Linear Algebra 2 – Practicals

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Contents

1 Solution of the last lecture exam of Analysis 1

1.1 Exam: Exercise 1

Exercise 1. Determine the limes of
$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\dots$$
does not help us. What about this representation?

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1}+\frac{b}{n-1} = \frac{a(n-1)+b(n+1)}{(n+1)(n-1)}$$
$$a(n-1)+b(n+1)=1$$
$$(a+b)n+(b-a)=1$$
$$\Rightarrow a+b=0 \wedge b-a=1$$
$$\Rightarrow a=-\frac{1}{2} \quad b=\frac{1}{2}$$

Followingly,
$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1}\right)$$

Okay, how to proceed? Let's build a pre-factor:

$$\frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$
$$=\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\dots$$
$$=\frac{1}{1}+\frac{1}{2}=\frac{3}{2}$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m := \frac{1}{2}\sum_{n=2}^{m}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n-1} - \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n+1}$$

Please be aware that we explicitly define S_m because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$= \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n-1} - \frac{1}{2}\sum_{n=4}^{m+2}\frac{1}{n-1}$$
$$= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{m}+\frac{1}{m+1}\right)$$

We already know $\frac{1}{m} \xrightarrow{m\to\infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m\to\infty} 0$. Followingly also $\frac{1}{2}\left(\frac{1}{m}+\frac{1}{m+1}\right) \xrightarrow{m\to\infty} 0$.

1.2 Exam: Exercise 2

Exercise 2. A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n\in\mathbb{N}}$$
$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \qquad a_1 = \frac{3}{2} \qquad a_2 = \frac{5}{4} \qquad a_3\frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \qquad a_1 = 4 \qquad a_2 = \frac{5}{2} \qquad a_3\frac{7}{4}$$

Exercise 3. a) Show that $1 \overset{!}{<} a_n \overset{!}{\leq} a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over n to prove this statement:

induction base $1 < a_0 \leq a_0$ holds trivially.

induction step We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &\leq \frac{1}{2}(a_0 + a_0) && \text{[induction hypothesis and } 1 < a_0\text{]} \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &> \frac{1}{2}(1 + 1) && \text{[induction hypothesis]} \\ &= 1 \end{aligned}$$

Exercise 4. b) Prove that $a_{n+1} \overset{!}{<} a_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &< \frac{1}{2}(a_n + a_n) && \text{[we have proven: } a_n > 1\text{]} \end{aligned}$$

Exercise 5. c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$b_n \coloneqq a_n - 1 \qquad \forall n \in \mathbb{N}$$
$$b_0 \coloneqq a_0 - 1$$
$$b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2}b_n$$

$$b_n = \frac{1}{2^n} b_0 \to 0 \cdot b_0 = 0$$
$$\Rightarrow b_n \to 0$$
$$\Rightarrow a_n = b_n + 1 \to 1$$

Does it work to just show: $1 = \frac{1}{2}(1+1)$? Nope, because in points of continuity this might be true even though 1 is not its limes.

Let $a_n \to a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$a_{n+1} \to a \qquad \frac{1}{2}(a_n + 1) \to \frac{1}{2}(a+1) \qquad a = \frac{1}{2}(a+1)$$

1.3 Exam: Exercise 3

Exercise 6. $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an ε-δ-proof.

If we don't need an ε-δ-proof, we would argue with the Algebraic Continuity Theorem: The function f is a composition of continuous functions, hence a continuous function itself.

ε-δ-definition:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If $|x - x_0| < \delta$,

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)\right| \\
&= \left|2x^2 + 5x - 2x_0^2 - 5x_0\right| \\
&\leq 2\left|x^2 - x_0^2\right| + 5|x - x_0| \\
&= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\
&= 2|x + x_0||x - x_0| + 5|x - x_0| \\
&\leq 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\
&\leq 2(|x_0| + \delta + |x_0|) + 5\delta
\end{aligned}$$

Our goal: we are able to claim $\overset{!}{<} \varepsilon$

$$\begin{aligned}
&= 4|x_0|\delta + 2\delta^2 + 5\delta \\
&= 2\delta^2 + (4|x_0| + 5)\delta
\end{aligned}$$

In general (here it does not apply), that x_0 might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$2\delta^2 < \frac{\varepsilon}{2}$$

$$\delta < \frac{\sqrt{\varepsilon}}{2}$$

$$(4\,|\,x_0\,| + 5)\delta < \varepsilon$$

$$\delta < \frac{\varepsilon}{4\,|\,x_0\,| + 5}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{5}, \frac{\varepsilon}{4|\,x_0\,|+6}\right)$. Then the ε-δ definition shows that f is continuous.

2 Exam: Exercise 4

Exercise 7. Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$h(0) = f(0) - f\left(\frac{1}{2}\right)$$
$$h(1) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0)$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x + \frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore h is continuous, because it is a composition of continuous functions.

Case 1: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\Rightarrow f(\xi) = f(\xi + \frac{1}{2})$$

Case 2: $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

Case 3: $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

3 Exercise 1

Exercise 8. Investigate the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(x\,|\,x\,|+x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

$$f'(x) = \begin{cases} 0 & x \le 0 \\ 2x & x > 0 \end{cases}$$

So this is differentiable, but in case of $x = 0$, it remains questionable.

We look at the definition of differentiability:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

$$f'(x) = \begin{cases} \lim_{x\to 0} \frac{0}{x} = 0 \\ \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0 \end{cases}$$

It follows that f is differentiable one time.

$$f''(x) = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

What about $x = 0$?

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} \begin{cases} \lim_{x\to 0} \frac{0}{x} = 0 \\ \lim_{x\to 0^+} \frac{2x}{x} = \lim_{x\to 0^+} 2 = 2 \end{cases}$$

Left and right limes differ. So it is not differentiable.

