Linear Algebra 2 – Practicals

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Contents

1 Exercise 1 3

2 Exercise 2 4

3 Exercise 3 5

4 Exercise 4 5

5 Exercise 5 6

6 Exercise 6 7

7 Exercise 7 8

8 Exercise 8 9

9 Exercise 10 10

10 Exercise 11 11

11 Exercise 12 11

12 Exercise 13 12

13 Exercise 14 12

14 Exercise 15 12

15 Exercise 17 13

16 Exercise 18 14

17 Exercise 19 15

18 Exercise 21 16

19 Exercise 22 **17**

20 Exercise 23 **19**

21 Exercise 25 **19**

Exercises, I did on the board:

1 Exercise 1

Exercise 1. Determine the matrix representation of the linear map

$$f : \mathbb{R}_2[x] \to \mathbb{R}_3[x]$$

$$p(x) \mapsto x \cdot p(x)$$

in terms of the bases $B = \{1, x, x^2 - 1\} \subseteq \mathbb{R}_2[x]$ and $C = \{1, x, x^2 - 1, x^3 - 2x\} \subseteq \mathbb{R}_3[x]$

1.1 Blackboard solution

$$\mathcal{L}(\left\{\underbrace{1, x, x^2 - 1}_{b_1, b_2, b_3}\right\}) \to \mathcal{L}(\left\{\underbrace{1, x, x^2 - 1, x^3 - 2x}_{c_1, c_2, c_3, c_4}\right\})$$
$$f : \alpha \mapsto x \cdot \alpha$$

$$\begin{aligned} f(1) &= x = 1c_2 \\ f(x) = x &= x^2 = 1c_3 + 1c_1 \\ f(x^2 - 1) &= x^3 - x = 1c_4 + 1c_2 \end{aligned}$$

	b_1	b_2	b_3
c_1	0	1	0
c_2	1	0	1
c_3	0	1	0
c_4		0	1

1.2 My solution

$$\begin{aligned} B &= \{1, x, x^2 - 1\} =: \{b_1, b_2, b_3\} \\ C &= \{1, x, x^2 - 1, x^3 - 2x\} =: \{c_1, c_2, c_3, c_4\}\ f(b_1) &&= x \cdot (1) = x \\ f(b_2) = x \cdot (x) = x^2 \\ f(b_3) = x \cdot (x^2 - 1) = x^3 - x \end{aligned}$$

$$\begin{aligned} x &= \lambda_1 \cdot 1 + \lambda_2 \cdot x + \lambda_3 \cdot (x^2 - 1) + \lambda_4 \cdot (x^3 - 2x) \\ &= \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3 \end{aligned}$$

By coefficient comparison, we get $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 - 2\lambda_4 = 1$ where $\lambda_4 \overset{!}{=} 0$. Hence $\lambda_2 = 1$.

$$\implies \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x^2 = \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3$$

By coefficient comparison, we get $\lambda_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_4 = 0$.

$$\implies \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x^3 - x = \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3$$

By coefficient comparison, we get $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 - 2\lambda_4 = -1$ with $\lambda_4 \overset{!}{=} 1$, hence $\lambda_2 = 1$.

$$\implies \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

So our solution is,

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Exercise 2

Exercise 2. A chain complex C is a sequence of linear maps

$$0 = V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} V_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} V_0 \xrightarrow{f_0} 0$$

with the property such that $\operatorname{im} f_{k+1} \subseteq \ker f_k$ for all $0 \leq k \leq n-1$, hence, $f_k \circ f_{k+1} = 0$. The quotient space $H_k(C) = \ker f_k / \operatorname{im} f_{k+1}$ is called k-th ***homology*** of the complex. Show that for finite-dimensional chain complexs (hence, $\dim V_k < \infty$ for all k) the following formula holds:

$$\sum_{k=0}^{n-1} (-1)^k \dim V_k = \sum_{k=0}^{n-1} (-1)^k \dim H_k(C)$$

2.1 Blackboard solution

$V \subset W$ vector spaces.

$$V = \mathcal{L}\{v_1, \ldots, v_n\} \qquad W = \mathcal{L}\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$$

${}^{W}\!/\!_{V} = \{[x]_V : x \in W\}$

$$[x]_n := \{x + v \mid v \in V\}$$

$[w_1]_v, \ldots, [w_n]_v$ is a basis of vector space ${}^{W}\!/\!_{V}$.

for $x, y \in W$,

$$x \sim_V y := x - y \in V$$

$$y + v_2 \in [y]_V$$

$$[x]_V \odot [y]_V = [x + v_1 + y + v_2]_V$$

$$[x]_V \odot [y]_V = [x+y]_V$$
$$\alpha[x]_V = [\alpha x]_V$$

$$\sum_{k=0}^{n-1}(-1)^k \dim V_k = \sum_{k=0}^{n-1}(-1)^k \dim H_k(C).$$

where $\dim(V_k) = \dim \text{kernel}(f_k) + \dim \text{image}(f_k)$ and $\dim(H_k) = \dim \text{kernel}(f_k) - \dim \text{image}(f_k) = \dim \text{kernel}(f_k) - \dim \text{image}(f_{k+1})$.

3 Exercise 3

Exercise 3. Let $A \in \mathbb{K}^{n\times n}$ be a nilpotent matrix, hence, there exists $k \in \mathbb{N}$ such that $A^k = 0$.

