

Introduction to Functional Analysis

Lecture notes, University of Technology, Graz
based on the lecture by Martin Holler

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Contents

0	Introduction	2
0.1	Application examples	2
0.2	Our first problem	3
1	Topological and metric spaces	7
2	Normed space	20
2.1	Fundamentals	20
2.2	Important examples of normed spaces	27
2.2.1	Basic properties of Lebesgue spaces	35
2.2.2	Separability of L^p requires a density result	35
3	Linear Operators	39
4	The Hahn-Banach Theorem and its consequences	43
5	Fundamental theorems for operators in Banach spaces	50
6	Dual spaces, reflexivity and weak convergence	56
7	Complementary subspace and adjoint operators	67
7.1	Adjoint operators	70
8	Hilbert spaces	74
8.1	Orthogonality	76

0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces U, V
 matrices $A \in \mathcal{M}^{n \times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
 functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto$ functionals $f : U \rightarrow \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
 - basis of Applied Mathematics
 - partial differential equations
 - physical modelling
 - inverse problems (operator A models some physical measurement process)
 - Optimization and optimal control

A motivating example was presented with slides.

0.1 Application examples

Let $K : U \rightarrow \mathbb{R}^m$ with U as vector space describe a physical model. For example, K is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0, 1] \rightarrow \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find u . Typically inversion of K is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$ where $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$ and $\|\cdot\|_u$ is a norm on U . Or alternatively, let $U = C^1([0, 1]^2)$ and solve $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$.

Other examples are JPEG compression and upsampling of images.

0.2 Our first problem

Let $U := C^1([0, 1]^2)$ be a normed space, $K : U \rightarrow \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \|Ku - d\| + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \Rightarrow u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ linear.

Proof. Case 1 (trivial model): Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \quad (1)$$

Take $(u_n)_{n \in \mathbb{N}}$ in \mathbb{R}^n such that $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \forall n \in \mathbb{N}$

$$\Rightarrow \lambda \|u_n\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i \in \mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \rightarrow u$ as $i \rightarrow \infty$.

Now: Show that u solves Problem (1). ∇ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \rightarrow \infty} \|\nabla u_{n_i}\|_2 + \lambda \|u_{n_i} - d\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that \hat{u} is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of ∇ , $\|\cdot\|_2$

Case 2 (K arbitrary): 1. K arbitrary does not provide boundedness anymore. Define $X := \text{kernel}(\nabla) \cap \text{kernel}(K)$ and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^n x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \quad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that X^\perp is called *orthogonal complement*.

Claim 0.1. *If \hat{u} solves $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. Then \hat{u} solves Problem (1).*

Proof. Let \hat{u} be a solution on X^\perp . Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^\perp$. Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \|\nabla(u_1 + u_2)\|_2 + \lambda \|k(u_1 + u_2) - d\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus \hat{u} solves our problem (1). \square

Take again $(u_n)_{n \in \mathbb{N}}$ be such that $u_n \in X^\perp \forall n$ and

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \ker(\nabla) + \ker(\nabla)^\perp$. $\nabla : \ker(\nabla)^\perp \rightarrow \text{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \ker(\nabla)^\perp \implies v \in \ker(\nabla) \implies \|v\|_2 = (v, v) = 0$. Thus, $\nabla^{-1} : \text{image}(\nabla) \rightarrow \ker(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^2\|_2 &= \|\nabla^{-1} \nabla u_n^2\|_2 = \|\nabla^{-1}\| \cdot \|\nabla u_n^2\|_2 \leq \|\nabla^{-1}\| \\ &\leq \|\nabla^{-1}\| (\|\nabla u_n^2\|_2 + \lambda \|Ku_n - d\|_2) \\ &= \|\nabla^{-1}\| \left(\underbrace{\|\nabla u_n\|_2}_{=\|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\ &< C \text{ for some } C > 0 \end{aligned}$$

Then $\|u_n^2\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \rightarrow \text{image}(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap \ker(\nabla) \implies v \in \ker(K)$. Hence $v \in \ker(K) \cap \ker(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$. Hence $K^{-1} : \text{image}(K) \rightarrow X^\perp \cap \ker(\nabla)$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^1\|_2 &= \|K^{-1}Ku_n^1\|_2 \leq \|K^{-1}\| \|Ku_n^1\|_2 \\ &= \frac{\|K\|}{\lambda} (\lambda \|K(u_1^n + u_2^n) - Ku_n^1\|_2 + \|\nabla u_n\|_2) \\ &\leq \frac{\|K\|}{\lambda} \left(\underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \|d - Ku_1^2\|_2}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\ &< D \text{ for some } D > 0 \end{aligned}$$

$$\implies (u_n^1)_n \text{ bounded} \implies (u_n) = (u_n^1 + u_n^2)_n \text{ is bounded}$$

$\implies (u_n)_n$ admits a subsequence converging to some \hat{u} . As in Case 1, \hat{u} is a solution to Problem (1).

In summary,

1. $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u|^2 dx}$ with $U = C^1([0,1]^2)$ relevant for application.
2. Discrete version: $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:
 - (a) complementary subspaces X^\perp
 - (b) boundedness and compactness
 - (c) continuity
 - (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ This lecture took place on 2019/03/07.

Define $U = C^1([0,1]^2)$. Let Y is some Banach space and $K : U \rightarrow Y$ is linear and continuous.

Consider the problem (P_∞) given by $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$.

Proposition 0.2. *There exists a solution of (P_∞) .*

Proof. Take $(u_n)_{n \in \mathbb{N}}$ as a sequence in U such that $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_n \rightarrow \inf_{u \in U} (\dots)$. Now we want to show that $(u_n)_{n \in \mathbb{N}}$ is bounded.

Case 1: Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\Rightarrow \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\Rightarrow (u_n)_{n \in \mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n \in \mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change U to $U = \left\{ u : [0,1]^2 \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]^2} |\nabla u|^2 dx} < \infty \right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u \in U} R(u) + \lambda \|K_{u-d}\|_2$ instead.

In this setting, $(u_n)_{n \in \mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i \in \mathbb{N}}$).

Our next step is to use continuity to show that \hat{u} is a solution.

Problem: $u \mapsto \|u - d\|_2$ is, in general, not continuous with respect to weak convergence.

But it is always true that $\|\hat{u} - d\|_2 \leq \liminf_{i \rightarrow \infty} \|u_{n_i} - d\|_2$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = - \int_0^1 u \partial_x \varphi \quad \forall \varphi \in C^\infty([0, 1]^2)$$

$\varphi = 0$ in $K \setminus [0, 1]^2$ for some $K \Subset (0, 1)^2$.

$$\begin{aligned} \implies \int_{[0,1]^2} \nabla u \varphi &= - \int_{[0,1]^2} u \cdot (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \\ \forall \varphi : (\varphi_1, \varphi_2) &= C^\infty([0, 1]^2, \mathbb{R}^2) + \text{zero on boundary} \end{aligned}$$

We define $w : [0, 1]^2 \rightarrow \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w \varphi = - \int_{[0,1]^2} u (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \text{ holds } \forall \varphi$$

Then w is called *weak gradient* of u . We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\|_2 &\leq \liminf_{i \rightarrow \infty} R(u_{n_i}) + \liminf_{i \rightarrow \infty} \lambda \|u_{n_i} - d\|_2 \\ &\leq \liminf_{i \rightarrow \infty} (R(u_{n_i}) + \lambda \|u_{n_i} - d\|_2) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

Case 2: Works as in the finite-dimensional setting using

- $X := \ker(A) \cap \ker(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \quad \forall u \in \ker(\nabla)^\perp$ is called *Poincare inequality*.

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces

2. Normal spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contemplementary subspaces
8. Hilbert spaces

↓ This lecture took place on 2019/03/12.

Remark. 1. Literature: UGU, in particular: Biezis, Werner
 2. In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified

1 Topological and metric spaces

Remark (Motivation). Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies

Definition 1.1 (Topology). Let X be a set and $\tau \subset \mathcal{P}(X) = \{\text{"set of subsets of } X\}$. We say that τ is a topology on X if

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. For any collection of sets $(U_i)_{i \in I}$ with I as some index set. We have $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$.

(X, τ) is called topological space.

A set $U \subset X$ is called open if $U \in \tau$ and is called closed if $U^c \in \tau$.

Remark. By the third property of topologies, $\bigcap_{i \in I} V_i$ is closed for any collection $(V_i)_{i \in I}$ of closed sets.

Definition 1.2 (Metric). Let X be a set, $d : X \times X \rightarrow \mathbb{R}$ be such that $\forall x, y, z \in X$

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric on X and (X, d) is called metric space.

Definition 1.3 (Norm). Let X be a vector space. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is called norm if $\forall x, y \in X, \lambda \in \mathbb{K}$

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Then $(X, \|\cdot\|)$ is called normed space.

Remark. If $\dim(x) < \infty$, all norms on X are equivalent.

Example. 1. Let X be a set then $\tau = \{\emptyset, X\}$ is a topology.

2. $(X, \mathcal{P}(X))$ is a topological space.
3. Define $S^{d-1} := \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\}$ and $d(x, y) := r$ where r is the length of the shortest connection between x and y on S^{d-1} . Then d is a metric on S^{d-1}
4. $X := \{u : [0, 1] \rightarrow \mathbb{R} \mid u \text{ is continuous}\}$ then $\|u\|_\infty := \sup_{x \in [0, 1]} |u(x)|$ is a norm on X
5. $l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u \text{ and } \sum_{i=1}^\infty |x_i|^p < \infty\}$ with $p \in [1, \infty)$ and $\|(x_i)_{i \in \mathbb{N}}\|_p := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$. Then $(l^p, \|\cdot\|_p)$ is a normed space (the proof will be done later).

Remark.

$$L^\infty := \left\{ (X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|(X_i)_{i \in \mathbb{N}}\| = \sup_i |X_i|$$

Proposition 1.4. Let X be a set.

1. If (X, d) is a metric space, define for $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ and $\tau = \{U \in \mathcal{P}(X) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\}$. Then (X, τ) is a topological space. We say that τ is the topology induced by d and we have that $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$
2. If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \rightarrow \mathbb{R}$ with $(x, y) \mapsto \|x - y\|$. Then (X, d) is a metric space and d is called the metric induced by $\|\cdot\|$.

Remark (Consequence). Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.

Definition 1.5. Let (X, τ) be a topological space. $U \subset X$. $x \in X$.

1. U is called a neighborhood of x if $\exists V \in \tau - x \in V \subset U : \mathcal{U}(x)$ is defined as the set of all neighborhoods of x
2. \bullet x is called interior point of U if $U \in \mathcal{U}$

- x is called adjacent point of U if $\forall V \in \tau$ such that $x \in V : V \cap U \neq \emptyset$
- x is called cluster point of U if it is an adjacent point of $U \setminus \{x\}$.

The third property is stronger.

3. Notational conventions:

$$\mathring{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \mathring{U}$$

Proposition 1.6. Let (X, τ) be a topological space, $U \in X$. Then

1. U is open $\iff \mathring{U} = U$
2. U is closed $\iff \overline{U} = U$
3. $\mathring{U} = \bigcup_{\substack{V \in \tau \\ V \subset U}} V$ and \mathring{U} is open [" \mathring{U} is the largest open set in U "]
4. $\overline{U} = \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$ and \overline{U} is closed [" \overline{U} is the smallest closed set containing U "]

Proof. 3. \subset Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V$

\supset Let $x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \subset U \implies x \in \mathring{U}$

\mathring{U} is open because it is the union of open sets.

1. $\implies \mathring{U} \subset U$ by definition. U is open, so $U \subset \bigcup_{\substack{V \in \tau \\ V \subset U}} V \stackrel{(3)}{=} \mathring{U}$
2. $\implies U \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \implies x \in U^c \in \tau$ and $U \cap U^c = \emptyset$. This contradicts to $x \in \overline{U}$.
 \Leftarrow Take $x \in U^c = \overline{U}^c$.
 $\stackrel{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$
 $\implies V \cap U = \emptyset \implies V \subset U^c$
 $\implies U^c$ open $\implies U$ closed

4. We prove the fourth property without the second.

\subset Take $x \in \overline{U}$. Take closed V such that $U \subset V$ if $x \notin V \implies x \in V^c$ which is open and $V^c \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.

\supset Take $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$. Suppose $x \notin \overline{U}$.

$\implies \exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$

$\implies U \subset Z^c, Z^c$ closed, $x \notin Z^c$. This contradicts to $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$

\overline{U} closed follows since the intersection of closed sets is closed.

□

Definition 1.7 (Limit). Let (X, τ) be a topological space, $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Henceforth, we write $(x_n)_n$ for $(x_n)_{n \in \mathbb{N}}$ and $\hat{x} \in X$. We say $x_n \rightarrow \hat{x}$ in τ as $n \rightarrow \infty$ (“ x_n converges to x ”, “ x is limit of x_n ”) if

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

Definition 1.8 (Proposition and definition). Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or Hausdorff) if

$$\forall x, y \in X \text{ with } x \neq y \exists U, V \in \tau : x \in U, y \in V \text{ and } U \cap V = \emptyset$$

- In a T_2 -sphere, the limit of any sequence is unique.
- If τ is induced by a metric, then (X, τ) is T_2 .

Proof. 1. Take $(x_n)_n$ to be a sequence and assume x_n converges to \hat{x} and \hat{y} with $\hat{x} \neq \hat{y}$. By T_2 , $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergence, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\{n_x, n_y\} : x_n \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x, y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology τ . Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

Definition 1.9. Let (X, τ) be a topological space, $U \subset V \subset X$. We say that U is dense in V , if $V \subset \overline{U}$. We say that X is separable if there exists a countable, dense subset.

Definition 1.10. Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \rightarrow Y$ a function. We say f is continuous at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. f is called continuous if it is continuous at any $x \in X$.

Proposition 1.11. With $(X, \tau_X), (Y, \tau_Y)$ and f as above, f is continuous $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$

Proof. Left as an exercise to the reader.

□

Definition 1.12. Let (X, τ) be a T_2 topological space, $M \subset X$ called compact if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ (“ $(U_i)_{i \in I}$ is an open covering of M ”), there exists U_{i_1}, \dots, U_{i_n} such that $M \subset \bigcup_{k=1}^n U_{i_k}$ (“there exists a finite subcover”).

Remark. Compactness can also be defined without T_2 , this is also referred to as quasi-compact.

Remark (Exercise). Reconsider the previous results for metric and normed spaces.

↓ This lecture took place on 2019/03/14.

Definition 1.13. Let (X, d) be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in X . Then we say,

1. V is bounded if $\exists x \in X, r > 0$ such that $U \in B_r(x)$
2. $(x_n)_n$ is a Cauchy sequence if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$
3. X is complete if any Cauchy sequence in X admits a limit point
4. X is a Banach space if it is a normed space and complete

Proposition 1.14. Let (X, d) be a metric space. $(x_n)_n$ be a sequence in X . Then

1. $x_n \rightarrow x$ in the induced topology $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$
2. If $x_n \rightarrow x$, then $(x_n)_n$ is bounded as subset of X and $(x_n)_n$ is Cauchy.
3. If $U \subset X$ is closed and X is complete. Then (U, d) is a complete metric space.

Proof. 1. We prove both directions:

\implies True, since $B_\varepsilon(x)$ is open $\forall \varepsilon > 0$

\impliedby Let $x \in V$ with V open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$

V open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$

$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1, \dots, n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in U , then $(y_n)_n$ is a Cauchy sequence in $X \implies \exists x \in X : y_n \rightarrow x$ as $n \rightarrow \infty$ if $x \notin U \implies x \in U^c \implies \exists n_0 \in \mathbb{N}$ such that $y_{n_0} \in U^c$ due to U^c open. This is a contradiction to $(y_n)_n$ in U

□

Proposition 1.15. Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \rightarrow Y$. The following are equivalent (TFAE):

- f is continuous (with respect to the induced topology)
- $\forall (x_n)_n$ such that $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$

Proof. Firstly, we prove that the first statement implies the second statement.
Take $(x_n)_n$ converging to x . Take $V \in \tau_Y$ such that $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$$\begin{aligned} &\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X \text{ such that } x \in \hat{U} \subset U \\ &\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_0) \rightarrow f(x) \end{aligned}$$

Remark. 1. \implies 2. holds true in any topological space

2. \implies 1. Not.

Secondly, we prove that the second statement implies the first statement.

Suppose f is not continuous, find $x_n \rightarrow x$ such that $f(x_n) \rightarrow f(x)$ is wrong. If f is not continuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(U) \not\subset V \forall U \in \mathcal{U}(x)$

$$\begin{aligned} &\implies \exists \hat{V} \in \tau_Y \text{ such that } f(U) \not\subset \hat{V} \forall U \in \mathcal{U}(x), f(x) \in \hat{V} \\ &\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V} \\ &\implies (x_n)_n \text{ converges to } x \text{ but } f(x_n) \notin \hat{V} \implies f(x_n) \not\rightarrow f(x). \text{ This gives a contradiction. } \square \end{aligned}$$

Definition 1.16. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$.
 f is uniformly continuous iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proposition 1.17. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$, $f : M \rightarrow Y$. If M is dense in X , Y is complete and f is uniformly continuous.

$$\implies \exists ! \hat{f} : X \rightarrow Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

Proof. Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \rightarrow x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(x_n), f(x_m)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in M , $\hat{y} \in Y$ such that $f(\hat{x}_n) \rightarrow \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X$, $(x_n)_n$ is a sequence in M such that $x_n \rightarrow x, f(x_n) \rightarrow y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0 :$

$$\begin{aligned} &d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0 \\ &\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d_Y(f(x_n), y) < 3\varepsilon \end{aligned}$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.
2. \hat{f} is uniformly continuous.

□

↓ This lecture took place on 2019/03/19.

Proposition 1.18. Let (X, d) be a metric space, $M \subset X$.

1. M is compact, so $\forall (X_i)_{i \in I}$ with X_i a closed set $\forall i$ such that $(\bigcap_{i \in I} X_i) \cap M = \emptyset$.

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \left(\bigcap_{i=1}^n X_{i_j} \right) \cap M = \emptyset$$

2. M is compact, so M is closed and bounded.

Proof. 1. We note that $\forall (X_i)_{i \in I}$ is a family of closed sets. $(X_i^C)_{i \in I}$ is a family of open sets and $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

Proposition 1.19. Let (X, d) be a metric space, $M \subset X$. TFAE:

1. M is compact.
2. Every infinite subset of M admits a cluster point.
3. Every sequence of M admits a convergent subsequence.
4. M is complete and totally bounded, where totally bounded is defined as

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$$

Remark. 1. totally bounded \implies bounded (proof is left as an exercise)

2. Assume $\dim(x) < \infty$. Compact \iff complete and bounded (see course Analysis I)

3. $\dim(x) < \infty \iff \overline{B_1(0)}$ is compact

where the last two items imply that X is a normed space.

Proof. 1 \rightarrow 2 If M is finite, (2) always holds true. So assume that M is infinite. Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists x \in C$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(x_n)_n$ is a sequence of distinct points and $x_n \rightarrow x$. This implies that x is a cluster point of C . This gives a contradiction.

Now $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$. If M is compact, then

$$\begin{aligned}
&\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i) \\
&\implies C \subset M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i) \\
&\implies C \text{ is finite}
\end{aligned}$$

This is a contradiction.

2 \rightarrow 3 Let $(x_n)_n$ be a sequence in M .

Case 1: $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies (x_n)_n$ admits a convergent sequence.

Case 2: $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid n \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

3 \rightarrow 4 Suppose that M is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in M as follows: Given x_1, \dots, x_n , choose $x_1 \in M$ arbitrary and $x_{i+1} \in M \setminus \bigcup_{j=1}^i B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in M and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convergent subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in M , then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \rightarrow x$ as $i \rightarrow \infty$. Since $(x_n)_n$ is Cauchy, $x_n \rightarrow x$ as $n \rightarrow \infty$ [left as an exercise]. Thus M is complete.

4 \rightarrow 1 Let $(U_i)_{i \in I}$ be an open covering of M and assume that $(U_i)_{i \in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}}(a)$. Define $\Omega := \{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } U_i\}$. We recursively define a sequence $(a_n)_n$ in M such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n-1}) \neq \emptyset$$

Goal: Show $(a_n)_n \rightarrow a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

Step 1 $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a \in E_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \rightarrow n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let

$$\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}. \text{ It holds that } M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a):$$

$$x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies \exists \hat{a} \in E_{n+1} : x \in B_{\frac{1}{2^{n+1}}}(\hat{a}) \implies B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(\hat{a}) \neq \emptyset \implies \hat{a} \in \tilde{E}_{n+1} \implies$$

Hence there exists $a_{n+1} \in E_{n+1}$ such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$

and

$B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

Step 2 Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) < d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. M is complete, so $\exists a \in M : a_n \xrightarrow{n \rightarrow \infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} \text{ and } \exists r > 0 : B_r(a) \subset U_{i_0}$$

Hence, for n sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

Proposition 1.20. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$ compact. Let $f : X \rightarrow Y$ be continuous. Then

1. $f(M)$ is compact
2. $f|_M : M \rightarrow Y$ is uniformly continuous.

Proof. 1. Let $(U_i)_{i \in I}$ be an open covering of $f(M)$

$$\implies (f^{-1}(U_i))_{i \in I} \text{ is an open covering of } M \text{ [why!]}$$

$$\implies \exists c_1, \dots, c_n \text{ such that } M \subset \bigcup_{j=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{j=1}^n U_{i_j}$$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon > 0 \forall n \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n, (y_n)_n$ sequences in M satisfying condition (*). M is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) < \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \rightarrow \infty} 0$$

□

↓ This lecture took place on 2019/03/21.

Proposition 1.21 (Proposition and definition). *Let (X, d_X) and (Y, d_Y) be metric spaces. $g : X \rightarrow Y$ is a function. g is called Lipschitz continuous if $\exists L > 0$ such that $d_Y(\varphi(x), \varphi(y)) \leq L d_X(x, y) \forall x, y \in X$. Any Lipschitz continuous function is uniformly continuous.*

Proof. Left as an exercise to the reader. \square

Theorem 1.22 (Arzelà-Ascoli theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Define $C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$ and $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$. Then*

1. d_C is well-defined and $(C(X, Y), d_C)$ is a complete metric space
2. A set $M \subset C(X, Y)$ is compact iff
 - (a) $\forall x \in X$ the set $M_x := \{f(x) \mid f \in M\}$ is compact
 - (b) M is equicontinuous, i.e. $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. 1. Show that: $d_C(f, g) < \infty$.

Pick $f, g \in C(X, Y)$. Because X is compact, $f(X), g(X)$ compact $\implies f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that d_C is a metric is left as an exercise.

Show that $(C(X, Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X, Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because Y is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to d_C : Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x)) \leq \lim_{m \rightarrow \infty} d(f_m, f_n) < \varepsilon$ (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X, Y) \implies f_n \rightarrow f$ with respect to d_C . Show that $f \in C(X, Y)$. Take $\varepsilon > 0$. Let n_0 such that $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x))$.

In general, we have $\forall x, y, z \in (Z, d_Z)$ with d_Z as a metric.

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Proof.

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y) \quad (2)$$

$$d(y, z) \leq d(y, x) + d(x, z) \implies d(y, z) - d(x, z) \leq d(x, y) \quad (3)$$

$$(2) \text{ and } (3) \implies |d(x, z) - d(y, z)| \leq d(x, y) \quad (4)$$

□

Consequently, $\forall z \in Z, x_n \rightarrow x$ in Z : $d(x_n, z) \rightarrow d(x, z)$ since $|d(x_n, z) - d(x, z)| \leq d(x_n, x) \rightarrow 0$.

2. We need to prove both directions.

$$\begin{aligned} \implies & \text{ (a) For } x \in X \text{ fixed, define } g_X : M \rightarrow Y \text{ with } f \mapsto f(x). \text{ Then} \\ & d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2) \\ \implies & g_X \text{ is Lipschitz continuous, in particular continuous} \\ \implies & M_X = g_X(M) \text{ compact} \\ & \text{(b) Take } \varepsilon > 0. \text{ } M \text{ is totally bounded, so } \exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i). \forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies \\ & d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}. \text{ Define } \delta := \min_i \delta_i > 0. \text{ Then } \forall x, y \in X : \\ & d(x, y) < \delta \text{ and } \forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0}) \\ \implies & d(f(x), f(y)) \leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon \end{aligned}$$

\Leftarrow We prove the other direction.

↓ This lecture took place on 2019/03/26.

M is complete since it is a closed subset of a Banach space.

Show: M is totally bounded.

Consider $\varepsilon > 0$. Show: $\exists f_1, \dots, f_n$ such that $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$.

- Because M is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X , $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{i=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{ii})$

Compare with Figure 1.

Now, for each tuple of indices $(y_{1,j_1}, \dots, y_{n,j_n})$ define $f_{y_{1,j_1}, \dots, y_{n,j_n}} \in C(x, y)$ to be such that $f_{y_{1,j_1}, \dots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an f exists. The set F of all such functions is finite. We show that $M \subset \bigcup_{q \in F} B_\varepsilon(q)$.

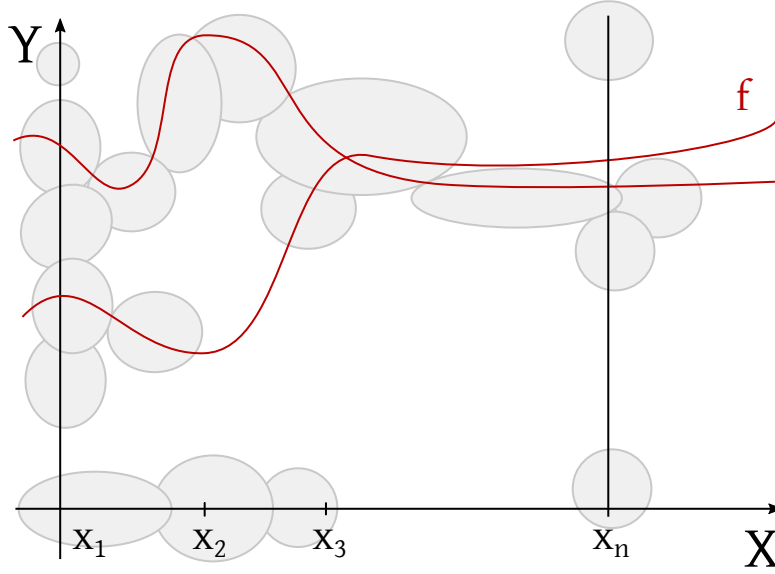


Figure 1: Covering of a function graph

Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \dots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_\alpha \in F$ accordingly.

Take $x \in X$ arbitrary and x_i such that $x \in B_\delta(x_i)$

$$\begin{aligned} \implies d(f(x), f_\alpha(x)) &\leq d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_\alpha) &= \sup_{x \in X} d(f(x), f_\alpha(x)) < \varepsilon \end{aligned}$$

□

Remark. Compare this to the fact that $B_1(0)$ in $C(X, Y)$ is not compact.

