Linear Algebra 2
List of potential exam questions

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7 Determinants

- Give the three determinant properties
- Give the definition of determinant forms and multilinearity
- Prove $\triangle(a_1, \ldots, a_k + \lambda a_i, \ldots, a_n) = \triangle(a_1, \ldots, a_k, \ldots, a_n) \forall \lambda \in \mathbb{K}, \forall i \neq k$
- Prove $\triangle(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = -\triangle(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n)$
- Define permutations
- Define transpositions
- Prove: Every permutation can be represented as product of transpositions
- Is the decomposition of a permutation as transpositions unique?
- Define and determine the signature of a permutation
- Prove $\operatorname{sign}(\pi) = \prod_{\substack{i,j \\ i<j}} \frac{\pi(j)-\pi(i)}{j-i}$
- Prove: every transposition has sign -1
- Prove: $\operatorname{sign}(\pi \circ \sigma) = \operatorname{sign}(\pi) \cdot \operatorname{sign}(\sigma)$
- Prove: $\forall \sigma \in \sigma_n : \triangle(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \operatorname{sign}(a) \cdot \triangle(a_1, \ldots, a_n)$
- Define Leibniz' formula for determinants
- Prove Leibniz' formula
- Prove: Let B and C be two bases of a vector space. The determinant of B is non-trivial iff the determinant of C is non-trivial.
- Prove: $\triangle$ non-trivial $\iff \triangle(b_1, \ldots, b_n) \neq 0$ for every basis
- Prove: Let $\triangle$ be a non-trivial determinant form $\triangle(v_1, \ldots, v_n) \neq 0 \iff v_1, \ldots, v_n$ is linearly independent.
- Prove: Two determinant forms are different only by some factor
- Prove: $f : V \to V$ is invertible $\iff \det(f) \neq 0$.
- Prove: For a matrix $A \in \mathbb{K}^{n \times n}$ it holds that $\det A \neq 0 \iff$ A has full rank.
- Prove: $f, g : V \to V$ linear. $\implies \det(f \circ g) = \det(f) \cdot \det(g)$
- Prove $\det(A \cdot B) = \det(A) \cdot \det(B)$ directly
- Prove: $\det(A^{-1}) = \frac{1}{\det(A)}$ if invertible
- Prove: $\det(A) = 0 \iff \operatorname{rank}(A) < n$
- Prove: $\det(A^t) = \det(A)$

- Define $\text{perm}(A)$
- Prove: Let A be an upper triangular matrix, hence $a_{ij} = 0$ if $i > j$. $\implies \det(A) = a_{11}a_{22}\dots a_{nn}$.
- Prove: $\begin{vmatrix} & & & 0 \\ & B & & 0 \\ & & & 0 \\ * & * & * & a_{nn} \end{vmatrix} = \det(B) \cdot a_{nn}$
- Define Laplace Expansion
- Define the cofactor of a matrix
- Define the complementary matrix
- Prove: $A^{-1} = \frac{1}{\det A}\hat{A}$
- Give and prove Cramer's rule

8 Inner products

- Give a geometrical proof of the Pythagorean theorem
- Define the scalar product in $\mathbb{R}^2$ and $\mathbb{R}^3$
- Prove $\langle a, b\rangle = \langle b, a\rangle$, $\langle \lambda a, b\rangle = \lambda\langle a, b\rangle = \langle a, \lambda b\rangle$ and $\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$ in $\mathbb{R}^2$
- Prove that the scalar product in $\mathbb{R}^3$ is the dot product.
- Give the law of Cosines
- Define an outer product (in $\mathbb{R}^3$)
- Prove $b \times a = -a \times b$, $(\lambda a) \times b = \lambda(a \times b) = a \times (\lambda b)$ and $(a + b) \times c = a \times c + b \times c$
- Define bilinearity
- Define antisymmetry
- Define an inner product
- The function value of an inner product is an element of which domain?
- Define positive/negative-(semi)definite and indefinite inner products (in terms of inner products and in terms of matrices).
- Define a scalar product in terms of inner products.
- What is a positive definite inner product in Hermitian form?
- What is a unitary product?
- When does the scalar product satisfy $\langle a, b\rangle = \langle b, a\rangle$?
- Why is the scalar product defined with sesquilinearity and not bilinearity?
- Define norms
- Every $\square$ induces a $\square$. How?
- Define l-norms
- Prove: $||x|| := \sqrt{\langle x, x\rangle}$ is a norm on V
- Give and prove: CBS inequality
- Prove: Let V be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $B = \{b_1, \dots, b_n\}$ be a basis. $\langle , \rangle$ is an inner product. There exists a unique matrix A in Hermitian form such that $\forall x, y \in V : \langle x, y\rangle = \Phi_B(x)^T \cdot A \cdot \overline{\Phi_B(y)}$. Additionally show: If $\langle , \rangle$ is positive definite, A is invertible.

