Florian Kruse

Sprechstunde: Tue, 14–15

# Exercise 01/1

**Exercise 1.** *The Euclidean norm of* $v = (v^1, v^2, \dots, v^n)^T \in \mathbb{R}^n$ *is defined as*

$$\|v\|_2 := \sqrt{(v^1)^2 + (v^2)^2 + \ldots + (v^n)^2}$$

*Show: A sequence* $(x_k) \subset \mathbb{R}^n$ *converges in regards of the Euclidean norm to* $x \in \mathbb{R}$ *iff they converge componentwise to* $x$

$$\lim_{k\to\infty} \|x_k - x\|_2 = 0 \iff \forall j \in \{1, \dots, n\} : \lim_{k\to\infty} x_k^j = x^j$$

Direction $\Rightarrow$.

Let $\lim_{k\to\infty} \|x_k - x\| = 0$.

Consider: $\left|x_{jk} - x_j\right|$ for arbitrary $j \in \{1, \dots, n\}$.

It holds that

$$0 \le \left|x_{jk} - x\right| = \sqrt{(x_{jk} - x_j)^2} \le \sqrt{(x_{1k} - x_1)^2 + \cdots + (x_{1k} - x_n)} = \|x_k - x\| \to 0$$

$$\implies \lim_{k\to\infty} \left|x_{jk} - x_j\right| = 0 \forall j$$

Direction $\Leftarrow$.

Let $\lim_{k\to\infty} x_{jk} = x_j \forall j \in \{1, \dots, n\}$.

The square root function is continuous.

$$\lim_{k\to\infty} \|x_k - x\| = \sqrt{(x_{1k} - x_1)^2 + \cdots + (x_{1k} - x_n)^2}$$

$$\sqrt{(\lim_{k\to\infty} x_{1k})^2 - 2(\lim_{k\to\infty} x_i k)x_1 + x_{1j}^2 + \cdots + (\lim_{k\to\infty} x_{n_k})^2 - 2(\lim x_{n_k})x_n + x_n^2}$$

$$= \sqrt{\underbrace{x_1^2 - 2x_1^2 + x_1^2}_{=0} + \cdots + \underbrace{x_n^2 - 2x_n^2 + x_n^2}_{=0}} = 0$$

**Remark:** In $\mathbb{R}^n$, all norms are equivalent. This exercise showed this property. So it you pick two numbers in $\mathbb{R}^n$ and they get "closer", they get "closer" in every norm.

## Exercise 01/2

**Exercise 2.** *In the lecture, we discussed the SCNF.* $d_{SCNF} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. *For some fixed* $p \in \mathbb{R}^2$ *it is defined as*

$$d_{SCNF} := \begin{cases} \left\|x - y\right\|_2 & \textit{if } \exists \lambda > 0 : y = p + \lambda(x - p) \\ \left\|x - p\right\|_2 + \left\|y - p\right\|_2 & \textit{else} \end{cases}$$

*For* $p := (0,0)^T$ *and* $x := (1,1)^T$, *sketch the set* $B_R(x)$ *for* $R = 1$ *and* $R = 2$.

$$B_R(x) := \left\{ y \in \mathbb{R}^2 \,\middle|\, d_{SCNF} < R \right\}$$

## Exercise 01/3

**Exercise 3.** *Let* $(M, d)$ *be a metric space and* $x \in M$. *Furthermore let* $(x_k) \subset M$ *be a sequence with property that every subsequence of* $(x_k)$ *contains a subsequence converging to x. Prove by contradiction, that* $(x_k)$ *converges to x.*

$x_0 \not\to x$.

There exists $\varepsilon_0 > 0$ for infinitely many $n \in \mathbb{N} : d(x_n, x) \geq \varepsilon_0$. Choose a subsequence $(x_{u_j})_{j \in \mathbb{N}}$ with $d(x_{n_j}, x) \geq \varepsilon_0 \forall j \in \mathbb{N}$. Then there does not exist a subsequence of $(x_{n_j})$ with limit $x$.

## Exercise 01/4

**Exercise 4.** *Let* $(M, d)$ *be a metric space and complete space. The diameter of a nonempty set* $A \subset M$ *is given by*

$$\operatorname{diam}(A) := \sup \left\{ d(x, y) \,\middle|\, x, y \in A \right\}$$

*Let* $(A_j)_{j \in \mathbb{N}}$ *be a sequence of nonempty, closed sets in M with* $A_{j+1} \subset A_j$ *for all* $j \in \mathbb{N}$. *Furthermore it holds that* $\operatorname{diam}(A_j) \to 0$ *for* $j \to \infty$. *Prove that* $x \in M$ *exists with* $\bigcap_{j=1}^{\infty} A_j = \{x\}$ *and that x is unique.*

$A_j \subseteq M$, because its a complete, metric space.

$$\implies \bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$$

Assume $\exists y_0 \in M : y_0 \neq x_0 \implies d(y_0, x_0) \geq \varepsilon > 0$

$$\forall j \in \mathbb{N} : \operatorname{diam}(A_j) \geq \varepsilon$$

This is a contradiction. However, this is not the equality, we are looking for. Assume $\bigcap_{j=1}^{\infty} A_j = \{x_0\} = \{y_0\} \implies x_0 = y_0$. This is the equality, that was meant to be proven.

