

# Introduction to Functional Analysis

Lecture notes, University of Technology, Graz  
based on the lecture by Martin Holler

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## 0 Introduction

↓ This lecture took place on 2019/03/05.

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g.  $\mathbb{R}^n, \mathbb{C}^n \mapsto$  vector spaces  $U, V$   
matrices  $A \in \mathcal{M}^{n \times m} \mapsto$  operators  $A \in \mathcal{L}(U, V)$   
functions  $f : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto$  functionals  $f : U \rightarrow \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
  - basis of Applied Mathematics
  - partial differential equations
  - physical modelling
  - inverse problems (operator  $A$  models some physical measurement process)
  - Optimization and optimal control

A motivating example was presented with slides.

## 0.1 Application examples

Let  $K : U \rightarrow \mathbb{R}^m$  with  $U$  as vector space describe a physical model. For example,  $K$  is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or  $Ku = y(1)$  where  $y : [0, 1] \rightarrow \mathbb{R}^m$  solves  $y'(t) = y(t) + u(t)$  and  $y(0) = 0$ .

Another example is the class of so-called *inverse problems*. Given  $d = ku$ , find  $u$ . Typically inversion of  $K$  is ill-constrained. Solution is typically non-unique.

Approach: Solve  $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$  where  $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$  and  $\|\cdot\|_u$  is a norm on  $U$ . Or alternatively, let  $U = C^1([0, 1]^2)$  and solve  $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$ .

Other examples are JPEG compression and upsampling of images.

## 0.2 Our first problem

Let  $U := C^1([0, 1]^2)$  be a normed space,  $K : U \rightarrow \mathbb{R}^m$  linear. Solve  $\min_{u \in U} \lambda \|Ku - d\| + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$ . The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let  $U : \mathbb{R}^n$  and  $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is a discrete gradient. In 1D, we have  $u = (u_i)_i \in \mathbb{R}^m$  and  $u_i = u(x_i) \implies u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$ . Consider  $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$  as problem.

Does there exist a solution to this problem assuming  $\lambda > 0$ ,  $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear and  $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$  linear.

*Proof. Case 1 (trivial model):* Let  $m = n$ .  $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \quad (1)$$

Take  $(u_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^n$  such that  $\lim_{n \rightarrow \infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$ . It holds that  $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$ . Without loss of generality, we can assume that  $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \forall n \in \mathbb{N}$

$$\implies \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$  is bounded. So the Bolzano-Weierstrass theorem applies and  $(u_n)_{n \in \mathbb{N}}$  admits a convergent subsequence  $(u_{n_i})_{i \in \mathbb{N}}$ . Take  $u \in \mathbb{R}^n$ .  $u_{n_i} \rightarrow u$  as  $i \rightarrow \infty$ .

Now: Show that  $u$  solves Problem (1).  $\nabla$  is continuous.  $\|\cdot\|_2$  is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \rightarrow \infty} \|\nabla u_{n_i}\|_2 + \lambda \|u_{n_i} - d\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that  $\hat{u}$  is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of  $\nabla$ ,  $\|\cdot\|_2$

**Case 2 ( $K$  arbitrary):** 1.  $K$  arbitrary does not provide boundedness anymore. Define  $X := \text{kernel}(\nabla) \cap \text{kernel}(K)$  and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^n x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \quad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that  $X^\perp$  is called *orthogonal complement*.

**Claim 0.1.** *If  $\hat{u}$  solves  $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ . Then  $\hat{u}$  solves Problem (1).*

*Proof.* Let  $\hat{u}$  be a solution on  $X^\perp$ . Take  $u \in \mathbb{R}^n$  arbitrary. We write  $u = u_1 + u_2 \in X \times X^\perp$ . Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|Ku - d\|_2 &= \|\nabla(u_1 + u_2)\|_2 + \lambda \|K(u_1 + u_2) - d\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|Ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus  $\hat{u}$  solves our problem (1). □

Take again  $(u_n)_{n \in \mathbb{N}}$  be such that  $u_n \in X^\perp \nabla n$  and

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_2 = \inf_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$$

Write  $u_1 = u_n^1 + u_n^2 \in \ker(\nabla) + \ker(\nabla)^\perp$ .  $\nabla : \ker(\nabla)^\perp \rightarrow \text{image}(\nabla)$  is bijective. Since  $\nabla v = 0$  for  $v \in \ker(\nabla)^\perp \implies v \in \ker(\nabla) \implies \|v\| = (v, v) = 0$ . Thus,  $\nabla^{-1} : \text{image}(\nabla) \rightarrow \ker(\nabla)^\perp$  exists and is continuous.

$$\begin{aligned} \implies \|u_n^2\|_2 &= \|\nabla^{-1} \nabla u_n^2\|_2 = \|\nabla^{-1}\| \cdot \|\nabla u_n^2\|_2 \leq \|\nabla^{-1}\| \\ &\leq \|\nabla^{-1}\| (\|\nabla u_n^2\|_2 + \lambda \|Ku_n - d\|_2) \\ &= \|\nabla^{-1}\| \left( \underbrace{\|\nabla u_n\|_2}_{= \|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\ &< C \text{ for some } C > 0 \end{aligned}$$

Then  $\|u_n^2\|_2$  bounded.

2. Show  $(u_n^1)_n$  is bounded.  $K : X^\perp \cap \ker(\nabla) \rightarrow \text{image}(K)$  is bijective. Since  $Kv = 0$  for  $v \in X^\perp \cap \ker(\nabla) \implies v \in \ker(K)$ . Hence  $v \in \ker(K) \cap \ker(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$ . Hence  $K^{-1} : \text{image}(K) \rightarrow X^\perp \cap \ker(\nabla)$  exists and is continuous.

$$\begin{aligned} \implies \|u_n^1\|_2 &= \|K^{-1} Ku_n^1\|_2 \leq \|K^{-1}\| \|Ku_n^1\|_2 \\ &= \frac{\|K\|}{\lambda} (\lambda \|K(u_1^n + u_2^n) - Ku_n^1\|_2 + \|\nabla u_n\|_2) \\ &\leq \frac{\|K\|}{\lambda} \left( \underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \|d - Ku_1^2\|_2}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\ &< D \text{ for some } D > 0 \end{aligned}$$

$$\implies (u_n^1)_n \text{ bounded} \implies (u_n) = (u_n^1 + u_n^2)_n \text{ is bounded}$$

$\implies (u_n)_n$  admits a subsequence converging to some  $\hat{u}$ . As in Case 1,  $\hat{u}$  is a solution to Problem (1).

In summary,

1.  $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 dx}$  with  $U = C^1([0,1]^2)$  relevant for application.
2. Discrete version:  $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$ . We have shown existence by using:
  - (a) complementary subspaces  $X^\perp$
  - (b) boundedness and compactness
  - (c) continuity

- (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ This lecture took place on 2019/03/07.

Define  $U = C^1([0, 1]^2)$ . Let  $Y$  is some Banach space and  $K : U \rightarrow Y$  is linear and continuous.

Consider the problem  $(P_\infty)$  given by  $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$  where  $d \in Y$  and  $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2}$ .

**Proposition 0.2.** *There exists a solution of  $(P_\infty)$ .*

*Proof.* Take  $(u_n)_{n \in \mathbb{N}}$  as a sequence in  $U$  such that  $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_n \rightarrow \inf_{u \in U} (\dots)$ . Now we want to show that  $(u_n)_{n \in \mathbb{N}}$  is bounded.

**Case 1:** Assume that  $Ku = u$ ,  $Y = U$  and  $\|\cdot\|_Y = \|\cdot\|_2$ .

$$\Rightarrow \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\Rightarrow (u_n)_{n \in \mathbb{N}} \text{ is bounded}$$

So does  $(u_n)_{n \in \mathbb{N}}$  admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change  $U$  to  $U = \left\{ u : [0, 1]^2 \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]^2} |\nabla u|^2} < \infty \right\}$ . Define, instead of  $\|\nabla u\|_2$ ,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider  $\min_{u \in U} R(u) + \lambda \|Ku - d\|_2$  instead.

In this setting,  $(u_n)_{n \in \mathbb{N}}$  admits a weakly convergent subsequence converging to some  $\hat{u} \in U$  (denoted by  $(u_{n_i})_{i \in \mathbb{N}}$ ).

Our next step is to use continuity to show that  $\hat{u}$  is a solution.

Problem:  $u \mapsto \|u - d\|_2$  is, in general, not continuous with respect to weak convergence.

But it is always true that  $\|\hat{u} - d\|_2 \leq \liminf_{i \rightarrow \infty} \|u_{n_i} - d\|_2$ . Yes. We consider that as first property.

Is it also true that  $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$ ? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = - \int_0^1 u \partial_x \varphi \quad \forall \varphi \in C^\infty([0, 1]^2)$$

$\varphi = 0$  in  $K \setminus [0, 1]^2$  for some  $K \in (0, 1)^2$ .

$$\begin{aligned} \implies \int_{[0,1]^2} \nabla u \varphi &= - \int_{[0,1]^2} u \cdot (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \\ \forall \varphi : (\varphi_1, \varphi_2) &= C^\infty([0, 1]^2, \mathbb{R}^2) + \text{zero on boundary} \end{aligned}$$

We define  $w : [0, 1]^2 \rightarrow \mathbb{R}^2$  is called *weak derivative* of  $u \in U$ .

$$\iff \int_{[0,1]^2} w \varphi = - \int_{[0,1]^2} u (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \text{ holds } \forall \varphi$$

Then  $w$  is called *weak gradient* of  $u$ . We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then  $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$ . We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i \rightarrow \infty} R(u_{n_i}) + \liminf_{i \rightarrow \infty} \lambda \|u_{n_i} - d\|_2 \\ &\leq \liminf_{i \rightarrow \infty} (R(u_{n_i}) + \lambda \|u_{n_i} - d\|_2) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

**Case 2:** Works as in the finite-dimensional setting using

- $X := \text{kernel}(A) \cap \text{kernel}(\nabla) \implies U = X \oplus X^\perp$  requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \forall u \in \text{kernel}(\nabla)^\perp$  is called *Poincare inequality*.

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normed spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Complementary subspaces
8. Hilbert spaces

↓ This lecture took place on 2019/03/12.

**Remark.** 1. Literature: UGU, in particular: Biezius, Werner  
2. In this lecture: always  $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$  if not further specified

## 1 Topological and metric spaces

**Remark** (Motivation). Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies

**Definition 1.1** (Topology). Let  $X$  be a set and  $\tau \subset \mathcal{P}(X) = \{\text{"set of subsets of } X\}$ . We say that  $\tau$  is a topology on  $X$  if

1.  $X, \emptyset \in \tau$
2.  $U, V \in \tau \implies U \cap V \in \tau$
3. For any collection of sets  $(U_i)_{i \in I}$  with  $I$  as some index set. We have  $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$ .

$(X, \tau)$  is called topological space.

A set  $U \subset X$  is called open if  $U \in \tau$  and is called closed if  $U^c \in \tau$ .

**Remark.** By the third property of topologies,  $\bigcap_{i \in I} V_i$  is closed for any collection  $(V_i)_{i \in I}$  of closed sets.

**Definition 1.2** (Metric). Let  $X$  be a set,  $d : X \times X \rightarrow \mathbb{R}$  be such that  $\forall x, y, z \in X$

1.  $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z)$

Then  $d$  is called a metric on  $X$  and  $(X, d)$  is called metric space.

**Definition 1.3** (Norm). Let  $X$  be a vector space. A function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is called norm if  $\forall x, y \in X, \lambda \in \mathbb{K}$

1.  $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2.  $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3.  $\|x + y\| \leq \|x\| + \|y\|$

Then  $(X, \|\cdot\|)$  is called normed space.

**Remark.** If  $\dim(x) < \infty$ , all norms on  $X$  are equivalent.

**Example.** 1. Let  $X$  be a set then  $\tau = \{\emptyset, X\}$  is a topology.

2.  $(X, \mathcal{P}(X))$  is a topological space.

3. Define  $S^{d-1} := \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\}$  and  $d(x, y) := r$  where  $r$  is the length of the shortest connection between  $x$  and  $y$  on  $S^{d-1}$ . Then  $d$  is a metric on  $S^{d-1}$

4.  $X := \{u : [0, 1] \rightarrow \mathbb{R} \mid u \text{ is continuous}\}$  then  $\|u\|_\infty := \sup_{x \in [0, 1]} |u(x)|$  is a norm on  $X$

5.  $l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u \text{ and } \sum_{i=1}^\infty |x_i|^p < \infty\}$  with  $p \in [1, \infty)$  and  $\|(x_i)_{i \in \mathbb{N}}\|_p := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$ . Then  $(l^p, \|\cdot\|_p)$  is a normed space (the proof will be done later).

**Remark.**

$$L^\infty := \left\{ (X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|(X_i)_{i \in \mathbb{N}}\| = \sup_i |X_i|$$

**Proposition 1.4.** Let  $X$  be a set.

1. If  $(X, d)$  is a metric space, define for  $\varepsilon > 0, x \in X$ .  $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$  and  $\tau = \{U \in \mathcal{P}(X) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\}$ . Then  $(X, \tau)$  is a topological space. We say that  $\tau$  is the topology induced by  $d$  and we have that  $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$
2. If  $(X, \|\cdot\|)$  is a normed space, define  $d : X \times X \rightarrow \mathbb{R}$  with  $(x, y) \mapsto \|x - y\|$ . Then  $(X, d)$  is a metric space and  $d$  is called the metric induced by  $\|\cdot\|$ .

**Remark (Consequence).** Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.

**Definition 1.5.** Let  $(X, \tau)$  be a topological space.  $U \subset X$ .  $x \in X$ .

1.  $U$  is called a neighborhood of  $x$  if  $\exists V \in \tau - x \in V \subset U$  :  $\mathcal{U}(x)$  is defined as the set of all neighborhoods of  $x$
2.
  - $x$  is called interior point of  $U$  if  $U \in \mathcal{U}(x)$
  - $x$  is called adjacent point of  $U$  if  $\forall V \in \tau$  such that  $x \in V : V \cap U \neq \emptyset$
  - $x$  is called cluster point of  $U$  if it is an adjacent point of  $U \setminus \{x\}$ .

The third property is stronger.

3. Notational conventions:

$$\overset{\circ}{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \overset{\circ}{U}$$

**Proposition 1.6.** Let  $(X, \tau)$  be a topological space,  $U \in X$ . Then



1.  $U$  is open  $\iff \mathring{U} = U$
2.  $U$  is closed  $\iff \overline{U} = U$
3.  $\mathring{U} = \bigcup_{\substack{V \in \tau \\ V \subset U}} V$  and  $\mathring{U}$  is open [" $\mathring{U}$  is the largest open set in  $U$ "]
4.  $\overline{U} = \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$  and  $\overline{U}$  is closed [" $\overline{U}$  is the smallest closed set containing  $U$ "]

*Proof.* 3.  $\subset$  Let  $x \in \mathring{U} \implies \exists \hat{V} \in \tau$  s.t.  $x \in \hat{V} \subset U \implies x \in \bigcup_{V \subset U} V$

$\supset$  Let  $x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V \implies x \in \hat{V}$  for some  $\hat{V} \in \tau, \hat{V} \subset U \implies x \in \mathring{U}$

$\mathring{U}$  is open because it is the union of open sets.

1.  $\implies \mathring{U} \subset U$  by definition.  $U$  is open, so  $U \subset \bigcup_{V \subset U} V \stackrel{(3)}{=} \mathring{U}$
2.  $\implies V \subset \overline{U}$  by definition. Take  $x_0 \in \overline{U}$ . If  $x \notin U \implies x \in U^C \in \tau$  and  $U \cap U^C = \emptyset$ . This contradicts to  $x \in \overline{U}$ .  
 $\Leftarrow$  Take  $x \in U^C = \overline{U}^C$ .  
 $\stackrel{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$   
 $\implies V \cap U = \emptyset \implies V \subset U^C$   
 $\implies U^C$  open  $\implies U$  closed
4. We prove the fourth property without the second.  
 $\subset$  Take  $x \in \overline{U}$ . Take closed  $V$  such that  $U \subset V$  if  $x \notin V \implies x \in V^C$  which is open and  $V^C \cap U = \emptyset$ . This contradicts to  $x \in \overline{U}$ .  
 $\supset$  Take  $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$ . Suppose  $x \notin \overline{U}$ .  
 $\implies \exists Z$  open such that  $x \in Z$  and  $Z \cap U = \emptyset$   
 $\implies U \subset Z^C, Z^C$  closed,  $x \notin Z^C$ . This contradicts to  $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$   
 $\overline{U}$  closed follows since the intersection of closed sets is closed.

□

**Definition 1.7** (Limit). Let  $(X, \tau)$  be a topological space,  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Henceforth, we write  $(x_n)_n$  for  $(x_n)_{n \in \mathbb{N}}$  and  $\hat{x} \in X$ . We say  $x_n \rightarrow \hat{x}$  in  $\tau$  as  $n \rightarrow \infty$  (" $x_n$  converges to  $x$ ", " $x$  is limit of  $x_n$ ") if

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

**Definition 1.8** (Proposition and definition). Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is  $T_2$  (or Hausdorff) if

$$\forall x, y \in X \text{ with } x \neq y \exists U, V \in \tau : x \in U, y \in V \text{ and } U \cap V = \emptyset$$

- In a  $T_2$ -sphere, the limit of any sequence is unique.
- If  $\tau$  is induced by a metric, then  $(X, \tau)$  is  $T_2$ .

*Proof.* 1. Take  $(x_n)_n$  to be a sequence and assume  $x_n$  converges to  $\hat{x}$  and  $\hat{y}$  with  $\hat{x} \neq \hat{y}$ . By  $T_2$ ,  $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$ . By convergenc,  $\exists n_x, n_y$  such that  $\forall n \geq n_x : x_n \in U$  and  $\forall n \geq n_y : x_n \in V$ .

$$\forall n \geq \max\{n_x, n_y\} : x_n \in U \cap V$$

This gives a contradiction.

2. Take  $x, y \in X : x \neq y$ . Define  $\varepsilon := d(x, y)$  and consider  $B_{\frac{\varepsilon}{2}}(x)$  and  $B_{\frac{\varepsilon}{2}}(y)$  which are open in the induced topology  $\tau$ . Also  $x \in B_{\frac{\varepsilon}{2}}(x)$  and  $y \in B_{\frac{\varepsilon}{2}}(y)$ . Assume that  $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$ .

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

**Definition 1.9.** Let  $(X, \tau)$  be a topological space,  $U \subset V \subset X$ . We say that  $U$  is dense in  $V$ , if  $V \subset \overline{U}$ . We say that  $X$  is separable if there exists a countable, dense subset.

**Definition 1.10.** Let  $(X, \tau_X), (Y, \tau_Y)$  be topological spaces and  $f : X \rightarrow Y$  a function. We say  $f$  is continuous at  $x \in X$  if  $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$ .  $f$  is called continuous if it is continuous at any  $x \in X$ .

**Proposition 1.11.** With  $(X, \tau_X), (Y, \tau_Y)$  and  $f$  as above,  $f$  is continuous  $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$

*Proof.* Left as an exercise to the reader.

□

**Definition 1.12.** Let  $(X, \tau)$  be a  $T_2$  topological space,  $M \subset X$  called compact if for any family  $(U_i)_{i \in I}$  with  $U_i \in \tau$  s.t.  $M \subset \bigcup_{i \in I} U_i$  (“ $(U_i)_{i \in I}$  is an open covering of  $M$ ”), there exists  $U_{i_1}, \dots, U_{i_n}$  such that  $M \subset \bigcup_{k=1}^n U_{i_k}$  (“there exists a finite subcover”).

**Remark.** Compactness can also be defined without  $T_2$ , this is also referred to as quasi-compact.

**Remark** (Exercise). Reconsider the previous results for metric and normed spaces.

↓ This lecture took place on 2019/03/14.

**Definition 1.13.** Let  $(X, d)$  be a metric space,  $V \subset X$  and  $(x_n)_n$  a sequence in  $X$ . Then we say,

1.  $V$  is bounded if  $\exists x \in X, r > 0$  such that  $U \in B_r(x)$
2.  $(x_n)_n$  is a Cauchy sequence if  $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$  such that  $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$

3.  $X$  is complete if any Cauchy sequence in  $X$  admits a limit point
4.  $X$  is a Banach space if it is a normed space and complete

**Proposition 1.14.** Let  $(X, d)$  be a metric space.  $(x_n)_n$  be a sequence in  $X$ . Then

1.  $x_n \rightarrow x$  in the induced topology  $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$
2. If  $x_n \rightarrow x$ , then  $(x_n)_n$  is bounded as subset of  $X$  and  $(x_n)_n$  is Cauchy.
3. If  $U \subset X$  is closed and  $X$  is complete. Then  $(U, d)$  is a complete metric space.

*Proof.* 1. We prove both directions:

$\implies$  True, since  $B_\varepsilon(x)$  is open  $\forall \varepsilon > 0$

$\impliedby$  Let  $x \in V$  with  $V$  open. Show that  $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$   
 $V$  open, then  $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$

$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get  $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$ . Let  $r := \max_{i=1, \dots, n_0} d(x, x_i) + 1$ . Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take  $(y_n)_n$  to be a Cauchy sequence in  $U$ , then  $(y_n)_n$  is a Cauchy sequence in  $X \implies \exists x \in X : y_n \rightarrow x$  as  $n \rightarrow \infty$  if  $x \notin U \implies x \in U^c \implies \exists n_0 \in \mathbb{N}$  such that  $y_{n_0} \in U^c$  due to  $U^c$  open. This is a contradiction to  $(y_n)_n$  in  $U$

□

**Proposition 1.15.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.  $f : X \rightarrow Y$ . The following are equivalent (TFAE):

- $f$  is continuous (with respect to the induced topology)
- $\forall (x_n)_n$  such that  $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$

*Proof.* Firstly, we prove that the first statement implies the second statement.

Take  $(x_n)_n$  converging to  $x$ . Take  $V \in \tau_Y$  such that  $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$$\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X \text{ such that } x \in \hat{U} \subset U$$

$$\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_n) \rightarrow f(x)$$

**Remark.** 1.  $\implies$  2. holds true in any topological space

2.  $\implies$  1. Not.

Secondly, we prove that the second statement implies the first statement.

Suppose  $f$  is not continuous, find  $x_n \rightarrow x$  such that  $f(x_n) \rightarrow f(x)$  is wrong. If  $f$  is not continuous, then  $\exists x \in X : \exists V \in \mathcal{U}(f(x))$  such that  $f(u) \notin V \forall U \in \mathcal{U}(x)$

$$\implies \exists \hat{V} \in \tau_Y \text{ such that } f(u) \notin \hat{V} \forall U \in \mathcal{U}(x), f(x) \in \hat{V}$$

$$\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\implies (x_n)_n$  converges to  $x$  but  $f(x_n) \notin \hat{V} \implies f(x_n) \not\rightarrow f(x)$ . This gives a contradiction.  $\square$

**Definition 1.16.** Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f : X \rightarrow Y$ .  $f$  is uniformly continuous iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

**Proposition 1.17.** Let  $(X, d_X), (Y, d_Y)$  be metric spaces.  $M \subset X$ ,  $f : M \rightarrow Y$ . If  $M$  is dense in  $X$ ,  $Y$  is complete and  $f$  is uniformly continuous.

$$\implies \exists ! \hat{f} : X \rightarrow Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

*Proof.* Take  $x \in X$ . By the practicals (and since  $\overline{M} = X$ ),  $\exists (x_n)_n$  such that  $x_n \rightarrow x$  and  $x_n \in M$ .