- Show that $I - A$ is invertible with $(I-A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$.
- Use the previous result to derive the inverse of the matrix:

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3.1 Blackboard solution

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1[= (x-1)(1 + x + x^2 + \cdots + x^{n-1}])}{x-1}$$

Just verify:

$$(I - A)(I - A + A^2 + \cdots + A^{n-1})$$

4 Exercise 4

Exercise 4. 1. Let A be an invertible $n \times n$ matrix over the field $\mathbb{K}$ and u, v are column vectors (hence, $n \times 1$ matrices), such that $\sigma = 1 + v^t A^{-1} u \neq 0$. Show that $(A + uv^t)$ is invertible and that

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} u v^t A^{-1}$$

2. Apply this formula, to determine the inverse of matrix

$$\begin{pmatrix} 5 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

efficiently.

4.1 Blackboard solution

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} u v^t A^{-1} \qquad \text{(Sherman-Morrison-Formula)}$$

$$\sigma = 1 + v^t A^{-1} u \neq 0$$

$$\begin{aligned}
(A + uv^t)(A^{-1} - \frac{1}{\sigma} A^{-1} uv^t A^{-1}) &= AA^{-1} + uv^t A^{-1} - \frac{1}{\sigma}(AA^{-1}uv^t A^{-1} + uv^t A^{-1} uv^t A^{-1}) \\
&= I + uv^t A^{-1} - \frac{1}{\sigma}(uv^t A^{-1} + (v^t A^{-1} u) uv^t A^{-1}) \\
&= I + uv^t A^{-1} - \frac{1}{\sigma}(1 + v^t A^{-1} u) uv^t A^{-1} \\
&= I + uv^t A^{-1} - \frac{\sigma}{\sigma} uv^t A^{-1} = I
\end{aligned}$$

These practicals took place on 2018/03/14.

5 Exercise 5

Exercise 5. a. Determine the dual basis of $(\mathbb{R})^4$ to B

$$B := \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

b. Determine the matrix of the distinct (why distinct?) projection map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ with

$$\operatorname{image} \varphi = \mathcal{L} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ and } \operatorname{kernel} \varphi = \mathcal{L} \left\{ \begin{pmatrix} -1 \\ -2 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

5.1 Blackboard solution

It must hold that

$$\begin{aligned}
\langle b_1, b_1^* \rangle &= 1 \\
\langle b_2, b_2^* \rangle &= 0
\end{aligned}$$

$$\begin{pmatrix}
1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\
-1 & -2 & 2 & -1 & 0 & 0 & 1 & 0 \\
2 & -1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix} \rightsquigarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 3 & -4 & -5 & 4 \\
0 & 1 & 0 & 0 & 1 & 2 & 1 & -1 \\
0 & 0 & 1 & 0 & 2 & 5 & 3 & -2 \\
0 & 0 & 0 & 1 & 5 & 15 & 8 & -6
\end{pmatrix}$$

Pay attention! We transposed the matrix initially. Now we can read the solution vectors in columns. You can also transpose it only in the end.

$$B^* = \{b_1^*, b_2^*, b_3^*, b_4^*\}$$

where e.g. $b_1^* = (3, 1, 2, 5)^T$.

Exercise b: $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$.

$$\begin{aligned}
\operatorname{image} \varphi &= L((b_1, b_2)) \\
\operatorname{kernel} \varphi &= L((b_3, b_4))
\end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot B^{*^T} = P$$

$$P = \begin{pmatrix} -12 & 3 & 7 & 20 \\ -6 & 2 & 4 & 10 \\ 6 & -1 & -3 & -10 \\ -4 & 2 & 5 & 15 \end{pmatrix}$$

Why distinct? The projection matrix is given with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where row i is b_i and column j is d_j where b and d are the bases of the two vector spaces.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_B = 1 \cdot b_1 + 2 \cdot b_2 + 3 \cdot b_3 + 4 \cdot b_4$$

$$P_{E,E} = \Phi_B^E \cdot P_{B,B} \cdot \overbrace{\underbrace{\Phi_E^B}_{(\Phi_B^E)^{-1}} v_E}^{v_B}$$

How to compute the inverse efficiently?

Let $A, B, C \in \mathbb{R}^{2\times 2}$.

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} A\alpha & A\beta + B\gamma \\ 0 & C\gamma \end{pmatrix} \overset{!}{=} \infty$$

$$\alpha = A^{-1} \qquad \gamma = C^{-1}$$

$$\beta = -A^{-1}B\gamma$$

6 Exercise 6

Exercise 6. Let $V = \mathbb{R}[x]_2$.

$$\xi_1 < \xi_2 < \xi_3 \in \mathbb{R}$$

6.1 Whiteboard solution

Exercise a:

$$\beta_i : V \to \mathbb{R}$$

$$p(x) \mapsto p(\xi_i)$$

$$\dim(V) = \dim(V^*) = 3$$

$$\sum a_i \beta_i = 0 \iff a_i = 0 \forall i$$

$$\forall p \in \mathbb{R}[x]_2 : \sum a_i \beta_i(p(x)) \overset{!}{=} 0$$

$$\forall p \in \mathbb{R}[x]_2 : \sum a_i \beta_i(\xi_i) \overset{!}{=} 0$$

$$\implies p_1(\xi_1) = p_1(x_2) = 0 \implies a_3 = 0 \dots a_i = 0 \forall i$$

hence linear independent.

