

Linear Algebra 2 – Practicals

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1 Solution of the last lecture exam of Analysis 1

1.1 Exam: Exercise 1

Exercise 1. Determine the limes of

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

does not help us. What about this representation?

$$\frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1} = \frac{a(n-1) + b(n+1)}{(n+1)(n-1)}$$

$$a(n-1) + b(n+1) = 1$$

$$(a+b)n + (b-a) = 1$$

$$\Rightarrow a+b=0 \wedge b-a=1$$

$$\Rightarrow a = -\frac{1}{2} \quad b = \frac{1}{2}$$

Followingly,

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right)$$

Okay, how to proceed? Let's build a pre-factor:

$$\begin{aligned} & \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\ &= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots \\ &= \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \end{aligned}$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m := \frac{1}{2} \sum_{n=2}^m \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \sum_{n=2}^m \frac{1}{n-1} - \frac{1}{2} \sum_{n=2}^m \frac{1}{n+1}$$

Please be aware that we explicitly define S_m because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$\begin{aligned} &= \frac{1}{2} \sum_{n=2}^m \frac{1}{n-1} - \frac{1}{2} \sum_{n=4}^{m+2} \frac{1}{n-1} \\ &= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1} \right) \end{aligned}$$

We already know $\frac{1}{m} \xrightarrow{m \rightarrow \infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m \rightarrow \infty} 0$. Followingly also $\frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1} \right) \xrightarrow{m \rightarrow \infty} 0$.

1.2 Exam: Exercise 2

Exercise 2. A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n \in \mathbb{N}}$$

$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \quad a_1 = \frac{3}{2} \quad a_2 = \frac{5}{4} \quad a_3 = \frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \quad a_1 = 4 \quad a_2 = \frac{5}{2} \quad a_3 = \frac{7}{4}$$

Exercise 3. a) Show that $1 < a_n \leq a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over n to prove this statement:

induction base $1 < a_0 \leq a_0$ holds trivially.

induction step We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &\leq \frac{1}{2}(a_0 + a_0) \quad [\text{induction hypothesis and } 1 < a_0] \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &> \frac{1}{2}(1 + 1) \quad [\text{induction hypothesis}] \\ &= 1 \end{aligned}$$

Exercise 4. b) Prove that $a_{n+1} < a_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &< \frac{1}{2}(a_n + a_n) \quad [\text{we have proven: } a_n > 1] \end{aligned}$$

Exercise 5. c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$b_n := a_n - 1 \quad \forall n \in \mathbb{N}$$

$$b_0 := a_0 - 1$$

$$b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2}b_n$$

$$b_n = \frac{1}{2^n}b_0 \rightarrow 0 \cdot b_0 = 0$$

$$\Rightarrow b_n \rightarrow 0$$

$$\Rightarrow a_n = b_n + 1 \rightarrow 1$$

Does it work to just show: $1 = \frac{1}{2}(1 + 1)$? Nope, because in points of continuity this might be true even though 1 is not its limit.

Let $a_n \rightarrow a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$a_{n+1} \rightarrow a \quad \frac{1}{2}(a_n + 1) \rightarrow \frac{1}{2}(a + 1) \quad a = \frac{1}{2}(a + 1)$$

1.3 Exam: Exercise 3

Exercise 6. $f : \mathbb{R} \rightarrow \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an ε - δ -proof.

If we don't need an ε - δ -proof, we would argue with the Algebraic Continuity Theorem: The function f is a composition of continuous functions, hence a continuous function itself.

ε - δ -definition:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &= |2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)| \\ &= |2x^2 + 5x - 2x_0^2 - 5x_0| \\ &\leq 2|x^2 - x_0^2| + 5|x - x_0| \\ &= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\ &= 2|x + x_0||x - x_0| + 5|x - x_0| \\ &\leq 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\ &\leq 2(|x_0| + \delta + |x_0|) + 5\delta \end{aligned}$$

Our goal: we are able to claim $\stackrel{!}{<} \varepsilon$

$$\begin{aligned} &= 4|x_0|\delta + 2\delta^2 + 5\delta \\ &= 2\delta^2 + (4|x_0| + 5)\delta \end{aligned}$$

In general (here it does not apply), that x_0 might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$\begin{aligned} 2\delta^2 &< \frac{\varepsilon}{2} \\ \delta &< \frac{\sqrt{\varepsilon}}{2} \\ (4|x_0| + 5)\delta &< \varepsilon \\ \delta &< \frac{\varepsilon}{4|x_0| + 5} \end{aligned}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{2}, \frac{\varepsilon}{4|x_0| + 5}\right)$. Then the ε - δ definition shows that f is continuous.

