Measure and integration theory

Lecture notes, University (of Technology) Graz
based on the lecture by Wolfgang Wöss

Lukas Prokop

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Contents

1 Course

1. Thursday, 16:15–18:00
2. Monday, 12:15–14:00
3. Exam: oral, date negotiation per email, 3 examinees at once
4. In this document, $\subset$ denotes $\subseteq$ or $\subsetneq$
5. Literature: "Measure Theory" by Paul R. Halmos

↓ *This lecture took place on 2018/10/01.*

2 Sigma algebras and measures

A measure represents the content of a set. In R^2, it represents the area. In R^3, it represents the volume. In R^d, we can consider the content of a subspace as dimensionwise combination of intervals:

$$[a_1, b_1] \times \cdots \times [a_d, b_d]$$

To determine the "size" of this space, we can use the product of the individual interval sizes:

$$(b_1 - a_1) \cdot \cdots \cdot (b_d - a_d)$$

Consider an geometric object as in Figure 1. We can approximate the size of B by considering inner or outer axis-parallel boundary. The approximation using the infimum of the outer and supremum of the inner boundary defines the Jordan measure.

The indicator function of this area (1_B) is Riemann-integrable.

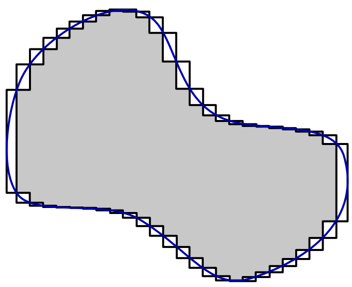

Figure 1: Jordan measurability of this area B

A one-point set is Jordan measurable with measure/content 0. However, $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable, because the indicator function is not Riemann integrable. It is desirable that the measure $\dot\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \text{measure}(A_n)$ (using pairwise disjoint union) holds true.

Modern measure theory was established by Lebesgue (1901):

1. Union of countable sets (σ-additivity)
2. arbitrary base set χ instead of $\mathbb{R}^d$, integration theory for $f : \chi \to \mathbb{R}$

2.1 Definition

Let (δ, ρ) be the non-empty base set. $\mathcal{A} \subset P(\chi)$. A set system of subsets of χ is called *sigma-algebra* (σ-algebra) if

1. $\chi \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^C = \chi \quad A \in \mathcal{A}$
3. $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$) $\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Properties 1 and 2 implies that $\emptyset \in \mathcal{A}$.

A *measurable space* is given by $(\chi, \mathcal{A})$. A *measure* $\mu : \mathcal{A} \to [0, \infty]$ is defined by

1. $\mu(\emptyset) = 0$
2. If $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$), pairwise disjoint, then

$$\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

A *measure space* is given with $(\chi, \mathcal{A}, \mu)$.

Remark.

- *μ is called probability space if* $\mu(\chi) = 1$
- *μ is called finite measure if* $\mu(\chi) < \infty$
- *μ is called σ-finite if* $\chi = \bigcup_{n=1}^{\infty} A_n$ *with* $A_n \in \mathcal{A}$ *and* $\mu(A_n) < \infty$ *(e.g. real axes decomposes into intervals of length* 1*)*

Examples:

1. χ is at most countable, then mostly $\mathcal{A} = \mathbb{P}(\chi)$. Then it suffices to know, $\mu(\{x\}) \in [0, \infty)$. Then we denote $\mu(x) = \mu(\{x\})$ with $x \in \chi$.

$$\mu(A) = \sum_{x \in A} \mu(x)$$

   e.g. $\mu(x) = 1 \forall x \in \chi$ in case of a *counting measure*.

2. If χ is uncountable, e.g. $\mathbb{R}^d$, then it is not recommended to use $\mathbb{P}(\chi)$. So what about A? Consider for example $\mathbb{R}^d$. All $[a_1, b_1] \times \cdots \times [a_d, b_d]$ should be elements of $\mathcal{A}$

2.2 Simple properties of sigma-algebras

1. $\emptyset \in \mathcal{A}$
2. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{k=1}^{n} A_k \in \mathcal{A}$

3. $A_n \in \mathcal{A}$ $(n \in \mathbb{N})$ or $, \dots, N \implies \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ (deMorgan) $\bigcap_n A_n = \left(\bigcup_n A_n^C\right)^C$
4. $A, B \in \mathcal{A} \implies A \quad B \in \mathcal{A}, A \triangle B = A \quad B \vee B \quad A$

Definition 2.1 (Generating set). *Let $\mathcal{E} \neq \emptyset$ with $\mathcal{E} \subset \mathbb{P}(\chi)$ be the* generator *(generating set) of the σ-algebra. $\sigma(\mathcal{E})$ is the smallest σ-algebra over χ which contains $\mathcal{E}$.*

$$= \bigcap \left\{ \tilde{A} : \tilde{A} \text{ is the } \sigma - \text{algebra over } \chi \text{ with } \mathcal{E} \subset \tilde{\mathcal{A}} \right\}$$

This set is non-empty because $\mathbb{P}(\chi)$ is the σ-algebra for all χ and $\mathcal{E} \subset \mathbb{P}(\chi)$

Lemma 2.2. *If $\mathcal{A}_i$ with $i \in I$ is a family of σ-algebras, then $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra over χ.*

Immediate:

1. if $\mathcal{E}_1 \subset \mathcal{E}_2$ ($\implies \mathcal{E}_1 \subset \sigma(\mathcal{E}_2)$), then $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$
2. if additionally $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$, then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$

Example:

$$\chi = \bigcup_{n\in I} E_n \neq \emptyset \qquad I = \mathbb{N} \text{ or } \{1, \dots, N\}$$

$$\mathcal{E} = \{E_n \mid n \in I\} \qquad \sigma(\mathcal{E}) = \left\{ \bigcup_{n\in J} E_n \mid J \subset I \right\}$$

1. Is a σ-algebra
2. If $\mathcal{E} \subset \tilde{\mathcal{A}}$, then $\left\{\bigcup_{n\in J} E_n \mid J \subset I\right\} \subset \tilde{\mathcal{A}}$

Definition. *(χ, d) is a metric space. Borel-σ-algebra $\sigma(\mathcal{O})$. $\mathcal{O}$ is the set of open sets in a metric space*

Example. *Consider $\mathbb{R}^d$. $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel σ-algebra.*

1. *$\mathcal{E}_1 = \{$open sets$\}$*
2. *$\mathcal{E}_2 = \{$closed sets$\}$*
3. *$\mathcal{E}_3 = \{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$ is a parallelepiped*
4. *$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$*
5. *$\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ because every open set is a countable union of open (or left half-open) parallelepipeds*

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$
$$\mathcal{E}_4 = \{(a_1, b_1] \times \cdots \times (a_d, b_d] \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$$

$$(a,b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$$

$$\mathcal{E}_5 = \{(-\infty, b_1) \times \cdots \times (-\infty, b_d] \mid b_i \in \mathbb{R}\} \text{ because } \mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$$

If $d = 1$, $(a,b] = (-\infty, b] \quad (-\infty, a]$. Recognize that $A \quad B = (A \cap B^C)$.
If $d = 2$,

$$(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \;\; (-\infty, a_1] \times (-\infty, b_2) \;\; (-\infty, b_1] \times (-\infty, a_2]$$

Definition. *$(\chi, \mathcal{A})$ is a measurable space, $B \in \mathbb{A}$ is a trace σ-algebra over B. $\{A \in \mathcal{A} \mid A \subset B\}$*

Definition. *$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$ is called measurable $\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$*

Remark. *In general φ is a map from χ_1 to χ_2. $\mathcal{A}_1$ and $\mathcal{A}_2$ are mentioned to clarify that the map depends on the chosen algebra.*