We show:  $(f(x_n))_n$  is Cauchy. Take  $\varepsilon > 0 \implies \exists \delta > 0$  such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

Now,  $(x_n)_n$  is Cauchy (why?)  $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(x_n), f(x_m)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe:  $\forall \hat{x} \in X$ , there exists  $(\hat{x}_n)_n$  in  $M$ ,  $\hat{y} \in Y$  such that  $f(\hat{x}_n) \rightarrow \hat{y}$ .

Now: for any  $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$  with  $x \in X$ ,  $(x_n)_n$  is a sequence in  $M$  such that  $x_n \rightarrow x, f(x_n) \rightarrow y$ . Now if  $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$ :

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d_Y(f(x_n), y) < 3\varepsilon$$

1. If  $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$  is well-defined.

2.  $\hat{f}$  is uniformly continuous.

$\square$

↓ This lecture took place on 2019/03/19.

**Proposition 1.18.** Let  $(X, d)$  be a metric space,  $M \subset X$ .

1.  $M$  is compact, so  $\forall (X_i)_{i \in I}$  with  $X_i$  a closed set  $\forall i$  such that  $(\bigcap_{i \in I} X_i) \cap M = \emptyset$ .

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \left( \bigcap_{i=1}^n X_{i_j} \right) \cap M = \emptyset$$

2.  $M$  is compact, so  $M$  is closed and bounded.

*Proof.* 1. We note that  $\forall (X_i)_{i \in I}$  is a family of closed sets.  $(X_i^C)_{i \in I}$  is a family of open sets and  $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

**Proposition 1.19.** Let  $(X, d)$  be a metric space,  $M \subset X$ . TFAE:

1.  $M$  is compact.
2. Every infinite subset of  $M$  admits a cluster point.
3. Every sequence of  $M$  admits a convergent subsequence.
4.  $M$  is complete and totally bounded, where totally bounded is defined as

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$$

**Remark.** 1. totally bounded  $\implies$  bounded (proof is left as an exercise)

2. Assume  $\dim(x) < \infty$ . Compact  $\iff$  complete and bounded (see course Analysis I)

3.  $\dim(x) < \infty \iff \overline{B_1(0)}$  is compact

where the last two items imply that  $X$  is a normed space.

*Proof.* 1  $\rightarrow$  2 If  $M$  is finite, (2) always holds true. So assume that  $M$  is infinite.

Now assume that (2) does not hold. Then there is  $C \subset M$  infinite which does not admit a cluster point.  $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$  contains at most one element of  $C]$ . If not,  $\exists x \in C$  such that  $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$  such that  $(x_n)_n$  is a sequence of distinct points and  $x_n \rightarrow x$ . This implies that  $x$  is a cluster point of  $C$ . This gives a contradiction.

Now  $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$ . If  $M$  is compact, then

$$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \subset M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$\implies C$  is finite

This is a contradiction.

2  $\rightarrow$  3 Let  $(x_n)_n$  be a sequence in  $M$ .

**Case 1:**  $\{x_n \mid n \in \mathbb{N}\}$  is finite  $\implies (x_n)_n$  admits a convergent sequence.

**Case 2:**  $\{x_n \mid n \in \mathbb{N}\}$  is infinite. By the second property, there is a cluster point of  $\{x_n \mid n \in \mathbb{N}\}$ . Thus  $(x_n)_n$  is a convergent subsequence to some  $x \in M$ .

3  $\rightarrow$  4 Suppose that  $M$  is not totally bounded.  $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$ . Construct a sequence  $(x_n)_n$  in  $M$  as follows: Given  $x_1, \dots, x_n$ , choose  $x_{n+1} \in M \setminus \bigcup_{j=1}^n B_\varepsilon(x_j)$  arbitrary. Then  $(x_i)_i$  is a sequence in  $M$  and  $d(x_i, x_j) > \frac{\varepsilon}{2}$  for  $i \neq j$ . Hence,  $(x_i)_i$  cannot admit a convergent subsequence.  $G \implies M$  totally bounded.

Completeness can be shown the following way: Let  $(x_n)_n$  be Cauchy in  $M$ , then there exists a subsequence  $(x_{n_i})_i$  and  $x \in M$  such that  $x_{n_i} \rightarrow x$  as  $i \rightarrow \infty$ . Since  $(x_n)_n$  is Cauchy,  $x_n \rightarrow x$  as  $n \rightarrow \infty$  [left as an exercise]. Thus  $M$  is complete.

4  $\rightarrow$  1 Let  $(U_i)_{i \in I}$  be an open covering of  $M$  and assume that  $(U_i)_{i \in I}$  does *not* admit a finite subsequence. For  $n \in \mathbb{N}$  let  $E_n \subset M$  be a finite set such that  $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}}(a)$ . Define  $\Omega := \{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } U_i\}$ . We recursively define a sequence  $(a_n)_n$  in  $M$  such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n-1}) \neq \emptyset$$

**Goal:** Show  $(a_n)_n \rightarrow a$  and then  $B_{\frac{1}{2^n}}(a_{n_0}) \subset U_{i_0}$ .

**Step 1**  $(a_n)_n$  is well defined.

$n = 1$  Since  $M \in \Omega$  and  $M \subset \bigcup_{a \in E_1} B_{\frac{1}{2}}(a)$ , we can pick  $a_1 \in E_1$  such that  $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$ .

$n \rightarrow n+1$  Let  $a_n \in E_n$  such that  $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$  be given. Let

$$\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}.$$

Since  $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$ . [Take  $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$ , but if  $B_{\frac{1}{2^{n+1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$$

Hence there exists  $a_{n+1}$  such that  $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$  and  $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \neq \emptyset$ . Thus  $(a_n)_n$  is well-defined.

**Step 2** Show that  $(a_n)_n$  converges. Take  $n \in \mathbb{N}$  and  $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$ .

$$\implies d(a_n, a_{n+1}) < d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus,  $(a_n)_n$  is Cauchy.  $M$  is complete, so  $\exists a \in M : a_n \xrightarrow{n \rightarrow \infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} \text{ and } \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for  $n$  sufficiently large such that  $d(a, a_n) < \frac{r}{2}$  and  $\frac{1}{2^n} < \frac{r}{2}$ . We take  $x \in B_{\frac{1}{2^n}}(a_n)$  and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to  $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ .

□

**Proposition 1.20.** Let  $(X, d_X), (Y, d_Y)$  be metric spaces.  $M \subset X$  compact. Let  $f : X \rightarrow Y$  be continuous. Then

1.  $f(M)$  is compact
2.  $f|_M : M \rightarrow Y$  is uniformly continuous.

*Proof.* 1. Let  $(U_i)_{i \in I}$  be an open covering of  $f(M)$

$$\implies (f^{-1}(U_i))_{i \in I} \text{ is an open covering of } M \text{ [why!]}$$

$$\implies \exists c_1, \dots, c_n \text{ such that } M \subset \bigcup_{j=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{j=1}^n U_{i_j}$$

2. If  $f|_M$  is not uniformly continuous, then  $\exists \varepsilon > 0 \forall n \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$  and  $d(f(x), f(y)) > \varepsilon$  (\*). Now take  $(x_n)_n, (y_n)_n$  sequences in  $M$  satisfying condition (\*).  $M$  is compact, so  $\exists (x_{n_i})_i$  subsequence converging to some  $x \in M$ .

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \rightarrow \infty} 0$$

□

↓ This lecture took place on 2019/03/21.

**Proposition 1.21** (Proposition and definition). Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.  $g : X \rightarrow Y$  is a function.  $g$  is called Lipschitz continuous if  $\exists L > 0$  such that  $d_Y(g(x), g(y)) \leq L d_X(x, y) \forall x, y \in X$ . Any Lipschitz continuous function is uniformly continuous.

*Proof.* Left as an exercise to the reader.

□

**Theorem 1.22** (Arzelà-Ascoli theorem). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and assume that  $X$  is compact. Define  $C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$  and  $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$ . Then*

1.  $d_C$  is well-defined and  $(C(X, Y), d_C)$  is a complete metric space
2. A set  $M \subset C(X, Y)$  is compact iff
  - (a)  $\forall x \in X$  the set  $M_x := \{f(x) \mid f \in M\}$  is compact
  - (b)  $M$  is equicontinuous, i.e.  $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

*Proof.* 1. Show that:  $d_C(f, g) < \infty$ .

Pick  $f, g \in C(X, Y)$ . Because  $X$  is compact,  $f(X), g(X)$  compact  $\implies f(X), g(X)$  bounded. Thus,  $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$ . Now for  $x \in X$ ,

$$\begin{aligned} d(f(X), g(X)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that  $d_C$  is a metric is left as an exercise.

Show that  $(C(X, Y), d_C)$  is a complete metric space.

Take  $(f_n)_n$  be Cauchy in  $C(X, Y) \implies (f_n(x))_n$  is Cauchy in  $Y \forall x \in X$ . Because  $Y$  is complete,  $(f_n(x))_n$  is convergent and we can define  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ . Convergence of  $(f_n)_n$  with respect to  $d_C$ : Take  $\varepsilon > 0$ , show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy,  $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$ . Consider  $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x)) \leq \lim_{m \rightarrow \infty} d(f_m, f_n) < \varepsilon$  (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if  $f \in C(X, Y) \implies f_n \rightarrow f$  with respect to  $d_C$ . Show that  $f \in C(X, Y)$ . Take  $\varepsilon > 0$ . Let  $n_0$  such that  $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$ . Take  $\delta > 0$  such that  $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$ . Then  $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show:  $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x))$ .

In general, we have  $\forall x, y, z \in (Z, d_Z)$  with  $d_Z$  as a metric.

$$|d(x, z) - d(y, z)| \leq d(x, y)$$



*Proof.*

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y) \quad (2)$$

$$d(y, z) \leq d(y, x) + d(x, z) \implies d(y, z) - d(x, z) \leq d(x, y) \quad (3)$$

$$(2) \text{ and } (3) \implies |d(x, z) - d(y, z)| \leq d(x, y) \quad (4)$$

□

Consequently,  $\forall z \in Z, x_n \rightarrow x$  in  $Z$ :  $d(x_n, z) \rightarrow d(x, z)$  since  $|d(x_n, z) - d(x, z)| \leq d(x_n, x) \rightarrow 0$ .

2. We need to prove both directions.

$$\begin{aligned} \implies & \text{ (a) For } x \in X \text{ fixed, define } g_X : M \rightarrow Y \text{ with } f \mapsto f(x). \text{ Then} \\ & d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2) \\ \implies & g_X \text{ is Lipschitz continuous, in particular continuous} \\ \implies & M_X = g_X(M) \text{ compact} \\ & \text{(b) Take } \varepsilon > 0. \text{ } M \text{ is totally bounded, so } \exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i). \forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies \\ & d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}. \text{ Define } \delta := \min_i \delta_i > 0. \text{ Then } \forall x, y \in X : \\ & d(x, y) < \delta \text{ and } \forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0}) \\ \implies & d(f(x), f(y)) \leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon \end{aligned}$$

$\Leftarrow$  We prove the other direction.

↓ *This lecture took place on 2019/03/26.*

$B$  is complete since it is a closed subset of a Banach space.

Show:  $M$  is totally bounded.

Consider  $\varepsilon > 0$ . Show:  $\exists f_1, \dots, f_n$  such that  $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$ .

- Because  $M$  is equicontinuous,  $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$ .
- By compactness of  $X$ ,  $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$  compact  $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{j=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{ij})$

Compare with Figure 1.

Now, for each tuple of indices  $(y_{1,j_1}, \dots, y_{n,j_n})$  define  $f_{y_{1,j_1}, \dots, y_{n,j_n}} \in C(x, y)$  to be such that  $f_{y_{1,j_1}, \dots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$  if such an  $f$  exists. The set  $F$  of all such functions is finite. We show that  $M \subset \bigcup_{q \in F} B_\varepsilon(q)$ .

Take  $f \in M$  arbitrary. Now choose  $\alpha = (y_{1,j_1}, \dots, y_{n,j_n})$  such that  $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$  and pick  $f_\alpha \in F$  accordingly.

Take  $x \in X$  arbitrary and  $x_i$  such that  $x \in B_\delta(x_i)$

$$\begin{aligned} \implies d(f(x), f_\alpha(x)) & \leq d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ & < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_\alpha) & = \sup_{x \in X} d(f(x), f_\alpha(x)) < \varepsilon \end{aligned}$$

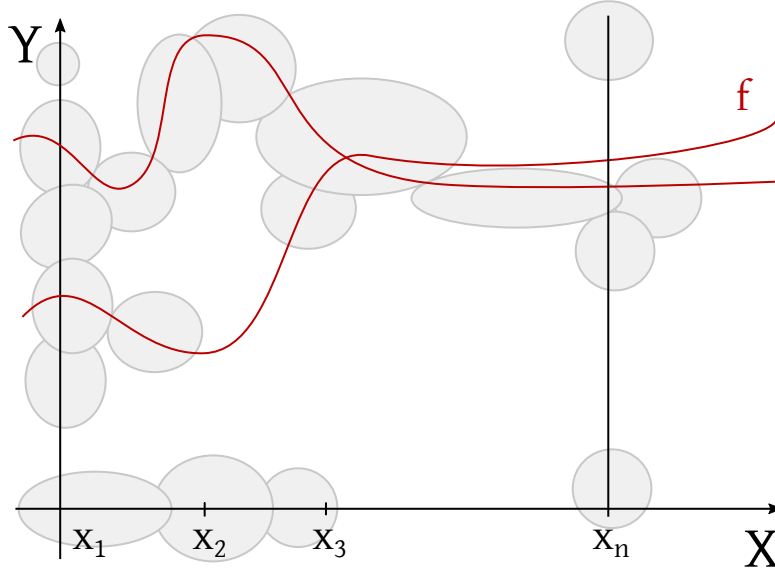


Figure 1: Covering of a function graph

□

**Remark.** Compare this to the fact that  $B_1(0)$  in  $C(X, Y)$  is not compact.

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

**Remark (Motivation).** In general, let  $(X, d)$  be a metric space. Let  $A$  and  $B$  be open and dense, then also  $A \cap B$  is dense.

*Proof.* Show  $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] \neq \emptyset$ . Take  $x \in X, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$ .  $A$  is dense.  $A$  is open, so  $\exists \varepsilon_1 > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$ .  $B$  is dense, so  $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$ .

$$\implies \exists z \in B_{\varepsilon_1}(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally,  $\forall A_1, \dots, A_n$  open, dense  $\implies \bigcap_{i=1}^n A_i$  is dense (this is left as an exercise). Does this also hold true for countably many  $A_i$ ?

**Theorem 1.23** (Baire theorem). Let  $(X, d)$  be a complete metric space. Let  $(O_n)_{n \in \mathbb{N}}$  be a sequence of dense sets. Then  $\bigcap O_n$  is dense.

*Proof.* Let  $D := \bigcap_{n \in \mathbb{N}} O_n$ . Show that for  $x \in X, \varepsilon > 0$  arbitrary we have  $B_\varepsilon(x) \cap D \neq \emptyset$ . We define iteratively a sequence  $(x_n)_{n \in \mathbb{N}}$ .

**n = 1** Take  $x_1, \varepsilon_1$  such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

**n - 1 → n** Given  $x_{n-1}, \varepsilon_{n-1}$ , take  $x_n, \varepsilon_n$  such that

$$\overline{B_{\varepsilon_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences  $(x_n)_n, (\varepsilon_n)_n$  such that  $\varepsilon_n < \frac{\varepsilon}{2^n}$  and  $x_n \in B_{\varepsilon_n}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete} \implies \exists x \in X : x_n \rightarrow x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_n}(x_N)} \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_N)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

**Definition 1.24.** Let  $(X, d)$  be a metric space,  $M \subset X$ . We say

- $M$  is nowhere dense (dt. “nirgends dicht”), if  $\overset{\circ}{M} = \emptyset$
- $M$  is of first category  $\iff M$  is a countable union of nowhere dense sets
- $M$  is of second category  $\iff M$  is not of first category

**Theorem 1.25** (Baire category theorem (weaker version)). Let  $(X, d)$  be a complete metric space. Then  $(X, d)$  is of second category.

In other words (which is a useful formulation): If  $X = \bigcup_{n \in \mathbb{N}} C_n \implies \exists n_0 : \overset{\circ}{C} \neq \emptyset$ . In particular, if

$$X = \bigcup_{n \in \mathbb{N}} C_n \text{ with } C_n \text{ closed} \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

*Proof.* Suppose that  $X = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O_n}$  with  $\overset{\circ}{O_n} = \emptyset \forall n$

$$\overset{\circ}{O_n} = \emptyset \implies \overline{\overset{\circ}{O_n}} = X$$

Why does this implication hold? Because consider  $x \in X, \varepsilon > 0$ .

$$B_\varepsilon(x) \cap \overset{\circ}{O_n} = \emptyset \implies B_\varepsilon(x) \subset \overline{O_n} \implies \overset{\circ}{O_n} \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O_n}^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O_n}^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n \in \mathbb{N}} \overline{O_n}^C \text{ is dense}$$

$$\bigcap_{n \in \mathbb{N}} \overline{O_n}^C = \left( \bigcup_{n \in \mathbb{N}} \overline{O_n} \right)^C = X^C = \emptyset$$

gives a contradiction

□

**Remark.** 1. This is a fundamental theorem in Functional Analysis

2. This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise, e.g. Weierstrass function)

## 2 Normed space

### 2.1 Fundamentals

**Definition 2.1.** Let  $X$  be a vector space. A function  $\|\cdot\| : X \rightarrow [0, \infty)$  is called seminorm if

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$

The first property differs between a norm and a seminorm.

The tuple  $(X, \|\cdot\|)$  is called a semi-normed space. We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.

**Example** (Not done in lecture). We found the following examples while studying:

$$f \text{ linear}, x \mapsto |f(x)| \quad \text{and} \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := |y|$$

**Definition 2.2** (Definition and proposition). Let  $(X, \|\cdot\|)$  be a semi-normed space and  $(x_n)_n$  be a sequence in  $X$ . We say that

- $\sum_{n=1}^{\infty} x_n$  converges to  $x \in X$  and write  $x = \sum_{n=1}^{\infty} x_n$  if  $\lim_{m \rightarrow \infty} \sum_{n=1}^m x_n = x$
- $\sum_{n=1}^{\infty} x_n$  is absolutely convergent if  $\sum_{n=1}^{\infty} \|x_n\|$  converges  $[\iff (\sum_{n=1}^m \|x_n\|)_m \text{ is bounded}]$

It holds that  $X$  is complete iff any absolutely converging series converges.

*Proof.*  $\implies$  Take  $m_1 < m_2$  arbitrary, then

$$\begin{aligned} \left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| &\leq \sum_{n=m_1+1}^{m_2} \|x_n\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_1} \|x_n\| \leq \left\| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \right\| \\ &\implies \left( \sum_{n=1}^m x_n \right)_m \text{ is Cauchy} \implies \text{convergent} \end{aligned}$$

$\Leftarrow$  Let  $(x_n)_n$  be Cauchy. Show that  $(x_n)_n$  converges. For  $\varepsilon_k = 2^{-k}$ , pick  $N_k$  such that  $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$$

Define  $y_k := x_{n_{k+1}} - x_{n_k} \implies \sum_k \|y_{n_w}\| \leq \sum_k 2^{-k} < \infty$

$$\implies \exists y \in X : \sum_{k=1}^n y_k \rightarrow y \text{ as } n \rightarrow \infty$$

$$\sum_{k=1}^n y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \rightarrow y - x_{n_1} \text{ as } n \rightarrow \infty$$

So  $(x_n)_n$  has a convergent subsequence and  $(x_n)_n$  is Cauchy, then  $(x_n)_n$  is convergent.

□

**Remark.** In  $\mathbb{R}^n$ ,  $\sum_n x_n$  is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction  $\implies$  is true.

↓ This lecture took place on 2019/03/28.

**Proposition 2.3** (Proposition and definition). Let  $X$  be a vector space and  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two norms on  $X$ . We say  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if

$$\exists m, M > 0 \forall x \in X : m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

*TFAE:*

1.  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.
2. For any sequence  $(x_n)_n$  and  $x \in X$ ,  $x_n \rightarrow x$  with respect to  $\|\cdot\|_1 \iff x_n \rightarrow x$  with respect to  $\|\cdot\|_2$
3. For any sequence  $(x_n)_n$  we have,

$$x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_2$$

*Proof.* (1)  $\implies$  (2)  $\implies$  (3) is immediate.

It remains to show that:

(3)  $\implies$  (1) Suppose no  $M > 0$  exists such that  $\|x\|_2 \leq M \cdot \|x\|_1 \forall x \in X$ .

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n \|x_n\|_1$$

Let  $y_n := \frac{x_n}{\|x_n\|_1 n}$ . Then  $\|y_n\|_1 = \frac{1}{n} \rightarrow 0$  hence  $y_n \rightarrow 0$ , but  $\|y_n\|_2 > n \|y_n\|_1 = 1$ .

$$\implies y_n \not\rightarrow 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

**Remark.** If  $\dim(X) < \infty$ , then any two norms on  $X$  are equivalent.

**Definition 2.4** (Quotient spaces). Let  $(X, \|\cdot\|)$  be a normed space and  $Y \subset X$  a subspace. Define a relation “ $\sim$ ” on  $X$  with  $x \sim y : \iff x - y \in Y$ .

Then  $\sim$  defines an equivalence relation on  $X$ . We define

- $[X]_{\sim} = \{y \in X \mid x \sim y\}$  is the equivalence class of  $x \in X$
- $X/Y := \{[x]_{\sim} \mid x \in X\}$  is the quotient space
- $\pi : \begin{cases} X \rightarrow X/Y \\ x \mapsto [x]_{\sim} \end{cases}$

Defining  $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \quad \hat{0} := [0]$$

We get that:

1.  $X/Y$  is a vector space
2.  $\|[x]\|_{X/Y} := \inf_{y \in [x]} \|y\|_X$  is a semi-norm.
3. If  $Y$  is closed, then  $\|\cdot\|_{X/Y}$  is a norm.
4. If  $X$  is complete and  $Y$  closed, then  $(X/Y, \|\cdot\|_{X/Y})$  is a Banach space.

*Proof.* • Equivalence relation

- Vector space with “+” and “ $\lambda[x]$ ” are well-defined

This is left as an exercise to the reader.