- Define the conjugate transpose of a matrix.
- Define self-adjoint matrices.
- Define symmetrical matrices.
- Define Hermitian matrices.
- Give one simple example each for a (positive/negative) (semi)definite matrix and indefinite matrix.
- Define congruent matrices.
- Prove: Every Hermitian matrix is congruent to a diagonal matrix D of form $\operatorname{diag}(D) = (1, \dots, 1, -1, \dots, -1, 0, \dots, 0)$.
- Give and prove Sylvester's law of inertia.
- Define the index and signature of a matrix.
- Prove: $A > 0 \iff A \hat{=} I \iff \operatorname{ind}(A) = n$
- Prove: $A \geq 0 \iff \operatorname{ind}(A) = \operatorname{sign}(A) = \operatorname{rank}(A)$
- Prove: $A \hat{=} B \iff \operatorname{ind}(A) = \operatorname{ind}(B) \land \operatorname{sign}(A) = \operatorname{sign}(B)$
- Prove: $\det(C^*) = \overline{\det(C)}$
- Prove: $A = A^* \implies \det(A) \in \mathbb{R}$
- Prove: $A = A^*, B = B^*, A\hat{B} \implies \operatorname{sign}\det(A) = \operatorname{sign}\det(B)$
- Prove: $A > 0 \implies \det(A) > 0$
- Define the minor of a matrix.
- Let $A = A^*$. Prove: $A > 0 \iff$ all first minors A_r satisfy $\det(A_r) > 0$
- All submatrices of a positive definite matrix are $\square$
- Let $A = A^*$. Prove: $A < 0 \iff (-1)^r \det(A_r) > 0 \forall r \in \{1, \dots, n\}$
- What is an Euclidean space? What is a unitary space?
- What is a Hilbert space?
- Give the parallelogram law.
- Define orthogonal and orthonormal families of vectors.
- Define orthonormal bases of vectors.
- Let $(v_i)_{i \in I} \subseteq V, v_i \neq 0 \forall i$. Prove: $(v_i)_{i \in I}$ is orthogonal, then $(v_i)_{i \in I}$ is linear independent.
- Let $B = (b_1, \dots, b_n)$ is an orthonormal basis of an finite-dimensional vector space over $\mathbb{K}$. For $v \in V$, let $\Phi_B(v) = \begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}^T$. For $w \in V$, let $\Phi_B(w) = \begin{pmatrix} \mu_1 & \dots & \mu_n \end{pmatrix}^T$. Prove:
  1. $\lambda_i = \langle v, b_i \rangle$
  2. $\langle v, w \rangle = \sum_{i=1}^n \lambda_i \overline{\mu_i}$
- Define orthogonal complements
- Let V be a vector space with scalar product. $M, N \subseteq V$ are partitions.
  1. $M^\perp$ is a subspace.
  2. $M \subseteq N \implies N^\perp \subseteq M^\perp$
     $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$
  3. $\{0\}^\perp = V$
  4. $V^\perp = \{0\}$

5. $M \cap M^\perp \subseteq \{0\}$
6. $M^\perp = \mathcal{L}(M)^\perp$
7. $M \subseteq (M^\perp)^\perp$

