# Analysis 3: Index of topics

Lecture notes, University of Graz
based on the lecture by Wolfgang Ring

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## Chapter 0: Connectedness in topological spaces

↓ This lecture took place on 2018/10/02.

- Define a topology
- Define an open set
- Define connectedness of topological spaces [Definition 1]
- Define a trace (= relative) topology
- Prove: Let $X, Y$ be topological spaces. Let $f$ be a continuous map. $f : X \to Y$. Let $X$ be connected. Then $f(X)$ is connected [Theorem 1]
- Prove: Let $A \subseteq \mathbb{R}$ such that $\forall a \leq b \in A$ and $\forall x : a \leq x \leq b : x \in A$. Then $A$ must be an interval [Lemma 1]
- Prove: Connected $A \subseteq \mathbb{R}$ is necessarily an interval [Lemma 2]
- Prove: Interval $I \subseteq \mathbb{R}$ is necessarily connected [Lemma 3]

↓ This lecture took place on 2018/10/03.

- Recognize: The connected subsets of $\mathbb{R}$ are exactly the intervals in $\mathbb{R}$
- Define continuous paths in a topological space [Definition 2]

- Define connectedness of two points in topological spaces [Definition 2]
- Define pathwise connectedness [Definition 2]
- Prove: Any pathwise connected topological space is also connected [Theorem 2]

## Chapter 1: $k$-dimensional surfaces in $\mathbb{R}^n$

- Define a regular parameterization of a $k$-dimensional surface in $\mathbb{R}^n$ [Definition 1]
- Define immersions [Definition 1]

↓ This lecture took place on 2018/10/09.

- Define reparameterization [Definition 2]
- Define embeddings, ie. embedded manifolds [Definition 3]

## Chapter 0: Connectedness in topological spaces

- Path connectedness is a transitive property [Definition 3]
- Prove: Every connected and open set $O \subseteq \mathbb{R}^n$ is pathwise connected [Theorem 3]

↓ This lecture took place on 2018/10/10.

## Chapter 1: $k$-dimensional surfaces in $\mathbb{R}^n$

- Define homeomorphisms and the homeomorphic property [Definition 4]
- Define localization and locally parametrized embedded surfaces [Definition 5]
- Define local charts [Definition 5]

- Define embedded manifolds in $\mathbb{R}^n$ [Definition 5]
- Define implicit surfaces [Theorem 1]
- Prove: Let $U \subseteq \mathbb{R}^n$ be an open domain, continuous differentiable $h : U \to \mathbb{R}^{n-k}$. $\operatorname{rank}(Dh(x)) = n - k \forall x \in U$. $\bar{x} \in U$ and $p = h(\bar{x})$. $S = \{x \in U \mid h(x) = p\}$ is a k-Lpes [Theorem 1]

↓ This lecture took place on 2018/10/16.

- Define the orthogonal group
- Give an example of a differentiable manifold
- Prove: Let $S$ be a Lpes of dimension $k$. Let $x \in S$ be given and appropriately ordered.

  $$x = \begin{bmatrix} x_1 & x_2 & \dots & x_k & x_{k+1} & \dots & x_n \end{bmatrix}^T = \begin{bmatrix} \tilde{x} & \hat{x} \end{bmatrix} \qquad \tilde{x} \in \mathbb{R}^n, \hat{x} \in \mathbb{R}^{n-k}$$

  Then there exists neighborhood $\tilde{D}$ of $\tilde{x} \in \mathbb{R}^k$ and neighborhood $U$ of $x$ in $\mathbb{R}^n$ and $C^1$-function $\varphi : \tilde{D} \to \mathbb{R}^{n-k}$ such that

  $$y = \begin{bmatrix} \tilde{y} \\ \hat{y} \end{bmatrix} \in S \cap U \iff \tilde{y} \in \tilde{D} \wedge \hat{y} = \varphi(\tilde{y})$$

