

# Linear Algebra 2

## List of potential exam questions

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### 7 Determinants

- Give the three determinant properties
- Give the definition of determinant forms and multilinearity
- Prove  $\Delta(a_1, \dots, a_k + \lambda a_i, \dots, a_n) = \Delta(a_1, \dots, a_k, \dots, a_n) \forall \lambda \in \mathbb{K}, \forall i \neq k$
- Prove  $\Delta(a_1, \dots, a_i, \dots, a_j, \dots, a_n) = -\Delta(a_1, \dots, a_j, \dots, a_i, \dots, a_n)$
- Define permutations
- Define transpositions
- Prove: Every permutation can be represented as product of transpositions
- Is the decomposition of a permutation as transpositions unique?
- Define and determine the signature of a permutation
- Prove  $\text{sign}(\pi) = \prod_{\substack{i,j \\ i < j}} \frac{\pi(j) - \pi(i)}{j - i}$
- Prove: every transposition has sign  $-1$
- Prove:  $\text{sign}(\pi \circ \sigma) = \text{sign}(\pi) \cdot \text{sign}(\sigma)$
- Prove:  $\forall \sigma \in \sigma_n : \Delta(a_{\sigma(1)}, \dots, a_{\sigma(n)}) = \text{sign}(\sigma) \cdot \Delta(a_1, \dots, a_n)$
- Define Leibniz' formula for determinants
- Prove Leibniz' formula
- Prove: Let  $B$  and  $C$  be two bases of a vector space. The determinant of  $B$  is non-trivial iff the determinant of  $C$  is non-trivial.
- Prove:  $\Delta$  non-trivial  $\iff \Delta(b_1, \dots, b_n) \neq 0$  for every basis
- Prove: Let  $\Delta$  be a non-trivial determinant form  $\Delta(v_1, \dots, v_n) \neq 0 \iff v_1, \dots, v_n$  is linearly independent.
- Prove: Two determinant forms are different only by some factor
- Prove:  $f : V \rightarrow V$  is invertible  $\iff \det(f) \neq 0$ .
- Prove: For a matrix  $A \in \mathbb{K}^{n \times n}$  it holds that  $\det A \neq 0 \iff A$  has full rank.
- Prove:  $f, g : V \rightarrow V$  linear.  $\implies \det(f \circ g) = \det(f) \cdot \det(g)$
- Prove  $\det(A \cdot B) = \det(A) \cdot \det(B)$  directly
- Prove:  $\det(A^{-1}) = \frac{1}{\det(A)}$  if invertible
- Prove:  $\det(A) = 0 \iff \text{rank}(A) < n$
- Prove:  $\det(A^t) = \det(A)$

- Define  $\text{perm}(A)$
- Prove: Let  $A$  be an upper triangular matrix, hence  $a_{ij} = 0$  if  $i > j$ .  $\implies \det(A) = a_{11}a_{22} \dots a_{nn}$ .
- Prove: 
$$\begin{vmatrix} & & 0 \\ & B & 0 \\ & & 0 \\ * & * & * & a_{nn} \end{vmatrix} = \det(B) \cdot a_{nn}$$
- Define Laplace Expansion
- Define the cofactor of a matrix
- Define the complementary matrix
- Prove:  $A^{-1} = \frac{1}{\det A} \hat{A}$
- Give and prove Cramer's rule

## 8 Inner products

- Give a geometrical proof of the Pythagorean theorem
- Define the scalar product in  $\mathbb{R}^2$  and  $\mathbb{R}^3$
- Prove  $\langle a, b \rangle = \langle b, a \rangle$ ,  $\langle \lambda a, b \rangle = \lambda \langle a, b \rangle = \langle a, \lambda b \rangle$  and  $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$  in  $\mathbb{R}^2$
- Prove that the scalar product in  $\mathbb{R}^3$  is the dot product.
- Give the law of Cosines
- Define an outer product (in  $\mathbb{R}^3$ )
- Prove  $b \times a = -a \times b$ ,  $(\lambda a) \times b = \lambda(a \times b) = a \times (\lambda b)$  and  $(a + b) \times c = a \times c + b \times c$
- Define bilinearity
- Define antisymmetry
- Define an inner product
- The function value of an inner product is an element of which domain?
- Define positive/negative-(semi)definite and indefinite inner products (in terms of inner products and in terms of matrices).
- Define a scalar product in terms of inner products.
- What is a positive definite inner product in Hermitian form?
- What is a unitary product?
- When does the scalar product satisfy  $\langle a, b \rangle = \langle b, a \rangle$ ?
- Why is the scalar product defined with sesquilinearity and not bilinearity?
- Define norms
- Every  $\square$  induces a  $\square$ . How?
- Define  $l$ -norms
- Prove:  $\|x\| := \sqrt{\langle x, x \rangle}$  is a norm on  $V$
- Give and prove: CBS inequality
- Prove: Let  $V$  be a vector space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $B = \{b_1, \dots, b_n\}$  be a basis.  $\langle, \rangle$  is an inner product. There exists a unique matrix  $A$  in Hermitian form such that  $\forall x, y \in V : \langle x, y \rangle = \Phi_B(x)^T \cdot A \cdot \overline{\Phi_B(y)}$ . Additionally show: If  $\langle, \rangle$  is positive definite,  $A$  is invertible.