Remark. *$(\chi_1, d_1) \to (\chi_1, d_2)$ on metric spaces is continuous iff $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ where $\mathcal{O}_1, \mathcal{O}_2$ are sets of open sets.*

Remark. *Measurable maps are a much stronger statement than continuity, because they cover much more sets than open ones.*

Lemma 2.3. *The composition of measurable maps is measurable.*

$$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2) \text{ measurable}$$

$$\psi : (\chi_2, \mathcal{A}_2) \to (\chi_2, \mathcal{A}_2) \text{ measurable}$$

$$\implies \psi \circ \varphi : (\chi_1, \mathcal{A}_1) \to (\chi_3, \mathcal{A}_3) \text{ measurable}$$

Proof. Show that $(\psi \circ \varphi)^{-1}(\mathcal{A}_3) \in \mathcal{A}_1$ is trivial. □

Theorem 2.3.1. *Let $\mathcal{E}_2$ be a generator of $\mathcal{A}_2$. Then $\varphi : (\chi_1, \mathcal{A}_1) \to \chi_2$ is measurable in regards of $\mathcal{A}_2$ iff $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$*

Proof. $\implies$ is immediate

$\impliedby$ $\tilde{\mathcal{A}}_2 = \{A_2 \in \mathcal{A}_2 \mid \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over χ_2. $\mathcal{E}_2$ is a TM of this σ algebra.

$$\implies \mathcal{A}_2 = \sigma(\mathcal{E}_2) \subset \tilde{\mathcal{A}}_2$$

□

Example 2.4. *$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing*

$$f^{-1}(-\infty, b] = \{x \mid f(x) \leq b\} = (-\infty, c) \in \mathcal{B}$$

Thus, f is measurable.

Remark. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

Example 2.5.

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{+\infty, -\infty\} \mid B \in \mathcal{B}_{\mathbb{R}}\}$$

$f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ *measurable in* $(\chi, \mathcal{A})$ *and* $f = \max\{f_1, \dots, f_n\}$

$$f : \mathbb{R} \to \overline{\mathbb{R}} \qquad x \mapsto \max\{f_1(x), \dots, f_n(x)\}$$

$$\begin{aligned} f^{-1}([-\infty, b]) &= \{x \mid f(x) \leq b\} \\ &= \{x \mid f_k(x) \leq b, k = 1, \dots, n\} \\ &= \bigcap_{k=1}^{n} \{x \mid f_k(x) \leq b\} \in \mathcal{B} \end{aligned}$$

Analogously for the minimum. Therefore f *is measurable.*

Example 2.6. *The same applies to countably many functions. Let* $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ *be measurable with* $n \in \mathbb{N}$*. Then* $f : \sup\{f_n \mid n \in \mathbb{N}\}$ *is measurable.*

$$\begin{aligned} f^{-1}([-\infty, b]) = \{x \mid \sup\{f_n(x)\} \leq b\} &= \{x \mid f_n(x) \leq b \forall n\} \\ = \bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in B} \in \mathcal{B} \end{aligned}$$

Analogously for the infimum.

Example 2.7.

$$\limsup_{n\to\infty} \sup f_n = \inf_n \underbrace{\sup_{k \geq n} f_k}_{\text{with } n\to\infty \text{ monotonically decreasing}} \quad \textit{is measurable}$$

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k \geq n} f_k \textit{ is measurable}$$

if all f_k *are measurable. Especially if* $f_n \to f$ *pointwise and all* f_n *are measurable, then* f *is measurable.*

Theorem 2.7.1 (Result from the previous example)**.**

$$f_n : (\chi, \mathcal{A}) \to \overline{\mathbb{R}} \textit{ measurable}, n \in \mathbb{N}$$

$$\implies \inf f_n, \sup f_n, \lim_{n\to\infty} \inf f_n, \lim_{n\to\infty} \sup f_n$$

are all measurable.

↓ *This lecture took place on 2018/10/04.*

1. Basic set χ [δ, ρ, ...]
2. σ-algebra $\mathcal{A} \subset p(\chi)$

   (a) $\chi \in \mathcal{A}$
   (b) $A \in \mathcal{A} \implies \mathcal{A}^C \in \mathcal{A}$
   (c) $A_n \in \mathcal{A}\ (n \in \mathbb{N}) \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

   $(\chi, \mathcal{A})$ is a measurable space

3. measure $\mu : \mathcal{A} \to [0, \infty]$
   (a) $\mu(\emptyset) = 0$
   (b) $A_n \in \mathcal{A}\ (n \in \mathbb{N}),\ A_n \cap A_m \neq 0 \forall n \neq m$

$$\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \qquad \sigma - \text{additivity}$$

   $(\chi, \mathcal{A}, \mu)$ is a measure space

4. $\mathcal{E} \subset p(\chi)$

$$\sigma(\mathcal{E}) = \bigcap \left\{ \tilde{A} : \tilde{A} \sigma - \text{algebra}, \mathcal{E} \subset \tilde{A} \right\}$$

   is the so-called $\mathcal{E}$-generated σ-algebra.

Recognize that $\mathcal{E}_1 \subset \mathcal{E}_2 \implies \sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$. If additionally, $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \implies \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

If X is a metric space, we commonly (sometimes implicitly) use the Borel-Sigma algebra as measure space.

Example: Let $\mathbb{R}^d$. Then $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel-sigma algebra.

Let $\mathcal{E}_1$ be the set of open sets. Let $\mathcal{E}_2$ be the set of closed sets. Let $\mathcal{E}_3 = \{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$. $\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ because every open set is a countable union of open (or left half-open) parallelepipeds (why countable?).

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$

$$\mathcal{E}_4 = \{(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]\}$$

$$(a, b) = \bigcup_{n=0}^{\infty} (a, b - \frac{1}{n})$$

$$\mathcal{E}_5 = \{(-\infty, b_1) \times (-\sigma, b_d) : b_1, \ldots, b_d \in \mathbb{R}\}$$

because $\mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$.

DeMorgan: $A \quad B = A \cap B^C$

Let $d = 1$, $(a, b] = (-\infty, b] \quad (-\infty, a]$.
Let $d = 2$, $(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \quad (-\infty, a_1] \times (-\infty, b_2] \quad (-\infty, b_2] \times (-\infty, a_1]$.

Definition 2.8. *Let $(\chi, \mathcal{A})$ be a measure space. $B \in \mathcal{A}$. trace σ-algebra over B is defined as $\{A \in \mathcal{A} : A \subset B\}$.*

Remark (Revision on continuity). *Let $\varphi : (\chi_1, d_1) \to (\chi_2, d_2)$ be a map between metric spaces. Let φ be continuous.*

On the one hand, we know the ε-δ definition, but we also consider $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ (set of open sets)

Definition 2.9 (Measurable maps). *Let $\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$*[1]

$$\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$$

Lemma 2.10. *The composition of measurable maps is measurable.*

$$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$$

$$\Psi : (\chi_2, \mathcal{A}_2) \to (\chi_3, \mathcal{A}_3)$$

with φ and Ψ measurable.

$$\implies \Psi \circ \varphi : (\chi_1, \mathcal{A}_1) \to (\chi_3, \mathcal{A}_3)$$

is measurable. (trivial to prove)

Theorem 2.10.1. *Let $\mathcal{E}_2$ be the generator of some algebra $\mathcal{A}_2$. Then $\varphi : (\chi_1, \mathcal{A}_1) \to \chi_2$ in regards of $\mathcal{A}_2$ is measurable if and only if $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$.*