2. – First of all,  $\|\cdot\|_{X/Y} \geq 0$  is trivial.

$$\|[0]\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|y\| \leq \|0\| = 0$$

- Secondly, consider  $\lambda \in \mathbb{K}$ ,  $[x] \in X/Y$ .  
Show that:  $\|\lambda[x]\|_{X/Y} = |\lambda| \|[x]\|_{X/Y}$ .  
Trivial, if  $\lambda = 0$ . Assume  $\lambda \neq 0$ ,

$$\|\lambda[x]\|_{X/Y} = \|[\lambda x]\|_{X/Y} = \inf_{y \in [\lambda x]} \|y\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \|y\| = \inf_{w \in [x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\|[x]\|_{X/Y}}$$

- Take  $[x_1], [x_2] \in X/Y$ ,  $\varepsilon > 0$ . We note that

$$\|[x]\|_{X/Y} = \inf_{\substack{y \in X \\ w \in Y \\ w := x \cdot y}} \|y\| = \inf_{w \in Y} \|x - w\|$$

Hence we can take  $y_1, y_2 \in Y$  such that  $\|x_1 - y_1\| < \|[x_1]\|_{X/Y} + \varepsilon$   
 $\varepsilon \in [1, 2)$ .

$$\begin{aligned} \Rightarrow \|[x_1] + [x_2]\|_{X/Y} &= \|[x_1 + x_2]\|_{X/Y} \leq \|x_1 + x_2 - (y_1 + y_2)\| \\ &\leq \|x_1 - y_1\| + \|x_2 - y_2\| \leq \|[x_1]\|_{X/Y} + \|[x_2]\|_{X/Y} + 2\varepsilon \end{aligned}$$

Since  $\varepsilon$  was arbitrary, the assertion follows.

3. Suppose  $Y$  is closed if  $\|[x]\|_{X/Y} = 0$ , then

$$\inf_{y \in Y} \|x - y\| = 0 \implies \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n \rightarrow \infty} \|x - y_n\| = 0$$

$$Y \text{ closed} \implies x \in Y \implies [x] = [0] = \hat{0}$$

4. Take  $([x_n])_n$  to be a sequence in  $X/Y$  and suppose that  $\sum_{i=1}^{\infty} \|[x_n]\|_{X/Y} < \infty$ . If we can show that  $\exists [x] \in X/Y$  such that  $\sum_{i=1}^{\infty} [x_n] = [x]$ , then by Proposition 2.2,  $X/Y$  is complete.

Choose  $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$  such that  $\|\tilde{x}_n\| \leq \|[x_n]\|_{X/Y} + 2^{-n}$

$$\implies \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} (\|[x_n]\|_{X/Y} + 2^{-n}) < c < \infty$$

$$X \text{ complete} \implies \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \quad \left\| [x] - \underbrace{\sum_{n=1}^m [x_n]}_{[x_n]} \right\|_{X/Y} \leq \left\| x - \underbrace{\sum_{k=0}^n \tilde{x}_k}_{\rightarrow 0} \right\|$$

□

↓ This lecture took place on 2019/04/02.

**Corollary 2.5.** Let  $X$  be a vector space with semi-norm  $\|\cdot\|_X : X \rightarrow [0, \infty)$ . Then

- $N = \{x \in X \mid \|x\|_X = 0\}$  is a subspace of  $X$
- $\|[X]\| := \|X\|_p$  is a norm on  $X/N$
- If  $X$  is complete, then  $(X/N, \|\cdot\|)$  is a Banach space.

*Proof.* The proof is left as an exercise. □

**Proposition 2.6.** Let  $(X, \|\cdot\|)$  be a normed space,  $U \subset X$  is a subspace. Then

- $\overline{U}$  is also a subspace.
- $X$  is separable iff  $\exists A \subset X$  complete such that  $X = \overline{\mathcal{L}(A)}$  where  $\mathcal{L}(A) = \{\sum_{i=1}^n \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathbb{K}, n \in \mathbb{N}\}$

*Proof.* • Left as an exercise

- $\implies$  True since  $\exists A \subset X$  countable such that  $\overline{A} = X \implies \underline{X} = \overline{A} \subset \overline{\mathcal{L}(A)} \subset X$

$\Leftarrow$  Let  $A \subset X$  countable such that  $\overline{\mathcal{L}(A)} = X$ . Define

$$B = \left\{ \sum_{i=1}^n (\lambda_i + i\mu_i)x_i \mid \lambda_i, \mu_i \in \mathbb{X}, x \in A, n \in \mathbb{N} \right\}$$

where  $i$  is the imaginary unit if  $\mathbb{K} = \mathbb{C}$  or  $i = 0$  if  $\mathbb{K} = \mathbb{R}$ . Then  $B$  is countable.

Show:  $\forall x \in X \forall \varepsilon \exists x \in B : \|x - y\| < \varepsilon$ .

Take  $x \in X, \varepsilon > 0 \implies \exists x_0 \in \mathcal{L}(A) : \|x - x_0\| < \frac{\varepsilon}{2}$  when  $x_0 = \sum_{i=1}^n (\lambda_i + i\mu_i)x_i$  with  $\lambda_i, \mu_i \in \mathbb{R}, x_i \in A$ . Choose  $\lambda'_i, \mu'_i \in \mathbb{Q}$  such that

$$\sqrt{(\lambda_i - \lambda'_i)^2 + (\mu_i - \mu'_i)^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^n \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let  $y := \sum_{i=1}^n (\lambda'_i + i\mu'_i)x_i \in B$ .

$$\begin{aligned} \implies \|x - y\| &\leq \|x - x_0\| + \|x_0 - y\| && \leq \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^n |(\lambda_i + i\varepsilon_i) - (\lambda'_i + i\mu'_i)| \|x_i\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^n \|x_i\|} = \varepsilon \end{aligned}$$

□

**Proposition 2.7** (Proposition and definition). *Let  $(X, \|\cdot\|_{x_i})$  for  $i = 1, \dots, n$  be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \dots \otimes X_n = \bigotimes_{i=1}^n X_i = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

For  $p \in [1, \infty]$ , define

$$\|(x_1, \dots, x_n)\|_{\otimes_i X_i, p} = \begin{cases} \left( \sum_{i=1}^n \|x_i\|_{x_i}^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1, \dots, n} \|x_i\|_{x_i} & \text{if } p = \infty \end{cases}$$

Then

- $(\bigotimes_i X_i, \|\cdot\|_{\otimes_i X_i, p})$  is a normed space with respect to componentwise addition and multiplication.
- If all  $X_i$  are complete, then  $\bigotimes_{i=1}^n X_i$  is complete.
- All norms  $\|\cdot\|_{\otimes_i X_i, p}$  are equivalent.

*Proof.* • Vector space properties: Left as an exercise

- Norm:  $\|x\|_{\otimes_i X_i, n} = 0 \iff x = 0$   
 $\|\lambda x\|_{\otimes_i X_i, p} = |\lambda| \|x\|_{\otimes_i X_i, p}$



- Triangle inequality:  $p = 1, p = \infty$   
 $p \in (1, \infty)$ . Take  $x, y \in \bigotimes_{i=1}^n X_i$  and we write  $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i, p}$ .

$$\begin{aligned}
\Rightarrow \|x + y\|_p^p &= \sum_{i=1}^n \|x_i + y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\
&\leq \sum_{i=1}^n \|x_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} + \sum_{i=1}^n \|y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\
&\leq \underbrace{\left( \sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}}}_{\text{Hölder ineq.}} \cdot \left( \sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\
&\quad + \left( \sum_{i=1}^n \|y_i\|_{X_i}^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \|x\|_p \|x + y\|_p^{p-1} + \|y\|_p \|x + y\|_p^{p-1} \\
&= (\|x\|_p + \|y\|_p) \cdot \|x + y\|_p^{p-1}
\end{aligned}$$

$$\Rightarrow \|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ if } x + y \neq 0 \text{ (trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in  $\mathbb{R}^n$ )

□

**Definition 2.8.** Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be normed spaces. If  $j : X \rightarrow Y$  is linear such that  $\|j(x)\|_Y = \|x\|_X$  (hence  $j$  is injective) then  $j$  is called *isometric embedding* from  $X$  to  $Y$ . If  $j$  is bijective, then  $j$  is called *isometric isomorphism* and we say  $X = Y$  up to isomorphism.

**Proposition 2.9.** Let  $(X, \|\cdot\|_X)$  be a normed space. Then  $\exists(\hat{X}, \|\cdot\|_{\hat{X}})$  a Banach space such that

1.  $\exists$  isometric embedding,  $i : X \rightarrow \hat{X}$  such that  $\overline{j(X)} = \hat{X}$  [ $\hat{X}$  can be regarded as completion of  $X$ ]
2. If  $j_1 : X \rightarrow Y$  is an isometric embedding on  $Y$ , a Banach space

$$\Rightarrow \exists i_2 : \hat{X} \rightarrow Y$$

an isometric embedding such that  $j_2 \circ i = j_1$  and if  $\overline{j_1(X)} = Y$  then  $j_2$  is an isometric isomorphism. Thus “the completion is essentially unique”.

*Proof.* 1. Set  $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is Cauchy}\}$ .  $\hat{X}$  is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \quad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define  $\|(x_n)_n\|_{\hat{X}} := \lim_{n \rightarrow \infty} \|x_n\|$  [well-defined since  $(\|x_n\|)_n$  is Cauchy in  $\mathbb{R}$ ]. Then  $\|\cdot\|_{\hat{X}}$  is a semi-norm (proof is left as an exercise). Setting  $N = \{(X_n)_n \mid \|(X_n)_n\|_{\hat{X}} = 0\}$ . By Corollary 2.5,  $\hat{X} := \hat{X} \setminus N$  with  $\|[(X_n)_n]\|_{\hat{X}} = \|(X_n)_n\|_{\hat{X}}$  is a normed space. Define

$$j : X \rightarrow \hat{X} \quad x \mapsto [(x)_n]$$

then  $j$  is linear and  $\|j(x)\|_{\hat{X}} = \|[(x)_n]\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x\| = \|x\|$ . So  $j$  is an isometric embedding.

Show:  $\overline{j(X)} = \hat{X}$ .

Take  $\hat{x} = [(X_n)_n] \in \hat{X}$ . Define  $y_n := j(x_n) \in \hat{X}$ .

$$\begin{aligned} \Rightarrow \|y_m - [(x_n)_n]\|_{\hat{X}} &= \|(x_m)_n - (x_n)_n\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x_m - x_n\| \\ &= \lim_{n \geq n_0} \|x_m - x_n\| < \varepsilon \end{aligned}$$

Now,  $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$ .

Show:  $\hat{X}$  is complete.

Let  $(y_n)_n$  be Cauchy in  $\hat{X}$ . Pick  $X_n \in X$  such that  $\|j(x_n) - y_n\|_{\hat{X}} \leq \frac{1}{n}$   
( $j(x) = \hat{x}$ )

$$\Rightarrow \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}} \leq \|j(x_n) - y_n\|_{\hat{X}} + \|y_n - y_m\|_{\hat{X}} + \|y_m - j(x_m)\|_{\hat{X}}$$

Take  $\varepsilon > 0$ . Then  $\exists n_0 \forall n, m \geq n_0 : \|y_n - y_m\|_{\hat{X}} < \frac{\varepsilon}{3}$ . Pick  $n_1$  such that  $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$ .

$$\Rightarrow \forall n, m > \max(n_0, n_0) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\Rightarrow (x_n)_n$  is Cauchy. Let  $y := (X_n)_n \in \tilde{X}$ . Then

$$\|y_n - [y]\|_{\hat{X}} \leq \|y_n - j(x_n)\|_{\hat{X}} + \|j(x_n) - [y]\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n \rightarrow \infty} \|x_n - x_m\|_X \xrightarrow{n \rightarrow \infty} 0$$

2.  $\downarrow$  This lecture took place on 2019/04/04.

Let  $\hat{x} \in \hat{X} \Rightarrow \exists (x_n)_n \in X$  such that  $j(x_n) \rightarrow \hat{x} \Rightarrow \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}}$ .

$\Rightarrow (x_n)_n$  is a Cauchy sequence.

$\Rightarrow j_1(x_n)$  is a Cauchy sequence in  $Y$ .

$\Rightarrow \exists \lim_{n \rightarrow \infty} j_1(x_n) := y$

Using this, we define  $j_2 : \hat{X} \rightarrow Y$  with  $\hat{x} \mapsto \lim_{n \rightarrow \infty} j_1(x_1)$  where  $j(x_1) \rightarrow \hat{x}$ .

Well-defined? Take  $\hat{x} \in \hat{X}$  and  $j(x_n) \rightarrow \hat{x}$ ,  $j(y_n) \rightarrow \hat{x}$ .

$$\Rightarrow \|j_1(x_n) - j_1(y_n)\| = \|x_n - y_n\| = \|j(x_n) - j(y_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\implies \lim_{n \rightarrow \infty} j_1(x_n) = \lim_{n \rightarrow \infty} j_1(y_n) \implies j_1 \text{ well-defined}$$

Show linearity is left as an exercise. By isometry, take  $\hat{x} \in \hat{X}$ ,

$$\underbrace{|i_2(\hat{x})|}_{j(x_n) \rightarrow \hat{x}} = \lim_{n \rightarrow \infty} \|j_1(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|i(x_n)\| = \|\hat{x}\|$$

Show:  $j_2 \circ j = j_1$ . Take  $x \in X \implies (x_n)$  is such that  $j(x) \rightarrow j(x) \implies j_2(j(x)) = \lim_{n \rightarrow \infty} j_1(x) = j_1(x)$ .

Assume that  $\overline{j_1(X)} = Y$ . Take  $y \in Y$ . Find  $\hat{x} \in \hat{X}$  such that  $i_2(\hat{x}) = y$ . By  $\overline{j_1(X)} = Y \implies \exists (x_n)_n$  in  $X$  such that  $i_1(x_n) \rightarrow Y \implies (j_1(x_n))_n$  is Cauchy.

$$\implies (x_n)_n \text{ Cauchy} \implies (j(x_n))_n \text{ Cauchy}$$

$$\xRightarrow{\hat{X} \text{ complete}} \exists \hat{x} \text{ such that } \lim_{n \rightarrow \infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n \rightarrow \infty} j_2(x_n) = Y$$

□

## 2.2 Important examples of normed spaces

**Definition 2.10** (Basic notation). Let  $\Omega \subset \mathbb{R}^N$ ,  $f : \Omega \rightarrow \mathbb{K}^M$  with  $N, M \in \mathbb{N}$ .

- We call  $\Omega$  a domain (dt. "Gebiet") if  $\Omega$  is open and connected, where connected means that  $\forall x, y \in \Omega$  there is a curve in  $\Omega$  connecting  $X$  and  $Y$ .
- For  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$  define  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$ . If  $f$  is  $r$ -times continuously differentiable, we set for  $\alpha \in \mathbb{N}_0^N$ ,  $\{\alpha\} \leq r$ .

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f$$

where  $\frac{\partial^{\alpha_1}}{\partial x_i^{\alpha_i}}$  is the partial derivative of  $f$  with respect to  $x_i$  of order  $\alpha_i$ .

**Example 2.11.** Let  $N = 2$  and  $\alpha = (1, 1)$ .

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

Let  $\alpha = (2, 0)$ .

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- For  $z \in \mathbb{K}^N$  we denote  $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$ .<sup>1</sup>
- We say  $E \subset \Omega$  is compact in  $\Omega$  and we write  $E \Subset \Omega$  if  $E$  is compact.

**Remark.** If  $E \Subset \Omega$ , then  $\exists \delta > 0 : \inf \{\|x - y\| \mid x \in E, y \in \partial\Omega\} > 0$ .

<sup>1</sup>This is an abuse of notation with  $|\alpha|$  for  $\alpha \in \mathbb{N}_0^N$

*Proof.* Left as an exercise (use compactness) □

- $f$  is compactly supported in  $\Omega$  if  $\text{supp}(f) \ll \Omega$ .
- $\text{supp}(f) := \overline{\{x \in \Omega \mid \|f(x)\| > 0\}}$

↓ This lecture took place on 2019/04/09.

**Definition 2.12** (Definition and proposition, Spaces of continuous functions).  
Let  $\Omega \subset \mathbb{R}^N$  be a domain. We define

$$\begin{aligned} C_b(\Omega, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ bounded}\} \text{ with } \|\varphi\|_{C_b} = \|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)| \\ C(\overline{\Omega}, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ can be continuously extended to } \overline{\Omega}\}, \|\varphi\|_C := \|\varphi\|_\infty \\ C^r(\overline{\Omega}, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid D^\alpha \varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \leq r\} \text{ and } \|\varphi\|_{C^r} = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq r}} \|D^\alpha \varphi\|_\infty \\ C_C^r(\Omega, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \text{supp}(\varphi) \ll \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M)\} \text{ and } \|\varphi\|_{C_C^r} = \|\varphi\|_{C^r} \\ C^\infty(\overline{\Omega}, \mathbb{K}^M) &= \bigcap_{r \in \mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M) \\ D(\Omega, \mathbb{K}^M) &= C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r \in \mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \text{ in } C^r(\overline{\Omega}, \mathbb{K}^M) \end{aligned}$$

Then for any bounded  $\Omega$ ,  $C^r, C_0^r, C_b$  are Banach spaces and  $C_C^r$  is a normed space.

Recall:  $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

*Proof.* The functions  $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$  are norms (proof is left as an exercise).

Show that  $C_b$  is complete: Take  $(\varphi_n)_n$  in  $C_b$  to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because  $|\varphi_n(x) - \varphi_m(x)| \leq \|\varphi_n - \varphi_m\|_\infty$ . Hence we can define  $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$ .

Show:  $\varphi_n \rightarrow \varphi$  in  $\|\cdot\|_\infty$ . Take  $\varepsilon > 0$ . Show that  $\exists n_0 \forall n \geq n_0 : \|\varphi - \varphi_n\|_\infty < \varepsilon$ .

Take  $n_0$  such that  $\forall n, m \geq n_0 : \|\varphi_n - \varphi_m\|_\infty < \varepsilon$ . Take  $m \geq n_0$ .

$$\implies \forall x \in \Omega : |\varphi(x) - \varphi_m(x)| = \lim_{\substack{n \rightarrow \infty \\ n \geq n_0}} |\varphi_n(x) - \varphi_m(x)| < \|\varphi_n - \varphi_m\|_\infty$$

Show:  $\varphi$  is bounded, i.e.  $\exists C > 0 : |\varphi(x)| \leq C < \|\varphi_n - \varphi_m\|_\varepsilon < \infty$ . Take  $n$  such that  $\|\varphi - \varphi_n\|_\infty < 1$

$$\implies \forall x \in \Omega : |\varphi(x)| > |\varphi(x) - \varphi_n(x)| + \underbrace{|\varphi_n(x)|}_{=C} \leq 1 + \|\varphi_n\|$$

Now  $C^r(\overline{\Omega}, \mathbb{K}^n)$  is a subspace of  $C^b(\Omega, \mathbb{K}^n)$ . Also  $C^r(\overline{\Omega}, \mathbb{K}^n)$  is closed, since the uniform limit of  $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$  with respect to  $\|\cdot\|_{C^r}$  is again in  $C^r(\overline{\Omega}, \mathbb{K}^M)$  [a result from Analysis].

$\implies C^r(\overline{\Omega}, \mathbb{K}^M)$  is a Banach space

$C_C^r(\overline{\Omega}, \mathbb{K}^M)$  is closed by definition, hence Banach.

$C_C^r(\Omega, \mathbb{K}^M)$  is a vector space, since  $\forall \lambda \in \mathbb{K} : \varphi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\lambda\varphi) = \text{supp}(\varphi)$  and for  $\varphi, \Psi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\varphi + \Psi) \ll \Omega$ .  $\square$

**Definition 2.13** (Definition and proposition). *Let  $(\Omega, \Sigma, \mu)$  with  $\Omega \subset \mathbb{R}^N$  be a measure space (i.e.  $\Sigma$  is a sigma algebra and  $\mu$  is a measure). For  $p \in [1, \infty)$ , we define*

$$\begin{aligned} \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) &= \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid f \mu - \text{measurable and } \int_{\Omega} |f(x)|^p d\mu(x) < \infty \right\} \\ \|f\|_p^* &= \left( \int_{\Omega} \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}} \\ \mathcal{L}^{\infty}(\Omega, \mathbb{K}^M, \mu) &:= \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} |f(x)| < \infty \right\} \\ \|f\|_{\infty}^* &= \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)| \end{aligned}$$

Our proposition is that these are semi-norms.

*Proof.* Show that  $\|\cdot\|_p^*$  for  $p \in [1, \infty]$  are seminorms.

They cannot be norms since  $\|f\|_p^* = 0$  for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0$ .  $\square$

**Proposition 2.14** (Hölder inequality). *Let  $p \in [1, \infty]$  and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \text{if } p \in (1, \infty) \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

*If  $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$  and  $g \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$  then for both*

$$\begin{aligned} f \cdot g : \Omega &\rightarrow \mathbb{K} \text{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)} \\ f \otimes g : \Omega &\rightarrow \mathbb{K}^M \text{ with } x \mapsto (f_i(x), g_i(x))_{i=1}^M \end{aligned}$$

we have that  $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$  and  $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$  and  $\|f \otimes g\|_1^* \leq \|fg\|_1^* \leq \|f\|_p^* \cdot \|g\|_q^*$ .

*Proof.* **Case  $p \in (1, \infty)$ :** Intermediate result:  $\forall \sigma, \tau \geq 0, r \in (0, 1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$  [AGM-inequality].

*Proof.*

Case  $\sigma = 0$  or  $\tau = 0$ : immediate

Case  $\sigma, \tau > 0$ :

$$\begin{aligned} \log(\sigma^r \tau^{1-r}) &= r \log(\sigma) + (1-r) \log(\tau) \leq \log(r\sigma + (1-r)\tau) \\ &\text{since } \log''(x) \leq 0 \text{ implies that } \log \text{ is concave} \\ \log \text{ is monotonic} &\implies \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau \end{aligned}$$

□

Let  $A := (\|f\|_p^*)^p$  and  $B := (\|g\|_q^*)^q$  with  $r = \frac{1}{p} \in (0, 1]$  we get

$$\begin{aligned} \forall x \in \Omega : \left( \frac{|f(x)|^p}{A} \right)^{\frac{1}{p}} \left( \frac{|g(x)|^q}{B} \right)^{\frac{1}{q}} &= \frac{1}{p} \frac{|f(x)|^p}{A} + \frac{1}{q} \frac{|g(x)|^q}{B} \\ \implies \frac{\int_{\Omega} |f(x)| |g(x)| d\mu(x)}{A^{\frac{1}{p}} B^{\frac{1}{q}}} &\leq \frac{1}{p} \frac{\int_{\Omega} |f(x)|^p d\mu(x)}{A} + \frac{1}{q} \frac{\int_{\Omega} |g(x)|^q d\mu(x)}{B} \\ \implies \int_{\Omega} |f(x)| |g(x)| d\mu(x) &\leq \|f\|_p^* \|g\|_q^* = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Now:  $\|f \cdot g\|_x^* \leq \|f\|_p^* \cdot \|g\|_q^*$  follows since  $|\langle x, y \rangle| \leq |x| |y| \forall x, y \in \mathbb{K}^M$ .