  locally $S$ is the graph of the function $\varphi$ [Theorem 2]
- Prove [kind of local extension]: Let $S \subseteq \mathbb{R}^n$ be a Lpes of dimension $k$. Let $x \in S$ and $f : D \subseteq \mathbb{R}^k \to \mathbb{R}^n$ be a local chart for $S$ with $x = f(u)$. Then there exists neighborhoods $\hat{U}$ of $\begin{bmatrix} U \\ 0 \end{bmatrix} \in \mathbb{R}^n$ and $U$ of $x \in \mathbb{R}^n$ and a diffeomorphism $F : \hat{U} \to U$ such that $F\left(\begin{bmatrix} V \\ 0 \end{bmatrix}\right) = f(v) \in S \forall \begin{bmatrix} V \\ 0 \end{bmatrix} \in \hat{U}$ [Theorem 3]

↓ This lecture took place on 2018/10/17.

## Tangent vectors and tangent space

- Define tangent vectors and the tangent space in a k-Lpes [Definition 6]

- Let $S$ be a k-Lpes, $x \in S$ and $f : D \to S$ be a local chart $f(u) = x$. Then $T_xS = \text{image}(Df(u))$ is a linear subspace of $\mathbb{R}^n$ and $(X_j)_{j=1,\dots,k}$ is a basis in $T_xS$ with $X_j := Df(u) \cdot e_j$. Furthermore $Df(u) : \mathbb{R}^k \to T_xS$ is a linear isomorphism [Lemma 2]
- Define the tangent to $S$ at $x$ [Definition 7]
- Define the orthogonal complement [Definition 7]
- Define the normal space to $S$ at $x$ [Definition 7]
- Prove: Let $S$ be a k-Lpes. $x = f(u) \in S$. $f : D \to S$ is a local chart. Then $W \in N_xS \iff (Df(u))^t \cdot W = 0 \iff W \in \text{kernel}(Df(u))^t$. $(Df(u))^t \in \mathbb{R}^{k \times n}$ [Lemma 3]
- Prove: Let $S$ be a k-Lpes. $x = f(u) \in S$ and $F : \hat{U} \to U$ is an extension of $f$ as in Theorem 3. Then $(W_{k+j})_{j=1}^{n-k}$ is a basis in $N_xS$ where $W_l$ is the $l$-th column of $(DF(\begin{bmatrix} u \\ 0 \end{bmatrix}))^{-t}$ [Lemma 4].
- Define the wedge product of $n-1$ vectors in $\mathbb{R}^n$ [Definition 8]
- Prove: Let $w = v_1 \wedge \cdots \wedge v_{n-1}$. Then [Lemma 5]
  1. For any $b \in \mathbb{R}^n$, $\langle b, w \rangle = \det([b, V])$
  2. $\left\langle w, v_j \right\rangle = 0$ for $j = 1, \dots, n-1$

↓ This lecture took place on 2018/10/23.

- Define orientability of $(n-1)$-Lpes [Definition 9]
- Give an example of a non-orientable surface
- Define the Gauss map of an $(n-1)$-Lpes [Definition 9a]
- Prove: Let $U \subseteq \mathbb{R}^n$ be a domain. $h : U \to \mathbb{R}^{n-k}$ be $C^1$. Let $\text{rank}(Dh(x)) = n-k \forall x \in U$. Let $S := \{x \in U \mid h(x) = 0\} \neq \emptyset$ is a $k$-Lpes. Then the tangent space of implicitly defined surfaces is given by $T_xS = \text{kernel}(Dh(x))$ with $Dh(x) \in \mathbb{R}^{(n-k)\times n}$ [Theorem 4]

## Maps into surfaces

- Prove: Let $S \subseteq \mathbb{R}^n$ be a $k$-Lpes. Let $f : D \to V \subseteq S$ be a local chart $f(u) = x$ $(V = f(D))$. Let $\varphi : W \subseteq \mathbb{R}^m \to V \subseteq S$ be given ($\varphi$ is a map into $S$). Then the following statements are equivalent [Theorem 5]:

- $\varphi : W \to \mathbb{R}^n$ is $C^1$
- $f^{-1} \circ \varphi : W \to D \subseteq \mathbb{R}^k$ is $C^1$

- Let $S$ be a $k$-Lpes and let $\varphi : S \to \mathbb{R}^m$ be continuous. Then the following statements are equivalent:
  - For any $x \in S$, there exists some neighborhood $U$ of $x$ in $\mathbb{R}^n$ and a $C^1$ function $\hat{\varphi} : U \to \mathbb{R}^m$ such that $\tilde{\varphi}\big|_{U \cap S} = \varphi$ ($\tilde{\varphi}$ is an extension of $\varphi$)
  - For any $x \in S$, there exists some local chart $f : D \to S$ with $f(u) = x$ such that $\varphi \circ f : D \to \mathbb{R}^m$ is $C^1$
  - For any $x \in S$ and any local chart $f : D \to S$ with $f(u) = x$ the composition $\varphi \circ f : D \to \mathbb{R}^m$ is $C^1$ [Theorem 6]

## Maps on surfaces

- Define continuously differentiable on $k$-Lpes $S$ [Definition 10]

## Maps between surfaces

- Define a map on $S_1$ and a map into $S_2$ [Definition 11]
- When is a map on a $k$-Lpes $C^1$ [Definition 11]

↓ This lecture took place on 2018/10/24.

## Derivative of a map $\varphi : S_1 \to S_2$

- Let $\varphi : S_1 \to S_2$ with $\varphi \in C^1$ and $S_1$ as $k$-Lpes in $\mathbb{R}^n$ and $S_2$ as $l$-Lpes in $\mathbb{R}^m$. Define the derivative $D\varphi(x)$ [Definition 12]
- Recognize that we calculate $D\varphi$ be choosing bases in $T_x S_1$ and in $T_{\varphi(x)} S_2$ and calculate the matrix representation of $D\varphi(x)$ wrt. these chosen bases.
- The matrix representation of $D\varphi(x)$ wrt. $(\partial_{u_i})_{i=1}^k$ is the basis in $T_x S_1$ and $(\partial_{v_j})_{j=1}^l$ is the basis in $T_{\varphi(x)} S_2$ is given by $D(f_2^{-1} \circ \varphi \circ f_1)(u)$.
- Let $D \subseteq \mathbb{R}^k$ be a domain. Then we can interpret $D$ as a $k$-dimensional Lpes in $\mathbb{R}^k$. Local parameterization is given by $f : D \to D$, $f(x) = x$ ($f = \mathrm{id}$).
- What is central projection?

- What does concentric mean?
- What are spherical coordinates?

↓ This lecture took place on 2018/10/30.

## Two-dimensional surfaces in $\mathbb{R}^3$

- Define the scalar product on tangent space $T_x S$ of $S \subseteq \mathbb{R}^3$ as 2-Lpes with $x \in S$ [Definition 13]
- Define the first fundamental form of $S$ at $x$ [Definition 13]
- Give the length of the tangent vector $X$ [Definition 14]
- Define the angle between two tangent vectors $X$ and $Y$ [Definition 14]
- Define an angle-preserving/conformal map $g$ [Definition 15]
- Prove: Let $g : S_1 \to S_2$ be a $C^1$ map. $S_i$ are 2-Lpes in $\mathbb{R}^3$. $g$ is conformal if one of the following equivalent conditions are met [Lemma 6]:
  - $\frac{I_{S_1}(X,Y)}{I_{S_1}(X) \cdot I_{S_1}(Y)} = \frac{I_{S_2}(D_{g(x)}X, D_{g(x)}Y)}{I_{S_2}(D_{g(x)}X) I_{S_2}(D_{g(x)}Y)}$
  - There exists a linear isometry $O : T_x S_1 \to T_{y(x)} S_2$, i.e. $\|OX\| = \|X\| \,\forall X \in T_x S_1$ and a real number $s > 0$ such that $Dg(x) = s \cdot O$
  - Let $G_1$ be the metric tensor to $S_1$ at $x$ wrt. the basis $\{\partial_{u_1}, \partial_{u_2}\}$ and $G_2$ the metric tensor for $S_2$ at $y = g(x)$ wrt. the basis $\{\partial_{v_1}, \partial_{v_2}\}$. Let moreover $M_u^v$ be the matrix representation of $Dg(x)$ wrt. $\{\partial_{u_1}, \partial_{u_2}\}$ ars basis in $T_x S_1$ and $\{\partial_{v_1}, \partial_{v_2}\}$ as basis in $T_{g(x)} S_2$. Then $\exists s > 0 : s^2 \cdot G_1 = (M_u^v)^t \cdot G_2 \cdot M_u^v$.