Exercise b:

$$\gamma : p(x) \mapsto p'(\xi_2)$$
$$\gamma(p(x)) = \sum a_i \beta_i(p(x)) = \sum a_i p(\xi_i) = p'(\xi_2)$$
$$p(x) = \alpha + \beta x + \delta x^2$$
$$\implies p'(\xi_2) = \beta + 2\delta\xi_2$$
$$p(x) = \alpha + \beta x + \delta x^2$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \end{pmatrix}}_{=A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2\xi_2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{\xi_2\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_1)} & \dots \\ -\frac{\xi_3\xi_1}{(\xi_2-\xi_1)(\xi_3-\xi_2)} & \dots \\ \frac{\xi_1\xi_2}{(\xi_3-\xi_1)(\xi_3-\xi_2)} & \dots \end{pmatrix}$$
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 1 \\ 2\xi_2 \end{pmatrix} = \begin{pmatrix} \frac{\xi_2-\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_1)} \\ \frac{\xi_1-2\xi_2+\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_2)} \\ \frac{\xi_2-\xi_1}{(\xi_3-\xi_1)(\xi_3-\xi_2)} \end{pmatrix}$$

Exercise c:

$$B = \{b_1(x), b_2(x), b_3(x)\}$$
$$l_i = \sum_{j=1}^{2} a_{ji} x^j$$
$$\beta_l(l_i(x)) = \delta_{li}$$

$$\begin{pmatrix} 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \\ 1 & \xi_3 & \xi_3^2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In essence, we look for $p(x) = \frac{(x_1-x)(x_2-x)}{(\xi_1-\xi_3)(\xi_2-\xi_3)}$. This is a Lagrange polynomial with $l_3 = p$.

7 Exercise 7

Exercise 7. Let V be a vector space with $\dim V = n < \infty$ and $U \subseteq V$ is a subspace with $\dim U = m$.

a. Show that $U^\perp = \{v^* \in V^*\}\, U \subseteq \operatorname{kernel} v^*$ is a subspace of dual space V^* and give $\dim U^\perp$.

b. Is $\{v^* \in V^* \mid U = \operatorname{kernel} v^*\}$ also a subspace?

Exercise a:

$$\left(U^\perp \quad = \{v^* \in V^* \mid U \subseteq \operatorname{kernel} v^*\} \quad = \{v^* \in V \mid \forall u \in U : v^*(u) = 0\}\right)$$

We prove subspace criteria:

1. $U^\perp \neq \emptyset$. Let $v^* : V \to \mathbb{K}$ with $v \mapsto 0$.

2.

$$\forall u_1^\perp, u_2^\perp \in U^\perp \forall \lambda, \mu \in \mathbb{K} : \lambda u_1^\perp + \mu u_2^\perp \in U^\perp$$

$$\lambda \underbrace{v_1^*(u)}_{0} + \mu v_2^*(u) = 0 \qquad \text{for } \forall v_1^*, v_2^* \in U^\perp, u \in U$$

Now, we need to determine the dimension $\dim U^\perp$.

Let $B_U = \{v_1, v_2, \ldots, v_m\}$.

$$B_V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$
$$B_{V^*} = \{v_1^*, v_2^*, \ldots, v_n^*\}$$
$$B_{U^\perp} = \{v_{m+1}^*, v_{m+2}^*, \ldots, v_n^*\} \text{ is basis of } U^\perp$$

$$\forall u \in U : v_j^*(u) = 0^C \forall j \in \{m+1, \ldots, n\}$$
$$B_U = \{v_1, v_2, \ldots, v_m\}$$
$$B_V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$
$$B_{V^*} = \{v_1^*, v_2^*, \ldots, v_n^*\}$$
$$B_{U^\perp} = \{v_{m+1}^*, v_{m+2}^*, \ldots, v_n^*\} \text{ is basis of } U^\perp$$
$$\implies \dim(U^\perp) = n - m$$

Exercise b:

$$W^\perp = \{v^* \in V^* \mid U = \text{kernel}(v^*)\}$$

The reason was given orally.

8 Exercise 8

Exercise 8. Let $f \in \text{Hom}(V, W)$ be a linear map between two finite-dimensional vector space with bases $B \subseteq V$ and $C \subseteq W$. We define the transposed map

$$f^T : W^* \to V^*$$
$$w^* \mapsto w^* \circ f$$

Hence $f^T(w^*)$ is a linear functional and $(f^T(w^*))(v) = w^*(f(v))$

a. Show that f^T is linear.

b. Show that the matrix representation, in regards of dual bases C^* and B^*, has the following matrix representation: $\Phi_{B^*}^{C^*}(f^T) = \Phi_C^B(f)^T$

Exercise a: Let $v \in V$ and $\lambda \in \mathbb{K}$, $w_1^*, w_2^* \in W^*$.

$$(f^T(w_1^* + w_2^*))(v) = (w_1^* + w_2^*)f(v) = w_1^*(f(v)) + w_2^*(f(w_1^*))(v) + (f^T(w_2^*))(v)$$
$$(f^T(\lambda w_1^*))(v) = (\lambda w_1^*)(f(v)) = \lambda w_1^*(f(v)) = \lambda(f^T(w_1^*))(v)$$

We proved $g(w_1 + \lambda w_2) = g(w_1) + \lambda g(w_2)$. Hence f^* is linear.