- Prove: Let $U \subseteq V$ be a subspace. $U + U^\perp$ is a direct sum in $\mathbb{R}^n$ such that $U + U^\perp = \mathbb{R}^n$.
- Define convexity of functions and sets.
- Let $V = U \dot{+} U^\perp$. Prove: $\forall x, y \in V : \langle x, \pi_U(y)\rangle = \langle \pi_U(x), y\rangle = \langle \pi_U(x), \pi_U(y)\rangle$
- Let $V = U \dot{+} U^\perp$. Prove: $\left\|\pi_U(x)\right\| \leq \|x\| \wedge \left\|\pi_U(x)\right\| = \|x\| \iff x \in U$
- Define the Gram matrix.
- Let $v_1, \dots, v_m \in V$. $G = \text{Gram}(v_1, \dots, v_m)$. Prove: $G = G^*$ is Hermitian, positive *semi*definite.
- Let $v_1, \dots, v_m \in V$. $G = \text{Gram}(v_1, \dots, v_m)$. Prove: $\xi \in \ker G \iff \sum_{i=1}^m \overline{\xi_i} v_i = 0$
- Let $v_1, \dots, v_m \in V$. $G = \text{Gram}(v_1, \dots, v_m)$. Prove: G is positive definite iff $(v_1, \dots, v_m)$ are linear independent.
- Give Bessel's inequality. Give an intuition what the inequality says/when it can be useful.
- Give Parseval's identity. Give an intuition what the inequality says/when it can be useful.
- Give and prove the Gram–Schmidt process for orthogonalization
- Define Laguerre polynomials
- Define Hermite polynomials
- How are Laguerre and Hermite polynomials related to the Gram–Schmidt process?
- Give Riesz representation theorem
- Prove Riesz representation theorem
- Does Riesz representation theorem hold for infinite-dimensional spaces?
- Prove: $v = 0 \iff \forall w \in V : \langle v, w\rangle = 0$
- Prove: $\langle x, y\rangle = \langle x, z\rangle \, \forall x \implies y = z$
- Prove: $\|v\| = \sup\{|\langle v, w\rangle| \mid \|w\| \leq 1\}$
- Define adjoint maps
- Prove: Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: For every $v \in V$ the map $w \mapsto \langle T(w), v\rangle_V$ is linear.
- Prove: Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: $\forall v \in V \exists! u \in W \forall w \in W : \langle T(w), v\rangle_V = \langle w, u\rangle_W$ and $T^*(v) = u$.
- Prove: Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Show: $T^* \in \text{Hom}(V, W)$
- Prove: Let $(V, \langle\cdot,\cdot\rangle_V)$ and $(W, \langle\cdot,\cdot\rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: the map $\text{Hom}(W, V) \mapsto \text{Hom}(V, W)$ with $T \mapsto T^*$ is antilinear and $T^{**} = T$.
- Does every linear map have an adjoint map?
- Define involutions
- Prove: Let $B \subseteq V, C \subseteq W$ be orthonormal bases. $f \in \text{Hom}(V, W)$.