2 Exam: Exercise 4

Exercise 7. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\begin{aligned} h(0) &= f(0) - f\left(\frac{1}{2}\right) \\ h(1) &= f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0) \end{aligned}$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x + \frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore h is continuous, because it is a composition of continuous functions.

Case 1: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\Rightarrow f(\xi) = f\left(\xi + \frac{1}{2}\right)$$

Case 2: $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

Case 3: $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

3 Exercise 1

Exercise 8. Investigate the function $f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \frac{1}{2}(x|x| + x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}$$

So this is differentiable, but in case of $x = 0$, it remains questionable.

We look at the definition of differentiability:

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x}$$
$$f'(x) = \begin{cases} \lim_{x \rightarrow 0} \frac{0}{x} = 0 \\ \lim_{x \rightarrow 0^+} \frac{x^2}{x} = \lim_{x \rightarrow 0^+} x = 0 \end{cases}$$

It follows that f is differentiable one time.

$$f''(x) = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

What about $x = 0$?

$$\lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} = \begin{cases} \lim_{x \rightarrow 0} \frac{0}{x} = 0 \\ \lim_{x \rightarrow 0^+} \frac{2x}{x} = \lim_{x \rightarrow 0^+} 2 = 2 \end{cases}$$

Left and right limes differ. So it is not differentiable.

4 Exercise 2

Exercise 9. Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$
2. $\lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1 - \sin x)}$
4. $\lim_{x \rightarrow 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4.1 Exercise 2.a

$$\lim_{x \rightarrow 1} \frac{\ln x}{x-1}$$

The conditions to apply l'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x \rightarrow 1} \frac{\frac{1}{x}}{1} = 1$$

4.2 Exercise 2.b

$$\lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x} = \lim_{x \rightarrow 0^+} \frac{-\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0$$

A nice hint to find out whether this function is differentiable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{\sin x - x}{x \sin x} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}} \approx x \rightarrow 0$$

This exploits, that it will take one run of L'Hôpital's rule (because each expression has at least degree 2) and its limes will be 0 (because of x).

4.3 Exercise 2.c

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\cos(x))}{\ln(1 - \sin(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied. We claim that $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} g(x) = \infty$ is fine.

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\cos(x)}{1 - \sin(x)}} = \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-\sin(x) \cdot (1 - \sin(x))}{\cos(x)(-\cos(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied.

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \frac{-\cos(x)(1 - \sin(x)) - \sin(x) \cdot (-\cos(x))}{-\sin(x)(-\cos(x)) + \cos(x) \cdot \sin(x)} = \frac{1}{2}$$

If we want to apply the previous estimate here, we should consider

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos(y) \quad y = \frac{\pi}{2} - x$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(y)$$

This gives us a different estimate of the result:

$$\lim_{y \rightarrow 0^+} \frac{\ln(\sin(y))}{\ln(1 - \cos(y))} \approx \lim_{y \rightarrow 0^+} \frac{\ln(y)}{\ln\left(\frac{y^2}{2}\right)} = \lim_{y \rightarrow 0^+} \frac{\ln(y)}{2 \ln(y) - \ln(2)} \approx \lim_{y \rightarrow 0^+} \frac{\ln(y)}{2 \ln(y)} = \frac{1}{2}$$

We define neighborhoods:

$$N_\delta(x_0) = \{x : |x - x_0| < \delta\}$$

$$N_R(\infty) = \{x : x > R\}$$

4.4 Exercise 2.d

$$\lim_{x \rightarrow 1^-} x^{\frac{1}{1-x}} = \lim_{x \rightarrow 1^-} e^{\ln(x) \frac{1}{1-x}} = \exp \left(\lim_{x \rightarrow 1^-} \underbrace{\frac{\ln(x)}{1-x}}_{(-1) \cdot \text{Exercise a}} \right) = \frac{1}{e}$$

4.5 Exercise 2.e

$$\lim_{n \rightarrow \infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \left(\exp \left(\frac{\ln n}{\sqrt{n}} \right) \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{\ln(n)}{\sqrt{n}} \right)$$

The conditions to apply L'Hôpital's rule are satisfied („ $\frac{\infty}{\infty}$ “)

$$\exp \left(\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} \right) = \exp \left(\lim_{n \rightarrow \infty} \frac{2\sqrt{n}}{n} \right) = \exp(0) = 1$$

4.6 Exercise 2.f

$$\lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \rightarrow \infty} \frac{e^x (1 - e^{-2x})}{e^x (1 + e^{-2x})} = \frac{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} \frac{1}{e^{2x}}}{\lim_{x \rightarrow \infty} 1 + \lim_{x \rightarrow \infty} \frac{1}{e^{2x}}}$$

