Mathematical analysis 1 – Lecture notes

course by Wolfgang Ring

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Oct 2015 to Jan 2016

Contents

First possible exam date: 4th February 2016 14:00.

1 Propositional logic

↓ This lecture took place on 1st of October 2015 with lecturer Wolfgang Ring

- Discussion about motivation for visiting university
- Kurt Gödel: Gödel's incompleteness theorem
- propositional logic (and/or/implication/equivalence operation)
  - $p \implies q$: "p implies q" ("notwendig"), "q requires p" ("hinreichend")
  - Indirect proof: $(\neg q \implies \neg p) \iff (p \implies q)$
  - Proof by contradiction: claim p, claim $\neg q$, show that $p \land \neg q$ is not possible
  - commutative law: $a \land b \iff b \land a$
  - associative law: $a \land (b \land c) = (a \land b) \land c$
  - distributive law: $(a \land b) \lor c = (a \lor c) \land (b \lor c)$
  - DeMorgan's law: $\neg(a \land b) \iff (\neg a) \lor (\neg b)$
- First-order logic
  - $\forall x \in \mathbb{N} : x \in \mathbb{R}$
  - $\forall x \in M : P(x)$
  - $\neg [(\forall x \in M) P(x)] \iff \exists x \in M : \neg P(x)$
- Peano's axioms: rationale for induction proofs

The lecture on 8th of October 2015 got cancelled spontaneously.

2 First-Order Logic

↓ This lecture took place on 12th of October 2015 with lecturer Wolfgang Ring

Literature recommendation:

- "Analysis 1 (Mathematik für das Lehramt)", Oliver Deiser

Let A and B be statements.

- Logical equivalence is given iff the truthtable of both expressions is the same.
- $\neg(\neg A) \iff A$
- $(A \lor B) \iff (B \lor A)$
- $(A \land B) \iff (B \land A)$
- $a \implies b$: implication

Boolean Laws:

$$\neg(A \implies B) \iff A \land \neg B \tag{1}$$
$$A \iff B \implies (A \implies B) \land (B \implies A) \tag{2}$$

"contraposition" or "indirect proof"

$$\neg B \implies \neg A \tag{3}$$
$$A \implies B \iff (\neg B \implies \neg A) \tag{4}$$
$$(A \iff B) \iff (\neg A \iff \neg B) \tag{5}$$
$$\neg(A \land B) \iff \neg A \lor \neg B \tag{6}$$
$$\neg(A \lor B) \iff \neg A \land \neg B \tag{7}$$
$$\neg(A \implies B) \iff (A \land \neg B) \tag{8}$$
$$A \land (B \lor C) \iff ((A \land B) \lor (A \land C)) \tag{9}$$
$$A \lor (B \land C) \iff ((A \lor B) \land (A \lor C)) \tag{10}$$
$$(A \implies B) \iff (\neg A \lor B) \tag{11}$$

"proof by contradiction"

$$((A \implies B) \land (A \implies \neg B)) \implies \neg A \tag{12}$$

"conclusion"

$$((A \implies B) \land (B \implies C)) \implies (A \implies C) \tag{13}$$

$$A \vee B \iff \neg(\neg A) \vee \neg(\neg B) \iff \neg(\neg A \wedge \neg B)$$
$$\neg(A \vee B) \iff \neg(\neg(\neg A) \vee (\neg B))$$

Distributive laws:

- $(A \vee B) \wedge C \iff (A \wedge C) \vee (A \wedge C)$
- $(A \wedge B) \vee C \iff (A \vee C) \wedge (B \vee C)$

2.1 Tautologies

A *tautology* is the composition of statements, which always yields the truth value true, independent of the truth value of its subexpressions.

Examples of tautologies:

"Law of excluded middle" $A \vee \neg A$

equivalences with itself are always tautologies $A \leftrightarrow \neg(\neg A)$

implication of itself $A \to A$

Tautology with multiple statements:

implication with or and not $(A \to B) \leftrightarrow (\neg A \vee B)$

proof by contradiction $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

chain inference $[(A \to B) \wedge (B \to C)] \to (A \to C)$

↓ This lecture took place on 14th of Oct 2015 with lecturer Wolfgang Ring

Proof. We prove, $[(A \to B) \wedge (A \to \neg B)] \to \neg A$.

$$\begin{aligned} (A \to B) \wedge (A \to \neg B) &\iff (\neg A \vee B) \wedge (\neg A \vee \neg B) \\ &\iff \underbrace{(B \wedge \neg B)}_{\perp} \vee \neg A \\ &\iff \neg A \end{aligned}$$

□

special case $A = B$.

$$(A \to A) \wedge (A \to \neg A) \to \neg A$$
$$(A \to \neg A) \to \neg A$$

□

2.2 Negation of a tautology

- is called *contradiction.*
- has always truth value false.

Proof.

$$(A \vee B) \to C \iff \neg(A \vee B) \vee C \iff (\neg A \wedge \neg B) \vee C$$
$$\iff (\neg A \vee C) \wedge (\neg B \vee C) \iff (A \to C) \wedge (B \to C)$$

□

Laws:

$$\begin{aligned} (A \vee B) \to C &\iff (A \to C) \wedge (B \to C) \\ (A \wedge B) \to C &\iff (A \to C) \vee (B \to C) \\ A \to (B \wedge C) &\iff (A \to B) \wedge (A \to C) \\ A \to (B \vee C) &\iff (A \to B) \vee (A \to C) \end{aligned}$$

Example proof by contradiction: Number of prime numbers. We prove a statement by Euklid of Alexandria, 300 BC:

> The number of prime numbers is infinite.

Assume the number of prime numbers is finite. Then there exists some $N \in \mathbb{N}$ such that $\mathbb{P} = \{p_1, p_2, \dots, p_N\}$ is the set of all prime numbers.

Every integer can be represented as product of prime numbers. Therefore for every integer there exists at least one prime number that divides this number (without remainder).

Let $m = p_1 \cdot p_2 \cdot \ldots \cdot p_N + 1$. Let a be a prime number that divides m.

It holds that: Every $p_i \in \mathbb{P}$ is not a divisor of m. Because when dividing $\frac{m}{p_i}$, the remainder is always one.

So $a \in \mathbb{P}$, so there exists more than N prime numbers (at least $N+1$). This contradicts with our assumption, that only N prime numbers exist.

Therefore always one more prime number exists. So the number of prime numbers is infinite. $\square$

2.3 Quantifiers

Quantified statements are statements, in which objects of a set occur.

Example: Let $P(x) = (x > 0)$. Its truth value cannot be determined if the set X is not defined.

Definition 1. Let M be a set, $x \in M$ and $P(x)$ a predicate.

For every $x \in M$, it holds that $P(x)$ is true, iff the truth value of $P(x)$ is always true independent of the selection of $x \in M$.

Example 1. Let $M = \mathbb{R}$ and $P(x) = (x^2 + 1 > 0)$.

This is true for all $x \in M$. We denote: $\forall x \in M : P(x)$.

Example 2. Let $M = \mathbb{R}$ and $P(x) = (x^2 - 1 > 0)$.

This is *not* true for all $x \in M$. We denote: $\exists x \in M : \neg P(x)$.

Definition 2. $\forall x \in M : P(x)$ does not hold if and only if $\exists x \in M : \neg P(x)$.

$\forall$ is called *all quantifier*. $\exists$ is called *existence quantifier*.

Negation works as follows:

$$\neg\left(\forall x \in M : P(x)\right) \iff \left(\exists x \in M : \neg P(x)\right)$$

$$\neg\left(\exists x \in M : P(x)\right) \iff \left(\forall x \in M : \neg P(x)\right)$$

↓ This lecture took place on 15th of Oct 2015 with lecturer Wolfgang Ring

$$\forall x \in M : (P(x) \land Q(x)) \iff (\forall x \in X : P(x)) \land (\forall y \in M : Q(y)))$$

Counterexample:

$$M = \mathbb{R} \quad P(x) := (x > 0)$$

A statement B is stronger than C if B implies at least the same propositions that C imply. "B is stronger than C" means "$\{D \mid C \to D\} \subsetneq \{D \mid B \to D\}$". In that sense the stronger statement covers more cases.

2.4 Composition of several quantifiers

Theorem 1. The order of quantifiers matters.

Proof. For every real number x, there exists $n \in \mathbb{N}$ with the property $n > x$:

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$$

The statement does not hold if the order is changed.

$$\exists n \in \mathbb{N} \forall x \in \mathbb{R} : n > x$$

$\square$

3 Sets

We consider objects, which we call *sets*. For every set M and every element x, it holds that

$$x \in M \lor \neg(x \in M)$$

3.1 Russell's paradox

Consider the set $L = \{M : M \text{ is a set and } M \notin M\}$. Does $L \notin L$ or $L \in L$ hold?

If $L \notin L$, then L satisfies the definition and therefore $L \in L$. If $L \in L$, then elements of L satisfy the property; therefore $L \notin L$.

Set operations:

- union: $x \in (L \cup M) \iff x \in L \vee x \in M$
- intersection: $x \in (L \cap M) \iff x \in L \wedge x \in M$
- subsets: $L \subseteq M \iff \forall x : x \in L \implies x \in M$
- $\forall S : \emptyset \subseteq S$

3.2 Complete induction

Theorem 2. (Pythagoreans, 450 BC)

$$\forall n \in \mathbb{N}_+ : \sum_{k=1}^{n} k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Proof. **Induction base $\mathbf{n = 1}$**

$$P(1) : 1 = \frac{1(1+1)}{2} \quad \checkmark$$

Induction step $\mathbf{n \to n+1}$

Assume $P(n)$ is true. So $(1 + 2 + \dots + n) = \frac{n(n+1)}{2}$.

$$[(1 + 2 + \dots + n) + (n+1)] = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1) \cdot \frac{(n+2)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark$$

So, it simply holds that:

$$s = 1 + 2 + 3 + \dots + n$$

$$2 \cdot s = \underbrace{n}_{\text{number of items}} \cdot \underbrace{(n+1)}_{\text{sum}} \implies s = \frac{n \cdot (n+1)}{2}$$

□

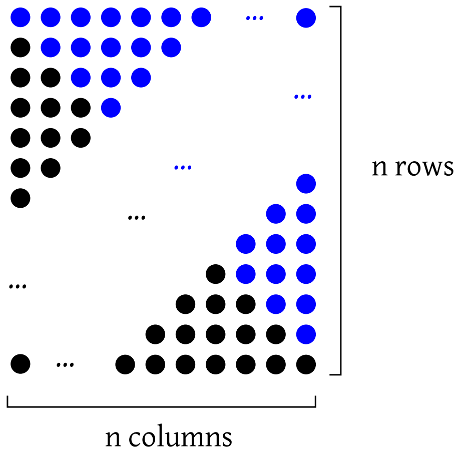

Figure 1: Illustration of the triangular number $\frac{n(n+1)}{2}$ (illustrative proof)

↓ This lecture took place on 21st of October 2015 with lecturer Wolfgang Ring

3.3 Notations to describe sets

- Let X be a set. $M = \{x \in X : P(x)\}$.
- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ ..."enumerating set representation"
- $M = \{x \in X \mid P(x)\}$, $N = \{x \in X \mid Q(x)\}$
- $M \cup N = \{x \in X \mid P(x) \vee Q(x)\}$
- Sets as union or intersection of sets:
  - Let X be a set. $A_0 \subseteq X$, $A_1 \subseteq X$, $A_2 \subseteq X$, etc
  - $\forall n \in \mathbb{N} : A_n \subseteq X$

- $A_0 \cup A_1 \cup A_2 \cup \dots = \bigcup_{n=1}^{\infty} A_n = \{x \in X \,|\, (x \in A_0) \vee (x \in A_1) \vee \dots\} = \{x \in X \,|\, \exists n \in \mathbb{N} : x \in A_n\}$
- $A_0 \cap A_1 \cap A_2 \cap \dots = \bigcap_{n=1}^{\infty} A_n = \{x \in X \,|\, \forall n \in \mathbb{N} : x \in A_n\}$

3.4 Cartesian product

Definition 3. Let A and B sets. The *cartesian product of A and B* is given as:

$$A \times B = \{(x, y) \,|\, x \in A, y \in B\}$$

This operation is *not* commutative!

Definition 4. We denote $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Example 3.

$$\begin{aligned} A &= \{a, b, c, d, e, f, g, h\} \\ B &= \{1, 2, 3, 4, 5, 6, 7, 8\} \\ A \times B &= \{(a, 1), (a, 2), (a, 3), \dots, (a, 8), (b, 1), (b, 2), \dots\} \end{aligned}$$

Example 4.

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$$

e.g. $(1, \frac{9}{8}) \in \mathbb{R} \times \mathbb{R}$.

Definition 5. Let $A_1, A_2, \dots, A_n$ be sets.

$$A_n = A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \,|\, a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

instead of $\underbrace{A \times A \times \dots \times A}_{n \text{ times}} = A^n$.

3.5 Power set

Definition 6. Let X be a set. Then $\mathcal{P}(X)$ is the *power set* of x, i.e. containing all subsets of X:

$$\mathcal{P}(X) = \{A \,|\, A \subseteq X\}$$

4 Mappings and functions

Definition 7. Let A and B be sets. A *mapping* f from A to B (denoted $f : A \to B$) is an assignment, such that for every $x \in A$ one $y \in B$ is assigned. We denote the corresponding $y \in B$ for some $x \in A$ with $y = f(x)$. A is called *domain*, B is called *co-domain*.

Definition 8 (Alternative definition of mappings)**.** A mapping f is a subset of $A \times B$ which fulfills the following properties:

- $\forall x \in A : (\exists y \in B : (x, y) \in f)$
- $\forall x \in A \wedge (y_1, y_2 \in B) : [(x_1, y_1) \in f \wedge (x_1, y_2) \in f] \implies y_1 = y_2$

Notation:

$$(x, y) \in f \iff y = f(x)$$
$$\{(x, f(x)) \in f \,|\, x \in A\} =: \text{graph of f}$$

Definition 9. Let $f : A \to B$ be a mapping.

- The mapping f is called *surjective*, if $\forall y \in B : \exists x \in A : y = f(x)$.
- The mapping f is called *injective*, if

  $$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \implies x_1 = x_2).$$

- Let $B' \subseteq B$. We call $f^{-1}(B') = \{x \in A \,|\, f(x) \in B'\}$ the *preimage of f*.

  Attention! The preimage distinguishes itself from the domain (it is a subset) and the inverse function f^{-1} (a function must not be invertible to have a preimage)!

- Let $A' \subseteq A$. Then we call $f(A') = \{f(x) \,|\, x \in A\} \subseteq B$ the *image of A'* under f.

  Special case: $A' = A$, then $f(A) \subseteq B$ is the image of A under f.

  Let $f : A \to B$ be a mapping. We define $\tilde{f} : A \to f(A) \subseteq B$ with $\tilde{f}(x) = f(x)$ for all $x \in A$. The mapping $\tilde{f}$ is surjective, i.e. $\forall y \in f(A)$ there exists one $x \in A$ such that $y = f(x)$.

- A mapping is called *bijective* iff the mapping is surjective and injective.

4.1 Bernoulli's inequality

Definition 10 (Bernoulli's inequality)**.** Let $x \in \mathbb{R}$ with $x > -1$ and $x \neq 0$. Let $n \in \mathbb{N}$ with $n > 1$. Then it holds that

$$(1+x)^n > 1 + nx$$

Proof. Proof by complete induction.

Induction base $\mathbf{n = 2}$

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x \quad \checkmark$$

because $x^2 > 0$ for $x \neq 0$.

Induction step $\mathbf{n \to n + 1}$

Assume $(1+x)^2 > 1 + n$, then $x > -1$ and $x \neq 0$.

$$(1+x)^{n+1} = (1+x)^n \cdot \underbrace{(1+x)}_{>0} \underbrace{>}_{\text{by ind. hypo.}} (1+nx)\cdot(1+x)$$

$$= (1 + nx + x + nx^2) = (1 + (n+1)\cdot x + \underbrace{nx^2}_{>0}) > 1 + (n+1)\cdot x$$

□

Back to sets and functions (notes are missing, but the topics we covered are):

- injective, surjective, bijective function
- composition of functions: Let $f : X \to Y$ and $g : Y \to Z$. $g \circ f : X \to Z$ is defined as $g(f(x))$ ("g after f").
- Let f and g be mappings. If f and g are injective, $f \circ g$ is injective. If f and g are surjective, $f \circ g$ is surjective. If f and g are bijective, $f \circ g$ is bijective.
- Identity function, $f \circ \mathrm{id} = \mathrm{id} \circ f = f$
- properties of an inverse function, $f \circ f^{-1} : X \to X$, $f^{-1} \circ f : X \to X$

5 About sums of integers

↓ This lecture took place on 21st of Oct 2015 with lecturer Wolfgang Ring

Definition 11. The summation notation is defined as,

$$\sum_{k=h}^{l} a_k$$

Iteration over all values from l to h (inclusive) and evaluation of the enclosed expression with k as iteration value. The resulting terms are added up and the sum gives the result of the summation expression.

Laws:

$$\sum_{k=l}^{h} a_k = \sum_{i=l}^{h} a_i \tag{14}$$

$$\sum_{k=l}^{h} (a_k + b_k) = \left(\sum_{k=l}^{h} a_k\right) + \left(\sum_{k=l}^{h} b_k\right) \tag{15}$$

$$\sum_{k=0}^{h} a_k = a_0 + \sum_{k=1}^{h} a_k \qquad \text{“Extraction of the initial value”} \tag{16}$$

$$\sum_{k=0}^{h} a_k = a_h + \sum_{k=0}^{h-1} a_k \qquad \text{“Extraction of the final value”} \tag{17}$$

$$\sum_{k=u+n}^{h+n} a_k = \sum_{k=u}^{h} a_{k+n} \qquad \text{“index shifting”} \tag{18}$$

$$\sum_{k=l}^{h} \lambda \cdot a_k = \lambda \cdot \sum_{k=l}^{h} a_k \qquad \text{“extraction of a constant } \lambda\text{”} \tag{19}$$

$$\sum_{k=0}^{n} n = \frac{n(n+1)}{2} \qquad \text{“triangular sum”} \tag{20}$$

5.1 Factorials

We consider $S_n = \{(a_1, a_2, \ldots, a_n) : a_i \in M_n \forall i = 1, \ldots, n \text{ with } a_i \neq a_j\} \subseteq M_n \times M_n \times \cdots \times M_n$. S_n is the set of all arrangements of the numbers $1, \ldots, n$.

Example: $\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

Theorem 3. It holds that $|\, S_n \,| = n!$ for all $n \in \mathbb{N}$

Proof. Proof by induction over n.

Induction base $n = 1$: $M_1 = \{1\}, S_1 = \{(1)\} \implies |\, S_1 \,| = 1 = 1!$ ✓

Induction step $n \to n+1$:

$$S_{n+1} = \{(a_1, a_2, \ldots, a_n) : a_i \in M_{n+1} \forall i \in M_{n+1}, a_i \neq a_j \text{ for } i \neq j\}$$

For $l \in M_{n+1}$:

$$W_l = \{(a_1, \ldots, a_{n+1}) \in S_{n+1} : a_l = n+1\}$$

It holds that $W_l \cap W_j = \emptyset$ for $l \neq j$ and $S_{n+1} = W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{n+1}$. Then it holds that $|\, S_{n+1} \,| = |\, W_1 \,| + |\, W_2 \,| + \ldots + |\, W_{n+1} \,| = \sum_{l=1}^{n+1} |\, W_l \,|$

Theorem 4. Claim: For every $l \in M_{n+1}$ it holds that $|\, W_l \,| = |\, S_n \,| = n!$.

Proof. We build a bijective map $\phi_l : W_l \to S_n$.

$$W_l = \{(a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}\}$$

$$: a_i \in M_n, \forall i \neq l, a_i \neq a_j \forall i \neq j$$

$$\phi\left((a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1})\right)$$

$$= (a_1, a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) \in S_n$$

S_n is surjective: Let $(b_1, \ldots, b_n) \in S_n$, then it holds that $(b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n) \in W_l$

$$\phi_l((b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n)) = (b_1, \ldots, b_n)$$

S_n is injective.

$$\phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$= \phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$\implies (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) = (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1})$$

Hence, ϕ is bijective. □

Therefore $|\, W_l \,| = |\, S_n \,| = n!$. Therefore $|\, S_{n+1} \,| = \sum_{l=1}^{n+1} |\, S_n \,| = \sum_{l=1}^{n+1} n! = (n+1)n! = (n+1)!$

□

Remark 1. Let $f : M_n \to M_n$. f is represented as

$$(1, 2, 3, 4, \ldots, n-1, n) \to (f(1), f(2), f(3), f(4), \ldots, f(n-1), f(n))$$

where $(a, b, c, \ldots)$ denotes a permutation. Therefore $(f(1), f(2), \ldots, f(n)) \in S_n$. Analogously every $(a_1, \ldots, a_n) \in S_n$ defined by $f(k) = a_k$ for $k = 1, \ldots, n$ is a bijective mapping $f : M_n \to M_n$. Therefore we set $S_n = \{f : M_n \to M_n : f \text{ is bijective}\}$. S_n is called *symmetric group of n elements.*

5.2 Binomial coefficients

Definition 12. Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ with $k \leq n$. We define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{“binomial coefficient } n \text{ choose } k\text{”}$$

It holds that

$$\binom{n}{k} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{(1 \cdot 2 \cdot \ldots \cdot k)(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))} = \frac{n(n-1) \cdot \ldots \cdot (k+1)}{(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))}$$

Factorial laws:

$$\binom{1}{0} = \frac{n!}{0!(n-0)!} = 1 \qquad \forall n \in \mathbb{N}$$
$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = 1$$
$$\binom{n}{n-k} = \frac{n!}{(n-k!)(n-n+k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \qquad \text{“symmetrical”}$$

A recursive definition is given by Pascal's Rule:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad n \geq 1, 1 \leq k \leq n-1$$

Proof.

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&\quad + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)! \cdot k}{k! \cdot (n-k)!} + \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} \\
&= \frac{k \cdot (n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{(n-k+k)(n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-k)!} = \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

□

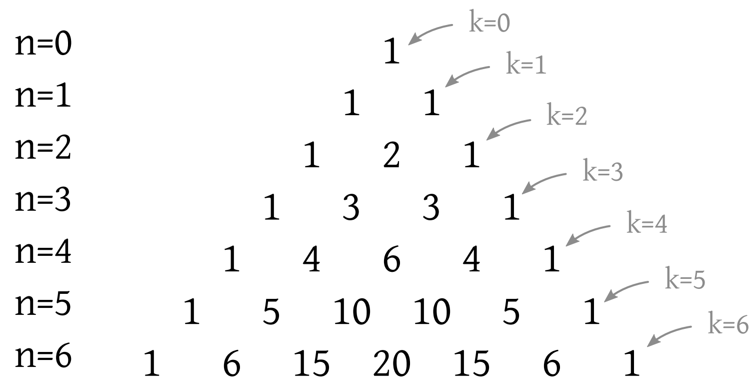

Figure 2: Pascal's triangle describes binomial coefficients. For every element of the triangle it holds that, it is adding up the two numbers above a number. The margins are defined by 1. For example, 5 is given by $\binom{5}{4}$.

5.3 Arrangement in Pascal's triangle

Theorem 5. Let $T_n^k = \{A \subseteq M_n : |\,A\,| = k\}$. Then it holds that $|\,T_n^k\,| = \binom{n}{k}$.

Example: $T_3^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$.

$|\,T_3^2\,| = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$

Proof. Let n be fixed. Complete induction over k.

Induction base $k = 0$

$$T_n^0 = \{\emptyset\}$$
$$|\,T_n^0\,| = 1 = \binom{n}{0}$$

Induction step $k \to k+1$

$$T_n^k = \underbrace{\{\{a_1,\dots,a_k\} : a_i \in M_n, (i=1,\dots,k), a_i \neq a_j \text{ for } i \neq j\}}_{A_1}$$
$$\cup \underbrace{\{\{a_1,\dots,a_{k-1}\} \cup [n] \in M_{n-1}\}}_{A_2}$$
$$\left|\, T_n^k \,\right| = \left|\, A_1 \,\right| + \left|\, A_2 \,\right|$$

□

↓ This lecture took place on 28th of October 2015 with lecturer Wolfgang Ring

Let A, B be sets and define

$$A \setminus B = \{x : x \in A \land x \notin B\}$$

Then $A \setminus B$ is called “A without B”.

Theorem 6.

$$T_n^k \coloneqq \{X \subseteq M_n : \left|\, X \,\right| = x\}$$

where $M_n = \{1, 2, \dots, n-1, n\}$. Let $k \in \mathbb{N}$ and $0 \leq k \leq n$. Then,

$$\left|\, T_n^k \,\right| = \binom{n}{k}$$

There are exactly $\binom{n}{k}$ k-ary subsets of M_n.

Example:

$$\begin{aligned} T_3^1 &= \{X \subseteq M_3 : \left|\, X \,\right| = 1\} \\ &= \{X \in \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} : \left|\, X \,\right| = 1\} \\ \left|\, T_3^1 \,\right| &= \left|\, \{1,2\}, \{1,3\}, \{2,3\} \,\right| = 3 \end{aligned}$$

Proof. Proof by complete induction over n of the following statement:

$$\forall n \in \mathbb{N} : \forall k \in \mathbb{N} \text{ with } 0 \leq k \leq n : \left|\, T_n^k \,\right| = \binom{n}{k}$$

Induction base $n = 0$ is fine.

$$M_0 = \emptyset \qquad T_0^0 = \{\emptyset\} \qquad \left|\, T_0^k \,\right| = 1 = \binom{0}{k}$$

For $n = 1$ there are two cases: $k = 0$ or $k = 1$.

$$M_1 = \{1\}$$

$$T_1^0 = \{\emptyset\} \qquad \left|\, T_1^0 \,\right| = 1 = \binom{1}{0}$$

$$T_1^1 = \{\{1\}\} \qquad \left|\, T_1^1 \,\right| = 1 = \binom{1}{1}$$

Also in this case, the induction base is satisfied.

Induction step The hypothesis is our assumption:

$$\forall 0 \leq k \leq 1 : \left|\, T_n^k \,\right| = \binom{n}{k}$$

Consider M_{n+1}. Special case $k = 0$:

$$T_{n+1}^0 = \{\emptyset\} \qquad \left|\, T_{n+1}^0 \,\right| = 1 = \binom{n+1}{0}$$

Special case $k = n+1$:

$$T_{n+1}^{n+1} = \{M_{n+1}\} \qquad \left|\, T_{n+1}^{n+1} \,\right| = 1 = \binom{n+1}{n+1}$$

Now we consider the more generic case. Let $1 \leq k \leq n$.

$$\begin{aligned} T_{n+1}^k &\coloneqq \{(a_1,\dots,a_k) : a_i \in M_{n+1}, \forall i \in \{1,\dots,k\}, a_i \neq a_j \forall (i,j) \in \{1,\dots,k\}\} \\ R_{n+1}^k &\coloneqq \left\{(a_1,\dots,a_{k-1}, n+1) : (a_1,\dots,a_{k-1}) \in T_n^k\right\} \\ S_{n+1}^k &\coloneqq \left\{(a_1,\dots,a_{l-1}, n+1, a_{l+1},\dots,a_{k-1}) : (a_1,\dots,a_{\hat{l}},\dots,a_k) \in T_n^k\right\} \end{aligned}$$

Union is disjoint $\implies \left|\, T_{n+1}^k \,\right| = \left|\, R_{n+1}^k \,\right| + \left|\, S_{n+1}^k \,\right|$

$$R_{n+1}^k = \{A \subseteq M_n : |\,A\,| = k\} = T_n^k$$

$$|\,R_{n+1}^k\,| = |\,T_n^k\,| = \binom{n}{k}$$

by induction hypothesis.

$$S_{n+1}^k = \{A \subseteq M_{n+1} : A = A' \cup \{n+1\} : A' \subseteq M_n : |\,A'\,| = k-1\}$$

We prove $|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,|$.

$$f : S_{n+1}^k \to T_n^{k-1}$$

$$f(A) = f(A' \cup \{n+1\}) = A'$$

f is bijective. f is surjective: Let $A' \in T_n^k$ define $A = A' \cup \{n+1\} \in S_{n+1}^k$ and $f(A) = A'$. f is injective: Let $f(A) = f(B)$ and $A = A' \cup \{n+1\} \in S_{n+1}^k$.
$B = B' \cup \{n+1\} \in S_{n+1}^k$. $A', B' \in T_n^{k-1}$.

$$f(A) = f(B) \implies A' = B' \implies A' \cup \{n+1\} = B' \cup \{n+1\} \implies A = B$$

$$|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,| \overset{\text{ind. hypo.}}{=} \binom{n}{k-1}$$

Therefore $|\,T_{n+1}^k\,| = \binom{n}{n} = \binom{n}{k-1} = \binom{n+1}{k}$. The last equality follows from the recursive definition of binomial coefficients.

□

5.4 Binomial theorem

Theorem 7 (Binomial theorem)**.** Let $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$). Then it holds that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof. 1. Proof by induction over n.

Induction step $n = 0$: $(a+b)^0 = 1$

$$\sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^0 = 1$$

Induction step $n \to n+1$

$$(a+b)^{n+1} = (a+b)^n \cdot (a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)(a+b)$$

$$= \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k}}_{\text{(I)}} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}}_{\text{(II)}}$$

$$= \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\substack{\text{index shift} \\ k+1=j, k=j-1 \\ k=0 \implies j=1 \\ k=n-1 \implies j=n}} + \underbrace{\binom{n}{n} a^{n+1} \cdot b^0}_{=a^{n+1}} \tag{I}$$

$$+ \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1} \tag{II}$$

$$= \sum_{j=1}^{n} \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

$$+ \binom{n+1}{n+1} a^{n+1} + \binom{n+1}{0} b^{n+1}$$

Renaming j to k. Then it holds that:

$$= \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\binom{n+1}{k} \text{ by recursive definition}} a^k b^{n+1-k}$$

$$+ \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a^0 b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

Therefore the binomial theorem holds for $n+1$.

□

↓ This lecture took place on 29th of October 2015 with lecturer Wolfgang Ring

$$\forall a, b \in \mathbb{R}, n \in \mathbb{N} : (a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k}$$

Induction base $n = 0, n = 1$ follows immediately

Induction step

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)(a+b)\dots(a+b)}_{n \text{ times}}$$

When multiplying, the products $a^n b^{n-k}$ are created $(0 \leq k \leq n)$. $a^n b^{n-k}$ are created iff a is the factor resulting from k parentheses groups and b originates from the remaining $(n-k)$ groups. There are exactly $\binom{n}{k}$ possibilities to select from n groups. $a^k b^{n-k}$ occurs $\binom{n}{k}$ times. Therefore

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

This is a rather informal proof, but suffices at this point.

6 Arithmetics of numbers

We consider two fundamental arithmetic operators and determine fundamental properties.

Definition 13. Let K be a set where two arithmetic operators are defined: Therefore $\forall a, b \in K$ let $a + b \in K$ and $a \cdot b \in K$.

We require the following properties:

A1 $\forall a, b \in K : a + b = b + a$

A2 $\forall a, b, c \in K : (a + b) + c = a + (b + c)$

A3 $\exists 0 \in K \forall a \in K : a + 0 = a$

A4 $\forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$

Then $(K, +)$ is a commutative group ("abelian group"). In general we denote $\tilde{a}$ as $-a$. We define $a - b = a + (-b)$ ("subtraction").

M1 $\forall a, b \in K : a \cdot b = b \cdot a$

M2 $\forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M3 $\exists 1 \in K : a \cdot 1 = a \forall a \in K$ (neutral element)

M4 $\forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$

In general we denote $\hat{a}$ as a^{-1}.

We set $\frac{a}{b} = a \cdot b^{-1}$.

$$\frac{1}{b} = 1 \cdot b^{-1} \text{ for } b \neq 0$$

Definition 14 (Composition)**.** Compatibility of $+$ and $\cdot$:

D $\forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$

Under these conditions K is called a *field*.

Example 5. Examples for fields: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

In every field it holds that

- the inverse element of a is unique ($\tilde{a}$ is unique). Let $-a$ be the inverse element of a and $a + b = 0 \implies b = -a$

  Proof. Let $\tilde{a}$ be the inverse of a. Let $\tilde{b}$ the inverse of a. Show $\tilde{a} = \tilde{b}$.

  $$\implies \tilde{a} + a = 0 \land \tilde{b} + a = 0$$

$$\begin{aligned} &\implies \tilde{a} + a = \tilde{b} + a \\ &\implies \tilde{a} = \tilde{b} \end{aligned}$$

□

- $0 \cdot a = 0$

  Proof.
  $$0 = 0 + 0$$
  follows from **D**.
  $$0 \cdot a = (0+0) \cdot a = 0 \cdot a + 0 \cdot a$$
  $$0 \cdot a + (-0 \cdot a) = 0 \cdot a + [0 \cdot a + (-0 \cdot a)]$$
  $$0 = 0 \cdot a$$
  □

- $-a = (-1) \cdot a$

  Proof.
  $$\begin{aligned} a + (-1) \cdot a = (1 + (-1))a &= 0 \\ a + (-1) \cdot a &= 0 \\ -a &= (-1) \cdot a \end{aligned}$$
  □

6.1 Integers and the field of rational numbers $\mathbb{Q}$

For $\mathbb{N}$, **A1**, **A2** and **A3** hold. If $n \geq m$, then also $n - m \in \mathbb{N}$. $n - m = k \in \mathbb{N}$ is defined in such a way that $n = m + k$.

Corollary 1. Extension:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \ldots\} = \mathbb{N}_+ \cup \{0\} \cup \{-n : n \in \mathbb{N}_0\}$$

We define $-0 \coloneqq 0$ and $\forall n \in \mathbb{N}_+$ let $n + (-n) \coloneqq 0$.

Therefore for every $z \in \mathbb{Z}$ exists some $\tilde{z}$ such that $z + \tilde{z} = 0$.

- $z \in \mathbb{Z}_+ \implies \tilde{z} = -z$
- $z = 0 \implies \tilde{z} = 0$
- $z = -n$ for $n \in \mathbb{N}_+$
- $\tilde{z} = n$

$$\forall z \in \mathbb{Z} \, \exists \tilde{z} \in \mathbb{Z} : z + \tilde{z} = 0$$

In general we denote $\tilde{z} = (-z)$. Also $-(-z) = z$.

For $z, w \in \mathbb{Z}$:

$$z + w = \begin{cases} z + w & z, w \in \mathbb{N} \\ (-z) + (-w) & -z, -w \in \mathbb{N} \\ z - (-w) & z, -w \in \mathbb{N} \text{ and } z > (-w) \\ -((-w) - z) & z, -w \in \mathbb{N} \text{ and } (-w) > z \end{cases}$$

$$z \cdot w = \begin{cases} z \cdot w & z, w \in \mathbb{N} \\ (-z)(-w) & -z, -w \in \mathbb{N} \\ -((-z) \cdot w) & -z \in \mathbb{N}, w \in \mathbb{N} \end{cases}$$

In $\mathbb{Z}$ the properties **A1**, **A2**, **A3**, **A4**, **M1**, **M2**, **M3** and **D** hold.

Definition 15.
$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$
where $\frac{m}{n} = \frac{m'}{n'} \iff m \cdot n' = n \cdot m'$. $\mathbb{Q}$ is called the set of rational numbers.

We define
$$\frac{m}{n} + \frac{k}{l} \coloneqq \frac{ml + nk}{nl}$$
$$\frac{m}{n} \cdot \frac{k}{l} = \frac{mk}{nl}$$

Show that
$$\frac{m}{n} = \frac{m'}{n'} \text{ and } \frac{k}{l} = \frac{k'}{l'}$$
$$\implies \frac{ml + nk}{nl} = \frac{m'l' + n'k'}{n'l'}$$

$$\implies (ml+nk)(n'l') = (m'l'+n'k')$$

$$\iff mn' \cdot ll' + nn' \cdot kl = m'n \cdot ll' + nn' \cdot k'l$$

Analogously for $\frac{m}{n} \cdot \frac{k}{l}$.

A1–A4, **M1–M4** and **D** hold for $\mathbb{Q}$.

For $z \in \mathbb{Z}$ we set $z = \frac{z}{1}$. Therefore it holds that $\mathbb{Z} \subseteq \mathbb{Q}$. $0 = \frac{0}{1}$ and $\frac{m}{n} + 0 = \frac{m}{n} + \frac{0}{1} = \frac{m \cdot 1 + n \cdot 0}{n \cdot 1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$. 0 is neutral in regards of addition in $\mathbb{Q}$.

Inverse element in regards of addition:

$$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{n^2} = \frac{(m + (-m))n}{n \cdot n} = \frac{0n}{n^2} = \frac{0}{1}$$

because $0 \cdot 1 = 0 \cdot n^2$.

Concerning multiplication:

$$1 = \frac{1}{1} \qquad \frac{m}{n} \cdot \frac{1}{1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$$

1 is a neutral element in regards of multiplication in $\mathbb{Q}$.

Let $\frac{m}{n} \in \mathbb{Q} \setminus \{0\} \implies m \neq 0 \implies \frac{n}{m} \in \mathbb{Q}$ and $\frac{m}{n}\frac{n}{m} = \frac{mn}{mn} = \frac{1}{1}$, because $m \cdot n \cdot 1 = 1 \cdot m \cdot n$.

Corollary 2.

$$\forall \frac{m}{n} \in \mathbb{Q} : -\frac{m}{n} = \frac{-m}{n}$$

$$\forall \frac{m}{n} \in \mathbb{Q} \setminus \{0\} : \left(\frac{m}{n}\right)^{-1} = \left(\frac{n}{m}\right)$$

Therefore $\mathbb{Q}$ is a field.

↓ This lecture took place on 30th of October 2015 with lecturer Wolfgang Ring

Literature:

- Eblinghaus et al., "Zahlen", Springer Verlag
- E. Landau: "Grundlagen der Analysis", uses Peano axioms to build calculus

6.2 Ordered fields

Definition 16. Let K be a field. We assume that K is taken from two sets: $K = K_+ \cup \{0\} \cup K_-$ with $0 \notin K_+, 0 \notin K_-$. It holds that

- $\forall a \in K$ it holds that either $a \in K_-$ or $a = 0$ or $a \in K_+$
  $a \in K_+ \iff -a \in K_-$
- $\forall a, b \in K_+$: $a + b \in K_+ \wedge a \cdot b \in K_+$

If those properties are satisfied, such a field is called an *ordered field*. Instead of $a \in K_+$ we write $a > 0$ (namely "positive numbers") and $a < 0$ for $a \in K_-$ correspondingly (namely "negative numbers").

For arbitrary $a, b \in K$ we define

$$a > b \iff a - b > 0$$

It holds that $a > b \iff b < a$.

$$a \geq b \iff a > b \vee a = b$$

Lemma 1. Let K be an ordered field. Then it holds that

1. $a \in K_+ \wedge b \in K_- \implies a \cdot b \in K_-$
   $a \in K_- \wedge b \in K_- \implies a \cdot b \in K_+$
2. $\forall a, b \in K$ one of the following relations hold:
   $$a > b \ \vee \ a = b \ \vee \ a < b$$
   Therefore $<$ defines a total order on K.
3. $\forall a, b, c \in K : [(a < b) \wedge (b < c) \implies a < c]$
   Therefore $<$ is transitive.
4. If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$ If $a > 0$ holds, then also $a^{-1} = \frac{1}{a} > 0$.
5. $\forall a, b, c \in K : a < b \implies a + c < b + c$
6. $\forall a, b \in K : \forall c > 0 : [a > b \implies ac > bc]$
   $\forall a, b \in K : \forall c < 0 : [a > b \implies ac < bc]$

7. $\forall a \in K \setminus \{0\} : a^2 = a \cdot a > 0$

Proof. 1. We know from the practicals: $\forall a, b \in K : (-a)(-b) = ab$

$$(-a)b = -(ab)$$

Let $a \in K_+, b \in K_-$, therefore $a \in K_+$, $(-b) \in K_-$, then it holds that $ab = (-a)(-b) = -(a(-b)) \in K_-$. Let $a \in K_-$ and $b \in K_-$ therefore $(-0) \in K_+ \wedge (-b) \in K_+ \implies ab = (-a)(-b) \in K_+$.

