Introduction to Functional Analysis

Lecture notes, University of Technology, Graz
based on the lecture by Martin Holler

Lukas Prokop

April 2, 2019

Contents

0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces U, V
  matrices $A \in \mathcal{M}^{n \times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
  functions $f : \mathbb{R}^n \to \mathbb{R} \mapsto$ functionals $f : U \to \mathbb{R}$

- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
  - basis of Applied Mathematics
  - partial differential equations
  - physical modelling
  - inverse problems (operator A models some physical measurement process)
  - Optimization and optimal control

A motivating example was presented with slides.

0.1 Application examples

Let $K : U \to \mathbb{R}^m$ with U as vector space describe a physical model. For example, K is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0,1] \to \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find u. Typically inversion of K is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$ where $\|z\|_2 \coloneqq \sqrt{\sum_{i=1}^n z_i^2}$ and $\|\cdot\|_u$ is a norm on U. Or alternatively, let $U = C^1([0,1]^2)$ and solve $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 \, dx}$.

Other examples are JPEG compression and upsampling of images.

0.2 Our first problem

Let $U \coloneqq C^1([0,1]^2)$ be a normed space, $K : U \to \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \|Ku - d\| + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 \, dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \implies u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \to \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ linear.

Proof. **Case 1 (trivial model)** Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \tag{1}$$

Take $(u_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim_{n \to \infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \, \forall n \in \mathbb{N}$

$$\implies \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i \in \mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \to u$ as $i \to \infty$.

Now: Show that u solves Problem (1). ∇ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \to \infty} \left\|\nabla u_{n_i}\right\| + \lambda \left\|u_{n_i} - d\right\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that $\hat{u}$ is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of ∇, $\|\cdot\|_2$

Case 2 (K arbitrary) 1. K arbitrary does not provide boundedness anymore. Define $X := \text{kernel}(\nabla) \cap \text{kernel}(k)$ and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^{n} x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \qquad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that $X^\perp$ is called *orthogonal complement*.

Claim 0.1. *If $\hat{u}$ solves* $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. *Then $\hat{u}$ solves Problem* (1).

Proof. Let $\hat{u}$ be a solution on $X^\perp$. Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^\perp$. Now we have:

$$\begin{aligned}
\|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \left\|\nabla(u_1 + u_2)\right\|_2 + \lambda \left\|k(u_1 + u_2) - d\right\|_2 \\
&= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\
&\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2
\end{aligned}$$

Thus $\hat{u}$ solves our problem (1). ∎

Take again $(u_n)_{n\in\mathbb{N}}$ be such that $u_n \in X^\perp \forall n$ and

$$\lim_{n\to\infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u\in X^\perp} \|\nabla_u\|_2 + \lambda \|ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \text{kernel}(\nabla) + \text{kernel}(\nabla)^\perp$. $\nabla : \text{kernel}(\nabla)^\perp \to \text{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \text{kernel}(\nabla)^\perp \implies v \in \text{kernel}(\nabla) \implies \|v_2\| = (v, v) = 0$. Thus, $\nabla^{-1} : \text{image}(\nabla) \to \text{kernel}(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^2\right\|_2 &= \left\|\nabla^{-1}\nabla u_n^2\right\|_2 = \left\|\nabla^{-1}\right\| \cdot \left\|\nabla u_n^2\right\|_2 \leq \left\|\nabla^{-1}\right\| \\
&\leq \left\|\nabla^{-1}\right\| \left(\left\|\nabla u_n^2\right\|_2 + \lambda \|Ku_n - d\|_2\right) \\
&= \left\|\nabla^{-1}\right\| \left(\underbrace{\|\nabla u_n\|_2}_{=\|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\
&< C \text{ for some } C > 0
\end{aligned}$$

Than $\left\|u_n^2\right\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \to \text{image}(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap \text{kernel}(\nabla) \implies v \in \text{kernel}(K)$. Hence $v \in \text{kernel}(K) \cap \text{kernel}(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$. Hence $K^{-1} : \text{image}(K) \to X^\perp \cap \text{kernel}(\nabla)$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^n\right\|_2 &= \left\|K^{-1}Ku_n^n\right\|_2 \leq \left\|K^{-1}\right\| \left\|Ku_n^n\right\|_2 \\
&= \frac{\|K\|}{\lambda} \left(\lambda \left\|K(u_1^n + u_2^n) - Ku_n^n\right\|_2 + \|\nabla u_n\|_2\right) \\
&\leq \frac{\|K\|}{\lambda} \left(\underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \left\|d - Ku_1^2\right\|}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\
&< D \text{ for some } D > 0
\end{aligned}$$

$$\implies (u_n^n)_n \text{bounded} \implies (u_n) = (u_n^n + u_n^n)_n \text{ is bounded}$$

$\implies (u_n)_n$ admits a subsequence converging to some $\hat{u}$. As in Case 1, $\hat{u}$ is a solution to Problem (1).

In summary,

1. $\min_{u\in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 \, dx}$ with $U = C^1([0,1]^2)$ relevant for application.

2. Discrete version: $\min_{u\in\mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:

(a) complementary subspaces $X^{\perp}$

(b) boundedness and compactness

(c) continuity

(d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ *This lecture took place on 2019/03/07.*

Define $U = C^1([0,1]^2)$. Let Y is some Banach space and $K : U \to Y$ is linear and continuous.

Consider the problem (P_∞) given by $\min_{u\in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2}$.

Proposition 0.2. *There exists a solution of* (P_∞).

Proof. Take $(u_n)_{n\in\mathbb{N}}$ as a sequence in U such that $\lim_{n\to\infty} \|\nabla u_1\|_2+\lambda \|Ku_n - d\|_n \to \inf_{u\in U}(\dots)$. Now we want to show that $(u_n)_{n\in\mathbb{N}}$ is bounded.

