Mathematical analysis 2 – Lecture notes
course by Wolfgang Ring

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March to July 2016

Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15

- Thursday, 45 minutes, beginning at 8:15

- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n \in \mathbb{N}}$ be a complex series with $\lim_{n \to \infty} z_n = z$ and $\lim_{n \to \infty} (1 + \frac{z_n}{n})^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\exp(z + w) = \exp(z) \cdot \exp(w)$$

$$\lim_{z \to 0} \frac{\exp(z) - 1}{z} = 1$$

$$\exp(1) = e \in \mathbb{R}$$

$$z = \frac{m}{n} \in \mathbb{Q} \wedge n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z \qquad \text{for } z \in \mathbb{C}$$

It holds that

$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$

$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^0 = 1$$

Let $n \in \mathbb{N}$

$$\lim_{x \to +\infty} \frac{e^x}{x^n} = \infty$$

$$\lim_{x \to -\infty} e^x \cdot x^n = 0$$

Figure 1: Graph of the exponential function
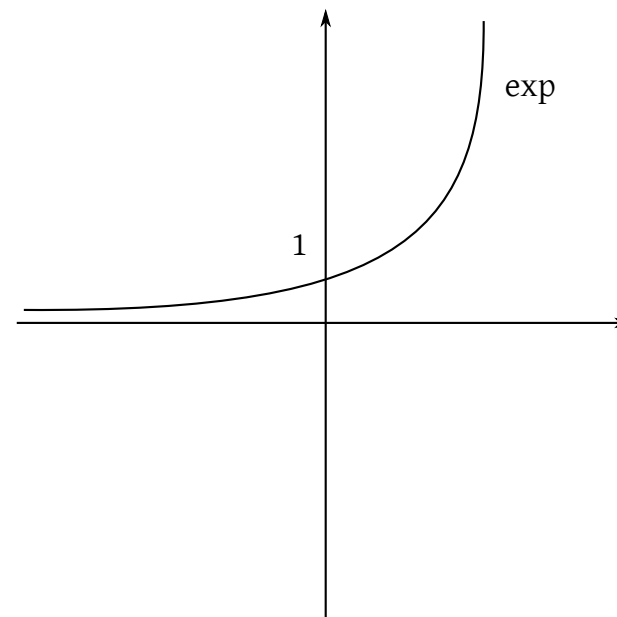
2 The natural logarithm

$$\exp : \mathbb{R} \to (0, \infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. $\exp : \mathbb{R} \to (0, \infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y] : \quad e^\xi = y$$

Case 2 Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w] : \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp : \mathbb{R} \to (0, \infty)$ is bijective. $\qquad \square$

Definition 1. We call the inverse function *natural logarithm*[1].

$$\exp^{-1} : (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R} : \ln(e^x) = x$ and $\forall y \in (0, \infty) : e^{\ln(y)} = y$.

- $\ln : (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

  Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xRightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! $\qquad \square$

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[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Limes:

$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Proof.
-
$$x \cdot y = e^{\ln(x \cdot y)}$$

$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$

Injectivity of exp:

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.

$$\lim_{n \to \infty} y_n = \ln(1) = 0$$

$$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$

where

$$e^0 = 1 \Rightarrow \ln(1) = 0$$

$\qquad \square$

Theorem 1 (Logarithmic growth). $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. $\qquad \square$

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.
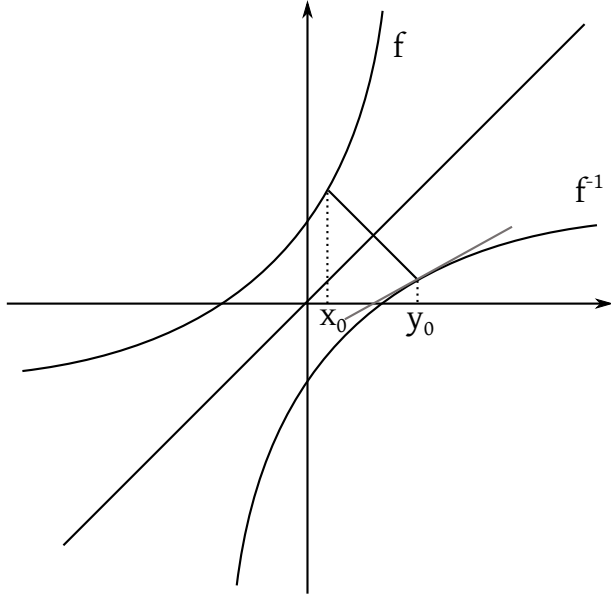

Figure 2: A geometric proof of differentiability

Proof. **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n \to \infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n \to \infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n \to \infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n \to \infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$. $\qquad \square$

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}}$$

$$= \frac{1}{x} \cdot \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x} \to 0$$

$1 + \frac{h}{x} = w$ then it holds that $h \to 0 \Rightarrow w \to 1$.

$$\frac{h}{x} = w - 1$$

$$\lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{=} \lim_{h \to 0} \frac{\ln(w)}{w - 1} = 1$$

$\square$

Remark 1. The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C} \to \mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C} \setminus \{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I \to \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I) \to \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$
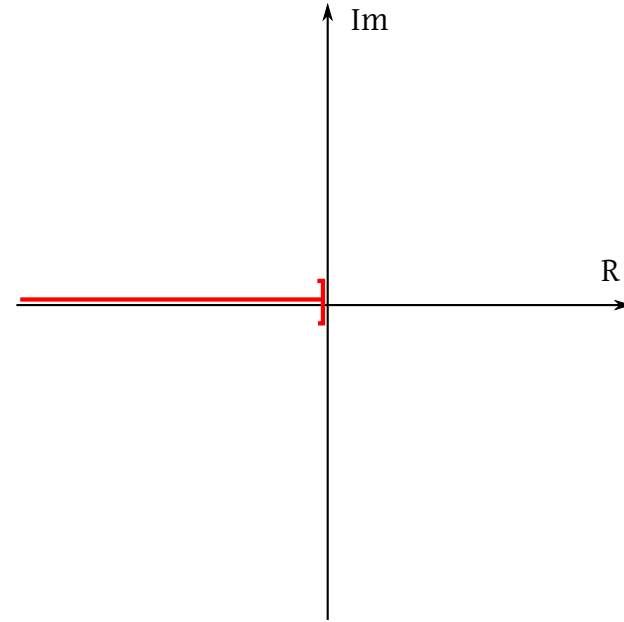
Figure 3: Continuous exponential function in $\mathbb{C}$

Proof. Let $y_n \to y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \Rightarrow x_n \neq x_0$.

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}$$

$$= \lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{\underbrace{\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}}_{\text{ex} = f'(x_0)}} = \frac{1}{f'(x_0)}$$

$\square$

Lemma 2. Let $f : I \to \mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \, \forall x \in I$$

Proof. $\Rightarrow$ Immediate.

$\Leftarrow$ Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

$\square$

Definition 2. Let I be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{const}$$

$\square$

Theorem 4. Let I be an interval. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n \to \infty} f_n(x)$ ($f_n \to f$ pointwise)

2. for every $x \in I$ let $(f_n'(x))_{n \in \mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n \to \infty} f_n'(x)$ exists for every x)

3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon |u - v| \, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n \to \infty} f_n'(x)$.

$$f'(x) = [\lim_{n \to \infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - \lim_{n \to \infty} f_N'(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0) \right| + \left| f_N'(x_0) - \lim_{n \to \infty} f_n'(x_0) \right| \forall N \in \mathbb{N}$$

$$\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right|$$

$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right| + |f_N'(x_0) - \varphi(x_0)|$$

1st term

$$\left| \frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0} \right| = \left| \frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0} \right|$$

$$\leq \frac{\varepsilon}{3} \frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left| \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| - f_N'(x_0)$$

Differentiability of f_N: Therefore for $|x - x_0| < \delta$.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$. $\qquad\square$

Theorem 5. Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ ($n \in \mathbb{N}$) and f_n is differentiable in I.

Assumption:

1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)

2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise

- $f_n'(x) \to \varphi(x)$ for every x

- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon |u - v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n \to \infty$

- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in I, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f_n'(x) - g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6.

We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence
$$\forall x \in I : f_n'(x) \to g(x)$$

3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:
$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$|f_n'(x) - f_m'(x)| = |f_n'(x) - g(x) + g(x) - f_m'(x)|$$

$$\leq |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $(f_n)_{n \in \mathbb{N}}$ is a uniform Cauchy sequence.