- Define the conjugate transpose of a matrix.
- Define self-adjoint matrices.
- Define symmetrical matrices.
- Define Hermitian matrices.
- Give one simple example each for a (positive/negative) (semi)definite matrix and indefinite matrix.
- Define congruent matrices.
- Prove: Every Hermitian matrix is congruent to a diagonal matrix  $D$  of form  $\text{diag}(D) = (1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$ .
- Give and prove Sylvester's law of inertia.
- Define the index and signature of a matrix.
- Prove:  $A > 0 \iff A \hat{=} I \iff \text{ind}(A) = n$
- Prove:  $A \geq 0 \iff \text{ind}(A) = \text{sign}(A) = \text{rank}(A)$
- Prove:  $A \hat{=} B \iff \text{ind}(A) = \text{ind}(B) \wedge \text{sign}(A) = \text{sign}(B)$
- Prove:  $\det(C^*) = \overline{\det(C)}$
- Prove:  $A = A^* \implies \det(A) \in \mathbb{R}$
- Prove:  $A = A^*, B = B^*, A\hat{=}B \implies \text{sign } \det(A) = \text{sign } \det(B)$
- Prove:  $A > 0 \implies \det(A) > 0$
- Define the minor of a matrix.
- Let  $A = A^*$ . Prove:  $A > 0 \iff$  all first minors  $A_r$  satisfy  $\det(A_r) > 0$
- All submatrices of a positive definite matrix are  $\square$
- Let  $A = A^*$ . Prove:  $A < 0 \iff (-1)^r \det(A_r) > 0 \forall r \in \{1, \dots, n\}$
- What is an Euclidean space? What is a unitary space?
- What is a Hilbert space?
- Give the parallelogram law.
- Define orthogonal and orthonormal families of vectors.
- Define orthonormal bases of vectors.
- Let  $(v_i)_{i \in I} \subseteq V, v_i \neq 0 \forall i$ . Prove:  $(v_i)_{i \in I}$  is orthogonal, then  $(v_i)_{i \in I}$  is linear independent.
- Let  $B = (b_1, \dots, b_n)$  is an orthonormal basis of a finite-dimensional vector space over  $\mathbb{K}$ . For  $v \in V$ , let  $\Phi_B(v) = (\lambda_1 \dots \lambda_n)^T$ . For  $w \in V$ , let  $\Phi_B(w) = (\mu_1 \dots \mu_n)^T$ . Prove:
  1.  $\lambda_i = \langle v, b_i \rangle$
  2.  $\langle v, w \rangle = \sum_{i=1}^n \lambda_i \overline{\mu_i}$
- Define orthogonal complements
- Let  $V$  be a vector space with scalar product.  $M, N \subseteq V$  are partitions.
  1.  $M^\perp$  is a subspace.
  2.  $M \subseteq N \implies N^\perp \subseteq M^\perp$   
 $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$
  3.  $\{0\}^\perp = V$
  4.  $V^\perp = \{0\}$