4 Exercise 2

Exercise 9. Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x\to 1} \frac{\ln x}{x-1}$
2. $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1-\sin x)}$
4. $\lim_{x\to 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4.1 Exercise 2.a

$$\lim_{x\to 1} \frac{\ln x}{x-1}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

4.2 Exercise 2.b

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

A nice hint to find out whether this function is differentiable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{\sin x - x}{x\sin x} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}} \approx x \to 0$$

This exploits, that it will take one run of L'Hôpital's rule (because each expression has at least degree 2) and its limes will be 0 (because of x).

4.3 Exercise 2.c

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\cos(x))}{\ln(1-\sin(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied. We claim that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = \infty$ is fine.

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\cos(x)}{1-\sin(x)}} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\sin(x)\cdot(1-\sin(x))}{\cos(x)(-\cos(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied.

$$\lim_{x\to\frac{\pi}{2}^-} \frac{-\cos(x)(1-\sin(x)) - \sin(x)\cdot(-\cos(x))}{-\sin(x)(-\cos(x)) + \cos(x)\cdot\sin(x)} = \frac{1}{2}$$

If we want to apply the previous estimate here, we should consider

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos(y) \qquad y = \frac{\pi}{2} - x$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(y)$$

This gives us a different estimate of the result:

$$\lim_{y\to 0^+} \frac{\ln(\sin(y))}{\ln(1-\cos(y))} \approx \lim_{y\to 0^+} \frac{\ln(y)}{\ln\left(\frac{y^2}{2}\right)} = \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y) - \ln(2)} \approx \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y)} = \frac{1}{2}$$

We define neighborhoods:

$$N_\delta(x_0) = \{x : |x - x_0| < \delta\}$$

$$N_R(\infty) = \{x : x > R\}$$

4.4 Exercise 2.d

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}} = \lim_{x\to 1^-} e^{\ln(x)\frac{1}{1-x}} = \exp\left(\lim_{x\to 1^-} \underbrace{\frac{\ln(x)}{1-x}}_{(-1)\cdot\text{Exercise a}}\right) = \frac{1}{e}$$

4.5 Exercise 2.e

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\left(\exp\left(\frac{\ln n}{\sqrt{n}}\right)\right) = \exp\left(\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}\right)$$

The conditions to apply L'Hôpital's rule are satisfied („$\frac{\infty}{\infty}$")

$$\exp\left(\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}\right) = \exp\left(\lim_{n\to\infty}\frac{2\sqrt{n}}{n}\right) = \exp(0) = 1$$

4.6 Exercise 2.f

$$\lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{n\to\infty}\frac{e^x\left(1-e^{-2x}\right)}{e^x\left(1+e^{-2x}\right)} = \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{1}{e^{2x}}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{e^{2x}}}$$

Remark:

$$\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\cosh(x)}{\sinh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}$$

$$y = \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} = \frac{1}{\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}} = \frac{1}{y}$$

5 Exercise 3

Exercise 10. Show that the function $f:\mathbb{R}\to\mathbb{R}$ with $x\mapsto x+e^x$ is bijective. Furthermore determine $(f^{-1})'(1)$ and $\lim_{y\to\infty}(f^{-1})'(y)$.

If the function is strictly monotonically increasing, it is injective.

$$f'(x) = 1 + e^x > 0 \qquad \forall x\in\mathbb{R}$$

We show that it is strictly monotonically increasing:

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\alpha) \qquad \text{with } \alpha\in(x_1,x_2)$$

$$f(x_2)-f(x_1) = f'(\alpha)(x_2-x_1) > 0$$

Is f surjective?

For an arbitrary $y_0\in\mathbb{R}$ it holds that $\exists x_0\in\mathbb{R} : f(x_0) = y_0$:

$$\exists f(a), f(b)\in\mathbb{R} : f(a) \le y_0 < f(b)$$

It holds that

$$\lim_{x\to-\infty} x + \underbrace{e^x}_{\to 0} = -\infty$$

$$\lim_{x\to+\infty} x + e^x = \infty$$

Formally:

$$\forall y_0 \exists x_0 : \forall x < x_0 : f(x) < y_0$$

From the Intermediate Value Theorem it follows that

$$\Rightarrow \exists c \in [a,b) : f(c) = y_0 \qquad c =: x_0$$

So it is surjective.

From injectivity and surjectivity it follows that it is bijective.

5.1 Determine $(f^{-1})'(1)$

$$f(x) = x + e^x$$
$$f'(x) = 1 + e^x$$

We apply the inverse function theorem:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$y = 1 = f(x)$$
$$x = f^{-1}(1)$$

An educated guess gives us that $x = 0$. In general determining x is more difficult.

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$$

5.2 Determine $\lim_{y\to\infty} \left(f^{-1}\right)'(y)$

$$\lim_{y\to\infty} \left(f^{-1}\right)'(y) = \lim_{y\to\infty} \frac{1}{1+e^x}$$

As x grows to infinity, also y grows to infinity. From bijectivity it follows that any value can be reached with x as well as $f(x)$.

$$\underbrace{f'(\underbrace{f^{-1}(\underbrace{y}_{\to\infty})}_{\to\infty})}_{\to\infty}$$

6 Exercise 4

Exercise 11. Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. Show

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2} = f'(x_0)$$

$$
\begin{aligned}
&= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0) + f(x_0) - f(x_0-h)}{2h} \\
&= \lim_{h' \to 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0) - f(x_0+h')}{-h'} \right) \\
&= \lim_{h' \to 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0+h') - f(x_0)}{h'} \right) \\
&= \frac{1}{2}(f'(x_0) + f'(x_0)) \\
&= f'(x_0)
\end{aligned}
$$