Proof. $\implies$ trivial

$\impliedby$ $\tilde{A}_2 := \{A_2 \in \mathcal{A}_2 : \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over χ_2 (why? left as an exercise). $\mathcal{E}_2$ is a subset of this σ-algebra. $\implies \mathcal{A}_2 = \sigma(\mathcal{E}_2) \in \tilde{A}_2 \subset \mathcal{A}_2$

□

Example. *$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. $f^{-1}(-\infty, b] = \{x : f(x) \leq b\}$ is in the Borel-sigma algebra $\mathcal{B}$. So f is measurable.*

Definition. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$
$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{\pm\infty\} : B \in \mathcal{B}_{\mathbb{R}}\}$

Example 2.11. *Let $f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ measurable. $f = \max\{f_1, \dots, f_n\}$.*

$$f^{-1}([-\infty, b]) = \{x : f(x) \leq b\} = \{x : f_k(x) \leq b, k = 1, \dots, n\} = \bigcap_{k=1}^{n} \underbrace{\{x : f_k(x) \leq b\}}_{f_k^{-1}[-\infty,b]} \in \mathcal{B}$$

Equivalently, $\min\{f_1, \dots, f_n\}$ *is measurable. Equivalently, $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable ($n \in \mathbb{N}$). $\implies f = \sup\{f_n : n \in \mathbb{N}\}$ is measurable.*

$$f^{-1}(\infty, b] = \{x : \sup f_n(x) \leq b\} = \{x : f_n(x) \leq b \forall n\}$$
$$f^{-1}[-\infty, b) = \{x : \sup f_n(x) < b\} \subset \{x : f_n(x) < b \forall n\}$$

$$\bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in \mathcal{B}} \in \mathcal{B}$$

[1]Actually, $\varphi : \chi_1 \to \chi_2$, but we don't want to forget about the associated σ-algebras

Let f_n be measurable functions.

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{k\geq n} f_k$$

The supremum of measurable functions is measurable (see Lemma 2.10). The infimum as well. So the result is measurable.

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k\geq n} f_k$$

Equivalently, the result is measurable.

Especially, if $f_n \to f$ pointwise, and all f_n are measurable, then also limit f is measurable.

How to determine measurability? Show that pre-images of generators are in the σ-algebra.

Theorem 2.11.1. *Let $f : (\chi, \mathcal{A}) \to \overline{\mathbb{R}}$ be measurable ($n \in \mathbb{N}$)*

$$\implies \inf f_n, \sup f_n, \liminf f_n, \limsup f_n$$

are also measurable.

↓ *This lecture took place on 2018/10/08.*

2.3 Simple properties of measures

A monotonically increasing sequence $(A_n)_{n\in\mathbb{N}}$ of sets is given by $A_1 \subset A_2 \subset A_3 \subset \dots$.

Theorem 2.11.2. *Let $(\chi, \mathcal{A}, \mu)$.*

1. $A_1, \dots, A_n \in \mathcal{A}, A_i \cap A_j = 0 \forall i \neq j \implies \mu\left(\biguplus_{k+1}^n A_n\right) = \sum_{k=1}^n \mu(A_n)$
2. $\mu(B) = \mu(A \cap B) + \mu(A^C \cap B)$ *for* $A, B \in \mathcal{A}$
3. $A \subset B \implies \mu(A) \leq \mu(B)$ *for* $A, B \in \mathcal{A}$
4. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
5. *Let* $(A_n)_{n\in\mathbb{N}}$ *be a monotonically increasing sequence of* $\mathcal{A}$ *and* $A = \biguplus_{n=1}^\infty A_n = \lim_{n\to\infty} A_n$*, then* $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ *"Continuity from below"*
6. *Let* A_n *be a monotonically decreasing sequence of* $\mathcal{A}$. $A = \bigcap_{n=1}^\infty A_n = \lim A_n$.
7. A_n *arbitrary* $\implies \mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n)$

Proof of continuity from below. Consider a monotonically increasing sequence of $\mathcal{A}$. Consider $B_1 = A_1, B_k = A_k \quad A_{k-1}$ and $k \geq 2$. Sets B_i and B_j are disjoint with $i \neq j$.

Then $B_1 \cup \dots \cup B_n = A_n$ and $\dot\bigcup_{k=1}^{\infty} B_k = A$.

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \mu(A_n)$$

$$A_n' = A_1 \quad A_n \qquad \nearrow A_1 \quad A$$

$$\mu(A_1 \quad A_n) = \mu(A_n') \qquad \nearrow \mu(A_1 \quad A)$$

□

What about the measure of intersected set in infinity? $A \cap B = A$ and $\mu(B) = \mu(A) + \mu(A^C \cap B)$. What happens if $\mu(A) = +\infty$ and $\mu(A^C \cap B) = -\infty$?

Remark. *How to compute algebraically with the extended real numbers?*

$$\pm\infty + a = \pm\infty \quad (a \in \mathbb{R})$$

$$+\infty \cdot a = \begin{cases} +\infty & a > 0 \\ 0 & a = 0 \\ -\infty & a < 0 \end{cases}$$

0 *for* $a = 0$ *makes sense in measure theory, but not in calculus.*

If $\mu(A_1) < \infty$, then $\mu(A_1 \quad A_n) = \mu(A_1) - \mu(A_n)$ and $\mu(A_1 \quad A) = \mu(A_1) - \mu(A)$.

Remark (Reminder).

$$\limsup a_n := \lim_{n\to\infty} \sup_{k\geq n} a_k$$

What about (A_n) arbitrary?

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k$$

Property 7 can be shown as generalization of $\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mu(A_n)$

Example (Simplest example). $\chi = \{x_n : n \in \mathbb{N}\}$. $\mathcal{A} = p(\chi)$. *Fix* $\mu(\{x_n\})$.

$$\rightsquigarrow \mu(A) = \sum_{n: x_n \in A} \mu(x_n)$$

$\mu(x_n) = 1$ *gives a counting measure.*

Let $\mathcal{E}$ be the generator of $\mathcal{A} = \sigma(\mathcal{E})$. A *stable set by intersection* is given by $E_1, E_2 \in \mathcal{E} \implies E_1 \cap E_2 \in \mathcal{E}$.

Theorem 2.11.3 (Uniqueness of measures). *Let μ, ν be measures on $\mathcal{A}$ with $\mu|_{\mathcal{E}} = \nu|_{\mathcal{E}}$. $\implies \mu = \nu$ on $\mathcal{A}$.*

$\chi \in \mathcal{E}$ and $\mu(\chi) = \nu(\chi) < \infty$ *or* $\chi = \bigcup_n E_n$ with $E_n \in \mathcal{E}$ and $\mu(E_n) = \nu(E_n) < \infty$.

Definition 2.12. *Let $\mathcal{E} \subset \mathcal{P}(\chi)$ be a semiring over χ. If*

1. $\emptyset \in \mathcal{E}$
2. $A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$
3. $A, B \in \mathcal{E}. \implies \exists C_1, \dots, C_k \in \mathcal{E}$ *pairwise disjoint* : $A \quad B = \dot\bigcup_{i=1}^k C_i$.

What is the difference compared to a *ring*? Let $A, B \in \mathcal{R} \implies (A \cap B \in \mathcal{E} \wedge A \triangle B \in \mathcal{E})$.

Theorem 2.12.1 (Extension theorem by Caratheodory). *$\mu : \mathcal{E}$ (semiring) $\to \{0, \infty\}$ with*

1. $\mu(\emptyset) = 0$
2. $(\chi \in \mathcal{E}$ *and* $\mu(\chi) < \infty)$ *or* $\left(\chi = \bigcup_{n=1}^\infty E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty\right)$
3. *μ is σ-additive on $\mathcal{E}$, hence (A_n) is a sequence in $\mathcal{E}$, pairwise disjoint and $\bigcup_{n=1}^\infty A_n \in \mathcal{E}$*
$$\implies \mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_n \mu(A_n)$$

Then μ has a (unique) continuation for a measure on $\mathcal{A} = \sigma(\mathcal{E})$.