Case 1 Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\implies \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2+\lambda \|d\| \le \|\nabla u_n\|_2+\lambda \|u_n - d\|_2+\lambda \|d\| < C \text{ for } C > 0$$

$$\implies (u_n)_{n\in\mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n\in\mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces.*

So we change U to $U = \left\{u : [0,1]^2 \to \mathbb{R} \mid \sqrt{\int_{[0,1]^2}} < \infty\right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } v \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u\in U} R(u) + \lambda \|K_{u-d}\|_2$ instead.

In this setting, $(u_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i\in\mathbb{N}}$).

Our next step is to use continuity to show that $\hat{u}$ is a solution.

Problem: $u \mapsto \|u-d\|_2$ is, in general, not continuous with respect to weak convergence.

But it is always true that $\|\hat{u}-d\|_2 \leq \liminf_{i\to\infty} \left\|u_{n_i}-d\right\|_L$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \leq \liminf_{i\to\infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u\varphi = -\int_0^1 u\partial_x\varphi \forall\varphi \in C^\infty([0,1]^2)$$

$\varphi = 0$ in $K \setminus [0,1]^2$ for some $K \Subset (0,1)^2$.

$$\implies \int_{[0,1]^2} \nabla u\varphi = -\int_{[0,1]^2} u\cdot(\partial_{x_i}\varphi_1 + \partial_{x_2}\varphi_2)$$
$$\forall\varphi : (\varphi_1,\varphi_2) = C^\infty([0,1]^2,\mathbb{R}^2) + \text{ zero on boundary}$$

We define $w : [0,1]^2 \to \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w\varphi = -\int_{[0,1]^2} u(\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2) \text{ holds } \forall\varphi$$

Then w is called *weak gradient* of u. We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i\to\infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u}-d\| &\leq \liminf_{i\to\infty} R(u_{n_i}) + \liminf_{i\to\infty} \lambda \left\|u_{n_i}-d\right\|_2 \\ &\leq \liminf_{i\to\infty}\Big(R(u_{n_i}) + \lambda\left\|u_{n_i}-d\right\|_2\Big) \\ &= \inf R(u) + \lambda\|u-d\|_2 \end{aligned}$$

Case 2 Works as in the finite-dimensional setting using

- $X := \text{kernel}(A) \cap \text{kernel}(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C\|\nabla u\|_2 \,\forall u \in \text{kernel}(\nabla)^\perp$ is called *Poincare inequality.*

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normal spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contemplementary subspaces
8. Hilbert spaces

↓ *This lecture took place on 2019/03/12.*

Remark. *1. Literature: UGU, in particular: Biezis, Werner*

2. In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified

1 Topological and metric spaces

Remark (Motivation). *Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies*

Definition 1.1 (Topology). *Let X be a set and $\tau \subset \mathcal{P}(X) = \{$"set of subsets of X"$\}$. We say that τ is a* topology *on X if*

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. *For any collection of sets $(U_i)_{i\in I}$ with I as some index set. We have $U_i \in \tau \forall i \in I \implies \bigcup_{i\in I} U_i \in \tau$.*

(X, τ) *is called* topological space.

A set $U \subset X$ is called open *if $U \in \tau$ and is called closed if $U^C \in \tau$.*

Remark. *By the third property of topologies, $\bigcap_{i\in I} V_i$ is closed for any collection $(V_i)_{i\in I}$ of closed sets.*

Definition 1.2 (Metric). *Let X be a set, $d : X\times X \to \mathbb{R}$ be such that $\forall x, y, z \in X$*

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$

2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric on X *and (X, d) is called* metric space.

Definition 1.3 (Norm)**.** *Let X be a vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called* norm *if $\forall x, y \in X, \lambda \in \mathbb{K}$*

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\left\|x + y\right\| \leq \|x\| + \left\|y\right\|$

Then $(X, \|\cdot\|)$ is called normed space.

Remark. *If $\dim(x) < \infty$, all norms on X are equivalent.*

Example. 1. *Let X be a set then $\tau = \{\emptyset, X\}$ is a topology.*

2. *$(X, \mathcal{P}(X))$ is a topological space.*
3. *Define $S^{d-1} := \left\{x \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i^2 = 1\right\}$ and $d(x, y) := r$ where r is the length of the shortest connection between x and y on S^{d-1}. Then d is a metric on S^{d-1}*
4. *$X := \{u : [0, 1] \to \mathbb{R} \mid u$ is continuous$\}$ then $\|u\|_\infty := \sup_{x \in [0,1]} \left|u(x)\right|$ is a norm on X*
5. *$l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u$ and $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with $p \in [1, \infty)$ and $\left\|(x_i)_{i \in \mathbb{N}}\right\|_p := (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. Then $(l^p, \|\cdot\|_p)$ is a normed space (the proof will be done later).*

Remark.

$$l^\infty := \left\{(X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty\right\}$$

$$\left\|(X_i)_{i \in \mathbb{N}}\right\| = \sup_i |X_i|$$

Proposition 1.4. *Let X be a set.*

1. *If (X, d) is a metric space, define for $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ and $\tau = \{U \in \mathcal{P}(x) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \in U\}$. Then (X, τ) is a* topological space. *We say that τ is the topology induced by d and we have that $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$*
2. *If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \to \mathbb{R}$ with $(x, y) \mapsto \left\|x - y\right\|$. Then (X, d) is a metric space and d is called the metric induced by $\|\cdot\|$.*

Remark (Consequence). *Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.*

Definition 1.5. *Let (X, τ) be a topological space. $U \subset X$. $x \in X$.*

1. *U is called a neighborhood of x if $\exists V \in \tau - x \in X \subset U$: $\mathcal{U}(x)$ is defined as the set of all neighborhoods of x*
2.
   - *x is called* interior point *of U if $U \in \mathcal{U}$*
   - *x is called* adjacent point *of U if $\forall V \in \tau$ such that $x \in V : V \cap U \neq \emptyset$*
   - *x is called* cluster point *of U if it is an adjacent point of $U \setminus \{x\}$.*

