Theorem 1. *The MWF problem and MST problem are equivalent.*

Theorem 2. *(Optimality conditions.) Let (G, i) be an instance of MST and T be a spanning tree in G. In this case the following statements are equivalent:*

- *T is optimal*
- *$\forall\, e = \{x, y\} \in E(G) \setminus E(T)$: no edge of the x-y-path in T has greater weight than e*
- *$\forall\, e \in E(T)$: If C is one of the connected components of $T \setminus \{e\}$, then e is an edge from $\delta(V(c))$ with minimal weight.*
- *$E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ can be ordered such that $\forall\, i \in \{1, 2, \ldots, n-1\}$ there is a set $X \subseteq V(G)$ such that $e_i \in \delta(X)$ with minimal weight ad $e_j \neq \delta(X)\, \forall\, j \in \{1, 2, \ldots, i-1\}$.*

Theorem 3. *$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$.*

Theorem 4. *Krukal's algorithm is correct.*

Theorem 5. *Let G be a digraph with n vertices. The following 7 statements are equivalent:*

1. *G is an arborescence with root r.*
2. *G is a branching with $n-1$ edges and $\deg^-(r) = 0$.*
3. *G has $n-1$ edges and every vertices is reachable from r.*
4. *Every vertex is reachable from r and removal of one edge destroys this property.*
5. *G satisfies $\delta^+(X) \neq 0\ \forall\, X \subset V(G)$ with $r \in X$. The removal of one arbitrary edge destroys this property.*
6. *$\delta^-(r) = 0$ and $\forall\, v \in V(G) \setminus \{r\}\, \exists$ one distinct directed $r - v$-path in G*
7. *$\delta^-(r) = 0$ and $|\delta^-(v)| = 1\, \forall\, v \in V(G) \setminus \{r\}$ and G is cycle-free.*

Theorem 6. *Kruskal's algorithm can be implemented with time complexity $\mathcal{O}(m \log n)$.*

Theorem 7. *Prim's algorithm is correct and can be implemented with time complexity of $\mathcal{O}(n^2)$. Correctness follows from theorem 2.2.d ($a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$): Spanning tree is optimal $\Leftrightarrow$ order of edges $e_1, \ldots, e_{n-1}$ such that $\forall\, i \in \{1, 2, \ldots, n-1\}\, \exists x_i \subset V(G)$ with $e_i \in \delta(X_i)$ is the minimum edge in $\delta(X_i)$ and $e_j \notin \delta(X_i)$ is the cheapest edge of $\delta(X_i)$ and $e_j \notin \delta(X_i)\, \forall\, 1 \leq j \leq i-1$. This is satisfied by construction.*

Theorem 8. *Is Prim's algorithm implemented with Fibonacci-Heaps we can solve the MST problem in $\mathcal{O}(m + n \log n)$ time.*

$$\mathcal{O}(n^2) \qquad \mathcal{O}(m + n\log n) \qquad m = \theta(n^2) \qquad G \text{ is dense}$$

Theorem 9. *(Arthur Cayley) The complete graph K_n has n^{n-2} spanning trees.*

Theorem 10. *Let B_0 be a subgraph of G with maximum weight and $\deg^-_{B_0}(v) \leq 1 \; \forall v \in V(G)$. Then $\exists$ an optimal branching $B \in G$ with properties $\forall$ cycle $C \in B_0 : |E(C) \setminus E(B)| = 1$.*

Theorem 11. *Edmonds' Branching Algorithm is correct and computes the branching in $\mathcal{O}(m \cdot n)$.*

Theorem 12. *Let G be a digraph with conservative weights. $c : E(G) \to \mathbb{R}$. Let $s, w \in V(G)$ and $k \in \mathbb{N}$. Let P be the shortest among all s-w-pathes with at most k edges. Let $e = (v, w)$ be the last edge of P. Then $P_{[s,w]}$ is the shortest s-v-path with at most $(k-1)$ edges.*

Theorem 13. *Dijkstra's algorithm is correct and can be implemented in $\mathcal{O}(n^2)$.*

Theorem 14. *(Fredman and Tarjan, 1987) A Fibonacci-Heap implementation of Dijkstra's algorithm runs in $\mathcal{O}(m + n \log n)$ time.*

Theorem 15. *The Moore-Bellman-Ford algorithm is correct and has runtime $\mathcal{O}(nm)$.*

Theorem 16. *Let G be a digraph with $c : E(G) \to \mathbb{R}$. A potential of (G, c) exists iff c is conservative.*

Theorem 17. *Let $G = (V, E)$ be a digraph with $c : E(G) \to \mathbb{R}$. The Moore-Bellman-Ford algorithm can either determine a desired potential or find a negative cycle in $\mathcal{O}(m \cdot n)$.*