6.1 Exercise 4.b

$$
\lim_{h \to 0} \frac{f(x_0+rh) - f(x_0+sh)}{h} = \lim_{h \to 0} \frac{f(x_0+rh) - f(x_0)}{h} + \lim_{h \to 0} \frac{f(x_0) - f(x_0+sh)}{h}
$$

$$
h_1 = rh \qquad h_2 = sh
$$

$$
\begin{aligned}
&= \lim_{h_1 \to 0} \frac{f(x_0+h_1) - f(x_0)}{\frac{1}{r} \cdot h_1} + \lim_{h_2 \to 0} \frac{f(x_0) - f(x_0+h_2)}{\frac{1}{s} \cdot h_2} \\
&= r \cdot f'(x_0) - s \cdot f'(x_0) \\
&= (r-s) \cdot f'(x_0)
\end{aligned}
$$

7 Exercise 5

Exercise 12. Let $D \subseteq \mathbb{R}$ be an open interval. $f : D \to \mathbb{R}$ is differentiable and f is twice differentiable in $x_0 \in D$.

7.1 Exercise 5.a

Exercise 13. Show that

$$
\lim_{h \to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0)
$$

f is differentiable, therefore continuous, and h goes to 0. So we have „$\frac{0}{0}$". All conditions to apply L'Hôpital's rule are satisfied.

$$
\lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} \approx \frac{0}{0}
$$

We can apply L'Hôpital's Rule again or just use the result of exercise 4a.

$$
\stackrel{4a}{\Longrightarrow} f''(x_0)
$$

7.2 Exercise 5.b

Exercise 14. Show that the limes from exercise 5.a can also exist, even if $f''(x_0)$ does not exist. Use the result from Exercise 1.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

We know that it is not twice differentiable. But we want to show that the limes exists.

We are only concerned with $x = 0$.

$$\lim_{h \to 0} f(x_0) = 0$$

$$\lim_{h \to 0} \frac{h^2 - h^2}{h^2} = \frac{0}{h^2} = 0$$

So if we traverse the graph from both sides at the same time $\frac{f(x_0+h)-f(x_0-h)}{h}$.

8 Exercise 6

Exercise 15. Determine the following limit for arbitrary $c \in \mathbb{R}$:

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right).$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right)$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right) = \lim_{n \to \infty} \frac{e^{\frac{c}{n} \cdot \ln n} - 1}{\frac{\ln n}{n}}$$

and

$$\left(e^{\frac{c}{n} \cdot \ln n}\right)' = e^{\frac{c}{n} \cdot \ln n} \cdot \left(-\frac{c}{n^2} \cdot \ln n + \frac{c}{n} \cdot \frac{1}{n} \right) = \frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln(n))$$

All conditions are satisfied to apply L'Hôpital's rule ("$\frac{0}{0}$"):

$$\lim_{n \to \infty} \frac{\frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln n)}{\frac{\frac{1}{n} \cdot n - \ln n}{n^2}}$$

$$= \lim_{n \to \infty} \frac{c \cdot e^{\frac{c}{n} \cdot \ln n}(1 - \ln(n))}{1 - \ln n} = \lim_{n \to \infty} c \cdot e^{\frac{c}{n} \cdot \ln n} = c \cdot 1$$

9 Exercise 7

Exercise 16.
- Show that $e^x \geq 1 + x$ holds for all $x \in \mathbb{R}$.
  Hint: On demand, use the Mean Value Theorem.
- Prove that for all $x > 0$, the following estimates hold:

  $$\ln x \leq x - 1$$

  and for all $k \in \mathbb{N}_+$ it holds that

  $$k \left(1 - \frac{1}{\sqrt[k]{x}} \right) \leq \ln x \leq k \left(\sqrt[k]{x} - 1 \right)$$

$x \geq 0$ Choose $f(x) = e^x$ in $[0, x)$. Mean value theorem:

$$\exists x_0 : f'(x_0) = \frac{f(b) - f(a)}{b - a} \qquad \text{for } a < x_0 < b$$

$$f'(x_0) = e^{x_0} \qquad e^{x_0} \geq 1 \qquad x_0 \geq 0$$

$$e^{x_0} = \frac{f'(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x} \Rightarrow \frac{e^x - 1}{x} \geq 1$$

Or alternatively: f is convex and therefoer $f''(x) > 0$.

Consider $f(x) = x - 1 - \ln x$

$$f'(x) = 1 - \frac{1}{x} \qquad f''(x) = \frac{1}{x^2}$$

$$f'(x) \stackrel{!}{=} 0$$

$$1 - \frac{1}{x} = 0 \Leftrightarrow x = -1$$

$$f''(1) = 1 > 0 \Rightarrow \text{ minimum and because } f(1) = 0 \Rightarrow \forall x : x - 1 - \ln x \geq 0$$

Or alternatively:

$$y := x - 1$$

$$x = y + 1$$

Show that $\ln(y + 1) \leq y \Leftrightarrow y + 1 \leq e^y$.

e^x is monotonically increasing $\Rightarrow x \leq y \Leftrightarrow e^x \leq e^y$.

And this has been proven previously.

9.1 Exercise 7.b

$$\ln(x) \leq k \left(\frac{[}{k}]x - 1 \right)$$

$$\ln(\sqrt[k]{x}) \leq \sqrt[k]{x} - 1 \Leftrightarrow \ln(y) \leq y - 1$$

And this has been proven in Exercise a.

The second part following analogously.

10 Exercise 8

Exercise 17. Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Show: If f is continuous in an environment U of $a \in D$, differentiable in $U \setminus \{a\}$ and there exists $\lim_{x \to a} f'(x)$, such that f in a differentiable and

$$f'(a) = \lim_{x \to a} f'(x).$$

Hint: On demand, use the Mean Value Theorem.