↓ This lecture took place on 2018/10/31.

- Let $g : S_1 \to S_2$ be $C^1$. Give three equivalent conditions when $g$ is an isometry [Definition 16]
- Define the Weingarten map/shape operator of $S$ in $x$
- Prove: Consider Weingarten map $W_x : T_x S \to T_x S$. Suppose $S$ is $C^2$. $I_x(X, Y) = \langle X, Y \rangle$ defines a scalar product on $T_x S$. $W_x$ is symmetric/self-adjoint on $T_x S$ wrt. $I_x$. This means $\forall X, Y \in T_x S : I_x(W_x X, Y) = I_x(X, W_x Y)$ [Theorem 7]

- Give the Rayleigh quotient
- Define $II_x(X, Y)$ (second fundamental form of $S$ at $x$) [Definition 18]
- Which properties does $II_x(X, Y)$ have? [Definition 18]
- What do you call the eigenvalues $\kappa_1$ and $\kappa_2$? [Definition 18]
- What do you call the eigenvectors $V_1$ and $V_2$? [Definition 18]
- What do we call the mean curvature of $S$ in $x$? [Definition 18]
- Define the Gauss curvature of $S$ in $x$? [Definition 18]

↓ This lecture took place on 2018/11/06.

- Give Meusnier's formula
- Define the minimal/maximal normal curvature
- Which quantities are defined without reference to a local chart $f : D \to S$?
- What is the metric tensor of $S$ in $x$? What does it depend on?
- Prove: Let $\{\partial_{u_1}, \partial_{u_2}\}$ be a local basis in $T_xS$. $X = x_{u_1} \cdot \partial_{u_1} + x_{u_2}\partial_{u_2}; Y = y_{u_1}\partial_{u_1} + y_{u_2}\partial_{u_2}$, ie. $\begin{bmatrix} x_{u_1} \\ x_{u_2} \end{bmatrix}$ and $\begin{bmatrix} y_{u_1} \\ y_{u_2} \end{bmatrix}$ are coordinate vectors of $X$ and $Y$ wrt. $\{\partial_{u_1}, \partial_{u_2}\}$.
  - We set $h_{ij} = II_x(\partial_{u_i}, \partial_{u_j}) = \langle \partial_{u_i}, W_x\partial_{u_j} \rangle$. $\kappa = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$. Then $II_x(X, Y) = \begin{bmatrix} x_{u_1} & x_{u_2} \end{bmatrix} \cdot \kappa \cdot \begin{bmatrix} y_{u_1} \\ y_{u_2} \end{bmatrix}$.
  - Let $W$ be the matrix representation of $W_p$, ie. $X = x_{u_1}\partial_{u_1} + x_{u_2}\partial_{u_2}$. Then $W_xX = x_{u_1}W_x\partial_{u_1} + x_{u_2}W_x\partial_{u_2}$. $W_x\partial_{u_i} = w_i^1\partial_{u_1} + w_i^2\partial_{u_2}$.
  - The matrix representation of $W_x$ has the form $W = \begin{bmatrix} W_1^1 & W_2^1 \\ W_1^2 & W_2^2 \end{bmatrix}$ and $W \cdot \begin{bmatrix} x_{u_1} \\ x_{u_2} \end{bmatrix}$ is a coordinate vector of $W_x \cdot X$ wrt. $\{\partial_{u_1}, \partial_{u_2}\}$.
  - $G$ is metric tensor with $g_{ij} = I_x(\partial_{u_i}, \partial_{u_j})$.