   The third property is stronger.
3. *Notational conventions:*

$$\mathring{U} := \{x \in U \mid x \textit{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \textit{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \mathring{U}$$

Proposition 1.6. *Let (X, τ) be a topological space, $U \in X$. Then*

1. *U is open $\iff \mathring{U} = U$*
2. *U is closed $\iff \overline{U} = U$*
3. *$\mathring{U} = \bigcup_{\substack{V \in \tau \\ V \subset U}} V$ and $\mathring{U}$ is open ["$\mathring{U}$ is the largest open set in U"]*
4. *$\overline{U} = \bigcap_{\substack{V closed \\ U \subset V}} V$ and $\overline{U}$ is closed ["$\overline{U}$ is the smallest closed set containing U"]*

Proof. 3. $\subset$ Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V$

$\supset$ Let $x \in \bigcup_{\substack{V \in \tau \\ V \in U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \in U \implies x \in \mathring{U}$

$\mathring{U}$ is open because it is the union of open sets.

1. $\implies \mathring{U} \subset U$ by definition. U is open, so $U \subset \bigcup_{\substack{V \subset \tau \\ V \subset U}} V \overset{(3)}{=} \mathring{U}$
2. $\implies V \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \implies x \in U^C \in \tau$ and $U \cap U^C = \emptyset$. This contradicts to $x \in \overline{U}$.

   $\impliedby$ Take $x \in U^C = \overline{U}^C$.

   $\overset{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$

   $\implies V \cap U = \emptyset \implies V \subset U^C$

$\implies U^C$ open $\implies U$ closed

4. We prove the fourth property without the second.

   $\subset$ Take $x \in \overline{U}$. Take closed V such that $U \subset V$ if $x \notin V \implies x \in V^C$ which is open and $V^C \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.

   $\supset$ Take $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}}$. Suppose $x \notin \overline{U}$.

   $\implies \exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$

   $\implies U \subset Z^C$, Z^C closed, $x \notin Z^C$. This contradicts to $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$

   $\overline{U}$ closed follows since the intersection of closed sets is closed.

$\square$

Definition 1.7 (Limit)**.** *Let (X, τ) be a topological space, $(X_n)_{n \in \mathbb{N}}$ be a sequence in X. Henceforth, we write $(X_n)_n$ for $(X_n)_{n \in \mathbb{N}}$ and $\hat{x} \in X$. We say $x_n \to x$ in τ as $n \to \infty$ ("x_n converges to x", "x is limit of x_n") if*

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

Definition 1.8 (Proposition and definition)**.** *Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or Hausdorff) if*

$$\forall x, y \in X \text{ with } x \neq y \, \exists U, V \in \tau : x \in U, v \in V \text{ and } U \cap V = \emptyset$$

- *In a T_2-sphere, the limit of any sequence is unique.*
- *If τ is induced by a metric, then (X, τ) is T_2.*

Proof. 1. Take $(x_n)_n$ to be a sequence and assume x_n converges to $\hat{x}$ and $\hat{y}$ with $\hat{x} \neq \hat{y}$. By T_2, $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergenc, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\left\{n_x, n_y\right\} : x_i \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x, y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology τ. Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

$\square$

Definition 1.9. *Let (X, τ) be a topological space, $U \subset V \subset X$. We say that U is* dense *in V, if $V \subset \overline{U}$. We say that X is* separable *if there exists a countable, dense subset.*

Definition 1.10. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \to Y$ a function. We say f is* continuous *at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. f is called* continuous *if it is continous at any $x \in X$.*

Proposition 1.11. *With $(X, \tau_X), (Y, \tau_Y)$ and f as above, f is continuous $\iff$ $f^{-1}(V) \in \tau_X \forall V \in \tau_Y$*

Proof. Left as an exercise to the reader. □

Definition 1.12. *Let (X, τ) be a T_2 topological space, $M \subset X$ called* compact *if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ ("$(U_i)_{i \in I}$ is an open covering of M"), there exists $U_{i_1}, \dots, U_{i_n}$ such that $M \subset \bigcup_{k=1}^{n} U_{i_k}$ ("there exists a finite subcover").*

Remark. *Compactness can also be defined without T_2, this is also referred to as* quasi-compact.

Remark (Exercise)**.** *Reconsider the previous results for metric and normed spaces.*

↓ *This lecture took place on 2019/03/14.*

Definition 1.13. *Let (X, d) be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in X. Then we say,*

1. *V is* bounded *if $\exists x \in X, r > 0$ such that $U \in B_r(x)$*
2. *$(x_n)_n$ is a* Cauchy sequence *if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$*
3. *X is* complete *if any Cauchy sequence in X admits a limit point*
4. *X is a* Banach space *if it is a normed space and complete*

Proposition 1.14. *Let (X, d) be a metric space. $(x_n)_n$ be a sequence in X. Then*

1. *$x_n \to x$ in the induced topology $\iff$ $\forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$*
2. *If $x_n \to x$, then $(x_n)_n$ is bounded as subset of X and $(x_n)_n$ is Cauchy.*
3. *If $U \subset X$ is closed and X is complete. Then (U, d) is a complete metric space.*

Proof. 1. We prove both directions:

$\Longrightarrow$ True, since $B_\varepsilon(x)$ is open $\forall \varepsilon 0$

$\Longleftarrow$ Let $x \in V$ with V open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$
V open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$
$\Longrightarrow \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1,\dots,n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\Longrightarrow y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in U, then $(y_n)_n$ is a Cauchy sequence in $X \Longrightarrow \exists x \in X : y_n \to x$ as $n \to x$ if $x \notin U \Longrightarrow x \in U^C \Longrightarrow \exists n_0 \in N$ such that $y_{n_0} \in U^C$ due to U^C open. This is a contradiction to $(y_n)_n$ in U

□

Proposition 1.15. *Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \to Y$. The following are equivalent (TFAE):*

- *f is continuous (with respect to the induced topology)*
- *$\forall (X_n)_n$ such that $x_n \to x \Longrightarrow f(x_n) \to f(x)$*

Proof. Firstly, we prove that the first statement implies the second statement.