Let $\varepsilon > 0$ be arbitrary and N such that $n, m \geq N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m \to \infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$= \lim_{m \to \infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)|$$

$$= \lim_{m \to \infty} |f_m'(\xi_{m,n}) - f_n'(\xi_{m,n})| \cdot |u - v|$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$
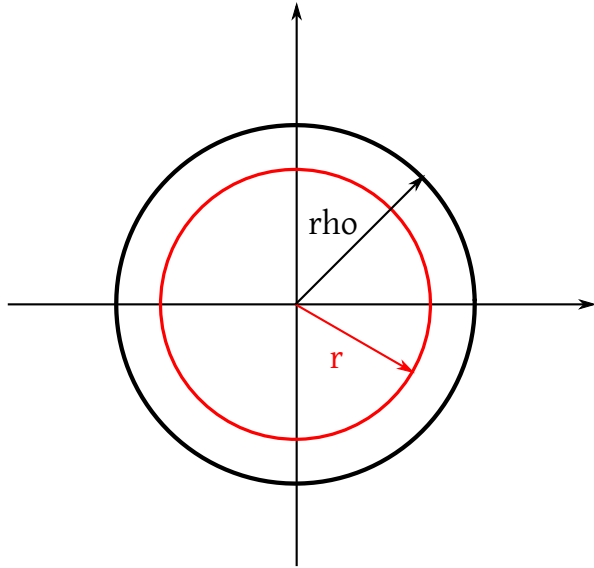
So the third condition of Theorem 6 is satisfied.

Figure 4: Convergence radius

Remark 2 (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

□ Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j \to \infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j \to \infty} |a_{j+1}|^{\frac{j+1}{j} \cdot \frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j} \cdot \frac{1}{j+1}}$$

$$= \limsup_{j \to \infty} \underbrace{\left(|a_{j+1}|^{\frac{1}{j+1}} \right)^{\frac{j+1}{j}} \overset{\to 1}{\overbrace{}}}_{L^1 = L} \cdot \underbrace{\lim_{j \to \infty} \left[(j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

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[2] Where overline means "closed"

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k \to \infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k \to \infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$

Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1 - (-x)) = \ln(1 + x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

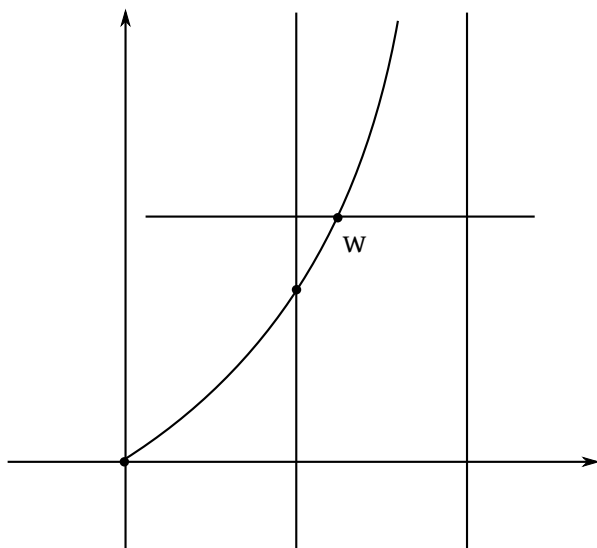$$f'(x) = \frac{1 - (-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\Rightarrow 1 + x = w - wx$$

$$x(1 + w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

Figure 5: Plot of $\frac{1+x}{1-x}$

3   Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left(\underbrace{\frac{1}{\mathbb{R}}}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$
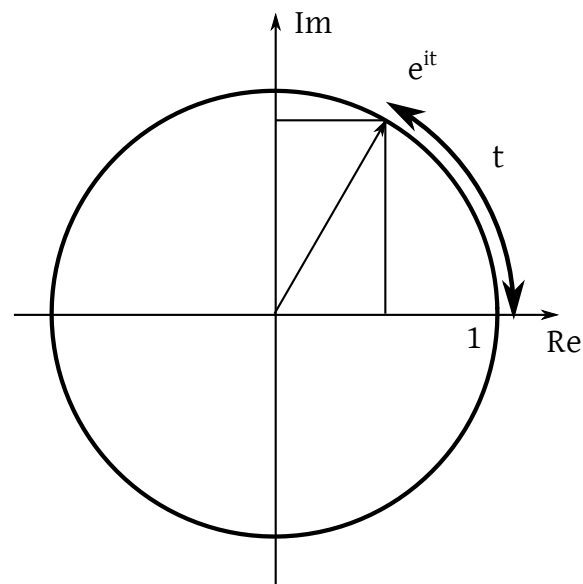
$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n} \right)^n = \lim_{n\to\infty} \left[\overline{\left(1 + \frac{it}{n} \right)} \right]^n$$

$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n} \right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n} \right)^n} = \overline{e^{it}}$$

$$\left| e^{it} \right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left| e^{it} \right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)

2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$

3.
$$\Re(z) = \frac{1}{2}(z + \bar{z})$$

$$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$

$$\Im(z) = \frac{1}{2i}[z - \bar{z}]$$

$$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$

4.
$$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$

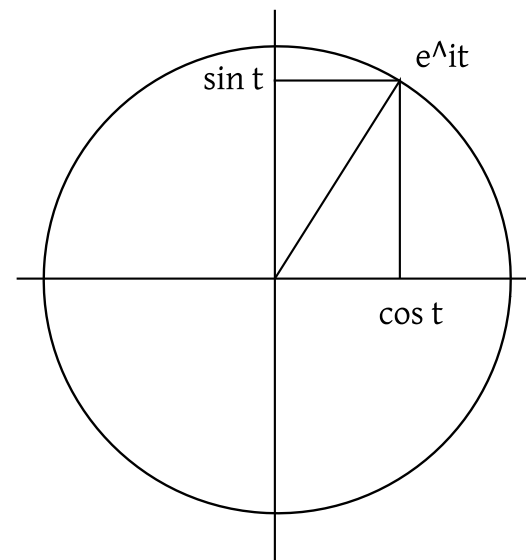$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$

Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence). Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

series sum. Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

$\square$

Example 1 (Application). Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0, \delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0, \delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \cdots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty}(-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \cdots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \cdots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \cdots$$

Followingly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots$$

$$= \sum_{l=0}^{\infty}(-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C}$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \cdots$$

$$= \sum_{l=0}^{\infty}(-1)^l \frac{z^{2l+1}}{(2l+1)!}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and substraction theorems)**.** We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$= (\cos(z+w) + i \sin(z+w))$$
$$\Rightarrow = (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z \cos w)$$

Analogously,

$$e^{-(z+w)} = e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w))$$
$$= \cos z \cdot \cos w - \sin z \sin w + i(-\cos z \sin w - \cos w \sin z)$$

but also

$$= (-\cos(z+w) + i \sin(-(z+w)))$$
$$\Rightarrow = \cos(z+w) - i \sin(z+w)$$

Addition:

$$2\cos(z+w) = 2(\cos z \cdot \cos w - \sin z \sin w)$$
$$\Rightarrow \cos(z+w) = \cos z \cos w - \sin z \sin w$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \,\forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\cos(z-w) = \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)}$$
$$\sin(z-w) = -\cos z \cdot \sin(w) + \sin(z)\cos(w)$$

Corollary 1.

$$z = \frac{1}{2}(z+w) + \frac{1}{2}(z-w)$$
$$\Rightarrow \cos z = \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2}$$
$$w = \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w)$$
$$\cos w = \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2}$$
$$\cos z - \cos w = -2\sin\frac{z+w}{2}\sin\frac{z-w}{2}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\lim_{\substack{z\to 0\\z\neq 0}} \frac{\sin z}{z} = \lim_{z\to 0}\frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right)$$
$$= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz}-1}{2iz}\right)$$
$$= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0 = 1} \cdot \underbrace{\lim_{z\to 0}\frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz; z\to 0 \Leftrightarrow w=0 \\ \lim_{w\to 0}\frac{e^w - 1}{w} = 1}}$$

So it holds that

$$\lim_{z\to 0}\frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled "definition of π" and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \ldots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left| \frac{c_{k+1}}{c_k} \right| = \left| \frac{x^2}{(2k+2)(2k+1)} \right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k \geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left| \frac{s_{k+1}}{s_k} \right| = \left| \frac{x^2}{(2k+2)(2k+3)} \right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$

Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1 - \frac{4}{6} = \frac{1}{3} > 0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = -2 \cdot \underbrace{\underbrace{\sin \frac{z+w}{2}}_{\in (0,2]} \cdot \underbrace{\sin \frac{z-w}{2}}_{\in (0,2]}}_{\substack{>0 \\ <0}}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that $\cos$ is a strictly monotonically decreasing function in $[0,2)$. Hence $\cos$ has only one root because it is continuous in $(0,2]$.