### Prove $\bigcap_{j=1}^{\infty} A_j \neq \emptyset \iff \exists x_0 \in M : \forall j$

**Hint:** If the assignment mentions that completeness must be proven, usually you have to construct a Cauchy sequence.

Construct $(x_j)_{j\in\mathbb{N}}$. Choose for $x_j$ some element of $A_j$. Choose $x_j \in A_j$ for $j \in \mathbb{N}$. This defines a Cauchy sequence $(x_j)_{j\in\mathbb{N}}$. Let $j \in \mathbb{N}$. $x_i \in A_j \supset A_{j+1}$ and $x_{j+1} \in A_{j+1} \forall i \in \mathbb{N}$.

$$\implies d(x_j, x_{j+i}) \leq \operatorname{diam}(A_j) \forall i \in \mathbb{N}$$

where $\operatorname{diam}(A_j) \to 0$ with $i \to \infty$.

$$\implies \exists x \in M : \lim_{j\to\infty}(x_j) = x$$

Because $(x_j)_{j\geq J} \subseteq A_j$ and $\lim_{j\to\infty}(x_j)_{j\geq J} = x$, it follows that $x \in A_j$ and then it follows that $x \in \bigcap_{j=1}^{\infty} A_j$.

*This lecture took place on 2018/03/22.*

## Exercise 02/1

### Blackboard solution

Let $B$ be bounded.

$$\operatorname{diam}(B) < \infty \qquad \operatorname{diam}(B) = \sup(\left\{ d(x,y) \,\middle|\, x, y \in B \right\})$$

$$d(B_k, B_{k+1}) = \inf(\left\{ d(x,y) \,\middle|\, x \in B_k, y \in B_{k+1} \right\})$$

Exercise (a).

Prove:

$$\sum_{k=1}^{\infty} \operatorname{diam}(B_k) < \infty \land \sum_{k=1}^{\infty} d(B_k, B_{k+1}) \implies \operatorname{diam}(\bigcup_{k=1}^{\infty} B_k) < \infty$$

$$\operatorname{diam}(B_k \cup B_{k+1}) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$$

We distinguish 3 cases:

1. $x \in B_k, y \in B_k : d(x,y) \leq \operatorname{diam}(B_k) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$
2. $x \in B_{k+1}, y \in B_{k+1}, d(x,y) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1})$

3. $\forall x \in B_k \forall y \in B_{k+1}$

Choose $x_0$ and $y_0$ on the border of sets $B_k$ and $B_{k+1}$ respectively. But $x_0, y_0$ do not necessarily exist if compactness is not given. But let $\varepsilon > 0$. Find $x_0, y_0$ with $d(x_0, y_0) \leq d(B_k, B_{k+1}) + \varepsilon$.

$$d(x,y) \leq \underbrace{d(x,x_0)}_{\leq \operatorname{diam}(B_k)} + \underbrace{d(x_0,y_0)}_{\leq d(B_k,B_{k+1})+\varepsilon} + \underbrace{d(x_0,y)}_{\leq \operatorname{diam}(\mathrm{B_k})} \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + \varepsilon$$

Laurent Pfeiffer continued the following solution (until Exercise 2):

$$\operatorname{diam}((B_k \cup B_{k+1}) \cup B_{k+2}) \leq \operatorname{diam}(B_k \cup B_{k+1}) + \underbrace{d((B_k \cup B_{k+1}), B_{k+2})}_{\leq d(B_{k+1},B_{k+2})} + \operatorname{diam}(B_{k+2})$$

$$\leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + d((B_k \cup B_{k+1}), B_{k+2}) + \operatorname{diam}(B_{k+2})$$

By induction it follows that

$$\operatorname{diam}(B_k \cup B_{k+1} \cup \cdots \cup B_n) \leq \operatorname{diam}(B_k) + d(B_k, B_{k+1}) + \operatorname{diam}(B_{k+1}) + d(B_{k+2}) + d(B_{n-1}, B_n) + \operatorname{diam}(B_n)$$

$$\operatorname{diam}(B_k \cup \cdots \cup B_n) \leq \underbrace{\sum_{i=1}^{n} \operatorname{diam}(B_i) + d(B_i, B_{i+1})}_{D}$$

Choose $x, y \in \bigcup_{i=1}^{\infty} B_i$. Then there exists some $k \in \mathbb{N}$ such that $x \in B_k$. There exists $n$ such that $y \in B_n$.

$$d(x,y) \leq \operatorname{diam}(B_k) + \cdots + \operatorname{diam}(B_n) \leq D$$

Exercise (b).

