Analysis 2 Practicals

Notes, University (of Technology) Graz
based on the lecture by Wolfgang Ring

Lukas Prokop

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Contents

1 Practicals

- Florian Kruse
- Analysis 2 practicals, every Thu, 15:00–16:30
- Sprechstunde: Tue, 14–15

2 Sheet 1, Exercise 1

Exercise 1. *The Euclidean norm of* $v = (v^1, v^2, \dots, v^n)^T \in \mathbb{R}^n$ *is defined as*

$$\|v\|_2 := \sqrt{(v^1)^2 + (v^2)^2 + \ldots + (v^n)^2}$$

Show: A sequence $(x_k) \subset \mathbb{R}^n$ *converges in regards of the Euclidean norm to* $x \in \mathbb{R}$ *iff they converge componentwise to* x

$$\lim_{k\to\infty} \|x_k - x\|_2 = 0 \iff \forall j \in \{1, \dots, n\} : \lim_{k\to\infty} x_k^j = x^j$$

Direction $\Rightarrow$.

Let $\lim_{k\to\infty} \|x_k - x\| = 0$.

Consider: $\left|x_{jk} - x_j\right|$ for arbitrary $j \in \{1, \dots, n\}$.

It holds that

$$0 \leq \left|x_{jk} - x\right| = \sqrt{(x_{jk} - x_j)^2} \leq \sqrt{(x_{1k} - x_1)^2 + \cdots + (x_{1k} - x_n)} = \|x_k - x\| \to 0$$

$$\implies \lim_{k\to\infty} \left|x_{jk} - x_j\right| = 0 \forall j$$

Direction $\Leftarrow$.

Let $\lim_{k\to\infty} x_{jk} = x_j \forall j \in \{1, \dots, n\}$.

The square root function is continuous.

$$\lim_{k\to\infty} \|x_k - x\| = \sqrt{(x_{1k} - x_1)^2 + \cdots + (x_{1k} - x_n)^2}$$

$$\sqrt{(\lim_{k\to\infty} x_{1k})^2 - 2(\lim_{k\to\infty} x_i k)x_1 + x_{1_j}^2 + \cdots + (\lim_{k\to\infty} x_{n_k})^2 - 2(\lim x_{n_k})x_n + x_n^2}$$

$$= \sqrt{\underbrace{x_1^2 - 2x_1^2 + x_1^2}_{=0} + \cdots + \underbrace{x_n^2 - 2x_n^2 + x_n^2}_{=0}} = 0$$

Remark: In $\mathbb{R}^n$, all norms are equivalent. This exercise showed this property. So it you pick two numbers in $\mathbb{R}^n$ and they get “closer”, they get “closer” in every norm.

3 Sheet 1, Exercise 2

Exercise 2. *In the lecture, we discussed the SCNF.* $d_{SCNF} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. *For some fixed* $p \in \mathbb{R}^2$ *it is defined as*

$$d_{SCNF} := \begin{cases} \left\|x - y\right\|_2 & \textit{if } \exists \lambda > 0 : y = p + \lambda(x - p) \\ \left\|x - p\right\|_2 + \left\|y - p\right\|_2 & \textit{else} \end{cases}$$

For $p := (0,0)^T$ and $x := (1,1)^T$, sketch the set $B_R(x)$ for $R = 1$ and $R = 2$.

$$B_R(x) := \left\{ y \in \mathbb{R}^2 \,\middle|\, d_{SCNF} < R \right\}$$

4 Sheet 1, Exercise 3

Exercise 3. *Let (M,d) be a metric space and $x \in M$. Furthermore let $(x_k) \subset M$ be a sequence with property that every subsequence of (x_k) contains a subsequence converging to x. Prove by contradiction, that (x_k) converges to x.*

$x_0 \not\to x$.

There exists $\varepsilon_0 > 0$ for infinitely many $n \in \mathbb{N} : d(x_n, x) \geq \varepsilon_0$. Choose a subsequence $(x_{u_j})_{j \in \mathbb{N}}$ with $d(x_{n_j}, x) \geq \varepsilon_0 \forall j \in \mathbb{N}$. Then there does not exist a subsequence of (x_{n_j}) with limit x.

5 Sheet 1, Exercise 4

Exercise 4. *Let (M,d) be a metric space and complete space. The diameter of a nonempty set $A \subset M$ is given by*

$$\operatorname{diam}(A) := \sup \left\{ d(x,y) \,\middle|\, x, y \in A \right\}$$

Let $(A_j)_{j \in \mathbb{N}}$ be a sequence of nonempty, closed sets in M with $A_{j+1} \subset A_j$ for all $j \in \mathbb{N}$. Furthermore it holds that $\operatorname{diam}(A_j) \to 0$ for $j \to \infty$. Prove that $x \in M$ exists with $\bigcap_{j=1}^{\infty} A_j = \{x\}$ and that x is unique.

$A_j \subseteq M$, because its a complete, metric space.

$$\implies \bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$$

Assume $\exists y_0 \in M : y_0 \neq x_0 \implies d(y_0, x_0) \geq \varepsilon > 0$

$$\forall j \in \mathbb{N} : \operatorname{diam}(A_j) \geq \varepsilon$$

This is a contradiction. However, this is not the equality, we are looking for. Assume $\bigcap_{j=1}^{\infty} A_j = \{x_0\} = \{y_0\} \implies x_0 = y_0$. This is the equality, that was meant to be proven.

5.1 Prove $\bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$

Hint: If the assignment mentions that completeness must be proven, usually you have to construct a Cauchy sequence.

Construct $(x_j)_{j\in\mathbb{N}}$. Choose for x_j some element of A_j. Choose $x_j \in A_j$ for $j \in \mathbb{N}$. This defines a Cauchy sequence $(x_j)_{j\in\mathbb{N}}$. Let $j \in \mathbb{N}$. $x_i \in A_j \supset A_{j+1}$ and $x_{j+1} \in A_{j+1} \forall i \in \mathbb{N}$.

$$\implies d(x_j, x_{j+i}) \leq \operatorname{diam}(A_j) \forall i \in \mathbb{N}$$

where $\operatorname{diam}(A_j) \to 0$ with $i \to \infty$.

$$\implies \exists x \in M : \lim_{j\to\infty}(x_j) = x$$

Because $(x_j)_{j\geq J} \subseteq A_j$ and $\lim_{j\to\infty}(x_j)_{j\geq J} = x$, it follows that $x \in A_j$ and then it follows that $x \in \bigcap_{j=1}^{\infty} A_j$.

This lecture took place on 2018/03/22.

6 Sheet 2, Exercise 1

6.1 Blackboard solution

Let B be bounded.

$$\operatorname{diam}(B) < \infty \qquad \operatorname{diam}(B) = \sup(\left\{ d(x,y) \,\middle|\, x, y \in B \right\})$$

$$d(B_k, B_{k+1}) = \inf(\left\{ d(x,y) \,\middle|\, x \in B_k, y \in B_{k+1} \right\})$$

Exercise (a).

Prove:

$$\sum_{k=1}^{\infty} \operatorname{diam}(B_k) < \infty \wedge \sum_{k=1}^{\infty} d(B_k, B_{k+1}) \implies \operatorname{diam}(\bigcup_{k=1}^{\infty} B_k) < \infty$$

$$\operatorname{diam}(B_k \cup B_{k+1}) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$$

We distinguish 3 cases:

1. $x \in B_k, y \in B_k : d(x,y) \leq \operatorname{diam}(B_k) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$
2. $x \in B_{k+1}, y \in B_{k+1}, d(x,y) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$
3. $\forall x \in B_k \forall y \in B_{k+1}$

Choose x_0 and y_0 on the border of sets B_k and B_{k+1} respectively. But x_0, y_0 do not necessarily exist if compactness is not given. But let $\varepsilon > 0$. Find x_0, y_0 with $d(x_0, y_0) \leq d(B_k, B_{k+1}) + \varepsilon$.

$$d(x,y) \leq \underbrace{d(x,x_0)}_{\leq \operatorname{diam}(B_k)} + \underbrace{d(x_0,y_0)}_{\leq d(B_k,B_{k+1})+\varepsilon} + \underbrace{d(x_0,y)}_{\leq \operatorname{diam}(\mathrm{B_k})} \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + \varepsilon$$

Laurent Pfeiffer continued the following solution (until Exercise 2):

$$\operatorname{diam}((B_k \cup B_{k+1}) \cup B_{k+2}) \leq \operatorname{diam}(B_k \cup B_{k+1}) + \underbrace{d((B_k \cup B_{k+1}), B_{k+2})}_{\leq d(B_{k+1}, B_{k+2})} + \operatorname{diam}(B_{k+2})$$

$$\leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + d((B_k \cup B_{k+1}), B_{k+2}) + \operatorname{diam}(B_{k+2})$$

By induction it follows that

$$\operatorname{diam}(B_k \cup B_{k+1} \cup \cdots \cup B_n) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + d(B_{k+2}) + d(B_{n-1}, B_n) + \operatorname{diam}(B_n)$$

$$\operatorname{diam}(B_k \cup \cdots \cup B_n) \leq \underbrace{\sum_{i=1}^{n} \operatorname{diam}(B_i) + d(B_i, B_{i+1})}_{D}$$

Choose $x, y \in \bigcup_{i=1}^{\infty} B_i$. Then there exists some $k \in \mathbb{N}$ such that $x \in B_k$. There exists n such that $y \in B_n$.

$$d(x, y) \leq \operatorname{diam}(B_k) + \cdots + \operatorname{diam}(B_n) \leq D$$

Exercise (b).

Let $x \in M$. We define: $B_{k+1} = B_{k+2} = \cdots = \{x\}$. For all $i \geq k$ it holds that

$$\operatorname{diam}(B_i) = 0$$

$$d(B_i, B_{i+1}) = 0$$

Therefore,

$$\sum_{i=1}^{\infty} \operatorname{diam}(B_i) = \sum_{i=1}^{k} \underbrace{\operatorname{diam}(B_i)}_{<+\infty} < +\infty$$

What about the distances?

$$\int_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^{k} d(B_i, B_{i+1}) < +\infty$$

By (a), it follows that

$$\left(\bigcup_{i=1}^{\infty} B_i\right) \text{ is bounded } \implies \left(\bigcup_{i=1}^{k} B_i\right) \subseteq \left(\bigcup_{i=1}^{\infty} B_i\right) \text{ is also bounded}$$

Exercise (c).

We define

$$B_i = \left[\sum_{j=1}^{i} \frac{1}{j}, \sum_{j=1}^{i+1} \frac{1}{j}\right]$$

Then it holds that

$$\operatorname{diam}(B_i) = \frac{1}{i+1} \xRightarrow{i\to\infty} 0$$

$$\sum_{i=1}^{\infty} \operatorname{diam}(B_i) = \infty$$

$$B_i \cap B_{i+1} = \left\{\sum_{j=1}^{i+1} \frac{1}{j}\right\} \implies d(B_i, B_{i+1}) = 0$$

$$B_1 \cup \dots \cup B_i = \left[1, \underbrace{\sum_{j=1}^{i+1} \frac{1}{j}}_{\to\infty}\right] \implies \underbrace{\bigcup_{i=1}^{\infty} B_i}_{\text{not bounded}} = [1, \infty)$$

We define $B_i = \left\{\sum_{j=1}^{i} \frac{1}{j}\right\}$. For all i:

- $\operatorname{diam}(B_i) = 0 \implies \sum_{i=1}^{\infty} \operatorname{diam}(B_i) = 0$
-
$$d(B_i, B_{i+1}) = \left(\sum_{j=1}^{i+1} \frac{1}{j}\right) - \left(\sum_{j=1}^{i} \frac{1}{j}\right) = \frac{1}{i+1} \xRightarrow{i\to\infty} 0$$

$$\sum_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty$$

The union is *not* bounded, because $\sum_{j=1}^{i} \frac{1}{j} \in \bigcup_{j=1}^{\infty} B_j$.