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos \frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left| \sin \frac{\pi}{2} \right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin \frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

TODO: table missing

$$e^{i\pi} = e^{i\frac{\pi}{2} + i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$

$$e^{i\frac{3}{2}\pi} = e^{i\pi + \frac{i}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$

$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\cos(z + 2\pi) = \frac{1}{2}\left(e^{iz+2\pi i} + e^{-iz-2\pi i}\right)$$

$$= \frac{1}{2}\left(e^{iz} + e^{-iz} \cdot \underbrace{\frac{1}{e^{2\pi i}}}_{=1}\right) = \cos z$$

Therefore the cosine is periodic in $\mathbb{C}$ with period 2π. Analogously, sine is periodic in $\mathbb{C}$ with period 2π.

This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.
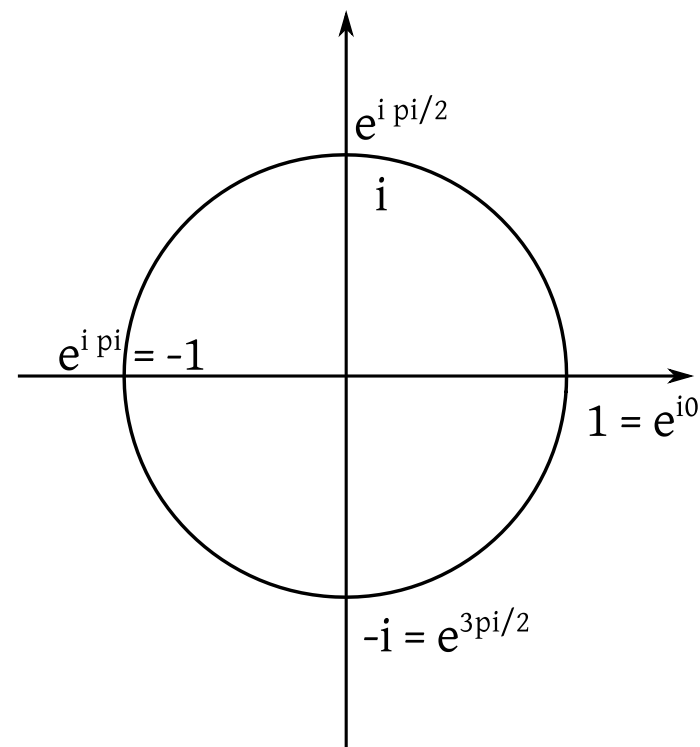
3.5 Periodicity and roots of trigonometric functions

TODO: equations missing

$$\cos(z + 2\pi) = \cos(z)$$

$$\sin(z + 2\pi) = \sin(z)$$

Remark 3. We will show: $\forall c \in (0, 2\pi)$, cos and sin are non-periodic with period c, hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x + c)$.

Definition 5.

$$f : \mathbb{C} \to \mathbb{C} \qquad (f : \mathbb{R} \to \mathbb{R})$$

is called *periodic* with period $c \in \mathbb{C}$ $(c \in \mathbb{R})$ if $\forall z \in \mathbb{C}$ it holds that

$$f(z + c) = f(z)$$

$$(\forall x \in \mathbb{R} : f(x + c) = f(x))$$

c is called *period of f*.

Remark 4. If f is periodic with period $c \in \mathbb{C}$, then f is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark 5.
$$z = u + iv$$
$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$
$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

Remark 6. Let $x \in \mathbb{R}$.
$$\cos\left(x + \frac{\pi}{2}\right) = \Re(e^{i(x+\frac{\pi}{2})})$$
$$= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}})$$
$$= \Re(ie^{ix})$$
$$= -\Im(e^{ix})$$
$$= -\sin(x)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \Im\left(e^{i(x+\frac{\pi}{2})}\right)$$
$$= \Im(ie^{ix})$$
$$= \Re(e^{ix})$$
$$= \cos(x)$$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= \sin(x)$$

$$\sin\left(x - \frac{\pi}{2}\right) = -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= -\cos(x)$$

Summary:
$$\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$$
$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$
$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$
$$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$$

Remark 7 (A remark on the name "cosine").
$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of x (Compare with Figure 8).

Remark 8.
$$\cos(x + \pi) = \Re(e^{i(x+\pi)})$$
$$= \Re(-e^{ix})$$
$$= -\cos(x)$$
$$\sin(x + \pi) = -\sin(x)$$

Remark 9. Let $0 < c < 2\pi$. Assume cos is periodic with period c. We know that cos has exactly one root in $[0, 2]$,
$$\cos(x) = \cos(-x)$$
cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.
$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$
$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

Therefore cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.
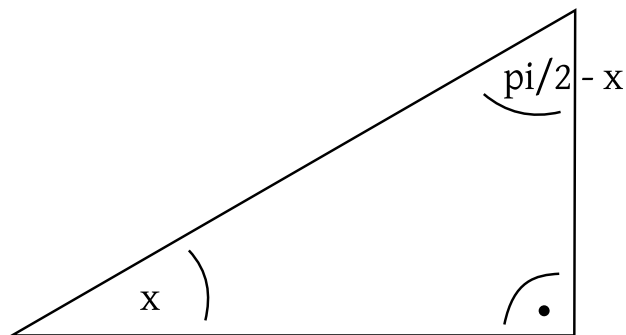
Figure 8: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$

c would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of cos in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period c. Hence 2π is indeed the smallest period of cos.

Analogously it holds for sin.

Remark 10 (Roots of cos)**.**

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$
$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \qquad \forall l \in \mathbb{Z}$$

3.6 Derivatives of trigonometric functions

It holds that

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

Furthermore it holds that

$$\lim_{z \to 0} \frac{1 - \cos z}{z} = 0$$

Proof.

$$\frac{1 - \cos z}{z} = \frac{1}{z}\left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots\right)$$

$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots$$

is convergent in $\mathbb{C}$ and (especially) continuous in 0

$$\lim_{z \to 0}\left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots\right) = 0$$

$\square$

$$\lim_{h \to 0} \frac{\cos(x + h) - \cos(x)}{h}$$

This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

$$\lim_{z \to 0} \frac{1 - \cos z}{z} = 0$$

Lemma 6. The trigonometric functions sin and cos are differentiable in $\mathbb{R}$ (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x) \qquad \sin'(x) = \cos(x)$$

Proof.

$$\lim_{h \to 0} \frac{\cos(x + h) - \cos(h)}{h} = \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos(h) - 1}{h} - \lim_{h \to 0} \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{=0} - \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{=1}$$

$$= -\sin(x)$$

Analogously:

$$\lim_{h \to 0} \frac{\sin(x + h) - \sin(h)}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h}$$

$$= \sin(x) \cdot \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{=0} + \cos(x) \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{n}}_{=1}$$

$$= \cos(x)$$

$\square$

TODO: incomplete graphics, verify text

Figure 9. We now use tools of integral calculus:

Let $I = [a, b]$ and $\gamma : I \to \mathbb{R}$ ($\mathbb{R}^2$).

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

Assumption: $\gamma_1 : [a, b] \to \mathbb{R}^n$.

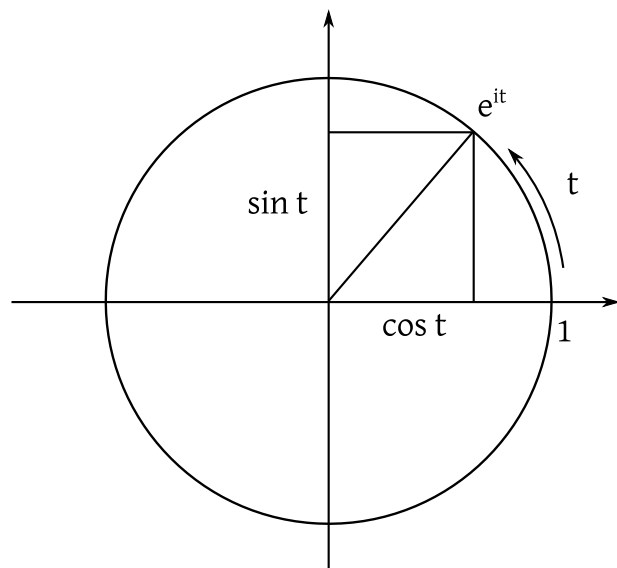$$\gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

Figure 9: The arc length is related to sin and cos

TODO: graphics missing

Let $t \in [a, b]$. Then the arc length of γ between a and t is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \, d\tau$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \to \mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$
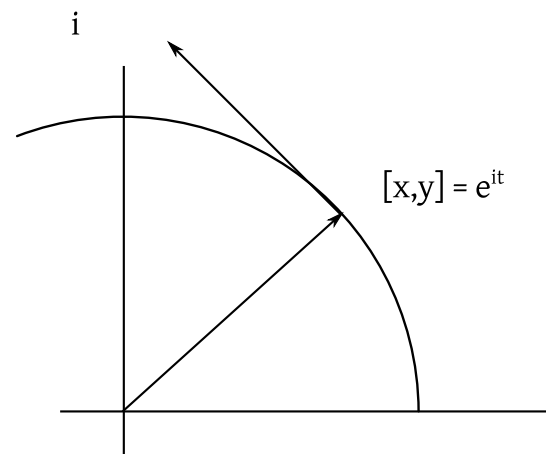
Figure 10: Derivative in $\mathbb{R}^2$

Compare with Figure 10.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \, d\tau = \int_0^t 1 \, d\tau = t$$

4 Integration calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses "Functional analysis" and "Measure and integration theory". For now, we look at the basis (as discussed by Königsberger).

Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \to \mathbb{R}$. We call φ a *step function*, if $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1}, x_j)} = c_j$ is constant. The points x_j define a partition of the interval $[a, b]$.

$\tau[a, b]$ defines the set of step functions of interval $[a, b]$. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 11).

Definition 6. Let $\varphi : [a, b] \to \mathbb{R}$ be a step function and $x_0 = a < x_1 < \ldots < x_n = b$ as partition of $[a, b]$ and let $\varphi|_{(x_{j-1}, x_j)} = c_j$ for $j = 1, \ldots, n$. Then we define

$$\int_a^b \varphi \, dx = \sum_{j=1}^n c_j \triangle x_j$$

where $\triangle x_j = x_j - x_{j-1}$ (for $j = 1, \ldots, n$).

$$\int_a^b \varphi \, dx \text{ is called } integral \text{ of } \varphi \text{ over } [a, b]$$

φ is the step function in terms of the partition $\{x_0, x_1, \ldots, x_5\}$.

It remains to show that if φ satisfies the definition of a step function in terms of partition $\{x_0, \ldots, x_n\}$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$ (TODO: text missing: "but …") and φ is a step function in terms of $\{w_0, w_1, \ldots, w_m\}$ and $\varphi|_{(w_{l-1}, w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^n c_j \triangle x_j = \sum_{l=1}^m c'_l \triangle w_l$$
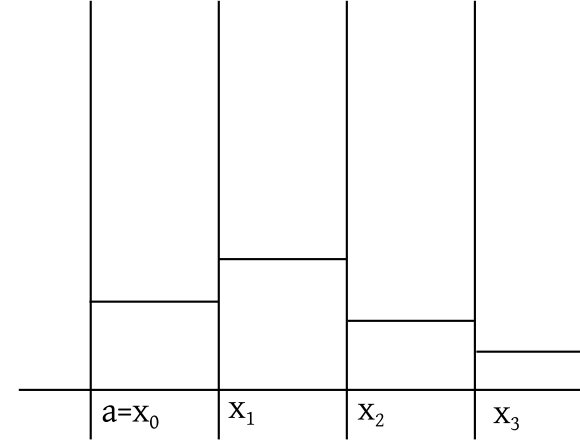
Compare with Figure 12.

Figure 11: Partition of an area into rectangles

Proof. Let $Z = \{x_0, \ldots, x_n\}$ and $Z' = \{w_0, \ldots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

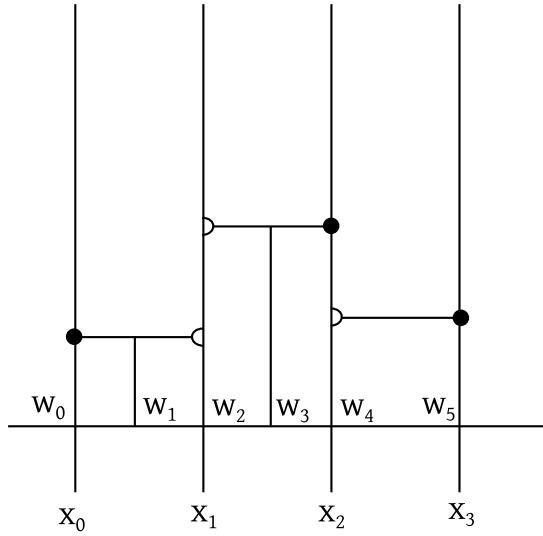$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly,

$$k_{j-1} < k_j$$

Let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$

Figure 12: Step function φ

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \ldots, L\}$ there exists $j \in \{1, \ldots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1}, \alpha_l)} \text{ is constant}$$

Hence φ is a step function in terms of the partition $\{\alpha_0, \ldots, \alpha_L\}$.

Let $l \in \{0, 1, \ldots, L\}$ and j such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c_l'' = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c_l'' = c_j$.

$$\sum_{l=1}^{L} c_l'' \cdot \triangle \alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c_l'' \triangle \alpha_l$$

$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}}^{k_j} \triangle \alpha_l$$

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle \alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1}) + (\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2})$$

$$+ \ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_j - x_{j-1} = \triangle x_j$$

Analogously,

$$\sum_{l=1}^{L} c_l'' \cdot \triangle \alpha_l = \sum_{k=1}^{m} c_k' \triangle w_k$$

So it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{k=1}^{m} c_k' \triangle w_k$$

$\square$

This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

Lemma 7. Let $\varphi \in \tau[a, b]$ be a step function in terms of partition $a = x_0 < x_1 < \ldots < x_n = b$. Let $a = \alpha_0 < \alpha_1 < \ldots < \alpha_L = b$ with $Z = \{x_0, \ldots, x_n\} \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = z'$ (z' has more intervals than Z').

Then also φ is step function in terms of partition z'.

$\square$

Proof. see above

$\square$

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Lemma 8. Let $\varphi_1, \varphi_2 \in \tau[a,b]$ and $\alpha, \beta \in \mathbb{C}$.

Then it holds that

- $\alpha\varphi + \beta\psi \in \tau[a,b]$ and

$$\int_a^b (\alpha\varphi + \beta\psi)\, dx = \alpha \int_a^b \varphi\, dx + \beta \int_a^b \psi\, dx$$

  Hence ("linearity"),

$$\int_a^b : \tau[a,b] \to \mathbb{R} \text{ is linear}$$

- $|\varphi| \in \tau[a,b]$ and it holds that

$$\left| \int_a^b \varphi\, dx \right| \leq \int_a^b |\varphi|\, dx \leq \|\varphi\|_\infty (b - a)$$

  Reminder: $\|\varphi\|_\infty = \max\{|\varphi(x)| : x \in [a,b]\}$
  This gives "boundedness".

- Let φ and ψ be real values and it holds that

$$\forall x \in [a,b] : \varphi(x) \leq \psi(x)$$

  Then TODO Monotonicity

Proof.
- Let $\varphi|_{(x_{k-1}, x_k)} = c_k \ \psi|_{(w_{j-1}, w_j)} = d_k$

$$z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = \{x_0, \ldots, x_n\} \cup \{w_0, \ldots, w_m\}$$

  where α_i is sorted ascendingly. φ and ψ are step functions in terms of z'', hence

$$\varphi|_{(\alpha_{i-1}, \alpha_i)} = c'_i \text{ and } \psi|_{(\alpha_{i-1}, \alpha_i)} = d'_i$$
$$\Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1}, \alpha_i)} = \alpha c'_i + \beta d'_i \text{ constant}$$

$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a,b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi)\, dx = \sum_{i=1}^L (\alpha c'_i + \beta d'_i) \cdot \triangle\alpha_i$$

$$= \alpha \sum_{i=1}^L c'_i \triangle\alpha_i + \beta \sum_{i=1}^L d'_i \triangle\alpha_i$$

$$= \alpha \int_a^b \varphi\, dx + \beta \int_a^b \psi\, dx$$

- Let $\varphi|_{(x_{i-1}, x_i)} = c_i \ (i = 1, \ldots, n)$. Then,

$$|\varphi| \,|_{(x_{i-1}, x_i)} = |c_i|$$

$$\left| \sum_{i=1}^n c_i \triangle x_i \right| \leq \sum_{i=1}^n |c_i| \cdot \underbrace{|\triangle x_i|}_{x_i - x_{i-1} > 0} = \sum_{i=1}^n |c_i| \cdot \triangle x_i = \int_a^b |\varphi|\, dx$$

$$\leq \sum_{i=1}^n \|\varphi\|_\infty \triangle x_i = \|\varphi\|_\infty \sum_{i=1}^n \triangle x_i$$

$$= \|\varphi\|_\infty ((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}))$$
$$= \|\varphi\|_\infty (x_n - x_0) = \|\varphi\|_\infty (b - a)$$

- Let φ, ψ and z'' as in the linearity statement.

$$\left.\begin{array}{ll} \varphi|_{(\alpha_{i-1}, \alpha_i)} & = c'_i \in \mathbb{R} \\ \psi|_{(\alpha_{i-1}, \alpha_i)} & = d'_i \in \mathbb{R} \end{array}\right\}$$

$$\int_a^b \varphi\, dx = \sum_{i=1}^L c'_i \underbrace{\triangle\alpha_i}_{>0} \leq \sum_{i=1}^L d'_i\, dx$$

$$\int_a^b \varphi\, dx$$

$\square$

Definition 7. Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\infty_A) : \mathbb{R} \to \mathbb{R}$ as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

the characteristic function of A. Hence $\chi_A(x)$ is 1 if and only if x is inside interval A.