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

Remark (Motivation). In general, let (X, d) be a metric space. Let A and B be open and dense, then also $A \cap B$ is dense.

Proof. Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] \neq \emptyset$. Take $x \in X, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. A is dense. A is open, so $\exists \varepsilon_1 > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. B is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_{\varepsilon_1}(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally, $\forall A_1, \dots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many A_i ?

Theorem 1.23 (Baire theorem). *Let (X, d) be a complete metric space. Let $(O_n)_{n \in \mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.*

Proof. Let $D := \bigcap_{n \in \mathbb{N}} O_n$. Show that for $x \in X$, $\varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n \in \mathbb{N}}$.

n = 1 Take x_1, ε_1 such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

n - 1 \rightarrow n Given $x_{n-1}, \varepsilon_{n-1}$, take x_n, ε_n such that

$$\overline{B_{\varepsilon_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_n}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete} \implies \exists x \in X : x_n \rightarrow x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_n}(x_N)} \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_N)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

Definition 1.24. *Let (X, d) be a metric space, $M \subset X$. We say*

- M is nowhere dense (dt. “nirgends dicht”), if $\overset{\circ}{M} = \emptyset$
- M is of first category $\iff M$ is a countable union of nowhere dense sets
- M is of second category $\iff M$ is not of first category

Theorem 1.25 (Baire category theorem (weaker version)). *Let (X, d) be a complete metric space. Then (X, d) is of second category.*

In other words (which is a useful formulation): If $X = \bigcup_{n \in \mathbb{N}} C_n \implies \exists n_0 : \overset{\circ}{C} \neq \emptyset$. In particular, if

$$X = \bigcup_{n \in \mathbb{N}} C_n \text{ with } C_n \text{ closed} \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

Proof. Suppose that $X = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O_n}$ with $\overset{\circ}{O_n} = \emptyset \forall n$

$$\overset{\circ}{O_n} = \emptyset \implies \overline{\overset{\circ}{O_n}} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overline{O_n}^C = \emptyset \implies B_\varepsilon(x) \subset \overline{O_n} \implies \overset{\circ}{O_n} \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O_n}^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\Rightarrow \overline{O_n}^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n \in \mathbb{N}} \overline{O_n}^C \text{ is dense}$$

$$\bigcap_{n \in \mathbb{N}} \overline{O_n}^C = \left(\bigcup_{n \in \mathbb{N}} \overline{O_n} \right)^C = X^C = \emptyset$$

gives a contradiction □

Remark. 1. *This is a fundamental theorem in Functional Analysis*

2. *This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise, e.g. Weierstrass function)*

2 Normed space

2.1 Fundamentals

Definition 2.1. Let X be a vector space. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is called seminorm if

- $x = 0 \Rightarrow \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$

The first property differs between a norm and a seminorm.

The tuple $(X, \|\cdot\|)$ is called a semi-normed space. We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.

Example (Not done in lecture). *We found the following examples while studying:*

$$f \text{ linear}, x \mapsto |f(x)| \quad \text{and} \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := |y|$$

Definition 2.2 (Definition and proposition). Let $(X, \|\cdot\|)$ be a semi-normed space and $(x_n)_n$ be a sequence in X . We say that

- $\sum_{n=1}^{\infty} x_n$ converges to $x \in X$ and write $x = \sum_{n=1}^{\infty} x_n$ if $\lim_{m \rightarrow \infty} \sum_{n=1}^m x_n = x$
- $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ converges [$\iff (\sum_{n=1}^m \|x_n\|)_m$ is bounded]

It holds that X is complete iff any absolutely converging series converges.

Proof. \implies Take $m_1 < m_2$ arbitrary, then

$$\left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| \leq \sum_{n=m_1+1}^{m_2} \|x_n\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \leq \left| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \right|$$

$$\implies \left(\sum_{n=1}^m x_n \right)_m \text{ is Cauchy} \implies \text{convergent}$$

\Leftarrow Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick N_k such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$$

$$\text{Define } y_k := x_{n_{k+1}} - x_{n_k} \implies \sum_k \|y_{n_k}\| \leq \sum_k 2^{-k} < \infty$$

$$\implies \exists y \in X : \sum_{k=1}^n y_k \rightarrow y \text{ as } n \rightarrow \infty$$

$$\sum_{k=1}^n y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \rightarrow y - x_{n_1} \text{ as } n \rightarrow \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, hence $(x_n)_n$ is convergent.

□

Remark. In \mathbb{R}^n , $\sum_n x_n$ is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction \implies is true.

↓ This lecture took place on 2019/03/28.

Proposition 2.3 (Proposition and definition). Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X . We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if

$$\exists m, M > 0 \forall x \in X : m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

TFAE:

1. $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. For any sequence $(x_n)_n$ and $x \in X$, $x_n \rightarrow x$ with respect to $\|\cdot\|_1 \iff x_n \rightarrow x$ with respect to $\|\cdot\|_2$
3. For any sequence $(x_n)_n$ we have,

$$x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_2$$

Proof. (1) \implies (2) \implies (3) is immediate.

It remains to show that:

(3) \implies (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \ \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n \|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\|y_n\|_1 = \frac{1}{n} \rightarrow 0$ hence $y_n \rightarrow 0$, but $\|y_n\|_2 > n \|y_n\|_1 = 1$.

$$\implies y_n \not\rightarrow 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

Remark. If $\dim(X) < \infty$, then any two norms on X are equivalent.

Definition 2.4 (Quotient spaces). Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation “ \sim ” on X with $x \sim y : \iff x - y \in Y$.

Then \sim defines an equivalence relation on X . We define

- $[X]_\sim = \{y \in X \mid x \sim y\}$ is the equivalence class of $x \in X$
- $X/Y := \{[x]_\sim \mid x \in X\}$ is the quotient space
- $\pi : \begin{cases} X \rightarrow X/Y \\ x \mapsto [x]_\sim \end{cases}$

Defining $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \quad \hat{0} := [0]$$

We get that:

1. X/Y is a vector space
2. $\|[x]\|_{X/Y} := \inf_{y \in [x]} \|y\|_X$ is a semi-norm.
3. If Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.
4. If X is complete and Y closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.

Proof. We need to show that the given relation is an equivalence relation and that the vector space with “ $+$ ” is well-defined. This is left to the reader as an exercise.

2. – First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\|[0]\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|y\| \leq \|0\| = 0$$

- Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$. Show that: $\|\lambda[x]\|_{X/Y} = |\lambda| \| [x] \|_{X/Y}$.

Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\|\lambda[x]\|_{X/Y} = \|[\lambda x]\|_{X/Y} = \inf_{y \in [\lambda x]} \|y\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \|y\| = \inf_{w \in [x], w = \frac{y}{\lambda}} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\|[x]\|_{X/Y}}$$

- Take $[x_1], [x_2] \in X/Y, \varepsilon > 0$. We note that

$$\|[x]\|_{X/Y} = \inf_{y \in [x]} \|y\| = \inf_{\substack{y \in [x] \\ w = x - y}} \|x - y\| = \inf_{w \in Y} \|x - w\|$$

Hence we can take $y_1, y_2 \in Y$ such that $\|x_1 - y_i\| < \|[x_i]\|_{X/Y} + \varepsilon, \quad i \in [1, 2)$.

$$\begin{aligned} \Rightarrow \|[x_1] + [x_2]\|_{X/Y} &= \|[x_1 + x_2]\|_{X/Y} \leq \|x_1 + x_2 - (y_1 + y_2)\| \\ &\leq \|x_1 - y_1\| + \|x_2 - y_2\| \leq \|[x_1]\|_{X/Y} + \|[x_2]\|_{X/Y} + 2\varepsilon \end{aligned}$$

Since ε was arbitrary, the assertion follows.

3. Suppose Y is closed if $\|[x]\|_{X/Y} = 0$, then

$$\inf_{y \in Y} \|x - y\| = 0 \Rightarrow \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n \rightarrow \infty} \|x - y_n\| = 0$$

$$Y \text{ closed} \Rightarrow x \in Y \Rightarrow [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in X/Y and suppose that $\sum_{i=1}^{\infty} \|[x_n]\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, X/Y is complete.

Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \|[x_n]\|_{X/Y} + 2^{-n}$

$$\Rightarrow \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} (\|[x_n]\|_{X/Y} + 2^{-n}) < c < \infty$$

$$X \text{ complete} \Rightarrow \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \quad \left\| [x] - \sum_{n=1}^m \underbrace{[x_n]}_{[\tilde{x}_n]} \right\|_{X/Y} \leq \left\| x - \sum_{k=0}^n \tilde{x}_k \right\| \rightarrow 0$$

□

↓ This lecture took place on 2019/04/02.

Corollary 2.5. Let X be a vector space with semi-norm $\|\cdot\|_X : X \rightarrow [0, \infty)$. Then

- $N = \{x \in X \mid \|x\|_X = 0\}$ is a subspace at X
- $\|[X]\| := \|X\|_p$ is a norm on X/N
- If X is complete, then $(X/N, \|\cdot\|)$ is a Banach space.

Proof. The proof is left as an exercise. \square

Proposition 2.6. Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then

- \overline{U} is also a subspace.
- X is separable iff $\exists A \subset X$ complete such that $X = \overline{\mathcal{L}(A)}$ where $\mathcal{L}(A) = \{\sum_{i=1}^n \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathbb{K}, n \in \mathbb{N}\}$

Proof. • Left as an exercise

- \Rightarrow True since $\exists A \subset X$ countable such that $\overline{A} = X \Rightarrow \underline{X} = \overline{A} \subset \overline{\mathcal{L}(A)} \subset X$
- \Leftarrow Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^n (\lambda_i + i\mu_i)x_i \mid \lambda_i, \mu_i \in \mathbb{K}, x_i \in A, n \in \mathbb{N} \right\}$$

where i is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then B is countable.

Show: $\forall x \in X \forall \varepsilon \exists x \in B : \|x - y\| < \varepsilon$.

Take $x \in X, \varepsilon > 0 \Rightarrow \exists x_0 \in \mathcal{L}(A) : \|x - x_0\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=1}^n (\lambda_i + i\mu_i)x_i$ with $\lambda_i, \mu_i \in \mathbb{K}, x_i \in A$. Choose $\lambda'_i, \mu'_i \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda'_i)^2 + (\mu_i - \mu'_i)^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^n \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let $y := \sum_{i=1}^n (\lambda'_i + i\mu'_i)x_i \in B$.

$$\begin{aligned} \Rightarrow \|x - y\| &\leq \|x - x_0\| + \|x_0 - y\| && \leq \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^n |(\lambda_i + i\mu_i) - (\lambda'_i + i\mu'_i)| \|x_i\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^n \|x_i\|} = \varepsilon \end{aligned}$$

\square

Proposition 2.7 (Proposition and definition). Let $(X_i, \|\cdot\|_{X_i})$ for $i = 1, \dots, n$ be a normed space. Denote by

$$X_1 \otimes X_1 \otimes \dots \otimes X_n = \bigotimes_{i=1}^n X_i = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

For $p \in [1, \infty]$, define

$$\|(x_1, \dots, x_n)\|_{\otimes_i X_i, p} = \begin{cases} \left(\sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1, \dots, n} \|x_i\|_{X_i} & \text{if } p = \infty \end{cases}$$

Then

- $(\otimes_i X_i, \|\cdot\|_{\otimes_i X_i, p})$ is a normed space with respect to componentwise addition and multiplication.
- If all X_i are complete, then $\otimes_{i=1}^n X_i$ is complete.
- All norms $\|\cdot\|_{\otimes_i X_i, p}$ are equivalent.

Proof. • Vector space properties: Left as an exercise

- Norm: $\|x\|_{\otimes_i X_i, p} = 0 \iff x = 0$
 $\|\lambda x\|_{\otimes_i X_i, p} = |\lambda| \|x\|_{\otimes_i X_i, p}$
- Triangle inequality: $p = 1, p = \infty$
 $p \in (1, \infty)$. Take $x, y \in \otimes_{i=1}^n X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\otimes_i X_i, p}$.

$$\begin{aligned} \implies \|x + y\|_p^p &= \sum_{i=1}^n \|x_i + y_i\|_{X_i}^p \|x_i + y_i\|_{X_i}^{p-1} \\ &\leq \sum_{i=1}^n \|x_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} + \sum_{i=1}^n \|y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\ &\leq \underbrace{\left(\sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}}}_{\text{H\"older ineq.}} \cdot \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\ &\quad + \left(\sum_{i=1}^n \|y_i\|_{X_i}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \|x\|_p \|x + y\|_p^{p-1} + \|y\|_p \|x + y\|_p^{p-1} \\ &= (\|x\|_p + \|y\|_p) \cdot \|x + y\|_p^{p-1} \end{aligned}$$

$$\implies \|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ if } x + y \neq 0 \text{ (trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in \mathbb{R}^n)

□

Definition 2.8. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \rightarrow Y$ is linear such that $\|j(x)\|_Y = \|x\|_X$ (hence j is injective) then j is called isometric embedding from X to Y . If j is bijective, then j is called isometric isomorphism and we say $X = Y$ up to isomorphism.

Proposition 2.9. Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_{\hat{X}})$ a Banach space such that

1. \exists isometric embedding, $i : X \rightarrow \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [\hat{X} can be regarded as completion of X]
2. If $j_1 : X \rightarrow Y$ is an isometric embedding on Y , a Banach space

$$\implies \exists i_2 : \hat{X} \rightarrow Y$$

an isometric embedding such that $j_2 \circ j = j_1$ and if $\overline{j_1(X)} = Y$ then j_2 is an isometric isomorphism. Thus “the completion is essentially unique”.

Proof. 1. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is Cauchy}\}$. \hat{X} is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \quad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define $\|(x_n)_n\|_{\hat{X}} := \lim_{n \rightarrow \infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in \mathbb{R}]. Then $\|\cdot\|_{\hat{X}}$ is a semi-norm (proof is left as an exercise). Setting $N = \{(X_n)_n \mid \|(X_n)_n\|_{\hat{X}} = 0\}$. By Corollary 2.5, $\hat{X} := \hat{X} \setminus N$ with $\|[(X_n)_n]\|_{\hat{X}} = \|(X_n)_n\|_{\hat{X}}$ is a normed space. Define

$$j : X \rightarrow \hat{X} \quad x \mapsto [(x)_n]$$

then j is linear and $\|j(x)\|_{\hat{X}} = \|[(x)_n]\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x\| = \|x\|$. So j is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n := j(x_n) \in \hat{X}$.

$$\begin{aligned} \implies \|y_m - [(x_n)_n]\|_{\hat{X}} &= \|(x_m)_n - (x_n)_n\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x_m - x_n\| \\ &= \lim_{n \geq n_0} \|x_m - x_n\| < \varepsilon \end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n_0 \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: \hat{X} is complete.

Let $(y_n)_n$ be Cauchy in \hat{X} . Pick $X_n \in X$ such that $\|j(x_n) - y_n\|_{\hat{X}} \leq \frac{1}{n}$
 $(\overline{j(X)} = \hat{X})$

$$\implies \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}} \leq \|j(x_n) - y_n\|_{\hat{X}} + \|y_n - y_m\|_{\hat{X}} + \|y_m - j(x_m)\|_{\hat{X}}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \|y_n - y_m\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick n_1 such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\implies \forall n, m > \max(n_0, n_1) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\implies (x_n)_n$ is Cauchy. Let $y := (X_n)_n \in \tilde{X}$. Then

$$\|y_n - [y]\|_{\hat{X}} \leq \|y_n - j(x_n)\|_{\hat{X}} + \|j(x_n) - [y]\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n \rightarrow \infty} \|x_n - x_m\|_X \xrightarrow{n \rightarrow \infty} 0$$

2. ↓ *This lecture took place on 2019/04/04.*

Let $\hat{x} \in \hat{X} \implies \exists (x_n)_n \in X$ such that $j(x_n) \rightarrow \hat{x} \implies \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}}$.

$\implies (x_n)_n$ is a Cauchy sequence.

$\implies j_1(x_n)$ is a Cauchy sequence in Y .

$\implies \exists \lim_{n \rightarrow \infty} j_1(x_n) := y$

Using this, we define $j_2 : \hat{X} \rightarrow Y$ with $\hat{x} \mapsto \lim_{n \rightarrow \infty} j_1(x_n)$ where $j(x_n) \rightarrow \hat{x}$.

Well-defined? Take $\hat{x} \in \hat{X}$ and $j(x_n) \rightarrow \hat{x}$, $j(y_n) \rightarrow \hat{x}$.

$$\implies \|i_1(x_n) - j_1(y_n)\| = \|x_n - y_n\| = \|j(x_n) - j(y_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\implies \lim_{n \rightarrow \infty} j_1(x_n) = \lim_{n \rightarrow \infty} j_1(y_n) \implies j_1 \text{ well-defined}$$

Show linearity is left as an exercise. By isometry, take $\hat{x} \in \hat{X}$,

$$|i_2(\hat{x})| \underbrace{=}_{j(x_n) \rightarrow \hat{x}} \lim_{n \rightarrow \infty} \|j_1(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|i(x_n)\| = \|\hat{x}\|$$

Show: $j_2 \circ j = j_1$. Take $x \in X \implies (x_n)$ is such that $j(x) \rightarrow j(x) \implies j_2(j(x)) = \lim_{n \rightarrow \infty} j_1(x) = j_1(x)$.

Assume that $\overline{j_1(X)} = Y$. Take $y \in Y$. Find $\hat{x} \in \hat{X}$ such that $i_2(\hat{x}) = y$.

By $\overline{j_1(X)} = Y \implies \exists (x_n)_n$ in X such that $i_1(x_n) \rightarrow Y \implies (j_1(x_n))_n$ is Cauchy.

$$\implies (x_n)_n \text{ Cauchy} \implies (j(x_n))_n \text{ Cauchy}$$

$$\xrightarrow{\hat{X} \text{ complete}} \exists \hat{x} \text{ such that } \lim_{n \rightarrow \infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n \rightarrow \infty} j_2(x_n) = Y$$

□

2.2 Important examples of normed spaces

Definition 2.10 (Basic notation). Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \rightarrow \mathbb{K}^M$ with $N, M \in \mathbb{N}$.

- We call Ω a domain (dt. "Gebiet") if Ω is open and connected, where connected means that $\forall x, y \in \Omega$ there is a curve in Ω connecting X and Y .
- For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$. If f is r -times continuously differentiable, we set for $\alpha \in \mathbb{N}_0^N$, $\{\alpha\} \leq r$.

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} f$$

where $\frac{\partial^{\alpha_1}}{\partial_{x_1}^{\alpha_1}}$ is the partial derivative of f with respect to x_i of order α_i .

Example. Let $N = 2$ and $\alpha = (1, 1)$.

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

Let $\alpha = (2, 0)$.

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- For $z \in \mathbb{K}^N$ we denote $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$.¹
- We say $E \subset \Omega$ is compact in Ω and we write $E \Subset \Omega$ if E is compact.

Remark. If $E \Subset \Omega$, then $\exists \delta > 0 : \inf \{ \|x - y\| \mid x \in E, y \in \partial\Omega \} > 0$.

Proof. Left as an exercise (use compactness) □

- f is compactly supported in Ω if $\text{supp}(f) \Subset \Omega$.
- $\text{supp}(f) := \overline{\{x \in \Omega \mid \|f(x)\| > 0\}}$

↓ This lecture took place on 2019/04/09.

Definition 2.11 (Definition and proposition, Spaces of continuous functions).
Let $\Omega \subset \mathbb{R}^N$ be a domain. We define

$$\begin{aligned} C_b(\Omega, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ bounded} \} \text{ with } \|\varphi\|_{C_b} = \|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)| \\ C(\overline{\Omega}, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ can be continuously extended to } \overline{\Omega} \}, \|\varphi\|_C := \|\varphi\|_\infty \\ C^r(\overline{\Omega}, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid D^\alpha \varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \leq r \} \text{ and } \|\varphi\|_{C^r} = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq r}} \|D^\alpha \varphi\|_\infty \end{aligned}$$

$$\begin{aligned} C_C^r(\Omega, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \text{supp}(\varphi) \Subset \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M) \} \text{ and } \|\varphi\|_{C_C^r} = \|\varphi\|_{C^r} \\ C^\infty(\overline{\Omega}, \mathbb{K}^M) &= \bigcap_{r \in \mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M) \\ D(\Omega, \mathbb{K}^M) &= C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r \in \mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \text{ in } C^r(\overline{\Omega}, \mathbb{K}^M) \end{aligned}$$

Then for any bounded Ω , C^r, C_0^r, C_b are Banach spaces and C_C^r is a normed space.

Recall: $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

¹This is an abuse of notation with $|\alpha|$ for $\alpha \in \mathbb{N}_0^N$

Proof. The functions $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$ are norms (proof is left as an exercise).

Show that C_b is complete: Take $(\varphi_n)_n$ in C_b to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because $|\varphi_n(x) - \varphi_m(x)| \leq \|\varphi_n - \varphi_m\|_\infty$. Hence we can define $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$.

Show: $\varphi_n \rightarrow \varphi$ in $\|\cdot\|_\infty$. Take $\varepsilon > 0$. Show that $\exists n_0 \forall n \geq n_0 : \|\varphi - \varphi_n\|_\infty < \varepsilon$.
Take n_0 such that $\forall n, m \geq n_0 : \|\varphi_n - \varphi_m\|_\infty < \varepsilon$. Take $m \geq n_0$.

$$\implies \forall x \in \Omega : |\varphi(x) - \varphi_m(x)| = \lim_{\substack{n \rightarrow \infty \\ n \geq n_0}} |\varphi_n(x) - \varphi_m(x)| < \|\varphi_n - \varphi_m\|_\infty$$

Show: φ is bounded, i.e. $\exists C > 0 : |\varphi(x)| \leq C < \|\varphi_n - \varphi_m\|_\varepsilon < \infty$. Take n such that $\|\varphi - \varphi_n\|_\infty < 1$

$$\implies \forall x \in \Omega : |\varphi(x)| > |\varphi(x) - \varphi_n(x)| + |\varphi_n(x)| \leq 1 + \underbrace{\|\varphi_n\|}_{=C}$$

Now $C^r(\overline{\Omega}, \mathbb{K}^n)$ is a subspace of $C^b(\Omega, \mathbb{K}^n)$. Also $C^r(\overline{\Omega}, \mathbb{K}^n)$ is closed, since the uniform limit of $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$ with respect to $\|\cdot\|_{C^r}$ is again in $C^r(\overline{\Omega}, \mathbb{K}^M)$ [a result from Analysis].

$$\implies C^r(\overline{\Omega}, \mathbb{K}^M) \text{ is a Banach space}$$

$C_c^r(\overline{\Omega}, \mathbb{K}^M)$ is closed by definition, hence Banach.

$C_c^r(\Omega, \mathbb{K}^M)$ is a vector space, since $\forall \lambda \in \mathbb{K} : \varphi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\lambda\varphi) = \text{supp}(\varphi)$ and for $\varphi, \Psi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\varphi + \Psi) \ll \Omega$. \square

Definition 2.12 (Definition and proposition). Let (Ω, Σ, μ) with $\Omega \subset \mathbb{R}^N$ be a measure space (i.e. Σ is a sigma algebra and μ is a measure). For $p \in [1, \infty)$, we define

$$\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) = \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid f \mu - \text{measurable and } \int_\Omega |f(x)|^p d\mu(x) < \infty \right\}$$

$$\|f\|_p^* = \left(\int_\Omega \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}}$$

$$\mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) := \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} |f(x)| < \infty \right\}$$

$$\|f\|_\infty^* = \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)|$$

Our proposition is that these are semi-norms.

Proof. Show that $\|\cdot\|_p^*$ for $p \in [1, \infty]$ are seminorms.

They cannot be norms since $\|f\|_p^* = 0$ for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0$. □

Proposition 2.13 (Hölder inequality). *Let $p \in [1, \infty]$ and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \text{if } p \in (1, \infty) \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

If $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $g \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$ then for both

$$f \cdot g : \Omega \rightarrow \mathbb{K} \text{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)}$$

$$f \otimes g : \Omega \rightarrow \mathbb{K}^M \text{ with } x \mapsto (f_i(x), \varphi_i(x))_{i=1}^M$$

we have that $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$ and $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$ and $\|f \otimes g\|_1^* \leq \|fg\|_1^* \leq \|f\|_p^* \cdot \|g\|_q^*$.

Proof. **Case $p \in (1, \infty)$:** Intermediate result: $\forall \sigma, \tau \geq 0, r \in (0, 1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$ [AGM-inequality].

Proof.

Case $\sigma = 0$ or $\tau = 0$: immediate

Case $\sigma, \tau > 0$:

$$\log(\sigma^r \tau^{1-r}) = r \log(\sigma) + (1-r) \log(\tau) \leq \log(r\sigma + (1-r)\tau)$$

since $\log''(x) \leq 0$ implies that \log is concave

$$\log \text{ is monotonic} \implies \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$$

□

Let $A := \left(\|f\|_p^*\right)^p$ and $B := \left(\|g\|_q^*\right)^q$ with $r = \frac{1}{p} \in (0, 1]$ we get

$$\forall x \in \Omega : \left(\frac{|f(x)|^p}{A}\right)^{\frac{1}{p}} \left(\frac{|g(x)|^q}{B}\right)^{\frac{1}{q}} = \frac{1}{p} \frac{|f(x)|^p}{A} + \frac{1}{q} \frac{|g(x)|^q}{B}$$

$$\implies \frac{\int_{\Omega} |f(x)| |g(x)| d\mu(x)}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \leq \frac{1}{p} \frac{\int_{\Omega} |f(x)|^p d\mu(x)}{A} + \frac{1}{q} \frac{\int_{\Omega} |g(x)|^q d\mu(x)}{B}$$

$$\implies \int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_p^* \|g\|_q^* = \frac{1}{p} + \frac{1}{q} = 1$$

Now: $\|f \cdot g\|_x^* \leq \|f\|_p^* \cdot \|g\|_q^*$ follows since $|\langle x, y \rangle| \leq |x| |y| \forall x, y \in \mathbb{K}^M$.