Remark:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\sinh(x)}{\cosh(x)} &\stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow \infty} \frac{\cosh(x)}{\sinh(x)} \stackrel{\text{L'Hôpital}}{=} \lim_{x \rightarrow \infty} \frac{\sinh(x)}{\cosh(x)} \\ y &= \lim_{x \rightarrow \infty} \frac{\sinh(x)}{\cosh(x)} = \frac{1}{\lim_{x \rightarrow \infty} \frac{\sinh(x)}{\cosh(x)}} = \frac{1}{y} \end{aligned}$$

5 Exercise 3

Exercise 10. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $x \mapsto x + e^x$ is bijective. Furthermore determine $(f^{-1})'(1)$ and $\lim_{y \rightarrow \infty} (f^{-1})'(y)$.

If the function is strictly monotonically increasing, it is injective.

$$f'(x) = 1 + e^x > 0 \quad \forall x \in \mathbb{R}$$

We show that it is strictly monotonically increasing:

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$.

$$\begin{aligned} \frac{f(x_2) - f(x_1)}{x_2 - x_1} &= f'(\alpha) \quad \text{with } \alpha \in (x_1, x_2) \\ f(x_2) - f(x_1) &= f'(\alpha)(x_2 - x_1) > 0 \end{aligned}$$

Is f surjective?

For an arbitrary $y_0 \in \mathbb{R}$ it holds that $\exists x_0 \in \mathbb{R} : f(x_0) = y_0$:

$$\exists f(a), f(b) \in \mathbb{R} : f(a) \leq y_0 < f(b)$$

It holds that

$$\lim_{x \rightarrow -\infty} x + \underbrace{e^x}_{\rightarrow 0} = -\infty$$

$$\lim_{x \rightarrow +\infty} x + e^x = \infty$$

Formally:

$$\forall y_0 \exists x_0 : \forall x < x_0 : f(x) < y_0$$

From the Intermediate Value Theorem it follows that

$$\Rightarrow \exists c \in [a, b) : f(c) = y_0 \quad c =: x_0$$

So it is surjective.

From injectivity and surjectivity it follows that it is bijective.

5.1 Determine $(f^{-1})'(1)$

$$f(x) = x + e^x$$

$$f'(x) = 1 + e^x$$

We apply the inverse function theorem:

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$y = 1 = f(x)$$

$$x = f^{-1}(1)$$

An educated guess gives us that $x = 0$. In general determining x is more difficult.

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}$$

5.2 Determine $\lim_{y \rightarrow \infty} (f^{-1})'(y)$

$$\lim_{y \rightarrow \infty} (f^{-1})'(y) = \lim_{y \rightarrow \infty} \frac{1}{1 + e^x}$$

As x grows to infinity, also y grows to infinity. From bijectivity it follows that any value can be reached with x as well as $f(x)$.

$$\underbrace{\underbrace{f'(f^{-1}(\underbrace{y}_{\rightarrow \infty}))}_{\rightarrow \infty}}_{\rightarrow \infty}$$

6 Exercise 4

Exercise 11. Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \rightarrow \mathbb{R}$ be differentiable in $x_0 \in D$. Show

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2} = f'(x_0)$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) + f(x_0) - f(x_0 - h)}{2h} \\
&= \lim_{h' \rightarrow 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0) - f(x_0 + h')}{-h'} \right) \\
&= \lim_{h' \rightarrow 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0 + h') - f(x_0)}{h'} \right) \\
&= \frac{1}{2} (f'(x_0) + f'(x_0)) \\
&= f'(x_0)
\end{aligned}$$

6.1 Exercise 4.b

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{f(x_0 + rh) - f(x_0 + sh)}{h} &= \lim_{h \rightarrow 0} \frac{f(x_0 + rh) - f(x_0)}{h} + \lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 + sh)}{h} \\
&\quad h_1 = rh \quad h_2 = sh \\
&= \lim_{h_1 \rightarrow 0} \frac{f(x_0 + h_1) - f(x_0)}{\frac{1}{r} \cdot h_1} + \lim_{h_2 \rightarrow 0} \frac{f(x_0) - f(x_0 + h_2)}{\frac{1}{s} \cdot h_2} \\
&= r \cdot f'(x_0) - s \cdot f'(x_0) \\
&= (r - s) \cdot f'(x_0)
\end{aligned}$$

7 Exercise 5

Exercise 12. Let $D \subseteq \mathbb{R}$ be an open interval. $f : D \rightarrow \mathbb{R}$ is differentiable and f is twice differentiable in $x_0 \in D$.