Also:

$$\begin{aligned} \forall x \in \Omega : |f \otimes g(x)| &= \sum_{i=1}^M |f_i(x)| |g_i(x)| = \begin{pmatrix} |f_1(x)| & |g_1(x)| \\ \vdots & \vdots \\ |f_n(x)| & |g_n(x)| \end{pmatrix} \leq |f(x)| |g(x)| \\ \implies \int_{\Omega} |f \otimes g(x)| d\mu(x) &\leq \|f\|_p^* \cdot \|g\|_q^* \end{aligned}$$

**Case  $p \in \{1, \infty\}$ :** Without loss of generality assume that  $p = 1, q = \infty$ .  $\forall N \in \Sigma$  with  $\mu(N) = 0$  we get

$$\begin{aligned} \int_{\Omega} |f(x)| |g(x)| d\mu(x) &= \int_{\Omega \setminus N} |f(x)| |g(x)| d\mu(x) \\ &\leq \int_{\Omega \setminus N} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| = \int_{\Omega} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| \end{aligned}$$

Taking the infimum over all such  $N$ , then

$$\int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_1^* \cdot \|g\|_{\infty}^*$$

And the result follows again from  $|\langle x, y \rangle| \leq |x| \cdot |y|$  and componentwise  $|\langle x_i, y_i \rangle_i| \leq |x| |y| \forall x, y \in \mathbb{K}^M$

□

**Proposition 2.15** (Minkowski inequality). *For  $p \in [1, \infty]$ ,  $f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ , we have that  $\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*$  with  $\|f\|_{\infty} := \inf_{\mu(N) \rightarrow 0} \sup_{x \in \Omega \setminus N} |f(x)|$ .*

*Proof.* **Case  $p = 1$ :** trivial

**Case  $p \in (1, \infty)$ :**

$$\begin{aligned} (\|f + g\|_p^*)^p &= \int_{\Omega} |f(x) + g(x)|^p d\mu(x) \\ &= \int_{\Omega} |f(x)| \cdot |f(x) + g(x)|^{p-1} d\mu(x) \\ &\quad + \int_{\Omega} |g(x)| |f(x) + g(x)|^{p-1} d\mu(x) \\ &\leq \|f\|_p^* \cdot \| |f + g|^{p-1} \|_q^* + \|g\|_p^* \cdot \| |f + g|^{p-1} \|_q^* \end{aligned}$$

Recognize that  $(\int |f + g|^p)^{\frac{1}{p}} = (\int |f + g|^{(p-1)q})^{\frac{1}{q}}$  because  $p = q \cdot (p-1)$

$$\begin{aligned} &= (\|f\|_p^* + \|g\|_p^*) \| |f + g|^{p-1} \|_q^* \\ \implies \|f + g\|_p^* &\leq \|f\|_p^* + \|g\|_p^* \end{aligned}$$

↓ This lecture took place on 2019/04/11.

**Case  $p = \infty$ :** First, note that  $\forall f \in \mathcal{L}^{\infty}(\Omega, \mathbb{K}^M, \mu) \exists N \in \Sigma$  such that  $\mu(N) = 0$  and  $\|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty} := \sup_{x \in \Omega \setminus N} |f(x)|$ .

**Claim 2.16.**

$$\|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty} := \sup_{x \in \Omega \setminus N} |f(x)| = \sup_{x \in \Omega \setminus \hat{N}} |f(x)| \text{ for } \mu(\hat{N}) = 0$$

*Proof.* For all  $n \in \mathbb{N}$ , define  $N_n \in \Sigma$  such that  $\mu(N_n) = 0$  and  $\|f|_{\Omega \setminus N_n}\|_{\infty} \leq \|f\|_{\infty}^* + \frac{1}{n}$ . Thus with  $N := \bigcup_{n \in \mathbb{N}} N_n \implies \mu(N) = 0$  and  $\|f\|_{\infty}^* \leq \|f|_{\Omega \setminus N}\|_{\infty} \leq \|f\|_{\infty}^* + \frac{1}{n}$ .  $n \rightarrow \infty \implies \|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty}$ . □

For  $f, g \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu)$ , pick  $N_f, N_g$  such that  $\mu(N_C) = \mu(N_g) = 0$  and  $\|f\|_\infty^* = \|f|_{\Omega \setminus N_f}\|_\infty$  and  $\|g\|_\infty^* = \|g|_{\Omega \setminus N_g}\|_\infty$ .

$$\begin{aligned} \Rightarrow \|f + g\|_\infty^* &\leq \|(f + g)|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus (N_f \cup N_g)}\|_\infty + \|g|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus N_f}\|_\infty + \|g|_{\Omega \setminus N_g}\|_\infty = \|f\|_\infty^* + \|g\|_\infty^* \end{aligned}$$

□

**Proposition 2.17.** *Let  $p \in [1, \infty]$ . Then  $\|\cdot\|_p^*$  is a seminorm on  $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$  and  $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$  is complete with the seminorm. With  $M := \{f \in \mathcal{L}^\infty \mid \|f\|_p^* = 0\}$ , we get that  $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)/M$  is a Banach space with respect to  $\|f\|_p := \|f\|_p^*$ .*

*Proof.* Seminorm is clear by Minkowski's inequality. Give completeness of  $f^p(\cdot)$ , the rest follows from Corollary 2.5.

Hence, show that  $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$  is complete.

Assume  $p < \infty$ . By Proposition 2.2, it suffices to show that for  $f_n(t_n)_n$  in  $\mathcal{L}^p(\cdot)$  such that  $a := \sum_{n=1}^\infty \|f_n\|_p^* < \infty$ .

$$\Rightarrow \exists f \in \mathcal{L}^p(\cdot) : f = \sum_{n=1}^\infty f_n$$

Define  $\hat{q}(x) := \sum_{n=1}^\infty |f_n(x)| \in [0, \infty]$ . Define  $\hat{q}_n(x) := \sum_{i=1}^n |f_i(x)|$ . Then  $q_n$  is measurable and by Minkowski's inequality,

$$\|q_n\|_p^* \leq \sum_{i=1}^n \|f_i\|_p^* \leq \sum_{i=1}^\infty \|f_i\|_p^* = a < \infty$$

Also  $\hat{q}_n^p : x \rightarrow \hat{q}_n(x)^n$  is a sequence of positive functions and it is monotonically increasing and converging to  $\hat{g}^p$ .

By Beppo-Levi (from measure theory):

$$\int_\Omega \hat{g}^p = \lim_{n \rightarrow \infty} \int_\Omega \hat{q}_n^p = \lim_{n \rightarrow \infty} (\|q_n\|_p^*)^p = a^p < \infty$$

$\Rightarrow \hat{g}^p < \infty$  almost everywhere (except for a  $\mu$  zero-set). Define  $g : \Omega \rightarrow \mathbb{R}$ ,

$$x \mapsto \begin{cases} \hat{g}(x) & \text{if } \hat{g}(x) < \infty \\ 0 & \text{else} \end{cases}$$

We get that  $g \in \mathcal{L}^p(\Omega, \mathbb{R}, \mu)$  and  $g(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n |f_i(x)|$   $\mu$ -almost everywhere. Furthermore, by completeness of  $\mathbb{K}^M$ ,  $f(x) := \sum_{i=1}^\infty f_i(x)$  exists for  $\mu$ -almost everywhere.  $x \in \Omega$ .



Show:  $f = \sum_{i=1}^{\infty} f_i$  in  $\mathcal{L}^n(\cdot)$ , i.e. show that  $\lim_{n \rightarrow \infty} \int_{\Omega} \left| \sum_{i=1}^{\infty} f_i \right|_{d_N}^p = \sigma$ .

$$\left\| \sum_{i=1}^{n-1} f_i - \sum_{i=1}^{\infty} f_i \right\|_p^* = \left\| \sum_{i=n}^{\infty} f_i \right\|_p^* \xrightarrow{!} 0$$

By contruction,  $|f| \leq q$  almost everywhere  $\implies \int_{\Omega} |f|^p \leq \int_{\Omega} q^p < \infty$ . Set  $h_n(x) = \left| \sum_{i=n}^{\infty} f_i(x) \right|^p$ . Then  $h_n(x) \rightarrow 0$  for  $\mu$ -almost everywhere  $x \in \Omega$  and  $h_n(x) \geq 0$  and

$$0 \leq h_n(x) \leq \left( \sum_{i=n}^{\infty} |f_i(x)| \right)^p \leq q(x)^p$$

Hence, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\Omega} h_n(x) = \int_{\Omega} \lim_{n \rightarrow \infty} h_n(x) = 0$$

This completes the assertion since

$$\int_{\Omega} h_n(x) = \int_{\Omega} \left| \sum_{i=n}^{\infty} f_i(x) \right|^p = \int_{\Omega} \left| \sum_{i=1}^{n-1} f_i(x) - f(x) \right|^p = \left( \left\| \sum_{i=1}^{n-1} f_i - f \right\|_p^* \right)^p$$

□

↓ This lecture took place on 2019/04/30.

**Proposition** (Proposition 2.15 again). *Let  $p \in [1, \infty]$ . Then  $\|\cdot\|_{L^p}$  is a seminorm,  $\mathcal{L}^p(\Omega, \mathbb{K}^n, \mu)$  is complete and  $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\cdot)/N$  where  $N = \{f \mid \|f\|_{L^p} = 0\}$  is a Banach space.*

*Proof.* Assume  $p \in [1, \infty]$ , then the proof of the last lecture is given.

Assume  $p = \infty$ . Let  $(f_n)_n$  be Cauchy in  $\mathcal{L}^{\infty}$ . Remember:  $\|f\|_{L^{\infty}} := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |f(x)|$ .

Pick  $N_{n,m}$  such that  $\mu(N_{n,m}) = 0$  and  $|f_n - f_m|_{\infty} = \left\| (f_n - f_m)|_{\Omega \setminus N_{n,m}} \right\|_{\infty}$ . Set  $N = \bigcup_{n,m} N_{n,m} \implies \mu(N) = 0$ .

Then  $\tilde{f}$  is the uniform limit of  $f_n \cdot \mathbf{1}_{\Omega \setminus N}$ . Hence  $\tilde{f}$  is measurable. Also  $\|\tilde{f}\|_{L^{\infty}} := \inf_{\mu(M)=0} \sup_{x \in \Omega \setminus M} |f(x)| \leq \|f\|_{\infty} \implies \tilde{f} \in L^{\infty}(\Omega, \mathbb{K}^n, \mu)$ . Also  $\|f_n - \tilde{f}\|_{L^{\infty}} = \left\| (f_n - f)|_{\Omega \setminus N} \right\|_{\infty} = \left\| f_n|_{\Omega \setminus N} - f \right\|_{\infty} \rightarrow 0$  as  $n \rightarrow \infty$ . □

Now  $(f_n|_N)_n$  is Cauchy with respect to  $\|\cdot\|_{\infty}$ . Since  $\forall n, m$ :

$$\begin{aligned} \|f_n|_{N^c} - f_m|_{N^c}\|_{\infty} &= \|(f_n - f_m)|_{N^c}\|_{\infty} \\ &\leq \|(f_n - f_m)|_{N_{m,n}^c}\|_{\infty} \\ &= \|f_n - f_m\|_{L^{\infty}} \end{aligned}$$

As in the proof of  $C_b$  being a Banach space:

$$\implies \exists f : \Omega \setminus N \rightarrow \mathbb{K}^M : \|f\|_\infty < \infty \text{ and } f_n|_{N^c} \rightarrow f \text{ w.r.t. } \|\cdot\|_\infty$$

**Remark** (Important special cases). **Case 1**  $\mu = \mathcal{L}^N$  is the Lebesgue measure on  $\Omega \subset \mathbb{R}^N$  (a domain). In this case we write  $L^p(\Omega, \mathbb{K}^M) := L^p(\Omega, \mathbb{K}^M, \lambda^M)$  and  $L^p(\Omega) := L^p(\Omega, \mathbb{K})$ . Here the space  $L^p(\Omega, \mathbb{K})$  is considered as functions which are defined almost everywhere.

**Case 2** Set  $\Omega = \mathbb{N}, \sigma = \mathbb{P}(\mathbb{N}), \mu_c(A) = |A|$ .

Then

- $f : \Omega \rightarrow \mathbb{K}^M$  is identified with a sequence  $(x_n)_n$  with  $x_n \in \mathbb{K}^M$ .
- $\int_\Omega f(x) d\mu(x) \sim \sum_{i \in \mathbb{N}} x_i \in \mathbb{K}^M$
- $\mu_c(A) = 0 \iff A = \emptyset$  and the equivalence class construction becomes obsolete.

And we denote,

$$\ell^p(\mathbb{N}, \mathbb{K}^M) = \mathcal{L}^p(\mathbb{N}, \mathbb{K}^M, \mu_c) \quad \ell^p := \ell^p(\mathbb{N}) = \ell^p(\mathbb{N}, \mathbb{K})$$

### 2.2.1 Basic properties of Lebesgue spaces

**Proposition 2.18.** The space  $\ell^p(\mathbb{N}, \mathbb{K}^M)$  is separable for  $p \in [1, \infty]$  and not separable for  $p = \infty$ .

*Proof.*  $p < \infty$  Define  $l_{i,j} \in \ell^p(\mathbb{N}, \mathbb{K}^M)$  as

$$(l_{ij})_k := \begin{cases} 0 & \text{if } i \neq k \\ \left(0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0\right)^T & \text{if } i = k \end{cases}$$

Then  $A := \{e_{ij} \mid i \in \mathbb{N}, j \in \{1, \dots, M\}\}$  is countable.

It suffices to show that  $\overline{\text{span}(A)} = \ell^p(\mathbb{N}, \mathbb{K}^M)$ .

This is true since  $\forall x \in \ell^p(\mathbb{N}, \mathbb{K}^M) : \forall \varepsilon > 0 \exists n_0 : \sum_{i=n_0+1}^\infty |x_i|^p < \varepsilon$  and hence

$$\left\|x - \sum_{i=1}^{n_0} \sum_{j=1}^M x_{ij} e_{ij}\right\|^p = \left(\sum_{i=n_0+1}^\infty |x_i|^p\right)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$$

$p = \infty$  It suffices to show that  $L^\infty(\mathbb{N})$  is not separable (why?). For  $M \subset \mathbb{N}$  define  $\mathbf{1}_M \in L^\infty$ . Then  $\Delta := \{\mathbf{1}_M \mid M \subset \mathbb{N}\}$  is uncountable.

For  $A \subset L^\infty$  countable and  $x \in A$  set  $M_x = \left\{y \in L^\infty \mid \|x - y\|_\infty < \frac{1}{3}\right\} = B_{\frac{1}{3}}(x)$ . Then each  $M_x$  contains at most one element of  $\Delta$  since if  $\mathbf{1}_M \neq \mathbf{1}_{M'}$  are such that  $\mathbf{1}_M, \mathbf{1}_{M'} \in M_x$ .

$$\implies 1 = \|\mathbf{1}_M - \mathbf{1}_{M'}\|_\infty \leq \|\mathbf{1}_M - x\| + \|\mathbf{1}_{M'} - x\| < \frac{2}{3}$$

This gives a contradiction.

$\Delta$  is uncountable,  $\{M_x \mid x \in A\}$  is countable.

$$\implies \exists \hat{M} \in \mathbb{N} : \mathbf{1}_{\hat{M}} \notin M_x \forall x \in A$$

$$\implies \|\mathbf{1}_{\hat{M}} - x\|_\infty \geq \frac{1}{3} \forall x \in A$$

Hence,  $A$  is not dense. Since  $A$  was arbitrary countable. Thus  $L^\infty$  is not separable.

□

### 2.2.2 Separability of $L^p$ requires a density result

**Proposition 2.19.** *Let  $f \in L^p(\mathbb{R}^N, \mathbb{K}^M)$ . Let  $p < \infty$ . Then  $\exists (f_n)_n \in \dots C_c(\mathbb{R}^N, \mathbb{R}^M)$  such that  $\|f_n - f\|_{L^p} \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof. Step 1* Reduction to step functions with  $E \in \Sigma$ .

$$\xi_E(x) := \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$$

Take  $f \in L^p(\dots)$ . For  $\varepsilon > 0$ , define

$$E_\varepsilon = \{x : \varepsilon \leq |f| \leq \frac{1}{\varepsilon}\}$$

Then  $E_\varepsilon \in \Sigma$  and  $\int_{\mathbb{R}^N} |f|^p \geq \varepsilon^p |E_\varepsilon|$  where  $|E_\varepsilon| := L^N(E_\varepsilon)$ .

$$|E_\varepsilon| < \infty \text{ and } \int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f| \leq \frac{1}{\varepsilon} \cdot |E_\varepsilon| < \infty$$

$$\implies \mathbf{1}_{E_\varepsilon} f \text{ is integrable} \implies \exists (q_{n,\varepsilon})_n \text{ step functions}$$

such that  $\int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f - q_{n,\varepsilon}| \rightarrow 0$  as  $n \rightarrow \infty$ . Define

$$f_{n,\varepsilon}(x) := \begin{cases} q_{n,\varepsilon}(x) & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| \leq \frac{2}{\varepsilon} \\ \frac{2}{\varepsilon} \frac{q_{n,\varepsilon}(x)}{|q_{n,\varepsilon}(x)|} & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon} \\ 0 & \text{else} \end{cases}$$

Hence  $(f_{n,\varepsilon})_n$  is a sequence of step functions. For  $x \in E_\varepsilon$  such that  $|q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon}$ .

$$\implies |f_{n,\varepsilon}(x) - f(x)| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} = \frac{3}{\varepsilon} \leq 3 \underbrace{(|q_{n,\varepsilon}(x)| - |f(x)|)}_{\geq \frac{1}{\varepsilon}} \leq 3 |q_{n,\varepsilon}(x) - f(x)|$$

$$\int_{\mathbb{R}^N} |f_{n,\varepsilon}(x) - X_{E_\varepsilon}(x) f(x)| dx \leq 3 \int_{\mathbb{R}^N} |g_{n,\varepsilon}(x) - \mathbf{1}_{E_\varepsilon}(x) f(x)| dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\int_{\mathbb{R}^N} |f - f_{n,\varepsilon}|^p \leq \int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p + \underbrace{\left(\frac{3}{\varepsilon}\right)^{p-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|}_{(*)} =: (X)$$

$$(*) = \int_{E_\varepsilon} |f - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \cdot \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| \left( \underbrace{|f \mathbf{1}_{E_\varepsilon}|}_{\leq \frac{1}{3}} + \underbrace{|f_{n,\varepsilon}|^{p-1}}_{\leq \frac{2}{3}} \right)$$

Now given  $\delta > 0$ , we first fix  $\varepsilon > 0$  such that  $\int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p < \frac{\delta}{2}$ . Then we find  $n_0$  such that  $\left(\frac{3}{\varepsilon}\right)^{n-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| < \frac{\delta}{2}$ . This is possible since  $\mathbb{R}^N = \bigcup_{\varepsilon>0} E_\varepsilon$  and  $\int_{\mathbb{R}^N} |f|^n < \infty$ .

$$\implies (X) < \delta$$

Now suppose  $\forall \varepsilon > 0 \forall E \in \Sigma : \exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K}^M)$  such that  $\|\mathbf{1}_E - \varphi\| < \varepsilon$ . We need to show that this is true. Then for  $f \in L^p(\mathbb{R}^N, \mathbb{K})$ ,  $\varepsilon > 0$ , we pick

$$g = \sum_{i=1}^n \underbrace{c_i}_{\in \mathbb{K}^M} \cdot \underbrace{\mathbf{1}_{E_i}}_{\in \Sigma}$$

such that  $\|f - g\|_p < \frac{\varepsilon}{2}$  (possible by what we just showed). For  $i \in \mathbb{N}$ , pick  $\varphi_i \in C_c(\mathbb{R}^N, \mathbb{R})$  such that  $\|\mathbf{1}_{E_i} - \varphi_i\|_p \leq \frac{2^{-i}\varepsilon}{|c_i|^2}$

$$\begin{aligned} \implies \left\| f - \underbrace{\sum_{i=1}^n c_i \cdot \varphi_i}_{\in C_c(\mathbb{R}^n, \mathbb{K}^n)} \right\|_p &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|c_i \mathbf{1}_{E_i} - c_i \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n |c_i| \cdot \|\mathbf{1}_{E_i} - \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n 2^{-i} \cdot \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

□

↓ This lecture took place on 2019/05/02.

**Proof. Step 1** It is sufficient to approximate  $f = \mathbf{1}_E$  for  $E \in \Sigma$

**Step 2** Reduce statement to  $f = \mathbf{1}_Q$  where  $Q = \times_{i=1}^N [a_i, b_i]$  with  $a_i, b_i \in \mathbb{R}$ . Take  $f = \mathbf{1}_E$ . Since  $\Sigma$  is generated by sets of the form  $\times_{i=1}^N [a_i, b_i] \forall \varepsilon > 0$  there exists  $(Q_i)_{i=1}^n, (\lambda_i)_{i=1}^n$  such that  $\|f - \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}\|_1 < \varepsilon$  [Alt, A1 10, axiom L5].

Define  $h_n(x) = \max(0, \min(1, q_n(x)))$  where  $q_n := \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}$ , also  $h_n$  is of the form of  $q_n$  and

$$\begin{aligned} |f(x) - h_n(x)| \leq 1 &\implies |f(x) - h_n(x)|^p \leq |f(x) - h_n(x)| \leq |f(x) - q_n(x)| \\ &\implies \|f - h_n\|_p \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

As in step 1, this reduces the assertion to  $f = \mathbf{1}_Q$  with  $Q = \times_{i=1}^N [a_i, b_i]$ . For such  $f = \mathbf{1}_Q$ , define

$$g_i(s) := \begin{cases} \frac{b_i - a_i}{2} + \left| s - \frac{b_i + a_i}{2} \right| & \text{if } s \in [a_i, b_i] \\ 0 & \text{else} \end{cases}$$

for  $i \in \{1, \dots, N\}$  and  $\tilde{g}_{i,\varepsilon}(x) = \prod_{i=1}^N g_{i,\varepsilon}(x_i)$ , we obtain that  $\|\mathbf{1}_Q - \hat{g}_\varepsilon\|_p \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbf{1}_Q - \hat{g}_\varepsilon|^p &= \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{i=1}^N |\mathbf{1}_{[a_i, b_i]}(x) - \tilde{g}_{i,\varepsilon}(x)|^p dx \\ &= \prod_{i=1}^N \int_{a_i}^{b_i} |\mathbf{1}_{[a_i, b_i]}(s) - \tilde{g}_{i,\varepsilon}(s)|^p ds \\ &\leq \prod_{i=1}^N |I_{i,\varepsilon}| \text{ where } |I_{i,\varepsilon}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

□

**Remark.** 1. If  $f \in L^p(\Omega, \mathbb{K}^M)$  with  $\Omega \subset \mathbb{R}^N$  a domain, defining

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & \text{else} \end{cases}$$

we get that  $\tilde{f} \in L^p(\mathbb{R}^N, \mathbb{K}^M)$  and using Proposition 2.19 for  $\tilde{f}$  we can approximate  $f$  by functions in  $C(\bar{\Omega}, \mathbb{K}^M) \cap C_c(\mathbb{R}^N, \mathbb{K}^M)$ .