Also:

$$\begin{aligned} \forall x \in \Omega : |f \otimes g(x)| &= \sum_{i=1}^M |f_i(x)| |g_i(x)| = \begin{pmatrix} |f_1(x)| & |g_1(x)| \\ \vdots & \vdots \\ |f_n(x)| & |g_n(x)| \end{pmatrix} \leq |f(x)| |g(x)| \\ \implies \int_{\Omega} |f \otimes g(x)| d\mu(x) &\leq \|f\|_p^* \cdot \|g\|_q^* \end{aligned}$$

Case $p \in \{1, \infty\}$: Without loss of generality assume that $p = 1, q = \infty$. $\forall N \in \Sigma$ with $\mu(N) = 0$ we get

$$\begin{aligned} \int_{\Omega} |f(x)| |g(x)| d\mu(x) &= \int_{\Omega \setminus N} |f(x)| |g(x)| \mu(x) \\ &\leq \int_{\Omega \setminus N} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| = \int_{\Omega} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| \end{aligned}$$

Taking the infimum over all such N , then

$$\int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_1^* \cdot \|g\|_{\infty}^*$$

And the result follows again from $|\langle x, y \rangle| \leq |x| \cdot |y|$ and componentwise $|\langle x_i, y_i \rangle_i| \leq |x| |y| \forall x, y \in \mathbb{K}^M$

□

Proposition 2.14 (Minkowski inequality). *For $p \in [1, \infty]$, $f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$, we have that $\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*$ with $\|f\|_{\infty} := \inf_{\mu(N) \rightarrow 0} \sup_{x \in \Omega \setminus N} |f(x)|$.*

Proof. **Case $p = 1$:** trivial

Case $p \in (1, \infty)$:

$$\begin{aligned} (\|f + g\|_p^*)^p &= \int_{\Omega} |f(x) + g(x)|^p d\mu(x) \\ &= \int_{\Omega} |f(x)| \cdot |f(x) + g(x)|^{p-1} d\mu(x) \\ &\quad + \int_{\Omega} |g(x)| |f(x) + g(x)|^{p-1} d\mu(x) \\ &\leq \|f\|_p^* \cdot \| |f + g|^{p-1} \|_q^* + \|g\|_p^* \cdot \| |f + g|^{p-1} \|_q^* \end{aligned}$$

Recognize that $\left(\int |f+g|^p\right)^{\frac{1}{p}} = \left(\int |f+g|^{(p-1)q}\right)^{\frac{1}{q}}$ because $p = q \cdot (p-1)$

$$\begin{aligned} &= \left(\|f\|_p^* + \|g\|_p^*\right) \|f+g\|_p^* \\ \Rightarrow \|f+g\|_p^* &\leq \|f\|_p^* + \|g\|_p^* \end{aligned}$$

↓ This lecture took place on 2019/04/11.

Case $p = \infty$: First, note that $\forall f \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) \exists N \in \Sigma$ such that $\mu(N) = 0$ and $\|f\|_\infty^* = \|f|_{\Omega \setminus N}\|_\infty := \sup_{x \in \Omega \setminus N} |f(x)|$.

Claim.

$$\|f\|_\infty^* = \|f|_{\Omega \setminus N}\|_\infty := \sup_{x \in \Omega \setminus N} |f(x)| = \sup_{x \in \Omega \setminus \hat{N}} |f(x)| \text{ for } \mu(\hat{N}) = 0$$

Proof. For all $n \in \mathbb{N}$, define $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $\|f|_{\Omega \setminus N_n}\|_\infty \leq \|f\|_\infty^* + \frac{1}{n}$. Thus with $N := \bigcup_{n \in \mathbb{N}} N_n \Rightarrow \mu(N) = 0$ and $\|f\|_\infty^* \leq \|f|_{\Omega \setminus N}\|_\infty \leq \|f\|_\infty^* + \frac{1}{n}$. $n \rightarrow \infty \Rightarrow \|f\|_\infty^* = \|f|_{\Omega \setminus N}\|_\infty$. \square

For $f, g \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu)$, pick N_f, N_g such that $\mu(N_f) = \mu(N_g) = 0$ and $\|f\|_\infty^* = \|f|_{\Omega \setminus N_f}\|_\infty$ and $\|g\|_\infty^* = \|g|_{\Omega \setminus N_g}\|_\infty$.

$$\begin{aligned} \Rightarrow \|f+g\|_\infty^* &\leq \|(f+g)|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus (N_f \cup N_g)}\|_\infty + \|g|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus N_f}\|_\infty + \|g|_{\Omega \setminus N_g}\|_\infty = \|f\|_\infty^* + \|g\|_\infty^* \end{aligned}$$

\square

Proposition 2.15. Let $p \in [1, \infty]$. Then $\|\cdot\|_p^*$ is a seminorm on $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ is complete with the seminorm. With $M := \{f \in \mathcal{L}^p \mid \|f\|_p^* = 0\}$, we get that $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)/M$ is a Banach space with respect to $\|[f]\|_p := \|f\|_p^*$.

Proof. Seminorm is clear by Minkowski's inequality. Give completeness of $f^p(\cdot)$, the rest follows from Corollary 2.5.

Hence, show that $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ is complete.

Assume $p < \infty$. By Proposition 2.2, it suffices to show that for $f_n(t_n)_n$ in $\mathcal{L}^p(\cdot)$ such that $a := \sum_{n=1}^{\infty} \|f_n\|_p^* < \infty$.

$$\implies \exists f \in \mathcal{L}^p(\cdot) : f = \sum_{n=1}^{\infty} f_n$$

Define $\hat{q}(x) := \sum_{n=1}^{\infty} |f_n(x)| \in [0, \infty]$. Define $\hat{q}_n(x) := \sum_{i=1}^n |f_i(x)|$. Then q_n is measurable and by Minkowski's inequality,

$$\|q_n\|_p^* \leq \sum_{i=1}^n \|f_i\|_p^* \leq \sum_{i=1}^{\infty} \|f_i\|_p^* = a < \infty$$

Also $\hat{q}_n^p : x \rightarrow \hat{q}_n(x)^p$ is a sequence of positive functions and it is monotonically increasing and converging to \hat{g}^p .

By Beppo-Levi (from measure theory):

$$\int_{\Omega} \hat{g}^p = \lim_{n \rightarrow \infty} \int_{\Omega} \hat{q}_n^p = \lim_{n \rightarrow \infty} (\|q_n\|_p^*)^p = a^p < \infty$$

$\implies \hat{g}^p < \infty$ almost everywhere (except for a μ zero-set). Define $g : \Omega \rightarrow \mathbb{R}$,

$$x \mapsto \begin{cases} \hat{g}(x) & \text{if } \hat{g}(x) < \infty \\ 0 & \text{else} \end{cases}$$

We get that $g \in \mathcal{L}^n(\Omega, \mathbb{R}, \mu)$ and $g(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n |f_i(x)|$ μ -almost everywhere. Furthermore, by completeness of \mathbb{K}^M , $f(x) := \sum_{i=1}^{\infty} f_i(x)$ exists for μ -almost everywhere. $x \in \Omega$.

Show: $f = \sum_{i=1}^{\infty} f_i$ in $\mathcal{L}^n(\cdot)$, i.e. show that $\lim_{n \rightarrow \infty} \int_{\Omega} |\sum_{i=1}^{\infty} f_i|_{d_N}^p = \sigma$.

$$\left\| \sum_{i=1}^{n-1} f_i - \sum_{i=1}^{\infty} f_i \right\|_p^* = \left\| \sum_{i=n}^{\infty} f_i \right\|_p^* \xrightarrow{!} 0$$

By contruction, $|f| \leq q$ almost everywhere $\implies \int_{\Omega} |f|^p \leq \int_{\Omega} q^p < \infty$. Set $h_n(x) = |\sum_{i=n}^{\infty} f_i(x)|^p$. Then $h_n(x) \rightarrow 0$ for μ -almost everywhere $x \in \Omega$ and $h_n(x) \geq 0$ and

$$0 \leq h_n(x) \leq \left(\sum_{i=n}^{\infty} |f_i(x)| \right)^p \leq q(x)^p$$

Hence, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\Omega} h_n(x) = \int_{\Omega} \lim_{n \rightarrow \infty} h_n(x) = 0$$

This completes the assertion since

$$\int_{\Omega} h_n(x) = \int_{\Omega} \left| \sum_{i=n}^{\infty} f_i(x) \right|^p = \int_{\Omega} \left| \sum_{i=1}^{n-1} f_i(x) - f(x) \right|^p = \left(\left\| \sum_{i=1}^{n-1} f_i - f \right\|_p^* \right)^p$$

□

↓ This lecture took place on 2019/04/30.

Proposition (Proposition 2.15 again). *Let $p \in [1, \infty]$. Then $\|\cdot\|_{L^p}$ is a seminorm, $\mathcal{L}^p(\Omega, \mathbb{K}^n, \mu)$ is complete and $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\cdot)/N$ where $N = \{f \mid \|f\|_{L^p} = 0\}$ is a Banach space.*

Proof. Assume $p \in [1, \infty]$, then the proof of the last lecture is given.

Assume $p = \infty$. Let $(f_n)_n$ be Cauchy in \mathcal{L}^∞ . Remember: $\|f\|_{L^\infty} := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |f(x)|$. Pick $N_{n,m}$ such that $\mu(N_{n,m}) = 0$ and $\|f_n - f_m\|_\infty = \|(f_n - f_m)|_{\Omega \setminus N_{n,m}}\|_\infty$. Set $N = \bigcup_{n,m} N_{n,m} \Rightarrow \mu(N) = 0$.

Then \tilde{f} is the uniform limit of $f_n \cdot \mathbf{1}_{\Omega \setminus N}$. Hence \tilde{f} is measurable. Also $\|\tilde{f}\|_{L^\infty} := \inf_{\mu(M)=0} \sup_{x \in \Omega \setminus M} |f(x)| \leq \|f\|_\infty \Rightarrow \tilde{f} \in L^\infty(\Omega, \mathbb{K}^n, \mu)$. Also $\|f_n - \tilde{f}\|_{L^\infty} = \|(f_n - f)|_{\Omega \setminus N}\|_\infty = \|f_n|_{\Omega \setminus N} - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. \square

Now $(f_n|_N)_n$ is Cauchy with respect to $\|\cdot\|_\infty$. Since $\forall n, m$:

$$\begin{aligned} \|f_n|_{N^c} - f_m|_{N^c}\|_\infty &= \|(f_n - f_m)|_{N^c}\|_\infty \\ &\leq \|(f_n - f_m)|_{N_{m,n}^c}\|_\infty \\ &= \|f_n - f_m\|_{L^\infty} \end{aligned}$$

As in the proof of C_b being a Banach space:

$$\Rightarrow \exists f : \Omega \setminus N \rightarrow \mathbb{K}^M : \|f\|_\infty < \infty \text{ and } f_n|_{N^c} \rightarrow f \text{ w.r.t. } \|\cdot\|_\infty$$

Remark (Important special cases). **Case 1** $\mu = \mathcal{L}^N$ is the Lebesgue measure on $\Omega \subset \mathbb{R}^N$ (a domain). In this case we write $L^p(\Omega, \mathbb{K}^M) := L^p(\Omega, \mathbb{K}^M, \lambda^M)$ and $L^p(\Omega) := L^p(\Omega, \mathbb{K})$. Here the space $L^p(\Omega, \mathbb{K})$ is considered as functions which are defined almost everywhere.

Case 2 Set $\Omega = \mathbb{N}, \sigma = \mathbb{P}(\mathbb{N}), \mu_c(A) = |A|$.

Then

- $f : \Omega \rightarrow \mathbb{K}^m$ is identified with a sequence $(x_n)_n$ with $x_n \in \mathbb{K}^M$.
- $\int_\Omega f(x) d\mu(x) \sim \sum_{i \in \mathbb{N}} x_i \in \mathbb{K}^M$
- $\mu_c(A) = 0 \iff A = \emptyset$ and the equivalence class construction becomes obsolete.

And we denote,

$$\ell^p(\mathbb{N}, \mathbb{K}^M) = \mathcal{L}^p(\mathbb{N}, \mathbb{K}^M, \mu_c) \quad \ell^p := \ell^p(\mathbb{N}) = \ell^p(\mathbb{N}, \mathbb{K})$$

2.2.1 Basic properties of Lebesgue spaces

Proposition 2.16. *The space $l^p(\mathbb{N}, \mathbb{K}^M)$ is separable for $p \in [1, \infty]$ and not separable for $p = \infty$.*

Proof. $p < \infty$ Define $l_{i,j} \in l^p(\mathbb{N}, \mathbb{K}^M)$ as

$$(l_{ij})_k := \begin{cases} 0 & \text{if } i \neq k \\ \left(0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0\right)^T & \text{if } i = k \end{cases}$$

Then $A := \{e_{ij} \mid i \in \mathbb{N}, j \in \{1, \dots, M\}\}$ is countable.

It suffices to show that $\overline{\text{span}(A)} = l^p(\mathbb{N}, \mathbb{K}^M)$.

This is true since $\forall x \in l^p(\mathbb{N}, \mathbb{K}^M) : \forall \varepsilon > 0 \exists n_0 : \sum_{i=n_0+1}^{\infty} |x_i|^p < \varepsilon$ and hence

$$\left\|x - \sum_{i=1}^{n_0} \sum_{j=1}^M x_{ij} e_{ij}\right\|^p = \left(\sum_{i=n_0+1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$$

$p = \infty$ It suffices to show that $L^\infty(\mathbb{N})$ is not separable (why?). For $M \subset \mathbb{N}$ define $\mathbf{1}_M \in L^\infty$. Then $\Delta := \{\mathbf{1}_M \mid M \subset \mathbb{N}\}$ is uncountable.

For $A \subset L^\infty$ countable and $x \in A$ set $M_x = \{y \in L^\infty \mid \|x - y\|_\infty < \frac{1}{3}\} = B_{\frac{1}{3}}(x)$. Then each M_x contains at most one element of Δ since if $\mathbf{1}_M \neq \mathbf{1}_{M'}$ are such that $\mathbf{1}_M, \mathbf{1}_{M'} \in M_x$.

$$\Rightarrow 1 = \|\mathbf{1}_M - \mathbf{1}_{M'}\|_\infty \leq \|\mathbf{1}_M - x\| + \|\mathbf{1}_{M'} - x\| < \frac{2}{3}$$

This gives a contradiction.

Δ is uncountable, $\{M_x \mid x \in A\}$ is countable.

$$\Rightarrow \exists \hat{M} \in \mathbb{N} : \mathbf{1}_{\hat{M}} \notin M_x \forall x \in A$$

$$\Rightarrow \|\mathbf{1}_{\hat{M}} - x\|_\infty \geq \frac{1}{3} \forall x \in A$$

Hence, A is not dense. Since A was arbitrary countable. Thus L^∞ is not separable.

□

2.2.2 Separability of L^p requires a density result

Proposition 2.17. *Let $f \in L^p(\mathbb{R}^N, \mathbb{K}^M)$. Let $p < \infty$. Then $\exists (f_n)_n \in \dots C_c(\mathbb{R}^N, \mathbb{R}^M)$ such that $\|f_n - f\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. **Step 1** Reduction to step functions with $E \in \Sigma$.

$$\xi_E(x) := \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$$

Take $f \in L^p(\dots)$. For $\varepsilon > 0$, define

$$E_\varepsilon = \lceil x : \varepsilon \leq |f| \leq \frac{1}{\varepsilon} \rceil$$

Then $E_\varepsilon \in \Sigma$ and $\int_{\mathbb{R}^N} |f|^p \geq \varepsilon^p |E_\varepsilon|$ where $|E_\varepsilon| := L^N(E_\varepsilon)$.

$$|E_\varepsilon| < \infty \text{ and } \int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f| \leq \frac{1}{\varepsilon} \cdot |E_\varepsilon| < \infty$$

$$\implies \mathbf{1}_{E_\varepsilon} f \text{ is integrable } \implies \exists (q_{n,\varepsilon})_n \text{ step functions}$$

such that $\int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f - q_{n,\varepsilon}| \rightarrow 0$ as $n \rightarrow \infty$. Define

$$f_{n,\varepsilon}(x) := \begin{cases} q_{n,\varepsilon}(x) & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| \leq \frac{2}{\varepsilon} \\ \frac{2}{\varepsilon} \frac{q_{n,\varepsilon}(x)}{|q_{n,\varepsilon}(x)|} & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon} \\ 0 & \text{else} \end{cases}$$

Hence $(f_{n,\varepsilon})_n$ is a sequence of step functions. For $x \in E_\varepsilon$ such that $|q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon}$.

$$\implies |f_{n,\varepsilon}(x) - f(x)| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} = \frac{3}{\varepsilon} \leq 3 \underbrace{(|q_{n,\varepsilon}(x)| - |f(x)|)}_{\geq \frac{1}{\varepsilon}} \leq 3 |q_{n,\varepsilon}(x) - f(x)|$$

$$\int_{\mathbb{R}^N} |f_{n,\varepsilon}(x) - X_{E_\varepsilon}(x) f(x)| dx \leq 3 \int_{\mathbb{R}^N} |g_{n,\varepsilon}(x) - \mathbf{1}_{E_\varepsilon}(x) f(x)| dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\int_{\mathbb{R}^N} |f - f_{n,\varepsilon}|^p \leq \int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p + \underbrace{\left(\frac{3}{\varepsilon}\right)^{p-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|}_{(*)} =: (X)$$

$$(*) = \int_{E_\varepsilon} |f - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \cdot \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| \left(\underbrace{|f \mathbf{1}_{E_\varepsilon}|}_{\leq \frac{1}{3}} + \underbrace{|f_{n,\varepsilon}|^{p-1}}_{\leq \frac{2}{3}} \right)$$

Now given $\delta > 0$, we first fix $\varepsilon > 0$ such that $\int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p < \frac{\delta}{2}$. Then we find n_0 such that $\left(\frac{3}{\varepsilon}\right)^{n-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| < \frac{\delta}{2}$. This is possible since $\mathbb{R}^N = \bigcup_{\varepsilon>0} E_\varepsilon$ and $\int_{\mathbb{R}^N} |f|^p < \infty$.

$$\implies (X) < \delta$$

Now suppose $\forall \varepsilon > 0 \forall E \in \Sigma : \exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K}^M)$ such that $\|\mathbf{1}_E - \varphi\| < \varepsilon$. We need to show that this is true. Then for $f \in L^p(\mathbb{R}^N, \mathbb{K})$, $\varepsilon > 0$, we pick

$$g = \sum_{i=1}^n \underbrace{c_i}_{\in \mathbb{K}^M} \cdot \underbrace{\mathbf{1}_{E_i}}_{\in \Sigma}$$

such that $\|f - g\|_p < \frac{\varepsilon}{2}$ (possible by what we just showed). For $i \in \mathbb{N}$, pick $\varphi_i \in C_c(\mathbb{R}^N, \mathbb{R})$ such that $\|\mathbf{1}_{E_i} - \varphi_i\|_p \leq \frac{2^{-i}\varepsilon}{|C_i|^2}$

$$\begin{aligned} \Rightarrow \left\| f - \underbrace{\sum_{i=1}^n c_i \cdot \varphi_i}_{\in C_c(\mathbb{R}^n, \mathbb{R}^n)} \right\|_p &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|c_i \mathbf{1}_{E_i} - c_i \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n |c_i| \cdot \|\mathbf{1}_{E_i} - \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n 2^{-i} \cdot \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

□

↓ This lecture took place on 2019/05/02.

Proof. Step 1 It is sufficient to approximate $f = \mathbf{1}_E$ for $E \in \Sigma$

Step 2 Reduce statement to $f = \mathbf{1}_Q$ where $Q = \times_{i=1}^N [a_i, b_i]$ with $a_i, b_i \in \mathbb{R}$. Take $f = \mathbf{1}_E$. Since Σ is generated by sets of the form $\times_{i=1}^N [a_i, b_i] \forall \varepsilon > 0$ there exists $(Q_i)_{i=1}^n, (\lambda_i)_{i=1}^n$ such that $\|f - \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}\|_1 < \varepsilon$ [Alt, A1 10, axiom L5].

Define $h_n(x) = \max(0, \min(1, q_n(x)))$ where $q_n := \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}$, also h_n is of the form of q_n and

$$\begin{aligned} |f(x) - h_n(x)| \leq 1 &\Rightarrow |f(x) - h_n(x)|^p \leq |f(x) - h_n(x)|^1 \leq |f(x) - q_n(x)| \\ &\Rightarrow \|f - h_n\|_p \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

As in step 1, this reduces the assertion to $f = \mathbf{1}_Q$ with $Q = \times_{i=1}^N [a_i, b_i]$. For such $f = \mathbf{1}_Q$, define

$$g_i(s) := \begin{cases} \frac{b_i - a_i}{2} + |s - \frac{b_i + a_i}{2}| & \text{if } s \in [a_i, b_i] \\ 0 & \text{else} \end{cases}$$

for $i \in \{1, \dots, N\}$ and $\tilde{g}_{i,\varepsilon}(x) = \prod_{i=1}^N g_{i,\varepsilon}(x_i)$, we obtain that $\|\mathbf{1}_Q - \hat{g}_\varepsilon\|_p \rightarrow 0$ as $\varepsilon \rightarrow 0$.

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbf{1}_Q - \hat{g}_\varepsilon|^p &= \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{i=1}^N |\mathbf{1}_{[a_i, b_i]}(x) - \tilde{g}_{i,\varepsilon}(x)|^p dx \\ &= \prod_{i=1}^N \int_{a_i}^{b_i} |\mathbf{1}_{[a_i, b_i]}(s) - \tilde{g}_{i,\varepsilon}(s)|^p ds \\ &\leq \prod_{i=1}^N |I_{i,\varepsilon}| \text{ where } |I_{i,\varepsilon}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

□

Remark. 1. If $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain, defining

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & \text{else} \end{cases}$$

we get that $\tilde{f} \in L^p(\mathbb{R}^N, \mathbb{K}^M)$ and using Proposition 2.17 for \tilde{f} we can approximate f by functions in $C(\bar{\Omega}, \mathbb{K}^M) \cap C_c(\mathbb{R}^N, \mathbb{K}^M)$.

2. Using “Mollification” Proposition 2.17 implies density of $\mathcal{D}(\Omega, \mathbb{K}^M)$ in $L^p(\Omega, \mathbb{K}^M)$ for $\Omega \subseteq \mathbb{R}^N$ a domain.

Proposition 2.18. Let $\Omega \subset \mathbb{R}^N$ measurable. Then $L^p(\Omega, \mathbb{K}^M)$ is separable for $1 \leq p < \infty$ and not separable for $p = \infty$.

Proof. **Case $p = \infty$** Similar to l^∞ , will be done in the Exercises.

Case $1 \leq p < \infty$ We show the result for $L^p(\mathbb{R}^N, \mathbb{K})$, the general case is a direct consequence. Denote $\mathcal{R} := \{Q \subseteq \mathbb{R}^N \mid Q = \prod_{i=1}^N [a_i, b_i] \text{ with } a_n, b_n \in \mathbb{Q}\}$. Then \mathcal{R} is countable and it suffices to show that $E := \mathcal{L}(\{\mathbf{1}_Q \mid Q \in \mathcal{R}\})$ is dense. Take $f \in L^p(\mathbb{R}^N, \mathbb{K})$, $\varepsilon > 0$. Then $\exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K})$ such that $\|f - \varphi\|_p \leq \frac{\varepsilon}{2}$. Now we need to find $h \in E$ such that $\|\varphi - h\|_p \leq \frac{\varepsilon}{2}$. Let $M \subseteq \mathbb{R}^N$ be closed, bounded hypercube such that $\text{supp}(\varphi) \subset M$. φ is uniformly continuous on M .

$$\implies \forall \delta > 0 \exists \rho > 0 \forall x, y \in M : |x - y| < \delta \implies |\varphi(x) - \varphi(y)| < \delta$$

Now we take $(Q_i)_{i=1}^K$ a disjoint covering of M with $Q_i \in \mathcal{R}$, such that $|x - y| < \delta \forall x, y \in Q_i$. Now define $\lambda_i = \varphi(z)$ for some $z \in Q_i$, $i = 1, \dots, K$. Define $h(x) := \sum_{i=1}^K \lambda_i \mathbf{1}_{Q_i}$.

$$\implies \forall x \in \mathbb{R}^M : |\varphi(x) - h(x)| \leq |\varphi(x) - \lambda_i| \leq \delta$$

$$\implies \|\varphi - h\|_p = \left(\int_{\mathbb{R}^N} |\varphi(x) - h(x)|^p \right)^{\frac{1}{p}} \leq \delta \cdot |M|^{\frac{1}{p}}$$

Choose $\delta := \frac{\varepsilon}{2 \cdot |M|^{\frac{1}{p}}}$, then the result follows.

□

↓ This lecture took place on 2019/05/09.

Proposition 2.19. Let $p \in [1, \infty]$, $(f_n)_n$, $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain such that $f_n \rightarrow f$ in L^p .

Then there exists a subsequence $(f_{n_k})_k$ such that

1. $f_{n_k}(x) \rightarrow f(x)$ for almost every $x \in \Omega$

2. $\exists h \in L^p(\Omega)$ such that $(f_{n_k}(x)) \leq |h(x)|$ for almost every $x \in \Omega$

Proof. **Case** $p = \infty$ Is left as an exercise to the reader.

Case $p \in [1, \infty)$ Pick $(n_k)_k$ such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq \frac{1}{2^k}$. Define $g_n := \sum_{k=1}^n |f_{n_{k+1}}(x) - f_{n_k}(x)|$.

Then $g_n(x)$ is increasing, $g_n(x) \geq 0 \forall n$.

$\implies g_n(x)$ is convergent for almost every $x \in \Omega$. Hence we can define $g(x) := \lim_{n \rightarrow \infty} g_n(x) \in [0, \infty]$.

Also, $\|g_n\|_p \leq \sum_{i=1}^n \|f_{n_{i+1}} - f_{n_i}\| \leq 1$. By Beppo-Levi,

$$\int_{\Omega} |g(x)|^p dx = \lim_{n \rightarrow \infty} \int_{\Omega} |g_n(x)|^p dx = \lim_{n \rightarrow \infty} \|g_n\|_p^p \leq 1 \implies g \in L^p(\Omega)$$

especially $g(x) < \infty$ for almost every $x \in \Omega$.

$$\forall l \geq k \geq 1 : |f_{n_l}(x) - f_{n_k}(x)| \leq \sum_{i=k}^{l-1} |f_{n_{i+1}}(x) - f_{n_i}(x)| \leq g_{l-1}(x) - g_{k-1}(x) \stackrel{\text{monot.}}{\leq} g(x) - g_{k-1}(x)$$

$\implies (f_{n_k}(x))_k$ is Cauchy for almost every $x \in \Omega$ such that we can define $\tilde{f}(x) := \lim_{k \rightarrow \infty} f_{n_k}(x)$.

$$|\tilde{f}(x) - f_{n_k}(x)| \leq g(x) \text{ for almost every } x \in \Omega$$

By the Dominated convergence theorem, $\|f_{n_k} - \tilde{f}\|_p \rightarrow 0$ for $k \rightarrow \infty$. $\implies f = \tilde{f}$ almost every and hence $f_{n_k}(x) \rightarrow f(x)$ for almost every $x \in \Omega \implies$ (1). Also

$$|f_{n_k}(x)| \leq |f_{n_k}(x) - f(x)| + |f(x)| \leq g(x) + |f(x)| =: h(x)$$

□

3 Linear Operators

Definition 3.1. Let X, Y be normed spaces and $D \subset X$ is a subspace. A linear operator with domain $\text{dom}(T) = D$ is a linear mapping $T : D \rightarrow Y$. We define: $\text{range}(T) = \text{rg}(T) := T(D)$. Graph of T , $\text{gr}(T) := \{(x, y) \mid x \in \text{dom}(T), y = Tx\} \subset X \times Y$.