$$\Phi^C_B(f^*) = \Phi^B_C(f)^* = \overline{\Phi^B_C(f)^T}$$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $(g \circ f)^* = f^* \circ g^*$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $f^{**} = f$
- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $\ker f = (\operatorname{im} f^*)^\perp$
- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $\operatorname{im} f = (\operatorname{kern} f^*)^\perp$
- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: f injective $\iff$ f^* surjective
- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: f surjective $\iff$ f^* injective
- Define self-adjoint matrices
- Define linear isometries (= unitary transformations)
- Prove: unitary transformations are injective
- If $\dim V = \dim W < \infty$ and $f : V \to W$ is linear and unitary, then f is invertible and $f^{-1} = f^*$.
- If $\dim V = \infty$, $f : V \to V$ is isometry, it does not imply that f is invertible.
- Define unitary matrices
- Let U be a unitary matrix. Prove: $UU^* = I \iff U^*U = I$.
- Define orthogonal matrices
- Eigenspaces are always □
- Prove: $T \in \mathbb{C}^{n\times n}$ is unitary $\implies$ T is unitary
- Prove: $T \in \mathbb{C}^{n\times n}$. $\forall x, y \in \mathbb{C}^n : \langle Tx, Ty\rangle = \langle x, y\rangle \iff$ the columns of T define an orthonormal basis of $\mathbb{C}^n$
- Define isometries in metric spaces.
- Is translation an isometry?
- Is translation unitary?
- What does the counterclockwise transformation matrix in $\mathbb{R}^2$ look like?
- Define the orthogonal group
- Define the unitary group
- Define the special orthogonal group
- Define the special unitary group
- Define the general linear group
- Define the special linear group
- Prove: $U \in \mathcal{U}(n) \implies \left|\det(U)\right| = 1$
- Give the general layout of a quaternion
- What is the multiplicative identity element of quaternions?

9 Polynomials

- Define algebras
- Does an algebra satisfy associativity?
- Does an algebra satisfy commutativity?
- Define Jordan algebras
- Define the polynomial algebra
- Define the algebra of formal power series
- Define the degree of polynomials. Give the degree of polynomials $0, 1, x$ and x^2.
- Prove: Let $p(x)$ and $q(x)$ be polynomials. $\deg(p(x)) + \deg(q(x)) = \deg(g(x) \cdot q(x))$
- Define zero division freedom for polynomials
- Prove: Every polynomial induces a polynomial function and the map $p(x) \mapsto p_f(x)$ is linear and multiplicative.
- Define algebra homomorphisms
- Give and prove the insertion theorem for polynomials
- Define the root of a polynomial
- There is a formula to find roots of any quadratic equation. What is the highest polynomial degree such that a formula exists?
- What is the idea of Cardanos cubic formula?
- Prove polynomial division with remainder: $p(x), q(x) \in \mathbb{K}[x]$, $q(x) \neq 0$. Then there exists exactly one polynomial $s(x), r(x) \in \mathbb{K}[x]$, $p(x) = s(x) \cdot q(x) + r(x)$. with $\deg r(x) < \deg q(x)$.
- Give Ruffini-Horner's method
- Define reducibility of polynomials
- Give the Fundamental Theorem of Algebra
- Define the greatest common divisor for polynomials
- Prove Bezout's identity for polynomials

10 Eigenvalues and eigenvectors

- Define eigenvalues and eigenvectors
- Give an intuitive notion of eigenvalues and eigenvectors
- Can 0 be an eigenvalue? Can $\vec{0}$ be an eigenvector?
- Does every matrix have eigenvalues and eigenvectors?
- Give simple examples in $\mathbb{R}^{2\times 2}$ and $\mathbb{Q}^{2\times 2}$ without eigenvectors and eigenvalues.
- Define spectrums of linear maps
- Let $A, B \in \mathbb{C}^{n\times n}$. Prove: $\text{spec}(AB) = \text{spec}(BA)$
- What kind of linear map do you need to consider eigenspaces (necessary, but not sufficient condition)?
- Define eigenspaces of linear maps
- What is the eigenvalue of the matrix $\lambda \cdot \text{id}$?
- Let $b_1, \dots, b_n$ be a basis of V. Let $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. Then there exists a linear map f such that $f(b_i) = \lambda_i \cdot b_i$ and this map is $\square$.