Let $x \in M$. We define: $B_{k+1} = B_{k+2} = \cdots = \{x\}$. For all $i \geq k$ it holds that

$$\operatorname{diam}(B_i) = 0$$

$$d(B_i, B_{i+1}) = 0$$

Therefore,

$$\sum_{i=1}^{\infty} \operatorname{diam}(B_i) = \sum_{i=1}^{k} \underbrace{\operatorname{diam}(B_i)}_{<+\infty} < +\infty$$

What about the distances?

$$\int_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^{k} d(B_i, B_{i+1}) < +\infty$$

By (a), it follows that

$$\left(\bigcup_{i=1}^{\infty} B_i\right) \text{ is bounded } \implies \left(\bigcup_{i=1}^{k} B_i\right) \subseteq \left(\bigcup_{i=1}^{\infty} B_i\right) \text{ is also bounded}$$

Exercise (c).

We define

$$B_i = \left[\sum_{j=1}^{i} \frac{1}{j}, \sum_{j=1}^{i+1} \frac{1}{j}\right]$$

Then it holds that

$$\operatorname{diam}(B_i) = \frac{1}{i+1} \xrightarrow{i\to\infty} 0$$

$$\sum_{i=1}^{\infty} \operatorname{diam}(B_i) = \infty$$

$$B_i \cap B_{i+1} = \left\{\sum_{j=1}^{i+1} \frac{1}{j}\right\} \implies d(B_i, B_{i+1}) = 0$$

$$B_1 \cup \cdots \cup B_i = \left[1, \underbrace{\sum_{j=1}^{i+1} \frac{1}{j}}_{\to\infty}\right] \implies \underbrace{\bigcup_{i=1}^{\infty} B_i}_{\text{not bounded}} = [1, \infty)$$

We define $B_i = \left\{\sum_{j=1}^{i} \frac{1}{j}\right\}$. For all $i$:

- $\operatorname{diam}(B_i) = 0 \implies \sum_{i=1}^{\infty} \operatorname{diam}(B_i) = 0$
-
$$d(B_i, B_{i+1}) = \left(\sum_{j=1}^{i+1} \frac{1}{j}\right) - \left(\sum_{j=1}^{i} \frac{1}{j}\right) = \frac{1}{i+1} \xrightarrow{i\to\infty} 0$$

$$\sum_{i=1}^{\infty} d(B_i, B_{i+1}) = \sum_{i=1}^{\infty} \frac{1}{i+1} = \infty$$

The union is *not* bounded, because $\sum_{j=1}^{i} \frac{1}{j} \in \bigcup_{j=1}^{\infty} B_j$.

## Exercise 02/2

**Exercise 5.** *Let $(X, d)$ be a sequentially compact, metric space. Show:*

*a. $X$ is bounded.*

*b.*

### Blackboard solution

Exercise (a).

Let $X$ be unbounded. Hence, there exists a tuple $(x_N, y_N) \in X \times X$ for every $N \in \mathbb{N}$ with $d(x_N, y_N) > N$. Because $(X, d)$ is sequentially compact, there exists a convergent subsequence $(x_{N_k}, y_{N_{k_i}})$ we can choose such that

$$\lim_{k\to\infty} x_{N_k} = \infty \qquad \lim_{i\to\infty} y_{N_{K_i}} = y_0 \qquad \lim_{i\to\infty}(x_{N_{k_i}}) = x_0$$

$$\implies \underbrace{N_{k_i}}_{\xrightarrow{i\to\infty}\infty} < d(x_{N_{k_i}}, y_{N_{k_i}}) \xrightarrow{i\to\infty} d(x_0, y_0)$$

By this contradiction, it follows that $X$ is bounded.

Exercise (b).

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $X$. Let $X$ be sequence compact $\implies$ there exists a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x \in X$. Show that $x_n \xrightarrow{n\to\infty} x$.

Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $\forall n, m \geq N : d(x_n, x_m) < \frac{\varepsilon}{2}$. Choose $k \in \mathbb{N}$ such that $n_k \geq N$ and $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

$$\forall n \geq n_k : d(x, x_n) \leq d(x, x_{n_k}) + d(x_{n_k}, x_n) < \varepsilon$$

Exercise (c).