2. Let $a, b \in K$. Then one of the following properties hold:

$$a - b > 0 \vee a - b = 0 \vee a - b < 0$$

Equivalently,

$$a > b \vee a = b \vee a < b$$

3. Let $a > b$ and $b > c$. Therefore $a - b > 0$ and $b - c > 0$.

$$\implies (a - b) + (b - c) > 0$$

$$a(-b + b) - c > 0$$

$$a - c > 0 \iff a > c$$

4. Let $a > 0 \implies a^{-1} \neq 0$. Assume $\frac{1}{a} = a^{-1} < 0 \implies a^{-1} \cdot a = 1 < 0$. Otherwise it holds that $1 = 1 \cdot 1 = 1^2 > 0$`

5. Let $a > b > 0$. Then it holds that

$$a^{-1}b^{-1}(b - a) = a^{-1}b^{-1}b - a^{-1}b^{-1}a = -a^{-1} \cdot b^{-1} = \frac{1}{a} \cdot \frac{1}{b} \implies a^{-1} < b^{-1}$$

6. $a < b$ therefore $a - b < 0 \implies a + c - c - b < 0 \implies (a + c) - (b + c) < 0$

$$\iff a + c < b + c$$

7. Let $a > b, c > 0 \implies (a - b) > 0 \implies (a - b) \cdot c > 0 \implies ac - bc > 0 \implies ac > bc$. For the second statement, it holds analogously: $a < b, c < 0 \implies (a - b) < 0 \implies (a - b) \cdot c < 0 \implies ac - bc < 0 \implies ac < bc$

8. $a > 0 \implies a \cdot a > 0$. Let $a < 0 \implies (-a) > 0$. It holds $a \cdot a = (-a)(-a) > 0$. Therefore the square of two numbers is always positive.

□

6.3 Remarks about some common fields

Remark 2. $\mathbb{C}$ is not an ordered field. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are ordered.

Remark 3. Let $q \in \mathbb{Q}$.

a) Let $m, n \in \mathbb{N}_+$ such that $q = \frac{m}{n}$ then $q > 0$.

b) Let $m, n \in \mathbb{N}_+$ such that $q = -\frac{m}{n}$ then $q < 0$.

We show that $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$. Every $q \in \mathbb{Q}$ has a representation of either a) or b), but not both. $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$.

$$q \neq 0 \implies q = \begin{cases} \frac{m}{n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{n} & m, n \in \mathbb{N}_+ \\ \frac{m}{-n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{-n} & m, n \in \mathbb{N}_+ \end{cases}$$

$$q = \frac{n}{-m} = \frac{-n}{m}$$

because $nm = (-n)(-m)$.

$$q = \frac{-m}{-n} = \frac{m}{n}$$

because $(-m) \cdot n = m \cdot (-n)$.

Remark 4. We want to show that $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$. Let $q \in \mathbb{Q}_+ \cap \mathbb{Q}_-$.

$$q = \frac{m}{n} = -\frac{m'}{n'} \qquad m, n, m', n' \in \mathbb{N}_+$$

$$\implies n \cdot n' = (-m')n$$

$$\implies \underbrace{\underbrace{mn'}_{\in \mathbb{N}_+} + \underbrace{m'n}_{\in \mathbb{N}_+}}_{\in \mathbb{N}_+} = 0 \qquad \lightning$$

Furthermore $p \in \mathbb{Q}_+ \wedge q \in \mathbb{Q}_+$

$$\implies p + q \in \mathbb{Q}_+ \wedge pq \in \mathbb{Q}_+$$

$$\implies p = \frac{k}{l} \qquad q = \frac{m}{n} \qquad k, l, m, n \in \mathbb{N}_+$$

$$p + q = \frac{\overbrace{kn + ml}^{\in\mathbb{N}_+}}{nm} \in \mathbb{Q}_+$$

$$pq = \frac{k}{l} \cdot \frac{m}{n} = \frac{\overbrace{km}^{\in\mathbb{N}_+}}{\underbrace{ln}_{\in\mathbb{N}_+}} \in \mathbb{Q}_+$$

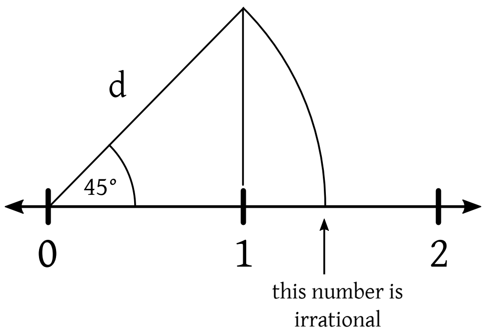

Figure 3: Illustration of an irrational number

Definition 17. Let K be an ordered field $a \in K$. The absolute value of a is defined as

$$|a| = \begin{cases} a & \text{if } a \in K_+ \\ 0 & \text{if } a = 0 \\ -a & \text{if } a \in K_- \end{cases}$$

Remark 5. Let K be an ordered field. Then it holds that

$$\mathbb{Q} \subseteq K \subseteq \mathbb{R}$$

except for isomorphism.

6.4 Triangle inequality

Theorem 8.

$$\forall a, b \in K : |a + b| \leq |a| + |b| \qquad \text{“Triangle inequality”}$$

Proof. **Case 1.** Let $a > 0 \land b > 0$

$$\implies a = |a| \land b = |b| \implies |a + b| = a + b = |a| + |b|$$

Case 2. Let $a > 0 \land b < 0$.

$$\implies a \cdot b < 0 \iff |ab| = -ab \qquad |a| \cdot |b| = a \cdot (-b)$$

$$b < 0 \iff -b > 0 \iff b < -b \iff \underbrace{a + b}_{|a+b|} < \underbrace{a + (-1 \cdot b)}_{|a|+|b|}$$

Case 3. Let $a < 0 \land b < 0$.

$$\implies a \cdot b > 0 \iff |ab| = ab \qquad |a| = -a \qquad |b| = -b$$

$$|a| \cdot |b| = -a \cdot -b = ab$$

$$|a + b| = |(-1)(a + b)| = |-a - b| = \left| \underbrace{|a|}_{\geq 0} + \underbrace{|b|}_{\geq 0} \right| = |a| + |b|$$

Case 4. Let $a < 0 \land b > 0$.

$$\implies a \cdot b < 0 \qquad |a| = -a \qquad -|a| = a \qquad |b| = b$$

$$a < 0 \iff -a > 0 \iff a < a \implies \underbrace{a + b}_{|a+b|} < \underbrace{(-1 \cdot a) + b}_{|a|+|b|}$$

□

$\downarrow$ This lecture took place on 4th of November 2015 with lecturer Wolfgang Ring

6.5 Laws for absolute values

Theorem 9. Let $y \geq 0$. Then it holds that $|x| \leq y \iff -y \leq x \wedge x \leq y$

Proof. First direction $\implies$:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Case 1 Let $x \geq 0$. Then

$$|x| \leq y \implies x \leq y \implies -y \leq x$$

because $-y \leq 0 \wedge x \geq 0$ anyways.

Case 2 Let $x < 0$, therefore $|x| = -x$. Because

$$-x \leq y \implies x \geq -y$$

$x \leq y$ holds anyways because $x < 0$ and $y \geq 0$.

Second direction $\impliedby$:

Let $-y \leq x \leq y$.

Case 1 $x \geq 0 : |x| = x \leq y$ because of the second inequality.

Case 2 $x < 0 : |x| = -x$

$$-(-1) \implies -(-y) \geq -x \text{ or equivalently } y \geq -x = |x|$$

□

Theorem 10.

$$|x| = 0 \iff x = 0$$

$$\forall a \in K : |a| = |-a|$$

$$\forall \varepsilon > 0 : |x - y| \leq \varepsilon \iff x = y$$

Proof. **First direction** $\implies$ Without loss of generality: $x \geq y$.

$$x \neq y \implies \exists \varepsilon > 0 : |x - y| > \varepsilon$$

Let $x \neq y$. Because $x \geq y$ holds, so does $x > y$. Therefore $x - y > 0$. We define $\varepsilon = \frac{x-y}{2} < x - y$

$$2 = 1 + 1 > 1$$

$$2^{-1} = \frac{1}{2} < 1 = 1^{-1}$$

Therefore it holds that $\varepsilon : |x - y| = x - y > \frac{1}{2}(x - y) = \varepsilon > 0$.

Second direction $\impliedby$ $x = y \implies |x - y| = 0 \leq \varepsilon \forall \varepsilon > 0$

□

Theorem 11 (Reversed triangle inequality)**.** Let $a, b \in K$. Then it holds that

$$||a| - |b|| \leq |a - b|$$

Proof. Show that $-|a - b| \leq |a| - |b| \leq |a - b|$.

First inequality

$$|b| = |b - a + a| \leq |b - a| + |a| \implies -|a - b| \leq |a| - |b|$$

Second inequality

$$|a| = |a - b + b| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|$$

□

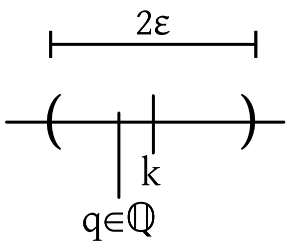

Figure 4: For every x and every ε-neighborhood, some element of the rational numbers exist within this neighborhood

6.6 Irrational numbers approximated by rational numbers

Additional remark from 14th of January 2016.

Theorem 12 ($\mathbb{Q}$ is dense in $\mathbb{R}$)**.** For all $x \in \mathbb{R}$ and for every $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ with $|x - q| < \varepsilon$.

Lemma 2. Let $A \subseteq \mathbb{N}$ and $A \neq \emptyset$. Then a minimum of A exists.

Proof by complete induction. We show: Let $A \subseteq \mathbb{N}$ such that no minimum exists. Then it holds that $A = \emptyset$.

Let $C = \{k \in \mathbb{N} \mid \forall n \in A : k < n\}$. C is the set of all lower bounds of A with operator $<$. We show: $0 \in C$ and $\forall k \in C \implies k + 1 \in C$.

Induction base Assume $0 \notin C$, hence for $k = 0$, $\exists n \in A \subset \mathbb{N} : n \leq 0 \implies n \geq 0$ anyways and hence $n = 0$ and $0 \in A$. $\implies 0 = \min A$ and A has a minimum.

This is a contradiction. So $0 \notin C$ does not hold. So $0 \in C$.

Induction step Let $k \in C$, hence $\forall n \in A : k < n$.

$$\implies \forall n \in A : k + 1 \leq n$$

Even $<$ holds. Assume $\exists n \in A : k + 1 = n$ and $\forall n' \in A : k + 1 \leq n'$.

$$\implies k + 1 \in A \land k + 1 \text{ is lower bound of } A$$

Therefore $k + 1 = \min A$. This is a contradiction to the assumption that $\min A$ does not exist. Therefore $\forall n \in A : k + 1 < n \implies k + 1 \in C$.

Due to the properties of induction: $\forall k \in \mathbb{N} : k \in C$ equivalently means $C = \mathbb{N}$. Therefore $A = \emptyset$ holds. Assume $m \in A$, so it holds that $m \notin C$, because $\neg(m < m)$.

□

Proof of Theorem 12. Case distinction:

Case $x > 0$. Let $\varepsilon > 0$ be arbitrary. Choose $n \in \mathbb{N}_+$ such that $\frac{1}{n} < \varepsilon$ and define $A = \{k \in \mathbb{N} \mid k > n \cdot x\}$.

We know that $A \neq \emptyset$ (by Archimedean's property). Let $m = \min A$.

$$\implies m > n \cdot x \land m - 1 \leq n \cdot x \qquad \implies x < \frac{m}{n} \land x \geq \frac{m-1}{n}$$

$$\left|x - \frac{m}{n}\right| = \left|(-1)\left(\frac{m}{n} - x\right)\right| = \frac{m}{n} - x \leq \frac{m}{n} - \frac{m-1}{n} = \frac{m - m + 1}{n} = \frac{1}{n} < \varepsilon$$

with $\frac{m}{n} = q \in \mathbb{Q}$.

Case $x < 0$. Therefore $-x > 0$. By the previous case, we know,

$$\forall x \in \mathbb{R}_+ \forall \varepsilon > 0 \exists q \in \mathbb{Q} : |x - q| < \varepsilon$$

$$\implies |-x - q| < \varepsilon \implies |(-1)(x + q)| < \varepsilon \implies |x - (-q)| < \varepsilon$$

Case $x = 0$. Let $q = 0 \in \mathbb{Q}$ and we're done.

□

Corollary 3. $\forall x \in \mathbb{R}$ and $\forall \varepsilon > 0$ it holds that

$$\mathbb{Q} \cap B(x, \varepsilon) = \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon) \neq \emptyset$$

Therefore x is a contact point of $\mathbb{Q}$.

Remark. It even holds that x is limit point of $\mathbb{Q}$.

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} \mid x \text{ is contact point of } \mathbb{Q}\} = \mathbb{R}$$

We say $\mathbb{Q}$ is *dense* (or: lies in) in $\mathbb{R}$.

Alternative characterization of contact points:

$$\forall x \in \mathbb{R} \exists (q_n)_{n\in\mathbb{N}} \text{ with } q_n \in \mathbb{Q} \text{ with } \lim_{n\to\infty} q_n = x$$

Every limit point is a contact point.

6.7 Intervals

↓ This lecture took place on 5th of November 2015 with lecturer Wolfgang Ring

Definition 18 (Intervals)**.** Let $a, b \in K$.

$$\begin{aligned}(a,b) &:= \{x \in K \mid (x > a) \wedge (x < b)\}\\ [a,b) &:= \{x \in K \mid (x \geq a) \wedge (x < b)\}\\ (a,b] &:= \{x \in K \mid (x > a) \wedge (x \leq b)\}\\ [a,b] &:= \{x \in K \mid (x \geq a) \wedge (x \leq b)\}\end{aligned}$$

Theorem 13 (Laws for intervals)**.**

$$(a,b) = \emptyset \text{ if } b \leq a \tag{21}$$
$$[a,b] = \emptyset \text{ if } b < a \tag{22}$$
$$[a,a] = \{a\} \tag{23}$$

If I is an non-empty interval (hence $I \neq \emptyset$), then $|I| = b - a$ is called *length of the interval*. Furthermore

$$(a,\infty) = \{x \in K \mid x > a\} \tag{24}$$
$$[a,\infty) = \{x \in K \mid x \geq a\} \tag{25}$$
$$(-\infty,a) = \{x \in K \mid x < a\} \tag{26}$$
$$(-\infty,a] = \{x \in K \mid x \leq a\} \tag{27}$$

Theorem 14. $\mathbb{Q}$ is arithmetically incomplete.

Proof. We will prove $\sqrt{2}$ to be irrational as an counterexample.

We define a mapping from $\mathbb{N}_+$ to $\mathbb{N}$: Let $n \in \mathbb{N}_+$ then we know that n can be represented distinctly as product of prime numbers. Let $\mathrm{Z}(n)$ be the number of twos in the prime product representation.

Examples:

$$\mathrm{Z}(14) = \mathrm{Z}(2 \cdot 7) = 1$$
$$\mathrm{Z}(15) = \mathrm{Z}(3 \cdot 5) = 0$$
$$\mathrm{Z}(24) = \mathrm{Z}(2 \cdot 2 \cdot 2 \cdot 3) = 3$$

It holds that $\mathrm{Z}(2n) = \mathrm{Z}(n) + 1 \forall n \in \mathbb{N}_+$ and $\mathrm{Z}(n^2) = \mathrm{Z}(n) \cdot 2 \forall n \in \mathbb{N}_+$.

We claim,

$$\nexists q : q = \frac{m}{n} \text{ with } q^2 = 2$$

Proof by contradiction:

1. Assume $\left(\frac{m}{n}\right)^2 = 2$.
2. Then $\frac{m^2}{n^2} = 2$.
3. Then $m^2 = 2 \cdot n^2$.
4. With $Z(m^2) = 2 \cdot \mathrm{Z}(m)$.
5. With $\mathrm{Z}(2 \cdot n^2) = \mathrm{Z}(n^2) + 1 = 2 \cdot \mathrm{Z}(n) + 1$.
6. If $m^2 = 2n^2$, then $\mathrm{Z}(m^2)$ must be even and $\mathrm{Z}(2 \cdot n^2)$ must be odd.
7. Then equality cannot be satisfied ↯

□

6.8 Archimedean property and Completeness axiom

Theorem 15. $\mathbb{Q}$ is geometrically incomplete.

We consider an infinite straight number line. We define $\mathbb{R}$ as ordered field with properties:

Archimedean property $\mathbb{N} \subseteq \mathbb{R}$ with $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : x < n$

$$\forall n \in \mathbb{N} : -n \in \mathbb{Z}$$

$$\forall n \in \mathbb{N}_+ : n^{-1} \in \mathbb{R}$$

$$\implies \mathbb{Z} \subseteq \mathbb{R}$$

Therefore $\forall m \in \mathbb{N} : m \cdot \frac{1}{n} = \frac{m}{n} \in \mathbb{R} \implies \mathbb{Q} \subseteq \mathbb{R}$.

Definition 19. Let $I_0, I_1, \dots$ $(I_n)_{n\in\mathbb{N}}$ is a sequence of closed intervals with

1. $\forall n \in \mathbb{N} : I_{n+1} \subseteq I_n$
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |I_n| < \varepsilon$

Completeness axiom Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals in $\mathbb{R}$. Then for all $n \in \mathbb{N}$ there exists *only one* $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}_+ : x \in I_n$ ("Nested interval theorem").

Proof by contradiction. Assume $x \in I_n$ and $y \in I_n \forall n \in \mathbb{N}$ and $x \neq y$. Let $I_n = [a, b]$ and $I_{n+1} = [\alpha, \beta]$. It holds that:

$$|I_{n+1}| = |\beta - \alpha| \leq |b - a| = |I_n|$$

Consider arbitrary small $\varepsilon > 0$ and $N \in \mathbb{N}$ sufficiently large, such that $|I_n| < \varepsilon \, \forall n > N$. Because $x, y \in I_n \implies |x - y| < \varepsilon \implies x = y$.[1] □

Corollary 4. From the Archimedean property it follows that,

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \implies \frac{1}{n} < \varepsilon$$

Proof. Let $x > \frac{1}{\varepsilon} \in \mathbb{R}$. Archimedean property: $\exists N \in \mathbb{N} : N > x$.

For $n \geq N$ it holds that $n > x > 0 \implies \frac{1}{n} < \frac{1}{x} = \varepsilon$. □

[1] Be aware, that this implication holds in general. An infinitesimal difference between two variables requires both variables to have the same value. In fact Georgi E. Shilov in "Elementary Real and Complex Analysis" defined that a *single point* is given iff this property holds.

Corollary 5. Let $p \in \mathbb{R}, p > 1 \forall x \in \mathbb{R} : n \geq N \implies p^n > x$.

Proof. $p > 1 + u$ with $u = p - 1$

$$p^n = (1+u)^n \underbrace{>}_{\text{Bernoulli}} 1 - nu = 1 + n(p-1)$$

Let $x \in \mathbb{R}$ arbitrary, select $N \in \mathbb{N} : \frac{x-1}{p-1} < N$.

Then it holds for $n \geq N$:

$$\frac{x-1}{\underbrace{p-1}_{>0}} \iff x - 1 < n \cdot (p-1) \iff x < 1 + n(p-1) < p^n$$

□

Theorem 16. Let $q \in \mathbb{R}$ with $|q| < 1$. Then it holds that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |q^n| = |q|^n < \varepsilon$$

Proof. Let $s = |q| \geq 0$.

Case $\mathbf{q > 0}$ Then,

$$q^n = 0, \qquad |q^n| = 0, \qquad |q|^n < \varepsilon \quad \forall \varepsilon > 0 \, \forall n \in \mathbb{N}$$

Case $\mathbf{q \neq 0}$ Then $0 < s < 1$. Let $p = \frac{1}{s} \implies p > 1$. Choose arbitrary $\varepsilon > 0$ and $x = \frac{1}{\varepsilon}$. Because of the Completeness axiom

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies p^n > x$$

So,

$$\forall n \geq N : \frac{1}{p^n} = S^n < \frac{1}{x} = \varepsilon \implies (|q|)^n = |q^n|$$

□

Theorem 17. Let $x \in \mathbb{R}, x > 0$ and let $k \in \mathbb{N}_+$. Then there exists a distinct $y \in \mathbb{R}$ with $y \geq 0$ such that

$$y^k = x$$

We denote $y = \sqrt[k]{x}$ and conclude there exists some k-th root numbers.

Proof. Idea: Construct nested intervals.

$(I_n)_{n\in\mathbb{N}}$ such that $y \in \bigcap_{n\in\mathbb{N}} I_n$ satisfies the property that $y^k = x$.

$$0 \leq y_1 < y_2 \implies y_1^k < y_2^k$$

We define $J_0 = [a_0, b_0]$ with $a_0 = 0$ and $b_0 = 1 + x$. Then it holds that

$$a_0^k = 0^k = 0 \leq x$$

$$b_0^k = (1+x)^k = 1 + k_n + \binom{k}{2} x^2 + \dots + x^k \geq 1 + kx > 0$$

□

↓ This lecture took place on 6th of November 2015 with lecturer Wolfgang Ring

Theorem 18. We prove:

$$\forall k \in \mathbb{N}_{\geq 0} : 0 \leq y_1 < y_2 \implies y_1^k \leq y_2^k$$

Proof. $\mathbf{k = 2}$

$$y_1^2 = y_1 \cdot y_1 < y_1 \cdot y_2 < y_2 \cdot y_2 = y_2^2$$

$\mathbf{k \to k+1}$

$$y_1^k \leq y_2^k \implies y_1^k \cdot y_1 \leq y_2^k \cdot y_1 < y_2^k \cdot y_2 \implies y_1^{k+1} \leq y_2^{k+1}$$

□

Theorem 19. Let $a, b \in K$ and $k \in \mathbb{N}$. Then,

$$a^k - b^k = (a-b)\left(\sum_{j=0}^{k-1} a^{k-1-j} b^j\right)$$

Examples:

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Proof.

$$\begin{aligned}
(a-b)\left(\sum_{j=0}^{k-1} a^{k-j-1} b^j\right) &= \sum_{j=0}^{j-1} a^{k-j} b^j - \sum_{j=0}^{k-1} a^{k-j-1} b^{j+1} \\
&= a^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \underbrace{b^{k-1}}_{j=k-1} - \sum_{j=0}^{k-2} a^{k-j-1} b^{j+1} \\
&= a^k - b^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \sum_{l=1}^{k-1} a^{k-l} b^l \\
&= a^k - b^k
\end{aligned}$$

□

Theorem 20. Let $y_2 > y_1$ then

$$y_2^k - y_1^k = \underbrace{(y_2 - y_1)}_{>0} \underbrace{\left(\sum_{j=0}^{k-1} y_2^{k-j-1} y_1^j\right)}_{>0} \implies y_2^k - y_1^k > 0$$

Proof.

$$\forall x \geq 0 \in \mathbb{R} : \exists y \geq 0 \in \mathbb{R} : y^k = x \text{ with } k \in \mathbb{N}_+$$

Special case: $x = 0$ and $y = 0$ is the solution.

Let $x > 0$: We construct y with $y \in \bigcap_{k=0}^{\infty} I_n$ where I_n are nested intervals. Specifically I_n must have the properties:

- $I_n = [a_1, b_n]$ with $a^k \leq x, b_n^k \geq x \quad \forall n \in \mathbb{N}$
- $I_{n+1} \subseteq I_n : |I_n| = \frac{1}{2}|I_{n+1}| = \left(\frac{1}{2}\right)^n |I_0|$

$$n = 0 \qquad I_0 = [0, x-1]$$
$$a_0 = b \qquad b_0 = x + 1$$
$$a_0^k = 0 < x \qquad \checkmark$$

$$b_0^k = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots + x^k > 1 + kx > x \text{ for } k \geq 1$$

Let I_n be given: $I_n = [a_n, b_n]$. Define $m_n = \frac{1}{2}(a_n + b_n)$

Case 1

$$m_n^k \geq x \implies \text{let } a_{n+1} = a_n, b_{n+1} = m$$

$$I_{n+1} = [a_n, m_n] \subseteq [a_n, b_n] = I_n$$

$$|I_{n+1}| = m_n - a_n = \frac{1}{2}a_n + \frac{1}{2}b_n - a_n$$

$$\frac{1}{2}(b_n - a_n) = \frac{1}{2}|I_n|$$

$$a_{n+1}^k = a^k \leq x \quad \checkmark$$

All conditions are satisfied.

Case 2 $m_n^k < x$: Let $a_{n+1} = m_1, b_{n+1} = b_n$. It holds that $a_{n+1} = m_n < x, b_{n+1} = b_n \geq x \quad \checkmark$. Furthermore it holds that $I_{n+1} \subseteq I$ and $|I_{n+1}| = \frac{1}{2}|I_n|$.

I_n is set of nested intervals. Let $\varepsilon > 0$ be arbitrary. Then

$$\exists N \in \mathbb{N} : n \geq N \implies \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{1+x}$$

For those $n \geq N$ it holds that

$$|I_n| = \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^n (x+1) < \frac{\varepsilon}{1+x} \cdot (1+x)$$

Let $y \in I_n \forall n \in \mathbb{N}$. Further nesting of intervals:

$$(I_n)_{n\in\mathbb{N}} \text{ with } I_n = [a_n^k, b_n^k]$$

It holds that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \text{ because } I_{n+1} \subseteq I_n \implies a_n^k \leq a_{n+1}^k < b_{n+1}^k \leq b_n^k$$

Length of I_n:

$$I_n = b_n^k - a_n^k = (b_n - a_n)\sum_{j=0}^{k-1} a_n^{k-1-j} b_n^j$$

Because $I_n \leq I_0 \implies a_n < b_0 \implies b_n \leq b_0$,

$$< (b_n - b_0)\sum_{j=0}^{k-1} b_0^{k-1-j} b_0^j$$

$$= (b_n - a_n)kb_0^k = (b_n - a_n)k(1+x)^k$$

Let $\varepsilon > 0$ be arbitrary. Find some $N \in \mathbb{N}$ with $n \geq N$:

$$|I_n| = (b_n - a_n) < \frac{\varepsilon}{k(1+x)^k}$$

For those n it holds that

$$|I_n| < |I_n| \cdot k(1-x)^k < \frac{\varepsilon}{k(1+x)^k} k(1+x)^k = \varepsilon$$

Therefore $(I_n)_{n\in\mathbb{N}}$ a set of nested intervals.

$\exists z \in \mathbb{R}$ with $z \in \lfloor a_n^k, b_n^k \rfloor : \forall n \in \mathbb{N}$ and z is unique. By construction of I_n it holds that $a_n^k \leq x \leq b_n^k$

$$\implies x \in I_n \forall n \in \mathbb{N} \implies x = z \in \bigcap_{n\in\mathbb{N}} I_n.$$

On the opposite side it holds that $y \in I_n$ (hence $a_n \leq y \leq b_n \implies a_n^k \leq y^k \leq b_n^k$). So $y^k \in I_n \forall n \in \mathbb{N} \implies y^k = z = x$. So we have found some y^k which is x. But is $y \geq 0$ with $y^k = x$ unique?

Let $y_1 \neq y_2$ with $y_1^k = y_2^k = x$ and without loss of generality,

$$0 \leq y_1 < y_2 \implies y_1^k < y_2^k \quad \grave{}$$

So, y is unique.

$\square$

7 Supremum property of $\mathbb{R}$

7.1 Boundedness in $\mathbb{R}$

Definition 20. Let $A \subseteq \mathbb{R}$.

- We call A to be *bounded above* if there exists some $u \in \mathbb{R}$ such that $\forall a \in A : a \leq u$.
- A number u with that property is called *upper bound of* A.
- We call A to be *bounded below* if there exists some $l \in \mathbb{R}$ such that $\forall a \in A : a \geq l$.
- A number l with that property is called *lower bound of* A.
- A is called *bounded* if there exists a lower and upper bound of A.

Corollary 6. Let (a, b) be bounded. Let u be its upper bound and let $v \geq u$. Then v is also an upper bound of (a, b).

↓ This lecture took place on 11th of November 2015 with lecturer Wolfgang Ring

7.2 Supremum and infimum in $\mathbb{R}$

Definition 21. Let A be bounded above. Assume $s \in \mathbb{R}$ has the properties

1. s is an upper bound for A
2. $\forall \sigma \in \mathbb{R} : \sigma < s$: σ is not an upper bound for A.

If those properties are satisfied, we call s *supremum of* A. A supremum s is always the smallest upper bound of A. We denote $s = \sup A$.

There exists at most one supremum for A. Let s_1 and s_2 be two suprema, then $s_1 \neq s_2$. So wlog. $\sigma_1 < \sigma_2$. This invalidates the supremum property of $s_2 \implies s_1$ is not a supremum of A ↯.

Analogously an *infimum of* A is the greatest lower bound of A. Let A be bounded below. $t \in \mathbb{R}$ is called *infimum of* A if

1. $\forall a \in A : t \leq a$ (t is a lower bound of A)
2. $\forall x > t$ so x is no lower bound of A

$$\iff \exists a \in A : a < x$$

We denote $t = \inf A$.

Definition 22. Let $A \subseteq \mathbb{R}$. We denote $u = \max A$ for the *maximum of* A if

1. $u \in A$ (is element of A)
2. $\forall a \in A : a \leq u$ (is an upper bound)

$l \in \mathbb{R}$ denoted $l = \min A$ is called minimum of A if

1. $l \in A$ (is element of A)
2. $\forall a \in A : l \leq a$ (l is a lower bound)

Theorem 21. Let $A \subseteq R$ and u be the maximum of A. Then it holds that $u = \sup A$. If $l = \min A \implies l = \inf A$.

Proof. We need to show, that l is an upper bound of A. This follows by definition. For $x < u$ it holds that x not an upper bound.

Let $x < u$, because $u \in A$ there exists some element y in A with $y > x$. Therefore x is not an upper bound of A. □

Example 6.

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}_+\right\}$$

Then it holds that $1 \in A$ and $1 \geq \frac{1}{n} \forall n \in \mathbb{N}_+$. Therefore $1 = \max A = \sup A$.

$0 = \inf A$, because 0 is a lower bound of A ($\frac{1}{n} > 0 \forall n \in \mathbb{N}_+$). Let $\varepsilon > 0$, then $\exists N \in \mathbb{N} : n \geq N \implies \frac{1}{n} \leq \varepsilon$. Therefore ε is not a lower bound of A.

So A does not have a minimum, because otherwise $l = \max A = \inf A = 0$.

Theorem 22. Let $A \neq \emptyset$ and $A \subseteq \mathbb{R}$ be bounded above. So some $s = \sup A \in \mathbb{R}$ exists (therefore $\mathbb{R}$ has a supremum property).

Proof. We construct nested intervals $(I_n)_{n\in\mathbb{N}}$ such that for $s \in \bigcap_{n\in\mathbb{N}} I_n$ we have $s = \sup A$. We construct I_{n+1} inductively using I_n.

Case $\mathbf{n = 0}$

Because $A \neq \emptyset$, we select $a_0 \in A$. Because A is bounded above, $\exists b_0 \in \mathbb{R}$ such that b_0 is an upper bound of A. We define $I_0 = [a_0, b_0]$.

Case $\mathbf{n \to n+1}$

Let $a_0 = b_0$, then it holds that b_0 is upper bound and $b_0 \in A$. We call that termination condition. Therefore $b_0 = \max A = \sup A$ and the supremum was found. Instead of n we use $n+1$. Let $I_0 = [a_n, b_n]$ with $a_n \neq b_n$ and $a_n \in A$, b_n is an upper bound of A. Furthermore it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider I_{n+1} such that the same properties are satisfied. Let $m_1 = \frac{1}{2}(a_1 + b_1)$. It holds that $a_n < m_n < b_n$.

Case $\mathbf{m_n}$ is an upper bound of A Then we set $a_{n+1} = a_n \in A$ and $b_{n+1} = m_n$ is an upper bound of A.

$$|I_{n+1}| = b_{n+1} - a_{n+1} = \frac{1}{2}(b_n + a_n) - a_n$$

$$= \frac{1}{2}b_n - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \frac{1}{2}\left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^{n+1} |I_0| \qquad \checkmark$$

Case $\mathbf{m_n}$ is not an upper bound of A Therefore $\exists x \in A$ with $x > m_n$.

Subcase $\mathbf{x = b_1}$ So b_1 is an upper bound. Therefore $x \in A$ and x is upper bound.

$$x = \max A = \sup A$$

We found the supremum.

Subcase $\mathbf{m_n < x < b_n}$ Let $a_{n+1} = x \in A$ and $b_{n+1} = b_n$ is an upper bound and

$$|I_{n+1}| = b_{n+1} - a_{n+1} = b_n - x < b_n - m_n = b_n - \frac{1}{2}a_n - \frac{1}{2}b_n$$

$$= \frac{1}{2}b_n - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

We have found supremum $s = \sup A$.

If in any case the termination condition holds, then we have found the supremum.

Now we show that we can create arbitrary small intervals with I_n. So we know, $\forall n \in \mathbb{N} : a_n < b_n, a_n \in A, b_n$ is upper bound of A.

$$|I_n| = b_n - a_n \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider $\varepsilon > 0$ and N such that $n \geq N \implies \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{|I_0|}$. For those n it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| < \frac{\varepsilon}{|I_0|} |I_0| = \varepsilon$$

Therefore $(I_n)_{n\in\mathbb{N}}$ are nested intervals.

We know: $s \in \mathbb{R}$ and $\forall n \in \mathbb{N} : s \in I_n$. It remains to show that $s = \sup A$.

$\downarrow$ This lecture took place on 12th of November 2015 with lecturer Wolfgang Ring

By completeness of $\mathbb{R}$:

$$\exists s \in \mathbb{R} : s \in I_n \forall n \in \mathbb{N}$$

We show: Every set with an upper bound has a supremum.

We construct $(I_n)_{n\in\mathbb{N}}$ with $I_n = [a_n, b_n]$ and $I_{n+1} \subseteq I_n$. $\forall n \in \mathbb{N} : a_n \in A$, b_n is the upper bound of A.

$$|I_{n+1}| \leq \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

Consider $I_{n+1} \subseteq I_n$ with $a_n < b_n \forall n \in \mathbb{N}$.

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

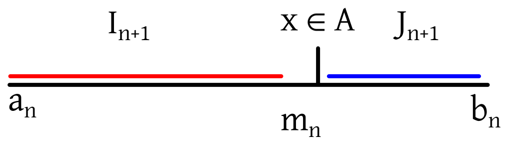

Figure 5: Possible relationship between a_n, b_n and J_{n+1} with $J_{n+1} \coloneqq I_n \setminus I_{n+1}$

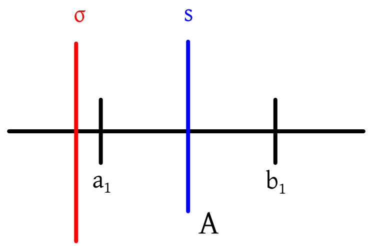

Figure 6: Illustration of s between a_n and b_n

1. Claim: s is $\sup A$.

   Proof by contradiction. s is upper bound of A. Assume $a \in A$ and $a > s$. Let $\varepsilon = a - s > 0$ and choose N sufficiently large such that

   $$|I_n| < \varepsilon = a - s$$

   Then it holds that

   $$b_N = \underbrace{b_n - a_n}_{\varepsilon} \not+ \underbrace{a_N}_{<s} < s + \varepsilon = a$$

   $$\implies b_N < a \in A \qquad ⚡$$

   Because b_n is an upper bound. □

2. $\forall \sigma < s$ it holds that σ is not an upper bound of A. Let $\sigma < s$ and $\varepsilon = s - \sigma > 0$ and choose $n \in N$ large enough such that $b_N - a_N < \varepsilon$. Then it holds that

   $$\begin{aligned} a_N &= a_N - b_N + b_N \\ &> -\varepsilon + s \\ &= -s + \sigma + s = \sigma \qquad \checkmark \end{aligned}$$

   Therefore it holds that s is smallest upper bound of A and therefore supremum.

□

Theorem 23. Every set with a lower bound in $\mathbb{R}$ has an infimum. Every set with an upper bound in $\mathbb{R}$ has an supremum.

Theorem 24. Remember that M has the same cardinality like A if $\varphi : M \to A$. φ is bijective, M is called countably infinite if M has the same cardinality like $\mathbb{N}$.

Let $\varphi : \mathbb{N} \to M$ be bijective therefore $M = \{\varphi(1), \varphi(2), \varphi(3), \dots\} = \{\varphi(n) \,|\, n \in \mathbb{N}\}$ and $\varphi(i) \neq \varphi(j)$ for $i \neq j$.

Notation. $\varphi(n) = m_n$.

$M = \{m_0, m_1, m_2, \dots\}$ with $m_i \neq m_j$ for $i \neq j$. φ is a complete enumeration of all elements of M.

Therefore every element of M has the structure: m_n with $i \in \mathbb{N}$.

Theorem 25.

$$\mathbb{Q}^+ = \left\{\frac{m}{n}, m \in \mathbb{N}, n \in \mathbb{N}_+\right\}$$

The set $\mathbb{Q}^+$ is countably infinite.

Proof. We enumerate the elements of $\mathbb{Q}^+$.

$$\begin{aligned} \mathbb{Q}_+ &= \{q_0, q_1, q_2, \dots\} \\ \mathbb{Q}_- &= \{-q_0, -q_1, -q_2, \dots\} \\ \mathbb{Q} &= \{0, q_0, -q_0, q_1, -q_1, \dots\} \end{aligned}$$

An enumeration exists. So $\mathbb{Q}$ is countably infinite. □

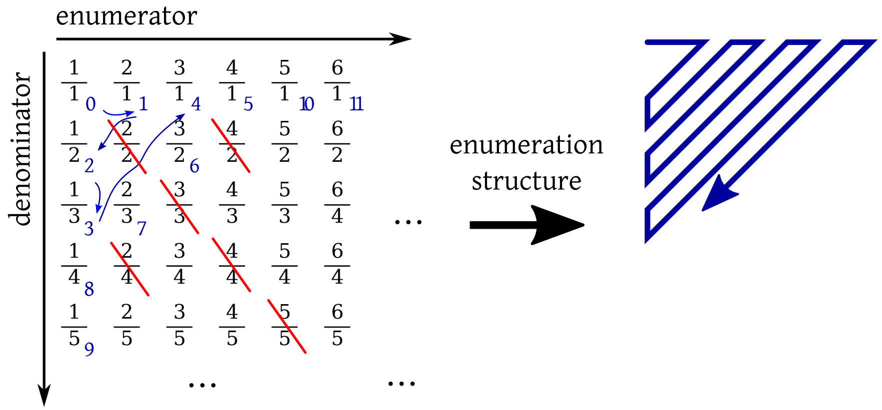

Figure 7: A complete enumeration of $\mathbb{Q}^+$ (diagonalization argument). We traverse the whole matrix diagonally. The blue numbers indicate the enumeration and red lines cross out values already enumerated. On the right-hand side the general order of the enumeration is illustrated.

Theorem 26. There is no bijective relation $\varphi : \mathbb{N} \to \mathbb{R}$. Therefore we call $\mathbb{R}$ *uncountable.*

Proof. We provide a proof by contradiction. Assume $\mathbb{R} = \{x_0, x_1, x_2, x_3, \dots\}$ is countable.

We construct nested intervals.

Case n = 0

$$I_0 = [x_0 + 1, x_0 + 2]$$

Let $|I_0| = 1$ and $x_0 \notin I_0$.

Case n → n + 1 Assume $I_0 \dots I_n$ were already defined with $x_k \notin I_k$ for $0 \leq k \leq n$.

$$I_{k+1} \subseteq I_k \text{ for } k = 0, \dots, n-1$$

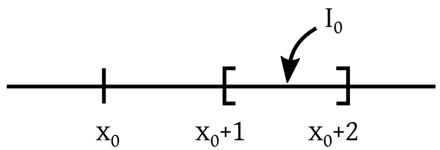

Figure 8: Construction of a nested interval and its I_0

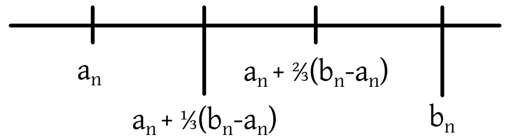

Figure 9: Construction of a nested interval and its I_n

$$|I_k| = \left(\frac{1}{3}\right)^k$$

We construct I_{n+1}. Let $I_n = [a_n, b_n]$.

$$I_n^1 = \left[a_n, \frac{2}{3}a_n + \frac{1}{3}b_n\right]$$
$$I_n^2 = \left[\frac{2}{3}a_n + \frac{1}{3}b_n, \frac{1}{3}a_n + \frac{2}{3}b_n\right]$$
$$I_n^3 = \left[\frac{1}{3}a_n + \frac{2}{3}b_n, b_n\right]$$

So x_n certainly is not contained in all three intervals I_n^1, I_n^2 and I_n^3 because

$I_n^1 \cap I_N^2 \cap I_N^3 = \emptyset$. Choose I_{n+1} as one of the three intervals I_n^l with $x_{n+1} \notin I_n^l = I_{n+1}$. $I_{n+1} < I_n$.

$$|I_{n+1}| = \frac{1}{3} I_n = \left(\frac{1}{3}\right)^{n+1}$$

For $\varepsilon > 0$ it holds that there exists some $N \in \mathbb{N}$ such that $n \geq N \implies |I_n| = \left(\frac{1}{3}\right)^n < \varepsilon$. Therefore nested intervals I_n are given.

Let $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : x \in I_n$ (because of completeness law). Then it holds that $\forall x_n : x \neq x_n$. $x \in I_n$ and $x_n \notin I_n$. Therefore $x \in \{x_0, x_1, x_2, \dots\} = \mathbb{R}$.

This contradicts with the assumption that $\mathbb{R}$ is countable.

