

# Analysis 2

Lecture notes, University (of Technology) Graz  
based on the lecture by Wolfgang Ring

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June 3, 2018

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This lecture took place on 2018/03/06.

# 1 Mathematical Redux and topological fundamentals

## 1.1 Metric

**Definition 1.1.** Let  $X \neq \emptyset$  be a set. We define a map  $d : X \times X \rightarrow [0, \infty)$ .  $d$  should behave like a geometrical distance. We require  $\forall x, y, z \in X$ :

- $d(x, y) = d(y, x)$  [called symmetry]
- $d(x, y) = 0 \iff x = y$  [called positive definiteness]
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$  [called triangle inequality]

Then  $d$  is called metric or distance function on  $X$ .  $(X, d)$  is called metric space.

**Example 1.1.**

- $X \subseteq \mathbb{C}$ ,  $d(x, y) = |x - y|$ . It satisfies  $|x - z| \leq |x - y| + |y - z|$
- $X \subseteq \mathbb{R}^n$ ,  $\|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}}$

**Claim.**

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^n x_i^2}$$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

It holds that  $\|x + y\| \leq \|x\| + \|y\|$  [triangle inequality].

*Proof.*

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 && \text{[see Cauchy-Schwarz inequality]} \\
&= (\|x\| + \|y\|)^2 \\
\|x - y\|^2 &= \langle x - y, x - y \rangle \\
&= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \\
\|x + y\|^2 + \|x - y\|^2 &= 2(\|x\|^2 + \|y\|^2)
\end{aligned}$$

□

## 1.2 Cauchy-Schwarz inequality

**Theorem 1.1** (Cauchy-Schwarz inequality).

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

*Proof.*

$$0 \leq \langle x - \lambda y, x - \lambda y \rangle = \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|y\|^2 \quad \forall \lambda \in \mathbb{R}$$

Let  $\lambda = \frac{\langle x, y \rangle}{\|y\|^2}$ . Then,

$$\begin{aligned}
0 &\leq \|x\|^2 - 2 \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^4} \cdot \|y\|^2 \\
&\implies 0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\
&\implies |\langle x, y \rangle|^2 \leq \|x\|^2 \cdot \|y\|^2
\end{aligned}$$

□

## 1.3 Euclidean norm

**Definition 1.2.**  $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$  is called Euclidean norm (length) of vector  $x \in \mathbb{R}^n$ .

$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$  It holds that

1.  $\|\lambda x\| = |\lambda| \|x\| \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}$
2.  $\|x\| = 0 \iff x = 0 \text{ in } \mathbb{R}^n$
3.  $\|x + y\| \leq \|x\| + \|y\|$

In general: Let  $V$  be a vector space over  $\mathbb{R}$ . A map  $\|\cdot\|$ , which assigns every vector  $x$  a non-negative real number satisfying the properties above, is called **norm** on  $V$ . Then  $(V, \|\cdot\|)$  is called a **normed vector space**.

Let  $X \subseteq \mathbb{R}^n$  ( $V$  is a normed vector space), then  $d(x, y) = \|x - y\|$  is a metric on  $X$ .

$$\|y - x\| = \|(-1)(x - y)\| = |-1| \cdot \|x - y\| = \|x - y\|$$

$$d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

$$d(x, z) = \|z - x\| = \|z - y + y - x\| \leq \|z - y\| + \|y - x\| = d(z, y) + d(y, x)$$

## 1.4 Metric space

**Example 1.2** (metric space). *Metric space, distance is not a norm. Consider an area in  $\mathbb{R}^3$ .*

$d(x, y)$  is the shortest path, connecting  $x$  and  $y$  in  $X$ . See Figure 1

**Example 1.3** (French railway). *All connections between two cities pass through Paris except one city is Paris.*

**Example 1.4.**  $X = \mathbb{R}^2$ . Let  $p \in \mathbb{R}^2$  be fixed.

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y, p \text{ are on one line} \\ |x - p| + |p - y| & \text{if } x, y, p \text{ are not on one line} \end{cases}$$

## 1.5 Open sets, convergence and accumulation points

Now we put some terminology into the context of a metric space.  $(X, d)$  is a metric space.

**Definition 1.3.** Let  $x \in X, r \geq 0$ .

$$K_r(x) = \{z \in X \mid d(x, z) < r\}$$

Is an open sphere with radius  $r$  and center  $x$ .

**Definition 1.4.**

$$\overline{K_r(x)} = \{z \in X \mid d(x, z) \leq r\}$$

Closed sphere with center  $x$  and radius  $r$ .

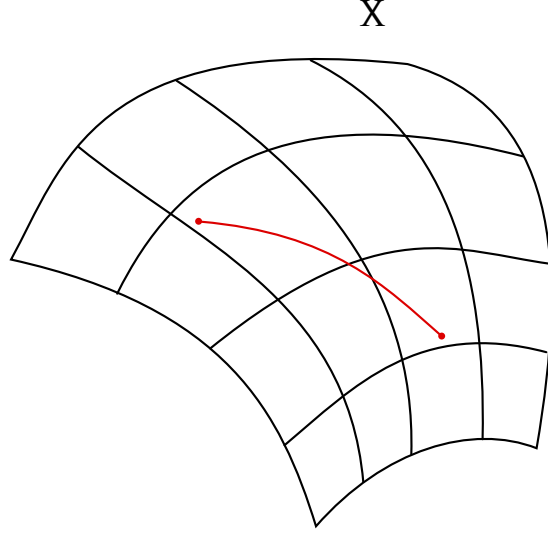


Figure 1: Example in  $\mathbb{R}^3$ . The red line illustrates the shortest path

**Definition 1.5** (Sequences in  $X$ ). Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  (hence,  $x_n \in X \forall n \in \mathbb{N}$ )

1.  $(x_n)_{n \in \mathbb{N}}$  is called convergent and limit  $x \in X$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies d(x_n, x) < \varepsilon$$

Denoted as  $\lim_{n \rightarrow \infty} x_n = x$ .

2.  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m \geq N \implies d(x_n, x_m) < \varepsilon$$

Every convergent sequence is also a Cauchy sequence.

*Proof.* Let  $(x_n)_{n \in \mathbb{N}}$  be convergent with limit  $x$ . Let  $\varepsilon > 0$  be arbitrary. Because  $(x_n)_{n \in \mathbb{N}}$  is convergent, there exists  $N \in \mathbb{N}$  such that  $n \geq N \implies d(x_n, x) < \frac{\varepsilon}{2}$ . Now let  $n, m \geq N$ . Then it holds that

$$d(x_n, x_m) \leq \underbrace{d(x_n, x)}_{< \frac{\varepsilon}{2}} + \underbrace{d(x, x_m)}_{< \frac{\varepsilon}{2}} < \varepsilon$$

□

**Definition 1.6.**  $(X, d)$  is called complete metric space if every Cauchy sequence in  $X$  is also convergent (has a limit).

$\mathbb{R}$  is complete.  $\mathbb{R}^n$  is also complete.  $\mathbb{Q} \subseteq \mathbb{R}$  is incomplete.

**Definition 1.7.** Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $X$  is called “accumulation point” (dt. Häufungspunkt) of the sequence.  $\forall \varepsilon > 0$ , it holds that  $K_\varepsilon(x)$  contains infinitely many sequence elements.

This lecture took place on 2018/03/08.

$(X, d)$  is called metric space.

$$\begin{aligned} d(x, y) = 0 &\iff x = y \\ \forall x, y \in X : d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \forall x, y, z \in X \end{aligned}$$

## 1.6 Norm

Let  $V$  be a vector space.  $\|\cdot\|$  is called norm on  $V$ .

$$\begin{aligned} \|x\| = 0 &\iff x = 0 \\ \forall \lambda \in \mathbb{R}, \mathbb{C} : \forall x \in V : \|\lambda x\| &= |\lambda| \|x\| \\ \forall x, y, z \in V : \|x + y\| &\leq \|x\| + \|y\| \end{aligned}$$

Let  $X \subseteq V$  be a subset of normed vector space  $V$ . Then  $X$  is a metric space with  $d(x, y) = \|x - y\|$ .

For  $V = \mathbb{R}^n$ . Then

$$\|x\| = \left( \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$$

is a norm on  $\mathbb{R}^n$ .  $\|x\|_2$  is called Euclidean norm on  $\mathbb{R}^n$ .

Other norms in  $\mathbb{R}^n$ :

$$\|x\|_\infty = \max \{ |x_i| \mid i = 1, \dots, n \}$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

for  $1 \leq p < \infty$ .

$$\|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

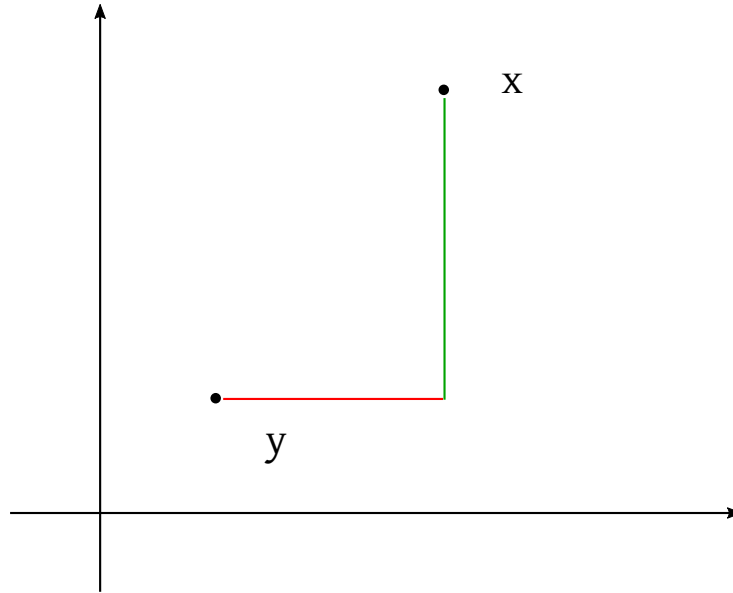


Figure 2: Visualizing  $\|x\|_1$

e.g.  $\|x\|_1$  in  $\mathbb{R}^2$

$$\|x - y\| = |x_1 - y_1| + |x_2 - y_2|$$

is the so-called *Manhattan metric*.

The concepts “subsequence”, “final element of a sequence”, “reordering of a sequence” correspond one-by-one to metric spaces.

**Definition 1.8** (Accumulation point). Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ .  $x \in X$  is called accumulation point of sequence  $X$  if  $\forall \varepsilon > 0$  the sphere  $K_\varepsilon(x)$  contains infinitely many elements.

**Lemma 1.1.**  $x \in X$  is accumulation point of sequence  $(x_n)_{n \in \mathbb{N}}$  if and only iff there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  such that  $x = \lim_{k \rightarrow \infty} x_{n_k}$ .

*Proof.* See Analysis 1 course

□

## 1.7 Contact point

Let  $B \subseteq X$ ,  $X$  is a metric space. Then  $B$  with  $d$  is a metric space itself.





Figure 3: Contact points in set  $B$

**Definition 1.9.** Let  $B \subseteq X$  and  $x \in X$ . We say,  $x$  is a contact point of  $B$  if  $\forall \varepsilon > 0 : K_\varepsilon(x) \cap B \neq \emptyset$ .

[  $y \in X$  is not a contact point of  $B \iff \exists \varepsilon > 0 : K_\varepsilon(y) \cap B = \emptyset$  ]

See Figure 3.

We let  $\bar{B} = \{x \in X \mid x \text{ is contact point of } B\}$ .

$\bar{B}$  is called closed hull of  $B$ .

$B$  is called closed if  $B = \bar{B}$ , hence, every contact point is also element of  $B$ .

**Remark 1.1.** Because  $\forall x \in B$  holds  $K_r(x) \cap B \supseteq \{x\} \forall r > 0$  is  $x$  always contact point of  $B$ . Also  $B \subseteq \bar{B}$  (always)

**Lemma 1.2.**  $x$  is contact point of  $B \iff \exists (x_n)_{n \in \mathbb{N}}$  with  $x_n \in B$  and  $\lim_{n \rightarrow \infty} x_n = x$ .

*Proof.* Let  $x$  be a contact point of  $B$ .

Direction  $\Rightarrow$ : Because  $K_{\frac{1}{n}}(x) \cap B \neq \emptyset$ , choose  $x_n \in K_{\frac{1}{n}}(x) \cap B$ . The sequence  $(x_n)_{n \in \mathbb{N}}$  has property  $d(x_n, x) < \frac{1}{n}$ . Let  $\varepsilon > 0$  be arbitrary. Choose  $N \in \mathbb{N}$  such that  $N > \frac{1}{\varepsilon}$  (consider the Archimedean axiom). Then for  $n \geq N$ ,  $d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ , hence  $\lim_{n \rightarrow \infty} x_n = x$ .

Direction  $\Leftarrow$ : Let  $x = \lim_{n \rightarrow \infty} x_n$  and  $x_n \in B$ . Let  $\varepsilon > 0$  be arbitrary and  $N \in \mathbb{N}$  such that  $d(x_n, x) < \varepsilon \forall n \geq N$ . Then  $d(x_N, x) < \varepsilon$ , hence

$$x_N \in \underbrace{K_\varepsilon(x) \cap B}_{\neq \emptyset}$$

So  $x$  is contact point of  $B$ . □

**Lemma 1.3.** It holds that  $\forall B \subseteq X : \bar{B} = \overline{\bar{B}}$ , hence  $\bar{B}$  itself is closed.

*Proof.* Show that  $x \in \overline{B}$ . Let  $x \in \overline{B}$ .

$$\iff \forall \varepsilon > 0 : K_\varepsilon(x) \cap \overline{B} \neq \emptyset$$

Therefore let  $\varepsilon > 0$  be arbitrary and  $x \in \overline{B}$ .

Show that  $K_\varepsilon(x) \cap B \neq \emptyset$ .

Because  $x \in \overline{B} : \exists y \in \overline{B} : y \in K_{\frac{\varepsilon}{2}}(x)$ . Because  $y \in \overline{B} : \exists z \in B : z \in K_{\frac{\varepsilon}{2}}(y)$ . Hence,

$$d(z, x) \leq \underbrace{d(z, y)}_{< \frac{\varepsilon}{2}} + \underbrace{d(y, x)}_{< \frac{\varepsilon}{2}} < \varepsilon$$

so  $z \in K(x, \varepsilon) \cap B$ . So  $x$  is contact point of  $B \implies x \in \overline{B}$ . □

**Lemma 1.4.** Let  $X$  be a metric space.

- $A_i \subseteq X$  be closed  $\forall i \in I$ . Then  $A = \bigcap_{i \in I} A_i = \{x \in X \mid x \in A_i \forall i \in I\}$  is closed itself.
- $A_1, \dots, A_n \subseteq X$  are closed. Then  $\bigcup_{k=1}^n A_k$  is closed in  $X$ .
- $\varphi$  is closed,  $X$  is closed.

*Proof.* See Analysis 1 course. □

**Definition 1.10.** Let  $x \in X$  is called accumulation point of set  $B \subseteq X$  if  $\forall \varepsilon > 0 : (K_\varepsilon(x) \setminus \{x\}) \cap B \neq \emptyset$ .

**Remark 1.2.** Accumulation points only exist in the context of sets. Accumulation values only exist in the context of sequences.

For example  $(+1, -1, +1, -1, +1, \dots)$  has accumulation values  $+1$  and  $-1$ .

**Lemma 1.5.** Let  $x \in X$  is accumulation point on  $B \iff$  every sphere  $K_\varepsilon(x)$  contains infinitely many points of  $B$ .

*Proof.* Direction  $\Leftarrow$  is trivial.

Direction  $\Rightarrow$ : Choose  $x_1 \in (K_1(x) \setminus \{x\}) \cap B$ , hence  $x_1 \neq x$ ,  $x_1 \in B$  and  $d(x_1, x) < 1$ . Let  $r_1 = 1$ .

Inductive: choose  $r_n = \min(\frac{1}{n}, d(x_{n-1}, x))$  and  $x_n \in (K_{r_n}(x) \setminus \{x\}) \cap B$ . Then  $d(x_n, x) > 0$  (because  $x_n \neq x$ ) where  $d(x_n, x) < r_n < \frac{1}{n}$ .

$$0 < d(x_n, x) < \frac{1}{n}$$

Furthermore,  $d(x_n, x) < r_n \leq d(x_{n-1}, x)$ . So  $x_n \neq x_{n-1}$ .

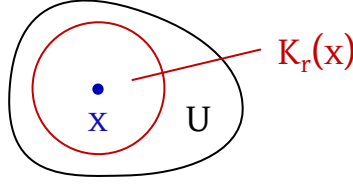


Figure 4:  $x$  is an inner point of  $U$  if  $\exists r > 0 : K_r(x) \subseteq U$

Inductive:  $x_n \neq x_{n-1} \neq x_{n-2} \neq \dots \neq x_1$ . Now consider arbitrary  $\varepsilon > 0$  and  $N$  large enough such that  $\frac{1}{N} < \varepsilon$ .

Then it holds that  $\forall n \geq N : 0 < d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ . So  $K_\varepsilon(x) \cap B$  contains infinitely many points  $x_N, x_{N+1}, x_{N+2}, \dots$   $\square$

**Definition 1.11.** Let  $U \subseteq X$  and  $x \in U$ . We say  $x$  is an inner point of  $U$  if  $\exists r > 0 : K_r(x) \subseteq U$ . We let  $\mathring{U} = \{x \in U \mid x \text{ is inner point of } U\}$  and call it interior of  $U$  (offenen Kern von  $U$  or das Innere von  $U$ ).  $O \subseteq X$  is called open (open set), if every point  $x \in O$  is also an inner point of  $O$ . Hence  $\mathring{O} = O$ . Compare with image 4.

**Example 1.5.** Let  $K_r(x)$  with  $r > 0$  be an open sphere in  $X$ . Then  $K_r(x)$  is an open set in  $X$ . Compare with image 5.

*Proof.* Why? Let  $y \in K_r(x)$ . Show that  $y$  is an inner point of the sphere.  $d(y, x) = s < r$ . Define  $r' = r - s > 0$ .

Claim:  $K_{r'}(y) \subseteq K_r(x)$ .

Let  $z \in K_{r'}(y)$ , hence  $d(z, y) < r'$ . Then,

$$d(z, x) \leq \underbrace{d(x, y)}_{< r'} + \underbrace{d(y, z)}_{= s} < r' + s = r$$

So it holds that  $z \in K_r(x)$  and therefore  $K_{r'}(y) \subseteq K_r(x)$ .  $\square$

**Lemma 1.6.** Let  $U \subseteq X$  be arbitrary. Then  $\mathring{U} \subseteq X$  be an open set in  $X$ .

*Proof.* Let  $x \in \mathring{U}$ , hence  $x$  is an inner point of  $U$ . Show that  $x$  is an inner point of  $\mathring{U}$ , also  $\exists r > 0 : K_r(x) \subseteq \mathring{U}$ .

Because  $x \in \mathring{U}$ ,  $r > 0$  exists:  $K_r(x) \subseteq U$ . Claim: Every point  $y \in K_r(x)$  is also an inner point of  $U$ . Obvious (previous example), because  $r' > 0$  exists such that  $K_{r'}(y) \subseteq K_r(x) \subseteq U$  so  $y \in \mathring{U}$  and  $K_r(x) \subseteq \mathring{U}$ .  $\square$

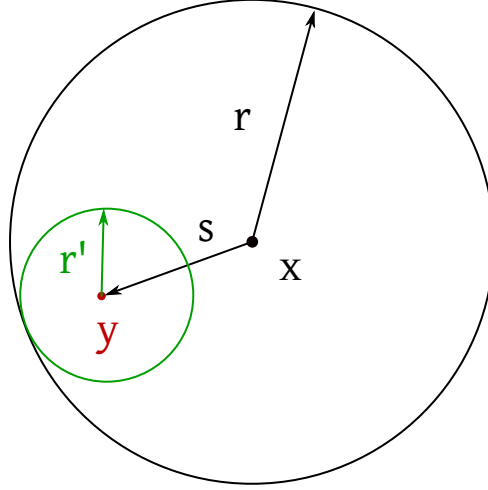


Figure 5: Let  $K_r(x)$  with  $r > 0$  be an open sphere in  $X$ . Then  $K_r(x)$  is an open set in  $X$ .

**Theorem 1.2.** *Let  $X$  be a metric space.*

$$A \subseteq X \text{ is closed in } X \iff O = X \setminus A = A^C \text{ is open}$$

*Proof.* Let  $A$  be closed and  $O = A^C$ . We choose  $x \in O$  and show that  $x$  is in the interior of  $O$ .

Assume the opposite.

$$\begin{aligned} \forall \varepsilon > 0 : \underbrace{\neg (K_\varepsilon(x) \subseteq O)}_{\iff K_\varepsilon(x) \cap O^C \neq \emptyset} \end{aligned}$$

where  $O^C = A$ .

Direction  $\Leftarrow$ . So  $x$  is contact point of  $A$ . Because  $A$  is closed, it holds that  $x \in A$ . This contradicts with  $x \in O = A^C$ . Thus  $O$  is open.

Direction  $\Rightarrow$ . Let  $O = A^C$  be open and let  $x$  be a contact point of  $A$ . Show that  $x \in A$ .

Assume the opposite, hence  $x \in A^C = O$  and  $O$  is open. So  $\exists r > 0 : K_r(x) \subseteq O$ , so  $K_r(x) \cap A = \emptyset$  where  $A = O^C$ . Hence  $x$  is not a contact point of  $A$ .

So every contact point of  $A$  is also an element of  $A$  and  $A$  is closed.  $\square$

**Theorem 1.3.** *Let  $X$  be a metric space. Then it holds that*

- If  $O_i \subseteq X$  is open in  $X \forall i \in I$ . Then also  $O = \bigcup_{i \in I} O_i$  is open in  $X$ .

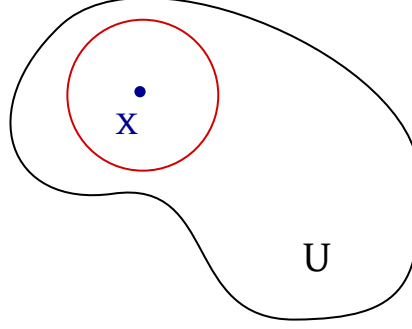


Figure 6: Neighborhood of  $x$

- If  $O_1, O_2, \dots, O_n$  is open in  $X$ , then  $\bigcap_{k=1}^n O_k$  is open in  $X$ .
- $X$  is open,  $\emptyset$  is open.

*Proof.* By Lemma 1.4, Theorem 1.2 and De Morgan's Laws:

$$\left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

□

## 1.8 Topology

**Definition 1.12.** Given a set  $X$ . If a subset  $T \subseteq \mathcal{P}(X)$  is defined such that the elements  $O \in T$  (hence  $O \subseteq X$ ) satisfy the conditions of Theorem 1.3, then  $T$  is called topology on  $X$ .  $(X, T)$  is called topological space.

The sets  $O \in T$  are called open sets in terms of  $T$ . The complements  $A = O^c$  for  $O \in T$  are called closed sets.

**Definition 1.13.** Let  $x \in U \subseteq X$ . We claim that  $U$  is a neighborhood of  $x$ , if  $r > 0$  exists such that  $x \in K_r(X) \subseteq U$

See Figure 6

**Remark 1.3.**  $O \subseteq X$  is open iff  $O$  is neighborhood of every point  $x \in O$ .

**Definition 1.14.** Let  $X$  and  $Y$  be metric spaces and  $x_0 \in X$ . Let  $f : X \rightarrow Y$  be given. We say  $f$  is continuous in  $x_0$  if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in X : d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

Here,  $d_X$  is a metric on  $X$  and  $d_Y$  is a metric on  $Y$ .

This lecture took place on 2018/03/13.

**Theorem 1.4.** Let  $X$  and  $Y$  be metric spaces.  $f : X \rightarrow Y$ . Let  $x_0 \in X$  be given. Then the following statements are equivalent:

1.  $f$  is continuous in  $x_0$
2. For every neighborhood  $V$  of  $y_0 = f(x_0)$  it holds that  $f^{-1}(V)$  is a neighborhood of  $x_0$
3. For every sequence  $(x_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} x_n = x_0$  it holds that  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$

*Proof.* See Analysis 1. □

**Definition 1.15.** Let  $f : X \rightarrow Y$  be called continuous on  $X$ , if  $f$  is continuous in every point  $x_0 \in X$ .

**Theorem 1.5.** Let  $f : X \rightarrow Y$  be given. Then  $f$  is continuous on  $X \iff \forall$  open  $O \subseteq Y : U = f^{-1}(O)$  open in  $X$ .

**Remark 1.4.** This characterization of continuity also works in topological spaces.

*Proof.* Direction  $\Rightarrow$ .

Let  $f$  be continuous in  $X$  and let  $O \subseteq Y$  be open. Let  $U = f^{-1}(O)$  and choose  $x_0 \in U$ . Then  $f(x_0) \in O$ , hence  $O$  is a neighborhood of  $f(x_0)$ . By Theorem 1.4 (b), it follows that  $U = f^{-1}(O)$  is a neighborhood of  $x_0$ .

Hence,  $U$  is neighborhood of every of its points, hence open in  $X$ .

Direction  $\Leftarrow$ .

Let the preimages of open sets be open and  $x_0 \in X$  and  $y_0 = f(x_0)$ . Let  $V$  be a neighborhood of  $y_0 = f(x_0)$ , hence  $\exists \varepsilon > 0 : K_\varepsilon(f(x_0)) \subseteq V$ . Because  $K_\varepsilon(f(x_0))$  is an open set, it holds that  $f^{-1}(K_\varepsilon(f(x_0))) \in x_0$  is open in  $X$ .

Therefore, there exists  $\delta > 0$  such that  $K_\delta(x_0) \subseteq f^{-1}(K_\varepsilon(f(x_0))) \subseteq f^{-1}(V)$ . Hence,  $f^{-1}(V)$  is a neighborhood of  $x_0$ . Then by Theorem 1.4 (b), it follows that  $f$  is continuous in  $x_0$  (chosen arbitrarily). Hence  $f$  is continuous on  $X$ . □

## 1.9 Variations of continuity notions

### 1.9.1 Uniform continuity

**Definition 1.16.** Let  $f : X \rightarrow Y$  be given. We call “ $f$  uniformly continuous on  $X$ ” if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X \wedge d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

**Remark 1.5.** Compare it with the definition of “continuous in  $X$ ”:

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : \forall y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

The difference is the location of the  $\forall x \in X$  quantifier.

Every uniformly continuous map is continuous.

Example:  $f : (0, \infty) \rightarrow (0, \infty)$  with  $f(x) = \frac{1}{x}$  is continuous, but not continuously continuous.

### 1.9.2 Lipschitz continuity

**Definition 1.17.**  $f : X \rightarrow Y$  is called Lipschitz continuous with Lipschitz constant  $L \geq 0$  if  $\forall x, y \in X : d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$ .

Rudolf Lipschitz [1832–1903], University of Bonn

**Theorem 1.6.** Every Lipschitz continuous function is uniformly continuous.

*Proof.* For  $\varepsilon > 0$ , choose  $\delta = \frac{\varepsilon}{L+1}$ . Then it holds that  $d_X(x, y) < \delta = \frac{\varepsilon}{L+1} \implies d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) < \frac{L}{L+1} \cdot \varepsilon < \varepsilon$ .  $\square$

- Most often  $X \subseteq V$ ,  $Y \subseteq W$ .  $V$  and  $W$  are normed vector spaces and  $d(x, y) = \|x - y\|$

### 1.10 Banach Fixed Point Theorem

**Definition 1.18.** A Lipschitz continuous map  $f : X \rightarrow X$  with Lipschitz constant  $L < 1$  is called contraction on  $X$ . Compare with Figure 7

**Theorem 1.7** (Banach fixed-point theorem). Let  $f : X \rightarrow X$  be a contraction and  $X$  be complete. Then there exists a uniquely defined  $\hat{x} \in X$  such that  $\hat{x} = f(\hat{x})$ .  $\hat{x}$  is called fixed point on  $f$ . Furthermore it holds that  $x_0 \in X$  is arbitrary and  $x_n = f(x_{n-1})$  for all  $n \geq 1$ . Compare with Figure 8.

$$\lim_{n \rightarrow \infty} x_n = \hat{x}$$

**Remark 1.6.** The following proof is a very common exam question.

*Proof.* Let  $x_0 \in X$  be arbitrary.  $x_n$  is constructed inductively by  $x_n = f(x_{n-1})$  for all  $n \geq 1$ .

**Claim.**  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ .

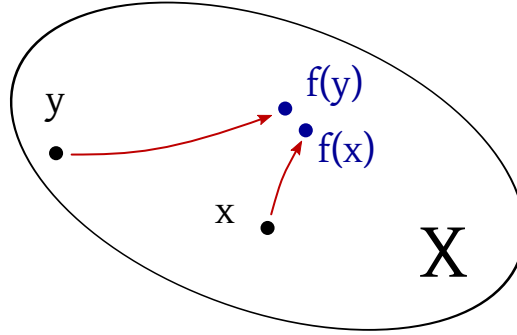


Figure 7: A contraction maps to points closer to each other

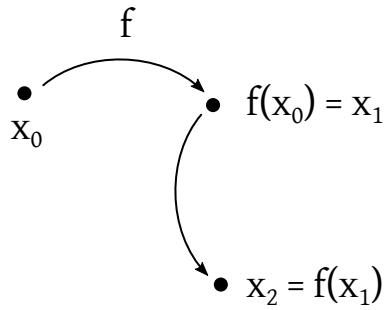


Figure 8: Banach's Fixed Point Theorem states that applying  $f$  iteratively gives a point coming closer and closer to the previous one

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k})$$

by triangle inequality

$$\begin{aligned} &= d(x_n, x_{n+1}) + d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) + \cdots + d(f(x_{n+k-2}), f(x_{n+k-1})) \\ &\leq d(x_n, x_{n+1}) + L(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-2}, x_{n+k-1})) \end{aligned}$$



this inequality is given by contraction

$$\begin{aligned}
&= d(x_n, x_{n+1})(1+L) + L(d(f(x_n), f(x_{n+1})) + \cdots + d(f(x_{n+k-3}), f(x_{n+k-2}))) \\
&\leq d(x_n, x_{n+1})(1+L) + L^2[d(x_n, x_{n+1}) + \cdots + d(x_{n+k-3}, x_{n+k-2})] \\
&\leq \cdots \leq d(x_n, x_{n+1})(1+L+L^2+\cdots+L^{k-1}) \\
&= d(f(x_{n-1}), f(x_n)) \left( \sum_{j=0}^{k-1} L^j \right) \leq Ld(x_{n-1}, x_n) \cdot \left( \sum_{j=0}^{k-1} L^j \right) \\
&\leq L^n d(x_0, x_1) \cdot \underbrace{\left( \sum_{j=1}^{k-1} L^j \right)}_{\leq \sum_{j=0}^{\infty} L^j = \frac{1}{1-L}} \\
&\leq \frac{L^n}{1-L} d(x_0, x_1) \\
d(x_n, x_{n+k}) &\leq \frac{L^n}{1-L} d(x_0, x_1) \forall n \in \mathbb{N} \forall k \in \mathbb{N}_0
\end{aligned}$$

with  $0 \leq L < 1$ .

$$\begin{aligned}
&\underbrace{\frac{L^n}{1-L}}_{>0} d(x_0, x_1) < \varepsilon \iff \\
L^n &< \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L) \quad (L > 0) \\
\iff n \underbrace{\ln L}_{<0} &< \ln \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L) \\
\iff n &> \frac{1}{\ln L} \ln \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L)
\end{aligned}$$

Hence  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $X$ .  $X$  is complete, hence  $\exists \hat{x} \in X$  :  $\hat{x} = \lim_{n \rightarrow \infty} x_n$ . Because  $\hat{x} = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(\hat{x})$  where the last equality is given by continuity of  $f$ . Therefore  $\hat{x} = f(\hat{x})$  is a fixed point on  $f$ .

It remains to prove uniqueness:

Let  $\tilde{x} = f(\tilde{x})$ . Then it holds that  $d(\hat{x}, \tilde{x}) = d(f(\hat{x}), f(\tilde{x})) \leq Ld(\hat{x}, \tilde{x})$  with  $L < 1$ . If  $d(\hat{x}, \tilde{x}) > 0$ , then  $1 \leq L$ . This is a contradiction. Hence  $d(\hat{x}, \tilde{x}) = 0$  must hold, hence  $\hat{x} = \tilde{x}$ .  $\square$

**Remark 1.7.** • The Fixed Point Theorem provides an algorithm for numeric computation of  $\hat{x}$ .

- It can reformulate problems  $f(x) = 0$  (in  $\mathbb{R}^n$ ) to

$$f(x) + x = g(x) = x$$

- *Attention: The conditions of the Fixed Point Theorem cannot be changed to the structure*

$$d(f(x), f(y)) < L \cdot d(x, y) \wedge L \leq 1$$

or

$$d(f(x), f(y)) \leq L \cdot d(x, y) \wedge L < 1$$

*This will be discussed in the practicals.*

**Lemma 1.7.** *Let  $X$  be a complete metric space. Let  $A \subseteq X$  be closed. Then  $(A, d)$  is itself a complete, metric space.*

*Proof.* Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $A$  ( $x_n \in A$ ). Then  $(x_n)_{n \in \mathbb{N}}$  is also a Cauchy sequence in  $X$ . Because  $X$  is complete, there exists  $\hat{x} = \lim_{n \rightarrow \infty} x_n$ . Therefore  $\hat{x}$  is a contact point of  $A$ . Because  $A$  is closed, it holds that  $\hat{x} \in A$ .

Therefore every Cauchy sequence in  $A$  has a limit point in  $A$ , hence  $A$  is complete.  $\square$

## 2 Compactness

### 2.1 Definition

**Definition 2.1.** *A metric space  $(X, d)$  is called compact if every sequence  $(x_n)_{n \in \mathbb{N}}$  has a convergent subsequence.*

*Specifically, this definition is called sequence compactness. The other definition defines compactness as closed and bounded subset of an Euclidean space. The latter definition only works for a subset of branches in mathematics. Therefore the generalization is recommended to be remembered.*

**Lemma 2.1.** *Let  $X$  be a compact, metric space. Then  $X$  is complete.*

*Proof.* Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $X$ . By compactness, it follows that  $\exists (x_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x}$ . Choose  $\varepsilon > 0$  arbitrary and  $L$  large enough such that  $k \geq L \implies d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2}$ . Furthermore choose  $N \in \mathbb{N}$  large enough such that  $n, m \geq N \implies d(x_n, x_m) < \frac{\varepsilon}{2}$  (satisfied, because  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence). Choose  $K \geq L$  and  $n_k \geq N$ . Let  $n_k$  be fixed this way. Then it holds  $\forall n \geq N : d(x_n, \hat{x}) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ . The first summand  $\frac{\varepsilon}{2}$  results from the Cauchy sequence property, the second summand  $\frac{\varepsilon}{2}$  results by convergence of  $(x_{n_k})$ . Hence  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $\hat{x}$ .  $\square$

### 2.2 Boundedness

**Definition 2.2.** *A metric space  $X$  is called bounded if there exists  $M \geq 0$ , such that  $d(x, y) \leq M \forall x, y \in X$ .*

It holds for arbitrary  $x \in X$  that  $\forall y \in X : y \in K_M(x)$ . So,  $X \subseteq K_M(x)$ . On the contrary, let  $X \subseteq \overline{K_M(x)}$  and let  $y \in X$  and  $z \in X$  be arbitrary. Then it holds that  $d(y, z) \leq d(y, x) + d(x, z) \leq M + M = 2M$ . Hence,  $X$  is bounded.