5.  $M \cap M^\perp \subseteq \{0\}$

6.  $M^\perp = \mathcal{L}(M)^\perp$

7.  $M \subseteq (M^\perp)^\perp$

- Prove: Let  $U \subseteq V$  be a subspace.  $U + U^\perp$  is a direct sum in  $\mathbb{R}^n$  such that  $U + U^\perp = \mathbb{R}^n$ .
- Define convexity of functions and sets.
- Let  $V = U \dot{+} U^\perp$ . Prove:  $\forall x, y \in V : \langle x, \pi_U(y) \rangle = \langle \pi_U(x), y \rangle = \langle \pi_U(x), \pi_U(y) \rangle$
- Let  $V = U \dot{+} U^\perp$ . Prove:  $\|\pi_U(x)\| \leq \|x\| \wedge \|\pi_U(x)\| = \|x\| \iff x \in U$
- Define the Gram matrix.
- Let  $v_1, \dots, v_m \in V$ .  $G = \text{Gram}(v_1, \dots, v_m)$ . Prove:  $G = G^*$  is Hermitian, positive semidefinite.
- Let  $v_1, \dots, v_m \in V$ .  $G = \text{Gram}(v_1, \dots, v_m)$ . Prove:  $\xi \in \ker G \iff \sum_{i=1}^m \overline{\xi_i} v_i = 0$
- Let  $v_1, \dots, v_m \in V$ .  $G = \text{Gram}(v_1, \dots, v_m)$ . Prove:  $G$  is positive definite iff  $(v_1, \dots, v_m)$  are linear independent.
- Give Bessel's inequality. Give an intuition what the inequality says/when it can be useful.
- Give Parseval's identity. Give an intuition what the inequality says/when it can be useful.
- Give and prove the Gram–Schmidt process for orthogonalization
- Define Laguerre polynomials
- Define Hermite polynomials
- How are Laguerre and Hermite polynomials related to the Gram–Schmidt process?
- Give Riesz representation theorem
- Prove Riesz representation theorem
- Does Riesz representation theorem hold for infinite-dimensional spaces?
- Prove:  $v = 0 \iff \forall w \in V : \langle v, w \rangle = 0$
- Prove:  $\langle x, y \rangle = \langle x, z \rangle \forall x \implies y = z$
- Prove:  $\|v\| = \sup \{ |\langle v, w \rangle| \mid \|w\| \leq 1 \}$
- Define adjoint maps
- Prove: Let  $(V, \langle \cdot, \cdot \rangle_V)$  and  $(W, \langle \cdot, \cdot \rangle_W)$  be spaces with a scalar product.  $\dim V, \dim W < \infty$ .  $T : W \rightarrow V$  linear. Prove: For every  $v \in V$  the map  $w \mapsto \langle T(w), v \rangle_V$  is linear.
- Prove: Let  $(V, \langle \cdot, \cdot \rangle_V)$  and  $(W, \langle \cdot, \cdot \rangle_W)$  be spaces with a scalar product.  $\dim V, \dim W < \infty$ .  $T : W \rightarrow V$  linear. Prove:  $\forall v \in V \exists! u \in W \forall w \in W : \langle T(w), v \rangle_V = \langle w, u \rangle_W$  and  $T^*(v) = u$ .
- Prove: Let  $(V, \langle \cdot, \cdot \rangle_V)$  and  $(W, \langle \cdot, \cdot \rangle_W)$  be spaces with a scalar product.  $\dim V, \dim W < \infty$ .  $T : W \rightarrow V$  linear. Show:  $T^* \in \text{Hom}(V, W)$
- Prove: Let  $(V, \langle \cdot, \cdot \rangle_V)$  and  $(W, \langle \cdot, \cdot \rangle_W)$  be spaces with a scalar product.  $\dim V, \dim W < \infty$ .  $T : W \rightarrow V$  linear. Prove: the map  $\text{Hom}(W, V) \mapsto \text{Hom}(V, W)$  with  $T \mapsto T^*$  is antilinear and  $T^{**} = T$ .
- Does every linear map have an adjoint map?
- Define involutions
- Prove: Let  $B \subseteq V, C \subseteq W$  be orthonormal bases.  $f \in \text{Hom}(V, W)$ .

$$\Phi_B^C(f^*) = \Phi_C^B(f)^* = \overline{\Phi_C^B(f)^T}$$

- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $(g \circ f)^* = f^* \circ g^*$

- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $f^{**} = f$
- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $\ker f = (\operatorname{im} f^*)^\perp$
- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $\operatorname{im} f = (\ker f^*)^\perp$
- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $f$  injective  $\iff f^*$  surjective
- Let  $U, V, W$  be finite-dimensional.  $U \xrightarrow{f} V \xrightarrow{g} W$ . Prove:  $f$  surjective  $\iff f^*$  injective
- Define self-adjoint matrices
- Define linear isometries (= unitary transformations)
- Prove: unitary transformations are injective
- If  $\dim V = \dim W < \infty$  and  $f : V \rightarrow W$  is linear and unitary, then  $f$  is invertible and  $f^{-1} = f^*$ .
- If  $\dim V = \infty$ ,  $f : V \rightarrow V$  is isometry, it does not imply that  $f$  is invertible.
- Define unitary matrices
- Let  $U$  be a unitary matrix. Prove:  $UU^* = I \iff U^*U = I$ .
- Define orthogonal matrices
- Eigenspaces are always  $\square$
- Prove:  $T \in \mathbb{C}^{n \times n}$  is unitary  $\implies T$  is unitary
- Prove:  $T \in \mathbb{C}^{n \times n}$ .  $\forall x, y \in \mathbb{C}^n : \langle Tx, Ty \rangle = \langle x, y \rangle \iff$  the columns of  $T$  define an orthonormal basis of  $\mathbb{C}^n$
- Define isometries in metric spaces.
- Is translation an isometry?
- Is translation unitary?
- What does the counterclockwise transformation matrix in  $\mathbb{R}^2$  look like?
- Define the orthogonal group
- Define the unitary group
- Define the special orthogonal group
- Define the special unitary group
- Define the general linear group
- Define the special linear group
- Prove:  $U \in \mathcal{U}(n) \implies |\det(U)| = 1$
- Give the general layout of a quaternion
- What is the multiplicative identity element of quaternions?