Show that $A \subset X$ is sequentially compact iff $A$ is closed.

$\Rightarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence, $(x_n)_{n\in\mathbb{N}} \subset A$, $\lim_{n\to\infty} x_n = x_0 \in X$. Show that $x_0 \in A$.

Set $A$ is sequentially compact. Choose subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset A$, $\lim_{k\to\infty} x_{n_k} = x_0 \in A \implies A$ is closed.

$\Leftarrow$ $A$ is closed. Show that $A$ is sequentially compact.

Let $(x_n)_{n\in\mathbb{N}} \subset A$ and there exists subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = x_0 \in X$, because $X$ is sequentially compact. $(x_{n_k})_{k\in\mathbb{N}} \subset A \implies A$ is sequentially compact.

## Exercise 02/2

**Exercise 6.** *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sqrt{1 + x^2}$.*

1. *Show that $\left|f(x) - f(y)\right| < \left|x - y\right| \, \forall x, y \in \mathbb{R}$ with $x \neq y$*
2. *Investigate which conditions of Banach's Fixed Point Theorem are [not] met.*
3. *Is Banach's Fixed Point Theorem applicable? Does $f$ have a fixed point?*

Exercise (a).

$$\left|f(x) - f(y)\right| < \left|x - y\right| \qquad x, y \in \mathbb{R}, x \neq y$$

$$\left|\sqrt{1 + x^2} - \sqrt{1 + y^2}\right| < \left|x - y\right|$$

$$1 + x^2 + 1 + y^2 - 2\sqrt{(1 + x^2)(1 + y^2)} < x^2 + y^2 - 2xy$$

$$2 - 2\sqrt{(1 + x^2)(1 + y^2)} < -2xy$$

$$1 + xy < \sqrt{(1 + x^2)(1 + y^2)}$$

We need to distinguish 2 cases here ($x$ and $y$ have same signum, $x$ and $y$ have different signum). This is trivial.

$$1 + 2xy + x^2y^2 < 1 + x^2 + y^2 + x^2y^2$$

$$0 < x^2 + y^2 - 2xy$$

$$0 < (x - y)^2$$

Exercise (b and c).

Let $x \in \mathbb{R}$.

$$f(x) = x$$

$$\sqrt{1 + x^2} = x$$

$$1 + x^2 = x^2$$

$$1 = 0$$

*This lecture took place on 2018/04/12.*

## Exercise 03/4

**Exercise 7.** *Let $(X, d)$ be a metric space and $x_0 \in X$. A function $f : X \to \mathbb{R}$ is called half-continuous from below in $x_0$, if for every $\varepsilon > 0$ some $\delta > 0$ exists, such that $d(x, x_0) < \delta$ implies $f(x_0) - f(x) < \varepsilon$. If $f$ is half-continuous from below in every $x_0 \in X$, then $f$ is called half-continuous from below.*

Obviously, continuity implies half-continuity.

### Exercise 03/4a

**Exercise 8.** *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$ such that $f$ is non-continuous.*

Let $f : [-1, 1] \to \mathbb{R}$.

$$x \mapsto \begin{cases} -1 & x = -1 \\ -x & x \neq -1 \end{cases}$$

$$\underbrace{f(-1)}_{=-1} - \underbrace{f(x)}_{\geq -1} \leq 0 < \varepsilon$$

### Exercise 03/4b

**Exercise 9.** *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$, but does not have a maximum.*

Same $f$ can be chosen.

### Exercise 03/4c

**Exercise 10.** *Give some half-continuous from below $f : [-1, 1] \to \mathbb{R}$, but does not have a minimum.*

$f$ as $f|_{[-1,1]}$ can be chosen.

### Exercise 03/4d

**Exercise 11.** *Prove that every half-continuous from below function in a compact set has a minimum.*

**Hint:** It is assumed that cover-compactness seems to be more cumbersome than sequential compactness.
**Remark:** This is a generalization of the theorem, that every continuous, compact function has a minimum and maximum.

Let $K \subseteq X$ be compact. $f : K \to \mathbb{R}$ is half-continuous from below.