□

8 Complex numbers $\mathbb{C}$

We introduce a new arithmetic unit denoted i, which extends the field $\mathbb{R}$. Elements of $\mathbb{C}$ are represented as $a + bi$ with $a, b \in \mathbb{R}$.

$$\forall a, b \in \mathbb{R} : a + bi = 0 \iff a = 0 \land b = 0 \tag{28}$$
$$i^2 = -1 \tag{29}$$
$$\text{associativity, commutativity, etc holds} \tag{30}$$

↓ This lecture took place on 13th of November 2015 with lecturer Wolfgang Ring

Definition 23. We consider an "arithmetic element" i extending $\mathbb{R}$ ("conjugate", dt. "adjungiert"). Arithmetic operations are well-defined for i. Associativity and commutativity holds. It holds that

- $a + ib = 0$ with $a, b \in \mathbb{R} \iff a = 0 \land b = 0$
- $i^2 = -1$ i.e. $i^2 + 1 = 0$.
- Arithmetic operations still hold.

By the first law,

$$a + ib = a' + ib' \iff (a - a') + i(b - b') = 0$$
$$\iff a - a' = 0 \land b - b' = 0 \implies a = a' \land b = b'$$

By the second law, i is the solution of the quadratic equation $i^2 + 1 = 0$.

Let $z = a + ib$ a complex number. We call i the "imaginary unit".

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$

$\mathbb{C}$ is the field of complex numbers with the following properties:

- For addition, it holds that
$$(a + ib) + (c + id) = (a + b) + i(b + d) \subseteq \mathbb{C}$$
and
$$(a + ib) + (-a - ib) = (a - a) + i(b - b) = 0 + i \cdot 0 = 0$$
- For multiplication, it holds that
$$(a + ib) \cdot (c + id) = (ac + \underbrace{(i)^2}_{=-1} bd) + i(bc + ad)$$
$$(ac - bd) + i(bc + ad)$$
- Laws $\mathbf{A_n}$ to $\mathbf{A_4}$, $\mathbf{M_1}$ to $\mathbf{M_3}$ and $\mathbf{D}$ hold.
- The one element exists:
$$1 = 1 + 0 \cdot i$$
$$(a + i \cdot b)(1 + i \cdot 0) = (a + (i)^2 \cdot 0) + i(b + 0) = a + ib$$
- **M4** holds: Let $z \in \mathbb{C} \setminus \{0\}$. Let $z = a + ib$ and $\neg(a = 0 \land b = 0) \iff a^2 + b^2 > 0$.

We define
$$w = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}$$

$$
\begin{aligned}
z \cdot w &= (a+ib)\left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right) \\
&= \left(\underbrace{\frac{a^2}{a^2+b^2} - \frac{b\cdot(-b)}{a^2+b^2}}_{=1}\right) + i\cdot\left(\underbrace{\frac{ba}{a^2+b^2} - \frac{a\cdot b}{a^2+b^2}}_{=0}\right) \\
&= 1 + i\cdot 0 = 1
\end{aligned}
$$

Therefore $w = z^{-1} = \frac{1}{z}$.

Therefore $\mathbb{C}$ is a field.

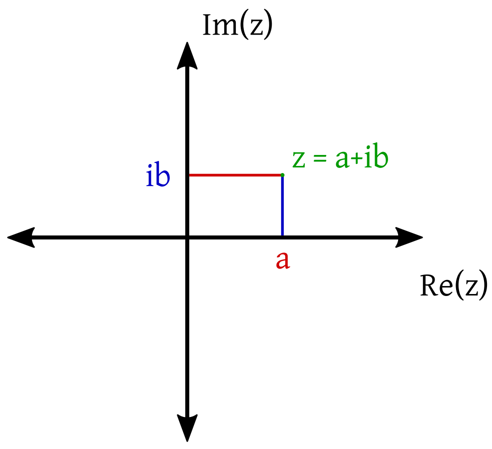

Figure 10: Illustration of complex numbers

Let $z = a + ib$. We denote

$$a =: \Re(z) \qquad b =: \Im(z) \qquad \overline{z} := a - ib \qquad |z| := \sqrt{a^2+b^2}$$

a is called *real part of* z. b is called *imaginary part of* z. z is called complex conjugate. $|z|$ is called absolute value of z.

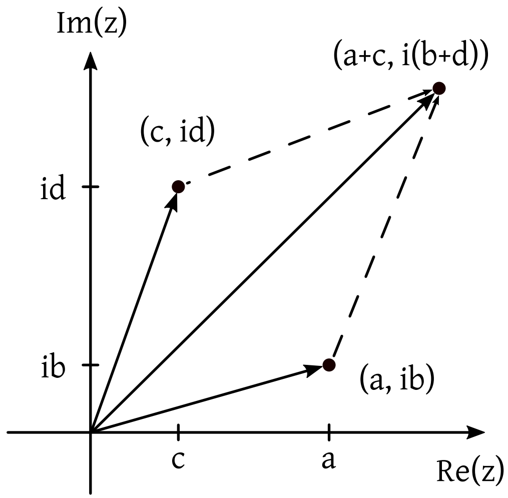

Figure 11: Illustration of complex number addition

Theorem 27. $\overline{(\overline{z})} = z$

Proof.

$$\overline{(\overline{z})} = \overline{(a-ib)} = (a-(-ib)) = a+ib = z$$

□

Theorem 28. $\Re(z) = \frac{1}{2}(z+\overline{z})$

Proof.

$$\frac{1}{2}(z+\overline{z}) = \frac{1}{2}(a+ib+a-ib) = \frac{1}{2}(2a) = a$$

□

Theorem 29. $\Im(z) = \frac{1}{2i}(z-\overline{z})$

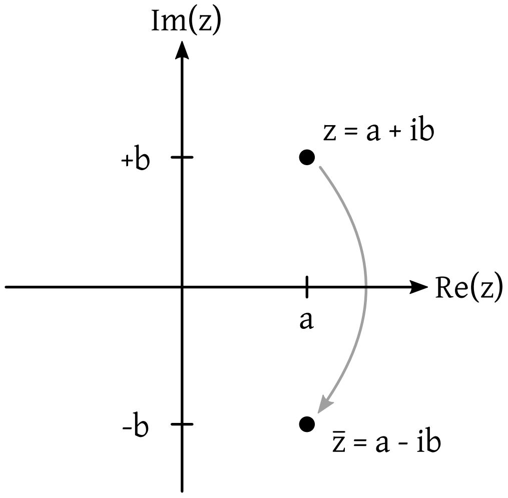

Figure 12: Illustration of the complex conjugate

Proof.
$$\frac{1}{2i}(a+ib-(a-ib)) = \frac{1}{2i}(2ib) = b$$
□

Theorem 30. $z \in \mathbb{R} \iff z = \overline{z}$

Proof.
$$z = a \in \mathbb{R} \implies \overline{z} = a = z$$
On the opposite, let $z = \overline{z}$ therefore
$$a = ib = a - ib \implies 2ib = 0 \implies b = 0$$
Therefore $z = a \in \mathbb{R}$. □

Theorem 31.
$$z \in (i\mathbb{R}) = \{ib : b \in \mathbb{R}\} \iff z = -\overline{z}$$
Proof follows analogously.

Theorem 32. It holds that $|z| = \sqrt{z \cdot \overline{z}}$.

Proof.
$$\begin{aligned}\sqrt{z \cdot \overline{z}} &= ((a+ib)(a-ib))^{\frac{1}{2}} \\ &= (a^2-(ib)^2)^{\frac{1}{2}} = (a^2-i^2b^2)^{\frac{1}{2}} \\ &= (a^2+b^2)^{\frac{1}{2}} = |z| \quad \checkmark\end{aligned}$$
□

Theorem 33. Let $z, w \in \mathbb{C}$. Then $\overline{(zw)} = \overline{z} \cdot \overline{w}$

Proof.
$$z = a+ib \qquad w = c+id$$
$$\begin{aligned}zw &= (ac-bd)+i(bc+ad) \\ \overline{zw} &= (ac-bd)-i(bc+ad) \\ \overline{zw} &= a-ib \qquad \overline{w} = c-id\end{aligned}$$
$$\overline{z} \cdot \overline{w} = (ac-(-b)(-d)) + i(-bc+a(-d)) = (ac-bd)-i(bc+ad)$$
□

Corollary 7.
$$\overline{z+w} = \overline{z} + \overline{w}$$

Theorem 34. $|zw| = |z| \cdot |w|$

Proof.
$$|z \cdot w| = (zw) \cdot (\overline{z \cdot w})^{\frac{1}{2}}$$
$$= (z \cdot \overline{z} \cdot w \cdot \overline{w})^{\frac{1}{2}} = (z \cdot \overline{z})^{\frac{1}{2}} \cdot (w \cdot \overline{w})^{\frac{1}{2}} = |z| \cdot |w|$$
□

Theorem 35.
$$z = 0 \iff |z| = 0 \in \mathbb{R}$$

Proof.

$$z = 0 = 0 + i0 \implies |z| = \sqrt{0^2 + 0^2} = 0$$

Let $|z| = \sqrt{a^2 + b^2} = 0 \implies a^2 + b^2 = 0$.

$$\implies a = 0 \wedge b = 0$$

□

Theorem 36.

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

$$|\Im(z)| = |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| =$$

Theorem 37. The triangle inequality holds:

$$\forall z, w \in \mathbb{C} : |z + w| \leq |z| + |w|$$

Remark 6. Let $0 \leq y_1 < y_2$ with $y_1, y_2 \in \mathbb{R}$. Let $k \in \mathbb{N}_+$. Then it holds that

$$\sqrt[k]{y_1} < \sqrt[k]{y_2}$$

Proof. Indirect proof: Let $\sqrt[k]{y_1} \geq \sqrt[k]{y_2} \geq 0$.

$$\implies \left(\sqrt[k]{y_1}\right)^k \geq \left(\sqrt[k]{y_2}\right)^k$$

therefore $y_1 \geq y_2$. This is the negation of our assumption. □

Proof of the triangle inequality. We show that $|z + w|^2 \leq (|z| + |w|)^2$.

$$\begin{aligned}|z + w|^2 &= (z + w)(\overline{z} + \overline{w}) \\ &= \underbrace{z\overline{z}}_{|z|^2} + w\overline{z} + z\overline{w} + \underbrace{w\overline{w}}_{|w|^2}\end{aligned}$$

$$2\Re(w\overline{z}) = (w\overline{z} + \underbrace{\overline{(w \cdot \overline{z})}}_{\overline{w} \cdot \overline{\overline{z}} = \overline{w} \cdot z})$$

$$\begin{aligned}|z + w|^2 &= |z|^2 + 2\Re(w \cdot \overline{z}) + |w|^2 \\ &\leq |z|^2 + 2\,|\Re(w \cdot \overline{z})| + |w|^2 \\ &\leq |z|^2 + 2 \cdot |w \cdot \overline{z}| + |w|^2 \\ &= |z|^2 + 2 \cdot |w| \cdot |\overline{z}| + |z|^2 \\ &= |z|^2 + 2 \cdot |w| \cdot |z| + |w|^2 \\ &= (|z| + |w|)^2\end{aligned}$$

□

Theorem 38. In our previous proof there was a small loop hole: We need to show that

$$|z| = |\overline{z}|$$

Proof.

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

□

8.1 Interpretation of multiplication

Multiplication with i. Let $z = a + ib$.

$$iz = i \cdot a + i^2 \cdot b = (-b) + ia$$

Multiplication with i rotates z counter-clockwise by 90° in the plane.

Let $z \in \mathbb{C}$ and $w = c + id$.

↓ This lecture took place on 18th of November 2015 with lecturer Wolfgang Ring

8.2 Taking roots

$$\forall a \in \mathbb{R} : a \geq 0 \forall n \in \mathbb{N}_+ : \exists x \geq 0 \in \mathbb{R} : x^n = a$$

Taking the n-th root only works for positive integers, because $\forall x \geq 0 : x^2 \geq 0$ and no solution in $\mathbb{R}$ exists for the equation $x^2 = -1$.

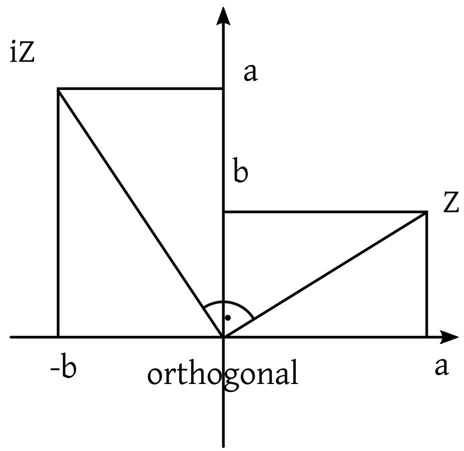

Figure 13: Multiplication corresponds to a rotation by 90°

In $\mathbb{C}$ it holds that $\forall w \in \mathbb{C} \setminus \{0\}$. $\forall n \in \mathbb{N}$ there exist exactly n different solutions of the equation $z^n = w$.

9 Sequences of real and complex elements

Definition 24. Let a be a mapping $\mathbb{N} \to \mathbb{R}$ is called *sequence* of real numbers.

$$\forall n \in \mathbb{N} : a(n) \in \mathbb{R}$$

We denote $a_n := a(n)$. Instead of $a : \mathbb{N} \to \mathbb{C}$ we write $(a_n)_{n\in\mathbb{N}} = (a_0, a_1, \ldots)$.

Analogously for the complex numbers $\mathbb{C}$ and general sets X.

Example 7. $a_n = \sqrt[n]{2}\frac{1}{n+1}$ with $(a_n)_{n\in\mathbb{N}}$. Or simply:

$$\left(\sqrt[n]{2}\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$

Example 8. Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals. Therefore $(I_n)_{n\in\mathbb{N}}$ is a sequence of elements in $X = \{[a, b] : a, b \in \mathbb{R}, a \le b\}$.

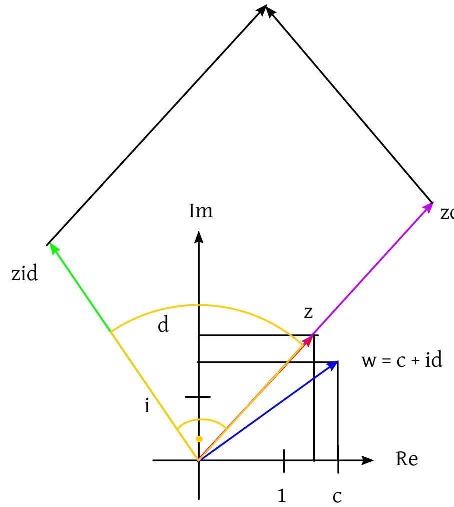

Figure 14: In regards of multiplication with w the complex number z is scaled by $|w|$ and then rotated by an angle which is given between w and the positive real axis.

Definition 25. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. $(a_n)_{n\in\mathbb{N}}$ is called *bounded above* if $o \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \le o$. $(a_n)_{n\in\mathbb{N}}$ is called *bounded below* if $u \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \ge u$.

$(a_n)_{n\in\mathbb{N}}$ is called bounded, if $(a_n)_{n\in\mathbb{N}}$ is bounded above and below.

Example 9. $(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{n}{n+1}$ is bounded below by 0 and bounded above by 1: $n \le n+1 \implies n\frac{1}{n+1} \le \frac{n+1}{n+1} = 1 \checkmark$.

9.1 Monotonicity

Definition 26.

- $(a_n)_{n\in\mathbb{N}}$ is called *monotonically increasing* if $\forall n \in \mathbb{N} : a_{n+1} \ge a_n$.

- $(a_n)_{n\in\mathbb{N}}$ is called *monotonically decreasing* if $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$.
- $(a_n)_{n\in\mathbb{N}}$ is called *monotonically strictly increasing* if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.
- $(a_n)_{n\in\mathbb{N}}$ is called *monotonically strictly decreasing* if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.

In $\mathbb{C}$, elements are not ordered, so we need to define an order explicitly. Let $(a_n)_{n\in\mathbb{N}}$ a complex sequence. We define:

- $(a_n)_{n\in\mathbb{N}}$ is called bounded if $(|a_n|)_{n\in\mathbb{N}}$ is a bounded real sequence. Hence $\exists o \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| \leq o$.
- The lower bound is implicitly given by 0.

Example 10. $a_n := i^n$ and $(a_n)_{n\in\mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, i, -1, \dots)$

$$|1| = 1 \qquad |-1| = 1 \qquad |i| = \sqrt{0^2 + 1^2} = 1 \qquad |-i| = \sqrt{0^2 + (-1)^2} = 1$$

So $(|a_n|)_{n\in\mathbb{N}} = (1, 1, 1, 1, 1, \dots)$. It holds that

$$|z| = |-z| = |\overline{z}|$$

Definition 27. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{C}$ and let $a \in \mathbb{C}$. We state: $(a_n)_{n\in\mathbb{N}}$ has a limit (lat. limes) a if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$$

We denote

$$\lim_{n\to\infty} a_n = a$$

The distance $|a_n - a|$ becomes arbitrary small, if n is sufficiently large.

A sequence, which has a limit, is called *convergent*. A sequence, which does not have a limit, is called *divergent*.

Remark 7. Sometimes we consider mappings $a : \mathbb{N}_+ \to \mathbb{C}$, which we also call sequences:

$$a \leftrightarrow (a_1, a_2, \dots)$$

Example 11.

$$a_n = \frac{1}{n}$$

We know:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies \frac{1}{n} < \varepsilon$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Let $q \in \mathbb{C}$, $|q| < 1$.

We know $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |q^n - 0| < \varepsilon$.

$$\lim_{n\to\infty} q^n = 0$$

↓ This lecture took place on 20th of November 2015 with lecturer Wolfgang Ring

Remark 8. Consider $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$ as a circle with radius ε. So if n is sufficiently large, all new sequence elements are located inside the circle.

Lemma 3. A sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$ can have at most one limit.

Proof. Assume a and b are limes of $(a_n)_{n\in\mathbb{N}}$. Then we prove:

$$\forall \varepsilon > 0 : |a - b| < \varepsilon$$

$$\implies a = b$$

Let $\varepsilon > 0$ arbitrary: Because $a = \lim_{n\to\infty} a_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2}\right]$$

Because $b = \lim_{n\to\infty} b_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \implies |b_n - b| < \frac{\varepsilon}{2}\right]$$

Let $N = \max(N_1, N_2)$, hence $N \geq N_1 \land N \geq N_2$.

$$\implies |a_N - a| < \frac{\varepsilon}{2} \land |a_N - b| < \frac{\varepsilon}{2}$$

$$|a-b| = |a \underbrace{-a_N + a_N}_{0} - b| \le \underbrace{|a-a_N|}_{<\frac{\varepsilon}{2}} + \underbrace{|a_N - b|}_{<\frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem 39 (Well-known convergent sequences.)**.**

1. Let $s = \frac{p}{q} \in \mathbb{Q}_+$ and $n \in \mathbb{N}_+$. Consider $\left(\frac{1}{n^s}\right)_{n\in\mathbb{N}}$.
$$n^s = n^{\frac{p}{q}} := \sqrt[q]{n^p}$$
It holds that
$$\lim_{n\to\infty} \frac{1}{n^s} = 0$$
2. Let $q \in \mathbb{C}, |q| < 1$. Then it holds that
$$\lim_{n\to\infty} q^n = 0$$
3. Let $a \in \mathbb{R}, a > 0, n \in \mathbb{N}_+$. Then it holds that
$$\lim_{n\to\infty} \sqrt[n]{a} = 1$$
4. It holds that $(n \in \mathbb{N}_+)$
$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$
5. Let $z \in \mathbb{C} : |z| > 1$. Let $k \in \mathbb{N}$. Then it holds that
$$\lim_{n\to\infty} \frac{n^k}{z^n} = 0$$

Remark 9 (Remark to sequence 5)**.** $|z^n|$ grows faster then n^k.

Proof of sequence 1. Let $0 \le x_1 < x_2$.

$$\implies 0 \le x_1^p < x_2^p \implies \sqrt[q]{x_1^p} < \sqrt[q]{x_2^p}$$

Therefore $f(x) = x^s$ is strictly monotonically increasing for $x \in (0,\infty)$. Let $\varepsilon > 0$ arbitrary and $N > \frac{1}{\varepsilon^{\frac{1}{s}}} = \varepsilon^{\frac{1}{s}} = \varepsilon^{-\frac{q}{p}}$. Then it holds that $n \ge N$:

$$\left|\frac{1}{n^s} - 0\right| = \frac{1}{n^s} \le \frac{1}{N^s}$$

$$\frac{1}{n^s} < \frac{1}{N^s} \implies n^s \ge N^s$$

$$\frac{1}{n^s} \le \frac{1}{N^s} < \frac{1}{\left(\frac{1}{\varepsilon^{\frac{1}{s}}}\right)^s} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

□

Proof of sequence 2. Already done. □

Proof of sequence 3. **Case $\mathbf{a > 1}$** Let $a > 1$. Consider $\varepsilon > 0$. Show that $|\sqrt[n]{a} - 1| < \varepsilon$ for sufficiently large n.

$$x_n = \sqrt[n]{a} - 1 = \left|\sqrt[n]{a} - 1\right|$$

$$a > 1 \implies \sqrt[n]{a} > \sqrt[n]{1} = 1 \implies \sqrt[n]{a} - 1 > 0$$

It holds that $x_n + 1 = \sqrt[n]{q}$, i.e. $(x_n+1)^n = a$.

$$a = (\underbrace{x_1}_{>0} + 1)^n \underbrace{>}_{\text{Bernoulli}} 1 + n \cdot x_n$$

$$\implies x_n < \frac{a-1}{n}$$

$$N > \frac{a-1}{\varepsilon} \xrightarrow{\text{for } x \ge N} \left|\sqrt[n]{a} - 1\right| = x_n$$

$$< \frac{a-1}{n} \le \frac{a-1}{N} < \frac{a-1}{\frac{a-1}{2}} = \varepsilon$$

Case $\mathbf{a = 1}$

$$\sqrt[n]{a} = \sqrt[n]{1} = 1$$

$$\left(\sqrt[n]{a}\right)_{n\in\mathbb{N}} = (1,1,1,1,\dots)$$

has the limit 1.

Case $0 < a < 1$ Let $0 < a < 1 \implies 0 < \sqrt[n]{q} < \sqrt[n]{1} = 1$.

$$x_n = 1 - \sqrt[n]{a} > 0$$

Show that $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies x_n < \varepsilon]$.

$$x_n = 1 - \sqrt[n]{a} = \sqrt[n]{a}\left(\frac{1}{\sqrt[n]{a}} - 1\right) = \sqrt[n]{a}\left(\sqrt[n]{\frac{1}{a}} - 1\right) < \left(\sqrt[n]{a'} - 1\right)$$

with $a' = \frac{1}{a} > 1$. From case $a > 1$ we already know

$$\exists N \in \mathbb{N} : \left[n \geq N \implies \left|\sqrt[n]{a'} - 1\right| = \sqrt[n]{a'} - 1 < \varepsilon\right]$$

$$\implies x_n < \varepsilon$$

□

Proof of sequence 4. This proof works similar to the proof of sequence 3.

$$x_n = \sqrt[n]{n} - 1 > 0 \text{ for } n \geq 2$$

Therefore $|x_n| = x_n$. Let $\varepsilon > 0$ be arbitrary.

$$x_n + 1 = \sqrt[n]{n} \quad \text{i.e.} \quad (x_n + 1)^n = n$$

$$n = (1 + x_n)^n = 1 + \underbrace{nx_n}_{>0} + \underbrace{\binom{n}{2}x_n^2}_{>0} + \underbrace{\binom{n}{3}x_n^3}_{>0} + \underbrace{\cdots + x_n^n}_{>0} > 1 + \binom{n}{2}x_n^2$$

All expressions we remove are positive (but we don't remove *all* positive expressions).

$$x_n^2 < \frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2 \cdot 1}} = \frac{2}{n}$$

$$x_n < \sqrt{\frac{2}{n}}$$

Choose $N > \frac{2}{\varepsilon^2}$. Then it holds for $n \geq N$ that

$$x_n < \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} < \sqrt{\frac{2}{\frac{2}{\varepsilon^2}}} = \varepsilon$$

Thus $\sqrt{\frac{2}{n}} < \varepsilon$, hence $\frac{2}{n} < \varepsilon^2$, hence $n > \frac{2}{\varepsilon^2}$. □

Proof of sequence 5.

$$x := |z| - 1 \land |z| > 1 \implies |z| = 1 + x \land x > 0$$

We show that for $\varepsilon > 0$ arbitrary, there exists $N \in \mathbb{N}$:

$$n \geq N \implies \left|\frac{n^k}{z^n} - 0\right| = \left|\frac{n^k}{z^n}\right| = \frac{n^k}{|z|^n} < \varepsilon$$

Let $\varepsilon > 0$ be given. For $n > 2k$ it holds that $n - k > n - \frac{n}{2} = \frac{n}{2}$.

$$|z|^n = (1 + x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j > \underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1}$$

$$n > 2k \geq k + 1$$

$$\underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1} = \frac{\overbrace{n}^{>\frac{n}{2}}\overbrace{(n-1)}^{>\frac{n}{2}}\overbrace{(n-2)}^{>\frac{n}{2}}\dots\overbrace{(n-k)}^{>\frac{n}{2}}}{(k+1)!}x^{k+1} > \frac{\frac{n^{k+1}}{2^{n+1}}}{(k+1)!}x^{n+1}$$

Therefore $|z|^n > \frac{n^{k+1}}{2^{k+1}(k+1)!}x^{k+1}$. So,

$$\frac{n^k}{|z|^n} < \frac{n^k \cdot 2^{k+1}(k+1)!}{n^{k+1} \cdot x^{k+1}} = \underbrace{\frac{2^{k+1}(k+1)!}{x^{n+1}}}_{\substack{= \text{constant} \land >0 \\ =: M}} \cdot \frac{1}{n} = M \cdot \frac{1}{n}$$

$$\frac{n^k}{|z|^n} < M \cdot \frac{1}{n} \text{ for } n > 2k$$

Consider N such that $N > \frac{M}{\varepsilon}$ and $N > 2k$. Then it holds that

$$\frac{n^k}{|z|^n} < M\frac{1}{n} \leq \frac{M}{N} < \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$$

□

Lemma 4. Every convergent sequence is bounded (in $\mathbb{C}$).

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. This means especially e.g. $\varepsilon = 13$.

$$\exists N \in \mathbb{N} \text{ s.t. } [n \geq N \implies |a_n - a| < 13]$$

Consider $O > 0$ such that

$$O = \max\{|a_0|, |a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 13\}$$

So $O \geq |a_n|$ for $n \in \{0, \ldots, N\}$. Then for $0 \leq n < N$ it holds that $|a_n| < O$. ✓

For $n \geq N$ it holds that

$$|a_n| = |a_n - a + a| \leq \underbrace{|a_n - a|}_{<13} + |a| < \underbrace{13 + |a|}_{\leq O}$$

Therefore $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and followingly $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$. □

Theorem 40. Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then the following laws hold:

1. $\lim_{n\to\infty}(a_n + b_n)$ is convergent with limes $a + b$
2. $\lim_{n\to\infty}(a_n \cdot b_n)$ is convergent with limes $a \cdot b$
3. $\lim_{n\to\infty} \frac{a_n}{b_n}$ is convergent with limes $\frac{a}{b}$ if $\forall n \in \mathbb{N} : b_n \neq 0 \land b \neq 0$.

Proof. 1. Let $\varepsilon > 0$ arbitrary. Because $(a_n)_{n\in\mathbb{N}}$ is convergent,

$$\exists N_1 : \left[n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2}\right]$$

(b_n) is convergent hence

$$\exists N_2 : \left[n \geq N_2 \implies |b_n - b| < \frac{\varepsilon}{2}\right]$$

$N = \max\{N_1, N_2\}$, hence for $n \geq N$ both statements above hold. Let $n \geq N$, then the triangle inequality holds:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

2. $(a_n)_{n\in\mathbb{N}}$ is convergent and therefore also bounded. Therefore,

$$\exists m \geq 0 : \forall n \in \mathbb{N} : |a_n| \leq m$$

$(b_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_1 : n \geq N_1 : |b_n - b| < \frac{\varepsilon}{2} \cdot \frac{1}{m+1}$$

$(a_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_2 \leq N : n \geq N_2 \implies |a_n - a| < \frac{\varepsilon}{2}\frac{1}{|b|+1}$$

$N = \max\{N_1, N_2\}$. For $n \geq N$ both relations above hold. Let $n \geq N$:

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$$

$$\leq |a_n(b_n - b)| + |b(a_n - a)| = |a_n| |b_n - b| + |b| |a_n - a|$$

$$\leq m\frac{\varepsilon}{2}\frac{1}{m+1} + |b| \frac{\varepsilon}{2}\frac{1}{|b|+1} < \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} \cdot 1 = \varepsilon$$

3. Left for the practicals.

□

9.2 Laws for convergent complex sequences

Theorem 41. Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a, $(a_n \to a)$. Then it holds that

- $(\Re(a_n))_{n\in\mathbb{N}}$ is convergent.
$$\lim_{n\to\infty} (\Re(a_n)) = \Re(a)$$
- $(\Im(a_n))_{n\in\mathbb{N}}$ is convergent.
$$\lim_{n\to\infty} (\Im(a_n)) = \Im(a)$$

- $(|a_n|)_{n\in\mathbb{N}}$ is a convergent real sequence.

$$\lim_{n\to\infty} |a_n| = |a|$$

- $(\overline{a_n})_{n\in\mathbb{N}}$ is convergent with

$$\lim_{n\to\infty} \overline{a_n} = \overline{a}$$

On the opposite, let $(a_n)_{n\in\mathbb{N}}$ with $a_n = \alpha_n + i\beta_n$ a sequence of complex numbers. Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be convergent with limes α i.e. β. Then $(a_n)_{n\in\mathbb{N}}$ is a convergent complex sequence with limes $a = \alpha + \beta i$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \implies |a_n - a| < \varepsilon$.

$$\underbrace{|a_n - a|}_{(\alpha_n-\alpha)+(\beta_n-\beta)i} = \sqrt{(\alpha_n-\alpha)^2 + (\beta_n-\beta)^2} \geq \begin{cases} \sqrt{(\alpha_n-\alpha)^2} = |\alpha_n - \alpha| \\ \sqrt{(\beta_n-\beta)^2} = |\beta_n - \beta| \end{cases}$$

hence $(\alpha_n) = (\Re(a_n))_{n\in\mathbb{N}}$ is convergent, $(\beta_n) = (\Im(a_n))_{n\in\mathbb{N}}$ is convergent.

Let $\varepsilon > 0$. Choose N such that $n \geq N \implies |a_n - a| < \varepsilon$.

$$||a_n| - |a|| \underbrace{\leq}_{\text{inverse triangular inequality}} |a_n - a| < \varepsilon \text{ for } n \geq N$$

Now we need to show $\alpha_n \to \alpha$ and $\beta_n \to \beta$

$$\implies a_n \to a$$

Let $\varepsilon > 0$ be arbitrary. Because $(\alpha_n)_{n\in\mathbb{N}}$ be convergent, there exists $N_1 \in \mathbb{N}$:

$$n \geq N_1 \implies |\alpha_1 - \alpha| < \frac{\varepsilon}{\sqrt{2}}$$

$(\beta_n)_{n\in\mathbb{N}}$ is convergent. So,

$$\exists N_2 \in \mathbb{N} : n \geq N_2$$

$$|\beta_n - \beta| < \frac{\varepsilon}{\sqrt{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$ both relations hold.

Let $n \geq N$:

$$|a_n - a| = |(\alpha_n - \alpha) + i(\beta_n - \beta)|$$

$$= \sqrt{(\alpha_n-\alpha)^2 + (\beta_n-\beta)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon$$

Let $a_n = \alpha_n + i\beta_n$ is convergent with limes $\alpha + i\beta$ which is a.

$$\implies \lim_{n\to\infty} \alpha_n = \alpha \wedge \lim_{n\to\infty} \beta_n = \beta$$

$$\implies \lim_{n\to\infty} (-\beta_n) = -\beta \qquad \text{“multiplication rule”}$$

$$\implies (\overline{a_n})_{n\in\mathbb{N}} = (\underbrace{\alpha_n}_{\text{convergent}} - \underbrace{i\beta_n}_{\text{convergent}})_{n\in\mathbb{N}}$$

$$\implies \lim_{n\to\infty} \overline{a_n} = \alpha - i\beta = \overline{a}$$

□

9.3 Further laws for sequences

Theorem 42. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$ with limes a and b respectively and it must hold that $\forall n \in \mathbb{N} : a_n \leq b_n$. Then also $a \leq b$.

Proof. Consider $a - b = \varepsilon > 0$.

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \implies |b_n - b| < \frac{\varepsilon}{2}$$

For $N = \max\{N_1, N_2\}$:

$$b_N = b_N - b + b \leq b + |b_N - b| < b\frac{\varepsilon}{2} = b + \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$a_N = \underbrace{a_N - a}_{\geq -|a_n - a|} + a \geq a - |a_n - a| > a - \frac{\varepsilon}{2} = a - \frac{a-b}{2} = \frac{1}{2}(a+b)$$

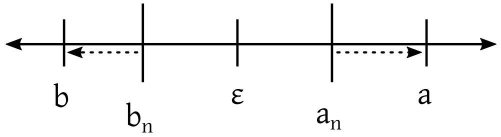

Figure 15: One possible relationship of the sequences a_n, b_n and limes a, b and ε

$$b_N < \frac{1}{2}(a+b) < d_N$$

Attention:

$$a_n < b_n \;\not\Longrightarrow\; a < b$$

Example: $a_n = 0$, $b_n = \frac{1}{n}$. $\square$

9.4 Convergence criteria

Are there criteria such that if the sequences have a specific structure, they are obviously convergent?

9.4.1 Squeeze theorem

Theorem 43. Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be convergent real sequences with $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = A$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $M \in \mathbb{N}$ such that

$$\forall n \geq M : A_n \leq a_n \leq B_n$$

Then $(a_n)_{n\in\mathbb{N}}$ is also convergent and $\lim a_n = A$.

Proof. Let $\varepsilon > 0$ be arbitrary. Consider N sufficiently large such that,

- $N \geq M$
- $n \geq N \implies |A_n - A| < \varepsilon$
- $n \geq N \implies |B_n - A| < \varepsilon$

Then for $n \geq N$:

$$\left.\begin{array}{l} A - a_n \leq A - A_n \leq |A - A_n| < \varepsilon \\ a_n - A \leq B_n - A \leq |B_n - A| < \varepsilon \end{array}\right\} = 1$$

$$\implies |a_n - A| < \varepsilon \qquad \iff \lim_{n\to\infty} a_n = A$$

$\square$

Example 12. Let $s \in \mathbb{Q}_+$. Show that,

$$\lim_{n\to\infty} \left(\sqrt[n]{n^s}\right) = 1$$

We apply the squeeze theorem:

$$n^2 \geq 1 \forall n \in \mathbb{N}$$

$$\implies \sqrt[n]{n^s} \geq 1$$

Let $k \in \mathbb{N}_+$. Then it holds that

$$\lim_{n\to\infty} \sqrt[n]{n^k} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}\,\sqrt[n]{n}\ldots\sqrt[n]{n}}_{k \text{ times}}$$

$$= 1 \cdot 1 \cdot 1 \cdots = 1$$

For the last two lines we actually need to read them from right to left.

Let $s = \frac{p}{q}$.

$$\implies n^s = n^{\frac{p}{q}} \leq q \cdot \left(n\frac{p}{q}\right)^q = n^p$$

$$q \geq 1 \implies \sqrt[n]{n^s} \leq \underbrace{\sqrt[n]{n^p}}_{\text{convergent with limes 1}} \qquad p \in \mathbb{N}$$

Then it holds that $\lim_{n\to\infty} \sqrt[n]{n^s} = 1$ with the squeezing theorem.

Remark 10. Let $A \subseteq \mathbb{R}$ be bounded above. Then it holds that

$$s = \sup A \iff s \text{ is upper bound of } A \land \forall \varepsilon > 0 \exists a \in A : a > s - \varepsilon$$

Proof. Implication from left to right: Let $s = \sup A$. Then it holds that s is upper bound of A and $s - \varepsilon < s$ is not an upper bound. Therefore $\exists a \in A : a > s - \varepsilon$.

Implication from right to left: Consider that both statements on the RHS hold. So s is an upper bound. We need to show that any t is not an upper bound with $t > s$. Let $t < s, s - t = \varepsilon > 0$. Therefore $t = s - \varepsilon$. Because of the right statement $\exists a \in A : a > s - \varepsilon = t$ therefore t is not an upper bound. □

Remark 11. Analogously:

$$\sigma = \inf A \iff \sigma \text{ is lower bound} \land \forall \varepsilon > 0 \exists a \in A : a < \sigma + \varepsilon$$

Theorem 44. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded monotonical sequence. Then $(a_n)_{n\in\mathbb{N}}$ has a limes a with

- $a = \sup\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing.
- $a = \inf\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be monotonically increasing. Let $a = \sup\{a_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ be arbitrary. Because a is a supremum, there exists $a_N \in \{a_n : n \in \mathbb{N}\}$ such that $a_N > a - \varepsilon$.

$$\implies \underbrace{a - a_N}_{\geq 0} < \varepsilon$$

because a is an upper bound. Therefore

$$|a - a_N| < \varepsilon$$

Let $n \geq N$ then it holds that

$$|a - a_n| \underbrace{=}_{a \text{ is upper bound}} a - a_n \leq a - a_N$$

because $a_N \leq a_n$ is increasing:

$$a - a_N < \varepsilon$$

Therefore $\lim_{n\to\infty} a_n = a$. □

↓ This lecture took place on 25th of November 2015 with lecturer Wolfgang Ring

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If $(a_n)_{n\in\mathbb{N}}$ is bounded and monotonous. Then $(a_n)_{n\in\mathbb{N}} \in \mathbb{N}$ is convergent.

Example: Wallis product John Wallis (1616–1703)

$$p_n = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \ldots \ldots \cdot (2n-1)} = \prod_{k=1}^{n} \frac{2k}{2k-1}$$

Consider

$$\alpha_n = \frac{p_n}{\sqrt{n}} \qquad \beta_n = \frac{p_n}{\sqrt{n+1}}$$

We need to show that

- (α_n) is monotonously decreasing
- (β_n) is monotonously increasing

$$\forall n \in \mathbb{N} : n \geq 1 : \alpha_n > \beta_n$$

Both are convergent.

1. Show that,

$$\alpha_{n+1} < \alpha_n \iff \frac{\alpha_{n+1}}{\alpha_n} < 1 \iff \frac{(\alpha_{n+1})^2}{(\alpha_n)^2} < 1$$

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2 = \left(\frac{\frac{2\cdot 4\cdot 6\cdot\ldots\cdot(2n+2)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\cdot(2n+1)}}{\frac{2\cdot 4\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)}} \cdot \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}}\right)^2$$

$$= \frac{(2n+2)^2 \cdot n}{(2n+1)^2(n+1)} = \frac{4n^3 + 8n^2 + 4n}{(4n^2 + 4n + 1) \cdot (n+1)} = \frac{4n^3 + 8n^2 + 4n}{4n^3 + 8n^2 + 5n + 1} < 1$$

2. We show,

$$\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 = \frac{(2n+2)^2 \cdot (n+1)}{(2n+1)^2 \cdot (n+2)} = \frac{(4n^2+8n+4)(n+1)}{(4n^2+2n+1)(n+2)}$$
$$= \frac{4n^3+12n^2+12n+4}{4n^3+12n^2+9n+2} > 1$$
$$\implies \beta_{n+1} > \beta_n \implies \beta_n \text{ is monotonically increasing}$$

Let $p = \lim_{n\to\infty} a_n$ and $p' = \lim_{n\to\infty} b_n$.

$$\beta_n = \frac{p_n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \alpha_n \cdot \sqrt{\frac{n}{n+1}}$$
$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha_n \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \alpha_n \cdot \underbrace{\lim_{n\to\infty} \sqrt{\frac{n}{n+1}}}_{=1}$$
$$\implies \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} a_n \implies p = p'$$

It holds that $p = \lim_{n\to\infty} \frac{p_n}{\sqrt{n}} = \sqrt{n}$.

9.5 On accumulation points and subsequences

Definition 28. Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence. The complex value a is called *limit point* (dt. "Häufungspunkt") of $(a)_{n\in\mathbb{N}}$ if $\forall\varepsilon > 0 : |a_n - a| < \varepsilon$ for infinitely many indices $n \in \mathbb{N}$. Hence infinitely many values of the sequence lie within a circle with center a and radius ε.

Remark 12. Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limit a. Then it holds that a is the only limit point of the sequence $(a_n)_{n\in\mathbb{N}}$.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. Let

$$\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |a_n - a|$$

Therefore $\forall n \in \{N, N+1, N+2, \dots\}$ it holds that $|a_n - a| < \varepsilon$. Assume $a' \in \mathbb{C}$ is another limit point with $a \neq a'$. Let

$$\varepsilon = \frac{|a - a'|}{2} > 0$$

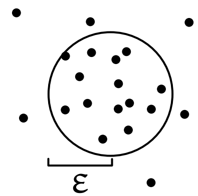

Figure 16: Illustration of a limit point in the Euclidean plane. The point is represented as circle with radius ε. Finitely many points lie outside the limit point; infinitely many inside.