7.1 Exercise 5.a

Exercise 13. Show that

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0)$$

f is differentiable, therefore continuous, and h goes to 0. So we have „ $\frac{0}{0}$ “. All conditions to apply L'Hôpital's rule are satisfied.

$$\lim_{h \rightarrow 0} \frac{f'(x_0 + h) - f'(x_0 - h)}{2h} \approx \frac{0}{0}$$

We can apply L'Hôpital's Rule again or just use the result of exercise 4a.

$$\stackrel{4a}{\Rightarrow} f''(x_0)$$

7.2 Exercise 5.b

Exercise 14. Show that the limes from exercise 5.a can also exist, even if $f''(x_0)$ does not exist. Use the result from Exercise 1.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

We know that it is not twice differentiable. But we want to show that the limit exists.

We are only concerned with $x = 0$.

$$\lim_{h \rightarrow 0} f(x_0) = 0$$

$$\lim_{h \rightarrow 0} \frac{h^2 - h^2}{h^2} = \frac{0}{h^2} = 0$$

So if we traverse the graph from both sides at the same time $\frac{f(x_0+h)-f(x_0-h)}{h}$.

8 Exercise 6

Exercise 15. Determine the following limit for arbitrary $c \in \mathbb{R}$:

$$\lim_{n \rightarrow \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right).$$

$$\lim_{n \rightarrow \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right)$$

$$\lim_{n \rightarrow \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right) = \lim_{n \rightarrow \infty} \frac{e^{\frac{c}{n} \cdot \ln n} - 1}{\frac{\ln n}{n}}$$

and

$$\left(e^{\frac{c}{n} \cdot \ln n} \right)' = e^{\frac{c}{n} \cdot \ln n} \cdot \left(-\frac{c}{n^2} \cdot \ln n + \frac{c}{n} \cdot \frac{1}{n} \right) = \frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln(n))$$

All conditions are satisfied to apply L'Hôpital's rule ($\frac{0}{0}$):

$$\lim_{n \rightarrow \infty} \frac{\frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln n)}{\frac{\frac{1}{n} \cdot n - \ln n}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{c \cdot e^{\frac{c}{n} \cdot \ln n} (1 - \ln(n))}{1 - \ln n} = \lim_{n \rightarrow \infty} c \cdot e^{\frac{c}{n} \cdot \ln n} = c \cdot 1$$

9 Exercise 7

Exercise 16. • Show that $e^x \geq 1 + x$ holds for all $x \in \mathbb{R}$.

Hint: On demand, use the Mean Value Theorem.

- Prove that for all $x > 0$, the following estimates hold:

$$\ln x \leq x - 1$$

and for all $k \in \mathbb{N}_+$ it holds that

$$k \left(1 - \frac{1}{\sqrt[k]{x}} \right) \leq \ln x \leq k \left(\sqrt[k]{x} - 1 \right)$$

$x \geq 0$ Choose $f(x) = e^x$ in $[0, x]$. Mean value theorem:

$$\begin{aligned}\exists x_0 : f'(x_0) &= \frac{f(b) - f(a)}{b - a} \quad \text{for } a < x_0 < b \\ f'(x_0) &= e^{x_0} \quad e^{x_0} \geq 1 \quad x_0 \geq 0 \\ e^{x_0} &= \frac{f'(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x} \Rightarrow \frac{e^x - 1}{x} \geq 1\end{aligned}$$

Or alternatively: f is convex and therefore $f''(x) > 0$.

Consider $f(x) = x - 1 - \ln x$

$$\begin{aligned}f'(x) &= 1 - \frac{1}{x} \quad f''(x) = \frac{1}{x^2} \\ f'(x) &\stackrel{!}{=} 0 \\ 1 - \frac{1}{x} &= 0 \Leftrightarrow x = -1 \\ f''(1) &= 1 > 0 \Rightarrow \text{minimum and because } f(1) = 0 \Rightarrow \forall x : x - 1 - \ln x \geq 0\end{aligned}$$

Or alternatively:

$$\begin{aligned}y &:= x - 1 \\ x &= y + 1\end{aligned}$$

Show that $\ln(y + 1) \leq y \Leftrightarrow y + 1 \leq e^y$.

e^x is monotonically increasing $\Rightarrow x \leq y \Leftrightarrow e^x \leq e^y$.

And this has been proven previously.

9.1 Exercise 7.b

$$\begin{aligned}\ln(x) &\leq k \left(\left\lceil \frac{1}{k} \right\rceil x - 1 \right) \\ \ln(\sqrt[k]{x}) &\leq \sqrt[k]{x} - 1 \Leftrightarrow \ln(y) \leq y - 1\end{aligned}$$

And this has been proven in Exercise a.