7 Sheet 2, Exercise 2

Exercise 5. *Let* (X, d) *be a sequentially compact, metric space. Show:*

a. X is bounded.

b.

7.1 Blackboard solution

Exercise (a).

Let X be unbounded. Hence, there exists a tuple $(x_N, y_N) \in X \times X$ for every $N \in \mathbb{N}$ with $d(x_N, y_N) > N$. Because (X, d) is sequentially compact, there exists a convergent subsequence $(x_{N_k}, y_{N_{k_i}})$ we can choose such that

$$\lim_{k\to\infty} x_{N_k} = \infty \qquad \lim_{i\to\infty} y_{N_{K_i}} = y_0 \qquad \lim_{i\to\infty}(x_{N_{k_i}}) = x_0$$

$$\implies \underbrace{N_{k_i}}_{\xrightarrow{i\to\infty}\infty} < d(x_{N_{k_i}}, y_{N_{k_i}}) \xrightarrow{i\to\infty} d(x_0, y_0)$$

By this contradiction, it follows that X is bounded.

Exercise (b).

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Let X be sequence compact $\implies$ there exists a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x \in X$. Show that $x_n \xrightarrow{n\to\infty} x$.

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $\forall n, m \geq N : d(x_n, x_m) < \frac{\varepsilon}{2}$. Choose $k \in \mathbb{N}$ such that $n_k \geq N$ and $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

$$\forall n \geq n_k : d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) < \varepsilon$$

Exercise (c).

Show that $A \subset X$ is sequentially compact iff A is closed.

$\Rightarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence, $(x_n)_{n\in\mathbb{N}} \subset A$, $\lim_{n\to\infty} x_n = x_0 \in X$. Show that $x_0 \in A$.

Set A is sequentially compact. Choose subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset A$, $\lim_{k\to\infty} x_{n_k} = x_0 \in A \implies A$ is closed.

$\Leftarrow$ A is closed. Show that A is sequentially compact.

Let $(x_n)_{n\in\mathbb{N}} \subset A$ and there exists subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = x_0 \in X$, because X is sequentially compact. $(x_{n_k})_{k\in\mathbb{N}} \subset A \implies A$ is sequentially compact.

8 Sheet 2, Exercise 2

Exercise 6. *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{1 + x^2}$.*

1. *Show that $\left|f(x) - f(y)\right| < \left|x - y\right| \,\forall x, y \in \mathbb{R}$ with $x \neq y$*
2. *Investigate which conditions of Banach's Fixed Point Theorem are [not] met.*

3. *Is Banach's Fixed Point Theorem applicable? Does f have a fixed point?*

Exercise (a).

$$\left|f(x) - f(y)\right| < \left|x - y\right| \qquad x, y \in \mathbb{R}, x \neq y$$

$$\left|\sqrt{1 + x^2} - \sqrt{1 + y^2}\right| < \left|x - y\right|$$

$$1 + x^2 + 1 + y^2 - 2\sqrt{(1 + x^2)(1 + y^2)} < x^2 + y^2 - 2xy$$

$$2 - 2\sqrt{(1 + x^2)(1 + y^2)} < -2xy$$

$$1 + xy < \sqrt{(1 + x^2)(1 + y^2)}$$

We need to distinguish 2 cases here (x and y have same signum, x and y have different signum). This is trivial.

$$1 + 2xy + x^2y^2 < 1 + x^2 + y^2 + x^2y^2$$

$$0 < x^2 + y^2 - 2xy$$

$$0 < (x - y)^2$$

Exercise (b and c).

Let $x \in \mathbb{R}$.

$$f(x) = x$$

$$\sqrt{1 + x^2} = x$$

$$1 + x^2 = x^2$$

$$1 = 0$$

This lecture took place on 2018/04/12.

9 Sheet 3, Exercise 4

Exercise 7. *Let (X, d) be a metric space and $x_0 \in X$. A function $f : X \to \mathbb{R}$ is called half-continuous from below in x_0, if for every $\varepsilon > 0$ some $\delta > 0$ exists, such that $d(x, x_0) < \delta$ implies $f(x_0) - f(x) < \varepsilon$. If f is half-continuous from below in every $x_0 \in X$, then f is called half-continuous from below.*

Obviously, continuity implies half-continuity.

9.1 Sheet 3, Exercise 4a

Exercise 8. *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$ such that f is non-continuous.*

Let $f : [-1, 1] \to \mathbb{R}$.

$$x \mapsto \begin{cases} -1 & x = -1 \\ -x & x \neq -1 \end{cases}$$

$$\underbrace{f(-1)}_{=-1} - \underbrace{f(x)}_{\geq -1} \leq 0 < \varepsilon$$

9.2 Sheet 3, Exercise 4b

Exercise 9. *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$, but does not have a maximum.*

Same f can be chosen.

9.3 Sheet 3, Exercise 4c

Exercise 10. *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$, but does not have a minimum.*

f as $f|_{[-1,1]}$ can be chosen.

9.4 Sheet 3, Exercise 4d

Exercise 11. *Prove that every half-continuous from below function in a compact set has a minimum.*

Hint: It is assumed that cover-compactness seems to be more cumbersome than sequential compactness.
Remark: This is a generalization of the theorem, that every continuous, compact function has a minimum and maximum.

Let $K \subseteq X$ be compact. $f : K \to \mathbb{R}$ is half-continuous from below.

Show that $f^k = \inf(f(K)) \in f(K)$.

$$\exists (x_n)_{n \in \mathbb{N}} \subseteq K \text{ with } f(x_n) - f^k < \frac{1}{n}$$

K is compact. Hence, there exists $(x_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} x_{n_k} := x^* \in K$. Let $\varepsilon > 0$ be arbitrary. By half-continuity from below, it follows that $\exists \delta > 0 : d(x^*, x) <$

$\delta \implies f(x^*) - f(x) < \varepsilon.$

$$\exists K \in \mathbb{N} \forall k \geq K : d(x^k, x_{n_k}) < \delta \implies f(x^k) - f(x_{n_k}) < \varepsilon \iff f(x^*) < f(x_{n_k}) + \varepsilon$$

$$\implies f(x^*) \leq \lim_{k\to\infty} f(x_{n_k}) \implies f(x^*) \leq \lim_{n\to\infty} f(x_n) = f^*$$

$$\implies f(x^*) = f^* \implies f^* \text{ is minimum of } f(X)$$

10 Sheet 3, Exercise 3

Exercise 12. *Let (X, d) and (Y, e) be metric spaces, where $d : X \to \mathbb{R}$ is a discrete metric, hence*

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

10.1 Sheet 3, Exercise 3a

Exercise 13. *Every map $f : X \to Y$ is continuous.*

Let $f : X \to Y$ be arbitrary. Let $x_0 \in X$ and $\varepsilon > 0$ be arbitrary. Show that

$$\exists \delta > 0 : d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$$

$$K_{\frac{1}{2}}(x_0) = \{x_0\}$$

10.2 Sheet 3, Exercise 3b

Exercise 14. *A map $f : X \to Y$ is not necessarily bounded.*

$M \geq 0$ arbitrary. $\exists x, y \in f(X) : e(x, y) > M$.

$$f : \mathbb{Z} \to \mathbb{Z} \qquad x \mapsto x$$

$$f(x) = \mathbb{Z} \qquad x = 0 \qquad y = M + 1$$

$e = |\cdot|$.

10.3 Sheet 3, Exercise 3c

Exercise 15. *Every map $g : Y \to X$ is bounded.*

Let $g : Y \to X$ be arbitrary. Show that $\exists M \geq 0 \forall x, y \in g(Y) : d(x, y) \leq M$. Choose $M = 2$. $\forall x, y \in X : d(x, y) \leq 1 \leq 2$.

10.4 Sheet 3, Exercise 3d

Exercise 16. *In case* $(Y, e) = (\mathbb{R}, |\cdot|)$*, every non-constant map* $g : Y \to X$ *is non-continuous.*

We show: continuity implies constant.

Let $g : \mathbb{R} \to X$ continuous. Let $x_0 \in \mathbb{R}$ be arbitrary and $\varepsilon = \frac{1}{2}$. $\exists \delta_0 > 0 : |x_0 - x| < \delta \implies d(g(x_0), g(x)) < \frac{1}{2}$ for $x_0 \in \mathbb{R}$ there exists δ_0 such that $\forall x \in (x_0 - \delta, x_0 + \delta) : g(x) = g(x_0)$.

$$\sup\{s \in [x_0, \infty) \mid g(x) = g(x_0) \forall x \in [x_0, s)\}$$

11 Sheet 3, Exercise 2

Exercise 17. *Let* V *be the vector space of bounded, complex sequences, hence*

$$V := \{(a_k)_{k\in\mathbb{N}} \subset \mathcal{C} \mid \exists M \in \mathbb{R} \text{ with } |a_k| \leq M \forall k \in \mathbb{N}\}$$

additionally with norm

$$\|(a_k)_{k\in\mathbb{N}}\|_\infty := \sup\{|a_k| \mid k \in \mathbb{N}\}$$

This solution was done by Mr. Kruse himself.

11.1 Sheet 3, Exercise 2b

Exercise 18. *The unit sphere in* $(V, \|\cdot\|_\infty)$*,*

$$B_1(0) = \{a \in V \mid \|a\|_\infty \leq 1\}$$

is closed and bounded, but not sequentially compact.

We need to prove boundedness.

Let $C, D \in B_1(0)$.

$$\implies \Bigg\| \underbrace{C}_{=(c_k)} - \underbrace{D}_{=(d_k)} \Bigg\|_\infty \leq 2$$

$$\sup\left\{ \left| \underbrace{c_k - d_k}_{\leq \underbrace{|c_k|}_{\leq 1 \forall k} + \underbrace{|d_k|}_{\leq 1 \forall k} \leq 2 \forall k} \right| : k \in \mathbb{N} \right\} \leq 2$$

We need to prove closedness.

$$(A^n)_{n\in\mathbb{N}} \subset B_1(0) \text{ with } \lim_{n\to\infty} A^n = A$$

Show that $A \in B_1(0)$.

$$\text{For every } A^n := (a^n_k)_{k\in\mathbb{N}} \text{ it holds that } \left\| \underbrace{(a^n_k)_{k\in\mathbb{N}}}_{=\sup\left\{\left|a^n_k\right| : k\in\mathbb{N}\right\}\leq 1} \right\|_\infty \leq 1$$

$$(A^n)_{n\in\mathbb{N}} \subset B_1(0) \text{ with } \lim_{n\to\infty} A^n = A$$
$$\iff \lim_{n\to\infty} \|A^n - A\|_\infty = 0$$

$\left|a^n_k\right|$ in

$$\sup\left\{\left|a^n_k\right| : k \in \mathbb{N}\right\}$$

converges to $|a_k| \leq 1$ for $n \to \infty$.