  Then $\kappa = W \cdot G$, and accordingly $W = \kappa \cdot G^{-1}$ and $G^{-1} = (g^{ij})_{i,j=1}^2$.

↓ This lecture took place on 2018/11/06.

# Chapter 2: Integration on surfaces

- Define a generalized parallelogram in $\mathbb{R}^n$
- What properties is a volume supposed to satisfy?
- Define the volume $\operatorname{vol}(P(v_1, \dots, v_n))$
- Define the $k$-dimensional volume in $\mathbb{R}^n$ [Definition 1]
- Prove: $\operatorname{vol}_k(v_1, \dots, v_k) = \sqrt{\det(V^t V)}$ with $V = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix} \in \mathbb{R}^{n \times k}$ [Lemma 1]
- Define integration on one signal chart [Definition 2]
- Define the Gram determinant [Definition 2]
- Prove: The definition of the integral (by Definition 1) is independent of the chosen parametrization [Lemma 2]
- Prove: Let $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ with $V := \begin{bmatrix} v_1 & \dots & v_{n-1} \end{bmatrix}$. Then $\det(V^t \cdot V) = \operatorname{vol}_{n-1}(v_1, \dots, v_{n-1})^2 = \|v_1 \wedge v_2 \wedge \dots \wedge v_{n-1}\|^2$

↓ This lecture took place on 2018/11/13.

- Define the base of a topology [Definition 3]
- Give the base for the standard topology in $\mathbb{R}^n$
- Define: second countable topological space [Definition 4]
- Define: $Q \subseteq X$ is dense in $X$ [Definition 4]
- Define separability of topologies [Definition 4]
- Define the interior of $M$
- Show: $Q^n$ is dense in $\mathbb{R}^n$
- Prove: Every separable metric space is second countable
- Prove: Every second countable space is separable

- Define neighborhood filters of $x$ [Definition 5]
- Define the base for neighborhood filters [Definition 5]
- Define: first countable topological space [Definition 5]
- Define: locally compact [Definition 6]
- Show: $\mathbb{R}^n$ is locally compact
- Define: Banach space
- Show: Let $S \subseteq \mathbb{R}^n$ be a k-Lpes. Then $S$ is locally compact [Lemma 4]
- Prove: For every $S \subseteq \mathbb{R}^n$ as $k$-Lpes, there exists a countable base for the topology on $S$: $\{V_i\}_{i\in\mathbb{N}}$ with the property that $\overline{V_i}$ is compact $\forall i$ [Lemma 5]

↓ This lecture took place on 2018/11/14.

- Prove: Let $S \subseteq \mathbb{R}^n$ be a $k$-Lpes. Then there exists an exhaustion of $S$ with compact sets $\{K_i\}_{i\in\mathbb{N}}$ [Lemma 6]
- Prove: Let $S$ be a $k$-Lpes in $\mathbb{R}^n$. Then there exist at most countably many charts $f_i : D_i \to f_i(D_i) =: V_i \subseteq S$ such that $S = \bigcup_{i=1}^{\infty} V_i$ [Corollary 1]
- Define: partition of unity [Definition 7]
- Prove: Let $S \subseteq \mathbb{R}^n$ be a $k$-Lpes. Let $\{U_s\}_{s\in I}$ be a given open cover of $S$. Then there exists a partition of unity $\{\varepsilon_i\}_{i\in\mathbb{N}}$ subordinate to $\{U_s\}_{s\in I}$ [Theorem 1]
- Define: Let $S \subseteq \mathbb{R}^n$ be a $k$-Lpes. Let $h : S \to \mathbb{R}$ be continuous. Define $\int_S h\, dS$ [Definition 8]

↓ This lecture took place on 2018/11/20+21.