## 9 Polynomials

- Define algebras
- Does an algebra satisfy associativity?
- Does an algebra satisfy commutativity?
- Define Jordan algebras
- Define the polynomial algebra
- Define the algebra of formal power series
- Define the degree of polynomials. Give the degree of polynomials  $0, 1, x$  and  $x^2$ .
- Prove: Let  $p(x)$  and  $q(x)$  be polynomials.  $\deg(p(x)) + \deg(q(x)) = \deg(g(x) \cdot q(x))$
- Define zero division freedom for polynomials
- Prove: Every polynomial induces a polynomial function and the map  $p(x) \mapsto p_f(x)$  is linear and multiplicative.
- Define algebra homomorphisms
- Give and prove the insertion theorem for polynomials
- Define the root of a polynomial
- There is a formula to find roots of any quadratic equation. What is the highest polynomial degree such that a formula exists?
- What is the idea of Cardanos cubic formula?
- Prove polynomial division with remainder:  $p(x), q(x) \in \mathbb{K}[x], q(x) \neq 0$ . Then there exists exactly one polynomial  $s(x), r(x) \in \mathbb{K}[x], p(x) = s(x) \cdot q(x) + r(x)$ . with  $\deg r(x) < \deg q(x)$ .
- Give Ruffini-Horner's method
- Define reducibility of polynomials
- Give the Fundamental Theorem of Algebra
- Define the greatest common divisor for polynomials
- Prove Bezout's identity for polynomials

## 10 Eigenvalues and eigenvectors

- Define eigenvalues and eigenvectors
- Give an intuitive notion of eigenvalues and eigenvectors
- Can 0 be an eigenvalue? Can  $\vec{0}$  be an eigenvector?
- Does every matrix have eigenvalues and eigenvectors?
- Give simple examples in  $\mathbb{R}^{2 \times 2}$  and  $\mathbb{Q}^{2 \times 2}$  without eigenvectors and eigenvalues.
- Define spectrums of linear maps
- Let  $A, B \in \mathbb{C}^{n \times n}$ . Prove:  $\text{spec}(AB) = \text{spec}(BA)$
- What kind of linear map do you need to consider eigenspaces (necessary, but not sufficient condition)?
- Define eigenspaces of linear maps
- What is the eigenvalue of the matrix  $\lambda \cdot \text{id}$ ?
- Let  $b_1, \dots, b_n$  be a basis of  $V$ . Let  $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ . Then there exists a linear map  $f$  such that  $f(b_i) = \lambda_i \cdot b_i$  and this map is  $\square$ .

- Prove: Left-sided eigenvalue  $\iff$  right-sided eigenvalue
- Write it down formally: The spectrum does not depend on the choice of the basis
- Prove: The spectrum does not depend on the choice of the basis
- Define characteristic polynomials of matrices
- Let  $A \in \mathbb{K}^{n \times n}$ . What is the degree of the characteristic polynomial of  $A$ ?
- Complete and prove:  $\lambda$  is eigenvalue  $\iff \chi_A(\lambda) = 0$
- Prove: A square matrix  $A$  is invertible if and only if 0 is not an eigenvalue of  $A$
- Define symmetrical minors
- Prove:  $\chi_{T^{-1}AT}(x) = \chi_A(x)$
- Define diagonalizable matrices
- Define equivalent matrices
- Define similar matrices
- Prove:  $A$  is diagonalizable  $\iff \exists$  basis of eigenvectors.
- Define exponentiation of matrices
- Define Fibonacci sequences
- Give an explanation of the relationship of rabbit populations and the Fibonacci sequence
- Give the iteration matrix of the Fibonacci sequence
- The golden ratio is the root of which polynomial?
- Give the golden ratio
- Prove: eigenvectors corresponding to different eigenvalues are linear independent.
- Prove: an  $n \times n$  matrix with  $n$  different eigenvalues is diagonalizable.
- Define the geometric and algebraic multiplicity of an eigenvalue
- Prove: A matrix is diagonalizable iff for different eigenvalues  $\lambda_1, \dots, \lambda_r$  it holds that  $\sum_{i=1}^r d(\lambda_i) = n$
- Give an argument why there are at most  $n$  eigenvalues for a matrix in  $\mathbb{K}^{n \times n}$
- Prove: For every eigenvalue  $\lambda$ , it holds that  $d(\lambda) \leq k(\lambda)$
- Define nilpotent matrices