Theorem 18. *The Floyd-Warshall algorithm works correctly and has a runtime of $\mathcal{O}(n^3)$*

Theorem 19. *(Karp 1978.) Let G be a digraph with $c : E(G) \to \mathbb{R}$. Let $s \in V(G)$ such that $\forall v \in V(G) \setminus \{s\} \; \exists$ directed s-v-path in G.*

$$\forall x \in V(G) \; \forall K \in \mathbb{Z}_+ :$$

$$F_K(x) := \min \left\{ \sum_{i=1}^{k} c(v_{i-1}, v_i) : v_0 = s, v_k = x, (v_{i-1}, v_i) \in E(G), \; \forall 1 \leq i \leq k \right\}$$

If there is no sequence of edges of length k from s to x, then $F_K(x) = \infty$. Set $\mu(G, c)$ be the minimal mean edge weight of a cycle in (G, i) and $\mu(G, c) = \infty$ if G is acyclic. Then it holds that

$$\mu(G, c) = \min_{x \in V(G)} \max_{0 \leq k \leq n-1} \frac{F_n(x) - F_k(x)}{n - k}$$

Theorem 20. *The minimal mean cycle works correctly and can be implemented with a runtime of $\mathcal{O}(n \cdot \max\{m, n\})$.*

Theorem 21. *MFP always has an optimal solution. Linear programming always provides an optimal solution and is limited by $\sum_{e \in E(G)} u_e$.*

Theorem 22. *$\forall A \subsetneqq V(G)$ with $s \in A, t \notin A$ and for every s-t-flow it holds that:*

1. $\text{value}(f) = \sum_{e \in \delta^+(A)} f(e) - \sum_{e \in \delta^-(A)} f(e)$
2. $\text{value}(f) \leq \sum_{e \in \delta^+(A)} u_e$

Theorem 23. *Let (G, u, s, t) be a network and f be a flow. If there is no s-t-path in G_f, then f is optimal. Hence* $\text{value}(f)$ *is at maximum.*

Theorem 24. *(*Max flow, min cut problem*, Ford & Fulkerson, 1956) Let (G, u, s, t) be a network than there exists a maximal s-t-flow f and a minimal cut (s-t-cut) $\delta^+(A)$ with* $\text{value}(f) = u(\delta^+(A))$*. Especially the value of a maximal flow and the capacity of a minimal s-t-cut is equal.*

Theorem 25. ***Flow decomposition theorem** (Galler 1956, Ford and Fulkerson 1962) Let (G, u, s, t) be a network and f be a s-t-flow. Then $\exists$ a family $\mathcal{P}$ of s-t-paths and a family $\mathcal{C}$ of cycles in G and the weights in $\mathcal{P} \cup \mathcal{C} \to \mathbb{R}_+$ $(P \mapsto w(P), C \mapsto w(C))$ such that*

$$f(e) = \sum_{P \in \mathcal{P} \cup \mathcal{C} : e \in E(P)} w(P) \; \forall \, e \in E(G)$$

$$\text{value}(f) = \sum_{p \in \mathcal{P}} w(P) \quad \textit{and} \quad |\mathcal{P}| + |\mathcal{C}| \leq |E(G)|$$

Theorem 26. *Let $f_0, f_1, \ldots, f_k, \ldots$ be a sequence of flows created by the E&K algorithm, where $f_{i+1} = f_i + P_i$ and P_i is a shortest s-t-path in G_{f_i} $\forall \, i$. Then it holds that*

- $|E(P_k)| \leq |E(P_{k+1})| \; \forall \, i$
- $|E(P_k) + z \leq |E(P_r)||$ *for all $k < r$ such that $P_k \cup P_r$ contains at least one pair of edges of opposing direction.*

Theorem 27. *(Edmonds and Karp, 1972) The algorithm of Edmonds and Karp requires at most $\frac{nm}{2}$ augmented paths (equals to the number of iterations) and determines a maximum flow correctly. The algorithm has a runtime complexity of $\mathcal{O}(m^2 \cdot n)$.*

Theorem 28. *Dinitz' algorithm finds a maximum flow in $\mathcal{O}(n^2 m)$ runtime.*

Theorem 29. *The push-relabel algorithm has two invariants:*

- *f is always an s-t-preflow*
- *ψ is always a corresponding distance marker*

Theorem 30. *Let f be a preflow and ψ be a distance marker in regards of f. Then the following statements hold:*

1. *s is reachable from every active vertex v in G_f.*
2. *If $v, w \in V(G)$ with w being reachable from v in G_f, then $\psi(v) \leq \psi(w) + n - 1$*
3. *t is not reachable in G_f*