Take $(x_n)_n$ converging to x. Take $V \in \tau_y$ such that $f(x) \in V \Longrightarrow V \in \mathcal{U}(f(x))$

$\Longrightarrow \exists U \in \mathcal{U} : f(U) \subset V \Longrightarrow \exists \hat{U} \in \tau_X$ such that $x \in \hat{U} \subset U$

$\Longrightarrow \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \Longrightarrow \forall n > n_0 : f(x_n) \in V \Longrightarrow f(x_0) \to f(x)$

Remark. 1. $\Longrightarrow$ 2. *holds true in any topological space*

2. $\Longrightarrow$ 1. *Not.*

Secondly, we prove that the second statement implies the first statement.

Suppose f is not continuous, find $x_n \to x$ such that $f(x_n) \to f(x)$ is wrong. If f is not contiuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(u) \not\subset V \forall U \in \mathcal{U}(x)$

$$\Longrightarrow \exists \hat{V} \in \tau_Y \text{ such that } f(u) \not\subset \hat{V} \forall U \in U(x), f(x) \in \hat{V}$$

$$\Longrightarrow \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\Longrightarrow$ $(x_n)_n$ converges to x but $f(x_n) \notin \hat{V} \Longrightarrow f(x_n) \not\to f(x)$. This gives a contradiction. □

Definition 1.16. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. Let* $f : X \to Y$.

f *is* uniformly continuous *iff*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proposition 1.17. *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces.* $M \subset X$, $f : M \to Y$. *If* M *is dense in* X, Y *is complete and* f *is uniformly continuous.*

$$\implies \exists! \hat{f} : X \to Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

Proof. Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \to x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(y_1), f(y_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(y_n), f(x_n)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in $M, \hat{y} \in Y$ such that $f(\hat{x}_n) \to \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X, (x_n)_n$ is a sequence in M such that $x_n \to x, f(x_n) \to y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$:

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d(f(x_n), f(x)) < 3\varepsilon$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.
2. $\hat{f}$ is uniformly continuous.

$\square$

↓ *This lecture took place on 2019/03/19.*

Proposition 1.18. *Let* (X, d) *be a metric space,* $M \subset X$.

1. *M is compact, so* $\forall (X_i)_{i \in I}$ *with* X_i *a closed set* $\forall i$ *such that* $\bigcap_{i \in I} X_i \cap M = \emptyset$.

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \bigcap_{i=1}^{n} X_{ij} \cap M = \emptyset$$

2. *M is compact, so M is closed and bounded.*

Proof. 1. We note that $\forall (X_i)_{i\in I}$ is a family of closed sets. $(X_i^C)_{i\in I}$ is a family of open sets and $\bigcap_{i\in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i\in I} X_i^C$

2. Is a special case of the next proposition.

□

Proposition 1.19. *Let (X,d) be a metric space, $M \subset X$. TFAE:*

1. *M is compact.*
2. *Every infinite subset of M admits a cluster point.*
3. *Every sequence of M admits a convergent subsequence.*
4. *M is complete and totally bounded, where totally bounded is defined as*

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

Remark. 1. *totally bounded $\implies$ bounded (proof is left as an exercise)*

2. *If $\dim(x) < \infty$, then compact $\iff$ complete and bounded (see course Analysis I)*

3. $\dim(x) < \infty \iff \overline{B_1}(0)$ *is compact*

where the last two items imply that X is a normed space.

Proof. $1 \to 2$ If M is finite, (2) always holds true. So assume that M is infinite. Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists x in C$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(X_n)_n$ is a sequence of distinct points and $x_n \to x$. This implies that x is a cluster point of C. This gives a contradiction.

Now $M \subset \bigcup_{x\in M} B_{\varepsilon_x}(x)$. If M is compact, then

$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$
$\implies C \subset M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$
$\implies C$ is finite

This is a contradiction.

$2 \to 3$ Let $(x_n)_n$ be a sequence in M.

Case 1 $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies (x_n)_n$ admits a convergent sequence.

Case 2 $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid x \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

$3 \to 4$ Suppose that M is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^{n} B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in M as follows: Given $x_1, \dots, x_n$, choose $x_1 \in M$ arbitrary and $x_{i+1} \in M \setminus \bigcup_{j=1}^{i} B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in M and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convenient subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in M, then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \to x$ as $i \to \infty$. Since $(x_n)_n$ is Cauchy, $x_n \to x$ as $n \to \infty$ [left as an exercise]. Thus M is complete.

$4 \to 1$ Let $(U_i)_{i \in I}$ be an open covering of M and assume that $(U_i)_{i \in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}(a)}$. Define $\Omega := \left\{ \tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } (U_i)_i \right\}$. We recursively define a sequence $(a_n)_n$ in M such that

$$\forall n \in \mathbb{N} : a_n \subset E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n} \cap B_{\frac{1}{2^{n-1}}}}(a_{n-1}) \neq \emptyset$$

Goal: Show $(a_n)_n \to a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

Step 1 $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a \in C_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \to n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let $\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}$.