- Prove: Left-sided eigenvalue $\iff$ right-sided eigenvalue
- Write it down formally: The spectrum does not depend on the choice of the basis
- Prove: The spectrum does not depend on the choice of the basis
- Define characteristic polynomials of matrices
- Let $A \in \mathbb{K}^{n \times n}$. What is the degree of the characteristic polynomial of A?
- Complete and prove: λ is eigenvalue $\iff$ $\chi_A(\lambda) = \square$
- Prove: A square matrix A is invertible if and only if 0 is not an eigenvalue of A
- Define symmetrical minors
- Prove: $\chi_{T^{-1}AT}(x) = \chi_A(x)$
- Define diagonalizable matrices
- Define equivalent matrices
- Define similar matrices
- Prove: A is diagonalizable $\iff$ $\exists$ basis of eigenvectors.
- Define exponentiation of matrices
- Define Fibonacci sequences
- Give an explanation of the relationship of rabbit populations and the Fibonacci sequence
- Give the iteration matrix of the Fibonacci sequence
- The golden ratio is the root of which polynomial?
- Give the golden ratio
- Prove: eigenvectors corresponding to different eigenvalues are linear independent.
- Prove: an $n \times n$ matrix with n different eigenvalues is diagonalizable.
- Define the geometric and algebraic multiplicity of an eigenvalue
- Prove: A matrix is diagonalizable iff for different eigenvalues $\lambda_1, \dots, \lambda_r$ it holds that $\sum_{i=1}^{r} d(\lambda_i) = n$
- Give an argument why there are at most n eigenvalues for a matrix in $\mathbb{K}^{n \times n}$
- Prove: For every eigenvalue λ, it holds that $d(\lambda) \leq k(\lambda)$
- Define nilpotent matrices

11 Jordan normal form

- Define invariant subspaces
- Show: Eigenspaces are invariant subspaces
- Prove: Let $A \in \mathbb{K}^{n \times n}, V = \mathbb{K}^n$. If $U \subseteq V$ is invariant and $p(x) \in \mathbb{K}[x]$, then U is invariant under $p(A)$.
- Prove: Let $A \in \mathbb{K}^{n \times n}, V = \mathbb{K}^n$. $U_1, \dots, U_k$ are invariant subspaces. Then $\bigcap_{i=1}^{k} U_i$ and $\sum_{i=1}^{k} U_i$ are invariant with respect to A.
- Prove: Let $f : V \to V$ and let $U \subseteq V$ be an invariant subspace. Then $f|_U : U \to U$ is a homomorphism
- Prove: Let $f : V \to V$. If V can be decomposed into a direct sum of invariant subspaces, then A can be transformed into block diagonal form.
- Let $\dim V = n, f \in \operatorname{End}(V)$. Prove: $\{0\} \subseteq \ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \dots$ and $\operatorname{im} f \supseteq \operatorname{im} f^2 \supseteq \operatorname{im} f^3 \supseteq \dots$

- Let $\dim V = n, f \in \mathrm{End}(V)$. Prove: $\exists m \leq n : \ker f^m = \ker f^{m+1}$
- Let $\dim V = n, f \in \mathrm{End}(V)$. Prove:

$$\ker f^m = \ker f^{m+1} \iff \operatorname{im} f^m = \operatorname{im} f^{m+1} \iff \ker f^m = \ker f^{m+k} \forall k \geq 1 \iff$$
$$\operatorname{im} f^m = \operatorname{im} f^{m+k} \forall k \geq 1 \iff \ker f^m \cap \operatorname{im} f^m = \{0\} \iff V = \ker f^m \dot{+} \operatorname{im} f^m$$

- Explain Fitting's Lemma in a few words.
- Define generalized spaces. Define generalized eigenvectors. How do generalized eigenvectors differ from eigenvectors?
- Prove: Let $\lambda_1, \dots, \lambda_k$ be different eigenvalues of A and $\ker(\lambda I - A)^{r_i}$ the corresponding generalized spaces where

$$\ker(\lambda_i I - A)^{r_{i-1}} \subsetneq \ker(\lambda_i I - A)^{r_i} = \ker(\lambda_i \cdot I - A)^{r_i + 1}$$
$$\implies \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i - A)^{r_i} \cap \ker((\lambda_1 I - A)^{r_1} (\lambda_2 I - A)^{r_2} \dots (\lambda_k I - A)^{r_k}) = \{0\}$$