## 11 Jordan normal form

- Define invariant subspaces
- Show: Eigenspaces are invariant subspaces
- Prove: Let  $A \in \mathbb{K}^{n \times n}$ ,  $V = \mathbb{K}^n$ . If  $U \subseteq V$  is invariant and  $p(x) \in \mathbb{K}[x]$ , then  $U$  is invariant under  $p(A)$ .
- Prove: Let  $A \in \mathbb{K}^{n \times n}$ ,  $V = \mathbb{K}^n$ .  $U_1, \dots, U_k$  are invariant subspaces. Then  $\bigcap_{i=1}^k U_i$  and  $\bigoplus_{i=1}^k U_i$  are invariant with respect to  $A$ .
- Prove: Let  $f : V \rightarrow V$  and let  $U \subseteq V$  be an invariant subspace. Then  $f|_U : U \rightarrow U$  is a homomorphism
- Prove: Let  $f : V \rightarrow V$ . If  $V$  can be decomposed into a direct sum of invariant subspaces, then  $A$  can be transformed into block diagonal form.
- Let  $\dim V = n$ ,  $f \in \text{End}(V)$ . Prove:  $\{0\} \subseteq \ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \dots$  and  $\text{im } f \supseteq \text{im } f^2 \supseteq \text{im } f^3 \supseteq \dots$

- Let  $\dim V = n, f \in \text{End}(V)$ . Prove:  $\exists m \leq n : \ker f^m = \ker f^{m+1}$
- Let  $\dim V = n, f \in \text{End}(V)$ . Prove:

$$\begin{aligned} \ker f^m = \ker f^{m+1} &\iff \text{im } f^m = \text{im } f^{m+1} \iff \ker f^m = \ker f^{m+k} \forall k \geq 1 \iff \\ \text{im } f^m &= \text{im } f^{m+k} \forall k \geq 1 \iff \ker f^m \cap \text{im } f^m = \{0\} \iff V = \ker f^m \dot{+} \text{im } f^m \end{aligned}$$

- Explain Fitting's Lemma in a few words.
- Define generalized spaces. Define generalized eigenvectors. How do generalized eigenvectors differ from eigenvectors?
- Prove: Let  $\lambda_1, \dots, \lambda_k$  be different eigenvalues of  $A$  and  $\ker(\lambda I - A)^{r_i}$  the corresponding generalized spaces where

$$\begin{aligned} \ker(\lambda_i I - A)^{r_{i-1}} &\subsetneq \ker(\lambda_i I - A)^{r_i} = \ker(\lambda_i \cdot I - A)^{r_i+1} \\ \implies \bigcap_{i=1}^k \text{im}(\lambda_i I - A)^{r_i} \cap \ker((\lambda_1 I - A)^{r_1} (\lambda_2 I - A)^{r_2} \dots (\lambda_k I - A)^{r_k}) &= \{0\} \end{aligned}$$

- Prove:  $\forall \lambda \neq \mu \in \text{spec}(A) \forall k, l \geq 1 : \ker(\lambda I - A)^k \cap \ker(\mu I - A)^l = \{0\}$
- Prove: The sum  $\sum_{i=1}^k \ker(\lambda_i I - A)^{r_i}$  is direct for arbitrary pairwise different  $\lambda_1, \dots, \lambda_k$ .
- Prove: Let  $\lambda_1, \dots, \lambda_k$  be pairwise different eigenvalues of  $A \in \mathbb{K}^{n \times n}$ . Let  $W := \bigcap_{i=1}^k \text{im}(\lambda_i I - A)^{r_i}$ .  $V = \ker(\lambda_1 I - A)^{r_1} \oplus \dots \oplus \ker(\lambda_k I - A)^{r_k} \oplus \bigcap_{i=1}^k \text{im}(\lambda_i I - A)^{r_i}$
- Prove:  $W$  is invariant under  $A$  and  $\lambda_i \notin \text{spec}(A|_W) \forall i \in \{1, \dots, k\}$
- Prove: Let  $\mathbb{K}$  be algebraically closed and let  $\lambda_1, \dots, \lambda_k$  be all eigenvalues of a matrix  $A \in \mathbb{K}^{n \times n}$ . Then  $\mathbb{K}^n = \ker(\lambda_1 I - A)^{r_1} \oplus \dots \oplus \ker(\lambda_k I - A)^{r_k}$ .
- What is the index of a linear map?
- Define nilpotent matrices.
- The sum of nilpotent matrices is  $\square$
- The product of nilpotent matrices is  $\square$
- Assume all generalized spaces are eigenspaces. What about diagonalizability?
- Prove: Let  $\ker(f^m) \subseteq \ker(f^{m+1}) \subseteq \ker(f^{m+2})$