2.4 Construction of the Lebesgue measures and similar ones

Let $\chi = \mathbb{R}$ or $\chi = \overline{\mathbb{R}}$.
$$\mathcal{E} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$$
is semiring.

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and right-sided continuous. Let $\mu(a, b] := F(b) - F(a)$. Properties 1 and 2 of the extension theorem are satisfied. We show finite additivity of property 3 in three steps:

1. If $(a, b] = \dot\bigcup_{k=1}^n (a_k, b_k]$ can be sorted. $a_1 = a, a_{k+1} = b_k$ for $k = 1, \dots, n-1$ and $b_n = b$. We get a telescoping sum such that
$$\sum_{b=1}^n (F(b_k) - F(a_k)) = F(b) - F(a)$$
2. Also $(a_1, b_1], \dots, (a_n, b_n]$. Disjoint subintervals of $(a, b]$ are
$$\implies \sum_{k=1}^n \mu(a_k, b_k] \leq \mu(a, b]$$

3. $(a,b] = \dot\bigcup_{n=1}^{\infty}(a_n, b_n]$

(a)

$$\dot\bigcup_{n=1}^{N}(a_n,b_n] \subset (a,b]$$

$$\sum_{n=1}^{N}\mu(a_n,b_n] \leq \mu(a,b] \forall N$$

$$\implies \sum_{n=1}^{\infty}\mu(a_n,b_n] \leq \mu(a,b]$$

(b) Let $\varepsilon > 0$, then $\exists a' \in (a,b] : F(a') - F(a) < \varepsilon$

$$\exists b_n' > b : F(b_n') - F(b_n) < \frac{\varepsilon}{2^n}$$

$$[a',b] \subseteq (a,b] \subset \bigcup_n (a_n,b_n] \subset \bigcup_n (a_n,b_n')$$

$$\implies \exists N : (a',b) \subset [a',b] \subset \bigcup_{n=1}^{N}(a_n,b_n') \subset \bigcup_{n=1}^{N}(a_n,b_n']$$

But these intervals in $\bigcup$ are not necessarily non-overlapping any more. But this is no problem as we can split them into disjoint sets.

$$\mu(a',b] \leq \sum_{n=1}^{N}\mu(a_n,b_n']$$

$$\mu(a',b] = F(b) - F(a') \leq \sum_{n=1}^{N} F(b_n') - F(a_n) \leq \sum_{n=1}^{N}\left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}\right)$$

with $F(b) - F(a') \geq F(b) - F(a) - \varepsilon$.

$$\mu(a,b] \leq \sum_{n=1}^{\infty}\mu(a_n,b_n] + 2\varepsilon \quad \forall\varepsilon > 0$$

↓ *This lecture took place on 2018/10/15.*

Theorem 2.12.2. *Let $\mathcal{E}$ be semiring over χ and $\mu : \mathcal{E} \to [0,\infty]$ on $\mathcal{E}$ be σ-additive and σ-finite. Then there exists exactly one continuaton for measure on $\sigma(\mathcal{E})$.*

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonic and right-sided continuous.

$$\mathcal{E} = \{(a,b] \mid a,b \in \mathbb{R}, a \leq b\} \qquad \mu(a,b] = F(b) - F(a)$$

Now consider the special case $F(x) = x$. This define the Lebesgue measure on $(\mathbb{R},\mathcal{B})$.

Theorem 2.12.3. *λ is the only measure on $(\mathbb{R}, \mathcal{B})$ with*

1. $\lambda(B+C) = \lambda(\{x+c \mid x \in B\}) = \lambda(B) \quad \forall B \in \mathcal{B} \forall c \in \mathbb{R}$
2. $\lambda(0,1] = 1$

Proof. Does λ satisfy these properties? Yes, λ has properties (1) and (2), because

(1) is correct $\forall (a,b] \in \mathcal{E}$

$$c \in \mathbb{R} : \{B \in \mathcal{B} \mid \lambda(B+c) = \lambda(B)\}$$

is σ-algebra and contains $\mathcal{E}$, so also $\sigma(\mathcal{E})$

(2) trivial

Is λ unique? Let λ be the measure with the two properties.

$$(0,1] = \dot{\bigcup_{k=1}^{n}} \left(\frac{k-1}{n}, \frac{k}{n}\right]$$

$$1 = \mu(0,1] = \sum_{k=1}^{n} \mu\left(\left(0, \frac{1}{n}\right] + \frac{k-1}{n}\right) = n\mu\left(0, \frac{1}{n}\right]$$

$$\mu\left(\frac{k-1}{n}, \frac{k}{n}\right] = \frac{1}{n} \quad \forall k \in \mathbb{Z}$$

$$\implies \mu(a,b] = b-a \qquad a, b \in \mathbb{Q}$$

$$\mu|_{\mathcal{E}_{\mathbb{Q}}} = \lambda_{\mathcal{E}_{\mathbb{Q}}} \qquad \mathcal{E}_{\mathbb{Q}} = \{(a,b] \mid a,b \in \mathbb{Q}, a \leq b\}$$

Closed under finite intersection, $\sigma(\mathcal{E}_{\mathbb{Q}}) = \mathcal{B}$:

$$(a,b) = \bigcup_n (a, b - \frac{1}{n}] \qquad (a,b] = \bigcap_n (a, b + \frac{1}{n}) \qquad \sigma\text{-finite}$$

$$\mu(-n,n] < \infty \qquad \bigcup_{n=1}^{\infty} (-n,n] = \mathbb{R} \qquad \sigma\text{-finite}$$

$$\implies \mu = \lambda \text{ (distinct extensionability)}$$

□

We apply the principle analogously to $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$.

$$\mathcal{E} = \{(a,b] = (a_1, b_1] \times \dots \times (a_n, b_n] \mid a_i \leq b_i \in \mathbb{R}\}$$

is semiring over $\mathbb{R}^d$. In $\mathbb{R}^2$, you can draw rectangles and their induced area based on their geometrical relation to each other. $F : \mathbb{R}^d \to \mathbb{R}$ complete is *monotonic* if

$$\mu(a,b] : \prod_{i=1}^{d} (F_i(b_i) - F_i(a_i)) = \sum_{x \in \{a_1,b_1\} \times \dots \times \{a_d,b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} \qquad F_1(x_1)F_2(x_2) \dots F_d(x_d)$$

Simplest case: $F_1, \dots, F_d : \mathbb{R} \to \mathbb{R}$ is monotonically right-sided continuous.

$$\sum_{x \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} F(x) \geq 0 \forall (a, b] \in \mathcal{E}$$
$$F(b_1, b_2) - F(a_1, b_2) - F(a_1, b_1) + F(a_1, a_2)$$

F is right-sided in every coordinate, thus $\mu(a, b] = \sum_{x \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{|\{i \mid x_i = a_i\}|}$

2.5 Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$

We can extend the previous definition from $\mathbb{R}$ to $\mathbb{R}^d$. Thus λ is the only measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ with

1. $\lambda^d(B + c) = \lambda(B) \forall B \in \mathbb{B}_{\mathbb{R}^d} \forall c \in \mathbb{R}^d$
2. $\lambda((0, 1]^d) = 1$

Theorem 2.12.4. *Let $H \subset \mathbb{R}^d$ be a hyperplane. Then $\lambda_d(H) = 0$.*

Proof. Without loss of generality, $\vec{O} \in H$ is subspace with dimension $d - 1$. Why is $H \in \mathcal{B}_d$ true? The Lebesgue measure is based on open sets. The σ-algebra requires the complement, thus closed sets are also given. The measure of closed sets is zero.