Let $N \in \mathbb{N}$ such that $\forall n \geq N : |a_n - a| < \varepsilon$.

$$\implies n \in \mathbb{N} : |a' - a_n| = |a' - a + a - a_n| = |a' - a - (a_n - a)| \geq |a' - a| - |a_n - a|$$
$$= 2\varepsilon - |a_n - a| > 2\varepsilon - \varepsilon = \varepsilon$$

At most for $n \in \{1, \dots, N-1\}$ it is possible that $|a_n - a'| < \varepsilon$. $\square$

Remark 13. $a_n = (-1)^n$ has the limit points $+1$ and -1.

The lecture on 26th of November 2015 got cancelled.

↓ This lecture took place on 27th of November 2015 with lecturer Wolfgang Ring

Definition 29. Let $a \in \mathbb{C}$ and $r > 0$ and

$$B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

and we call $B(a, r)$ an *open* circle with center a and radius r. So the circle itself is not part of the set, unlike the following set:

$$B'(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}$$

Let a be a limit point of $(a_n)_{n\in\mathbb{N}} \iff \forall\varepsilon > 0$. $B(a, \varepsilon)$ contains infinitely many sequence values.

Example 13.

$$a_n = \frac{1}{2}\left[1 + (-1)^n \left(\frac{1-n}{n}\right)\right] \qquad n \geq 1$$

$$\implies a_1 = \frac{1}{2} \quad a_2 = \frac{1}{4} \quad a_3 = \frac{5}{6}$$

$$a_4 = \frac{1}{8} \quad a_5 = \frac{9}{10} \quad a_6 = \frac{1}{12} \quad a_7 = \frac{13}{14}$$

"$\frac{5}{6}$? Ah, passt ma eh bessa." (Wolfgang Ring)

Estimated limit points: $a = 0, b = 1$.

Proof. Let $\varepsilon > 0$ and $a = 0$. We consider sequence values of even index. So for indices it holds that $n = 2k$.

$$\begin{aligned}
|a_{2k} - 0| &= \left|\frac{1}{2}(1 + \underbrace{(-1)^{2k}}_{+1}\left(\frac{1-2k}{2k}\right))\right| \\
&= \frac{1}{2}\left|1 + \frac{1-2k}{2k}\right| \\
&= \frac{1}{2}\left|\frac{2k+1-2k}{2k}\right| \\
&= \frac{1}{4k} < \varepsilon \text{ if } \quad \underbrace{k > \frac{1}{4\varepsilon}}_{\substack{\text{infinitely many ks} \\ \text{satisfy the relation}}}
\end{aligned}$$

Let $\varepsilon > 0$ and $b = 1$. We consider sequence values of structure $n = 2k + 1$.

$$\begin{aligned}
|a_{2k+1} - 1| &= \left|\frac{1}{2}\left[1 + \underbrace{(-1)^{2k+1}}_{=-1}\left[\frac{1-(2k+1)}{2k+1}\right]\right] - 1\right| \\
&= \left|\frac{1}{2}\left[1 - \frac{-2k}{2k+1}\right] - 1\right| = \left|\frac{1}{2}\frac{2k+1+2k}{2k+1} - 1\right| \\
&= \left|\frac{4k+1}{4k+2} - 1\right| = \left|\frac{4k+1-4k-2}{4k+2}\right| \\
&= \frac{1}{4k+2} < \varepsilon
\end{aligned}$$

$$\text{if } 4k + 2 > \frac{1}{\varepsilon} \implies \underbrace{k}_{\substack{\text{infinitely many} \\ \text{indices}}} > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$$

□

Example 14. $(c_n)_{n\in\mathbb{N}}$ is defined with $c_n = i^n$.

$$(c_n)_{n\in\mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1 \dots)$$

What are its accumulation points?

Definition 30. Let $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$. For example,

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots\right)$$

We remove some elements

$$\left(1, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \dots\right)$$

A *subsequence* is created. We also reenumerate the numbers:

$$\left(\underbrace{1}_{n_0}, \underbrace{\frac{1}{3}}_{n_1}, \underbrace{\frac{1}{4}}_{n_2}, \underbrace{\frac{1}{6}}_{n_3}, \dots\right)$$

Let $n : \mathbb{N} \to \mathbb{N}$ be strictly monotonically increasing. Therefore

$$\forall k \in \mathbb{N} : n(k+1) > n(k) \implies n_{k+1} > n_k$$

We call $(n_k)_{n\in\mathbb{N}}$ an *index subsequence* and $(a_{n_k})_{k\in\mathbb{N}}$ is called subsequence of $(a_n)_{n\in\mathbb{N}}$.

Lemma 5. Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a and $(a_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(a_n)_{n\in\mathbb{N}}$. Then also the subsequence is convergent and has the same limes a.

Proof. For every subsequence index n_k with $k \in \mathbb{N}$ it holds that $n_k \geq k$.

Proof by induction: $k = 0$

$\mathbf{n_0} \in \mathbb{N}$

$$n_0 \geq 0 = k \qquad \checkmark$$

$\mathbf{n_k \geq k}$ Because $\underbrace{n_{k+1}}_{\in\mathbb{N}} > n_k$ (strictly monotonic). Therefore,

$$n_{k+1} \geq n_k + 1 > k + 1$$

Proof of limes: $\lim_{k\to\infty} a_{n_k} = a$. Let $\varepsilon > 0$. Because $(a_n)_{n\in\mathbb{N}}$ is convergent, it holds that $\exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \varepsilon$. Let $k \geq N$. This holds because $n_k \geq k \geq N : |a_{n_k} - a| < \varepsilon$. Therefore $(a_{n_k})_{k\in\mathbb{N}}$ has limes a. $\square$

Lemma 6. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. Then it holds that $a \in \mathbb{C}$ is accumulation point if and only if there exists some subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a$.

Proof. We first prove direction $\Longleftarrow$.

Assume $(a_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ with limes a. Let $\varepsilon > 0$.

$$\exists N \in \mathbb{N} : k \geq N \implies |a_{n_k} - a| < \varepsilon$$

Therefore $B(a, \varepsilon)$ has infinitely many sequence elements of $(a_{n_k})_{k\in\mathbb{N}}$ and therefore also infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$.

We prove direction $\implies$.

We build a convergent subsequence. Consider $k \in \mathbb{N}$ with $k \geq 1$.

$$\varepsilon_k \coloneqq \frac{1}{k}$$

We define $n_0 = 0$ and $a_{n_0} = a_0$. Assume $a_{n_0}, a_{n_1}, \ldots, a_{n_{k-1}}$ are already defined.

By definition of a_{n_k}, there are infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$ in $B(a, \varepsilon_k)$. We consider $n_k > n_{k-1}$ and $a_{n_k} \in B(a, \varepsilon_k)$.

Then it holds that $\lim_{k\to\infty} a_{n_k} = a$. Let $\varepsilon > 0$ be arbitrary. Consider $K > \frac{1}{\varepsilon}$. Hence $\varepsilon > \frac{1}{K} = \varepsilon_K$ for all $k \geq K$ it holds that $n_k \geq n_K$ and $|a_{n_k} - a| < \varepsilon_k = \frac{1}{k} \leq \frac{1}{K} < \varepsilon$. $\square$

9.6 Bolzano-Weierstrass theorem

$\boxed{\text{Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897)}}$

Theorem 45. Every bounded sequence of real numbers has an accumulation point in $\mathbb{R}$.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$, hence $\exists M > 0$ such that all sequence elements a_n in $I_0 = [-M, M]$ and let $F_0 = \{n \in \mathbb{N} \mid a_n \in I_0\} = \mathbb{N}$ (index set). F_0 is infinite. We build nested intervals with the properties:

- $I_{n+1} \subseteq I_n$
- $|I_{n+1}| = \frac{1}{2} |I_n|$
- $F_n = \{k \in \mathbb{N} \mid a_k \in I_n\}$ is infinite.

This construction is inductive:

induction base $I_0 \quad \checkmark$

induction step Let $I_n = [A_n, B_n]$ be given and $M_n = \frac{1}{2}(A_n + B_n)$. Let $J_n = [A_n, M_n]$ and $L_n = [M_n, B_n]$. It holds that $J_n \subseteq I_n \land L_n \subseteq I_n$ and $|\,J_n\,| = \frac{1}{2} |I_n| \land |L_n| = \frac{1}{2} |I_n|$. Because there are infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$ in I_n and $I_n = J_n \cup L_n$, in at least one subinterval there have to be infinitely many sequence elements.

Therefore select $I_{n+1} = J_n$ if J_n contains infinitely many sequence elements and consider $I_{n+1} = L_n$ if J_n contains only finitely many sequence elements. Therefore I_{n+1} contains infinitely many sequence elements.

$$F_{n+1} = \{k \in \mathbb{N} \mid a_k \in I_{n+1}\}$$

is infinite. So $(I_n)_{n\in\mathbb{N}}$ is a nested interval.

Let $a \in \bigcap_{n\in\mathbb{N}} I_n$ (completeness of $\mathbb{R}$).

Claim: a is limit point of $(a_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be given and n sufficiently large, such that $|I_n| = B_n - A_n < \varepsilon$. Then it holds that for every $x \in I_n$ that $|x - a| \leq B_n - A_n < \varepsilon$ (with $x \in I_n, a \in I_n$). Because I_n contains infinitely many sequence elements of $(a_n)_{n\in\mathbb{N}}$, it holds that infinitely many sequence elements a_k satisfy the relation $|a_n - a| < \varepsilon$. Therefore a is limit point of $(a_n)_{n\in\mathbb{N}}$. $\square$

Corollary 8 (typical definition of the Bolzano-Weierstrass theorem)**.** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem 46 (Bolzano-Weierstrass theorem in $\mathbb{C}$)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. Then $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence and therefore also at least one limit point in $\mathbb{C}$.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be bounded. $a_n = \alpha_n + i\beta_n$. So $(\alpha_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ as well as $(\beta_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$.

Consider a convergent subsequence of $(\alpha_n)_{n\in\mathbb{N}}$, $(\alpha_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\alpha_{n_k} = \alpha$. Now consider bounded $(\beta_{n_k})_{k\in\mathbb{N}}$. From the Bolzano-Weierstrass theorem it follows that there exists a convergent subsequence $\left(\beta_{n_{k_l}}\right)_{l\in\mathbb{N}}$ with $\beta = \lim_{l\to\infty}\beta_{n_{k_l}}$.

$\left(\alpha_{n_{k_l}}\right)_{l\in\mathbb{N}}$ is subsequence of $(\alpha_{n_k})_{k\in\mathbb{N}}$ convergent with limit point α.

Let $a_{n_{k_l}} = \alpha_{n_{k_l}} + i\beta_{n_{k_l}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Real and imaginary parts are convergent, therefore $\lim_{l\to\infty} a_{n_{k_l}} = a = \alpha + i\beta$. Therefore $(a_n)_{n\in\mathbb{N}}$ contains a convergent subsequence. $\square$

↓ This lecture took place on 2nd of December 2015 with lecturer Wolfgang Ring

Theorem 47 (Weierstrass-Bolzano theorem)**.** Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.

Theorem 48 (Convergence)**.** Let $(x_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{C}$ with limes x.

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : |x_n - x| < \varepsilon$$

Definition 31 (Metric space)**.** Let X be a set. We call $d : X \times X \to \mathbb{R}$ a *distance function* (or *metric*) on X if,

- $\forall x \in X : d(x,x) = 0$
- $\forall x, y \in X : d(x,y) = d(y,x)$ (symmetry)
- $\forall x, y, z \in X : d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

(X,d) is called *metric space.*

Example 15. $X = \mathbb{C}$, $d(x,y) = |x-y|$.

Definition 32 (Convergence with metric spaces)**.** Let X be a metric space. $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements in X. Let $x \in X$. We call $(x_n)_{n\in\mathbb{N}}$ convergent with limes x if

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : d(x_n, x) < \varepsilon$$

Definition 33. Let $K \subseteq X$ be a subset of the metrical space X. We call K *pre-compact* if every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in K$ has a convergent subsequence. K is called compact if the limes a of the convergent subsequence is also in K.

Definition 34. In $\mathbb{C}$ it holds that every bounded set is pre-compact.

9.7 Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$

Augustin-Louis Cauchy (1789–1857)

Definition 35. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We call $(a_n)_{n\in\mathbb{N}}$ a *Cauchy sequence* (fundamental sequence) if

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \land m \geq N \implies |a_n - a_m| < \varepsilon$$

Definition 36 (Cauchy sequence in a metric space)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. We call $(a_n)_{n\in\mathbb{N}}$ a *Cauchy sequence* (fundamental sequence) if

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \land m \geq N \implies d(a_n, a_m) < \varepsilon$$

Lemma 7. Every convergent sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is a Cauchy sequence.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a. Let $\varepsilon > 0$ be arbitrary.

Convergence implies that $\exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \frac{\varepsilon}{2}$. For $m, n \geq N$ it holds that

$$|a_n - a_m| = |a_n - a + a - a_m| \leq \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2}\text{ because }n\geq N} + \underbrace{|a - a_m|}_{<\frac{\varepsilon}{2}\text{ because }m\geq N}$$

$\square$

Lemma 8. Every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is bounded.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$. The Cauchy condition for $\varepsilon = 1$ states:

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |a_n - a_m| < 1$$

specifically $m = N : \forall n \geq N$

$$|a_n - a_N| < 1$$

Therefore $|a_n| = |a_n - a_N + a_N| \leq \underbrace{|a_n - a_N|}_{<1} + |A_N| < |a_N| + 1$.

Let $m = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|\}$ and $M = \max\{m, |A_N| + 1\}$.

Then for $n \leq N - 1$ it holds that

$$|a_n| \leq m \leq M$$

and for $n \geq N$ it holds that

$$|a_n| \leq |a_N| + 1 \leq M$$

Therefore $\forall n \in \mathbb{N} : |a_n| \leq M$. Therefore $(a_n)_{n\in\mathbb{N}}$ is bounded. □

9.8 Is $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ complete?

Theorem 49 (Cauchy sequences and limes)**.** Every Cauchy sequence in $\mathbb{C}$ has a limes and is therefore convergent. Followingly we call $\mathbb{C}$ to be *complete*.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$. We know that $(a_n)_{n\in\mathbb{N}}$ is bounded. From the Bolzano-Weierstrass theorem it follows that a limit point a of $(a_n)_{n\in\mathbb{N}}$ exists. Let $\varepsilon > 0$ be arbitrary.

1. We choose $N \leq \mathbb{N}$ sufficiently large such that
$$n, m \geq N \implies |a_n - a_m| < \frac{\varepsilon}{2}$$
2. Because $B(a, \frac{\varepsilon}{2})$ contains infinitely many sequence elements (a is limit point), $K \geq N$ exists with $|a - a_K| < \frac{\varepsilon}{2}$.

Let $n \geq N$. Then

$$|a_n - a| = |a_n - a_K + a_K - a| \leq \underbrace{|a_n - a_K|}_{<\frac{\varepsilon}{2}\text{ (Cauchy seq.)}} + \underbrace{|a_K - a|}_{<\frac{\varepsilon}{2}\text{ (limit point } a)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is convergent with limes a.

We have proven that if $(a_n)_{n\in\mathbb{N}}$ has a limit point, this limit point is also its limes.

We concluded: nested intervals $\implies$ compactness / Bolzano-Weierstrass theorem $\implies$ completeness.

Actually nested intervals are equivalent to completeness. □

↓ This lecture took place on 3rd of December 2015 with lecturer Wolfgang Ring

Corollary 9. $\mathbb{C}$ is complete.

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$.

$$z_n = a_n + ib_n$$

Then $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$.

Show that this property: Let $\varepsilon > 0$. Because $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it holds that

$$\exists N \in \mathbb{N} : n, m \geq N \implies |z_n - z_m| < \varepsilon$$

Because $|a_n - a_m| \leq |z_n - z_m|$ and $|b_n - b_m| \leq |z_n - z_m|$ hold, it follows that for $n, m \geq N : |a_n - a_m| < \varepsilon \land |b_n - b_m| < \varepsilon$. Therefore $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences.

Because $\mathbb{R}$ is complete, it follows that $\exists a \in \mathbb{R}$ such that

$$a = \lim_{n\to\infty} a_n \text{ and } \exists b \in \mathbb{R}$$

with $b = \lim_{n\to\infty} b_n$. Because $\lim_{n\to\infty} z_n = z = a + ib$,

$$\iff a = \lim_{n\to\infty} a_n \land b = \lim_{n\to\infty} b_n$$

□

Example 16. We show a counterexample for the completeness of $\mathbb{Q}$. So we have Cauchy sequences with limes, which lie outside $\mathbb{Q}$.

We define a recursion:

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) & \text{if } n > 0 \end{cases}$$

We observe, $\forall n \in \mathbb{N} : a_n > 0 \land a_n \in \mathbb{Q}$.

Proof by complete induction:

Induction base: $n = 0$

$$a_0 = 2 > 0 \land 2 \in \mathbb{Q} \quad \checkmark$$

Induction step: $n \to n+1$ Let $a_n > 0$ and $a_n \in \mathbb{Q}$.

$$a_{n+1} = \frac{1}{2}\left(\underbrace{a_n}_{>0} + \underbrace{\frac{2}{a_n}}_{>0}\right) > 0$$

and $a_{n+1} \in \mathbb{Q}$.

We prove by induction: $\forall n \in \mathbb{N} : a_n^2 > 2$.

Induction base: $n = 0$

$$a_0 = 2 \qquad a_0^2 = 4 > 2 \quad \checkmark$$

Induction step: $n \to n+1$ It holds that $a_n^2 - 2 > 0$.

$$a_{n+1}^2 - 2 = \frac{1}{4}\left(a_n^2 + 4 + \frac{4}{a_n^2}\right) - 2 = \frac{1}{4a_n^2}\left(a_n^4 + 4a_n^2 + 4 - 8a_n^2\right)$$

$$= \frac{1}{4a_n^2}\left(a_n^4 - 4a_n^2 + 4\right) = \frac{1}{4a_n^2}\underbrace{\left(a_n^2 - 2\right)^2}_{>0} > 0$$

Furthermore it holds that $a_{n+1} < a_n$.

$$2a_{n+1} = a_n + \frac{2}{a_n} \implies 2(a_{n+1} - a_n) = -a_n + \frac{2}{a_n} = \frac{\overbrace{2 - a_n^2}^{<0}}{a_n} < 0$$

$$\implies a_{n+1} - a_n < 0 \implies a_{n+1} < a_n$$

Therefore the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly monotonically decreasing and is bound by below. Therefore some $a \in \mathbb{R}$ exists with $a = \lim_{n\to\infty} a_n$.

Monotonicity really depends on the completeness of $\mathbb{R}$. We cannot argue equivalently to Theorem 44 with the supremum.

For this example we know that $(a_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$. $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. So $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$.

For the limes a it holds that,

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

$$= \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{2}\frac{2}{\lim_{n\to\infty} a_n} = \frac{1}{2}a + \frac{1}{a}$$

$$a = \frac{1}{2}a + \frac{1}{a} \implies \frac{1}{2}a = \frac{1}{a}$$

$$a^2 = 2 \implies a = +\sqrt{2} \notin \mathbb{Q}$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is *not* convergent in $\mathbb{Q}$. We found a convergent Cauchy sequence whose limes is not in $\mathbb{Q}$ which immediately means that $\mathbb{Q}$ is incomplete.

Definition 37 (Tending towards infinity)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- We state $(a_n)_{n\in\mathbb{N}}$ *tends to infinity* with limes $+\infty$:

$$\lim_{n\to\infty} a_n = +\infty$$

$$\text{if } \forall M > 0 \exists N \in \mathbb{N} : n \geq N \implies a_n > M$$

- We state $(a_n)_{n\in\mathbb{N}}$ *tends to negative infinity* with limes $-\infty$:

$$\lim_{n\to\infty} a_n = -\infty$$

$$\forall M > 0 \exists N \in \mathbb{N} : n \geq N \implies a_n < -M$$

Example 17.

$$a_n = \frac{n^2+2}{n+1}$$

has limes $+\infty$. The proof is given in the practicals. We show that ...

$$\frac{n^2+2}{n+1} > M \iff \dots$$

Definition 38 (Limes superior, Limes inferior)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence which is bounded above and

$$H = \left\{\xi \in \mathbb{R} \,|\, \xi \text{ is limit point of } (a_n)_{n\in\mathbb{N}}\right\} \neq \emptyset$$

Then H is also bounded by above and we call $S^* = \sup H$ a *limes superior* of the sequence $(a_n)_{n\in\mathbb{N}}$. We denote:

$$S^* = \limsup_{n\to\infty} a_n$$

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence which is bounded below and

$$H = \left\{\xi \in \mathbb{R} \,|\, \xi \text{ is limit point of } (a_n)_{n\in\mathbb{N}}\right\} \neq \emptyset$$

Then H is also bounded by below and we call $S_* = \inf H$ a *limes inferior* of the sequence $(a_n)_{n\in\mathbb{N}}$. We denote:

$$S_* = \liminf_{n\to\infty} a_n$$

Theorem 50. If $(a_n)_{n\in\mathbb{N}}$ is bounded by above by M, $H \neq \emptyset$, then M is also an upper bound of H.

Proof. Assume $\exists s \in H$ with $s > M$. Choose $\varepsilon = S - M > 0$. Because S is a limit point of $(a_n)_{n\in\mathbb{N}}$ it holds that $(s-\varepsilon, s+\varepsilon)$ contains infinitely many sequence elements. So for infinitely many indices n it holds that,

$$a_n > s - \varepsilon = s - (s - M) = M$$

This contradicts with M being the upper bound of the sequence. ☐

Lemma 9. Let $(a_n)_{n\in\mathbb{N}}$ be bounded by above. $a_n \in \mathbb{R}$. Let $H \neq \emptyset$ be defined as above. Then it holds that

$$S^* = \limsup_{n\to\infty}(a_n) = \max H$$

ie. S^* is a limit point itself of the sequence.

Proof. Show that S^* itself is a limit point of the sequence. Let $\varepsilon > 0$: Choose $\xi \in H$ such that

$$\xi > S^* - \frac{\varepsilon}{2} \implies S^* - \xi = |S^* - \xi| < \frac{\varepsilon}{2}$$

Because ξ is a limit point of the sequence, in $(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2})$ there are infinitely many sequence elements.

Let $x \in (\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}) \iff |x - \xi| < \frac{\varepsilon}{2}$. Then it holds that

$$|x - S^*| = |x - \xi + \xi - S^*| \leq \underbrace{|x - \xi|}_{<\frac{\varepsilon}{2}} + \underbrace{|\xi - S^*|}_{=S^*-\xi<\frac{\varepsilon}{2}}$$

$$\implies x \in (S^* - \varepsilon, S^* + \varepsilon)$$

Followingly,

$$\underbrace{\left(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}\right)}_{\text{contains infinitely many sequence elements}} \subseteq \underbrace{(S^* - \varepsilon, S^* + \varepsilon)}_{\text{contains infinitely many sequence elements}}.$$

☐

Remark 14. The analogous statement holds for the limes inferior.

$$S^* = \limsup_{n\to\infty} a_n \iff$$

1. $S^* \in H$, therefore S^* is limit point of $(a_n)_{n\in\mathbb{N}}$.
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

Proof. Let $S^* = \limsup_{n\to\infty} a_n$.

1. The first property holds immediately.
2. We use an indirect proof.

$$\implies \exists \varepsilon > 0 : \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Therefore infinitely many sequence elements a_n exist with $a_n \geq S^* + \varepsilon$. We sort the sequence elements in a subsequence $(a_{n_k})_{k\in\mathbb{N}}$. It holds that

$$S^* + \varepsilon \leq a_{n_k} \leq M$$

$(a_{n_k})_{k\in\mathbb{N}}$ is bounded and has a limit point S with $S^* + \varepsilon < S \implies S > S^*$. S is also a limit point of the original sequence $(a_n)_{n\in\mathbb{N}}$ with $S > S^* = \max H$. This is a contradiction.

□

↓ This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring

Theorem 51 (Repetition of the theorem)**.** Let $(a_n)_{n\in\mathbb{N}}$ be bounded above and let $(a_n)_{n\in\mathbb{N}}$ has a limit point. Then it holds that $S^* = \limsup_{n\to\infty} a_n \iff$

1. S^* is limit point of $(a_n)_{n\in\mathbb{N}}$.
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

Therefore above $S^* + \varepsilon$ there are only finitely many sequence elements.

Proof. We prove the first direction $\implies$.

Let $S^* = \limsup_{n\to\infty} a_n$. Let $\varepsilon > 0$ be arbitrary. The first property follows immediately. The second property needs to be shown.

Proof by contradiction for the second property.

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Then we build a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ from $(a_n)_{n\in\mathbb{N}}$ with $a_{n_k} \geq S^* + \varepsilon$.

The subsequence is built inductively:

$n = 0$ then (because the second property holds negated) there exists $x_n \geq 0$: $a_{n_0} \geq S^* + \varepsilon$.

$k \to k+1$ Let $a_{n_0}, a_{n_1}, \ldots, a_{n_k}$ be found with $a_{n_l} \geq S^* + \varepsilon$ with $l = 0, \ldots, n$ and $n_l < n_{l+1}$. Let $N = n_k + 1$. Because the second property holds negated, $n_{k+1} \geq N > n_k$ such that $a_{n_{k+1}} \geq S^* + \varepsilon$.

The subsequence's elements have the properties:

- $a_{n_k} \geq S^* + \varepsilon \qquad \forall k \in \mathbb{N}$
- Because $(a_n)_{n\in\mathbb{N}}$ is bounded above, also $(a_{n_k})_{k\in\mathbb{N}}$ is bounded above

From the Bolzano-Weierstrass theorem it follows that $(a_{n_k})_{k\in\mathbb{N}}$ has a limit point $S \geq S^* + \varepsilon$. Because every limit point of $(a_{n_k})_{k\in\mathbb{N}}$ is a limit point of $(a_n)_{n\in\mathbb{N}}$, it holds that S is limit point of $(a_n)_{n\in\mathbb{N}}$ and $S > S^* + \varepsilon > S^*$. This is a contradiction. □

We prove the second direction $\impliedby$.

Assume properties 1 and 2 hold. It remains to show that S^* is the largest limit point. Assume $S > S^*$. We need to show that S cannot be a limit point.

$$\varepsilon = \frac{S - S^*}{2} > 0 \implies 2\varepsilon = S - S^* \implies S^* + \varepsilon = S - \varepsilon$$

Because the second property holds, there exists some $N \in \mathbb{N}$ such that $\forall n \geq N \implies a_n < S^* + \varepsilon$. Therefore only finitely many sequence elements are larger than $S^* + \varepsilon = S - \varepsilon$. Therefore at most finitely many sequence elements $(S - \varepsilon, S + \varepsilon)$. Followingly S is not a limit point. □

Theorem 52 (Analogous result for limes inferior)**.**

$$S_* = \liminf_{n\to\infty} a_n \iff$$

1. S_* is limit point of $(a_n)_{n\in\mathbb{N}}$.
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n > S_* - \varepsilon$

Theorem 53. Let $(a_n)_{n\in\mathbb{N}}$ be bounded above and $(a_n)_{n\in\mathbb{N}}$ has a limit point.

- Let $k \in \mathbb{N}$. We define
$$A_k = \{a_k, a_{k+1}, a_{k+2}, \dots\} = \{a_j : j \geq k\}$$
- It holds that $A_{k+1} \subseteq A_k$ and A_k is bounded above[2].

We define $S_k = \sup A_k$. Then $(S_k)_{k\in\mathbb{N}}$ is a monotonically decreasing sequence in $\mathbb{R}$ and $(S_k)_{k\in\mathbb{N}}$ is bounded below. Therefore $(S_k)_{k\in\mathbb{N}}$ is convergent and it holds that
$$\lim_{n\to\infty} S_k = \inf\{S_k : k \in \mathbb{N}\} = S^*$$
It turns out that
$$S^* = \limsup_{n\to\infty} a_n$$
We denote
$$\lim_{k\to\infty} \sup A_k = \lim_{k\to\infty} \sup\{a_j : j \geq k\} = \inf\{\sup A_k : k \in \mathbb{N}\} = \limsup_{n\to\infty} a_n$$

Proof.
$$A_{k+1} \subseteq A_k \implies \sup A_{k+1} \leq \sup A_k \implies S_{k+1} \leq S_k$$
$(S_k)_{k\in\mathbb{N}}$ is bounded below. Choose $\xi \in H$ and ξ is limit point of $(a_n)_{n\in\mathbb{N}}$. Then $\xi - 1$ is a lower bound for $(S_k)_{k\in\mathbb{N}}$ because infinitely many sequence elements are in $(\xi - 1, \xi + 1)$. Therefore,
$$\forall k \in \mathbb{N} : \exists n \geq k : a_n > \xi - 1 \quad \implies S_k = \sup A_k > \xi - 1 \quad \checkmark$$
We know that $(S_k)_{k\in\mathbb{N}}$ is convergent. Let $S^* = \lim_{n\to\infty} S_k$. We show the first property:

S^* is limit point of $(a_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be given. We need to show that infinitely many sequence elements are in $(S^* - \varepsilon, S^* + \varepsilon)$.

[2] Obviously.

Because $\lim_{k\to\infty} S_k = S^*$ there exists some
$$N \in \mathbb{N} : k \geq N \implies \underbrace{|S_k - S^*|}_{-S^*} < \frac{\varepsilon}{2}.$$
We build a subsequence of $(a_n)_{n\in\mathbb{N}}$ inductively, which is entirely inside $(S^* - \varepsilon, S^* + \varepsilon)$. Because $S_N = \sup\{a_N, a_{N+1}, a_{N+2}, \dots\}$ exists, there exists $a_j \geq S_N - \frac{\varepsilon}{2}$ with $j \geq N$.
$$\implies \underbrace{S_N - a_j}_{=|S_N - a_j|} \leq \frac{\varepsilon}{2}$$

$k = 0$ Choose $n_0 = j \geq N$ (j from above), therefore it holds that
$$|S^* - a_{n_0}| = |S^* - S_N + S_N - a_{n_0}| \leq |S^* - S_N| + |S_N - a_j|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Therefore $a_{n_0} \in (S^* - \varepsilon, S^* + \varepsilon)$.

$k \to k+1$ Consider $a_{n_0}, a_{n_1}, \dots, a_{n_k}$ such that $n_k > n_{k-1} > \dots > n_0 \geq N$ holds and $|a_{n_l} - S^*| < \varepsilon$. Because $n_k + 1 > N$ holds
$$|S^* - S_{n_k+1}| < \frac{\varepsilon}{2}$$
because $S_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \dots\}$, exists $j' \geq n_k + 1 > n_k$ such that
$$|S_{n_k+1} - a_{j'}| = S_{n_k+1} - a_{j'} < \frac{\varepsilon}{2}$$
Choose $n_{k+1} = j'$ from above.
$$n_{k+1} \geq n_k + 1 > n_k \text{ and } \left|S^* - a_{n_{k+1}}\right| = |S^* - S_{n_k+1} + S_{n_k+1} - a_{j'}|$$
$$\leq |S^* - S_{n_k+1}| + |S_{n_k+1} - a_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}1 = \varepsilon$$
Therefore we have found a subsequence $(a_n)_{n\in\mathbb{N}}$ such that
$$\forall k \in \mathbb{N} : a_{n_k} \in (S^* - \varepsilon, S^* + \varepsilon)$$
$\implies S^*$ is limit point of the sequence.

We show that S^* is the largest limit point. Let $S < S^*$. We show that S is not a limit point.

Let $\varepsilon = \frac{1}{2}(S - S^*) > 0$ such that $S^* + \varepsilon = S - \varepsilon$. Choose $k \in \mathbb{N}$ such that $S_k - S^* = |S_k - S^*| < \varepsilon$. $\forall n \geq K$ it holds that $a_n \leq S_k < S^* + \varepsilon = S - \varepsilon$. Therefore there are at most finitely many sequence elements in $(S-\varepsilon, S+\varepsilon)$. Therefore S is not a limit point.

□

The analogous result for the limes inferior also holds and is given in the practicals.

10 Infinite series

Definition 39. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex values. We define

- $S_0 = a_0$
- $S_1 = a_0 + a_1$
- $S_2 = a_0 + a_1 + a_2$
- ...
- $S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$

We call $(S_n)_{n\in\mathbb{N}}$ an *infinite series* with a_k sequence elements. We call S_n the *n-th partial sum* of the series. The series is called *convergent* if $(S_n)_{n\in\mathbb{N}}$ is a convergent series in $\mathbb{C}$. For a convergent series instead of

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} a_k}_{=S_n}$$

we denote

$$S = \sum_{k=0}^{\infty} a_k$$

Actually a series must be denoted like a sequence with $(S_n)_{n\in\mathbb{N}}$. But we also say "let $\sum_{k=0}^{\infty} a_k$ be a series" (but actually the sum of partial sums is meant). So this an ambiguous definition (per default always assume that the sum of partial sums is considered).

10.1 The geometric series

Theorem 54. Let $q \in \mathbb{C}$ with $q \neq 1$. Consider $\sum_{k=0}^{\infty} q^k$ hence $S_n = \sum_{k=0}^{n} q^n$. The limes of this series is given with $\frac{1-q^{n+1}}{1-q}$ for $|q| < 1$.

Proof. We find a simple equation for S_n:

$$S_n - q \cdot S_n = (1-q)S_n$$

$$\begin{aligned}
&(1 + q + q^2 + \cdots + q^n) - q(1 + q + q^2 + \cdots + q^n) \\
&= (1 + q + q^2 + \cdots + q^n) - (q + q^2 + \cdots + q^n + q^{n+1}) \\
&= (1 - q^{n+1})
\end{aligned}$$

Therefore $(1-q) \cdot S_n = 1 - q^{n+1}$. That is,

$$S_n = \frac{1-q^{n+1}}{1-q}$$

If $|q| < 1$ it holds that

$$\lim_{n\to\infty} q^{n+1} = q \lim_{n\to\infty} q^n = q \cdot 0 = 0$$

$$\lim_{n\to\infty} S_n = \frac{1 - \lim_{n\to\infty} q^{n+1}}{1-q} = \frac{1}{1-q}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

If $|q| > 1$ it holds that

$$|S_n| = \frac{1}{|1-q|} \cdot \left|1 - q^{n+1}\right| \geq \frac{1}{|1-q|} \left(\left|q^{n+1}\right| - 1\right)$$

This is the reversed triangle inequality.

$$= \frac{1}{|1-q|}\left(\underbrace{|q|^{n+1}}_{\to\infty} - 1\right)$$

Hence $(S_n)_{n\in\mathbb{N}}$ is unbounded and therefore not convergent. □

Theorem 55. Let $a_n = \frac{1}{n}$ hence $\sum_{n=1}^{\infty}\frac{1}{n}$.

$$\sum_{n=1}^{\infty}\frac{1}{n} \text{ is divergent}$$

Proof. Consider

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots\\
&> \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\frac{1}{8}+\frac{1}{8}+\dots\\
&= \frac{1}{2}+2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\dots\right)\\
&= \frac{1}{2}+2\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right)\\
&= \frac{1}{2}+\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right)\\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n}
\end{aligned}$$

So we have,

$$\sum_{n=1}^{\infty}\frac{1}{n} > \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n}$$

Let $\sum_{n=1}^{\infty}\frac{1}{n} = H$, then $H > \frac{1}{2}+H$ must hold for some real value. This is impossible, so H cannot exist. Hence $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. □

↓ This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring

10.2 Remark about notation of convergence

Let $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$ be convergent with limes a.
Notation: $a_n = \lim_{n\to\infty} a$
or even shorter: $a_n \to a$ for $n\to\infty$

$$a_n \xrightarrow{n\to\infty} a$$

We call $(a_n)_{n\in\mathbb{N}}$ a zero sequence if $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} a_n = 0$.

10.3 Convergence tests

Theorem 56. Let $(a_k)_{k\in\mathbb{N}}$ with $a_k \in \mathbb{R}$ and $a_k > 0$ be a *real* sequence. Then $\sum_{k=0}^{\infty} a_k$ is convergent if and only if $s_n = \sum_{k=0}^{n} a_n$ is a *bounded* sequence in $\mathbb{R}$.

Proof. $\implies$ Let $(s_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$, then it holds that $(s_n)_{n\in\mathbb{N}}$ is also bounded.

$\impliedby$ $(s_n)_{n\in\mathbb{N}}$ is bounded.

$$\begin{aligned}
s_n - s_{n-1} &= (a_0+\dots+a_{n-1}+a_n)-\\
&\quad (a_0+\dots+a_{n-1})\\
&= a_n \geq 0
\end{aligned}$$

Hence, $s_n \geq s_{n-1}$ so $(s_n)_{n\in\mathbb{N}}$ is monotonically increasing and therefore also convergent.

□

Theorem 57. Let $\alpha \in \mathbb{Q}_+$. Then it holds that: The series $\sum_{k=1}^{\infty}\frac{1}{k^\alpha}$ is

convergent if $\alpha > 1$

divergent if $\alpha \leq 1$

Case 1: $\alpha > 1$ We know: Map $f(x) = x^\alpha$ is montonically increasing.

$$x < y \implies x^\alpha < y^\alpha$$

Let $S_{\alpha,n} = \sum_{k=1}^{n} \frac{1}{k^\alpha}$ be the n-th partial sum. $n = 2^k - 1$.

$$S_{\alpha,2^\alpha-1} = \underbrace{1}_{2^0 \text{ terms}} + \underbrace{\underbrace{\frac{1}{2^\alpha}}_{=\frac{1}{2^\alpha}} + \underbrace{\frac{1}{3^\alpha}}_{<\frac{1}{2^\alpha}}}_{2^1 \text{ terms}} + \underbrace{\frac{1}{4^\alpha} + \underbrace{\frac{1}{5^\alpha}}_{<\frac{1}{4^\alpha}} + \underbrace{\frac{1}{6^\alpha}}_{<\frac{1}{4^\alpha}} + \underbrace{\frac{1}{7^\alpha}}_{<\frac{1}{4^\alpha}}}_{2^2 \text{ terms}}$$

$$+ \underbrace{\frac{1}{8^\alpha} + \dots + \underbrace{\frac{1}{15^\alpha}}_{<\frac{1}{8^\alpha}}}_{2^3 \text{ terms}} + \dots + \underbrace{\frac{1}{(2^{k-1})^\alpha} + \dots + \underbrace{\frac{1}{(2^k-1)^\alpha}}_{<\frac{1}{(2^{k-1})^\alpha}}}_{2^{k-1} \text{ terms}}$$

$$\begin{aligned}
&< 1 + 2\frac{1}{2^\alpha} + 4\frac{1}{4^\alpha} + 8\frac{1}{8^\alpha} + \dots + 2^{k-1}\frac{1}{(2^{k-1})^\alpha} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \frac{1}{8^{\alpha-1}} + \dots + \frac{1}{(2^{n-1})^{\alpha-1}} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \left(\frac{1}{3^{\alpha-1}}\right)^3 + \dots \\
&= \underbrace{\sum_{j=0}^{k-1} \left(\frac{1}{2^{\alpha-1}}\right)^2}_{\substack{\text{geometric} \\ \text{series}}} \\
&= \frac{1 - \left(\frac{1}{2^{\alpha-1}}\right)^2}{1 - \frac{1}{2^{\alpha-1}}} \\
&< \frac{1}{1 - \frac{1}{2^{\alpha-1}}} = \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}
\end{aligned}$$

Therefore $(S_{\alpha,2^k-1})$ is bounded. Let $n \in \mathbb{N}$ be arbitrary and choose a sufficiently large K such that $2^k > n + 1$. Therefore $2^k - 1 > n$. Because $\frac{1}{j^\alpha} > 0$ for all $j \geq 1$, it holds that $S_{2^k-1} > S_n$. At the same time $S_{2^k-1} < \frac{2^{\alpha-1}}{2^{\alpha-1}-1}$. So $(S_n)_{n\in\mathbb{N}}$ is bounded. Hence $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ is convergent.

Case 2: $\alpha \leq 1$ Then it holds that $k^\alpha \leq k$ and therefore $\frac{1}{k^\alpha} \geq \frac{1}{k}$. Because $S_{\alpha,n} \geq S_{1,n}$ and because $S_{1,n}$ is unbounded, it holds that $(S_{\alpha,n})_{n\in\mathbb{N}}$ is unbounded and followingly $\sum_{k=0}^{\infty} \frac{1}{k^\alpha}$ is divergent.

Remark 15. $\alpha \in \mathbb{Q}_+$ can be replaced by $\alpha \in \mathbb{R}_+$. It is even possible to choose $\alpha \in \mathbb{C}$. Then we can define $\zeta : M \subseteq \mathbb{C} \to \mathbb{C}$ with $\xi(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$. This is Riemann's Zeta function.