$$\begin{aligned} u_1 \dots u_p \dots &\text{ basis of } \ker f^n \\ u_1 \dots u_p v_1 \dots v_k \dots &\text{ basis of } \ker f^{m+1} \\ u_1 \dots u_p v_1 \dots v_k w_1 \dots w_r \dots &\text{ basis of } \ker f^{m+2} \end{aligned}$$

Then  $(u_1 \dots u_p, f(w_1), \dots, f(w_r))$  is linear independent.

- What kind of matrix is the Jordan normal form of a matrix?
- Prove: Let  $\dim V = n$ .  $f : V \rightarrow V$  is nilpotent of index  $p$  ( $f^p = 0$ ).  $d = \dim \ker f$ . Then there exists a basis  $B$  of  $V$  such that

$$\text{diag}(\Phi_B^B(f)) = ([N_1], [N_2], \dots, [N_d])$$

where

$$N_i = \begin{bmatrix} 0 & 1 & \ddots & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}_{n_i \times n_i} \quad p = n_1 \geq n_2 \geq \dots \geq n_d \geq 1 \quad n_1 + \dots + n_d = n$$

- Define Jordan blocks of length  $k$  of a given eigenvalue  $\lambda$ .
- Prove: Let  $\mathbb{K}$  be an algebraically closed field. Then every matrix  $A \in \mathbb{K}^{n \times n}$  is similar to a matrix of Jordan normal form.



- Let  $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$  be a Jordan normal form with  $J_i = J_{k_i}(\lambda_i)$ . Prove:  $\sum_{i=1}^q k_i = n$
- Let  $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$  be a Jordan normal form with  $J_i = J_{k_i}(\lambda_i)$ . Prove: Geometric multiplicity of  $\lambda$  equals the number of corresponding Jordan blocks. Algebraic multiplicity of  $\lambda$  equals the sum of sizes of corresponding Jordan blocks.
- Let  $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$  be a Jordan normal form with  $J_i = J_{k_i}(\lambda_i)$ . Show: The smallest exponent  $r$  such that  $\ker((\lambda I - A)^r) = \ker((\lambda I - A)^{r+1})$  is the largest length of a corresponding Jordan block.
- Let  $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$  be a Jordan normal form with  $J_i = J_{k_i}(\lambda_i)$ . Prove: Let  $k \in \mathbb{N}$ .  $\#\{i : \lambda_i = \lambda \wedge k_i \geq k + 1\} = \text{rank}(\lambda I - A)^k - \text{rank}(\lambda I - A)^{k+1}$
- Let  $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$  be a Jordan normal form with  $J_i = J_{k_i}(\lambda_i)$ . Prove: The Jordan blocks are uniquely determined (except for the order)

- Let  $A \in \mathbb{K}^{n+n}$  matrix.  $\Psi_A : \frac{\mathbb{K}[x] \rightarrow \mathbb{K}^{n+n}}{p(x) \mapsto p(A)}$  and  $a_0 + a_1x + \dots + a_kx^k \mapsto a_0 \cdot I + a_1A + \dots + a_kA^k$ . Prove:  $p \left( \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right) =$
- Let  $A \in \mathbb{K}^{n+n}$  matrix.  $\Psi_A : \frac{\mathbb{K}[x] \rightarrow \mathbb{K}^{n+n}}{p(x) \mapsto p(A)}$  and  $a_0 + a_1x + \dots + a_kx^k \mapsto a_0 \cdot I + a_1A + \dots + a_kA^k$ . Prove:  $p \left( \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \right) =$   