So,  $X$  is bounded  $\iff \exists x \in X \wedge M \geq 0 : X \subseteq \overline{K_M(x)}$ .

**Lemma 2.2.** *Every compact, metric space is also bounded.*

*Proof.* Assume  $X$  is unbounded.

We construct a sequence of points  $(x_n)_{n \in \mathbb{N}}$  with  $d(x_n, x_m) \geq 1 \forall n, m \in \mathbb{N}$  with  $n \neq m$ .

We use the following auxiliary result: Let  $B = \bigcup_{j=1}^n K_1(z_j)$  for arbitrary  $n \in \mathbb{N}$  and arbitrary  $z_j \in X$ . Then  $B$  is bounded. This result will be part of the practicals.

We construct  $(x_n)_{n \in \mathbb{N}}$  inductively. Choose arbitrary  $x_0 \in X$ . Assume  $(x_1, \dots, x_{n-1})$  are already found. Then it holds that

$$\underbrace{X}_{\text{unbounded}} \not\subseteq \underbrace{\bigcup_{j=1}^{n-1} K_1(x_j)}_{\text{bounded}}$$

hence  $\exists x_n \in X \setminus \bigcup_{j=1}^{n-1} K_1(x_j)$ . Because  $x_n \notin K_1(x_j)$  for  $j = 0, \dots, n-1$  it holds that  $d(x_n, x_j) \geq 1 \forall j < n$ . We get  $(x_n)_{n \in \mathbb{N}}$  with  $d(x_n, x_m) \geq 1 \forall n \in \mathbb{N} \forall m < n$ , hence  $m \neq n$ . Because  $d(x_n, x_m) \geq 1$ , i.e.  $(x_n)_{n \in \mathbb{N}}$  does not contain any Cauchy sequence as subsequence,  $(x_n)_{n \in \mathbb{N}}$  does not have a convergent subsequence. Therefore  $X$  is not compact.  $\square$

*This lecture took place on 2018/03/15.*

Every compact metric space is bounded. Every compact metric space is complete. In  $C(\mathbb{R}^n)$  it holds that  $A \subseteq C$  is closed. Then  $A$  with metric  $d(x, y) = |x - y|$  is complete as metric space.

If  $A$  is additionally bounded, then  $A$  is compact (see course Analysis 1, Bolzano-Weierstrass).

Attention! Let  $V$  be an infinite-dimensional, complete, normed vector space. For example,  $V = C([a, b], \mathbb{R}) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous in } [a, b]\}$  with norm  $\|f\|_\infty = \max\{|f(x)| : x \in [a, b]\}$  and metric  $\|f - g\|_\infty = \max\{|f(x) - g(x)| : x \in [a, b]\}$ .  $C([a, b], \mathbb{R})$  is a complete, normed vector space. It holds that  $\overline{K_1(0)}$  is not compact in  $C([a, b], \mathbb{R})$  (i.e.  $V$ , for every infinite-dimensional vector space).

Again: do not remember “compactness” not as closed and bounded, as this only holds in the finite-dimensional case.

In the last proof, we have shown: If a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in X$  and  $d(x_n, x_m) \geq 1$  (or  $\geq \varepsilon$ )  $\forall n \neq m \implies X$  is not compact.

## 2.3 Total boundedness

**Definition 2.3.**  $X$  is called totally bounded, if for every  $\varepsilon > 0$ , finitely many points  $X_1^\varepsilon, X_2^\varepsilon, \dots, X_{N(\varepsilon)}^\varepsilon$  such that  $X \subseteq \bigcup_{i=1}^{N(\varepsilon)} K_\varepsilon(X_i^\varepsilon)$ .

Hence, for every  $x \in X$ , there exists some  $X_j^\varepsilon$  such that  $d(x, X_j^\varepsilon) < \varepsilon$ .

**Remark 2.1** (For the practicals). Let  $X$  be totally bounded, then there does not exist some sequence  $(x_n)_{n \in \mathbb{N}}$  with  $d(x_n, x_m) \geq \varepsilon \forall n \neq m$ . It holds, that  $X$  is compact if and only if  $X$  is totally bounded and complete.

## 2.4 Compactness, continuity and openness

**Theorem 2.1.** Let  $f : X \rightarrow Y$  be continuous. Let  $X$  be compact. Then image  $f(X) \subseteq Y$  is also compact.

Be aware, that this proof is a common exam question and students often begin with the wrong order.

*Proof.* Let  $(y_n)_{n \in \mathbb{N}}$  be an arbitrary sequence in  $f(X)$ . Show that  $(y_n)_{n \in \mathbb{N}}$  has a convergent subsequence. Because  $y_n \in f(X)$ , there exists at least one  $x_n$  with  $y_n = f(x_n)$ . Then  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $X$ ,  $X$  is compact, hence there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x} \in X$ . Because  $f$  is continuous, it holds that  $\lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} y_{n_k} = f(\hat{x}) =: \hat{y}$ . So  $(y_n)_{n \in \mathbb{N}}$  has a convergent subsequence. Hence  $f(X) \subseteq Y$  is compact.  $\square$

**Theorem 2.2** (Conclusion). Let  $X$  be compact,  $f : X \rightarrow \mathbb{R}$  continuous on  $X$ . Then there exists  $\underline{x}$  and  $\bar{x} \in X$ , such that

$$f(\underline{x}) \leq f(x) \leq f(\bar{x}) \quad \forall x \in X$$

Hence,  $f$  has a maximum and a minimum.

*Proof.*  $f(X) \subseteq \mathbb{R}$  is compact (Theorem 2.1), hence  $f(X)$  is bounded and complete, hence closed in  $\mathbb{R}$ . There exists  $\xi \in \mathbb{R}$  with  $\xi = \sup f(X)$ , because  $f(X)$  is complete and  $\xi$  is a contact point of  $f(X)$ , it holds that  $\xi \in f(X)$ , hence  $\exists \bar{x} \in X : \xi = f(\bar{x})$ . Furthermore,  $\xi$  is an upper bound of  $f(X) \rightarrow f(x) \leq \xi = f(\bar{x}) \forall x \in X$ .

For  $\underline{x}$ , it works the same way.  $\square$

**Theorem 2.3.** Let  $f : X \rightarrow Y$  is continuous on  $X$  and  $X$  is compact. Then  $f$  is uniformly continuous on  $X$ .

*Indirect proof.* Assume  $X$  is compact,  $f : X \rightarrow Y$  is continuous, but not uniformly continuous. Uniform continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Not uniformly continuous:

$$\exists \varepsilon > 0 \forall \delta_n = \frac{1}{n} (n \in \mathbb{N}) \exists x_n, y_n \in X : d_X(x_n, y_n) < \frac{1}{n} \wedge d_Y(f(x_n), f(y_n)) \geq \varepsilon$$

Now choose some  $(x_n)$  and  $(y_n)$ . We will use a specific  $\varepsilon$  later. Because  $X$  is compact, there exists a convergent subsequence of  $(x_n)_{n \in \mathbb{N}}$ , hence  $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x}$ . The sequence  $(y_{n_k})_{k \in \mathbb{N}}$  has a convergent subsequence itself:

$$\lim_{l \rightarrow \infty} y_{(n_k)_l} = \hat{y}$$

Because  $(x_{n_k})_{k \in \mathbb{N}}$  is convergent, the subsequence  $(x_{(n_k)_l})_{l \in \mathbb{N}}$  converges towards the same limit  $\hat{x}$ .

$$\tilde{x}_l := x_{n_{k_l}} \quad \tilde{y}_l := y_{n_{k_l}}$$

because  $l \leq n_{k_l}$  and

$$d_X(\tilde{x}_l, \tilde{y}_l) = d_X(x_{n_{k_l}}, y_{n_{k_l}}) \underbrace{\quad}_{\text{by assumption}} < \frac{1}{n_{k_l}} \leq \frac{1}{l}$$

**Claim.** For  $\hat{x} = \lim_{l \rightarrow \infty} \tilde{x}_l$  and  $\hat{y} = \lim_{l \rightarrow \infty} \tilde{y}_l$ , it holds that  $\hat{x} = \hat{y}$ .

*Proof.* Let  $\varepsilon' > 0$  be arbitrary,  $l$  large enough such that

- $\frac{1}{l} < \frac{\varepsilon'}{3}$
- $d_X(\tilde{x}_l, \hat{x}) < \frac{\varepsilon'}{3}$
- $d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3}$

Therefore it holds that

$$d_X(\hat{x}, \hat{y}) \leq d_X(\hat{x}, \tilde{x}_l) + d_X(\tilde{x}_l, \tilde{y}_l) + d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3} + \frac{1}{l} + \frac{\varepsilon'}{3} < \varepsilon'$$

Therefore it holds that  $d_X(\hat{x}, \hat{y}) = 0$ , hence  $\hat{x} = \hat{y}$ . □

Because  $f$  is continuous and  $\tilde{x}_l \rightarrow \hat{x}$  and  $\tilde{y}_l \rightarrow \hat{x}$ , there exists  $l \in \mathbb{N}$  such that

$$d_Y(f(\tilde{x}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$

and also

$$d_Y(f(\tilde{y}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$



Figure 9: Subsets of  $(\mathbb{R}^n, \|\cdot\|)$  as metric spaces

where  $\varepsilon$  is the epsilon from the very beginning of the proof.

$$\implies d_Y(f(\tilde{x}_l), f(\hat{x})) + d_Y(f(\tilde{y}_l), f(\hat{x})) < \varepsilon$$

This contradicts to

$$d_Y(f(\tilde{x}_l), f(\tilde{y}_l)) = d_Y(f(x_{n_{k_l}}), f(y_{n_{k_l}})) \geq \varepsilon$$

Hence,  $f$  is uniformly continuous.  $\square$

Subsets of  $(\mathbb{R}^n, \|\cdot\|)$  (or  $(V, \|\cdot\|)$ ) as metric spaces.

We consider  $\Omega \subseteq V$  where  $V$  is a normed vector space.  $(\Omega, d)$  is  $d(x, y) = \|x - y\|$  is a metric space.

$$K_r^\Omega(x) = \{y \in \Omega \mid \|y - x\| < r\}$$

is a sphere with center  $x$  and radius  $r$  in  $\Omega$ .

$$K_r^V(x) = \{y \in V \mid \|y - x\| < r\}$$

obvious:  $K_r^\Omega(x) = \Omega \cap K_r^V(x)$ .

**Lemma 2.3.** Let  $O' \subseteq \Omega \subseteq V$ .

Then it holds that  $O'$  is open in  $\Omega \iff$  there exists  $O \subseteq V$  is open in  $V$  such that  $O' = O \cap \Omega$ .

*Proof.*  $\Rightarrow$  Let  $O' \subseteq \Omega$  be open in  $\Omega$  and  $x \in O'$  be arbitrary. Then there exists  $r(x) > 0 : x \in K_{r(x)}^\Omega(x) = K_{r(x)}^V(x) \cap \Omega \subseteq O'$ . Then it holds that

$$O' = \bigcup_{x \in O'} \{x\} \subseteq \bigcup_{x \in O'} K_{r(x)}^\Omega(x) = \left( \bigcup_{x \in O'} (K_{r(x)}^V(x)) \right) \cap \Omega = \underbrace{\left( \bigcup_{x \in O'} K_{r(x)}^V(x) \right)}_{=O \subseteq V \text{ is open in } V} \cap \Omega \subseteq O'$$

So every  $\subseteq$  in this inclusion chain is actually an equality. So  $O' = O \cap \Omega$ .

$\Leftarrow$  Let  $O' = O \cap \Omega$  and  $x \in O'$  be chosen arbitrarily. Because  $x \in O$  and  $O$  is open in  $V$ .

$$\exists r > 0 : K_r^V(x) \subseteq O \implies \underbrace{K_r^V(x) \cap \Omega}_{=K_r^\Omega(x)} \subseteq O \cap \Omega = O'$$

So  $O'$  is open in  $\Omega$ .

□

**Remark 2.2.**  $A' \subseteq \Omega$  is closed in  $\Omega \iff \exists A \subseteq V$  closed in  $V$  with  $A' = A \cap \Omega$ .

**Remark 2.3.** Let  $T$  be an arbitrary topological space with topology  $\tau$  on  $T$  (a system of open sets). Furthermore let  $\Omega \subseteq T$ .

Then  $\Omega$  itself is a topological space with  $O' \subseteq \Omega$  is open  $\iff \exists O \subset T$  open in  $T$  with  $O' = O \cap \Omega$ .

Also called “subspace topology”, “trace topology” or “relative topology”.

Attention!

$$O' \subseteq \Omega \text{ open in } \Omega \implies O' \text{ open in } V$$

does *not* hold in general.

**Example 2.1.**

$$\Omega = [0, 1] \cap [0, 1)$$

$K_{\frac{1}{2}}(p) \cap \Omega$  is open in  $\Omega$  but not open in  $\mathbb{R}^2$ .

Analogously,

$$A' \subseteq \Omega \text{ is closed} \implies A' \text{ closed in } V$$

does *not* hold in general.

**Remark 2.4.**  $K$  is compact in  $\Omega \implies K$  is compact in  $V$

Let  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $K$ . Compactness  $\implies \exists (x_{n_k})_{k \in \mathbb{N}} : x_{n_k} \rightarrow \hat{x}$  for  $k \rightarrow \infty$  and  $K \subseteq \Omega \subseteq V$ .

Then  $(x_n)_{n \in \mathbb{N}}$  also has a convergent subsequence in  $V$ .

## 2.5 Normed vector spaces

**Definition 2.4.** Let  $V$  be a vector space and  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are normed on  $V$ . We say,  $\|\cdot\|_1$  is equivalent to norm  $\|\cdot\|_2$ , if  $0 < m \leq M$  exist such that

$$m \|v\|_1 \leq \|v\|_2 \leq M \|v\|_1 \quad \forall v \in V$$

**Remark 2.5.** Equivalence of norms is an equivalence relation.

**reflexivity**  $\|\cdot\|_1$  is equivalent to  $\|\cdot\|_1$  with  $m = M = 1$ .

**symmetry**

$$\begin{aligned} m \|v\|_1 \leq \|v\|_2 &\implies \|v\|_1 \leq \frac{1}{m} \|v\|_2 \wedge \|v\|_2 \leq M \cdot \|v\|_1 \implies \frac{1}{M} \|v\|_2 \leq \|v\|_1 \\ &\implies \underbrace{\frac{1}{M}}_{m'} \|v\|_2 \leq \|v\|_1 \leq \underbrace{\frac{1}{m}}_{M'} \|v\|_2 \end{aligned}$$

hence the equivalence relations of norms are symmetrical.

**transitivity** Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be equivalent. Let  $\|\cdot\|_2$  and  $\|\cdot\|_3$  be equivalent.

$$\begin{aligned} m \cdot \|v\|_1 &\leq \|v\|_2 \leq M \|v\|_1 \quad \forall v \in V \\ m' \cdot \|v\|_2 &\leq \|v\|_3 \leq M' \|v\|_2 \quad \forall v \in V \\ \implies m \cdot m' \|v\|_1 &\leq m' \|v\|_2 \leq \|v\|_3 \leq M' \|v\|_2 \leq M \cdot M' \|v\|_1 \end{aligned}$$

This lecture took place on 2018/03/20.

Addendum:

- Let  $(x_n)_{n \in \mathbb{N}}$  be in  $(X, d)$ , then it holds that

$$\underbrace{x = \lim_{n \rightarrow \infty} x_n}_{\text{in } X} \iff \underbrace{\lim_{n \rightarrow \infty} d(x_n, x) = 0}_{\text{in } \mathbb{R}}$$

$$(\iff \lim_{n \rightarrow \infty} \|x_n - x\| = 0 \text{ in normed vector spaces } V)$$

- Reversed triangle inequality: Let  $V$  be a normed vector space. Let  $x, y \in V$ .

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

Hence,

$$\|x\| - \|y\| \leq \|x - y\|$$



By exchanging  $x$  and  $y$ ,

$$\|y\| - \|x\| \leq \|x - y\|$$

Hence, it holds that

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|$$

- Define the map  $n : V \rightarrow [0, \infty)$  on  $(V, \|\cdot\|)$  with  $n(x) = \|x\|$ . Then  $n$  is continuous on  $V$  because

$$|n(x_1) - n(x_2)| = \left| \|x_1\| - \|x_2\| \right| \leq \|x_1 - x_2\|$$

Hence,  $n$  is Lipschitz continuous with constant 1.

Regarding the equivalence of norms:

**Lemma 2.4.** *Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be equivalent norms on  $V$ . Then it holds that*

1.  $\lim_{n \rightarrow \infty} \|x_n - x\|_1 = 0 \iff \lim_{n \rightarrow \infty} \|x_n - x\|_2 = 0$ , hence  $(x_n)_{n \in \mathbb{N}}$  is convergent with limit  $x$  in regards of  $\|\cdot\|_1 \iff (x_n)_{n \in \mathbb{N}}$  is convergent with limit  $x$  in regards of  $\|\cdot\|_2$ .
2.  $O \subseteq V$  is open in regards of  $\|\cdot\|_1 \iff O$  is open in regards of  $\|\cdot\|_2$ , hence  $\tau_1 = \tau_2$  (topologies are equivalent).
3.  $K \subseteq V$  is compact in regards of  $\|\cdot\|_1 \iff K$  is compact in regards of  $\|\cdot\|_2$ .

*Proof.* Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent, hence  $\exists m, M > 0 : m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \forall x \in V$ .

1. Let  $\varepsilon > 0$  and  $\lim_{n \rightarrow \infty} \|x_n - x\|_1 = 0$ . Choose  $N \in \mathbb{N}$  such that  $n \geq N \implies \|x_n - x\|_1 < \frac{\varepsilon}{M}$ . For those  $n$  it holds that

$$\|x_n - x\|_2 \leq M\|x_n - x\|_1 < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

Hence,  $\lim_{n \rightarrow \infty} \|x_n - x\|_2 = 0$ .

2.  $K_r^2(x) = \{y \in V \mid \|y - x\|_2 < r\}$ . For  $y \in K_r^2(x)$  it holds that

$$m\|y - x\|_1 \leq \|y - x\|_2 < r$$

hence,

$$\|y - x\|_1 < \frac{r}{m} \implies y \in K_{\frac{r}{m}}^1(x)$$

hence  $K_r^2(x) \subseteq K_{\frac{r}{m}}^1(x)$ . Let  $y \in K_{\frac{r}{m}}^1(x)$ . Then it holds that,

$$\|y - x\|_2 \leq M\|y - x\|_1 < M \cdot \frac{r}{M} = r$$

hence  $y \in K_r^2(x) \implies K_{\frac{r}{M}}^1(x) \subseteq K_r^2(x)$ . Now let  $O$  be open in regards of  $\|\cdot\|_2$ , hence

$$\forall x \in O \exists r > 0 : K_r^2(x) \subseteq O \implies K_{\frac{r}{M}}^1(x) \subseteq K_r^2(x) \subseteq O$$

so  $O$  is open in regards of  $\|\cdot\|_1 \implies O$  is open in regards of  $\|\cdot\|_2$  analogously.

3. Let  $K$  be compact in regards of  $\|\cdot\|_1$  and  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $K$ . Then there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  with  $\|x_{n_k} - x\|_1 \rightarrow 0$  for  $k \rightarrow \infty$ .  
 $\xrightarrow{\text{by the first property}} \|x_{n_k} - x\|_2 \rightarrow 0$ . Hence  $(x_{n_k})_{k \in \mathbb{N}}$  is also a convergent subsequence in regards of  $\|\cdot\|_2$ .

□

**Remark 2.6** (Proven in the practicals). Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}^k$

$$\|x\|_\infty = \max \{ |x^i| \mid i = 1, \dots, n \}$$

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix}$$

It holds that  $\lim_{n \rightarrow \infty} \|x_n - x\|_\infty = 0 \iff \lim_{n \rightarrow \infty} |x_n^i - x^i| = 0$  for all  $i \in \{1, \dots, k\}$ .

**Theorem 2.4** (Bolzano-Weierstrass theorem in  $\mathbb{R}^k$ ). Let  $K \subseteq \mathbb{R}^k$  be closed and bounded. Then  $K$  is compact in  $(\mathbb{R}^k, \|\cdot\|_\infty)$ .

*Proof.* Let  $\|x\|_\infty \leq M \forall x \in K \iff |x^i| \leq M \forall x \in K$  and  $i \in \{1, \dots, k\}$ . Choose  $(x_n)_{n \in \mathbb{N}}$  an arbitrary sequence in  $K$ .  $(x_n^i)_{n \in \mathbb{N}}$  is a bounded sequence in  $\mathbb{R}$ . Because  $(x_n^1)_{n \in \mathbb{N}}$  is bounded, there exists a convergent subsequence  $(x_{n_{l_1}}^1)_{l_1 \in \mathbb{N}}$

$$\lim_{l_1 \rightarrow \infty} x_{n_{l_1}}^1 = x^1$$

Consider  $(x_{n_{l_1}}^2)_{l_1 \in \mathbb{N}}$ , a subsequence of a bounded sequence, hence bounded itself. By the Bolzano-Weierstrass theorem in  $\mathbb{R}$ , there exists a convergent subsequence  $(x_{n_{l_1 l_2}}^2)_{l_2 \in \mathbb{N}}$  with  $\lim_{l_2 \rightarrow \infty} x_{n_{l_1 l_2}}^2 = x^2$ . Consider  $x_{n_{l_1 l_2}}^1$  as subsequence of  $x_{n_{l_1}}^1$  is already convergent, hence  $\lim_{l_2 \rightarrow \infty} x_{n_{l_1 l_2}}^1 = x^1$ . Furthermore, up to index  $i$ , it holds that:

$$\lim_{l_k \rightarrow \infty} x_{n_{l_1 l_2 \dots l_k}} = x^i \quad \text{for } i = 1, \dots, k$$

Hence, with  $\tilde{x}_{l_k} = x_{n_{l_1 l_2 \dots l_k}}$  gives a subsequence of  $x_n$ , converging by each coordinate. Thus,

$$\lim_{l_k \rightarrow \infty} \|\tilde{x}_{l_k} - x\|_\infty = 0$$

Because  $\tilde{x}_{l_k} \in K$  and  $K$  be closed, it holds that  $x \in K$ . Hence  $K$  is compact. □

**Theorem 2.5** (Norm equivalence in  $\mathbb{R}^k$ ). *In  $\mathbb{R}^k$ , all norms are equivalent.*

*Proof.* We show: Let  $\|\cdot\|$  be an arbitrary norm on  $\mathbb{R}^n$ . Then  $\|\cdot\|$  is equivalent to  $\|\cdot\|_\infty$ . By transitivity of norm equivalence, two arbitrary norms are equivalent to each other.

1. Let  $(e_1, e_2, \dots, e_k)$  be the canonical basis in  $\mathbb{R}^k$ .

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^k \end{bmatrix} = \sum_{j=1}^k x^j e_j$$

Furthermore let  $M' = \max \{\|e_j\| : j = 1, \dots, k\}$  with  $\|e_j\| \neq 0$  and  $M' > 0$ . Then it holds that

$$\|x\| = \left\| \sum_{j=1}^k x^j e_j \right\| \leq \sum_{j=1}^k \|x^j e_j\| = \sum_{j=1}^k |x^j| \|e_j\| \leq M' \sum_{j=1}^k \underbrace{|x_j|}_{\leq \|x\|_\infty} \leq \underbrace{M' \cdot k}_M \|x\|_\infty = M \|x\|_\infty$$

2. We consider  $v : \mathbb{R}^k \rightarrow [0, \infty)$ .  $v(x) = \|x\|$  as map on  $(\mathbb{R}^k, \|\cdot\|_\infty)$ .

**Claim.**  $v$  is continuous on  $(\mathbb{R}^k, \|\cdot\|_\infty)$ .

*Proof.* Show that,

$$|v(x) - v(y)| = \left| \|x\| - \|y\| \right| \underbrace{\leq}_{\text{reversed triangle ineq.}} \|x - y\| \underbrace{\leq}_{\text{because of (1)}} M \|x - y\|$$

Hence  $v$  is Lipschitz continuous.  $\square$

We consider  $S_\infty^{k-1} = \{x \in \mathbb{R}^k \mid \|x\|_\infty = 1\} = \text{boundary}(K_1^\infty(0))$ .  $S_\infty^{k-1}$  is bounded.

Let  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $S_\infty^{k-1}$  with  $x = \lim_{n \rightarrow \infty} x_n$ . Because  $n(x) = \|x\|_\infty$  is continuous, it holds that

$$\lim_{n \rightarrow \infty} \underbrace{\|x_n\|_\infty}_{=1} = \underbrace{\|x\|}_{=1}$$

Hence  $x \in S_\infty^{k-1}$ . Hence,  $S_\infty^{k-1}$  is closed in  $(\mathbb{R}^k, \|\cdot\|_\infty)$ . Hence  $S_\infty^{k-1}$  is compact in  $(\mathbb{R}^k, \|\cdot\|_\infty)$ ,  $v : S_\infty^{k-1} \rightarrow [0, \infty)$ , with  $S_\infty^{k-1}$  compact, is continuous. Has

a minimum  $n$  on  $S_\infty^{k-1}$ . Thus there exists  $\bar{x} \in S_\infty^{k-1} : \underbrace{m}_{>0} = \left\| \underbrace{\bar{x}}_{\neq 0} \right\| \leq$



Figure 10: Illustration of a partition

$\|x\| \forall x \in S_{\infty}^{-1}$ . Let  $x \in \mathbb{R}^k$  be arbitrary with  $x \neq 0$ . Then it holds that  $\frac{x}{\|x\|_{\infty}} \in S_{\infty}^{k-1}$  and it holds that

$$m \leq \left\| \frac{x}{\|x\|_{\infty}} \right\| = \frac{1}{\|x\|_{\infty}} \|x\| \implies m \|x\|_{\infty} \leq \|x\|$$

Inequality also holds true for  $x = 0$ .

□

### 3 Integration calculus

#### 3.1 Partitions and refinements

**Definition 3.1.** Let  $a < b$  with  $a, b \in \mathbb{R}$ . We consider functions of  $[a, b]$ . We call  $(x_j)_{j=0}^n$  a partition of  $[a, b]$  if  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ .  $x_j$  decomposes  $[a, b]$  in subintervals  $(x_{j-1}, x_j)$ .  $\varphi : [a, b] \rightarrow \mathbb{R}$  is called step function in  $[a, b]$  in regards of partition  $(x_j)_{j=0}^n$  if  $\varphi|_{(x_{j-1}, x_j)} = c_j$ , so constant for  $j = 1, \dots, n$ .

$\varphi$  is called step function in  $[a, b]$  if there exists a partition such that  $\varphi$  is a subsequence.

$$\tau[a, b] = \{\varphi : [a, b] \rightarrow \mathbb{R} : \varphi \text{ is subsequence}\}$$

- Let  $(\xi_i)_{i=0}^m$  be a partition of  $[a, b]$  and  $(x_j)_{j=0}^n$  is a partition as well. Then we call  $(\xi_i)_{i=0}^m$  a refinement of  $[a, b]$  and  $(x_j)_{j=1}^n$  as well. Then  $(\xi_i)_{i=0}^m$  is a refinement of  $(x_j)_{j=0}^n$  if  $\{x_0, x_1, \dots, x_n\} \subseteq \{\xi_0, \xi_1, \dots, \xi_m\}$

Compare with Figure 11. Functions values in boundaries  $x_{j-1}$  and  $x_j$  do not have any constraints and will be relevant for an integral. A  $\varphi$  can be a step function in terms of many, various partitions.

**Lemma 3.1.** Let  $\varphi \in \tau[a, b]$  be a step function in terms of partition  $(x_j)_{j=0}^n$  and let  $(x_i)_{i=0}^n$  be a refinement of  $(x_j)_{j=0}^n$  in terms of  $(x_i)_{i=0}^m$ .



Figure 11: (top) Refinement (middle) function values in points  $x_j$  are unrestricted (bottom) step functions on a refinement

*Proof.* Refinement: For every  $j \in \{0, \dots, n\}$  there exists  $i_j \in \{0, \dots, m\}$  such that  $X_j = \xi_{i_j}$ .  $i_0 = 0, i_n = m$ .  $i_{j-1} < i_j$ .

Let  $i \in \{1, \dots, m\}$ . Then there exists a uniquely determined  $j \in \{1, \dots, n\}$  such that  $\xi_{i_{j-1}} < \xi_i \leq \xi_{i_j}$  Compare with Figure 12.

Then it holds that  $(\xi_{i_{j-1}}, \xi_i) \subseteq \underbrace{(\xi_{i_{j-1}}, \xi_{i_j})}_{=(x_{j-1}, x_j)}$  and  $\varphi|_{(\xi_{i_{j-1}}, \xi_{i_j})} = c_j = \text{const}$ . So  $\varphi$  is a subsequence in regards of  $(\xi_i)_{i=0}^m$ .  $\square$

**Definition 3.2.** Let  $\varphi \in \tau[a, b]$  in terms of partition  $(X_j)_{j=0}^n$  with  $\varphi|_{(X_{j-1}, X_j)} = c_j$  and  $\Delta X_j = X_j - X_{j-1} > 0$  for  $j = 1, \dots, n$ . Then we define ...

$$\int_a^b \varphi dx = \sum_{j=1}^n c_j \Delta x_j$$



Figure 12:  $\xi$  on a refinement  $x_{i_j}$

is called integral of  $\varphi$  in terms of partition  $(x_j)_{j=0}^n$

This lecture took place on 2018/03/22.

Step function  $\varphi$ .  $\varphi|_{x_{j-1}, x_j} = c_j$

$$\delta x_j = x_j - x_{j-1}$$

$$\int_a^b \varphi dx = \sum_{j=1}^n c_j \cdot \delta x_j$$

**Lemma 3.2.** Let  $(x_i)_{i=0}^n$  be a partition of  $[a, b]$  and  $(\xi_i)_{i=0}^m$  be a refinement of  $(x_j)_{j=0}^n$ . Furthermore let  $\varphi$  be a subsequence with respect to  $(x_j)_{j=0}^n$  (so also with respect to  $(\xi_i)_{i=0}^m$ ). Then the integrals of  $\varphi$  with respect to  $(x_j)_{j=0}^n$  and  $(\xi_i)_{i=0}^m$  are equal.

*Proof.* There exist indices  $i_j$  for  $j = 0, n$  such that  $x_j = \xi_{i_j}$ .

$$i_0 = 0 \quad i_n = m \quad i_{j-1} < i_j$$

$$\delta x_j = x_j - x_{j-1} = \xi_{i_j} - \xi_{i_{j-1}} = \xi_{i_j} - \xi_{i_{j-1}} = \underbrace{\sum_{i=i_{j-1}+1}^{i_j} (\xi_i - \xi_{i-1})}_{\text{telescoping sum}} = \sum_{i=i_{j-1}+1}^{i_j} \delta \xi_i$$

$$\varphi|_{x_{j-1}, x_j} = c_j \implies \varphi|_{(\xi_{i-1}, \xi_i)} = c_j \text{ for } i = i_{j-1} + 1, \dots, i_j$$

$$\tilde{c}_i = \varphi|_{(\xi_{i-1}, \xi_i)}$$

$$\underbrace{\sum_{i=1}^m \tilde{c}_i \delta \xi_i}_{\text{integral of } \varphi \text{ w.r.t } (\xi_i)_{i=0}^m} = \sum_{j=1}^n \sum_{i=i_{j-1}+1}^{i_j} \tilde{c}_i \delta \xi_i = \sum_{j=1}^n c_j \underbrace{\sum_{i=i_{j-1}+1}^{i_j} \delta \xi_i}_{=x_j} = \sum_{j=1}^n c_j \delta x_j$$

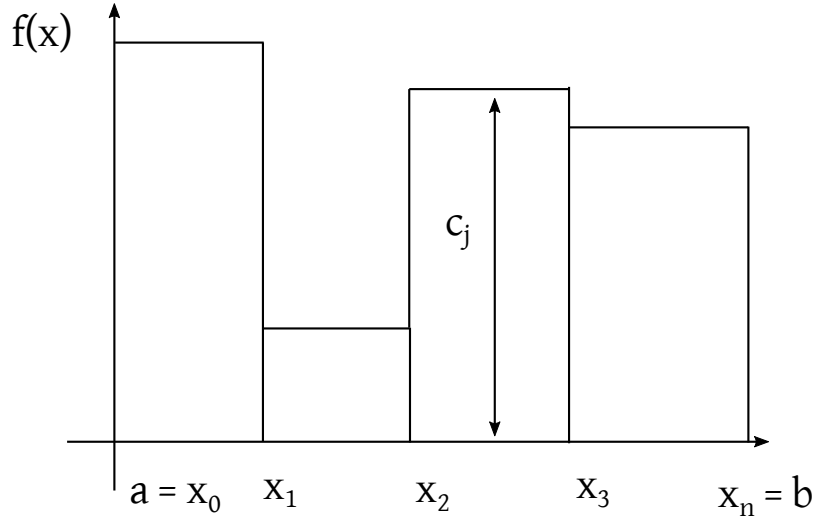


Figure 13: Integral of a step function as sum of areas of rectangles

This is the integral of  $\varphi$  with respect to  $(x_j)_{j=0}^n$ . □

**Lemma 3.3.** *Let  $\varphi$  be a step function with respect to  $(x_j)_{j=0}^n$  and  $(w_l)_{l=0}^L$ . Then the integrals of  $\varphi$  with respect to  $(x_j)_{j=0}^n$  and with respect to  $(w_l)_{l=0}^L$  equal.*

*Proof.* Let  $\{\xi_i \mid i = 1, \dots, m\} = \{x_j \mid j = 0, \dots, n\} \cup \{w_l \mid l = 0, \dots, L\}$  with  $\xi_0 = a$ ,  $\xi_m = x_n = w_L = b$  and  $\xi_{i-1} < \xi_i$  for  $i = 1, \dots, m$ . Then  $(\xi_i)_{i=0}^m$  is a refinement of  $(x_j)_{j=0}^n$  as well as  $(w_l)_{l=0}^L$ . By Lemma 3.2, the integral of  $\varphi$  with respect to  $(x_j)_{j=0}^n$  = integral of  $\varphi$  with respect to  $(\xi_i)_{i=1}^m$  = integral of  $\varphi$  with respect to  $(w_l)_{l=0}^L$ . Here we discard the statement “with respect to  $(x_j)_{j=0}^n$ ”. □

**Lemma 3.4.** *Let  $f, g$  be step functions on  $[a, b]$ .  $f, g \in \tau[a, b]$ .*

- for  $\alpha, \beta \in \mathbb{R}$ , let  $\alpha f + \beta g \in \tau[a, b]$  and

$$\int_a^b (\alpha f + \beta g) dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Hence, the integral is linear on  $[a, b]$ .  $\tau[a, b]$  is a vector space.

- $f \leq g$  in  $[a, b]$ , then  $\int_a^b f dx \leq \int_a^b g dx$  (monotonicity).