Let h_n be an arbitrary zero-sequence (with $h_n(x) > 0 \quad \forall x \in D$) and due to Mean Value Theorem $\exists \xi_n \in D$ with $f'(\xi_n) = \frac{f(a+hn) - f(a)}{h_n}$.

$$\lim_{n \to \infty} f'(\xi_n) = \lim_{x \to a} f'(x) = \lim_{n \to \infty} \frac{f(a + h_n) - f(a)}{h_n} = f'(a)$$

$$\lim_{n \to \infty} \frac{f(a + h_n) - f(a)}{h_n} = \lim_{n \to \infty} f'(\xi_n) = \lim_{x \to a} f'(x) = z$$

For the arbitrary zero-sequence, we really need to consider it arbitrary (otherwise we just show it for the one sequence). Consider this counterexample:

$$f(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

10.1 Alternative approach

Application of "Schrankensatz".

$$\exists \lim f'(x) = \alpha$$

Hence for arbitrary $\varepsilon > 0 : \exists \delta > 0 \forall x \in (a-\delta, a+\delta) \setminus \{a\} : |f'(x) - \alpha| < \varepsilon$. Hence $\alpha - \varepsilon < f'(x) < \alpha + \varepsilon$.

- $$\forall x \in (a, a+\delta) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$
- $$\forall x \in (a-\delta, a) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall x \in (a-\delta, a+\delta) \setminus \{a\} : \left| \frac{f(x) - f(a)}{x - a} - \alpha \right| \leq \varepsilon$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha$$

10.2 Second alternative approach

$$\lim_{f(a+h)-f(a)} h$$

If I know f is continuous, then $f(a+h) \to f(a)$. So,

$$\text{“}\frac{0}{0}\text{”}$$

$$\lim_{h \to 0} \frac{f'(a+h) - 0}{1} = \lim_{h \to 0} f'(a+h) = \lim_{x \to a} f'(x)$$

11 Exercise 9

Exercise 18. Let $f : [a,b] \to \mathbb{R}$, $a < b$, differentiable with $f(a) > 0$, $f'(a) > 0$ and $f(b) = 0$. Prove that there exists $\xi \in (a,b) : f'(\xi) = 0$.

First, we want to show that $f'(a) > 0 \Rightarrow \exists \delta > 0 \forall x \in (a, a+\delta) : f(x) > f(a)$.

$$\exists \delta > 0 \forall x \in (a, a+\delta) : \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0$$

$$\Rightarrow f(x) - f(a) > \frac{f'(a)}{2}(x - a) > 0$$

Indeed, $f(x)$ satisfies this property.

Secondly, we want to show that,

$$\exists \eta \in (a+\delta, b) : f(a) = f(\eta)$$

$$\exists \xi \in [a,\eta] \forall x_1 \in [a,\eta] : f(\xi) \geq f(x_1)$$
$$\exists \xi \in (a,\eta) : \frac{f(\eta) - f(a)}{\eta - a} = f'(\eta) = 0$$

There might be more than this one ξ, so the ξ between the second and third line might be different. Anyways, we found a ξ with the desired property.

12 Exercise 10

Exercise 19. Determine the pointwise limit of the following function sequences $f_n : [0,\infty) \to \mathbb{R}$ and determine its uniform convergence:

- $f_n(x) = \sqrt[n]{x}$
- $f_n(x) = \frac{1}{1+nx}$
- $f_n(x) = \frac{x}{1+nx}$

12.1 Exercise 10.a

If $x \neq 0$, $\lim_{n\to\infty} \sqrt[n]{x} = 1$.
If $x = 0$, $\lim_{n\to\infty} \sqrt[n]{x} = \lim_{n\to\infty} 0^{\frac{1}{n}} = 0$.

In terms of uniform convergence:

$$\left| \sqrt[n]{x} - 1 \right| < \varepsilon$$
$$\lim_{x\to\infty} \sqrt[n]{x} = \infty$$

Example:

$$\begin{aligned} \left| \sqrt[n]{x} - 1 \right| &< \varepsilon \\ \sqrt[n]{x} - 1 &< \varepsilon \\ \sqrt[n]{x} &< \varepsilon + 1 \\ \sqrt[n]{100} &< \varepsilon + 1 \end{aligned}$$

12.2 Exercise 10.b

$$f_n(x) = \frac{1}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{1}{1+nx} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{1}{1+n\cdot 0} = 1$$

Assume it it continuously convergent. Show that:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists x \in [0,\infty) : n \geq N \wedge |f_n(x) - f(x)| \geq \varepsilon$$

Does not hold for $\frac{9}{n} \geq x$.

12.3 Exercise 10.c

$$f_n(x) = \frac{x}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{x}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{x}+n} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{0}{1+n\cdot 0} = 0$$

$$\left|\frac{x}{1+nx} - 0\right| < \varepsilon$$

$$\left|\frac{x}{1+nx}\right| < \left|\frac{x}{nx}\right| = \left|\frac{1}{n}\right|$$

Convergence is given. Uniform convergence is not given.

Advice: The simplest approach to show convergence is to show:

$$|f_n(x) - f(x)| \leq a_n \to 0$$

where a_n is independent from x.

13 Exercise 11

Exercise 20. Determine $\cos\alpha$, $\sin\alpha$ and $\tan\alpha$ for $\alpha \in \left\{\frac{\pi}{5}, \frac{2\pi}{5}\right\}$.