We say that T is decently define, if $\overline{\text{dom}(T)} = X$.

Example. 1. $X = Y = C([0, 1], \mathbb{R})$ and $\text{dom}(T) := C^1([0, 1], \mathbb{R})$ $T : \text{dom}(T) \rightarrow Y$ with $u \mapsto u'$.

2. $X = Y = \mathbb{R}^N, T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $x \mapsto Ax$ with $A \in \mathbb{R}^{n \times n}$

3. Fixed $u \in L^p(\Omega)$ and $p \in [1, \infty)$.

$$q := \begin{cases} \frac{p}{p-1} & p \neq 1 \\ \infty & \text{else} \end{cases}$$

$$T : L^q(\Omega) \rightarrow \mathbb{R} \quad v \mapsto \int_{\Omega} u \cdot v$$

4. $X = L^2(\Omega), Y = \mathbb{R}, \text{dom}(T) = C(\overline{\Omega})$ with $x \in \Omega$ fixed, $T : \text{dom}(T) \rightarrow Y$ with $u \mapsto u(x_0)$

Definition 3.2. Let X, Y be normed spaces and $T : X \rightarrow Y$ a linear operator ($\text{dom}(T) = X$). We say that T is bounded $\iff \exists M > 0 \forall x \in X : \|Tx\|_Y \leq M \|x\|_X$. In this case, we define $\|T\| = \|T\|_{\mathcal{L}(X, Y)} := \inf \{M > 0 \mid \|Tx\| \leq M \|x\| \forall x\}$.

$$\mathcal{L}(X, Y) := \{T : X \rightarrow Y \mid T \text{ bounded, linear operator}\}$$

$$\mathcal{L}(X) := \mathcal{L}(X, X)$$

Proposition 3.3. Let X, Y be normed spaces, $T : X \rightarrow Y$ be linear. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. $\exists M > 0$ such that $\|Tx\| \leq M \|x\| \forall x \in X$ (T bounded)
4. T is uniformly continuous

Also:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \leq 1} \|T(x)\| \quad \text{and} \quad \|Tx\| \leq \|T\| \|x\| \forall x \in X$$

Proof. (3) \rightarrow (4) Is true since $\forall x, y \in X : \|Tx - Ty\| = \|T(x - y)\| \leq M \|x - y\|$

(4) \rightarrow (1) \rightarrow (2) trivial

(2) \rightarrow (3) Assume (3) is not true, then

$$\exists (x_n)_n \text{ in } X : \forall n \in \mathbb{N} : \|Tx_n\| > n \|x_n\|$$

Define $y_n := \frac{x_n}{\|x_n\|n} \implies \|y_n\| = \frac{1}{n} \implies y_n \rightarrow 0$ but $\|Ty_n\| = \frac{\|Tx_n\|}{\|x_n\|n} > 1$. This gives a contradiction to continuity at 0 since $T0 = 0$.

□

Additionally,

$$M := \sup_{x \neq 0} \frac{\|Tx\| \|x\|}{\|x\|^2} = \sup_{x \neq 0} \left\| T \left(\frac{x}{\|x\|} \right) \right\| \leq \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\| \leq 1} \|Tx\|$$

But also,

$$\sup_{\|x\| \leq 1} \|Tx\| = \sup_{\lambda \in [0,1]} \sup_{\|x\|=1} \|T(\lambda x)\| = \sup_{\lambda \in [0,1]} \lambda \left(\sup_{\|x\|=1} \|Tx\| \right) = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \left\| \frac{Tx}{\|x\|} \right\|$$

We also get that

$$M_0 \geq \frac{\|Tx\|}{\|x\|} \forall x \in X, x \neq 0$$

$$\Rightarrow \|Tx\| \leq M_0 \|x\| \forall x \in X : x \neq 0 \text{ and also for } x = 0 \Rightarrow \|T\| \leq M_0$$

$$M_0(1 - \varepsilon) \leq \frac{\|Tx_\varepsilon\|}{\|x_\varepsilon\|}$$

For $\varepsilon > 0$ pick $x_\varepsilon \neq 0$ such that

$$\|Tx_\varepsilon\| \geq M_0(1 - \varepsilon) \|x_\varepsilon\|$$

$$\Rightarrow \|T\| \geq M_0(1 - \varepsilon)$$

since $\varepsilon > 0$ was arbitrary $\Rightarrow \|T\| \geq M_0$.

↓ This lecture took place on 2019/05/10.

Proposition 3.4. Let X and Y be normed spaces. Then

1. $\mathcal{L}(X, Y)$ is a vectorspace with

$$(T + S)(x) := T(x) + S(x) \quad (\lambda T)(x) := \lambda T(x) \quad 0(x) := 0$$

2. $T \mapsto \|T\|$ is a norm on $\mathcal{L}(X, Y)$ (the operator norm)

3. If Y is complete, then $\mathcal{L}(X, Y)$ is complete. In particular, $\mathcal{L}(X, \mathbb{K})$ is complete for any X and is also called the space of bounded linear functionals

Proof. 1. Left as an exercise to the reader

2. (N1) $\|0\| = \sup_{\|x\| \leq 1} \|0(x)\| = 0$.

$$\text{Also } \|T\| = 0 \Rightarrow \|Tx\| \leq 0 \|x\| = 0 \forall x \Rightarrow T = 0$$

(N2)

$$\|\lambda T\| = \sup_{\|x\| \leq 1} \|\lambda T(x)\| = \sup_{\|x\| \leq 1} \underbrace{|\lambda|}_{\geq 0} \|Tx\| = |\lambda| \cdot \|T\|$$

(N3)

$$\begin{aligned} \forall x : \|(T + S)(x)\| &= \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|) \|x\| \\ \Rightarrow \|T + S\| &\leq \|T\| + \|S\| \end{aligned}$$

3. Let $(T_n)_n$ be Cauchy in $\mathcal{L}(X, Y)$ and Y a Banach space. Since $\|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \|x\| \Rightarrow (T_n x)_n$ is Cauchy in $Y \forall x \in X \Rightarrow Tx := \lim_{n \rightarrow \infty} T_n x$ is well defined.

Furthermore, we want to show

Linearity:

$$\forall x, y \in X, \lambda \in \mathbb{K} : T(\lambda x + y) = \lim_{n \rightarrow \infty} T_n(\lambda x + y) = \lim_{n \rightarrow \infty} \lambda T_n x + \lim_{n \rightarrow \infty} T_n y = \lambda Tx + Ty$$

$\|\mathbf{T}_n - \mathbf{T}\| \rightarrow 0$: Take $\varepsilon > 0$, $n_0 \in \mathbb{N} : \|T_n - T_m\| \leq \varepsilon \forall n, m \geq n_0$

Show: $\exists n_1 \forall n \geq n_1 : \|T_n - T\| \leq 2\varepsilon$. For $x \in X : \|x\| \leq 1$ fix $m_x \geq n_0$:
 $\|T_{m_x}x - Tx\| \leq \varepsilon \implies \forall n \geq n_1 =: n_0 :$

$$\begin{aligned} \|T_nx - Tx\| &\leq \|T_nx - T_{m_x}x\| + \|T_{m_x}x - Tx\| \\ &\leq \|T_n - T_{m_x}\| + \varepsilon \leq 2\varepsilon \\ \implies \|T_n - T\| &= \sup_{\|x\| \leq 1} \|T_nx - Tx\| < 2\varepsilon \end{aligned}$$

$$\implies \forall x \in X : \|Tx\| \leq \|T_nx - Tx\| + \|T_nx\| \leq \|T_n - T\| + \|T_n\| \forall n \text{ fixed}$$

□

Proposition 3.5. Let X, Y be normed spaces. $D \subset X$ is a subspace such that $\overline{D} = X$, $T \in \mathcal{L}(D, Y)$.

$$\exists! \hat{T} \in \mathcal{L}(X, Y) : \hat{T}|_D = T$$

In addition: $\|\hat{T}\| = \|T\|$.

Proof. Unique extension is clear for T is uniformly continuous.

Also:

$$\|\hat{T}\| = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\|\hat{T}x\|}{\|x\|} \stackrel{\text{by density}}{=} \sup_{\substack{x \in D \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|$$

To show the density equality is left as an exercise to the reader. □

Proposition 3.6. Let X, Y, Z be normed spaces. $S \in \mathcal{L}(X, Y)$. $T \in \mathcal{L}(Y, Z)$. Then $T_0S \in \mathcal{L}(X, Z)$ and $\|T_0S\| \leq \|T\| \|S\|$.

Proof. T_0S is linear (show as an exercise).

$$\text{Take } x \in X. \quad \|T_0S(x)\| = \|T(Sx)\| = \|T\| \|Sx\| \leq \|T\| \|Sx\| \leq \|T\| \|S\| \|x\|. \implies \|T_0S\| \leq \|T\| \|S\| \quad \square$$

Remark. If $\dim(X) < \infty$, $T : X \rightarrow Y$ is linear, then $T \in \mathcal{L}(X, Y)$ (left as an exercise).

Proposition 3.7 (Neumann series). Let X be a normed space. $T \in \mathcal{L}(X)$. If $\sum_{n=0}^{\infty} T^n$ is convergent in $\mathcal{L}(X)$, then $(I - T)$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$.

Here: $T^n := T_0 \cdot T_0 \cdot T_0 \cdot \dots$ n times

In particular, if X is Banach and $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} =: a < 1$ then $\sum_{i=0}^{\infty} T^n$ is convergent. Also if $\|T\| < 1$, then $a < 1$ holds true. In case of $a < 1$, then $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$.

Proof. Let $S_m := \sum_{n=0}^m T^n$ and $S := \lim_{m \rightarrow \infty} S_m$. Then $(I - T)S_m = I - T^{m+1} = S_m(I - T)$ (compute!).

$$\|T^m\| = \left\| \sum_{n=0}^m T^n - \sum_{n=0}^{m-1} T^n \right\| = \|S_m - S_{m-1}\| \rightarrow 0$$

for $m \rightarrow \infty$ since $(S_m)_n$ is Cauchy. ($RS := R_0S$)

Now note that for fixed $R \in \mathcal{L}(X)$ the mappings

$$S \mapsto RS \quad S \mapsto SR$$

are continuous since $\|S_n R - SR\| \leq \|S_n S\| \|R\| \rightarrow 0$ for $S_n \rightarrow S$. Continuity implies that

$$\begin{aligned} I = \lim_{m \rightarrow \infty} I - T^{m+1} &= \begin{cases} \lim_{m \rightarrow \infty} (I - T)S_m = (I - T)S \\ \lim_{m \rightarrow \infty} S_m(I - T) = S(I - T) \end{cases} \\ &\Rightarrow (I - T)^{-1} = S \end{aligned}$$

Now if $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq a < 1 \forall \varepsilon > 0 \Rightarrow \exists n_0 \forall n \geq n_0 : \|T^n\| \leq (a + \varepsilon)^n$

$$\Rightarrow \sum_{n=0}^{\infty} \|T^n\| \leq c + \sum_{n=0}^{\infty} (a + \varepsilon)^n = \frac{1}{1 - (a + \varepsilon)} + c < \infty \text{ for } c > 0$$

X is Banach, so $\sum_{n=0}^{\infty} T^n$ is convergent and

$$\|(I - T)^{-1}\| = \left\| \sum_{n=0}^{\infty} T^n \right\| \leq \frac{1}{1 - (a + \varepsilon)}$$

Since ε was arbitrary, $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$.

If $\|T\| < 1$, then

$$\begin{aligned} \limsup_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} &\leq \limsup_{m \rightarrow \infty} (\|T\| \cdot \|T\| \dots \|T\|)^{\frac{1}{m}} \\ &\leq \limsup_{m \rightarrow \infty} (\|T\|^m)^{\frac{1}{m}} \\ &= \limsup_{m \rightarrow \infty} \|T\| \\ &= \|T\| \end{aligned}$$

□

Remark. $(I - T)^{-1}$ is linear (left as an exercise). However: $(I - T)^{-1} \notin \mathcal{L}(X)$ in general!

4 The Hahn-Banach Theorem and its consequences

Apparently the Hahn-Banach Theorem of this chapter is very central to Functional Analysis. This section deals with an extension of linear functionals and separation of sets.

First, consider $\mathbb{K} = \mathbb{R}$.

Definition 4.1. Let X be a vector space. $p : X \rightarrow \mathbb{R}$ is called sublinear iff

1. $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0, x \in X$

$$2. p(x+y) \leq p(x) + p(y) \forall x, y \in X$$

Example. $p(x) = \|x\|$, p linear and p is a seminorm.

Theorem 4.2 (Hahn-Banach Theorem, real version). *Let X be a vector space over \mathbb{R} , $U \subset X$, a subspace. $p : X \rightarrow \mathbb{R}$ be sublinear and $l : U \rightarrow \mathbb{R}$ is linear such that $l(x) \leq p(x) \forall x \in U$*

Then $\exists L : X \rightarrow \mathbb{R}$ is linear such that

$$L|_U = l \quad L(x) \leq p(x) \forall x \in X$$

Proof. This proof consists of two steps:

1. Method to extend l from U to $U + \text{span}(x_0)$, $x_0 \notin U$
2. Iterate this step and get maximal extension (Zorn)

Step 1 For $x_0 \in X \setminus U$, let $V = U + \text{span}(x_0) = \{u + \lambda x_0 \mid u \in U, \lambda \in \mathbb{R}\}$. Any $v \in V$ can be written uniquely as $v = u + \lambda x_0$ for $u \in U, \lambda \in \mathbb{R}$ (why? left as an exercise). Thus for any $r \in \mathbb{R}$, we can define $L_r : V \rightarrow \mathbb{R}$. $v = u + \lambda x_0 \mapsto l(u) + \lambda r$. L_r is linear (why? left as an exercise).

Also: $L_r(x) \leq p(x) \forall x \in V \iff l(u) + \lambda r \leq p(u + \lambda x_0) \forall \lambda, u$ (let this statement be (*)).

$\lambda = 0$ (*) holds true

$\lambda > 0$ (*)

$$\begin{aligned} &\iff r \leq p\left(\frac{u}{\lambda} + x_0\right) - l\left(\frac{u}{\lambda}\right) \forall u \in U \\ &\iff r \leq \inf_{u \in U} p(u + x_0) - l(u) \end{aligned}$$

$\lambda < 0$

$$\begin{aligned} &\iff -r \leq p\left(\frac{u}{-\lambda} - x_0\right) - l\left(\frac{u}{-\lambda}\right) \forall u \in U \iff r \geq -p(u - x_0) + l(u) \forall u \in U \\ &\iff r \geq \sup_{u \in U} l(u) - p(u - x_0) \end{aligned}$$

Thus, (*) holds for $r = \sup_{u \in U} l(u) - p(u - x_0)$ if $\sup_{u \in U} l(u) - p(u - x_0) \leq \inf_{u \in U} p(u + x_0) - l(u) \iff l(w) - p(w - x) \leq p(u + x_0) - l(u) \forall w, u \in U \iff l(w) + l(u) \leq p(u + x_0) + p(w - x_0)$.

But this holds since:

$$l(w) + l(u) = l(w+u) \leq p(w+u) = p(w-x_0+x_0+u) \leq p(w-x_0) + p(u+x_0)$$

Step 2

Revision (Zorn's Lemma). *Let (A, \leq) be a partially ordered set such that every chain (every subset R of $A \forall a, b \in R : a \leq b \vee b \leq a$) admits an upper bound (i.e. $\exists c \in A : b \leq c \forall b \in R$), then A has a maximal element, i.e. $\exists z \in A$ such that $\forall a \in A : z_0 \leq a \implies a = z_0$*

Let A be a set of (V, L_V) tuples where $V \subset X$ is a subspace with $U \subset V$ and $L_V : V \rightarrow \mathbb{R}$ such that $L_V \leq p$ on V and $L_V|_U = l$.

For (V_1, L_{V_1}) and $(V_2, L_{V_2}) \in A$, we say that $(V_1, L_{V_1}) \leq (V_2, L_{V_2})$ if $V_1 \subset V_2$ and $L_{V_2}|_{V_1} = L_{V_1}$. Now $A \neq \emptyset$ since $(U, l) \in A$. If $(V_i, L_{V_i})_{i \in I} := R$ is a chain, define $V := \bigcup_{i \in I} V_i$, $L_V(x) := L_{V_i}(x)$ if $x \in V_i$.

This is well-defined.

$\Rightarrow (V, L_V)$ is an upper bound for R .

□

↓ This lecture took place on 2019/05/14.

Proof of Theorem 4.2. Let $U \subset X$, $x_0 \notin U$, $V = U + \text{span}(x_0)$.

$$\Rightarrow \exists L_V : V \rightarrow \mathbb{R} : L_V|_U = l, L_V(v) = p(v) \forall v \in V$$

$$R = \{(V, L_V) \mid U \subset V, L_V|_U = l, L_V = p \text{ on } V\}$$

$$(V_1, L_{V_1}) \leq (V_2, L_{V_2}) : \iff V_1 \subset V_2, L_{V_2}|_{V_1} = L_{V_1}$$

Remark. Any chain has an upper bound.

Let $(V_i, L_{V_i})_{i \in I}$ be a chain in R . Then we define $V = \bigcup_{i \in I} V_i$. $L_V : V \rightarrow \mathbb{R}$ with $v \mapsto L_{V_i}(v)$ if $v \in V_i$. Thus we showed well-definedness.

Then (V, L_V) is an upper bound of $(V_i, L_{V_i})_{i \in I}$ since $V_i \subset V$, $L_V|_{V_i} = L_{V_i} \forall i \in I$. By Zorn, there exists (V_0, L_{V_0}) a maximal element of R . It is left to show that $V_0 = X$. If not: Take some $x_0 \in X \setminus V_0$, define $\tilde{V} := V_0 + \text{span}(x_0)$ and $L_{\tilde{V}}$ as an extension of L_{V_0} as in step 1.

$$\Rightarrow (V_0, L_{V_0}) \leq (\tilde{V}, L_{\tilde{V}})$$

This contradicts the maximality of (V_0, L_{V_0}) .

□

Remark. If U is not dense, then the extension is unique.

Next: Hahn-Banach Theorem for $\mathbb{K} = \mathbb{C}$.

Approach: Establish bijection between \mathbb{R} vector space and \mathbb{C} vector space.

Proposition 4.3. Let X be a \mathbb{C} vector space (vector space over the complex numbers).

1. If $l : X \rightarrow \mathbb{R}$ is \mathbb{R} -linear (i.e. $l(x+y) = l(x) + l(y)$ and $l(\lambda x) = \lambda l(x) \forall \lambda \in \mathbb{R}$). We set $\hat{l} : X \rightarrow \mathbb{C}$ with $x \mapsto l(x) - i \cdot l(ix)$. Then \hat{l} is \mathbb{C} -linear and $\Re(\hat{l}) = l$.
2. If $h : X \rightarrow \mathbb{C}$ is \mathbb{C} -linear and we let $l := \Re(h)$ and \hat{l} as in (1), then l is \mathbb{R} -linear and $\hat{l} = h \upharpoonright l \rightarrow \hat{l}$ is surjective

3. If $p : X \rightarrow \mathbb{R}$ is a seminorm and $l : X \rightarrow \mathbb{C}$ is \mathbb{C} -linear. Then

$$|l(x)| \leq p(x) \forall x \iff |\Re(l(x))| \leq p(x) \forall x$$

4. If X is normed, $l \in \mathcal{L}(X, \mathbb{C})$, then $\|l\| = \|\Re(l)\|$

Remark. This means that $l \mapsto [x \mapsto l(x) - il(ix)]$ is bijective and an isometry if X is normed.

Proof. 1. By construction \hat{l} is \mathbb{R} -linear and $\Re(\hat{l}) = l$ is obvious.

Show: $\tilde{l}(ix) = i\tilde{l}(x)$.

$$\begin{aligned} \tilde{l}(ix) &= l(ix) - il(iix) = l(ix) - il(-x) \\ &= i(l(x) - il(ix)) = i\tilde{l}(x) \end{aligned}$$

2. Define $l := \Re(h)$. Show: $\tilde{l} = h$.

Note: $\forall z \in \mathbb{C} : \Im(z) = -\Re(iz)$.

$$\begin{aligned} h(x) &= \Re(h(x)) + i \cdot \Im(h(x)) = \Re(h(x)) - i \cdot \Re(i \cdot h(x)) \\ &= \Re(h(x)) - i \cdot \Re(h(ix)) = l(x) - i \cdot l(ix) = \tilde{l}(x) \end{aligned}$$

Hence $l \mapsto \tilde{l}$ is bijective.

3. Since $|\Re(z)| \leq |z|$,

\implies holds trivially

$$\begin{aligned} \iff \text{Write } l(x) &= \lambda_X |l(x)| \text{ with } |\lambda_X| = 1. \text{ Then } \forall x \in X : |l(x)| = \lambda_X^{-1} l(x) = \\ l(\lambda_X^{-1} x) &= |\Re(l(\lambda_X^{-1} x))| \leq p(\lambda_X^{-1} x) = |\lambda_X^{-1}| p(x) = p(x) \end{aligned}$$

4. Consequence of (3) with $p(x) := \|l\| \|x\|$

□

Theorem 4.4 (Hahn-Banach Theorem, complex version). *Let X be a \mathbb{C} vector space. $U \subset X$. $p : X \rightarrow \mathbb{R}$ sublinear and $l : U \rightarrow \mathbb{C}$ be linear such that $\Re l(u) < p(u) \forall u \in U$.*

$$\exists L : X \rightarrow \mathbb{C} \text{ linear such that } L|_U = l, \Re L(x) \leq p(x) \forall x$$

Proof. Applying Theorem 4.2 to $\Re l \implies \exists F : X \rightarrow \mathbb{R}$ r -linear such that $F|_U = \Re(l)$ and $F(x) \leq p(x) \forall x \in X$

Proposition 4.3 implies there exists some $L : X \rightarrow \mathbb{C}$ that is \mathbb{C} -linear such that $F = \Re(L)$. Now $\Re(L)|_U = F|_U = \Re(l) \implies L = l$ by Proposition 4.3 (2) and also $\Re L(x) = F(x) = p(x) \forall x \in X$. □

Proposition 4.5 (Consequence). *If X is a normed space, $U \subset X$ be a subspace, $u' \in \mathcal{L}(U, \mathbb{K})$, then $\exists x' \in \mathcal{L}(X, \mathbb{K})$ such that $x'|_U = u'$ with $\|x'\| = \|u'\|$.*

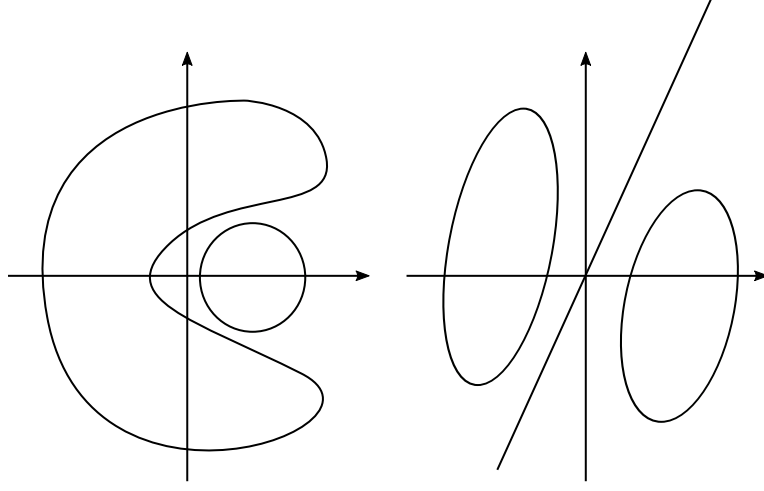


Figure 2: Inseparable convex sets (left) and separable convex sets (right)

Proof. **Case $\mathbb{K} = \mathbb{R}$:** Let $p(x) := \|u'\| \|x\|$. Then p is sublinear and $u'(x) \leq |u'(x)| \leq p(x) \forall x \in U$. By Theorem 4.2, there exists $x' : X \rightarrow \mathbb{R}$ linear such that $x'|_U = u'$ and $x'(x) \leq p(x) \forall x \in X$.

$$\implies -x'(x) = x'(-x) \leq p(-x) = p(x) \implies |x'(x)| \leq p(x) = \|u'\| \|x\| \implies \|x'\| \leq \|u'\|$$

Also:

$$\|u'\| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} |u'(u)| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} |x'(u)| \leq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|x'(x)\| = \|x'\| \implies \|x'\| = \|u'\|$$

Case $\mathbb{K} = \mathbb{C}$: As before, $\exists x' : X \rightarrow \mathbb{C} : x'|_U = u'$ and $\|\Re x'\| \leq \|u'\|$. By proposition 4.3, $\|x'\| = \|\Re(x')\|$

□

Remark (Next application: Separation of convex sets). Motivation: *Given two (convex) sets $A, B \subset \mathbb{R}^2$. When can we find a line L separating these sets*

Compare with Figure 2.

Remark. In \mathbb{R}^2 , any line L can be separable as $L = \{x \in \mathbb{R}^2 : (x, n) = \alpha \mid \alpha \in \mathbb{R}, n \in \mathbb{R}^2, \|n\| = 1\}$.

Definition 4.6. Let X be a vector space $H \subset X$ is called a hyperplane if it is of the form $H = \{x \in X \mid \Re(f(x)) = \alpha\}$ with $\alpha \in \mathbb{R}$, $f : X \rightarrow \mathbb{K}$ linear.

Lemma 4.7. Let X be a normed space, $H \subset X$ be a hyperplane of the form $H = \{x \in X \mid \Re(f(x)) = \alpha\}$ with $\alpha \in \mathbb{R}$, $f : X \rightarrow \mathbb{K}$ linear.

Then H is closed iff $f \in \mathcal{L}(X, \mathbb{K})$.

Proof. Compare with the practicals. \square

Remark (Goal). Given X as a normed vector space. $A, B \subset X$ where does some closed hyperplane H exist represented by $f \in \mathcal{L}(X, \mathbb{K})$ and α separating A and B , e.g. $\Re(f(a)) \leq \alpha \leq \Re(f(b)) \forall a \in A, b \in B$.

To this aim associate a set $U \subset X$ to a sublinear functional $p : X \rightarrow \mathbb{R}$.

Definition 4.8. Let X be a vector space. $A \subset X$. The Minkovsky functional $p_A : X \rightarrow [0, \infty]$ is defined as $p_A(x) = \inf \left\{ \lambda > 0 \mid \frac{x}{\lambda} \in A \right\}$. A is called absorbing if $p_A(x) < \infty \forall x \in X$.

Theorem 4.9. Let X be a normed space. $U \subset X$ convex such that $0 \in \text{interior}(U) = \overset{\circ}{U}$. Then,

1. U is absorbing and $\forall \varepsilon > 0 : B_\varepsilon(0) \subseteq U \implies p_U(x) \leq \frac{1}{\varepsilon} \|x\|$ [no convexity needed]
2. p_U is sublinear
3. If U is open, then $U = p_U^{-1}([0, 1))$.