The second part following analogously.

10 Exercise 8

Exercise 17. Let $f : D \rightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}$. Show: If f is continuous in an environment U of $a \in D$, differentiable in $U \setminus \{a\}$ and there exists $\lim_{x \rightarrow a} f'(x)$, such that f in a differentiable and

$$f'(a) = \lim_{x \rightarrow a} f'(x).$$

Hint: On demand, use the Mean Value Theorem.

Let h_n be an arbitrary zero-sequence (with $h_n(x) > 0 \quad \forall x \in D$) and due to Mean Value Theorem $\exists \xi_n \in D$ with $f'(\xi_n) = \frac{f(a+h_n) - f(a)}{h_n}$.

$$\begin{aligned}\lim_{n \rightarrow \infty} f'(\xi_n) &= \lim_{x \rightarrow a} f'(x) = \lim_{n \rightarrow \infty} \frac{f(a + h_n) - f(a)}{h_n} = f'(a) \\ \lim_{n \rightarrow \infty} \frac{f(a + h_n) - f(a)}{h_n} &= \lim_{n \rightarrow \infty} f'(\xi_n) = \lim_{x \rightarrow a} f'(x) = z\end{aligned}$$

For the arbitrary zero-sequence, we really need to consider it arbitrary (otherwise we just show it for the one sequence). Consider this counterexample:

$$f(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

10.1 Alternative approach

Application of “Schranksatz”.

$$\exists \lim f'(x) = \alpha$$

Hence for arbitrary $\varepsilon > 0 : \exists \delta > 0 \forall x \in (a - \delta, a + \delta) \setminus \{a\} : |f'(x) - \alpha| < \varepsilon$. Hence $\alpha - \varepsilon < f'(x) < \alpha + \varepsilon$.

•

$$\forall x \in (a, a + \delta) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

•

$$\forall x \in (a - \delta, a) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall x \in (a - \delta, a + \delta) \setminus \{a\} : \left| \frac{f(x) - f(a)}{x - a} - \alpha \right| \leq \varepsilon$$

$$\Rightarrow \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \alpha$$

10.2 Second alternative approach

$$\lim_{f(a+h)-f(a)} h$$

If I know f is continuous, then $f(a + h) \rightarrow f(a)$. So,

$$\frac{0}{0}$$

$$\lim_{h \rightarrow 0} \frac{f'(a + h) - 0}{1} = \lim_{h \rightarrow 0} f'(a + h) = \lim_{x \rightarrow a} f'(x)$$

11 Exercise 9

Exercise 18. Let $f : [a, b] \rightarrow \mathbb{R}$, $a < b$, differentiable with $f(a) > 0$, $f'(a) > 0$ and $f(b) = 0$. Prove that there exists $\xi \in (a, b) : f'(\xi) = 0$.

First, we want to show that $f'(a) > 0 \Rightarrow \exists \delta > 0 \forall x \in (a, a + \delta) : f(x) > f(a)$.

$$\begin{aligned} \exists \delta > 0 \forall x \in (a, a + \delta) : \frac{f(x) - f(a)}{x - a} &> \frac{f'(a)}{2} > 0 \\ \Rightarrow f(x) - f(a) &> \frac{f'(a)}{2}(x - a) > 0 \end{aligned}$$

Indeed, $f(x)$ satisfies this property.

Secondly, we want to show that,

$$\exists \eta \in (a + \delta, b) : f(a) = f(\eta)$$

$$\exists \xi \in [a, \eta] \forall x_1 \in [a, \eta] : f(\xi) \geq f(x_1)$$

$$\exists \xi \in (a, \eta) : \frac{f(\eta) - f(a)}{\eta - a} = f'(\eta) = 0$$

There might be more than this one ξ , so the ξ between the second and third line might be different. Anyways, we found a ξ with the desired property.

12 Exercise 10

Exercise 19. Determine the pointwise limit of the following function sequences $f_n : [0, \infty) \rightarrow \mathbb{R}$ and determine its uniform convergence:

- $f_n(x) = \sqrt[n]{x}$
- $f_n(x) = \frac{1}{1+nx}$
- $f_n(x) = \frac{x}{1+nx}$

12.1 Exercise 10.a

If $x \neq 0$, $\lim_{n \rightarrow \infty} \sqrt[n]{x} = 1$.

If $x = 0$, $\lim_{n \rightarrow \infty} \sqrt[n]{x} = \lim_{n \rightarrow \infty} 0^{\frac{1}{n}} = 0$.