Hint: Show that $u := \cos\frac{2\pi}{5}$ and $v := \cos\frac{\pi}{5}$ satisfy the equations $u = 2v^2 - 1$ and $-2u^2 + 1 = v$. Determine u, v this way.

$$\begin{aligned}
u = \cos\left(\frac{2\pi}{5}\right) &= \cos\left(\frac{\pi}{5}+\frac{\pi}{5}\right) \\
&= \cos^2\left(\frac{\pi}{5}\right) - \sin^2\left(\frac{\pi}{5}\right) \\
&= 2\cos^2\left(\frac{\pi}{5}\right) - 1 \\
&= 2v^2 - 1
\end{aligned}$$

To show: $v + 2u^2 - 1 = 0$, $\cos\left(\frac{\pi}{5}\right) + 2\cos^2\left(\frac{\pi}{5}\right) - 1 = 0$.

$$\begin{aligned}
\cos\left(\frac{\pi}{5}\right) + 2\cos\frac{2\pi}{5} - 1 &= \cos\frac{\pi}{5} + \cos\frac{4\pi}{5} \\
&= \cos\frac{\pi}{5} + \cos\left(\pi - \frac{1}{5}\pi\right) \\
&= \cos\frac{\pi}{5} + \cos\pi\cdot\cos\left(\frac{\pi}{5}\right) + \sin\pi\cdot\sin\frac{\pi}{5} - \cos\frac{\pi}{5}\cdot\cos\frac{\pi}{5} \\
&= 0
\end{aligned}$$

For $u + v > 0$:

$$2v^2 - 1 = u$$

$$-2u^2 + 1 = v$$

$$2v^2 - 2u^2 = u + v$$
$$2(v+u)(v-u) = u + v$$
$$2(v-u) = 1 \Leftrightarrow v - u = \frac{1}{2}$$

$$v - 2v^2 + \frac{1}{2} = 0$$
$$v^2 - \frac{1}{2}v - \frac{1}{4} = 0$$
$$v_{1,2} = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{4}{16}} = \frac{1 \pm \sqrt{5}}{4}$$

$$0 < \cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{4}$$
$$u = \cos\frac{2\pi}{5} = v - \frac{1}{2} = \frac{-1+\sqrt{5}}{4}$$

$$\cos\left(\frac{2\pi}{5}\right) = \cos^2\frac{\pi}{5} - \sin^2\frac{\pi}{5}$$
$$\Leftrightarrow \frac{-1+\sqrt{5}}{4} = \left(\frac{\sqrt{5}+1}{4}\right)^2 - \sin^2\left(\frac{\pi}{5}\right)$$
$$\Leftrightarrow \sin^2\left(\frac{\pi}{5}\right) = \frac{5+2\sqrt{5}+1}{16} - \frac{-4+4\sqrt{5}}{16}$$
$$= \frac{5+2\sqrt{5}+1+4-4\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16} = \frac{5-\sqrt{5}}{8}$$
$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}} \approx 0.59$$

$$\sin\frac{2\pi}{5} = \sin\left(\frac{\pi}{5}+\frac{\pi}{5}\right) = \sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\cdot\sin\frac{\pi}{5} = 2\sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5}$$
$$= 2\frac{1+\sqrt{5}}{4}\sqrt{\frac{5-\sqrt{5}}{8}} = \frac{1+\sqrt{5}}{2}\cdot\frac{5-\sqrt{5}}{8} = \sqrt{\frac{5+\sqrt{5}}{8}} \approx 0.95$$

$$\tan\frac{\pi}{5} = \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = \frac{\sqrt{\frac{5-\sqrt{5}}{8}}}{\frac{\sqrt{5}+1}{4}} = \frac{\sqrt{2(5-\sqrt{5})}}{1+\sqrt{5}} = \sqrt{5-2\sqrt{5}} \approx 0.73$$
$$\tan\left(\frac{2\pi}{5}\right) = \frac{\sin\frac{2\pi}{5}}{\cos\frac{2\pi}{5}} = \frac{4}{-1+\sqrt{5}}\cdot\frac{1+\sqrt{5}}{2}\cdot\sqrt{\frac{5-\sqrt{5}}{8}} = \sqrt{5+2\sqrt{5}} \approx 3.05$$

14 Exercise 12

Exercise 21. To which order do you have to consider values in the series expansion of cosine, to approximate $\cos 1$ with an error smaller 10^{-7}? Furthermore show that $\cos 1$ is irrational.

Hint: To show irrationality of $\cos 1$, assume, $p, q \in \mathbb{N}_+$ with $\cos 1 = \frac{p}{q}$. Replace that in the estimated error of

$$\cos 1 - \sum_{k=0}^{q} \frac{(-1)^k}{(2k)!},$$

multiply with $(2q)!$ and derive a contradiction.

14.1 Exercise 12.a

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \cdot (-1)^k$$

Consider,

$$S_{2m} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k$$

$$S_{2k+1} = \sum_{k=0}^{2k+1} \frac{1}{(2k)!} \cdot (-1)^k$$

So S_{2k+1} has a negative, last expression. S_{2m} has a positive last expression.

$$S_{2k+1} < \cos 1 < S_{2m}$$

$$S_{2m} - S_{2m+1} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k - \sum_{k=0}^{2m+1} \frac{1}{(2k)!}(-1)^k$$

$$\triangle \cos(1) = -\frac{1}{(2(2m+1))!} \cdot (-1)^{2m+1} = \frac{1}{(2 \cdot (2m+1))!} \overset{!}{<} 10^{-7}$$

$$N! > 10^7 \Rightarrow N > 11$$

$$2 \cdot (2m+1) > 11$$

$$2m + 1 > \frac{11}{2} = 5.5$$

$\Rightarrow$ 10-th order because every odd expression is cancelled out.

Consider paper: "The irrationality of e and Others".