$\left\{\vec{b}_1, \dots, \vec{b}_{d-1}\right\}$ is an orthonormal basis of H.

$$Q = \left\{c_1 \vec{b}_1 + \dots + c_{d-1} \vec{b}_{d-1} \mid 0 \leq c_i \leq 1\right\} \in \mathcal{B}_{\mathbb{R}^d}$$

$\vec{b}_d \perp \vec{b}_i$ $(i = 1, \dots, d - 1)$, $\left\|\vec{b}_d\right\| = 1$.

$$Q + q \cdot \vec{b}_d \qquad q \in \mathbb{Q} \cap [0, 1] \text{ pairwise disjoint}$$
$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} Q + q\vec{b}_d \subset \left\{c_1 \vec{b}_1 + \dots + c_d \vec{b}_d \mid 0 \leq c_i \leq 1\right\} \text{ compact}$$
$$\infty > \lambda_d \left(\dot{\bigcup_{q \in \mathbb{Q} \cap [0,1]}} Q + q \cdot \vec{b}_d \right) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda_d(Q)$$
$$\implies \lambda_d(Q) = 0 \qquad H \subset \bigcup_{\vec{x} \in \mathbb{Z}^d} (Q + \vec{x})$$
$$\lambda_d(H) \leq \sum_{\vec{x} \in \mathbb{Z}^d} \lambda_d(Q + \vec{x}) = 0$$

□

Theorem 2.12.5. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be linear and bijective. $\varphi(\vec{x}) = M \cdot \vec{x}$ with M as regular matrix.*

Linear implies continuous in finite dimensions. Every continuous map is measurable.

$\implies$ *φ is measurable and $\lambda_d(\varphi(B)) = \det(\varphi) \cdot \lambda_d(B)$. This holds even if φ is not bijective, because then $\det(\varphi) = 0$ and thus we have a factor zero. If φ is not bijective,* *then the matrix has lower rank. The image is a hyperplane or is contained in a hyperplane. So the measure is zero.*

Proof. $\mu_\varphi(B) := \lambda_d(\varphi(B))$ is measure on $\mathcal{B}_{\mathbb{R}^d}$ (why? left as an exercise to the reader).

$$\mu_\varphi(B + \vec{c}) = \lambda_d(\varphi(B + \vec{c})) = \lambda_d(\varphi(B) + \underbrace{\varphi(\vec{c})}_{X}) = \mu_\varphi(B)$$

$$\frac{\mu_\varphi}{\mu_\varphi((0,1]^d)} = \lambda_d$$

Show that: $\mu_\varphi((0,1]^d] = \left|\det \varphi\right|$

Case 1 $\varphi(M)$ is orthogonal $M^* = M^{-1}$.

$$\varphi(B_1(\vec{0})) = B_1(\vec{0}) \qquad 0 < \lambda_d(B_1(\vec{0})) < \infty$$

$$\frac{\lambda_d(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \frac{\mu_\varphi(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \lambda_d(B_1(\vec{0}))$$

$$\mu_\varphi = \lambda_d$$

Case 2

$$M = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_d \end{pmatrix} \qquad d_i > 0$$

$$\varphi(\vec{e}_i) = d_i \cdot e_i$$

$$\varphi((0,1]^d) = (0, d_1] \times (0, d_2] \times \ldots (0, d_d]$$

$$\varphi_\varphi((0,1]^d) = \det D$$

Generic case Let M be any matrix. We consider the singular value decomposition $M = O_1 \cdot D \cdot O_2$ with O_1, O_2 orthogonal and D is a non-negative diagonal matrix.

$$M^*M \rightsquigarrow O^*D^2O$$

Then $\varphi = \varphi_1 \circ \psi \circ \varphi_2$. φ_1 and φ_2 are orthogonal. Let D be the representation matrix of ψ. Diagonal entries are positive because it is regular.

$$\left|\det \varphi\right| = \det(\psi)$$

Combining these results gives us the theorem.

□

↓ *This lecture took place on 2018/10/16.*

2.6 Sigma-algebra generated by maps

Definition 2.13. *$\mathcal{A}_i$ ($i \in I$) is σ-algebra over χ.*

$$\bigvee_{i \in I} \mathcal{A}_i = \sigma\left(\bigcup_{i \in I} \mathcal{A}_i\right)$$

Definition 2.14 (Image σ-algebra and Push-forward measure)**.** *Push-forward measures are called Bildmaß $(\chi, \mathcal{A})$ is a measure space. $\varphi : \chi \to \chi'$.*

$$\varphi(\mathcal{A}) = \{A' \subset \chi' \mid \varphi^{-1}(A') \in \mathcal{A}\}$$

$\varphi(\chi, \mathcal{A}) \to (\chi', \mathcal{A}')$ is measurable $\iff \varphi(\mathcal{A}) \supset \mathcal{A}'$.

$(\chi, \mathcal{A}, \mu)$ is a measure space, $\varphi : \chi \to \chi'$. μ_φ is the push-forward measure on $(\chi', \varphi(\mathcal{A}))$.

$$\mu_\varphi(A') = \mu(\varphi^{-1}(A'))$$

Definition 2.15 (Generated σ-algebra)**.**

1. *χ, $(\chi', \mathcal{A}')$ is a measurable space. $\varphi : \chi \to \chi'$*

   $$\sigma(\varphi) = \{\varphi^{-1}(A') \mid A' \in \mathcal{A}'\}$$

   Iff $\varphi : (\chi, \mathcal{A}) \to (\chi', \mathcal{A}')$ is measurable, $\sigma(\varphi) \subset \mathcal{A}$.

2. *χ, $(\chi_i, \mathcal{A}_i)$, $i \in I$ are measure spaces*

   $$\psi_i : \chi \to \chi_i \forall i$$

   The σ-algebra generated by ψ_i ($i \in I$) is the smallest σ-algebra that contains such a set. $\bigvee_{i \in I} \sigma(\psi_i)$. Is the smallest σ-algebra on χ which are measurable for all ψ_i.

Example. $\varphi : (\mathbb{R}^2, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$.

$$\varphi(x, y) = \sqrt{x^2 + y^2} \qquad \sigma(\varphi) = \{B \subset \mathbb{R}^2 \mid B \textit{ rotation invariant in } 0.000001\}$$

Theorem 2.15.1. *Let $(\chi, \mathcal{A})$ be a measure space. Let $(\chi', \mathcal{A}')$ be another one. Let $(\chi_i, \mathcal{A}_i)$ be measure spaces with ($i \in I$). Then we can map from $(\chi, \mathcal{A})$ to $(\chi', \mathcal{A}')$ with measurable φ and we can map $(\chi', \mathcal{A}')$ to $(\chi_i, \mathcal{A}_i)$ with ψ_i such that $\mathcal{A}' = \sigma(\psi_i : i \in I)$. Then φ is measurable iff $\psi_i \circ \varphi$ is measurable $\forall i \in I$.*

Proof. $\implies$ immediate.