We need to prove sequentially non-compact of $B_1(0)$. So we only need to find some sequence that does not have some converging subsequence.

We define

$$A^n := (a^n_k)_{k\in\mathbb{N}} := \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

for every $n \in \mathbb{N}$. As such we get a sequence

$$\implies (A^n)_{n\in\mathbb{N}} \subset B_1(0)$$

but it holds that $\|A^n - A^m\|_\infty = 1 \forall n \neq m$. This is also not a Cauchy sequence.

12 Sheet 3, Exercise 1

Exercise 19. *Let (X, d) be a metric space. A set $K \subset X$ is called cover-compact, if for every family of open sets $(U_i)_{i\in I} \subset X$ with $K \subset \bigcup_{i\in I} U_i$ it holds that: There exists a finite set $J \subset I$ with $K \subset \bigcup_{i\in J} U_i$. Let $K \subset X$ be cover-compact.*

12.1 Sheet 3, Exercise 1a

Exercise 20. *Show that K is totally bounded, hence for every $r > 0$, there exists $x_1, \ldots, x_n$ in K with $K \subset \bigcup_{i=1}^n B_r(x_i)$.*

Construct a family of open spheres $((\mathcal{B}_r(x))_{x\in K} \subset K$ covering $K)$. By cover-compactness it follows there exists some finite $J \subset K$ with $K \subset \bigcup_{x\in J} B_r(x)$.

12.2 Sheet 3, Exercise 1b

Exercise 21. *Prove that K is sequentially compact.*

Proof by contradiction: Assume K is not sequentially compact.

Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \in K$ which has a subsequence $(x_{n_k})_{k\in\mathbb{N}} \to c \notin K$.

$$\forall x \in K : \exists r_x > 0 : B_{r_x}(x) \text{ contains finitely many sequence elements}$$

Because $\bigcup_{x\in K} B_{r_x}(x) \supset K$ it holds: there exists $J \subset K$ finite $\bigcup_{x\in J} B_{r_x}(x) \supset K$. This contradicts with $(x_n)_{n\in\mathbb{N}} \subset K$.

12.3 Sheet 4, Exercise 1

Exercise 22. *Let (M, d) be a complete metric space and $(A_k)_{k\in\mathbb{N}} \subset M$ is a sequence of closed sets. Use Cantor's Theorem to prove: $\bigcup_{k\in\mathbb{N}} A_k$ contains an open set if at least one A_k contains an open set. Illustrate this statement for $(M, d) = (\mathbb{R}, |\cdot|)$.*

First we illustrate it in $\mathbb{R}$.

$$(A_k) = \{a_k\}$$

where $a_k \in \mathbb{R}$.

Consider some

13 Sheet 4, Exercise 2

Exercise 23. *Let $f : [-1, 1] \to \mathbb{C}$ be continuous and $O \subset \mathbb{C}$ is an open set. In the lecture we have seen that $f^{-1}(O)$ is open. Review the result and prove for $O = \mathbb{C}$.*

1. *The set O is open.*
2. *It holds that $f^{-1}(O) = [-1, 1]$*
3. *The set $[-1, 1] \subset \mathbb{R}$ is not open.*
4. *The statement of the lecture about $f^{-1}(O)$ is still correct.*

13.1 Sheet 4, Exercise 2a

Show that $\mathbb{C}$ is open.

Let $z \in \mathbb{C}$. $\exists \varepsilon > 0$,

$$B(z, \varepsilon) \subseteq \mathbb{C}$$

13.2 Sheet 4, Exercise 2b

Follows from the definition of a function.

13.3 Sheet 4, Exercise 2c

If it is an open set, there must be a neighborhood of arbitrary ε such that this neighborhood is completely in the set.

Let $\varepsilon > 0$. Choose $x \in B(1, \varepsilon)$ with $x = 1 + \frac{\varepsilon}{2}$.

$$\implies x \in B(1, \varepsilon) \land x \notin [-1, 1]$$

13.4 Sheet 4, Exercise 2d

Let (X, d) and (Y, e) be metric spaces and $f : X \to Y$ continuous then $f^{-1}(O)$ is open $\forall O \subseteq Y$ open.

Show:

$$\forall x \in [-1, 1] \exists \varepsilon > 0 : \quad \underbrace{B(x, \varepsilon)}_{=\{z \in [-1,1] \mid d(x,z) < \varepsilon\}} \quad \subseteq [-1, 1]$$

So the difference is the domain of z ($[-1, 1]$ unlike exercise c, where we used $\mathbb{R}$).

The point was to illustrate how to read the theorem properly.

14 Sheet 4, Exercise 3

Exercise 24. *Let Ω be a non-empty set and $B(\Omega)$ the vector space of real-valued bounded functions on Ω. Hence,*

$$B(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \exists M \in \mathbb{R} \text{ with } \left|f(x)\right| \leq M \forall x \in \Omega \right\}$$

with norm

$$\left\|f\right\|_\infty := \sup \left\{ \left|f(x)\right| \,\middle|\, x \in \Omega \right\}$$

Prove the following statements:

1. *$(B(\Omega), \|\cdot\|_\infty)$ is a complete normed vector space.*
2. *The unit circle U in $B(\Omega)$ is closed and bounded.*

$$U = \left\{ f \in B(\Omega) \,\middle|\, \left\|f\right\|_\infty \leq 1 \right\}$$

3. *The unit circle is sequentially compact if and only if Ω is finite.*

14.1 Sheet 4, Exercise 3a

Given $\Omega \neq 0$.

$$B(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \exists M \in \mathbb{R} : \left|f(x)\right| \leq M \quad \forall x \in \Omega \right\}$$

First, we show that $\|\cdot\|_\infty$ is indeed a norm. We just show absolute homogeneity for illustrative purposes:

$$\begin{aligned}
\left\|\lambda f\right\|_\infty &= \sup \left\{ \left|\lambda \cdot f(x)\right| \,\middle|\, x \in \Omega \right\} \\
&= \sup \left\{ |\lambda| \cdot \left|f(x)\right| \,\middle|\, x \in \Omega \right\} \\
&= |\lambda| \cdot \sup \left\{ \left|f(x)\right| \right\} x \in \Omega \\
&= |\lambda| \cdot \left\|f\right\|
\end{aligned}$$

We show completeness of $(B(\Omega), \|\cdot\|_\infty)$. Equivalently, all Cauchy sequences in $B(\Omega)$ are convergent. Equivalently, for all Cauchy sequences $(f_n)_{n\in\mathbb{N}} : \exists f \in B(\Omega) : \left\|f_n - f\right\|_\infty \to 0$ for $n \to \infty$.
Let $(f_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence. Hence,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m > N \implies \left\|f_n - f_m\right\|_\infty = \sup \left\{ (f_n - f_m)(x) \,\middle|\, x \in \Omega \right\} < \varepsilon$$

$$\forall \varepsilon > 0 : n, m > N$$

$$\forall x \in \Omega : \left|(f_n - f_m)(x)\right| < \varepsilon$$

$$\implies \forall x \in \Omega : (f_n(x))_{n\in\mathbb{N}} \subseteq R$$

is a Cauchy sequence in $\mathbb{R}$.

$$\iff \forall x \in \Omega : (f_n(x))_{n\in\mathbb{N}} \text{ converges}$$

$$\forall x \in \Omega : (f_n(x)))_{n\in\mathbb{N}} \to f(x) \forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \implies \left|f_n(x) - f(x)\right| < \varepsilon$$

$$\exists N \in \mathbb{N} \forall n > N : \left\|f_n - f\right\|_\infty < 1$$

$$\left\|f\right\|_\infty = \left\|f - f_N + f_N\right\|_\infty \leq \underbrace{\left\|f - f_N\right\|_\infty}_{<1} + \underbrace{\left\|f_N\right\|}_{\leq M} < 1 + M$$

14.2 Sheet 4, Exercise 3b

Let $K_1 := \left\{ f \in B(\Omega) \,\middle|\, \left\|f\right\|_\infty \leq 1 \right\}$. Show K_1 is bounded and closed.

14.2.1 K_1 is bounded

Let $f, g \in K_1$ be arbitrary.

$$\left\|f - g\right\|_\infty \leq \left\|f\right\|_\infty + \left\|g\right\|_\infty \leq 1 + 1 = 2$$

2 is a boundary and therefore K_1 is bounded.

14.2.2 K_1 is closed

Let $(f_n)_{n\in\mathbb{N}}$ be a convergent sequence in K_1 with $\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty} \left\|f_n - f\right\| = 0$.

Show $f \in K_1$.

$$\forall f_n \in K_1 : \left\|f_n\right\| \leq 1$$
$$\left\|f\right\|_\infty = \left\|f - f_n\right\|_\infty \leq \underbrace{\left\|f - f_n\right\|_\infty}_{\xrightarrow{n\to\infty} 0} + \underbrace{\left\|f_n\right\|_\infty}_{\leq 1} \leq 1$$
$$\implies \left\|f\right\|_\infty \leq 1 \implies f \in K_1$$

14.3 Sheet 4, Exercise c

f is sequentially compact if and only if Ω is finite? Equivalently, every sequence $(f_n)_{n\in\mathbb{N}} \subseteq K_1$ has a convergent subsequence with limit in K_1.

Direction $\implies$.

Let Ω be infinite. Then $\exists$ a sequence $(f_n)_{n\in\mathbb{N}}$ without convergent subsequence. We build a sequence $(f_n)_{n\in\mathbb{N}}$ in K_1.