- Prove: $\forall \lambda \neq \mu \in \operatorname{spec}(A) \forall k, l \geq 1 : \ker(\lambda I - A)^k \cap \ker(\mu I - A)^l = \{0\}$
- Prove: The sum $\sum_{i=1}^{k} \ker(\lambda_i I - A)^{r_i}$ is direct for arbitrary pairwise different $\lambda_1, \dots, \lambda_k$.
- Prove: Let $\lambda_1, \dots, \lambda_k$ be pairwise different eigenvalues of $A \in \mathbb{K}^{n \times n}$. Let $W := \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i I - A)^n$. $V = \ker(\lambda_1 I - A)^n \oplus \dots \oplus \ker(\lambda_k I - A)^n \oplus \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i I - A)^n$
- Prove: W is invariant under A and $\lambda_i \notin \operatorname{spec}(A|_W) \forall i \in \{1, \dots, k\}$
- Prove: Let $\mathbb{K}$ be algebraically closed and let $\lambda_1, \dots, \lambda_k$ be all eigenvalues of a matrix $A \in \mathbb{K}^{n \times n}$. Then $\mathbb{K}^n = \ker(\lambda_1 I - A)^n \oplus \dots \oplus \ker(\lambda_k I - A)^n$.
- What is the index of a linear map?
- Define nilpotent matrices.
- The sum of nilpotent matrices is $\square$
- The product of nilpotent matrices is $\square$
- Assume all generalized spaces are eigenspaces. What about diagonizability?
- Prove: Let $\ker(f^m) \subseteq \ker(f^{m+1}) \subseteq \ker(f^{m+2})$

$$\begin{aligned} u_1 \dots u_p \dots &\text{ basis of } \ker f^n \\ u_1 \dots u_p v_1 \dots v_k \dots &\text{ basis of } \ker f^{m+1} \\ u_1 \dots u_p v_1 \dots v_k w_1 \dots w_r \dots &\text{ basis of } \ker f^{m+2} \end{aligned}$$

  Then $(u_1 \dots u_p, f(w_1), \dots, f(w_r))$ is linear independent.
- What kind of matrix is the Jordan normal form of a matrix?
- Prove: Let $\dim V = n$. $f : V \to V$ is nilpotent of index p ($f^p = 0$). $d = \dim \ker f$. Then there exists a basis B of V such that

$$\operatorname{diag}\left(\Phi_B^B(f)\right) = ([N_1], [N_2], \dots, [N_d])$$

  where

$$N_i = \begin{bmatrix} 0 & 1 & \ddots & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & \ddots & 0 \end{bmatrix}_{n_i \times n_i} \qquad p = n_1 \geq n_2 \geq \dots \geq n_d \geq 1 \qquad n_1 + \dots + n_d = n$$

- Define Jordan blocks of length k of a given eigenvalue λ.
- Prove: Let $\mathbb{K}$ be an algebraically closed field. Then every matrix $A \in \mathbb{K}^{n+m}$ is similar to a matrix of Jordan normal form.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: $\sum_{i=1}^{q} k_i = n$

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: Geometric multiplicity of λ equals the number of corresponding Jordan blocks. Algebraic multiplicity of λ equals the num of sizes of corresponding Jordan blocks.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Show: The smallest exponent r such that $\ker((\lambda I - A)^r) = \ker((\lambda I - A)^{r+1})$ is the largest length of a corresponding Jordan block.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: Let $k \in \mathbb{N}$. $\#\{i : \lambda_i = \lambda \wedge k_i \geq k+1\} = \operatorname{rank}(\lambda I - A)^k - \operatorname{rank}(\lambda I - A)^{k+1}$

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: The Jordan blocks are uniquely determined (except for the order)

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : {}^{\mathbb{K}[x] \to \mathbb{K}^{n+n}}_{p(x) \mapsto p(A)}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)$.

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : {}^{\mathbb{K}[x] \to \mathbb{K}^{n+n}}_{p(x) \mapsto p(A)}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix}$.