2. Using “Mollification” Proposition 2.19 implies density of  $\mathcal{D}(\Omega, \mathbb{K}^M)$  in  $L^p(\Omega, \mathbb{K}^M)$  for  $\Omega \subseteq \mathbb{R}^N$  a domain.

**Proposition 2.20.** Let  $\Omega \subset \mathbb{R}^N$  measurable. Then  $L^p(\Omega, \mathbb{K}^M)$  is separable for  $1 \leq p < \infty$  and not separable for  $p = \infty$ .

*Proof.* **Case**  $p = \infty$  Similar to  $l^\infty$ , will be done in the Exercises.

**Case**  $1 \leq p < \infty$  We show the result for  $L^p(\mathbb{R}^N, \mathbb{K})$ , the general case is a direct consequence. Denote  $\mathcal{R} := \{Q \subseteq \mathbb{R}^N \mid Q = \prod_{i=1}^N [a_i, b_i] \text{ with } a_n, b_n \in \mathbb{Q}\}$ . Then  $\mathcal{R}$  is countable and it suffices to show that  $E := \mathcal{L}(\{\mathbf{1}_Q \mid Q \in \mathcal{R}\})$  is dense. Take  $f \in L^p(\mathbb{R}^N, \mathbb{K}), \varepsilon > 0$ . Then  $\exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K})$  such that  $\|f - \varphi\|_p \leq \frac{\varepsilon}{2}$ . Now we need to find  $h \in E$  such that  $\|\varphi - h\|_p \leq \frac{\varepsilon}{2}$ . Let

$M \subseteq \mathbb{R}^N$  be closed, bounded hypercube such that  $\text{supp}(\varphi) \subset M$ .  $\varphi$  is uniformly continuous on  $M$ .

$$\implies \forall \delta > 0 \exists \rho > 0 \forall x, y \in M : |x - y| < \delta \implies |\varphi(x) - \varphi(y)| < \delta$$

Now we take  $(Q_i)_{i=1}^K$  a disjoint covering of  $M$  with  $Q_i \in \mathcal{R}$ , such that  $|x - y| < \delta \forall x, y \in Q_i$ . Now define  $\lambda_i = \varphi(z)$  for some  $z \in Q_i$ ,  $i = 1, \dots, K$ . Define  $h(x) := \sum_{i=1}^K \lambda_i \mathbf{1}_{Q_i}$ .

$$\implies \forall x \in \mathbb{R}^M : |\varphi(x) - h(x)| \leq |\varphi(x) - \lambda_i| \leq \delta$$

$$\implies \|\varphi - h\|_p = \left( \int_{\mathbb{R}^N} |\varphi(x) - h(x)|^p \right)^{\frac{1}{p}} \leq \delta \cdot |M|^{\frac{1}{p}}$$

Choose  $\delta := \frac{\varepsilon}{2 \cdot |M|^{\frac{1}{p}}}$ , then the result follows.

□

↓ This lecture took place on 2019/05/09.

**Proposition 2.21.** Let  $p \in [1, \infty]$ ,  $(f_n)_n$ ,  $f \in L^p(\Omega, \mathbb{K}^M)$  with  $\Omega \subset \mathbb{R}^N$  a domain such that  $f_n \rightarrow f$  in  $L^p$ .

Then there exists a subsequence  $(f_{n_k})_k$  such that

1.  $f_{n_k}(x) \rightarrow f(x)$  for almost every  $x \in \Omega$
2.  $\exists h \in L^p(\Omega)$  such that  $(f_{n_k}(x)) \leq |h(x)|$  for almost every  $x \in \Omega$

*Proof.* **Case  $p = \infty$**  Is left as an exercise to the reader.

**Case  $p \in [1, \infty)$**  Pick  $(n_k)_k$  such that  $\|f_{n_{k+1}} - f_{n_k}\|_p \leq \frac{1}{2^k}$ . Define  $g_n := \sum_{k=1}^n |f_{n_{k+1}}(x) - f_{n_k}(x)|$ . Then  $g_n(x)$  is increasing,  $g_n(x) \geq 0 \forall n$ .

$\implies g_n(x)$  is convergent for almost every  $x \in \Omega$ . Hence we can define  $g(x) := \lim_{n \rightarrow \infty} g_n(x) \in [0, \infty]$ .

Also,  $\|g_n\|_p \leq \sum_{i=1}^n \|f_{n_{i+1}} - f_{n_i}\|_p \leq 1$ . By Beppo-Levi,

$$\int_{\Omega} |g(x)|^n dx = \lim_{n \rightarrow \infty} \int_{\Omega} |g_n(x)|^n = \lim_{n \rightarrow \infty} \|g_n\|_p^n \leq 1 \implies g \in L^p(\Omega)$$

especially  $g(x) < \infty$  for almost every  $x \in \Omega$ .

$$\forall l \geq k \geq 1 : |f_{n_l}(x) - f_{n_k}(x)| \leq \sum_{i=k}^{l-1} |f_{n_{i+1}}(x) - f_{n_i}(x)| \leq g_{l-1}(x) - g_{k-1}(x) \stackrel{\text{monot.}}{\leq} g(x) - g_{k-1}(x)$$

$\implies (f_{n_k}(x))_k$  is Cauchy for almost every  $x \in \Omega$  such that we can define  $\hat{f}(x) := \lim_{k \rightarrow \infty} f_{n_k}(x)$ .

$$|\hat{f}(x) - f_{n_k}(x)| \leq g(x) \text{ for almost every } x \in \Omega$$

By the Dominated convergence theorem,  $\|f_{n_k} - \tilde{f}\|_n \rightarrow 0$  for  $k \rightarrow \infty$ .  $\implies f = \tilde{f}$  almost every and hence  $f_{n_k}(x) \rightarrow f(x)$  for almost every  $x \in \Omega \implies (1)$ . Also

$$|f_{n_k}(x)| \leq |f_{n_k}(x) - f(x)| + |f(x)| \leq q(x) + |f(x)| =: h(x)$$

□

### 3 Linear Operators

**Definition 3.1.** Let  $X, Y$  be normed spaces and  $D \subset X$  is a subspace. A linear operator with domain  $\text{dom}(T) = D$  is a linear mapping  $T : D \rightarrow Y$ . We define:  $\text{range}(T) = \text{rg}(T) := T(D)$ . Graph of  $T$ ,  $\text{gr}(T) := \{(x, y) \mid x \in \text{dom}(T), y = Tx\} \subset X \times Y$ .

We say that  $T$  is decently define, if  $\overline{\text{dom}(T)} = X$ .

**Example 3.2.** 1.  $X = Y = C([0, 1], \mathbb{R})$  and  $\text{dom}(T) := C^1([0, 1], \mathbb{R})$   $T : \text{dom}(T) \rightarrow Y$  with  $u \mapsto u'$ .

2.  $X = Y = \mathbb{R}^n, T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  with  $x \mapsto Ax$  with  $A \in \mathbb{R}^{n \times n}$

3. Fixed  $u \in L^p(\Omega)$  and  $p \in [1, \infty)$ .

$$q := \begin{cases} \frac{p}{p-1} & p = 1 \\ \infty & \text{else} \end{cases}$$

$$T : L^q(\Omega) \rightarrow \mathbb{R} \quad v \mapsto \int_{\Omega} u \cdot v$$

4.  $X = L^2(\Omega), Y = \mathbb{R}, \text{dom}(T) = C(\overline{\Omega})$  with  $x \in \Omega$  fixed,  $T : \text{dom}(T) \rightarrow Y$  with  $u \mapsto u(x_0)$

**Definition 3.3.** Let  $X, Y$  be normed spaces and  $T : X \rightarrow Y$  a linear operator ( $\text{dom}(T) = X$ ). We say that  $T$  is bounded  $\iff \exists M > 0 \forall x \in X : \|Tx\|_Y \leq M \|x\|_X$ . In this case, we define  $\|T\| = \|T\|_{\mathcal{L}(X, Y)} := \inf \{M > 0 \mid \|Tx\| \leq M \|x\| \forall x\}$ .

$$\mathcal{L}(X, Y) := \{T : X \rightarrow Y \mid T \text{ bounded, linear operator}\}$$

$$\mathcal{L}(X) := \mathcal{L}(X, X)$$

**Proposition 3.4.** Let  $X, Y$  be normed spaces,  $T : X \rightarrow Y$  be linear. The following are equivalent:

1.  $T$  is continuous
2.  $T$  is continuous at 0
3.  $\exists M > 0$  such that  $\|Tx\| \leq M \|x\| \forall x \in X$  ( $T$  bounded)
4.  $T$  is uniformly continuous

Also:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|T(x)\| \quad \text{and} \quad \|Tx\| \leq \|T\| \|x\| \quad \forall x \in X$$

*Proof.* (3)  $\rightarrow$  (4) Is true since  $\forall x, y \in X : \|Tx - Ty\| = \|T(x - y)\| \leq M \|x - y\|$

(4)  $\rightarrow$  (1)  $\rightarrow$  (2) trivial

(2)  $\rightarrow$  (3) Assume (3) is not true, then

$$\exists (x_n)_n \text{ in } X : \forall n \in \mathbb{N} : \|Tx_n\| > n \|x_n\|$$

Define  $y_n := \frac{x_n}{\|x_n\|n} \Rightarrow \|y_n\| = \frac{1}{n} \Rightarrow y_n \rightarrow 0$  but  $\|Ty_n\| = \frac{\|Tx_n\|}{\|x_n\|n} > 1$ .  
This gives a contradiction to continuity at 0 since  $T0 = 0$ .

□

Additionally,

$$M := \sup_{x \neq 0} \frac{\|Tx\| \|x\|}{\|x\|^2} = \sup_{x \neq 0} \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|\leq 1} \|Tx\|$$

But also,

$$\sup_{\|x\|\leq 1} \|Tx\| = \sup_{\lambda \in [0,1]} \sup_{\|x\|=1} \|T(\lambda x)\| = \sup_{\lambda \in [0,1]} \lambda \left( \sup_{\|x\|=1} \|Tx\| \right) = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \left\| \frac{Tx}{\|x\|} \right\|$$

We also get that

$$\begin{aligned} M_0 &\geq \frac{\|Tx\|}{\|x\|} \quad \forall x \in X, x \neq 0 \\ \Rightarrow \|Tx\| &\leq M_0 \|x\| \quad \forall x \in X : x \neq 0 \quad \text{and also for } x = 0 \Rightarrow \|T\| \leq M_0 \\ M_0(1 - \varepsilon) &\leq \frac{\|Tx_\varepsilon\|}{\|x_\varepsilon\|} \end{aligned}$$

For  $\varepsilon > 0$  pick  $x_\varepsilon \neq 0$  such that

$$\begin{aligned} \|Tx_\varepsilon\| &\geq M_0(1 - \varepsilon) \|x_\varepsilon\| \\ \Rightarrow \|T\| &\geq M_0(1 - \varepsilon) \end{aligned}$$

since  $\varepsilon > 0$  was arbitrary  $\Rightarrow \|T\| \geq M_0$ .

↓ This lecture took place on 2019/05/10.

**Proposition 3.5.** Let  $X$  and  $Y$  be normed spaces. Then

1.  $\mathcal{L}(X, Y)$  is a vectorspace with

$$(T + S)(x) := T(x) + S(x) \quad (\lambda T)(x) := \lambda T(x) \quad 0(x) := 0$$

2.  $T \mapsto \|T\|$  is a norm on  $\mathcal{L}(X, Y)$  (the operator norm)



3. If  $Y$  is complete, then  $\mathcal{L}(X, Y)$  is complete. In particular,  $\mathcal{L}(X, \mathbb{K})$  is complete for any  $X$  and is also called the space of bounded linear functionals

*Proof.* 1. Left as an exercise to the reader

2. (N1)  $\|0\| = \sup_{\|x\| \leq 1} \|0(x)\| = 0$ .  
Also  $\|T\| = 0 \implies \|Tx\| \leq 0\|x\| = 0 \forall x \implies T = 0$

$$(N2) \quad \|\lambda T\| = \sup_{\|x\| \leq 1} \|\lambda T(x)\| = \sup_{\|x\| \leq 1} \underbrace{|\lambda|}_{\geq 0} \|Tx\| = |\lambda| \cdot \|T\|$$

$$(N3) \quad \forall x : \|(T + S)(x)\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|)\|x\| \\ \implies \|T + S\| \leq \|T\| + \|S\|$$

3. Let  $(T_n)_n$  be Cauchy in  $\mathcal{L}(X, Y)$  and  $Y$  a Banach space. Since  $\|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \|x\| \implies (T_n x)_n$  is Cauchy in  $Y \forall x \in X \implies Tx := \lim_{n \rightarrow \infty} T_n x$  is well defined.

Furthermore, we want to show

**Linearity:**

$$\forall x, y \in X, \lambda \in \mathbb{K} : T(\lambda x + y) = \lim_{n \rightarrow \infty} T_n(\lambda x + y) = \lim_{n \rightarrow \infty} \lambda T_n x + \lim_{n \rightarrow \infty} T_n y = \lambda Tx + Ty$$

$\|T_n - T\| \rightarrow 0$ : Take  $\varepsilon > 0$ ,  $n_0 \in \mathbb{N} : \|T_n - T_m\| \leq \varepsilon \forall n, m \geq n_0$

Show:  $\exists n_1 \forall n \geq n_1 : \|T_n - T\| \leq 2\varepsilon$ . For  $x \in X : \|x\| \leq 1$  fix  $m_x \geq n_0$  :  
 $\|T_{m_x} x - Tx\| \leq \varepsilon \implies \forall n \geq n_1 =: n_0 :$

$$\|T_n x - Tx\| \leq \|T_n x - T_{m_x} x\| + \|T_{m_x} x - Tx\| \\ \leq \|T_n - T_{m_x}\| + \varepsilon \leq 2\varepsilon \\ \implies \|T_n - T\| = \sup_{\|x\| \leq 1} \|T_n x - Tx\| < 2\varepsilon$$

$$\implies \forall x \in X : \|Tx\| \leq \|T_n x - Tx\| + \|T_n x\| \leq \|T_n - T\| + \|T_n\| \forall n \text{ fixed}$$

□

**Proposition 3.6.** Let  $X, Y$  be normed spaces.  $D \subset X$  is a subspace such that  $\overline{D} = X$ ,  $T \in \mathcal{L}(D, Y)$ .

$$\exists! \hat{T} \in \mathcal{L}(X, Y) : \hat{T}|_D = T$$

In addition:  $\|\hat{T}\| = \|T\|$ .

*Proof.* Unique extension is clear for  $T$  is uniformly continuous.

Also:

$$\|\hat{T}\| = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\|\hat{T}x\|}{\|x\|} \stackrel{\text{by density}}{=} \sup_{\substack{x \in D \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|$$

To show the density equality is left as an exercise to the reader.

□

**Proposition 3.7.** Let  $X, Y, Z$  be normed spaces.  $S \in \mathcal{L}(X, Y)$ .  $T \in \mathcal{L}(Y, Z)$ . Then  $T_0 S \in \mathcal{L}(X, Z)$  and  $\|T_0 S\| \leq \|T\| \|S\|$ .

*Proof.*  $T_0 S$  is linear (show as an exercise).

Take  $x \in X$ .  $\|T_0 S(x)\| = \|T(Sx)\| = \|T\| \|Sx\| \leq \|T\| \|S\| \|x\|$ .  $\Rightarrow \|T_0 S\| \leq \|T\| \|S\|$   $\square$

**Remark.** If  $\dim(X) < \infty$ ,  $T : X \rightarrow Y$  is linear, then  $T_C \mathcal{L}(X, Y)$  (left as an exercise).

**Proposition 3.8** (Neumann series). Let  $X$  be a normed space.  $T \in \mathcal{L}(X)$ . If  $\sum_{n=0}^{\infty} T^n$  is convergent in  $\mathcal{L}(X)$ , then  $(I - T)$  is invertible and  $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ .

Here:  $T^n := T_0 \cdot T_0 \cdot T_0 \cdot \dots$   $n$  times

In particular, if  $X$  is Banach and  $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} =: a < 1$  then  $\sum_{i=0}^{\infty} T^n$  is convergent. Also if  $\|T\| < 1$ , then  $a < 1$  holds true. In case of  $a < 1$ , then  $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$ .

*Proof.* Let  $S_m := \sum_{n=0}^m T^n$  and  $S := \lim_{m \rightarrow \infty} S_m$ . Then  $(I - T)S_m = I - T^{m+1} = S_m(I - T)$  (compute!).

$$\|T^m\| = \left\| \sum_{n=0}^m T^n - \sum_{n=0}^{m-1} T^n \right\| = \|S_m - S_{m-1}\| \rightarrow 0$$

for  $m \rightarrow \infty$  since  $(S_m)_n$  is Cauchy. ( $RS := R_0 S$ )

Now note that for fixed  $R \in \mathcal{L}(X)$  the mappings

$$S \mapsto RS \quad S \mapsto SR$$

are continuous since  $\|S_n R - SR\| \leq \|S_n S\| \|R\| \rightarrow 0$  for  $S_n \rightarrow S$ . Continuity implies that

$$\begin{aligned} I &= \lim_{m \rightarrow \infty} I - T^{m+1} = \begin{cases} \lim_{m \rightarrow \infty} (I - T)S_m = (I - T)S \\ \lim_{m \rightarrow \infty} S_m(I - T) = S(I - T) \end{cases} \\ &\Rightarrow (I - T)^{-1} = S \end{aligned}$$

Now if  $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq a < 1 \forall \varepsilon > 0 \Rightarrow \exists n_0 \forall n \geq n_0 : \|T^n\| \leq (a + \varepsilon)^n$

$$\Rightarrow \sum_{n=0}^{\infty} \|T^n\| \leq c + \sum_{n=0}^{\infty} (a + \varepsilon)^n = \frac{1}{1 - (a + \varepsilon)} + c < \infty \text{ for } c > 0$$

$X$  is Banach, so  $\sum_{n=0}^{\infty} T^n$  is convergent and

$$\|(I - T)^{-1}\| = \left\| \sum_{n=0}^{\infty} T^n \right\| \leq \frac{1}{1 - (a + \varepsilon)}$$

Since  $\varepsilon$  was arbitrary,  $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$ .

If  $\|T\| < 1$ , then

$$\begin{aligned}\limsup_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} &\leq \limsup (\|T\| \cdot \|T\| \cdots \|T\|)^{\frac{1}{n}} \\ &\leq \limsup (\|T\|^n)^{\frac{1}{n}} \\ &= \limsup \|T\| \\ &= \|T\|\end{aligned}$$

□

**Remark.**  $(I - T)^{-1}$  is linear (left as an exercise). However:  $(I - T)^{-1} \notin \mathcal{L}(X)$  in general!

## 4 The Hahn-Banach Theorem and its consequences

Apparently the Hahn-Banach Theorem of this chapter is very central to Functional Analysis. This section deals with an extension of linear functionals and separation of sets.

First, consider  $\mathbb{K} = \mathbb{R}$ .

**Definition 4.1.** Let  $X$  be a vector space.  $p : X \rightarrow \mathbb{R}$  is called sublinear iff

1.  $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0, x \in X$
2.  $p(x + y) \leq p(x) + p(y) \forall x, y \in X$

**Example 4.2.**  $p(x) = \|x\|$ ,  $p$  linear and  $p$  is a seminorm.

**Theorem 4.3** (Hahn-Banach Theorem, real version). Let  $X$  be a vector space over  $\mathbb{R}$ ,  $U \subset X$ , a subspace.  $p : X \rightarrow \mathbb{R}$  be sublinear and  $l : U \rightarrow \mathbb{R}$  is linear such that  $l(x) \leq p(x) \forall x \in U$

Then  $\exists L : X \rightarrow \mathbb{R}$  is linear such that

$$L|_U = l \quad L(x) \leq p(x) \forall x \in X$$

*Proof.* This proof consists of two steps:

1. Method to extend  $l$  from  $U$  to  $U + \text{span}(x_0)$ ,  $x_0 \notin U$
2. Iterate this step and get maximal extension (Zorn)

**Step 1** For  $x_0 \in X \setminus U$ , let  $V = U + \text{span}(x_0) = \{u + \lambda x_0 \mid u \in U, \lambda \in \mathbb{R}\}$ . Any  $v \in V$  can be written uniquely as  $v = u + \lambda x_0$  for  $u \in U, \lambda \in \mathbb{R}$  (why? left as an exercise). Thus for any  $r \in \mathbb{R}$ , we can define  $L_r : V \rightarrow \mathbb{R}$ .  $v = u + \lambda x_0 \mapsto l(u) + \lambda r$ .  $L_r$  is linear (why? left as an exercise).

Also:  $L_r(x) \leq p(x) \forall x \in V \iff l(u) + \lambda r \leq p(u + \lambda x_0) \forall \lambda, u$  (let this statement be  $(*)$ ).

$\lambda = 0$  (\*) holds true

$\lambda > 0$  (\*)

$$\begin{aligned} &\iff r \leq p\left(\frac{u}{\lambda} + x_0\right) - l\left(\frac{u}{\lambda}\right) \forall u \in U \\ &\iff r \leq \inf_{u \in U} p(u + x_0) - l(u) \end{aligned}$$

$\lambda < 0$

$$\begin{aligned} &\iff -r \leq p\left(\frac{u}{-\lambda} - x_0\right) - l\left(\frac{u}{-\lambda}\right) \forall u \in U \iff r \geq -p(u - x_0) + l(u) \forall u \in U \\ &\iff r \geq \sup_{u \in U} l(u) - p(u - x_0) \end{aligned}$$

Thus, (\*) holds for  $r = \sup_{u \in U} l(u) - p(u - x_0)$  if  $\sup_{u \in U} l(u) - p(u - x_0) \leq \inf_{u \in U} p(u + x_0) - l(u) \iff l(w) - p(w - x) \leq p(u + x_0) - l(u) \forall w, u \in U \iff l(w) + l(u) \leq p(u + x_0) + p(w - x_0)$ .

But this holds since:

$$l(w) + l(u) = l(w + u) \leq p(w + u) = p(w - x_0 + x_0 + u) \leq p(w - x_0) + p(u + x_0)$$

## Step 2

**Revision 4.4** (Zorn's Lemma). *Let  $(A, \leq)$  be a partially ordered set such that every chain (every subset  $R$  of  $A$   $\forall a, b \in R : a \leq b \vee b \leq a$ ) admits an upper bound (i.e.  $\exists c \in A : b \leq c \forall b \in R$ ), then  $A$  has a maximal element, i.e.  $\exists z \in A$  such that  $\forall a \in A : z_0 \leq a \implies a = z_0$*

Let  $A$  be a set of  $(V, L_V)$  tuples where  $V \subset X$  is a subspace with  $U \subset V$  and  $L_V : V \rightarrow \mathbb{R}$  such that  $L_V \leq p$  on  $V$  and  $L_V|_U = l$ .

For  $(V_1, L_{v_1})$  and  $(V_2, L_{v_2}) \in A$ , we say that  $(V_1, L_{v_1}) \leq (V_2, L_{v_2})$  if  $V_1 \subset V_2$  and  $L_{v_2}|_{V_1} = L_{v_1}$ . Now  $A \neq \emptyset$  since  $(U, l) \in A$ . If  $(V_i, L_{v_i})_{i \in I} := R$  is a chain, define  $V := \bigcup_{i \in I} V_i$ ,  $L_V(x) := L_{v_i}(x)$  if  $x \in V_i$ .