In terms of uniform convergence:

$$| \sqrt[n]{x} - 1 | < \varepsilon$$

$$\lim_{x \rightarrow \infty} \sqrt[n]{x} = \infty$$

Example:

$$| \sqrt[n]{x} - 1 | < \varepsilon$$

$$\sqrt[n]{x} - 1 < \varepsilon$$

$$\sqrt[n]{x} < \varepsilon + 1$$

$$\sqrt[n]{100} < \varepsilon + 1$$

12.2 Exercise 10.b

$$f_n(x) = \frac{1}{1+nx}$$

If $x \neq 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{1+nx} = 0$$

If $x = 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{1+n \cdot 0} = 1$$

Assume it is continuously convergent. Show that:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists x \in [0, \infty) : n \geq N \wedge |f_n(x) - f(x)| \geq \varepsilon$$

Does not hold for $\frac{9}{n} \geq x$.

12.3 Exercise 10.c

$$f_n(x) = \frac{x}{1+nx}$$

If $x \neq 0$,

$$\lim_{n \rightarrow \infty} \frac{x}{1+nx} = \lim_{n \rightarrow \infty} \frac{1}{\frac{1}{x} + n} = 0$$

If $x = 0$,

$$\lim_{n \rightarrow \infty} \frac{0}{1+n \cdot 0} = 0$$

$$\left| \frac{x}{1+nx} - 0 \right| < \varepsilon$$

$$\left| \frac{x}{1+nx} \right| < \left| \frac{x}{nx} \right| = \left| \frac{1}{n} \right|$$

Convergence is given. Uniform convergence is not given.

Advice: The simplest approach to show convergence is to show:

$$|f_n(x) - f(x)| \leq a_n \rightarrow 0$$

where a_n is independent from x .

13 Exercise 11

Exercise 20. Determine $\cos \alpha$, $\sin \alpha$ and $\tan \alpha$ for $\alpha \in \{\frac{\pi}{5}, \frac{2\pi}{5}\}$.

Hint: Show that $u := \cos \frac{2\pi}{5}$ and $v := \cos \frac{\pi}{5}$ satisfy the equations $u = 2v^2 - 1$ and $-2u^2 + 1 = v$. Determine u, v this way.

$$\begin{aligned} u &= \cos\left(\frac{2\pi}{5}\right) = \cos\left(\frac{\pi}{5} + \frac{\pi}{5}\right) \\ &= \cos^2\left(\frac{\pi}{5}\right) - \sin^2\left(\frac{\pi}{5}\right) \\ &= 2\cos^2\left(\frac{\pi}{5}\right) - 1 \\ &= 2v^2 - 1 \end{aligned}$$

To show: $v + 2u^2 - 1 = 0$, $\cos\left(\frac{\pi}{5}\right) + 2\cos^2\left(\frac{\pi}{5}\right) - 1 = 0$.

$$\begin{aligned} \cos\left(\frac{\pi}{5}\right) + 2\cos\frac{2\pi}{5} - 1 &= \cos\frac{\pi}{5} + \cos\frac{4\pi}{5} \\ &= \cos\frac{\pi}{5} + \cos\left(\pi - \frac{1}{5}\pi\right) \\ &= \cos\frac{\pi}{5} + \cos\pi \cdot \cos\left(\frac{\pi}{5}\right) + \sin\pi \cdot \sin\frac{\pi}{5} - \cos\frac{\pi}{5} \cdot \cos\frac{\pi}{5} \\ &= 0 \end{aligned}$$

For $u + v > 0$:

$$\begin{aligned} 2v^2 - 1 &= u \\ -2u^2 + 1 &= v \end{aligned}$$

$$2v^2 - 2u^2 = u + v$$

$$2(v+u)(v-u) = u+v$$

$$2(v-u) = 1 \Leftrightarrow v-u = \frac{1}{2}$$

$$v - 2v^2 + \frac{1}{2} = 0$$

$$v^2 - \frac{1}{2}v - \frac{1}{4} = 0$$

$$v_{1,2} = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{4}{16}} = \frac{1 \pm \sqrt{5}}{4}$$

$$0 < \cos\left(\frac{\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}$$

$$u = \cos \frac{2\pi}{5} = v - \frac{1}{2} = \frac{-1 + \sqrt{5}}{4}$$

$$\cos\left(\frac{2\pi}{5}\right) = \cos^2 \frac{\pi}{5} - \sin^2 \frac{\pi}{5}$$

$$\Leftrightarrow \frac{-1 + \sqrt{5}}{4} = \left(\frac{\sqrt{5} + 1}{4}\right)^2 - \sin^2\left(\frac{\pi}{5}\right)$$

$$\begin{aligned} \Leftrightarrow \sin^2\left(\frac{\pi}{5}\right) &= \frac{5 + 2\sqrt{5} + 1}{16} - \frac{-4 + 4\sqrt{5}}{16} \\ &= \frac{5 + 2\sqrt{5} + 1 + 4 - 4\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16} = \frac{5 - \sqrt{5}}{8} \end{aligned}$$