14.2 Exercise 12.b

$$\cos(1) \notin \mathbb{Q}$$

Assume $\exists p \in \mathbb{Z}$, $q \in \mathbb{N}$:

$$\cos(1) = \frac{p}{q}$$

$$\left| \cos(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \right|$$

$$= \left| \frac{p}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k}{(2k)!} \right| < \frac{1}{(2q)!}$$

$$= \left| \frac{p(2q)!}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k \cdot (2q)!}{(2k)!} \right| < 1$$

$$|x - y| < 1 \Rightarrow 0 \qquad \text{because } x \in \mathbb{Z}, y \in \mathbb{Z}$$

Leibniz criterion requires that the limes is not achieved in the sequence, because the functions need to be strictly monotonical.

15 Exercise 13

Exercise 22. Let $f : [\frac{\pi}{2}, \frac{3\pi}{2}] \to [-1,1]$, $x \mapsto \sin x$. Show that f is bijective and compute (using the formula for the derivative of the inverse function $(f^{-1})'(y)$ at all possible points $y \in [-1,1]$). Also give an explicit representation for f^{-1}

$$\ldots = -\frac{1}{\sqrt{1-y^2}}$$

It is important to recognize the negative sign.

16 Exercise 14

Exercise 23. Let $w, z \in \mathbb{R}$ with $w, z, w+z \notin \left\{\frac{\pi}{2} + k\pi \,\middle|\, k \in \mathbb{Z}\right\}$. Prove the addition theorem of the tangens function:

$$\tan(w+z) = \frac{\tan(w) + \tan(z)}{1 - \tan(w)\tan(z)}.$$

Let $x, y \in \mathbb{R}$ with $xy < 1$. Show that $\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and use it to prove the addition theorem for the arcustangens function:

$$\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}.$$

1. Show that $\tan(w+z) = \frac{\tan(w)+\tan(z)}{1-\tan(w)\tan(z)}$.

$$\tan(w+z) = \frac{\sin(w+z)}{\cos(w+z)} = \frac{\cos(w)\cdot\sin(z) + \sin(w)\cos(z)}{\cos(w)\cos(z) - \sin(w)\sin(z)}$$

$$\frac{\frac{\cos(w)\sin(w)}{\cos(w)\cos(z)} + \frac{\sin(w)\cdot\cos(z)}{\cos(w)\cdot\cos(z)}}{1 - \frac{\sin(w)\sin(z)}{\cos(w)\cos(z)}}$$

$$= \frac{\tan(z) + \tan(w)}{1 - \tan(w)\tan(z)}$$

2.

$$\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$x, y \in \mathbb{R}, xy < 1$.

Let $x = \tan(z)$ and $y = \tan(w)$.

$$xy = \tan(z)\cdot\tan(w) = \frac{\sin(z)\cdot\sin(w)}{\cos(z)\cdot(w)} < 1$$

$$\sin(z)\cdot\sin(w) < \cos(z)\cos(w)$$

$$\Leftrightarrow 0 < \cos(z)\cdot\cos(w) - \sin(z)\cdot\sin(w)$$

$$\Leftrightarrow 0 \overset{!}{<} \cos(z+w) \Leftarrow z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \vee w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This proof is insufficient! A case distinction for $\cos(z)\cos(w) > 0$ is required.

3. Show that $\arctan(x) + \arctan(y) = \arctan \frac{x+y}{1-xy}$. Let $x = \tan(z)$ and $y = \tan(w)$.

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan\left(\frac{\tan(z)+\tan(w)}{1-\tan(z)\tan(w)}\right) = \arctan(\tan(z+w)) = z + w = \arctan(x) + \arctan(y)$$

17 Exercise 15

Exercise 24. Compute the following integrals by approximating the integrands using a sequence of step functions with the given points. Let $a, b \in \mathbb{R}$ with $a < b$.

1. $\int_a^b e^x \, dx$ with points $x_k := a + k(b-a)/n$.
2. $\int_a^b x^p \, dx$ with points $x_k := aq^k$, $q := \sqrt[n]{b/a}$ and $p \in \mathbb{R} \setminus \{-1\}$.

17.1 Exercise 15.a

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} e^{a+\frac{k(b-a)}{n}} \\
&= \lim_{n\to\infty} \frac{b-a}{n} e^a \sum_{k=0}^{n-1} \left(e^{\frac{b-a}{n}}\right)^k \\
&= \lim_{n\to\infty} e^a \cdot \frac{b-a}{n} \frac{e^{\frac{b-a}{n}} - 1}{e^{\frac{b-a}{n}} - 1} \\
&= \lim_{n\to\infty} e^a \left(e^{b-a} - 1\right) \cdot \underbrace{\frac{\frac{b-a}{n}}{e^{\frac{b-a}{n}} - 1}}_{\to 1} \\
&= e^a \cdot \frac{e^b}{e^a} - e^a = e^b - e^a
\end{aligned}$$

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in [a,b]) : |\varphi_n(x) - e^x| < \varepsilon$$

$$\begin{aligned}
e^{a+(b-a)\frac{n-1}{n}} - e^b &= e^{a+(b-a)(1-\frac{1}{n})} - e^b = e^{a+b-\frac{b}{n}-a+\frac{a}{n}} - e^b \\
&= e^{b-\frac{b}{n}+\frac{a}{n}-e^b}
\end{aligned}$$

17.2 Exercise 15.b

$$x_k := aq^k \qquad q := \left(\frac{b}{a}\right)^{\frac{1}{n}}$$

$$p \neq -1$$

$$\begin{aligned}
y_k &:= x_{k+1} - x_k \\
&= aq^{k+1} - aq^k \\
&= aq^k(q-1)
\end{aligned}$$

$$\sum_{k=0}^{n-1} y_k x_k^p = \sum_{k=0}^{n-1} aq^k(q-1)(aq^k)^p = a^{p+1}(q-1)\sum_{k=0}^{n-1}(q^{p+1})^k$$