Proof. 1. Trivial

2. • $p_u(\lambda x) = \lambda p_u(x)$ for $\lambda > 0$. Compare with the practicals.
- Take $x, y \in X$. Show: $p_u(x + y) \leq p_u(x) + p_u(y)$.

Take $\varepsilon > 0$ and choose λ, μ :

$$\begin{aligned} \lambda &\leq p_u(x) + \varepsilon & \frac{x}{\lambda} &\in U \\ \mu &\leq p_u(y) + \varepsilon & \frac{y}{\mu} &\in U \end{aligned}$$

Since U is convex,

$$\begin{aligned} \frac{x+y}{\lambda+\mu} &= \frac{\lambda}{\lambda+\mu} \left(\frac{x}{\lambda} \right) + \frac{\mu}{\lambda+\mu} \left(\frac{y}{\mu} \right) \in U \\ \implies p_u(x+y) &\leq \lambda + \mu = p_u(x) + p_u(y) + 2\varepsilon \end{aligned}$$

ε can be arbitrary, thus the proof is complete.

3. Direction \supset . If $p_u(x) < 1 \implies \exists \lambda > 0 : \lambda < 1$ and $\frac{x}{\lambda} \in U$. Since $0 \in U$,

$$\begin{aligned} \implies x &= \lambda \left(\frac{x}{\lambda} \right) + (1-\lambda)0 \in U \\ \implies p_u^{-1}([0, 1)) &\subset U \end{aligned}$$

Direction \subset . If $p_u(x) \geq 1$, then $\frac{x}{\lambda} \notin U \forall \lambda < 1$

$$\implies x = \lim_{\substack{\lambda \rightarrow 1 \\ \lambda < 1}} \frac{x}{\lambda} \in U^c$$

\square

Lemma 4.10 (Fundamental lemma). *Let X be a normed vector space. $V \subset X$ be convex and open. $0 \notin V$.*

$$\implies \exists x' : X \rightarrow \mathbb{K} \text{ continuous}$$

linear such that $\Re x'(x) < 0 \forall x \in V$.

Proof. Define $A \mp B = \{a + b \mid a \in A, b \in B\}$.

Case $\mathbb{K} = \mathbb{R}$: Take $x_0 \in V \setminus \{0\}$, define $y_0 := -x_0$ and $U := V - \{x_0\}$.

$$\implies U \text{ is open, convex, } 0 \in U, y_0 \notin U$$

We consider $p_u : X \rightarrow \mathbb{R}$ which is sublinear, finite and $p_u(y_0) \geq 1$. On $Y := \text{span}(y_0)$ we define $y' : Y \rightarrow \mathbb{R}$ with $ty_0 \mapsto tp_u(y_0)$ and $t \in \mathbb{R}$.

$$\implies y'(y) \leq p_u(y) \forall y \in Y$$

since

$$y'(y) = y'(ty_0) = tp_u(y_0)$$

- $t \leq 0$: $\leq 0 \leq p_u(y)$
- $t > 0$: $= p_u(ty_0) = p_u(y)$

↓ *This lecture took place on 2019/05/16.*

Now by Hahn-Banach Theorem, $\exists x' : X \rightarrow \mathbb{R}$ linear such that $x'|_Y = y'$ and $x'(x) \leq p_u(x) \forall x \in X$. For $\varepsilon > 0 : B_\varepsilon(0) \subseteq U$:

$$\forall x \in X : |x'(x)| = \max \left\{ x'(x), \underbrace{-x'(x)}_{=x'(-x)} \right\} \leq \min(p_u(x), -p_u(-x)) \leq \frac{1}{2} \|x\|$$

$$\implies x' \in \mathcal{L}(X, \mathbb{R})$$

Also $x'(y_0) = y'(y_0) = p_u(y_0) \geq 1$.

$$\implies \forall v \in V \text{ we can write } v = u - y_0 \text{ with } u \in U$$

$$\implies x'(v) = x'(u) - x'(y_0) \leq p_u(u) - 1 < 0$$

Case $\mathbb{K} = \mathbb{C}$ Lemma 4.3. Left as an exercise.

□

Theorem 4.11 (Separation 1). *Let X be normed. Let $V_1, V_2 \subset X$ be convex and V_1 open. $V_1 \cap V_2 = \emptyset$*

$$\implies \exists x' \in \mathcal{L}(X, \mathbb{K}) \text{ s.t. } \Re(x'(u_1)) \leq \Re(x'(x_2)) \forall u_1 \in V_1, v_2 \in V_2$$

Proof. Define $V := V_1 - V_2$. Then V is convex (why?) and open since $V = \bigcup_{x \in V_2} V_1 - \{x\}$ since $V_1 \cap V_2 = \emptyset$. Thus $0 \in V$. By Lemma 4.10,

$$\begin{aligned} \exists x' \in \mathcal{L}(X, \mathbb{K}) : \Re x'(v_1 - v_2) < 0 \forall v_1 \in V_1, v_2 \in V_2 \\ \implies \Re x'(v_1) < \Re x'(v_2) \end{aligned}$$

□

Remark. V being open is sufficient.

Theorem 4.12 (Separation 2). *Let X be a normed spaces. $V \subset X$ is closed and convex.*

$$\begin{aligned} \hat{x} \notin V &\implies \exists x' \in \mathcal{L}(X, \mathbb{K}) \\ \Re(x'(\hat{x})) &< \inf_{v \in V} \Re(x'(v)) \end{aligned}$$

$$i.e. \exists \varepsilon > 0 : \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \varepsilon \leq \inf_{v \in V} \Re(x'(v))$$

Proof.

$$V \text{ closed} \iff \exists \varepsilon > 0 : \underline{B_\varepsilon(\hat{x})}_{V_1} \cap V = \emptyset$$

By Theorem 4.11, $\exists x' \in \mathcal{L}(X, \mathbb{K})$:

$$\Re(x'(\hat{x} + u)) < \Re(x'(v)) \forall v \in V, u \in X : \|u\| < \varepsilon$$

$$\Re(x'(\hat{x})) + \Re(x'(u)) < \Re(x'(v)) \forall v \in V, u \in X, \|u\| \leq \frac{\varepsilon}{2}$$

Taking the sum over u .

$$\Re(x'(\hat{x})) + \left\| \Re(x') \right\| \frac{\varepsilon}{2} \leq \Re(x'(v)) \forall v \in V$$

since

$$\begin{aligned} \left\| \Re(x') \right\| &= \sup_{\|\lambda\| \leq 1} \left| \Re(x'(x)) \right| \frac{\varepsilon}{2} = \sup_{\|x\| \leq \frac{\varepsilon}{2}} \left| \Re(x'(x)) \right| = \sup_{\|x\| \leq \frac{\varepsilon}{2}} \Re(x'(x)) \\ &\implies \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \|x'\| \frac{\varepsilon}{2} \leq \inf_{v \in V} \Re(x'(v)) \end{aligned}$$

□

5 Fundamental theorems for operators in Banach spaces

In this chapter we are going to discuss the Baire theorem.

Theorem 5.1 (Banach-Steinhaus, uniform boundedness principle). *Let X be a Banach space, Y normed. Let I be an index set. For all $i \in I$, let $T_i \in \mathcal{L}(X, Y)$.*

Then if $\forall x \in X : \sup_{i \in I} \|T_i x\| < \infty \implies \sup_{i \in I} \|T_i\| < \infty$

Proof. Define $E_n := \{x \in X \mid \sup_{i \in I} \|T_i x\| \leq n\}$ since all T_i are continuous

$$\implies E_n = \bigcap_{i \in I} \|T_i(\cdot)\|^{-1}([0, n])$$

since $x \mapsto \|T_i x\|$ is continuous \rightarrow closed.

$\implies E_n$ is closed as the intersection of closed sets

Also: $X = \bigcup_{n \in \mathbb{N}} E_n$

By Baire's theorem, $\exists E_n : \overset{\circ}{E}_{n_0} \neq \emptyset$.

$$\implies \exists \varepsilon > 0, y \in E_{n_0} \text{ fixed such that } \forall x \in X : \|x - y\| \leq \varepsilon \implies x \in E_{n_0}$$

Now take $x \in X : \|x + y\| \leq \varepsilon$.

$$\|x + y\| = \|x - (-y)\| = \|-x - y\| \implies -x \in E_n \implies x \in E_n$$

Also, $\forall x_1, x_2 \in X, \lambda \in [0, 1], x_1, x_2 \in E_{n_0}$.

$$\implies \lambda x_1 + (1 - \lambda)x_2 \in E_{n_0}$$

since $\forall i : \|T_i(\lambda x_1 + (1 - \lambda)x_2)\| \leq \lambda \|T_i x_1\| + (1 - \lambda) \|T_i x_2\| < n_0$.

$$\forall x \in X : \|x\| \leq \varepsilon \implies x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \in E_{n_0}$$

since $x + y \in E_{n_0}$ and $x - y \in E_{n_0}$.

$$\begin{aligned} &\implies \forall i \in I : \|T_i x\| \leq n_0 \forall \|x\| \leq \varepsilon \\ &\implies \|T_i\| = \frac{1}{\varepsilon} \sup_{\|x\| \leq 1} \|T_i(\varepsilon x)\| \leq \frac{1}{\varepsilon} n_0 \\ &\implies \sup_{i \in I} \|T_i\| \leq \frac{\varepsilon}{n_0} \end{aligned}$$

□

↓ This lecture took place on 2019/05/21.

$$\forall \sup_{i \in I} \|T_i x\| < \infty \implies \sup_{i \in I} \|T_i\| < \infty$$

Remark. With $d = \{(x_i)_i \mid x_i = 0 \text{ for all but finitely many } i, x_i \in \mathbb{R}\}$, $(d, \|\cdot\|_\infty)$ is a normed space. Define $T_n : d \rightarrow \mathbb{R}, (x_i)_i \mapsto n \cdot x_n$ for given n .

Then $\forall x \in d : \sup_{n \in \mathbb{N}} |T_n x| = \sup_{n \in \mathbb{N}} |n x_n| \leq |n_0 x_{n_0}|$ for some $n_0 \in \mathbb{N}$. However $\forall n : \|T_n\| > |T_n e_n| = n \implies \sup_{n \in \mathbb{N}} \|T_n\| = \infty$. Thus, X is Banach space is necessary for Theorem 5.1 (also $(d, \|\cdot\|_\infty)$ is not Banach).

Corollary 5.2. Let X be Banach, Y normed $\forall n \in \mathbb{N} : T_n \in \mathcal{L}(X, Y)$ and suppose that $\lim_{n \rightarrow \infty} T_n x = T_x$ exists and is finite for all $x \in X$. Then $T \in \mathcal{L}(X, Y)$.

Proof. Left as an exercise to the reader. \square

Now: Continuous invertibility of linear operators, or, if $T \in \mathcal{L}(X, Y)$ bijective such that $T^{-1} : Y \rightarrow X$ exists and is linear. When does $T^{-1} \in \mathcal{L}(Y, X)$ hold?

Definition 5.3 (f maps open sets to open sets). Let (X, τ_X) and (Y, τ_Y) be two spaces and $f : X \rightarrow Y$. f is called open if $f(A) \in \tau_Y \forall A \in \tau_X$.

Remark. f is continuous $\iff f^{-1}(A) \in \tau_X \forall A \in \tau_Y$. If f is open and invertible, then $(f)^{-1}(A) = f(A) \in \tau_Y \forall A \in \tau_X \implies f^{-1}$ is continuous.

Lemma 5.4. Let $T : X \rightarrow Y$ be linear. X, Y be normed. TFAE:

1. T is open
2. $\forall r > 0 \exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_r(0))$
3. $\exists \varepsilon > 0 : B_\varepsilon \subset T(B_1(0))$

Proof. $1 \rightarrow 2$ True since $0 \in T(B_r(0))$ and $T(B_r(0))$ is open.

$2 \rightarrow 1$ Let $O \subset X$ be open, $y \in T(O) \implies x \in O$. $Tx = y$. $\exists r > 0 : x + B_r(0) \subset O$.

$$\implies T(x) + T(B_r(0)) \subset T(O). \text{ By (2), } \exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_r(0))$$

$$\implies Tx + T_\varepsilon(0) \subset Tx + T(B_r(0)) \subset O$$

$$\implies B_\varepsilon(Tx) = B_\varepsilon(y) \in O \text{ since } y \text{ was arbitrary, thus } O \text{ is open.}$$

$2 \rightarrow 3$ Left as an exercise to the reader \square

Remark. If X, Y is normed and $T : X \rightarrow Y$ linear, then T is injective.

Proof. Take $y \in Y \setminus \{0\}$, $\varepsilon > 0$ such that $B_\varepsilon(0) \subset T(B_1(0))$.

$$\implies \frac{\varepsilon y}{2\|y\|} \in B_\varepsilon(0) \implies \exists x : Tx = \frac{\varepsilon y}{2\|y\|} \implies T\left(\frac{2x\|y\|}{\varepsilon}\right) = y$$

\square

Theorem 5.5 (Open mapping theorem). Let X and Y be Banach. $T \in \mathcal{L}(X, Y)$ injective $\implies T$ open.

Proof. Here B_r denotes $B_r(0)$.

Show $\exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_\varepsilon(0))$.

Part 1 Show $\exists \varepsilon > 0 : B_\varepsilon \subset T(B_1)$.

We have $Y = \bigcup_{n \in \mathbb{N}} T(B_n)$ since T is surjective. By Baire's category theorem, $\exists N \in \mathbb{N} : \overline{T(B_N)} \neq \emptyset$.

$$\implies \exists y_0 \in \overline{T(B_N)}, \varepsilon > 0 \forall z \in Y : \|z - y_0\| < \varepsilon \implies z \in \overline{T(B_N)}$$

As in the proof of Theorem 5.1, $B_\varepsilon \subset \overline{T(B_N)}$ is implied.

Part 2 Show: If $B_\varepsilon \subset \overline{T(B_1)} \implies B_\varepsilon \subset T(B_1)$.

Let $\|y\| < \varepsilon$. Show: $y \in T(B_1)$. Choose $\tilde{\varepsilon} > 0 : \|y\| < \tilde{\varepsilon} < \varepsilon$ define $\tilde{y} := \frac{\varepsilon}{\tilde{\varepsilon}}y$. $\|\tilde{y}\| < \varepsilon \implies \exists y_0 \in Y, x_0 \in X : y_0 = Tx_0$ and $\|\tilde{y} - y_0\| < \alpha \tilde{\varepsilon}$ where $0 < \alpha < 1$ is

$$\implies \frac{\tilde{y} - (y_0 + \alpha y_1)}{\alpha^2} \in B_\varepsilon \implies \exists y_2 \in Y, x_2 \in B_1 : Tx_2 = y_2$$

$$\text{and } \|\tilde{y} - (y_0 + \alpha y_1 + \alpha^2 y_2)\| < \alpha^3 \varepsilon \text{ by } \left\| \frac{\tilde{y} - (y_0 + \alpha y_1)}{\alpha^2} - y_2 \right\| < \alpha \varepsilon$$

We can construct a sequence (by induction) $(x_n)_n$ such that $\|x_n\| < 1 \forall n$ and $\|\tilde{y} - T(\sum_{i=0}^n \alpha^i x_i)\| < \alpha^{n+1} \varepsilon$. Since $\alpha < 1$, $\|\sum_{i=1}^n \alpha^i x_i\| \leq \sum_{i=0}^n \alpha^i < (1 - \alpha)^{-1} < \infty \implies \sum_{i=0}^\infty \alpha^i x_i$ is absolutely convergent. X is Banach space, thus $\exists \hat{x} := \lim_{n \rightarrow \infty} \sum_{i=0}^n \alpha^i x_i \in X$. $\|\tilde{y} - T(\sum_{i=0}^n \alpha^i x_i)\| < \alpha^{n+1} \varepsilon \implies T(\sum_{i=0}^n \alpha^i x_i) \rightarrow \tilde{y} \implies T\hat{x} = \tilde{y}$. With $x := \frac{\tilde{\varepsilon}}{\varepsilon} \hat{x} \implies Tx = \frac{\tilde{\varepsilon}}{\varepsilon} T\hat{x} = \frac{\tilde{\varepsilon}}{\varepsilon} \tilde{y} = y$. Also,

$$\|x\| = \frac{\tilde{\varepsilon}}{\varepsilon} \|\hat{x}\| \leq \frac{\tilde{\varepsilon}}{\varepsilon} \sum_{i=0}^\infty \alpha^i \|x_i\| \leq \frac{\tilde{\varepsilon}}{\varepsilon} \frac{1}{1 - \alpha} < 1$$

□

Corollary 5.6 (Consequence 1). *Let X, Y be Banach. $T \in \mathcal{L}(X, Y)$. T is bijective, then $T^{-1} \in \mathcal{L}(Y, X)$.*

Corollary 5.7 (Consequence 2). *Let X, Y be Banach. $T \in \mathcal{L}(X, Y)$. T injective. $\text{range}(T)$ closed $\iff T^{-1} : \text{range}(T) \rightarrow X$ is linear and bounded.*

Proof. \implies Immediate since $T \in \mathcal{L}(X, \text{range}(T))$ and $\text{range}(T)$ is Banach.

\Leftarrow Assume that $T^{-1} : \text{range}(T) \rightarrow X$ is continuous. Let $(x_n)_n$ in $\text{range}(T)$ be Cauchy. Then $(T^{-1}(x_n))_n$ is Cauchy. Let X be Banach, then $\exists y \in X : T^{-1}(x_n) \rightarrow y$. Let T be continuous, then $x_n = T(T^{-1}x_n) \rightarrow Ty \implies x \in \text{range}(T)$ ($x \in Y$ since Y is Banach).

□

Corollary 5.8. *Let X and Y be Banach. $T \in \mathcal{L}(X, Y)$. $\text{range}(T)$ closed. Define $\tilde{X} := X \setminus \text{kernel}(T)$. $\tilde{T} : \tilde{X} \rightarrow Y$ with $[x] \mapsto Tx$. Then \tilde{T} is well-defined. $\tilde{T} \in \mathcal{L}(\tilde{X}, Y)$ injective. $\text{range}(\tilde{T}) = \text{range}(T)$, $\|\tilde{T}\| = \|T\|$ and $\tilde{T}^{-1} : \text{range}(T) \rightarrow \tilde{X}$ is continuous and linear.*

Proof. The proof is left as an exercise to the reader.

□

Corollary 5.9. Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on X such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach and $\exists M > 0 : \|x\|_1 \leq M \|x\|_2 \ \forall x \in X$. Thus $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proof. Apply Corollary 5.6 to $\text{id} : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$ with $x \mapsto x$. □

A second class of consequences: When is $T : X \rightarrow Y$ linear already bounded?

Definition 5.10. Let X and Y be normed. $D \subset X$ be a subspace. $T : D \rightarrow Y$ be linear. T is called closed if $\forall (x_n)_n \in D : x_n \rightarrow x$ and $Tx_n \rightarrow y$ it follows that $x \in D, Tx = y$.

Remark. This is weaker than continuity: Let $T : X \rightarrow Y$ be linear. Continuity means that $x_n \rightarrow x \implies Tx_n \rightarrow y$ and $Tx = y$. Closed means that $x_n \rightarrow x$ and $Tx_n \rightarrow y \implies Tx = y$.

Remark. Differential operators are often closed but not continuous.

e.g. $D := C_1([0, 1]) \subset L^\infty([0, 1])$.

$$T : \begin{matrix} D \rightarrow L^\infty([0, 1]) \\ f \mapsto f' \end{matrix}$$

Then T is not closed but not continuous.

Not continuous: Define $f_n(x) = \frac{1}{n} \cos(2\pi nx)$. Then $\|f_n\|_\infty = \frac{1}{n} \rightarrow 0$, hence $f_n \rightarrow 0 \in D$, but

$$\|Tf_n\|_\infty = \|Tf'_n\|_\infty = \|f'_n\|_\infty \geq \left| f'_n \frac{1}{8n} \right| = \left| 2\pi \sin\left(\frac{\pi}{2}\right) \right| = 2\pi$$

and hence $Tf_n \not\rightarrow Tf = 0$.

Remark. $T : D \subset X \rightarrow Y$ linear, $\text{graph}(T) := \{(x, Tx) \mid x \in D\} \subset X \times Y$.

↓ This lecture took place on 2019/05/23.

Lemma 5.11. Let X and Y be normed and $D \subset X$ be a subspace. Let $T : D \rightarrow Y$ be linear. Then

- $\text{graph}(T) \subset X \times Y$ is a subspace.
- T closed iff $\text{graph}(T)$ is closed

Proof. • Immediate.

- T closed $\iff \forall (x_n)_n \in D, x \in X, z \in Y : x_n \rightarrow x. \ Tx_n \rightarrow y \implies x \in D \wedge Tx = y$.

$$\iff \forall (x_n, y_n) \in \text{graph}(T), (x, y) \in X \times Y : (x_n, y_n) \rightarrow (x, y) \implies (x, y) \in \text{graph}(T) \iff \text{graph}(T) \text{ is closed}$$

□

Closed operators are continuous in the right topology.

Lemma 5.12. *Let X and Y be Banach spaces. $D \subset X$ be subspaces and $T : D \rightarrow Y$ be closed and linear. Then:*

1. $(D, \|\cdot\|_T)$ where $\|\cdot\|_T := \|x\|_X + \|Tx\|_Y$ is Banach (graph norm)
2. $T : (D, \|\cdot\|_T) \rightarrow Y$ is continuous.

Proof. 1. $\|\cdot\|_T$ is indeed a norm.

Let $(x_n)_n$ be Cauchy in D w.r.t. $\|\cdot\|_T$.

$\Rightarrow (x_n)_n$ and $(Tx_n)_n$ are Cauchy sequences in X and Y respectively

Thus $\exists x := \lim_{n \rightarrow \infty} x_n$ and $y := \lim_{n \rightarrow \infty} Tx_n$. T closed implies $x \in D$ and $Tx = y$. Hence $(x_n)_n \rightarrow x \in D$ for $n \rightarrow \infty$ w.r.t. $\|\cdot\|_T$.

2. $\forall x \in D : \|Tx\| \leq \|x\| + \|Tx\| \leq \|x\|_T$

Extension of the open mapping theorem for closed operators. \square

Lemma 5.13. *Let X and Y be a Banach space. $D \subset X$ be a subspace. $T : D \rightarrow Y$ be linear, closed and surjective. $\Rightarrow T$ is open, in particular if T is injective, $T^{-1} : Y \rightarrow D$ is continuous.*

Proof. By Lemma 5.12, $T : D \rightarrow Y$ is continuous wrt. $\|\cdot\|_T$ in D and D, Y are Banach.

By Theorem 5.5, T is open from D to Y . Show that T is open wrt. $\|\cdot\|_X$ and $\|\cdot\|_Y$. Take $0 \subseteq D$ to be open wrt. $\|\cdot\|_X$. Because $\|\cdot\|_X \subset \|\cdot\|_T \Rightarrow 0$ is open wrt. $\|\cdot\|_T \Rightarrow T(0)$ is open in Y . \square

Corollary 5.14. *Let X, Y be Banach. $D \subseteq X$ be a subspace. $T : D \rightarrow Y$ closed, linear and has closed range. Then with $\tilde{D} := D \setminus \text{kernel}(T)$. $\tilde{T} : \tilde{D} \rightarrow Y$ with $[x] \mapsto Tx$. We get that \tilde{T} is bijective from \tilde{D} to $\text{range}(T)$ and $\exists \tilde{T}^{-1} : \text{range}(T) \rightarrow \tilde{D}$ and is continuous. In particular, if T is injective*

$$\Rightarrow T^{-1} : \text{range}(T) \rightarrow D \text{ is continuous}$$

Proof. Similar to the proof above. Thus this is left as an exercise to the reader. \square

Theorem 5.15 (Closed graph theorem). *Let X and Y be Banach spaces. $T : X \rightarrow Y$ be linear and closed. [e.g. $X \subset \hat{X}$ with X closed, \hat{X} Banach, $T : D := X \rightarrow Y$] $\Rightarrow T$ is continuous.*

Proof. By Lemma 5.12, $T : (X, \|\cdot\|_T) \rightarrow (Y, \|\cdot\|_Y)$ is continuous and $(X, \|\cdot\|_T)$ is Banach. Also $\|x\|_X \leq \|x\|_T \forall x \in X$. By Corollary 5.9, $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent, thus $\exists M > 0 : \|x\|_T \leq \|x\|_X \forall x \in X$.

$$\Rightarrow \forall x \in X : \|Tx\|_Y < C \|x\|_T \leq CM \|x\|_X$$

$$\Rightarrow T \text{ is continuous wrt. } \|\cdot\|_X$$

□

Remark. For differential operators, the domain is usually not closed. (C^1 is not closed in the C^0 -norm)

↓ This lecture took place on 2019/05/28.

6 Dual spaces, reflexivity and weak convergence

Remark. Obtain “Bolzano-Weierstrass” in infinite-dimensional spaces

Definition 6.1. Let X be a normed space. Then $X^* := \mathcal{L}(X, \mathbb{K})$ is called the dual space of X . We denote $\|x^*\|_{X^*} := \|x^*\|_{\mathcal{L}(X, \mathbb{K})}$.

Corollary 6.2. Let X be a normed space. Then X^* is complete.

Lemma 6.3. Let X be normed. Then $\forall x \in X \setminus \{0\} \exists x^* \in X^* : \|x^*\|_X = 1 \vee x^*(x) = \|x\|$. In particular,

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \implies \exists x^* \in X^* : x^*(x_1) \neq x^*(x_2)$$

Proof. Take $x \in X, x \neq 0$ fixed. Define $u^* : \text{span}(x) \rightarrow \mathbb{K}$ with $\lambda x \mapsto u^*(\lambda x) := \lambda \|x\|$. Then,

$$\|u^*\| = \sup_{\|\lambda x\| \leq 1} |u^*(\lambda x)| = \sup_{\|\lambda x\| \leq 1} |\lambda \|x\|| = \sup_{\|\lambda x\| \leq 1} \|\lambda x\| = 1$$

Also $u^*(x) = \|x\|$. By the Hahn-Banach Theorem, existence of x^* , as claimed, follows.

In particular, if $x_1 \neq x_2$ we define $x^* = x^*(x_1 - x_2) = \|x_1 - x_2\| \implies x^*(x_1) - x^*(x_2) \neq 0$. □

Lemma 6.4. Let X be normed. Then

$$\forall x \in X : \|x\| = \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} |x^*(x)|$$

Proof. Let $x \in X$. If $x = 0$, then trivial. If $x \neq 0$, then

$$\sup_{\|x^*\| \leq 1} \|x^*(x)\| \leq \sup_{\|x^*\| \leq 1} \|x^*\| \|x\| \leq \|x\|$$

Also, $\exists \hat{x} \in X : \|\hat{x}\| = 1 \implies \hat{x}^*(x) = \|x\| \implies \sup_{\|x^*\| \leq 1} |x^*(x)| \geq |\hat{x}^*(x)| = \|x\|$ □

Lemma 6.5. Let X be normed. $U \subset X$ be a closed subspace. $x \notin U \implies \exists \hat{x} \in X^* : \hat{x}|_U = 0$ with $\hat{x}(x) \neq 0$.