$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5 - \sqrt{5}}{8}} \approx 0.59$$

$$\sin \frac{2\pi}{5} = \sin\left(\frac{\pi}{5} + \frac{\pi}{5}\right) = \sin \frac{\pi}{5} \cdot \cos \frac{\pi}{5} + \cos \frac{\pi}{5} \cdot \sin \frac{\pi}{5} = 2 \sin \frac{\pi}{5} \cdot \cos \frac{\pi}{5}$$

$$= 2 \frac{1 + \sqrt{5}}{4} \sqrt{\frac{5 - \sqrt{5}}{8}} = \frac{1 + \sqrt{5}}{2} \cdot \frac{5 - \sqrt{5}}{8} = \sqrt{\frac{5 + \sqrt{5}}{8}} \approx 0.95$$

$$\tan \frac{\pi}{5} = \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} = \frac{\sqrt{\frac{5 - \sqrt{5}}{8}}}{\frac{\sqrt{5} + 1}{4}} = \frac{\sqrt{2(5 - \sqrt{5})}}{1 + \sqrt{5}} = \sqrt{5 - 2\sqrt{5}} \approx 0.73$$

$$\tan\left(\frac{2\pi}{5}\right) = \frac{\sin \frac{2\pi}{5}}{\cos \frac{2\pi}{5}} = \frac{4}{-1 + \sqrt{5}} \cdot \frac{1 + \sqrt{5}}{2} \cdot \sqrt{\frac{5 - \sqrt{5}}{8}} = \sqrt{5 + 2\sqrt{5}} \approx 3.05$$

14 Exercise 12

Exercise 21. To which order do you have to consider values in the series expansion of cosine, to approximate $\cos 1$ with an error smaller 10^{-7} ? Furthermore show that $\cos 1$ is irrational.

Hint: To show irrationality of $\cos 1$, assume, $p, q \in \mathbb{N}_+$ with $\cos 1 = \frac{p}{q}$. Replace that in the estimated

error of

$$\cos 1 - \sum_{k=0}^q \frac{(-1)^k}{(2k)!},$$

multiply with $(2q)!$ and derive a contradiction.

14.1 Exercise 12.a

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \cdot (-1)^k$$

Consider,

$$S_{2m} = \sum_{k=0}^{2m} \frac{1}{(2k)!} (-1)^k$$

$$S_{2k+1} = \sum_{k=0}^{2k+1} \frac{1}{(2k)!} \cdot (-1)^k$$

So S_{2k+1} has a negative, last expression. S_{2m} has a positive last expression.

$$S_{2k+1} < \cos 1 < S_{2m}$$

$$S_{2m} - S_{2m+1} = \sum_{k=0}^{2m} \frac{1}{(2k)!} (-1)^k - \sum_{k=0}^{2m+1} \frac{1}{(2k)!} (-1)^k$$

$$\Delta \cos(1) = -\frac{1}{(2(2m+1))!} \cdot (-1)^{2m+1} = \frac{1}{(2 \cdot (2m+1))!} \stackrel{!}{<} 10^{-7}$$

$$N! > 10^7 \Rightarrow N > 11$$

$$2 \cdot (2m+1) > 11$$

$$2m+1 > \frac{11}{2} = 5.5$$

\Rightarrow 10-th order because every odd expression is cancelled out.

Consider paper: “The irrationality of e and Others”.

14.2 Exercise 12.b

$$\cos(1) \notin \mathbb{Q}$$

Assume $\exists p \in \mathbb{Z}, q \in \mathbb{N}$:

$$\cos(1) = \frac{p}{q}$$

$$\begin{aligned} & \left| \cos(1) - \sum_{k=0}^n \frac{(-1)^k}{(2k)!} \right| \\ &= \left| \frac{p}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k}{(2k)!} \right| < \frac{1}{(2q)!} \end{aligned}$$

$$= \left| \frac{p(2q)!}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k \cdot (2q)!}{(2k)!} \right| < 1$$

$$|x - y| < 1 \Rightarrow 0 \quad \text{because } x \in \mathbb{Z}, y \in \mathbb{Z}$$

Leibniz criterion requires that the limit is not achieved in the sequence, because the functions need to be strictly monotonical.