This is well-defined.

$\implies (V, L_V)$  is an upper bound for  $R$ .

□

↓ This lecture took place on 2019/05/14.

*Proof of Theorem 4.3.* Let  $U \subset X$ ,  $x_0 \notin U$ ,  $V = U + \text{span}(x_0)$ .

$$\implies \exists L_V : V \rightarrow \mathbb{R} : L_V|_U = l, L_V(v) = p(v) \forall v \in V$$

$$R = \{(V, L_V) \mid U \subset V, L_V|_U = l, L_V = p \text{ on } V\}$$

$$(V_1, L_{V_1}) \leq (V_2, L_{V_2}) : \iff V_1 \subset V_2, L_{V_2}|_{V_1} = L_{V_1}$$

**Remark.** Any chain has an upper bound.

Let  $(V_i, L_{V_i})_{i \in I}$  be a chain in  $R$ . Then we define  $V = \bigcup_{i \in I} V_i$ .  $L_V : V \rightarrow \mathbb{R}$  with  $v \mapsto L_{V_i}(v)$  if  $v \in V_i$ . Thus we showed well-definedness.

Then  $(V, L_V)$  is an upper bound of  $(V_i, L_{V_i})_{i \in I}$  since  $V_i \subset V$ ,  $L_V|_{V_i} = L_{V_i} \forall i \in I$ . By Zorn, there exists  $(V_0, L_{V_0})$  a maximal element of  $R$ . It is left to show that  $V_0 = X$ . If not: Take some  $x_0 \in X \setminus V_0$ , define  $\tilde{V} := V_0 + \text{span}(x_0)$  and  $L_{\tilde{V}}$  as an extension of  $L_{V_0}$  as in step 1.

$$\implies (V_0, L_{V_0}) \leq (\tilde{V}, L_{\tilde{V}})$$

This contradicts the maximality of  $(V_0, L_{V_0})$ .  $\square$

**Remark.** If  $U$  is not dense, then the extension is unique.

*Next:* Hahn-Banach Theorem for  $\mathbb{K} = \mathbb{C}$ .

*Approach:* Establish bijection between  $\mathbb{R}$  vector space and  $\mathbb{C}$  vector space.

**Proposition 4.5.** Let  $X$  be a  $\mathbb{C}$  vector space (vector space over the complex numbers).

1. If  $l : X \rightarrow \mathbb{R}$  is  $\mathbb{R}$ -linear (i.e.  $l(x+y) = l(x) + l(y)$  and  $l(\lambda x) = \lambda l(x) \forall \lambda \in \mathbb{R}$ ). We set  $\hat{l} : X \rightarrow \mathbb{C}$  with  $x \mapsto l(x) - i \cdot l(ix)$ . Then  $\hat{l}$  is  $\mathbb{C}$ -linear and  $\Re(\hat{l}) = l$ .
2. If  $h : X \rightarrow \mathbb{C}$  is  $\mathbb{C}$ -linear and we let  $l := \Re(h)$  and  $\hat{l}$  as in (1), then  $l$  is  $\mathbb{R}$ -linear and  $\hat{l} = h$  [  $l \rightarrow \hat{l}$  is surjective ]
3. If  $p : X \rightarrow \mathbb{R}$  is a seminorm and  $l : X \rightarrow \mathbb{C}$  is  $\mathbb{C}$ -linear. Then

$$|l(x)| \leq p(x) \forall x \iff |\Re(l(x))| \leq p(x) \forall x$$

4. If  $x$  is normed,  $l \in \mathcal{L}(X, \mathbb{C})$ , then  $\|L\| = \|\Re(x)\|$

**Remark.** This means that  $l \mapsto [x \mapsto l(x) - il(ix)]$  is bijective and an isometry if  $X$  is normed.

*Proof.* 1. By construction  $\hat{l}$  is  $\mathbb{R}$ -linear and  $\Re(\hat{l}) = l$  is obvious.

Show:  $\tilde{l}(ix) = i\tilde{l}(x)$ .

$$\begin{aligned} \tilde{l}(ix) &= l(ix) - il(iix) = l(ix) - il(-x) \\ &= i(l(x) - il(ix)) = i\tilde{l}(x) \end{aligned}$$

2. Define  $l := \Re(h)$ . Show:  $\tilde{l} = h$ .

Note:  $\forall z \in \mathbb{C} : \Im(z) = -\Re(iz)$ .

$$\begin{aligned} h(x) &= \Re(h(x)) + i \cdot \Im(h(x)) = \Re(h(x)) - i \cdot \Re(i \cdot h(x)) \\ &= \Re(h(x)) - i \cdot \Re(h(ix)) = l(x) - i \cdot l(ix) = \tilde{l}(x) \end{aligned}$$

Hence  $l \mapsto \tilde{l}$  is bijective.

3. Since  $|Re(z)| \leq |z|$ ,

$\implies$  holds trivially

$\Leftarrow$  Write  $l(x) = \lambda_X |l(x)|$  with  $|\lambda_X| = 1$ . Then  $\forall x \in X : |l(x)| = \lambda_X^{-1} l(x) = l(\lambda_X^{-1} x) = |\Re l(\lambda_X^{-1} x)| \leq p(\lambda_X^{-1} x) = |\lambda_X^{-1}| p(x) = p(x)$

4. Consequence of (3) with  $p(x) := \|l\| \|x\|$

□

**Theorem 4.6** (Hahn-Banach Theorem, complex version). *Let  $X$  be a  $\mathbb{C}$  vector space.  $U \subset X$ .  $p : X \rightarrow \mathbb{R}$  sublinear and  $l : U \rightarrow \mathbb{C}$  be linear such that  $\Re l(u) < p(u) \forall u \in U$ .*

$$\exists L : X \rightarrow \mathbb{C} \text{ linear such that } L|_U = l, \Re L(x) \leq p(x) \forall x$$

*Proof.* Applying Theorem 4.3 to  $\Re l \implies \exists F : X \rightarrow \mathbb{R}$   $r$ -linear such that  $F|_U = \Re(l)$  and  $F(x) \leq p(x) \forall x \in X$

Proposition 4.5 implies there exists some  $L : X \rightarrow \mathbb{C}$  that is  $\mathbb{C}$ -linear such that  $F = \Re(L)$ . Now  $\Re(L)|_U = F|_U = \Re(l) \implies L = l$  by Proposition 4.5 (2) and also  $\Re L(x) = F(x) = p(x) \forall x \in X$ . □

**Proposition 4.7** (Consequence). *If  $X$  is a normed space,  $U \subset X$  be a subspace,  $u' \in \mathcal{L}(U, \mathbb{K})$ , then  $\exists x' \in \mathcal{L}(X, \mathbb{K})$  such that  $x'|_U = u'$  with  $\|x'\| = \|u'\|$ .*

*Proof. Case  $\mathbb{K} = \mathbb{R}$ :* Let  $p(x) := \|u'\| \|x\|$ . Then  $p$  is sublinear and  $u'(x) \leq |u'(x)| \leq p(x) \forall x \in U$ . By Theorem 4.3, there exists  $x' : X \rightarrow \mathbb{R}$  linear such that  $x'|_U = u'$  and  $x'(x) \leq p(x) \forall x \in X$ .

$$\implies -x'(x) = x'(-x) \leq p(-x) = p(x) \implies |x'(x)| \leq p(x) = \|u'\| \|x\| \implies \|x'\| \leq \|u\|$$

Also:

$$\|u'\| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} |u'(u)| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} |x'(u)| \leq \sup_{\substack{x \in X \\ \|x\| \leq 1}} |x'(x)| = \|x'\| \implies \|x'\| = \|u'\|$$

**Case  $\mathbb{K} = \mathbb{C}$ :** As before,  $\exists x' : X \rightarrow \mathbb{C} : x'|_U = u'$  and  $\|\Re x'\| \leq \|u'\|$ . By proposition 4.5,  $\|x'\| = \|\Re(x')\|$

□

**Remark** (Next application: Separation of convex sets). *Motivation: Given two (convex) sets  $A, B \subset \mathbb{R}^2$ . When can we find a line  $L$  separating these sets*

*Compare with Figure 2.*

**Remark.** In  $\mathbb{R}^2$ , any line  $L$  can be separable as  $L = \{x \in \mathbb{R}^2 : (x, n) = \alpha \mid \alpha \in \mathbb{R}, n \in \mathbb{R}^2, \|n\| = 1\}$ .

**Definition.** Let  $X$  be a vector space  $H \subset X$  is called a hyperplane if it is of the form  $H = \{x \in X \mid \Re(f(x)) = \alpha\}$  with  $\alpha \in \mathbb{R}$ ,  $f : X \rightarrow \mathbb{K}$  linear.

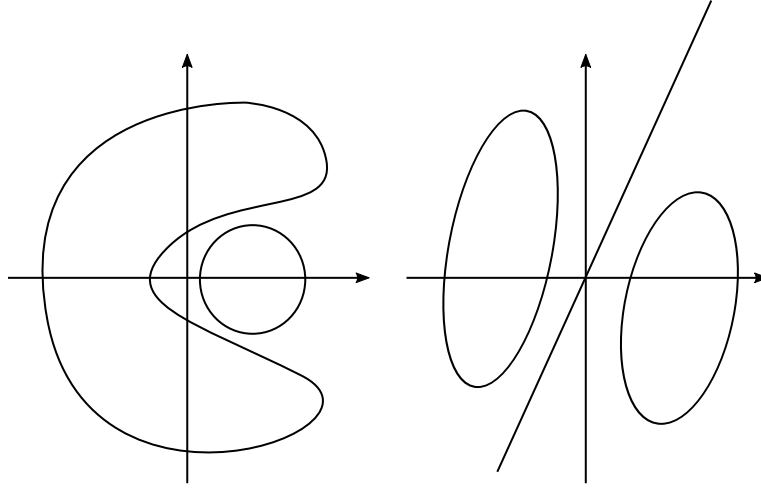


Figure 2: Inseparable convex sets (left) and separable convex sets (right)

**Lemma 4.8.** *Let  $X$  be a normed space,  $H \subset X$  be a hyperplane of the form  $H = \{x \in X \mid \Re(f(x)) = \alpha\}$  with  $\alpha \in \mathbb{R}$ ,  $f : X \rightarrow \mathbb{K}$  linear.*

*Then  $H$  is closed iff  $f \in \mathcal{L}(X, \mathbb{K})$ .*

*Proof.* Compare with the practicals. □

**Remark (Goal).** *Given  $X$  as a normed vector space.  $A, B \subset X$  where does some closed hyperplane  $H$  exist represented by  $f \in \mathcal{L}(X, \mathbb{K})$  and  $\alpha$  separating  $A$  and  $B$ , e.g.  $\Re(f(a)) \leq \alpha \leq \Re(f(b)) \forall a \in A, b \in B$ .*

*To this aim associate a set  $U \subset X$  to a sublinear functional  $p : X \rightarrow \mathbb{R}$ .*

**Definition 4.9.** *Let  $X$  be a vector space.  $A \subset X$ . The Minkovsky functional  $p_A : X \rightarrow [0, \infty]$  is defined as  $p_A(x) = \inf \left\{ \lambda > 0 \mid \frac{x}{\lambda} \in A \right\}$ .  $A$  is called absorbing if  $p_A(x) < \infty \forall x \in X$ .*

**Theorem 4.10.** *Let  $X$  be a normed space.  $U \subset X$  convex such that  $0 \in \text{interior}(U) = \overset{\circ}{U}$ . Then,*

1.  *$U$  is absorbing and  $\forall \varepsilon > 0 : B_\varepsilon(0) \subseteq U \implies p_U(x) \leq \frac{1}{\varepsilon} \|x\|$  [no convexity needed]*
2.  *$p_U$  is sublinear*
3. *If  $U$  is open, then  $U = p_U^{-1}([0, 1))$ .*

*Proof.* 1. Trivial

2. •  $p_u(\lambda x) = \lambda p_u(x)$  for  $\lambda > 0$ . Compare with the practicals.  
 • Take  $x, y \in X$ . Show:  $p_u(x + y) \leq p_u(x) + p_u(y)$ .

Take  $\varepsilon > 0$  and choose  $\lambda, \mu$ :

$$\begin{aligned} \lambda &\leq p_u(x) + \varepsilon & \frac{x}{\lambda} &\in U \\ \mu &\leq p_u(y) + \varepsilon & \frac{y}{\mu} &\in U \end{aligned}$$

Since  $U$  is convex,

$$\begin{aligned} \frac{x+y}{\lambda+\mu} &= \frac{\lambda}{\lambda+\mu} \left( \frac{x}{\lambda} \right) + \frac{\mu}{\lambda+\mu} \left( \frac{y}{\mu} \right) \in U \\ \Rightarrow p_u(x+y) &\leq \lambda + \mu = p_u(x) + p_u(y) + 2\varepsilon \end{aligned}$$

$\varepsilon$  can be arbitrary, thus the proof is complete.

3. Direction  $\supset$ . If  $p_u(x) < 1 \Rightarrow \exists \lambda > 0 : \lambda < 1$  and  $\frac{x}{\lambda} \in U$ . Since  $0 \in U$ ,

$$\begin{aligned} \Rightarrow x &= \lambda \left( \frac{x}{\lambda} \right) + (1-x)0 \in U \\ \Rightarrow p_u^{-1}([0, 1)) &\subset U \end{aligned}$$

Direction  $\subset$ . If  $p_u(x) \geq 1$ , then  $\frac{x}{\lambda} \notin U \forall \lambda < 1$

$$\Rightarrow x = \lim_{\substack{\lambda \rightarrow 1 \\ \lambda < 1}} \frac{x}{\lambda} \in U^c$$

□

**Lemma** (Fundamental lemma). *Let  $X$  be a normed vector space.  $V \subset X$  be convex and open.  $0 \notin V$ .*

$$\Rightarrow \exists x' : X \rightarrow \mathbb{K} \text{ continuous}$$

linear such that  $\Re x'(x) < 0 \forall x \in V$ .

*Proof.* Define  $A \mp B = \{a + b \mid a \in A, b \in B\}$ .

**Case  $\mathbb{K} = \mathbb{R}$ :** Take  $x_0 \in V \setminus \{0\}$ , define  $y_0 := -x_0$  and  $U := V - \{x_0\}$ .

$$\Rightarrow U \text{ is open, convex, } 0 \in U, y_0 \notin U$$

We consider  $p_u : X \rightarrow \mathbb{R}$  which is sublinear, finite and  $p_u(y_0) \geq 1$ . On  $Y := \text{span}(y_0)$  we define  $y' : Y \rightarrow \mathbb{R}$  with  $ty_0 \mapsto tp_u(y_0)$  and  $t \in \mathbb{R}$ .

$$\Rightarrow y'(y) \leq p_u(y) \forall y \in Y$$

since

$$y'(y) = y'(ty_0) = tp_u(y_0)$$

- $t \leq 0$ :  $\leq 0 \leq p_u(y)$
- $t > 0$ :  $= p_u(ty_0) = p_u(y)$



↓ This lecture took place on 2019/05/16.

Now by Hahn-Banach Theorem,  $\exists x' : X \rightarrow \mathbb{R}$  linear such that  $x'|_Y = y'$  and  $x'(x) \leq p_u(x) \forall x \in X$

$$\forall x \in X : |x'(x)| = \max \left\{ x'(x), \underbrace{-x'(x)}_{=x'(-x)} \right\} \leq \min(p_u(x), -p_u(-x)) \leq \frac{1}{2} \|x\| \quad \text{for } \varepsilon > 0 : B_\varepsilon(0) \subseteq U$$

$$\implies x' \in \mathcal{L}(X, \mathbb{R})$$

Also  $x'(y_0) = y'(y_0) = p_u(y_0) \geq 1$ .

$\implies \forall v \in V$  we can write  $v = u - y_0$  with  $u \in U$

$\implies x'(v) = x'(u) - x'(y_0) \leq p_u(u) - 1 < 0$

**Case  $\mathbb{K} = \mathbb{C}$**  Lemma 4.5. Left as an exercise.

□

**Theorem 4.11** (Separation 1). *Let  $X$  be normed. Let  $V_1, V_2 \subset X$  be convex and  $V_1$  open.  $V_1 \cap V_2 = \emptyset$*

$$\implies \exists x' \in \mathcal{L}(X, \mathbb{K}) \text{ s.t. } \Re(x'(u_1)) \leq \Re(x'(x_2)) \forall v_1 \in V_1, v_2 \in V_2$$

*Proof.* Define  $V := V_1 - V_2$ . Then  $V$  is convex (why?) and open since  $V = \bigcup_{x \in V_2} V_1 - \{x\}$  since  $V_1 \cap V_2 = \emptyset$ . Thus  $0 \in V$ . By Lemma 4,

$$\exists x' \in \mathcal{L}(X, \mathbb{K}) : \Re x'(v_1 - v_2) < 0 \forall v_1 \in V_1, v_2 \in V_2$$

$$\implies \Re x'(v_1) < \Re x'(v_2)$$

□

**Remark.**  $V$  being open is sufficient.

**Theorem 4.12** (Separation 2). *Let  $X$  be a normed spaces.  $V \subset X$  is closed and convex.*

$$\hat{x} \notin V \implies \exists x' \in \mathcal{L}(X, \mathbb{K})$$

$$\Re(x'(\hat{x})) < \inf_{v \in V} \Re(x'(v))$$

i.e.  $\exists \varepsilon > 0 : \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \varepsilon \leq \inf_{v \in V} \Re(x'(v))$

*Proof.*

$$V \text{ closed} \iff \exists \varepsilon > 0 : \underline{B_\varepsilon(\hat{x})}_{V_1} \cap V = \emptyset$$

By Theorem 4.11,  $\exists x' \in \mathcal{L}(X, \mathbb{K})$ :

$$\Re(x'(\hat{x} + u)) < \Re(x'(v)) \forall v \in V, u \in X : \|u\| < \varepsilon$$

$$\Re(x'(\hat{x})) + \Re(x'(u)) < \Re(x'(v)) \forall v \in V, u \in X, \|u\| \leq \frac{\varepsilon}{2}$$

Taking the sum over  $u$ .

$$\Re(x'(\hat{x})) + \|\Re(x')\| \frac{\varepsilon}{2} \leq \Re(x'(v)) \forall v \in V$$

since

$$\begin{aligned} \|\Re(x')\| &= \sup_{\|\lambda\| \leq 1} |\Re(x'(x))| \frac{\varepsilon}{2} = \sup_{\|x\| \leq \frac{\varepsilon}{2}} |\Re(x'(x))| = \sup_{\|x\| \leq \frac{\varepsilon}{2}} \Re(x'(x)) \\ &\implies \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \|x'\| \frac{\varepsilon}{2} \leq \inf_{v \in V} \Re(x'(v)) \end{aligned}$$

□

## 5 Fundamental theorems for operators in Banach spaces

In this chapter we are going to discuss the Baire theorem.

**Theorem 5.1** (Banach-Steinhaus, uniform boundedness principle). *Let  $X$  be a Banach space,  $Y$  normed. Let  $I$  be an index set. For all  $i \in I$ , let  $T_i \in \mathcal{L}(X, Y)$ .*

*Then if  $\forall x \in X : \sup_{i \in I} \|T_i x\| < \infty \implies \sup_{i \in I} \|T_i\| < \infty$*

*Proof.* Define  $E_n := \{x \in X \mid \sup_{i \in I} \|T_i x\| \leq n\}$  since all  $T_i$  are continuous

$$\implies E_n = \bigcap_{i \in I} \|T_i(\cdot)\|^{-1}([0, n])$$

since  $x \mapsto \|T_i x\|$  is continuous  $\rightarrow$  closed.

$\implies E_n$  is closed as the intersection of closed sets

Also:  $X = \bigcup_{n \in \mathbb{N}} E_n$

By Baire's theorem,  $\exists E_n : \overset{\circ}{E}_{n_0} \neq \emptyset$ .

$$\implies \exists \varepsilon > 0, y \in E_{n_0} \text{ fixed such that } \forall x \in X : \|x - y\| \leq \varepsilon \implies x \in E_{n_0}$$

Now take  $x \in X : \|x + y\| \leq \varepsilon$ .

$$\|x + y\| = \|x - (-y)\| = \|-x - y\| \implies -x \in E_n \implies x \in E_n$$

Also,  $\forall x_1, x_2 \in X, \lambda \in [0, 1], x_1, x_2 \in E_{n_0}$ .

$$\implies \lambda x_1 + (1 - \lambda)x_2 \in E_{n_0}$$

since  $\forall i : \|T_i(\lambda x_1 + (1 - \lambda)x_2)\| \leq \lambda \|T_i x_1\| + (1 - \lambda) \|T_i x_2\| < n_0$ .

$$\forall x \in X : \|x\| \leq \varepsilon \implies x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \in E_{n_0}$$

since  $x + y \in E_{n_0}$  and  $x - y \in E_{n_0}$ .

$$\begin{aligned} &\Rightarrow \forall i \in I : \|T_i x\| \leq n_0 \forall \|x\| \leq \varepsilon \\ &\Rightarrow \|T_i\| = \frac{1}{\varepsilon} \sup_{\|x\| \leq 1} \|T_i(\varepsilon x)\| \leq \frac{1}{\varepsilon} n_0 \\ &\Rightarrow \sup_{i \in I} \|T_i\| \leq \frac{\varepsilon}{n_0} \end{aligned}$$

□

↓ This lecture took place on 2019/05/21.

$$\forall \sup_{i \in I} \|T_i x\| < \infty \Rightarrow \sup_{i \in I} \|T_i\| < \infty$$

**Remark.** With  $d = \{(x_i)_i \mid x_i = 0 \text{ for all but finitely many } i, x_i \in \mathbb{R}\}$ ,  $(d, \|\cdot\|_\infty)$  is a normed space. Define  $T_n : d \rightarrow \mathbb{R}$ ,  $(x_i)_i \mapsto n \cdot x_n$  for given  $n$ .

Then  $\forall x \in d : \sup_{n \in \mathbb{N}} |T_n x| = \sup_{n \in \mathbb{N}} |n x_n| \leq |n_0 x_{n_0}|$  for some  $n_0 \in \mathbb{N}$ . However  $\forall n : \|T_n\| > |T_n e_n| = n \Rightarrow \sup_{n \in \mathbb{N}} \|T_n\| = \infty$ . Thus,  $X$  is Banach space is necessary for Theorem 5.1 (also  $(d, \|\cdot\|_\infty)$  is not Banach).

**Corollary 5.2.** Let  $X$  be Banach,  $Y$  normed  $\forall n \in \mathbb{N} : T_n \in \mathcal{L}(X, Y)$  and suppose that  $\lim_{n \rightarrow \infty} T_n x = T_x$  exists and is finite for all  $x \in X$ . Then  $T \in \mathcal{L}(X, Y)$ .

*Proof.* Left as an exercise to the reader. □

Now: Continuous invertibility of linear operators, or, if  $T \in \mathcal{L}(X, Y)$  bijective such that  $T^{-1} : Y \rightarrow X$  exists and is linear. When does  $T^{-1} \in \mathcal{L}(Y, X)$  hold?