$\impliedby$

$$\mathcal{E}' = \bigcup \sigma(\psi_i) \text{ generates } \mathcal{A}'$$

$$A' \subset \mathcal{E}' \implies \exists i : A' \in O(\psi_i), \text{so } A' = \psi_i^{-1}(A_i) \text{ with } A_i \in \mathcal{A}_i.$$

$$\varphi^{-1}(A) = \psi^{-1}(\psi_i^{-1}(A_i)) = (\underbrace{\psi_i \circ \varphi}_{\in \mathbb{R}})^{-1}(A_i)$$

$\square$

2.7 Product space

Let $\chi_n, \mathcal{A}_n$ and $n = 1, \dots, N$ with $N < \infty$. Let $\chi = \prod_{n=1}^{N} \chi_n$ ("product $sigma$-algebra") generated by $\mathcal{E} = \left\{\prod_{n=1}^{N} A_n \cdot A_n \subset A_n \forall n\right\}$.

Consider $N = 2$. $\chi = \chi_1 \times \chi_2$. $\mathcal{E} = \{A_1 \times A_2 \mid A_n \in \mathcal{A}_n, n = 1, 2\}$. Product σ-algebra: $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Commonly, we use the notation $(\chi, \otimes \mathcal{A}_n) = \otimes(\chi_n, \mathcal{A}_n)$

Lemma 2.16.

$$\oplus_{n=1}^{N} \mathcal{A}_n = \sigma\left(\pi_n : n = 1, \dots, N\right)$$

where $\pi_n : \chi \to \chi_n$ is the n-th projection.

Hint: $\mathcal{E}_0 = \left\{\prod_{n=1}^{N} A_n$ with $A_n = \chi_n \forall n$ expect for one and this $A_n \in \mathcal{A}_n\right\}$ also generates $\otimes \mathcal{A}_n$.

This lemma holds obviously.

Theorem 2.16.1. *$\varphi : (\chi, \mathcal{A}) \to \otimes_{n=1}^{N} (\chi_n, \mathcal{A}_n)$, where N denotes finite or countable, is measurable $\iff$ $\pi_n \circ \varphi : (\chi, \mathcal{A}) \to (\chi_n, \mathcal{A}_n)$ is measurable $\forall n$. This is a special case of Theorem 2.15.1.*

Prospect: Product measure.

Let $(\chi_1, \mathcal{A}_1) \otimes (\chi_2, \mathcal{A}_2, \mu_2)$. How to generate this? Well,

$$= (\chi_1 \times \chi_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$$

on $\mathcal{E} : \mu_1 \otimes \mu 2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ (compare it to the trivial case of the area of a rectangle in $\mathbb{R}^2$) where $\mathcal{E}$ is a semiring.

3 Integration of non-negative functions

Let $(\chi, \mathcal{A}, \mu)$ be a measure space. Consider $f : (\chi, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B})$ How about $\int_\chi f \, d\mu$?

First of all, $f : (\chi, \mathcal{A}) \to [0, \infty]$. We know construct the Lebesgue integral:

First step Consider simple functions (like step functions). f takes up only finitely many different values. $z_1, \dots, z_n (\geq 0) : [f = z_k] := \{x \in \chi \mid f(x) = z_k\} = f^{-1}(\{z_k\}) \in \mathcal{A}$. We restrict $z_i \geq 0$ to avoid issues like $+\infty + (-\infty)$.

$$\chi = \dot{\bigcup_{k=1}^{n}} [f = z_k]$$

Definition 3.1.

$$\int f \, d\mu = \sum_{k=1}^{n} z_k \mu[f = z_k]$$

Consider that $z_k \mu[f = z_k]$ might go to infinity. We commonly denote $\sum_{z \in \mathbb{R}} z\mu[f = z]$ in the real-valued case to avoid indices.

Second step Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.

$$\int_\chi f \, d\mu \coloneqq \sup \left\{ \int_\chi s \, d\mu : s \text{ simple}, 0 \leq s \leq f \right\}$$

So the Riemann integral approximates the area with upper and lower bounds for rectangles. For the Lebesgue integral, we split the function into horizontal slices in $\mathbb{R}$. Then we consider the differences of the function values between two consecutive slices. The important point is that this does not require $\mathbb{R}$, but some χ and therefore is more generic.

Third step Let $f : (\chi, \mathcal{A}) \to \mathbb{R}$ and $f = f^+ - f^-$. Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. If $\int_\chi f^+ \, d\mu = \int_\chi f^- \, d\mu = \infty : \int_\chi f \, d\mu$ is not defined. Otherwise $\int_\chi f \, d\mu = \int_\chi f^+ \, d\mu - \int_\chi f^- \, d\mu$.

Does this definition/construction of the Lebesgue integral satisfy the desired properties of linearity/monotonicity/…? In the following, we will denote "simple" functions always as s.

Definition 3.2. *Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Let $A \in \mathcal{A}$.*

$$\int_A f \, d\mu \coloneqq \int \mathbf{1}_A f \, d\mu$$

Lemma 3.3. *Let $s : (\chi, \mathcal{A}) \to [0, \infty]$ be a simple function. Then $\nu_s(A) = \int_A s \, d\mu$ is a measure on $(\chi, \mathcal{A})$.*

$$\nu_s(A) = \sum_{k=1}^n z_k \mu([s = z_k] \cap A)$$

because $\mathbf{1}_A \cdot s = \sum_{k=1}^n z_k \mathbf{1}_{[s=z_k]} \mathbf{1}_A + 0 \cdot \mathbf{1}_{A^C}$.

$A \mapsto \mu([s = z_k] \cap A)$ *is a measure* $\forall k$.

↓ *This lecture took place on 2018/10/22.*

Definition 3.4. *Let $(\chi, \mathcal{A}, \mu)$ be a measure space. $s : (\chi, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is called* simple *if $s(\chi)$ is finite.*
$s \geq 0$.

$$\int_\chi s \, d\mu \coloneqq \sum_z z \cdot \mu[s = z]$$

Trivial: If $s = \sum_{j=1}^m c_j \cdot \mathbf{1}_{A_j}, A_j \in \mathcal{A}$ then s is simple. A_j are not necessarily pairwise disjoint and $\int_\chi s \, d\mu = \sum_{j=1}^n c_j \mu(A_j)$.

Proof. $\vec{\varepsilon} \in \{-1, 1\}^m$ with $A^1 \coloneqq A, A^{-1} \coloneqq A^C$. $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$. E.g. $A_1 \cap A_2 \cap A_3^C = B_{1,1,-1}$.

$$B_{\vec{\varepsilon}} = A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \dots \cap A_m^{\varepsilon_m}$$

is pairwise disjoint. On $B_{\vec{\varepsilon}}$, s has value $\sum_{\varepsilon_j=1} c_j = b_{\vec{\varepsilon}}$

$$\implies s = \sum b_{\vec{\varepsilon}} \mathbf{1}_{B_{\vec{\varepsilon}}}$$

and $\int \leq d\mu = \sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}}) = \cdots = \sum c_j \mu(A_j)$ (where $\sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}})$ is the disjoint case and $\sum c_j \mu(A_j)$ is generic).

$$\sum_{\vec{\varepsilon}} \sum_{j:\varepsilon_j=1} c_j \cdot \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \sum_{\vec{\varepsilon}:\varepsilon_j=1} \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \mu(A_j)$$

□

Corollary 3.5. *Let $s_1, s_2 : \chi \to [0,\infty]$ be simple. Then $s = \alpha \cdot s_1 + \beta \cdot s_2$ ($\alpha, \beta \geq 0$) is simple and $\int s\,d\mu = \alpha \cdot \int s_1\,d\mu + \beta \int s_2\,d\mu$.*

Theorem 3.5.1 (Markov inequality). *Let $z \in \mathbb{R}$. Let $f \geq 0$.*

$$z \cdot \mu[\underbrace{f \geq z}_{\{x \in \chi \mid f(x) \geq z\}}] \leq \int f\,d\mu$$

Proof.

$$s = z \cdot \mathbf{1}_{[f \geq z]} \leq f$$

If $x \in [f \leq z] : z \cdot 1 \leq f(x)$.
If $x \notin [f \leq z] : z \cdot 0 \leq f(x)$.