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : {}^{\mathbb{K}[x] \to \mathbb{K}^{n+n}}_{p(x) \mapsto p(A)}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix}\right) = \begin{bmatrix} p(A_1) & & \\ & \ddots & \\ & & p(A_n) \end{bmatrix}$

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : {}^{\mathbb{K}[x] \to \mathbb{K}^{n+n}}_{p(x) \mapsto p(A)}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $A = T^{-1}BT \implies p(A) = T^{-1}p(B)T$

- For some Jordan block $J_k(\lambda)$ it holds that,

$$p(J_k(\lambda))_{i,j} = \begin{cases} \frac{p^{(j-i)}(\lambda)}{(j-i)} & j > i \\ p(\lambda) & j = i \\ 0 & j < i \text{ (below the diagonal)} \end{cases}$$

- Let $A \in \mathbb{K}^{N \times N}$. Prove: $\exists p(x) \in \mathbb{K}[x] : p(A) = 0$

- Define the annihilator of a matrix.

- Let $A \in \mathbb{K}^{N\times N}$. Prove: $\exists$ a unique polynomial $m_A(x) \in \mathbb{K}[x]$ with minimal degree and leading coefficients 1 and $p(x) \in \mathrm{Ann}(A) \iff m_A(x) \big| p(x)$
- Let $A \in \mathbb{K}^{N\times N}$. Prove: $m_A(\lambda) = 0 \forall \lambda \in \mathrm{spec}(A)$
- Give and prove the Cayley-Hamilton Theorem
- Prove: $m_A(x) \big| \chi_A(x)$
- Conclude: the roots of $m_A(x)$ are the eigenvalues of A
- Recognize: The minimal polynomial has the structure $m_A(x) = \prod(\lambda - \lambda_i)^{m_i}$ where m_i is the smallest exponent for $\ker(\lambda_i - A)^m = \ker(\lambda_i - A)^{m+1}$, hence this equals the largest length of a Jordan block for eigenvalue λ_i.
- Recognize: A is diagonalizable $\iff$ all $m_i = 1 \iff m_A(x) = \prod_{i=1}^k(\lambda - \lambda_i) \iff m_A(x)$ has only simple roots.
- $p(A) \cdot x = p(\lambda) \cdot x \qquad$ if $\lambda \in \mathrm{spec}(A) \implies p(\lambda) \in \mathrm{spec}(p(A))$
- What is an important precondition for the spectral mapping theorem?
- Give and prove the spectral mapping theorem

12 Normal matrices

- Define normal matrices
- If a matrix is self-adjoint, then is it necessarily □
- A real-valued self-adjoint matrix is called □
- Are unitary matrices normal?
- Prove: A is normal $\iff$ A is unitarily diagonalizable.
- The sum of normal matrices is □
- The product of normal matrices is □
- $A \in \mathbb{C}^{n\times n}$ is normal. Prove: $\ker A = \ker A^*$
- $A \in \mathbb{C}^{n\times n}$ is normal. Prove: $\ker A = \ker A^2$
- Let $A \in \mathbb{C}^{n\times n}$ is normal. Prove: $\ker(\lambda I - A) = \ker(\overline{\lambda} I - A^*)$
- Let $A \in \mathbb{C}^{n\times n}$ is normal. Prove: $\ker(\lambda I - A)^2 = \ker(\lambda I - A)$
- Let A be normal. Prove: $\lambda \neq \mu \in \mathrm{spec}\, A \implies \ker(\lambda I - A) \perp \ker(\mu I - A)$
- Matrix A is normal. What can you say about generalized eigenspaces?
- Let $A \in \mathbb{C}^{n\times n}$ be normal. Prove: Eigenvectors of different eigenvalues are orthogonal to each other.
- Let $A \in \mathbb{C}^{n\times n}$. Prove: A is normal $\implies \exists$ orthonormal basis of eigenvectors
- Let $A \in \mathbb{C}^{n\times n}$. Prove: $\exists$ orthonormal basis of eigenvectors $\implies \exists$ unitary matrix U such that $U^*AU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.
- Let $A \in \mathbb{C}^{n\times n}$. Prove: $\exists$ unitary matrix U such that $U^*AU = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \implies A$ is normal.
- Define Schur's decomposition
- If $A \in \mathbb{R}^{n\times n}$ and $\chi_A(\lambda)$ decomposes into linear factors. $\exists U \in \mathcal{U}(n) : U^*AU = R$ is an upper triangular matrix. Then U □
- Prove: A matrix is normal $\iff$ Schur normal form = diagonal matrix.
- Prove: Let $A \in \mathbb{C}^{n\times n}, A = A^* \implies \mathrm{spec}(A) \subseteq \mathbb{R}$.
- Let $A \in \mathbb{C}^{n\times n}$ self-adjoint. Prove: $A \geq 0 \iff \mathrm{spec}(A) \subseteq [0, \infty)$.
- For which matrices is the square root of a matrix defined?