Show that $f^k = \inf(f(K)) \in f(K)$.

$$\exists (x_n)_{n\in\mathbb{N}} \subseteq K \text{ with } f(x_n) - f^k < \frac{1}{n}$$

$K$ is compact. Hence, there exists $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} := x^* \in K$. Let $\varepsilon > 0$ be arbitrary. By half-continuity from below, it follows that $\exists \delta > 0 : d(x^*, x) < \delta \implies f(x^*) - f(x) < \varepsilon$.

$$\exists K \in \mathbb{N} \forall k \geq K : d(x^k, x_{n_k}) < \delta \implies f(x^k) - f(x_{n_k}) < \varepsilon \iff f(x^*) < f(x_{n_k}) + \varepsilon$$

$$\implies f(x^*) \leq \lim_{k\to\infty} f(x_{n_k}) \implies f(x^*) \leq \lim_{n\to\infty} f(x_n) = f^*$$

$$\implies f(x^*) = f^* \implies f^* \text{ is minimum of } f(X)$$

# Exercise 03/3

**Exercise 12.** *Let $(X, d)$ and $(Y, e)$ be metric spaces, where $d : X \to \mathbb{R}$ is a discrete metric, hence*

$$d(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

## Exercise 03/3a

**Exercise 13.** *Every map $f : X \to Y$ is continuous.*

Let $f : X \to Y$ be arbitrary. Let $x_0 \in X$ and $\varepsilon > 0$ be arbitrary. Show that

$$\exists \delta > 0 : d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$$

$$K_{\frac{1}{2}}(x_0) = \{x_0\}$$

## Exercise 03/3b

**Exercise 14.** *A map $f : X \to Y$ is not necessarily bounded.*

$M \geq 0$ arbitrary. $\exists x, y \in f(X) : e(x, y) > M$.

$$f : \mathbb{Z} \to \mathbb{Z} \qquad x \mapsto x$$
$$f(x) = \mathbb{Z} \qquad x = 0 \qquad y = M + 1$$

$e = |\cdot|$.

### Exercise 03/3c

**Exercise 15.** *Every map $g : Y \to X$ is bounded.*

Let $g : Y \to X$ be arbitrary. Show that $\exists M \geq 0 \forall x, y \in g(Y) : d(x, y) \leq M$. Choose $M = 2$. $\forall x, y \in X : d(x, y) \leq 1 \leq 2$.

### Exercise 03/3d

**Exercise 16.** *In case $(Y, e) = (\mathbb{R}, |\cdot|)$, every non-constant map $g : Y \to X$ is non-continuous.*

We show: continuity implies constant.

Let $g : \mathbb{R} \to X$ continuous. Let $x_0 \in \mathbb{R}$ be arbitrary and $\varepsilon = \frac{1}{2}$. $\exists \delta_0 > 0 : |x_0 - x| < \delta \implies d(g(x_0), g(x)) < \frac{1}{2}$ for $x_0 \in \mathbb{R}$ there exists $\delta_0$ such that $\forall x \in (x_0 - \delta, x_0 + \delta) : g(x) = g(x_0)$.

$$\sup \{s \in [x_0, \infty) \mid g(x) = g(x_0) \forall x \in [x_0, s)\}$$

## Exercise 03/2

**Exercise 17.** *Let $V$ be the vector space of bounded, complex sequences, hence*

$$V := \{(a_k)_{k \in \mathbb{N}} \subset C \mid \exists M \in \mathbb{R} \text{ with } |a_k| \leq M \forall k \in \mathbb{N}\}$$

*additionally with norm*

$$||(a_k)_{k \in \mathbb{N}}||_\infty := \sup \{|a_k| \mid k \in \mathbb{N}\}$$

This solution was done by Mr. Kruse himself.

### Exercise 03/2b

**Exercise 18.** *The unit sphere in $(V, ||\cdot||_\infty)$,*

$$B_1(0) = \{a \in V \mid ||a||_\infty \leq 1\}$$

*is closed and bounded, but not sequentially compact.*

We need to prove boundedness.

Let $C, D \in B_1(0)$.

$$\implies \left\| \underbrace{C}_{=(c_k)} - \underbrace{D}_{=(d_k)} \right\|_\infty \leq 2$$

$$\sup \left\{ \left| \underbrace{c_k - d_k}_{\leq \underbrace{|c_k|}_{\leq 1 \forall k} + \underbrace{|d_k|}_{\leq 1 \forall k} \leq 2 \forall k} \right| : k \in \mathbb{N} \right\} \leq 2$$

We need to prove closedness.

$$(A^n)_{n \in \mathbb{N}} \subset B_1(0) \text{ with } \lim_{n \to \infty} A^n = A$$

Show that $A \in B_1(0)$.

$$\text{For every } A^n := (a_k^n)_{k \in \mathbb{N}} \text{ it holds that } \left\| \underbrace{(a_k^n)_{k \in \mathbb{N}}}_{=\sup\{|a_k^n| : k \in \mathbb{N}\} \leq 1} \right\|_\infty \leq 1$$

$$(A^n)_{n \in \mathbb{N}} \subset B_1(0) \text{ with } \lim_{n \to \infty} A^n = A$$
$$\iff \lim_{n \to \infty} \|A^n - A\|_\infty = 0$$

$\left|a_k^n\right|$ in

$$\sup \left\{ \left|a_k^n\right| : k \in \mathbb{N} \right\}$$

converges to $|a_k| \leq 1$ for $n \to \infty$.