Exercise b:

$$\Phi_{B^*}^{C^*}(f^T) = \Phi_C^B(f)^T$$
$$\{v_1 \ldots, v_n\} = B \qquad \{w_1, \ldots, w_m\} = C$$
$$f(v_j) = \sum_{i=1}^m m_{ij} w_i$$

$$(f^t(w_i^*))(v_k) = w_j^*(f(v_k)) = w_j^*\left(\sum_{l=1}^{m} lkw_l\right) = m_{jk}$$

$$m_{jk} = \sum_{l=1}^{n} m_{jl} \underbrace{v_l^*(v_k)}_{\delta_{lk}}$$

$$= \sum_{l=1}^{n} m_{jl} v_l^*(v_k)$$

$$\implies f^T(w_j^*) = \sum_{l=1}^{n} A_{l,j} v_l^*$$

$$\implies A = \Phi_C^B(f)^T$$

$$A = \Phi_{B^*}^{C^*}(f^T)$$

These practicals took place on 2018/03/21.

9 Exercise 10

Exercise 9. A permutation $\pi \in \sigma_n$ is called cyclic, if there exists some $k \geq 1$ and a sequence $i_1, i_2, \ldots, i_k$ such that $\pi(i_j) = i_{j+1}$ for $1 \leq j \leq k-1$, $\pi(i_k) = i_1$ and $\pi(i) = i$ for $i \notin \{i_1, i_2, \ldots, i_k\}$, hence

$$i_1 \to i_2 \to \cdots \to i_1.$$

and all other i are fixed. Common notation: $\pi = (i_1, i_2, \ldots, i_k)$.

- Show, that two cyclic permutations $\pi = (i_1, i_2, \ldots, i_k)$ and $\rho = (j_1, j_2, \ldots, j_l)$ commutate ($\pi \circ \rho = \rho \circ \pi$), if $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_l\} = \emptyset$.
- Decompose the cycle into a product of transpositions and show that for a cyclic permutation, it holds that $\operatorname{sign}(\pi) = (-1)^{k-1}$.

For the first part,

Let $\operatorname{supp}(\pi) \cap \operatorname{supp}(\rho) = \emptyset$ where $\operatorname{supp}(\pi)$ defines the elements in the cycle of permutation π.

$i \notin \operatorname{supp}(\pi) \cup \operatorname{supp}(\rho)$

$$\implies \rho(i) = i = \pi(i) = i$$

$$\implies \pi(\rho(i)) = \rho(\pi(i)) = i$$

$i \in \operatorname{supp}(\pi)$ $\;i \in \operatorname{supp}(\pi) \implies \pi(i) \in \operatorname{supp}(\pi)$

$$\rho(\pi(i)) = \pi(i) \implies \rho(\pi(i)) = \pi(i) = \pi(\rho(i))$$

For the second part,

$$\pi = \tau_1 \cdot \tau_2 \cdot \cdots = (i_1, i_2)(i_2, i_3) \ldots (i_{k-1}, i_k)(i_k, i_1)$$

giving $k-1$ transposition.

$$\implies \operatorname{sign}(\pi) = (-1)^{k-1}$$

$$\tau_{24} = 1432$$

$$T_{34}^{2341} T_{23}^{2314} T_{42}^{2134}$$

10 Exercise 11

Exercise 10. Let $\pi \in \sigma_n$ be a permutation and $i \in \{1, 2, \ldots, n\}$.

1. Show that the sequence $i, \pi(i), \pi^2(i), \ldots$ is periodic and that the first number occuring twice is i.
2. The sequence $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$, where k is the smallest exponent such that $\pi^k(i) = i$, is called *cycle of i*. Show that the relation $i \sim j :\iff$ (j is in inside the cycle) defines an equivalence relation in $\{1, 2, \ldots, n\}$.
3. Show that every permutation can be written as product of commutative cycles.
4. Apply this decomposition to permutation π in Exercise 9.

Exercise (a).

k is certainly finite, because of the pidgeonhole principle. Furthermore smaller than n, because there are at most n numbers it can be mapped to. We have n distinct elements. i is the first element, which is not mapped to any number. So i is the first number which will occur for the second time. This implies that the map is bijective, which is given for any permutation.

Exercise (b).

Reflexivity is trivial. Symmetry: Let $\pi^l(i) = j$, then $\pi^{k-l}(j) = i$. This shows that both are in the same cycle and symmetry is given. If $i \sim j \land j \sim m \implies i \sim m$.

$$\pi(i) = j \qquad \pi^p(j) = m \iff \pi^p(\pi^l(i)) = m \iff \pi^{p+l}(i) = m$$

$$\pi^p \circ \pi^l(i) = m$$

Exercise (c).

$$1 \quad \pi(1) \quad \pi(\pi(1)) \quad \pi(\pi(\pi(1))) \quad \ldots$$

$$\pi = (\quad)(1, \ldots, \pi^{k-1})a_2\pi(a_2) \neq a_2$$

Exercise (d).