- Prove: The square root of a positive definite matrix exists and is unique.
- Define Cholesky decompositions
- Define Schur complements of D in M. Give the dimensions of D and M.
- Let $A \in \mathbb{C}^{n \times n}$. $A > 0, b \in \mathbb{C}^n, \gamma > 0$.
$$\det \left[\begin{array}{c|c} A & B \\ \hline b^* & \gamma \end{array}\right] = \det A \cdot (\gamma - b^* A^{-1} b)$$
- Cholesky decomposition is a □ decomposition
- Prove: The lower triangular matrix in the Cholesky decomposition is unique
- Prove: If $A > 0$, then $\det A \leq a_{11} a_{22} \dots a_{nn}$.
- Give and prove Hadamard's inequality
- Which matrices have polar decompositions?
- Define the polar decomposition of a matrix A.
- Prove: Let $A \in \mathbb{C}^{n \times n}$. $|A| := (A^* A)^{\frac{1}{2}}$. Then $\exists U \in \mathcal{U}(n)$ such that $A = U \cdot |A|$.
- Which matrices have singular values?
- Define singular value decompositions of matrices.

13 Eigenvalue estimates

- Define the numerical range of a matrix A
- Define the numerical radius of matrix A
- Prove: $\text{spec}(A) \subseteq W(A)$
- Give and prove the Theorem by Toeplitz–Hausdorff
- Define Rayleigh quotients.
- Which requirements are given for matrix A in the Rayleigh-Ritz Theorem and the Courant–Fischer–Weyl min–max principle?
- Give and prove the Rayleigh-Ritz Theorem.
- Give and prove the Courant–Fischer–Weyl min–max principle.
- Which one is more generic: Rayleigh-Ritz Theorem or Courant–Fischer–Weyl min–max principle?
- Give and prove the Cauchy interlacing theorem
- Prove: A, B are self-adjoint $\in \mathbb{C}^{n \times n}$. $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$
- Define Geršgorin discs
- Give and prove the Geršgorin theorem
- What is the spectral gap of a matrix?

14 Matrix norms

- Give one scalar product for matrices in $\mathbb{C}^{m \times n}$.
- Define Frobenius norms (= Schur norms) (= Hilbert-Schmidt norms)
- Let V, W be normed vectorspaces. $f \in \mathrm{Hom}(V, W)$. Defined the so-called induced norm.
- Define optimal norms
- Define the spectral radius of a matrix
- What does every compatible matrix norm satisfy with respect to the spectral radius.
- Prove: The spectral radius is not a norm.
- Prove: $\forall A \in \mathbb{C}^{n \times n}\ \forall \varepsilon > 0\ \exists$ norm on $\mathbb{C}^n$: the induced matrix norm satisfies $||A|| \leq \rho(A) + \varepsilon$
- Define condition numbers of matrices

15 Non-negative matrices

- Define non-negative matrices
- Give the Perron–Frobenius theorem

16 Generic/recall question

- Which matrices satisfy $AB = BA$?
- Which matrices satisfy $A^{-1} = A$?
- Which matrices satisfy $A^T = A$?
- Which matrices satisfy $AA^* = I$?
- Which matrices satisfy $AA^* = A^*A$?
- Which matrices satisfy $A = A^*$?