- Prove: Let $S$ be a $k$-Lpes. $f : D \to V = f(D)$ be a local chart. Let $h : S \to \mathbb{R}$ be a such that $\operatorname{supp}(h) \subseteq V$. Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a partition of unity on $S$. Then $h$ is integrable on $S$ iff the following two conditions are satisfied:
  - $\forall i \in \mathbb{N}$ the product $\varepsilon_i h$ is integrable on $S$
  - $\sum_{i\in\mathbb{N}} \int_S |\varepsilon_i \cdot h|\, ds < \infty$

If true, then $\int_S h\,ds = \sum_{i\in\mathbb{N}} \int_S \varepsilon_i h\,dS$ [Lemma 7].

- Define: atlas of a topology
- Define the integral over some $k$-Lpes $S$ [Definition 8]
- Let $S$ be a $k$-Lpes, $h : S \to \mathbb{R}$. Let $(f_j, D_j, V_j)$ be an atlas for $S$; $f_j : D_j \to V_j = f(D_j) \subseteq S$ and let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a partition of unity subordinate to $(V_j)_{j\in\mathbb{N}}$. We suppose
  - $\forall i \in \mathbb{N}$ ($\exists j \in \mathbb{N}$ such that $\mathrm{supp}(\varepsilon_i h) \subseteq \mathrm{supp}(\varepsilon_i) \subseteq V_j$): $\varepsilon_i h$ is integrable on $S$
  - $\sum_{i\in\mathbb{N}} \int_S |h|\,\varepsilon_i\,ds < \infty$

  Then for any other atlas $(g_l, E_l, W_l)$ and any partition of unity $(\eta_m)_{m\in\mathbb{N}}$ subordinate to $(W_l)_{l\in\mathbb{N}}$ the same two conditions hold. Moreover the value $\sum_{i\in\mathbb{N}} \int_S \varepsilon_i \cdot h\,ds$ does not depend on the specific choice of the Atlas or on the choice of the partition of unity $(\varepsilon_i)_{i\in\mathbb{N}}$. We define $\sum_{i\in\mathbb{N}} \int_S \varepsilon_i \cdot h\,ds := \int_S h\,dS$. If both conditions hold, we say that $h$ is integrable on $S$ [Theorem 2]
- Define: $k$-dimensional zero set [Definition 9]

## Polar coordinates and the unit ball in $\mathbb{R}^n$

- Prove: $\det(DP_n) = r^{n-1} \prod_{k=1}^{n-1} \cos^{k-1}(\varphi_k)$ [Lemma 8]
- Define spherical shells
- Give the transformation theorem for integrals

↓ This lecture took place on 2018/11/27.

- Define: Euler's Gamma function
- Euler's Gamma function is the only function satisfying which 3 properties?
- Give Fubini's Theorem
- Give the volume of the $n$-dimensional unit sphere

## The Divergence Theorem of C.F. Gauss

- (Give Gauss' Divergence theorem?) (occurs later in other form?!)
- Define $C^1$-smooth boundaries $\partial\Omega$ where $\Omega \subseteq \mathbb{R}^n$ is a bounded domain [Definition 10]

> ↓ This lecture took place on 2018/11/28.

- Define vector fields [Definition 11]
- Define $C^1$ vector fields [Definition 11]
- Define tangential vector fields
- Define: Continuous vector field $F$ is integrable on $(n-1)$-Lpes $S$ [Definition 12]
- Define: Divergence of $F : \Omega \to \mathbb{R}^n$ as $C^1$ vector field on open domain $\Omega \subset \mathbb{R}^n$ [Definition 13]
- Give Gauss' Divergence theorem [Theorem 3]
- Recognize: Gauss' Divergence theorem is a generalization of the fundamental theorem of calculus into $n$ dimensions
- Prove: Let $\Omega \subseteq \mathbb{R}^n$ be an open domain $f \in C_0^1(\Omega)$, ie. $\operatorname{supp}(f) \subset \Omega$ where $\operatorname{supp}(f)$ is compact. Let $C_0^k(\Omega) = \left\{g \in C^k(\Omega) \mid \operatorname{supp}(g) \subseteq \Omega \text{ is compact}\right\}$. Then we have $\forall i \in \{1, \dots, n\} : \int_\Omega f_{x_i}(x)\,dx = 0$ [Lemma 9]
- Give Cramer's rule
- What can Cramer's rule be used for (besides solving a linear equation system)?
- Define the cofactor matrix [Definition 14]
- Define the adjunct matrix [Definition 14]

> ↓ This lecture took place on 2018/12/04.