Let $(x_i)_{i\in\mathbb{N}}$ be an arbitrary sequence in Ω with $x_i \neq x_j \forall i \neq j$.

$$f_n(x) := \begin{cases} 1 & \text{if } x = x_n \\ 0 & \text{else} \end{cases}$$

Then it holds that $\forall n \neq m$,

$$\left\|f_n - f_m\right\|_\infty = 1$$

Assume there exists a convergent subsequence in $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ with limit f.

$$\implies \exists M > 0 : k > M : \left\|f_{n_k} - f\right\|_\infty < \frac{1}{2}$$

Let $k, l > M$ with $k \neq l$

$$\implies \left\|f_{n_k} - f_{n_l}\right\|_\infty \leq \left\|f_{n_k} - f\right\|_\infty + \left\|f_{n_l} - f\right\|_\infty < \frac{1}{2} + \frac{1}{2} = 1$$

This is a contradiction to $\left\|f_n - f_m\right\|_\infty = 1$.
Direction $\Longleftarrow$.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in K_1 without limit. Let $n \in \mathbb{N}$.

$$\Omega = \{x_1, \ldots, x_n\} \implies \left|\{f_n(x_1), \ldots, f_n(x_n)\}\right| < \infty$$

Let $f_n \in K_1 \implies \left|f_n(x_i)\right| \leq 1 \forall i \in \{1, \ldots, m\} \, \forall n \in \mathbb{N}$.
Consider $x_1 \in \Omega$.

$$(f_n(x_1)) = y_n^1 \in [-1, 1]$$

$$[-1, 1] \text{ compact } \implies (y_n^1)_{n\in\mathbb{N}} \text{ has convergent subsequence } (y_{n_k}^1)_{k\in\mathbb{N}} \to \tilde{y}^1$$

$$(f_{n_k}(x_1))_{k\in\mathbb{N}} = (y_{n_k}^1)_{k\in\mathbb{N}} \to \tilde{y}_1 := f(x_1)$$

and this goes on up to

$$(f_{n_{\ddots_z}}(x_m))_{z\in\mathbb{N}} \to f(x_m)$$

For every $\varepsilon > 0$

$$\exists N_1 : \forall n \in N_1 : \left|f_{n_{\ddots_2}}(x_1) - f(x_1)\right| < \varepsilon$$

$$\vdots$$

$$\exists N_m : \forall n \in N_m : \left|f_{n_{\ddots_2}}(x_m) - f(x_m)\right| < \varepsilon$$

Choose $N := \max N_1, \ldots, N_m$. For all $n \geq N$,

$$\implies \left\|f_{n_{\ddots_2}}\right\|_\infty < \varepsilon$$

15 Sheet 4, Exercise 4

Exercise 25. *Let $k \in \mathbb{N}$. Show: $\exists \phi_k : \sqrt{k\pi} \leq \xi_k \leq \sqrt{(k+1)\pi}$ such that*

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2)\, dx = \frac{(-1)^k}{\xi_k}$$

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2)\, dx = \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \frac{x \cdot \sin(x^2)}{x}\, dx = \frac{1}{\xi_k} \cdot \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} x \cdot \sin(x^2)\, dx$$

But this IVT is unconventional.

$$= \frac{1}{\xi_k} \cdot \left(-\frac{1}{2} \cdot \cos(x^2)\right)\Bigg|_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}$$

If k is even:

$$\frac{1}{\xi_k}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{\xi_k}$$

If k is odd:

$$\frac{1}{\xi_k}\left(-\frac{1}{2} - \frac{1}{2}\right) = -\frac{1}{\xi_k}$$

This implies a boundary of

$$\frac{(-1)^k}{\xi_k}$$

This lecture took place on 2018/04/26.

16 Sheet 5, Exercise 1

Exercise 26. *Let $\mathcal{R}[a,b]$ be the vector space of real-valued regulated functions on $[a,b] \subseteq \mathbb{R}$, hence*

$$\mathcal{R}[a,b] := \left\{ f : [a,b] \to \mathbb{R} \,\middle|\, f \text{ is a regulated function} \right\}$$

annotated with a norm $\|\cdot\|_\infty$ of Sheet 4 Exercise 3. Prove that $(\mathcal{R}[a,b], \|\cdot\|_\infty)$ is a complete normed vector space with a sequentially non-compact unit sphere.

17 Sheet 5, Exercise 2

Exercise 27. *Let $f, b \in \mathcal{R}[a,b]$ with*

$$f_+(x) = g_+(x) \qquad \forall x \in [a,b)$$

$$f_-(x) = g_-(x) \qquad \forall x \in (a,b]$$

1. *For $\alpha, \beta \in [a,b] : \int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta g(x)\,dx$ holds.*
2. *For every antiderivative $F : [a,b] \to \mathbb{R}$ of f there exists an antiderivative $G : [a,b] \to \mathbb{R}$ of g with $F(x) = G(x)$ for all $x \in [a,b]$.*

17.1 Sheet 5, Exercise 2a

Let $f, g \in \mathcal{R}[a, b]$.

$$F'_+(x) := f_+(x) = g_+(x)$$
$$F'_-(x) := f_-(x) = g_-(x)$$

Show: $\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta g(x)\,dx$.
In general $f_+(x) \neq f(x) \neq f_-(x)$.

$$F := \int f(x)\,dx$$
$$G := \int g(x)\,dx$$

$$\int_\alpha^\beta f(x)\,dx = F|_\alpha^\beta \overset{(b)}{=} \underbrace{F(\beta) + K}_{G(\beta)} - \underbrace{(F(\alpha) - K)}_{G(\alpha)} = \int_\alpha^\beta g(x)\,dx$$

17.2 Sheet 5, Exercise 2b

F is an antiderivative of f if and only if

$$F = \int f(x)\,dx$$

$$F'_+(x) = f_+(x) = g_+(x) = g_+(x) \qquad \forall x \in [a, b)$$
$$F'_-(x) = f_-(x) = g_-(x) = g_-(x) \qquad \forall x \in (a, b]$$

18 Sheet 5, Exercise 3

Exercise 28. 1. *Let $f : [a, b] \to \mathbb{R}$ continuously differentiable with $f(x) \neq 0 \forall x \in [a, b]$. Show that*

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \ln\left|f(b)\right| - \ln\left|f(a)\right|$$

2. *Determine the value of I using* $\cos(x) = \frac{1}{2}(\sin x + \cos x + \cos x - \sin x)$

$$I := \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\,dx$$

3. *Determine I using the substitution $y(x) = \frac{\pi}{2} - x$.*

18.1 Sheet 5, Exercise 3a

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \left|\begin{matrix} t = f(x) \\ dt = f'(x)\,dx \end{matrix}\right| = \int_{f(a)}^{f(b)} \frac{1}{t}\,dt$$
$$= [\ln|t|]_{f(a)}^{f(b)} = \ln\left|f(b)\right| - \ln\left|f(a)\right|$$

18.2 Sheet 5, Exercise 3b

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)+\cos(x)} = \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin(x)+\cos(x)}{\sin(x)+\cos(x)}}_{\frac{\pi}{4}} + \underbrace{\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}}_{f(x)}$$
$$= \frac{\pi}{4} + \ln\left|\cos(\frac{\pi}{4}) + \sin(\frac{\pi}{2})\right| - \ln|\cos(0)+\sin(0)|$$
$$= \frac{\pi}{4} + 0$$

18.3 Sheet 5, Exercise 3c

$$u(x) = \frac{\pi}{2} - x$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)+\cos(x)}\,dx = \int_{\frac{\pi}{2}}^0 -\frac{\cos(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}\,du$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{2}-u)}{\sin(\frac{\pi}{2}-u)+\cos(\frac{\pi}{2}-u)}\,du$$
$$= \int_0^{\frac{\pi}{2}} \frac{\sin(u)}{\sin(u)+\cos(u)}\,du$$
$$\implies 2I = \int_0^{\frac{\pi}{2}} \frac{\sin(u)}{\sin(u)+\cos(u)}\,du + \int_0^{\frac{\pi}{2}} \frac{\cos(u)}{\sin(u)+\cos(u)}\,du$$
$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin(u)+\cos(u)}{\sin(u)+\cos(u)}\,du$$
$$2I = \frac{\pi}{2} \iff I = \frac{\pi}{4}$$

19 Sheet 5, Exercise 4

Exercise 29. 1. *Evaluate using integration by parts:* $\int_0^{\pi} (\sin x)^2\,dx$

2. *Determine (for $n \in \mathbb{N}$) by integration by parts:* $\int_0^{\frac{\pi}{2}} (\cos x)^{2n}\, dx$

3. *Determine by integration by parts followed by substitution:* $\int_0^1 \log(x+1)\, dx$

19.1 Sheet 5, Exercise 4a

Let $u := \sin(x)$, $u' = \cos(x)$, $v' := \sin(x)$ and $v = -\cos(x)$.

$$\begin{aligned}
\int_0^{\pi} (\sin(x))^2\, dx &= [-\sin(x)\cos(x)]_0^{\pi} - \int_0^{\pi} -\cos(x)\cos(x)\, dx \\
&= \int_0^{\infty} 1 - \int_0^{\pi} \sin(x)^2\, dx \\
\iff \int_0^{\pi} 2 \cdot \sin(x)^2\, dx = \int_0^{\infty} 1 = \pi \\
&= \frac{\pi}{2}
\end{aligned}$$

19.2 Sheet 5, Exercise 4b

Let $n \in \mathbb{N} \setminus \{0\}$.

$$\int_0^{\frac{\pi}{2}} (\cos(x))^{2n}\, dx$$

We prove by complete induction: Consider $n = 0$.

$$\int_0^{\frac{\pi}{2}} (\cos(x))^{2n}\, dx = \frac{\pi}{2}$$

Consider $n - 1 \to n$.

$$\int_0^{\frac{\pi}{2}} \cos(x)^{2n+2}\, dx = \int_0^{\frac{\pi}{2}} \underbrace{\cos(x)^{2n+1}}_{u} \underbrace{\cos(x)}_{v'}\, dx$$

$$\int_0^{\frac{\pi}{2}} (\cos(x))^2 = \frac{\pi}{4}$$

$$\begin{aligned}
\text{By induction hypothesis } \int_0^{\frac{\pi}{2}} \cos(x)^{2n}\, dx &= \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos(x)^{2(n-1)} \\
&= \left| \begin{matrix} u' & = -(2n+1)\sin(x)\cos(x)^{2n} \\ v & = \sin(x) \end{matrix} \right|
\end{aligned}$$

$$[\cos(x)^{2n+1} \cdot \sin(x)]_0^{\frac{\pi}{2}} + (2n+1) \cdot \int_0^{\frac{\pi}{2}} \cos(x)^{2n} \cdot \sin(x)^2 \, dx = (2n+1) \cdot \int_0^{\frac{\infty}{2}} \cos(x)^{2n} \, dx - (2n+1) \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} \, dx$$

$$\implies (2n+2) \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} \, dx = (2n+1) \int_0^{\frac{\pi}{2}} \cos(x)^{2n} \, dx$$

$$\implies \int_0^{\frac{\pi}{2}} \cos(x)^{2n+2} \, dx = \frac{(2n+1)}{2n+2} \int_0^{\frac{\pi}{2}} \cos(x)^{2n} \, dx$$

$$\frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \ldots \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

19.3 Sheet 5, Exercise 4c

$$\int_0^1 x \cdot \log(x+1) \, dx = \left| \begin{matrix} u' = x & u = \frac{x^2}{2} \\ v = \log(x+1) & v' = \frac{1}{1+x} \end{matrix} \right|$$

$$\left[\frac{x^2}{2} \log(x+1) \right]_0^1 - \int_0^1 \left(\frac{x^2}{2} \cdot \frac{1}{1+x} \right) dx \qquad u(x) = 1 + x$$

$$= \left[\frac{x^2}{2} \log(x+1) \right]_0^1 - \frac{1}{2} \underbrace{\int_1^2 (u-1)^2 \cdot \frac{1}{u} \, du}_{\int_1^2 \left(\frac{u^2+1-2u}{u} \right) du = \int_1^2 u + \frac{1}{u} - 2 \, du}$$

$$\frac{\log(2)}{2} - \frac{1}{2} \left[\frac{u^2}{2} + \log(u) - 2u \right]_1^2 = \frac{1}{4}$$

It is valid to assume that log = ln in this exercise, because it is not specified otherwise. But you can also consider a factor a, which normalizes it to ln.