s is simple, so $z\mu[f \geq z] = \int s\,d\mu \leq \int f\,d\mu$. $s = 0 : \mathbf{1}_{[f<z]} \times z \cdot \mathbf{1}_{[f \geq z]}$. □

Definition 3.6. *A statement holds* almost everywhere *if* $\forall x \in \mathcal{A} : \mu(A^C) = 0$. *So A^C is a null set, i.e. of measure zero.*

Theorem 3.6.1.

$$\forall f, g : \chi \to [0,\infty] \textit{ measurable}$$

$f \leq g$ *almost everywhere* $\implies \int f\,d\mu \leq \int g\,d\mu$

2. $f = g$ *almost everywhere* $\implies \int f\,d\mu = \int g\,d\mu$

3. $\int f\,d\mu = 0 \implies f = 0$ *almost everywhere*

4. $\int f\,d\mu < \infty \implies f < \infty$ *almost everywhere*

Proof. 1. Let s be simple, $0 \leq s \leq f$. $s \cdot \mathbf{1}_{[f \leq g]} \leq g$ where $s \cdot \mathbf{1}_{[f \leq g]}$ is simple. $\int s \cdot \mathbf{1}_{[f \leq g]}\,d\mu \leq \int g\,d\mu$. $\int s \cdot \mathbf{1}_{[f \leq g]}\,d\mu = \int s\,d\mu$.
If $\forall s$ simple, $0 \leq s \leq f$, then

$$\int f\,d\mu = \sup\left\{\int s\,d\mu \mid 0 \leq s \leq f, s \text{ simple}\right\} \leq \int g\,d\mu$$

2. $f \leq g$ almost everywhere and $f \geq g$ almost everywhere $\implies \int f\,d\mu = \int g\,d\mu$.

3. Markov inequality with $z = \frac{1}{n}$.

$$\frac{1}{n}\mu\left[f \geq \frac{1}{n}\right] \leq \int f\,d\mu = 0 \implies \mu\left[f \geq \frac{1}{n}\right] = 0 \forall n \in \mathbb{N}$$

$x \in [f \geq \frac{1}{n}] \implies x \in [f \geq \frac{1}{n+1}]$

$$\implies \mu\left[f \geq \frac{1}{n}\right] \to \mu\left[\bigcup\left[f \geq \frac{1}{n}\right]\right] = \mu\left[f > 0\right] = 0$$

4. $z > 0$, $s = z \cdot \mathbf{1}_{[f=\infty]} \leq f$.

$$z\mu[f = \infty] = \int s\,d\mu \leq \int f\,d\mu = M < \infty$$

$$\mu[f = \infty] \leq \frac{M}{z} \qquad \forall z > 0 \implies \mu[f = \infty] = 0$$

□

Theorem 3.6.2 (Levi's theorem about monotonic convergence). *If* $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ *is measurable and pointwise monotonically increasing* ($f_1 \leq f_2 \leq \dots$) *and* $f = \lim_{n\to\infty} f_n$ *then* $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$

Proof. Because of (1) in the previous theorem, $\int f_n\,d\mu$ is monotonically increasing and $\leq \int f\,d\mu$, so $\lim \int f_n\,d\mu \leq \int f\,d\mu$.

$(y)^+$ denotes the function y if $y \geq 0$ and 0 otherwise.

Show "$\geq$". Let $0 \leq s \leq f$ be simple. Let $\varepsilon > 0$. $s_{n,\varepsilon} := (s-\varepsilon)^+\mathbf{1}_{[f_n \geq f-\varepsilon]}$ is a simple function. $s - \varepsilon \leq f - \varepsilon \leq f_n$. $s_{n,\varepsilon} \leq f_n$.

$$\sum_z (z-\varepsilon)^+\mu\left[s = z, f_n > f - \varepsilon\right] = \int s_{n,\varepsilon}\,d\mu \leq \int f_n\,d\mu \leq \lim \int f_n\,d\mu$$

$$s_{n,\varepsilon} = \underbrace{\sum_{z \text{ (values of s)}} (z-\varepsilon)^+\mathbf{1}_{[s-z]}}_{(s-\varepsilon)^+} \mathbf{1}_{[f_n > f-\varepsilon]}$$

$$[f_n > f - \varepsilon] \nearrow \chi \qquad [s = z, f_n > f - \varepsilon] \nearrow [s - z]$$

$$\implies \sum_z (z-\varepsilon)^+\mu[s = z] \leq \lim \int f_n\,d\mu$$

$$\varepsilon \to 0 \implies \sum_z z\mu[s = z] \leq \lim \int f_n\,d\mu$$

If $z > 0$, such that $\mu[s = z] = +\infty$. $0 < \varepsilon < z$.

Let $s_{n,\varepsilon} = (s-\varepsilon)^+\mathbf{1}_{[f_n \geq M \wedge (f-\varepsilon)]}$, where $a \wedge b$ denotes the minimum of a and b. Let $M \geq \max s$. □

↓ *This lecture took place on 2018/10/29.*

Remark (Revision). *Let s be a simple function.* $s = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$.
$s = \sum_z z\mathbf{1}_{[s=z]}$ *is a finite sum*
$\int s\, d\mu = \sum_z \mu[s = z] = \sum_{i=1}^n c_i \mu(A_i)$
This is independent of the representation.

Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Then we can approximate the integral of f using the integrals of simple functions.

$$\int f\, d\mu = \sup \left\{ \int s\, d\mu \mid 0 \le s \le f, \text{ simple} \right\}$$

Remark (Properties).

1. $0 \le f \le f$ *almost everywhere (wrt. μ)* $\implies \int f\, d\mu \le \int g\, d\mu$
2. $f = g$ *almost everywhere (wrt. μ)* $\implies \int f\, d\mu = \int g\, d\mu$
3. $\int f\, d\mu = 0 \iff f = 0$ *almost everywhere (wrt. μ)*
4. $\int f\, d\mu < \infty \implies f < \infty$ *almost everywhere*

It is obvious if s is simple, then $\int s\, d\mu = \max \{\int t\, d\mu \mid 0 \le t \le s \text{ simple}\}$

Theorem (Monotonic convergence). *Let $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$f_n \le f_{n+1} \forall n \qquad f = \lim_{n\to\infty} f_n \qquad \implies \int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu$$

Lemma (Lemma by Fatou). *Let $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$\implies \int \left(\lim_{n\to\infty} \inf f_n \right) d\mu \le \liminf_{n\to\infty} \int f_n\, d\mu$$

Proof.

$$\lim_{n\to\infty} \underbrace{\inf_{m\ge n} f_m}_{g_n} \qquad g_n \nearrow \liminf f_n$$

By the theorem of monotonic convergence,

$$\implies \int (\liminf f_n)\, d\mu = \int \lim g_n\, d\mu = \lim \int g_n\, d\mu$$

□

Lemma 3.7. *Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ with countable $f(\chi)$.*

$$\implies \int f\, d\mu = \sum_{z\in f(\chi)} z\mu[f = z]$$

Proof.

$$f(\chi) = \{z_n \mid n \in \mathbb{N}\}$$

$$f_n = \sum_{k=1}^{n} z_k \mathbf{1}_{[f=z_k]} \nearrow f \implies \int f \, d\mu = \lim \int f_n \, d\mu = \lim \sum_{k=1}^{n} z_k \mu[f = z_k]$$

□

The integral should be linear. We expect this for any integral.