- $\left| \int_a^b f dx \right| \leq \int_a^b |f| dx$  ( $|f(x)|$  is also a step function)

*Proof.* 1. Let  $f, g \in \tau[a, b]$ . Let  $(\xi_i)_{i=0}^m$  be a partition such that  $f|_{(\xi_{i-1}, \xi_i)} = c_i$  and  $g|_{(\xi_{i-1}, \xi_i)} = d_i$ . Then

$$\int_a^b (\alpha f + \beta g) dx = \sum_{i=1}^m (\alpha c_i + \beta d_i) \delta \xi_i = \alpha \sum_{i=1}^m c_i \delta \xi_i + \beta \sum_{i=1}^m d_i \delta \xi_i = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Furthermore,

$$(\alpha f + \beta g)|_{(\xi_{i-1}, \xi_i)} = \alpha c_i + \beta d_i = \text{const.}$$

Thus,

$$\alpha f + \beta g \in \tau[a, b]$$

2. Let  $h \in \tau[a, b]$  and  $h(x) \geq 0 \forall x \in [a, b]$ , then it holds that  $v_i = h|_{(\xi_{i-1}, \xi_i)} \geq 0$  and

$$\int_a^b h dx = \sum_{i=1}^m \underbrace{h_i}_{\geq 0} \underbrace{\Delta \xi_i}_{\geq 0} \geq 0$$

Now let  $f, g \in \tau[a, b]; f \leq g$ . Then  $h = g - f \in \tau[a, b]$  (membership because of (1.)) and  $h \geq 0$ . Therefore,

$$0 \leq \int_a^b h dx = \int_a^b (g - f) dx = \int_a^b g dx - \int_a^b f dx$$

3.  $f \leq |f|$ , hence  $\int_a^b f dx \leq \int_a^b |f| dx$  and also  $-f \leq |f|$ , so

$$\begin{aligned} \int_a^b (-f) dx &= - \int_a^b f dx \leq \int_a^b |f| dx \\ \implies \left| \int_a^b f dx \right| &\leq \int_a^b |f| dx \end{aligned}$$

It is left to prove:  $|f| \in \tau[a, b]$  (i.e.  $|f|$  is a step function)

Let  $f|_{(\xi_{i-1}, \xi_i)} = c_i \implies |f|_{(\xi_{i-1}, \xi_i)} = |c_i| = \text{constant}$ . Hence  $|f| \in \tau[a, b]$ .

□

## 3.2 Characteristic functions

**Definition 3.3.** Let  $A \subseteq \mathbb{R}^k$ . We call  $\chi_A : \mathbb{R}^n \rightarrow \mathbb{R}$  with

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

a characteristic function (indicator function) of set  $A$ . Often denoted as  $\chi_A = \mathbb{1}$ . Compare with Figure 14.



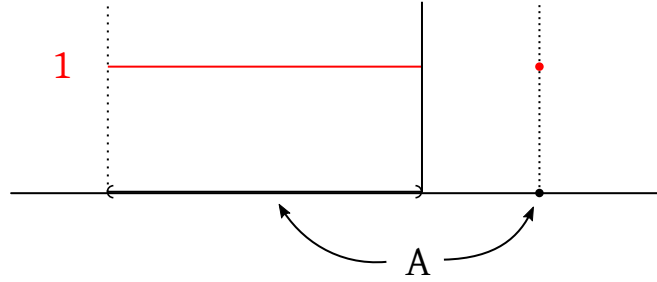


Figure 14: A characteristic function takes value 1 inside a set  $A$  which can be an interval (left) or a single point (right)

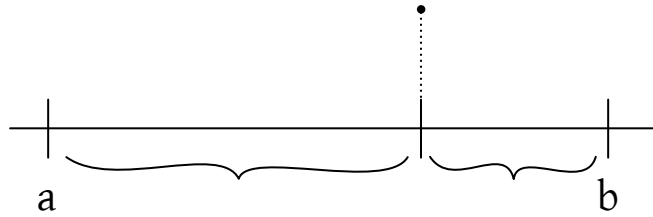


Figure 15: Linear combinations of characteristic functions are step functions

**Remark 3.1.** Let  $A = (a', b')$  with  $a \leq a' < b' \leq b$ . Then  $\chi_{(a', b')} \in \tau[a, b]$ . Also for  $x \in [a, b]$ , it holds that  $\chi_{\{x\}} = \tau[a, b]$ . Therefore

- every linear combination of characteristic functions of open subintervals  $(a', b')$  of  $[a, b]$  or
- characteristic functions of one-point sets  $\chi_{\{x\}}, x \in [a, b]$

are step functions on  $[a, b]$ . Compare with Figure 15.

$$\sum_{j=1}^n \alpha_j \chi_{(a_j, b_j)} + \sum_{k=1}^m \beta_k \chi_{\{x_k\}} \in \tau[a, b]$$

On the opposite,  $f \in \tau[a, b]$ , hence

$$f|_{(x_{j-1}, x_j)} \underbrace{=}_{j=1, \dots, n} c_j \text{ and } f(x_j) \underbrace{=}_{j=0, \dots, n} d_j$$

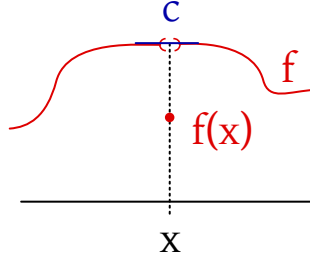


Figure 16: The limit value must not be necessarily the function value

$$f = \sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^n d_j \chi_{\{x_j\}} = (*)$$

for  $x \in (x_{j-1}, x_j)$  it holds that  $\chi_{(x_{j-1}, x_j)}(x) = 1$ .

$$\chi_{(x_{l-1}, x_l)}(x) = 0 \text{ for } l \neq j$$

$$\chi_{\{x_l\}}(x) = 0 \text{ for } l = 0, \dots, n$$

i.e.  $\sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)}(x) + \sum_{l=0}^n d_l \chi_{\{x_l\}}(x) = c_j \cdot 1 + 0 = c_j$  hence  $(*) = c_j$  on  $(x_{j-1}, x_j)$ . Therefore  $f \in \tau[a, b] \iff f$  is linear combination of characteristic functions of open intervals or one-pointed sets.

### 3.3 Limit points

**Definition 3.4.** Let  $X$  be a metric space  $A \subseteq X$  and  $x \in X$  is an accumulating point<sup>1</sup> of  $A$ . Let  $f : A \rightarrow \mathbb{R}$ . We say,  $f$  has limit  $c \in \mathbb{R}$  in  $x$  ( $\lim_{\xi \rightarrow x} f(\xi) = c$ ) if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi \neq x \text{ and } d(\xi, x) < \delta : |f(\xi) - c| < \varepsilon$$

**Remark 3.2.**  $x \in A$  and  $c = f(x) \implies f$  is continuous in  $x$ .

We usually consider  $A = [a, b] \subseteq \mathbb{R}$ ,  $x \in [a, b]$ .

It is possible, that  $f$  in  $x$  has a limit,  $x \in A$  and  $c = \lim_{\xi \rightarrow x} f(\xi) \neq f(x)$ . Compare with Figure 16.

**Definition 3.5.** Now let  $A \subseteq \mathbb{R}$  and  $x$  is a accumulation point of  $A$ . Let  $f : A \rightarrow \mathbb{R}$  be given. We say  $f$  has a right-sided limit  $c$  in  $x$  with  $c = \lim_{\xi \rightarrow x^+} f(\xi) = c$  if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi > x : |\xi - x| = \xi - x < \delta \implies |f(\xi) - c| < \varepsilon$$

<sup>1</sup>An accumulation point has 3 equivalent definitions (sequence, intersection, infinitely many elements in sphere).

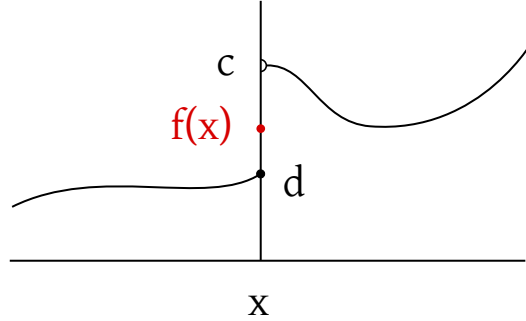


Figure 17: Left-sided and right-sided limit

Compare with Figure 17. The left-sided limit follows analogously (with  $\xi < x$ ).

$$c = \lim_{\xi \rightarrow x^+} f(\xi) \quad d = \lim_{\xi \rightarrow x^-} f(\xi)$$

**Lemma 3.5** (Sequence criterion for limits of functions). Let  $f : A \subseteq X \rightarrow \mathbb{R}$  be given.  $x$  is an accumulation point of  $A$ . Then it holds that

$$\lim_{\xi \rightarrow x} f(\xi) = c \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi_n \in A, \xi_n \neq x \text{ and } \lim_{n \rightarrow \infty} \xi_n = x \text{ it holds that } \lim_{n \rightarrow \infty} f(\xi_n) = c$$

For one-sided limits  $A \subseteq \mathbb{R}$  it holds that

$$c = \lim_{\xi \rightarrow x^+} f(\xi) \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi_n \in A \quad \xi_n > x \text{ with } \lim_{n \rightarrow \infty} \xi_n = x \text{ it holds that } \lim_{n \rightarrow \infty} f(\xi_n) = c$$

*Proof.* See Analysis 1 lecture notes. □

**Remark 3.3.** Attention! We, therefore, use two different definitions of limits.

**Lemma 3.6** (Cauchy criterion of limits of functions). Let  $f : A \subseteq X \rightarrow \mathbb{R}$ . Let  $x$  be an accumulation point of  $A$ . Let  $X$  be a metric space. Then it holds that  $f$  has a limit in  $x$  if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi, \eta \in A : \xi \neq x, \eta \neq x$$

with  $d(\xi, x) < \delta$  and  $d(\eta, x) < \delta$  it holds that  $|f(\xi) - f(\eta)| < \varepsilon$ . Analogously for one-sided limits with  $A \subseteq \mathbb{R}$ . Additionally, we need the constraint that  $\xi > x$  and  $\eta > x$  for  $\lim_{\xi \rightarrow x^+} f(\xi)$ . Or equivalently,  $\xi < x$  and  $\eta < x$  for  $\lim_{\xi \rightarrow x^-} f(\xi)$ .

*Proof.*  $\Leftarrow$  Let  $c = \lim_{\xi \rightarrow x} f(\xi)$  and let  $\varepsilon > 0$  be chosen arbitrarily. Then there exists  $\delta > 0$  such that  $d(\xi, x) < \delta$  and  $\xi \neq x$

$$\implies |f(\xi) - c| < \frac{\varepsilon}{2}$$

For  $\xi, \eta$ :  $d(\xi, x) < \delta$  and  $d(\eta, x) < \delta$  with  $\xi, \eta \neq x$  is therefore

$$|f(\xi) - f(\eta)| = |f(\xi) - c + c - f(\eta)| \leq |f(\xi) - c| + |f(\eta) - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\Rightarrow$  Assume the Cauchy criterion holds. We show that

1. for every sequence  $(\xi_n)_{n \in \mathbb{N}}$ ,  $\xi_n \in A \setminus \{x\}$  with  $\lim_{n \rightarrow \infty} \xi_n = x$  it holds that  $(f(\xi_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$  and therefore convergent in  $\mathbb{R}$ .
2. all Cauchy sequences have the *same* limit  $c$ .

We prove (1.)

Let  $(\xi_n)_{n \in \mathbb{N}}$  be as above. Let  $\varepsilon > 0$  be arbitrary. and  $N_\varepsilon$  large enough such that  $\forall n \in N_\varepsilon$  it holds that  $d(\xi_n, x) < \delta$  ( $\delta$  chosen appropriately to  $\varepsilon$  according to the Cauchy criterion).

By the Cauchy criterion,  $|f(\xi_n) - f(\xi_m)| < \varepsilon$  for all  $m, n \geq N_\varepsilon$ . Therefore  $(f(\xi_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}$ . If  $\mathbb{R}$  is complete, then there exists  $c = \lim_{n \rightarrow \infty} f(\xi_n)$ . QED.

We prove (2.)

Let  $\xi_n \rightarrow x$  as above and  $\xi'_n \rightarrow x$  as above and  $c = \lim_{n \rightarrow \infty} f(\xi_n)$  as well as  $c' = \lim_{n \rightarrow \infty} f(\xi'_n)$ . Let  $\varepsilon > 0$  be arbitrary,  $N_\varepsilon$  such that  $n \geq N_\varepsilon \implies |f(\xi_n) - c| < \frac{\varepsilon}{3}$  and  $N'_\varepsilon \in \mathbb{N}$  such that  $n \geq N'_\varepsilon \implies |f(\xi'_n) - c'| < \frac{\varepsilon}{3}$ .

Furthermore choose  $\delta > 0$  such that

$$d(\xi, x) < \delta \wedge d(\eta, x) < \delta \implies |f(\xi) - f(\eta)| < \frac{\varepsilon}{3}$$

(because of the Cauchy criterion).  $M_\varepsilon$  such that

$$n \geq M_\varepsilon \implies d(\xi_n, x) < \delta \wedge M'_\varepsilon : n \geq M'_\varepsilon \implies d(\xi'_n, x) < \delta$$

Let  $n \geq \max\{N_\varepsilon, N'_\varepsilon, M_\varepsilon, M'_\varepsilon\}$ .

*This lecture took place on 2018/04/10.*

Then it holds that

$$|c - c'| \leq \underbrace{|c - f(\xi_n)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(\xi_n) - f(\xi'_n)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(\xi'_n) - c'|}_{< \frac{\varepsilon}{3}} \quad \forall \varepsilon > 0$$

Hence,  $c = c'$ . We have shown that  $\exists c \in \mathbb{R} : \forall (\xi_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \xi_n = x$  it holds that  $\lim_{n \rightarrow \infty} f(\xi_n) = c$ . So  $\lim_{\xi \rightarrow x} f(\xi) = c$  because of Lemma 3.5. QED.

□

### 3.4 Regulated functions

**Definition 3.6** (Regulated function). Let  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$ . We call  $f$  a regulated function on  $[a, b]$  if

1.  $\forall x \in (a, b)$ ,  $f$  in  $x$  has a right-sided and a left-sided limit.
2. in  $x = a$ ,  $f$  has a right-sided limit.
3. in  $x = b$ ,  $f$  has a left-sided limit.

$$\mathcal{R}[a, b] = \{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is a regulated function} \}$$

**Definition 3.7** (Equivalent definition). 1.  $\forall x \in [a, b]$ ,  $f$  has a right-sided limit in  $x$

2.  $\forall x \in (a, b]$ ,  $f$  has a left-sided limit in  $x$

**Example 3.1.** Let  $f$  be continuous in  $[a, b]$ . Let  $\varphi \in \tau[a, b]$  be a regulated function. Then  $\varphi \in \mathcal{R}[a, b]$ .

*Rationale:*

Let  $x_0 = a < x_1 < \dots < x_n = b$  and  $\varphi|_{(x_{j-1}, x_j)} = c_j$ .

Let  $x \in [a, b]$  be chosen arbitrarily.

**Case 1** Let  $x \in (x_{j-1}, x_j)$  for some  $j \in \{1, \dots, n\}$

$$\implies \lim_{\xi \rightarrow x} \varphi(\xi) = c_j$$

Choose  $\delta$  small enough such that  $(x - \delta, x + \delta) \subseteq (x_{j-1}, x_j)$ .  $\forall \xi$  with  $\xi \in (x - \delta, x + \delta)$  it holds that

$$|\varphi(\xi) - c_j| = 0$$

**Case 2** Let  $x = x_j$  for  $j = 1, \dots, n - 1$ .

$$\implies \lim_{\xi \rightarrow x_j^+} \varphi(\xi) = c_{j+1}$$

$$\lim_{\xi \rightarrow x_j^-} \varphi(\xi) = c_j$$

Compare with Figure 18.

**Case 3** Let  $x = x_0 = a \implies \lim_{\xi \rightarrow a^+} \varphi(\xi) = c_1$ .

$$x = x_n = b \implies \lim_{\xi \rightarrow b^-} \varphi(\xi) = c_n$$

Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotonically increasing oder monotonically decreasing. Then  $f \in \mathcal{R}[a, b]$ . The proof will be done in the practicals.

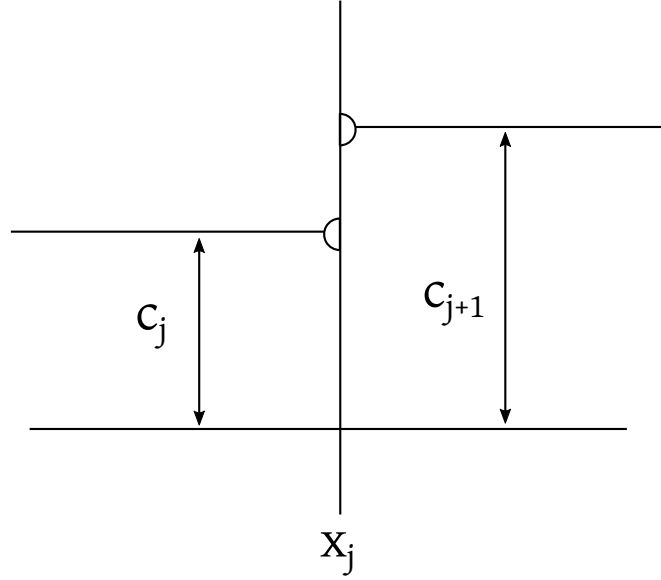


Figure 18: Regulated function

### 3.5 Bounded functions on bounded sets

**Definition 3.8** (Boundedness). Let  $X \neq \emptyset$  be a set.  $f : X \rightarrow \mathbb{K}$  with  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . We say:  $f$  is bounded on  $X$ , if  $f(X) \subseteq \mathbb{K}$  is a bounded set in  $\mathbb{K}$ . Hence,  $\exists m \geq 0 : |f(x)| \leq m \forall x \in X$ . We let,

$$\mathcal{B}(X) = \{f : X \rightarrow \mathbb{K} \mid f \text{ is bounded}\}$$

$\mathcal{B}(X)$  has vector space structure.  $f, g \in \mathcal{B}(X), \lambda \in \mathbb{K}$ .

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda \cdot f)(x) = \lambda \cdot f(x)$$

$f + g \in \mathcal{B}(X)$  and  $\lambda f \in \mathcal{B}(X)$ . Let  $|f(x)| \leq m \forall x \in X$  and  $|g(x)| \leq m' \forall x \in X$ . Then it holds that

$$|(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq m + m'$$

**Remark 3.4.** It is very interesting, that  $X$  does not require any kind of algebraic structure.

We let

$$\|f\|_{\infty} = \sup \underbrace{\{|f(x)| \mid x \in X\}}_{\text{bounded in } \mathbb{R}} = \min \{m \geq 0 \mid |f(x)| \leq m \forall x \in X\}$$

Some work is required to show that  $\|\cdot\|_\infty$  is a norm on  $\mathcal{B}(X)$ .

Hence,  $(\mathcal{B}(X), \|\cdot\|_\infty)$  is a normed vector space. Convergence in  $\mathcal{B}(X)$ : It holds that  $f_n \rightarrow f$  in  $(\mathcal{B}(X), \|\cdot\|_\infty)$  if and only if  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies \|f_n - f\|_\infty < \varepsilon$ .

$$\begin{aligned} \|f_n - f\|_\infty < \varepsilon &\iff \sup \{|f_n(x) - f(x)| : x \in X\} \\ &\iff |f_n(x) - f(x)| \leq \varepsilon \forall x \in X \end{aligned}$$

Hence,  $f_n \rightarrow f$  in  $(\mathcal{B}(X), \|\cdot\|_\infty) \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |f_n(x) - f(x)| \leq \varepsilon \forall x \in X$ . We say " $f_n$  converges uniformly to  $f$  on  $X$ ".

**Theorem 3.1** (Approximation theorem for regulated function). *Let  $f : [a, b] \rightarrow \mathbb{R}$ . Then it holds that  $f \in \mathcal{R}[a, b] \iff \forall \varepsilon > 0$  there exists some step function  $\varphi \in \tau[a, b]$  such that  $|\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]$  ( $\|\varphi - f\|_\infty < \varepsilon$ ).*

*Especially  $\varepsilon_n = \frac{1}{n}$  and  $\varphi_n$  as above. Then it holds that  $\|\varphi_n - f\|_\infty < \frac{1}{n}$ , hence  $f = \lim_{n \rightarrow \infty} \varphi_n$  uniformly on  $[a, b]$ .*

*Proof.*

**Direction  $\implies$ .** Let  $f \in \mathcal{R}[a, b]$ .

Proof by contradiction. We negate our hypothesis:

$$\exists \varepsilon > 0 : \forall \varphi \in \tau[a, b] \exists x \in [a, b] : |\varphi(x) - f(x)| \geq \varepsilon \quad (1)$$

Assume (1) holds for  $f \in [a, b]$ . We construct nested intervals  $[a_n, b_n]$  with  $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$  and  $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$  and (1) holds on  $[a_n, b_n] \forall n \in \mathbb{N}$ . Hence  $\forall \varphi \in \tau[a_n, b_n] \exists x \in [a_n, b_n]$  such that  $|\varphi(x) - f(x)| \geq \varepsilon$ . This is what we want to show.

Let  $a_0 = a$  and  $b_0 = b$ . Then (1) holds on  $[a_0, b_0]$  by assumption.  $n \rightarrow n+1$ : Construction of  $[a_{n+1}, b_{n+1}]$ . Let  $m_n = \frac{1}{2}(a_n + b_n)$ .

**Claim.** (1) holds either on  $[a_n, m_n]$  or on  $[m_n, b_n]$ .

*Proof.* Because if the opposite of (1) holds on  $[a_n, m_n]$  as well as  $[m_n, b_n]$ , then there exists  $\varphi_n^1 \in \tau[a_n, m_n]$  with  $|\varphi_n^1(x) - f(x)| < \varepsilon \forall x \in [a_n, m_n]$  and if the opposite of (1) holds on  $[m_n, b_n]$ :

$$\exists \varphi_n^2 \in \tau[m_n, b_n] : |\varphi_n^2(x) - f(x)| < \varepsilon \forall x \in [m_n, b_n]$$

$$\varphi^n(x) := \begin{cases} \varphi_n^1(x) & \text{if } x \in [a_n, m_n] \\ \varphi_n^2(x) & \text{if } x \in (m_n, b_n] \end{cases}$$

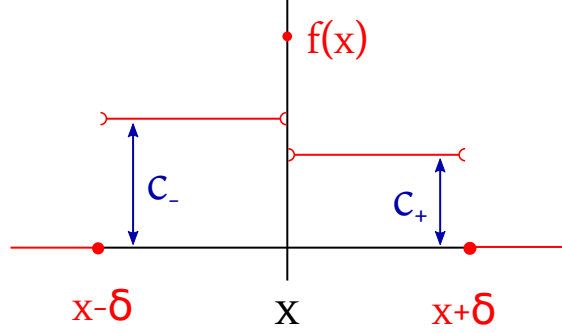


Figure 19: Construction of  $\hat{\varphi}(\xi)$

Then  $\varphi^n$  is piecewise constant, hence  $\varphi^n \in \tau[a_n, b_n]$  and it holds that

$$|\varphi^n(x) - f(x)| = \begin{cases} \underbrace{|\varphi_1^n(x) - f(x)|}_{< \varepsilon} & \text{for } x \in [a_n, m_n] \\ \underbrace{|\varphi_2^n(x) - f(x)|}_{< \varepsilon} & \text{for } x \in [m_n, b_n] \end{cases}$$

Therefore,  $|\varphi^n(x) - f(x)| < \varepsilon$ , which contradicts with (1) on  $[a_n, b_n]$ . We conclude: (1) holds on  $[a_n, m_n]$  or on  $[m_n, b_n]$ .  $\square$

Now, choose  $[a_{n+1}, b_{n+1}]$  as one of the subintervals in which (1) holds. Let  $X \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$  (by completeness of  $\mathbb{R}$ ).

**Case  $x \in (a, b)$**  Let  $x \in (a, b)$ . Let  $\varepsilon$  as above such that (1) holds on every interval  $[a_n, b_n]$ . Let  $c_+ = \lim_{\xi \rightarrow x^+} f(\xi)$  and  $c_- = \lim_{\xi \rightarrow x^-} f(\xi)$  (feasible, because  $f \in \mathcal{R}[a, b]$ ).

By the limit property,  $\exists \delta > 0 : |\xi - x| < \delta$  and  $\xi < x$ , then  $|f(\xi) - c_-| < \varepsilon$  and  $|\xi - x| < \delta$  and  $x < \delta$  then  $|f(\xi) - c_+| < \varepsilon$ .

Additionally, choose  $\delta$  sufficiently small enough such that  $(x - \delta, x + \delta) \subseteq [a, b]$ .

$$\hat{\varphi}(\xi) := \begin{cases} 0 & \text{for } \xi \in [a, b] \setminus (x - \delta, x + \delta) \\ c_- & \text{for } \xi \in (x - \delta, x) \\ c_+ & \text{for } \xi \in (x, x + \delta) \\ f(x) & \text{for } \xi = x \end{cases}$$



Compare with Figure 19.  $\hat{\varphi} \in \tau[a, b]$  and it holds that

$$\forall \xi \in (x - \delta, x + \delta) : |\hat{\varphi}(\xi) - f(\xi)| = \begin{cases} \underbrace{|c_- - f(\xi)|}_{< \varepsilon} & \text{for } \xi \in (x - \delta, x) \\ |f(x) - f(x)| & \text{for } \xi = x \\ \underbrace{|c_+ - f(\xi)|}_{< \varepsilon} & \text{for } \xi \in (x, x + \delta) \end{cases}$$

Hence,  $|\hat{\varphi}(\xi) - f(\xi)| < \varepsilon$ . Now let  $N$  be sufficiently large enough such that  $[a_N, b_N] \subseteq (x - \delta, x + \delta)$  (possible because  $([a_n, b_n])_{n \in \mathbb{N}}$  gives nested intervals tightening on  $x$ ). Then for  $[a_N, b_N]$  it holds that:

$$\hat{\varphi}|_{[a_N, b_N]} \in \tau[a_N, b_N]$$

and  $\forall \xi \in [a_N, b_N] \subseteq (x - \delta, x + \delta)$  it holds that  $|\hat{\varphi}(\xi) - f(\xi)| < \varepsilon$ . This contradicts with (1) on  $[a_N, b_N]$ .

**Case  $x = a$  and  $x = b$**  Is analogous to one-sided limits.

**Direction  $\Leftarrow$ .** Let  $f = \lim_{n \rightarrow \infty} \varphi_n$  be uniform on  $[a, b]$ .

**Claim.**  $\forall x \in [a, b)$  there exists a right-sided limit of  $f$  in  $x$ .

*Proof.* Let  $\varepsilon > 0$  be arbitrary.  $N \in \mathbb{N}$  sufficiently large such that  $|f(\xi) - \varphi_N(\xi)| < \frac{\varepsilon}{2} \forall \xi \in [a, b]$ .  $\varphi_N$  is piecewise constant. Choose  $\delta > 0$  such that  $\varphi_N|_{(x, x+\delta)} = c$ . Now let  $\xi, \eta \in (x, x + \delta)$  be chosen arbitrarily. Then it holds that

$$|f(\xi) - f(\eta)| \leq \underbrace{|f(\xi) - c|}_{= \varphi_N(\xi)} + \underbrace{|c - f(\eta)|}_{= \varphi_N(\eta)} = \underbrace{|f(\xi) - \varphi_N(\xi)|}_{< \frac{\varepsilon}{2}} + \underbrace{|\varphi_N(\eta) - f(\eta)|}_{< \frac{\varepsilon}{2}} < \varepsilon$$

Therefore  $f$  has a right-sided limit in  $x$  by the Cauchy criterion.  $f$  has left-sided limit in every point.  $\square$

$x \in (a, b]$  follows analogously.  $\square$

**Corollary.** Every regulated function  $f \in \mathcal{R}[a, b]$  is bounded. Let  $\varphi \in \tau[a, b]$  with  $\|f - \varphi\|_\infty < 1$ .  $\varphi$  is bounded, hence  $\exists m \in [0, \infty)$ :  $|\varphi(x)| \leq m \forall x \in [a, b]$ . Then it holds that  $|f(x)| \leq |f(x) - \varphi(x)| + |\varphi(x)| < 1 + m \forall x \in [a, b]$ , hence  $f \in \mathcal{B}[a, b]$ .

$$\mathcal{R}[a, b] \subseteq \mathcal{B}[a, b]$$

**Corollary.** Let  $f \in \mathcal{R}[a, b] \iff f = \sum_{j=0}^{\infty} \psi_j$  with  $\psi_j \in \tau[a, b]$  and the series converges uniformly on  $[a, b]$ .

*Proof.*

**Direction  $\Leftarrow$ .** Let  $f = \sum_{j=0}^{\infty} \psi_j$  with uniform convergence. Let  $\varphi_n = \sum_{j=0}^n \psi_j \in \tau[a, b]$  and  $f = \lim_{n \rightarrow \infty} \varphi_n$  uniform on  $[a, b] \xrightarrow{\text{Theorem 3.1}} f \in \mathcal{R}[a, b]$ .

**Direction  $\Rightarrow$ .** Let  $f \in \mathcal{R}[a, b]$  and  $f = \lim_{n \rightarrow \infty} \varphi_n$  with  $\varphi_n \in \tau[a, b]$  (by Theorem 3.1).

$$\begin{aligned} \psi_0 &= \varphi_0 \\ \psi_j &= \varphi_j - \varphi_{j-1} \quad \text{for } j \geq 1 \\ \sum_{j=0}^n \psi_j &= \varphi_0 + \sum_{j=1}^n (\varphi_j - \varphi_{j-1}) = \varphi_0 + \sum_{j=1}^n \varphi_j - \sum_{j=0}^{n-1} \varphi_j = \varphi_n \end{aligned}$$

converges uniformly to  $f$ .

□

## 4 Integration of regulated functions

**Definition 4.1** (Definition with a theorem). Let  $f \in \mathcal{R}[a, b]$  and  $\varphi_n \in \tau[a, b]$  with  $f = \lim_{n \rightarrow \infty} \varphi_n$  is uniform on  $[a, b]$ . We let

$$\int_a^b f \, dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n \, dx$$

for the integral of  $f$  on  $[a, b]$ .

*Theorem:* This limit (on the right-hand side) always exists and is independent of the particular choice of the approximating sequence.

*Proof.*  $\varphi_n$  is chosen as above.

$$i_n = \int_a^b \varphi_n \, dx$$

Show:  $i_n$  is cauchy sequence in  $\mathbb{R}$ .

*This lecture took place on 2018/04/12.*

Let  $\varepsilon > 0$  be chosen arbitrary. Choose  $N \in \mathbb{N}$  such that

$$n \geq N \implies \|f - \varphi_n\|_{\infty} < \frac{\varepsilon}{2(b-a)}$$

For  $n, m \geq N$  it holds for  $x \in [a, b]$  that

$$\begin{aligned} |\varphi_n(x) - \varphi_m(x)| &\leq |\varphi_n(x) - f(x)| + |f(x) - \varphi_m(x)| \\ &\leq \|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty < \frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{b-a} \end{aligned}$$

$|\varphi_n - \varphi_m|$  is a step function.

$$|\varphi_n - \varphi_m| \leq \frac{\varepsilon}{b-a} \cdot \underbrace{\chi_{[a,b]}}_{\in \tau[a,b]}$$

Integral for subsequence is monotonous:

$$\begin{aligned} |i_n - i_m| &= \left| \int_a^b \varphi_n dx - \int_a^b \varphi_m dx \right| = \left| \int_a^b (\varphi_n - \varphi_m) dx \right| \leq \int_a^b |\varphi_n - \varphi_m| dx \\ &\underbrace{\leq}_{\text{by monotonicity}} \int_a^b \frac{\varepsilon}{b-a} \cdot \chi_{[a,b]} dx = \frac{\varepsilon}{b-a} \underbrace{\int_a^b \chi_{[a,b]} dx}_{1 \cdot (b-a)} = \varepsilon \end{aligned}$$

So  $(i_n)_{n \in \mathbb{N}}$  is a Cauchy sequence.  $\mathbb{R}$  is complete, hence  $i = \lim_{n \rightarrow \infty} i_n$  exists.

Uniqueness: (dt. mithilfe des Reissverschlussprinzips)

Let  $(\varphi_n)_{n \in \mathbb{N}}, (\Phi_n)_{n \in \mathbb{N}}$  be two sequences of step functions, converging uniformly towards  $f$ .

$$\begin{aligned} i_n &= \int_a^b \varphi_n dx \quad \text{and} \quad j_n = \int_a^b \Phi_n dx \\ i &= \lim_{n \rightarrow \infty} i_n \quad j = \lim_{n \rightarrow \infty} j_n \end{aligned}$$

Show that  $i = j$ .

Now we construct a sequence  $(\mu_n)_{n \in \mathbb{N}}$  of step functions.

$$\underbrace{(\varphi_1, \Phi_1, \varphi_2, \Phi_2, \dots)}_{(\mu_n)_{n \in \mathbb{N}}}$$

$\mu_n$  is a sequence of step functions converging uniformly towards  $f$  (the proof is left as an exercise to the reader).

Because of part 1 of the proof:

$$m_n = \int_a^b \mu_n dx \text{ converges with limit } m$$

$(i_n)_{n \in \mathbb{N}}$  as well as  $(j_n)_{n \in \mathbb{N}}$  are subsequences of  $(m_n)_{n \in \mathbb{N}}$ . Hence it holds that  $i = \lim_{n \rightarrow \infty} i_n = m = \lim_{n \rightarrow \infty} j_n = j$ .  $\square$

**Theorem 4.1** (Elementary properties of an integral). *Let  $f, g \in \mathcal{R}[a, b]$ ,  $\lambda, \mu \in \mathbb{R}$ . Then it holds that*

**Linearity**

$$\lambda f + \mu g \in \mathcal{R}[a, b] \text{ and } \int_a^b (\lambda f + \mu g) dx = \lambda \int_a^b f dx + \mu \int_a^b g dx$$

**Monotonicity** *If  $f(x) \leq g(x) \forall x \in [a, b]$  ( $f \leq g$ ) it holds that*

$$\int_a^b f dx \leq \int_a^b g dx$$

**Boundedness**  $|f| \in \mathcal{R}[a, b]$  and

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx$$

*Proof.*

**Linearity.** Let  $x \in [a, b]$  and  $c_+ = \lim_{\xi \rightarrow x_+} f(\xi)$  as well as  $d_+ = \lim_{\xi \rightarrow x_+} g(\xi)$  ( $f, g \in \mathcal{R}[a, b]$ ). Then it holds that

$$\lim_{\xi \rightarrow x^+} (\lambda f(\xi) + \mu g(\xi)) = \lambda \lim_{\xi \rightarrow x^+} f(\xi) + \mu \lim_{\xi \rightarrow x^+} g(\xi) = \lambda c_+ + \mu d_+$$

exists. Analogously for the left side, hence  $\lambda f + \mu g \in \mathcal{R}[a, b]$ .

**Claim.** Let  $\varphi_n, \Phi_n \in \tau[a, b]$  with  $\varphi_n \rightarrow f$  and  $\Phi_n \rightarrow g$  is uniform on  $[a, b]$ . Hence  $\lambda \varphi_n + \mu \Phi_n \rightarrow \lambda f + \mu g$  is continuous on  $[a, b]$ .