20 Sheet 6, Exercise 1

Exercise 30. *Let $\mathcal{R}[a,b]$ be the set of regulated functions, $C[a,b]$ be the set of continuous functions and $\mathcal{M}[a,b]$ be the set of montonic functions on $[a,b] \subset \mathbb{R}$. Show:*

1. $f \in C[a,b] \implies f \in \mathcal{R}[a,b]$
2. $f \in \mathcal{M}[a,b] \implies f \in \mathcal{R}[a,b]$
3. $f \in C[a,b], g \in \mathcal{R}[a,b] \wedge g([a,b]) \subset [a,b] \implies f \circ g \in \mathcal{R}[a,b]$

20.1 Sheet 6, Exercise 1a

Assume $f \in C[a, b]$. For all $x \in [a, b]$, f has one-sided limits.

20.2 Sheet 6, Exercise 1b

Let $x \in [a, b]$. Consider $x_{n\in\mathbb{N}} \nearrow x$. Show that $\lim_{n\to\infty} f(x_n)$ exists. We consider a monotonic subsequence

$$f(x_{n_k}) \geq f(x_{n_{k+1}}) \forall k \in \mathbb{N}$$

$$f(x) \leq f(x_{n_k}) \forall k \in \mathbb{N}$$

20.3 Sheet 6, Exercise 1c

$(x_n)_{n\in\mathbb{N}} \nearrow x$.

$$\lim_{n\to\infty} f(g(x_n)) \text{ exists}$$

$$\lim_{n\to\infty} \underbrace{g(x_n)}_{=:y_n} = y \in \mathbb{R}$$

$$\lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty}) = f(\lim_{n\to\infty} y_n) TODO$$

$g : [a, b] \to [a, b]$. $f \in \mathcal{R}[a, b]$, $g \in C([a, b]), g([a, b]) \subset [a, b]$.

21 Sheet 6, Exercise 2

Exercise 31. *Determine all antiderivatives:*

$$\int \frac{1}{x(\ln x)^3} \, dx \qquad (x > 0) \tag{1}$$

$$\int \sin^3(x) \cos^4(x) \, dx \tag{2}$$

$$\int \operatorname{arsinh}(x) \, dx \tag{3}$$

21.1 Sheet 6, Exercise 2a

We apply integration by substitution:

$$\int_{g(a)}^{g(b)} f(x) \, dx = \int_a^b f(g(u)) \cdot g'(u) \, du$$

We consider:

$$f(x) = \left(\frac{1}{x^3}\right) = \frac{1}{x^3}$$

$$g(x) = \ln(x) \qquad g'(x) = \frac{1}{x}$$

$$\int \frac{1}{x(\ln x)^3}\, dx = \int \left(\frac{1}{u^3}\right) du = \int u^{-3}\, du = \frac{u^{-2}}{-2} + c = \frac{1}{-2 \cdot u^2} + c = \frac{1}{-2 \cdot \ln(x)^2} + c$$

Hint. Because we apply Backsubstitution, we do what we usually do by computing the integral over some specified limits. Therefore the improper integral is exact as well.

21.2 Sheet 6, Exercise 2b

$$\begin{aligned}
\int \sin(x)^2 \cdot \sin(x) \cdot \cos(x)^4\, dx &= \int (1 - \cos(x)^2) \cdot \cos(x)^4 \cdot \sin(x)\, dx \\
&= \int (\cos(x)^4 - \cos(x)^6) \cdot \sin(x)\, dx \\
&\left| \begin{array}{c} u = \cos(x) \\ u' = -\sin(x) \\ du = dx \cdot u' \end{array} \right| \\
&= \int (u^4 - u^6) \cdot (-1)\, du = \int (-u^4 + u^6)\, du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + c = \frac{\cos(x)^7}{7} - \frac{\cos(x)^5}{5} + c
\end{aligned}$$

21.3 Sheet 6, Exercise 2c

$$\int \operatorname{arsinh}(x)\,dx = \int \ln(x + \sqrt{x^2+1})\,dx$$

$$\left| \begin{array}{c} u = \ln(x + \sqrt{x^2+1}) \\ v' = 1 \\ v = x \\ u' = \frac{1}{\sqrt{x^2+1}} \end{array} \right|$$

$$= \ln(x + \sqrt{x^2+1})x - \int \frac{1}{\sqrt{x^2+1}} x\,dx$$

$$\left| \begin{array}{c} u = x^2+1 \\ u' = 2x \\ du = 2x\,dx \end{array} \right|$$

$$= \operatorname{arsinh}(x) \cdot x - \int \frac{1}{\sqrt{u}} \frac{1}{2}\,du$$

$$= \operatorname{arsinh}(x) \cdot x - \sqrt{u} + c$$

$$= \operatorname{arsinh}(x) \cdot x - \sqrt{x^2+1} + c$$

22 Sheet 6, Exercise 3

Exercise 32. *For $a = 0$ and $a > 0$, determine all antiderivatives:*

$$\int \frac{\ln(x)}{\sqrt{a+x}}\,dx \qquad (x > 0)$$

Case $a = 0$:

$$\int \frac{ln(x)}{\sqrt{x}} \left| \begin{array}{cc} u' = \frac{1}{\sqrt{x}} & u = 2\sqrt{x} \\ v = \ln(x) & v' = \frac{1}{x} \end{array} \right|$$

$$= \ln(x) \cdot 2\sqrt{x} ...$$

$$= \ln(x) \cdot \sqrt{x} - 4\sqrt{x} + c$$

Case $a > 0$:

$$
\begin{aligned}
\int \frac{\ln(x)}{\sqrt{x+a}} &= \int \frac{\ln(x)}{\sqrt{x+a}} \cdot 2\sqrt{x+a}\, du \\
&\left| \begin{array}{c} u = \sqrt{x+a} \\ \frac{du}{dx} = \frac{1}{2\sqrt{x+a}} \implies dx = 2\sqrt{x+a}\, du \\ u = \sqrt{x+a} \implies x = u^2 - a \end{array} \right| \\
&= 2\int \ln(x)\, du \\
&= 2\ln(u^2 - a)\, du \\
&= 2\int \ln(u + \sqrt{a}) + \ln(u - \sqrt{a})\, du \\
&= 2\left(\int (u + \sqrt{a})\, du + \int \ln(u - \sqrt{a})\, du\right)
\end{aligned}
$$

We compute separately:

$$
\begin{aligned}
\int \ln(x+c)\, dx &= \int 1 \cdot \ln(x+c)\, dx \\
&\left| \begin{array}{c} u' = 1 \implies u = x \\ v = \ln(x+c) \implies v' = \frac{1}{x+c} \end{array} \right| \\
&= x\ln(x+c) - \int \frac{x+c-c}{x+c} \\
&= x\ln(x+c) - x + c\ln(x+c) \\
&= (x+c)\ln(x+c) - x + c
\end{aligned}
$$

with

$$
\int \frac{x+c}{x+c} - \frac{c}{x+c} = \int 1 - \frac{c}{x+c} = x - c\ln(x+c) + c
$$

We continue:

$$
\begin{aligned}
&= 2((u + \sqrt{a})\ln(u + \sqrt{a}) - (u + \sqrt{a}) + (u - \sqrt{a})\ln(u - \sqrt{a}) - (u - \sqrt{a})) + c \\
&= 2(u\ln(u^2 - a) + \sqrt{a}\ln\left(\frac{u + \sqrt{a}}{u - \sqrt{a}}\right) - 2u) + c \\
&= 2\sqrt{x+a}\ln(x) + \sqrt{a}\ln\left(\frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} - \sqrt{a}}\right) - 4\sqrt{x+a} + c
\end{aligned}
$$

23 Sheet 6, Exercise 4

Exercise 33. *Let $k \in \mathbb{Z}$, $I_k := ((2k-1)\pi, (2k+1)\pi)$ and*

$$
f : \mathbb{R} \to \mathbb{R}, \qquad f(x) := \frac{1}{3\cos(x) + 5}
$$

1. *Prove for all $x \in I_k$ the identity*

$$\cos(x) = \frac{1 - \tan(x/2)^2}{1 + \tan(x/2)^2}$$

2. *Determine all antiderivatives:*

$$\int f(x)\,dx, x \in I_k$$

   Begin by integration by substitution with $u(x) = \tan(\frac{x}{2})$.

3. *Construct a continuous function $F : \mathbb{R} \to \mathbb{R}$, that is an antiderivative of f on every compact interval.*

23.1 Sheet 6, Exercise 4a

$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos(x)}$$

Proof: Let $u = \frac{x}{2}$ and $x = 2u$.

$$\tan(u) = \frac{\sin 2u}{1 + \cos(2u)} = \frac{2\sin(u)\cos(u)}{1 + \cos^2(u) - \sin^2(u)} = \frac{2\sin(u)\cos(u)}{2\cos^2(u)} = \frac{\sin(u)}{\cos(u)} = \tan(u)$$

Then,

$$\begin{aligned}
\frac{1 - \tan(x/2)^2}{1 + \tan(x/2)^2} &= \frac{1 - \frac{\sin^2(x)}{1+\cos(x)}}{1 + \frac{\sin^2(x)}{(1+\cos(x))^2}} \\
&= \frac{(1 + \cos(x))^2 - \sin^2(x)}{(1 + \cos(x))^2 + \sin^2(x)} \\
&= \frac{1 + 2\cos(x) + \cos(x)^2 - \sin(x)}{1 + 2\cos(x) + \underbrace{\cos(x)^2 + \sin^2(x)}_{=1}} \\
&= \frac{2\cos(x)(1 + \cos(x))}{2(1 + \cos(x))} \\
&= \cos(x)
\end{aligned}$$

23.2 Sheet 6, Exercise 4b

Let $x \in I_k$.

$$\begin{aligned}
\int f(x)\,dx &= \int \frac{1}{3\cos(x)+5}\,dx \\
&= \int \frac{1}{3\left(\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)+5}\,dx \\
&\left|\begin{aligned} u &= \tan(x/2) \\ du &= \tfrac{1}{2\cos^2(x/2)}\,dx \end{aligned}\right| \\
&= \int \frac{1}{3\left(\frac{1-u^2}{1+u^2}+5\right)} 2\cos^2(x/2)\,du \\
&= 2\int \frac{1}{\left(3\left(\frac{1-u^2}{1+u^2}+5\right)+5\right)(1+u^2)}\,du
\end{aligned}$$

$$\cos(x) = \frac{1}{1+\tan^2(x)}$$

We compute separately:

$$\left(\frac{3(1-u^2)+5}{1+u^2}+5\right)(1+u^2) = \frac{3(1-u^2)}{1+u^2}(1+u^2)+5(1+u^2) = 2(4+u^2)$$

$$= 2\int \frac{1}{2}\frac{1}{4+u^2}\,du = \int \frac{1}{4+u^2}\,du = \left|\begin{aligned} t &= \tfrac{u}{2} \\ dt &= \tfrac{1}{2}du \end{aligned}\right| = 2\int \frac{1}{4+4t^2}\,dt = \frac{2}{4}\int \frac{1}{1+t^2}\,dt$$

$$\frac{1}{2}\arctan(t)+c = \frac{1}{2}\arctan\left(\frac{u}{2}\right)+c = \frac{1}{2}\arctan\left(\frac{\tan(x/2)}{2}\right)+c$$

Is expected to be continuously differentiable.