Since $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$. [Take $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$, but if $B_{\frac{1}{2^{n-1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{a^{n+1}}}(a)$$

Hence there exists a_{n+1} such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$ and $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

Step 2 Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) \leq d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. M is complete, so $\exists a \in M : a_n \xrightarrow{n\to\infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} and \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for n sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

Proposition 1.20. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$ compact. Let $f : X \to Y$ be continuous. Then*

1. *$f(M)$ is compact*
2. *$f|_M : M \to Y$ is uniformly continuous.*

Proof. 1. Let $(U_i)_{i\in I}$ be an open covering of $f(M)$

$\implies$ $(f^{-1}(U_i))_{i\in I}$ is an open covering of M [why!]
$\implies$ $\exists c_1, \ldots, c_n$ such that $M \subset \bigcup_{i=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{i=1}^n U_{i_j}$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n, (y_n)_n$ sequences in M satisfying condition (*). M is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i\to\infty} 0$$

□

↓ *This lecture took place on 2019/03/21.*

Proposition 1.21 (Proposition and definition)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. $g : X \to Y$ is a function. g is called* Lipschitz continuous *if* $\exists L > 0$ *such that $d_Y(\varphi(x), \varphi(y)) \leq L d_X(x, y) \forall x, y \in X$. Any Lipschitz continuous function is uniformly continuous.*

Proof. Left as an exercise to the reader. □

Theorem 1.22 (Arzelà-Ascoli theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Define $C(X,Y) := \{f : X \to Y \mid f$ continuous$\}$ and $d_C(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$. Then*

1. *d_C is well-defined and $(C(X,Y), d_C)$ is a complete metric space*
2. *A set $M \subset C(X,Y)$ is compact iff*
   (a) *$\forall x \in X$ the set $M_X := \{f(x) \mid f \in M\}$ is compact*
   (b) *M is* equicontinuous, *i.e.* $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. 1. Show that: $d_C(f,g) < \infty$.

Pick $f, g \in C(X,Y)$. Because X is compact, $f(X), g(X)$ compact $\implies$ $f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that d_C is a metric is left as an exercise.

Show that $(C(X,Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X,Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because Y is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n\to\infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to d_C: Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x)) \leq \lim_{m\to\infty} d(f_m, f_n) < \infty$ (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X,Y) \implies f_n \to f$ with respect to d_C. Show that $f \in C(X,Y)$. Take $\varepsilon > 0$. Let n_0 such that $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x,y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x,y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x))$.
In general, we have $\forall x, y, z \in (Z, d_Z)$ with d_Z as a metric.

$$\left|d(x,z) - d(y,z)\right| \leq d(x,y)$$

Proof.

$$d(x,z) \leq d(x,y) + d(y,z) \implies d(x,z) - d(y,z) \leq d(x,y) \tag{2}$$
$$d(y,z) \leq d(y,x) + d(x,z) \implies d(y,z) - d(x,z) \leq d(x,y) \tag{3}$$
$$(2) \text{ and } (3) \implies \left|d(x,z) - d(y,z)\right| \leq d(x,y) \tag{4}$$

□

Consequently, $\forall z \in Z, x_n \to x$ in Z: $d(x_n, z) \to d(x,z)$ since $\left|d(x_n,z) - d(x,z)\right| \leq d(x_n, x) \to 0$.

2. We need to prove both directions.

$\implies$ (a) For $x \in X$ fixed, define $g_X : M \to Y$ with $f \mapsto f(x)$. Then $d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2)$
$\implies$ g_X is Lipschitz continuous, in particular continuous
$\implies$ $M_X = g_X(M)$ compact

(b) Take $\varepsilon > 0$. M is totally bounded, so $\exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$. $\forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x,y) < \delta_i \implies d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$. Define $\delta := \min_i \delta_i > 0$. Then $\forall x, y \in X$: $d(x,y) < \delta$ and $\forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0})$

$$\implies d(f(x), f(y)) \leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon$$

$\impliedby$ We prove the other direction.

↓ *This lecture took place on 2019/03/26.*

B is complete since it is a closed subset of a Banach space.
Show: M is totally bounded.
Consider $\varepsilon > 0$. Show: $\exists f_1, \dots, f_n$ such that $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$.

- Because M is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x,y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X, $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{i=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{i_i})$

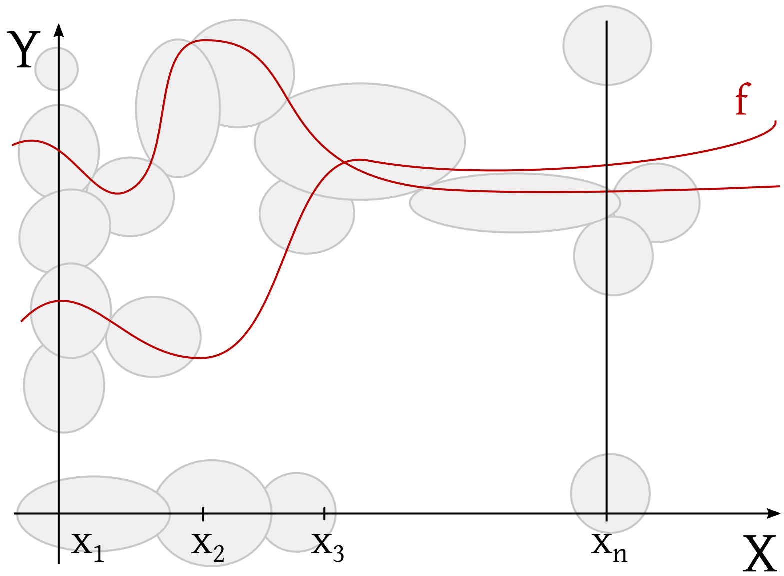

Figure 1: Covering of a function graph

Compare with Figure 1.
Now, for each tuple of indices $(y_{1,j_1}, \ldots, y_{n,j_n})$ define $f_{y_{1,j_1},\ldots,y_{n,j_n}} \in C(x,y)$ to be such that $f_{y_{1,j_1},\ldots,y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an f exists. The set F of all such functions is finite. We show that $M \subset \bigcup_{q\in F} B_\varepsilon(q)$.
Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \ldots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_\alpha \in F$ accordingly.
Take $x \in X$ arbitrary and x_i such that $x \in B_\delta(x_i)$

$$\begin{aligned}\implies d(f(x), f_\alpha(x)) &\leq d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_\alpha) &= \sup_{x\in X} d(f(y), f_\alpha(x)) < \varepsilon\end{aligned}$$

□

Remark. *Compare this to the fact that $B_1(0)$ in $C(X,Y)$ is* not *compact.*

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

Remark (Motivation)**.** *In general, let (X,d) be a metric space. Let A and B be open and dense, then also $A \cap B$ is dense.*

Proof. Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] = \emptyset$. Take $x \in Y, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. A is dense. A is open, so $\exists \varepsilon > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. B is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_\varepsilon(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally, $\forall A_1, \dots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many A_i?