Definition 40. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with $a_n \geq 0$. Then we call $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n = (-1)^n a_n$, or equivalently $\alpha_n = (-1)^{n+1} a_n$, an .

A series of structure $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k \geq 0$ is called *alternating series*.

10.4 Leibniz convergence criterion

Gottfried Wilhelm Leibniz (1646–1716)

Theorem 58 (Leibniz convergence criterion)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a real, monotonically zero sequence with $a_n \geq a_{n+1} \geq 0 \quad \forall n \in \mathbb{N}$. Then $\sum_{k=0}^{\infty} (-1)^k a_k$ is convergent.

Proof.

$$\begin{aligned}
S_{2n-1} &= \sum_{k=0}^{2n-1} (-1)^k a_k \\
S_{2n} &= \sum_{k=0}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} \\
&= S_{2n-1} + a_{2n} \\
S_{2n+1} &= S_{2n-1} + \underbrace{a_{2n} - a_{2n-1}}_{\geq 0} \\
S_{2n+2} &= \underbrace{S_{2n-1} + a_{2n}}_{S_{2n}} \underbrace{-a_{2n+1} + a_{2n+2}}_{=-(a_{2n+1}-a_{2n+2})\geq 0}
\end{aligned}$$

Therefore it holds that $S_{2n+1} \geq S_{2n-1}$, $S_{2n+2} \leq S_{2n}$ and $S_{2n} \geq S_{2n-1}$.

$(S_{2n})_{n\in\mathbb{N}}$ is monotonically decreasing. $(S_{2n+1})_{n\in\mathbb{N}}$ is monotonically increasing.

It holds that: $\forall m, n \in \mathbb{N} : S_{2n} \geq S_{2m-1}$.

Proof. **Case 1: m > n**

$$S_{2m+1} \leq S_{2n} \leq S_{2n} \qquad \checkmark$$

Case 2: m ≤ n

$$S_{2m+1} \leq S_{2n+1} \underbrace{\leq}_{\alpha<1} S_{2n}$$

So $(S_{2n})_{n\in\mathbb{N}}$ is monotonically decreasing and bounded by below (for example by S_1). Therefore $S_{2n} \to S^*$ for $n \to \infty$ (S_{2n+1}) is monotonically increasing and bounded by above by S_*:

$$S_{2n+1} \to S_* \text{ for } n \to \infty$$

It holds that $S_* \leq S^*$ because $S_{2n+1} \leq S_{2n}$. □

↓ This lecture took place on 10th of December 2015 with lecturer Wolfgang Ring

Given $S_* \leq S^*$, we show that $S^* = S_*$ and we prove that $\forall \varepsilon > 0 : S^* - S_* < \varepsilon$.

Let $\varepsilon > 0$ and choose N sufficiently large, such that $a_{2N} < \varepsilon$.

$$a_{2N} = S_{2N} - S_{2N-1} > S^* - S_*$$

$$a_{2N} < \varepsilon$$

So $\forall \varepsilon > 0$, it holds that

$$S^* - S_* = |\, S^* - S_* \,| < \varepsilon$$

$$\implies S^* = S_* = S$$

So it holds that,

$$\lim_{n\to\infty} S_n = S^* = S_* = S$$

and the series converges. □

Example 18.

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k} \text{ is convergent}$$

10.5 Series in $\mathbb{C}$ and absolute convergence

Theorem 59 (Cauchy convergence criterion)**.** The complex series $\sum_{k=0}^{\infty} a_k$ is convergent if and only if the partial sums $(s_n)_{n\in\mathbb{N}}$ are a Cauchy sequence in $\mathbb{C}$.

Remark 16. Therefore

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n, m > N$$

$$\implies |S_n - S_m| < \varepsilon$$

Therefore without loss of generality, $n \geq m$.

$$S_n - S_m = \sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k = \sum_{k=m+1}^{n} a_k$$

Hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N$.

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

Equivalently, with $m + 1 = n$ and $n - m = l$.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \text{ and } l \in \mathbb{N}$$

$$\left| \sum_{k=0}^{l} a_{n+k} \right| < \varepsilon$$

Proof by $(S_n)_{n\in\mathbb{N}}$ being convergent.

$$(S_n)_{n\in\mathbb{N}} \iff \text{Cauchy sequence}$$

Lemma 10. If $\sum_{k=0}^{\infty} a_n$ is convergent in $\mathbb{C}$, then $(a_n)_{n\in\mathbb{N}}$ is a zero sequence.

Proof. Follows directly from the Cauchy criterion for $l = 0$.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N : \underbrace{\left| \sum_{n=0}^{0} a_{n+k} \right|}_{|a_n|} < \varepsilon \quad \text{hence } a_n \to 0$$

□

□

Definition 41. The complex series $\sum_{k=0}^{\infty} a_k$ is called *absolute convergent* if the real series $\sum_{k=0}^{\infty} |a_k|$ is convergent.

Example 19.

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{n^2} \qquad \text{absolute convergent}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{n} \qquad \text{\sout{absolute} convergent (Leibniz)}$$

Lemma 11. Let $\sum_{k=0}^{\infty} a_k$ be absolute convergent. Then $\sum_{k=0}^{\infty} a_k$ is also convergent.

Proof. Let $\sum_{k=0}^{\infty} |a_k|$ be convergent. From the Cauchy criterion it follows that,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N :$$

$$\left|\sum_{k=m+1}^{n} |a_k|\right| = \sum_{k=m+1}^{n} |a_k| \geq \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon$$

$\implies \sum_{k=0}^{\infty} a_k$ is convergent according to Cauchy criterion. □

10.6 Direct comparison test

Theorem 60 (Direct comparison test)**.**

1. (dt. Majorantenkriterium) Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be complex series. Let $\sum_{k=0}^{\infty} b_k$ be absolute convergent and $\exists N \in \mathbb{N} : k \geq N \implies |a_k| \leq |b_k|$.

   Then $\sum_{k=0}^{\infty} a_k$ is absolute convergent. $\sum_{k=0}^{\infty} b_k$ is called *majorant* of $\sum_{k=0}^{\infty} a_k$.

2. (dt. Minorantenkriterium) Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be complex series. Let $\sum_{k=0}^{\infty} a_k$ be divergent. Assume $\exists N \in \mathbb{N} : k \geq N \implies |a_k| \leq |b_k|$. Then also $\sum_{k=0}^{\infty} b_k$ is divergent. $\sum_{k=0}^{\infty} a_k$ is *minorant* of $\sum_{k=0}^{\infty} b_k$.

Proof. 1. We need to show that $\sum_{n=0}^{\infty} \underbrace{|a_k|}_{\geq 0}$ is convergent. It suffices to show that $\sum_{k=0}^{n} |a_k| = \sigma_n$. $(\sigma_n)_{n\in\mathbb{N}}$ is bounded. Let $n \geq N$.

$$\begin{aligned}
\sigma_n &= \sum_{n=0}^{n} |a_k| \\
&= |a_0| + |a_1| + \cdots + |a_{N-1}| + \sum_{k=N}^{n} |a_k| \\
&\leq \underbrace{|a_0| + \cdots + |a_{N-1}| + \underbrace{\sum_{k=N}^{\infty} |b_k|}_{s\geq 0}}_{M}
\end{aligned}$$

Therefore $(\sigma_n)_{n\in\mathbb{N}}$ is bounded and therefore $\sum_{n=0}^{\infty} a_n$ is absolute convergent.

2. Let $\sum_{k=0}^{\infty} a_k$ be divergent. Then also $\sum_{k=0}^{\infty} |a_k|$ is divergent. Otherwise $\sum_{k=0}^{\infty} a_k$ is absolute convergent and therefore convergent.

$$\implies \sigma_n = \sum_{k=0}^{n} |a_k|$$

$(\sigma_n)_{n\in\mathbb{N}}$ is unbounded. Because

$$\begin{aligned}
\sum_{k=0}^{n} |b_k| &= |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^{n} |b_k| \\
&\geq |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^{N} |a_k| \\
&= \underbrace{|b_0| + \cdots + |b_{N-1}| - (|a_0| + \cdots + |N-1|) + \sum_{k=0}^{n} |a_k|}_{z} \\
&= z + \sigma_n
\end{aligned}$$

$z + \sigma_n$ is unbounded. Therefore $\sum_{k=0}^{\infty} |b_k|$ is not convergent. Therefore $\sum_{k=0}^{\infty} b_k$ is not absolute convergent.

□

10.7 Ratio test

Theorem 61 (Ratio test (dt. Quotientenkriterium))**.** 1. Let $\sum_{k=0}^{\infty} a_k$ be a complex series. Assume $\exists q \in [0,1)$ with $(0 \leq q < 1)$ and $N \in \mathbb{N}$ such that

- $\frac{|a_{n+1}|}{|a_n|} < q \quad \forall n \geq N$ with $|a_n| \neq 0$, or "Ratio test"
- $\sqrt[n]{|a_n|} < q \quad \forall n \geq N$ "Root test"

Then the series $\sum_{k=0}^{\infty} a_k$ is absolute convergent.

2. Assume there exists $q > 1$ and $N \in \mathbb{N}$ such that

- $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$
- $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$

Then $\sum_{k=0}^{\infty} a_k$ is divergent.

Proof. This follows from the direct comparison criterion. Compare with geometric series $\sum_{k=0}^{\infty} q^k$.

1. Assume the second statement of the ratio test holds. Therefore $\forall n \geq N$ it holds that $\sqrt[n]{|a_n|} \leq q \iff |a_n| \leq q^n$. $|a_n| \leq q^n$. Due to the direct comparison test, $\sum_{k=0}^{\infty} q^k \checkmark$.

   Assume the first statement of the ratio test does not hold.

$$\frac{|a_{n+1}|}{|a_n|} \leq q (< 1)$$

   Then it holds that $\forall k \in \mathbb{N}$:

$$|a_{k+N}| \leq |a_N| \cdot q^k$$

   Proof by induction over k:

   $\mathbf{k = 0}$

$$|a_N| \leq |a_N| \cdot q^0 \qquad \checkmark$$

   $\mathbf{k \to k+1}$ Assume $|a_{N+k}| \leq |a_N| \cdot q^k$. Because

$$\frac{|a_{N+k+1}|}{|a_{N+k}|} \leq q \implies |a_{N+K-1}| \leq q\,|a_{N+k}| \leq q{\cdot}|a_N|{\cdot}q^k = |a_N|\,q^{k+1} \qquad \checkmark$$

   We set

$$b_K = \begin{cases} 0 & \text{for } k = 0, 1, 2, \ldots, N-1 \\ |a_N| \cdot q^{K-n} & \text{for } n \geq N \end{cases}$$

$$\sum_{k=0}^{\infty} b_k = 0 + 0 + 0 + \cdots + 0 + |a_N| \cdot q^0 + |a_N| \cdot q^1 + |a_N|\,q^2 + \ldots$$

$$= |a_N| \sum_{j=0}^{\infty} q_j \text{ is absolute convergent}$$

   $\sum_{k=0}^{\infty} b_k$ is an absolute convergent majorant for $\sum_{k=0}^{\infty} a_k$.

$$\implies \sum_{k=0}^{\infty} a_k \text{ is convergent}$$

2. Assume the second statement (square root test) holds: $\sqrt[n]{|a_n|} \geq q$ or equivalently $\underbrace{|a_n|}_{\text{unbounded}} \geq \underbrace{q^n}_{\text{unbounded}}$. Therefore $(a_n)_{n\in\mathbb{N}}$ is no zero sequence. Therefore $\sum_{k=1}^{\infty} a_k$ is divergent.

   Assume the first statement holds.

$$\implies |a_{N+k}| \geq |a_N| \cdot q^k$$

   Because $|a_N| \cdot q^k$ is unbounded, $|a_{N+k}|$ is unbounded. $(a_k)_{k\in\mathbb{N}}$ are not zero sequences.

□

Remark 17. Assume $\frac{|a_{n+1}|}{|a_n|}$ is bounded and $q = \limsup_{n\to\infty} \left(\frac{|a_{n+1}|}{a_n}\right) < 1$. Let $2\varepsilon = 1 - q > 0$.

$$\implies \exists N \in \mathbb{N} : n \geq N : \frac{|a_{n+1}|}{|a_n|} < q + \varepsilon$$

$$= q + \frac{1}{2}(1-q) = \frac{1}{2}(1+q) = 1 - \varepsilon < 1$$

Due to the ratio test, the series $\sum_{k=0}^{\infty} a_k$ is absolute convergent.

Lemma 12. Let $\sum_{k=0}^{\infty} a_k$ be a complex series with $a_k \neq 0 \forall k \in \mathbb{N}$ and if it holds that

$$q = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

Then $\sum_{k=0}^{\infty} a_k$ is absolute convergent.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = q$$

↓ This lecture took place on 11th of December 2015 with lecturer Wolfgang Ring

10.8 Revision

So $\sum_{k=0}^{\infty}$ is absolute convergent if $\exists q \in [0,1) \exists N \in \mathbb{N}$.

- $\frac{|a_{n+1}|}{|a_n|} \leq q \quad \forall n \geq N$
- $\sqrt[n]{|a_n|} \leq q \quad \forall n \geq N$

If $q > 1$ and either $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$ or $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$, then this series is convergent.

Corollary 10. Let $q = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{k=0}^{\infty} a_k$ is absolute convergent. Let $q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then $\sum_{k=0}^{\infty} a_k$ is absolute convergent. Let $q = \limsup \sqrt[n]{|a_n|} > 1$. Then $\sum_{k=0}^{\infty} a_k$ is divergent.

Proof. Let $q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$$2\varepsilon = 1 - q > 0$$

Then there exists some $N \in \mathbb{N} : n \geq N$

$$\implies \sqrt[n]{|a_n|} \leq q + \varepsilon = 1 - \varepsilon < 1$$

Is absolute convergent according to the square root theorem.

We also need to show divergence: Let $q > 1$ be limit point of $\sqrt[n]{|a_n|}$. So there exists some subsequence $\left(\sqrt[n_k]{|a_{n_k}|}\right)_{k\in\mathbb{N}}$ with $\lim_{n\to\infty} \sqrt[n_k]{|a_{n_k}|} = q > 1 \implies \varepsilon = \frac{1}{2}(q-1) > 0$.

$$\sqrt[n_k]{|a_{n_k}|} > q - \varepsilon \qquad \forall k \geq K$$

$$\implies |a_{n_k}| > (q-\varepsilon)^{n_k} = (1+\varepsilon)^{n_k} > 1$$

$$\implies (|a_{n_k}|)_{k\in\mathbb{N}} \text{ is not a zero sequence}$$

$$\implies (|a_n|)_{n\in\mathbb{N}} \text{ is also not a zero sequence}$$

$$\implies \sum_{k=0}^{\infty} a_k \text{ is divergent}$$

□

Example 20 (Binomial series). Let $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, n\}$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1 \cdot 2 \cdot \ldots \cdot (n-k)(n-k+1) \cdot \cdots \cdot n}{k! \cdot 1 \cdot 2 \cdot \ldots \cdot (n-k)}$$

$$= \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}$$

Let $s \in \mathbb{C}$. We define the binomial coefficient $\binom{s}{k} = \frac{s \cdot (s-1) \cdot (s-2) \cdot \ldots \cdot (s-k+1)}{k!}$. Also let $\binom{s}{0} = 1$ and $\binom{s}{1} = s$. Let $k > n$ and $n \in \mathbb{N}$, then

$$\binom{n}{k} = \frac{n(n-1) \cdot \ldots \cdot \overbrace{(n-n)}^{0} \cdot \ldots \cdot (n-k+1)}{k!} = 0$$

Example 21. We define the binomial series for $s, z \in \mathbb{C}$ with

$$B_S(z) = \sum_{k=0}^{\infty} \underbrace{\binom{s}{k} z^k}_{:=a_k}$$

What about convergence? Well,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\frac{s\cdot(s-1)\cdot\ldots\cdot(s-(k+1)+1)}{(k+1)!} z^{k+1}\right|}{\left|\frac{s(s-1)(s-2)\cdot\ldots\cdot(s-k+1)}{k!} z^k\right|}$$

$$\frac{|a_{k+1}|}{|a_k|} = \left|\frac{(s-k)z}{k+1}\right| = \left|\frac{\left(\frac{s}{k}-1\right)\cdot z}{1+\frac{1}{k}}\right| \to |z| \text{ as } \frac{s}{k} \to 0 \text{ and } \frac{1}{k} \to 0$$

Therefore $B_S(z)$ is convergent for $|z| < 1$ and divergent for $|z| > 1$. So geometrically, it is convergent within a circle of radius 1 or i (at center $(0,0)$) and divergent outside.

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

We know, for $s \in \mathbb{N}$:

$$B_S(z) = \sum_{k=0}^{\infty} \binom{n}{k} z^k = \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n$$

Remember that $\binom{n}{k} = 0$ for $k > n$.

Therefore

$$(1+z)^s := \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

This is the definition of a power function i.e.

$$z = \xi - 1 \qquad 1 + z = \xi$$

$$\xi^S = \sum_{k=0}^{\infty} \binom{s}{k} (\xi - 1)^k$$

is convergent for $|\xi - 1| < 1$.

Geometrically, this is a circle of radius 1 or i (at center $(1,0)$).

11 Power series

Definition 42. A power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

So we have one free variable c as part of the coefficients of the series.

- In $\sum_{k=1}^{\infty} \frac{1}{k^2}$ all summands are fixed.
- $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ with $|z| < 1$ has one variable z.

Example 22.

$$f : B(0,1) \to \mathbb{C} \qquad \underbrace{f(z) = \sum_{n=0}^{\infty} z^k}_{\sum_{k=0}^{\infty} \binom{s}{k} z^k} = \frac{1}{1-z}$$

$$B_S : B(0,1) \to \mathbb{C} \qquad \underbrace{B_S(z)}_{\text{map}} = \sum_{k=0}^{\infty} \binom{s}{k} \underbrace{z^k}_{\text{variable}}$$

$$\varepsilon : \mathbb{C} \to \mathbb{C} \qquad \varepsilon(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

How about convergence? Let $z \in \mathbb{C}$ arbitrary.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\frac{z^{k+1}}{(k+1)!}\right|}{\left|\frac{z^k}{k!}\right|} = \left|\frac{z}{k+1}\right| \xrightarrow{k\to\infty} 0$$

$\implies$ $\varepsilon(z)$ is convergent for all $z \in \mathbb{C}$.

$$\varepsilon : \mathbb{C} \to \mathbb{C}$$

Corollary 11. Using series sum we can define mappings (functions).

Definition 43. Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence and let $z \in \mathbb{C}$. Then $\sum_{k=0}^{\infty} a_k \cdot z^k$ is called *power series with coefficient sequence* $(a_k)_{k\in\mathbb{N}}$.

Its convergence property depends on z. For $z = 0$ every power series is convergent.

$$\sum_{k=0}^{\infty} a_k \cdot 0^k$$

Because we define $0^0 := 1$ here, the constant series a_0 is given.

Lemma 13. Let $\sum_{k=0}^{\infty} a_k z^k$ is a power series in $\mathbb{C}$ and $z_0 \in \mathbb{C} \setminus \{0\}$ such that $\sum_{k=0}^{\infty} a_k z_0^k$ is convergent. Then the power series is absolute convergent for all z with $|z| < |z_0|$.

Geometrically, if the series is convergent at one point z_0 at the circle, it is convergent in all points of the circle.

Proof. Direct comparison test: Because $\sum_{k=0}^{\infty} a_k z_0^k$ is convergent, it holds that $\lim_{k\to\infty} a_k z_0^k = 0$. Therefore $(a_k z_0^k)_{n\in\mathbb{N}}$ is also bounded and there exists some $m \geq 0$ such that $\left|a_k z_0^k\right| \leq m \quad \forall k \in \mathbb{N}$.

Let $|z| < |z_0|$. Then,

$$\left|a_k z^k\right| = \left|a_k \frac{z^k}{z_0^k} \cdot z_0^k\right| = \left|a_k z_0^k\right| = \underbrace{\left|a_k z_0^k\right|}_{\leq m} \underbrace{\left|\frac{z}{z_0}\right|^k}_{:=q} \leq m \cdot q^k$$

with $0 \leq q < 1$. Therefore $\sum_{k=0}^{\infty} a_k z^k$ is convergent because of the direct comparison test with $\sum_{k=0}^{\infty} m \cdot q^k = m \cdot \sum_{k=0}^{\infty} q^k$. □

Definition 44. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series in $\mathbb{C}$. We define

$$\rho(P) = \sup\{r \geq 0, r \in \mathbb{R} : P(r) \text{ is convergent}\}$$

$\rho(P)$ is called convergence radius of P. If $\{r \geq 0 : P(r) \text{ is convergent}\}$ is unbounded, then we define $P(r) = \infty$.

Lemma 14. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series in $\mathbb{C}$ and let $\rho(P)$ be its convergence radius of P. Then $P(z)$ is absolute convergent for all $z \in \mathbb{C}$ with $|z| < \rho(P)$.

Proof. For $\rho(P) = 0$, nothing has to be shown.

Let $\rho(P) > 0$ and $|z| < \rho(P)$, then $\varepsilon := \rho(P) - |z|$. Because $\rho(P) = \sup\{r \geq 0 : P(r) \text{ is convergent}\}$, there exists some $r \in \mathbb{R}$ such that $\rho(P) - \varepsilon < r \leq \rho(P)$ and $P(r)$ is convergent. $\rho(P) - \varepsilon = |z| < r$. So $P(z)$ is absolute convergent according to Lemma 13.

Geometrically, $\rho(P)$ is a circle and its interior is convergent. On the outside the power series is divergent. The convergence property at the circle itself is unknown (not generally uniform). □

Lemma 15. Let $z \in \mathbb{C}$, P is a power series and $|z| > \rho(P)$. Then $\sum_{k=0}^{\infty} a_k z^k$ is divergent for this point.

Proof. Proof by contradiction. Assume $P(z)$ is convergent and $|z| > \rho(P)$. Let $\varepsilon = 2(|z| - \rho(P))$. Then $\rho(P) + \varepsilon < |z|$ with $\rho(P) + \varepsilon > \rho(P)$. From the previous lemma it follows that $P(\rho(P) + \varepsilon)$ is convergent. But this contradicts with $\rho(P) = \sup\{r \geq 0 : P(r) \text{ is convergent}\}$. □

Remark 18. $B(0, \rho(P))$ is called *convergence circle of* P.

Theorem 62 (Formulas to compute $\rho(P)$)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series. Then it holds in every case that,

- $\rho(P) = \frac{1}{L}$ with $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (for $L = \infty$ if $\left(\sqrt[n]{|a_n|}\right)_{n\in\mathbb{N}}$ is unbounded and $\frac{1}{\infty} := 0$) (Cauchy & Hadamard)
- If $q := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then the convergence disk of this power series is $\frac{1}{q}$:

$$\rho(P) = \frac{1}{q}$$

with $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$.

↓ This lecture took place on 16th of December 2015 with lecturer Wolfgang Ring

11.1 Equations for $\rho(P)$

Theorem 63.

$$P(z) = \sum_{k=0}^{\infty} a_k z^k$$

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$\rho(P) = \frac{1}{L} \qquad \text{“Cauchy-Hadamard theorem”}$$

If $q = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it holds that $\rho(P) = \frac{1}{q}$ (Euler).

Proof. 1. Let $z \neq 0$ and let $L^* = \limsup \sqrt[k]{|a_k z^k|} = \limsup_{k\to\infty} |z| \sqrt{a_k} = |z| \cdot k$. Due to the square root criterion it holds that:

- If $|z| L < 1$, then $\sum_{k=0}^{\infty} a_k z^k$ is absolute convergent.
- If $|z| L > 1$, then $\sum_{k=0}^{\infty} a_k z^k$ is absolute divergent.

Therefore for $|z| < \frac{1}{L}$, P is convergent. For $|z| > \frac{1}{L}$, P is divergent.

$$\Longrightarrow \rho(P) = \frac{1}{L}$$

2. Ratio test: Assume $q = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ exists. The ratio test for $P(z) = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_k z^k$ gives us

$$\lim_{k\to\infty} \left|\frac{a_{k+1}\cdot z^{k+1}}{a_k \cdot z^k}\right| = |z| \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = |z| \cdot q$$

Therefore P is convergent, if $|z| \cdot q < 1 \iff |z| < |z| < \frac{1}{q}$. And P is divergent, if $|z| \cdot q > 1 \iff |z| > \frac{1}{q}$.

$\square$

Remark 19. What happens for $|z| = \rho(P)$? We need a different approach for convergence/divergence.

1.

$$G(z) = \sum_{k=0}^{\infty} z^k \qquad L = \limsup_{k\to\infty} \sqrt[k]{|1|}$$

$$\rho(G) = 1$$

2.

$$H(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$$

$$q = \lim_{k\to\infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k\to\infty} \left| \frac{k}{k+1} \right| = \lim_{k\to\infty} \left| \frac{1}{1 + \underbrace{\frac{1}{k}}_{\to 0}} \right|$$

3.

$$Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

$$q = \lim_{n\to\infty} \left(\frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right) = 1$$

$$\rho(Q) = 1$$

Case 1 Let $z \in \mathbb{C}$ with $|z| = 1$. Then $G(z)$ is not convergent because $(z^k)_{k\in\mathbb{Z}}$ is not a zero sequence because $\left|z^k\right| = |z|^k = 1$. So geometrically, the circle itself of the convergence circle is divergent.

Case 2 Consider $H(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$. H is divergent for $z = 1$. For $z = -1$, $H(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ is convergent according to the Leibniz criterion.

Case 3 For $Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ and let $|z| = 1$. Then it holds that $\left|\frac{z^k}{k^2}\right| \leq \frac{1}{k^2}$. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is absolute convergent. The direct comparison test tells us that $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ is absolute convergent.

12 Functions and their regularity properties

Recall: Let $D \subseteq \mathbb{C}$ (or $\subseteq \mathbb{R}$). A mapping $f : D \implies \mathbb{C}$ (or $f : D \to \mathbb{R}$) is a function. Depending on the domain, we call the function *complex* or *real*.

12.1 Fundamental topological terminology

Recall: $B(z,r) = \{\zeta \in \mathbb{C} : |z-\zeta| < r\}$. Geometrically this corresponds to an open circular disk with center z and radius r.

Analogously, $B(x,r) = \{y \in \mathbb{R} : |y-x| < r\} = (x-r, x+r)$ in $\mathbb{R}$.

Definition 45. Let $U \subseteq \mathbb{C}$ ($U \subseteq \mathbb{R}$) and $z_0 \in U$. Then U is called surrounding of z_0 in $\mathbb{C}$, if $\exists r > 0 : B(z_0, r) \subseteq U$.

- $O \subseteq \mathbb{C}$ if called *open set* if $\forall z \in O$: O is surrounding of z.

$$\iff \forall z \in O : \exists r = r(z) : B(z,r) \subseteq O$$

- $A \subseteq \mathbb{C}$ is called *closed set*, if $\mathbb{C} \setminus A$ is an open set.

Theorem 64.
1. Let I be a set and $\forall i \in I$ let O_i be an open set in $\mathbb{C}$. Then $\bigcup_{i\in I} O_i = \{z \in \mathbb{C} : \exists i \in I : z \in O_i\}$ is an open set.
2. Let $O_1, O_2, \dots, O_n$ be open sets. Then $\bigcap_{k=1}^n O_k = O_1 \cap O_2 \cap \ldots \cap O_n$ is open.
3. If $\emptyset$ is open, then $\mathbb{C}$ is open.
4. I is a set $\forall i \in I$. Let A_i be closed. Then $\bigcap_{i\in I} A_i$ is closed.
5. Let $A_1, A_2, \dots, A_n$ be closed, then $A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{k=1}^n A_k$ is closed.

Proof.
1. Let $z \in \bigcup_{i\in I} O_i$. Show that $\exists r > 0 : B(z,r) \subseteq \bigcup_{i\in I} O_i$.

   Let $z \in \bigcup_{i\in I} O_i$, therefore $\exists j \in I : z \in O_j$. Because O_i is open, $\exists r > 0$: $B(z,r) \subseteq O_j \subseteq \bigcup_{j\in I} O_j$.
2. Let $O_1, \dots, O_n$ and let $z \in O_k$. Hence $\forall k \in \{1, \dots, m\} : z \in O_k$ with O_k as open set. $\exists r_k > 0 : B(z, r_k) \subseteq O_k$. Let $r = \min\{r_1, r_2, \dots, r_n\} > 0$. Then it holds that $B(z,r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\} \subseteq \{\zeta \in \mathbb{C} : |\zeta - z| < r_k\} = B(z, r_k) \subseteq O_k$ because $r \le r_k$.

   So $\forall k \in \{1, \dots, n\} : B(z,r) \subseteq O_k$. Otherwise $B(z,r) \subseteq \bigcap_{k=1}^n O_k \implies \bigcap_{k=1}^n O_k$ is open.
3. Let $O = \emptyset$. Then it holds that $\forall z \in \emptyset : B(z,1) \subseteq \emptyset$. So $\emptyset$ is open. For $O = \mathbb{C}$ it holds that $\forall z \in \mathbb{C} : B(z,1) \subseteq \mathbb{C}$, therefore $\mathbb{C}$ is open.
4. Let A_i be closed and $A = \bigcap_{i\in I} A_i$ and $O = \mathbb{C} \setminus A = \left\{z \in \mathbb{C} : z \notin \bigcap_{i\in I} A\right\}$. $O = \mathbb{C} \setminus A = \left\{z \in \mathbb{C} : z \notin \bigcap_{i\in I} A_i\right\}$. $\{z \in \mathbb{C} : \exists j \in I : z \notin A_j\} = \bigcap_{j\in I} \{z \in \mathbb{C} : z \notin A_j\} = \bigcup_{j\in I} (\mathbb{C} \setminus A_0) \to$ open. So $\mathbb{C} \setminus A$ is open, therefore A is closed.

$$\mathbb{C} \setminus \bigcap_{j\in I} A_j = \bigcup_{j\in I} (\mathbb{C} \setminus A_j)$$

   The last statement was proven by DeMorgan.
5. Let $A = \bigcup_{k=1}^n A_k$.

$$\mathbb{C} \setminus A = \mathbb{C} \setminus \bigcup_{k=1}^n A_k = \bigcap_{k=1}^n (\mathbb{C} \setminus A_n)$$

   where $\mathbb{C} \setminus A_n$ is an open set. So A is closed.

□

Theorem 65. $A \subseteq \mathbb{C}$ is closed $\iff \forall (a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ and $(a_n)_{n\in\mathbb{N}}$ is convergent with limes $a \in \mathbb{C}$, then $a \in A$.

Proof. $\implies$ Let A be closed ($\mathbb{C} \setminus A$ is open) and $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} a_n = a$. Show that $a \in A$.

Proove by contradiction: Assume $a \notin A$, so $a \in \mathbb{C} \setminus A$.

Because $\mathbb{C} \setminus A$ is an open set, $\exists r > 0 : B(a,r) \subseteq \mathbb{C} \setminus A$. And $B(a,r) \cap A = \emptyset$ so it holds that $\forall n \in \mathbb{N} : a_n \notin B(a,r)$ with $a_n \in A$. So it holds that $\forall n \in \mathbb{N} : |a_n - a| \ge r > 0$. This is contradiction to the assumption that a_n converges to a for $n \to \infty$.

$\impliedby$ Assume the limes of every convergent sequence with sequence elements in A, is again in A. We show that for $z \notin A$ ($z \in \mathbb{C} \setminus A$) there exists $\varepsilon > 0 : B(z,\varepsilon) \cap A = \emptyset \iff B(z,\varepsilon) \subseteq \mathbb{C} \setminus A$.

We prove the existence of such an ε by contradiction: So we assume such a ε does not exist:

$$\forall \varepsilon > 0 : B(z, \varepsilon) \cap A \neq \emptyset$$

Especially: $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}_+$.

$$B(z, \frac{1}{n}) \cap A \neq \emptyset \text{ therefore } \exists a_n \in A \cap B(z, \frac{1}{n})$$

therefore $a_n \in A \wedge |a_n - z| < \frac{1}{n}$. So this constructed sequence $(a_n)_{n\in\mathbb{N}}$ satisfies:

$$a_n \in A : |a_n - z| < \frac{1}{n} \implies \lim_{n\to\infty} a_n = z$$

By hypothesis, it holds that $z \in A$, but this is a contradiction to $z \in \mathbb{C} \setminus A$. So it is shown that $\mathbb{C} \setminus A$ is an open set. So A is closed.

□

↓ This lecture took place on 17th of December 2015 with lecturer Wolfgang Ring

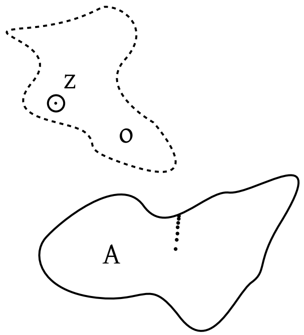

Figure 17: Contact point z in o and accumulation point z in A

Definition 46. Let $M \subseteq \mathbb{C}$ $(\mathbb{R})$. A point $z \in \mathbb{C}$ $(\mathbb{R})$ is called *contact point* of a set M, if $\forall r > 0 : B(z, r) \cap M \neq \emptyset$. A point $z \in \mathbb{C}$ $(\mathbb{R})$ is called *accumulation point* of a set M if $\forall r > 0$ it holds that $B(z, r)$ contains a point $w \in M$ with $w \neq z$.

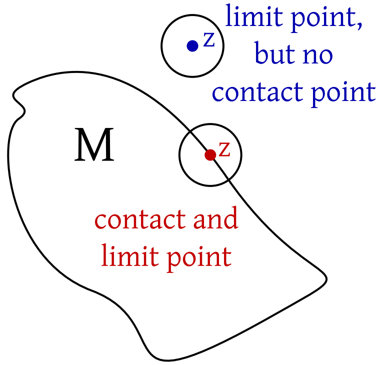

Figure 18: Illustration of a limit point and contact point

Every limit point is also a contact point.

Remark 20. An accumulation value exists in the context of sequences. An accumulation point exists in the context of open sets and topologies.

Lemma 16. Let $M \subseteq \mathbb{C}$ $(\mathbb{R})$. It holds that

1. $z \in \mathbb{C}$ is a contact point of M if and only if $\exists (z_n)_{n\in\mathbb{N}} : z_n \in M$ and $\lim_{n\to\infty} z_n = z$.
2. $z \in \mathbb{C}$ is an accumulation point of M if and only if $\exists (z_n)_{n\in\mathbb{N}} : z_n \in M$ with $z_n \neq z \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} z_n = z$.

Proof. $\Longrightarrow$ Let z be a contact point of M. Choose $r_n = \frac{1}{n}$, due to contact point property, there exists sequence $z_n \in M$ to r_n with $z_n \in B(z, \frac{1}{n})$ hence $|z_n - z| < \frac{1}{n}$. Then it holds $\lim_{n\to\infty} z_n = z$, then for $\varepsilon > 0$ arbitrary let N be arbitrary large, such that $\frac{1}{N} < \varepsilon$. Then for $n \geq N$:

$$|z_n - z| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Equivalently, the accumulation point property, $z_n \in B(z, \frac{1}{n}) \setminus \{z\}$ and $z_n \neq z$ allows the same proof for accumulation points.

$\Longleftarrow$ Assume $\exists (z_n)_{n\in\mathbb{N}}$ with limes z. $z_n \in M$. Choose $r > 0$ arbitrary. Due to convergenze of $(z_n)_{n\in\mathbb{N}}$ there exists some $N \in \mathbb{N} : n \geq N$ such that $|z_n - z| < r$.

$$\implies z_n \in M \wedge z_n \in B(z, r) \implies z \text{ is contact point of } M$$

Also,

$$\implies z_n (\neq z) \in M \wedge z_n \in B(z, r) \implies z \text{ is limit point of } M$$

□

Theorem 66. $A \subseteq \mathbb{C}$ (or $\mathbb{R}, \mathbb{R}^n$) is closed if and only if for every contact point z of A it holds that $z \in A$.

Proof. **Direction** $\implies$ Let A be closed and z is a contact point of A. Due to Lemma 16 there exists $(z_n)_{n\in\mathbb{N}}$ with $z_n \in A$ and $\lim_{n\to\infty} z_n = z$. By the Lemma before the last, it holds that $z \in A$.

Direction $\Longleftarrow$ Assume for all contact points z of A it holds that $z \in A$. By the Lemma before the last: Let $(z_n)_{n\in\mathbb{N}}$ be a convergent sequence with $z_n \in A$ and $\lim_{n\to\infty} z_n = z$.

Show that $z \in A$.

This follows immediately. By the previous Lemma, $z = \lim_{n\to\infty} z_n$ is a contact point of A and by assumption $z \in A$.

□

Remark 21. In general it holds that $z \in M$, then z is a contact point of M. Because $\{z\} \subseteq B(z, r) \cap M$ with $B(z, r) \cap M \neq 0$.

Definition 47. Let $M \subseteq \mathbb{C}$ ($\mathbb{R}$). We define $\overline{M} = \{z \in \mathbb{C} : z \text{ is contact point of } M\}$. $\overline{M}$ is called *closed hull*. It holds that $M \subseteq \overline{M}$ and M is closed $\iff M = \overline{M}$.

Definition 48. A set $K \subseteq \mathbb{C}$ ($\mathbb{R}$, $\mathbb{R}^n$) is called *compact*, if for each sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in K$, a subsequence $(z_{n_l})_{l\in\mathbb{N}}$ exists which is convergent and its limes is inside K.

Remark 22. There are equivalent definitions which do not use sequences (e.g. using open covers).

Theorem 67 (Bolzano-Weierstrass theorem for sets)**.** $K \subseteq \mathbb{C}$ is compact if and only if K is bounded and closed.

Proof. **Direction** $\Longleftarrow$ Let K be bounded and closed and let $(z_n)_{n\in\mathbb{N}}$ be a sequence of elements in K. Then $(z_n)_{n\in\mathbb{N}}$ is a bounded sequence. Due to the Bolzano-Weierstrass Theorem for sequences, there exists some convergent subsequence $(z_{n_l})_{l\in\mathbb{N}}$ with $\lim_{l\to\infty} z_{n_l} = z$ where $z_{n_l} \in K$. Followingly z is contact point in K. Because K is closed, it holds that $z \in K$.

Direction $\implies$ Let K be compact. Assume K is not bounded. Therefore for $m = 1, 2, \ldots, 5$, there exists $z_m \in K$ with $|z_m| > m$. $(z_m)_{m\in\mathbb{N}}$ has certainly no convergent subsequence, because every subsequence $(z_m)_{m\in\mathbb{N}}$ is also unbounded and therefore not convergent. This is a contradiction.

It remains to show that K is closed (for this, we have to show that $\overline{M} \subseteq M$). Let $z \in \overline{K}$ (z is a contact point of K). There exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in K$ and $z = \lim_{n\to\infty} z_n$. Because K is compact, there exists a subsequence $(z_{n_k})_{l\in\mathbb{N}}$ of $(z_n)_{n\in\mathbb{N}}$ with $\lim_{l\to\infty} z_{n_l} = w$ and $w \in K$. Because $(z_n)_{n\in\mathbb{N}}$ is already convergent, every subsequence is convergent with limes z. It follows that $w \in K$ and $w = z$, so $z \in K$. So K is closed.

□

13 Continous functions

Definition 49. Let $D \subseteq \mathbb{C}$ ($D \subseteq \mathbb{R}$) and $f : D \to \mathbb{C}$ be a function. We say "f is continous" (dt. "stetig") iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in D \text{ with } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

Intuitively, the difference of function values are arbitrary close to each other if the difference of the arguments is sufficiently small.

Example 23. 1. D is "strange". Specifying the codomain and discussion of continuity in regards of this codomain is very important! Compare with Figure 20.

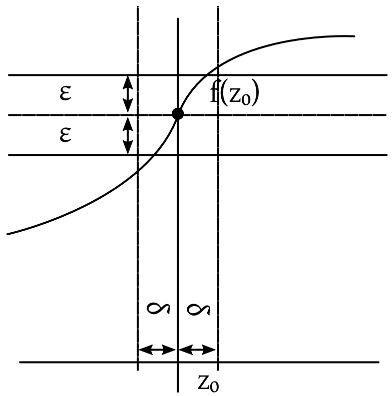

Figure 19: Illustration of continuity as a local property

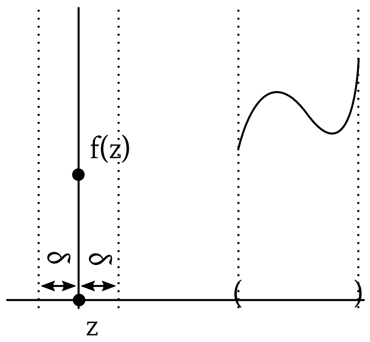

Figure 20: “Strange” D

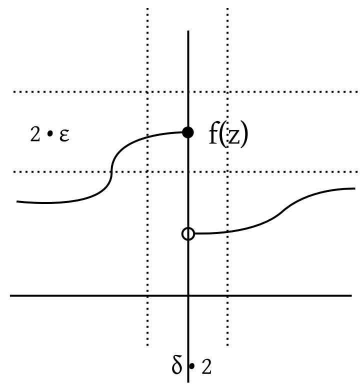

Figure 21: Example of a non-continuous function: function with a jump discontinuity

2. A non-continous function f has a discontinuity in z_0. Compare with Figure 21. So ε cannot be arbitrary small.

3. $f : \mathbb{C} \to \mathbb{C}$. $f(z) = z^2$. Let $z_0 \in \mathbb{C}$ arbitrary. Then f is continuous in z_0.