Theorem 31. *When PR algorithm terminates, f is a maximal s-t-flow.*

Theorem 32. *(number of relabel operations)*

- *$\forall v \in V(G) : \psi(v)$ is increased in every relabel operation by at least one (strong monotonicity, no decrement)*
- *$\psi(v) \leq 2n - 1$ is an invariant $\forall v \in V(G)$*
- *No vertex exists which is relabelled more than $2n - 1$ times. Hence the maximum number of relabel operations is $2n^2 - n$*

Theorem 33. *The number of saturating push operations is $2nm$.*

Theorem 34. Number of non-saturating push operations. *The number of non-saturating push operations is $\mathcal{O}(n^2 m)$.*

Theorem 35. Better analysis for number of non-saturating push operations. Cheriyan and Mehlhorn 1999. *If the algorithm always select an active vertex with maximum $\psi(v)$, then the push-and-relabel algorithm only requires $8n^2\sqrt{m}$ non-saturating push operations.*

Theorem 36. *The push-and-relabel algorithm solves the maximum-flow problem correctly and can be implemented with $\mathcal{O}(n^2\sqrt{m})$ runtime. (with selection of active vertices as in Theorem 35)*

Theorem 37. *For every triple of vertices $i, j, k \in V(G)$ (G is an undirected graph) it holds that*

$$\lambda_{i,k} \geq \min\{\lambda_{i,j}, \lambda_{j,k}\}$$

Theorem 38. *Let G be an undirected graph and $u : E(G) \to \mathbb{R}_+$. Let $s, t \in V(G)$ and $\delta(A)$ a minimal s-t-cut in (G', u'). (G', u') results from (G, u) by contraction of A by a single vertex K. Let $s', t' \in V(G) \setminus A$. Then it holds that*

$$\forall \min s\text{'-}t\text{'-cuts} : \delta(K \cup \{A\}) \text{ is } \delta(K \cup A) \text{ a minimal } s\text{'-}t\text{'-cut in } (G, u)$$

Theorem 39. *After every iteration of step 4, the following conditions hold:*

- $A \dot{\cup} B = V(G)$
- $E(A,B)$ *is a minimal s-t-cut in* (G,u)

$$A, B \subseteq V(G) \qquad E(A,B) := \{e \in E(G) : e = (x,y) \quad x \in A, y \in B\}$$

Theorem 40. *Invariant of the algorithm:*

$$w(e) = u(\delta_G(\bigcup_{z \in C_e} Z)) \,\forall\, e \in E(T)$$

where c_e *and* $V(T) \setminus c_e$ *are the two connected components of* $T - e$*. Furthermore it holds that*

$$\forall e = \{P, Q\} \in E(T) \quad \exists p \in P \quad \exists q \in Q \text{ with } \lambda_{p,q} = w(e)$$

Theorem 41. *The Gomory-Hu algorithm works correctly. Every undirected graph contains a Gomory-Hu tree which can be computed in runtime* $\mathcal{O}(n^3\sqrt{m})$*.*

Theorem 42. *In an undirected graph* G *with* $u : E(G) \to \mathbb{R}_+$ *we can compute a MA-order in* $\mathcal{O}(m + n \log n)$ *time.*

Theorem 43. *Let* G *be an undirected graph with* $u : E(G) \to \mathbb{R}_+$ *and MA-order* $u_1, \ldots, u_n$*. Then it holds that*

$$\lambda_{v_{n-1},v_n} = \sum_{e \in E(\{v_n\},\{v_1,\ldots,v_{n-1}\})}$$

Theorem 44. *A cut of minimal capacity in an undirected graph* G *with* $u : E(G) \to \mathbb{R}_+$ *can be computed with* $\mathcal{O}(nm + n^2 \log m)$ *runtime.*

Theorem 45. *Let* G *be a digraph with capacity* $u : E(G) \to \mathbb{R}_+$*. Let* f *and* f' *be b-flows in* G*. Then* $g : \overleftrightarrow{E}(G) \to \mathbb{R}$ *with* $g(e) = \max\{0, f'(e) - f(e)\}$ *and* $g(\overleftarrow{e}) = \max\{0, f(e) - f'(e)\} \;\forall\, e \in E(G)$ *is a* circulation *in* $\overleftrightarrow{G} := (V(G), \overleftrightarrow{E}(G))$*. Furthermore it holds that* $g(e) = 0 \;\forall\, e \in \overleftrightarrow{E}(G) \setminus E(G_f)$ *and* $c(g) = c(f') - c(f)$*.*