**Definition 5.3** (f maps open sets to open sets). Let  $(X, \tau_X)$  and  $(Y, \tau_Y)$  be two spaces and  $f : X \rightarrow Y$ .  $f$  is called open if  $f(A) \in \tau_Y \forall A \in \tau_X$ .

**Remark.**  $f$  is continuous  $\iff f^{-1}(A) \in \tau_X \forall A \in \tau_Y$ . If  $f$  is open and invertible, then  $(f)^{-1}(A) = f(A) \in \tau_Y \forall A \in \tau_X \implies f^{-1}$  is continuous.

**Lemma 5.4.** Let  $T : X \rightarrow Y$  be linear.  $X, Y$  be normed. TFAE:

1.  $T$  is open
2.  $\forall r > 0 \exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_r(0))$
3.  $\exists \varepsilon > 0 : B_\varepsilon \subset T(B_1(0))$

*Proof.*  $1 \rightarrow 2$  True since  $0 \in T(B_r(0))$  and  $T(B_r(0))$  is open.

$2 \rightarrow 1$  Let  $O \subset X$  be open,  $y \in T(O) \implies x \in O$ .  $Tx = y$ .  $\exists r > 0 : x + B_r(0) \subset O$ .

$\implies T(x) + T(B_r(0)) \subset T(O)$ . By (2),  $\exists \varepsilon > 0 : B_r(0) \subset T(B_r(0))$

$$\begin{aligned} &\Rightarrow Tx + T_\varepsilon(0) \subset Tx + T(B_r(0)) \subset O \\ &\Rightarrow B_\varepsilon(Tx) = B_\varepsilon(y) \in O \text{ since } y \text{ was arbitrary, thus } O \text{ is open.} \end{aligned}$$

2  $\rightarrow$  3 Left as an exercise to the reader

□

**Remark.** If  $X, Y$  is normed and  $T : X \rightarrow Y$  linear, then  $T$  is injective.

*Proof.* Take  $y \in Y \setminus \{0\}$ ,  $\varepsilon > 0$  such that  $B_\varepsilon(0) \subset T(B_1(0))$ .

$$\Rightarrow \frac{\varepsilon y}{2\|y\|} \in B_\varepsilon(0) \Rightarrow \exists x : Tx = \frac{\varepsilon y}{2\|y\|} \Rightarrow T\left(\frac{2x\|y\|}{\varepsilon}\right) = y$$

□

**Theorem 5.5** (Open mapping theorem). Let  $X$  and  $Y$  be Banach.  $T \in \mathcal{L}(X, Y)$  injective  $\Rightarrow T$  open.

*Proof.* Here  $B_r$  denotes  $B_r(0)$ .

Show  $\exists \varepsilon > 0 : B_\varepsilon(0) \subset T(B_1(0))$ .

**Part 1** Show  $\exists \varepsilon > 0 : B_\varepsilon \subset T(B_1)$ .

We have  $Y = \bigcup_{n \in \mathbb{N}} T(B_n)$  since  $T$  is surjective. By Baire's category theorem,  $\exists N \in \mathbb{N} : \overline{T(B_N)} \neq \emptyset$ .

$$\Rightarrow \exists y_0 \in \overline{T(B_N)}, \varepsilon > 0 \forall z \in Y : \|z - y_0\| < \varepsilon \Rightarrow z \in \overline{T(B_N)}$$

As in the proof of Theorem 5.1,  $B_\varepsilon \subset \overline{T(B_N)}$  is implied.

**Part 2** Show: If  $B_\varepsilon \subset \overline{T(B_1)} \Rightarrow B_\varepsilon \subset T(B_1)$ .

Let  $\|y\| < \varepsilon$ . Show:  $y \in T(B_1)$ . Choose  $\tilde{\varepsilon} > 0 : \|y\| < \tilde{\varepsilon} < \varepsilon$  define  $\tilde{y} := \frac{\varepsilon}{\tilde{\varepsilon}}y$ .  $\|\tilde{y}\| < \varepsilon \Rightarrow \exists y_0 \in Y, x_0 \in X : y_0 = Tx_0$  and  $\|\tilde{y} - y_0\| < \alpha\tilde{\varepsilon}$  where  $0 < \alpha < 1$  is

$$\Rightarrow \frac{\tilde{y} - (y_0 + \alpha y_1)}{\alpha^2} \in B_\varepsilon \Rightarrow \exists y_2 \in Y, x_2 \in B_1 : Tx_2 = y_2$$

$$\text{and } \|\tilde{y} - (y_0 + \alpha y_1 + \alpha^2 y_2)\| < \alpha^3 \varepsilon \text{ by } \left\| \frac{\tilde{y} - (y_0 + \alpha y_1)}{\alpha^2} - y_2 \right\| < \alpha \varepsilon$$

We can construct a sequence (by induction)  $(x_n)_n$  such that  $\|x_n\| < 1 \forall n$  and  $\|\tilde{y} - T(\sum_{i=0}^n \alpha^i x_i)\| < \alpha^{n+1} \varepsilon$ . Since  $\alpha < 1$ ,  $\|\sum_{i=1}^n \alpha^i x_i\| \leq \sum_{i=0}^n \alpha^i < (1 - \alpha)^{-1} < \infty \Rightarrow \sum_{i=0}^\infty \alpha^i x_i$  is absolutely convergent.  $X$  is Banach space, thus  $\exists \hat{x} := \lim_{n \rightarrow \infty} \sum_{i=0}^n \alpha^i x_i \in X$ .  $\|\tilde{y} - T(\sum_{i=0}^n \alpha^i x_i)\| < \alpha^{n+1} \varepsilon \Rightarrow T(\sum_{i=0}^n \alpha^i x_i) \rightarrow \tilde{y} \Rightarrow T\hat{x} = \tilde{y}$ . With  $x := \frac{\tilde{\varepsilon}}{\varepsilon} \hat{x} \Rightarrow Tx = \frac{\tilde{\varepsilon}}{\varepsilon} T\hat{x} = \frac{\tilde{\varepsilon}}{\varepsilon} \tilde{y} = y$ . Also,

$$\|x\| = \frac{\tilde{\varepsilon}}{\varepsilon} \|\hat{x}\| \leq \frac{\tilde{\varepsilon}}{\varepsilon} \sum_{i=0}^\infty \alpha^i \|x_i\| \leq \frac{\tilde{\varepsilon}}{\varepsilon} \frac{1}{1 - \alpha} < 1$$

□

**Corollary 5.6** (Consequence 1). *Let  $X, Y$  be Banach.  $T \in \mathcal{L}(X, Y)$ .  $T$  is bijective, then  $T^{-1} \in \mathcal{L}(Y, X)$ .*

**Corollary 5.7** (Consequence 2). *Let  $X, Y$  be Banach.  $T \in \mathcal{L}(X, Y)$ .  $T$  injective.  $\text{range}(T)$  closed  $\iff T^{-1} : \text{range}(T) \rightarrow X$  is linear and bounded.*

*Proof.*  $\implies$  Immediate since  $T \in \mathcal{L}(X, \text{range}(T))$  and  $\text{range}(T)$  is Banach.

$\impliedby$  Assume that  $T^{-1} : \text{range}(T) \rightarrow X$  is continuous. Let  $(x_n)_n$  in  $\text{range}(T)$  be Cauchy. Then  $(T^{-1}(x_n))_n$  is Cauchy. Let  $X$  be Banach, then  $\exists y \in X : T^{-1}(x_n) \rightarrow y$ . Let  $T$  be continuous, then  $x_n = T(T^{-1}x_n) \rightarrow Ty \implies x \in \text{range}(T)$  ( $x \in Y$  since  $Y$  is Banach).

□

**Corollary 5.8.** *Let  $X$  and  $Y$  be Banach.  $T \in \mathcal{L}(X, Y)$ .  $\text{range}(T)$  closed. Define  $\tilde{X} := X \setminus \text{kernel}(T)$ .  $\tilde{T} : \tilde{X} \rightarrow Y$  with  $[x] \mapsto Tx$ . Then  $\tilde{T}$  is well-defined.  $\tilde{T} \in \mathcal{L}(\tilde{X}, Y)$  injective.  $\text{range}(\tilde{T}) = \text{range}(T)$ ,  $\|\tilde{T}\| = \|T\|$  and  $\tilde{T}^{-1} : \text{range}(T) \rightarrow \tilde{X}$  is continuous and linear.*

*Proof.* The proof is left as an exercise to the reader. □

**Corollary 5.9.** *Let  $X$  be a vector space and  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be norms on  $X$  such that  $(X, \|\cdot\|_1)$  and  $(X, \|\cdot\|_2)$  are Banach and  $\exists M > 0 : \|x\|_1 \leq M \|x\|_2 \forall x \in X$ . Thus  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent.*

*Proof.* Apply Corollary 5.6 to  $\text{id} : (X, \|\cdot\|_2) \rightarrow (X, \|\cdot\|_1)$  with  $x \mapsto x$ . □

A second class of consequences: When is  $T : X \rightarrow Y$  linear already bounded?

**Definition 5.10.** *Let  $X$  and  $Y$  be normed.  $D \subset X$  be a subspace.  $T : D \rightarrow Y$  be linear.  $T$  is called closed if  $\forall (x_n)_n \in D : x_n \rightarrow x$  and  $Tx_n \rightarrow y$  it follows that  $x \in D, Tx = y$ .*

**Remark.** *This is weaker than continuity: Let  $T : X \rightarrow Y$  be linear. Continuity means that  $x_n \rightarrow x \implies Tx_n \rightarrow y$  and  $Tx = y$ . Closed means that  $x_n \rightarrow x$  and  $Tx_n \rightarrow y \implies Tx = y$ .*

**Remark.** *Differential operators are often closed but not continuous.*

*e.g.  $D := C_1([0, 1]) \subset L^\infty([0, 1])$ .*

$$T : \begin{matrix} D \rightarrow L^\infty([0, 1]) \\ f \mapsto f' \end{matrix}$$

*Then  $T$  is not closed but not continuous.*

*Not continuous: Define  $f_n(x) = \frac{1}{n} \cos(2\pi nx)$ . Then  $\|f_n\|_\infty = \frac{1}{n} \rightarrow 0$ , hence  $f_n \rightarrow 0 \in D$ , but*

$$\|Tf_n\|_\infty = \|Tf'_n\|_\infty = \|f'_n\|_\infty \geq \left| f'_n \frac{1}{8n} \right| = \left| 2\pi \sin\left(\frac{\pi}{2}\right) \right| = 2\pi$$

*and hence  $Tf_n \not\rightarrow Tf = 0$ .*

**Remark.**  $T : D \subset X \rightarrow Y$  linear,  $\text{graph}(T) := \{(x, Tx) \mid x \in D\} \subset X \times Y$ .

↓ This lecture took place on 2019/05/23.

**Lemma 5.11.** Let  $X$  and  $Y$  be normed and  $D \subset X$  be a subspace. Let  $T : D \rightarrow Y$  be linear. Then

- $\text{graph}(T) \subset X \times Y$  is a subspace.
- $T$  closed iff  $\text{graph}(T)$  is closed

*Proof.* • Immediate.

- $T$  closed  $\iff \forall (x_n)_n \in D, x \in X, z \in Y : x_n \rightarrow x. Tx_n \rightarrow y \implies x \in D \wedge Tx = y.$

$$\iff \forall (x_n, y_n) \in \text{graph}(T), (x, y) \in X \times Y : (x_n, y_n) \rightarrow (x, y) \implies (x, y) \in \text{graph}(T) \iff \text{graph}(T) \text{ is closed}$$

□

Closed operators are continuous in the right topology.

**Lemma 5.12.** Let  $X$  and  $Y$  be Banach spaces.  $D \subset X$  be subspaces and  $T : D \rightarrow Y$  be closed and linear. Then:

1.  $(D, \|\cdot\|_T)$  where  $\|\cdot\|_T := \|x\|_X + \|Tx\|_Y$  is Banach (graph norm)
2.  $T : (D, \|\cdot\|_T) \rightarrow Y$  is continuous.

*Proof.* 1.  $\|\cdot\|_T$  is indeed a norm.

Let  $(x_n)_n$  be Cauchy in  $D$  w.r.t.  $\|\cdot\|_T$ .

$\implies (x_n)_n$  and  $(Tx_n)_n$  are Cauchy sequences in  $X$  and  $Y$  respectively

Thus  $\exists x := \lim_{n \rightarrow \infty} x_n$  and  $y := \lim_{n \rightarrow \infty} Tx_n$ .  $T$  closed implies  $x \in D$  and  $Tx = y$ . Hence  $(x_n)_n \rightarrow x \in D$  for  $n \rightarrow \infty$  w.r.t.  $\|\cdot\|_T$ .

2.  $\forall x \in D : \|Tx\| \leq \|x\| + \|Tx\| \leq \|x\|_T$

Extension of the open mapping theorem for closed operators. □

**Lemma 5.13.** Let  $X$  and  $Y$  be a Banach space.  $D \subset X$  be a subspace.  $T : D \rightarrow Y$  be linear, closed and surjective.  $\implies T$  is open, in particular if  $T$  is injective,  $T^{-1} : Y \rightarrow D$  is continuous.

*Proof.* By Lemma 5.12,  $T : D \rightarrow Y$  is continuous wrt.  $\|\cdot\|_T$  in  $D$  and  $D, Y$  are Banach.

By Theorem 5.5,  $T$  is open from  $D$  to  $Y$ . Show that  $T$  is open wrt.  $\|\cdot\|_X$  and  $\|\cdot\|_Y$ . Take  $0 \subseteq D$  to be open wrt.  $\|\cdot\|_X$ . Because  $\|\cdot\|_X \subset \|\cdot\|_T \implies 0$  is open wrt.  $\|\cdot\|_T \implies T(0)$  is open in  $Y$ . □

**Corollary 5.14.** *Let  $X, Y$  be Banach.  $D \subseteq X$  be a subspace.  $T : D \rightarrow Y$  closed, linear and has closed range. Then with  $\tilde{D} := D \setminus \ker(T)$ .  $\tilde{T} : \tilde{D} \rightarrow Y$  with  $[x] \mapsto Tx$ . We get that  $\tilde{T}$  is bijective from  $\tilde{D}$  to  $\text{range}(T)$  and  $\exists \tilde{T}^{-1} : \text{range}(T) \rightarrow \tilde{D}$  and is continuous. In particular, if  $T$  is injective*

$$\implies T^{-1} : \text{range}(T) \rightarrow D \text{ is continuous}$$

*Proof.* Similar to the proof above. Thus this is left as an exercise to the reader.  $\square$

**Theorem 5.15.** *Let  $X$  and  $Y$  be Banach spaces.  $T : X \rightarrow Y$  be linear and closed. [e.g.  $X \subset \hat{X}$  with  $X$  closed,  $\hat{X}$  Banach,  $T : D := X \rightarrow Y$ ]  $\implies T$  is continuous*

*Proof.* By Lemma 5.12,  $T : (X, \|\cdot\|_T) \rightarrow (Y, \|\cdot\|_Y)$  is continuous and  $(X, \|\cdot\|_T)$  is Banach. Also  $\|x\|_X \leq \|x\|_T \forall x \in X$ . By Corollary 5.9,  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  are equivalent, thus  $\exists M > 0 : \|x\|_T \leq \|x\|_X \forall x \in X$ .

$$\implies \forall x \in X : \|Tx\|_Y < C \|x\|_T \leq CM \|x\|_X$$

$$\implies T \text{ is continuous wrt. } \|\cdot\|_X$$

$\square$

**Remark.** *For differential operators, the domain is usually not closed. ( $C^1$  is not closed in the  $C^0$ -norm)*

$\downarrow$  This lecture took place on 2019/05/28.

## 6 Dual spaces, reflexivity and weak convergence

**Remark.** *Obtain “Bolzano-Weierstrass” in infinite-dimensional spaces*

**Definition 6.1.** *Let  $X$  be a normed space. Then  $X^* := \mathcal{L}(X, \mathbb{K})$  is called the dual space of  $X$ . We denote  $\|x^*\|_{X^*} := \|x^*\|_{\mathcal{L}(X, \mathbb{K})}$ .*

**Corollary 6.2.** *Let  $X$  be a normed space. Then  $X^*$  is complete.*

**Lemma 6.3.** *Let  $X$  be normed. Then  $\forall x \in X \setminus \{0\} \exists x^* \in X^* : \|x^*\|_X = 1 \vee x^*(x) = \|x\|$ . In particular,*

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \implies \exists x^* \in X^* : x^*(x_1) \neq x^*(x_2)$$

*Proof.* Take  $x \in X, x \neq 0$  fixed. Define  $u^* : \text{span}(x) \rightarrow \mathbb{K}$  with  $\lambda x \mapsto u^*(\lambda x) := \lambda \|x\|$ . Then,

$$\|u^*\| = \sup_{\|\lambda x\| \leq 1} |u^*(\lambda x)| = \sup_{\|\lambda x\| \leq 1} |\lambda \|x\|| = \sup_{\|\lambda x\| \leq 1} \|\lambda x\| = 1$$

Also  $u^*(x) = \|x\|$ . By the Hahn-Banach Theorem, existence of  $x^*$ , as claimed, follows.

In particular, if  $x_1 \neq x_2$  we define  $x^* = x^*(x_1 - x_2) = \|x_1 - x_2\| \implies x^*(x_1) - x^*(x_2) \neq 0$ .  $\square$

**Lemma 6.4.** *Let  $X$  be normed. Then*

$$\forall x \in X : \|x\| = \sup_{\substack{x^* \in X^* \\ \|x^*\| \leq 1}} |x^*(x)|$$

*Proof.* Let  $x \in X$ . If  $x = 0$ , then trivial. If  $x \neq 0$ , then

$$\sup_{\|x^*\| \leq 1} \|x^*(x)\| \leq \sup_{\|x^*\| \leq 1} \|x^*\| \|x\| \leq \|x\|$$

Also,  $\exists \hat{x} \in X : \|\hat{x}\| = 1 \implies \hat{x}^*(x) = \|x\| \implies \sup_{\|x^*\| \leq 1} |x^*(x)| \geq |\hat{x}^*(x)| = \|x\|$   $\square$

**Lemma 6.5.** *Let  $X$  be normed.  $U \subset X$  be a closed subspace.  $x \notin U \implies \exists \hat{x} \in X^* : \hat{x}|_U = 0$  with  $\hat{x}^*(x) \neq 0$ .*

*Proof.* Define  $w : X \rightarrow X/U$  with  $x \mapsto [x] = \{y \in X \mid x - y \in U\}$

$$w(u) = 0 \forall u \in U, w(x) \neq 0$$

Choose  $l \in (X/U)^*$  such that  $l^*(w(x)) \neq 0$  and define  $x^* := l \circ w$ . Thus  $x^*(x) = l^*(w(x)) \neq 0$ .  $x^*(u) = l^*(0) = 0$ .  $\square$

**Lemma 6.6.** *Let  $X^*$  be normed.  $U \subset X$  be a subspace. TFAE:*

- $U$  is dense in  $X$
- $\forall x^* \in X^* : x^*|_U = 0 \implies x^* = 0$

*Proof.* (1)  $\rightarrow$  (2) Obvious by continuity.

(2)  $\rightarrow$  (1)  $\bar{U}$  is closed. If  $\bar{U} \neq X \implies \exists x^* \in X^* : x^*|_{\bar{U}} = 0$  and  $x^* \neq 0$ . This gives a contradiction.  $\square$

**Remark.** (2)  $\rightarrow$  (1) is often useful to show density

**Theorem 6.7.** *Let  $1 \leq p \leq \infty$ .  $a \in [1, \infty] : \frac{1}{p} + \frac{1}{a} = 1$  and  $(\Omega, \Sigma, \mu)$  be a  $\sigma$ -finite measure space. Define  $T : L^a(\Omega, \mathbb{K}^M, \mu) \rightarrow L^*p(\Omega, \mathbb{K}^M, \mu)^*$  with  $g \mapsto T_g$  with  $T^g(f) := \int_{\Omega} (f, g) d\mu$*

*Then  $T$  is well-defined, linear and isometric ( $\implies$  injective). If  $p < \infty$ , then  $T$  is surjective and  $L^p(\dots)^* \cong L^q$ .*



*Proof.* **Well-defined** Linear is obvious. By Hölder:  $|T_g(f)| = \left| \int_{\Omega} (f \cdot g) d\mu \right| \leq \|g\|_q \|f\|_p \implies T_g$  bounded and  $\|T_g\| \leq \|g\|_q$ .

Next, assume  $p < \infty$ . Show:  $T_g$  is surjective. Here we show the result only for  $L^p(\Omega, \mu)$  and  $|\mu(\Omega)| < \infty$  (the rest is left as an exercise to the reader). Take  $y^* \in (L^p)^*$ . Construct  $q \in L^q : T_q = y^*$ . We consider  $\nu : \Sigma \rightarrow \mathbb{K} : \nu(E) := y^*(\chi_E)$ . Then  $\nu(\emptyset) = y^*(0) = 0$ . Furthermore, for  $(E_i)_i$  in  $\Sigma$  pairwise-disjoint, we get that

$$\sum_{i=1}^n \chi_{E_i} = \chi_{\bigcup_{i=1}^n E_i} \rightarrow \chi_{\bigcup_{i=1}^{\infty} E_i} \text{ pointwise, } E := \bigcup_{i=1}^{\infty} E_i$$

Furthermore,

$$\int_{\Omega} \|X_{\bigcup_{i=1}^n E_i} - X_E\|^p d\mu \leq \int_{\Omega} |\chi_E|^p d\mu$$

Lebesgue dominated convergence theorem implies

$$\begin{aligned} X_{\bigcup_{i=1}^n E_i} &\rightarrow \chi_E \text{ in } L^p \\ \implies \sum_{i=1}^{\infty} u(E_i) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n v(E_i) = \lim_{n \rightarrow \infty} y^*(\chi_{\bigcup_{i=1}^n E_i}) = y^*(X_E) = v(E) \end{aligned}$$

Thus  $\nu$  is a complex measure. Also  $\mu(E) = 0 \implies \nu(E) = 0$ . The Radon-Nikodym Theorem (compare with Werner, Prop. A 4.6) implies  $\exists q \in L^q(\Omega, \mu)$  such that

$$u(E) = \int_E q d\mu = \int_{\Omega} \langle \chi_E, g \rangle d\mu$$

Hence:  $y^*(\chi_E) = \int_{\Omega} \langle \chi_E, g^* \rangle d\mu \forall E \in \Sigma$ .

By linearity,  $y^*(f) = \int_{\Omega} \langle f, g \rangle d\mu \forall f$  as step functions.

Now since  $\mu(\Omega) < \infty$ ,  $L^{\infty}(\Omega, \mu) \subset L^p(\Omega, \mu)$  (by Hölder).