24 Sheet 7, Exercise 1

Exercise 34. *Use the direct comparison criterion to determine the convergence of these integrals:*

$$(a)\int_1^\infty \frac{1}{x^2+5x+1}\,dx \qquad (b)\int_0^\infty \frac{1}{x^s+x^{\frac{1}{s}}}\,dx \quad s \in \mathbb{R}\setminus\{0\}$$

24.1 Sheet 7, Exercise 1a

$$\int_1^\infty \frac{1}{x^2+5x+1}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$$

$$\int_1^\infty \frac{1}{x^p} < \infty \iff p > 1$$

24.2 Sheet 7, Exercise 1b

Case $s = 1$

$$\begin{aligned}\int_0^\infty \frac{1}{x+x}\,dx = \frac{1}{2}\int_0^\infty \frac{1}{x} &= \frac{1}{2}\left(\int_0^1 \frac{1}{x} + \int_1^\infty \frac{1}{x}\right)\\ &= \frac{1}{2}\lim_{t\to\infty}\int_0^t \frac{1}{x}\,dx\\ &= \frac{1}{2}\left(\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx + \lim_{t\to\infty}\int_t^1 \frac{1}{x}\,dx\right)\end{aligned}$$

Case $s < 0$

$$\int_0^\infty \frac{1}{x^s + x^{\frac{1}{s}}}\,dx$$

Because $s < 0$, $x^s + x^{\frac{1}{s}}$ is monotonically decreasing and positive.

$$\frac{1}{x^s + x^{\frac{1}{s}}}$$

is monotonically increasing. More specifically:

$$\int_0^1 \underbrace{\frac{1}{x^s + x^{\frac{1}{s}}}}_{\geq 0} + \int_1^\infty \underbrace{\frac{1}{x^s + x^{\frac{1}{s}}}}_{\geq 1}$$

$$\int_1^\infty 1\,dx = \infty$$

25 Sheet 7, Exercise 2

Exercise 35. *Prove the following statements:*

1. $\forall k \in \mathbb{N} \cup \{0\} : \int_{k\pi}^{(k+1)\pi} |\mathrm{sinc}(x)|\;dx \geq \frac{2}{(k+1)\pi}$.
2. *The improper integral* $\int_0^\infty |\mathrm{sinc}(x)|\;dx$ *does not exist.*

25.1 Sheet 7, Exercise 2a

We apply the Mean Value Theorem:

$$\exists \xi \in [k\pi, (k+1)\pi] : I = \frac{1}{\xi} \int_{u\pi}^{(u+1)\pi} |\sin(x)| \; dx$$

$$\int_{k\pi}^{(k+1)\pi} |\sin(x)| \; dx = \left| \int_{k\pi}^{(k+1)\pi} \sin(x) \, dx \right| = \left| - \cos(x)|_{k\pi}^{(k+1)\pi} \right| = 2$$

$$\implies I = \frac{1}{\xi} 2 \geq \frac{2}{(k+1)\pi} \forall n \in \mathbb{N}$$

$$\overbrace{\operatorname{sinc}(0)}^{=1} \geq \frac{\overbrace{\sin(0)}^{=0}}{\pi}$$

Let $k = 0$:

$$\int_0^\infty \operatorname{sinc}(x) \, dx \overbrace{\implies}^{\text{for } x \neq 0} \operatorname{sinc}(x) = \frac{\sin(x)}{x} \geq \frac{\sin(x)}{\pi} \forall x \in (0, \pi]$$

We can exclude the case $x = 0$, because individual finitely many values don't matter.

$$\geq \int_0^\pi \frac{\sin(x)}{\pi} \, dx = \frac{2}{\pi} = \frac{2}{(k+1)\pi}$$

$$\implies \operatorname{sinc}(x) \geq \frac{\sin(x)}{\pi} \forall x \in [0, \pi]$$

25.2 Sheet 7, Exercise 2b

Sketch of the proof (but it lacks details acc. to the tutor)

$$\int_0^\infty |\sin(x)| \; dx = \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi} |\operatorname{sinc}(x)| \; dx$$

$$\geq \lim_{n\to\infty} \sum_{k=0}^N \underbrace{\frac{2}{(k+1)\pi}}_{\to\infty} = \sum_{k=0}^\infty \frac{2}{k\pi + \pi}$$

$$\lim_{N\to\infty} \frac{2}{\pi} \sum_{k=1}^N \frac{1}{k}$$

$$\int_{k\pi}^{(k+1)\pi} |\sin(x)| \geq \frac{2}{(k+1)\pi}$$

We add some details:

$$\lim_{N\to\infty} \sum_{k=0}^{N} T_n \cdot \triangle x =: \int f$$

$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^b f\,dx$$

$$\lim_{R\to\infty} \int_0^{R\pi} |\mathrm{sinc}(x)|\,dx \geq \lim_{N\to\infty} \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} |\mathrm{sinc}(x)|\,dx$$

26 Sheet 7, Exercise 3

Exercise 36.

27 Sheet 7, Exercise 4

Exercise 37. *Let $n \in \mathbb{N}$. For $k \in \{0, 1, \ldots, n\}$, we define $x_k := \frac{k}{n}$ and the step function*

$$T_n : [0,1] \to \mathbb{R} \qquad T_n(x) := \begin{cases} x_k^2 & \text{if } x \in [x_{k-1}, x_k) \\ 1 & \text{if } x = 1 \end{cases}$$

1. *Show that: For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left\| T_n(x) - x^2 \right\| < \varepsilon$ for all $n \geq N$.*
2. *Determine $\int_0^1 x^2\,dx$ using sequence $(T_n)_{n\in\mathbb{N}}$.*

27.1 Sheet 7, Exercise 4a

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n \geq N : \left\| T_n(x) - x^2 \right\|_\infty < \varepsilon$$

$$\left\| T_n(x) - x^2 \right\|_\infty = 1 - x_{n-1}^2$$

$$\left\| T_n(x) - x^2 \right\|_\infty = 1 - x_{n-1}^2 = 1 - \left(\frac{n-1}{n}\right)^2 = \frac{2n-1}{n^2}$$

1. $\forall x \in [x_{k-1}, x_k) : \left| T_n(x) - x^2 \right| \leq x_k^2 - x_{k-1}^2 = \left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2 = \frac{2k-1}{n^2}$.
Remark: Also $\frac{2k-1}{n^2} \leq \frac{2n-1}{n^2} \to 0$.
Remark: $x_k^2 - x_{k-1}^2 = (x_k - x_{k-1})(x_k + x_{l-1}) = \frac{1}{n}\delta$ with $0 \leq \delta \leq 2$.

2. $\forall k \in \{0, 1, \dots, n-2\} : x_{k+1}^2 - x_k^2 < x_{k+2}^2 - x_{k+1}^2$

$$\left(\frac{k+1}{n}\right)^2 - \left(\frac{k}{n}\right)^2 = \frac{k^2 + 2k + 1 - k^2}{n^2} < \frac{2k+3}{n^2}$$

$$= \frac{k^2 + 4k + 4 - (k^2 + 2k + 1)}{n^2} = \left(\frac{k+2}{n}\right)^2 - \left(\frac{k+1}{n}\right)^2$$

27.2 Sheet 7, Exercise 4b

$$\int_0^1 x^2 \, dx$$

By exercise (4a), it follows that $\lim_{n\to\infty} \left\|T_n - x^2\right\|_\infty = 0$.

$$\int_0^1 x^2 \, dx = \lim_{n\to\infty} \int_0^1 T_n(x) \, dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n x_k^2 = \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{1}{3}$$

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{6} \cdot \left[1 \cdot \left(1 + \frac{1}{n}\right) \cdot \left(2 + \frac{1}{n}\right)\right]$$

The integral is independent of the particular chosen approximating sequence (see lecture notes).

27.3 Remark on integrals

You are allowed to change a regulated function in countable infinite many points. Its limit won't change.

$$\int_a^b f \, dx$$

$$\tilde{f} := \begin{cases} f(x) & x \in (a, b] \\ 0 & x = a \end{cases}$$

Then $\int_a^b f \, dx = \int \tilde{f} \, dx$.

This lecture took place on 2018/05/24.

28 Sheet 8, Exercise 1

Exercise 38. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be given by* $f(x) := \cosh(2x)$.

1. *Determine* $f^{(n)}(x)$ *and* $T_f^n(x;0)$ *for* $n \geq 0$ *and furthermore* $T_f(x;0)$
2. *Show that for all* $x \in \mathbb{R}$ *it holds that* $R_f^{n+1}(x;0) \to 0$ *for* $n \to \infty$*. You can use the Lagrange representation of the Taylor remainder* R_f^{n+1}*.*

28.1 Sheet 8, Exercise 1a

$$T_f^n(x;0) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) x^k = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \frac{1}{k!} 2^k \underbrace{\cosh(0)}_{=1} x^k + \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \frac{1}{k!} 2^k \underbrace{\sinh(0)}_{=0} x^k$$

$$= \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \frac{2^k}{k!} x^{-k}$$

$$T_f(x;0) = \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} x^{2k}$$

28.2 Sheet 8, Exercise 1b

$$R_p^{n+1}(x;0) = \frac{1}{(n+1)!} f^{n+1}(\xi) x^{n+1}$$

$$\xi \in (x,0) \cup (0,x)$$

$$\left| \frac{1}{(n+1)!} f^{n+1}(\xi) x^{n+1} \right| \leq \frac{x^{n+1}}{(n+1)!} 2^{n+1} \left|\cosh(2\xi)\right|$$

$$\leq \frac{\left|x^{n+1}\right|}{(n+1)!} 2^{n+1} \underbrace{\cosh(2x)}_{\text{constant}} \xrightarrow{n \to \infty} 0$$

28.3 Sheet 8, Exercise 1c

$$T_f(x;0) = \sum_{k=0}^{\infty} \frac{2^{(2k)}}{(2k)!} x^{2k}$$

$$\underbrace{\lim_{n \to \infty} R_f^{n+1}(x;0)}_{0} = \lim_{n \to \infty} (f(x) - T_f^n(x;0)) = f(x) - T_f(x;0)$$

$$0 = f(x) - \lim_{n \to \infty} T_f^n(x;0)$$

with $\lim_{n\to\infty}(f(x) - T_f^n(x;0)) = \lim_{n\to\infty} T_f^n(x;0)$. As $\lim_{n\to\infty}(f(x) - T_f^n(x;0))$ converges, it holds that $\lim_{n\to\infty}(f(x) - T_f^n(x;0)) = \lim_{n\to\infty} f(x) - \lim_{n\to\infty} T_f^n(x;0)$. So we do not need to show convergence of $\lim_{n\to\infty} T_f^n(x;0)$.