Theorem 46. *For every circulation* f *in a digraph* G *there is a family* $\mathcal{C}$ *of at most* $E(G)$ *cycles in* G *and positive numbers* $h(C) \;\forall\, c \in \mathcal{C}$ *with*

$$f(e) = \sum_{c \in \mathcal{C}, e \in E(C)} h(e)$$

Theorem 47. *(Klein, 1967) Let* (G, u, b, c) *be an instance of MKFP. A b-flow* g *has minimum costs exactly iff there are no* f*-augmented cycles with negative costs in* G_f*.*

Theorem 48. *(Corollary.) A b-flow has minimum costs iff* (G_f, C_f) *has a (valid) potential function.*

Theorem 49. *x optimal $\Rightarrow \exists$ optimal solution $(2_e)_{e \in E(G)}, (y_v)_v \in V(G)$ of DLP with non-satisfied complementary slack.*

Theorem 50. *Let $f_1, f_2, \ldots, f_K$ be a sequence of b-flows such that for all $i = 1, 2, \ldots, k-1$: $\mu(f_i) < 0$ and f_{i+1} originates from f_i by augmenting f_i along cycle K_i in G_{f_i} ($f_{i+1} = f_i \oplus K_i$).*

For now let K_i be a cycle with minimal average weight in G_f. Then the following statements hold:

$$\mu(f_i) \leq \mu(f_{i+1}) \; \forall \, i$$

$$\mu(f_i) \leq \frac{n}{n-2} \mu(f_c) \; \forall \, i < l$$

with property that $K_i \cup K_l$ contains at least one pair of edges of opposing direction.

Theorem 51. ***(Corollary)*** *During the MMCC algorithm $|\mu(f)|$ is decremented all $m \cdot n$ iterations by at least factor $\frac{1}{2}$.*

Theorem 52. *Assume $c : E(G) \to Q$ (without loss of generality: $c : E(G) \to \mathbb{Z}$) it holds that: after $\mathcal{O}(nm \log_2 n \, |c_{min}|)$ iterations the MMCC algorithm terminates with $c_{min} = \min \{\pm c_e | e \in E(G)\}$.*

Theorem 53. *(Tarjan, Goldberg, 1989) The MMCC algorithm can be implemented with $\mathcal{O}(m^3 n^2 \log n)$ runtime.*

Theorem 54. *Let (G, u, b, c) an instance of MKFP and f be a b-flow with minimum costs. Let P be a shortest s-t-path in regards of c_f in G_f for any $s, t \in V(G_f)$. f' results from f by augmentation along P by $\gamma \leq \min \{u_f(e) : e \in E(P)\}$, hence*

$$f'(e) = \left\{ \begin{array}{ll} f(e) & e \notin E(P), \overleftarrow{e} \notin E(P) \\ f(e) + \gamma & e \in E(P) \\ f(e) - \gamma & \overleftarrow{e} \in E(P) \end{array} \right\}$$

Then f' is a b'-flow with minimum costs where

$$b'(v) = \left\{ \begin{array}{ll} b(v) & \forall \, v \notin \{s, t\} \\ b(v) + \gamma & v = s \\ b(v) - \gamma & v = t \end{array} \right\}$$

Theorem 55. *Let G be a digraph with $u : E(G) \to \mathbb{R}_+$ and $b : V(G) \to \mathbb{R}$*

$$\sum_{v \in V(G)} b(v) = 0$$

$$\exists \text{ b-flow in } G \Leftrightarrow \forall X \subseteq V(G) \text{ it holds that:}$$

$$\sum_{e \in \delta^+(X)} u(e) \geq \sum_{v \in V(X)} b(v)$$

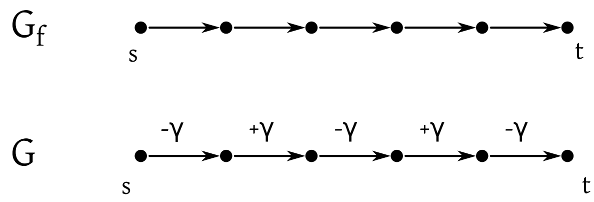


Figure 1: Proof of theorem 54

Theorem 56. *If the algorithm terminates with "there does not exist a b-flow in G", this statement is correct.*

Theorem 57. *If $u : E(G) \to \mathbb{Z}_+, b : V(G) \to \mathbb{Z}$ and c is conservative, the successive shortest path algorithm can be implemented in $\mathcal{O}(nm + B(m + n \log n))$.*

Theorem 58. *In every i-th iteration of the algorithm a potential function π exists:*

$$\pi : V(G) \to \mathbb{R} \text{ in } G_{f_i}(c_{f_i}(u,v) + \pi(u) - \pi(v) \geq 0) \, \forall \, e \in E(G_{f_i})$$