We need to prove sequentially non-compact of $B_1(0)$. So we only need to find some sequence that does not have some converging subsequence.

We define

$$A^n := (a_k^n)_{k \in \mathbb{N}} := \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$$

for every $n \in \mathbb{N}$. As such we get a sequence

$$\implies (A^n)_{n \in \mathbb{N}} \subset B_1(0)$$

but it holds that $\|A^n - A^m\|_\infty = 1 \forall n \neq m$. This is also not a Cauchy sequence.

## Exercise 03/1

**Exercise 19.** *Let $(X, d)$ be a metric space. A set $K \subset X$ is called cover-compact, if for every family of open sets $(U_i)_{i \in I} \subset X$ with $K \subset \bigcup_{i \in I} U_i$ it holds that: There exists a finite set $J \subset I$ with $K \subset \bigcup_{i \in J} U_i$. Let $K \subset X$ be cover-compact.*

### Exercise 03/1a

**Exercise 20.** *Show that $K$ is totally bounded, hence for every $r > 0$, there exists $x_1, \ldots, x_n$ in $K$ with $K \subset \bigcup_{i=1}^{n} B_r(x_i)$.*

Construct a family of open spheres ($(\mathcal{B}_r(x))_{x \in K} \subset K$ covering $K$). By cover-compactness it follows there exists some finite $J \subset K$ with $K \subset \bigcup_{x \in J} B_r(x)$.

### Exercise 03/1b

**Exercise 21.** *Prove that $K$ is sequentially compact.*

Proof by contradiction: Assume $K$ is not sequentially compact.

Then there exists a sequence $(x_n)_{n \in \mathbb{N}} \in K$ which has a subsequence $(x_{n_k})_{k \in \mathbb{N}} \to c \notin K$.

$$\forall x \in K : \exists r_x > 0 : B_{r_x}(x) \text{ contains finitely many sequence elements}$$

Because $\bigcup_{x \in K} B_{r_x}(x) \supset K$ it holds: there exists $J \subset K$ finite $\bigcup_{x \in J} B_{r_x}(x) \supset K$. This contradicts with $(x_n)_{n \in \mathbb{N}} \subset K$.

## Exercise 04/1

**Exercise 22.** *Let $(M, d)$ be a complete metric space and $(A_k)_{k \in \mathbb{N}} \subset M$ is a sequence of closed sets. Use Cantor's Theorem to prove: $\bigcup_{k \in \mathbb{N}} A_k$ contains an open set if at least one $A_k$ contains an open set. Illustrate this statement for $(M, d) = (\mathbb{R}, |\cdot|)$.*

First we illustrate it in $\mathbb{R}$.

$$(A_k) = \{a_k\}$$

where $a_k \in \mathbb{R}$.

Consider some

## Exercise 04/2

**Exercise 23.** *Let $f : [-1, 1] \to \mathbb{C}$ be continuous and $O \subset \mathbb{C}$ is an open set. In the lecture we have seen that $f^{-1}(O)$ is open. Review the result and prove for $O = \mathbb{C}$.*

1. *The set $O$ is open.*
2. *It holds that $f^{-1}(O) = [-1, 1]$*
3. *The set $[-1, 1] \subset \mathbb{R}$ is not open.*
4. *The statement of the lecture about $f^{-1}(O)$ is still correct.*

### Exercise 04/2a

Show that $\mathbb{C}$ is open.

Let $z \in \mathbb{C}$. $\exists \varepsilon > 0$,

$$B(z, \varepsilon) \subseteq \mathbb{C}$$

### Exercise 04/2b

Follows from the definition of a function.

### Exercise 04/2c

If it is an open set, there must be a neighborhood of arbitrary $\varepsilon$ such that this neighborhood is completely in the set.

Let $\varepsilon > 0$. Choose $x \in B(1, \varepsilon)$ with $x = 1 + \frac{\varepsilon}{2}$.

$$\implies x \in B(1, \varepsilon) \land x \notin [-1, 1]$$

### Exercise 04/2d

Let $(X, d)$ and $(Y, e)$ be metric spaces and $f : X \to Y$ continuous then $f^{-1}(O)$ is open $\forall O \subseteq Y$ open.