Remark 11. Let $a \leq a' < b' \leq b$. Then

$$\chi_{(a',b')} \in \tau[a,b] \qquad \int_a^b \chi_{(a',b')}\,dx = 1 \cdot (b' - a')$$

Every linear combination of characteristic functions is also in $\tau[a,b]$.

On the opposite side, let $\varphi \in \tau[a,b]$ with $\varphi|_{(x_{i-1},x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \leq i \leq n$ and $0 \leq j \leq n$.

$$\Rightarrow \varphi = \sum_{i=1}^n c_i \chi_{(x_{i-1},x_i)} + \sum_{j=0}^n r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_a^b \varphi\,dx = \sum_{i=1}^n c_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^n c_j \int_a^b \chi_{(x_{i-1},x_i)}\,dx$$

5 Regulated functions

Definition 8. Let $D \subseteq \mathbb{R}$. Let x_0 be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n \in \mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n \to \infty} z_n = x_0$. Let $f : D \to \mathbb{C}$ be given.

We state that f has left-sided limit y_0 in x_0 if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \cap (-\infty, x_0) \wedge |x - x_0| < \delta]$$

$$\Rightarrow |f(x) - y_0| < \varepsilon$$

Equivalently $\forall (z_n)_{n \in \mathbb{N}}$ with $z_n \in D$ and $z_n < x_0$ and $\lim_{n \to \infty} z_n = x_0$ $\forall n \in \mathbb{N}$

$$\lim_{n \to \infty} f(x_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by (x_0, ∞).

We denote: y_0 is left-sided limit of f in x_0:

$$y_0 = \lim_{x \to x_0^-} f(x)$$

and right-sided limit of f in x_0:

$$y_0 = \lim_{x \to x_0^+} f(x)$$

Definition 9. Let $a, b \in \mathbb{R}$ and $a < b$. A function $f : [a,b] \to \mathbb{C}$ is called *regulated functions* if

- $\forall x \in (a,b)$ f has a left-sided and a right-sided limes in x

- f has a right-sided limes in a

- f has a left-sided limes in b

Examples for regulated functions:

- Every continuous function in $[a,b]$ is a regulated function.

- Every step function is a regulated function.
  Why? Consider $x \in (x_{i-1}, x_i)$. Then

$$\lim_{\xi \to x^+} \varphi(\xi) = c_i = \lim_{\xi \to x^-} \varphi(\xi)$$

Let $x = x_i$ be a partitioning point.

$$\lim TODO \text{ and } \lim TODO$$

So $\tau[a,b] \subseteq R[a,b]$. Compare with Figure 13.

- Let $f : [a,b] \to \mathbb{R}$ be monotonically. Then it holds that

$$f \in R[a,b]$$

5.1 Approximation theorem for regulated functions

Let $f : [a,b] \in \mathbb{C}$. Then it holds that $f \in R[a,b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a,b]$ such that $\|f - \varphi\|_\infty < \varepsilon$. Hence $\forall x \in [a,b] : |f(x)|$ TODO

We build nested intervals such that the desired property $|\varphi(x) - f(x)| \geq \varepsilon$ holds on every subinterval $[a_n, b_n]$.

Induction:

$n = 0$ Let $a_0 = a$ and $b_0 = b$, hence the property holds in $[a_0, b_0]$.

$n \mapsto n + 1$ Let $m = \frac{1}{2}(a_n + b_n)$. In $[a_n, b_n]$ the property holds.

Then the property either holds in $[a_n, m]$ or $[m, b_n]$. If the property does not hold in $[a_n, m]$:

$$\exists \varphi_1 \in \tau[a_n, m] \text{ with } |\varphi_1(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, m]$$

If the property does not hold in $[m, b_n]$:

$$\exists \varphi_2 \in \tau[m, b_n] \text{ with } |\varphi_2(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [m, b_n]$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in [a_n, m] \\ \varphi_2(x) & \text{for } x \in [m, b_n] \end{cases}$$

$$\Rightarrow \varphi \in \tau[a, b] \text{ and } |\varphi(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1}, b_{n+1}]$.

$([a_n, b_n])_{n \in \mathbb{N}}$ are nested intervals. Let $\varphi \in \bigcap_{n \in \mathbb{N}}[a_n, b_n]$.

Case $\xi \in (\mathbf{a}, \mathbf{b})$ Let ε satisfy the desired property. $f \in R[a, b]$, hence f has left-sided limit c_- in ξ and right-sided limit c_+. Hence $\exists \delta > 0$ such that

- $|x - \xi| < \delta \wedge a \leq x < \xi \Rightarrow |f(x) - c_-| < \varepsilon$
- $|x - \xi| < \delta \wedge \delta < x \leq b \Rightarrow |f(x) - c_+| < \varepsilon$

Choose δ sufficiently small such that

$$a < \xi - \delta < \xi + \delta < b$$

Let

$$\varphi(x) = \begin{cases} c_- & \text{for } x \in (\xi - \delta, \xi) \\ f(\xi) & \text{for } x = \xi \\ c_+ & \text{for } x \in (\xi, \xi + \delta) \end{cases}$$

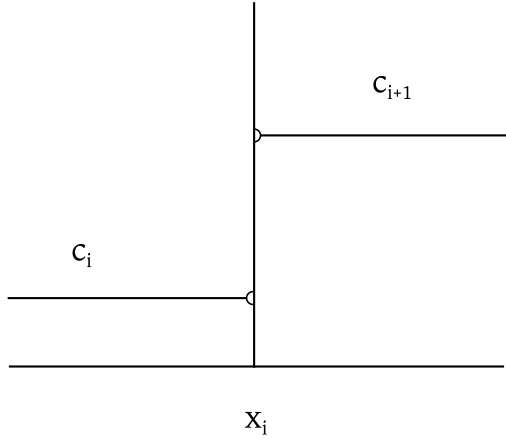
Figure 13: Step functions are also regulated functions

$$\Leftrightarrow \underbrace{\sup\{|f(x) - \varphi(x)| : x \in [a, b]\}}_{\|f - \varphi\|_\infty} < \varepsilon$$

S $\varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a, b]$ such that

$$|\varphi_n(x) - f(x)| < \varepsilon \forall x \in [a, b]$$

hence f is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges continuously towards f.

Proof. $\Rightarrow$ Let $f \in R[a, b]$. Assume $\exists \varepsilon > 0$ fixed such that $\forall \varphi \in \tau[a, b]$

$$\exists x \in [a, b] : |\varphi(x) - f(x)| \geq \varepsilon$$

φ is necessarily a step function in $(\xi - \delta, \xi + \delta)$ and it holds that $\forall x \in (\xi - \delta, \xi + \delta) : |\varphi(x) - f(x)| < \varepsilon$.

Let n be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi - \delta, \xi + \delta)$$

then

$$\varphi|_{[a_n, b_n]} \in \tau[a_n, b_n] \text{ and } |\varphi(x) - f(x)| < \varepsilon \quad \forall x \in [a_n, b_n]$$

This is a contradiction to our desired property.

For $\xi = a$ or $\xi = b$ only with one-sided limit.

$\square$

This lecture took place on 17th of March 2016 with lecturer Wolfgang Ring.

We learned: All regulated functions can be approximated with step functions.

$f \in R[a, b]$ in the proof $\Leftrightarrow f$ is uniform limit of step functions. We have prove direction $\Rightarrow$.

Lemma 9 (Cauchy criterion for limits of functions)**.** Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and z_0 is a limit point of D. Then f has a limit in z_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : v, w \in D \setminus \{z_0\} \wedge |v - z_0| < \delta \wedge |w - z_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$.

If $D \subseteq \mathbb{R}$ and x_0 is limit point of $D \cap (x_0, \infty)$, then f has a *right-sided limit* in x_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : [v, w \in D \cap (x_0, \infty) \wedge |v - x_0| < \delta \wedge |w - x_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon]$.

Analogously for left-sided limit.

Proof. This proof is done only for the first point.

$\Rightarrow$

Assume f has a limit η in z_0. Choose δ such that $v, w \in D$ with $|v - z_0| < \delta$ and $|w - z_0| < \delta$ implies that $|f(v) - \eta| < \frac{\varepsilon}{2}$ and $|f(w) - \eta| < \frac{\varepsilon}{2}$. Then $|f(v) - f(w)| \leq |f(v) - \eta| + |\eta - f(w)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Leftarrow$

Assume the Cauchy criterion holds. Show: There exists $\eta \in \mathbb{C}$ such that

for every sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in D \setminus \{z_0\}$ with $\lim_{n \to \infty} w_n = z_0$ it holds that $\lim_{n \to \infty} f(w_n) = \eta$.