15 Exercise 13

Exercise 22. Let $f : [\frac{\pi}{2}, \frac{3\pi}{2}] \rightarrow [-1, 1]$, $x \mapsto \sin x$. Show that f is bijective and compute (using the formula for the derivative of the inverse function $(f^{-1})'(y)$) at all possible points $y \in [-1, 1]$. Also give an explicit representation for f^{-1}

$$\dots = -\frac{1}{\sqrt{1-y^2}}$$

It is important to recognize the negative sign.

16 Exercise 14

Exercise 23. Let $w, z \in \mathbb{R}$ with $w, z, w+z \notin \{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\}$. Prove the addition theorem of the tangens function:

$$\tan(w+z) = \frac{\tan(w) + \tan(z)}{1 - \tan(w)\tan(z)}.$$

Let $x, y \in \mathbb{R}$ with $xy < 1$. Show that $\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and use it to prove the addition theorem for the arcustangens function:

$$\arctan(x) + \arctan(y) = \arctan \frac{x+y}{1-xy}.$$

1. Show that $\tan(w+z) = \frac{\tan(w)+\tan(z)}{1-\tan(w)\tan(z)}$.

$$\begin{aligned} \tan(w+z) &= \frac{\sin(w+z)}{\cos(w+z)} = \frac{\cos(w) \cdot \sin(z) + \sin(w) \cos(z)}{\cos(w) \cos(z) - \sin(w) \sin(z)} \\ &= \frac{\frac{\cos(w) \sin(w)}{\cos(w) \cos(z)} + \frac{\sin(w) \cdot \cos(z)}{\cos(w) \cdot \cos(z)}}{1 - \frac{\sin(w) \sin(z)}{\cos(w) \cos(z)}} \\ &= \frac{\tan(z) + \tan(w)}{1 - \tan(w) \tan(z)} \end{aligned}$$

2.

$$\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$x, y \in \mathbb{R}, xy < 1$.

Let $x = \tan(z)$ and $y = \tan(w)$.

$$xy = \tan(z) \cdot \tan(w) = \frac{\sin(z) \cdot \sin(w)}{\cos(z) \cdot \cos(w)} < 1$$

$$\sin(z) \cdot \sin(w) < \cos(z) \cos(w)$$

$$\Leftrightarrow 0 < \cos(z) \cdot \cos(w) - \sin(z) \cdot \sin(w)$$

$$\Leftrightarrow 0 < \cos(z+w) \Leftrightarrow z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \vee w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This proof is insufficient! A case distinction for $\cos(z) \cos(w) > 0$ is required.

3. Show that $\arctan(x) + \arctan(y) = \arctan \frac{x+y}{1-xy}$. Let $x = \tan(z)$ and $y = \tan(w)$.

$$\arctan \left(\frac{x+y}{1-xy} \right) = \arctan \left(\frac{\tan(z) + \tan(w)}{1 - \tan(z)\tan(w)} \right) = \arctan(\tan(z+w)) = z+w = \arctan(x) + \arctan(y)$$

17 Exercise 15

Exercise 24. Compute the following integrals by approximating the integrands using a sequence of step functions with the given points. Let $a, b \in \mathbb{R}$ with $a < b$.

1. $\int_a^b e^x dx$ with points $x_k := a + k(b-a)/n$.
2. $\int_a^b x^p dx$ with points $x_k := aq^k$, $q := \sqrt[n]{b/a}$ and $p \in \mathbb{R} \setminus \{-1\}$.

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=0}^{n-1} e^{a + \frac{k(b-a)}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} e^a \sum_{k=0}^{n-1} \left(e^{\frac{b-a}{n}} \right)^k \\ &= \lim_{n \rightarrow \infty} e^a \cdot \frac{b-a}{n} \frac{e^{\frac{b-a}{n}} - 1}{e^{\frac{b-a}{n}} - 1} \\ &= \lim_{n \rightarrow \infty} e^a \left(e^{\frac{b-a}{n}} - 1 \right) \cdot \underbrace{\frac{\frac{b-a}{n}}{e^{\frac{b-a}{n}} - 1}}_{\rightarrow 1} \\ &= e^a \cdot \frac{e^b}{e^a} - e^a = e^b - e^a \end{aligned}$$

$$\begin{aligned} &(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in [a, b]) : |\varphi_n(x) - e^x| < \varepsilon \\ e^{a+(b-a)\frac{n-1}{n}} - e^b &= e^{a+(b-a)(1-\frac{1}{n})} - e^b = e^{a+b-\frac{b}{n}-a+\frac{a}{n}} - e^b \\ &= e^{b-\frac{b}{n}+\frac{a}{n}} - e^b \end{aligned}$$