*Proof.* Let  $\varepsilon > 0$  be arbitrary,  $N$  such that  $n \geq N \implies \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(|\lambda|+1)}$  and  $M$  such that  $n \geq M \implies \|\Phi_n - g\|_\infty < \frac{\varepsilon}{2(|\mu|+1)}$ .

Then it holds that

$$\begin{aligned} \|\lambda \varphi_n + \mu \Phi_n - \lambda f - \mu g\|_\infty &\leq |\lambda| \|\varphi_n - f\|_\infty + |\mu| \|\Phi_n - g\|_\infty \\ &< \frac{|\lambda|}{2(|\lambda|+1)} \cdot \varepsilon + \frac{|\mu|}{2(|\mu|+1)} \cdot \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

□

We continue:

$$\begin{aligned} \int_a^b (\lambda f + \mu g) dx &= \lim_{n \rightarrow \infty} \int_a^b (\lambda \varphi_n + \mu \Phi_n) dx = \lim_{n \rightarrow \infty} \left( \lambda \int_a^b \varphi_n dx + \mu \int_a^b \Phi_n dx \right) \\ &= \underbrace{\lambda \lim_{n \rightarrow \infty} \int_a^b \varphi_n dx}_{\text{exists}} + \underbrace{\mu \lim_{n \rightarrow \infty} \int_a^b \Phi_n dx}_{\text{exists}} = \lambda \int_a^b f dx + \mu \int_a^b g dx \end{aligned}$$

**Monotonicity.** Show: Let  $h \in \mathcal{R}[a, b]$  with  $h \geq 0$  in  $[a, b]$ . Then it holds that

$$\int_a^b h \, dx \geq 0.$$

**Claim.** There exists  $(\tilde{\varphi}_n)_{n \in \mathbb{N}}$  with  $\tilde{\varphi}_n \rightarrow h$  uniform on  $[a, b]$  and  $\tilde{\varphi}_n \geq 0$ .

*Proof.* Let  $(\varphi_n)_{n \in \mathbb{N}}$ ,  $\varphi_n \in \tau[a, b]$  with  $\varphi_n \rightarrow h$  uniform on  $[a, b]$ . We define  $\tilde{\varphi}_n$  such that

$$\begin{aligned} \varphi_n &= \sum_{j=1}^{m_n} c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^{m_n} d_j \chi_{\{x_j\}} \\ \tilde{\varphi}_n &:= \sum_{j=1}^{m_n} \underbrace{\tilde{c}_j}_{\geq 0} \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^{m_n} \underbrace{h(x_j)}_{\geq 0} \chi_{\{x_j\}} \end{aligned}$$

and  $\tilde{c}_j := \max\{c_j, 0\} \geq 0$ . So it holds that  $\tilde{\varphi}_n \geq 0$ .

For  $x = x_l$  ( $l \in \{0, \dots, m_n\}$ ) it holds that

$$\begin{aligned} |\tilde{\varphi}_n(x_l) - h(x_l)| &= \left| \sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1}, x_j)}(x_l)}_{=0 \text{ bc. } x_l \notin (x_{j-1}, x_j)} + \sum_{j=0}^{m_n} h(x_j) \underbrace{\chi_{\{x_j\}}(x_l)}_{=\delta_{j,l}} - h(x_l) \right| \\ &= |h(x_l) - h(x_l)| = 0 \leq |\varphi_n(x_l) - h(x_l)| \end{aligned}$$

For  $x \in (x_{j-1}, x_j)$  it holds that

$$\begin{aligned} |\tilde{\varphi}_n(x) - h(x)| &= \left| \sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1}, x_j)}(x)}_{\delta_{l,j}} + \sum_{j=0}^{m_n} h(x) \cdot \underbrace{\chi_{\{x_j\}}(x)}_{=0 \text{ bc. } x \neq x_j} - h(x) \right| \\ &= |\tilde{c}_l - h(x)| = \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ |h(x)| = h(x) & \text{if } c_l < 0 \end{cases} \\ &\leq \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ h(x) - c_l & \text{if } c_l < 0 \end{cases} \\ &= \begin{cases} |\varphi_n(x) - h(x)| & \text{if } c_l = \varphi_n(x) \geq 0 \\ |h(x) - \varphi_n(x)| & \text{if } c_l = \varphi_n(x) < 0 \end{cases} \\ &= |\varphi_n(x) - h(x)| \end{aligned}$$

hence,  $|\tilde{\varphi}_n(x) - h(x)| \leq |\varphi_n(x) - h(x)|$  for  $x \in (x_{l-1}, x_l)$  as well as  $x = x_l$ , hence

$$\|\tilde{\varphi}_n - h\|_{\infty} \leq \underbrace{\|\varphi_n - h\|_{\infty}}_{\rightarrow 0 \text{ for } n \rightarrow \infty}$$

Hence  $\|\tilde{\varphi}_n - h\|_\infty \rightarrow 0$  for  $n \rightarrow \infty$ , hence  $\tilde{\varphi}_n$  converges uniformly to  $h$ .  
There exists

$$\int_a^b h \, dx = \lim_{n \rightarrow \infty} \underbrace{\int_a^b \underbrace{\tilde{\varphi}_n}_{\geq 0} \, dx}_{\geq 0} \geq 0$$

□

To show monotonicity, we let  $f \leq g$  in  $[a, b]$ , hence  $h = g - f \geq 0$  in  $[a, b]$

$$\begin{aligned} \implies 0 &\leq \int_a^b h \, dx = \int_a^b g \, dx - \int_a^b f \, dx \\ &\implies \int_a^b f \, dx \leq \int_a^b g \, dx \end{aligned}$$

**Boundedness.** Consider  $|f|$ . Proving  $|f| \in \mathcal{R}[a, b]$  is left as an exercise to the reader.

$$\begin{aligned} f &\leq |f| \text{ on } [a, b] \xrightarrow{\text{monotonicity}} \int_a^b f \, dx \leq \int_a^b |f| \, dx \\ -f &\leq |f| \text{ on } [a, b] \xrightarrow{\text{monotonicity}} \int_a^b (-f) \, dx = - \int_a^b f \, dx \leq \int_a^b |f| \, dx \\ &\implies \left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx \end{aligned}$$

□

**Remark 4.1.**  $\mathcal{R}[a, b]$  is a vector space.

1.  $f, g \in \mathcal{R}[a, b] \implies \lambda f + \mu g \in \mathcal{R}[a, b]$ .  $\|\cdot\|_\infty$  is a norm on  $\mathcal{R}[a, b]$ .  $(\mathcal{R}[a, b], \|\cdot\|_\infty)$  is a normed vector space. Subspace of  $(\mathcal{B}[a, b], \|\cdot\|_\infty)$ . We will show in the practicals that  $(\mathcal{R}[a, b], \|\cdot\|_\infty)$  is complete.

**Theorem 4.2** (Mean value theorem of integration calculus). *Let  $f$  be continuous on  $[a, b]$  and  $p \in \mathcal{R}[a, b]$  and  $p \geq 0$  in  $[a, b]$ . Then  $f \cdot p \in \mathcal{R}[a, b]$  and there exists  $\xi \in [a, b]$  such that*

$$\int_a^b f \cdot p \, dx = f(\xi) \cdot \int_a^b p \, dx$$

*Proof.* Let  $m = \min \{f(z) : z \in [a, b]\}$  (exists because  $f$  is continuous and  $[a, b]$  is compact).

$$M = \max \{f(z) : z \in [a, b]\}$$

$$f([a, b]) = [m, M] \text{ (by the mean value theorem)}$$

It holds that

$$\underbrace{m \cdot p(x)}_{\geq 0} \leq f(x) \cdot p(x) \leq M \cdot p(x)$$

By monotonicity,

$$m \int_a^b p(x) dx \leq \int_a^b f p dx \leq M \int_a^b p dx$$

Therefore, there exists  $\eta \in [m, M]$ .

$$\eta \cdot \int_a^b p(x) dx = \int_a^b f p dx$$

Mean value theorem: For  $\eta \in [m, M]$  there exists  $\xi \in [a, b]$  such that

$$\eta = f(\xi) \text{ (f is continuous!)}$$

Hence,

$$f(\xi) \int_a^b p dx = \int_a^b f \cdot p dx$$

$f \cdot p$  is regulated function (over one-sided limits). □

**Lemma 4.1.** Let  $f \in \mathcal{R}[a, b]$  and  $a \leq \alpha < \beta < \gamma \leq b$ . Then

$$f|_{[\alpha, \beta]} \in \mathcal{R}[\alpha, \beta], f|_{[\beta, \gamma]} \in \mathcal{R}[\beta, \gamma]$$

$$f|_{[\alpha, \gamma]} \in \mathcal{R}[\alpha, \gamma] \text{ (immediate over onesided limit)}$$

and it holds that

$$\int_{\alpha}^{\gamma} f dx = \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f dx$$

Compare with Figure 20.

*Proof.* Show that this statement holds for  $\varphi \in \tau[a, b]$ . Without loss of generality,  $\alpha = a, \gamma = b$ .

$$\gamma' = \sum_{j=1}^m c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^m 0 \cdot \chi_{x_j}$$

The zero represents that this term is not relevant for the integral in any way.

**Case 1**  $\beta = x_l$  for some  $l \in \{1, \dots, m-1\}$

$$\int_{\alpha}^{\gamma} \varphi dx = \sum_{j=1}^m c_j (x_j - x_{j-1})$$

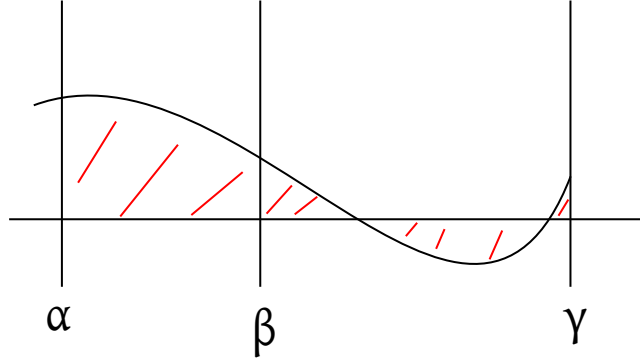


Figure 20: Positive and negative area covered by the integral

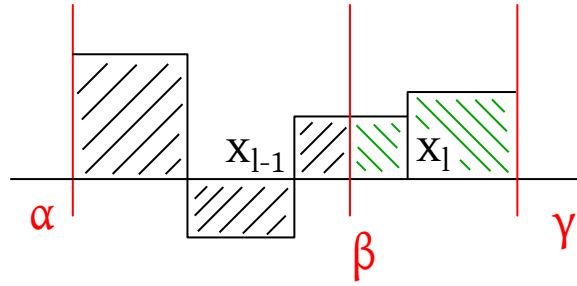


Figure 21: Setting for  $\beta \in (x_{l-1}, x_l)$

$$\int_{\alpha}^{\beta} \varphi dx = \int_{\alpha}^{x_l} \varphi dx = \sum_{j=1}^l c_j(x_j - x_{j-1})$$

$$\int_{\beta}^{\gamma} \varphi dx = \int_{x_l}^{\gamma} \varphi dx = \sum_{j=l+1}^m c_j(x_j - x_{j-1})$$

And now,

$$\sum_{j=1}^l c_j(x_j - x_{j-1}) + \sum_{j=l+1}^m c_j(x_j - x_{j-1}) = \sum_{j=1}^m c_j(x_j - x_{j-1})$$

**Case 2**  $\beta \in (x_{l-1}, x_l)$  for some  $l \in \{1, \dots, m\}$ . Compare with Figure 21.

$$\int_{\alpha}^{\beta} \varphi dx = \sum_{j=1}^{l-1} c_j(x_j - x_{j-1}) + c_l \cdot (\beta - x_{l-1})$$



$$\int_{\beta}^{\gamma} \varphi dx = c_l(x_l - \beta) + \sum_{j=l+1}^m c_j(x_j - x_{j-1})$$

Now we consider the addition of these two expressions:

$$\begin{aligned} & \int_{\alpha}^{\beta} \varphi dx + \int_{\beta}^{\gamma} \varphi dx \\ &= \sum_{j=1}^{l-1} c_j(x_j - x_{j-1}) + \underbrace{c_l(\beta - x_{l-1}) + c_l(x_l - \beta)}_{=c_l(x_l - x_{l-1})} + \sum_{j=l+1}^m c_j(x_j - x_{j-1}) \\ &= \sum_{j=1}^m c_j(x_j - x_{j-1}) = \int_{\alpha}^{\gamma} \varphi dx \end{aligned}$$

Let  $\varphi_n \in \tau[\alpha, \beta]$  with  $\varphi_n \rightarrow f$  uniform on  $[\alpha, \beta] \implies \varphi_n|_{[\alpha, \beta]} \rightarrow f|_{[\alpha, \beta]}$  uniform on  $[\alpha, \beta]$  and also  $\varphi_n|_{[\beta, \gamma]} \rightarrow f|_{[\beta, \gamma]}$  uniform on  $[\beta, \gamma]$ .

$$\begin{aligned} \int_{\alpha}^{\gamma} f dx &= \lim_{n \rightarrow \infty} \int_{\alpha}^{\gamma} \varphi_n dx = \lim_{n \rightarrow \infty} \left( \int_{\alpha}^{\beta} \varphi_n dx + \int_{\beta}^{\gamma} \varphi_n dx \right) \\ &= \underbrace{\lim_{n \rightarrow \infty} \int_{\alpha}^{\beta} \varphi_n dx}_{\substack{\text{exists because} \\ \varphi_n|_{[\alpha, \beta]} \rightarrow f|_{[\alpha, \beta]} \text{ uniform}}} + \lim_{n \rightarrow \infty} \int_{\beta}^{\gamma} \varphi_n dx = \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f dx \end{aligned}$$

□

**Remark 4.2** (Notation). Let  $\alpha < \beta$ ,  $\alpha, \beta \in [a, b]$  and  $f \in \mathcal{R}[a, b]$ . We let

$$\int_{\beta}^{\alpha} f dx := - \int_{\alpha}^{\beta} f dx$$

By this convention, it holds that

$$\int_{\alpha}^{\alpha} f dx = - \int_{\alpha}^{\alpha} f dx \implies \int_{\alpha}^{\alpha} f dx = 0$$

**Lemma 4.2.** Let  $f \in \mathcal{R}[a, b]$  and  $\alpha, \beta, \gamma \in [a, b]$  (without particular order). Then it holds that

$$\int_{\alpha}^{\gamma} f dx = \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f dx$$

*Proof.* Special case: 2 points are equal

$$\alpha = \gamma \implies \int_{\alpha}^{\alpha} f dx = 0$$

$$\int_{\alpha}^{\beta} f dx + \int_{\beta}^{\alpha} f dx = \int_{\alpha}^{\beta} f dx - \int_{\alpha}^{\beta} f dx = 0$$

$\beta = \gamma \quad \beta = \alpha$

Case:  $\alpha < \beta < \gamma$  follows immediately

And just as a representative other case:  $\alpha < \gamma < \beta$

$$\begin{aligned} \int_{\alpha}^{\beta} f dx & \underbrace{=}_{\text{by Lemma 1.7}} \int_{\alpha}^{\gamma} f dx + \underbrace{\int_{\gamma}^{\beta} f dx}_{-\int_{\beta}^{\gamma} f dx} \\ \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f dx &= \int_{\alpha}^{\gamma} f dx \end{aligned}$$

□

*This lecture took place on 2018/04/17.*

**Lemma 4.3.** *Let  $f \in \mathcal{R}[a, b]$ . Then there exists an at most countable set  $A \subseteq [a, b]$  such that  $f$  is continuous in every point  $x \in [a, b] \setminus A$ .*

*Proof.* Let  $f \in \mathcal{R}[a, b]$  and  $(\varphi_n)_{n \in \mathbb{N}}$  with  $\varphi_n \in \tau[a, b]$  and  $\varphi \rightarrow f$  converging uniformly on  $[a, b]$ .

$$\begin{aligned} \varphi_n &= \sum_{j=1}^{m_n} c_j^n \chi_{(X_{j-1}^n, X_j^n)} + \sum_{j=0}^{m_n} d_j^n \chi_{\{x_j^n\}} \\ x_0^n &= a < x_1^n < \dots < x_{m_n}^n = b \end{aligned}$$

are separating points for  $\varphi_n$

$$A = \{X_j^n : n \in \mathbb{N}, j \in \{0, \dots, m_n\}\}$$

$A$  is a countable union of finite sets  $A_n = \{x_0^n, x_{m_n}^n\}$ .  $A$  is countable (as unions of finite sets are).

Now we show:  $f$  is continuous in every point  $x \in [a, b] : x \notin A$ . Let  $\varepsilon > 0$  be arbitrary. Choose  $N \in \mathbb{N}$  sufficiently large such that  $\|\varphi_N - f\|_{\infty} < \frac{\varepsilon}{2}$ . Because  $x \in A$ , there exists  $j \in \{1, \dots, m_N\}$  such that  $x \in (x_{j-1}^N, x_j^N)$  is open. Choose  $\delta > 0$  such that  $(x - \delta, x + \delta) \subset (x_{j-1}^N, x_j^N)$ , hence  $\forall \xi \in (x - \delta, x + \delta)$  it holds that  $\varphi_N(\xi) = c_j^N$ . Now consider  $\xi \in (x - \delta, x + \delta)$ , hence  $|\xi - x| < \delta$ . Then it holds that

$$|f(\xi) - f(x)| = \left| f(\xi) - \underbrace{\varphi_N(x)}_{c_j^N = \varphi_N(\xi)} + \varphi_N(x) - f(x) \right|$$

$$\leq \underbrace{|f(\xi) - \varphi_N(\xi)|}_{\leq \|f - \varphi_N\|_\infty} + \underbrace{|\varphi_N(x) - f(x)|}_{\leq \|\varphi_N - f\|_\infty} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence  $f$  is continuous in  $x$ .  $\square$

**Remark 4.3** (Notation). Let  $f \in \mathcal{R}[a, b]$ . For  $x \in [a, b)$ , there exists  $f_+(x) := \lim_{\xi \rightarrow x_+} f(\xi)$ . For  $x \in (a, b]$ , there exists  $f_-(x) := \lim_{\xi \rightarrow x_-} f(\xi)$ . Because of Lemma 4.3, it holds that  $f_+(x) = f_-(x) = f(x)$  for all  $x \in [a, b] \setminus A$  and  $A$  is at most countable.

**Definition 4.2** (One-sided derivatives). Let  $g : [a, b] \rightarrow \mathbb{R}$  and  $x \in [a, b)$ . We say  $g$  has the right-sided derivative  $g'_+(x)$  if

$$\lim_{\xi \rightarrow x_+} \frac{g(\xi) - g(x)}{\xi - x} =: g'_+(x)$$

exists. Analogously we define the left-sided derivative

$$g'_-(x) = \lim_{\xi \rightarrow x_-} \frac{g(\xi) - g(x)}{\xi - x}$$

for  $x \in (a, b]$ . Compare with Figure 22.

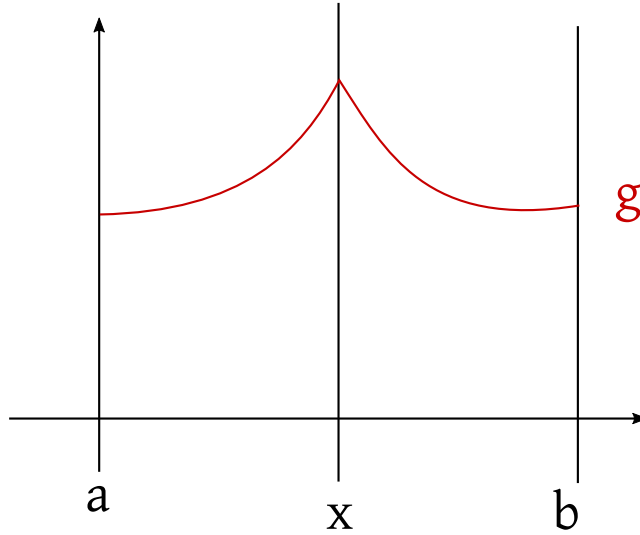


Figure 22: In this example, the left- and right-sided derivatives are not equal.  $f'_+(x) \neq f'_-(x)$

**Remark 4.4.** If  $g$  in  $x$  has a one-sided derivative, then it holds that

$$\lim_{\xi \rightarrow x_{\pm}} (g(\xi) - g(x)) = 0$$

Hence  $g$  is continuous in  $x$ .

**Remark 4.5.**  $g : [a, b] \rightarrow \mathbb{R}$  is differentiable in point  $x \in (a, b)$  with derivative  $g'(x) \iff g$  has a left- and right-sided derivative in  $x$  and it holds that  $g'_-(x) = g'_+(x) (= g'(x))$ .

**Theorem 4.3** (Fundamental theorem of differential/integration calculus, variation 1). Isaac Barrow (1630–1677), Isaac Newton (1642–1726), Gottfried Wilhelm von Leibniz (1646–1716).

Let  $f \in \mathcal{R}[a, b]$ ,  $\alpha \in [a, b]$  and we define

$$F(x) = \int_{\alpha}^x f d\xi$$

Then  $F$  is right-sided differentiable in every point  $x \in [a, b)$  and in every  $x \in (a, b]$  left-sided differentiable. Furthermore it holds that

$$F'_+(x) = f_+(x) \forall x \in [a, b) \quad (2)$$

$$F'_-(x) = f_-(x) \forall x \in (a, b] \quad (3)$$

**Remark 4.6.**

$$\frac{d}{dx} \left( \int_{\alpha}^x f d\xi \right) = f(x)$$

for all  $x$  such that  $f$  is continuous in  $x$ . For those  $x$ ,  $F'(x)$  is differentiable in  $x$  with  $F'(x) = f(x)$ .

**Definition 4.3.** Let  $f \in \mathcal{R}[a, b]$  and  $\varphi : [a, b] \rightarrow \mathbb{R}$  such that  $\varphi$  is one-sided differentiable on  $[a, b]$ . If  $\Phi'_+(x) = f_+(x) \forall x \in [a, b)$  and  $\Phi'_-(x) = f_-(x) \forall x \in (a, b]$  then we call  $\Phi$  an antiderivative of regulated function  $f$ .

*Proof of the Theorem 4.3.* Let  $x_1, x_2 \in [a, b]$  be arbitrary. Let  $F$  be defined as above. Then it holds that

$$\begin{aligned} |F(x_2) - F(x_1)| &= \left| \int_{\alpha}^{x_2} f d\xi - \int_{\alpha}^{x_1} f d\xi \right| \\ &= \left| \int_{\alpha}^{x_2} f d\xi + \int_{x_1}^{\alpha} f d\xi \right| = \left| \int_{x_1}^{x_2} f d\xi \right| \\ &\leq \int_{x_1}^{x_2} |f| d\xi \leq \int_{x_1}^{x_2} \underbrace{\|f\|_{\infty}}_{\text{const independent of } \xi} d\xi = \|f\|_{\infty} \cdot |x_2 - x_1| \end{aligned}$$

Hence  $F$  is Lipschitz continuous with Lipschitz constant  $\|f\|_\infty$ . So  $F$  is continuous in  $[a, b]$ .

One-sided derivatives: Let  $x \in [a, b)$  and  $\varepsilon > 0$  be arbitrary. Choose  $\delta > 0$  such that  $\forall \xi \in [x, x + \delta)$  it holds that  $|f(\xi) - f_+(x)| < \varepsilon$ . For  $\xi \in (x, x + \delta)$  it holds that

$$\begin{aligned} \left| \frac{F(\xi) - F(x)}{\xi - x} - f_+(x) \right| &= \frac{1}{|\xi - x|} \left| \underbrace{\int_x^\xi f \, dy}_{F(\xi) - F(x)} - \underbrace{f_+(x)(\xi - x)}_{\int_x^\xi \underbrace{f_+(x)}_{\text{const.}} \, dy} \right| \\ &= \frac{1}{|\xi - x|} \left| \int_x^\xi (f - f_+(x)) \, dy \right| \leq \frac{1}{|\xi - x|} \int_x^\xi \underbrace{|f(y) - f_+(x)|}_{< \varepsilon} \, dy \\ &\quad y \in (x, \xi) \subseteq (x, x + \delta) \\ &< \frac{1}{\xi - x} \varepsilon \cdot \underbrace{\int_x^\xi 1 \, dy}_{|\xi - x|} = \varepsilon \end{aligned}$$

Hence,  $F'_+(x) = f_+(x)$ . Analogously,  $F'_-(x) = f_-(x)$  for  $x \in (a, b]$ .  $\square$

**Theorem 4.4** (Fundamental theorem of differential/integration calculus, variation 2). *Let  $f \in \mathcal{R}[a, b]$  and  $\phi$  is an arbitrary antiderivative of  $f$  according to Definition 4.3. For  $\alpha, \beta \in [a, b]$  arbitrary, it holds that*

$$\int_\alpha^\beta f \, dx = \phi(\beta) - \phi(\alpha)$$

**Remark 4.7.** *Let  $f$  be continuous and  $\phi$  be an antiderivative of  $f$ . Hence,  $\Phi'(x) = f(x) \forall x \in [a, b]$ . Then it holds that*

$$\int_\alpha^\beta \Phi' \, dx = \Phi(\beta) - \Phi(\alpha)$$

“Integral of a derivative of  $\Phi$  gives  $\Phi(\beta) - \Phi(\alpha)$ ”.

**Lemma 4.4.** *Let  $A \subseteq [a, b]$  countable.  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f$  is differentiable in every point  $x \in [a, b] \setminus A$ . Furthermore let  $|f'(x)| \leq L$  ( $L \geq 0$ ) for all  $x \in [a, b] \setminus A$ . Then  $f$  is Lipschitz continuous on  $[a, b]$  with constant  $L$ , hence*

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1| \forall x_1, x_2 \in [a, b]$$

**Remark 4.8.** Some people call it differentiable almost everywhere, but this expression collides with a different definition pronounced the same way from measure theory.

*Proof.* Let  $x_1, x_2 \in [a, b]$ , without loss of generality:  $x_1 < x_2$ . Let  $\varepsilon > 0$  be arbitrary. We define

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L + \varepsilon)(x - x_1)$$

for  $x \in [x_1, b]$ .

Let  $\varepsilon > 0$  be arbitrary. We prove:  $F_\varepsilon(x) \leq 0 \forall x \in [x_1, b]$ . In particular:  $F_\varepsilon(x_2) \leq 0$ . Hence,

$$|f(x_2) - f(x_1)| \leq (L + \varepsilon) \underbrace{(x_2 - x_1)}_{|x_2 - x_1|}$$

We prove by contradiction: Assume there exists  $\varepsilon > 0$  and  $x_\varepsilon > x_1$  such that

$$F_\varepsilon(x_\varepsilon) = \eta > 0 \tag{4}$$

We recognize: Let  $A' = [x_1, b] \cap A$  be countable.

1. hence  $F_\varepsilon(A') \subseteq \mathbb{R}$  is countable
2.  $F_\varepsilon(x_1) = 0, F_\varepsilon(x_\varepsilon) > 0 \implies x_\varepsilon > x_1$
3.  $F_\varepsilon$  is continuous on  $[x_1, b]$ . It holds that  $0 \in F_\varepsilon([x_1, x_\varepsilon])$  because  $0 = F_\varepsilon(x_1)$  and  $\eta \in F_\varepsilon([x_1, x_\varepsilon])$  because  $\eta = F_\varepsilon(x_\varepsilon)$ .

By the Intermediate Value Theorem, it follows that  $[0, \eta] \subseteq F_\varepsilon([x_1, x_\varepsilon])$  where  $[0, \eta]$  is uncountable.  $F_\varepsilon(A')$  is countable, hence there exists  $\gamma \in (0, \eta]$  such that  $\gamma = F_\varepsilon(y)$  and  $\gamma \notin A'$  ( $\gamma > 0$ ); compare with inequality (4). Hence,  $y \notin A'$ . So  $f$  in  $y$  is differentiable. Let  $B := F_\varepsilon^{-1}(\{\gamma\}) \cap ([x_1, x_\varepsilon] \setminus A')$ , where we can skip  $A'$ . Then  $B \neq \emptyset$ .

$B \subseteq [x_1, x_\varepsilon]$  is therefore bounded,  $B \neq \emptyset$ . Hence,  $B$  has a supremum. Let  $x = \sup B$ . Choose  $(y_n)_{n \in \mathbb{N}}$  with  $y_n \in B$  and  $y_n \rightarrow x$  for  $n \rightarrow \infty$ . Because  $F_\varepsilon$  is continuous, it holds that

$$\lim_{n \rightarrow \infty} \underbrace{F_\varepsilon(y_n)}_\gamma = F_\varepsilon(x)$$

hence  $F_\varepsilon(x) = \gamma$ . This implies  $x \notin A$ .

Furthermore it holds for  $w \in (x, x_\varepsilon]$  that  $F_\varepsilon(w) > \gamma$ . Because assume the opposite ( $F_\varepsilon(w) \leq \gamma$  for  $w > x$ ). Furthermore it holds that  $F_\varepsilon(x_\varepsilon) = \eta \geq \gamma$ . Because of the Intermediate Value Theorem,  $\exists y \geq w$  with  $F_\varepsilon(y) = \gamma$ . This contradicts with the supremum property of  $x$ .

Now let  $y \in (x, x_\varepsilon]$ .

$$\begin{aligned}
\varphi(y) &= \frac{F_\varepsilon(y) - F_\varepsilon(x)}{y - x} \\
&\stackrel{\substack{\text{definition of} \\ F_\varepsilon}}{=} \frac{|f(y) - f(x_1)| - |f(x) - f(x_1)|}{y - x} - \frac{(L + \varepsilon)(y - x_1 - x + x_1)}{y - x} \\
&\stackrel{\substack{\leq \\ \text{reversed triangle ineq.}}}{\leq} \frac{f(y) - f(x)}{y - x} - (L + \varepsilon)
\end{aligned}$$

Because  $F_\varepsilon(y) > \gamma = F_\varepsilon(x)$  it holds that  $\varphi(y) > 0$  for  $y > x$ . So,

$$\frac{|f(y) - f(x)|}{y - x} \geq L + \varepsilon$$

$$|f'(x)| = \lim_{y \rightarrow x_+} \left| \frac{f(y) - f(x)}{y - x} \right| \geq L + \varepsilon$$

This contradicts with the boundedness of the derivative by  $L$  and  $f$  is in  $x \notin A$  differentiable.

So, inequality (4) does not hold. Therefore  $\forall x_1, x_2$  with  $x_1 < x_2$  in  $[a, b]$  and  $\forall \varepsilon > 0$ ,

$$\begin{aligned}
|f(x_2) - f(x_1)| &\leq (L + \varepsilon)|x_2 - x_1| \\
\implies |f(x_2) - f(x_1)| &\leq L|x_2 - x_1|
\end{aligned}$$

□

**Corollary** (Corollary to Lemma 4.4). *Let  $f, g : [a, b] \rightarrow \mathbb{R}$  differentiable for all points  $x \in [a, b] \setminus A$  and  $A$  is countable. Furthermore let  $f'(x) = g'(x) \forall x \notin A$ . Then there exists  $K \in \mathbb{R}$  such that  $f(x) = g(x) + K \forall x \in [a, b]$ .*

*Proof.* Let  $h = f - g$ . Then it holds that

$$h'(x) = f'(x) - g'(x) = 0 \forall x \in [a, b] \setminus A$$

By Lemma 4.4 with  $L = 0$ , it follows that

$$\begin{aligned}
|h(x_1) - h(x_2)| &\leq 0 \cdot |x_1 - x_2| = 0 \\
\implies h(x_1) &= h(x_2) \forall x_1, x_2 \in [a, b]
\end{aligned}$$

Hence,  $h(x) = K \in \mathbb{R}$ .

$$\implies f(x) = g(x) + h(x) = g(x) + K$$

□

This lecture took place on 2018/04/19.

By reference (\*),  $\gamma \in [0, \eta]$  (uncountable) and  $\gamma \notin f(A)$  (countable).

$$\implies \forall u \in [x_1, b) \text{ with } F_\varepsilon(u) = \gamma$$

it holds that  $u \notin A$ , hence  $f$  is differentiable in  $u$ .

*Proof of Theorem 4.4.* Let  $f \in \mathcal{R}[a, b]$ ,  $\phi$  is an antiderivative of  $f$ , hence  $\phi'_+ = f_+$ ,  $\phi'_- = f_-$ . Let  $\alpha \in [a, b]$  be arbitrary. By the Theorem variant 1,  $F(x) = \int_\alpha^x f d\xi$  is also an antiderivative of  $f$ . By Lemma 4.4,  $\exists K \in \mathbb{R} : F(x) = \int_\alpha^x f d\xi = \phi(x) + K$ . Determine  $K$ : Let  $x = \alpha \implies F(\alpha) = \int_\alpha^\alpha f dx = 0 = \phi(\alpha) - K$  hence  $K = \phi(\alpha)$ . Hence,

$$\int_\alpha^x f d\xi = \phi(x) - \phi(\alpha)$$

Let  $x = \beta$ . □

**Remark 4.9** (Remark for the previous corollary).  $F, \phi$  are differentiable on all points  $x$  for which  $f$  is continuous (all of them except for countable many). For those  $x$ , it holds that  $F'(x) = \phi'(x) = f(x)$ .

**Remark 4.10** (Notation). Let  $f \in \mathcal{R}[a, b]$ . Then

$$\int f dx$$

- is some particular antiderivative of  $f$  (usually some arbitrary chosen)
- the set of all antiderivatives of  $f$

$$\int f dx = \{F : F \text{ is antiderivative of } f\}$$

If  $F_0$  is some fixed antiderivative, then

$$\int f dx = \{F_0 + K : K \in \mathbb{R}\}$$

Then  $\int f dx$  is the so-called indefinite integral of  $f$ . Notation:

$$\int x^k dx = \frac{x^{k+1}}{k+1} + c \quad (k \neq -1)$$



$f$	$F$	remark
$x^\alpha$	$\frac{x^{\alpha+1}}{\alpha+1} + c$	$\alpha \in \mathbb{R} \setminus \{-1\}$ ; restrict $x$ such that $x^\alpha$ and $x^{\alpha+1}$ are defined
$x^{-1}$	$\ln x + c \ (x > 0)$	
$\left(\frac{1}{-x}\right) \cdot (-1) = x^{-1}$	$\ln -x + c \ (x < 0)$	
$e^x$	$e^x$	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\frac{1}{1+x^2}$	$\arctan x$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$ x  < 1$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	

Table 1: Table of antiderivatives

## 4.1 Integration methods

In this chapter, we discuss how to determine the antiderivative of a function. Usually they are composites of basic functions. Some of these are given in Table 1.

**Remark 4.11.** Let  $F, G : [a, b] \rightarrow \mathbb{R}$  in  $x \in [a, b)$  right-sided differentiable. Then also  $F \cdot G$  in  $x$  is right-sided differentiable and it holds that

$$(F \cdot G)'_+(x) = F'_+(x) \cdot G(x) + F(x) \cdot G'_+(x)$$

hence the product law holds.

Analogously, the same holds for the left-sided derivative.

Look up the proof in the course Analysis 1.