Let $(w_n)_{n \in \mathbb{N}}$ be as above. Show: $(f(w_n))_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and δ as above. Choose $N \in \mathbb{N}$ such that $n, m \geq N$

$$\Rightarrow |w_n - z_0| < \delta \wedge |w_m - z_0| < \delta$$

The Cauchy criterion holds for $n, m \geq N$:

$$|f(w_n) - f(w_m)| < \varepsilon$$

So $(f(w_n))_{n \in \mathbb{N}}$ is a Cauchy sequence and (because $\mathbb{C}$ is complete) is also convergent. So $\exists \eta' \in \mathbb{C} : \lim_{n \to \infty} f(w_n) = \eta'$.

It remains to show: η' is unique.

Let $(v_n)_{n \in \mathbb{N}}$ be another sequence with $\lim_{n \to \infty} v_n = z_0$ and $v_n \in D \setminus \{z_0\}$. As above: $\exists \eta'' \in \mathbb{C}$ such that $\lim_{n \to \infty} f(v_n) = \eta''$.

We construct:

$$(\xi_n)_{n \in \mathbb{N}} = (w_0, v_0, w_1, v_1, w_2, v_2, \ldots)$$

Then it holds that $\lim_{n \to \infty} \xi_n = z_0$.

We use the argument from above: $(f(\xi_n))_{n \in \mathbb{N}}$ is convergent, hence $\lim_{n \to \infty} f(\xi_n) = \eta$. Both subsequences $(f(w_n))_{n \in \mathbb{N}}$ and $(f(v_n))_{n \in \mathbb{N}}$ must have the same limit, hence $\eta' = \eta = \eta''$.

$\square$

Proof of approximation theorem. $\Leftarrow$

Let $f = \lim_{n \to \infty} \varphi_n$ be uniform on $[a, b]$. Let $\varphi_n \in \tau[a, b]$ and let $x_0 \in [a, b)$. Show: f has a right-sided limit in x_0. Let $\varepsilon > 0$ arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that

$$|f(x) - \varphi_N(x)| < \frac{\varepsilon}{2} \forall x \in [a, b]$$

φ_N is a step function (hence interval-wise constant). Choose $\delta > 0$ such that $\varphi_N|_{(x_0, x_0 + \delta)} = c$ constant. Let $v, w \in (x_0, x_0 + \delta)$. Then it holds that

$$|f(v) - f(w)| \leq |f(v) - c| + |c - f(w)|$$

$$= |f(v) - \varphi_N(v)| + |f(w) - \varphi_N(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The Cauchy criterion implies that f has a right-sided limit in x_0.

$\square$

Corollary 2. $f \in R[a,b]$ if and only if $f(x) = \sum_{j=0}^{\infty} \psi_j(x)$ with $\psi_j \in \tau[a,b]$ and the series converges uniformly in $[a,b]$.

Proof. $\Leftarrow$
Let $\varphi_n = \sum_{j=0}^{n} \psi_j \in \tau[a,b]$ and $\varphi_n \to f$ continuously in $[a,b]$. From the approximation theorem it follows that $f \in R[a,b]$.

$\Rightarrow$

Let $f \in R[a,b]$. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of step functions with $\varphi_n \to f$ uniform in $[a,b]$. Let $\psi_0 = \varphi_0$.

$$\psi_j := \varphi_j - \varphi_{j-1} \text{ for } j \geq 1$$

Then it holds that

$$\sum_{j=0}^{n} \psi_j = \varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \ldots + (\varphi_{n-1} - \varphi_{n-2}) + (\varphi_n - \varphi_{n-1}) = \varphi_n$$

and $(\varphi_n)_{n \in \mathbb{N}}$ converges uniform if and only if the series is uniformly convergent.

$\square$

Lemma 10 (Sidenote). Let $(f_n)_{n \in \mathbb{N}}$ with $f_n : D \to \mathbb{C}$ a sequence of functions in D, let $z_0 \in D$ and $\forall n \in \mathbb{N}$ f_n is continuous in z_0. Furthermore let $f : D \to \mathbb{C}$ and $f_n \to f$ is uniform in D. Then f is continuous in z_0.

Proof. Let $\varepsilon > 0$ arbitrary. Choose N sufficiently large such that $|f(z) - f_w(z)| < \frac{\varepsilon}{3}$ $\forall z \in D$ (uniform convergence). Because f_N is continuous in z_0, $\exists \delta > 0$ such that $z \in D$ and $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$.

Then for $|z - z_0| < \delta$ (with $z \in D$)

$$\underbrace{|f(z) - f(z_0)|}_{} \leq \underbrace{|f(z) - f_N(z)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(z) - f_N(z_0)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{< \frac{\varepsilon}{3}}$$

$\square$

This lecture took place on 18th of March 2016 with lecturer Wolfgang Ring.

Theorem 9. Let f be a regulated function in $[a,b]$. Then f is in at most countable infinite points of $[a,b]$ non-continuous.

Proof.

$$f = \sum_{k=0}^{\infty} \psi_k$$

where ψ_k is a sequence of step functions and and the series is uniformly convergent. $\psi_k \in \tau[a,b]$.

Let $\{x_0^k, \ldots, x_{n(k)}^k\}$ be the partition points of ψ_k. Then ψ_k is continuous in $[a,b] \setminus Z_k$. Let $Z = \bigcup_{k=0}^{\infty} Z_k$ be countable. Let $x \in [a,b] \setminus Z$ and $\varphi_n = \sum_{k=0}^{n} \psi_k$. Then it holds that $\varphi_n \to f$ is uniform in $[a,b]$ and φ_n is continuous in x, because $x \notin Z$.

From Lemma 10 it follows that f is continuous in x. $\square$

5.2 Norms and vector spaces

Definition 10 (Normed vector spaces). Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). A map $n : V \mapsto [0, \infty)$ is called *norm* in V, if

1. $n(V) = 0 \Leftrightarrow V = 0$ (V is null vector)
   "definity"

2. $\forall \lambda \in \mathbb{C}$ ($\mathbb{R}$) $\forall v \in V : n(\lambda v) = |\lambda| \cdot n(v)$ "positive homogeneity"

3. $\forall v, w \in V : n(v + w) \leq n(v) + n(w)$
   "triangle inequality"

Common notation: $\|v\|$ for $n(v)$ ("norm of v")
A vector space satisfying the norm properties is called *Normed vector space*

Example 2.
- $|x|$ is a norm in $\mathbb{R}$.
- $|z|$ is a norm in $\mathbb{C}$.

$\|\vec{x}\|$ is norm in $\mathbb{R}^n$.

Let $D \subseteq \mathbb{C}$.
$$B(D) = \{f : D \to \mathbb{C} : f \text{ limited to } D\}$$

$B(D)$ is a vector space. For $f \in B(D)$ we define:
$$\|f\|_\infty = \sup \{|f(z)| : z \in D\}$$

"supremum norm" of ∞-norm of f in D.

It holds that $\|\cdot\|_\infty$ is a norm in $B(D)$.

$$\|f\|_\infty = 0 \Leftrightarrow \sup \left\{ \underbrace{|f(z)|}_{\geq 0} : z \in D \right\} = 0$$

$$\Leftrightarrow |f(z)| = 0 \quad \forall z \in D$$
$$\Rightarrow f = 0 \text{ in } B(D)$$

Homogeneity:
$$|\lambda \cdot f|_\infty = \sup \{|\lambda f(z)| : z \in D\}$$
$$= \sup \{|\lambda| \, |f(z)| : z \in D\}$$
$$= \sup \{|f(z)| : z \in D\} \cdot |\lambda|$$
$$= |\lambda| \cdot \|f\|_\infty$$

Triangle inequality: Let $f, g \in B(D)$.

$$\|f + g\|_\infty = \sup \{|f(z) + g(z)| : z \in D\}$$

$$= \sup \left\{ \underbrace{|f(z)|}_{\leq \|f\|_\infty} + \underbrace{|g(z)|}_{\leq \|g\|_\infty} : \right\}$$

$$\leq TODO \qquad\qquad = \|f\|_\infty + \|g\|_\infty$$

Remark 12. Let $V \subseteq B(D)$ be an arbitrary subvectorspace of $B(D)$. So $\|\cdot\|_\infty$ is also a norm in V.