Theorem 1.23 (Blaire theorem)**.** *Let (X, d) be a complete metric space. Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.*

Proof. Let $D := \bigcap_{n\in\mathbb{N}} O_n$. Show that for $x \in X$, $\varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n\in\mathbb{N}}$.

n = 1 Take x_1, ε_1 such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

n − 1 → n Given $x_{n-1}, \varepsilon_{n-1}$, take x_n, ε_n such that

$$\overline{B_{C_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_N}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete } \implies \exists x \in X : x_n \to x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_N}}(x_N) \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_n)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

Definition 1.24. *Let (X, d) be a metric space, $M \subset X$. We say*

- *M is* nowhere dense*(dt. "Nirgends dicht"), if $\overset{\circ}{\overline{M}} = \emptyset$*
- *M is of* first category $\iff$ *M is the countable union of nowhere dense sets*
- *M is of* second category $\iff$ *M is not of first category*

Theorem 1.25 (Blaire category theorem (weaker version))**.** *Let (X,d) be a complete metric space. Then (X,d) is of second category.*

In otheor words (which is a useful formulation): If $X = \bigcup_{n\in\mathbb{N}} C_n \implies \exists n_0 :$ $\overset{\circ}{\overline{C}} \neq \emptyset$. In particular, if

$$X = \bigcup_{n\in\mathbb{N}} C_n \text{ with } C_n \text{ closed } \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

Proof. Suppose that $X = \bigcup_{n\in\mathbb{N}} O_n = \bigcup_{n\in\mathbb{N}} \overline{O_n}$ with $\overset{\circ}{\overline{O}}_n = \emptyset \forall n$

$$\overset{\circ}{\overline{O}}_n = \emptyset \implies \overline{\overline{O}_n^C} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overline{O}_n^C = \emptyset \implies B_\varepsilon(x) \subset \overline{O}_n \implies \overset{\circ}{\overline{O}}_n \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O}_n^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O_n}^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n\in\mathbb{N}} \overline{O}_n^C \text{ is dense}$$

$$\bigcap_{n\in\mathbb{N}} \overline{O}_n^C = \left(\bigcup_{n\in\mathbb{N}} \overline{O}_n\right)^C = X^C = \emptyset$$

gives a contradiction ◻

Remark.
1. *This is a fundamental theorem in Functional Analysis*
2. *This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise)*

2 Normed space

2.1 Fundamentals

Definition 2.1. *Let X be a vector space. A function $\|\cdot\| : X \to [0,\infty)$ is called* seminorm *if*

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, yinX$

The first property differs between a norm and a seminorm.

The tuple $(X, \|\cdot\|)$ *is called a* semi-normed space. *We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.*

Definition 2.2 (Definition and proposition)**.** *Let* $(X, \|\cdot\|)$ *be a semi-normed space and* $(x_n)_n$ *be a sequence in* X*. We say thet*

- $\sum_{n=1}^{\infty} x_n$ converges *to* $x \in X$ *and write* $x = \sum_{n=1}^{\infty} x_n$ *if* $\lim_{m\to\infty} \sum_{n=1}^{m} x_n = x$
- $\sum_{n=1}^{\infty} x_n$ *is* absolutely convergent *if* $\sum_{n=1}^{\infty} \|x_n\|$ *converges [* $\iff$ $(\sum_{n=1}^{m} \|x_n\|)_m$ *is bounded]*

It holds that X *is complete iff any absolutely converging series converges.*

Proof. $\implies$ Take $m_1 < m_2$ arbitrary, then

$$\left\|\sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n\right\| \leq \sum_{n=m_1+1}^{m_2} \|x_1\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\| \leq \left\|\sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\|\right\|$$

$$\implies \left(\sum_{n=1}^{m} x_n\right)_m \text{ is Cauchy } \implies \text{ convergent}$$

$\impliedby$ Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick N_k such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \left\|x_{n_{k+1}} - x_{n_k}\right\| \leq 2^{-k}$$

Define $y_k := x_{n_{k-1}} - x_{n_k} \implies \sum_k \left\|y_{n_w}\right\| \leq \sum_k 2^{-k} < \infty$

$$\implies \exists y \in X : \sum_{k=1}^{n} y_k \to y \text{ as } n \to \infty$$

$$\sum_{k=1}^{n} y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \to y - x_{n_1} \text{ as } n \to \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is convergent.

□

Remark. *In* $\mathbb{R}^n$*,* $\sum_n x_n$ *is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction* $\implies$ *is true.*

↓ *This lecture took place on 2019/03/28.*

Proposition 2.3 (Proposition and definition)**.** *Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if*

$$\exists m, M > 0 \forall x \in X : m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

TFAE:

1. $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent.*
2. *For any sequence* $(x_n)_n$ *and* $x \in X, x_n \to x$ *with respect to* $\|\cdot\|_1 \iff x_n \to x$ *with respect to* $\|\cdot\|_2$
3. *For any sequence* $(x_n)_n$ *we have,*

$$x_n \to 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \to 0 \text{ with respect to } \|\cdot\|_2$$

Proof. (1) $\implies$ (2) $\implies$ (3) is immediate.

