)^4} - \frac{2}{(\xi+1)^4}}{3} x^3$$