### 4.1.1 Partial integration

**Definition 4.4** (Partial integration). Let  $f, g$  be given. Let  $F, G$  be its antiderivatives respectively. Then  $F \cdot G$  is an antiderivative of  $F \cdot g + f \cdot G$ .

This is immediate, because

$$(F \cdot G)'_+ = F'_+ \cdot G + F \cdot G'_+ = f_+ \cdot G + F \cdot g_+ = f_+ G_+ + F_+ \cdot g_+$$

Hence, it holds that

$$\int_a^b (Fg + fG) dx = \underbrace{F(b) \cdot G(b) - F(a)G(a)}_{=: F \cdot G|_a^b}$$

Usually, this is rewritten as

$$\int_a^b F \cdot g \, dx = F \cdot G|_a^b - \int_a^b fG \, dx$$

If  $F = u$  is continuously differentiable and  $G = v$  as well, then  $f = u'$  and  $g = v'$  and the law has the structure

$$\int_a^b uv' \, dx = u \cdot v|_a^b - \int_a^b u'v \, dx$$

**Example 4.1.** Let  $a \neq -1$  and  $x > 0$ .

$$\begin{aligned} \int \underbrace{x^a}_{v'} \cdot \underbrace{\ln x}_u \, dx &= \underbrace{\left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^a & v = \frac{x^{\alpha+1}}{\alpha+1} \end{array} \right|}_{\text{scribble notes}} \cdot \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^{\alpha+1}}{\alpha+1} \, dx \\ &= \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{\alpha+1} \int x^{\alpha} \, dx = \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{(\alpha+1)^2} x^{\alpha+1} \end{aligned}$$

**Example 4.2.** Let  $k \in \{2, 3, 4, \dots\}$ .

$$\begin{aligned} \int \cos^k(x) \, dx &= \left| \begin{array}{ll} u = \cos^{k-1}(x) & u' = (k-1) \cdot \cos^{k-2}(x) \cdot (-\sin x) \\ v' = \cos x & v = \sin x \end{array} \right| \\ &= \cos^{k-1}(x) \sin x + (k-1) \int \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{(1-\cos^2 x)} \, dx \\ &= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx - (k-1) \int \cos^k(x) \, dx \end{aligned}$$

Then we can use the following identity:

$$k \int \cos^k(x) \, dx = \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx$$

This gives a recursive formula:

$$\int \cos^k(x) \, dx = \frac{1}{k} \cos^{k-1}(x) \cdot \frac{k-1}{k} \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx$$

Analogously,

$$\int \sin^k(x) \, dx = -\frac{1}{k} \sin^{k-1}(x) \cdot \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x) \, dx$$

Let  $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x) dx$ . Then the following formula holds:

$$\begin{aligned} c_{2n} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2} \\ c_{2n+1} &= \prod_{k=1}^n \frac{2k}{2k+1} \end{aligned}$$

*Proof by induction.* Let  $n = 1$ .

$$\begin{aligned} c_2 &= \int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2} \cos x \sin x \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 dx = 0 - 0 + \frac{\pi}{4} \\ &= \underbrace{\prod_{k=1}^1 \frac{2k-1}{2k}}_{\frac{1}{2}} \cdot \frac{\pi}{2} \end{aligned}$$

$$c_1 = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$$\underbrace{\prod_{k=1}^0 \frac{2k}{2k+1}}_{\text{empty product}} = 1$$

We make the induction step  $n \rightarrow n+1$ :

$$\begin{aligned} c_{2(n+1)} &= \frac{1}{2n+2} \cdot \underbrace{\cos^{2n+1}(x)}_{=0 \text{ for } x=\frac{\pi}{2}} \cdot \underbrace{\sin(x)}_{=0 \text{ for } x=0} \Big|_0^{\frac{\pi}{2}} + \frac{2n+1}{2n+2} \int_0^{\frac{\pi}{2}} \cos^{2n}(x) dx \\ &= \frac{2n+1}{2n+2} \prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2} = \prod_{k=1}^{n+1} \frac{2k-1}{2k} \cdot \frac{\pi}{2} \end{aligned}$$

$c_{2(n+1)+1}$  analogously. □

**Theorem 4.5** (Wallis product). *John Wallis (1616–1703), result from 1655*

Let  $w_n = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots$ . Then it holds that  $\lim_{n \rightarrow \infty} w_n = \frac{\pi}{2}$ .

*Proof.*

$$\frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} \cdot \prod_{k=1}^n \frac{\frac{2k}{2k+1}}{\prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2}} = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = w_n$$

It remains to show that  $\lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = 1$  in  $[0, \frac{\pi}{2}]$  it holds that  $0 \leq \cos x \leq 1$ .

$$\implies \cos^{2n+2}(x) \leq \cos^{2n+1}(x) \leq \cos^{2n}(x)$$

So,  $c_{2n+2} \leq c_{2n+1} \leq c_{2n}$  for  $n \geq 1$ .

$$\begin{aligned} 1 &\geq \frac{c_{2n+1}}{c_{2n}} \\ \implies 1 &\geq \frac{c_{2n+1}}{c_{2n}} \geq \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1} \frac{2k-1}{2k} \frac{\pi}{2}}{\prod_{k=1}^n \frac{2k-1}{2k} \frac{\pi}{2}} \\ &= \frac{2n+2-1}{2n+2} \rightarrow 1 \text{ for } n \rightarrow \infty \end{aligned}$$

Because of the sandwich lemma for convergent sequences, the intermediate expression must also converge to 1, hence

$$\lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = 1 \quad \wedge \quad \frac{\pi}{2} \cdot \lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = \underbrace{\lim_{n \rightarrow \infty} w_n}_{=1}$$

□

#### 4.1.2 Integration by substitution

**Definition 4.5** (Integration by substitution). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Let  $t : [\alpha, \beta] \rightarrow [a, b]$  be continuously differentiable. Let  $F$  be an antiderivative of  $f$  ( $F$  is therefore continuously differentiable). Then  $F \circ t : [\alpha, \beta] \rightarrow \mathbb{R}$  is also continuously differentiable and the chain rule holds:

$$(F \circ t)' = (F' \circ t) \cdot t' = (f \circ t) \cdot t'$$

Hence  $F \circ t$  is an antiderivative of  $(f \circ t) \cdot t'$ . We apply it to integration:

$$\int_{\alpha}^{\beta} (f \circ t)(u) \cdot t'(u) du = (F \circ t)(\beta) - (F \circ t)(\alpha) = F(t(\beta)) - F(t(\alpha)) = \int_{t(\alpha)}^{t(\beta)} f(x) dx$$

Then we get the substitution integration method:

$$\int_{t(\alpha)}^{t(\beta)} f(x) dx = \int_{\alpha}^{\beta} f(t(u)) \cdot t'(u) du$$

**Remark 4.12** (Mnemonic). Consider the left-hand side and right-hand side simultaneously. Let  $x = t(u)$  (expressions inside parentheses). Then  $dx = t'(u) \cdot du$  (expressions on the right). Let  $u = \alpha \implies x = t(\alpha)$  and  $u = \beta \implies x = t(\beta)$  (interval boundaries).

**Example 4.3.**

$$\int_0^1 2x \sqrt{1-x^2} dx$$

Usually we have some expression, we want to substitute with  $u$ .

$$1 - x^2 = u \quad x = \sqrt{1-u} = t(u)$$

$$x = 0 = t(1) \quad x = 1 = t(0)$$

$$dx = \frac{1}{2} \cdot \frac{1}{\sqrt{1-u}} \cdot (-1) du$$

$$\int_0^1 2x \sqrt{1-x^2} dx = \int_1^0 2 \cdot \sqrt{1-u} \cdot u \cdot \frac{1}{2}(-1) \frac{1}{\sqrt{1-u}} du = \int_0^1 \sqrt{u} du = \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^1 = \frac{2}{3}$$

$$\int_0^1 2x \sqrt{\underbrace{1-x^2}_u} dx = \left| \begin{array}{ll} u = 1-x^2 & \\ x = 0 & \Leftrightarrow u = 1 \\ x = 1 & \Leftrightarrow xu = 0 \\ 1 \cdot du & = -2x dx \end{array} \right| = - \int_1^0 \sqrt{u} du = \int_0^1 \sqrt{u} du$$

In general: we set  $h(u) = g(x)$ , then it holds that  $h'(u) du = g'(x) dx$ .

**Theorem 4.6.** Let  $f, \tilde{f} \in \mathcal{R}[a, b]$  and  $A \subseteq [a, b]$  countable. Furthermore  $f(x) = \tilde{f}(x) \forall x \in [a, b] \setminus A$ . Then it holds that

$$\int_a^b |f - \tilde{f}| dx = 0$$

Then it follows especially that

$$\int_a^b f dx = \int_a^b \tilde{f} dx$$

This lecture took place on 2018/04/24.

*Proof.* Show:  $r \in \mathcal{R}[a, b], r \geq 0$ .  $\int_a^b r dx = 0$  and  $r(x) = 0$  for  $x \in [a, b] \setminus A$ . Then it holds that  $\int_a^b r dx = 0$ . Let  $r$  be as above. First, we show:  $r_+(x) = \lim_{\xi \rightarrow x+} r(\xi) = 0 \forall x \in [a, b)$  and also  $r_-(x) = 0 \forall x \in (a, b]$ .

Proof of that: Let  $x \in [a, b)$  and  $y = r_+(x)$  (exists because  $r \in \mathcal{R}[a, b]$ ). Choose  $\delta_n = \frac{1}{n}$ .  $(x, x + \frac{1}{n}) \cap [a, b)$  is an open interval with uncountable many points, so

there is certainly one point in  $A$ . So there exists  $\xi_n \in ((x, x + \frac{1}{n}) \cap [a, b]) \setminus A$  and  $|\xi_n - x| < \delta_n = \frac{1}{n}$ . Hence,  $\lim_{n \rightarrow \infty} \xi_n = x$  and  $r(\xi_n) = 0$ . Therefore,  $\lim_{n \rightarrow \infty} r(\xi_n) = 0$  where  $r(\xi_n) = y = r_+(x)$ .

Analogously,  $r_-(x) = 0$  on  $(a, b]$ .

Let  $\varepsilon > 0$  be arbitrary. We let  $A_\varepsilon = \{w \in [a, b] \mid r(w) > \varepsilon\}$ . We show:  $A_\varepsilon$  is finite.

Assume  $A_\varepsilon$  would have infinitely many points. Choose a sequence  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in A_\varepsilon$  and  $w_n \neq w_m$  for  $n \neq m$  (works because  $A_\varepsilon$  is infinite).  $(w_n)_{n \in \mathbb{N}}$  is bounded, hence there exists a convergent subsequence  $(w_{n_k})_{k \in \mathbb{N}}$  with  $x = \lim_{k \rightarrow \infty} w_{n_k} \in [a, b]$  and  $w_{n_k} \in [a, b]$ .

Either  $(w_{n_k})$  contains infinitely many sequence element  $w_{n_k} < x$  (variant (a)) or infinitely many  $w_{n_k} > x$  (variant (b)). Let variant b hold without loss of generality.

Combine all  $w_{n_k} > x$  to one subsequence  $(w_{n_{k_l}})_{l \in \mathbb{N}}$ . This gives  $\lim_{l \rightarrow \infty} w_{n_{k_l}} = x$  and  $w_{n_{k_l}} > x$ , thus  $\lim_{l \rightarrow \infty} \underbrace{r(w_{n_{k_l}})}_{\geq \varepsilon \text{ because } w_{n_{k_l}} \in A_\varepsilon} = r_+(x) = 0$ . This gives a contradiction.

$A_\varepsilon$  must be finite.

Consider

$$A_{\frac{1}{n}} = \{w_1^n, \dots, w_{m_n}^n\}$$

finite. Let  $\varphi_n = \sum_{k=1}^{m_n} r(w_k^n) \cdot \chi_{\{w_k^n\}} \in \tau[a, b]$ .

For  $x = w_k^n \in A_{\frac{1}{n}}$  it holds that

$$\varphi_n(w_k^n) = \sum_{k=1}^{m_n} r(w_k^n) \cdot \underbrace{\chi_{\{w_k^n\}}(w_j^n)}_{\delta_{jk}} = r(w_j^n)$$

so  $|\varphi_n(x) - r(x)| = 0 \forall x \in A_{\frac{1}{n}}$ . Let  $x \in [a, b] \setminus A_{\frac{1}{n}}$ . Then it holds  $0 \leq r(x) < \frac{1}{n}$  and for  $x \notin A_{\frac{1}{n}}$  it holds that  $\varphi(x) = 0$ . Therefore,

$$|r(x) - \varphi(x)| = r(x) < \frac{1}{n}$$

hence  $\|r - \varphi_n\|_\infty < \frac{1}{n}$ . This means that  $\varphi_n \rightarrow r$  uniformly on  $[a, b]$ . Therefore

$$\lim_{n \rightarrow \infty} \underbrace{\int_a^b \varphi_n dx}_{=0} = \int_a^b r dx = 0$$

Now we want to finish the proof of our theorem: Let  $r(x) = |f(x) - \tilde{f}(x)| \geq 0$  and  $r(x) = 0$  for  $x \notin A$ . So,  $\int_a^b |f - \tilde{f}| dx = 0$  (first part proven).

$$\left| \int_a^b f dx - \int_a^b \tilde{f} dx \right| = \left| \int_a^b (f - \tilde{f}) dx \right| \leq \int_a^b |f - \tilde{f}| dx = 0$$

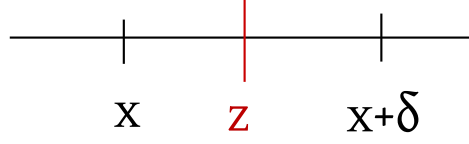


Figure 23:  $\xi$  must be sufficiently close enough to  $z$  such that  $|f(\xi) - f_+(z)| \leq \frac{\varepsilon}{2}$ .

$$\implies \int_a^b f \, dx = \int_a^b \tilde{f} \, dx$$

Second part proven. □

**Lemma 4.5.** *Let  $f \in \mathcal{R}[a, b]$ . Then it holds that  $f_+ \in \mathcal{R}[a, b]$  and also  $f_- \in \mathcal{R}[a, b]$ .*

*Proof.* Only for  $f_+$ : First, we show: Let  $x \in [a, b]$ .

$$f_+(x) = \lim_{\xi \rightarrow x_+} f(\xi) = \lim_{\xi \rightarrow x_+} f_+(x)$$

(the plus is important on the right-hand side!).

Proof of this: Let  $\varepsilon > 0$  be arbitrary. Then there exists  $\delta > 0$  such that  $\forall \xi \in (x, x + \delta)$ :  $|f(\xi) - f_+(x)| < \frac{\varepsilon}{2}$ . Now let  $z \in (x, x + \delta)$  be arbitrary chosen. For  $z$  there exists  $\xi \in (z, x + \delta)$  because  $f_+(z)$  exists. Compare with Figure 23.

$$|f_+(z) - f_+(x)| \leq |f_+(z) - f(\xi)| + |f(\xi) - f_+(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

so  $f_+(x) = \lim_{z \rightarrow x_+} f_+(z)$ .

It remains to show:  $f_+$  has left-sided limits. Let  $x \in (a, b]$  be arbitrary and  $f_-(x) = \lim_{\xi \rightarrow x_-} f(\xi)$ . We show:  $f_-(x) = \lim_{\xi \rightarrow x_-} f_+(x)$  (again: the plus is important).

Let  $\varepsilon > 0$  be arbitrary. Choose  $\delta > 0$  such that  $\forall z \in (x - \delta, x)$  it holds that  $|f(z) - f_-(x)| < \frac{\varepsilon}{2}$ .

Now let  $\xi \in (x - \delta, x)$  (compare with Figure 24) and choose  $x > z > \xi$  with the property that  $|f(z) - f_+(\xi)| < \frac{\varepsilon}{2}$  (feasible because  $f$  in  $\xi$  has a right-sided limit):

$$|f_+(\xi) - f_-(x)| \leq \underbrace{|f_+(\xi) - f(z)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(z) - f_-(x)|}_{< \frac{\varepsilon}{2}}$$

because of the choice of  $\delta$  and  $z \in (\xi, x) \subseteq (x - \delta, x)$ .

Hence,  $\lim_{\xi \rightarrow x_-} f_+(\xi) = f_-(x)$ . Analogously for  $f_-$  □

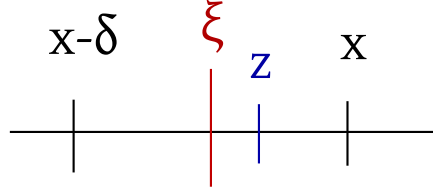


Figure 24:  $\xi$  and  $z$

**Remark 4.13.**

$$\lim_{\xi \rightarrow x_+} f_+(\xi) = f_+(x)$$

$$\lim_{\xi \rightarrow x_-} f_-(\xi) = f_-(x)$$

from the proof. So  $f_+$  is right-sided continuous and  $f_-$  is left-sided continuous.

**Lemma 4.6.** Let  $f \in \mathcal{R}[a, b]$ . Then it holds that

$$\int_a^b f \, dx = \int_a^b f_+ \, dx = \int_a^b f_- \, dx$$

*Proof.* For  $f_+$ :

$$f, f_+ \in \mathcal{R}[a, b]$$

$\forall x \in [a, b]$  with  $f$  is continuous in  $x$  it holds that

$$f(x) = \lim_{\xi \rightarrow x} f(\xi) = \lim_{\xi \rightarrow x_+} f(\xi) = f_+(x)$$

$f$  has at most countable many discontinuity points. By Satz 4.6,

$$\int_a^b |f - f_+| \, dx = 0 \quad \text{or equivalently} \quad \int_a^b f \, dx = \int_a^b f_+ \, dx$$

□

## 4.2 Improper integrals

Let  $I$  be an interval in  $\mathbb{R}$  with marginal points  $a$  and  $b$  with  $-\infty \leq a < b \leq +\infty$ . Let  $f$  be a regulated function on  $I$ . We define

$$1. \text{ If } I = [a, b), \int_a^b f \, dx = \lim_{\beta \rightarrow b_-} \int_a^\beta f \, dx$$



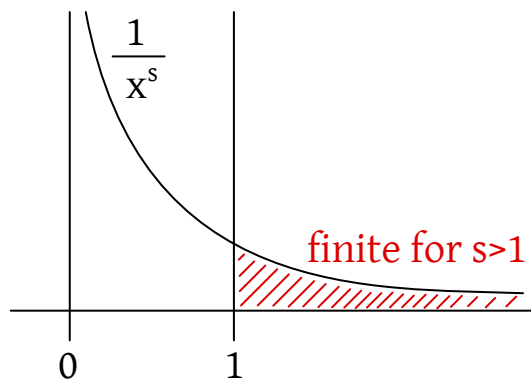


Figure 25: The function  $\frac{1}{x^s}$  for  $s > 1$

2. If  $I = (a, b]$ ,  $\int_a^b f dx = \lim_{\alpha \rightarrow a+} \int_\alpha^b f dx$

3. If  $I = (a, b)$ ,  $\int_a^b f dx = \lim_{\alpha \rightarrow a+} \int_\alpha^c f dx + \lim_{\beta \rightarrow b-} \int_c^\beta f dx$

for an arbitrarily chosen  $c \in (a, b)$  under the constraint that the corresponding limits in  $\mathbb{R}$  exist.

Standard examples will follow:

**Example 4.4.** Let  $s > 1$ .

$$\begin{aligned} \int_1^\infty x^{-s} dx &= \lim_{\beta \rightarrow \infty} \int_1^\beta x^{-s} dx = \lim_{\beta \rightarrow \infty} \left( \frac{1}{-s+1} x^{-s+1} \right) \Big|_1^\beta \\ &= \frac{1}{1-s} \cdot \underbrace{\lim_{\beta \rightarrow \infty} \frac{1}{s-1}}_{\substack{>0 \\ =0}} - \frac{1}{1-s} \cdot 1 = \frac{1}{s-1} \end{aligned}$$

Compare with Figure 25.

**Example 4.5.** Let  $s < 1$ .

$$\begin{aligned} \int_0^1 x^{-s} dx &= \lim_{\alpha \rightarrow 0+} \int_\alpha^1 x^{-s} dx = \lim_{\alpha \rightarrow 0+} \frac{1}{-s+1} x^{-s+1} \Big|_\alpha^1 \\ &= \frac{1}{1-s} - \frac{1}{1-s} \cdot \underbrace{\lim_{\alpha \rightarrow 0} \alpha^{1-s}}_{\substack{>0 \\ =0}} = \frac{1}{1-s} \end{aligned}$$

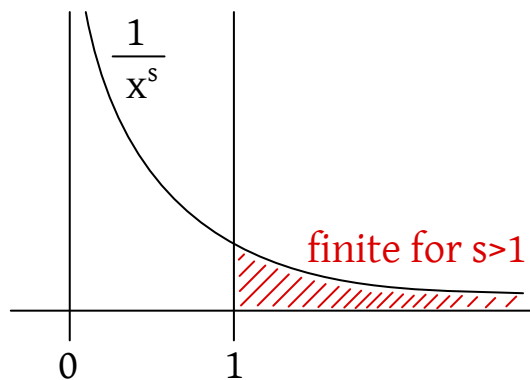


Figure 26: The function  $x^s$  for  $s < 1$

Compare with Figure 26.

For  $s = 1$ , neither  $\int_0^1 \frac{1}{x} dx$  nor  $\int_1^\infty \frac{1}{x} dx$  exists.

**Example 4.6.** For  $c > 0$ ,

$$\int_0^\infty e^{-cx} dx = \lim_{\beta \rightarrow \infty} \int_0^\beta e^{-cx} dx = \lim_{\beta \rightarrow \infty} \left( -\frac{1}{c} \right) \cdot e^{-cx} \Big|_0^\beta - \frac{1}{c} \cdot \underbrace{\lim_{\beta \rightarrow \infty} e^{-c\beta}}_{=0} + \frac{1}{c} = \frac{1}{c}$$

**Theorem 4.7** (Direct comparison test for improper integrals). In German, “Majorantenkriterium für uneigentliche Integrale”.

Let  $f, g$  be regulated functions on  $I$  and it holds that

$$|f(x)| \leq g(x) \forall x \in I$$

Assume  $\int_a^b g dx$  exists as improper integral. Then also the following improper integrals exist:

$$\int_a^b |f| dx \text{ and } \int_a^b f dx$$

In German,  $g$  is called Majorante of  $f$  (there is no equivalent terminology in English).

*Proof.* Without loss of generality, let  $I = [a, b)$ . Let  $G(\beta) = \int_a^\beta g dx$ . We know that  $\lim_{\beta \rightarrow b-} G(\beta)$  exists. By Lemma 3.6 (Cauchy criterion for existence of limits): Let  $\varepsilon > 0$  be arbitrary, then there exists a right-sided neighborhood  $U$  of  $b$  ( $U = (b - \delta, b)$  if  $b < \infty$  and  $U = (M, \infty)$  if  $b = \infty$ ) with  $u, v \in U$ , then it holds that  $|G(v) - G(u)| < \varepsilon$ .

$$|G(v) - G(u)| = \left| \int_a^v g dx - \int_a^u g dx \right| = \left| \int_u^a g dx + \int_a^v g dx \right| = \left| \int_u^v g dx \right|$$

Let  $F(\beta) = \int_a^\beta |f| dx$ . Analogously as for  $G$ , it holds that  $F(v) - F(u) = \int_u^v |f| dx$ . Let  $u, v \in \mathcal{U}$ . Then it holds that

$$|F(v) - F(u)| = \left| \int_u^v |f| dx \right| \leq \left| \int_u^v g dx \right| = |G(v) - G(u)| < \varepsilon$$

hence by the Cauchy criterion for  $F$ :  $\lim_{\beta \rightarrow b_-} F(\beta)$  exists, so there exists  $\int_a^b |f| dx$  as improper integral. The same applies for the existence of  $\int_a^b f dx$ .  $\square$

**Example 4.7.** The cardinal sine function is defined as

$$\text{sinc}(x) = \frac{\sin x}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{sinc}(0) = 1$$

So  $\text{sinc}(x)$  is continuous on  $\mathbb{R}$ .

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \underbrace{\frac{\sin x}{x} dx}_{\substack{\text{continuous} \\ \text{exists}}} + \int_1^\infty \frac{\sin x}{x} dx$$

How about  $\int_1^\infty \frac{\sin(x)}{x} dx$ ?

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\sin x}{x} dx &= \left| \begin{array}{ll} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{array} \right| = \lim_{\beta \rightarrow \infty} \left[ -\frac{1}{x} \cos x \right]_1^\beta - \int_1^\beta \frac{\cos x}{x^2} dx \\ &= \cos(1) - \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\cos(x)}{x^2} dx \\ &\quad \left| \frac{\cos(x)}{x^2} \right| \leq \frac{1}{x^2} \text{ on } [1, \beta] \end{aligned}$$

and  $\int_1^\infty \frac{1}{x^2} dx$  exists. So  $g(x) = \frac{1}{x^2}$  is a majorant of  $\frac{\cos(x)}{x^2}$  and by Theorem 4.7,  $\lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\cos(x)}{x^2} dx$  exists.

Attention!  $\int_0^\infty \left| \frac{\sin(x)}{x} \right| dx$  does not exist. Is not Lebesgue integrable.

**Definition 4.6.** Let  $x > 0$ . We call  $\Gamma$  Euler's Gamma function.

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

**Remark 4.14.** The improper integral in the definition of the  $\Gamma$ -function exists for all  $x > 0$ .

This lecture took place on 2018/04/26.

Euler's  $\Gamma$ -function exists for all  $x > 0$ :

$$\int_0^1 t^{x-1} e^{-t} dt$$

for  $0 < x \leq 1$ :

$$t^{x-1} \cdot \underbrace{e^{-t}}_{\leq 1} \leq t^{x-1} \text{ and } \underbrace{\int_0^1 \overbrace{t^{x-1}}^{\in(-1,0]} dt}_{\text{exists}}$$

also exists  $\int_0^1 t^{x-1} e^{-t} dt$  because of the direct comparison criterion. For  $x \geq 1$ ,  $t^{x-1} e^{-t}$  is continuous on  $[0, 1]$ , hence  $\int_0^1 t^{x-1} e^{-t} dt$  exists.

**Claim.** Let  $x > 0$  be fixed.  $\exists c > 0$  such that

$$t^{x-1} e^{-t} \leq c \cdot e^{-\frac{t}{2}} \forall t \in [0, \infty)$$

*Proof.*

$$\lim_{t \rightarrow \infty} \underbrace{t^{x-1}}_{\text{polynomially in } t} \cdot \underbrace{e^{-t}}_{\text{exponentially } \rightarrow 0} = 0$$

Also there exists  $L > 1$ , such that  $\forall x > L$  it holds that  $t^{x-1} e^{-t/2} < 1$  on  $[1, L]$  (which is a compact interval) continuous. So there exists  $M > 0$  such that  $t^{x-1} e^{-\frac{t}{2}} \leq M \forall t \in [1, L]$ . Let  $c = \max\{M, 1\}$ . Therefore it holds on  $[1, L]$  and also on  $(L, \infty)$ .

$$t^{x-1} e^{-\frac{t}{2}} \leq c$$

Multiply with  $e^{-\frac{t}{2}} > 0$ , then it holds that  $t^{x-1} \cdot e^{-t} \leq c e^{-\frac{t}{2}} \forall t \in [1, \infty)$ .

$$c \int_1^\infty e^{-\frac{t}{2}} dt$$

exists. By the direct comparison test, we get  $\int_1^\infty t^{x-1} e^{-t} dt$  exists.  $\square$

**Lemma 4.7.** For all  $x > 0$  it holds that

$$\Gamma(x+1) = x \cdot \Gamma(x) \quad (\text{functional equation of the } \Gamma\text{-function})$$

*Epecially with  $\Gamma(1) = 1$  it holds that  $\Gamma(n+1) = n!$  for all  $n \in \mathbb{N}_0$ .*

*Proof.*

$$\Gamma(x+1) = \int_0^\infty t^{x+1-1} e^{-t} dt = \int_0^\infty t^x e^{-t} dt$$

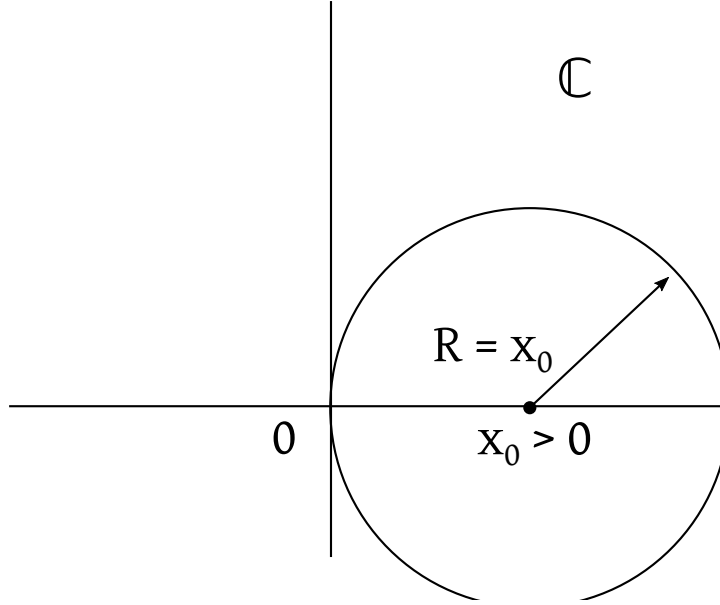


Figure 27:  $\Gamma$  on  $\mathbb{C}$

$$\begin{aligned}
 &= \left| \begin{array}{ll} u = t^x & u' = x \cdot t^{x-1} \\ v' = e^{-t} & v = -e^{-t} \end{array} \right| \\
 &= \underbrace{-t^x \cdot e^{-t} \Big|_0^\infty}_{\substack{=0 \text{ on the upper bound} \\ =0 \text{ on the lower bound}}} + \int_0^\infty x \cdot t^{x-1} \cdot e^{-t} dt = x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)
 \end{aligned}$$

$$\Gamma(1) = \int_0^\infty \underbrace{t^{1-1}}_{=1} \cdot e^{-t} dt = -e^{-t} \Big|_0^\infty = 1$$

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1)\Gamma(n-1) = n \cdot (n-1) \cdot \dots \cdot 1 \cdot \underbrace{\Gamma(1)}_{=1} = n!$$

□

**Remark 4.15.** *There exists a power series  $\Gamma(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$ .  $\Gamma(z)$  is also defined for  $z \in \mathbb{C}$  with  $\Re z > 0$ . Compare with Figure 27.*

### 4.3 Young's inequality

Some important inequalities in integration theory follow.

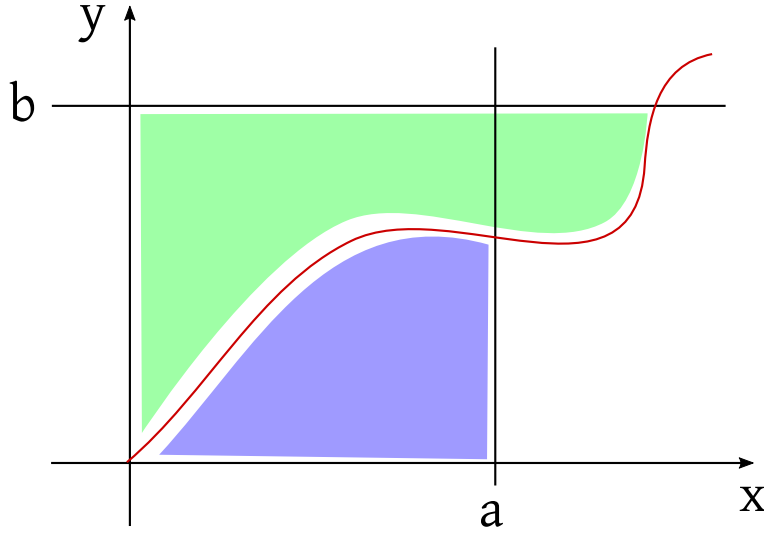


Figure 28: Young's inequality visualized. The blue area denotes  $\int_0^a f(x) dx$  and  $\int_0^b f^{-1}(y) dy$  is the green area.

**Theorem 4.8** (Young's inequality). *Let  $f : [0, \infty) \rightarrow [0, \infty)$  be continuous differentiable, strictly monotonically increasing with  $f(0) = 0$  and  $f$  is unbounded. Then  $f : [0, \infty) \rightarrow [0, \infty)$  bijective and  $f^{-1} : [0, \infty) \rightarrow [0, \infty)$  is strictly monotonically increasing and continuous. Let  $a, b \geq 0$  be given. Then it holds that*

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy$$

*Equality is given if and only if,  $b = f(a)$  or  $a = f^{-1}(b)$ . Compare with Figure 28.*

*Proof.* Let  $f : [0, \infty) \rightarrow [0, \infty)$  be as above. Let  $x_1 \neq x_2$ . Without loss of generality  $x_1 < x_2$ . Then it holds that  $f(x_1) < f(x_2) \implies f$  is injective. Surjectivity:  $f(0) = 0$ , hence  $0 \in f([0, \infty))$ . Let  $\eta > 0$  be arbitrary. Because  $f$  is unbounded, there exists  $z \in (0, \infty)$  with  $f(z) > \eta$ .  $f(0) = 0 < \eta < f(z)$ .

By the Intermediate Value Theorem ( $f$  is continuous), there exists  $\xi \in (0, z)$  with  $f(\xi) = \eta$ . So  $f$  is surjective.

$$f^{-1} : [0, \infty) \rightarrow [0, \infty)$$

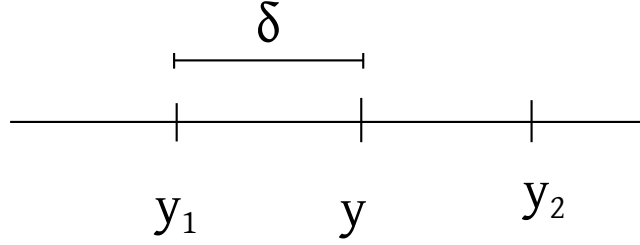


Figure 29:  $\delta$ ,  $y$ ,  $y_1$  and  $y_2$

*Monotonicity:* Let  $y_1 < y_2$ . Then it holds that  $x_1 = f^{-1}(y_1) < x_2 = f^{-1}(y_2)$ . If this would not be true (hence,  $x_2 \leq x_1$ ) then  $y_2 = f(x_2) \leq y_1 = f(x_1)$  gives a contradiction.

*Continuity of  $f^{-1}$ :* Let  $\varepsilon > 0$  be arbitrary. Let  $y \in (0, \infty)$  be chosen arbitrarily. We show  $f^{-1}$  is continuous in  $y$ . Let  $x = f^{-1}(y) > 0$  and choose  $\varepsilon = \min\left\{\frac{x}{2}, \frac{\varepsilon}{2}\right\}$ .

$$x_1 = x - \varepsilon > 0 \quad x_2 = x + \varepsilon > 0$$

Let  $y_1 = f(x_1)$ ,  $y_2 = f(x_2)$ ,  $x_1 = f^{-1}(y_1)$  and  $x_2 = f^{-1}(y_2)$ . By monotonicity of  $f$ :  $x_1 < x < x_2 \implies y_1 < y < y_2$ .