Theorem 59. *(Edmonds and Karp, 1972) The capacity scaling algorithm solves the MKFP with integers b, infinite capacities and conservative weights correctly. The algorithm can be implemented in $\mathcal{O}(n(m + n \log n) \log b_{max})$ runtime where $b_{max} := \max \{b(v) : v \in V(G)\}$.*

Theorem 60. *(Ford, Fulkerson, 1958) The MFoTP can be solved with the same time complexity like MKFP.*

Theorem 61. *(Berge, 1957) Let M be a matching in (G, E). M is maximal if and only if there is no M-augmenting path in G.*

Theorem 62. *Let $G = (v_1 \mathbin{\dot\cup} v_2, E)$ be a bipartite graph. Then it holds $v(G) = \zeta(G)$.*

Theorem 63. *(Hall's marriage condition.) Let G be a bipartite graph $(A \mathbin{\dot\cup} B, E)$ then G has a covering matching for A if and only if $|\Gamma(X)| \geq |X| \;\; \forall X \subseteq A$ where $\Gamma(X) = \{b \in B : \exists a \in X, (a,b) \in E(G)\}$.*

Theorem 64. *(Marriage corollary.) Let G be a bipartite graph with $V(e) = A \mathbin{\dot\cup} B$ and $|A| = |B|$. G has a perfect matching if and only if $\forall X \subseteq A$ with $|\Gamma(X)| \geq |X|$ holds.*

Theorem 65. *Let G be a graph, then*

$$q_G(X) - |X| \equiv |V(G)| \mod 2 \; \forall\, X \subseteq V(G)$$

Theorem 66. *Let G be a graph. G contains a perfect matching if and only if the Tutte condition is satisfied, hence $q_G(X) \leq |X| \; \forall\, X \subseteq V(G)$.*

Theorem 67. *(Theorem by Tutte.) Let G be a graph with a perfect matching $\Leftrightarrow q_G(x) \leq |X| \; \forall\, X \subseteq V(G)$ (tutte condition).*

Less formally: A graph $G = (V, E)$ has a perfect matching if and only if every subgraph G' of any $U \subseteq V(G)$ has at most $|U|$ connected components with an odd number of vertices.

Theorem 68. *Let M be a matching in M in G and T be an alternating degenerated tree. Then G has no perfect matching.*

Theorem 69. *Let C be an odd cycle in G and let G' be a graph which results by contraction of C. Let M' be a matching in G'. Then there exists a matching M in G with*

- *$M \subset M' \cup E(C)$*
- *the number of non-matched vertices of M in G equals the number of non-matched vertices of M' in G'*

Theorem 70. *Let G' be a graph constructed by iterative contraction of odd cycles as in Edmonds Blossom Algorithm. Let M' be a matching in G' and T be a M'-alternating tree in G, such that $\forall\, w \in A(T)$ is w a contracted vertex.*

It follows if T becomes atrophied (no edges left), then G has no perfect matching.

Theorem 71. *Edmonds Blossom Algorithm terminates after $\mathcal{O}(n)$ matching augmentations, $\mathcal{O}(n^2)$ contractions and $\mathcal{O}(n^2)$ extensions of the tree. It decides correct whether a perfect matching exists.*

Theorem 72. *Edmonds Blossom Algorithm can be implemented with runtime $\mathcal{O}(nm \log n)$.*

Theorem 73. *The assignment problem can be solved with $\mathcal{O}(nm + n^2 \log n)$ runtime.*

Theorem 74. *(Hoffman & Kruskal, 1956) Let $A \in \mathbb{Z}^{m \times n}$. The following statements are equivalent:*

1. *A is total unimodular.*
2. *Polyeder $P(b) := \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is integral $\forall\, b \in \mathbb{Z}^m$*
3. *Every quadratic regular submatrix of A has an integral inverse*

Theorem 75. *(Heller & Tompkins, 1959) Let* $A \in \{0, \pm 1\}^{m \times n}$ *with at most two non-zero eintries per column. A is total unimodular if there exists a partition* (R, T) *of the rows in A* ($R \cup T = \{1, 2, \dots, m\}$) *such that*

- *if column j contains two* ± 1 *entries, then the corresponding rows belong to different parts of the partition.*
- *if column j contains one* $+1$ *and one* -1 *entry, then the corresponding rows belong to the same part of the partition.*

Theorem 76. *(Corollary by Hoffman and Kruskal) Let A be total unimodular with* $A \in \{0, \pm 1\}^{m \times n}$.