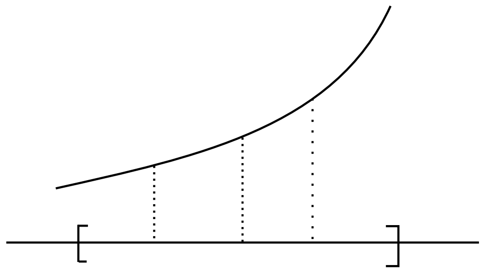

Figure 1: Illustration of 15b

Is a geometric series:

$$= a^{p+1}(q-1)\frac{1-(q^{p+1})^{n-1}}{1-q^{p+1}}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} y_k x_k^p = a^{p+1}\lim_{n\to\infty}\left(\left(\frac{b}{a}\right)^{\frac{1}{n}}-1\right)\frac{1-(\frac{b}{a})^{\frac{n-1}{n}(p+1)}}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = a^{p+1}\left(1-\left(\frac{b}{a}\right)^{p+1}\right)\underbrace{\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}}}_{\text{“}0/0\text{”}}$$

$$\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \text{“}0/0\text{”}$$

L’Hópital’s Rule:

$$= \lim_{n\to\infty}\frac{\exp\left(\frac{1}{n}\log\left(\frac{b}{a}\right)\right)-1}{1-\exp\left(\frac{p+1}{n}\log\left(\frac{b}{a}\right)\right)} = \lim_{n\to\infty}\frac{\log\left(\frac{b}{a}\right)\cdot\frac{-1}{n^2}\exp\left(\frac{1}{n}\log\frac{b}{a}\right)}{-(p+1)\log\frac{b}{a}\cdot\frac{-1}{n^2}\exp\left(\frac{p+1}{n}\log\frac{b}{a}\right)}$$

$$= \lim_{n\to\infty}\frac{-1}{p+1}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}}{\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \frac{-1}{p+1}$$

$$\Rightarrow = (a^{p+1}-b^{p+1})\cdot\frac{-1}{p+1} = \frac{b^{p+1}-a^{p+1}}{p+1}$$

The assignment explicitly asks for a step function. This approach only verifies that

$$\int_a^b x^p\,dx$$

$$\left.\frac{x^{p+1}}{p+1}\right|_{x=a}^{x=b} = \frac{b^{p+1}}{p+1}-\frac{a^{p+1}}{p+1}$$

We only did the approximation from one side (also upper bound is needed which works analogously):

$$\sum_{k=0}^{n-1} y_k x_{k+1}^p = \dots$$

18 Exercise 16

Exercise 25. For an interval $I \subseteq \mathbb{R}$ let $f_n : I \to \mathbb{R}$ be a sequence of functions which are uniformly continuous converging towards $f : I \to \mathbb{R}$. Show that the following statements hold or provide a counterexample:

- If all f_n are uniformly continuous, then f is uniformly continuous.
- If all f_n are Lipschitz continuous, then f is Lipschitz continuous.

18.1 Exercise 16.a

It holds. So a proof is given in the following.

We want to show:

$$\forall \varepsilon \exists \delta : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f(x_0) + f_n(x_0) - f(x_0)| \\ &\leq \underbrace{|f(x) - f_n(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x) - f_n(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} \end{aligned}$$

We need to elaborate: For which n does $\frac{\varepsilon}{3}$ hold?

$$\forall \varepsilon > 0 \exists \overset{\text{depends on } \varepsilon}{n_0} : \forall n \geq n_0 \forall x \in I : |f(x) - f_n(x)| < \frac{\varepsilon}{3}$$

$$\forall \varepsilon > 0 \forall n \exists \delta = \delta(n, \varepsilon) : \forall x, x_0 : |x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

18.2 Exercise 16.b

This does not hold. So we provide a counterexample.

Consider $f(x) = \sqrt{x}$. It is not differentiable at $x = 0$, but $f(0) = 0$ is defined. The function cannot be Lipschitz-continuous, because the Lipschitz constant grows as we tend towards 0. We need functions f_n.

Consider $f(x) = \sqrt{x + \frac{1}{n}}$. The function f_n looks like f, but is shifted slightly to the left. As n tends towards infinity, f_n becomes f and we get the problem at $x = 0$.

You can also consider:

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq \frac{1}{n} \\ \sqrt{\frac{1}{n}} & \text{for } x < \frac{1}{n} \end{cases}$$

19 Exercise 17

Exercise 26. Let $f : [0,1] \to \mathbb{R}$ be a regulated function continuous in 0. Show the following relation:

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\,ds = f(0).$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\,ds = f(0) = \lim_{n\to\infty} n \cdot \left(F\left(\frac{1}{n}\right) - F(0)\right) = \lim_{n\to\infty} \frac{F\left(\frac{1}{n}\right) - F(0)}{\frac{1}{n}} = \lim_{h\to 0} \frac{F(h) - F(0)}{h} = f(0)$$

19.1 Other approach

Continuity at $x = 0$:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall |x| < \delta : |f(x) - f(0)| < \varepsilon$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\,dx \le \lim_{n\to\infty} n \int_0^{\frac{1}{n}} (f(0) + \varepsilon)\,dx$$

For $\frac{1}{n} < \delta$ it holds that $f(x) < f(0) + \varepsilon$ for $x \in [0, \frac{1}{n}]$.

$$= \lim_{n\to\infty} n(f(0) + \varepsilon)\frac{1}{n} = f(0) + \varepsilon$$

holds for all $\varepsilon > 0$.

$$\Rightarrow \lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\,dx \le f(0)$$