Proof. Let $\varepsilon > 0$ be arbitrary. Find $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| = \left|z^2 - z_0^2\right| < \varepsilon$$

Let $\delta := \min\left(1, \frac{\varepsilon}{1+2|z_0|}\right) < \varepsilon$. For $|z - z_0| < \delta$,

$$\begin{aligned}
|f(z) - f(z_0)| = \left|z^2 - z_0^2\right| &= |(z - z_0)(z + z_0)| \\
&= |z - z_0| \cdot |z + z_0| \\
&= |\, z \underbrace{- z_0 + z_0}_{=0} + z_0 \,| \cdot |z - z_0| \\
&\leq (\underbrace{|z - z_0|}_{<1} + 2\,|z_0|)\, \underbrace{(z - z_0)}_{\frac{\varepsilon}{1+2|z_0|}} \\
&= (1 + 2\,|z_0|)\, \frac{\varepsilon}{1 + 2\,|z_0|} = \varepsilon
\end{aligned}$$

Example 24. Let $D = [0, \infty) \subseteq \mathbb{R}$. Let $f(x) = \sqrt[k]{x}$ be continuous in every point $x_0 \in D$.

Let $\varepsilon > 0$ be given. Claim: It holds that $\left|\sqrt[k]{x} - \sqrt[k]{x_0}\right| \leq \sqrt[k]{|x - x_0|}$.

Proof: Show that for $a, b \geq 0$, it holds that $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$.

Assume $\sqrt[k]{a+b} > \sqrt[k]{a} + \sqrt[k]{b}$. Taking the k-th power keeps monotonicity:

$$(\sqrt[k]{a+b})^k = a + b > \left(\sqrt[k]{a} + \sqrt[k]{b}\right)^k$$

$$= a + \underbrace{\sum_{j=1}^{k-1} \binom{k}{j} a^{\frac{k-j}{k}} b^{\frac{j}{k}}}_{>0} + b \geq a + b$$

↓ This lecture took place on 18th of December 2015 with lecturer Wolfgang Ring

We prove $\left|\sqrt[k]{x} - \sqrt[k]{x_0}\right| \leq \sqrt[k]{|x - x_0|}$ using $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$.

$$|x| = \left|\underbrace{x - x_0}_{a} + \underbrace{x_0}_{b}\right| \leq \underbrace{|x - x_0|}_{a} + \underbrace{|x_0|}_{b}$$

$$\sqrt[k]{|x|} \leq \sqrt[k]{|x - x_0| + |x_0|} \leq \sqrt[k]{|x - x_0|} + \sqrt[k]{|x_0|}$$

$$\sqrt[k]{|x|} - \sqrt{|x_0|} \leq \sqrt[k]{|x - x_0|}$$

Analogously:

$$|x_0| = |x_0 - x + x| \leq \underbrace{|x_0 - x|}_{a} + \underbrace{|x|}_{b}$$

$$\implies \sqrt[k]{|x_0|} - \sqrt[k]{|x|} \leq \sqrt[k]{|x - x_0|}$$

$$\left|\underbrace{\sqrt[k]{|x|}}_{f(x)} - \underbrace{\sqrt[k]{|x_0|}}_{f(x_0)}\right| \leq \sqrt[k]{|x - x_0|}$$

□ Let $\varepsilon > 0$ arbitrary. Let $\delta \coloneqq \varepsilon^k$. For $|x - x_0| < \delta = \varepsilon^k$ it holds that

$$|f(x) - f(x_0)| = \left|\sqrt[k]{|x|} - \sqrt[k]{|x_0|}\right|$$

$$\leq \sqrt[k]{|x - x_0|} < \sqrt[k]{\delta} = \sqrt[k]{\varepsilon^k} = \varepsilon \quad \checkmark$$

Theorem 68 (Sequence criterion for continuity)**.** Let $f : D \subset \mathbb{C} \to \mathbb{C}$ (or $D \subseteq \mathbb{R}$). Then it holds that f is continuous in $z_0 \in D$ if and only if for *every convergent* sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} w_n = z_0$ it holds that $(f(w_n))_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} f(w_n) = f(z_0)$.

In a different way, this theorem states:

$$w_n \xrightarrow{n\to\infty} z_0 \implies f(w_n) \xrightarrow{n\to\infty} f(z_0)$$

Proof. **Direction** $\implies$ Let f be continuous in z_0 and $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D$ with $\lim_{n\to\infty} w_n = z_0$. Show that $f(w_n) \xrightarrow{n\to\infty} f(z_0)$.

Let $\varepsilon > 0$ arbitrary. Because f is continuous, there exists some $\delta > 0$ such that $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$ $(z \in D)$. $(w_n)_{n\in\mathbb{N}}$ converges to z_0. So there exists $N \in \mathbb{N} : n \geq N \implies |w_n - z_0| < \delta$. For those indices it holds that: $|f(w_n) - f(z_0)| < \varepsilon$. Hence $\lim_{n\to\infty} f(w_n) = f(z_0)$.

Direction $\impliedby$ Proof by contradiction: For every sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D$ and $w_n \to z_0$ it holds that: $f(w_n) \to f(z_0)$ for $n \to \infty$. Assume f is not continuous in z_0.

So $\exists \tilde{\varepsilon} > 0 : \forall \delta > 0 \exists z_\delta \in D$ with

$$|z_\delta - z_0| < \delta \wedge |f(z_\delta) - f(z_0)| \geq \varepsilon$$

We choose $\delta_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots$.

$$w_n \coloneqq z_{\delta_n}$$

So it holds that

$$\forall n \in \mathbb{N} : |w_n - z_0| < \frac{1}{n} \wedge |f(w_n) - f(z_0)| \geq \tilde{\varepsilon}$$

Hence $w_n \in D$ and $\lim_{n\to\infty} w_n = z_0$ and for $\varepsilon > 0$ we choose N such that $\frac{1}{N} < \varepsilon$. Then it holds for $n \geq N$: $\frac{1}{n} < \frac{1}{N} < \varepsilon$ and therefore $|w_n - z_0| < \frac{1}{n} <$

ε, but $f(w_n)$ does not converge to $f(z_0)$, because $|f(w_n) - f(z_0)| \geq \tilde{\varepsilon} > 0$. This is a contradiction to our assumption.

□

Definition 50. Let $f : D \to \mathbb{C}$ ($D \subseteq \mathbb{C}$ or $D \subseteq \mathbb{R}$). We call f "*continuous on D*" if f is continuous in every point $z \in D$.

13.1 Laws for continuous functions

Theorem 69. Let $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ be functions and f and g are continuous in $z_0 \in D$. Then it holds that

1. $(f+g) : D \to \mathbb{C}$ and $(f+g)(z) = f(z) + g(z)$.
   So the sum function $(f+g)$ is continuous in z_0.
2. $(f \cdot g) : D \to \mathbb{C}$ and $(f \cdot g)(z) = f(z) \cdot g(z)$.
   The product function is continuous in z_0.
3. Let $g(z) \neq 0 \forall z \in D$. Then $\left(\frac{f}{g}\right) : D \to \mathbb{C}$ with $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$.
   The quotient function $\left(\frac{f}{g}\right)$ is continuous in z_0.

Proof. Let $(w_n)_{n\in\mathbb{N}}$ be an arbitrary sequence with $w_n \in D$ and $\lim_{n\to\infty} w_n = z_0$. Due to the sequence criterion it holds that $f(w_n) \xrightarrow{n\to\infty} f(z_0)$ and $g(w_n) \xrightarrow{n\to\infty} g(z_0)$. The laws for convergent sequences state that,

$$f(w_n) \cdot g(w_n) \xrightarrow{n\to\infty} f(z_0) \cdot g(z_0)$$

$$f(w_n) + g(w_n) \xrightarrow{n\to\infty} f(z_0) + g(z_0)$$

$$\frac{f(w_n)}{g(w_n)} \xrightarrow{n\to\infty} \frac{f(z_0)}{g(z_0)}$$

Hence $(f+g), (f \cdot g)$ and $\left(\frac{f}{g}\right)$ is continuous in z_0. □

Corollary 12.

- $k : \mathbb{C} \to \mathbb{C}$, $k(z) = c \in \mathbb{C}$ is a constant function. k is continuous in $\mathbb{C}$.

- The function $f(z) = z$ is continuous in $\mathbb{C}$, because we can choose $\delta = \varepsilon$.
  $$|z - z_0| < \varepsilon \implies |f(z) - f(z_0)| = |z - z_0| < \varepsilon$$
- The functions $p_n(z) = z^n$ for $n = 0, 1, 2, \ldots$ are continuous in $\mathbb{C}$ as products of continuous functions.
- All polynomials $P(z) = \sum_{k=0}^{n} a_k z^k$ with $a_k \in \mathbb{C}$ are continuous in $\mathbb{C}$.
- Let $D = B(0, \rho(P))$ with $\rho(P)$ is convergence radius of the power series
  $$P(z) = \sum_{k=0}^{\infty} a_k z^k$$
  Then $P(z)$ is continuous in $B(0, \rho(P))$.
- Let $P(z) = \sum_{k=0}^{n} a_k z^k$ and $Q(z) = \sum_{l=0}^{m} b_l z^l$ be polynomials. And let $D = \{z \in \mathbb{C} : Q(z) \neq 0\}$. Then $\left(\frac{P}{Q}\right) : D \to \mathbb{C}$ is continuous in D.
  Therefore all rational functions are continuous in all points except for the roots of the denominator:
  $$\frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \ldots + b_m z^m}$$

Theorem 70. Let $f : D \to U \subseteq \mathbb{C}$ and $g : U \to \mathbb{C}$ be two functions. Let f be continuous in $z_0 \in D$ and let g be continuous in $y_0 = f(z_0) \in U$. Then $g \circ f : D \to \mathbb{C}$ is continuous in z_0.

Proof. Due to the sequence criterion: Let $(w_n)_{n\in\mathbb{N}}$ ($w_n \in D$) with $\lim_{n\to\infty} w_n = z_0$. The sequence criterion for f yields

$$\lim_{n\to\infty} \underbrace{f(w_n)}_{\in U} = f(z_0) = y_0$$

The sequence criterion for g states that

$$\lim_{n\to\infty} \underbrace{g(f(w_n))}_{g \circ f(w_n)} = g(y_0) = \underbrace{g(f(z_0))}_{g \circ f(z_0)}$$

So $g \circ f$ is continuous in z_0. □

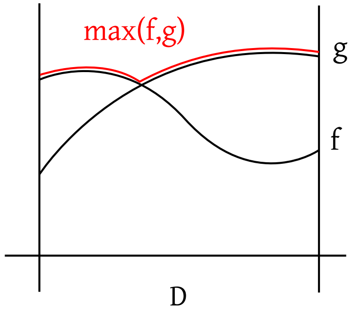

Figure 22: Maximum of two continuous functions

We know $w_k(x) = \sqrt[k]{x}$ is continuous in $[0, \infty)$.

$$P_l(x) = x^l \text{ is continuous in } \mathbb{C}$$

$$\implies P_l \circ w_k \text{ is continuous in } [0, \infty)$$

$p_0 \circ w_k(x) = p_l(\sqrt[k]{x}) = (\sqrt[k]{x})^l = x^{\frac{l}{k}}$ is continuous.

- $n(z) = |z|$ is continuous in $\mathbb{C}$.
  Let $\varepsilon > 0$ be arbitrary. It holds that
  $$|n(z) - n(z_0)| = ||z| - |z_0|| \leq |z - z_0|$$
  Choose $\delta = \varepsilon$. Then for $|z - z_0| < \delta = \varepsilon$ it holds that $|n(z) - n(z_0)| < \varepsilon$.
- $\Re : \mathbb{C} \to \mathbb{C}$ and $\Im : \mathbb{C} \to \mathbb{C}$ are continuous in $\mathbb{C}$. Because $|\Re(z) - \Re(z_0)| \leq |z - z_0|$ ✓.
- Let $f, g : D \to \mathbb{R}$. Then $\max(f, g) : D \to \mathbb{R}$ $(\max(f, g))(z) = \max\{f(z), g(z)\}$ is continuous in D. because $\max f(z), g(z) = \frac{1}{2}(|f(z) - g(z)| + f(z) + f(z))$.

↓ This lecture took place on 7th of January 2016 with lecturer Wolfgang Ring

13.2 Revision of the continuity definition

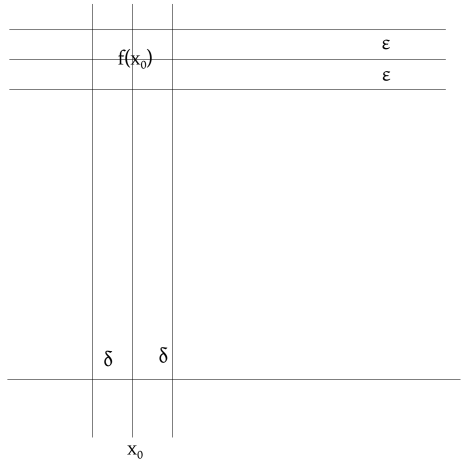

Figure 23: The notion of continuity

f is continuous in x_0 if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \wedge |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon]$$

Reminder: Let $z_0 \in U \subseteq \mathbb{C}$. U is called neighborhood of z_0 if $r > 0$ exists such that $B(z_0, r) \subset U$.

Definition 51. Let $D \subseteq \mathbb{C}$ and $z_0 \in U \subseteq D$. We call U *neighborhood* of z_0 *in* D if $\exists r > 0$ such that $B(z_0, r) \cap D \subseteq U$.

Theorem 71. Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$. Let $z_0 \in D$. Then f is continuous in z_0 if and only if for every neighborhood U of $y_0 = f(z_0)$ it holds that $V = f^{-1}(u)$ is an neighborhood of $z_0 \in D$ (where f^{-1} denotes the preimage).

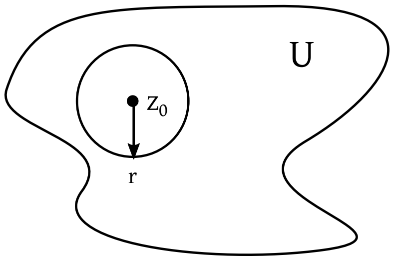

Figure 24: Neighborhood with radius r

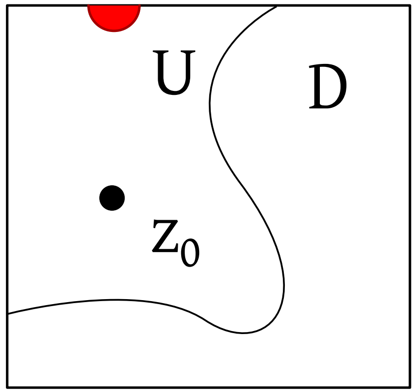

Figure 25: Neighborhood U

Proof. $\Longrightarrow$

Let f be continuous in z_0 and let U be an neighborhood of $y_0 = f(z_0)$, hence $\exists \varepsilon > 0 : B(y_0, \varepsilon) \subseteq U$ with $y_0 = f(z_0)$. Because f is continuous in z_0, it holds that

$$\exists \delta > 0 : |z - z_0| < \delta \wedge z \in D \implies \underbrace{|f(z) - f(z_0)| < \varepsilon}_{f(z) \in \underbrace{B(f(z_0), \varepsilon)}_{B(y_0, \varepsilon \subseteq U)}}.$$

This requires:

$$|z - z_0| < \delta \wedge z \in D \iff z \in B(z_0, \delta) \wedge z \in D$$

$$\implies z \in B(z_0, \delta) \cap D$$

Therefore we can redefine continuity as:

$$z \in B(z_0, \delta) \cap D \implies f(z) \in B(y_0, \varepsilon)$$

So it holds that

$$\forall z \in B(z_0, \delta) \cap D \implies z \in f^{-1}(B(y_0, \varepsilon)) \subseteq f^{-1}(U)$$

So it holds that $B(z_0, \delta) \cap D \subseteq f^{-1}(U)$.

$\Longleftarrow$

Let the preimage of every neighborhood in y_0 be an neighborhood of z_0 in D. Let $\varepsilon > 0$ arbitrary. Then it holds that $B(y_0, \varepsilon)$ is an neighborhood of y_0. By assumption it holds that $V = f^{-1}(B(y_0, \varepsilon))$ is an neighborhood of z_0 in D, hence

$$\exists \delta > 0 : B(z_0, \delta) \cap D \subseteq f^{-1}(B(y_0, \varepsilon)).$$

Therefore for $z \in B(z_0, \delta) \cap D$ it holds that $f(z) \in B(y_0, \varepsilon)$.

In other words:

$$|z - z_0| < \delta \wedge z \in D \implies |f(z) - \underbrace{f(z_0)}_{=y_0}| < \varepsilon$$

So f is continuous in z_0.

This notion of continuity is the most general one accepted by the mathematical community. It can be used in all topological spaces. $\square$

13.3 Variants of continuity

Definition 52. Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be a function f called uniformly continuous in D if

$$\forall \varepsilon > 0 \exists \delta > 0 : [\forall z_0, z_1 \in D \text{ with } |z_1 - z_0| < \delta \implies |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that δ only depends on ε, meaning that it can be arbitrarily shifted on the x-axis ($\delta = \delta(\varepsilon)$).

Reminder: f is continuous in D

$$\iff \forall z_0 \in D \forall \varepsilon > 0 \exists \delta > 0 : [\forall z_1 \in D \wedge |z_1 - z_0| < \delta \implies |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that δ depends on z_0 and ε ($\delta = \delta(\varepsilon, z_0)$). Therefore this second definition provides more freedom to parameter δ. So uniform continuity implies continuity in D.

Example 25. Let $f : (0, 1]$ and $f(x) = \frac{1}{x}$. f is continuous in every point $x_0 \in (0, 1]$. However, f is not uniformly continuous.

$$\forall \varepsilon > 0 \exists \delta > 0 : \left[\forall x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \implies \left|\frac{1}{x_0} - \frac{1}{x_1}\right| < \varepsilon\right]$$

The negation is given with:

$$\exists \varepsilon > 0 \forall \delta > 0 : \left[\exists x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \wedge \left|\frac{1}{x_0} - \frac{1}{x_1}\right| \geq \varepsilon\right]$$

We look at $\varepsilon = 1$. Let $\delta > 0$ arbitrary. We choose $x_0 = \frac{1}{n}$ and $x_1 = \frac{1}{n+1}$ for appropriate $n \in \mathbb{N}_+$. Then it holds that

$$|x_0 - x_1| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \underbrace{<}_{\text{for } n \in \mathbb{N}_+} \frac{1}{n} < \delta$$

if $n > \frac{1}{\delta}$

$$\left|\frac{1}{x_0} - \frac{1}{x_1}\right| = \left|\frac{1}{\frac{1}{n}} - \frac{1}{\frac{1}{n+1}}\right| = |n - (n+1)| = |-1| = 1$$

Therefore $f(x) = \frac{1}{x}$ is not uniformly continuous in $(0, 1]$.

Remark: $f(x) = \frac{1}{x}$ is uniformly continuous in $D = [\frac{1}{100}, 1]$, but not in $\mathbb{R}$.

Definition 53 (Lipschitz continuity)**.** Another notion of continuity is given by Rudolf Lipschitz (1832–1903).

$f : D \subseteq \mathbb{C} \mapsto \mathbb{C}$ is called Lipschitz continuous if $k \geq 0$ exists such that

$$\forall z_1, z_2 \in D : f(z_1) - f(z_2) \leq k\,|z_1 - z_2|$$

The value k is called Lipschitz constant for f.

Definition 54 (Hölder continuity)**.** Yet another notion of continuity is given by Otto Hölder (1859–1937).

f is called Hölder continuous with exponent $H \in (0, 1]$ if there exists $k > 0$ such that

$$\forall z_1, z_2 \in D : |f(z_1) - f(z_2)| \leq k\,|z_1 - z_2|^H$$

Corollary 13. A hierarchy for those continuity notion is given:

Lipschitz continuous $\subseteq$ uniformly continuous $\subseteq$ continuous in D.

Theorem 72. Let $K \subseteq \mathbb{C}$ be compact. Let $f : K \to \mathbb{C}$ be continuous in K. Then $f(K) = \{y = f(z) : z \in K\} \subset \mathbb{C}$ is compact in $\mathbb{C}$.

Proof. Every sequence $(y_n)_{n\in\mathbb{N}}$, with $y_n = f(z_n)$ and $z_n \in K$ where $y_n \in f(K)$, has a convergent subsequence. The sequence of preimage values $(z_n)_{n\in\mathbb{N}}$ is a sequence in K which, followingly, has a convergent subsequence. Let $(z_{n_k})_{k\in\mathbb{N}}$ $\lim_{k\to\infty} z_{n_k} = z \in K$. Because of the sequence criterion for continuity if holds that

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(z_{n_k}) = f(z) \in f(K-1)$$

with $y = f(k)$. So $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence with limes $y \in f(K)$. Therefore $f(K)$ is compact. $\square$

Definition 55. Let $f : D \to \mathbb{R}$ and $D \subseteq \mathbb{C}$. A point $z_{\max} \in D$ is called global maximum of f if $f(z_{\max}) \geq f(z) \quad \forall z \in D$.

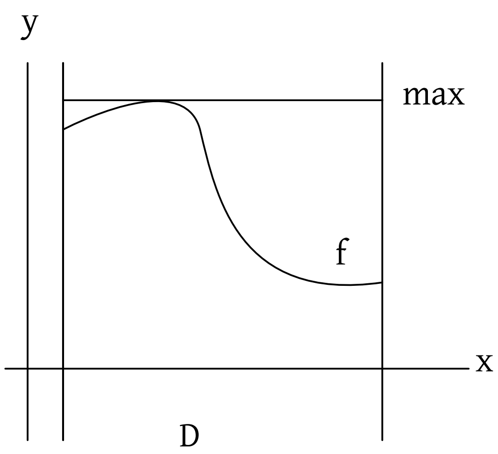

Figure 26: Illustration of a global maximum

Theorem 73. Let $f : K \to \mathbb{R}$ ($K \subseteq \mathbb{C}$) is continuous in K and K is compact in $\mathbb{C}$. Then f has a global maximum and a global minimum.

Remark 23. For non-compact definition sets this statement is not generally true. For example, $f(x) = \frac{1}{x}$ in $D = (0, 1)$ has neither a global maximum nor a global minimum.

Proof.

$$f(K) \subseteq \mathbb{R}$$

is compact (because of the previous theorem) and therefore bounded and closed in $\mathbb{R}$ (by Theorem by Bolzano-Weierstrass). Because $f(K)$ is bounded, $f(K)$ has a supremum ζ^* and an infimum ζ_* (supremum property). Supremum and infimum are contact points of $f(K)$. Because $f(K)$ is closed it holds that

$$\zeta^* \in f(K) \text{ and } \zeta_* \in f(K)$$

Therefore there exists $z_{\min} \in K$ with $f(z_{\min}) = \zeta_*$ and $z_{\max} \in K$ with $f(z_{\max}) = \zeta^*$. Because $f(K)$ is closed, it holds that $\zeta^* \in f(K)$ and $\zeta_* \in f(K)$. Therefore there exists $z_{\min} \in K$ with $f(z_{\min}) = \zeta_*$. and $f(z_{\max}) \in K$ with $f(z_{\max}) \geq y$, therefore $\forall z \in K : f(z_{\max}) \geq f(z)$.

Therefore $z_{\max}$ is a global maximum. The analogous statement holds for ζ_* and a global minimum. □

Theorem 74 (A very universal theorem about maxima)**.** A continuous function has a global maximum in a compact domain.

Using this method to show existence of a value is called "direct method of variation computations".

↓ This lecture took place on 8th of January 2016 with lecturer Wolfgang Ring

"Direct method of calculus of variations" (dt. "direkte Methode der Variationsrechnung").

Theorem 75 (Intermediate value theorem for continuous functions)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous with $a \leq b$. Let

$$m^* = \max \{f(x) : x \in [a, b]\}$$
$$m_* = \min \{f(x) : x \in [a, b]\}$$

m^* and m_* exist because $[a, b]$ is compact (bounded and closed).

Let $m_* \leq \eta \leq m^*$.

Then there exists $\xi \in [a, b]$ with $f(\xi) = \eta$. The function f takes any value for some x in m_* and m^*. Compare with Figure 27.

Proof. Let $a_0 \in [a, b]$ such that $f(a_0) = m_*$ and $b_0 \in [a, b]$ such that $f(b_0) = m^*$.

Without loss of generality: $a_0 \leq b_0$.

Case $\mathbf{a_0 = b_0}$: Then $\max \{f(x) : x \in [a, b]\} = f(b_0) = f(a_0) = \min \{f(x) : x \in [a, b]\}$. So f is constant, hence $f(x) = m_* = m^* \quad \forall x \in [a, b]$ and we're done.

$$m_* \leq \eta \leq m^* \implies m_* = \eta = m^* \implies f(x) = \eta \quad \forall x \in [a, b]$$

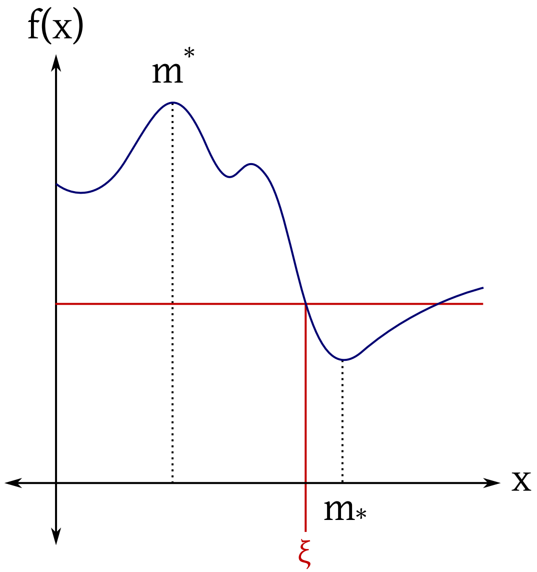

Figure 27: ζ in $[a,b]$

Case $\mathbf{a_0 < b_0}$: We know, $f(a_0) = m_* \leq \eta \leq m^* = f(b_0)$. We use nested intervals:

Assume $I_n = [a_n, b_n]$ for $n \in \mathbb{N}$ was already found with the property $f(a_n) \leq \eta \leq f(b_n)$. Let $m_n = \frac{1}{2}(a_n + b_n)$ be the midpoint of I_n.

Case $\mathbf{f(m_n)} \geq \eta$: Let $b_{n+1} = m_n$ and $a_{n+1} = a_n$ (compare Figure 28). Then $f(a_{n+1}) = f(a_n) \leq \eta$ and $f(b_{n+1}) = f(m_n) \geq \eta$. Furthermore $|I_{n+1}| = \frac{1}{2}|I_n|$.

Case $\mathbf{f(m_n)} < \eta$: Let $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

$$I_{n+1} = [a_{n+1}, b_{n+1}]$$

$$f(a_{n+1}) = f(m_n) < \eta$$

$$f(b_{n+1}) = f(b_n) \geq \eta$$

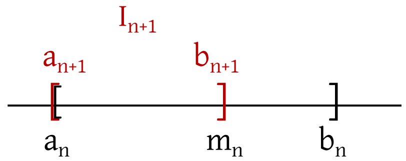

Figure 28: Interval I_{n+1}

$$|I_{n+1}| = \frac{1}{2}|I_n|$$

$(I_n)_{n\in\mathbb{N}}$ satisfy properties:

$$I_{n+1} \subseteq I_n \qquad |I_n| = \left(\frac{1}{2}\right)^n \cdot |I_0| = \left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)$$

and $f(a_n) \leq \eta \leq f(b_n)$. So, $(I_n)_{n\in\mathbb{N}}$ are nested intervals. Let $\xi \in \bigcap_{n\in\mathbb{N}} I_n$ and it holds that $|\xi - a_n| \leq |b_n - a_n| = \left(\frac{1}{2}\right)^n \cdot (b_0 - a_0) \xrightarrow{n\to\infty} 0$. Therefore $\lim_{n\to\infty} a_n = \xi$ and

$$|b_n - \xi| \leq |b_n - a_n| = \left(\frac{1}{2}\right)^n (b_0 - a_0) \xrightarrow{n\to\infty} 0.$$

So $\lim_{n\to\infty} b_n = \xi$.

Because f is continuous on $[a,b]$, it holds that

$$\begin{aligned} \eta \leq f(b_n) \quad \forall n \in \mathbb{N} &\implies \eta \leq \lim_{n\to\infty} f(b_n) \\ \text{continuity} &\implies \lim_{n\to\infty} f(b_n) = f(\xi) \\ \text{continuity} &\implies f(\xi) = \lim_{n\to\infty} \underbrace{f(a_n)}_{\leq\eta} \end{aligned}$$

So,

$$\eta \leq \lim_{n\to\infty} f(b_n) = f(\xi) = \lim_{n\to\infty} f(a_n) \leq \eta.$$

Therefore $\eta = f(\xi)$.

□

Remark 24. From this we can derive continuity for a numerical algorithm for solving $f(x) = \eta$. It's called *bisection method*.

Remark 25. Often the intermediate value theorem is defined as:

Let η be between $f(a)$ and $f(b)$. Then there exists $\xi \in [a, b]$ such that $f(\xi) > \eta$. Obviously because $m_* \leq f(a)$ and $f(b) \leq m^*$.

Definition 56 (Limes of a function)**.** Let $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$. Let z be a limit point of D. We say, that f in z has the limes w if the function

$$\hat{f} : D \cup \{z\} \to \mathbb{C}$$

$$\hat{f}(\xi) = \begin{cases} f(\xi) & \text{if } \xi \neq z \\ w & \text{if } \xi = z \end{cases}$$

is continuous. We denote $\lim_{\xi \to z} f(\xi)$.

Example 26. See Figures 29, 30 and 31.

Lemma 17. Let $f : D \to \mathbb{C}$ given and z is a limit point of $D \subseteq \mathbb{C}$. Then f has a limes $w \in \mathbb{C}$ if and only if one of the equivalent conditions hold.

- $\forall \varepsilon > 0 \exists \delta > 0 \forall \xi \in D : |z - \xi| < \delta \implies |\underbrace{f(\xi)}_{\hat{f}(\xi)} - \underbrace{w}_{\hat{f}(z)}| < \varepsilon$

  "Continuity of $\hat{f}$"

- $\forall (\xi)_{n \in \mathbb{N}}$ with $\xi_n \in D \setminus \{z\}$ and $\lim_{n \to \infty} \xi_n = z$ holds.

  $$\lim_{n \to \infty} f(\xi_n) = w$$

  "Sequence criterion for $\hat{f}$"

Example 27. $f : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ with

$$f(z) = \frac{z^2 - 1}{z - 1}$$

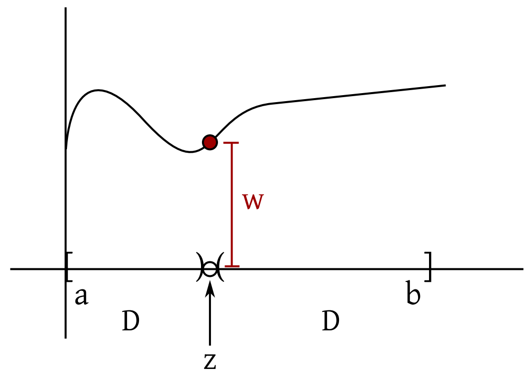

Figure 29: Example 1 with $D = [a, b] \setminus \{z\}$ and $w = \lim_{\xi \to z} f(\xi)$

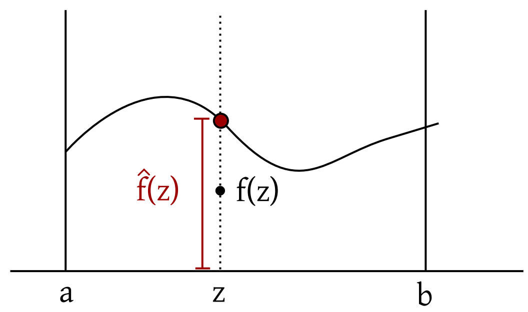

Figure 30: Example 2 which defines function new in point z with $D = [a, b]$ and $\lim_{\xi \to z} f(\xi)$. f is not continuous in z, but $\hat{f}$ is continuous in z.

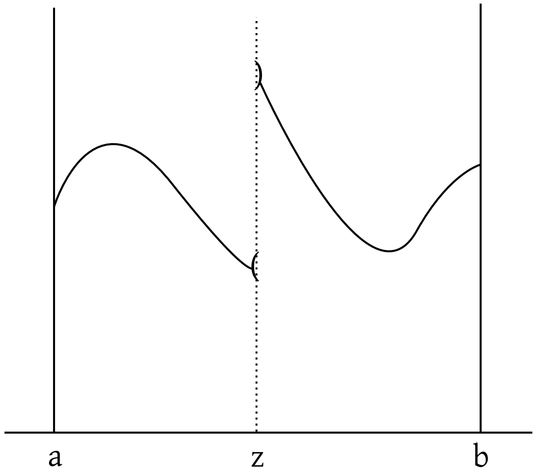

Figure 31: Example 3 with $D = [a,b] \setminus \{z\}$. f does not have a limes in z. Due to the jumping point, it is not a continuous function. Therefore we cannot find ε. We say $\hat{f}$ is a continuous continuation of f in point z.

For $z \neq 1$ it holds that:

$$f(z) = \frac{(z-1)(z+1)}{(z-1)} = (z+1)$$

Let

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq 1 \\ 2 & \text{if } z = 1 \end{cases}$$

$\hat{f}(z) = z+1$ in $\mathbb{C}$ is continuous. f has limes $w = 2$ in point $z > 1$.

Example 28. Let $s \in \mathbb{Q} \setminus \{0\}$ and $D = (-1, \infty) \setminus \{0\}$

$$f(x) = \frac{(1+x)^s - 1}{x}$$

It holds that $\lim_{x \to 0} f(x) = s$.

for $|x| < 1$.

$$(1+x)^s = \sum_{k=0}^{\infty} \binom{s}{k} x^k \implies \frac{(1+x)^s - 1}{x}$$

$$\implies \frac{(1+x)^s - 1}{x} = \frac{\sum_{k=1}^{\infty} \binom{s}{k} x^k}{x} = \sum_{k=1}^{\infty} \binom{s}{k} \cdot x^{k-1}$$

$$\lim_{x \to 0} \underbrace{\left(\sum_{k=1}^{\infty} \binom{s}{k} x^{k-1} \right)}_{f(x)} = \sum_{k=1}^{\infty} \binom{s}{k} 0^{k-1} = \binom{s}{1} = s$$

We need the following theorem: A power series is in its convergence radius a continuous function. □

14 Differential calculus

Let $f : (a,b) \to \mathbb{R}$ be given. with $a < b$.

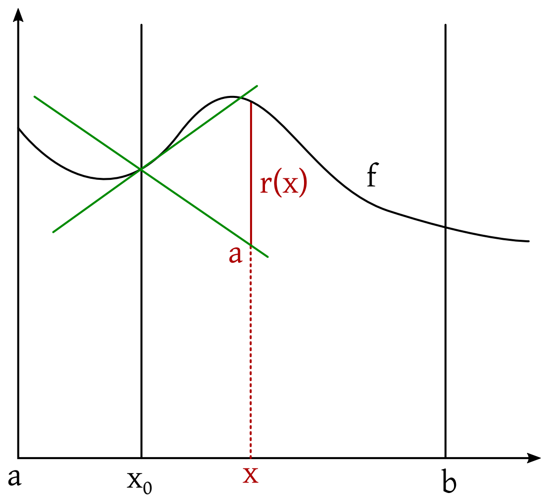

Figure 32: Differential f in x_0

Idea: We want f close to point $x_0 \in (a, b)$ be approximated by a linear-affine function $a(x) = k(x - x_0) + d$.

$$a(x) = k(x - x_0) + d = kx + \underbrace{(-kx_0 + d)}_{\tilde{d}} = kx + \tilde{d}$$

$\tilde{a}(x) = kx$ is linear. Linear and constant functions are linear affine. a should (at least) cross point x_0, ie. $f(x_0)$. Compare with Figure 32.

$$\implies a(x_0) = k(\underbrace{x_0 - x_0}_{0}) + a \stackrel{!}{=} f(x_0) \implies d = f(x_0)$$
$$\implies a(x) = k(x - x_0) + f(x_0)$$

How should we select k such that the approximation of f is best possible by selection of a. We consider the deviation.

$$f(x) = f(x) - a(x)$$

$r(x)$ should be as small as possible in x_0. Therefore $\lim_{x \to x_0} r(x) = 0$.

$$\lim_{x \to x_0} r(x) = \lim_{x \to x_0} [f(x) - f(x_0) - k \cdot (x - x_0)] = 0 \quad \forall k$$

We need: $r(x)$ should converge to 0 very quickly for $x \to x_0$.

Idea: Require that $\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0$. $\frac{1}{x - x_0}$ is unbounded close to x_0. $\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0$ means $\lim_{x \to x_0} \left| \frac{r(x)}{x - x_0} - 0 \right| = 0$

$$\implies \lim_{x \to x_0} \left| \frac{f(x) - f(x_0) - k \cdot (x - x_0)}{x - x_0} \right| = \lim_{x \to x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - k \right|$$

Hence,

$$\implies k = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

with

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0,$$

k is uniquely identified with

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

↓ This lecture took place on 13th of January 2016 with lecturer Wolfgang Ring

$$\begin{aligned} y &= kx + d \\ d &= f(x_0) - k \cdot x_0 \text{ and accordingly} \\ y &= k(x - x_0) + f(x_0) \end{aligned}$$

k is defined such that

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0 \qquad r = f(x) - (f(x_0) + k(x - x_0))$$

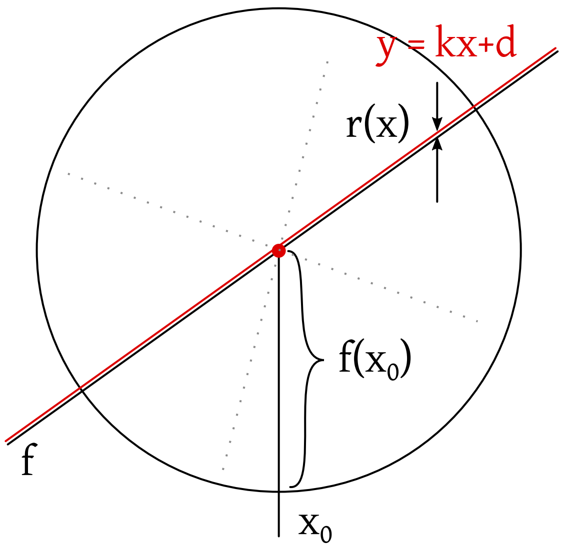

Figure 33: The derivative is a linear approximation

$$\implies k = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

Definition 57 (Landau's symbols)**.** Let $g : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$. Let z_0 be a limit point of g and assume g has a limit point 0 for $z \to z_0$. Therefore,

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall z \in D \land |z - z_0| < \delta$$

where $z \neq z_0$.

$$\implies |g(z) - 0| < \varepsilon$$

We say that y is *of order* $\mathcal{O}(n)$ in point z_0, if $k \geq 0$ and some $r > 0$ such that

$$|g(z)| \leq K\,|z - z_n|^n \quad \forall z \in D \text{ with } |z - z_0| < r \land z \neq z_0$$

We denote it with $g(z) = \mathcal{O}(|z - z_0|^n)$.

We say that g is of order $o(n)$ if $r > 0$ and some function $k : (0, r) \to \mathbb{R}^+$ with $\lim_{x\to 0} k(x) = 0$ exists, such that

$$|g(z)| \leq k(|z - z_0|) \cdot |z - z_0|^n \,\forall z \in D \text{ with } |z - z_0| < r \land z \neq z_0$$

We denote,

$$g(z) = o(|z - z_0|^n)$$

Corollary 14. It holds that, $g : \mathcal{O}(|z - z_0|^n) \iff \exists r > 0$ such that

$$\frac{|g(z)|}{|z - z_0|^n}$$

is bounded in $B(z_0, z) \setminus \{z_0\}$ and $g = o(|z - z_0|^n)$, if $\exists r > 0$ such that $\frac{|g(z)|}{|z-z_0|^n}$ in point z_0 has limit point 0.

Corollary 15. For determination of the slope k for the best-achievable linear-affine approximation of f it must hold that

$$f(x) - (f(x_0) - k(x - x_0)) = o(|x - x_0|)$$

Definition 58. Let $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$. We claim that f in x_0 is *differentiable*, if the limit point of the function $\frac{f(x)-f(x_0)}{x-x_0}$ exists. The corresponding limit point $k = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ is called *derivative of f in* x_0.