$$\begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix}.$$
- Let  $A \in \mathbb{K}^{n+n}$  matrix.  $\Psi_A : \frac{\mathbb{K}[x] \rightarrow \mathbb{K}^{n+n}}{p(x) \mapsto p(A)}$  and  $a_0 + a_1x + \dots + a_kx^k \mapsto a_0 \cdot I + a_1A + \dots + a_kA^k$ . Prove:  $p \left( \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix} \right) =$   

$$\begin{bmatrix} p(A_1) & & \\ & \ddots & \\ & & p(A_n) \end{bmatrix}$$
- Let  $A \in \mathbb{K}^{n+n}$  matrix.  $\Psi_A : \frac{\mathbb{K}[x] \rightarrow \mathbb{K}^{n+n}}{p(x) \mapsto p(A)}$  and  $a_0 + a_1x + \dots + a_kx^k \mapsto a_0 \cdot I + a_1A + \dots + a_kA^k$ . Prove:  $A = T^{-1}BT \implies p(A) = T^{-1}p(B)T$
- For some Jordan block  $J_k(\lambda)$  it holds that,

$$p(J_k(\lambda))_{i,j} = \begin{cases} \frac{p^{(j-i)}(\lambda)}{(j-i)!} & j > i \\ p(\lambda) & j = i \\ 0 & j < i \text{ (below the diagonal)} \end{cases}$$

- Let  $A \in \mathbb{K}^{N \times N}$ . Prove:  $\exists p(x) \in \mathbb{K}[x] : p(A) = 0$
- Define the annihilator of a matrix.

- Let  $A \in \mathbb{K}^{N \times N}$ . Prove:  $\exists$  a unique polynomial  $m_A(x) \in \mathbb{K}[x]$  with minimal degree and leading coefficients 1 and  $p(x) \in \text{Ann}(A) \iff m_A(x) \mid p(x)$
- Let  $A \in \mathbb{K}^{N \times N}$ . Prove:  $m_A(\lambda) = 0 \forall \lambda \in \text{spec}(A)$
- Give and prove the Cayley-Hamilton Theorem
- Prove:  $m_A(x) \mid \chi_A(x)$
- Conclude: the roots of  $m_A(x)$  are the eigenvalues of  $A$
- Recognize: The minimal polynomial has the structure  $m_A(x) = \prod (\lambda - \lambda_i)^{m_i}$  where  $m_i$  is the smallest exponent for  $\ker(\lambda_i - A)^{m_i} = \ker(\lambda_i - A)^{m_i+1}$ , hence this equals the largest length of a Jordan block for eigenvalue  $\lambda_i$ .
- Recognize:  $A$  is diagonalizable  $\iff$  all  $m_i = 1 \iff m_A(x) = \prod_{i=1}^k (\lambda - \lambda_i) \iff m_A(x)$  has only simple roots.
- $p(A) \cdot x = p(\lambda) \cdot x$  if  $\lambda \in \text{spec}(A) \implies p(\lambda) \in \text{spec}(p(A))$
- What is an important precondition for the spectral mapping theorem?
- Give and prove the spectral mapping theorem

## 12 Normal matrices

- Define normal matrices
- If a matrix is self-adjoint, then is it necessarily  $\square$
- A real-valued self-adjoint matrix is called  $\square$
- Are unitary matrices normal?
- Prove:  $A$  is normal  $\iff A$  is unitarily diagonalizable.
- The sum of normal matrices is  $\square$
- The product of normal matrices is  $\square$
- $A \in \mathbb{C}^{n \times n}$  is normal. Prove:  $\ker A = \ker A^*$
- $A \in \mathbb{C}^{n \times n}$  is normal. Prove:  $\ker A = \ker A^2$
- Let  $A \in \mathbb{C}^{n \times n}$  is normal. Prove:  $\ker(\lambda I - A) = \ker(\bar{\lambda} I - A^*)$
- Let  $A \in \mathbb{C}^{n \times n}$  is normal. Prove:  $\ker(\lambda I - A)^2 = \ker(\lambda I - A)$
- Let  $A$  be normal. Prove:  $\lambda \neq \mu \in \text{spec } A \implies \ker(\lambda I - A) \perp \ker(\mu I - A)$
- Matrix  $A$  is normal. What can you say about generalized eigenspaces?
- Let  $A \in \mathbb{C}^{n \times n}$  be normal. Prove: Eigenvectors of different eigenvalues are orthogonal to each other.
- Let  $A \in \mathbb{C}^{n \times n}$ . Prove:  $A$  is normal  $\implies \exists$  orthonormal basis of eigenvectors
- Let  $A \in \mathbb{C}^{n \times n}$ . Prove:  $\exists$  orthonormal basis of eigenvectors  $\implies \exists$  unitary matrix  $U$  such that  $U^* A U = \text{diag}(\lambda_1, \dots, \lambda_n)$ .
- Let  $A \in \mathbb{C}^{n \times n}$ . Prove:  $\exists$  unitary matrix  $U$  such that  $U^* A U = \text{diag}(\lambda_1, \dots, \lambda_n) \implies A$  is normal.
- Define Schur's decomposition
- If  $A \in \mathbb{R}^{n \times n}$  and  $\chi_A(\lambda)$  decomposes into linear factors.  $\exists U \in \mathcal{U}(n) : U^* A U = R$  is an upper triangular matrix. Then  $U \square$
- Prove: A matrix is normal  $\iff$  Schur normal form = diagonal matrix.
- Prove: Let  $A \in \mathbb{C}^{n \times n}, A = A^* \implies \text{spec}(A) \subseteq \mathbb{R}$ .
- Let  $A \in \mathbb{C}^{n \times n}$  self-adjoint. Prove:  $A \geq 0 \iff \text{spec}(A) \subseteq [0, \infty)$ .
- For which matrices is the square root of a matrix defined?