28.4 Sheet 8, Exercise 1d

Show that

$$\left|f(x) - T_f^8(x;0)\right| < \frac{|x|^9}{700}\left|\sinh(2x)\right| < \frac{|x|^9}{1400}e^{2|x|}$$

$$\left|R_f^9(x;0)\right| = \frac{1}{9!}\left|f^{(9)}(\xi)x^9\right| \qquad \xi \in (0,x) \vee (x,0)$$

$$= \frac{|x|^9}{9!}2^9\left|\sinh(2\xi)\right| < \frac{|x|^9}{700}\left|\sinh(2\xi)\right|$$

Show:

1. $$\frac{2^9}{9!} \overset{!}{<} \frac{1}{700}$$

   $$\frac{2\cdot 2^2\cdot 2^3\cdot 2^3\cdot 4}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9} = \frac{4}{81\cdot 35} < \frac{4}{80\cdot 35} = \frac{1}{700}$$

2. $$\left|\sinh(2\xi)\right| < \left|\sinh(2x)\right|$$

   $$x > 0 \implies \xi > 0$$

   because of monotonicity.

   $$x < 0 \implies x < \xi < 0$$

28.5 Sheet 8, Exercise 1e

$$\sinh(2x) = \frac{1}{2}(\underbrace{e^{2x}}_{>0} - \underbrace{e^{-2x}}_{>0}) < \frac{1}{2}e^{2x}$$

$$\frac{1}{2}\left|e^{2x}\right| = \frac{1}{2}$$

$$x < 0 : \left|\sinh(2x)\right| < +\frac{e^{2|x|}}{2} \qquad \forall x \in [-1,1]$$

$$\left|f(x) - T_f^8(x;0)\right| < \frac{6}{1000} \qquad x = 0$$

$$\left|f(x) - T_f^8(x;0)\right| < \frac{|x|^9}{1400}e^{2|x|}$$

$$\frac{|x|^9}{1400}e^{2|x|} \le \frac{|x|^9}{1400}e^2 < \frac{2.8^2}{1400} = \frac{28\cdot 26}{140000} = \frac{7}{1250} < \frac{6}{1000}$$

29 Sheet 8, Exercise 2

Exercise 39. *Let $n \in \mathbb{N} \cup \{0\}$, $a > 0$, $I := [-a, a]$ and $f : I \to \mathbb{R}$ n-times differentiable.*

1. *Show: If f is even, i.e. $f(x) = f(-x) \forall x \in I$, $T_f^n(x; 0)$ is even*
2. *Show: If f is odd, i.e. $f(x) = -f(-x) \forall x \in I$, $T_f^n(x; 0)$ is odd*
3. *Prove that the inverse statements of (a) and (b) are wrong. Use $g : I \to \mathbb{R}, g(x) := x^{n+1}$ for $x > 0$, $g(x) = 0$.*
4. *Prove that a and b also hold for $T_f(x; 0)$ instead of $T_f^n(x; 0)$ if f is arbitrary often differentiable.*
5. *Show that the inverse of statements (a) and (b) are also wrong for $T_f(x; 0)$ instead of $T_f^n(x; 0)$, if f is arbitrarily often differentiable.*

29.1 Sheet 8, Exercise 2a

$$T_f^n = \ln(x_0) + \sum_{k=1}^{n} \underbrace{\frac{(-1)^{k+1}}{x_0^k \cdot k}}_{a_k} (x - x_0)^k$$

29.2 Sheet 8, Exercise 2b

$$\text{Cauchy-Hadamard} \implies \rho = \left(\limsup_{k \to \infty} \sqrt[k]{|a_k|} \right)^{-1}$$

Area of convergence: $(0, 2x)$

Outside the area of convergence, the series diverges.

$$\left(\limsup_{k \to \infty} \frac{1}{|x_0| \cdot \sqrt[k]{k}} \right)^{-1} = \left(\frac{1}{|x_0|} \right)^{-1} = x_0$$

Consider $x = 2x_0$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{x_0^k \cdot k} \cdot x_k^k \implies \text{converges}$$

Consider $x = 0$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{2k}(-1)}{x_0^k \cdot k} x_0^k \implies \text{diverges}$$

Thus, the actual area of converge is $(0, 2x_0]$.

29.3 Sheet 8, Exercise 2c

Show that:

$$\lim_{n\to\infty} R_f^{n+1}(x;x_0) = 0$$

$$\begin{aligned}
\left|R_f^{n+1}(x;x_0)\right| &= \left|\frac{1}{n!}\int_{x_0}^{x}(x-t)^n \cdot f^{(n+1)}(t)\,dt\right| \\
&= \left|\frac{1}{n!}\int_{x_0}^{x}(x-t)^n \frac{(-1)^n n!}{t^{n+1}}\,dt\right| \\
&= \left|\int_{x_0}^{x}\frac{(x-t)^n}{t^{n+1}}\,dt\right| \\
&= \left|\int_{x_0}^{x}\frac{1}{t}\cdot\left(\underbrace{\frac{x}{t}-1}_{=:q}\right)^n\,dt\right|
\end{aligned}$$

$$\begin{aligned}
&\sup\left\{\frac{x}{t}-1\middle|*\right\} \\
&t \in [x_0, x] \\
&x \in [x_0, 2x_0) \\
&= \underbrace{\frac{x}{x_0}}_{<2} - 1 < 1
\end{aligned}$$

Whence, consider $x = x_0$,

$$\left|\int_{x_0}^{x}\frac{1}{t}\cdot(q)^n\;dt\right| \le \left|\tilde{q}^n\right|\cdot|\ln(x)-\ln(x_0)| \xrightarrow{n\to\infty} 0$$

The identity in the assignment implies that $T_f(x;x_0)$ converges.

$T_f(x;x_0)$ does not converge at $x = 0$.

30 Sheet 8, Exercise 3

Exercise 40. *Let $n \in \mathbb{N}\cup\{0\}, a > 0, I := [-a,a]$ and $f : I \to \mathbb{R}$ n-times differentiable.*

1. *Show: If f is even, i.e. $f(x) = f(-x)\forall x \in I$, $T_f^n(x;0)$ is even.*
2. *Show: If f is odd, i.e. $f(x) = -f(-x)\forall x \in I$, $T_f^n(x;0)$ is odd.*

30.1 Sheet 8, Exercise 3a

$f(x)$ is even, then $f'(x)$ is odd $\iff f'(x) = -f'(-x)$. How?

$$f(x) = f(-x) \iff f(x) = f((-1)\cdot(x)) \implies f'(x) = -f'(-x)$$

$$\begin{aligned}
f(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
T_f^n(x;0) &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0) \cdot x^k \\
&= \sum_{\substack{k=0 \\ k \bmod 2 \equiv 0}}^{n} \frac{1}{k!} f^{(k)}(0) \cdot x^k + \sum_{\substack{k=1 \\ k \bmod 2 = 1}}^{n} \overbrace{\frac{1}{k!} f^{(k)}(0) \cdot x^k}^{=0} \\
&= \sum_{\substack{k=0 \\ k \bmod 2 \equiv 0}}^{n} \frac{1}{k!} f^{(k)}(0) \cdot (x)^k = T_f^n(-x, 0)
\end{aligned}$$

30.2 Sheet 8, Exercise 3b

Analogous to Exercise 3a.

30.3 Sheet 8, Exercise 3c

$$g : I \to \mathbb{R}$$

$$x \mapsto \begin{cases} x^{n+1} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(0) \cdot x^k$$

$$g^{(0)} = 0 \qquad g^{(k)}(0) = 0 \forall k \leq n$$

Do not skip to show that $x = 0$ in all derivatives is zero.

30.4 Sheet 8, Exercise 3d

$$f(x) = f(-x) \overset{!}{\implies} T_f(x_0) = T_f(-x, 0)$$

$$T_f(x, 0) = \lim_{n\to\infty} T_f^n(x, 0) = \lim_{n\to\infty} \left(T_f^n(-x, 0)\right)$$

This implies that the Taylor series converges.

30.5 Sheet 8, Exercise 3e

Find a function that is differentiable infinitely often, is even and odd and $f(0) = 0$.

$$k(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

31 Sheet 9, Exercise 3

31.1 Sheet 9, Exercise 3a

$$n \in \mathbb{N}, t \in \mathbb{R} : \frac{1}{1+t^2} = \frac{(-t^2)^{n+1}}{1+t^2} + \sum_{k=0}^{n} (-t^2)^k$$

Let $z := -t^2$ and we are done (the domains of $-t^2$ and z also match).

31.2 Sheet 9, Exercise 3b

We already know:

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

By the fundamental theorem of differential and integration calculus, variant 1:

$$\implies \int_0^x \left(\frac{d}{dt} \arctan(x) \right) dt = \arctan(x)$$

$$\begin{aligned} \arctan(x) &= \int_0^x \left[\frac{(-t^2)^{n+1}}{1+t^2} + \sum_{k=0}^{n} (-t^2)^k \right] \\ &= \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} \, dt + \sum_{k=0}^{n} \int_0^x (-t^2)^k \, dt \\ &= \int_0^x \frac{(-t^2)^{n+1}}{1+t^2} \, dt + \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} \end{aligned}$$

31.3 Sheet 9, Exercise 3c

$$\forall x \in [-1, 1] : \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

Show that $\lim_{n\to\infty} \int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\, dt = 0$.

$$\left|\int_0^x \frac{(-t^2)^{n+1}}{1+t^2}\, dt\right| \leq \left|\int_0^x (-t^2)^{n+1}\, dt\right| = \left|\int_0^x \left|-t^2\right|^{n+1}\, dt\right|$$

$$= \left|\int_0^x t^{2n+2}\, dt\right| = \left|\frac{t^{2n+3}}{2n+3}\Big|_0^x\right| = \frac{|x|^{2n+3}}{2n+3} \leq \frac{1}{2n+3} \xrightarrow{n\to\infty} 0$$

31.4 Sheet 9, Exercise 3d

$$1 - \frac{1}{3} + \frac{1}{5} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 1^{2k+1}}{2k+1} = \arctan(x)$$

32 Sheet 9, Exercise 4

32.1 Sheet 9, Exercise 4a

$$P(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \qquad Q(z) = \sum_{k=0}^{\infty} a_k (f(z) - z_0)^k$$

$$f : \substack{\mathbb{C}\to\mathbb{C} \\ \hat{z}\mapsto\overline{z}}$$

Show that $P(z)$ converges $\iff$ $Q(\hat{z})$ converges.

$$P(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (\overline{z} - z_0)^k = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k (f(\hat{z}) - z_0)^k$$
$$= \sum_{k=0}^{\infty} a_k (f(\hat{z}) - z_0)^k = Q(\hat{z})$$

32.2 Sheet 9, Exercise 4b

$$P(z) = \sum_{k=0}^{\infty} z^k \qquad Q(z) = \sum_{k=0}^{\infty} (-1)^k (z^2)^k = \sum_{k=0}^{\infty} (-z^2)^k$$

with $P(z) = \frac{1}{1-z}$.

$$f(z) = -z^2$$

$$|z| < 1 \iff \left|z^2\right| < 1 \iff \left|-z^2\right| < 1$$

32.3 Sheet 9, Exercise 4c

Determine the root function of $Q(z)$ with $z \in \mathbb{R}$.

$$\int Q(z)\,dz = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k-1} + C$$

These are all root functions. But are these all root functions? Yes. There is some C such that this integral becomes arctan, specifically $C = 0$.