Proof. Define $w : X \rightarrow X/U$ with $x \mapsto [x] = \{y \in X \mid x - y \in U\}$

$$w(u) = 0 \forall u \in U, w(x) \neq 0$$

Choose $l \in (X/U)^*$ such that $l^*(w(x)) \neq 0$ and define $x^* := l \circ w$. Thus $x^*(x) = l^*(w(x)) \neq 0$. $x^*(u) = l^*(0) = 0$. \square

Lemma 6.6. *Let X^* be normed. $U \subset X$ be a subspace. TFAE:*

- U is dense in X
- $\forall x^* \in X : x^*|_U = 0 \implies x^* = 0$

Proof. (1) \rightarrow (2) Obvious by continuity.

(2) \rightarrow (1) \bar{U} is closed. If $\bar{U} \neq X \implies \exists x^* \in X : x^*|_{\bar{U}} = 0$ and $x^* \neq 0$. This gives a contradiction. \square

Remark. (2) \rightarrow (1) is often useful to show density

Theorem 6.7. *Let $1 \leq p \leq \infty$. $a \in [1, \infty]$: $\frac{1}{p} + \frac{1}{a} = 1$ and (Ω, Σ, μ) be a σ -finite measure space. Define $T : L^a(\Omega, \mathbb{K}^M, \mu) \rightarrow L^p(\Omega, \mathbb{K}^M, \mu)^*$ with $g \mapsto T_g$ with $T_g(f) := \int_{\Omega} (f, g) d\mu$*

Then T is well-defined, linear and isometric (\implies injective). If $p < \infty$, then T is surjective and $L^p(\dots)^ \cong L^q$.*

Proof. **Well-defined** Linear is obvious. By Hölder: $|T_g(f)| = |\int_{\Omega} (f, g) d\mu| \leq \|g\|_q \|f\|_p \implies T_g$ bounded and $\|T_g\| \leq \|g\|_q$.

Next, assume $p < \infty$. Show: T_g is surjective. Here we show the result only for $L^p(\Omega, \mu)$ and $|\mu(\Omega)| < \infty$ (the rest is left as an exercise to the reader). Take $y^* \in (L^p)^*$. Construct $q \in L^q : T_q = y^*$. We consider $\nu : \Sigma \rightarrow \mathbb{K} : \nu(E) := y^*(\chi_E)$. Then $\nu(\emptyset) = y^*(0) = 0$. Furthermore, for $(E_i)_i$ in Σ pairwise-disjoint, we get that

$$\sum_{i=1}^n \chi_{E_i} = \chi_{\bigcup_{i=1}^n E_i} \rightarrow \chi_{\bigcup_{i=1}^{\infty} E_i} \text{ pointwise, } E := \bigcup_{i=1}^{\infty} E_i$$

Furthermore,

$$\int_{\Omega} \|X_{\bigcup_{i=1}^n E_i} - X_E\|^p d\mu \leq \int_{\Omega} |\chi_E|^p d\mu$$

Lebesgue dominated convergence theorem implies

$$X_{\bigcup_{i=1}^n E_i} \rightarrow X_E \text{ in } L^p$$

$$\implies \sum_{i=1}^{\infty} u(E_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n v(E_i) = \lim_{n \rightarrow \infty} y^*(\chi_{\bigcup_{i=1}^n E_i}) = y^*(X_E) = v(E)$$

Thus ν is a complex measure. Also $\mu(E) = 0 \implies \nu(E) = 0$. The Radon-Nikodym Theorem (compare with Werner, Prop. A 4.6) implies $\exists q \in L^*(\Omega, \mu)$ such that

$$u(E) = \int_E q d\mu = \int_\Omega \langle \chi_E, g \rangle d\mu$$

Hence: $y^*(\chi_E) = \int_\Omega \langle \chi, g^* \rangle d\mu \forall E \in \Sigma$.

By linearity, $y^*(f) = \int_\Omega \langle f, g \rangle d\mu \forall f$ as step functions.

Now since $\mu(\Omega) < \infty$, $L^\infty(\Omega, \mu) \subset L^p(\Omega, \mu)$ (by Hölder).

Since $\forall h \in L^\infty : \int_\Omega |h|^p = \int_\Omega |h| |h|^{p-1} = \|h\|_\infty \int_\Omega |h|^{p-1}$.

$$\implies |y^*(f)| \leq \|y^*\| \cdot \|f\|_p \leq \|y^*\| \cdot c \cdot \|f\|_\infty \forall f \in L^\infty$$

Hence, $y^* \in \mathcal{L}(L^\infty, \mathbb{K})$. Also $f \mapsto \int_\Omega \langle f, g \rangle d\mu$ for $f \in L^\infty$ is continuous wrt. L^∞ -convergence since $\forall f \in L^\infty : \int_\Omega \langle f, g \rangle d\mu \leq \|f\|_\infty \cdot \|g\|_1$ with $\|g\|_1 < \infty$.

$$\implies \forall f \in L^\infty : y^*(f) = \int_\Omega \langle f, g \rangle d\mu$$

by density of step functions, we know that from measure theory.

Now, show that $g \in L^q$.

Case $q < \infty$: Define

$$f(x) = \begin{cases} \frac{|g(x)|^\infty}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{else} \end{cases}$$

$$E_n := \{x \in \Omega \mid |g(x)| \leq n\} \implies X_{E_n} f \in L^\infty$$

Further:

$$\int_{E_n} |g|^q d\mu = \int_\Omega X_{E_n} \langle f, g \rangle d\mu \quad (5)$$

$$= y^*(X_{E_n} f) \quad (6)$$

$$\leq \|y^*\| \|X_{E_n} \cdot f\|_p \quad (7)$$

$$= \|y^*\| \left(\int_{E_n} |f|^p d\mu \right)^{\frac{1}{p}} \quad (8)$$

$$= \|y^*\| \left(\int_{E_n} |g|^{(q-1)p} d\mu \right)^{\frac{1}{p}} \quad (9)$$

with $(q-1)p = q$ because $p = \frac{q}{q-1}$.

$$\implies \left(\int_{E_n} |g|^q d\mu \right)^{\frac{1}{q}} = \left(\int_{E_n} |g|^q d\mu \right)^{1-\frac{1}{p}} \leq \|y^*\|$$

By the Beppo-Levi Theorem,

$$\left(\int_\Omega |g|^q d\mu \right)^{\frac{1}{q}} \leq \|y^*\|$$

Hence, $g \in L^q$ and $\|g\|_q \leq \|y^*\|$.

Case $q = \infty$: Define $E := \{x \in \Omega \mid |q(x)| > \|y^*\|\}$. $f := \chi_E \cdot \frac{|g|}{g} \in L^\infty$. If $\mu(E) > 0$, then

$$\mu(E) \|y^*\| < \int_E |g| d\mu = \int_\Omega (fg) d\mu = y^*(f) \leq \|y^*\| \|f\|_1 = \|y^*\| \mu(E)$$

Gives a contradiction. $\Rightarrow \mu(E) = 0$,

$$\Rightarrow |q(x)| \leq \|y^*\| \text{ almost everywhere}$$

$$\Rightarrow \|g\|_\infty \leq \|y^*\| \text{ and } q \in L^\infty$$

$$\int_\Omega \langle f, g \rangle d\mu = y^*(f) \forall f \in L^\infty \text{ and } g \in L^q$$

hence $f \mapsto \int_\Omega \langle f, g \rangle d\mu$ is continuous on L^p (Hölder).

Hence,

$$\int_\Omega \langle f, g \rangle d\mu = y^*(f) \forall f \in \overline{C_c(\Omega)}^{L^p} = L^p(\Omega, \mu)$$

since $C_c \subset L^\infty$.

We know $\|Tg\| \leq \|g\|_q$ and $\|g\|_q \leq \|y^*\|$ with $\|Tg\| = \|y^*\|$

$$\Rightarrow \|y^*\| = \|g\|_q$$

Final open point: Show that $\|g\|_q \leq \|Tg\|$ for $p = \infty$.

$$\begin{aligned} \|Tg\| &= \sup_{\substack{f \in L^\infty \\ \|f\|_\infty \leq 1}} |T_g(f)| = \sup_{\|f\|_\infty = 1} \int_\Omega \langle f, g \rangle d\mu \\ &\geq \int_\Omega |g| d\mu = \|g\|_q = \|g\|_1 \end{aligned}$$

□

Corollary 6.8. Let $p, q \in [1, \infty] : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$T_1 : l^q \rightarrow (l^p)^* \quad y \mapsto T_y(x) := \sum_{i=1}^{\infty} x_i \overline{y_i}$$

$$T_2 : L^q(\Omega) \rightarrow L^p(\Omega)^* \quad g \mapsto Tg(f) := \int_\Omega \langle f, g \rangle d\mu$$

are well-defined, isometric, linear and surjective if $p < \infty$.

Theorem 6.9 (Riesz-Representation Theorem). Let K be a compact metric space. Then $C(K)^* \cong M(K)$ where $M(K)$ is the set of Radon measures, is regular, finite. Borel measures on K and $T : M(K) \rightarrow C(K)^*$.

Proof. Radin 1986, see the book in the literature list

□

↓ This lecture took place on 2019/06/04.

Revision 6.10. $p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1 \implies (L^p)^* = (L^q)$

For which spaces does this hold true?

$$(L^q)^* \cong L^p \implies ((L^p)^*)^* \cong L^p$$

Definition 6.11 (Proposition and definition). Let X be a normed space. We call $X^{**} = (X^*)^*$ to the bidual space of X . Define

$$i = i_X : X \rightarrow X^{**} \quad x \mapsto i_X(x) : X^* \rightarrow \mathbb{K}$$

Then i_X is linear and isometric. $x^* \mapsto i_X(x)(x^*) := x^*(x)$. We call i_X to be the canonical embedding of X into its bidual space.

Proof. Linearity Show: $i(\lambda x + z) = \lambda i(x) + i(z) \forall x, z \in X, \lambda \in \mathbb{K}$

$$x^* \in X^*.$$

$$\begin{aligned} i(\lambda x + z) &= x^*(\lambda x + z) \\ &= \lambda x^*(x) + x^*(z) \\ &= \lambda i(x) + i(z) \end{aligned}$$

Recall that $x^* \in X^*$ and thus linear.

Isometric

$$\forall x \in X, x^* \in X^* : |i(x)(x^*)| = |x^*(x)| = \|x^*\| \|x\| \implies \|i(x)\| \leq \|x\|$$

Lemma 6.4: If $x = 0$, then $\|i(x)\| = \|x\| = 0$. If $x \neq 0$, then $\exists \hat{x}^* \in X^* : \hat{x}^* \in X^* : \hat{x}^*(x) = \|x\|, \|\hat{x}^*\| = 1$.

$$\implies \|i(x)\| = \sup_{\|x^*\| \leq 1} \|i(x)(x^*)\| \geq |i(x)(\hat{x}^*)| = |\hat{x}^*(x)| = \|x\|$$

□

Remark. Hence X can be identified with a subspace of X^{**} . In particular, if X is a Banach space $i(X)$ is a closed subspace.

Definition 6.12. A Banach space X is called reflexive if $i_X : X \rightarrow X^{**}$ is surjective.

Remark. • Dual spaces are always complete, hence only Banach spaces can be reflexive.

- We already know: L^p is reflexive for $p \in (1, \infty)$
- Alternative definition: $\overline{X \text{ reflexive}}$ iff $X \cong X^{**}$.

$$\implies \text{Reflexive} \implies \overline{\text{reflexive}} \text{ (requires a particular isomorphism).}$$

\Leftarrow Is not true. Our definition is far more common since it is useful to have the isometry explicitly.

Corollary 6.13. Let (Ω, Σ, μ) be a sigma-finite measure space. $p \in (1, \infty)$. Hence $L^p(\Omega, \mathbb{K}^M, \mu)$ is reflexive, in particular $L^p(\Omega)$, \mathbb{P} are reflexive.

Proposition 6.14. Let X be normed. Then

1. If X is reflexive, $U \subset X$ a closed subspace $\Rightarrow U$ is reflexive.
2. If X is Banach: X reflexive $\iff X^*$ is reflexive.

Proof. 1. Take $u^{**} \in U^{**}$. Show: $\exists u \in U : i_U(u) = u^{**}$. The mapping $x^* \mapsto u^{**}(x^*|_U)$ is in X^* since

$$|u^{**}(x^*|_U)| \leq \|u^{**}\|_{U^{**}} \|x^*|_U\|_{V^*} \leq \|u^{**}\|_{V^*} \|x^*\|_{X^*}$$

$$X \text{ reflexive} \Rightarrow \exists x \in X : i_X(x) = f$$

$$\Rightarrow x^*(x) = u^{**}(x^*|_U) \forall x^* \in X^*$$

Show: $x \in U$. If $x \notin U$, then $x^* \in X^* : x^*(x) = 1, x^*|_U = 0 \Rightarrow 1 = u^{**}(x^*|_U) = u^{**}(0) = 0$ gives a contradiction. Hence $x \in U$. Define $u = x$.

Show: $u^*(u) = u^{**}(u^*|_U) \forall u^* \in U^*$. Take $u^* \in U^*$. Take x^* to be an extension of u^* by Hahn-Banach.

$$\Rightarrow u^{**}(u^*) = u^{**}(x^*|_U) = x^*(u) = u^*(u) = i_U(u)(u^*) \Rightarrow u^{**} = i_U(u)$$

2. Assume that X is reflexive.

Show: $i_{X^*} : X^* \rightarrow X^{***}$ is surjective.

Take $x^{***} \in X^{***}$. Define $x^* : X \rightarrow \mathbb{K}$ and $x \mapsto x^{***}(i_X(x))$. Then $x^* \in X^*$.

Show: $i_{X^*}(x^*) = x^{***}$.

Since X is reflexive, any x^{**} can be written as $x^{**} = i_X(x)$.

$$\begin{aligned} \Rightarrow \forall x^{**} \in X^{**} : x^{***}(x^{**}) &= x^{***}(i_X(x)) = \\ x^*(x) &= i_X(x)(x^*) = x^{**}(x^*) = i_{X^*}(x^*)(x^{**}) \\ \Rightarrow x^{***} &= i_{X^*}(x^*) \end{aligned}$$

Now if X^* is reflexive, then X^{**} is reflexive. $i(X) \subset X^{**}$ is reflexive as closed subspace of X^{**} . Hence X is reflexive.

□

Proposition 6.15. Let X be normed.

- If X^* is separable, then X is separable.
- If X is reflexive, then $(X \text{ reflexive} \iff X^*$ is separable)

Remark. 1. “ \Leftarrow ” in the first item is not true since L^1 is separable, but $L^\infty = (L^1)^*$ is not separable.

2. By the second item, L^1 is not reflexive, since otherwise L^∞ would be separable.

Proof. • follows from item 1

- Assume X^* is separable.

$$S_1^{X^*} = \{x^* \in X^* \mid \|x^*\| = 1\}$$

is separable as being a subset. Every subset of a separable set is separable (left as an exercise). Take $(x_n^*)_n$ to be dense in S_1 . For all $n \in \mathbb{N}$ pick $x_n \in X : \|x_n\| = 1$ and $|x_n^*(x_n)| > \frac{1}{2}$. Set $U = \text{span}((x_n)_{n \in \mathbb{N}})$.

Show: U is dense in X .

Let $x^* \in X^*$ such that $x^*|_U = 0$.

Show: $x^* = 0$ ($\Rightarrow \overline{U} = X$)

If not, wlog. assume $\|x^*\| = 1$.

$$\Rightarrow \exists x_{n_0}^* = S_1^{X^*} : \|x^* - x_{n_0}^*\| < \frac{1}{4}$$

$$\Rightarrow \frac{1}{2} < |x_{n_0}^*(x_{n_0})| = |x_{n_0}^*(x_{n_0}) - x^*(x_{n_0})| \leq \|x_{n_0}^* - x^*\| \|x_{n_0}\| < \frac{1}{4} \cdot 1$$

Contradiction.

□

Fundamental difficulty in ∞ -dimensional spaces. Closed and bounded does not imply sequentially compact. In particular, bounded sequences do not admit convergent subsequences in general.

Solution: A weaker notion of convergence.

Definition 6.16. Let X be normed, $(x_n)_n$ in X , $x \in X$. We say x_n converges weakly to x (denoted $(x_n) \rightharpoonup x$) iff

$$x^*(x_n) \rightarrow x^*(x) \forall x^* \in X^*$$

Remark. • This is obviously weaker than norm-convergence (also called strong convergence)

- All $x^* \in X^*$ are still sequentially continuous wrt. weak convergence. i.e. $x^*(x_n)_n \rightarrow x^*(x) \quad \forall x_n \rightarrow x$

Proposition 6.17. Let X be normed, $(x_n)_n$ in X , $x \in X$. Then

1. $(x_n)_n \rightharpoonup x \Rightarrow (x_n)_n \rightarrow x$
2. If $(x_n)_n \rightharpoonup x$ then $(x_n)_n$ is bounded
3. Weak limits are unique. i.e. if $(x_n)_n \rightharpoonup x$ and $(x_n)_n \rightharpoonup y \Rightarrow x = y$

Proof. 1. Immediate

2. follows from Lemma 6 below

3. Assume $x_n \rightharpoonup x$, $x_n \rightharpoonup y$

$$\implies \forall x^* \in X : x^*(x) = \lim_{n \rightarrow \infty} x^*(x_n) = x^*(y)$$

$$\implies x^*(x - y) = 0 \forall x^* \in X^* \implies x - y = 0 \implies x = y$$

□

Remark. \Leftarrow in item 1 does not hold true, e.g. with $e_i = (0, \dots, 0, 1, 0, \dots)$ (the 1 is at the i -th position). We have that $(e_n)_n \rightharpoonup 0$ in l^p for $p \in (1, \infty)$. Since $\forall (u_n)_n \in l^q = (l^p)^*$ with $\frac{1}{p} + \frac{1}{q} = 1$.

$$(u_n)_n(e_m) = \sum_{n=1}^{\infty} u_n(e_m)_n = u_m \rightarrow 0 \text{ as } m \rightarrow \infty$$

since $(u_m)_m \in l^q$.

Hence $(e_n)_n \rightharpoonup 0$ in l^p .

But $\|e_n\|_p = 1 \forall n \implies (e_n)_n \not\rightharpoonup 0$

Remember that $\forall v \in l^p$ and $u \in (l^p)^*$ we write $v = (v_n)_n$ and $u = (u_n)_n \in l^q$.

Then $u(v) = \sum_{n=0}^{\infty} v_n u_n$

Lemma. Let X be normed. $M \subset X$. TFAE

1. M is bounded

2. $\forall x^* \in X^* : x^*(M) \subset \mathbb{K}$ is bounded

Also if X is Banach, $M^* \subset X^*$. TFAE:

1'. M^* is bounded

2'. $\forall x \in X : \{x^*(x) \mid x^* \in M^*\}$ is bounded

Proof. 1 \rightarrow 2: Let $c > 0 : \|x\| \leq c \forall x \in M$.

$$\implies |x^*(x)| \leq \|x^*\| \cdot c$$

$$\implies x^*(M) \text{ bounded for } x^* \text{ fixed.}$$

2 \rightarrow 1: Consider $i_X(x) \in X^{**}$ for $x \in X$. We have that $\sup_{x \in M} |x^*(x)| = \sup_{x \in M} |i_X(x)(x^*)| < \infty \forall x^* \in X^*$ by assumption. By uniform boundedness principle,

$$\|x\| = \|i_X(x)\| < c < \infty \quad \forall x \in M$$

$$\implies M \text{ bounded}$$

1' \rightarrow 2': True since $|x^*(x)| \leq \|x^*\| \|x\| \leq C \|x\| \forall x^* \in M^*, x \in X$

2' \rightarrow 1': Direct application of uniform boundedness principle.

□

Theorem 6.18. *Let X be reflexive. Then every bounded sequence in X admits a weakly-convergent subsequence.*

Proof. Take $(x_n)_n$ be a bounded sequence in X . Assume first that X is separable. Hence X^* is separable, e.g. $\exists (x_n^*)_n$ such that $X^* = \overline{\{x_n^* \mid n \in \mathbb{N}\}}$

Idea: Construct subsequence $(y_m)_m$ of $(x_n)_n$ such that $(x_i^*(y_m))_m$ converges $\forall i \in \mathbb{N}$.

Claim. $\forall i \in \mathbb{N} \exists (x_{n_j^i})_j$ subsequence of $(x_n)_n$ such that

1. $(x_{n_j^i})_j$ is a subsequence of $(x_{n_j^k})_j \forall k \leq i$
2. $(x_k^*(x_{n_j^i}))_j$ is convergent $\forall k \leq i$

Proof by induction. Case $i = 1$:

$$\begin{aligned} |x_1^*(x_n)| &\leq \|x_1^*\| \cdot \|x_n\| \leq \|x_1^*\| C \quad \text{for } C > 0 : \|x_n\| \leq C \forall n \\ \implies (x_1^*(x_n))_n &\text{ is bounded by } \mathbb{K} \implies \exists \text{ convergent subsequence } (x_{n_j^1})_j \end{aligned}$$

Case $i \rightarrow i + 1$: Let $(x_{n_j^i})_j$ be given as claimed. Again $|x_{i+1}^*(x_{n_j^i})| \leq \|x_{i+1}^*\| C \implies \exists$ subsequence $(x_{n_j^{i+1}})_j$ such that $(x_{i+1}^*(x_{n_j^{i+1}}))_j$ is convergent. Subsequence implies that both assertions are true.

□

Now, we define $y_j = x_{n_j^i} \forall j \in \mathbb{N} \implies (y_j)_j$ is a subsequence of $(x_n)_n$. Also, for $k \in \mathbb{N}$, $\lim_{i \rightarrow \infty} x_k^*(y_i) = \lim_{\substack{j \rightarrow \infty \\ j \geq k}} x_k^*(y_j)$ exists.

Next: $\forall x^* \in X^* : \lim_{j \rightarrow \infty} x^*(y_j)$ exists. Take $\varepsilon > 0, x^* \in X^*$ pick $i : \|x_i^* - x^*\| < \varepsilon$

$$\begin{aligned} \implies \forall n, m \in \mathbb{N} : &\|x^*(y_n) - x^*(y_m)\| \\ &\leq \|x^*(y_n) - x_i^*(y_n)\| + \|x_i^*(y_n) - x_i^*(y_m)\| + \|x_i^*(y_m) - x^*(y_m)\| \\ &\leq \|x^* - x_i^*\| \|y_n\| + \|x_i^*(y_n) - x_i^*(y_m)\| + \|x_i^* - x^*\| \|y_m\| \\ &\leq 2\varepsilon c + \|x_i^*(y_n) - x_i^*(y_m)\| \leq 3\varepsilon c \rightarrow 0 \text{ for } n, m \rightarrow \infty \end{aligned}$$

$\implies (x^*(y_n))_n$ is Cauchy, thus convergent.

Show: $\exists y \in X : x^*(y_m) \rightarrow x^*(y) \forall x^* \in X^*$

Define $l : X^* \rightarrow \mathbb{K}$ well-defined and linear with $x^* \mapsto \lim_{n \rightarrow \infty} x^*(y_n)$. Furthermore $|l(x^*)| = \lim_{n \rightarrow \infty} |x^*(y_n)| \leq \|x^*\| c \implies l \in (X^*)^*$. $\implies \exists y \in X : i_X(y) = l$. This means that $\forall x^* \in X^* : x^*(y) = i_X(y)(x) = l(x^*) = \lim_{n \rightarrow \infty} x^*(y_n)$

$$\implies y_n \rightarrow y$$

Now without separability: Take again $(x_n)_n$ to be bounded. Define $Y := \text{span}((x_n)_n)$.

Hence Y is separable, reflexive as closed subset of X (reflexive). x_n is a sequence in Y . Thus use the previous case.

$\implies \exists (y_n)_n$ subsequence of $(x_n)_n, y \in Y$ such that $x^*(y_n) \rightarrow x^*(y) \forall x^* \in Y^*$.

For $x^* \in X^*, x^*|_Y \in Y^* \implies x^*(y) \rightarrow x^*(y)$. \square

Remark. Further important question: When are closed sets also closed wrt. weak convergence?

Not always true! Remember that $(l_n)_n$ in $\ell^p : \|e_n\| = 1 \implies e_n = \{x \mid |x| = 1\}$ but $e_n \rightharpoonup 0$

Theorem 6.19. Let X be normed, $V \subset X$ closed and convex. Then $\forall (x_n)_n$ in V such that $x_n \rightharpoonup x \in X \implies x \in V$ (" V is weakly closed").

In particular, any closed subspace is also weakly closed.

Proof. Assume $x \notin V \implies \exists x^* \in X^* : x^*|_V = 0, x^*(x) \neq 0 \implies 0 = \lim_{n \rightarrow \infty} x^*(x_n) = x^*(x)$ gives a contradiction. \square

Remark (Consequence). $B_1(0)$ in X reflexive is weakly sequentially compact but not strongly sequentially compact if $\dim(X) = \infty$.

Corollary 6.20. Let X be normed. $(x_n)_n$ in X such that $x_n \rightharpoonup x \in X$. Then there exists a sequence $(y_n)_n$ where each y_n is a convex combination of the $(x_n)_n$ s.t. $y_n \rightarrow x$.

Remark (i.e.).

$$\exists N^n, (\lambda_i^n)_{i=1}^{N^n} : \lambda_i^n \geq 0, \sum_{i=1}^{N^n} \lambda_i^n = 1 \text{ such that } y_n := \sum_{i=1}^{N^n} \lambda_i^n x_i \rightarrow x \text{ as } n \rightarrow \infty$$

Proof. Apply Theorem 6.19 to the closed, convex hull of $\{x_n \mid n \in \mathbb{N}\}$. \square

↓ This lecture took place on 2019/06/07.

Theorem 6.21. Let X, Y be Banach spaces. Let $T : X \rightarrow Y$ be a linear operator. T is sequentially continuous wrt. norm convergence in $X, Y \iff T$ is sequentially continuous wrt. weak norm convergence in X, Y

Proof. \implies Let $x_n \rightharpoonup x$. Show: $\forall y \in Y^* : y^*(Tx_n) \rightarrow y(Tx)$. But the mapping $f : X \rightarrow \mathbb{K}$ with $x \mapsto y^*(Tx)$ $\|f(x)\| \leq \|y^*\| \|T\| \|x\| \implies f(x_n) \rightarrow f(x)$.

\Leftarrow Consider graph(t). Show: $\text{gr}(T)$ is closed $\implies T \subseteq \mathcal{L}(X, Y)$.