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix} = (1253)(467)$$

11 Exercise 12

Exercise 11. Show that every permutation $\pi \in \sigma_n$ can be written as composition of permutations $\gamma = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & n-1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 1 & \ldots & n-1 & n \end{pmatrix}$

From the lecture:

Every permutation $\sigma \in \sigma_n$ with $\sigma \neq \mathrm{id}$ can be denoted as a product of transpositions.

1. Consider the theorem from the lecture.
2. Every transposition can be represented as composition of swapping two neighbors.

$$\tau_{ij} = (i, i+1)(i+1, i+2)\ldots(j-1, j)(j-2, j-1)\ldots(i, i+1)$$

3.

$$\tau_{i,i+1} = \gamma^{i-1} \cdot \tau \cdot \gamma^{-(i-1)}$$

12 Exercise 13

Exercise 12. In the sliding 6-puzzle, which permutations can be reached?

We begin with the initial position (right-bottom shows the vacant field) and need to end with the initial position as well. We can only do transpositions with the vacant field.

1. even number of transpositions
2. signature $\pi = (-1)^{\#\text{ transpositions}}$
3. no permutation with $\operatorname{sign} -1$

The second item is wrong.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

Any permutation is a product of π_1 and π_2.

We can permute in a shape of the infinity symbol.

13 Exercise 14

Exercise 13. Determine the determinant using three different methods (Leibniz, Laplace, Gauss-Jordan) of the matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

TODO

14 Exercise 15

Exercise 14. The numbers 18270, 16128, 63042, 17304 and 17934 are divisible by 42. Show that the determinant

$$\det(A) = \begin{vmatrix} 1 & 8 & 2 & 7 & 0 \\ 1 & 6 & 1 & 2 & 8 \\ 6 & 3 & 0 & 4 & 2 \\ 1 & 7 & 3 & 0 & 4 \\ 1 & 7 & 9 & 3 & 4 \end{vmatrix}$$

is divisible by 42 without explicit evaluation.

$$\begin{vmatrix} 1 & 8 & 2 & 7 & 0 \\ 1 & 6 & 1 & 2 & 8 \\ 6 & 3 & 0 & 4 & 2 \\ 1 & 7 & 3 & 0 & 4 \\ 1 & 7 & 9 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 2 & 7 & 18270 \\ 1 & 6 & 1 & 2 & 16128 \\ 6 & 3 & 0 & 4 & 63042 \\ 1 & 7 & 3 & 0 & 17304 \\ 1 & 7 & 9 & 3 & 17934 \end{vmatrix}$$

$$\det(A) = \sum_{k=1}^{5} a_{k,5} \underbrace{(-1)^{k+5} \det A_{k,5}}_{\in\mathbb{Z}}$$

$\det(A)$ consists of 5 summands, which are divisible by 42 each, hence the sum is divisible

These practicals took place on 2018/04/11.

15 Exercise 17

Exercise 15. Evaluate the determinants:

15.1 Exercise 17a

Exercise 16.

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\begin{vmatrix} 0 & -x & -x & y+xy-x \\ 0 & -x & 0 & y \\ 0 & 0 & y & y \\ 1 & 1 & 1 & 1-y \end{vmatrix} = -1 \cdot \begin{vmatrix} -x & -x & y+xy-x \\ -x & 0 & y \\ 0 & y & y \end{vmatrix}$$
$$= (-1)(-xy^2 - (xy)^2 + x^2y - x^2y + xy^2) = (xy)^2$$

15.1.1 A simpler solution

Assume $C \in \mathrm{GL}(\mathbb{R})$ and $\vec{V}, \vec{W} \in \mathbb{R}^n$ where GL is the set of invertible matrices. Then it holds that

$$\det(C + \vec{v}\vec{w}^t) = \det C \left(1 + \langle C^{-1}\vec{v}, \vec{w}\rangle\right)$$

where $\langle \cdot, \cdot \rangle$ is an inner product with $\langle \vec{v}, \vec{w} \rangle = v_1 \cdot w_1 + \ldots + v_n \cdot w_n$.

$$A\vec{x} = \vec{b}$$
$$x_i = \frac{\det(A_j)}{\det A}$$

15.2 Exercise 17b

Exercise 17.

$$\begin{vmatrix} x & 0 & \ldots & a_0 \\ -1 & x & \ldots & a_1 \\ & -1 & \ddots & \\ & \ddots & \ddots & \\ 0 & & -1 & x + a_{n-1} \end{vmatrix}$$

Alternative approach: Use Laplace expansion theorem along the last column.

Always consider: A division by x requires a case distinction!