Choose  $\delta = \min\{y - y_1, y_2 - y\} > 0$  (compare with Figure 29). Hence  $(y - \delta, y + \delta) \subseteq (y_1, y_2) \forall \eta \in (y - \delta, y + \delta)$  it holds that

$$f^{-1}(\eta) < f^{-1}(y + \delta) < f^{-1}(y_2) = x_2 = x + \varepsilon$$

$$f^{-1}(\eta) < f^{-1}(y - \delta) < f^{-1}(y_1) = x_1 = x - \varepsilon$$

So  $f^{-1}(\eta) \in (x - \varepsilon, x + \varepsilon)$ , or equivalently

$$|\eta - y| < \delta \implies \left| f^{-1}(\eta) - \underbrace{f^{-1}(y)}_{=x} \right| < c \leq \frac{\varepsilon}{2} < \varepsilon$$

So  $f^{-1}$  is continuous in  $y$  and  $f^{-1}$  is continuous in  $y_0$  analogously.  $\square$

Consider

$$\begin{aligned} \int_0^b f^{-1}(y) dy &= \left| \begin{array}{l} y \\ y=0 \\ y=b \end{array} \right| \begin{array}{l} = f(x) \\ \Rightarrow x = f^{-1}(0) = 0 \\ \Rightarrow x = f^{-1}(b) \end{array} \left| \begin{array}{l} = f'(x) dx \\ = f'(x) dx \\ = f'(x) dx \end{array} \right| = \int_0^{f^{-1}(b)} \underbrace{f^{-1}(f(x))}_{=x} \cdot f'(x) dx = \int_0^{f^{-1}(b)} x \cdot f'(x) dx \\ &\stackrel{\text{integration by parts}}{=} x \cdot f(x) \Big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} 1 \cdot f(x) dx \\ &= f^{-1}(b) \cdot b - \int_0^{f^{-1}(b)} f(x) dx \end{aligned}$$

So

$$\begin{aligned} I &= \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy = \int_{f^{-1}(b)}^0 f(x) dx + b \cdot f^{-1}(b) \\ &= \int_{f^{-1}(b)}^a f(x) dx + b \cdot f^{-1}(b) \end{aligned}$$

**Case 1**  $a = f^{-1}(b)$

$$\Rightarrow I = \underbrace{\int_a^a f(x) dx}_{=0} + b \cdot a$$

**Case 2**  $b < f(a)$ , or equivalently  $f^{-1}(b) < a$

$$\Rightarrow \int_{f^{-1}(b)}^a \underbrace{f(x)}_{f(f^{-1}(b)) \text{ for } x > f^{-1}(b)} dx > \overbrace{b}^{\text{minimal value}} \cdot \underbrace{(a - f^{-1}(b))}_{\text{length of integration interval}}$$

Therefore  $I > b(a - f^{-1}(b)) + b \cdot f^{-1}(b) = ab$ .

**Case 3**  $b > f(a)$ , or equivalently  $f^{-1}(b) > a$

$$\begin{aligned} \int_{f^{-1}(b)}^a f(x) dx &= \int_a^{f^{-1}(b)} \underbrace{(-f(x))}_{\substack{\text{monotonically decreasing} \\ > -f(f^{-1}(b)) \forall x \in [a, f^{-1}(b)]}} dx > -f(f^{-1}(b)) \cdot (f^{-1}(b) - a) \\ &= -b(f^{-1}(b) - a) \\ I &> -b(f^{-1}(b) - a) + b \cdot f^{-1}(b) = ab \end{aligned}$$



**Remark 4.16.** Young's inequality also holds without requiring differentiability of  $f$  (but the proof is more complex).

**Lemma 4.8** (Special case of Young's inequality). Let  $A, B \geq 0$  and  $p, q > 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = p \cdot q$ . Then  $p$  and  $q$  are called conjugate exponents. Then it holds that  $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$ .

*Proof.*

$$f(x) = x^{p-1} \text{ in Young's inequality}$$

$$y = x^{p-1} \iff x = y^{\frac{1}{p-1}}$$

$$\frac{1}{p-1} = q-1 \text{ is immediate, because}$$

$$\frac{1}{p-1} = q-1 \iff 1 = pq - p - q + 1 \iff p + q = pq$$

So  $f^{-1}(y) = y^{\frac{1}{p-1}} = y^{q-1}$ . By Young's inequality:

$$\begin{aligned} AB &\leq \int_0^A x^{p-1} dx + \int_0^B y^{q-1} dy \\ &= \frac{x^p}{p} \Big|_0^A + \frac{y^q}{q} \Big|_0^B = \frac{A^p}{p} + \frac{B^q}{q} \end{aligned}$$

□

**Remark 4.17.**

$$AB = \frac{A^p}{p} + \frac{B^q}{q}$$

Equality holds if and only if  $B = A^{p-1} \iff B^q = \overbrace{A^{pq} - q}^p = A^p$ .

## 4.4 Hölder's inequality

**Theorem 4.9** (Hölder's inequality). Let  $I$  be an interval with boundary values  $a$  and  $b$ .  $-\infty \leq a < b \leq +\infty$ . Let  $p$  and  $q$  be conjugate exponents. Let  $f_1$  and  $f_2$  be regulated function on  $I$  such that

$$\int_a^b |f_1(x)|^p dx < \infty$$

$$\int_a^b |f_2(x)|^q dx < \infty$$

both exist.

We let  $\|f_1\|_p := \left( \int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}}$  and  $\|f_2\|_q := \left( \int_a^b |f_2(x)|^q dx \right)^{\frac{1}{q}}$ . They are called  $L^p$ -norm of  $f_1$  and  $L^q$ -norm of  $f_2$ .

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)| dx < \infty$$

exists and

$$\int_a^b |f_1(x) \cdot f_2(x)| dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

*Proof.* Assume that  $\|f_1\|_p > 0$  and  $\|f_2\|_q > 0$ . Let  $A = \frac{|f_1(x)|}{\|f_1\|_p}$  and  $B = \frac{|f_2(x)|}{\|f_2\|_q}$ . By Lemma 4.8,

$$\frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q} \leq \frac{1}{q} \cdot \frac{|f_1(x)|^p}{\|f_1\|_p^p} + \frac{1}{q} \cdot \frac{|f_2(x)|^q}{\|f_2\|_q^q}$$

We integrate the inequality,

$$\begin{aligned} & \frac{1}{\|f_1\|_p \cdot \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| dx \\ & \leq \frac{1}{p} \cdot \frac{1}{\|f_1\|_p^p} \cdot \underbrace{\int_a^b |f_1(x)|^p dx}_{=\|f_1\|_p^p} + \frac{1}{q} \cdot \frac{1}{\|f_2\|_q^q} \underbrace{\int_a^b |f_2(x)|^q dx}_{=\|f_2\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

$$\frac{1}{\|f_1\|_p \cdot \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| dx \implies \int_a^b |f_1(x) f_2(x)| dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

Special case: Let  $\|f_1\|_p = 0$

$$\implies \left( \int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}} = 0 \implies \int_a^b \underbrace{|f_1(x)|^p}_{\geq 0} dx = 0$$

By Theorem 4.6,  $f_1(x) = 0 \forall x \in [a, b] \setminus A$  and  $A$  is at most countable.

$$\implies f_1(x) \cdot f_2(x) = 0 \forall x \in [a, b] \setminus A$$

$$\implies \int_a^b |f_1(x) \cdot f_2(x)| dx = 0$$

$$\implies 0 = 0 \text{ in Hölder's inequality}$$

□

**Remark 4.18** (Special case of Hölder's inequality). Let  $p = q = 2$ ,  $\frac{1}{2} + \frac{1}{2} = 1$ .

$$\int_a^b |f_1(x) \cdot f_2(x)| dx \leq \|f_1\|_2 \|f_2\|_2$$

is called *Cauchy-Schwarz inequality* for  $L^2$  functions.

$$\int_a^b f_1(x)f_2(x) dx = \langle f_1, f_2 \rangle_2 = \langle f_1, f_2 \rangle_{L^2}$$

is an inner product on a proper space of functions.

## 5 Elaboration on differential calculus

We consider a metric space  $X$  and functions  $f : X \rightarrow \mathbb{C}$ . We define a concept of uniform convergence of such sequences:

$$f_n : X \rightarrow \mathbb{C} \quad (n \in \mathbb{N}) \text{ and } f : X \rightarrow \mathbb{C}$$

We say,  $(f_n)_{n \in \mathbb{N}}$  converges uniformly towards  $f$  if  $\forall \varepsilon > 0 \forall N \in \mathbb{N}$  such that  $\forall x \in X$  and  $\forall n \geq N$  it holds that

$$\underbrace{|f_n(x) - f(x)|}_{\text{absolute value in } \mathbb{C}} < \varepsilon$$

$$\iff \sup \{|f_n(x) - f(x)| : x \in X\} < \varepsilon$$

**Remark 5.1.** Do not use  $\|f\|_\infty$  for the definition of uniform convergence, because  $f_n$  and  $f$  must not be necessarily bounded. Hence,

$$\|f\|_\infty = \{ |f(x)| : x \in X \}$$

must not be finite.

**Theorem 5.1.** Let  $X$  be a metric space,  $f_n : X \rightarrow \mathbb{C}$  be a sequence of continuous functions and  $f : X \rightarrow \mathbb{C}$  such that  $f_n \rightarrow f$  uniform on  $X$ . Then  $f$  is also continuous on  $X$ .

*This lecture took place on 2018/05/03.*

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Choose  $x \in X$ . Show:  $f$  is continuous in  $x$ .

Compare with Figure 30.

Because of uniform convergence  $f_n \rightarrow f$ , there exists  $N \in \mathbb{N}$  such that  $|f_N(z) - f(z)| < \frac{\varepsilon}{3} \forall z \in X$ . Let  $N$  be fixed. Because  $f_N$  is continuous in  $x$ , there exists  $\delta > 0$  such that  $d(x, \xi) < \delta \implies |f_N(\xi) - f_N(x)| < \frac{\varepsilon}{3}$ .

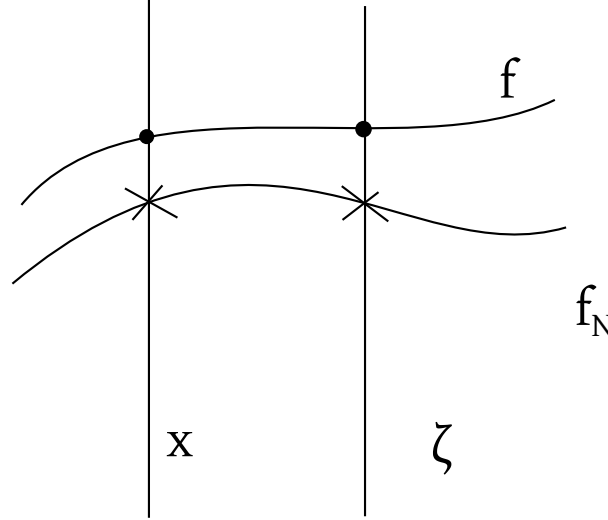


Figure 30: Uniform convergence of  $f_N$  to  $f$

We consider now  $\xi \in X$  with  $d_X(x, \xi) < \delta$ . Then it holds that

$$\begin{aligned}
 |f(x) - f(\xi)| &= |f(x) - f_N(x) + f_N(x) - f_N(\xi) + f_N(\xi) - f(\xi)| \\
 &\leq \underbrace{|f(x) - f_N(x)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f_N(x) - f_N(\xi)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(\xi) - f(\xi)|}_{< \frac{\varepsilon}{3}} \\
 &= \varepsilon
 \end{aligned}$$

by uniform convergence, by continuity and by uniform convergence respectively.

Thus,  $f$  is continuous in  $x$ . □

**Theorem 5.2.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series in  $\mathbb{C}$  with convergence radius  $\rho_P > 0$ . Furthermore, let  $0 < r < \rho_P$ . Let  $P_n(z) = \sum_{k=0}^n a_k z^k$  ( $n$ -th partial sum of  $P$ ). Then  $P_n \rightarrow P$  uniformly on  $\overline{K_r(0)}$ .

*Proof.* Approximation theorem for power series. Lettl Analysis 1, lecture notes, section 5, theorem 10.

Let  $0 < r < \rho_P$ . Choose  $\bar{r}$  with  $r < \bar{r} < \rho_P$ . Then it holds for  $z \in \overline{K_r(0)}$  that

$$|P(z) - P_n(z)| < \frac{\bar{r}}{\bar{r} - r} \cdot \left(\frac{r}{\bar{r}}\right)^n$$

Remark 5.2.

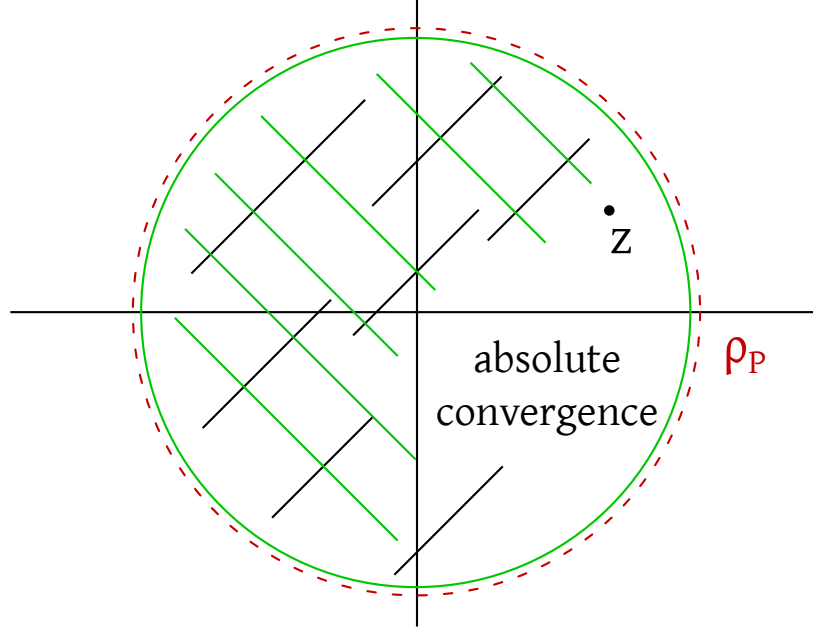


Figure 31: We cannot make a general statement about convergence/divergence. But on every small closed sphere  $P$  converges absolutely for every  $z$

$$\frac{r}{\bar{r}} < 1$$

hence  $\left(\frac{r}{\bar{r}}\right)^n$  is arbitrary small, for every  $n$  sufficiently large.

$$\Rightarrow \sup \left\{ |P(z) - P_n(z) : z \in \overline{K_r(0)} \right\} \leq \underbrace{\frac{\bar{r}}{\bar{r} - r}}_{\text{fixed}} \cdot \underbrace{\left(\frac{r}{\bar{r}}\right)^n}_{\text{arbitrary small for } n \text{ sufficiently large}}$$

Hence,  $P_n \rightarrow P$  uniform on  $\overline{K_r(0)}$ . □

**Corollary.**  $P$  is continuous on  $K_{\rho_P}(0)$ .

**Theorem 5.3.** Let  $I \subseteq \mathbb{R}$  be an interval. Let  $f_n : I \rightarrow \mathbb{R}$  be continuously differentiable on  $I \forall n \in \mathbb{N}$ . It holds that

1.  $\exists g : I \rightarrow \mathbb{R}$  such that  $f'_n \rightarrow g$  uniform on  $I$
2.  $\exists f : I \rightarrow \mathbb{R}$  such that  $\forall x \in I$  it holds that  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  ("pointwise convergence").

Then it holds that  $f$  is continuously differentiable on  $I$  and  $g = f'$ .

*Proof.*  $g$  is continuous as uniform limit of continuous  $f'_n$  (Theorem 5.1). For  $f_n$ , the Fundamental Theorem of Differential Calculus can be applied ( $f'_n$  is continuous, hence a regulated function). Let  $x_0 \in I$ . Then it holds that

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f'_n(\xi) d\xi$$

Convergence for  $n \rightarrow \infty$ :

$$f_n(x) \rightarrow f(x) \quad f_n(x_0) \rightarrow f(x_0)$$

(Pointwise convergence)

$$\int_{x_0}^x f'_n(\xi) d\xi \rightarrow \int_{x_0}^x g(\xi) d\xi$$

Therefore, for  $n \rightarrow \infty$ ,

$$f(x) = f(x_0) + \int_{x_0}^x g(\xi) d\xi$$

The right-hand side is continuously differentiable by  $x$  according to the Fundamental Theorem, variant 1, with

$$\left( f(x_0) + \int_{x_0}^x g(\xi) d\xi \right)'(x) = g(x)$$

Hence, by  $f(x) = f(x_0) + \int_{x_0}^x g(\xi) d\xi$  it follows that

$$f'(x) = g(x) \quad \forall x \in I$$

□

To finish our proof, we need a result we missed in the section about Integrals.

**Lemma 5.1.** *Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of regulated functions on  $[a, b]$  and  $f_n \rightarrow f$  uniform on  $[a, b]$ . Then it holds that*

$$\int_a^b |f_n - f| dx \rightarrow 0 \quad \text{for } n \rightarrow \infty \quad \text{especially } \int_a^b f_n dx \rightarrow \int_a^b f dx$$

*Proof.*  $f$  as a uniform limit of regulated functions is a regulated function. The proof has been done in the practicals.

Let  $N \in \mathbb{N}$  large enough such that

$$\forall n \geq N \forall x \in [a, b] : |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$$

Then it holds that

$$\int_a^b |f_n(x) - f(x)| dx < \int_a^b \frac{\varepsilon}{b-a} dx = \frac{\varepsilon}{b-a} (b-a) = \varepsilon$$

Hence,

$$\lim_{n \rightarrow \infty} \int_a^b |f_n(x) - f(x)| dx = 0$$

$$\underbrace{\left| \int_a^b f_n dx - \int_a^b f dx \right|}_{\Rightarrow \rightarrow 0} \leq \underbrace{\int_a^b |f_n - f| dx}_{\rightarrow 0}$$

So,

$$\int_a^b f dx = \lim_{n \rightarrow \infty} \int_a^b f_n dx$$

□

## 5.1 Higher derivatives and Taylor's Theorem

**Definition 5.1.** Let  $f : I \rightarrow \mathbb{R}$ ,  $I \subseteq \mathbb{R}$  is an interval. We define inductively:

$$f^{(0)}(x) = f(x)$$

Assume  $f^{(n-1)}$  is defined continuously on  $I$  and differentiable in  $x \in I$ . Then we let

$$f^{(n)}(x) = \left( f^{(n-1)} \right)'(x)$$

$f^{(n)}(x)$  is called  $n$ -th derivative of  $f$  in  $x$ .

Notational remark:

$$f^{(0)} = f \quad f^{(1)} = f' \quad f^{(2)} = f'' \quad f^{(3)} = f''' \quad f^{(4)} = f''''$$

Furthermore, we let

$$C^n(I) := \left\{ f : I \rightarrow \mathbb{R} : f^{(k)}(x) \text{ exists } \forall x \in I \text{ and } x \mapsto f^{(k)}(x) \text{ is continuous } \forall 0 \leq k \leq n \right\}$$

We call  $C$  the space of  $n$ -times continuously differentiable functions on  $I$ .

**Remark 5.3.**  $C^n(I)$  is a vector space. If  $I = [a, b]$  is compact, then

$$\|f\|_{C^n} = \max \left\{ \sup |f^{(k)}(x)| : x \in I : 0 \leq k \leq n \right\}$$

defines a norm on  $C^n(I)$  with  $\sup |f^{(k)}(x)| : x \in I = \|f^{(k)}\|_{\infty}$ .

**Remark 5.4** (New topic). Let  $f \in C^n(I)$  and  $x_0 \in I$ . Find an appropriate polynomial  $T$  which approximated  $f$  in an environment of  $x_0$  in the "best" way.

**Definition 5.2.** Let  $P(x) = \sum_{k=0}^n a_k x^k$  be a polynomial with  $a_n \neq 0$  (hence degree of  $P$  is  $n$ ).

$$P \in \mathbb{R}[x] \dots \text{ set of all polynomials with coefficients in } \mathbb{R}$$

This set of polynomials is a ring.

$x_0 \in \mathbb{R}$  is called  $k$ -times root of  $P$  ( $k \in \mathbb{N}$ ) if  $Q \in \mathbb{R}[x]$  exists such that  $P(x) = (x - x_0)^k Q(x)$  with  $Q(x_0) \neq 0$ .

**Remark 5.5.**  $P(x) = (x - x_0)^k \cdot Q(x)$  means that division of  $P$  by  $(x - x_0)^k$  gives no remainder. Recall that division with remainder means that  $\exists \hat{Q}, \hat{R}$  that are polynomials of degree  $\hat{R} < k$ ,

$$P(x) = (x - x_0)^k \cdot \hat{Q}(x) + \hat{R}(x)$$

$\hat{Q}, \hat{R}$  is unique. If  $P(x) = (x - x_0)^k \cdot Q(x) \implies \hat{R} = 0, \hat{Q} = Q$ .

**Lemma 5.2.** Let  $P(x) = \sum_{l=0}^n a_l x^l$  with  $a_n \neq 0$ . Let  $1 \leq k \leq n$ . Then it holds that  $x_0 \in \mathbb{R}$  is a  $k$ -times root of polynomial  $P \iff P^{(j)}(x_0) = 0$  for  $j = 0, \dots, k-1$  and  $P^{(k)}(x_0) \neq 0$ .

*Proof.* Proof by complete induction.

**Induction begin** Consider  $k = 1$ . Direction  $\implies$ .

Let  $x_0$  be a simple root of  $P$ , then it holds that  $P(x) = (x - x_0) \cdot Q(x)$  and  $Q(x_0) \neq 0$ . Hence,  $P(x_0) = (x_0 - x_0) \cdot Q(x_0) = 0$  and  $P'(x) = Q(x) + (x - x_0) \cdot Q'(x)$ . Thus,  $P'(x_0) = Q(x_0) + (x_0 - x_0) \cdot Q'(x_0) = Q(x_0) \neq 0$ .

Direction  $\impliedby$ .

Let  $P(x_0) = 0$  and  $P'(x_0) \neq 0$ . Division with remainder:  $P(x) = (x - x_0) \cdot Q(x) + R(x)$  with  $\text{degree}(R) \leq \text{degree}(x - x_0) = 1$ . Thus,  $R$  is constant. We insert  $x_0$ . This gives  $P(x_0) = (x_0 - x_0) \cdot Q(x_0) + R$  with  $P(x_0) = 0$  and  $(x_0 - x_0) = 0$ . Hence,  $R = 0$  is the zero polynomial and  $P(x) = (x - x_0) \cdot Q(x)$ . It remains to show that  $Q(x_0) \neq 0$ .  $P'(x) = 1 \cdot Q(x_0) + (x - x_0) \cdot Q'(x_0)$ . We insert  $x = x_0 \implies 0 \neq P'(x_0) = Q(x_0) + (x_0 - x_0) \cdot Q'(x_0)$ . Thus it holds that  $Q(x_0) = P'(x_0) \neq 0$ .

**Induction step**  $k \rightarrow k + 1$

**Claim** (Auxiliary claim). Let  $P(x) = (x - x_0) \cdot \tilde{P}(x)$ . Let  $P, \tilde{P}$  be polynomials. Then it holds  $\forall j \in \mathbb{N}$  that

$$P^{(j)}(x) = (x - x_0) \cdot \tilde{P}^{(j)}(x) + j \cdot \tilde{P}^{(j-1)}(x)$$

*Proof.* Proof by complete induction.

Let  $j = 1$ .

$$P'(x) = 1 \cdot \underbrace{\tilde{P}(x)}_{\tilde{P}^{(0)}(x)} + (x - x_0) \cdot \underbrace{\tilde{P}'(x)}_{\tilde{P}^{(1)}(x)}$$

Consider  $j \rightarrow j + 1$ .

$$\begin{aligned} P^{(j+1)}(x) &= \left( P^{(j)} \right)'(x) \\ &= \underbrace{((x - x_0) \cdot \tilde{P}^{(j)}(x))}_{\text{induction assumption}} \\ &\quad + j \tilde{P}^{(j-1)}(x)'(x - x_0) \tilde{P}^{(j+1)}(x) + \tilde{P}^{(j)}(x) + j \cdot \tilde{P}^{(j)}(x) \\ &= (x - x_0) \tilde{P}^{(j+1)}(x) + (j + 1) \cdot \tilde{P}^{(j)}(x) \end{aligned}$$



□

We continue with the induction step after verifying our auxiliary claim.  
Direction  $\implies$ .

Let  $x_0$  be an  $k+1$  times zero of  $P$ . Hence  $P(x) = (x - x_0)^{k+1} \cdot Q(x)$ .  $Q(x_0) \neq 0$ .  
Let  $\tilde{P}(x) = (x - x_0)^k \cdot Q(x)$ . We can apply the induction assumption on  $\tilde{P}$ .  
Hence

$$\tilde{P}^{(j)} = 0 \quad \text{for } j = 0, \dots, k-1 \quad \text{and} \quad \tilde{P}^{(k)}(x_0) \neq 0$$

$$P(x) = (x - x_0) \cdot \tilde{P}(x)$$

By the auxiliary claim,  $P^{(j)}(x) = (x - x_0) \cdot \tilde{P}^{(j)}(x) + j \cdot \tilde{P}^{(j-1)}(x)$ . Therefore

$$P^{(j)}(x_0) = j \cdot \tilde{P}^{(j-1)}(x_0) = \begin{cases} 0 & \text{for } j = 0, \dots, k \\ (k+1)\tilde{P}^{(k)}(x_0) \neq 0 & \text{for } j = k+1 \end{cases}$$

Hence, our claim about the derivatives is true (all derivatives are zero).

Direction  $\impliedby$ .

Let  $P^{(j)}(x_0) = 0$  for  $j = 0, \dots, k$  and  $P^{(k+1)}(x_0) \neq 0$  and induction assumption holds for  $k$ . Division with remainder and  $P^{(0)}(x_0) = 0 \implies P(x) = (x - x_0) \cdot \tilde{P}(x)$ . By our auxiliary claim, we get

$$P^{(j)}(x) = (x - x_0) \cdot \tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x)$$

we insert  $x = x_0$  and use  $P^{(j)}(x_0) = 0$  for  $j = 0, \dots, k$

$$\implies \tilde{P}^{(j)}(x_0) = 0 \quad \text{for } j = 0, \dots, k-1$$

By the induction assumption,  $\tilde{P}(x) = (x - x_0)^k Q(x)$  with  $Q(x_0) \neq 0$

$$\implies P(x) = (x - x_0) \cdot \tilde{P}(x) = (x - x_0)^{k+1} Q(x)$$

□

*This lecture took place on 2018/05/08.*

**Definition 5.3.** Let  $I \subseteq \mathbb{R}$  be an interval,  $f \in C^n(I)$ . We let

$$T_f^n(x; x_0) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0) (x - x_0)^k$$

$T_f^n(x; x_0)$  is a polynomial in  $x$  with  $\text{degree}(T_f^n) \leq n$ .  $T_f^n(x; x_0)$  is called Taylor polynomial of  $f$  of order  $n$  in  $x_0$ .

Brook Taylor (1685–1731)

**Lemma 5.3.** *The premise is the same like in Definition 5.3. The Taylor polynomial of  $T_f^n(x; x_0)$  is the only polynomial of degree  $\leq n$  which satisfies*

$$(T_f^n)^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, \dots, n$$

*Proof.* Claim:

$$(T_f^n)^{(k)}(x; x_0) = \sum_{l=k}^n \frac{f^{(l)}(x_0)}{(l-k)!} (x - x_0)^{l-k} \quad \text{for } 0 \leq k \leq n$$

Proof of the claim by complete induction:

**Induction base**  $n = 0$

$$(T_f^n)^{(0)}(x; x_0) = \sum_{l=0}^n \frac{f^{(l)}(x_0)}{l!} (x - x_0)^l$$

**Induction step**  $k \rightarrow k + 1$  Let  $(T_f^n)^{(k)}(x; x_0)$

$$= \sum_{l=k}^n \frac{f^{(l)}(x_0)}{(l-k)!} (x - x_0)^{(l-k)}$$

by induction hypothesis. Then,

$$\begin{aligned} &= \sum_{l=k+1}^n \frac{f^{(l)}(x_0)}{(l-k)!} (l-k) \cdot (x - x_0)^{l-k-1} \\ &= \sum_{l=k+1}^n \frac{f^{(l)}(x_0)}{(l-(k+1))!} (x - x_0)^{l-(k+1)} \end{aligned}$$

We apply insertion:  $x = x_0$  into  $(T_f^n)^{(k)}(x; x_0)$

$$(T_f^n)^{(k)}(x; x_0) = \sum_{l=k}^n \frac{f^{(l)}(x_0)}{(l-k)!} (x - x_0)^{l-k} = \frac{f^{(k)}(x_0)}{0!} = f^{(k)}(x_0)$$

We need to prove uniqueness: Let  $T, \tilde{T}$  be polynomials with  $T^{(k)}(x_0) = \tilde{T}^{(k)}(x_0) = f^{(k)}(x_0)$  for  $k = 0, \dots, n$ . Assume  $T \neq \tilde{T}$ , hence  $T - \tilde{T} \neq 0$  (where 0 is the zero polynomial). For  $P = T - \tilde{T}$  it holds that

$$P^{(k)}(x_0) = T^{(k)}(x_0) - \tilde{T}^{(k)}(x_0) = 0 \quad (\text{for } 0 \leq k \leq n)$$

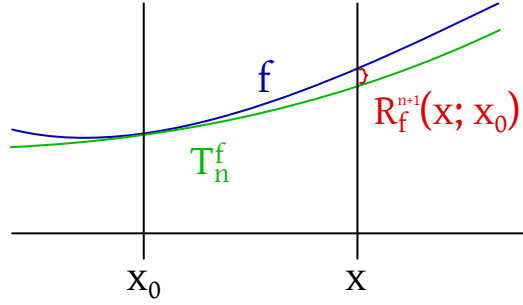


Figure 32: Visualization of the remainder term of a Taylor polynomial

By Lemma 5.2 it holds that  $x_0$  is an  $n + 1$ -times root of  $P$ . Thus, there exists a polynomial  $Q \neq 0$  with  $Q(x_0) \neq 0$  such that

$$\underbrace{P(x)}_{\text{degree} \leq n} = \underbrace{(x - x_0)^{(n+1)} \cdot Q(x)}_{\text{degree} \geq n+1}$$

This is a contradiction. Hence it holds that  $T - \tilde{T} = 0$ .  $\square$

**Definition 5.4.** Let  $f \in C^n(I)$ ,  $x_0 \in I$ . Furthermore let  $T_f^n(x; x_0)$  be the Taylor polynomial of  $n$ -th degree of  $f$  in  $x_0$ . We let  $R_f^n(x; x_0) = f(x) - T_f^n(x; x_0)$ . We call  $R_f^{n+1}(x; x_0)$  the approximation error of the Taylor polynomial. Also called remainder term of  $n + 1$ -th order. Compare with Figure 32.

**Theorem 5.4.** Let  $f^{(n+1)}(I)$ ,  $x \in I$ ,  $x_0 \in I$ . Then it holds that

$$R_f^{n+1}(x; x_0) = \frac{1}{n!} \int_{x_0}^x (x - t)^n f^{(n+1)}(t) dt$$

We call it the integral form of the remainder term.

*Proof.* Complete induction over  $n$ .

**Induction base  $n = 0$**

$$T_f^0(x; x_0) = f(x_0)$$

$$\begin{aligned}
R_f^1(x; x_0) &= \underbrace{f(x) - f(x_0)}_{f \in C^1} \\
&= \int_{x_0}^x f'(t) dt \\
&= \frac{1}{0!} \int_{x_0}^x (x-t)^0 f^{(1)}(t) dt
\end{aligned}$$

**Induction step  $n-1 \rightarrow n$**

$$\begin{aligned}
R_f^n(x; x_0) &= f(x) - T_f^{n-1}(x; x_0) \\
&\stackrel{\text{ind. hypothesis}}{=} \frac{1}{(n-1)!} \int_{x_0}^x (x-t)^{n-1} f^{(n)}(t) dt \\
&= \left| \begin{array}{ll} u' = (x-t)^{n-1} & v = f^{(n)}(t) \\ u = -\frac{1}{n}(x-t)^n & v' = f^{(n+1)}(t) \end{array} \right| \\
&= \frac{1}{(n-1)!} \left[ \underbrace{-\frac{1}{n}(x-t)^n \cdot f^{(n)}(t)}_{=\frac{1}{n!}(x-x_0)^n \cdot f^{(n)}(x_0)} \right] \bigg|_{t=x_0}^x + \underbrace{\frac{1}{(n-1)!} \int_{x_0}^x \frac{1}{n}(x-t)^n \cdot f^{(n+1)}(t) dt}_{=\frac{1}{n!} \int_{x_0}^x (x-t)^n \cdot f^{(n+1)}(t) dt}
\end{aligned}$$

So,

$$\begin{aligned}
&\underbrace{f(x) - T_f^{n-1}(x; x_0) - \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n}_{-T_f^n(x; x_0)} \\
&= \frac{1}{n!} \int_{x_0}^x (x-t)^n \cdot f^{(n+1)}(t) dt
\end{aligned}$$

Therefore,

$$R_f^{(n+1)}(x; x_0) = \frac{1}{n!} \int_{x_0}^x (x-t)^n f^{(n+1)}(t) dt$$

□

**Theorem 5.5** (Lagrange form of the remainder term). *Let  $f \in C^{n+1}(I)$ ,  $n \in \mathbb{N}_0$ ,  $x, x_0 \in I$ ,  $x \neq x_0$ . Then there exists some  $\xi$  between  $x_0$  and  $x$  (hence,  $\xi \in (x_0, x)$  if  $x > x_0$  or  $\xi \in (x, x_0)$  if  $x < x_0$ ) such that*

$$R_f^{(n+1)}(x; x_0) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

*Proof.* Idea: we apply the Mean Value Theorem for definite integrals on the Taylor remainder.

**Case 1** Let  $x_0 < x$ .

$$\begin{aligned}
R_f^{n+1}(x; x_0) &= \frac{1}{n!} \int_{x_0}^x \underbrace{(x-t)^n}_{\substack{\geq 0 \\ \text{regulated function}}} \underbrace{f^{(n+1)}(t)}_{\text{continuous in } t} dt \\
&\stackrel{\text{MVT}}{=} \frac{1}{n!} f^{(n+1)}(\xi) \cdot \int_{x_0}^x (x-t)^n dt \\
&= \frac{1}{n!} f^{(n+1)}(\xi) \left[ -\frac{1}{n+1} (x-t)^{n+1} \right]_{t=x_0}^x \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x-x_0)^{n+1}
\end{aligned}$$

where MVT is the Mean Value Theorem for definite integrals (Theorem 4.2).