- Define the whole group in $\mathbb{R}^{n\times n}$ [Lemma 10]
- Is $\mathrm{GL}(n) \subseteq \mathbb{R}^{n\times n}$ open in $\mathbb{R}^{n\times n}$?

- Prove: Jacobi Formula [Lemma 11]
- Give the Jacobi Formula in the special case $A \in \mathrm{GL}(n)$ [Lemma 11]
- Prove: Let $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ (column vectors) and $A \in \mathrm{GL}(n)$. Let $\hat{A} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n-1}$. $\hat{A}^t \hat{A} \in \mathbb{R}^{(n-1)\times(n-1)}$. Then $\det(\hat{A}^t\hat{A}) = \left\| A^{-t} e_n \right\|^2 \det(A^t A)$ [Lemma 12]
- Prove: Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open domain with $C^2$-smooth boundaries. Let $\Psi : U \to V$ be a local diffeomorphism with the properties in Definition 10. Let $f : U \to \mathbb{R}$ be contiuously differentiable with supp $\Subset U$. Then
$$\int_{U^+} f_{x_i} \, dx = \int_{U \cap \Omega} f_{x_i} \, dx = \int_{U \cap \partial\Omega} f \cdot \nu_i \, ds$$
where $\nu_i$ is the i-th component of $\nu(x)$ und $\nu(x)$ is the normalized exterior normal vector field to $\partial\Omega$ [Lemma 13]

↓ This lecture took place on 2018/12/05.

Apparently, we only did proofs.

↓ This lecture took place on 2018/12/11.

- Assume $\partial\Omega$ is only $C^1$. Does Gauss' divergence theorem still hold?
- Define a regular boundary point [Definition 15]
- Define a singular boundary point [Definition 15]
- Define a regular $\mathbb{C}^1$ polyhedron [Definition 15]
- Define the Laplace operator [Definition 16]
- Define harmonic functions [Definition 16]
- Give and prove Greens formula [Theorem 4]
- Give and prove the mean value theorem of harmonic functions [Theorem 5]
- Define radially symmetric functions.

↓ This lecture took place on 2018/12/12.

- Define the Maximum Principle for Harmonic Functions [Corollary]

## Surfaces (manifolds) with boundary

- Define a $k$-dimensional locally parameterized embedded surface with boundary ($k$-Lpesb) [Definition 17]
- Define a interior point [Definition 17]
- Define a boundary point [Definition 17]
- Define boundary $\partial S$ [Definition 17]
- When does $\partial S$ correspond to the topological boundary?
- Define the curl of F [Definition 18]
- Give Stokes' Theorem [Theorem 6]

↓ This lecture took place on 2019/01/08.

- Define the exterior unit normal vector field to $\partial\Omega$ [Lemma 14]
- Let $\Omega \subseteq \mathbb{R}^n$ be bounded, connected, $C^1$-smooth boundary, $f, g \in C^1(\overline{\Omega})$. Let $[v_1, \dots, v_n]^T$ be the exterior unit normal vector field. Show [Lemma 14]:

$$\int_\Omega f_{x_i} \cdot g \, dx = -\int_\Omega f \cdot g_{x_i} \, dx + \int_{\partial\Omega} f \cdot g \cdot \nu_i \, dS$$

- Prove Stokes' Theorem

↓ This lecture took place on 2019/01/09.

## Classical Theory of surfaces

- What is Einstein's summation convention?
- Define a metric tensor.
- Define the second fundamental form on $S$ at $x$.
- Define the principal curvatures of $S$ in $x$.
- Define the Christoffel symbol of second kind.
- Derive the Gauss-Weingarten equations.