Case 1: $x \neq 0$:

$$\begin{vmatrix} x & & \ldots & a_0 \\ 0 & x & \ldots & a_1 + \frac{a_0}{x} \\ & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & & -1 & x + a_{n-1} \end{vmatrix} = \begin{vmatrix} x & & a_0 \\ & \ddots & a_1 + \frac{a_0}{x} \\ & & x + a_{n-1} + \frac{a_{n-2}}{x} + \ldots + \frac{a_0}{x^{n-1}} \end{vmatrix}$$

$$= x^{n-1}(x + a_{n-1} + \frac{a_{n-2}}{x} + \ldots) = x^n + x^{n-1}a_{n-1} + \ldots + a_0 = x^n + \sum_{i=1}^{n} a_{n-i}x^{n-i}$$

Case 2: $x = 0$.

$$\begin{vmatrix} 0 & & a_0 \\ -1 & \ddots & \\ & & -1 \cdot a_{n-1} \end{vmatrix} = (-1)^{n+1} \cdot a_0 \cdot \begin{vmatrix} -1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & -1 \end{vmatrix} = (-1)^{n+1} \cdot a_0 \cdot (-1)^{n-1} = (-1)^{2n} \cdot a_0 = a_0$$

15.3 Exercise 17c

Exercise 18.

$$\begin{vmatrix} 0 & 0 & \dots & & a_n & & \\ 0 & 0 & \dots & a_{n-1} & * & & \\ \vdots & & & & \vdots & & \\ 0 & a_2 & & & \vdots a_1 & * & * \end{vmatrix}$$

Case distinction: n is even.

$$= (-1)^{\frac{n}{2}} \begin{vmatrix} a_1 & * & & * \\ & a_2 & * & \vdots \\ & & \ddots & \\ 0 & & & a_n \end{vmatrix} = (-1)^{\frac{n-1}{2}} \prod_{i=1}^{n} a_i$$

You can skip the case distinction if you use the Gaussian bracket: $(-1)^{\lfloor \frac{n}{2} \rfloor}$

16 Exercise 18

Exercise 19. Show: There exists some matrix $A \in \mathbb{R}^{n \times n}$ with entries $a_{ij} = \pm 1$ such that $\det(A) = n!$ if and only if $n < 3$.

Hint: For $n = 2$, it is easy. For $n = 3$, consider why no all summands in Leibniz' formula for determinants have the same sign. The case $n > 3$ can be reduced to the case $n = 3$.

For $n = 2$,

$$2! = 2 \qquad \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 - (-1) = 2$$

For $n = 3$, we consider the Rule of Sarrus and assume such a matrix A exists. Because $n! = 6$, we need all summands of the Rule of Sarrus to be positive. We consider the diagonals given in the Rule of Sarrus and recognize, that both diagonals use the same elements. Consider the diagonals with positive sign. All of them must either use zero or two -1. At the same time, all diagonals with negative sign must either use three or one -1. This contradicts assuming they use the same elements. The proof by contradiction has been completed.

Now we look for the generalization of $n \to n + 1$ for $n \geq 3$.

This will be proven by complete induction.

Induction hypothesis $A \in \mathbb{R}^{n \times n}$ with $a_{ij} = \pm 1$

Induction base $n = 3$ has been proven

Induction step We apply Laplace expansion along one row. Let $\varepsilon^{(i)}$ be the value of $\det(A_n^{(i)})$ where A_n is a square matrix of dimension $n \times n$.

$$\det(A_{n+1}) = +\underbrace{\det(A_n^{(1)})}_{<n!} - \underbrace{\det(A_n^{(2)})}_{<n!} + \underbrace{\det(A_n^{(3)})}_{<n!} - \dots$$
$$= \sum_{i=1}^{n+1} \det(A_n^{(i)}) = \sum_{i=1}^{n+1} \varepsilon^{(i)} < (n+1)n! = (n+1)!$$

Hence $\det(A_{n+1}) < (n+1)n!$.

17 Exercise 19

Exercise 20. (a) Let $\mathbb{K}$ be a field and $a_1, a_2, \dots, a_n \in \mathbb{K}$. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \prod_{i<j} (a_j - a_i)$$

(b) Conclude from this, that for given pairwise different numbers $x_0, x_1, \dots, x_n \in \mathbb{K}$ and arbitrary $y_0, y_1, \dots, y_n \in \mathbb{K}$ there exists exactly one polynomial $p(x) \in \mathbb{K}[x]$ with degree n, such that $p(x_i) = y_i$ for all i.

(c) Extra point to be solved on a computer: Determine for each different n, one polynomial $p(x) \in \mathbb{R}[x]$, such that $p(x_k) = |\, x_k \,|$, $k = -n, \dots, n$, with $x_k = \frac{k}{n}$.

17.1 Exercise 19a

Induction base: $n = 2$.

$$\begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = (a_2 - a_1)$$

Induction step: $n - 1 \to n$.

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & a_1 & \dots & & a_1^{n-1} \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 & \dots & a_2^{n-1} - a_1^{n-1} \\ \dots & & & & \\ 0 & a_n - a_1 & a_2^2 - a_1^2 & & a_n^{n-1} - a_1^{n-1} \end{vmatrix}$$

The following equation holds:

$$(x^n - y^n) = (x - y) \sum_{i=0}^{n-1} x^{n-1-i} y^i$$

$$= \begin{vmatrix} (a_2 - a_1) & (a_2^2 - a_1^2) & (a_2^{n-1} - a_1^{n-1}) \\ \vdots & \vdots & \vdots \\ (a_n - a_1) & (a_n^2 - a_n^2) & (a_n^{n-1} - a_1^{n-1}) \end{vmatrix} = \prod_{i=2}^{n} (a_j - a_1) \cdot \begin{vmatrix} 1 & (a_2 + a_1) & (a_2^{n-2} + a_2^{n-3} a_1 + \dots + a_1^{n-2}) \\ 1 & (a_3 + a_1) & \vdots \\ \vdots & \vdots & \vdots \\ 1 & (a_n + a_1) & (a_n^{n-2} + \dots + a_1^{n-2}) \end{vmatrix}$$

$$= \prod_{j=2}^{n} (a_j - a_1) \cdot \begin{vmatrix} 1 & a_2 & a_2^2 & \dots & a_2^{n-2} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} \end{vmatrix}$$

$$= \prod_{j=2}^{n} (a_j - a_1) \prod_{\substack{i<j \\ i,j \neq 1}}^{n} (a_j - a_i) = \prod_{i<j}^{n} (a_j - a_i)$$