We can compute k using $k = f'(x_0)$.

Alternatively: f is differentiable in x_0 if $x \in \mathbb{R}$ exists, such that $r : (a, b) \setminus \{0\} \to \mathbb{R}$ with $r(x) = f(x) - f(x_0) - k(x - x_0)$ is of order $o(1)$ in x_0.

$$f(x) - f(x_0) - k(x - x_0) = \mathcal{O}(|x - x_0|)$$

The second definition is more general and can also be applied for functions $f : \mathcal{O} \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

Corollary 16. Let $f : (a, b) \to \mathbb{R}$ be differentiable in $x_0 \in (a, b)$. Then the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \in (a, b) \setminus \{x_0\} \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

$\varphi : (a, b) \to \mathbb{R}$ and φ is continuous in x_0.

Show that $\lim_{x \to x_0} \varphi(x) = \varphi(x_0)$.

$$f(x) = f(x_0) + \varphi(x)(x - x_0)$$

because $\varphi(x) = \frac{f(x)-f(x_0)}{x-x_0}$ for $x \neq x_0$. $f(x)$ is constant, $\varphi(x)$ is continuous in x_0 and $(x - x_0)$ is continuous in (a, b). For $x = x_0$, $f(x) = f(x_0) + \varphi(x)(x - x_0)$ holds as well.

Therefore all expressions of $f(x_0)+\varphi(x)(x-x_0)$ are continuous in x_0, followingly f is continuous in x_0.

Lemma 18. Let $f : (a, b) \to \mathbb{R}$ be differentiable in $x_0 \in (a, b)$.

$$k = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is slope of affine function, which approximates f in x_0.

Plot of this function:

$$y(x) = f'(x_0)(x - x_0) - f(x_0)$$

is called *tangent* of f in x_0.

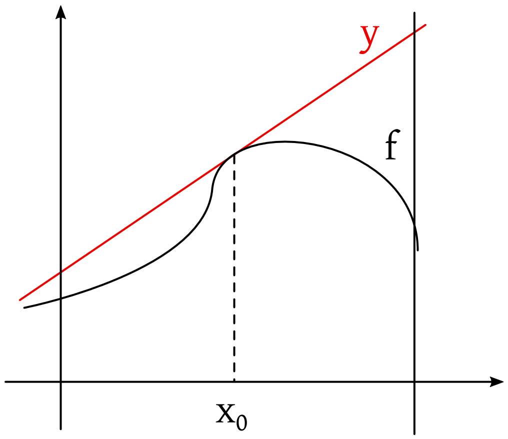

Figure 34: Tangent of f in x_0

↓ This lecture took place on 14th of Jan 2016 with lecturer Wolfgang Ring

Theorem 76 (Convergence, limes and differentiable functions)**.** Let $f : (a, b) \to \mathbb{R}$ be differentiable in $x_0 \in (a, b)$. Therefore the equivalent defining properties hold.

1. $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in (a, b)$ with $|x - x_0| < \delta$ and $x \neq x_0$ it holds that $\left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \varepsilon$. This constitutes a definition of the limes.
2. For all $(\xi_n)_{n\in\mathbb{N}}$ with $\xi_n \in (a, b)$ and $\xi_n \neq x_0$ and $\lim_{n\to\infty} \xi_n = x_0$, it holds that
$$\left(\frac{f(\xi_n) - f(x_0)}{\xi_n - x_0}\right)_{n\in\mathbb{N}} \text{ is convergent towards } f'(x_0)$$
This is the sequence criterion for the limes.
3. For all $\varepsilon > 0$, there exists some $\delta > 0$ such that $\forall x \in (a, b)$ with $|x - x_0| < \delta$ it holds that
$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \varepsilon\,|x - x_0|$$
holds also for $x = x_0$.

The (3) implies the (1): Assume (3) holds and choose δ such that $\forall\,|x - x_0| < \delta$ it holds that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{\varepsilon}{2}\,|x - x_0| \underbrace{<}_{\text{for } x\neq x_0} \varepsilon\,|x - x_0| \underbrace{\Longrightarrow}_{\text{divide by } x - x_0} \quad (1)$$

14.1 Derivatives of common functions

1. Let $p_n : \mathbb{R} \to \mathbb{R}$, $p_n(x) = x^n$. Let $x_0 \in \mathbb{R}$ and $x \neq x_0$ and $n \in \mathbb{N}$. Then,

$$\begin{aligned}\frac{p_n(x) - p_n(x_0)}{x - x_0} &= \frac{x^n - x_0^n}{x - x_0} = \frac{(x - x_0) \cdot \sum_{k=0}^{n-1} x^k x_0^{n-1-k}}{x - x_0} \\ &= \sum_{k=0}^{n-1} x^k x_0^{n-1-k} \xrightarrow{x \to x_0} \sum_{k=0}^{n-1} x_0^k x_0^{n-1-k} \\ &= \sum_{k=0}^{n-1} x_0^{n-1} = n x_0^{n-1}\end{aligned}$$

Therefore p_n is differentiable in x_0 and $p_n'(x_0) = n x_0^{n-1}$.

$$(x^n)' = n x^{n-1} \qquad \forall n \in \mathbb{N}$$

2. Let $f(x) = a^x$ with $a > 0$. This function is called *exponential function* with basis a. It holds that:

$$\begin{aligned}\frac{a^x - a^{x_0}}{x - x_0} &= \frac{a^{x_0} \cdot a^{x-x_0} - a^{x_0}}{x - x_0} = a^{x_0} \cdot \frac{a^{x-x_0} - 1}{x - x_0} \\ &\xrightarrow{x \to x_0} a^{x_0} \cdot \lim_{x \to x_0} \frac{a^{x-x_0} - 1}{x - x_0} = \left| \begin{array}{c} x - x_0 = h \\ x \to x_0 \iff h \to 0 \end{array} \right| \\ &= a^{x_0} \lim_{h \to 0} \underbrace{\frac{a^h - 1}{h}}_{=c \in \mathbb{R}}\end{aligned}$$

Therefore $|a^x|' = c \cdot a^k$ with $c = \lim_{h \to 0} \frac{a^k - 1}{h}$.
In the special case that this constant a is the Eulerian number e, we have:

$$\lim_{h \to 0} \frac{e^k - 1}{h} = 1 \qquad (e^x)' = e^x$$

3. $\log : (0, \infty) \to \mathbb{R}$ with $e^{\log x} = x \; \forall x > 0$ or equivalently $\log(e^y) = y \; \forall y \in \mathbb{R}$.

$$\frac{\log x - \log x_0}{x - x_0} = \frac{\log \frac{x}{x_0}}{x - x_0} = \frac{1}{x_0} \frac{\log \frac{x}{x_0}}{\frac{x}{x_0} - 1} \to \frac{1}{x_0} \cdot \underbrace{\lim_{h \to 1} \frac{\log h}{h - 1}}_{=1} = \frac{1}{x_0}$$

Therefore $(\log x)' = \frac{1}{x}$ for $x > 0$.

14.2 Derivation laws

Theorem 77. Let $f, g : (a, b) \to \mathbb{R}$. Let $x_0 \in (a, b)$ and let f, g be differentiable in x_0. Then it holds that

- $f + g : (a, b) \to \mathbb{R}$ is differentiable in x_0 and the derivative is given by $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.
- Let $\lambda \in \mathbb{R}$. Then it holds that $\lambda \cdot f : (a, b) \to \mathbb{R}$ is differentiable in x_0 and it holds that $(\lambda f)'(x_0) = \lambda \cdot (f'(x_0))$.
- Let $f \cdot g : (a, b) \to \mathbb{R}$ be differentiable and it holds that

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + g'(x_0) \cdot f(x_0)$$

This is the so-called *product law for derivatives.*

Proof.

- Addition holds:

$$\begin{aligned}f'(x_0) + g'(x_0) &= \lim_{x \to x_0} x - x_0 + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} = (f + g)'(x_0)\end{aligned}$$

- Multiplication with a scalar holds:

$$\lambda f'(x_0) = \lambda \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\lambda f(x) - \lambda f(x_0)}{x - x_0} = (\lambda f)'(x_0)$$

- The product law holds:

$$\begin{aligned}&f'(x_0) g(x_0) + f(x_0) g'(x_0) \\ &= g(x_0) \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \underbrace{f(x_0)}_{=\lim_{x \to x_0} f(x)} \cdot \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}\end{aligned}$$

because f is differentiable and therefore continuous in x_0.

$$= \lim_{x\to x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0)}{x - x_0} + \lim_{x\to x_0} \frac{f(x)\cdot g(x) - f(x)\cdot g(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0) + g(x)f(x) - g(x_0)f(x)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{f(x)\cdot g(x) - f(x_0)g(x_0)}{x - x_0} = (f\cdot g)'(x_0)$$

□

Definition 59. Let $f : (a, b) \to \mathbb{R}$ be given. Assume f is differentiable in *every* point $x_0 \in (a, b)$, then we call f is *differentiable on interval* (a, b). The mapping $f' : (a, b) \to \mathbb{R}$ which assigns $x \in (a, b)$ its $f'(x)$, is called *derivative function.*

f is called *continuously* differentiable if f' is a continuous function on (a, b).

↓ This lecture took place on 15th of Jan 2016 with lecturer Wolfgang Ring

Remark 26. Let $D \subseteq \mathbb{R}$ and let $x_0 \in D$ be limit point of D. Then the function

$$\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0} \text{ in } D \setminus \{x_0\}$$

can be investigated and the question of existence of a limes of φ (theoretically) answered.

Therefore the function $f : [a, b] \to \mathbb{R}$ can be discussed in term of convergence and $f'(a)$ and $f'(b)$ can be defined (under the assumption that the limes exists)

$$k = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \iff \forall(\xi)_{n\in\mathbb{N}}, \min \xi_n \geq a, \lim_{n\to\infty} \xi_n = a$$

$$\implies \lim_{n\to\infty} \frac{f(\xi_n) - f(a)}{\xi_n - a} = k$$

The derivative in a is *right-sided*. The derivative in b is *left-sided.*

Remark 27. Functions that are not differentiable:

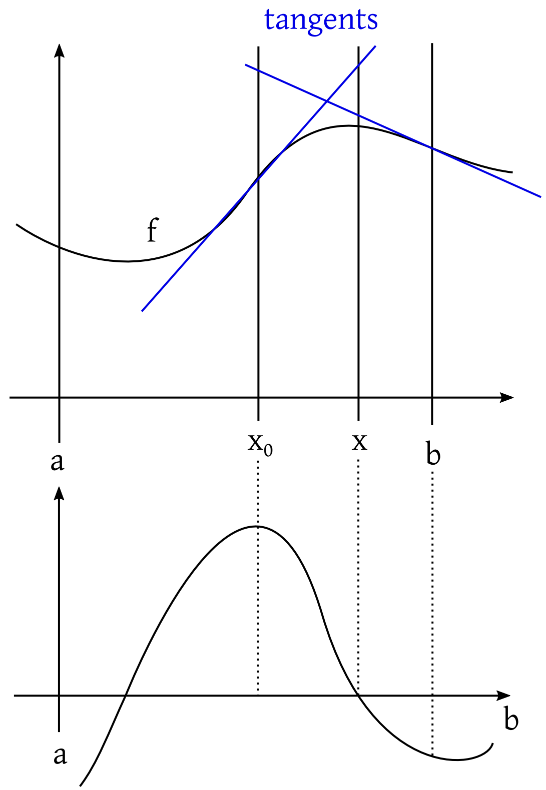

Figure 35: Slopes and tangents of two functions

- $f(x) = x$ is not differentiable in $x = 0$.

  Proof. Let $\varepsilon_1 = \frac{1}{n}$.

  $$\lim_{n\to\infty} \frac{f(\xi_n) - f(0)}{\xi_n - 0} = \lim_{n\to\infty} \frac{\left|\frac{1}{n}\right| - |0|}{\frac{1}{n} - 0} = 1 \overset{n\to\infty}{\to} 1$$

  "right-sided limes"

Let $\eta_n = -\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{f(\eta_n) - f(0)}{\eta_n - 0} = \frac{\left|-\frac{1}{n}\right| - 0}{-\frac{1}{n} - 0} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1 \overset{n\to\infty}{-} 1$$

"left-sided limes"

Therefore limes of $f(\xi_n)$ and $f(\eta_n)$ are different even though both sequences $(\xi_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ have the same limes. Therefore it is not differentiable in $x = 0$. □

- Consider $g : [a, b] \to \mathbb{R}$ with $g(x) = \sqrt{x}$. Claim: g is not differentiable in $x = 0$.

Proof. Let $(\xi)_{n\in\mathbb{N}}$ and $\xi_n = \frac{1}{n} \implies \lim_{n\to\infty} \xi_n = 0$.

$$\frac{g(\xi_n) - g(0)}{\xi_n - 0} = \frac{\sqrt{\frac{1}{n}} - \sqrt{0}}{\frac{1}{n} - 0} = \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

$(\sqrt{n})_{n\in\mathbb{N}}$ is unbounded, therefore not convergent. □

14.3 Computing with the limes of functions

We actually used that already (for example, when proving the product law for derivatives).

Theorem 78. Let $f, g : D \to \mathbb{C}$ with $d \subseteq \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be limit point of D and f has limes $a \in \mathbb{C}$ in z_0 and g has limes b in z_0. Then

- $(f + g)$ has limes $a + b$ in z_0.
- $(f \cdot g)$ has limes $a \cdot b$ in z_0
- If $g(z) \neq 0 \quad \forall z \in D$ and $b \neq 0$, then $\frac{f}{g}$ has the limes $\frac{a}{b}$ in z_0.

Proof. Sequence criterion and laws for convergent sequences. Let $(\xi)_{n\in\mathbb{N}}$ and $\xi_n \in D$ and $\lim_{n\to\infty} \xi_n = z_0$. Because f has limes a and g has limes b, it holds that

$$\lim_{n\to\infty} f(\xi_n) = a \wedge \lim_{n\to\infty} g(\xi_n) = b$$

Due to the laws for convergent sequences:

$$\underbrace{\lim_{n\to\infty} f(\xi_n) + \lim_{n\to\infty} g(\xi_n)}_{a+b}$$

$$= \lim_{n\to\infty} (f(\xi_n) + g(\xi_n)) = \lim_{n\to\infty} (f + g)(\xi_n)$$

Therefore $\lim_{\xi\to z_0}(f + g)(\xi) = a + b$.

The proofs work analogously for $\cdot$ and $/$. □

14.4 Other equivalent definitions of differential calculus

Theorem 79.

$$f : [a, b] \to \mathbb{R} \text{ or } f : (a, b) \to \mathbb{R}$$

In general, let I be an interval, $f : I \to \mathbb{R}$ and $x_0 \in I$. Then f is differentiable in x_0 if and only if there exists $\varphi : I \to \mathbb{R}$ such that φ is continuous in x_0 and $f(x) = f(x_0) + \varphi(x)(x - x_0)$.

If φ exists with such properties, $f'(x_0) = \varphi(x_0)$.

Proof. $\impliedby$ Let $x \neq x_0$, $x \in I$ and it holds that $f(x) = f(x_0) + \varphi(x)(x - x_0)$, then

$$\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

because φ is continuous, there exists some limes

$$\lim_{x\to x_0} \varphi(x) = \varphi(x_0)$$

Hence f is differentiable and $f'(x_0)$.

$\implies$ Let f be differentiable. Then we define

$$\varphi(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

then φ is continuous in x_0 and

$$f(x) = f(x_0) + \varphi(x)(x - x_0) \text{ for } x \neq x_0$$

$$f(x_0) = f(x_0) + \varphi(x_0)\underbrace{(x_0 - x_0)}_{0} \text{ for } x = x_0$$

□

Theorem 80. Let J, I be intervals.

$$f : I \to J \qquad g : J \to \mathbb{R}$$

f is differentiable in $x_0 \in I$ and let g be differentiable in $y_0 = f(x_0)$. Then $g \circ f : I \to \mathbb{R}$ is differentiable in x_0 and it holds that

$$(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Proof. f is differentiable implies $\exists \varphi : I \to \mathbb{R}$ is continuous in x_0 with $f(x) = f(x_0) + \varphi(x)(x - x_0)$.

g is differentiable implies $\exists \psi : J \to \mathbb{R}$ with $g(y) = g(y_0) + \psi(y)(y - y_0)$ is continuous.

Let $y \in f(I)$, hence $y = f(x)$ and $y_0 = f(x_0)$. It follows (due to the previous theorems) that

$$g(f(x)) = g(f(x_0)) + \psi(f(x))\underbrace{(f(x) - f(x_0))}_{\varphi(x)(x-x_0)}$$

$$= g(f(x_0)) + \psi(f(x))\varphi(x)(x - x_0)$$

$$g \circ f(x) = g \circ f(x_0) + (\psi \cdot f)(x) \cdot \varphi(x) \cdot (x - x_0)$$

$$\vartheta(x) = \psi \circ f(x) \cdot \varphi(x)$$

with $\vartheta : I \to \mathbb{R}$ and f is continuous in x_0, because it is differentiable, ψ is continuous in $y_0 = f(x_0)$ and φ is continuous in x_0. Therefore ϑ is continuous in x_0 and $g \circ f(x) = g \circ f(x_0) + \vartheta(v)(x - x_0)$. Therefore $g \circ f$ is differentiable in x_0 and

$$(g \circ f)'(x_0) = \vartheta(x_0) = \underbrace{\psi(f(x_0))}_{g'(f(x_0))} \cdot \underbrace{\varphi(x_0)}_{f'(x_0)}.$$

□

Example 29.

$$f : \mathbb{R} \to \mathbb{R}^+, f(x) = x^2$$

$$g : \mathbb{R} \to \mathbb{R}, g(x) = e^x$$

$$g \circ f : \mathbb{R} \to \mathbb{R}$$

$$g \circ f(x) = e^{f(x)} = e^{x^2}$$

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

$$g'(y) = e^y, g'(f(x_0)) = e^{f(x_0)} = e^{x_0^2}$$

$$f'(x_0) = 2x_0$$

$$(e^{x^2})' = \underbrace{e^{x^2}}_{\text{outer derivative}} \cdot \underbrace{2x}_{\text{inner derivative}}$$

$$f \circ g : \mathbb{R} \to \mathbb{R}$$

$$(f \circ g)(x) = (e^x)^2$$

$$(f \circ g)'(x) = \underbrace{f'(g(x))}_{2(y(x))=2e^x} \circ \underbrace{g'(x)}_{=e^x}$$

$$\implies 2 \cdot e^x \cdot e^x = 2e^{2x}$$

Example 30. We decompose this function h.

$$h(x) = \cos(\sqrt{x^2 + 1})$$

$$h(x) = g \circ f(x)$$

So we either get

$$g(y) = \cos(\sqrt{y})$$

$$f(x) = x^2 + 1$$

or

$$g(y) = \cos(y)$$

$$f(x) = \sqrt{x^2 + 1}$$

Both are correct. Not the second decomposition is way more useful.

Theorem 81. Consider $r : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ and $r(x) = \frac{1}{x}$. Then it holds that r is differentiable for all $x_0 \neq 0$ and $r'(x_0) = -\frac{1}{x_0^2}$.

Proof.

$$\lim_{x \to x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\frac{x_0 - x}{x - x_0}}{x - x_0}$$

$$= - \lim_{x \to x_0} \frac{1}{x - x_0}$$

$$= \frac{1}{x_0^2}$$

□

Theorem 82. Let $g : I \to \mathbb{R}$ with $g(x) \neq 0 \quad \forall x \in I$ where I is an interval. Let g be differentiable in $x_0 \in I$. Then $\frac{1}{g} : I \to \mathbb{R}$ is differentiable in x_0 and it holds that $\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}$.

Furthermore let $f : I \to \mathbb{R}$ differentiable in x_0. Then the quotient $\left(\frac{f}{g}\right)$ is differentiable in x_0 and it holds that

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{(g(x_0))^2}$$

"Quotient law"

Proof. To be done rigurously next Wednesday.

Idea: $\frac{1}{g} = r \circ g$ and quotient law

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

□

↓ This lecture took place on 20th of January 2016 with lecturer Wolfgang Ring

Proof.

$$\frac{1}{g} = r \circ g \qquad r(y) = \frac{1}{y}$$

Chain rule: $x_n \in I$ and g differentiable in x_0, $y_0 = g(x_0) \neq 0$ and $r(y) = \frac{1}{y}$ in y_0. Therefore $g \circ y$ is in x_0 and

$$(r \circ g)' x_0 = r'(g(x_0)) \cdot g'(x_0) = -\frac{1}{g(x_0)^2} \cdot r'(x_0)$$

$$\frac{f}{g} = f \cdot 1g$$

Product law:

$$\left(\frac{f}{g}\right)'(x_0) = f'(x_0) \cdot \frac{1}{g(x_0)} + f(x_0) \cdot \left(-\frac{g'(x_0)}{(g(x_0))^2}\right) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$$

□

Remark 28. What is differential calculus good for?

Geometrical investigation of functions.

Definition 60. Let $f : I \to \mathbb{R}$ be a function. I is an interval. We call $x_0 \in I$ a *local maximum* of f, if $\varepsilon > 0$ exists such that

$$[x \in I \wedge |x - x_0| < \varepsilon] \implies f(x) \leq f(x_0)$$

We call $x_0 \in I$ a *local minimum* of f, if $\varepsilon > 0$ exists such that

$$[x \in I \wedge |x - x_0| < \varepsilon] \implies f(x) \geq f(x_0)$$

Theorem 83 (Necessary optimality criterion)**.** Let $f : I \to \mathbb{R}$ be differentiable and I is an interval. Let $x_0 \in I$ be a local maximum of f. Then there exists $\varepsilon > 0$ such that for all $x \in I$ with $|x - x_0| < \varepsilon$ the following relation holds:

$$f'(x_0)(x - x_0) \leq 0.$$

Remark 29. This is a more general statement than $f'(x_0) = 0$.

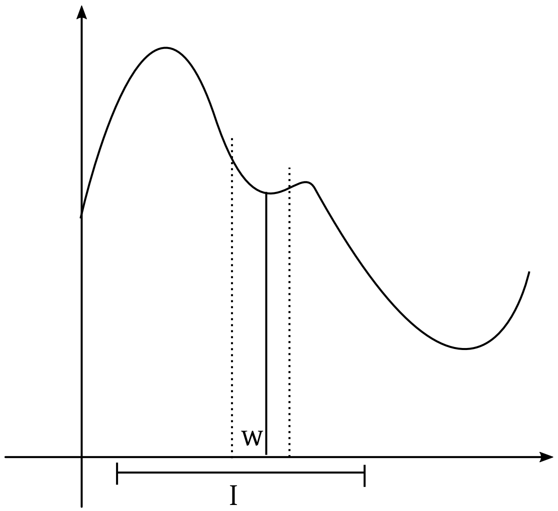

Figure 36: Local minimum w

Proof. Let x_0 be a local maximum. Assume

$$\forall \varepsilon > 0 \exists x_\varepsilon : |x_\varepsilon - x_0| < \varepsilon \wedge f'(x_0)(x_\varepsilon - x_0) > 0$$

Especially: $\varepsilon = \frac{1}{n}$, $x_\varepsilon = x_n$. Therefore it holds that $\lim_{n\to\infty} x_n = x_0$ and $f'(x_0)(x_n - x_0) > 0$. Followingly both factors must be non-zero, hence $f'(x_0) \neq 0$. □

Theorem 84 (Differentiability of f in x_0)**.**

$$f(x_0) = f(x_0) - f'(x_0)(x_n - x_0) + \underbrace{r(x_0)(x_n - x_0)}_{\mathcal{O}(|x_n - x_0|)}$$

$$\lim_{x\to x_0} r(x) = 0$$

Let n sufficiently large such that

$$|f(x_n)| \leq \frac{1}{2} \underbrace{|f'(x_0)|}_{>0} \quad \forall n \geq N$$

Then it holds that

$$\begin{aligned} f(x_n) - f(x_0) &= \overbrace{f'(x_0)(x_n - x_0)}^{>0} + r(x_n)(x_n - x_0) \\ &= |f'(x_0)(x_n - x_0)| + r(x_n)(x_n - x_0) \\ &\geq |f'(x_0)|\,|x_n - x_0| - |r(x_0)|\,|x_n - x_0| \\ &= (\Big| f'(x_0) - \underbrace{|r(x_n)|}_{\leq \frac{1}{2}|f'(x_0)|} \Big|) \cdot |x_n - x_0| \geq \frac{1}{2} \\ &= \frac{1}{2} \underbrace{f'(x_0)(x_n - x_0)}_{>0} > 0 \end{aligned}$$

and therefore $f(x_n) > f(x_0) \quad \forall n \geq N$. This is a contradiction to the assumption that x_0 is a local maximum.

Remark 30. x_0 is a local minimum. Therefore

$$f'(x_0)(x - x_0) \geq 0 \quad \forall |x - x_0| < \varepsilon \text{ where } x \in I$$

Corollary 17. Let I be an interval and x_0 an inner point of I (therefore $\exists \varepsilon > 0 : (x_0 - \varepsilon, x_0 + \varepsilon) < I$). Assume $f : I \to \mathbb{R}$ has a local maximum (or minimum) in x_0 and let f be differentiable. Then it holds that

$$f'(x_0) = 0$$

Proof. Let $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \in I$ and let $x = x_0 + \frac{\varepsilon}{2} \in I$.

The optimality criterion is given with:

$$f'(x_0) \cdot (x - x_0) = f'(x_0) \left(x_0 + \frac{\varepsilon}{2} - x_0 \right) = \frac{\varepsilon}{2} f'(x_0) \leq 0$$

$$w = x_0 - \frac{\varepsilon}{2} \in I$$

Necessary optimality criterion:

$$f'(x_0)(w - x_0) = f'(x_0)\left(x_0 - \frac{\varepsilon}{2} - x_0\right) = -\frac{\varepsilon}{2} f'(x_0) \le 0$$

$$f'(x_0) \le 0 \text{ and } f'(x_0) \ge 0 \implies f'(x_0) = 0$$

□

↓ This lecture took place on 21st of January 2016 with lecturer Wolfgang Ring

Theorem 85 (Consideration of optimal points at the margins of I)**.** Let $I = [a, b]$ and $x_0 = a$ is a local maximum. Then the necessary optimality criterion (NOC) yields:

$$\text{NOC:} \qquad f'(a)(x - a) \le 0 \qquad x \in [a, b]$$

and x sufficiently close to a. Choose $\varepsilon > 0$ small enough such that for $x = a + \varepsilon$ (necessary optimality criterion), we have

$$\implies f'(a)(a - \varepsilon - a) = \underbrace{\varepsilon}_{>0} f'(a) \le 0 \qquad \implies f'(a) \ge 0$$

Analogously:
$x_0 = a$ is a local minimum. So $f'(a) \ge 0$.
$x_0 = b$ is a local maximum. So $f'(b) \ge 0$.
$x_0 = b$ is a local minimum. So $f'(b) \le 0$.

Michel Rolle (1652–1719)

Theorem 86 (Rolle's theorem)**.** Let $I = [a, b]$ and $f : I \to \mathbb{R}$ is differentiable in I. Furthermore it holds that $f(a) = f(b)$. Then there exists some $\xi \in [a, b]$ with $f'(\xi) = 0$ (even with $\xi \in (a, b)$).

Proof. **Case 1: f constant** Therefore $f(x) = f(a) = f(b) = c \forall x \in [a, b]$

$$\implies f'(x) = 0 \forall x \in [a, b]$$

Case 2: f is non-constant Therefore $\exists x \in (a, b)$ with $f(x) \ne f(a)$. Without loss of generality: $f(x) > f(a) = f(b)$. $[a, b]$ is a compact interval. f is

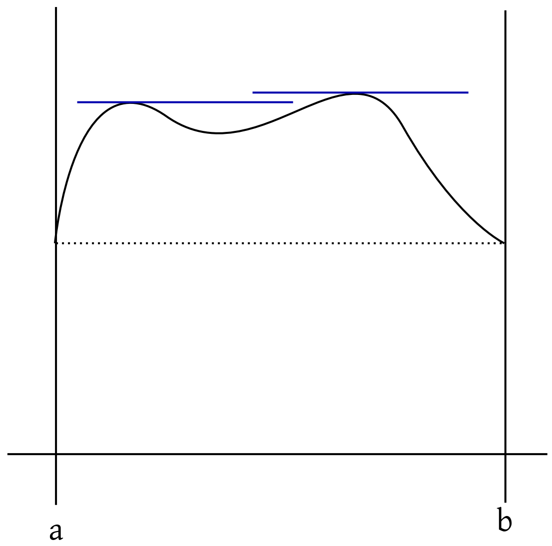

Figure 37: Rolle's theorem says that one x with $f'(x) = 0$ must exist between two points x_1 and x_2 with $f(x_1) = f(x_2)$ and $x_1 \ne x_2$

continuous in $[a, b]$ (because it's differentiable). The theorem about the existence of a global maximum tells us:

$$\exists \xi \in [a, b] : f(\xi) \ge f(z) \quad \forall z \in [a, b]$$

$$f(\xi) \ge f(x) > f(a) = f(b)$$

$$\implies \xi \ne a \wedge \xi \ne b$$

So ξ is an inner point of $[a, b]$, hence $\xi \in (a, b)$.

Analogously the same holds for a minimum: Without loss of generality: $f(x) < f(a) = f(b)$. And the same proof works for a global minimum.

□

Theorem 87 (Mean value theorem)**.** Let $I = [a, b]$ be a compact interval with $a < b$ and let $f : I \to \mathbb{R}$ be differentiable in $[a, b]$. Then there exists some $\xi \in [a, b]$ (even $\xi \in (a, b)$) such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Equivalently,

$$f(b) = f(a) + f'(\xi)(b - a)$$
$$f(a) = f(b) + f'(\xi)(a - b)$$

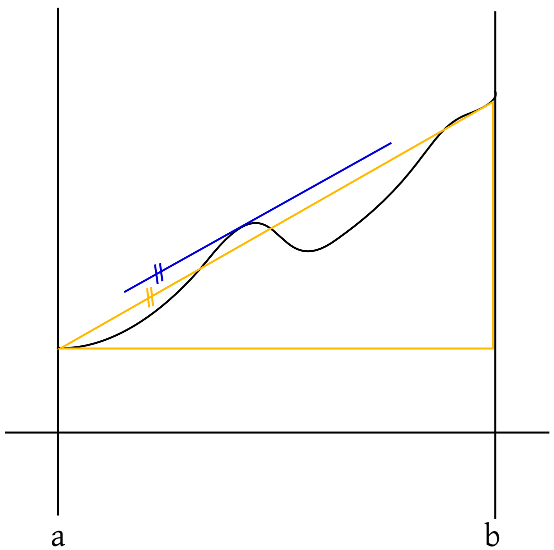

Figure 38: The Intermediate Value Theorem (IVT) claims that some tangent exists which is parallel to the line connecting $f(a)$ and $f(b)$

Proof. Let

$$g(x) := f(x) - \underbrace{\left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]}_{\text{linear, hence differentiable}}$$

$$\implies g(a) = f(a) - [f(a) - 0] = 0$$
$$g(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(b - a)\right] = 0$$

By the Rolle's Theorem it follows that

$$\exists \xi \in [a, b] \text{ with } g'(\xi) = 0$$

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$g'(\xi) = 0 \implies f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

□

Definition 61 (Monotonicity for functions)**.** Let I be an interval, $f : I \to \mathbb{R}$. Then f is called *monotonically increasing* in I if

$$x_1, x_2 \in I \wedge x_1 \le x_2 \implies f(x_1) \le f(x_2)$$

f is called *monotonically decreasing* in I if

$$x_1, x_2 \in I \wedge x_1 \le x_2 \implies f(x_1) \ge f(x_2)$$

f is called *strictly monotonically increasing* in I

$$x_1, x_2 \in I \wedge x_1 \le x_2 \implies f(x_1) < f(x_2)$$

f is called *strictly monotonically decreasing* in I

$$x_1, x_2 \in I \wedge x_1 \le x_2 \implies f(x_1) > f(x_2)$$

Theorem 88. Let $f : I \to \mathbb{R}$ be differentiable in I where I is some interval. Then

- f is monotonically increasing in $I \iff f'(x) \geq 0 \quad \forall x \in I$
- f is monotonically decreasing in $I \iff f'(x) \leq 0 \quad \forall x \in I$

Proof. We only show the proof for monotonically increasing functions. It follows analogously for monotonically decreasing functions.

$\implies$ Let f be monotonically increasing and $x_0 \in I$. Let $(w_n)_{n\in\mathbb{N}}$ and $w_n \in I$ with $\lim_{n\to\infty} w_n = x_0$, $w_1 \neq x_0 \quad \forall n \in \mathbb{N}$. Then it holds that

$$f'(x_0) = \lim_{n\to\infty} \underbrace{\frac{f(w_n) - f(x_0)}{x_n - x_0}}_{S_n}$$

- If $w_n > x_0$, then $f(w_n) \geq f(x_0)$ due to monotonicty.

$$\implies S_n \neq 0$$

- If $w_n < x_0$ (hence $w_n - x_0 < 0$), then $f(w_n) \leq f(x_0)$ hence $f(w_n) - f(x_0) \leq 0$, due to monotonicity.

$$\implies S_n \geq 0$$

$$\implies f'(x_0) = \lim_{n\to\infty} S_n \geq 0$$

$\impliedby$ Let $f'(x) \geq 0 \forall x \in I$. Show that f is monotonically increasing.

Proof by contradiction: Assume the opposite. f is not monotonically increasing, so there exist $x_1, x_2 \in I$ with $x_1 \leq x_2$ and $f(x_1) > f(x_2)$. f is differentiable in $[x_1, x_2] \subseteq I$. The Intermediate Value Theorem tells us that $\exists \xi \in (x_1, x_2)$ with

$$f'(\xi) = \frac{\overbrace{f(x_2) - f(x_1)}^{<0}}{\underbrace{x_2 - x_1}_{>0}}$$

$$\implies f'(\xi) < 0$$

This contradicts with our assumption that $f'(x) \geq 0 \forall x \in I$.

□

Lemma 19. Let $f : I \to \mathbb{R}$ where I is an interval. Let f be differentiable in I. Assume

$$f'(x) > 0 \qquad \forall x \in I$$

Then it follows that f is strictly monotonically increasing.

Assume

$$f'(x) > 0 \qquad \forall x \in I$$

Then it follows that f is strictly monotonically decreasing.

Attention! This is a necessary, but not sufficient condition! $f(x) = x^3$ is strictly monotonically increasing in $\mathbb{R}$, but $f'(x) = 3x^2$ and therefore $f'(0) = 0$.

Proof. See the previous proof, part $\impliedby$, and use $f(x_1) \geq f(x_2)$ and $f'(\xi) \leq 0$ in contradiction to $f'(x) > 0 \quad \forall x \in I$. □

Theorem 89 (Generalization of the IVT)**.** Let $f, g : [a, b] \to \mathbb{R}$ be differentiable in $[a, b]$ and $g'(x) \neq 0$ for all $x \in [a, b]$. Then it holds that

$$g(a) \neq g(b)$$

and there exists $\xi \in (a, b)$ with

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(If $g(x) = x$, the IVT is given as special case)

Proof.

$$F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$$

It holds that $g(a) \neq g(b)$, because $g(a) = g(b) \implies g'(\xi) = 0$ for some $\xi \in (a, b)$ by Rolle's Theorem. This is a contradiction to our assumption.

F is well-defined and differentiable in $[a, b]$.

$$F(a) = f(a) - 0$$

$$\begin{aligned}F(b) &= f(b) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(b)-g(a)) \\ &= f(b) - f(b) + f(a) \\ &= f(a)\end{aligned}$$

By Rolle's Theorem it follows that

$$\exists \xi \notin (a,b) \text{ with } F'(\xi) = 0$$

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} \cdot g'(x)$$

$$F'(\xi) = 0 \implies \frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

□

Guillaume Francois Antoine Marquis de l'Hôpital (1661–1704)

Example 31 (Application of this generalization)**.** Assume f, g are differentiable in I. Let $x_0 \in I$ with $f(x_0) = g(x_0)$. Therefore $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = y_0$.

If $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)}$ exists, then

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{\xi\to x_0} \frac{f'(\xi)}{g'(\xi)}$$

"L'Hôpital's rule"

Proof. Assuming the generalization of the IVT, we have:

$$\exists \xi \in [x, x_0] \text{ wlog. } x < x_0 : \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(\xi)}{g'(\xi)}$$

and for $|x - x_0| < \varepsilon$ it holds that

$$|\xi - x_0| < \varepsilon$$

$$\implies x \to x_0 \implies \xi \to x_0$$

□

Example 32.

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{\xi\to 0} \frac{e^\xi}{1} = 1$$

This holds only if the limit actually exists.

Corollary 18. Corollaries following this monotonicity criterion:

- Let $f : I \to \mathbb{R}$ differentiable in I and let $x_0 \in I$ be a local maximum. Then there exists $\varepsilon > 0$ such that for all $x \in I$ with $x \in (x_0 - \varepsilon, x_0]$ it holds that

$$f(x) \leq f(x_0) \wedge \forall w \in I \text{ with } w \in [x_0, x_0 + \varepsilon) : f(w) \leq f(x_0)$$

- Assume f is monotonically increasing in $(x_0 - \varepsilon, x_0]$ and f is monotonically decreasing in $[x_0, x_0 + \varepsilon)$

$$\implies \exists x, \tilde{x} \in (x_0 - \varepsilon, x_0] : f(x) \leq f(\tilde{x}) \wedge \forall w, \tilde{w} \in [x_0, x + \varepsilon)$$

  with $\tilde{w} \leq w$ it holds that $f(\tilde{w}) \geq f(w)$. Especially for $\tilde{x} = x_0$ and $\tilde{w} = x_0$ it holds that

$$f(x) \leq f(x_0) \wedge f(x_0) \geq f(w)$$

  Condition for local maximum: Therefore if $\varepsilon > 0$ exists, such that f in $I \cap (x_0 - \varepsilon, x_0]$ monotonically increasing and f in $I \cap [x_0, x + \varepsilon)$ is monotonically decreasing, then f has a local maximum in x_0.

  This is a sufficient condition for a maximum. So if this condition holds, a maximum is given.

↓ This lecture took place on 23rd of Jan 2016 with lecturer Wolfgang Ring

Theorem 90. Let $(w_n)_{n\in\mathbb{N}}$ with $w_n \in I$ such that $\lim_{n\to\infty} w_n = x_0$ and

$$\xi_n \in \begin{cases} [w_n, x_0] & \text{if } w_n < x_0 \\ [x_0, w_n] & \text{if } x_0 < w_n \end{cases}$$

with

$$\frac{f(w_n) - f(x_0)}{g(w_n) - g(x_0)} = \frac{f'(\xi_n)}{g'(\xi_n)}$$

Because $|\xi_n - x_0| < \underbrace{|w_n - x_0|}_{\to 0}$ it holds that

$$\lim_{n\to\infty} \xi_n = x_0.$$

If $\lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = d$.

$$\implies \lim_{n\to\infty} \frac{f(w_n) - f(x_0)}{g(w_n) - g(x_0)} = d$$

14.5 Sufficient optimality criteria

Theorem 91. Let $f : I \to \mathbb{R}$, $x_0 \in I$. If $\varepsilon > 0$ exists, such that f is monotonically increasing in $(x_0 - \varepsilon, x_0] \cap I$ and f is monotonically decreasing in $[x_0, x_0 + \varepsilon) \cap I$, then f has a local maximum in f.

Remark 31. Informal: Increasing to the right, decreasing to the left? So it must be a local maximum.

Remark 32. This is a sufficient, but not necessary condition. Compare with Figure 39.

Remark 33. Let f be differentiable in I and $x_0 \in I$. If $\varepsilon > 0$ exists with $\forall x \in (x_0 - \varepsilon, x_0] \cap I : f'(x) \geq 0$ and $\forall x \in [x_0, x_0 + \varepsilon) : f'(x) \leq 0$, then f is monotonically increasing to the left of x_0 and monotonically decreasing to the right. So, x_0 is a local maximum.

Hence, the point when f' changes its sign, a maximum (or minimum) is given.

All statements hold analogously for the minimum (negate the operators).

14.6 Behavior of curvatures in functions

Remark 34. Assume the line on the graph defines our road. Do we need to drive to the left or right in a curvature?