↓ This lecture took place on 2019/01/15.

"Vector fields and the covariant derivative"

- Give the Riesz representation theorem
- Define the Gradient (also called "contravariant") representation of $D\varphi$.
- Define the directional derivative of $\varphi$ in direction $V$ [Definition 1].
- Prove: Let $X, Y$ be smooth tangential vector fields on $S$. Let $\varphi : S \to \mathbb{R}$ be an arbitrary smooth scalar function. Then $\partial_X(\partial_y \varphi) - \partial_Y(\partial_X \varphi)$ is a smooth scalar function on $S$ and there exists a unique, tangential vector field $Z : S \to \mathbb{R}^n$ such that $\partial_X(\partial_y \varphi) - \partial_Y(\partial_X \varphi) = \partial_Z \varphi$ ($\forall \varphi : S \to \mathbb{R}$ smooth). If $X = \xi^i X_{u_i}$ and $Y = \eta^i X_{u_i}$, then $Z = \alpha^j X_{u_j}$ with $\alpha^j = \xi^i \eta^j_{u_i} - \eta^i \cdot \xi^j_{u_i}$ [Lemma 1].
- Define the Lie bracket [Definition 2]

↓ This lecture took place on 2019/01/16.

- Define the vector field on $S$ along $\gamma$ [Definition 3].
- Define the covariant derivative of $V$ along $\gamma$: $\frac{\nabla}{dt} V(t)$ [Definition 4]
- Let $S$ be a 2-Lpes in $\mathbb{R}^3$, $\gamma : I \to S$ be a regular curve. Let $V, W : I \to \mathbb{R}^3$ be vector fields along $\gamma$. Let $\gamma$ be smooth and let $\varphi : J \to I$ be a diffeomorphism, i.e. $\tilde{\gamma} : J \to S$. Let $\tilde{\gamma} = \gamma \circ \varphi$ be a reparametrization of $\gamma$. Give the 4 laws of covariant derivatives. [Lemma 2]

↓ This lecture took place on 2019/01/22.

- Define the covariant derivative of the vector field $V$ in direction $X$ [Definition 5]
- Give a characterization of $\nabla_x V$
- Define the tangential vector field [Definition 6]
- Rules for covariant derivatives [Lemma 3]
- Define the second covariant derivative of smooth vector field $Z$ [Definition 7]
- Prove: Let $V, W$ and $Z$ be smooth vector fields on $S$. Then with $V = v^i X_i$, $W = w^i X_i$, $Z = z^i X_i$ [Lemma 4].

$$\begin{aligned}\nabla^2_{v,w} Z = \Big[ & z^m_{u_i u_j} v^i w^j + \Gamma^m_{ij} z^i_{u_k} (v^i w^k + v^k w^i) - \Gamma^k_{ij} z^m_{u_k} v^i w^j \\ & + \left( (\Gamma^m_{kj})_{u_i} + (\Gamma^m_{li} \Gamma^l_{kj} - \Gamma^l_{kl} \Gamma^l_{ij}) \right) \cdot v^i w^j z^k \Big]\end{aligned}$$

- Define the second covariant derivative of the vector field $Z$ on $S$ [Definition 8]

↓ This lecture took place on 2019/01/23.

- Define the Riemann curvature tensor on a surface [Definition 9]

## Inner geometry of surfaces

- Define geometric quantities [Definition 10]
- Define geometric quantities of an inner geometry on $S$ [Definition 10]
- Prove: $R(V, W)Z = I_2(W, Z) \cdot S_x V - I_2(V, Z) \cdot S_x W$ [Theorem 1]

↓ This lecture took place on 2019/01/29.

- Give and prove the Theorema Egregium by Gauss [Theorem 2]

- Give a consequence of the Theorema Egregium
- Give and prove the symmetries of the Riemannian curvatures [Lemma 5]
- What is the result for local coordinates? [Lemma 6]
- Give $R(V,W)X$ in matrix and coordinate form. What does it determine? Prove it [Lemma 7]