Since  $\forall h \in L^{\infty} : \int_{\Omega} |h|^p = \int_{\Omega} |h| |h|^{p-1} = \|h\|_{\infty} \int_{\Omega} |h|^{p-1}$ .

$$\implies |y^*(f)| \leq \|y^*\| \cdot \|f\|_p \leq \|y^*\| \cdot c \cdot \|f\|_{\infty} \forall f \in L^{\infty}$$

Hence,  $y^* \in \mathcal{L}(L^{\infty}, \mathbb{K})$ . Also  $f \mapsto \int_{\Omega} \langle f, g \rangle d\mu$  for  $f \in L^{\infty}$  is continuous wrt.  $L^{\infty}$ -convergence since  $\forall f \in L^{\infty} : \int_{\Omega} \langle f, g \rangle d\mu \leq \|f\|_{\infty} \cdot \|g\|_1$  with  $\|g\|_1 < \infty$ .

$$\implies \forall f \in L^{\infty} : y^*(f) = \int_{\Omega} \langle f, g \rangle d\mu$$

by density of step functions, we know that from measure theory.

Now, show that  $g \in L^q$ .

**Case  $q < \infty$ :** Define

$$f(x) = \begin{cases} \frac{|g(x)|^q}{g(x)} & \text{if } g(x) \neq 0 \\ 0 & \text{else} \end{cases}$$

$$E_n := \{x \in \Omega \mid |q(x)| \leq n\} \implies X_{E_n} f \in L^\infty$$

Further:

$$\int_{E_n} |g|^q d\mu = \int_{\Omega} X_{E_n} \langle f, g \rangle d\mu = y^*(\chi_{E_n} f) \leq \|y^*\| \|\chi_{E_n} \cdot f\|_p = \|y^*\| \left( \int_{E_n} |f|^p d\mu \right)^{\frac{1}{p}} = \|y^*\| \left( \int_{E_n} |g|^{(q-1)p} d\mu \right)^{\frac{1}{p}}$$

with  $(q-1)p = q$  because  $p = \frac{q}{q-1}$ .

$$\implies \left( \int_{E_n} |g|^q d\mu \right)^{\frac{1}{q}} = \left( \int_{E_n} |g|^q d\mu \right)^{1 - \frac{1}{p}} \leq \|y^*\|$$

By the Beppo-Levi Theorem,

$$\left( \int_{\Omega} |g|^q d\mu \right)^{\frac{1}{q}} \leq \|y^*\|$$

Hence,  $g \in L^q$  and  $\|g\|_q \leq \|y^*\|$ .

**Case  $q := \infty$ :** Define  $E := \{x \in \Omega \mid |q(x)| > \|y^*\|\}$ .  $f := \chi_E \cdot \frac{|g|}{g} \in L^\infty$ . If  $\mu(E) > 0$ , then

$$\mu(E) \|y^*\| < \int_E |g| d\mu = \int_{\Omega} (fg) d\mu = y^*(f) \leq \|y^*\| \|f\|_1 = \|y^*\| \mu(E)$$

Gives a contradiction.  $\implies \mu(E) = 0$ ,

$$\implies |q(x)| \leq \|y^*\| \text{ almost everywhere}$$

$$\implies \|g\|_\infty \leq \|y^*\| \text{ and } q \in L^\infty$$

$$\int_{\Omega} \langle f, g \rangle d\mu = y^*(f) \forall f \in L^\infty \text{ and } g \in L^q$$

hence  $f \mapsto \int_{\Omega} \langle f, g \rangle d\mu$  is continuous on  $L^p$  (Hölder).

Hence,

$$\int_{\Omega} \langle f, g \rangle d\mu = y^*(f) \forall f \in \overline{C_C(\Omega)}^{L^p} = L^p(\Omega, \mu)$$

since  $C_C \subset L^\infty$ .

We know  $\|Tg\| \leq \|g\|_q$  and  $\|g\|_q \leq \|y^*\|$  with  $\|Tg\| = \|y^*\|$

$$\implies \|y^*\| = \|g\|_q$$

Final open point: Show that  $\|g\|_q \leq \|Tg\|$  for  $p = \infty$ .

$$\begin{aligned} \|Tg\| &= \sup_{\substack{f \in L^\infty \\ \|f\|_\infty \leq 1}} |T_g(f)| = \sup_{\|f\|_\infty = 1} \int_{\Omega} \langle f, g \rangle d\mu \\ &\geq \int_{\Omega} |g| d\mu = \|g\|_q = \|g\|_1 \end{aligned}$$

□

**Corollary 6.8.** Let  $p, q \in [1, \infty] : \frac{1}{p} + \frac{1}{q} = 1$ . Then

$$T_1 : l^q \rightarrow (l^p)^* \quad y \mapsto T_y(x) := \sum_{i=1}^{\infty} x_i \overline{y_i}$$

$$T_2 : L^q(\Omega) \rightarrow L^p(\Omega)^* \quad g \mapsto Tg(f) := \int_{\Omega} \langle f, g \rangle d\mu$$

are well-defined, isometric, linear and surjective if  $p < \infty$ .

**Theorem 6.9** (Riesz-Representation Theorem). Let  $K$  be a compact metric space. Then  $C(K)^* \cong M(K)$  where  $M(K)$  is the set of Radon measures, is regular, finite. Borel measures on  $K$  and  $T : M(K) \rightarrow C(K)^*$ .

*Proof.* Radin 1986, see the book in the literature list

□

↓ This lecture took place on 2019/06/04.

**Revision 6.10.**  $p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1 \implies (L^p)^* = (L^q)$

For which spaces does this hold true?

$$(L^q)^* \cong L^p \implies ((L^p)^*)^* \cong L^p$$

**Definition 6.11** (Proposition and definition). Let  $X$  be a normed space. We call  $X^{**} = (X^*)^*$  to the bidual space of  $X$ . Define

$$i = i_X : X \rightarrow X^{**} \quad x \mapsto i_X(x) : X^* \rightarrow \mathbb{K}$$

Then  $i_X$  is linear and isometric.  $x^* \mapsto i_X(x)(x^*) := x^*(x)$ . We call  $i_X$  to be the canonical embedding of  $X$  into its bidual space.

*Proof.* **Linearity** Show:  $i(\lambda x + z) = \lambda i(x) + i(z) \forall x, z \in X, \lambda \in \mathbb{K}$

$$x^* \in X^*.$$

$$\begin{aligned} i(\lambda x + z) &= x^*(\lambda x + z) \\ &= \lambda x^*(x) + x^*(z) \\ &= \lambda i(x) + i(z) \end{aligned}$$

Recall that  $x^* \in X^*$  and thus linear.

**Isometric**

$$\forall x \in X, x^* \in X^* : |i(x)(x^*)| = |x^*(x)| = \|x^*\| \|x\| \implies \|i(x)\| \leq \|x\|$$

Lemma 6.4: If  $x = 0$ , then  $\|i(x)\| = \|x\| = 0$ . If  $x \neq 0$ , then  $\exists \hat{x}^* \in X^* : \hat{x}^* \in X^* : \hat{x}^*(x) = \|x\|, \|\hat{x}^*\| = 1$ .

$$\implies \|i(x)\| = \sup_{\|x^*\| \leq 1} \|i(x)(x^*)\| \geq |i(x)(\hat{x}^*)| = |\hat{x}^*(x)| = \|x\|$$

□

**Remark.** Hence  $X$  can be identified with a subspace of  $X^{**}$ . In particular, if  $X$  is a Banach space  $i(X)$  is a closed subspace.

**Definition 6.12.** A Banach space  $X$  is called reflexive if  $i_X : X \rightarrow X^{**}$  is surjective.

**Remark.** • Dual spaces are always complete, hence only Banach spaces can be reflexive.

• We already know:  $L^p$  is reflexive for  $p \in (1, \infty)$

• Alternative definition:  $\overline{X \text{ reflexive}} \text{ iff } X \cong X^{**}$ .

$\Rightarrow$  Reflexive  $\Rightarrow \overline{\text{reflexive}}$  (requires a particular isomorphism).

$\Leftarrow$  Is not true. Our definition is far more common since it is useful to have the isometry explicitly.

**Corollary 6.13.** Let  $(\Omega, \Sigma, \mu)$  be a sigma-finite measure space.  $p \in (1, \infty)$ . Hence  $L^p(\Omega, \mathbb{K}^M, \mu)$  is reflexive, in particular  $L^p(\Omega)$ ,  $\mathbb{P}$  are reflexive.

**Proposition 6.14.** Let  $X$  be normed. Then

1. If  $X$  is reflexive,  $U \subset X$  a closed subspace  $\Rightarrow U$  is reflexive.

2. If  $X$  is Banach:  $X$  reflexive  $\iff X^*$  is reflexive.

*Proof.* 1. Take  $u^{**} \in U^{**}$ . Show:  $\exists u \in U : i_U(u) = u^{**}$ . The mapping  $x^* \mapsto u^{**}(x^*|_U)$  is in  $X^{**}$  since

$$|u^{**}(x^*|_U)| \leq \|u^{**}\|_{U^{**}} \|x^*|_U\|_{V^*} \leq \|u^{**}\|_{V^*} \|x^*\|_{X^*}$$

$X$  reflexive  $\Rightarrow \exists x \in X : i_X(x) = f$

$$\Rightarrow x^*(x) = u^{**}(x^*|_U) \forall x^* \in X^*$$

Show:  $x \in U$ . If  $x \notin U$ , then  $x^* \in X^* : x^*(x) = 1, x^*|_U = 0 \Rightarrow 1 = u^{**}(x^*|_U) = u^{**}(0) = 0$  gives a contradiction. Hence  $x \in U$ . Define  $u = x$ .

Show:  $u^*(u) = u^{**}(u^*|_U) \forall u^* \in U^*$ . Take  $u^* \in U^*$ . Take  $x^*$  to be an extension of  $u^*$  by Hahn-Banach.

$$\Rightarrow u^{**}(u^*) = u^{**}(x^*|_U) = x^*(u) = u^*(u) = i_U(u)(u^*) \Rightarrow u^{**} = i_U(u)$$

2. Assume that  $X$  is reflexive.

Show:  $i_{X^*} : X^* \rightarrow X^{***}$  is surjective.

Take  $x^{***} \in X^{***}$ . Define  $x^* : X \rightarrow \mathbb{K}$  and  $x \mapsto x^{***}(i_X(x))$ . Then  $x^* \in X^*$ .

Show:  $i_{X^*}(x^*) = x^{***}$ .

Since  $X$  is reflexive, any  $x^{**}$  can be written as  $x^{**} = i_X(x)$ .

$$\begin{aligned} \Rightarrow \forall x^{**} \in X^{**} : x^{***}(x^{**}) &= x^{***}(i_X(x)) = \\ x^*(x) &= i_X(x)(x^*) = x^{**}(x^*) = i_{X^*}(x^*)(x^{**}) \\ \Rightarrow x^{***} &= i_{X^*}(x^*) \end{aligned}$$

Now if  $X^*$  is reflexive, then  $X^{**}$  is reflexive.  $i(X) \subset X^{**}$  is reflexive as closed subspace of  $X^{**}$ . Hence  $X$  is reflexive.

□

**Proposition 6.15.** *Let  $X$  be normed.*

- If  $X^*$  is separable, then  $X$  is separable.
- If  $X$  is reflexive, then  $(X \text{ reflexive} \iff X^* \text{ is separable})$

**Remark.** 1. “ $\Leftarrow$ ” in the first item is not true since  $L^1$  is separable, but  $L^\infty = (L^1)^*$  is not separable.

2. By the second item,  $L^1$  is not reflexive, since otherwise  $L^\infty$  would be separable.

*Proof.* • follows from item 1

- Assume  $X^*$  is separable.

$$S_1^{X^*} = \{x^* \in X^* \mid \|x^*\| = 1\}$$

is separable as being a subset. Every subset of a separable set is separable (left as an exercise). Take  $(x_n^*)_n$  to be dense in  $S_1$ . For all  $n \in \mathbb{N}$  pick  $x_n \in X : \|x_n\| = 1$  and  $|x_n^*(x_n)| > \frac{1}{2}$ . Set  $U = \text{span}((x_n)_{n \in \mathbb{N}})$ .

Show:  $U$  is dense in  $X$ .

Let  $x^* \in X^*$  such that  $x^*|_U = 0$ .

Show:  $x^* = 0$  ( $\implies \bar{U} = X$ )

If not, wlog. assume  $\|x^*\| = 1$ .

$$\implies \exists x_{n_0}^* \in S_1^{X^*} : \|x^* - x_{n_0}^*\| < \frac{1}{4}$$

$$\implies \frac{1}{2} < |x_{n_0}^*(x_{n_0})| = |x_{n_0}^*(x_{n_0}) - x^*(x_{n_0})| \leq \|x_{n_0}^* - x^*\| \|x_{n_0}\| < \frac{1}{4} \cdot 1$$

Contradiction.

□

Fundamental difficulty in  $\infty$ -dimensional spaces. Closed and bounded does not imply sequentially compact. In particular, bounded sequences do not admit convergent subsequences in general.

*Solution:* A weaker notion of convergence.

**Definition 6.16.** *Let  $X$  be normed,  $(x_n)_n$  in  $X$ ,  $x \in X$ . We say  $x_n$  converges weakly to  $x$  (denoted  $(x_n) \rightharpoonup x$ ) iff*

$$x^*(x_n) \rightarrow x^*(x) \forall x^* \in X^*$$

**Remark.** • This is obviously weaker than norm-convergence (also called strong convergence)

- All  $x^* \in X^*$  are still sequentially continuous wrt. weak convergence. i.e.  $x^*(x_n)_n \rightarrow x^*(x) \quad \forall x_n \rightarrow x$

**Proposition 6.17.** *Let  $X$  be normed,  $(x_n)_n$  in  $X$ ,  $x \in X$ . Then*

1.  $(x_n)_n \rightharpoonup x \implies (x_n)_n \rightarrow x$
2. If  $(x_n)_n \rightarrow x$  then  $(x_n)_n$  is bounded
3. Weak limits are unique. i.e. if  $(x_n)_n \rightarrow x$  and  $(x_n)_n \rightarrow y \implies x = y$

*Proof.* 1. Immediate

2. follows from Lemma 6 below

3. Assume  $x_n \rightarrow x, x_n \rightarrow y$

$$\begin{aligned} &\implies \forall x^* \in X^* : x^*(x) = \lim_{n \rightarrow \infty} x^*(x_n) = x^*(y) \\ &\implies x^*(x - y) = 0 \forall x^* \in X^* \implies x - y = 0 \implies x = y \end{aligned}$$

□

**Remark.**  $\Leftarrow$  in item 1 does not hold true, e.g. with  $e_i = (0, \dots, 0, 1, 0, \dots)$  (the 1 is at the  $i$ -th position). We have that  $(e_n)_n \rightarrow 0$  in  $l^p$  for  $p \in (1, \infty)$ . Since  $\forall (u_n)_n \in l^q = (l^p)^*$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

$$(u_n)_n(e_m) = \sum_{n=1}^{\infty} u_n(e_m)_n = u_m \rightarrow 0 \text{ as } m \rightarrow \infty$$

since  $(u_m)_m \in l^q$ .

Hence  $(e_n)_n \rightarrow 0$  in  $l^p$ .

But  $\|e_n\|_p = 1 \forall n \implies (e_n)_n \not\rightarrow 0$

Remember that  $\forall v \in l^p$  and  $u \in (l^p)^*$  we write  $v = (v_n)_n$  and  $u = (u_n)_n \in l^q$ . Then  $u(v) = \sum_{n=0}^{\infty} v_n u_n$

**Lemma.** Let  $X$  be normed.  $M \subset X$ . TFAE

1.  $M$  is bounded
2.  $\forall x^* \in X^* : x^*(M) \subset \mathbb{K}$  is bounded

Also if  $X$  is Banach,  $M^* \subset X^*$ . TFAE:

- 1'.  $M^*$  is bounded
- 2'.  $\forall x \in X : \{x^*(x) \mid x^* \in M^*\}$  is bounded

*Proof.* 1  $\rightarrow$  2: Let  $c > 0 : \|x\| \leq c \forall x \in M$ .

$$\implies |x^*(x)| \leq \|x^*\| \cdot c$$

$$\implies x^*(M) \text{ bounded for } x^* \text{ fixed.}$$

2  $\rightarrow$  1: Consider  $i_X(x) \in X^*$  for  $x \in X$ . We have that  $\sup_{x \in M} |x^*(x)| = \sup_{x \in M} |i_X(x)(x^*)| < \infty \forall x^* \in X^*$  by assumption. By uniform boundedness principle,

$$\|x\| = \|i_X(x)\| < c < \infty \quad \forall x \in M$$

$\Rightarrow M$  bounded

1'  $\rightarrow$  2': True since  $|x^*(x)| \leq \|x^*\| \|x\| \leq C \|x\| \forall x^* \in M^*, x \in X$

2'  $\rightarrow$  1': Direct application of uniform boundedness principle.

□

**Theorem 6.18.** *Let  $X$  be reflexive. Then every bounded sequence in  $X$  admits a weakly-convergent subsequence.*

*Proof.* Take  $(x_n)_n$  be a bounded sequence in  $X$ . Assume first that  $X$  is separable. Hence  $X^*$  is separable, e.g.  $\exists (x_n^*)_n$  such that  $X^* = \overline{\{x_n^* \mid n \in \mathbb{N}\}}$

*Idea:* Construct subsequence  $(y_m)_m$  of  $(x_n)_n$  such that  $(x_i^*(y_m))_m$  converges  $\forall i \in \mathbb{N}$ .

**Claim.**  $\forall i \in \mathbb{N} \exists (x_{n_j^i})_j$  subsequence of  $(x_n)_n$  such that

1.  $(x_{n_j^i})_j$  is a subsequence of  $(x_{n_j^k})_j \forall k \leq i$
2.  $(x_k^*(x_{n_j^i}))_j$  is convergent  $\forall k \leq i$

*Proof by induction. Case  $i = 1$ :*

$$|x_1^*(x_n)| \leq \|x_1^*\| \cdot \|x_n\| \leq \|x_1^*\| C \quad \text{for } C > 0 : \|x_n\| \leq C \forall n$$

$$\Rightarrow (x_1^*(x_n))_n \text{ is bounded by } \mathbb{K} \Rightarrow \exists \text{ convergent subsequence } (x_{n_j^1})_j$$

**Case  $i \rightarrow i + 1$ :** Let  $(x_{n_j^i})_j$  be given as claimed. Again  $|x_{i+1}^*(x_{n_j^i})| \leq \|x_{i+1}^*\| C \Rightarrow \exists$  subsequence  $(x_{n_j^{i+1}})_j$  such that  $(x_{i+1}^*(x_{n_j^{i+1}}))_j$  is convergent. Subsequence implies that both assertions are true.

□

Now, we define  $y_j = x_{n_j^i} \forall j \in \mathbb{N} \Rightarrow (y_j)_j$  is a subsequence of  $(x_n)_n$ . Also, for  $k \in \mathbb{N}$ ,  $\lim_{i \rightarrow \infty} x_k^*(y_i) = \lim_{j \rightarrow \infty} x_k^*(y_j)$  exists.

Next:  $\forall x^* \in X^* : \lim_{j \rightarrow \infty} x^*(y_j)$  exists. Take  $\varepsilon > 0, x^* \in X^*$  pick  $i : \|x_i^* - x^*\| < \varepsilon$

$$\begin{aligned} \Rightarrow \forall n, m \in \mathbb{N} : & \|x^*(y_n) - x^*(y_m)\| \\ & \leq \|x^*(y_n) - x_i^*(y_n)\| + \|x_i^*(y_n) - x_i^*(y_m)\| + \|x_i^*(y_m) - x^*(y_m)\| \\ & \leq \|x^* - x_i^*\| \|y_n\| + \|x_i^*(y_n) - x_i^*(y_m)\| + \|x_i^* - x^*\| \|y_m\| \\ & \leq 2\varepsilon C + \|x_i^*(y_n) - x_i^*(y_m)\| \leq 3\varepsilon C \rightarrow 0 \text{ for } n, m \rightarrow \infty \end{aligned}$$

$\Rightarrow (x^*(y_n))_n$  is Cauchy, thus convergent.

Show:  $\exists y \in X : x^*(y_m) \rightarrow x^*(y) \forall x^* \in X^*$

Define  $l : X^* \rightarrow \mathbb{K}$  well-defined and linear with  $x^* \mapsto \lim_{n \rightarrow \infty} x^*(y_n)$ . Furthermore  $|l(x^*)| = \lim_{n \rightarrow \infty} |x^*(y_n)| \leq \|x^*\| c \Rightarrow l \in (X^*)^*$ .  $\Rightarrow \exists y \in X : i_X(y) = l$ . This means that  $\forall x^* \in X^* : x^*(y) = i_X(y)(x) = l(x^*) = \lim_{n \rightarrow \infty} x^*(y_n)$

$$\Rightarrow y_n \rightarrow y$$

Now without separability: Take again  $(x_n)_n$  to be bounded. Define  $Y := \text{span}((x_n)_n)$ .

Hence  $Y$  is separable, reflexive as closed subset of  $X$  (reflexive).  $x_n$  is a sequence in  $Y$ . Thus use the previous case.

$\Rightarrow \exists (y_n)_n$  subsequence of  $(x_n)_n, y \in Y$  such that  $x^*(y_n) \rightarrow x^*(y) \forall x^* \in Y^*$ .

For  $x^* \in X^*, x^*|_Y \in Y^* \Rightarrow x^*(y) \rightarrow x^*(y)$ .  $\square$

**Remark.** Further important question: When are closed sets also closed wrt. weak convergence?

Not always true! Remember that  $(l_n)_n$  in  $l^p : \|e_n\| = 1 \Rightarrow e_n = \{x \mid |x| = 1\}$  but  $e_n \rightarrow 0$

**Theorem 6.19.** Let  $X$  be normed,  $V \subset X$  closed and convex. Then  $\forall (x_n)_n$  in  $V$  such that  $x_n \rightharpoonup x \in X \Rightarrow x \in V$  (" $V$  is weakly closed").

In particular, any closed subspace is also weakly closed.

*Proof.* Assume  $x \notin V \Rightarrow \exists x^* \in X^* : x^*|_V = 0, x^*(x) \neq 0 \Rightarrow 0 = \lim_{n \rightarrow \infty} x^*(x_n) = x^*(x)$  gives a contradiction.  $\square$

**Remark** (Consequence).  $B_1(0)$  in  $X$  reflexive is weakly sequentially compact but not strongly sequentially compact if  $\dim(X) = \infty$ .

**Corollary 6.20.** Let  $X$  be normed.  $(x_n)_n$  in  $X$  such that  $x_n \rightharpoonup x \in X$ . Then there exists a sequence  $(y_n)_n$  where each  $y_n$  is a convex combination of the  $(x_n)_n$  s.t.  $y_n \rightarrow x$ .

**Remark** (i.e.).

$$\exists N^n, (\lambda_i^n)_{i=1}^{N^n} : \lambda_i^n \geq 0, \sum_{i=1}^{N^n} \lambda_i^n = 1 \text{ such that } y_n := \sum_{i=1}^{N^n} \lambda_i^n x_i \rightarrow x \text{ as } n \rightarrow \infty$$

*Proof.* Apply Theorem 6.19 to the closed, convex hull of  $\{x_n \mid n \in \mathbb{N}\}$ .  $\square$



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