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1} = \arctan(z) \qquad \forall z \in (-1, 1)$$

33 Sheet 9, Exercise 2

33.1 Sheet 9, Exercise 2a

$$f : \mathbb{R} \setminus \{-1, 2\} \to \mathbb{R}$$
$$f(x) = \frac{x+3}{x^2 - x - 2}$$
$$\frac{-\frac{2}{3}}{x+1} + \frac{\frac{5}{3}}{x-2}$$
$$f^{(n)}(x) = \frac{\frac{2}{3}n! \cdot (-1)^n}{(x+1)^{n+1}} + \frac{\frac{5}{3}n!(-1)}{(x-2)^{n+1}}$$
$$T_f^n(x; 0) = \sum_{k=0}^{n} \frac{-\frac{2}{3}k!(-1)^k + \frac{5}{3}k!(-1)^k}{(-2)^{k+1}} k! \cdot x^k$$

33.2 Sheet 9, Exercise 2b

$$T_f^2(x; 0) = -\frac{3}{2} + \frac{x}{4} - \frac{7}{8}x^2$$

$\xi \in (0, x)$.

$$R_3 = \left| \frac{f^3(\xi)}{3!} x^3 \right| = \left| \frac{\frac{4}{(\xi+1)^4} - \frac{10}{(\xi-2)^4}}{6} x^3 \right| = \frac{\frac{5}{(\xi-2)^4} - \frac{2}{(\xi+1)^4}}{3} x^3$$
$$\frac{5}{(1-2)^4} - \frac{2}{(1+1)^4} = \frac{\frac{39}{8}}{3}$$
$$\left| \frac{16}{375} - \frac{32}{3} \right| = \frac{10.624}{1000}$$

33.3 Sheet 9, Exercise 2c

$$T_f(x,0) = f(x) \forall x \in [0,1)$$
$$T_f^n(x;0) - f(x) = R_f^{n+1}(x;0)$$
$$R^{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{-\frac{2}{3}}{(\xi+1)^{n+2}} + \frac{\frac{5}{3}}{(\xi-2)^{n+2}} |x|^{n+1} = 0$$

33.4 Sheet 9, Exercise 2d

Not so easy.

33.5 Sheet 9, Exercise 2e

If convergence radius > 1, then function is continuous and series is continuous in all points (smaller the radius). Contradicts with $f(-1)$ is excluded from the set. But another approach works better: Continuous functions on compact sets are bounded.

Cauchy-Hadamard:

$$\rightsquigarrow \sqrt[n]{\left|-\frac{5}{3} \cdot \frac{(-1)(-1)^n}{2^{n+1}} + \frac{2}{3}\right|}^{-1} \leq \sqrt[n]{\frac{2}{3}} \underbrace{\sqrt[n]{1 + \frac{5}{2^{n+2}}}}_{\leq \sqrt[n]{3}}$$

34 Sheet 10, Exercise 2

Exercise 41. *Show that the following functions are differentiable and determine the corresponding Jacobi matrix.*

1. $f : \mathbb{R} \to \mathbb{R}^4, f(x) := \begin{pmatrix} 3x^2 \\ \sin(3x) \\ 42 \\ \cos(x^2) \end{pmatrix}$
2. $g : \mathbb{R}^3 \to \mathbb{R}^2, g(x) := \begin{pmatrix} 4x^2y^3 \\ xye^z + e^x y \end{pmatrix}$
3. $h : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}, h(x,y,z) := \sin(zx)\ln(x+y^2)$

34.1 Sheet 10, Exercise 2a

$$f'(x) = \begin{pmatrix} 6x \\ \cos(3x) \cdot 3 \\ -\sin(x^2) \cdot 2x \end{pmatrix}$$

34.2 Sheet 10, Exercise 2b

$$g'(x, y, z) = \begin{pmatrix} 8xy^3 & 12x^2y^2 & 0 \\ ye^z + e^x y & xe^z + e^x & xye^z \end{pmatrix}$$

34.3 Sheet 10, Exercise 2c

$$h'(x, y, z) = \begin{pmatrix} \cos(zx) \cdot z \cdot \ln(x + y^2) + \frac{\sin(zx)}{x+y^2} \\ \frac{\sin(zx) \cdot 2y}{x+y^2} \\ \cos(zx) \cdot x \cdot \ln(x + y^2) \end{pmatrix}$$

34.4 Remark on differentiability

Continuous partial differentiability implies total differentiability implies continuity. Continuous partial differentiability implies total differentiability implies partial differentiability.

35 Sheet 10, Exercise 3

Exercise 42. *Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$f(x, y) := \begin{cases} \frac{xy^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that,

1. *f is continuous and continuously partially differentiable.*
2. *f is two times partially differentiable and $\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0)$.*
3. *$f \in C^2(\mathbb{R}^2 \setminus \{0\})$.*
4. *Computationally: One of the second partial derivatives of f is non-continuous in $(0, 0)$.*
5. *The statement of (d) without calculations.*

35.1 Sheet 10, Exercise 3a

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{\overbrace{f(x,0)}^{0} - \overbrace{f(0,0)}^{0}}{x} = \lim_{x\to 0} = 0$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{y\to 0} \frac{\overbrace{f(0,y)}^{0} - \overbrace{f(0,0)}^{0}}{y} = \lim_{y\to 0} 0 = 0$$

Let $(x,y) \neq (0,0)$.

$$\frac{\partial f}{\partial x}(x,y) = \frac{y^3(x^2+y^2) - 2x(xy^3)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} \frac{3xy^2(x^2+y^2) - 2y(xy^3)}{(x^2+y^2)^2}$$

The partial derivatives exist everywhere except $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{y^3(x^2+y^2) - 2x(xy^3)}{(x^2+y^2)^2} = \lim_{(x,y)\to(0,0)} \frac{x^2y^3 + y^5 - 2x^2y^3}{x^4 + 2x^2y^2 + y^4}$$

Continuity follows from total differentiability.

35.2 Sheet 10, Exercise 3b

It is not obvious that this fraction is two times differentiable. $\frac{1}{x}$ is not differentiable on

$$(x,y) \neq (0,0) \qquad \frac{\partial f^2}{\partial x \partial y} = \frac{-3x^4y^2 + 6x^2y^4 + y^6}{(x^2+y^2)^3} = \frac{\partial f^2}{\partial y \partial x}$$

The Schwarz' Theorem requires that $f \in C^2$.

Either we only determine one partial derivative and copy it for the second derivative. Or we compute both of them, but then they need to equate and we can use this for verification.

$$\lim_{y\to 0} \frac{-3x^4y^2 + 6x^2y^4 + y^6}{(x^2+y^2)^3}$$

Continuous function, because $x \neq 0$ and $y \neq 0$.

$$\partial x \partial y f(0,0) = \lim_{h\to 0} \frac{1}{h}(\partial x f(h,0) - \partial x f(0,0)) = \lim_{h\to 0} \frac{1}{h}\left(\frac{-3h^4 \cdot 0^2 + 6h^2 \cdot 0^4 + 0^4}{(h^2+0^2)^3}\right) = \lim_{h\to 0} \frac{1}{h} 0 = 0$$

$$\partial y \partial x f(0,0) = \lim_{h\to 0} \frac{1}{h}(\partial x f(0,h) - \partial x f(0,0)) = \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^5}{h^4} = 1$$

35.3 Sheet 10, Exercise 3c

We prove it using polar coordinates.

$$1 = \sin(\varphi_n^2) + \cos(\varphi_n^2)$$

$$x_n = r_n \cos(\varphi_n) \qquad y_n = r_n \sin(\varphi_n)$$

$$(x_n, y_n) \to \vec{0} \iff (r_n \cos(\varphi_n), r_n \sin(\varphi_n)) \to \vec{0} \iff r_n \to 0$$

Let $\{(x_n, y_n)\}$ be a sequence with limit $\vec{0}$. Show: $\lim_{n\to\infty} \frac{\partial f}{\partial x}(x_n, y_n) = \frac{\partial f}{\partial x} f(0,0) = 0$.

$$\begin{aligned}\frac{\partial f}{\partial x}(x_n, y_n) &= \frac{r_n^5 \sin(\varphi_n^5) - r_n^2 \cos(\varphi_n^2) r_n^3 \sin(\varphi_n^3)}{(rn^2 \cos(\varphi_n^2) + r_n^2 \sin(\varphi_n^2))^2} \\ &= \underbrace{r_n}_{\to 0} \underbrace{(\sin(\varphi_n^5) - \cos(\varphi_n^2)\sin(\varphi_n^3))}_{\text{bounded}} \to 0\end{aligned}$$

$$\left| \frac{y^3(x^2+y^2) - 2x(xy^3)}{(x^2+y^2)^2} \right|$$

35.4 Sheet 10, Exercise 3d

$$\partial x \partial y f(x,y) = \frac{-3x^4y^2 + 6x^2y^4 + y^6}{(x^2+y^2)^3}$$

Let $\{(X_h, X_h)\}$ be a sequence of $(X_n, X_n) \to \vec{0}$ for $n \to \infty$.

Show that: $\lim_{x_n \to 0} \partial_{xy} f(x_n, x_n) = \partial_{xy} f(0,0) = 0$.

35.5 Sheet 10, Exercise 3e

We know $f \in C^2(\mathbb{C} \setminus \{0\})$ and $\partial_{xy} f(0,0) \neq \partial yx f(0,0)$.

We use the inverse ($p \implies q$ also gives $\not q \implies \not p$) of the Schwarz' Theorem: Because symmetry of derivatives ($f \notin C^2(\mathbb{R}^2)$) is not given, the two-times partial derivative is non continuous in $(0,0)$.

Why is the point of non-continuity $(0,0)$? Well, we showed continuity for $\mathbb{R}\setminus\left\{\vec{0}\right\}$, so $(0,0)$ remains.

$$\exists (x,y) \in \mathbb{R} : \partial x \partial y f(x,y) \neq \partial y \partial x f(x,y)$$

36 Sheet 10, Exercise 4

Exercise 43. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by*

$$f(x, y) := \sqrt{|xy|} \qquad g(x, y) := |x| + |y|$$

1. *Determine the level sets $\left\{ (x, y)^T \in \mathbb{R}^2 \;\middle|\; f(x, y) = c \right\}$ and accordingly $\left\{ (x, y)^T \in \mathbb{R}^2 \;\middle|\; g(x, y) = c \right\}$ for $c \in \mathbb{R}$.*
2. *Determine the gradients ∇f and ∇g, assuming they exist.*

37 Sheet 10, Exercise 4a

$$y = \pm(c - |x|)$$

Consider $c = 0$.

$$|x| + |y| = 0 \iff x = 0 \land y = 0$$

Consider $c < 0$.

$$\Gamma_0 = \emptyset$$

Consider $c > 0$.

$$|x| + |y| = c$$

$$\Gamma = \left\langle (x, y) \in \mathbb{R}^2 \middle| |x| + |y| = c \right\rangle$$

38 Sheet 10, Exercise 4b

The gradients do not exist on $\mathbb{R}^2$. But the gradients exist if $x \neq 0 \lor y \neq 0$.

$$\nabla g(x, y) = \begin{bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{bmatrix}$$

$$\frac{\partial g}{\partial x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The gradients exist if $x \neq 0 \land y \neq 0$:

$$\frac{\partial g}{\partial x}(0, y) = \frac{d}{dx} |x|$$

$$f(x) = |x| \qquad \frac{df}{dx}(0)$$