Assume $(x_n, Tx_n) \rightarrow (x, y)$.

$$\implies \left. \begin{matrix} x_n \rightarrow x \\ Tx_n \rightarrow y \end{matrix} \right\} \implies \left\{ \begin{matrix} x_n \rightarrow x \implies Tx_n \rightarrow Tx \\ Tx_n \rightarrow y \end{matrix} \right.$$

\square

What about non-reflexive spaces?

Example. Consider $(e_j)_j$ in l^1 when l_j is a zero row vector with 1 at position j . Then $\|e_j\| = 1 \implies (e_j)_j$ is bounded in l^1 . Assume there exists a subsequence $(e_{j_k})_k : e_{j_k} \rightharpoonup x \in l^1$. Define e_n^* as zero vector with 1 at position n in l^∞ .

Weak convergence

$$\implies e_k^*(e_{n_j}) \rightarrow e_k(x) = x_n \forall k \in \mathbb{N}$$

Recall that $x^*(x) = \sum x_k^* x_k$ for $x^* \in l^\infty, x \in l^1$.

Now $e_k^*(e_{n_j}) = 0 \forall n_j > k \implies x = 0$ (by convergence: $x_k = 0 \forall k$).

But $z^* := (1, 1, \dots) \in l^1 \implies 1 = \lim_{j \rightarrow \infty} z^*(l_{n_j}) = z^*(x) = 0$ giving a contradiction.

We need an ever weaker notion of convergence.

Definition 6.22. Let X be a normed space. $(x_n^*)_n$ in X^* with $x^* \in X^*$. We say $(x_n^*)_n$ weak*-converges to x^* and write $x_n^* \xrightarrow{*} x^*$ if $x_n^*(x) \rightarrow x^*(x) \forall x \in X$

Proposition 6.23. Let X be a normed space. $(x_n^*)_n, x^* \in X^*$. Then

1. $x_n^* \rightarrow x^* \implies x_n^* \xrightarrow{*} x^*$
2. If X is reflexive, $x_n^* \rightarrow x^* \iff x_n^* \xrightarrow{*} x^*$
3. If X is a Banach space, $x_n^* \xrightarrow{*} x^* \implies (\|x_n^*\|)_n$ is bounded.
4. If $x_n^* \xrightarrow{*} x^*$ and $x_n^* \xrightarrow{*} y^* \implies x^* = y^*$

Proof. Left as an exercise. □

Remark (Remark with huge consequences). In general: closed, convex $\not\implies$ weak* closed.

Theorem 6.24. Let X be separable, $(x_n^*)_n$ in X^* bounded. Then $(x_n^*)_n$ has a weak* convergent subsequence.

Remark. Applies to sequences in $L^\infty, l^\infty, M(\Omega)$. Not to $L^1, l^1 \rightarrow$ no duals.

Proof. Consider $(x_n^*)_n$ in X^* bounded. $(x_n)_n$ in X such that $\overline{\{x_n \mid n \in \mathbb{N}\}} = X \implies |x_n^*(x_k)| \leq \|x_n^*\| \|x_k\|$ is bounded $\forall k$ fixed. As in the proof with weak convergence $\implies \exists (y_n^*)_n$ a subsequence of $(x_n^*)_n$ s.t. $y_n^*(x)$ converges $\forall x \in X$. Define $l : X \rightarrow \mathbb{K}$ with $x \mapsto \lim_{n \rightarrow \infty} y_n^*(x)$. Hence l is well-defined, linear and bounded. Thus $l \in X^*$. By definition, $l(x) = \lim_{n \rightarrow \infty} y_n^*(x) \implies y_n^* \xrightarrow{*} l$. □

Remark. Why not continue for non-separable spaces?

7 Complementary subspace and adjoint operators

1. Let X be normed, $U \subset X$ subspace. When can we project on U ?
2. \implies characterization of closed-range operators

Definition 7.1. Let X be normed, $U \subset X$, $V \subset X^*$. Define

$$U^\perp = \{x^* \in X^* \mid x^*(x) = 0 \forall x \in U\} \quad V_\perp = \{x \in X \mid x^*(x) = 0 \forall x^* \in V\}$$

U^\perp, V_\perp are called annihilators of U and V .

Proposition 7.2. Let X be a Banach space. $G, L \subset X$ be two closed subspaces such that $G + L$ is closed $[G + L = \{g + l : g \in G, l \in L\}] \implies \exists c > 0 : z \in G + L \exists x \in G, y \in L : z = x + y$ and $\|x\| \leq c \|z\|, \|y\| \leq c \|z\|$.

Proof. Consider $G \times L$ with $\|(x, y)\|_{G \times L} := \|x\| + \|y\|$. Define $T : G \times L \rightarrow G + L$ with $(x, y) \mapsto x + y$. Thus T is linear, surjective. By the open mapping theorem, $\exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_1(0))$.

$$\implies \forall z \in G + L : \|z\| < \varepsilon \implies z = x + y \text{ with } \|x\| + \|y\| \leq 1$$

$$\implies \forall z \in G + L : \frac{\varepsilon z}{2 \|z\|} \in B_\varepsilon(0) \implies \frac{\varepsilon z}{2 \|z\|} = x + y \text{ with } \|x\| + \|y\| \leq 1$$

$$\implies z = \frac{x \|z\| 2}{\varepsilon} + \frac{y \|z\| 2}{\varepsilon} = \hat{x} + \hat{y} \quad \text{with } \|x\| + \|y\| \leq 1$$

and

$$\|\hat{x}\| = \frac{\|x\| \|z\| 2}{\varepsilon} \leq \frac{2}{\varepsilon} \|z\| \quad \|\hat{y}\| = \frac{\|y\| \|z\| 2}{\varepsilon} \leq \frac{2}{\varepsilon} \|z\|$$

□

Proposition 7.3. Let X be normed, $P : X \rightarrow X$ is called projection if $P \circ P = P$. P is called linear and continuous projection if it is linear and continuous. P is called projection to $U \subset X$ if $P(X) \subset U$. Also, we write $X = A \oplus B$ for $A, B \subset X$ subspaces if $X = A + B$ and $A \cap B = \{0\}$.

$$\implies \forall x \in X \exists ! a \in A, b \in B : x = a + b$$

If P is a continuous, linear projection, then

1. $P = 0$ on $\|P\| \geq 1$
2. $\text{kernel}(P)$ and $\text{range}(P)$ are closed
3. $X = \text{kernel}(P) \oplus \text{range}(P)$ ["projection yields decomposition of X "]

Proof. 1. $\|P\| = \|P \circ P\| \leq \|P\| \|P\|$.

2. $\ker(P)$ closed since P is continuous. Also $(\text{id} - P)$ is a projection. Linear and continuous since

$$(\text{id} - P) \circ (\text{id} - P) = \text{id} - P - P(\text{id} - P) = \text{id} - P - P \circ \text{id} + P \circ P = \text{id} - P$$

Also if $\text{range}(P) = \ker(I - P) \implies \text{range}(P)$ closed since $I - P$ is continuous. Since:

$$\subset: \text{ If } x = Py \implies (I - P)(x) = x - Px = Py - PP_y = 0$$

$$\supset: \text{ If } 0 = (I - P)(x) \implies Px = x$$

3. $\forall x \in X : x = P(x) + x - P(x) \in \text{range}(P) + \ker(P) \implies \text{"+"}$. If $x \in \ker(P) \cap \text{range}(P) \implies x = Py \implies 0 = Px = PP_y = Py = x$

□

Remark. If $x_n = a_n + b_n \in \ker(P) + \text{range}(P)$ and $x = a + b$. Then $x_n \rightarrow x \iff a_n \rightarrow a$ and $b_n \rightarrow b$.

Proof. \implies Immediate

$$\Leftarrow \|b_n - b\| = \|P(x_n - x)\| \leq C \|x_n - x\| \text{ and the same for } \|a_n - a\|$$

□

Proposition 7.4. Let X be Banach. $X = G \oplus L$ with G, L closed. Then $\exists P : X \rightarrow X$ as a continuous, linear projection such that $\ker(P) = G$ and $\text{range}(P) = L$.

Proof. Define $P : X \rightarrow X$ with $x \mapsto a$ when $x = a + b \in G \oplus L$. Hence P is well-defined and $PPx = Px$. Linear $\lambda x + y = \lambda(a_1 + b_1) + (a_2 + b_2)$ with $x = a_1 + b_1$ and $y = a_2 + b_2$

$$\implies P(\lambda x + y) = \lambda a_1 + a_2 = \lambda P(x) + P(y)$$

Continuity: By Proposition 7.2, $\exists C > 0$

$$\|Px\| = \|a\| \leq C \|x\| \forall x \in X$$

Hence P is continuous, $\text{range}(P) = G$ since $Pa = a \forall a \in G$.

Show: $\ker(P) = L$.

$$\supset: \text{ If } x \in L \implies x = 0 + x \in G + L \implies Px = 0$$

$$\subset: \text{ If } Px = 0, \text{ then } x = 0 + b \in G + L \implies x \in L$$

□

In finite dimensions, given $G \subset X \implies \exists L : X = G + L$ ($L = G^\perp$) $\implies \exists P : X \rightarrow X$ continuous linear projection such that $\text{range}(P) = G$.

Definition 7.5. Let X be normed, $G \subset X$ a closed subspace. We say “ G admits a complement in X ” (denoted by G^C) if $\exists L \subset X$ closed subspace such that $X = G \oplus L$.

Remark. • G admits a complement $\iff \exists P : X \rightarrow X, P = G$ a continuous, linear projection.

- In finite dimensions: By Linear Algebra, $\forall G \subset X$ subspace, G admits a complement.

↓ This lecture took place on 2019/06/13.

Lemma 7.6. Let X be normed. $U \subseteq X$ a subspace. $\dim U < \infty$. Thus U admits a complement in X .

Proof. Let $(e_i)_{i=1}^n$ be a basis of U , $\|e_i\| = 1$. Define $\varphi_i : U \rightarrow \mathbb{K}$. $u = \sum \lambda_i e_i \mapsto \lambda_i$ with $i = 1, \dots, n$.

φ_i is linear and bounded ($|\varphi_i(u)| = \lambda_i \leq \|u\|_2 \leq c_i \cdot \|u\|$, equivalence of norms on U). Each φ_i can be extended to $\varphi_i \in \mathcal{L}(X, \mathbb{K})$ by Hahn-Banach.

Define $P : X \rightarrow U$ with $x \mapsto \sum_{i=1}^n \varphi_i(x) e_i$.

Then P is linear. $P(x) \in U \forall u \in U$ and $P(u) = u \forall u \in U$. Thus P is linear projection.

$$\|P(x)\| \leq \sum \|\varphi_i(x) \cdot e_i\| \leq \sum \|\varphi_i\| \|x\| \|e_i\| \leq n \cdot \|x\| \cdot c$$

Hence U admits a complement (and $\|P\| \leq n$, but not true in our setting. We know that $|\varphi_i(x)| \leq c_i \cdot \|x\|$) (left as an exercise: when is $\|P\| = n$ true?). \square

Definition 7.7. Let X be normed and $U \subseteq X$ be a subspace. We say, U has finite co-dimension if $\exists V \subseteq X : \dim(V) < \infty$ and $U + V = X$.

Proposition 7.8. Let X be normed and $U \subseteq X$ be subspace of finite co-dimension. Then U admits a complement.

Proof. Left to the reader as an exercise. \square

Now: Further results on U^\perp , V_\perp as key to characterize closed range operators.

Proposition 7.9. Let X be normed. $U \subseteq X$ and $V \subseteq X^*$. Then:

1. U^\perp and V_\perp are closed subspaces.
2. $(U_\perp)_\perp = \overline{U}$ and $(V_\perp)^\perp \supset \overline{V}$ and equality if X is reflexive.

Proof. 1. Left as an exercise

2. \subseteq Let $u \in \overline{U}$. Take $u^* \in U^\perp$. By definition, $u^*(u) = 0$ and $u \in U^\perp$. Thus $U \subseteq U^\perp$ and $\overline{U} \subseteq U_\perp^\perp$.

\supseteq Let $\hat{u} \in U_\perp^\perp$ (since U_\perp^\perp is closed. Assume that $\hat{u} \notin \overline{U}$. By Hahn-Banach,

$$\exists x^* \in X^* : \Re(x^*(u)) < \alpha < \Re(x^*(\hat{u})) \forall u \in U$$

U is a subspace, thus $x^*(u) = 0$ and hence $x^* \in U^\perp$ and $x^*(\hat{u}) \neq 0$.

The remaining parts are left as an exercise. \square

Proposition 7.10. *Let X be normed. Let G and L be closed subspaces.*

1. $G \cap L = (G^\perp + L^\perp)_\perp$
2. $G^\perp \cap L^\perp = (G + L)^\perp$
3. $(G \cap L)^\perp \supseteq G^\perp + L^\perp$
4. $(G^\perp + L^\perp)_\perp = \overline{G + L}$

Those results will be important later.

Proof. 1. First statement is proven in two directions:

$$\subseteq x \in G \cap L. \text{ Let } x^* \in G^\perp + L^\perp. \text{ Show: } x^*(x) = 0.$$

$$\implies x^* = x_1^* + x_2^*, x_1^* \in G^\perp, x_2^* \in L^\perp$$

$$\implies x^*(x) = x_1^*(x) + x_2^*(x) = 0 + 0$$

$$\implies x \in (G^\perp + L^\perp)_\perp$$

$$\supseteq G^\perp \subseteq G^\perp + L^\perp \implies (G^\perp + L^\perp)_\perp \subseteq G_\perp^\perp. \text{ (In general: } A, B \in X^\perp \text{ and } A \subseteq B \text{ then } B_\perp \leq A_\perp.) \text{ Similar: } (G^\perp + L^\perp)_\perp \subseteq L^\perp$$

$$\implies (G^\perp + L^\perp)_\perp \subseteq L_\perp^\perp \cap G_\perp^\perp = L \cap G$$

2. Left as an exercise.

$$3. (G \cap L)^\perp \stackrel{(1.)}{=} ((G^\perp + L^\perp)_\perp)^\perp \stackrel{7.10}{\supseteq} \overline{G^\perp + L^\perp}$$

$$4. (G^\perp \cap L^\perp)_\perp \stackrel{(2.)}{=} (G + L)_\perp^\perp \stackrel{7.9}{=} \overline{G + L}$$

□

7.1 Adjoint operators

Motivation: Consider $T : X \rightarrow Y$ linear and bounded. Can we associate a dual operator to T as we can associate X with X^* and Y with Y^* ?

Definition 7.11 (Definition and proposition). *Let X, Y be normed and $T \in \mathcal{L}(X, Y)$. We define a dual operator or adjoint operator to T as $T^* : Y^* \rightarrow X^*$*

$$y^* \mapsto T^* y^* : X \rightarrow \mathbb{K} \text{ with } x \mapsto y^*(Tx)$$

Then $T^* \in \mathcal{L}(Y^*, X^*)$.

Proof. **Linear** Immediate.

Bounded

$$\begin{aligned} |(T^* y^*)(x)| &= |y^*(Tx)| \leq \|y^*\| \|Tx\| = \|T\| \|x\| \|y^*\| \\ &\implies \|T^* y^*\| \leq \|T\| \|y^*\| \\ &\implies \|T^*\| \leq \|T\| \end{aligned}$$

□

Example. $T : l^p \rightarrow l^p$. $x = (x_i)_{i=1}^\infty \mapsto (x_{i+1})_{i=1}^\infty$. $p \in (1, \infty)$.
 $\implies T \in \mathcal{L}(l^p, l^p)$

$T^* = ?$.

Let $y^* \in l^{p^*} = l^q$ and $\frac{1}{q} + \frac{1}{p} = 1$. Take $x \in l^p$.

$$\begin{aligned} \implies (T^* y^*)(x) &= y^*(Tx) = y^*((x_{i+1})_i) \\ &= \sum_{i=1}^\infty y_i^*(x_{i+1}) = \sum_{i=1}^\infty \tilde{y}_i x_i \text{ where } \tilde{y}_i = y_{i-1}^* \text{ or } 0 \\ \implies \tilde{y}^* &:= (\tilde{y}_i) \end{aligned}$$

$Ty^* = \tilde{y}^*$

$$\implies T^*(y_1, y_2, \dots) = (0, y_1, y_2, \dots)$$

↓ This lecture took place on 2019/06/14.

Theorem 7.12. Let X, Y, Z be normed spaces.

1. $T : \mathcal{L}(X, Y) \rightarrow \mathcal{L}(Y^*, X^*)$ with $T \mapsto T^*$ is linear and isometric.
2. $T \in \mathcal{L}(X, Y), S \in \mathcal{L}(Y, Z) \implies (S \circ T)^* = T^* \circ S^*$
3. $T \in \mathcal{L}(X, Y) \implies T^{**} \circ i_X = i_Y \circ T$

Proof. Isometric property: We already know that $\|T^*\| \leq \|T\|$.

$$\begin{aligned} \|T\| &= \sup_{\|x\| \leq 1} \|Tx\| \\ &= \sup_{\|x\| \leq 1} \sup_{\|y^*\| \leq 1} |y^*Tx| \\ &= \sup_{\|y^*\| \leq 1} \sup_{\|x\| \leq 1} |y^*Tx| \\ &= \sup_{\|y^*\| \leq 1} \sup_{\|x\| \leq 1} |T^*y^*(x)| \\ &= \sup_{\|y^*\| \leq 1} \|T^*y^*\| \\ &= \|T^*\| \end{aligned}$$

The remaining parts are left as an exercise. □

Theorem 7.13. Let X, Y be Banach. $T \in \mathcal{L}(X, Y)$. TFAE:

1. $\text{kernel}(T) = (\text{range}(T^*))^\perp$
2. $\text{kernel}(T^*) = (\text{range}(T))^\perp$

$$3. (\ker(T))^\perp \supseteq \overline{\operatorname{range}(T^*)}$$

$$4. (\ker(T^*))^\perp = \overline{\operatorname{range}(T)}$$

(1) and (2) relates injectivity and surjectivity of T and T^* .

Proof. Corollary of a previous results on $(G + L)^\perp$ etc. See book by Brezis (Corollary 7.18) \square

Theorem 7.14. Let X, Y be Banach. Let $T \in \mathcal{L}(X, Y)$. TFAE:

1. $\operatorname{range}(T)$ closed
2. $\operatorname{range}(T^*)$ closed
3. $\operatorname{range}(T) = \ker(T^*)^\perp$
4. $\operatorname{range}(T^*) = \ker(T)^\perp$

First, we need two lemmas.

Lemma 7.15. Let X and Y be Banach. $T \in \mathcal{L}(X, Y)$ such that $\exists c > 0 : c \|y^*\| \leq \|T^*y^*\| \forall y^* \in Y^*$. Then T is open, in particular surjective (see Remark before the Open Mapping Theorem).

Proof. It suffices to show that $B_C^Y(0) \subset T(B_1^X(0))$ for which it suffices to show that $B_C^Y \subset \overline{T(B_1^X)} := D$ [as in the proof of the open mapping theorem]. Take $y_0 \in B_C^Y$ such that $\|y_0\| < c$. If $y_0 \notin D \implies \exists y^* \in Y^*$, then

$$\exists y^* \in Y^* : \Re(y^*(y)) \leq x < \Re(y^*(y_0)) \leq |y^*(y_0)| \quad \forall y \in D$$

Since $0 \in D$ and $\pm iy \in D$ for $y \in D$ ($\tilde{y}^* = \frac{y^*}{\alpha}$, we know that $|y^*(y)| \leq 1 < |y^*(y_0)|$.

$$\implies \forall x \in X : \|x\| \leq 1 \quad |T^*y^*(x)| = |y^*(Tx)| \leq 1$$

$$\implies \|Ty^*\| \leq 1$$

but on the other hand, $1 < |y^*(y_0)| \leq \|y^*\| \|y_0\| < c \|y^*\|$ contradicts $c \|y^*\| \leq \|T^*y^*\|$. \square

Lemma 7.16. Let X, Y be Banach and $T \in \mathcal{L}(X, Y)$ such that $\operatorname{range}(T)$ is closed. Thus

$$\exists c > 0 \forall y \in \operatorname{range}(T) \exists x \in X : Tx = y \text{ and } \|x\| \leq c \|y\|$$

Informally, $\|T^{-1}y\| \leq c \|y\|$.

Proof. True by corollary of the open mapping theorem. Consider $\tilde{T} : X \setminus \ker(T) \rightarrow \operatorname{range}(T)$ bijective between Banach spaces. \square

Proof of theorem 7.14. The equivalence of statement (1) and statement (3) follows from Theorem 7.13 (4).

We prove that (4) follows from (1).

$$\text{range}(T^*) \subseteq \overline{\text{range}(T^*)} \stackrel{\text{Theorem 7.13}}{\subseteq} (\text{kernel}(T))^\perp$$

⊃ Take $x^* \in \text{kernel}(T)^\perp$. Find $y^* : T^*y^* = x^*$. Consider $z^* : \text{range}(T) \rightarrow \mathbb{K}$ with $y \mapsto x^*(x)$ with $Tx = y$.

Well-defined Assume $Tx_1 = Tx_2 \implies x_1 - x_2 \in \text{kernel}(T)$. Hence

$$\implies x^*(x_1) = x^*(x_1 - (x_1 - x_2)) = x^*(x_2)$$

Linear Continuous. Take $y \in \text{range}(T)$. $|z^*(y)| = |x^*(x)|$ with x as in Lemma 7.16

$$\implies |x^*(x)| < \|x^*\| \|x\| \leq \|x^*\| c \|Tx\| = c \|x^*\| \|y\|$$

Take y^* to be a Hahn-Banach extension at z^*

$$\implies \forall x \in X : x^*(x) = z^*(Tx) = y^*(Tx) = T^*y^*(x)$$

$$\implies x^* = T^*y^*$$

Proof statement (2) using (4), trivial since U^\perp is closed. To prove statement (1) using (2), assume $\text{range}(T^*)$ is closed. Define $Z = \overline{\text{range}(T)}$ and $S \in \mathcal{L}(X, Z)$, $Sx := Tx$.

Idea: Show S is surjective.

To this aim, show that $\text{range}(S^*)$ is closed. For $y^* \in Y^*, x \in X$, we have that $T^*y^*(x) = y^*(Tx) = y^*|_Z(Tx) = [S^*(y^*|_Z)](x)$. So, $T^*y^* = S^*(y^*|_Z) \implies \text{range}(T^*) \subset \text{range}(S^*)$ (why?). But also conversely, for $S^*z^* \in \text{range}(S^*)$ and y^* is a Hahn-Banach extension of z^* .

$$\implies T^*y^* = S^*(y^*|_Z) = S^*z^* \implies \text{range}(T^*) = \text{range}(S^*)$$

By assumption, $\text{range}(S^*)$ is closed. Also, S^* is injective. Since $\text{kernel}(S^*) = \text{range}(S)^\perp = \{0\}$ by Proposition 7.13. Hence, S^* is bijective from Z^* to $\text{range}(S^*)$, i.e. between B-spaces.

Open mapping implies

$$\exists c > 0 : \|z^*\| \leq c \|S^*z^*\| \quad \forall z^* \in Z^*$$

By Lemma 7.15, S is surjective, thus $\text{range}(T) = \text{range}(S) = Z = \overline{\text{range}(T)}$.

□

Refer to the book by Brezis to study consequences of this Lemma.

Corollary 7.17. Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X, Y)$. Then

- T is bijective if and only if T^* is bijective and

- T is isometry if and only if T^* is isometry.

Proof. This is a consequence of Theorem 7.14 and Theorem 7.13 and of the fact that $\|T\| = \|T^*\|$ and $\|T\| = 1 \iff T$ is an isometry (proof is left as an exercise to the reader). \square

Corollary 7.18. *Let X, Y be Banach spaces. $T \in \mathcal{L}(X, Y)$ an isomorphism. Then X is reflexive iff Y is reflexive.*

In particular, $i_X(X)$ is reflexive iff X is reflexive.

Proof. Without loss of generality, assume that X is reflexive. T is isomorphic, thus T^* is isomorphic, thus T^{**} is isomorphic. Also, $T^{**} \circ i_X = i_Y \circ T$. Hence i_Y is bijective if i_X is bijective. Thus Y is reflexive. \square

8 Hilbert spaces

Definition 8.1. *Let X be a vector space. A mapping $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$ is called inner (or scalar) product if*

1. $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle \forall x_1, x_2, y \in X$
2. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \forall x, y \in X, \lambda \in \mathbb{K}$
3. $\langle x, y \rangle = \overline{\langle y, x \rangle} \forall x, y \in X$ ($\langle x, x \rangle \in \mathbb{R}$)
4. $\langle x, x \rangle \geq 0$
5. $\langle x, x \rangle = 0 \iff x = 0$

Remark. *Consequences:*

- $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle \forall x, y_1, y_2 \in X$
- $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle \forall x, y \in X, \lambda \in \mathbb{K}$

Proposition 8.2. *Let X be a inner product space. Then*

1. $|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \forall x, y \in X$ [Cauchy-Schwarz inequality]
Equality is given iff $\exists \lambda \in \mathbb{K} : x = \lambda y$ or $y = \lambda x$
2. The mapping $x \mapsto \|x\| := \sqrt{\langle x, x \rangle}$ is a norm and $|\langle x, y \rangle| \leq \|x\| \|y\| \forall x, y \in X$

Proof. Compare with Linear Algebra \square

Definition 8.3. *A normed space $(X, \|\cdot\|_X)$ is called inner product space if \exists an inner product $\langle \cdot, \cdot \rangle$ such that $\|x\|_X = \sqrt{\langle x, x \rangle}$. A Hilbert space is a complete inner product space.*

Remark (Example). Consider $L^2(\Omega, \mathbb{K}^m, \mu)$ with $(f, g) := \int_{\Omega} f \cdot \bar{g} d\mu$.

$$\sqrt{\langle f, f \rangle} = \sqrt{\int_{\Omega} |f|^2 d\mu} = \|f\|_{L^2}$$

L^2 is a typical example of an inner product space. L^2 admits H^m for $m \in \mathbb{N}$ (by definition of an inner product) discussed in courses like Advanced Functional Analysis.

Remark (Note). $x \mapsto \langle x, y \rangle \in X^*$ (see later).

Lemma 8.4. Let X be an inner product space, $U \subset X$ is a dense subspace such that $\langle x, y \rangle = 0 \forall y \in U$ implies that $x = 0$

Proof. Define $Y = \{y \in X \mid \langle x, y \rangle = 0\}$ for x fixed such that $\langle x, u \rangle = 0 \forall u \in U$. $U \subset Y$ and Y is closed $\implies X = \overline{U} \subset X \rightarrow Y = X$.

$$\implies x \in Y \implies \langle x, x \rangle = 0 \implies x = 0$$

□

Lemma 8.5. If X is an inner product space, then

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right) \text{ if } \mathbb{K} = \mathbb{R}$$

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \right)$$

i.e. The inner product space can be expressed via the norm.