17.2 Exercise 19b

Show: there exists exactly one polynomial $p \in \mathbb{K}_n[x] (\forall i \in \{0, \dots, n\}) : p(x_i) = y_i$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\begin{array}{ccccccc} a_0 & + & a_1 x_0 & + & \dots & a_n x_0^n & = & y_0 \\ a_0 & + & a_1 x_1 & + & \dots & a_n x_1^n & = & y_1 \\ \vdots & & & & & & & \\ a_0 & + & a_1 x_n & + & \dots & a_n x_n^n & = & y_n \end{array} \iff \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\det(M) = \prod_{i<j} (x_j - x_i)$$

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & \dots & x_0^1 \\ \vdots & & \vdots \\ 1 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

17.3 Exercise 20

Exercise 21. Let $A, B \in \mathbb{K}^{n \times n}$. Show by elementary row- and column transformations, that the following identity for block matrices holds:

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$$

Derive an alternative proof for the multiplication law of determinants ($\det(AB) = \det(A) \cdot \det(B)$).

1. We consider the left-hand side.
2. We add the $n+1$-th row to the first row multiplied by a_{11} and use the result as row $n+1$. As a result, the value in $a_{n+1,1}$ becomes 0.
3. We add the $n+2$-th row to the first row multiplied by a_{21} and use the result as row $n+2$. As a result, the value in $a_{n+2,1}$ becomes 0.
4. We also do this process for columns and the second row.
5. As a result we get $\begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$.

$$\det(AB) = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix} = \begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = (-1)^n \begin{vmatrix} I & B \\ A & 0 \end{vmatrix} = (-1)^n (-1)^n \begin{vmatrix} A & 0 \\ I & B \end{vmatrix} = (-1)^{2n} \det(A) \det(B)$$

18 Exercise 21

Exercise 22. Prove by induction:

$$A := \begin{vmatrix} \alpha & \beta & \beta & \dots & \beta \\ \beta & \alpha & \beta & \dots & \beta \\ \vdots & & & \ddots & \vdots \\ \beta & \beta & \beta & \dots & \alpha \end{vmatrix} = (\alpha - \beta)^{n-1} (\alpha + (n-1)\beta)$$

Induction base For $n = 1$, it holds that $|\alpha| = \alpha$. Induction base satisfied.

Induction step

$$\frac{1}{\alpha^n}\begin{vmatrix} \alpha & \alpha\beta & \alpha\beta & \dots & \\ \beta & \alpha^2 & & & \\ \vdots & & \ddots & & \\ & & & & a^2 \end{vmatrix}$$

$$= \frac{1}{\alpha^n}\begin{vmatrix} \alpha & \alpha\beta & \alpha\beta & \dots & \\ \beta & \alpha^2 & & & \\ \vdots & & \ddots & & \\ & & & & a^2 \end{vmatrix}$$

$$= \frac{1}{\alpha^n}\begin{vmatrix} \alpha & 0 & 0 & \dots & \\ \beta & \alpha^2-\beta^2 & & & \\ \vdots & \beta(\alpha-\beta) & \ddots & & \\ & & & & \end{vmatrix}$$

$$= \alpha\frac{1}{alpha^n}\begin{vmatrix} \alpha^2-\beta^2 & & \beta(\alpha-\beta) \\ & \ddots & \\ \beta(\alpha-\beta) & & \alpha^2-\beta^2 \end{vmatrix} =: d$$

$$\begin{aligned} d &= \frac{1}{\alpha^{n-1}}(\alpha^2-\beta^2-\alpha\beta+\beta^2)^{n-1}\left(\alpha^2-\beta^2+(n-1)(\alpha\beta-\beta^2)\right) \\ &= \frac{1}{\alpha^{n-1}}(\alpha(\alpha-\beta))^{n-1}(\alpha+\beta)(\alpha-\beta)+(n-1)\beta(\alpha-\beta) \\ &= \frac{1}{\alpha^{n-1}}\alpha^{n-1}(\alpha-\beta)^{n-1}\cdot(\alpha-\beta)(\alpha+\beta+(n-1)\beta) \\ &= (\alpha-\beta)^n(\alpha+n\beta) \end{aligned}$$

Again: the division by α implies that $\alpha \neq 0$. It is important to consider $\alpha = 0$. It is easy to show this case, but if you skip it, points are lost.

19 Exercise 22

Exercise 23. Let $P_i = (x_i, y_i)$ are pairwise different points in $\mathbb{R}^2$.