It remains to show that:

(3) $\implies$ (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \; \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n \|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\left\|y_n\right\|_1 = \frac{1}{n} \to 0$ hence $y_n \to 0$, but $\left\|y_n\right\|_2 > n \left\|y_n\right\|_1 = 1$.

$$\implies y_n \not\to 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

Remark. *If* $\dim(X) < \infty$, *then any two norms on* X *are equivalent.*

Definition 2.4 (Quotient spaces)**.** *Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation "$\sim$" on X with $x \sim y :\iff x - y \in Y$.*

Then $\sim$ defines an equivalence relation on X. We define

- $[X]_\sim = \{y \in X \mid x \sim y\}$ *...the equivalence class of* $x \in X$
- $X/Y := \{[x]_\sim \mid x \in X\}$ *...the quotient space*
- $\pi : \begin{cases} X \to X/y \\ x \mapsto [x]_\sim \end{cases}$

Defining $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \qquad \hat{0} := [0]$$

We get that:

1. *X/Y is a vector space*
2. *$\left\|[x]\right\|_{X/Y} := \inf_{y \in [x]} \left\|y\right\|_X$ is a semi-norm.*
3. *If Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.*
4. *If X is complete and Y closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.*

Proof.
- Equivalence relation
- Vector space with "+" and "$\lambda[x]$" are well-defined

This is left as an exercise to the reader.

2. – First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\left\|[0]\right\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|Y\| \leq \|0\| = 0$$

   – Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$.
   Show that: $\left\|\lambda[x]\right\|_{X/Y} = |\lambda| \left\|[x]\right\|_{X/Y}$.
   Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\left\|\lambda[x]\right\|_{X/Y} = \left\|[\lambda x]\right\|_{X/Y} = \inf_{y \in [\lambda x]} \left\|y\right\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \left\|y\right\| = \inf_{w \in [x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\left\|[x]\right\|_{X/Y}}$$

   – Take $[x_1], [x_2] \in X/Y, \varepsilon > 0$. We note that

$$\left\|[x]\right\|_{X/Y} = \inf_{\substack{y \in X \\ w \in Y \\ w := x \cdot y}} \left\|y\right\| = \inf_{w \in Y} \|x - w\|$$

   Hence we can take $y_1, y_2 \in Y$ such that $\left\|x_1 - y_i\right\| < \left\|[x_i]\right\|_{X/Y} + \varepsilon \qquad \varepsilon \in [1, 2)$.

$$\Longrightarrow \left\|[x_1] + [x_2]\right\|_{X/Y} = \left\|[x_1 + x_2]\right\|_{X/Y} \leq \left\|x_1 + x_2 - (y_1 + y_2)\right\|$$
$$\leq \left\|x_1 - y_1\right\| + \left\|x_2 - y_2\right\| \leq \left\|[x_1]\right\|_{X/Y} + \left\|[x_2]\right\|_{X/Y} + 2\varepsilon$$

   Since ε was arbitrary, the assertion follows.

3. Suppose Y is closed if $\left\|[x]\right\|_{X/Y} = 0$, then

$$\inf_{y \in Y} \left\|x - y\right\| = 0 \implies \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n\to\infty} \left\|x - y_n\right\| = 0$$

$$Y \text{ closed } \implies x \in Y \implies [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in X/Y and suppose that $\sum_{i=1}^{\infty} \left\|[x_n]\right\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, X/Y is complete.
   Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \left\|[x_n]\right\|_{X/Y} + 2^{-n}$

$$\implies \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} \left(\left\|[x_n]\right\|_{X/Y} + 2^{-n}\right) < c < \infty$$

$$X \text{ complete } \implies \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \qquad \left\|[x] - \sum_{n=1}^{m} [x_n]\right\|$$

□

↓ *This lecture took place on 2019/04/02.*

Corollary 2.5. *Let X be a vector space with semi-norm $\|\cdot\|_X : X \to [0, \infty)$. Then*

- $N = \{x \in X \mid \|x\|_X = 0\}$ *is a subspace at* X
- $\left\|[X]\right\| := \|X\|_p$ *is a norm on* X/N
- *If X is complete, then $(X/N, \|\cdot\|)$ is a Banach space.*

Proof. The proof is left as an exercise. □

Proposition 2.6. *Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then*

- $\overline{U}$ *is also a subspace.*
- *X is separable iff $\exists A \subset X$ complete such that $X = \overline{\mathcal{L}(A)}$ where $L(A) = \{\sum_{i=1}^{n} \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathcal{K}, n \in \mathbb{N}\}$*

Proof.
- Left as an exercise
- $\implies$ True since $\exists A \subset X$ countable such that $\overline{A} = X \implies \underline{X} = \overline{A} \subset \overline{L(A)} \subset X$

$\Longleftarrow$ Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^{n} (\lambda_i + i\mu_i) x_i \mid \lambda_i, \mu_i \in \mathbb{X}, x \in A, n \in \mathbb{N} \right\}$$

where i is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then B is countable.
Show: $\forall x \in X \forall \varepsilon \exists x \in B : \left\| x - y \right\| < \varepsilon$.
Take $x \in X, \varepsilon > 0 \implies \exists x_0 \in \mathcal{L}(A) : \|x - x_i\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=0}^{n} (\lambda_i + i\mu_i) x_i$ with $\lambda_i, \mu_i \in \mathbb{R}, x_i \in A$. Choose $\lambda', \mu_i' \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda_i')^2 + (\mu_i - \mu_i')^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^{n} \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let $y := \sum_{j=1}^{n} (\lambda_i' + \mu_i') x_i \subset B$.

$$\begin{aligned}
\implies \left\| x - y \right\| \leq \|x - x_0\| + \left\| x_0 - y \right\| & \qquad\qquad \leq \frac{\varepsilon}{2} \\
\leq \sum_{i=1}^{n} \left| (\lambda_i + i\varepsilon_i) - (\lambda_i' + i\mu_i') \right| \|x_i\| & \\
\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^{n} \|x_i\|} = \varepsilon &
\end{aligned}$$

□

Proposition 2.7 (Proposition and definition). *Let $(X, \|\cdot\|_{x_i})$ for $i = 1, \dots, n$ be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \ldots \otimes X_n = \bigotimes_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

For $p \in [1, \infty]$, define

$$\left\| (x_1, \dots, x_n) \right\|_{\otimes_i X_i, p} = \begin{cases} \left(\sum_{i=1}^{n} \|x_i\|_{x_i}^{p} \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1,\dots,n} \|x_i\|_{x_i} & \text{if } p = \infty \end{cases}$$

Then

- *$(\bigotimes_i X_i, \|\cdot\|_{\bigotimes_i X_i, p})$ is a normed space with respect to componentwise addition and multiplication.*
- *If all X_i are complete, then $\bigotimes_{i=1}^{n} X_i$ is complete.*
- *All norms $\|\cdot\|_{\bigotimes_i X_i, p}$ are equivalent.*

Proof.