1. *Then it holds that*

$$\forall\, c \in \mathbb{Z}^n,\ \forall\, b \in \mathbb{Z}^m: \begin{array}{ll} P_p & = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\} \\ P_d & = \{y \in \mathbb{R}^m : A^t y \geq c, y \geq 0\} \end{array}$$

   P_p *and* P_d *are integral.*

2. *Polyeder* $S = \left\{x \in \mathbb{R}^n : \underline{b} \leq Ax \leq \overline{b}, 0 \leq x \leq d\right\}$ *is integral if* $\underline{b}, \overline{b} \in \mathbb{Z}^m$ *and* $d \in \mathbb{Z}^n_+$.

Theorem 77. *(Theorem by Birckhoff) The permutation matrices correspond to the corners of an assignment polytop and every double-stochastic matrix can be represented as convex combination of permutation matrices.*

Theorem 78. *The following IDS are matroids*

1. *E is set of column vectors of a matrix A over an arbitrary field K.*

   $\mathcal{F} := \{F \subseteq E : \text{vectors of } F \text{ are linearly independent in } K\}$ *"vector matroid"*

$$Y = \{col_1, col_2, \dots col_k\}\ \forall \in \mathcal{F}$$

$$X = \left\{ \underbrace{\overline{col_1}, \overline{col_2}, \dots, \overline{col_l}}_{\text{linear indep.}} \right\} \in \mathcal{F} \qquad l > k$$

   Consider $X \cup Y$*:* $\operatorname{rank}(X \cup Y) \geq l$ *and* $\operatorname{rank}(Y) = k < \operatorname{rank}(X \cup Y)$*. Then it follows that*

$$\exists \text{vector } v \in X \cup Y \text{ with } Y \cup \{u\} \text{ linearly independent } v \in X \setminus Y$$

2. *IDS of exercise* 6*. "*Graphical matroids*".* X, Y *forests in* $G : |X| > |Y|$ *with (M3) condition. Show that* $\exists x \in \mathcal{X} : Y \cup \{x\}$ *is forest.*

   Assumption: $\forall\, x \in X : Y \cup \{x\}$ *is not a forest* $\Leftrightarrow$ *x is in a connected component of Y* $\forall\, x \in X$.

$\Rightarrow$ *every connected component of forest X is a subset of a connected component of forest Y.*

For any $G = (V, E)$ if G is cycle-free it holds that

$$|\textit{connected components}| = |V(G)| - |E(G)|$$

$$p := |\textit{connected components of } X|$$
$$q := |\textit{connected components of } Y|$$

$$p \geq q$$
$$p = |V(G)| - |X| \geq |V(G)| - |Y|$$

As far as $|X| \leq |Y|$, this is a contradiction.

> **Tree** *number of connected components $= n - (n-1)$.*
>
> **Forest** *number of connected components $= |V(G)| - |E(G)|$ if G is cycle-free.*

3. "Uniform matroid".

$$E = \{e_1, \ldots, e_n\} \quad \mathcal{F} := \{F \subseteq E : |F| \leq k\}$$

with $k \in \mathbb{N}$. $(M3)$ is trivial to show.

4. *$G = (V, E)$ is graph. $S \subseteq V(G)$ stable. $\forall s \in S : k_s \in \mathbb{N}$.*

$$E = E(G) \quad \mathcal{F} := \{F \subseteq E(G) : \delta_F(s) \leq k_s \ \forall\, s \in S\}$$

$$F = \{(1,2), (1,3), (4,5), \cancel{(4,2)}\}$$
$$F = \{(1,2), (1,3), (4,5), (4,3)\}$$

See figure 2.

(M3) $X, Y \in \mathcal{F} : |X| > |Y|$.

$$S' = \{s \in S : \delta_Y(s) = k_s\}$$

$|X| > |Y|$ *and* $\delta_X(s) \leq k_s \ \forall\, s \in S$

$$\xRightarrow{\textit{to show}} \exists e \in S \setminus Y\, e \notin \delta(s) \ \forall\, s \in S'$$

If such an edge exists, we can append it.

$$\Rightarrow Y \cup \{e\} \in \mathcal{F}$$

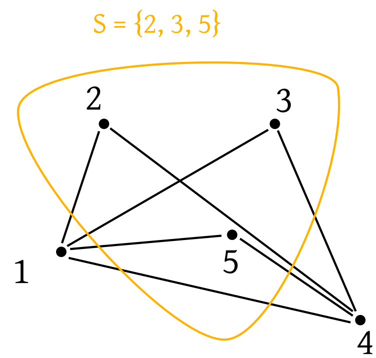
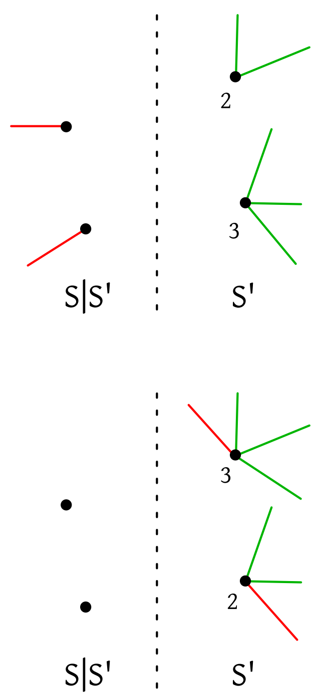