Proof. Compare with the book by Werner (direct computation)

□

Proposition 8.6 (Parallelogram law). Let $(X, \|\cdot\|)$ is an inner product space iff $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof. \implies direct computation

\Leftarrow Define $\langle \cdot, \cdot \rangle$ as in Proposition 8.5 + computation (compare with the book by Werner)

□

Lemma 8.7. Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space. Then $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{K}$ is continuous.

Proof.

$$\begin{aligned} \forall (x_1, y_1), (x_2, y_2) \in X \times X : |\langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle| &= |\langle x_1 - x_2, y_1 \rangle + \langle x_2, y_1 - y_2 \rangle| \\ &\leq \|x_1 - x_2\| \|y_1\| + \|x_2\| \|y_1 - y_2\| \end{aligned}$$

□

↓ This lecture took place on 2019/06/25.

Revision. $(X, \|\cdot\|)$ is inner product space iff $\forall x, y \in X : \|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Proposition 8.8. Let X be a normed space. Then

1. X is an inner product space iff every 2-dimensional subspace of X is an inner product space
2. Subspaces of inner product spaces are inner product spaces with the same inner product
3. The completion of an inner product space is a Hilbert space.

Proof. 1. Lemma 8.7
2. Restrict inner product
3. Follows from continuity of inner product.

□

8.1 Orthogonality

Definition 8.9. Let X be an inner product space.

- For $x, y \in X$, we write $x \perp y$ (“ x is orthogonal to y ”) $\iff \langle x, y \rangle = 0$
- For $A, B \subseteq X$, we write $A \perp B$ (“ A is orthogonal to B ”) $\iff x \perp y \forall x \in A, y \in B$
- For $A \subseteq X$, we define the orthogonal complement of A .

$$A^\perp = \{y \in X \mid x \perp y \forall x \in A\}$$

Remark. We will see later that this is consistent with $A^\perp \subset X^*$

Proposition 8.10. Let X be an inner product space. Then

1. If $x, y \in X : x \perp y \implies \|x\|^2 + \|y\|^2 = \|x + y\|^2$
2. $\forall A \subseteq X : A^\perp$ is a closed subspace of X
3. $\overline{A} \subset (A^\perp)^\perp$ for any $A \subseteq X$
4. $A^\perp = \overline{\mathcal{L}(A)}^\perp$

Proof. Just some calculations. Left as an exercise.

□

Theorem 8.11 (Major result). *Let H be a Hilbert spaces. $K \subset H$ be closed and convex. Then $\forall x_0 \in H \exists ! x \in K : \|x - x_0\| = \inf_{y \in K} \|y - x_0\|$*

Proof. Take $x_0 \in H$. If $x_0 \in K$.

Now without loss of generality assume $x_0 = 0$. This is valid, because otherwise apply the result to 0 and $k - \{x_0\}$.

$$\begin{aligned} \implies \exists ! z \in K - \{x_0\} : \|z - 0\| &= \inf_{y \in K - \{x_0\}} \|y\| \\ \implies \exists ! \hat{z} \in K : \|\hat{z} - x_0\| &= \inf_{y \in K} \|y - x_0\| \end{aligned}$$

Let $d := \inf \{\|y\| \mid y \in K\}$. Show $\exists y : \|y\| = d$.

Let $(y_n) \in K$. $d = \lim \{y_n\}$ is possible.

Show $(y_n)_n$ is Cauchy. We have (by Proposition 8.6):

$$\begin{aligned} \forall n, m \in \mathbb{N} : \left\| \frac{y_n + y_m}{2} \right\|^2 + \left\| \frac{y_n - y_m}{2} \right\|^2 &= \frac{1}{2} (\|y_n\|^2 + \|y_m\|^2) \xrightarrow{d^2 \text{ as } n, m \rightarrow \infty} d^2 \\ \frac{y_n + y_m}{2} &\in K \text{ (since } K \text{ is convex)} \\ \implies \left\| \frac{y_n + y_m}{2} \right\|^n &\geq d^2 \end{aligned}$$

$$\begin{aligned} \implies 0 &\leq \left\| \frac{y_n + y_m}{2} \right\|^2 + \left\| \frac{y_n - y_m}{2} \right\|^2 - d^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty \\ &= \left\| \frac{y_n - y_m}{2} \right\|^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty \implies (y_n)_n \text{ is Cauchy} \end{aligned}$$

K closed, $\implies \exists y \in K : y = \lim_{n \rightarrow \infty} y_k \implies \|y\| = \lim_{n \rightarrow \infty} \|y_n\| = d$

What about uniqueness? Let $x, \hat{x} \in K$ be such that $\|x\| = \|\hat{x}\| = d$ and assume $x \neq \hat{x}$.

$$\implies \left\| \frac{x + \tilde{x}}{2} \right\|^2 < \left\| \frac{x + \tilde{x}}{2} \right\|^2 + \left\| \frac{x - \tilde{x}}{2} \right\|^2 = \frac{1}{2}(d^2 + d^2) = d^2$$

gives a contradiction and thus uniqueness is given. \square

Proposition 8.12. *Let A be Hilbert and $K \subset H$ be closed and convex. $x_0 \in H$. TFAE:*

1. $\|x_0 - x\| = \inf_{y \in K} \|x_0 - y\|$
2. $\Re(x_0 - x, y - x) \leq 0 \forall y \in K$

Proof. (2) \rightarrow (1) Take $y \in K$.

$$\begin{aligned} \implies \|x_0 - y\|^2 &= \|x_0 - x + (x - y)\|^2 \\ &= \|x_0 - x\|^2 + 2\Re(x_0 - x, x - y) + \underbrace{\|x - y\|^2}_{\geq 0} \\ &\geq \|x_0 - x\|^2 \end{aligned}$$

(1) \rightarrow (2) Take $y \in K$ and for $t \in (0, 1]$, let $y_t := (1 - t)x + ty$ with $x \in K$ such that (1) holds.

$$y_t \in K \forall t \in (0, 1] \implies \|x_0 - x\|^2 < \|x_0 - y_t\|^2$$

$$\begin{aligned} \|x_0 - y_t\|^2 &= \langle x_0 - x + t(x - y), x_0 - x + t(x - y) \rangle \\ &= \|x_0 - x\|^2 + 2\Re \langle x_0 - x, t(x - y) \rangle + \|t(x - y)\|^2 \\ &\implies \Re \langle x_0 - x, y - x \rangle \leq \frac{t}{2} \|x - y\|^2 \forall t \in (0, 1] \end{aligned}$$

$$\text{Taking } t \rightarrow 0 \implies \Re \langle x_0 - x, y - x \rangle \leq 0$$

□

Proposition 8.13. *Let H be Hilbert, $K \subset H$ be closed and convex. Define $P_K : H \rightarrow H$ with $x \mapsto \operatorname{argmin}_{y \in K} \|x - y\|$. Then P_K is well-defined, a projection and Lipschitz continuous with Lipschitz constant 1.*

Proof. Well-definedness property and projection are trivial. To prove Lipschitz continuity, take $x_1, x_2 \in H$ and let $y_1 = P_K x_1$ and $y_2 = P_K x_2$.

$$\begin{aligned} &\implies \Re \langle x_1 - y_1, z - y_1 \rangle \leq 0 \forall z \in K \\ &\Re \langle x_2 - y_2, z - y_2 \rangle \leq 0 \forall z \in K \\ z = y_2 &\implies \Re \langle x_1 - y_1, y_2 - y_1 \rangle \leq 0 \\ z = y_1 &\implies \Re \langle x_2 - y_2, y_1 - y_2 \rangle \leq 0 \\ \|y_1 - y_2\|^2 &= \langle y_1 - y_2, y_1 - y_2 \rangle = \langle y_1, y_1 - y_2 \rangle - \langle y_2, y_1 - y_2 \rangle \\ &= \langle y_1 - x_1, y_1 - y_2 \rangle + \langle x_1, y_1 - y_2 \rangle + \langle x_2 - y_2, x_1 - y_1 \rangle - \langle x_2, y_1 - y_2 \rangle \\ &= \langle x_1 - y_1, y_2 - y_1 \rangle + \langle x_2 - y_2, y_1 - y_2 \rangle + \langle x_1 - x_2, y_1 - y_2 \rangle \\ &= \Re(\dots) + \Re \langle x_1 - x_2, y_1 - y_2 \rangle \\ &\leq \Re \langle x_1 - x_2, y_1 - y_2 \rangle \leq \|x_1 - x_2\| \|y_1 - y_2\| \end{aligned}$$

If $y_1 = y_2$ then done. Else $\|y_1 - y_2\| \leq \|x_1 - x_2\|$ then done. □

Proposition 8.14. *Let H be a Hilbert space. Let $U \subset H$ be a closed subspace and P_K as in Proposition 8.13. Then:*

1. $y = P_K(x) \iff y - x \in U^\perp$
2. P_U is continuous, linear projection with $\|P_U\| = 1$
3. $\operatorname{kernel}(P_K) = U^\perp$, $\operatorname{range}(P_U) = U$. In particular $U \oplus U^\perp = H$.
4. $I - P_U$ is a continuous, linear projection on U^\perp and $\|I - P_U\| = 1$

Proof. 1.

$$\begin{aligned}
y = P_U x &\stackrel{8.12}{\iff} \Re \langle x - y, z - y \rangle \leq 0 \forall z \in U \\
&\stackrel{\hat{z}=z-y}{\iff} \Re \langle x - y, z \rangle = 0 \forall z \in U \\
&\stackrel{\hat{z}=iz}{\iff} \Re \langle x - y, z \rangle = 0 \forall z \in U \\
&\iff x - y \in U^\perp
\end{aligned}$$

2. It is only left to show linearity. Note that U^\perp is a subspace. Take $x_1, x_2 \in H$ and $\lambda \in \mathbb{K}$.

Show: $P_U(\lambda x_1 + x_2) = \lambda P_U(x_1) + P_U(x_2)$.

$$(\lambda x_1 - x_2) - (\lambda P_U(x_1) - P_U(x_2)) \in U^\perp$$

Then the equality to show follows from (1).

$$(\lambda x_1 - x_2) - (\lambda P_U(x_1) - P_U(x_2)) = \underbrace{\lambda(x_1 - P_U(x_1))}_{\in U^\perp} + \underbrace{(x_2 - P_U(x_2))}_{\in U^\perp} \in U^\perp$$

3.

$\text{range}(P_U) = U$ clear since $P_U(x) = x \forall x \in U$

$\text{kernel}(P_U) = U^\perp$ is true since by (1) $P_U(x) = 0 \iff 0 - x \in U^\perp$

$$\implies x \in U^\perp$$

$$\implies U \oplus U^\perp = K \text{ by previous results}$$

4. $I - P_U$ is continuous and linear.

$$(I - P_U)(x) = x - P_U(x) \in U^\perp \text{ by (1)}$$

$$x \in U^\perp \implies (I - P_U)(x) = x - 0 = x$$

$$\begin{aligned}
x_0 \in H, \|x_0\|^2 &= \|x_0 \pm P_U(x_0)\|^2 = \|x_0 - P_U(x_0)\|^2 + \|P_U(x_0)\|^2 \geq \|(I - P_U)(x_0)\|^2 \\
&\implies \|I - P_U\| \leq 1
\end{aligned}$$

is an equality since it is a projection.

□

Corollary 8.15. *Let H be a Hilbert space. $U \subset H$ is a subspace. Then $\overline{U} = (U^\perp)^\perp$.*

Proof. Consider $P_{\overline{U}} : H \rightarrow H$. Then $I - P_{\overline{U}} = P_{\overline{U}^\perp}$. Also

$$\begin{aligned}
\overline{U}^\perp &= U^\perp = \underbrace{I}_{=P_{\overline{U}}} - P_{\overline{U}^\perp} = P_{(\overline{U}^\perp)^\perp} \\
&\implies P_{\overline{U}} = P_{(\overline{U}^\perp)^\perp}
\end{aligned}$$

since $\forall x \in (\overline{U}^\perp)^\perp \Rightarrow x = P_{(\overline{U}^\perp)^\perp} x = P_{\overline{U}} x \subset \overline{U}$

$$\Rightarrow \overline{U} = \left(\overline{U}^\perp \right)^\perp = (U^\perp)^\perp$$

□

Theorem 8.16. *Let H be Hilbert. Then the mapping $T : H \rightarrow H^\perp$ with $y \mapsto \langle \cdot, y \rangle : H \rightarrow \mathbb{K}$ such that $x \mapsto \langle x, y \rangle$ is well-defined, conjugate linear (i.e. $T(\lambda y_1 + y_2) = \bar{\lambda} T y_1 + T y_2$), isometric and bijective.*

In other words: $\forall x^ \in H^* \exists! \hat{x} \in H : x^*(y) = \langle y, \hat{x} \rangle \forall y \in H$. In particular, the notation H^\perp is consistent (assuming that $H = H^\perp$).*

Proof. Conjugate linearity and well-definedness are trivial.

Isometry For $x, y \in K$,

$$\begin{aligned} |(Ty)(x)| &= |\langle x, y \rangle| \leq \|x\| \|y\| \\ &\Rightarrow \|Ty\| \leq \|y\| \\ y \neq 0 &\Rightarrow \|Ty\| \geq \left| Ty \left(\frac{y}{\|y\|} \right) \right| = \frac{1}{\|y\|} \|y\|^2 = \|y\| \\ &\Rightarrow \|Ty\| \geq \|y\| \\ &\Rightarrow \|Ty\| = \|y\| \\ &\Rightarrow T \text{ is injective} \end{aligned}$$

□

↓ This lecture took place on 2019/06/27.

TODO

Corollary 8.17. *TODO*

TODO

8.2 Orthogonal bases

Definition 8.18. *Let H be Hilbert. $S \subset H$ is called orthonormal system if*

1. $\forall e \in S : \|e\| = 1$
2. $\forall e, f \in S : \langle e, f \rangle_H = 0$

An orthonormal system T is called orthonormal basis [or complete orthonormal system] if $\forall S \subset T$ and T is orthonormal system. Thus $T = S$ is the “maximal element” (equivalent to unexpandable, linear system in Linear Algebra terminology).

Example. • $H = l^2$. $S = (e_n)_n$ where $e_n = (0, \dots, 0, 1, 0, \dots, 0)$ is 1 at index n .

• $H = L^2([0, 2\pi])$. Then

$$S = \left\{ x \mapsto \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ x \mapsto \frac{1}{\sqrt{\pi}} \cos(nx) \mid n \in \mathbb{N} \right\} \cup \left\{ x \mapsto \frac{1}{\sqrt{\pi}} \sin(nx) \mid n \in \mathbb{N} \right\}$$

is a (complete) orthonormal system.

↓ This lecture took place on 2019/06/28.

Proposition 8.19 (Gram-Schmid method). Let $(x_n)_{n=1}^\infty$ be linearly independent in H , a Hilbert space. Then $\exists S \subset H$ is an orthonormal system such that $\text{span } S = \text{span}(x_n : n \in \mathbb{N})$.

Proof. Define $e_1 = \frac{x_1}{\|x_1\|}$ and for $n \in \mathbb{N}$:

$$\begin{cases} f_{n+1} := x_{n+1} - \sum_{i=1}^n \langle x_{n+1}, e_i \rangle e_i \\ e_{n+1} = \frac{f_{n+1}}{\|f_{n+1}\|} \end{cases}$$

Proof by induction is left as an exercise. □

Proposition 8.20 (Bessel inequality). Let H be Hilbert. Let $(e_n)_n$ be an orthonormal system in H , $x \in H$. Then $\sum_{n=1}^\infty |\langle x, e_n \rangle|^2 \leq \|x\|^2$.

Proof. Let $x_N := x - \sum_{i=1}^N \langle x, e_i \rangle e_i \implies x_N \perp e_k \forall k \in \{1, \dots, N\}$.

$$\implies \|x\|^2 = \|x_N\|^2 + \left\| \sum_{i=1}^N \langle x, e_i \rangle e_i \right\|^2 = \|x_N\|^2 + \sum_{i=1}^N |\langle x, e_i \rangle|^2 \geq \sum_{i=1}^N |\langle x, e_i \rangle|^2$$

□

Lemma 8.21. Let $(e_n)_n$ be an orthonormal system in Hilbert space H . Then $\forall x, y \in H : \sum_{n=1}^\infty |\langle x, e_n \rangle \langle e_n, y \rangle| < \infty$.

Proof. Hölder's inequality for $p = 2$,

$$\sum_{n=1}^\infty |\langle x, e_n \rangle \langle e_n, y \rangle| \leq \sqrt{\sum_{n=1}^\infty |\langle x, e_n \rangle|^2} \sqrt{\sum_{n=1}^\infty |\langle e_n, y \rangle|^2} \leq \|x\| \|y\| < \infty$$

□

Lemma 8.22. *Let H be a Hilbert space. Let $S \subset H$ be an orth. system. Let $x \in H$. Then $S_x = \{e \in S \mid \langle x, e \rangle \neq 0\}$ is at most countable.*

Proof. For $n \in \mathbb{N}$, define $S_{x,n} = \{l \in S \mid |\langle x, l \rangle| \geq \frac{1}{n}\}$. Then $S_{x,n}$ is finite due to Bessel's inequality.

$$\implies S_x = \bigcup_{n \in \mathbb{N}} S_{x,n} \text{ is at most countable}$$

□

Definition 8.23 (Basis representation with uncountable systems). *Let X be a normed space. Let I be an index set and $x_i \in X \forall i \in I$. We say that $\sum_{i \in I} x_i$ converges unconditionally to $x \in X$ if*

- $I_0 := \{i \in I \mid x_i \neq 0\}$ is countable
- For each numbering $(n_i)_{i=1}^\infty$ of I_0

$$\implies \sum_{i=1}^\infty x_{n_i} = x$$

In this case, we write $\sum_{i \in I} x_i = x$.

Remark. • Motivation: Give meaning to

$$\sum_{i \in I} \langle x_i, e_i \rangle e_i = x \text{ for } (e_i)_{i \in I}$$

any orthonormal system.

- Point here: Convergence does not depend on ordering
- Remember: $\dim(X) < \infty$ and $(x_i)_{i \in I} = (x_i)_{i \in \mathbb{N}}$ Then unconditional convergence \iff absolute convergence (in general, absolute convergence implies only convergence).
- Note: In this chapter: Difference between $\sum_{i=1}^\infty x_i$ and $\sum_{i \in \mathbb{N}} x_i$.

Corollary 8.24 (Generalized Bessel inequality). *Let H be a Hilbert space $S \subset H$ on orth. system. Let $x \in H$. Then*

$$\sum_{e \in S} |\langle x, e \rangle|^2 \leq \|x\|^2$$

In particular, $\sum_{e \in S} |\langle x, e \rangle|^2$ is absolutely convergent.

Proof. Only countably many $\langle x, e \rangle$ are non-zero. For any counting $(n_i)_{i=1}^\infty$ of the non-zero coefficients, Bessel implies $\sum_{i=1}^\infty |\langle x, e_{n_i} \rangle|^2 \leq \|x\|^2$. □

Proposition 8.25. *Let H be a Hilbert space. Let $S \subset H$ be an orth. system. Then*

1. $\forall x \in H : \sum_{e \in S} \langle x, e \rangle e$ is unconditionally convenient.
2. With $P : X \mapsto \sum_{e \in S} \langle x, e \rangle e$ we have that $Px = \operatorname{argmin}_{y \in \operatorname{span}(S)} \|x - y\|$ is $P = \overline{P_{\operatorname{span}(S)}}$.

Proof. 1. Write $\{e \in S \mid \langle x, e \rangle \neq 0\}$ as $(e_{n_i})_{i \in \mathbb{N}}$ with $(n_i)_{i \in \mathbb{N}}$ some numbering.
First, show that $\sum_{i=1}^{\infty} \langle x, e_{n_i} \rangle e_{n_i}$ is Cauchy. Consider for $M, N \in \mathbb{N}$:

$$\left\| \sum_{i=N+1}^M \langle x, e_{n_i} \rangle e_{n_i} \right\|^2 = \sum_{i=N+1}^M |\langle x, e_{n_i} \rangle|^2 \rightarrow 0 \text{ as } N, M \rightarrow \infty$$

$$\implies \exists y := \lim_{N \rightarrow \infty} \sum_{i=1}^N \langle x, e_{n_i} \rangle e_{n_i}$$

Now for $\pi : \mathbb{N} \rightarrow \mathbb{N}$ TODO define $y_\pi := \lim_{N \rightarrow \infty} \langle x, e_{\pi(n_i)} \rangle e_{\pi(n_i)}$.

Show: $y_\pi = y$

Take $z \in H$. Then,

$$\begin{aligned} \langle y, z \rangle &:= \sum_{i=1}^{\infty} \langle x, e_{n_i} \rangle \langle e_{n_i}, z \rangle = \sum_{i=1}^{\infty} \langle x, e_{\pi(n_i)} \rangle \langle e_{\pi(n_i)}, z \rangle \\ &= (y_\pi, z) \\ &\implies y = y_\pi \end{aligned}$$

2. It suffices to show:

$$\left(x - \sum_{i=1}^{\infty} \langle x, e_{n_i} \rangle e_{n_i}, e \right) = 0 \quad \forall e \in S$$

Because then $x - \sum_{i=1}^{\infty} \langle x, e_{n_i} \rangle e_{n_i} \in \overline{\operatorname{span}(S)}^\perp \iff \sum_{i=1}^{\infty} \langle x, e_{n_i} \rangle e_{n_i} = P_{\overline{\operatorname{span}(S)}}(x)$. This true for both $e \notin \{e_{n_i} : i \in \mathbb{N}\}$ and also for $e = e_{n_{i_0}}$.

□

Proposition 8.26. Let H be a Hilbert space. Let $S \subset H$ be an ortho. system. Then:

1. there exists a orth. basis $\tilde{S} \subset H$ such that $S \subset \tilde{S}$ (in particular, any $H \neq \{0\}$ has orth. basis)
2. TFAE:
 - (a) S is orth. basis
 - (b) $\forall x \in H : x \perp S \implies x = 0$
 - (c) $H = \overline{\operatorname{span}(S)}$
 - (d) $\forall x \in H : x = \sum_{e \in S} \langle x, e \rangle e$
 - (e) $\langle x, y \rangle = \sum_{e \in S} \langle x, e \rangle \langle e, y \rangle$

$$(f) \|x\|^2 = \sum_{e \in S} |\langle x, e \rangle|^2 \text{ (Parseval's identity)}$$

Proof. 1. Application of Lemma of Zorn ($S_1 \subset S_2$ as ordering of bases)

2. We make a cyclic proof:

(1) \rightarrow (2) If not, i.e. $\exists x \neq 0$ such that $S \perp x \implies S \cup \left(\frac{x}{\|x\|}\right) := \bar{S}$ is an orth.

basis $S \subsetneq \bar{S}$. This gives a contradiction.

(2) \rightarrow (3) $\text{span}(S) = (S^\perp)^\perp$ but by (2): $S^\perp = \{0\}$. Hence $(S^\perp)^\perp = H$.

(3) \rightarrow (4) Proposition 8.25 since by (3) $Px = x \forall x \in X$

(4) \rightarrow (5) Follows from testing with y

(5) \rightarrow (6) Let $y = x$

(6) \rightarrow (1) If (1) is not true, $\exists x \in H : \|x\| = 1$ and $x \perp S$.

$$\implies 1 = \|x\|^2 \stackrel{(6)}{=} \sum_{e \in S} \underbrace{|\langle x, e \rangle|^2}_{=0} = 0$$

gives a contradiction.

□

Corollary 8.27. *Let H be a Hilbert space. TFAE:*

1. H is separable
2. All orthonormal basis of H are countable
3. There exists a countable orth. basis

Proof. (1) \rightarrow (2) Let S be an orth. basis of $H \implies \forall e, f \in S : e \neq f$

$$\|e - f\|^2 = \|e\|^2 + \|f\|^2 = 2$$

$$\|e - f\| = \sqrt{2}$$

As in the proof of non-separability l^∞ , $L^\infty \implies S$ is countable.

(2) \rightarrow (3) immediate.

(3) \rightarrow (1) immediate, because if U is countable such that $\bar{U} = H \implies H$ is separable.

□

Proposition 8.28. *Let H be a separable Hilbert space, $\dim(H) = \infty$. Then $H \cong l^2$ [" H is isometrically equivalent to l^2 "]*

Proof. Define $T : H \rightarrow l^2$ with $x \mapsto (\langle x, e_n \rangle)_{n \in \mathbb{N}}$ when $(e_n)_n$ is an orthonormal basis of H . Then $\|Tx\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 = \|x\|^2 \implies T$ is isometric and injective. Thus T is linear.

For surjectivity, take $(\lambda_i)_{i \in \mathbb{N}}$ in l^1 . For $M, N \in \mathbb{N} : M > N : \left\| \sum_{i=N+1}^M \lambda_i e_i \right\|^2 = \sum_{i=N+1}^M |\lambda_i|^2 \rightarrow 0$ for $N, M \rightarrow \infty \implies \left(\sum_{i=1}^N \lambda_i e_i \right)_{N \in \mathbb{N}}$ is Cauchy and thus converges and hence $T \left(\sum_{i=1}^{\infty} \lambda_i e_i \right) = (\lambda_i)_{i \in \mathbb{N}}$. □

Index

- Absolutely convergent, 20
- Absorbing subspace, 48
- Adjacent point, 9
- Adjoint operator, 70
- Annihilators, 67

- Baire theorem, 19
- Banach space, 11
- Bessel inequality, 81
- Bidual space, 60
- Bounded sequence, 11

- Canonical embedding, 60
- Cauchy sequence, 11
- Closed operator, 54
- Cluster point, 9
- Co-dimension, 69
- Compactly supported, 28
- Compactness, 10
- Complete space, 11
- Continuity, 10
- Continuous function, 10
- Convergence in semi-normed spaces, 20
- Convergent sequence, 10

- Dense space, 10
- Domain, 27
- Dual operator, 70
- Dual space, 56

- Equicontinuous set, 16
- Equivalence class, 22

- First category, 19

- Hausdorff space, 10
- Hilbert spaces, 6
- Hyperplane, 47

- Interior point, 8
- Inverse problems, 2
- Isometric isomorphism, 25

- Limit, 10
- Lipschitz continuity, 16

- Metric, 7
- Metric space, 7

- Norm, 8
- Normed space, 8
- Nowhere dense, 19

- Open map, 52
- Orthogonal complement, 3
- Orthonormal basis, 81
- Orthonormal system, 80

- Parseval's identity, 84
- Projection, 67

- Quasicompactness, 10
- Quotient space, 22

- Reflexive space, 5, 60

- Second category, 19
- Semi-normed space, 20
- Seminorm, 20
- Separable space, 10
- Set of first category, 19
- Set of second category, 19
- Space of bounded linear functionals, 41
- Strong convergence, 62
- Sublinearity, 43

- T_2 space, 10
- Topological space, 7, 8
- Topology, 7

- Uniform continuity, 12

- Weak convergence, 5, 62
- Weak derivative, 6
- Weak gradient, 6
- Weak* convergence, 66