20 Exercise 18

Exercise 27. Prove the Riemann-Lebesgue Lemma: For every regulated function $f : [a,b] \to \mathbb{R}, a < b$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx = 0.$$

Hint: Show the following partial results:

- For all intervals $[\alpha, \beta] \subseteq [a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_\alpha^\beta \sin(\lambda x)\,dx = 0.$$

- For all step functions $g \in \tau[a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b g(x)\sin(\lambda x)\,dx = 0.$$

20.1 Exercise 18.a

$$-\frac{1}{\lambda}\cos(\lambda x)\Big|_\alpha^\beta = \underbrace{\frac{1}{\lambda}}_{\to 0} \underbrace{(-\cos(\beta\lambda) + \cos(\alpha\lambda))}_{\text{bounded}}$$

20.2 Exercise 18.b

Because g is a step function of $[a,b]$, there exists a decomposition

$$a = x_0 < x_1 < \ldots < x_n = b$$

such that $g(x)$ has a constant value c_i in every subinterval $[x_{i-1}, x_i)$.

$$\lim_{\lambda\to\infty}\int_a^b f(x)\sin(\lambda x)\,dx = \lim_{\lambda\to\infty}\sum_{i=1}^{n} c_i \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx$$

This can be done, because we consider a finite sum.

$$\sum_{i=1}^{n} c_i \underbrace{\int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx}_{\to 0 \forall \text{ subintervals } H(i)}$$

$$= \sum_{i=1}^{n} c_i \cdot \underbrace{\lim \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx = 0}_{\to 0}$$

20.3 Conclusion

Because $f(x)$ is a regulated function $\forall\varepsilon > 0$, there exists a step function $g_\varepsilon(x)$ with $|f(x) - g_\varepsilon(x)| < \varepsilon \quad \forall x \in [a,b]$.

$$\left|\int_a^b f(x)\cdot\sin(\lambda x)\,dx\right| \le \underbrace{\int_a^b \left|\underbrace{f(x) - g_\varepsilon(x)}_{<\varepsilon}\right|}_{<\varepsilon(b-a)} \cdot \left|\underbrace{\sin(\lambda x)}_{\le 1}\right| dx + \underbrace{\left|\int_a^b g_\varepsilon(x)\sin(\lambda x)\,dx\right|}_{\to 0 \text{ for } \lambda\to\infty}$$

$$\lim_{\lambda\to\infty}\left|\int_a^b f(x)\sin(\lambda x)\,dx\right| \le \varepsilon(b-a)$$

We can choose ε arbitrary, so it must tend towards 0.

21 Exercise 19

Exercise 28. Let $I, J \subseteq \mathbb{R}$ be intervals, $f : I \to \mathbb{R}$ continuous and $g, h : J \to I$ differentiable. Furthermore it holds that $g \le h$ in J. Prove that

$$A : J \to \mathbb{R}, \quad x \mapsto \int_{g(x)}^{h(x)} f(\xi)\,d\xi$$

is differentiable and determine its derivative.

21.1 Exercise 19.a

Show differentiability.

So

$$\lim_{x\to x_0}\frac{A(x) - A(x_0)}{x - x_0}$$

exists.

$$\lim_{x\to x_0}\frac{A(x) - A(x_0)}{x - x_0} = \lim_{x\to x_0}\frac{\int_{g(x)}^{h(x)} f(\xi)\,d\xi - \int_{g(x_0)}^{h(x_0)} f(\xi)\,d\xi}{x - x_0} = \lim_{x\to x_0}\frac{F(h(x)) - F(g(x)) - F(h(x_0)) + F(g(x_0))}{x - x_0}$$

$F(h(x))$ and $F(g(x))$ exists, because $h(x)$ is continuous, so a regulated function and regulated functions always have a primitive function.

$$\lim_{x\to x_0}\frac{F(h(x)) - F(h(x_0))}{x - x_0} - \lim_{x\to x_0}\frac{F(g(x)) - F(g(x_0))}{x - x_0}$$

If $h(x)$ is continuous, then $F(h(x))$ is differentiable (analogously for $g(x)$). And the composition is also differentiable.

21.2 Exercise 19.b

Determine its derivative.

$$(F \circ h)'(x) - (F \circ g)'(x_0) = f(h(x_0)) \cdot h'(x_0) - f(g(x_0)) \cdot g'(x_0)$$

22 Exercise 20

Exercise 29. Determine the following integrals for arbitrary $a, b \in \mathbb{R}, a < b$:

- $\int_a^b \frac{d}{dx}\left(x^5 \cdot e^x\right)\, dx$
- $\int_a^b x^4 e^{x^5}\, dx$

22.1 Exercise 20.a

$$\int_a^b \frac{d}{dx}\left(x^5 e^x\right)\, dx = \int_a^b 5x^4 e^x - x^5 e^x\, dx = \int_a^b \underbrace{e^x}_{g'(x)} \underbrace{(5x^4 - x^5)}_{=f(x)}\, dx$$

$$= e^x(5x^4 - x^5)\Big|_a^b - \int_a^b e^x(20x^3 + 5x^4) = e^b b^5 - e^a a^5$$

22.2 Exercise 20.b

$$\int_a^b x^4 e^{x^5}\, dx$$

$$u := x^5 \Rightarrow \frac{du}{dx} = 5x^4 \qquad dx = \frac{du}{5x^4}$$

$$= \int_{a^5}^{b^5} x^4 e^u \frac{du}{5x^4} = \int_{a^5}^{b^5} e^u \frac{du}{5} = \frac{1}{5}\int_{a^5}^{b^5} e^u\, du = \frac{1}{5}\left(e^{b^5} - e^{a^5}\right)$$

Other approach for 20.b:

$$F = \frac{1}{5}e^{x^5}$$

$$F' = x^4 e^{x^5} = f$$