- Prove: The square root of a positive definite matrix exists and is unique.
- Define Cholesky decompositions
- Define Schur complements of  $D$  in  $M$ . Give the dimensions of  $D$  and  $M$ .
- Let  $A \in \mathbb{C}^{n \times n}$ .  $A > 0$ ,  $b \in \mathbb{C}^n$ ,  $\gamma > 0$ .

$$\det \left[ \begin{array}{c|c} A & B \\ \hline b^* & \gamma \end{array} \right] = \det A \cdot (\gamma - b^* A^{-1} b)$$

- Cholesky decomposition is a  $\square$  decomposition
- Prove: The lower triangular matrix in the Cholesky decomposition is unique
- Prove: If  $A > 0$ , then  $\det A \leq a_{11} a_{22} \dots a_{nn}$ .
- Give and prove Hadamard's inequality
- Which matrices have polar decompositions?
- Define the polar decomposition of a matrix  $A$ .
- Prove: Let  $A \in \mathbb{C}^{n \times n}$ .  $|A| := (A^* A)^{\frac{1}{2}}$ . Then  $\exists U \in \mathcal{U}(n)$  such that  $A = U \cdot |A|$ .
- Which matrices have singular values?
- Define singular value decompositions of matrices.
- Define the numerical range of a matrix  $A$
- Define the numerical radius of matrix  $A$
- Prove:  $\text{spec}(A) \subseteq W(A)$
- Give and prove the Theorem by Toeplitz–Hausdorff
- Define Rayleigh quotients.
- Which requirements are given for matrix  $A$  in the Rayleigh–Ritz Theorem and the Courant–Fischer–Weyl min–max principle?
- Give and prove the Rayleigh–Ritz Theorem.
- Give and prove the Courant–Fischer–Weyl min–max principle.
- Which one is more generic: Rayleigh–Ritz Theorem or Courant–Fischer–Weyl min–max principle?
- Give and prove the Cauchy interlacing theorem
- Prove:  $A, B$  are self-adjoint  $\in \mathbb{C}^{n \times n}$ .  $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$
- Define Geršgorin discs
- Give and prove the Geršgorin theorem
- What is the spectral gap of a matrix?

## 13 Matrix norms

- Give one scalar product for matrices in  $\mathbb{C}^{m \times n}$ .
- Define Frobenius norms (= Schur norms) (= Hilbert–Schmidt norms)
- Let  $V, W$  be normed vectorspaces.  $f \in \text{Hom}(V, W)$ . Define the so-called induced norm.
- Define optimal norms
- Define the spectral radius of a matrix
- What does every compatible matrix norm satisfy with respect to the spectral radius.
- Prove: The spectral radius is not a norm.
- Prove:  $\forall A \in \mathbb{C}^{n \times n} \forall \varepsilon > 0 \exists$  norm on  $\mathbb{C}^n$  : the induced matrix norm satisfies  $\|A\| \leq \rho(A) + \varepsilon$
- Define condition numbers of matrices

## 14 Non-negative matrices

- Define non-negative matrices
- Give the Perron–Frobenius theorem

## 15 Generic/recall question

- Which matrices satisfy  $AB = BA$ ?
- Which matrices satisfy  $A^{-1} = A$ ?
- Which matrices satisfy  $A^T = A$ ?
- Which matrices satisfy  $AA^* = I$ ?
- Which matrices satisfy  $AA^* = A^*A$ ?
- Which matrices satisfy  $A = A^*$ ?