Theorem 3.7.1. *Let $f, g : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Let $\alpha \geq 0$.*

1. $\int (\alpha f) \, d\mu = \alpha \int f \, d\mu$ *(trivial to prove)*
2. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Proof. 1. trivial

2. We represent f_n

$$f_n = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbf{1}_{[\frac{k}{2^n} \leq f < \frac{k+1}{2^n})} + n \cdot \mathbf{1}_{[f \geq n]} \nearrow f$$

Compare with Figure 2. g analogously $\nearrow g$

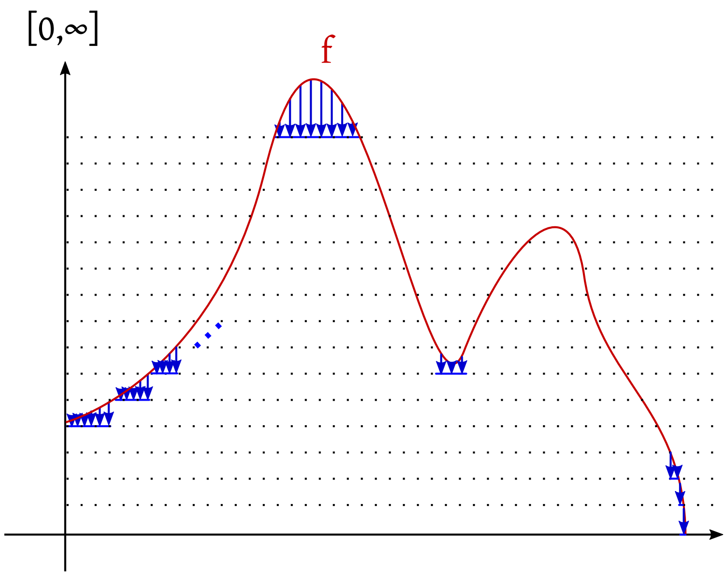

Figure 2: Illustration of the lebesgue integral

Let f_n, g_n be simple. $f_n + g_n \nearrow f + g$

$$\int (f + g) \, d\mu \overset{\text{monotonic convergence}}{=} \lim \int (f_n + g_n) \, d\mu$$

$$= \lim(\int f_n \, d\mu + \int g_n \, d\mu) \overset{\text{monotonic convergence}}{=} \int f \, d\mu + \int g \, d\mu$$

□

Unlike the Riemann integral, we use horizontal lines instead of vertical lines. Thus we partition the image, not the domain.

4 Integrable functions

Definition 4.1. *Let $f : (\chi, d) \to \overline{\mathbb{R}}$ is measurable. If not $\int f^+ \, d\mu = \int f^- \, d\mu = +\infty$, integral exists:*

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

$$f^+ = \max\{f, 0\} \qquad f^- = \max\{-f, 0\} \qquad f = f^+ - f^- \qquad |f| = f^+ + f^-$$

f is called integrable, if $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$

$$\iff \int f \, d\mu \text{ exists and is finite}$$

Remark 4.2 (Properties)**.**

1. *f is integrable $\iff$ $|f|$ is integrable and $|\int f \, d\mu| \leq \int |f| \, d\mu$*
2. *f, g are integrable with $f \leq g$ almost everywhere wrt. $\mu \implies \int f \, d\mu \leq \int g \, d\mu$*
3. *f is integrable, $\alpha \in \mathbb{R} \implies \alpha \cdot f$ is integrable and $\int (\alpha \cdot f) \, d\mu = \alpha \cdot \int f \, d\mu$*
4. *f, g are integrable $\implies f + g$ is integrable and $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$*

Proof. 1. f is integrable

$$:\iff \int f^{\pm} \, d\mu < \infty \iff \underbrace{\int f^+ \, d\mu + \int f^- \, d\mu < \infty}_{\int |f| \, d\mu < \infty}$$

$$\begin{aligned} \left| \int f \, d\mu \right| &= \left| \int f^+ \, d\mu - \int f^- \, d\mu \right| \\ &\leq \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int |f| \, d\mu \end{aligned}$$

2. $f^+ - f^- \overset{\substack{\text{almost} \\ \text{everywhere}}}{\leq} g^+ - g^- \implies f^+ + g^- \overset{\text{a.e.}}{\leq} f^- + g^+$

$$\int f^+ \, d\mu + \int g^- \, d\mu = \int (f^+ + g^-) \, d\mu \leq \int (f^- + g^+) \, d\mu = \int f^- \, d\mu + \int g^+ \, d\mu$$

$$\int f^+ \, d\mu - \int f^- \, d\mu \leq \int g^+ \, d\mu - \int g^- \, d\mu$$

It is important to recognize that all integrals are finite.

3. For $\alpha = 0$, the statement is true. Consider $\alpha > 0$.

   $(\alpha f)^\pm = \alpha \cdot f^\pm \qquad \int \alpha f \, d\mu = \int \alpha \cdot f^+ \, d\mu - \int \alpha \cdot f^- \, d\mu = \alpha \int f^+ \, d\mu - \alpha \int f^- \, d\mu$

   Now consider $\alpha < 0$, or more simply $\alpha = -1$ (any negative number is the product of a positive number and -1):

   $$(-f)^+ = f^- (-f)^- = f^+ \qquad \ldots$$

4. $(f+g)^+ - (f+g)^- = f + g = f^+ + g^+ - (f^- + g^-)$

   $$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

   $$\int (f+g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int (f+g)^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu$$

   $$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu$$

□

Riemann integral only works for $\mathbb{R}^n$. The Lebesgue integral works for any measure space.

Example 4.3. *We consider the Riemann integral:*

$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \overset{\text{Riemann}}{=} \lim_{c,d \to \infty} \int_{-c}^{d} \frac{\sin x}{x} \, dx \text{ exists}$$

If you consider $\frac{\sin x}{x}$ for one π, we have a positive and negative area. By Leibniz criterion, we have an alternating series and its limit is zero.

We consider the Lebesgue integral:

$$\int_{\mathbb{R}} \left| \frac{\sin x}{x} \right| \, dx = +\infty$$

$\frac{\sin x}{x}$ is not Lebesgue-integrable. Because in case of the Lebesgue integral, we don't consider an alternating series, but need to consider $|f|$, which is non-negative and the series does not converge.

Theorem 4.3.1 (Dominated convergence theorem by Lebesgue)**.** *Let $f_n : (\chi, \mathcal{A}) \to \overline{\mathbb{R}}$ be a sequence of measurable functions. $f_n \to f$ pointwise [almost everywhere wrt. μ]. There exists $g : (\chi, \mathcal{A}) \to [0, \infty]$ integrable [$\int g \, d\mu < \infty$].*

$$|f_n| \leq g \text{ almost everywhere wrt. } \mu \forall n \implies \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proof. Without loss of generality, almost everywhere implies everywhere.

$$|f| = \lim |f_n| \leq g \qquad \text{all of them are integrable}$$
$$g_n = 2g - |f_n - f| \geq 0 \qquad g_n \to 2g$$
$$\liminf \int g_n \, d\mu \geq \int (\liminf g_n) \, d\mu \overset{g_n \to 2g}{=} \int (\lim g_n) \, d\mu = 2 \int g \, d\mu$$
$$\int g \, d\mu - \limsup \int |f_n - f| \, d\mu = \liminf \int g_n \, d\mu = 2 \int g \, d\mu$$
$$\limsup \left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \limsup \int |f_n - f| \, d\mu = 0$$

Again:

$$\int g_n = \left(\int 2g - \int |f_n - f| \right)$$
$$\implies \limsup \int g_n = \limsup \left(\int 2g - \int |f_n - f| \right)$$
$$= \int 2g + \limsup \left(- \int |f_n - f| \right) = \int 2g - \liminf \left(\int |f_n - f| \right)$$

□

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