**Case 2** Let  $x < x_0$  and  $n$  odd.

$$\begin{aligned}
R_f^{n+1}(x; x_0) &= -\frac{1}{n!} \int_x^{x_0} \underbrace{(x-t)^n}_{= (-1)^n (t-x)^n} \cdot f^{(n+1)}(t) dt \\
&= \frac{1}{n!} \int_x^{x_0} \underbrace{(t-x)^n}_{\geq 0} \cdot \underbrace{f^{(n+1)}(t)}_{\text{continuous}} dt \\
&= \frac{f^{(n+1)}(\xi)}{n!} \int_x^{x_0} (t-x)^n dt \\
&= \frac{f^{(n+1)}(\xi)}{n!} \left[ \frac{1}{n+1} (t-x)^{n+1} \right]_x^{x_0} \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x_0-x)^{n+1}
\end{aligned}$$

$n+1$  is even

$$= \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x-x_0)^{n+1}$$

**Case 3** Let  $x < x_0$  and  $n$  even.

$$\begin{aligned}
R_f^{n+1}(x; x) &= -\frac{1}{n!} \int_x^{x_0} \underbrace{(x-t)^n}_{\geq 0} \cdot \underbrace{f^{(n+1)}(t)}_{\text{continuous}} dt \\
&= -\frac{1}{n!} f^{(n+1)}(\xi) \cdot \int_x^{x_0} (x-t)^n dt \\
&= -\frac{1}{n!} f^{(n+1)}(\xi) \cdot \left[ -\frac{1}{n+1} (x-t)^{n+1} \right]_x^{x_0} \\
&= \frac{1}{(n+1)!} f^{(n+1)}(\xi) (x-x_0)^{n+1}
\end{aligned}$$

□

Any extreme value satisfies that its derivative is zero. But not every point with derivative zero is an extreme value. We now consider conditions to select extreme values from all value satisfying derivative zero.

**Corollary** (Sufficient conditions for existence of extreme values). *Let  $I$  be an open interval. Let  $x_0 \in I$  and  $f \in C^{n+1}(I)$ . Assume*

$$f^{(1)}(x_0) = f^{(2)}(x_0) = \dots = f^{(n)}(x_0) = 0$$

*and  $f^{(n+1)}(x_0) \neq 0$ . Then  $f$  in  $x_0$  has*

1. *a strict local maximum if  $n$  is even and  $f^{(n+1)}(x_0) < 0$*
2. *a strict local minimum if  $n$  is odd and  $f^{(n+1)}(x_0) > 0$*
3. *no extreme value in  $x_0$  if  $n$  is even.*

*Proof. Case a* Let  $f^{(n+1)}(x_0) < 0$  and  $f^{(n+1)}$  is continuous, then  $\exists \varepsilon > 0$  such that  $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq I$  ( $I$  is open) and  $f^{(n+1)}(\xi) < 0 \forall \xi \in (x_0 - \varepsilon, x_0 + \varepsilon)$ . Now let  $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ . Then by Theorem 5.5,

$$\frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} = R_f^{n+1}(x; x_0) = f(x) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x_0) (x-x_0)^k = f(x) - f(x_0)$$

for  $k = 1, \dots, n$ . So,

$$f(x) - f(x_0) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{<0} \underbrace{(x-x_0)^{n+1}}_{\substack{\text{even} \\ >0 \text{ for } x \neq x_0}}$$

hence  $f(x) - f(x_0) < 0$ , or equivalently

$$f(x) < f(x_0) \quad \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon), x \neq x_0$$

So  $f$  is a strict local maximum.

**Case b** Analogously.

**Case c** We apply the same idea as in Case a up to the point, where we consider  $f(x) - f(x_0)$ .

$$f(x) - f(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

$f^{(n+1)}(\xi)$  has the same sign as  $\underbrace{f^{(n+1)}(x_0)}_{\neq 0} \quad \forall \xi \in (x_0 - \varepsilon, x_0 + \varepsilon)$ . This is

feasible due to continuity of  $f^{(n+1)}$  for sufficiently small  $\varepsilon$ .

$$f(x) - f(x_0) = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{has constant sign indep. of } x} \cdot \underbrace{(x - x_0)}_{\text{changes its sign}} \overbrace{n+1}^{\text{odd}}$$

Therefore  $f(x) - f(x_0)$  changes its sign for  $x = x_0$ . Hence  $f$  has no extreme value in  $x = x_0$ .

□

**Theorem 5.6** (Qualitative Taylor equation). *Let  $f \in C^n(I)$ ,  $x, x_0 \in I$ . Then there exists some function  $r \in C(I)$  with  $r(x_0) = 0$  such that*

$$f(x) = T_f^n(x; x_0) + (x - x_0)^n \cdot r(x)$$

or equivalently,

$$R_f^{n+1}(x; x_0) = (x - x_0)^n \cdot r(x)$$

**Remark 5.6.** For some function  $r$  with  $\lim_{x \rightarrow x_0} r(x) = 0$ , we also denote  $o(x - x_0)$  instead of  $r(x)$ . This general notation is called Landau's Big-Oh notation.

$$f(x) = T_f^n(x; x_0) + (x - x_0)^n \cdot o(x - x_0)$$

*Proof.* Let  $r(x) = \frac{f(x) - T_f^n(x; x_0)}{(x - x_0)^n}$  for  $x \neq x_0$  and  $r(x_0) := 0$ . Then  $f$  is continuous and  $T_f^n$  is continuous in every point  $x \neq x_0$ . It remains to show that  $r$  is continuous in  $x = x_0$ .

$$\begin{aligned} r(x) &= \frac{1}{(x - x_0)^n} \underbrace{(f(x) - T_f^{n-1}(x; x_0))}_{R_f^n(x; x_0)} - \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \\ &\stackrel{\text{Lagrange}}{=} \frac{1}{(x - x_0)} \left[ \frac{1}{n!} (x - x_0)^n \cdot f^{(n-1)}(\xi) - \frac{1}{n!} f^{(n)}(x_0) (x - x_0)^n \right] \end{aligned}$$

$\xi \in (x_0, x)$

$$= \frac{1}{n!} [f^{(n)}(\xi) - f^{(n)}(x_0)] \rightarrow 0 \text{ for } x \rightarrow x_0 \text{ because } f^{(n)} \text{ is continuous}$$

as  $x_0 < x < x$ , hence  $\xi \rightarrow x_0$  for  $x \rightarrow x_0$

So  $\lim_{x \rightarrow x_0} r(x) = 0 = r(x_0)$ , so  $r$  in  $x_0$  is continuous.  $\square$

*This lecture took place on 2018/05/15.*

## 5.2 Taylor series

Assume  $f : I \rightarrow \mathbb{R}$  is infinitely often differentiable on  $I$ ,  $x_0 \in I$ . Then there exists  $T_f^n(x; x_0)$  for arbitrary  $n \in \mathbb{N}$ .

$$T_f(x; x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

$T_f(x; x_0)$  defines the *Taylor series* on  $f$  in  $x_0$ . Power series in  $\xi = x - x_0$ .  $T_f$  has a convergence radius,

$$\rho(T_f) = \left[ \limsup_{k \rightarrow \infty} \sqrt[k]{\frac{|f^{(k)}(x_0)|}{k!}} \right]^{-1}$$

If  $\rho(T_f) > 0$ , then it holds that

$$f(x) = T_f(x; x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (x - x_0)^k$$

in  $(x_0 - \rho(T_f), x_0 + \rho(T_f))$ ? Compare with Figure 33.

**Example 5.1** (Counterexample). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ .

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

It holds for  $x > 0$ ,

$$f^{(n)}(x) = \frac{P(x)}{Q(x)} \cdot e^{-\frac{1}{x}}$$

where  $P, Q$  are polynomials. So the function value (of an infinitely often differentiable function) must not equate with its Taylor series value.



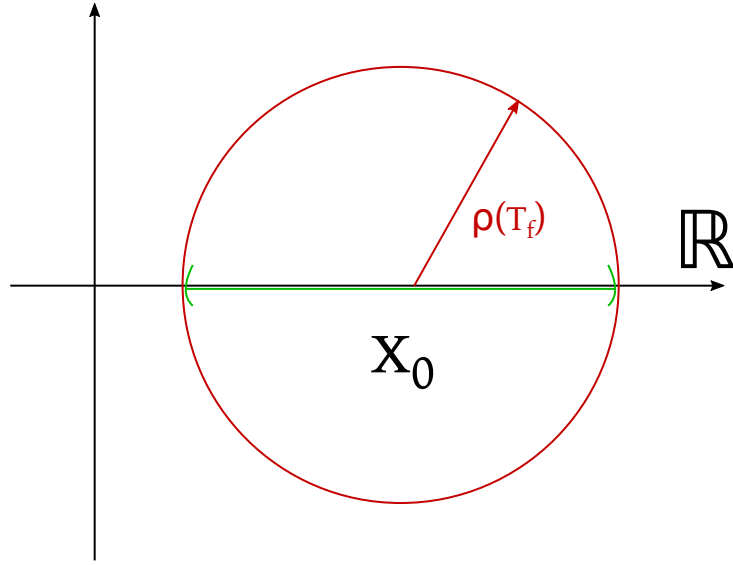


Figure 33: Taylor series

*Proof.* Proof by complete induction over  $n$ .

**Case**  $n = 0$  immediate with  $P = Q = 1$ .

**Case**  $n \mapsto n + 1$

$$\begin{aligned}
 f^{(n+1)}(x) &= \left[ \underbrace{\frac{P(x)}{Q(x)} \cdot e^{-\frac{1}{x}}}_{f^{(n)}(x) \text{ by induction hypothesis}} \right]' \\
 &= \frac{P' \cdot Q - Q' \cdot P}{Q^2} \cdot e^{-\frac{1}{x}} + \frac{P}{Q} \cdot \frac{1}{x^2} \cdot e^{-\frac{1}{x}} \\
 &= \frac{(P'Q - Q'P)x^2 + PQ}{Q^2x^2} \cdot e^{-\frac{1}{x}}
 \end{aligned}$$

It holds that  $\lim_{x \rightarrow 0^+} \frac{P(x)}{Q(x)} \cdot e^{-\frac{1}{x}} = 0$ . Immediately,  $\lim_{x \rightarrow 0^-} f^{(n)}(x) = 0$ , hence  $f^{(n)}(0) = 0 \forall n \in \mathbb{N}$ .  $f$  is arbitrarily often continuously differentiable on  $\mathbb{R}$ . Thus,

$$T_f(x; 0) = \sum_{k=0}^{\infty} \overbrace{\frac{f^{(k)}(0)}{k!}}^{=0} x^k = 0$$

but  $f(x) \neq 0$  on  $\mathbb{R}$ . Thus, it holds that  $f \neq T_f(x; 0)$ . But it holds that  $R_f = f - T_f(x; 0) = f$ .

$$|R_f(x)| \leq c_n |x|^n \quad \forall n \in \mathbb{N}$$

□

**Theorem 5.7.** Let  $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$  be an analytical<sup>2</sup> function with convergence radius  $\rho(f) > 0$ . Then  $f$  is infinitely often continuously differentiable on  $I := (x_0 - \rho(f), x_0 + \rho(f))$  and it holds that  $a_k = \frac{f^{(k)}(x_0)}{k!}$ , hence the given power series is the Taylor series of the function.

*Proof.* See Analysis 1 lecture notes, chapter 8, theorem 1 by G. Lettl.

$f$  is differentiable on  $I = (x_0 - \rho(f), x_0 + \rho(f))$  and it holds that  $f'(x) = \sum_{k=0}^{\infty} k a_k(x-x_0)^{k-1}$ . Thus,  $f'$  is also analytical and the power series of  $f'$  converges on  $K(x_0) \implies \rho(f') \geq \rho(f)$  (if you consider the Cauchy-Hadamard Theorem, then  $\rho(f') = \rho(f)$ ).

Induction:  $f^{(n)}(x)$  is analytical on  $I$  and it holds that

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k \cdot (k-1) \dots (k-n+1) \cdot a_k \cdot (x-x_0)^{k-n}$$

We insert:  $x = x_0$

$$f^{(n)}(x_0) = n \cdot (n-1) \dots 1 \cdot a_n \implies a_n = \frac{f^{(n)}(x_0)}{n!}$$

□

Revision: Expansion on a different point ( $\xi_0$  instead of  $x_0$ ):

$$f(z) = \sum_{k=0}^{\infty} a_k(z-x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (z-x_0)^k$$

with  $a_k = \frac{f^{(k)}(x_0)}{k!}$ .  $f(z) = \sum_{k=0}^{\infty} \tilde{a}_k(z-\xi_0)^k$  with  $\tilde{a}_k = \frac{f^{(k)}(\xi_0)}{k!}$ . Compare with Figure 34.

## 6 Multidimensional differential calculus

Let  $V, W$  be vector space over  $\mathbb{K}$  ( $\mathbb{R}, \mathbb{C}$ ).

$$\underbrace{\mathcal{L}(V, W)}_{\text{Hom}(V, W)} = \{\varphi : V \rightarrow W : \varphi \text{ is linear}\}$$

---

<sup>2</sup>Reminder: A function is analytical if it is locally given by a convergent power series.

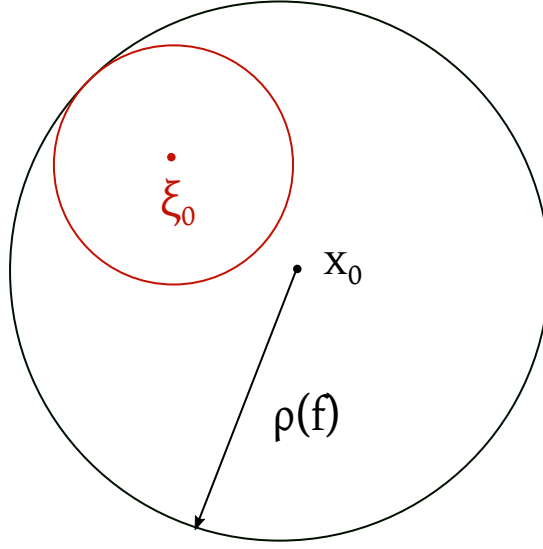


Figure 34: Expansion on a different point

$\text{Hom}(V, W)$  has vector space properties.  $\varphi, \psi \in \mathcal{L}(V, W)$ ,  $\lambda, \mu \in \mathbb{K}$ . Then it holds that  $\lambda\varphi + \mu\psi \in \mathcal{L}(V, W)$ . In general, it is feasible that to define a norm on  $\mathcal{L}(V, W)$ . Hence,  $\|\cdot\| : \mathcal{L}(V, W) \rightarrow [0, \infty)$  with

1.  $\|\varphi\| = 0 \iff \varphi = 0$  (zero mapping)
2.  $\forall \lambda \in \mathbb{K}, \varphi \in \mathcal{L}(V, W)$  it holds that  $\|\lambda\varphi\| = |\lambda| \cdot \|\varphi\|$ .
3.  $\forall \varphi, \psi \in \mathcal{L}(V, W)$  it holds that  $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$ .

## 6.1 Frobenius and matrix norm

**Example 6.1.** Let  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^{n \times m}$ . (identify linear maps with its matrix representation in regards of the canonical basis)

$$A \in \mathbb{R}^{n \times m} \quad A = (a_{ij})_{\substack{i=1, \dots, n \\ j=1, \dots, m}}$$

$$\|A\|_F = \left( \sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2 \right)^{\frac{1}{2}} \quad \text{"Frobenius norm"}$$

It basically works by appending the next column to the previous one. Hence, this gives a column vector. We square every entries, sum it up and take its square root (a common norm procedure). A norm on  $\mathbb{R}^{n \times m}$  (a matrix) is called matrix norm ( $\mathbb{C}^{n \times m}$ ).

## 6.2 Operator norm and bounded linear operators

**Definition 6.1.** Let  $V, W$  be normed vector spaces over  $\mathbb{K}$ . A linear map  $\varphi : V \rightarrow W$  is called *bounded* if  $\exists m \geq 0 : \|\varphi(x)\|_W < m \cdot \|x\|_V$  (we call this the *boundedness criterion*) such that  $\|\varphi(x)\|_W \leq m \cdot \|x\|_V$  for all  $x \in V$ .

The set  $\mathcal{L}_b(V, W) = \{\varphi : V \rightarrow W : \varphi \text{ is linear and bounded}\}$  is a subvectorspace of  $\mathcal{L}(V, W)$ . We let

$$\|\varphi\| = \inf \{m \geq 0 : \|\varphi(x)\|_W \leq m \cdot \|x\|_V \forall x \in V\}$$

and call  $\|\varphi\|$  the operator norm on  $\varphi$  in regards of  $\|\cdot\|_V$  and  $\|\cdot\|_W$ .

Regarding the subvector space property:

Let  $\varphi, \psi \in \mathcal{L}_b(V, W)$ ,  $\lambda, \mu \in \mathbb{K}$ . Show that  $\lambda\varphi + \mu\psi \in \mathcal{L}_b(V, W)$ .

$$\begin{aligned} \|(\lambda\varphi + \mu\psi)(x)\|_W &= \|\lambda \cdot \varphi(x) + \mu \cdot \psi(x)\|_W \\ &\leq |\lambda| \|\varphi(x)\|_W + |\mu| \|\psi(x)\|_W \\ &\leq \underbrace{|\lambda| m + |\mu| m'}_{\text{because } \varphi, \psi \text{ are bounded}} \|x\|_V \\ &= \underbrace{(|\lambda| m + |\mu| m')}_{=:\tilde{m} \geq 0} \|x\|_V \end{aligned}$$

hence  $\lambda\varphi + \mu\psi \in \mathcal{L}_b(V, W)$ .  $\mathcal{L}_b(V, W) \neq \emptyset$ .

**Lemma 6.1.** Let  $V, W$  be normed vector spaces. Then it holds for any  $\varphi \in \mathcal{L}_b(V, W)$

1.  $\|\varphi(x)\|_W \leq \|\varphi\| \cdot \|x\|_V \forall x \in V$ . Hence,  $m = \|\varphi\|$  satisfies the boundedness criterion, hence informally  $\inf$  equals  $\min$  in Definition 6.1.

2.

$$\|\varphi\| = \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \in V \setminus \{0\} \right\} = \sup \left\{ \|\varphi(x)\|_W : x \in V \text{ with } \|x\|_V = 1 \right\}$$

3.  $\|\cdot\|$  is a norm on  $\mathcal{L}_b(V, W)$ .

*Proof.* 1. Let  $m_n \geq 0$  with  $m_n$  satisfies the boundedness criterion, hence

$$\|\varphi(x)\|_W \leq m_n \cdot \|x\|_V \forall x \in V$$

and  $m_n \rightarrow \|\varphi\|$ . The inequality retains in the limit. Thus,  $\|\varphi(x)\|_W \leq \|\varphi\| \cdot \|x\|_V$ .

2. Let  $\tilde{m} = \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\}$ . Hence,  $\frac{\|\varphi(x)\|_W}{\|x\|_V} \leq \tilde{m} \forall x \in V$ , because  $\tilde{m}$  is an upper bound. So,  $\|\varphi(x)\|_W \leq \tilde{m} \|x\|_V$ . Thus  $\tilde{m}$  satisfies the boundedness criterion and  $\|\varphi\| \leq \tilde{m}$ .

On the opposite: Let  $m$  such that the boundedness criterion is satisfied  $\Rightarrow \|\varphi(x)\|_W \leq m \cdot \|x\|_V \forall x \in V, x \neq 0$  or equivalently,  $\frac{\|\varphi(x)\|_W}{\|x\|_V} \leq m$ . Hence,  $m$  is upper bound of  $\left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\}$ , hence  $m \geq \tilde{m} = \sup \{ \cdot \}$ . Hence,  $m \geq \tilde{m} = \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\}$ . The statement above also holds for the infimum of  $m - s$ , hence  $\|\varphi\| \geq \tilde{m}$ , hence  $\|\varphi\| = \tilde{m} = \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\}$ . Because  $\{x \in V : \|x\| = 1\} \subseteq \{x \in V : x \neq 0\}$  it holds that  $\sup \|\varphi(x)\|_W : \|x\| = 1 = \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : \|x\|_V = 1 \right\} \leq \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\} = \|\varphi\|$ .

On the opposite: Let  $x \neq 0$ . Then  $\tilde{x} = \frac{x}{\|x\|_V}$  defines a *unit vector*.

$$\|\tilde{x}\|_V = \left\| \frac{x}{\|x\|_V} \right\| = \frac{1}{\|x\|_V} \cdot \|x\|_V = 1$$

and it holds that

$$\begin{aligned} \frac{\|\varphi(x)\|_W}{\|x\|_V} &= \frac{1}{\|x\|_V} \|\varphi(x)\|_W = \left\| \frac{1}{\|x\|_V} \varphi(x) \right\|_W \underbrace{=}_{\varphi \text{ is linear}} = \left\| \varphi\left(\frac{x}{\|x\|_V}\right) \right\|_W = \|\varphi(\tilde{x})\| \\ \Rightarrow \forall x \neq 0 : \frac{\|\varphi(x)\|_W}{\|x\|_V} &= \|\varphi(\tilde{x})\| \leq \sup \left\{ \|\varphi(z)\|_W : \|z\|_V = 1 \right\} \\ \Rightarrow \sup \left\{ \frac{\|\varphi(x)\|_W}{\|x\|_V} : x \neq 0 \right\} &\leq \sup \left\{ \|\varphi(z)\|_W : \|z\|_V = 1 \right\} \end{aligned}$$

3. Show that  $\|\varphi\|$  is a norm.

$$\|\varphi\| = 0 \iff \forall x \in V : \|\varphi(x)\|_W \leq 0 \cdot \|x\|_W$$

hence  $\varphi(x) = 0 \forall x \in V$  or equivalently,  $\varphi = 0$  in  $\mathcal{L}(V, W)$ .

$$\begin{aligned} \|\lambda\varphi\| &= \sup \left\{ \|\lambda\varphi(x)\|_W : \|x\|_V = 1 \right\} = \sup \left\{ |\lambda| \|\varphi(x)\|_W : \|x\|_V = 1 \right\} \\ &= |\lambda| \sup \left\{ \|\varphi(x)\|_W : \|x\|_V = 1 \right\} = |\lambda| \|\varphi\| \end{aligned}$$

Triangle inequality: Let  $\varphi, \psi \in \mathcal{L}_b(V, W)$ .

$$\begin{aligned} \|\varphi(x) + \psi(x)\|_W &\underbrace{\leq}_{\text{triangle inequality in } W} \|\varphi(x)\|_W + \|\psi(x)\|_W \\ &\leq \|\varphi\| \cdot \|x\|_V + \|\psi\| \cdot \|x\|_V = (\|\varphi\| + \|\psi\|) \cdot \|x\| \end{aligned}$$

By (1.),  $\|\varphi\| + \|\psi\|$  satisfies the boundedness criterion for the linear map  $\varphi + \psi$ . Hence,  $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$ .

□

**Remark 6.1.** •  $\|A\|_F$  is no operator norm on  $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ .

- Boundedness of linear mappings is required to define  $\|\varphi\|$ .

We consider special case  $V = \mathbb{R}^m, W = \mathbb{R}^n$ .

$$\|\cdot\|_V = \|\cdot\|_\infty \quad \|\cdot\|_W = \|\cdot\|_\infty$$

Let  $A \in \mathbb{R}^{n \times m}$  ( $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ ). Then it holds that

$$\begin{aligned} \|Ax\|_\infty &= \max \{|(Ax)_i| : i = 1, \dots, n\} \\ &= \max \left\{ \left| \sum_{j=1}^m a_{ij}x_j \right| : i = 1, \dots, n \right\} \\ &\leq \max \left\{ \sum_{j=1}^m |a_{ij}| \cdot \underbrace{|x_j|}_{\leq \|x\|_\infty} : i = 1, \dots, n \right\} \\ &\leq \max \left\{ \|x\|_\infty \cdot \sum_{j=1}^m |a_{ij}| : i = 1, \dots, n \right\} \\ &= \underbrace{\max \left\{ \sum_{j=1}^m |a_{ij}| : i = 1, \dots, n \right\}}_{=m} \cdot \|x\|_\infty \\ &= m \cdot \|x\|_\infty \end{aligned}$$

Hence the boundedness criterion is satisfied.  $A$  is bounded in regards of  $\|\cdot\|_\infty$  in the preimage and image space. By the norm equivalence theorem, it follows that  $A$  is bounded in regards of arbitrary norms on  $\mathbb{R}^m$ , or equivalently  $\mathbb{R}^n$ .

This lecture took place on 2018/05/17.

Further remarks:

**Remark 6.2.** A linear map  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is always bounded. Thus,

$$\|Ax_1 - Ax_2\|_{\mathbb{R}^n} = \|A(x_1 - x_2)\|_{\mathbb{R}^n} \leq \|A\| \|x_1 - x_2\|_{\mathbb{R}^m}$$

So every linear map  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is Lipschitz continuous with Lipschitz constant  $\|A\|$ .

The considerations above hold for arbitrary finite-dimensional normed vector spaces  $V$  and  $W$  (over  $\mathbb{K} = \mathbb{R}, \mathbb{C}$ ).

**Lemma 6.2.** *Let  $X, Y$  and  $Z$  be normed vector spaces. Let  $A : X \rightarrow Y$  be linear and bounded. Let  $B : Y \rightarrow Z$  be linear and bounded. Then*

$$B \cdot A : X \rightarrow Z$$

*is also bounded and it holds that*

$$\|B \cdot A\| \leq \|B\| \cdot \|A\|$$

*Proof.* Let  $x \in X$  be arbitrary and  $BAx = B(Ax) \in Z$  and

$$\|BAx\|_Z = \|B(Ax)\|_Z \overset{\substack{\leq \\ B \text{ is bounded}}}{\leq} \|B\| \|Ax\|_Y \overset{\substack{\leq \\ A \text{ is bounded}}}{\leq} \|B\| \cdot \|A\| \|x\|_X$$

Hence  $m = \|B\| \|A\|$  satisfies the boundedness criterion for the linear map  $B \cdot A : X \rightarrow Z$ . Because  $\|B \cdot A\|$  is the smallest constant for which the boundedness criterion holds, it follows that  $\|BA\| \leq \|B\| \|A\|$ .  $\square$

### 6.3 Landau notation

**Definition 6.2** (Landau  $\mathcal{O}$  symbols). *Let  $h, g : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $D$  is open,  $a \in D$ .*

1. *We denote  $h = \mathcal{O}(g)$  in  $a$  (German pronunciation:  $h$  ist groß  $\mathcal{O}$  von  $g$ ) iff  $\exists U \subseteq D$  environment of  $a$  in  $D$  and  $\exists r : U \rightarrow \mathbb{R}$  with  $r$  bounded such that  $h(x) = r(x) \cdot g(x) \forall x \in U$ . Thus,*

$$\left| \frac{h(x)}{g(x)} \right| = |r(x)| \leq M \quad \forall x \in U$$

$$(g(x) = 0 \text{ iff } h(x) = 0)$$

2. *We denote  $h = o(g)$  in  $a$  (German pronunciation:  $h$  ist klein  $o$  von  $g$ ) iff  $\exists U \subseteq D$ , with  $U$  being the environment of  $a$ , and  $r : U \rightarrow \mathbb{R}$  such that  $\lim_{x \rightarrow a} r(x) = 0$  and  $h(x) = r(x) \cdot g(x) \forall x \in U$ . In that sense,*

$$\lim_{x \rightarrow a} \frac{h(x)}{g(x)} = 0$$

*Most often,*

$$\underbrace{\mathcal{O}(\|x - x_0\|^n)}_{g(x)}$$

*is used and  $a = x_0$ .*

## 6.4 Multidimensional derivative of a function

**Definition 6.3** (Definition of the derivative of a function).

$$f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n \quad D \text{ is open, } x_0 \in D$$

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \in \mathbb{R}^n$$

$$f_i(x) = f_i\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right) = f_i(x_1, x_2, \dots, x_m) \in \mathbb{R}$$

**Remark 6.3.** First trial to define of a the derivative:

$$f'(x_0) := \lim_{x \rightarrow x_0} \frac{\overbrace{f(x) - f(x_0)}^{\in \mathbb{R}^n}}{\underbrace{x - x_0}_{\in \mathbb{R}^m}}$$

Does not work because of incompatibility of dimensions (and we cannot divide vectors).

**Remark 6.4.** We use Taylor's Theorem to characterize  $f'(x_0)$ . A Taylor polynomial of 1st degree is given by

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_f^2(x; x_0)$$

and  $R_f^2(x; x_0) = r(x)(x - x_0)$  with  $\lim_{x \rightarrow x_0} r(x) = 0$ . Hence,

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = O(x - x_0)$$

or we insert the absolute operators:

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| = |r(x)| \cdot |x - x_0| = o(|x - x_0|)$$

where  $(f(x) - f(x_0)) \in \mathbb{R}^n$  for the multidimensional case and  $(x - x_0) \in \mathbb{R}^m$  for the multidimensional case. Thus,

$$= O(|x - x_0|)$$

**Definition 6.4.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $D$  is open and  $x_0 \in D$ . We say, “ $f$  is differentiable in  $x_0$ ” (specifically, “Frechét differentiable”) if there exists  $A \in \mathbb{R}^{n \times m}$  such that

$$\|f(x) - f(x_0) - A(x - x_0)\|_{\mathbb{R}^n} = o(\|x - x_0\|_{\mathbb{R}^m})$$

We call this condition differentiability condition.  $A$  is a linear approximation of  $f$  in  $x_0$ . Because of norm equivalence on  $\mathbb{R}^n$ , or equivalently  $\mathbb{R}^m$ , it is irrelevant which norm  $\|\cdot\|_{\mathbb{R}^n}$  and  $\|\cdot\|_{\mathbb{R}^m}$  is chosen.



**Lemma 6.3.** *Let  $f$  be, as in Definition 6.4, differentiable in  $x_0$ . Then the linear approximation  $A$ , by the differentiability condition, is uniquely determined.*

*Proof.* Assume  $A, B \in \mathbb{R}^{n \times m}$  satisfy the differentiability condition. Let  $r > 0$  such that  $K_r(x_0) \subseteq D$  (feasible because  $D$  is open and  $x_0 \in D$ ). Furthermore let  $v \in \mathbb{R}^m$  and  $\|v\| < r$ , hence  $x = x_0 + v \in K_r(x_0) \subset D$  and  $v = x - x_0$ .

$$\begin{aligned} \|(A - B)v\| &= \|Av - Bv\| = \|A(x - x_0) - B(x - x_0)\| \\ &= \|f(x) - f(x_0) - B(x - x_0) - (f(x) - f(x_0) - A(x - x_0))\| \\ &\leq \|f(x) - f(x_0) - B(x - x_0)\| + \|f(x) - f(x_0) - A(x - x_0)\| \end{aligned}$$

by the differentiability criterion

$$r(x) \cdot \|x - x_0\| + \tilde{r}(x) \|x - x_0\|$$

with  $\lim_{x \rightarrow x_0} r(x) = \lim_{x \rightarrow x_0} \tilde{r}(x) = 0$ . Hence, for  $\hat{r}(x) = r(x) + \tilde{r}(x)$  it holds that

$$\|(A - B)v\| \leq \hat{r}(x) \|x - x_0\| = \hat{r}(x) \|v\| = \mathcal{O}(\|v\|) \text{ in } x_0$$

(Thus  $\lim_{x \rightarrow x_0} \hat{r}(x) = 0$ )

Show:  $(A - B)w = 0 \forall w \in \mathbb{R}^m$ . Assume  $\exists w \in \mathbb{R}^m, w \neq 0$  with  $(A - B)w \neq 0$ . For  $|\alpha| < \frac{r}{\|w\|}$  (with  $r$  as radius of the sphere) it holds that

$$\|\alpha w\| = |\alpha| \|w\| < \frac{r}{\|w\|} \cdot \|w\| = r$$

Let  $v = \alpha w$ . Then it holds that

$$\begin{aligned} \|(A - B)v\| &= |\alpha| \|(A - B)w\| \leq \hat{r}(x) \|\alpha w\| = |\alpha| \hat{r}(x) \cdot \|w\| \\ \implies \|(A - B)w\| &\leq \underbrace{\hat{r}(x)}_{\rightarrow 0 \text{ for } x \rightarrow x_0} \cdot \underbrace{\|w\|}_{\text{constant}} \\ \implies (A - B)w &= 0 \end{aligned}$$

This contradicts with our assumption.

Therefore,  $Aw = Bw \forall w \in \mathbb{R}^m$ , hence  $A = B$ . □

## 6.5 Frechét derivative

**Definition 6.5** (Part 2 of Definition 6.4). *If  $f$  is differentiable in  $x_0$ , then we call the uniquely determined linear map  $A$  the “Frechét derivative” of  $f$  in  $x_0$  and denote  $A = Df(x_0)$ . An alternative notations are  $f'(x_0)$  and  $D_{x_0}f$ .*

**Lemma 6.4.** *Let  $f : D \rightarrow \mathbb{R}^n, D \subseteq \mathbb{R}^m$  open,  $x_0 \in D$ . If  $f$  is differentiable in  $x_0$ , then  $f$  is also continuous in  $x_0$ .*

*Proof.* Let  $x \in D$ . Then it holds that

$$\begin{aligned}
\|f(x) - f(x_0)\|_{\mathbb{R}^n} &= \|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0) + Df(x_0)(x - x_0)\| \\
&\leq \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| + \|Df(x_0)(x - x_0)\| \\
&\leq \underbrace{r(x)}_{\rightarrow 0 \text{ for } x \rightarrow x_0} \cdot \underbrace{\|x - x_0\|}_{\rightarrow 0 \text{ for } x \rightarrow x_0} + \overbrace{\left\| \underbrace{Df(x_0)}_{\in \mathbb{R}^{n \times m}} \right\|}_{\text{constant}} \cdot \underbrace{\|x - x_0\|}_{\rightarrow 0 \text{ for } x \rightarrow x_0} \\
&\rightarrow 0 \text{ for } x \rightarrow x_0
\end{aligned}$$

Hence  $f$  is continuous in  $x_0$ .  $\square$

**Lemma 6.5.** Let  $f, g : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $f$  and  $g$  be differentiable in  $x_0 \in D$ . Let  $\lambda \in \mathbb{R}$ . Then it holds that

1.  $f + g$  is differentiable in  $x_0$  with

$$D(f + g)(x_0) = Df(x_0) + Dg(x_0)$$

2.  $\lambda f$  is differentiable in  $x_0$  and

$$D(\lambda f)(x_0) = \lambda Df(x_0)$$

Thus differentiability is a linear operation on the vector space's appropriate differentiable functions.

*Proof.* Let  $F := \left\| (f + g)(x) - (f + g)(x_0) - \underbrace{[Df(x_0) + Dg(x_0)](x - x_0)}_{D(f+g)(x_0)} \right\|$ . Show that  $F = o(\|x - x_0\|)$ .

$$\begin{aligned}
F &\leq \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| + \|g(x) - g(x_0) - Dg(x_0)(x - x_0)\| \\
&= o(\|x - x_0\|) + o(\|x - x_0\|) \\
&= o(\|x - x_0\|)
\end{aligned}$$

For  $\lambda f$  it holds analogously.  $\square$

**Lemma 6.6.** Let  $C : D \rightarrow \mathbb{R}^n$ .  $c(x) = k \in \mathbb{R}^n$  is constant. Then  $c$  be differentiable in every point  $x_0 \in D$  and it holds that  $DC(x_0) = 0 \in \mathbb{R}^{n \times m}$ . Let  $A : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be linear. Then  $A$  is differentiable in every point  $x_0 \in \mathbb{R}^m$  and it holds that  $DA(x_0) = A$ .

Let  $f(x) = k + Ax : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be linear affine, then  $f$  is differentiable in every point  $x_0 \in \mathbb{R}^m$  with  $Df(x_0) = A$ .