Show:

$$\forall x \in [-1, 1] \exists \varepsilon > 0 : \underbrace{B(x, \varepsilon)}_{=\{z \in [-1,1] \mid d(x,z) < \varepsilon\}} \subseteq [-1, 1]$$

So the difference is the domain of $z$ ($[-1, 1]$ unlike exercise c, where we used $\mathbb{R}$).

The point was to illustrate how to read the theorem properly.

## Exercise 04/3

**Exercise 24.** *Let $\Omega$ be a non-empty set and $B(\Omega)$ the vector space of real-valued bounded functions on $\Omega$. Hence,*

$$B(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \exists M \in \mathbb{R} \textit{ with } \left|f(x)\right| \leq M \forall x \in \Omega \right\}$$

*with norm*

$$\left\|f\right\|_\infty := \sup \left\{ \left|f(x)\right| \,\middle|\, x \in \Omega \right\}$$

*Prove the following statements:*

1. *$(B(\Omega), \|\cdot\|_\infty)$ is a complete normed vector space.*
2. *The unit circle $U$ in $B(\Omega)$ is closed and bounded.*

$$U = \left\{ f \in B(\Omega) \,\middle|\, \left\|f\right\|_\infty \leq 1 \right\}$$

3. *The unit circle is sequentially compact if and only if $\Omega$ is finite.*

### Exercise 04/3a

Given $\Omega \neq 0$.

$$B(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \exists M \in \mathbb{R} : \left|f(x)\right| \leq M \quad \forall x \in \Omega \right\}$$

First, we show that $\|\cdot\|_\infty$ is indeed a norm. We just show absolute homogeneity for illustrative purposes:

$$\begin{aligned} \left\|\lambda f\right\|_\infty &= \sup \left\{ \left|\lambda \cdot f(x)\right| \,\middle|\, x \in \Omega \right\} \\ &= \sup \left\{ |\lambda| \cdot \left|f(x)\right| \,\middle|\, x \in \Omega \right\} \\ &= |\lambda| \cdot \sup \left\{ \left|f(x)\right| \right\} x \in \Omega \\ &= |\lambda| \cdot \left\|f\right\| \end{aligned}$$

We show completeness of $(B(\Omega), \|\cdot\|_\infty)$. Equivalently, all Cauchy sequences in $B(\Omega)$ are convergent. Equivalently, for all Cauchy sequences $(f_n)_{n\in\mathbb{N}} : \exists f \in B(\Omega) : \left\|f_n - f\right\|_\infty \to 0$ for $n \to \infty$.

Let $(f_n)_{n\in\mathbb{N}}$ be an arbitrary Cauchy sequence. Hence,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m > N \implies \left\|f_n - f_m\right\|_\infty = \sup \left\{ (f_n - f_m)(x) \,\middle|\, x \in \Omega \right\} < \varepsilon$$

$$\forall \varepsilon > 0 : n, m > N$$

$$\forall x \in \Omega : \left|(f_n - f_m)(x)\right| < \varepsilon$$

$$\implies \forall x \in \Omega : (f_n(x))_{n\in\mathbb{N}} \subseteq R$$

is a Cauchy sequence in $\mathbb{R}$.

$$\iff \forall x \in \Omega : (f_n(x))_{n\in\mathbb{N}} \text{ converges}$$

$$\forall x \in \Omega : (f_n(x)))_{n\in\mathbb{N}} \to f(x) \forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \implies \left|f_n(x) - f(x)\right| < \varepsilon$$

$$\exists N \in \mathbb{N} \forall n > N : \left\|f_n - f\right\|_\infty < 1$$

$$\left\|f\right\|_\infty = \left\|f - f_N + f_N\right\|_\infty \leq \underbrace{\left\|f - f_N\right\|_\infty}_{<1} + \underbrace{\left\|f_N\right\|}_{\leq M} < 1 + M$$

## Exercise 04/3b

Let $K_1 := \left\{ f \in B(\Omega) \,\middle|\, \left\|f\right\|_\infty \leq 1 \right\}$. Show $K_1$ is bounded and closed.

### $K_1$ is bounded

Let $f, g \in K_1$ be arbitrary.

$$\left\|f - g\right\|_\infty \leq \left\|f\right\|_\infty + \left\|g\right\|_\infty \leq 1 + 1 = 2$$

2 is a boundary and therefore $K_1$ is bounded.

### $K_1$ is closed

Let $(f_n)_{n\in\mathbb{N}}$ be a convergent sequence in $K_1$ with $\lim_{n\to\infty} f_n = f \iff \lim_{n\to\infty} \left\|f_n - f\right\| = 0$.