Figure 2: Example for Theorem 78 bullet point 4. $k_2 = 1, k_3 = 2, k_5 = 1$



$$\textit{Assumption:} \xRightarrow{\textit{to show}} \textit{does} \text{ not } \textit{hold:} \;\; \forall\, e \in X \setminus Y : \exists s \in S' : e \in \delta(s)$$

$$\Rightarrow |X| = \sum_{s \in S'} \delta_X(s) \leq \sum_{s \in S'} ks = \sum_{s \in S'} \delta y(s) = |Y|$$

$$|X| \leq |Y|$$

Contradiction to the assumption.

5. *Let* $G = (V, E)$ *be a digraph.* $S \subseteq V(E)$. $k_s \in \mathbb{N}\, \forall\, s \in S$. $E = E(G)$.

$$\mathcal{F} := \left\{ F \subseteq E : \delta_k^-(s) \leq k_s \right\}$$

(M3) analogous as in the previous item #4, but replace δ *with* δ^-. *Stability is relevant for the rational in item #4, but because a direction is given here, it is not required.*

Theorem 79. *Let* $(E, \mathcal{F})$ *be a IDS. Then the following statements are equivalent:*

M3: *Let* $X, Y \in \mathcal{F}, |X| > |Y| \Rightarrow \exists x \in X \setminus Y \quad Y \cup \{x\} \in \mathcal{F}$

M3': *Let* $X, Y \in \mathcal{F}, |X| = |Y| + 1 \Rightarrow \exists x \in X \setminus Y \quad Y \cup \{x\} \in \mathcal{F}$

M3": *For every* $X \subseteq E$ *the bases of* X *have the same cardinality.*

Theorem 80. *Let* $(E, \mathcal{F})$ *be an IDS. Then it holds that* $q(E, \mathcal{F}) \leq 1$. *Furthermore iff* $q(E, \mathcal{F}) = 1$ *then* $(E, \mathcal{F})$ *is a matroid.*

Theorem 81. *(Hausmann, Jenkyns, Korte, 1980) Let* $(E, \mathcal{F})$ *be an IDS. If* $\forall\, A \in \mathcal{F}\, \forall\, e \in E$, $A \cup \{e\}$ *contains at most* ρ *cycles, then it holds that*

$$q(E, \mathcal{F}) \geq \frac{1}{\rho}$$

Theorem 82. *(bases) Let* E *be a finite set and* $\mathcal{B} \subseteq 2^E$. *Family* $\mathcal{B}$ *is the set of bases of a matroid if and only if the following base axioms are satisfied*

(B1) $B \neq \varnothing$

(B2) $\forall\, B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \setminus B_2 : \exists y \in B_2 \setminus B_1$ *with* $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

If (B_1) *satisfies* (B_2), *then* $(E, \mathcal{F})$ *is the matroid with base set* $\mathcal{B}$ *where*

$$\mathcal{F} = \{F \subseteq E : \exists B \in \mathcal{B} \text{ with } F \subseteq B\}$$

Theorem 83. *Let* E *be a finite set and* $r : 2^E \to \mathbb{Z}_+$. *Then the following 3 statements are equivalent:*

- r *is the rank function of a matroid* $(E, \mathcal{F})$ *(with* $\mathcal{F} = \{F \subseteq E : r(F) = |F|\}$*).*
- $\forall\, X, Y \subseteq E$ *it holds that*

  (R1) $r(X) \leq |X|$

  (R2) $X \subseteq Y \Rightarrow r(X) \leq r(Y)$

(R3) $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ *(submodular)*

- $\forall X \subseteq E$ *and* $x, y \in E$ *it holds that*

(R1') $r(\varnothing) = 0$

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$

(R3') $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X) \Rightarrow r(X \cup \{x, y\}) = r(X)$

Theorem 84. *(Closure) Let E be a finite set with $r : 2^E \to 2^E$. σ is the closure function of a matroid if $\forall X, Y \subseteq E$ and $\forall x, y \in E$ it holds that*

(S1) $X \subseteq \sigma(X)$

(S2) $X \subseteq Y \Rightarrow \sigma(X) \subseteq \sigma(Y)$

(S3) $\sigma(\sigma(x)) = \sigma(x)$

(S4) $[y \notin \sigma(X) \wedge y \in \sigma(X \cup \{x\})] \Rightarrow x \in \sigma(X \cup \{y\})$