- Vector space properties: Left as an exercise
- Norm: $\|x\|_{\bigotimes_i X_i,n} = 0 \iff x = 0$
  $\|\lambda x\|_{\bigotimes_i X_i,p} = |\lambda| \, \|x\|_{\bigotimes_i X_i,p}$
- Triangle inequality: $p = 1$, $p = \infty$
  $p \in (1, \infty)$. Take $x, y \in \bigotimes_{i=1}^n X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i,p}$.

$$\begin{aligned}
\implies \left\|x + y\right\|_p^p &= \sum_{i=1}^n \left\|x_i + y_i\right\|_{X_i} \left\|x_i + y_i\right\|_{X_i}^{p-1} \\
&\leq \sum_{i=1}^n \|x_i\|_{X_i} \left\|x_i + y_i\right\|_{X_i}^{p-1} + \sum_{i=1}^n \left\|y_i\right\|_{X_i} \left\|x_i + y_i\right\|_{X_i}^{p-1} \\
&\underbrace{\leq}_{\text{Hölder ineq.}} \left(\sum_{i=1}^n \|X_i\|_{X_i}^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n \left\|x_i + y_i\right\|_{X_i}^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^n \left\|y_i\right\|_{x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left\|x_i + y_i\right\|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \|x\|_p \left\|x + y\right\|_p^{p-1} + \left\|y\right\|_p \left\|x + y\right\|_p^{p-1} \\
&= \left(\|x\|_p + \left\|y\right\|_p\right) \cdot \left\|x + y\right\|_p^{p-1} \\
&= \left(\|x\|_p + \left\|y\right\|_p\right)\left(\left\|x + y\right\|_n^{n-1}\right)
\end{aligned}$$

$$\implies \left\|x + y\right\|_p \leq \|x\|_p + \left\|y\right\|_p \text{ if } x + y \neq 0 \text{(trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in $\mathbb{R}^n$)

□

Definition 2.8. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \to Y$ is linear such that $\left\|j(x)\right\|_Y = \|x\|_X$ (hence j is injective) then j is called isometric embedding from X to Y. If j is bijective, then j is called* isometric isomorphism *and we say $X = Y$ up to isomorphism.*

Proposition 2.9. *Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_X)$ a Banach space such that*

1. *$\exists$ isometric embedding, $i : X \to \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [$\hat{X}$ can be regarded as completion of X]*
2. *If $j_x : X \to Y$ is an isometric embedding on Y, a Banach space*

$$\implies \exists i_2 : \hat{X} \to Y$$

an isometric embedding such that $j_2 \circ j = j_1$ and if $\overline{j_1(x)} = y$ then j_2 is an isometric isomorphism. Thus "the completion is essentially unique".

Proof. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is } Cauchy\}$. $\hat{X}$ is a vector space by

$$(x_n)_n + (y_n)_n \coloneqq (x_n + y_n)_n \qquad \lambda(x_n)_n \coloneqq (\lambda x_n)_n \quad \hat{0} \coloneqq (0)_n$$

Define $\left\|(x_n)_n\right\|_{\tilde{X}} \coloneqq \lim_{n\to\infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in $\mathbb{R}$]. Then $\|\cdot\|_{\tilde{X}}$ is a semi-norm (proof is left as an exercise). Setting $N = \left\{(X_n)_n \mid \left\|(X_n)_n\right\|_{\hat{X}} = 0\right\}$. By Corollary 2.5, $\hat{X} \coloneqq \hat{X} \setminus N$ with $\left\|[(X_n)_n]\right\|_{\hat{X}} = \left\|(X_n)_n\right\|_{\hat{X}}$ is a normed space. Define

$$j : X \to \hat{X} \qquad x \mapsto [(x)_n]$$

then j is linear and $\left\|j(x)\right\|_{\hat{x}} = \left\|[(x)_n]\right\|_{\hat{x}} = \lim_{n\to\infty} \|x\| = \|x\|$. So j is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n \coloneqq j(x_n) \in \hat{X}$.

$$\begin{aligned}\implies \left\|y_m - [(x_n)_n]\right\|_{\hat{X}} &= \left\|(x_m)_n - (x_n)_n\right\|_{\hat{X}} = \lim_{n\to\infty} \|x_m - x_n\| \\ &= \lim_{n\geq n_0} \|x_m - x_n\| < \varepsilon\end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: $\hat{X}$ is complete.

Let $(y_n)_n$ be Cauchy in $\hat{X}$. Pick $X_n \in X$ such that $\left\|j(x_n) - y_n\right\|_{\hat{X}} \leq \frac{1}{n}$ $(\overline{j(x)} = \hat{x})$

$$\begin{aligned}\implies \|x_n - x_m\|_X &= \left\|j(x_n) - j(x_m)\right\|_{\hat{X}} \\ &\leq \left\|j(x_n) - y_n\right\|_{\hat{X}} + \left\|y_n - y_m\right\|_{\hat{X}} + \left\|y_n - j(x_n)\right\|_{\hat{X}}\end{aligned}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \left\|y_n - y_m\right\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick n_1 such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\implies \forall n, m > \max(n_0, n_0) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\implies$ $(x_n)_n$ is Cauchy. Let $y \coloneqq (X_n)_n \in \tilde{X}$. Then

$$\left\|y_n - [y]\right\|_{\hat{X}} \leq \left\|y_n - j(x_n)\right\|_{\hat{X}} + \left\|j(x_n) - [y]\right\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n\to\infty} \|x_n - x_m\|_X \xrightarrow{n\to\infty} 0$$

$\square$

Index