Show $f \in K_1$.

$$\forall f_n \in K_1 : \left\|f_n\right\| \leq 1$$

$$\left\|f\right\|_\infty = \left\|f - f_n\right\|_\infty \leq \underbrace{\left\|f - f_n\right\|_\infty}_{\xrightarrow{n\to\infty} 0} + \underbrace{\left\|f_n\right\|_\infty}_{\leq 1} \leq 1$$

$$\implies \left\|f\right\|_\infty \leq 1 \implies f \in K_1$$

## Exercise 04/c

$f$ is sequentially compact if and only if $\Omega$ is finite? Equivalently, every sequence $(f_n)_{n\in\mathbb{N}} \subseteq K_1$ has a convergent subsequence with limit in $K_1$.

Direction $\implies$.

Let $\Omega$ be infinite. Then $\exists$ a sequence $(f_n)_{n\in\mathbb{N}}$ without convergent subsequence. We build a sequence $(f_n)_{n\in\mathbb{N}}$ in $K_1$.

Let $(x_i)_{i\in\mathbb{N}}$ be an arbitrary sequence in $\Omega$ with $x_i \neq x_j \forall i \neq j$.

$$f_n(x) := \begin{cases} 1 & \text{if } x = x_n \\ 0 & \text{else} \end{cases}$$

Then it holds that $\forall n \neq m$,

$$\left\| f_n - f_m \right\|_\infty = 1$$

Assume there exists a convergent subsequence in $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ with limit $f$.

$$\implies \exists M > 0 : k > M : \left\| f_{n_k} - f \right\|_\infty < \frac{1}{2}$$

Let $k, l > M$ with $k \neq l$

$$\implies \left\| f_{n_k} - f_{n_l} \right\|_\infty \leq \left\| f_{n_k} - f \right\|_\infty + \left\| f_{n_l} - f \right\|_\infty < \frac{1}{2} + \frac{1}{2} = 1$$

This is a contradiction to $\left\| f_n - f_m \right\|_\infty = 1$.

Direction $\impliedby$.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $K_1$ without limit. Let $n \in \mathbb{N}$.

$$\Omega = \{x_1, \dots, x_n\} \implies \left| \{f_n(x_1), \dots, f_n(x_n)\} \right| < \infty$$

Let $f_n \in K_1 \implies \left| f_n(x_i) \right| \leq 1 \forall i \in \{1, \dots, m\} \, \forall n \in \mathbb{N}$.

Consider $x_1 \in \Omega$.

$$(f_n(x_1)) = y_n^1 \in [-1, 1]$$

$$[-1, 1] \text{ compact } \implies (y_n^1)_{n\in\mathbb{N}} \text{ has convergent subsequence } (y_{n_k}^1)_{k\in\mathbb{N}} \to \tilde{y}^1$$

$$(f_{n_k}(x_1))_{k\in\mathbb{N}} = (y_{n_k}^1)_{k\in\mathbb{N}} \to \tilde{y}_1 := f(x_1)$$

and this goes on up to

$$(f_{n_{\ddots_z}}(x_m))_{z\in\mathbb{N}} \to f(x_m)$$

For every $\varepsilon > 0$

$$\exists N_1 : \forall n \in N_1 : \left| f_{n_{\ddots_2}}(x_1) - f(x_1) \right| < \varepsilon$$

$$\vdots$$

$$\exists N_m : \forall n \in N_m : \left| f_{n_{\ddots_2}} (x_m) - f(x_m) \right| < \varepsilon$$

Choose $N := \max N_1, \ldots, N_m$. For all $n \geq N$,

$$\implies \left\| f_{n_{\ddots_2}} \right\|_\infty < \varepsilon$$

# Exercise 04/4

**Exercise 25.** *Let $k \in \mathbb{N}$. Show: $\exists \phi_k : \sqrt{k\pi} \leq \xi_k \leq \sqrt{(k+1)\pi}$ such that*

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2)\, dx = \frac{(-1)^k}{\xi_k}$$

$$\int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \sin(x^2)\, dx = \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} \frac{x \cdot \sin(x^2)}{x}\, dx = \frac{1}{\xi_k} \cdot \int_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}} x \cdot \sin(x^2)\, dx$$

But this IVT is unconventional.

$$= \frac{1}{\xi_k} \cdot \left( -\frac{1}{2} \cdot \cos(x^2) \right) \Big|_{\sqrt{k\pi}}^{\sqrt{(k+1)\pi}}$$

If $k$ is even:

$$\frac{1}{\xi_k} \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{1}{\xi_k}$$

If $k$ is odd:

$$\frac{1}{\xi_k} \left( -\frac{1}{2} - \frac{1}{2} \right) = -\frac{1}{\xi_k}$$

This implies a boundary of

$$\frac{(-1)^k}{\xi_k}$$