Theorem 85. *(Cycles) Let E be a finite set and $\mathcal{C} \subseteq 2^E$. $\mathcal{C}$ is the set of cycles of an IDS $(E, \mathcal{F})$ with $\mathcal{F} := \{F \subseteq E : \nexists C \in \mathcal{C} \text{ with } C \subseteq F\}$ if and only if the following conditions are satisfied:*

(C1) $\varnothing \notin \mathcal{C}$

(C2) $\forall C_1, C_2 \in \mathcal{C} : C_1 \subseteq C_2 \Rightarrow C_1 = C_2$

Furthermore for the set $\mathcal{C}$ of cycles of an IDS it holds that:

a) $(E, \mathcal{F})$ *is a matroid*

b) $\forall X \in \mathcal{F}\ \forall e \in E : X \cup \{e\}$ *contains at most one cycle. Denote this number of cycles as $C(X, e)$. If no cycle exists, let $C(X, e) = \varnothing$.*

where $a \Leftrightarrow b$.

Furthermore this statement is equivalent b)

(C3) $\forall C_1, C_2 \in \mathcal{C}$ *with* $C_1 \neq C_2\ \forall e \in C_1 \cap C_2, \exists C_3 \in \mathcal{C}$ *with* $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$

(C4) $\forall C_1, C_2 \in \mathcal{C},\ \forall e \in C_1 \cap C_2,\ \forall f \in C_1 \setminus C_2$ *exists* $C_3 \in \mathcal{C}$ *with* $f \in C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Theorem 86. *It holds that* $(E, \mathcal{F}^{**}) = (E, \mathcal{F})$

Theorem 87. *B^* base of $(E, \mathcal{F}^*) \Leftrightarrow \exists$ base B of $(E, \mathcal{F})$ with $B^* = B^C$ and $(E, \mathcal{F}^*)$ its dual. Let r and r^* be the corresponding rank functions. Then it holds that*

a) $(E, \mathcal{F})$ *is a matroid* $\Leftrightarrow (E, \mathcal{F}^*)$ *is matroid*

b) *If $(E, \mathcal{F})$ is a matroid, then it holds that $r^*(F) = |F| + r(E \setminus F) - r(E) \; \forall \, F \subseteq E$*

Theorem 88. *Let G be a connected planar graph with an arbitrary planar embedding. Let G^* be the planar duality of G. Let $M(G)$ be the graphical matroid of G. It holds that*

$$M^*(G) = (M(G))^* = M(G^*)$$

Furthermore G is planar if and only if $(M(G))^*$ is graphical; hence if a graph G' with $M(G') = (M(G))^*$ exists.

If G is planar, then G' is isomorphic to a planar embedding of G^*.

Theorem 89. *(Jenkyns, Korte, Hausmann, 1978) Let $(E, \mathcal{F})$ be an IDS and $c : E \to \mathbb{R}_+$. Denote $G(E, \mathcal{F}, c)$ as the costs of a solution determined by the BEST-IN-GREEDY algorithm. Denote $\mathrm{OPT}(E, \mathcal{F}, c)$ as the costs of an optimal solution (both for the maximization problem the GREEDY-IN algorithm is tackling).*

Then it holds that

$$q(E, \mathcal{F}) \leq \frac{G(E, \mathcal{F}, c)}{\mathrm{OPT}(E, \mathcal{F}, c)} \underbrace{\leq}_{trivial} 1 \; \forall \, c : E \to \mathbb{R}_+$$

Theorem 90. *(Edmonds, Rado, 1971) An IDS $(E, \mathcal{F})$ is a matroix if and only if the BEST-IN-GREEDY algorithm provides an optimal solution for the maximization problem $\forall \, c : E \to R_+$.*

Theorem 91. *(Edmonds 1971, polyedric representation) Let $(E, \mathcal{F})$ be a matroid and $r : E \to \mathbb{Z}_+$ be a rank function. Then the matroid polytop $P(E, \mathcal{F})$ (convex hull of incidence vectors of all independent sets) is given by:*

$$P(E, \mathcal{F}) = \left\{ x \in \mathbb{R}^{|E|} : x \geq 0, \sum_{e \in A} x_e \leq r(A) \; \forall \, A \subseteq E \right\}$$

$$F \in \mathcal{F} \qquad \underbrace{x^F(e)}_{incidence\ vectors} = \begin{cases} 1 & e \in F \\ 0 & else \end{cases} \qquad \sum_{e \in A} x_e^F = |A \cap F| \leq r(A)$$

We conclude: $x^F \in P(E, \mathcal{F}) \; \forall \, F \in \mathcal{F}$.