*Proof.*

$$\|c(x) - c(x_0) - 0 \cdot (x - x_0)\|$$

where 0 denotes the zero-matrix.

$$= \|k - k\| = \|0\| = 0 = o(\|x - x_0\|)$$

Hence 0 satisfies the differentiability condition for  $c$ .

$$\implies 0 = Dc(x_0)$$

in the linear case

$$\|Ax - Ax_0 - A(x - x_0)\| = \|Ax - Ax_0 - Ax + Ax_0\| = 0 = o(\|x - x_0\|)$$

hence  $DA(x_0) = A$ . Affine: use Lemma 6.5.

$$D(k + A)(x_0) = \underbrace{Dk(x_0)}_{=0} + \underbrace{DA(x_0)}_{=A} = A$$

This is analogous to the one-dimensional case:

$$(k + ax)' = a$$

□

## 6.6 Chain rule

**Theorem 6.1** (Chain rule in multiple dimensions). *Let  $D \subseteq \mathbb{R}^l$  be open. Let  $E \subseteq \mathbb{R}^m$  be open. Let  $f : D \rightarrow \mathbb{R}^m$  such that  $f(D) \subseteq E$  and  $g : E \rightarrow \mathbb{R}^n$ . Compare with Figure 35.*

*Let  $f$  in  $x_0$  be differentiable and  $g$  in  $y_0 = f(x_0)$  is differentiable. Then also  $g \circ f : D \rightarrow \mathbb{R}^n$  is differentiable in  $x_0$  and it holds that*

$$\underbrace{D(g \circ f)(x_0)}_{\in \mathbb{R}^{n \times l}} = \underbrace{Dg(f(x_0))}_{\in \mathbb{R}^{n \times m}} \underbrace{Df(x_0)}_{\in \mathbb{R}^{m \times l}}$$

(The dimensions match.)

*Proof.* Let  $\varepsilon > 0$  be arbitrary, but  $\varepsilon < 2$ . Show that

$$\frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(y_0) \cdot Df(x_0)(x - x_0)\| < \varepsilon$$

for sufficiently small  $\|x - x_0\|$  ( $x \neq x_0$ ).

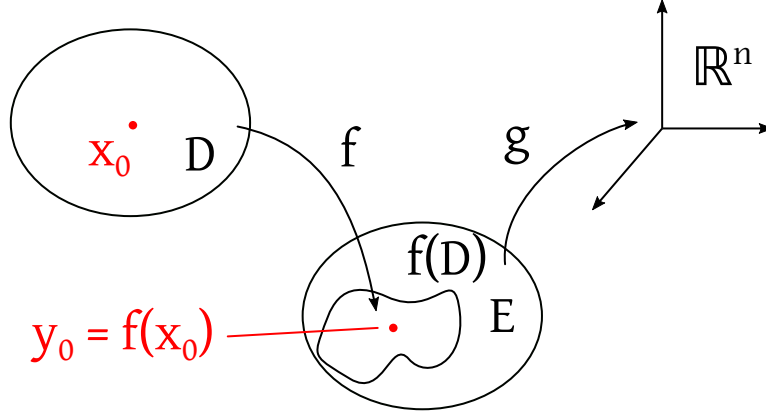


Figure 35: Chain rule in multiple dimensions

$$\begin{aligned}
& \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(y_0) \cdot Df(x_0)(x - x_0)\| \\
&= \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot (f(x) - f(x_0)) \\
&\quad + Dg(f(x_0))(f(x) - f(x_0)) - Dg(f(x_0))Df(x_0)(x - x_0)\|
\end{aligned}$$

recognize that we have a common factor  $Dg(f(x_0))$

$$\begin{aligned}
&\leq \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0))\| \\
&\quad + \frac{1}{\|x - x_0\|} \|Dg(y_0)\| \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \\
&=: (I) + (II)
\end{aligned}$$

Choose  $\delta_1 > 0$  such that  $\|x - x_0\| < \delta_1 \implies$

$$\frac{1}{\|x - x_0\|} \cdot \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| < \frac{\varepsilon}{2} \frac{1}{\|Dg(y_0)\| + 1}$$

is feasible, because  $f$  is differentiable in  $x_0$ .

$$\frac{\varepsilon}{2} \cdot \frac{1}{\|Dg(y_0)\| + 1} < \frac{2}{2} = 1$$

so it also holds that

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| < 1 \cdot \|x - x_0\|$$

By the reverse triangle inequality,

$$\begin{aligned} \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| &\geq \|f(x) - f(x_0)\| - \|Df(x_0)(x - x_0)\| \\ &\geq \|f(x) - f(x_0)\| - \|Df(x_0)\| \cdot \|x - x_0\| \\ &\implies \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} \leq \|Df(x_0)\| + 1 \end{aligned}$$

*This lecture took place on 2018/05/24.*

$$\|f(x) - f(x_0)\| - \|Df(x_0)\| \|x - x_0\| < 1 \|x - x_0\|$$

hence, for  $x \neq x_0$

$$\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < \|Df(x_0)\| + 1$$

$g$  is differentiable in  $y_0 = f(x_0)$ . Hence, we can choose  $\delta_g > 0$  such that  $\forall y \in E$  with  $\|y - y_0\| < \delta_g$  it holds that

$$\|g(y) - g(y_0) - Dg(y_0) \cdot (y - y_0)\| < \frac{\varepsilon}{2(\|Df(x_0)\| + 1)} \|y - y_0\|$$

Because  $f$  is continuous in  $x_0$ , there exists  $\delta_2 > 0$  such that  $x \in D$  and  $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_g$ . Now let  $\delta = \min(\delta_1, \delta_2) > 0$ . Then it holds that

$$I = \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot (f(x) - f(x_0))\|$$

Let  $y = f(x)$ ,  $y_0 = f(x_0)$ . Because  $\|f(x) - f(x_0)\| < \delta_g$  gives  $\|x - x_0\| < \delta_2$

$$\implies I < \frac{\varepsilon}{2(\|Df(x_0)\| + 1)} \underbrace{\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|}}_{< \|Df(x_0)\| + 1} < \frac{\varepsilon}{2}$$

$$\begin{aligned} II &= \|Dg(y_0)\| \frac{1}{\|x - x_0\|} \|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \\ &< \|Dg(y_0)\| \cdot \frac{\varepsilon}{2} \cdot \frac{1}{\|Dg(y_0)\| + 1} < \frac{\varepsilon}{2} \end{aligned}$$

hence,  $I + II < \varepsilon$  for  $\|x - x_0\| < \delta$ . □

## 6.7 Differentiability on $D$

**Definition 6.6.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  be differentiable in every point  $x \in D$ . Then we are used to say “ $f$  is differentiable on  $D$ ”.

In this case, we call the map

$$x \mapsto Df(x)$$

$$D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m}$$

the mapping function of  $f$  (dt. Abbildungsfunktion). If this function is continuous (in terms of  $\|\cdot\|_{\mathbb{R}}$  or  $\|\cdot\|_{\mathbb{R}^{n \times m}}$  . . . operator norm), then  $f$  is called continuously differentiable on  $D$ .

**Remark 6.5.** To define the differentiability of  $f$ , we require  $x_0$  to be an accumulation point of  $D$ . So  $x_0$  might also be a point on the boundary of  $D$ .

### 6.7.1 Determination of $Df(x_0) \in \mathbb{R}^{n \times m}$

**Definition 6.7.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  be given.  $D$  is open,  $x_0 \in D$ ,  $v \in \mathbb{R}^m$  is arbitrary, but  $v \neq 0$ . We consider  $t \mapsto f(x_0 + tv)$  defined on  $(-\frac{r}{\|v\|}, \frac{r}{\|v\|})$  for  $r > 0$  such that  $K_r(x_0) \subseteq D$ .

$$\left(-\frac{r}{\|v\|}, \frac{r}{\|v\|}\right) \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$$

Compare with Figure 36.

We define  $df(x_0; v) = \lim_{t \rightarrow 0} \frac{1}{t}(f(x_0 + tv) - f(x_0))$  if this limit exists.  $df(x_0, v)$  is called directional derivative of  $f$  in  $x_0$  in direction  $v$ . It is also called Gateaux derivative of  $f$  in  $x_0$  in direction  $v$ .

**Remark 6.6.** How does it go together?

Derivative  $Df(x_0)$  and  $df(x_0; \cdot)$ . Assumption: Let  $f$  be differentiable in  $x_0$ . We define  $l_{x_0, v}(t) = x_0 + tv$  where  $tv$  is the linear part.

$$l_{x_0, v} : \left(-\frac{r}{\|v\|}, \frac{r}{\|v\|}\right) \rightarrow D$$

$l_{x_0, v}$  is linear affine from  $\mathbb{R}$  to  $\mathbb{R}^m$ .

$$Dl_{x_0, v}(0) = V \in \mathbb{R}^{m \times 1}$$

with  $V$  as linear part of  $l_{x_0, v}$ .

$$l_{x_0, v}(0) = x_0$$

$$f(x_0 + tv) = f \circ l_{x_0, v}(t)$$

Therefore it holds that (chain rule)

$$D(f \circ l_{x_0, v})(0) = Df(l_{x_0, v}(0)) \cdot Dl_{x_0, v}(0) = Df(x_0) \cdot v$$

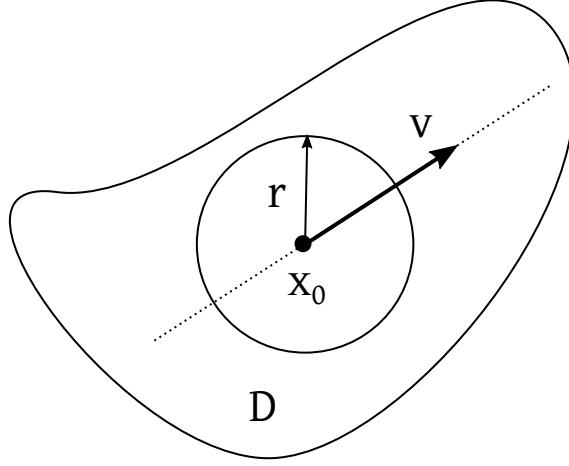


Figure 36: Setting in Definition 6.7. The line is given by  $g = \{x_0 + tv : t \in \mathbb{R}\}$

On the other side, it holds that

$$\begin{aligned}
 0 &= \lim_{t \rightarrow 0} \underbrace{\frac{1}{|t|} |f_{x_0,v}(t) - f_{x_0,v}(0) - Df_{x_0,v}(0) \cdot t|}_{=0} \\
 &= \lim_{t \rightarrow 0} \frac{1}{|t|} |f(x_0 + tv) - f(x_0) - Df_{x_0,v}(0) \cdot t| \\
 &= \lim_{t \rightarrow 0} \left| \frac{f(x_0 + tv) - f(x_0)}{t} - Df_{x_0,v}(0) \right| = 0
 \end{aligned}$$

therefore it holds that

$$Df_{x_0,v}(0) = \lim_{t \rightarrow 0} \frac{f(x_0 + tv) - f(x_0)}{t} = df(x_0; v)$$

**Lemma 6.7.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  in  $x_0 \in D$  (Frechét) differentiable with derivative  $Df(x_0)$ . Then also the directional derivative  $df(x_0; v)$  for every direction  $v \in \mathbb{R}^m \setminus \{0\}$  and it holds that

$$df(x_0; v) = Df(x_0) \cdot v$$

**Remark 6.7.**  $v \mapsto df(x_0; v)$  is linear. We can derive the structure of the derivative matrix. Let  $f$  as above. Let  $\mathcal{B} = \{e_1, \dots, e_m\}$  be the canonical basis in  $\mathbb{R}^m$ . Then it holds

that:  $Df(x_0) \cdot e_j$  is the  $j$ -th column of  $Df(x_0)$  for  $j = 1, \dots, m$ . On the other hand,

$$\begin{aligned} Df(x_0) \cdot e_j &= df(x_0; e_j) = \lim_{t \rightarrow 0} \frac{1}{t} [f(x_0 + te_j) - f(x_0)] \\ &= \begin{bmatrix} \lim_{t \rightarrow 0} \frac{1}{t} (f_1(x_0 + te_j) - f_1(x_0)) \\ \vdots \\ \lim_{t \rightarrow 0} \frac{1}{t} (f_n(x_0 + te_j) - f_n(x_0)) \end{bmatrix} \text{ for } f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \in \mathbb{R}^n \end{aligned}$$

**Remark 6.8** (Notation). Consider  $x$  instead of  $x_0$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = [x_1, \dots, x_m]^t$$

Instead of  $f(x) = f\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right)$  we also write  $f(x_1, x_2, \dots, x_m)$ .

## 6.8 Partial derivative and Jacobi matrix

**Definition 6.8.** Let  $f : D \rightarrow \mathbb{R}^n$ ,  $x \in D$  as above.  $f$  is differentiable in  $x$ . Then we let

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x) &= df(x; e_j) = \lim_{t \rightarrow 0} \frac{1}{t} [f(x + te_j) - f(x)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} [f(x_1, \dots, x_j + t, \dots, x_m) - f(x_1, \dots, x_j, \dots, x_m)] \end{aligned}$$

and we call  $\frac{\partial f}{\partial x_j}(x)$  the partial derivative of  $f$  of variable  $x_j$  in point  $x$ .

Notations for  $\frac{\partial f}{\partial x_j}$ :

$$f_{x_j} \quad f_j \quad \partial_j f$$

The second notation is ambiguous. We will prefer the last one.

**Remark 6.9.**

$$\frac{\partial f}{\partial x_j}(x_0) = \begin{bmatrix} df_1(x; e_j) \\ df_2(x; e_j) \\ \vdots \\ df_n(x; e_j) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_j}(x) \\ \vdots \\ \frac{\partial f_n}{\partial x_j}(x) \end{bmatrix} \in \mathbb{R}^n$$

Because  $\frac{\partial f}{\partial x_j}(x)$  is the  $j$ -th column of  $Df(x_0)$ , we get

$$(Df(x))_{i,j} = \frac{\partial f_i}{\partial x_j}(x) = \partial_j f_i(x)$$



We say,  $Df(x) = (\partial_j f_i(x))_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$  is the Jacobi matrix of  $f$ .

**Remark 6.10.** The Jacobi matrix can exist even though the derivative does not exist. If the derivative exists, the Jacobi matrix exists for sure.

**Remark 6.11.**

$$\frac{\partial f}{\partial x_j} = \lim_{t \rightarrow 0} \frac{1}{t} \left[ f(x_1, \dots, x_j + t, \dots, x_m) - f(x_1, \dots, x_j, \dots, x_m) \right]$$

Thus, consider  $x_j$  as derivation variable and all  $x_k$  for  $k \neq j$  as constant parameters.

**Example 6.2.** Consider  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  with

$$\begin{aligned} f(x_1, x_2, x_3) &= \begin{bmatrix} x_1 x_3^2 + \sin(x_1 x_3) \\ \frac{x_2^2}{x_1^2 + 1} \end{bmatrix} \\ \partial_1 f(x_1, x_2, x_3) &= \begin{bmatrix} 1 \cdot x_3^2 \\ -x_2^2 (x_1^2 + 1)^{-2} \cdot 2x_1 \end{bmatrix} = \begin{bmatrix} x_3^2 \\ -2 \frac{x_1 x_2^2}{(x_1^2 + 1)^2} \end{bmatrix} \\ \partial_2 f(x_1, x_2, x_3) &= \begin{bmatrix} x_3 \cos(x_2 x_3) \\ \frac{2x_2}{x_1^2 + 1} \end{bmatrix} \\ \partial_3 f(x_1, x_2, x_3) &= \begin{bmatrix} 2x_1 x_3 + x_2 \cos(x_2 x_3) \\ 0 \end{bmatrix} \end{aligned}$$

Jacobi-Matrix

$$Df(x) = \left[ \frac{\partial f_i}{\partial x_j} \right]_{\substack{i=1,\dots,2 \\ j=1,\dots,3}} = \begin{bmatrix} x_3^2 & x_3 \cos(x_2 x_3) & 2x_1 x_3 + x_2 \cos(x_2 x_3) \\ -\frac{2x_1 x_2^2}{(x_1^2 + 1)^2} & \frac{2x_2}{x_1^2 + 1} & 0 \end{bmatrix}$$

**Remark 6.12.** Existence of partial derivatives of  $f$  does not suffice to ensure Frechét-differentiability.

This lecture took place on 2018/05/29.

Usually, we always have to point out which norm is used to define differentiability. Of course, in  $\mathbb{R}$  itself, all norms are equivalent.

**Remark 6.13.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$  be given. Let  $\|\cdot\|_{1,m}$  and  $\|\cdot\|_{2,m}$  be two equivalent norms on  $\mathbb{R}^m$  (norm equivalence theorem) and  $\|\cdot\|_{1,n}$  and  $\|\cdot\|_{2,n}$  are equivalent norms on  $\mathbb{R}^n$ .

Let  $f$  in  $x_0 \in D$  be differentiable in regards of  $\|\cdot\|_{1,m}$  and  $\|\cdot\|_{1,n}$ . Then also  $f$  is differentiable in  $x_0$  in regards of  $\|\cdot\|_{2,m}$  and  $\|\cdot\|_{2,n}$ .

Rationale: Let  $c \|x\|_{2,m} \leq \|x\|_{1,m} \leq C \|x\|_{2,m}$  and  $k \|y\|_{2,n} \leq \|y\|_{1,n} \leq K \|y\|_{2,n}$ , then

$$\begin{aligned} & \frac{\|f(x) - f(x_0) - A(x - x_0)\|_{2,n}}{\|x - x_0\|_{2,m}} \\ & \leq \frac{\frac{1}{k} \|f(x) - f(x_0) - A(x - x_0)\|_{1,n}}{\frac{1}{c} \|x - x_0\|_{1,m}} \\ & = \frac{c}{k} \underbrace{\frac{\|f(x) - f(x_0) - A(x - x_0)\|_{1,n}}{\|x - x_0\|_{1,m}}}_{\xrightarrow{x \rightarrow x_0} 0} \end{aligned}$$

Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$ . Then  $f$  is differentiable in  $x_0 \in D \iff f_k : D \rightarrow \mathbb{R}$  is differentiable for all  $k \in \{1, \dots, n\}$ .

Rationale: Let  $f$  be differentiable in  $x_0$ . Let  $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$  be the derivative of  $f$ . Let  $a_k$  be the rows of  $A$ .  $f$  is differentiable in  $x_0$ , so

$$\begin{aligned} & \frac{\|f(x) - f(x_0) - A(x - x_0)\|_{\infty}}{\|x - x_0\|} \xrightarrow{x \rightarrow x_0} 0 \\ \iff & \frac{|f_n(x) - f_k(x) - a_k(x - x_0)|}{\|x - x_0\|} \xrightarrow{x \rightarrow x_0} 0 \end{aligned}$$

where  $a_k$  is a row vector and  $x - x_0$  is a column vector and  $k \in \{1, \dots, n\}$ .

$$\iff f_k \text{ is differentiable in } x_0$$

**Example 6.3** (Counterexample). We define  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2 x_2}{x_1^2 + x_2^2} & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

Partial derivatives exist in every point  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .

$$\partial_1 f(x_1, x_2) = \frac{2x_1 x_2 (x_1^2 + x_2^2) - x_1^2 x_2 2x_1}{(x_1^2 + x_2^2)^2} = \frac{2x_1 x_2^3}{(x_1^2 + x_2^2)^2}$$

$$\partial_2 f(x_1, x_2) = \frac{x_1^2(x_1^2 + x_2^2) - x_1^2 x_2 \cdot 2x_2}{(x_1^2 + x_2^2)^2} = \frac{x_1^2(x_1^2 x_2^2)}{(x_1^2 + x_2^2)^2}$$

$$\partial_1 f(0, 0) = \lim_{x_1 \rightarrow 0} \frac{1}{x_1} \underbrace{[f(x_1, 0)]}_{=0} - \underbrace{f(0, 0)}_{=0} = 0$$

$$\partial_2 f(0, 0) = \lim_{x_2 \rightarrow 0} \frac{1}{x_2} \underbrace{[f(0, x_2)]}_{=0} - \underbrace{f(0, 0)}_{=0} = 0$$

If  $f$  would be differentiable in  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$  then  $Df(0) = [00]$ ,  $df(0, v) = Df(0) \cdot v$ . Choose  $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies df(0; v) = [00] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$ . But!

$$\begin{aligned} df(0; v) &= \lim_{t \rightarrow 0} \frac{1}{t} [f(tv) - f(0)] = \lim_{t \rightarrow 0} \frac{1}{t} [f(t, t) - f(0, 0)] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \left[ \frac{t^2 \cdot t}{t^2 + t^2} - 0 \right] = \lim_{t \rightarrow 0} \frac{t^3}{2t^3} = \frac{1}{2} \end{aligned}$$

## 6.9 Total differentiability, nabla operator and gradients

**Remark 6.14.** Notation:  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ . Often, we denote  $df(x_0)$  ( $\in \mathbb{R}^{1 \times m}$ , row vector) instead of  $Df(x_0)$  and we call  $df(x_0)$  the total differential of  $f$ . We let

$$\nabla f(x_0) = df(x_0)^t = [Df(x_0)]^t = \begin{bmatrix} \partial_1 f(x_0) \\ \partial_2 f(x_0) \\ \vdots \\ \partial_n f(x_0) \end{bmatrix}$$

$\nabla f(x_0)$  is called gradient of  $f$  in  $x_0$ . We call  $\nabla$  the nabla operator. It is also denoted  $\text{grad}(f)$  instead of  $\nabla f$ . It holds that

$$df(x_0; v) = df(x_0) \cdot v = \nabla f(x_0)^t \cdot v = \langle \nabla f(x_0), v \rangle_{\mathbb{R}^m}$$

**Remark 6.15.** Let  $f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$  continuously differentiable be given on  $D$ . Consider  $\Gamma_s = \{x \in D \mid f(x) = s\}$ . Compare with Figure 37.

Assume  $\Gamma_s$  is a graph of a family of curves (dt. Parametrisierte Kurve)  $\gamma_s : I \rightarrow D$ . Let  $I$  be an interval.  $\Gamma_s = \{\gamma_s(t) \mid t \in I\}$ . We assume, that  $\gamma_s$  is regular, hence  $\gamma$  is differentiable and  $\gamma'_s(t) = \begin{bmatrix} \gamma'_1(t) \\ \gamma'_2(t) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \forall t \in I$ .  $\gamma'_s(t)$  is tangential vector on  $\Gamma_s$  in point  $x = \gamma_s(t)$ .

It holds that

$$\langle \nabla f(\gamma_s(t)), \gamma'_s(t) \rangle = \langle \nabla f(x), v \rangle = 0$$

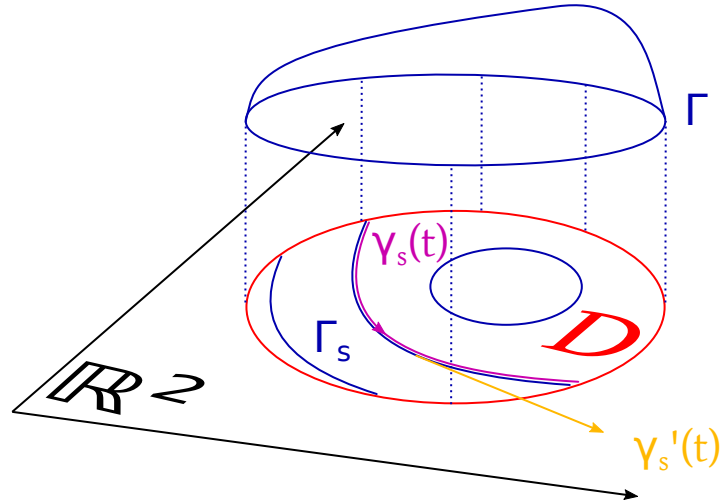


Figure 37:  $\Gamma_s$  is the niveau level of  $f$

$$\underbrace{f(\gamma_s(t))}_{\in \Gamma_s} = s \quad \dots \text{constant}$$

$$\Rightarrow \frac{d}{dt}[f(\gamma_s(t))] = 0$$

where

$$\begin{aligned} \frac{d}{dt}f(\gamma_s(t)) &= \underbrace{Df(x)}_{\nabla f(x)^t} \cdot \underbrace{D\gamma_s(x)}_{\gamma'_s(t)} = \nabla f(x)^t \cdot \gamma'_s(t) \\ &= \langle \nabla f(x), \gamma'_s(t) \rangle \end{aligned}$$

Compare with Figure 38.

**Theorem 6.2.** This theorem establishes the relation of differentiability and partial derivatives.

Let  $f : D \rightarrow \mathbb{R}$ ,  $D \subseteq \mathbb{R}^m$  is open. Assume  $\forall x \in D$  exist all partial derivatives  $\partial_i f(x)$  for  $j = 1, \dots, m$  and the functions  $x \mapsto \partial_j f(x)$  on  $D \rightarrow \mathbb{R}$  are continuous for  $j = 1, \dots, m$ . Then  $f$  is differentiable in every point  $x_0 \in D$  with  $Df(x_0) = [\partial_1 f(x_0), \dots, \partial_m f(x_0)]$ . Then  $f$  is also continuously differentiable.

*Proof.* Proof idea: We approximate  $x_0$  (starting from  $x_0 \in D$ ) along lines parallel to the coordinate system axes. Compare with Figure 39.

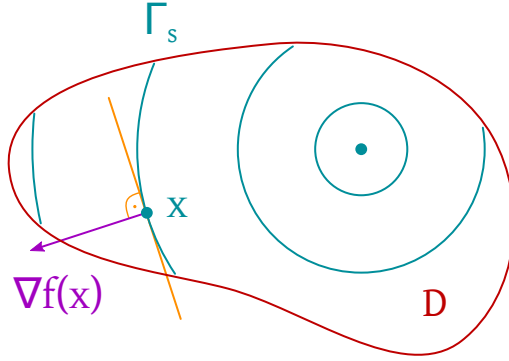


Figure 38:  $Df(x) = \nabla f(x)^T$

$$\begin{aligned}
 x_0 &:= \begin{bmatrix} x_1^0 \\ \vdots \\ x_m^0 \end{bmatrix} & x &:= \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} & \xi_0 &:= x_0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_m^0 \end{bmatrix} & \xi_1 &:= \begin{bmatrix} x_1 \\ x_2^0 \\ \vdots \\ x_m^0 \end{bmatrix} & \xi_2 &:= \begin{bmatrix} x_1 \\ x_2 \\ x_3^0 \\ \vdots \\ x_m^0 \end{bmatrix} \\
 & & & & \xi_k &:= \begin{bmatrix} x_1 \\ \vdots \\ x_{k_0} \\ x_{k+1} \\ \vdots \\ x_m^0 \end{bmatrix} & \dots & \xi_m &:= \begin{bmatrix} x_0 \\ \vdots \\ x_m \end{bmatrix} = x
 \end{aligned}$$

where  $\xi_i$  are “intermediate” points. It holds that  $\xi_k + (x_{k+1} - x_{k+1}^0) \cdot e_{k+1} = \xi_{k+1}$  for  $k = 0, \dots, m-1$ . Define  $\varphi_k : [0, 1] \rightarrow \mathbb{R}$ .

$$\varphi_k(t) = f(\xi_k + t(x_{k+1} - x_{k+1}^0) \cdot e_{k+1})$$

then it holds that  $\varphi_k(0) = f(\xi_k)$ ;  $\varphi_k(1) = f(\xi_{k+1})$ .  $\varphi'_k(t) = ?$

$$\underbrace{f(\xi_k + t(x_{k+1} - x_{k+1}^0) \cdot e_{k+1})}_{=\varphi_k(t)} = f(x_1, \dots, x_k, \underbrace{x_{k+1}^0 + t(x_{k+1} - x_{k+1}^0)}_{(k+1)\text{-th variable}}, x_{k+2}^0, \dots, x_m^0)$$

$$\varphi'_k(t) = \frac{d}{dt} \left[ f(x_1, \dots, x_k, \underbrace{x_{k+1}^0 + t(x_{k+1} - x_{k+1}^0)}_{(k+1)\text{-th variable}}, x_{k+2}^0, \dots, x_m^0) \right]$$

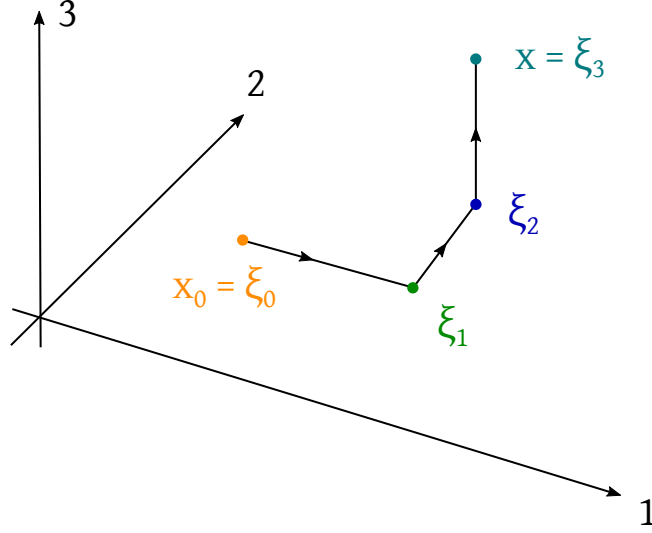


Figure 39:  $x_0$  approximates  $x$  along lines parallel to the coordinate system axes

$$\begin{aligned}
 &= \partial_{k+1} f(x_1, \dots, x_k, x_{k+1}^0 + t(x_{k+1} - x_{k+1}^0), x_{k+2}^0, \dots, x_m^0) \cdot (x_{k+1} - x_{k+1}^0) \\
 &= \partial_{k+1} f(\xi_k + t(x_{k+1} - x_{k+1}^0) \cdot e_{k+1}) \cdot (x_{k+1} - x_{k+1}^0)
 \end{aligned}$$

$\varphi_k$  is continuously differentiable on  $[0, 1]$  because  $\partial_{k+1} f$  is continuous. By the mean value theorem of differential calculus, it follows that some  $\tau_{k+1} \in (0, 1)$  exists such that

$$\begin{aligned}
 \varphi(1) - \varphi(0) &= \varphi'(\tau_{k+1}) \cdot (1 - 0) \\
 \implies f(\xi_{k+1}) - f(\xi_k) &= \partial_{k+1} f(\xi_k + \tau_{k+1}(x_{k+1} - x_{k+1}^0) \cdot e_{k+1}) \cdot (x_{k+1} - x_{k+1}^0)
 \end{aligned}$$

For differentiability, we have to show:

$$\lim_{x \rightarrow x_0} \frac{1}{\|x - x_0\|} \left| f(x) - f(x_0) - [\partial_1 f(x_0), \dots, \partial_m f(x_0)] \begin{bmatrix} x_1 - x_1^0 \\ x_2 - x_2^0 \\ \vdots \\ x_m - x_m^0 \end{bmatrix} \right| = 0 \quad (*)$$

Choose  $\|x\| = \|x\|_\infty$  on  $\mathbb{R}^m$ .

$$\frac{1}{\|x - x_0\|} \left| \underbrace{f(x)}_{\xi_m} - \underbrace{f(x_0)}_{\xi_0} - \sum_{k=1}^m \partial_k f(x_0)(x_k - x_k^0) \right|$$

$$\begin{aligned}
&= \frac{1}{\|x - x_0\|} \left| \underbrace{\sum_{k=1}^m (f(\xi_k) - f(\xi_{k-1}) - \partial_k f(x_0)(x_k - x_k^0))}_{\text{telescoping sum}} \right| \\
&\stackrel{\text{triangle ineq.}}{\leq} \frac{1}{\|x - x_0\|} \sum_{k=1}^m |\partial_k f(\xi_{k-1} + \tau_k(x_k - x_k^0) \cdot e_k)(x_k - x_k^0) - \partial_k f(x_0)(x_k - x_k^0)| \\
&= \sum_{k=1}^m \underbrace{\frac{|x_k - x_k^0|}{\|x - x_0\|}}_{\leq 1} \cdot |\partial_k f(\xi_{k-1} + \tau_k(x_k - x_k^0) \cdot e_k) - \partial_k f(x_0)| \quad (\#)
\end{aligned}$$

Choose  $\varepsilon > 0$  arbitrary. By continuity of  $\partial_k f(x)$ , there exists  $\delta > 0$  such that  $\|y - y_0\|_\infty < \delta \implies |\partial_k f(y) - \partial_k f(x_0)| < \frac{\varepsilon}{m}$  for every  $k \in \{1, \dots, m\}$ . Choose  $x$  such that  $\|x - x_0\| \leq \delta$ . We consider

$$\begin{aligned}
\xi_{k-1} + \tau_k(x_k - x_k^0) \cdot e_k - x_0 &= \begin{bmatrix} x_1 \\ \vdots \\ x_{k-1} \\ x_k^0 \\ \vdots \\ x_m \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \tau_k(x_k - x_k^0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} x_1^0 \\ \vdots \\ x_{k-1}^0 \\ x_k^0 \\ \vdots \\ x_m^0 \end{bmatrix} = \begin{bmatrix} x_1 - x_1^0 \\ \vdots \\ x_{k-1} - x_{k-1}^0 \\ \tau_k(x_k - x_k^0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&\implies \|\xi_{k-1} + \tau_k(x_k - x_k^0) \cdot e_k - x_0\|_\infty \\
&= \max \left\{ |x_1 - x_1^0|, |x_2 - x_2^0|, \dots, |x_{k-1} - x_{k-1}^0|, \underbrace{\tau_k}_{\in (0,1)} |x_k - x_k^0| \right\} \\
&\leq \|x - x_0\|_\infty < \delta \\
&\implies |\partial_k f(\xi_{k-1} + \tau_k(x - x_0^k)e_k) - \partial_k f(x_0)| < \frac{\varepsilon}{m} \\
(\#) &\leq 1 \cdot \sum_{k=1}^m \frac{\varepsilon}{m} = m \cdot \frac{\varepsilon}{m} = \varepsilon
\end{aligned}$$

Thus,  $f$  is differentiable in  $x_0$ .

Because  $Df(x) = [\partial_1 f(x), \dots, \partial_m f(x)]$  depends continuously on  $x$ ,  $f$  is continuously differentiable on  $D$   $\square$

**Corollary.** Let  $f : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$ .

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

such that all partial derivatives  $\partial_k f_i(x)$  exist and are continuous in  $x$  (for  $k = 1, \dots, m$ ;  $i = 1, \dots, n$ ). Then  $f$  is continuously differentiable on  $D$ .

*Rationale.* Use  $f$  is continuously differentiable on  $D \iff f_i$  is continuously differentiable for  $i = 1, \dots, n$  and use Theorem 6.2.  $\square$

**Example 6.4** (Counterexample).

$$f(x_1, x_2) = \frac{x_1^2 x_2}{x_1^2 + x_2^2} \quad \partial_1 f(x) = \frac{2x_1 x_2^3}{(x_1^2 + x_2^2)^2} \quad \partial_2 f(x) = \frac{x_1^2(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2}$$

$$\partial_1 f(0) = \partial_2 f(0) = 0$$

$$\partial_1 f\left(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}\right) = \frac{2\varepsilon \varepsilon^3}{(\varepsilon^2 + \varepsilon^2)^2} = \frac{2\varepsilon^4}{4\varepsilon^4} = \frac{1}{2} \not\rightarrow 0$$

Therefore,  $\partial_1 f$  is not continuous in 0.



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