Important example:
$$V = \mathcal{C}_b(D) = \{f : D \to \mathbb{C} : f \text{ is continuous and bounded in } D\}$$

Special case: $D = K$ compact in $\mathbb{C}$. Then every continuous function is also bounded.

$$\mathcal{C}(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\}$$
$$\subseteq B(K) \qquad \text{(sub vector space)}$$

Another special case: $D = [a, b] \subseteq \mathbb{C}$

$$\tau[a, b] \subseteq B([a, b]) \text{ and}$$
$$R[a, b] \subseteq B([a, b])$$

Remark 13 (Further properties of the norm)**.** The inverse triangle inequality holds:
$$\forall v, w \in V : |\|v\| - \|w\|| \leq \|v - w\|$$

Proof.
$$v = (v - w) + w$$

From triangle inequality it follows that
$$\|v\| \leq \|v - w\| + \|w\|$$
$$w = (w - v) + w$$
$$\|w\| \leq \|w - v\| + \|w\|$$

$$= \|(-1) \cdot (v - w)\| + \|v\|$$
$$= |(-1)| \cdot \|v - w\| + \|v\|$$
$$= \|v - w\| + \|v\|$$

requirement 1 $\Rightarrow \|v\| - \|w\| \leq \|v - w\|$
requirement 2 $\Rightarrow \|w\| - \|v\| \leq \|v - w\|$
requirements $\Rightarrow TODO$

$\square$

Definition 11. Let V be a normed vector space, $(v_n)_{n \in \mathbb{N}}$ be a sequence of elements in V and $v \in V$. We define $(v_n)_{n \in \mathbb{N}}$ is convergent with limit V if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq \mathbb{N} \Rightarrow \|V_n - V\| \leq \varepsilon]$$

Remark 14 (Metric on V).

$$d(v, w) = \|v - w\|$$

defines a metric on V. Properties of a metric:

1. $d(v, w) \geq 0$

2. $d(v, w) = 0 \Leftrightarrow v = w$

3. $\|v - w\| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$

Triangle inequality of metrics: Let $v, w, u \in V$.

$$d(v, u) = \|v - u\| = \|v - w + w - u\|$$

$$\leq \|v - w\| + \|w - u\| = d(v, w) + d(w, u)$$

Works only if $d(v, w) = d(w, v)$ and can be simply proven:

$$d(v, w) = \|v - w\| = \|w - v\| = d(w, v)$$

Remark 15. $(V_n)_{n \in \mathbb{N}}$ is called *Cauchy sequence in V* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|v_n - v_m\| < \varepsilon]$$

V is called *complete normed vector space* if every Cauchy sequence in V is also a convergent sequence in V.

A complete normed vector space is called *Banach space*.

5.3   Integration of regulated functions

Theorem 10. Let $f \in \mathcal{B}[a, b]$ and $(\varphi_n)_{n \in \mathbb{N}}$ with $\varphi_n \in \tau[a, b]$ and $\varphi_n \to_{n \to \infty} f$ uniform in $[a, b]$ ($\Leftrightarrow \|\varphi_n - f\| \to 0$ for $n \to \infty$).

Then we define

$$\int_a^b f \, dx = \lim_{n \to \infty} \int_a^b \varphi_n \, dx$$

for the integral of f in $[a, b]$. The right-sided limit exists for every sequence $(\varphi)_{n \in \mathbb{N}}$ with the property above and is independent of the choice of the sequence $(\varphi_n)_{n \in \mathbb{N}}$.

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [\underbrace{n \geq N \Rightarrow |\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]}_{\sup\{|\varphi_n(x) - f(x)| : x \in [a, b] \leq \varepsilon\}}]$$

$$\Rightarrow \|\varphi_n - f\|_\infty \leq \varepsilon$$

So φ_n converges towards f in terms of $\|\cdot\|_\infty$ in $\mathcal{B}[a, b]$.

Let N be sufficiently large such that

$$\forall n \geq N : \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(b - a)}$$

Then it holds for $i_n = \inf_a^b \varphi_n \, dx$ and $n, m \geq N$,

$$|i_n - i_m| = \left| \int_a^b \varphi_n \, dx - \int_a^b \varphi_m \, dx \right|$$

$$= \left| \int_a^b (\varphi_n - \varphi_m) \, dx \right|$$

$$\leq \|\varphi_n - \varphi_m\|_\infty (b - a)$$

$$= \|\varphi_n - f + f - \varphi_m\|_\infty (b - a)$$

$$\leq (\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty)(b - a)$$

$$< \left(\frac{\varepsilon}{2(b - a)} + \frac{\varepsilon}{2(b - a)} \right) (b - a)$$

$$= \varepsilon$$

So $(i_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and therefore convergent. So there exists

$$\lim_{n \to \infty} \int_a^b \varphi_n \, dx$$

Let $i = \lim_{n\to\infty} \int_a^b \varphi_n \, dx$. Let $(\psi_n)_{n\in\mathbb{N}}$ be another sequence of step functions with $\psi_n \to_{n\to\infty} f$ is uniform in $[a, b]$. Analogously as above:

$$j_n = \int_a^b \psi_n \, dx$$

$(j_n)_{n\in\mathbb{N}}$ is convergent and has limes j.

Show that $i = j$. We again use a zip-like construction:

$$F = (\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \dots)$$

F is a sequence of step functions, which converge towards f uniformly. Let l be the limit of integrals of this sequence of step functions. Then it holds that TODO (subsequences have the same limit)

$$i = l = j$$

$\square$

Theorem 11 (Elementary properties of the integral). Let $f, g \in \mathcal{B}[a, b]$ and $\alpha, \beta \in \mathbb{C}$. Then it holds that

linearity

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx$$

boundedness

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx \leq \|f\|_\infty (b - a)$$

monotonicity Let $f, g \in \mathcal{B}[a, b]$ with values in $\mathbb{R}$ and it holds that

$$f(x) \leq g(x) \qquad \forall x \in [a, b]$$

Then it holds that

$$\int_a^b f \, dx \leq \int_a^b g \, dx$$

Proof. • Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences of step functions with $\varphi_n \to f$ and $\psi_n \to g$ uniform in $[a, b]$. Then it holds that

$$\alpha\varphi_n + \beta\psi_n \to_{n\to\infty} \alpha f + \beta g$$

(proof left as exercise to the reader) uniform in $[a, b]$. So it holds that

$$\int_a^b (\alpha f + \beta g) \, dx = \lim_{n\to\infty} \int_a^b (\alpha\varphi_n + \beta\psi_n) \, dx$$
$$= \alpha \lim_{n\to\infty} \int_a^b \varphi_n \, dx + \beta \lim_{n\to\infty} \int_a^b \varphi_n \, dx$$
$$= \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx$$

• Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to_{n\to\infty} f$ continuous in $[a, b]$. Then also $(|\varphi_n|)_{n\in\mathbb{N}}$ is a sequence of step functions and it holds that

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a, b]$$

Proof. Let N be sufficiently large such that $\forall n \geq N \forall x \in [a, b]$:

$$|\varphi_n(x) - f(x)| < \varepsilon \Rightarrow ||\varphi_n(x)| - |f(x)|| \leq |\varphi_n(x) - f(x)| < \varepsilon$$

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a, b]$$

So it holds that

$$\left| \int_a^b f \, dx \right| = \left| \lim_{n\to\infty} \int_a^b \varphi_n \, dx \right| = \lim_{n\to\infty} \left| \int_a^b \varphi_n \, dx \right| \leq \lim_{n\to\infty} \int_a^b |\varphi_n| \, dx = \int_a^b |f| \, dx$$

Because $|f - \varphi_n|_\infty \to_{n\to\infty} 0$ it follows that

$$|\|f\|_\infty - \|\varphi_n\|_\infty| \leq \|f - \varphi_n\|_\infty \to 0$$

hence $\|f\|_\infty = \lim_{n\to\infty} \|\varphi_n\|_\infty$.

$\square$

Hence,

$$\int_a^b |f|\ dx = \lim_{n\to\infty} \int_a^b |\varphi_n|\ dx$$
$$\leq \lim_{n\to\infty} \|\varphi_n\|_\infty (b-a)$$
$$= \|f\|_\infty (b-a)$$

Remark 16. We have proven that $\|\cdot\| : V \to [0,\infty)$ is a continuous map, hence $v_n \to v \Rightarrow \|v_n\| \to \|v\|$.

$\square$

This lecture took place on 12th of April 2016 with lecturer Wolfgang Ring.

Definition 12. Let $f : [a,b] \to \mathbb{R}$ be given. Let $x_0 \in [a,b)$. We claim that f has a right-sided derivative $f'_+(x_0)$ in x_0 if the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

has a right-sided limit in x_0. Then f is denoted with $f'_+(x_0)$.

$$f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$$

Analogously for the left-sided derivative: Let $x_0 \in (a,b]$. $f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ if the limit exists.

Theorem 12 (Mean value theorem of calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and $p : [a,b] \to \mathbb{R}$ is a regulated function with $p(x) \geq 0 \quad \forall x \in [a,b]$.

Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x) \cdot p(x)\ dx = f(\xi) \cdot \int_a^b p(x)\ dx$$

Proof. Let $M = \max\{f(x) : x \in [a,b]\}$ and $m = \min\{f(x) : x \in [a,b]\}$

$$mp(x) \leq f(x)\underbrace{p(x)}_{\geq 0} \leq Mp(x) \qquad \forall x \in [a,b]$$

Due to monotonicity of the integral it holds that

$$m \int_a^b p(x)\ dx \leq \int_a^b f(x)p(x)\ dx \leq M \int_a^b p(x)\ dx$$

hence $\exists \eta \in [m,M]$ such that $\eta \cdot \int_a^b p(x)\ dx = \int_a^b f(x)p(x)\ dx$. From the Intermediate Value Theorem it follows that $\exists \xi \in [a,b] : \eta = f(\xi)$.

$$\Rightarrow f(\xi) : \int_a^b p(x)\ dx = \int_a^b f(x)p(x)\ dx$$

$\square$

Remark 17. Consider $p \equiv 1$.

$$\exists \xi \in [a,b] : \int_a^b f(x) \cdot 1\ dx = f(\xi) \cdot \int_a^b 1\ dx = f(\xi) \cdot (b-a)$$

Lemma 11. Let $I = [a,b]$ and $f \in R[a,b]$ and $a \leq \alpha < \beta < \gamma \leq b$ (compare with Figure 15). Then $f|_{[\alpha,\gamma]} \in R[\alpha,\gamma]$.

Furthermore it holds that

$$\int_\alpha^\beta f(x)\ dx = \int_\alpha^\beta f(x)\ dx + \int_\beta^\gamma f(x)\ dx$$

Proof. Let φ be a step function in $[\alpha,\gamma]$. Then $\varphi|_{[\alpha,\beta]} \in \tau[\alpha,\beta]$ and $\varphi|_{[\beta,\gamma]} \in \tau[\beta,\gamma]$. Furthermore it holds (proof not given here)

$$\int_\alpha^\gamma \varphi\ dx = \int_\alpha^\beta \varphi\ dx = \int_\beta^\gamma \varphi\ dx$$

For $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of subsequences with $\varphi_n \to f$ continuous in $[\alpha,\gamma]$.

$$\Rightarrow \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]} \text{ uniform in } [\alpha,\beta]$$

analogously for $[\beta,\gamma]$.

$$\int_\alpha^\gamma f\,dx = \lim_{n\to\infty} \int_\alpha^\gamma \varphi_n\ dx = \lim_{n\to\infty}\left[\int_\alpha^\beta \varphi_n\ dx + \int_\beta^\gamma \varphi_n\ dx\right]$$

Figure 14: Mean value Theorem
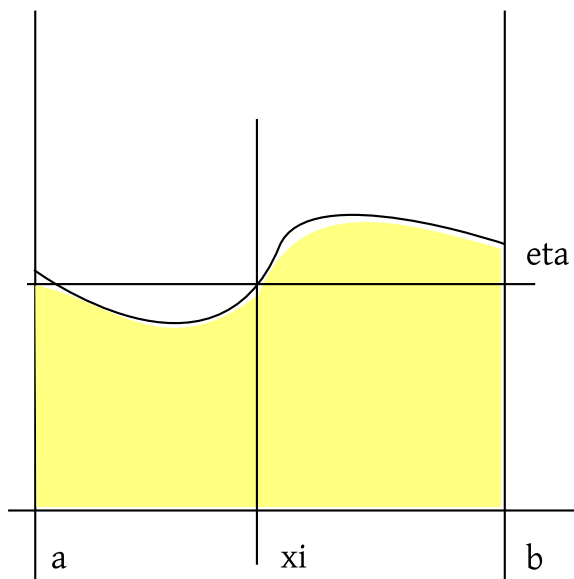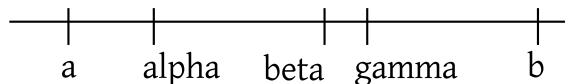

Figure 15: Relation of $a \leq \alpha < \beta < \gamma \leq b$

$$= \lim_{n \to \infty} \int_\alpha^\beta \varphi_n \, dx + \lim_{n \to \infty} \int_\beta^\gamma \varphi_n \, dx$$
$$\underbrace{\phantom{\lim_{n \to \infty} \int_\alpha^\beta \varphi_n \, dx}}_{= \int_\alpha^\beta f \, dx} \quad \underbrace{\phantom{\lim_{n \to \infty} \int_\beta^\gamma \varphi_n \, dx}}_{= \int_\beta^\gamma f \, dx}$$

Remark 18. Notation ($\alpha, \beta \in [a, b]$):

$$\int_\beta^\alpha f(x) \, dx = - \int_\alpha^\beta f(x) \, dx$$

So it follows that

$$\int_\alpha^\alpha f(x) \, dx = - \int_\alpha^\alpha f(x) \, dx = 0$$

With this notation it holds that $\forall \alpha, \beta, \gamma \in I$:

$$\int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f(x) \, dx$$

independent of the relation of α, β, γ towards each other. For $\alpha < \beta < \gamma$ everything is fine.

Let's also look at $\beta < \gamma < \alpha$ as an exercise.

Then it holds that

$$\int_\beta^\alpha f \, dx = \int_\beta^\gamma f \, dx + \int_\gamma^\alpha f \, dx$$

$$- \int_\alpha^\beta f \, dx = \int_\beta^\gamma f \, dx - \int_\alpha^\gamma f \, dx$$

$$\Rightarrow \int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx$$

Case $\alpha = \beta$ or $\beta = \gamma$ is trivial.

Theorem 13 (Fundamental theorem of Calculus). Originally formulated by Isaac Barrow (1630–1677). Followingly popularized by Newton (1642–1727) and Leibniz (1646–1716).

Let $f : I \to \mathbb{R}$ be a regulated function. I is an interval and $a \in I$ is fixed. For $x \in I$ we define

$$F(x) = \int_a^x f(\xi) \, d\xi$$

Then it holds that (two variants/characterizations)

1. F is right-sided derivable and also left-sided derivable for every $x_0 \in I$ and it holds that

$$F'_+(x) = f_+(x_0) = \lim_{x \to x_0^+} f(x) \qquad \text{and}$$

$$F'_-(x) = f_-(x_0) = \lim_{x \to x_0^-} f(x)$$

Especially if f is continuous in x_0, then F is differentiable in x_0 with derivative $F'(x_0) = f(x_0)$.

We call a function with the properties of F above a *primitive function* of the regulated function f.

2. Let $\Phi : I \to \mathbb{R}$ be an arbitrary primitive function of f and let $a, b \in I$. Then it holds that

$$\int_a^b f(\xi)\, d\xi = \Phi(b) - \Phi(a)$$

The first characterization claims informally that the derivative for the upper limit of the integral of f gives f.

Let $f = \Phi'$ (Φ is our primitive function of f). The second characterization claims that the integral of a derivative of Φ gives Φ.

$$\int_a^b \Phi'\, dx = \Phi(b) - \Phi(a)$$

Proof. 1. Let $x_1, x_2 \in I$ and wlog $x_1 \le x_2$.

$$|F(x_1) - F(x_2)| = \left| \int_a^{x_1} f(\xi)\, d\xi - \int_a^{x_2} f(\xi)\, d\xi \right|$$

$$= \left| \int_a^{x_1} f(\xi)\, d\xi + \int_{x_2}^a f(\xi)\, d\xi \right|$$

$$= \left| \int_{x_2}^{x_1} f(\xi)\, d\xi \right| = \left| \int_{x_1}^{x_2} f(\xi)\, d\xi \right|$$

$$\le \int_{x_1}^{x_2} |f(\xi)|\, d\xi \le |x_2 - x_1| \cdot \|f\|_\infty$$

hence F is Lipschitz continuous in I. So F is continuous in I.

One-sided limits:

Let $\varepsilon > 0$ arbitrary and $x_0 \in I$ and δ such that $\forall x \in (x_0, x_0 + \delta)$ it holds that:

$$|f(x) - f_+(x_0)| < \varepsilon$$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f_+(x_0) \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_a^x f(\xi)\, d\xi - \int_a^{x_0} f(\xi)\, d\xi - f_+(x_0) \cdot (x - x_0) \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(\xi)\, d\xi - f_+(x_0) \int_{x_0}^x 1\, d\xi \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(\xi)\, d\xi - \int_{x_0}^x f_+(x_0)\, d\xi \right|$$

$$= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(\xi) - f_+(x_0)\, d\xi \right|$$

$$\le \frac{1}{|x - x_0|} \int_{x_0}^x |f(\xi) - f_+(x_0)|\, d\xi$$

$\xi \in (x_0, x) \subseteq (x_0, x_0 + \delta)$

$$< \frac{1}{|x - x_0|} \cdot \varepsilon \underbrace{\int_{x_0}^x 1\, d\xi}_{|x - x_0|}$$

$$= \varepsilon$$

$$\Rightarrow F'_+(x_0) = f_+(x_0)$$

Analogously $F'_-(x_0) = f_-(x_0)$.

$\square$

German keywords

English keywords