Definition 62. Let I be an interval $f : I \to \mathbb{R}$. Then f is called *convex* in I if $\forall a, b \in I$ with $a < b$ and for all $\lambda \in [0, 1]$ it holds that

$$f((1-\lambda)\cdot a + \lambda \cdot b) \leq (1-\lambda)f(a) + \lambda f(b)$$

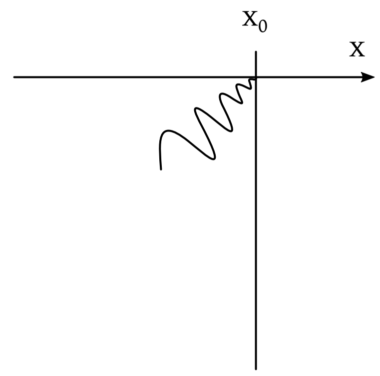

Figure 39: This is not a local maximum, but Theorem 91 holds

f is called *concave* if the following holds:

$$f((1-\lambda)\cdot a + \lambda \cdot b) \geq (1-\lambda)f(a) + \lambda f(b)$$

f is called *strictly convex* if the following holds:

$$f((1-\lambda)\cdot a + \lambda \cdot b) < (1-\lambda)f(a) + \lambda f(b)$$

f is called *strictly concave* if the following holds:

$$f((1-\lambda)\cdot a + \lambda \cdot b) > (1-\lambda)f(a) + \lambda f(b)$$

Remark 35. Let $\lambda \in [0, 1]$.

$$(1-\lambda)\cdot a + \lambda \cdot b \leq (1-\lambda)\cdot b + \lambda \cdot b = b$$
$$(1-\lambda)\cdot a + \lambda \cdot a = a$$

$(1-\lambda)\cdot a + \lambda \cdot b$ defines an arbitrary point in $[a, b]$. It's called *convex combination* of a and b.

Remark 36. In case of convexness, the function graph lies underneath the function. Compare with Figure 40.

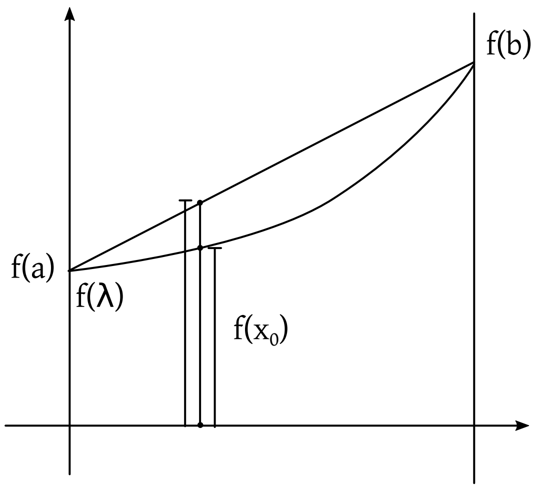

Figure 40: Convex combination

Theorem 92. Let $f : I \to \mathbb{R}$ be differentiable and I an interval. Then it holds that f is convex in $I \iff f' : I \to \mathbb{R}$ is monotonically increasing.
Analogously for concave and monotonically decreasing.

Proof. $\Longleftarrow$ Let $f' : I \to \mathbb{R}$ be monotonically increasing. Let $a, b \in I$, $a < b$ and let $\lambda \in [0, 1]$.

Hence convexity condition for $\lambda \in \{0, 1\}$ is satisfied. Let $\lambda = 0$, then

$$f((1-0)\cdot a + 0 \cdot b) = f(a) \leq (1-0)\cdot f(a) + 0 \cdot f(b)$$

Analogously for $\lambda = 1$, then

$$f((1-1)\cdot a + 1 \cdot b) = f(b) = (1-1)\cdot f(a) + 1 \cdot f(b)$$

Let $\lambda \in (0, 1)$

$$\begin{aligned}
&(1-\lambda)f(a) + \lambda f(b) - \underbrace{1}_{((1-\lambda)+\lambda)} \cdot f((1-\lambda)\cdot a + \lambda \cdot b) \\
&= (1-\lambda)f(a) - (1-\lambda)f((1-\lambda)a + \lambda b) + \lambda f(b) - \lambda f((1-\lambda)\cdot a + \lambda b) \\
&= (1-\lambda)\left[f(a) - f\left((1-\lambda)a + \lambda b\right)\right] + \lambda\left[f(b) - f((1-\lambda)a + \lambda b)\right]
\end{aligned}$$

Let $x_\lambda \coloneqq (1-\lambda)\cdot a + \lambda b$.

$$\begin{aligned}
&= \lambda[f(b) - f(x_\lambda)] - (1-\lambda)[f(x_\lambda) - f(a)] \\
&=: E
\end{aligned}$$

$\exists \xi_2 \in (x_\lambda, b)$ such that (by the Mean Value Theorem),

$$\begin{aligned}
f(b) - f(x_\lambda) &= f'(\xi_2)(b - x_\lambda) \\
&= f'(\xi_2)(b - (1-\lambda)a - \lambda b) \\
&= f'(\xi_2)(1-\lambda)(b-a)
\end{aligned}$$

$\exists \xi_1 \in (a, x_\lambda)$ such that

$$\begin{aligned}
f(x_\lambda) - f(a) &= f'(\xi_1)(x_\lambda - a) \\
&= f'(\xi_1)((1-\lambda)\cdot a + \lambda b - a) \\
&= f'(\xi_1)\lambda(b-a)
\end{aligned}$$

$$\begin{aligned}
E &= \lambda(1-\lambda)(b-a)\cdot f'(\xi_2) - (1-\lambda)\lambda(b-a)f'(\xi_1) \\
&= \underbrace{\lambda(1-\lambda)(b-a)}_{>0}\underbrace{[f'(\xi_2) - f'(\xi_1)]}_{\geq 0} \geq 0
\end{aligned}$$

because f' is monotonically increasing and $\xi_1 < x_\lambda < \xi_2$ holds.

Therefore, $(1-\lambda)f(a) + \lambda f(b) \geq f(x_\lambda)$

$\Longrightarrow$ Let f be convex and differentiable in I. Let $x_1 < x_2$ with $x_1, x_2 \in I$. Show that $f'(x_1) \leq f'(x_2)$. Choose $n \in \mathbb{N}$ with $n \geq 2$. Let $w_n = x_n + \frac{1}{n}(x_2 - x_1)$ and $z_n = x_2 - \frac{1}{n}(x_2 - x_1)$.

$$\lim_{n\to\infty} w_n = x_1 \text{ and } \lim_{n\to\infty} z_n = x_2$$

We consider

$$\frac{f(x_2)-f(z_n)}{x_2-z_n}-\frac{f(w_n)-f(x_1)}{w_n-x_1}$$
$$= n\cdot\frac{1}{x_2-x_1}\left(f(x_2)-f(z_n)\right)-n\cdot\frac{1}{x_2-x_1}\left(f(w_n)-f(x_1)\right) =: E$$

with $f(z_n)\le(1-\mu)f(x_1)+\mu f(x_2)$ and $f(w_n)\le(1-\lambda)f(x_1)+\lambda f(x_2)$

$$z_n = x_2-\frac{1}{n}(x_2-x_1)=\frac{1}{n}x_1+\left(1-\frac{1}{n}\right)x_2$$
$$w_n = x_1+\frac{1}{n}(x_2-x_1)=\left(1-\frac{1}{n}\right)x_1+\frac{1}{n}x_2$$
$$=(1-\lambda)x_1+\lambda x_2 \text{ with } \lambda:=\frac{1}{n}$$
$$z_n=\left(1-(1-\frac{1}{n})\right)x_1+\left(1-\frac{1}{n}\right)x_2$$
$$=(1-\mu)x_1+\mu x_2 \text{ with } \mu:=1-\frac{1}{n}$$

By convexity,

$$E\ge n\cdot\frac{1}{x_2-x_1}\left[f(x_2)-((1-\mu)\cdot f(x_1)+\mu f(x_2))\right]$$
$$-n\cdot\frac{1}{x_2-x_1}\left[(1-\lambda)f(x_1)+\lambda f(x_2)-f(x_1)\right]$$
$$=\frac{n}{x_2-x_1}\left[(1-\mu)(f(x_2)-f(x_1))\right]-\frac{n}{x_2-x_1}\left[\lambda(f(x_2)-f(x_1))\right]$$
$$\left[\lambda:=\frac{1}{n}\qquad \mu:=1-\frac{1}{n}\right]$$
$$=\frac{n}{x_2-x_1}\frac{1}{n}\left(f(x_2)-f(x_1)\right)-\frac{n}{x_2-x_1}\frac{1}{n}(f(x_2)-f(x_1))=0$$

So

$$\underbrace{\frac{f(x_2)-f(z_n)}{x_2-z_n}}_{\xrightarrow{n\to\infty} f'(x_2)}\ge\underbrace{\frac{f(w_1)-f(x_1)}{w_n-x_1}}_{\xrightarrow{n\to\infty} f'(x_1)}$$

So $f'(x_2)\ge f'(x_0)$.

□

Definition 63. Let $f:I\to\mathbb{R}$ and $x_0\in I$. Assume x_0 is an inner point of I and $\exists\varepsilon>0$ such that $(x_0-\varepsilon,x_0+\varepsilon)\subseteq I$ and f in $(x_0-\varepsilon,x_0]$ is convex and f in $[x_0,x_0+\varepsilon)$ is concave, then x_0 is called *inflection point*.

If f is concave in $(x_0-\varepsilon,x_0]$ and convex in $[x_0,x_0+\varepsilon)$, then x_0 is also an inflection point.

Definition 64 (Higher derivatives)**.** Assume $f:I\to\mathbb{R}$ is differentiable in I and the derivative $f':I\to\mathbb{R}$ in a point $x_0\in I$ itself is differentiable. Then $f''(x_0)=(f')'(x_0)$ is called *second derivative* of f in x_0.

Analogously for higher derivatives: Let the derivative function of order n ($n\in\mathbb{N}$) be already defined and let itself be differentiable in x_0, then

$$f^{n-1}:I\to\mathbb{R}$$

is called derivative function of $(n-1)$-th order where

$$f^{(0)}=f, f^{(1)}=f'$$

Then we let

$$f^{(n)}(x_0)=\left(f^{n-1}\right)(x_0)$$

Remark 37. We can use the second derivative to check the monotonicity of the first derivative.

$$f^{(2)}:I\to\mathbb{R},\quad f^{(2)}(x)\ge 0\quad\forall x\in I$$
$$\implies f^{(1)}=f' \text{ is monotonical in } I$$
$$\implies f \text{ is convex in } I$$

Remark 38. Let f be convex in I and differentiable in x_0. Then it holds with $t:I\to\mathbb{R}$ and $t(f)=f(x_0)+f'(x_0)(x-x_0)$, which is the tangent of f in x_0, that

$$\forall x\in I: t(x)\le f(x)$$

↓ This lecture took place on 27th of January 2016 with lecturer Wolfgang Ring

$$P(z)=\sum_{x=0}^{\infty}a_nz^n$$

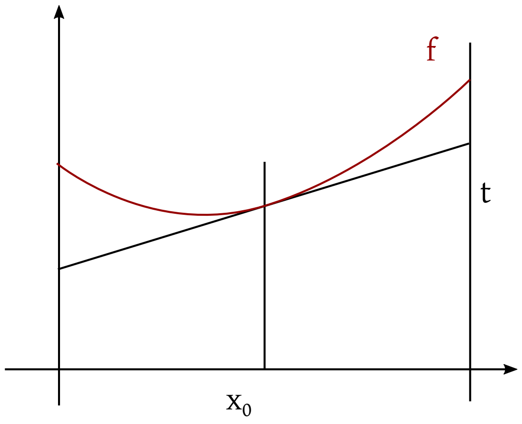

Figure 41: Tangent in x_0

$$L = \limsup_{k\to\infty} \sqrt[n]{|a_n|}$$

$$\delta = \frac{1}{L} \qquad \text{P(z) is convergent}$$

14.7 Function sequences and uniform convergence

Sequences, we know:

$$(z_n)_{n\in\mathbb{N}} \qquad z_n \in \mathbb{C} \qquad \text{sequence of complex numbers}$$

$$(I_n)_{n\in\mathbb{N}} \qquad I_{n+1} \in I_n \qquad \text{sequence of intervals}$$

Function sequences: Consider $(f_n)_{n\in\mathbb{N}}$ with $f : D \to \mathbb{C}$ with $D \subseteq \mathbb{C}$. Then $(f_n)_{n\in\mathbb{N}}$ is called *function sequence*. It is important to recognize that all functions have the same co-domain.

Definition 65. Let $D \subseteq \mathbb{C}$ and $f_n : D \to \mathbb{C}$ for $n \in \mathbb{N}$ and $f : D \to \mathbb{C}$. We say, a function sequence $(f_n)_{n\in\mathbb{N}}$ is *uniformly convergent* with f if

$$\forall\varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} : [n \geq N_\varepsilon \implies |f_n(z) - f(z)| < \varepsilon \forall z \in D]$$

Lemma 20. Let $(f)_{n\in\mathbb{N}}$ be a function sequence in $D \subseteq \mathbb{C}$ and $f : D \to \mathbb{C}$. Then it holds $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent in D towards limes f if and only if

$$\lim_{n\to\infty} \sup \{|f_n(z) - f(z)| : z \in D\} = 0$$

Proof. $\implies$ Let f be a uniform limes of $(f_n)_{n\in\mathbb{N}}$. Then $\forall\varepsilon > 0 \exists N_\varepsilon : [n \geq N_\varepsilon \implies |f_n(z) - f(z)| < \varepsilon \forall z \in D]$

$$\text{for } n \geq N_\varepsilon \text{ it holds that } \sup \{|f_n(z) - f(z)| : z \in D\}$$

So it holds that

$$\sup \{|f_n(z) - f(z)| : z \in D\} \xrightarrow{n\to\infty} 0$$

$\impliedby$ Let $\varepsilon > 0$. Convergence of supremum sequence implies that

$$\exists N_\varepsilon \in \mathbb{N} : [n \geq N_\varepsilon \implies \sup |f_n(z) - f(z)| : z \in D < \varepsilon]$$

for those n and for every $z \in D$ it holds that

$$|f_n(z) - f(z)| < \varepsilon$$

□

Remark 39. Let $B(D) = \{f : D \to \mathbb{C} \text{ with } f \text{ is bound to } D\}$ and

$$\|f\|_\infty = \sup \{|f(z)| : z \in D\}$$

Then it holds that $(f_n)_{n\in\mathbb{N}}$ converges uniformly towards f (with $f_n \in B(D)$ and $f \in B(D)$)

$$\iff \|f_n - f\|_\infty \to 0 \text{ for } n \to \infty$$

Remark 40. It can be shown that $B(D)$ is a vector space and $\|\cdot\|_\infty$ is a *norm* in $B(D)$, hence

$$\begin{aligned}\|f\|_\infty = 0 &\iff f(z) = 0 \forall z \in D\\ \|\alpha f\|_\infty &= |\alpha| \, \|f\|_\infty\\ \|f+g\|_\infty &\leq \|f\|_\infty - \|g\|_\infty \qquad \forall f, g \in B(D), \alpha \in \mathbb{C}\end{aligned}$$

$\|\cdot\|_\infty$ is called *supremum norm* in D.

$$C_b(D) := \{f : D \to \mathbb{C}, f \in B(D) \text{ and } f \text{ is continuous in } D\} \subseteq B(D)$$

The supremum norm can also be defined on $C_b(D)$.

If $D = K \subseteq \mathbb{C}$ is compact in $\mathbb{C}$, it follows immediately that every continuous function is bounded.

Show that $\{|f(z) : z \in K|\}$ is a bounded set in $\mathbb{R}$.

$$|f| : D \to \mathbb{R}$$

$|f|$ is the composition of two functions, namely f and the absolute value function. Both are continuous. $|f|$ has a maximum, hence $\exists z_0 \in K : |f(z)| \leq |f(z_0)| \forall z \in K$. So $|f(z_0)|$ is upper bound of $\{|f(z)| : z \in K\}$.

$$C(K) = \{f : K \to \mathbb{C} : f \text{ is continuous}\} \subseteq B(K)$$

and for $f \in C(K)$ it holds that

$$\|f\|_\infty = \sup\{|f(z)| : z \in K\} = \max\{|f(z)| : z \in K\}$$

Theorem 93. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in D with all f_n continuous in D. Assume $f : D \to \mathbb{C}$ and $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on f in D. Then f is continuous in D.

Proof. Let $\varepsilon > 0$ be given and $z_0 \in D$. Show that $\exists \delta > 0$ such that for all $z \in D$ with

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \varepsilon$$

1. Because $(f_n)_{n\in\mathbb{N}}$ converges uniformly towards f, there exists some

$$N \in \mathbb{N} : |f_N(w) - f(w)| < \frac{\varepsilon}{3} \forall w \in D$$

2. If f_N is continuous on its own, then

$$\exists \delta > 0 \text{ such that } z \in D \text{ and } |z - z_0| < \delta \implies |f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$$

Let $z \in D$ and $|z - z_n| < \delta$ (with δ properties as above). Then it holds that

$$|f(z) - f(z_0)| = \left| f(z) - \underbrace{f_N(z) + f_N(z)}_{=0} - \underbrace{f_N(z_0) + f_N(z_0)}_{=0} - f(z_0) \right|$$

$$\underbrace{\leq}_{\text{triangle inequality}} \underbrace{|f(z) - f_N(z)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(z) - f_N(z_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{<\frac{\varepsilon}{3}}$$

The middle term is $< \frac{\varepsilon}{3}$ because f is continuous. The other terms are $< \frac{\varepsilon}{3}$ because of convergence and selection of N.

So overall $< \varepsilon$. So f is continuous in z_0. Because $z_0 \in D$ is arbitrary, it holds for all z_0. So f is continuous in D.

□

↓ This lecture took place on 28th of January 2016 with lecturer Wolfgang Ring

"The continuous limit of a sequence of continuous functions is continuous"

15 Power series

$$\sum_{n=0}^{\infty} a_n z^n \qquad \text{absolute convergent } \forall z \in B(0, \rho)$$

where ρ is the convergence radius. $\rho = \frac{1}{L}$ with

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Lemma 21 (Remaining term estimate). Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with convergence radius $\rho > 0$ and let

$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k \qquad (k \in \mathbb{N})$$

Assume $0 \le |z| \le r < \rho$. Then there exists a constant $c = c(r)$ such that

$$|R_n(z)| \le c \left(\frac{|z|}{r}\right)^n$$

Proof.

$$|R_n(z)| = \left|\sum_{k=n}^{\infty} a_k z^k\right| \le \sum_{k=n}^{\infty} |a_k| \, |z|^k = \sum_{k=n}^{\infty} |a_k| \cdot r^k \cdot \underbrace{\frac{|z|^k}{r^k}}_{\le \frac{|z|^n}{r^n}}$$

$$\le \frac{|z|^n}{r^n} \sum_{k=n}^{\infty} |a_k| \, r^k \le \frac{|z|^n}{r^n} \frac{|z|^n}{r^n} \underbrace{\sum_{k=0}^{\infty} |a_k| \, r^k}_{=c(r)}$$

Is $c(r)$, because the series is absolute convergent and so the series has some value c. $r \in B(0, \rho)$. ☐

Theorem 94. Let $P(z) = \sum_{k=0}^{\infty} a_n z^k$ be a power series with convergence radius $\rho > 0$ and let $0 \le r < \rho$. We define

$$P_n(z) = \sum_{k=0}^{n} a_n z^k$$

(n-th partial sum of the series)

Then $(P_n)_{n \in \mathbb{N}}$ converges uniformly towards P in $B(0, r)$.

Proof. Let $\hat{r} = \frac{1}{2}(r + \rho)$, hence $r < \hat{r} < \rho$. Then it holds that $P(\hat{r})$ is convergent (because $\hat{r} \in B(0, \rho)$)

So $\forall z \in B(0, r)$, the remaining term estimation theorem holds.

$$\exists c(\hat{r}) : \left|\sum_{k=n+1}^{\infty} a_k z^k\right| \le \frac{|z|^{n+1}}{\hat{r}^{n+1}} \cdot c(\hat{r})$$

$$\le c(\hat{r}) \cdot \frac{r^{n+1}}{\hat{r}^{n+1}} = c(\hat{r}) \left(\frac{r}{\hat{r}}\right)^{n+1}$$

Let $\varepsilon > 0$ be arbitrary and N sufficiently large such that

$$\left(\underbrace{\frac{r}{\hat{r}}}_{<1}\right)^{N+1} < \frac{\varepsilon}{c(\hat{r})}$$

Then for all $n \ge N$ and for all $z \in B(0, r)$ it holds that

$$|P(z) - P_n(z)| = \left|\sum_{k=0}^{\infty} a_k z^k - \sum_{k=n}^{\infty} a_n z^k\right|$$

$$\left|\sum_{k=n+1}^{\infty} a_k z^k\right| \le \left(\frac{r}{\hat{r}}\right)^{n+1} \cdot c(\hat{r})$$

$$\le \left(\frac{r}{\hat{r}}\right)^{N+1} \cdot c(\hat{r}) < \frac{\varepsilon}{c(\hat{r})} \cdot c(\hat{r}) = \varepsilon$$

So it holds that $P_n \to P$ is uniform on $B(0, r)$. ☐

Corollary 19. $P_n(z)$ is continuous in $\overline{B(0, r)}$

$$\implies P : \overline{B(0, r)} \to \mathbb{C} \text{ is continuous}$$

Let $z \in B(0, \rho)$, hence $|z| < \rho$. Let $r = \frac{1}{2}(|z| + \rho)$.

P is continuous in $\overline{B(0, r)}$ and $z \in B(0, r)$. Hence it holds that P is continuous in z. So it holds that P is continuous in $B(0, \rho)$. Compare with Figure 42.

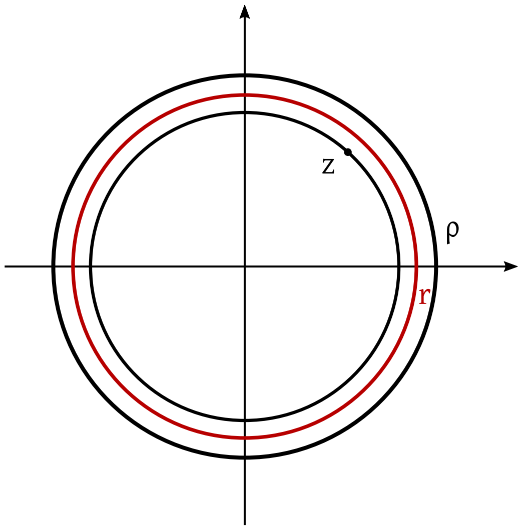

Figure 42: Convergence radius of power series

15.1 The exponential function and its relatives

We want to define the function $f_{\text{ex}} : \mathbb{C} \to \mathbb{C}$, which behaves like $z \mapsto b^z$. We want to achieve the power laws in f_{ex} as well. We require:

$$(F) \quad f_{\text{ex}}(z_1) \cdot f_{\text{ex}}(z_2) = f_{\text{ex}}(z_1 + z_2) \qquad \forall z_1, z_2 \in \mathbb{C}$$

"Functional equation of the exponential function"

Corollary 20.

$$f_{\text{ex}}(z) = f_{\text{ex}}(z+0) = f_{\text{ex}}(z) \cdot f(0)$$

Let $z \in \mathbb{C}$ such that $f_{\text{ex}}(z) \neq 0$. We divide, followingly,

$$f_{\text{ex}}(0) = 1$$

Corollary 21. Let $z \in \mathbb{C}$ be arbitrary and $k \in \mathbb{N}_+$. Then

$$z = \underbrace{\frac{z}{k} + \frac{z}{k} + \ldots + \frac{z}{k}}_{k \text{ times}}$$

$$f_{\text{ex}}(z) = f_{\text{ex}}\left(\frac{z}{k} + \ldots + \frac{z}{k}\right) = \left(f_{\text{ex}}\left(\frac{z}{k}\right)\right)^k$$

Corollary 22. Assume: f_{ex} is continuous in 0. Let $z \in \mathbb{C}$ fixed, $k \in \mathbb{N}$, then it holds that

$$\frac{z}{k} \xrightarrow{k \to \infty} 0$$

So it holds that

$$f_{\text{ex}}\left(\frac{z}{k}\right) \to f_{\text{ex}}(0) = 1$$

Remark 41. Approach: Consider $f_{\text{ex}}(\frac{z}{k}) = 1 + \frac{w_k}{k}$ where w_k as enumerator is undefined, small and not really important.

Corollary 23.

$$w_k = K \cdot \left(f_{\text{ex}}\left(\frac{z}{k}\right) - 1\right)$$

$$f_{\text{ex}}(z) = \left(1 + \frac{w_k}{k}\right)^k$$

Desired:

$$w = \lim_{k \to \infty} w_k$$

$$f_{\text{ex}}(z) = \lim_{k \to \infty} \left(1 + \frac{w_k}{k}\right)^k = \lim_{k \to \infty} \left(1 + \frac{w}{k}\right)^k$$

If the limit of w_k actually exists, then w_k depends on z

$$\lim_{k \to \infty} w_k = \lim_{k \to \infty} \frac{f_{\text{ex}}\left(\frac{z}{k}\right) - 1}{\frac{1}{k}} = \lim_{k \to \infty} z \cdot \frac{f_{\text{ex}}\left(\frac{z}{k}\right) - 1}{\frac{z}{k}} = z \cdot \underbrace{\lim_{w \to 0} \frac{f(w) - 1}{w}}_{= c \in \mathbb{C}}$$

With the assumption that this limes actually exists. Then it follows that,

$$w = \lim_{k \to \infty} w_k = c \cdot z$$

$$\boxed{f_{\text{ex}}(z) = \lim_{k\to\infty}\left(1+\frac{c\cdot z}{k}\right)^k}$$

As a general toolbox to define exponential functions.

Corollary 24. For $c = 1$ we get the definition of e^z.

15.2 Fundamental lemma of exponential function

For every convergent complex sequence $(w_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} w_k = w$ it holds that

$$\lim_{k\to\infty}\left(1+\frac{w_k}{k}\right)^k = \sum_{n=0}^{\infty}\frac{1}{n!}w^n$$

Remark 42. The constant sequence $z_n = w \quad \forall k\in\mathbb{N}$ has limes w and therefore it holds that

$$\lim_{k\to\infty}\left(1+\frac{z_k}{k}\right)^k = \underbrace{\lim_{k\to\infty}\left(1+\frac{w}{k}\right)^k}_{\text{with } w} = \sum_{n=0}^{\infty}\frac{1}{n!}w^n = \underbrace{\lim_{k\to\infty}\left(1+\frac{w_k}{k}\right)^k}_{\text{with } w_k}$$

Proof of the fundamental lemma. Let $\varepsilon > 0$ arbitrary. We choose $K\in\mathbb{N}$, such that $n \geq K \implies |w_k| \leq |w|+1$ (this theorem holds because $|w_k| \xrightarrow{k\to\infty} |w|$). At the same time let K be sufficiently large such that

$$\sum_{k=K}^{\infty}\frac{(|w|+1)^k}{k!} < \frac{\varepsilon}{3}$$

This is possible, because the series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ converges in $\mathbb{C}$.

Let $n \geq K$. Then

$$\left|\left(1+\frac{w_n}{n}\right)^n - \sum_{k=0}^{\infty}\frac{w^k}{k!}\right| \overset{\text{triangle inequality}}{\leq} \left|\underbrace{\left(1+\frac{w_n}{n}\right)^n}_{\text{apply binomial theorem}} - \sum_{k=0}^{K-1}\frac{w^k}{k!}\right| + \left|\sum_{k=K}^{\infty}\frac{w^k}{k!}\right|$$

$$\left|\sum_{k=0}^{n}\binom{n}{k}\frac{w_n^k}{n^k} - \sum_{n=0}^{k-1}\frac{w^k}{k!}\right| + \left|\sum_{k=K}^{\infty}\frac{w^k}{k!}\right|$$

$$\leq \left|\sum_{k=0}^{k-1}\left(\binom{n}{k}\frac{w_n^k}{n^k} - \frac{w^k}{k!}\right)\right| + \left|\sum_{k=K}^{n}\binom{n}{k}\frac{w_n^k}{n^k}\right| + \underbrace{\sum_{n=K}^{\infty}\frac{(|w|+1)^k}{k!}}_{<\frac{\varepsilon}{3}}$$

Second expression:

$$\binom{n}{k}\cdot\frac{1}{n^k} = \frac{1}{k!}\underbrace{\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-k+1}{n}}_{k \text{ times}} < \frac{1}{k!}$$

$$= \left|\sum_{k=K}^{n}\binom{n}{k}\frac{w_n^k}{k!}\right| \leq \sum_{k=K}^{n}\binom{n}{k}\frac{|w_n|^k}{n^k} < \sum_{k=K}^{\infty}\frac{1}{k!}(|w|+1)^k < \frac{\varepsilon}{3}$$

First expression:

$$\lim_{n\to\infty}\binom{n}{k}\frac{1}{n^k} = \lim_{n\to\infty}\frac{1}{k!}\cdot\frac{n}{n}\frac{n-1}{n}\cdot\ldots\cdot\frac{n-k+1}{n}$$

$$= \frac{1}{k!}\underbrace{\lim_{n\to\infty}\frac{n-1}{n}}_{=1}\cdot\underbrace{\lim_{n\to\infty}}_{=1}\underbrace{\frac{n-2}{n}}_{=1}\cdot\ldots\cdot\underbrace{\lim_{n\to\infty}\frac{n-k+1}{n}}_{=1} = \frac{1}{k!}$$

Therefore it holds that,

$$\lim_{n\to\infty}\sum_{k=0}^{K-1}\underbrace{\binom{n}{k}\frac{1}{n^k}}_{\to\frac{1}{k!}}\underbrace{w_n^k}_{\to w} = \sum_{k=0}^{K}\frac{1}{k!}w^k$$

Therefore some $N\in\mathbb{N}$ exists such that for $n \geq N$ it holds that,

$$\left|\sum_{k=0}^{K-1}\binom{n}{k}\frac{1}{n^k}w_n^k - \sum_{k=0}^{K-1}\frac{1}{k!}w^k\right| < \frac{\varepsilon}{3}$$

So it holds for $n \geq N$:

$$\left|\left(1+\frac{w_n}{n}\right)^n - \sum_{k=0}^{n}\frac{w^k}{k!}\right| < \varepsilon$$

$$\implies \lim_{n\to\infty}\left(1+\frac{w_n}{n}\right)^n = \sum_{k=0}^{\infty}\frac{w^k}{k!}$$

□

Definition 66 (Exponential function)**.** We define for some $z \in \mathbb{C}$

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

For every sequence $z_n \in \mathbb{C}$ with $\lim_{n\to\infty} z_n = z$ it holds that

$$\exp(z) = \lim_{n\to\infty} \left(1 + \frac{z_n}{n}\right)^n$$

Especially for $z_n = z$ it holds that

$$\exp(z) = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$$

↓ This lecture took place on 29th of Jan 2016 with lecturer Wolfgang Ring

$$w_n \to w \in \mathbb{C}$$

$$\implies \lim_{n\to\infty} \left(1 + \frac{w_n}{n}\right)^n = \sum_{k=0}^{\infty} \frac{w^k}{k!} \quad \text{Fundamental lemma}$$

$$\exp(z) = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

We desire an exponential function satisfying:

$$f_{\text{ex}}(z) \cdot f_{\text{ex}}(w) = f_{\text{ex}}(z + w)$$

Theorem 95. The exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is defined on entire $\mathbb{C}$ and it holds that

(F) $\forall z, w \in \mathbb{C} : \exp(z) \cdot \exp(w) = \exp(z + w)$

(A) $\lim_{\zeta\to 0} \frac{\exp(\zeta)-1}{\zeta} = 1$

Furthermore the exponential function is the *only* function satisfiying properties (A) and (F).

Proof. The power series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ has convergence radius $\rho = \infty$, hence the exponential function is defined on entire $\mathbb{C}$.

What about property (F)?

$$\exp(z)\exp(x) = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n \cdot \lim_{n\to\infty} \left(1 + \frac{w}{n}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{z}{n}\right)\left(1 + \frac{w}{n}\right)\right]^n = \lim_{n\to\infty} \left(1 + \frac{z + w + \frac{zw}{n}}{n}\right)^n$$

It holds that $\zeta_n = z + w + \frac{zw}{n} \to 0$, hence $\lim_{n\to\infty} \zeta_n = z + w$. So,

$$= \lim_{n\to\infty} \left(1 + \frac{\zeta_n}{n}\right) \underset{\substack{\text{fundamental}\\\text{theorem}}}{=} \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = \exp(z + w)$$

What about property (A)?

$$\exp(\zeta) - 1 = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} - 1 = \sum_{k=1}^{\infty} \frac{\zeta^k}{k!} = \zeta \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!}$$

for $\zeta \neq 0$ it is,

$$\frac{\exp(\zeta) - 1}{\zeta} = \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!} = \underbrace{\sum_{l=0}^{\infty} \frac{\zeta^l}{(l+1)!}}_{Q(\zeta)} \qquad \text{power series converging in } \mathbb{C}$$

So $\rho = \infty$. Theorem about continuity of power series:

$$\lim_{\zeta\to 0} Q(\zeta) = Q(0) = \frac{1}{1!} = 1$$

So it holds that

$$\lim_{\zeta\to 0} \frac{\exp(\zeta) - 1}{\zeta} = 1$$

Proof for uniqueness: Let f_{ex} be a function which satisfies (A) and (F). Let $z \in \mathbb{C}$ arbitrary.

Approach:

$$f_{\text{ex}}\left(\frac{z}{n}\right) = 1 + \frac{w_n}{n}$$

Then it holds that

$$\lim_{n\to\infty} f_{\text{ex}}\left(\frac{z}{n}\right) = f_{\text{ex}}(0) = 1$$

$$f_w = \frac{f_{\text{ex}}\left(\frac{z}{n} - 1\right)}{\frac{1}{n}}$$

Because of (F) it holds that

$$f(z) = \left(f\left(\frac{z}{n}\right)\right)^n = \left(1 + \frac{w_n}{n}\right)^n$$

$$w_n = z \cdot \frac{f_{\text{ex}}\left(\frac{z}{n}\right) - 1}{\frac{z}{n}}$$

and

$$\lim_{n\to\infty} w_n = z \underbrace{\lim_{n\to\infty} \frac{f_{\text{ex}}\left(\frac{z}{n}\right) - 1}{\frac{z}{n}}}_{=1} = z$$

$$f_{\text{ex}}(z) = \left(1 + \frac{w_n}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{w_n}{n}\right)^n \underset{\substack{\text{fundamental}\\ \text{theorem}}}{=} \sum_{n=0}^{\infty} \frac{z^k}{k!} = \exp(z)$$

Let $n \in \mathbb{N}$.

$$\exp(n) = \exp(\underbrace{1 + 1 + \ldots + 1}_{n \text{ times}}) = \exp(1)$$

We let

$$\exp(1) = e = \sum_{n=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \in \mathbb{R}$$

e is the Eulerian number (irrational, $\approx 2.718281828459045$).

$\boxed{\text{Leonard Euler (1707–1783)}}$

Let $m \in \mathbb{N}^+$, then it holds that

$$\underbrace{\frac{1}{m} + \frac{1}{m} + \ldots + \frac{1}{m}}_{m \text{ times}} = 1$$

Therefore

$$\exp\left(\frac{1}{m} + \frac{1}{m} + \ldots + \frac{1}{m}\right) = \exp\left(\frac{1}{m}\right)^m = \underbrace{e}_{\exp(1)}$$

$$\exp\left(\frac{1}{m}\right) = \sqrt{m}e = e^{\frac{1}{m}}$$

$$\exp\left(\frac{n}{m}\right) = \exp\left(\underbrace{\frac{1}{m} + \frac{1}{m} + \ldots + \frac{1}{m}}_{n\text{times}}\right) = \exp(\frac{1}{m})^n = \left(e^{\frac{1}{m}}\right)^n = e^{\frac{n}{m}}$$

Let $z \in \mathbb{C}$, then it holds that $z - z = 0$.

$$1 = \exp(0) = \exp(z + (-z)) = \exp(z) \cdot \exp(-z)$$

$$\implies \forall z \in \mathbb{C} : \exp(z) \neq 0$$

and

$$\exp(-z) = \frac{1}{\exp(z)} = \exp(z)^{-1}$$

□ The exponential function does not have roots (i.e. x such that $f(x) = 0$).

So for $\frac{n}{m} \in \mathbb{Q}_-$, $n < 0$, $m > 0$ it holds that

$$\exp\left(\frac{n}{m}\right) = \frac{1}{\underbrace{\exp\left(-\frac{n}{m}\right)}_{\in\mathbb{Q}_+}} = \frac{1}{e^{-\frac{n}{m}}} = e^{\frac{n}{m}}$$

So it holds that

$$\forall q \in \mathbb{Q} : \exp(q) = e^q$$

We denote for $z \in \mathbb{C}$:

$$\exp(z) = e^z$$

15.3 The exponential function for real arguments

Theorem 96. $\exp : \mathbb{R} \to \mathbb{R}$ is differentiable in $\mathbb{R}$ and it holds that $\exp' = \exp$.

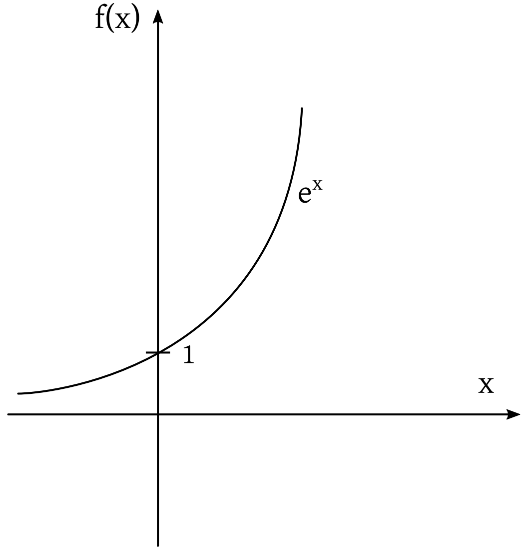

Figure 43: Plot of the general exponential function e^x

Proof. Let $x_0 \in \mathbb{R}$ and consider

$$\begin{aligned}\lim_{x\to x_0} \frac{\exp(x)-\exp(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{\exp(x-x_0+x_0)-\exp(x_0)}{x-x_0}\\ &= \lim_{x\to x_0} \frac{\exp(x-x_0)\cdot\exp(x_0)-\exp(x_0)}{x-x_0}\\ &= \exp(x_0)\cdot \lim_{x\to x_0} \frac{\exp(x-x_0)-1}{x-x_0}\\ &= \exp(x_0)\cdot \underbrace{\lim_{x\to x_0\to 0} \frac{\exp(x-x_0)-1}{x-x_0}}_{=1 \text{ because of (A)}} = \exp(x_0)\end{aligned}$$

So it has been proved that

$$\exp'(x_0) = \exp(x_0)$$

□

Corollary 25.
- $e^x > 0 \quad \forall x \in \mathbb{R}$
- exp is strictly monotonically increasing in $\mathbb{R}$
- exp is strictly convex in $\mathbb{R}$

Proof.
- We already know that $e^x \neq 0 \quad \forall x \in \mathbb{R}$.

$$e^x = e^{\frac{x}{2}+\frac{x}{2}} = \underbrace{\left(e^{\frac{x}{2}}\right)^2}_{\geq 0 \text{ as square}}$$

$$e^x \neq 0 \implies e^x > 0$$

- So it holds that $\forall x \in \mathbb{R} : \exp'(x) > 0$

$$\underbrace{\implies}_{\substack{\text{monotonic}\\ \text{property}}} \exp \text{ is strictly monotonically increasing}$$

- The derivative $\exp'$ of exp is strictly monotonically increasing. Hence exp is strictly convex (Convexity criterion)

□

Definition 67 (Review of improper limits of functions)**.** Let $f : \mathbb{R} \to \mathbb{R}$. We say f tends to infinity $a \in \mathbb{R}$ for x to infinity if

$$\forall \varepsilon > 0 \exists M \in \mathbb{R} : x > M \implies |f(x) - a| < \varepsilon \iff \lim_{x\to\infty} f(x) = a$$

We say f for x to ∞ tends to (improper) limit $+\infty$ if $\forall L > 0 \exists M \in \mathbb{R} : x > M \implies f(x) > L$, denoted $\lim_{x\to\infty} f(x) = +\infty$.

Theorem 97 (Exponential growth)**.** Let $n \in \mathbb{N}$. Then it holds that

- $\lim_{x\to\infty} \frac{e^x}{x^n} = +\infty$
  exp with $x \to \infty$ grows stronger than any x^n
- $\lim_{x\to-\infty} e^x \cdot x^n = 0$
  exp with $x \to -\infty$ drops stronger towards zero than any x^n grows

Proof.

- Let $L > 0$ arbitrary, $n \in \mathbb{N}$ is fixed. For $x > 0$ it holds that

$$e^x = \sum_{k=0}^{\infty} \underbrace{\frac{x^k}{k!}}_{>0} > \frac{x^{n+1}}{(n+1)!}$$

Hence

$$\frac{e^x}{x^n} > \frac{\frac{x^{n+1}}{(n+1)!}}{x^n} = \frac{x}{(n+1)!} > L \text{ if } x > \underbrace{L \cdot (n+1)!}_{=M}$$

- Let $\xi = -x$.

$$\lim_{x \to -\infty} e^x \cdot x^n = \lim_{\xi \to +\infty} e^{-\xi} \cdot (-\xi)^n = -\lim_{\xi \to +\infty} \frac{\xi^n}{e^{\xi}} = 0$$

$\square$

German keywords

English keywords