1. Show that the uniquely determined line g crossing points P_1 and P_2 can be described by the following equation:

$$g = \left\{(x,y) \in \mathbb{R}^2 : \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} = 0\right\}$$

2. Show that the uniquely determined circle k crossing points P_1, P_2 and P_3, can be described by:

$$k = \left\{(x,y) \in \mathbb{R}^2 : \begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x & y & x^2+y^2 \end{vmatrix}\right\} = 0$$

   What is the result, if the points are colinear?

3. Determine the center of the circle crossing points $(-4, 1)$, $(-2, -3)$ and $(4, 5)$.

19.1 Exercise 22a

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

Again, consider: $x_2 = x_1$ separately!

Laplace expansion along the last row:

$$1 \cdot (x_1 y_2 - x_2 y_1) - x(y_2 - y_1) + y(x_2 - x_1) \stackrel{!}{=} 0$$

$$\underbrace{\frac{(x_1 y_2 - x_2 y_1)}{x_2 - x_1}}_{d} - x \underbrace{\frac{(y_2 - y_1)}{x_2 - x_1}}_{k}$$

$$y_0 = \frac{y_2 - y_1}{x_2 - x_1} x_1 + d$$

$$d = y_1 - \frac{(y_2 - y_1)x_1}{(x_2 - x_1)x_1} = \frac{y_1 x_2 - y_1 x_1 - y_2 x_1 + y_2 x_1}{x_2 - x_1}$$

This corresponds to the slope of the line. Hence, our model matches the formula (the one involving the determinant).

What about $x_2 = x_1$? Then the second column is a linear combination of the others. Hence, determinant equals 0.

19.2 Exercise 22b

Consider 3 points P_1, P_2 and P_3. Consider point A half-way of $\overline{P_1 P_2}$. Consider point B half-way of $\overline{P_1 P_3}$. If the line g_1, orthogonal to $P_1 P_2$ and crossing A, crosses with the line g_2, orthogonal to $P_1 P_3$ and crossing B, meet this crosspoint M is the center of the circumference circle of P_1, P_2 and P_3.

$$v_1 = P_2 - P_1 = (2, -4) \to A = P_1 + \frac{v_1}{2} = (-3, -1)$$
$$v_2 = P_3 - P_1 = (8, 4) \to B = P_1 + \frac{v_2}{2} = (0, 3)$$

$$n_1 = \perp v_1 = (4, 2)$$
$$n_2 = \perp v_2 = (4, -8)$$

$$g_1 = A + t \cdot n_1$$
$$g_2 = B + s \cdot n_2$$

19.3 Exercise 22c

$$\begin{pmatrix} -3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

Gives $t = 1$ and

$$\begin{pmatrix} -3 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = M$$

19.4 Exercise 22b: What if all points are colinear?

A generic circle equation is given by

$$(x - \overline{x})^2 + (y - \overline{y})^2 = r^2$$

$$x^2 - 2x\overline{x} + \overline{x}^2 + y^2 - 2y\overline{y} + \overline{y}^2 = r^2$$

$$x^2 + y^2 = \underbrace{r^2 - \overline{x}^2 - \overline{y}^2}_{K} + 2\overline{y}y + 2\overline{x}x$$

$$M \cdot \begin{pmatrix} K \\ 2\overline{x} \\ 2\overline{y} \end{pmatrix} = V$$

where M are the first three columns and V is the last column.

20 Exercise 23

Exercise 24. Let $A, B, C, D \in \mathbb{K}_{n\times n}$ be matrices. D is invertible and M is a $2n \times 2n$ block matrix.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

1. Show: M is invertible iff $A - BD^{-1}C$ is invertible
2. Show: $\det(M) = \det(A - BD^{-1}C)\det(D)$

20.1 Exercise 23a

$$\det(M) = \underbrace{\det(A - BD^{-1}C)}_{\neq 0 \text{ if invertible}} \quad \underbrace{\det(D)}_{\neq 0 \text{ if invertible}}$$

$\det(D)$ is invertible by the exercise specification.

$$\det(A - BD^{-1}C) \neq 0 \implies A - BD^{-1}C = \text{ invertible}$$

20.2 Exercise 23b

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & D \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{bmatrix}$$

$$\begin{vmatrix} I & B \\ 0 & D \end{vmatrix} \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{vmatrix} = \det(D) \cdot \det(A - BD^{-1}C)\det(I)$$

21 Exercise 25

Exercise 25. Let A be a $m \times n$ matrix. Show that $\operatorname{rank}(A)$ is identical with the largest number $k \in \{1, 2, \ldots, \min(m, n)\}$ for which a non-vanishing subdeterminant of order k exists, hence index sets $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$, such that

$$|A_{i_k,j_k}| := \begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \ldots & a_{i_1,j_k} \\ a_{i_2,j_1} & a_{i_2,j_2} & \ldots & a_{i_2,j_k} \\ \ldots & \ldots & \ddots & \vdots \\ a_{i_k,j_1} & a_{i_k,j_2} & \ldots & a_{i_k,j_k} \end{vmatrix} \neq 0$$

Assume $k \geq \operatorname{rank}(A)$.

$$A \to \tilde{A}$$

$m - \operatorname{rank}(A)$ rows and $n - \operatorname{rank}(A)$ columns. $\operatorname{rank}(A)$ is the number linear independent rows (or equivalently, columns)

$$\implies k \leq \operatorname{rank}(A) \implies k = \operatorname{rank}(A)$$