# Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

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March to July 2016

## Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

# 1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z\in\mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$
$$\exp(z+w) = \exp(z)\cdot\exp(w)$$
$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n}\in\mathbb{Q}\wedge n\neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z \qquad \text{for } z\in\mathbb{C}$$

It holds that

$$\exp(z)\neq 0 \qquad \forall z\in\mathbb{C}$$

$\exp(x)$ for $x\in\mathbb{R}$

$$e^x > 0 \qquad \forall x\in\mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^0 = 1$$

Let $n\in\mathbb{N}$

$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$
$$\lim_{x\to-\infty} e^x\cdot x^n = 0$$



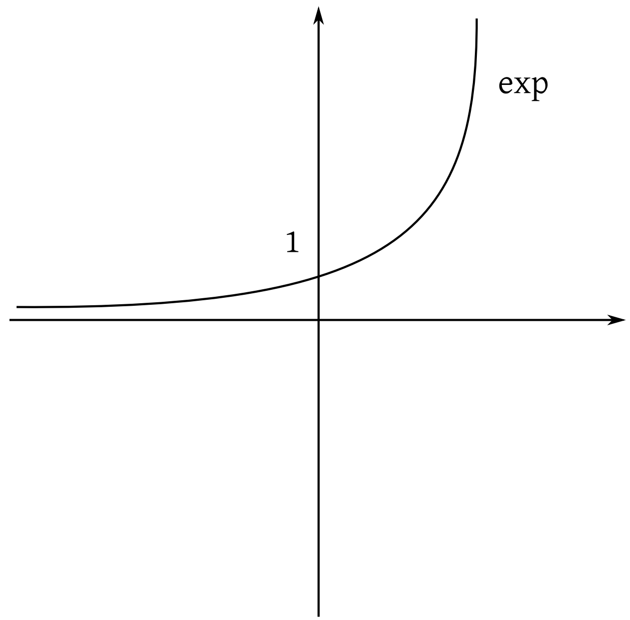


Figure 1: Graph of the exponential function

# 2 The natural logarithm

$$\exp:\mathbb{R}\to(0,\infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

**Lemma 1.** $\exp:\mathbb{R}\to(0,\infty)$ is surjective.

*Proof.* We need to show that the equation $e^x = y$ has some solution for every $y>0$. We will use the Intermediate Value Theorem, we discussed in the previous course “Analysis 1”.

**Case 1** First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y]: \quad e^\xi = y$$

**Case 2** Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w]: \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp: \mathbb{R} \to (0, \infty)$ is bijective. □

**Definition 1.** We call the inverse function *natural logarithm*[1].

$$\exp^{-1}: (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R}: \ln(e^x) = x$ and $\forall y \in (0, \infty): e^{\ln(y)} = y$.
- $\ln: (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

  *Proof.* Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xRightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! □

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.



## 2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that
  $$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Limes:
  $$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

*Proof.*
- $$x \cdot y = e^{\ln(x \cdot y)}$$
  $$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$
  Injectivity of exp:
  $$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.
  $$\lim_{n \to \infty} y_n = \ln(1) = 0$$
  $$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$
  where
  $$e^0 = 1 \Rightarrow \ln(1) = 0$$

□

**Theorem 1** (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

*Proof.* Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. □

**Theorem 2.** The logarithm function is differentiable in $(0,\infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

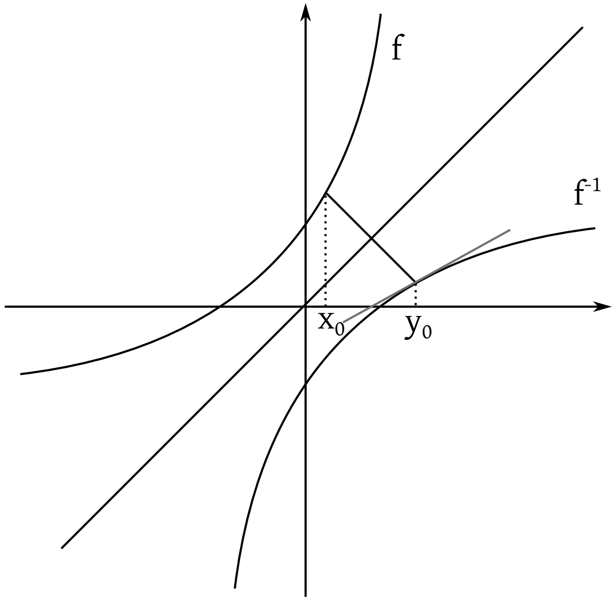


Figure 2: A geometric proof of differentiability

*Proof.* **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$



$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

**Second approach using chain rule** Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

**Third approach** Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

## 2.2 Extension of the functional equation of logarithm

## 2.3 A different proof for the derivative of logarithm

*Proof.*

$$[\ln(x)]' = \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$= \frac{1}{x}\cdot \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x}\to 0$$

$1+\frac{h}{x} = w$ then it holds that $h\to 0 \Rightarrow w\to 1$.

$$\frac{h}{x} = w-1$$

$$\lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{=} \lim_{h\to 0} \frac{\ln(w)}{w-1} = 1$$

□

**Remark 1.** The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C}\to\mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C}\setminus\{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C}\setminus\{x\in\mathbb{R} : x\le 0\}$

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This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

## 2.4 Further remarks on differential calculus

**Theorem 3.** Let $f : I\to\mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where $I$ is an interval. Then $f^{-1} : f(I)\to\mathbb{R}$ is defined and the inverse function.

Let $f$ in $x_0\in I$ be differentiable and $f'(x_0)\neq 0$. Then $f^{-1}$ is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$



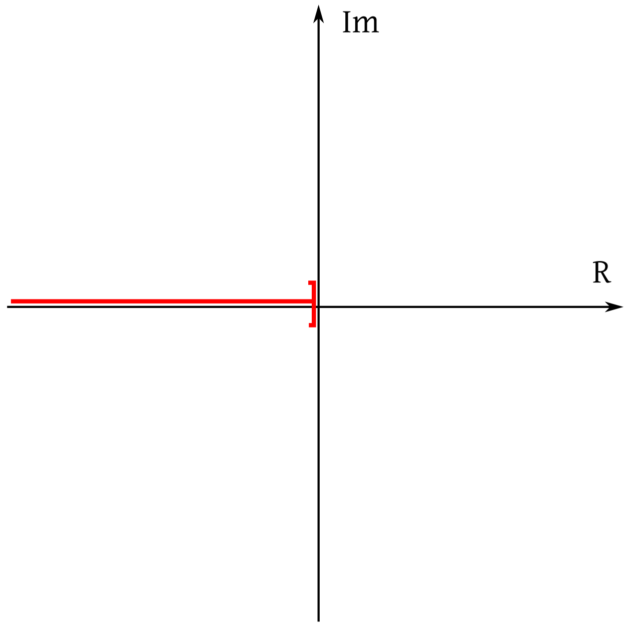


Figure 3: Continuous exponential function in $\mathbb{C}$

*Proof.* Let $y_n\to y_0$ and $y_n\in f(I)$; $y_0 = f(x_0)$; $y_0\in f(I)$; $y_n = f(x_n)$. $y_n\neq y_0\Rightarrow x_n\neq x_0$.

$$\lim_{n\to\infty} \frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}$$

$$= \lim_{n\to\infty} \frac{x_n-x_0}{f(x_n)-f(x_0)} = \frac{1}{\lim_{n\to\infty} \underbrace{\frac{f(x_n)-f(x_0)}{x_n-x_0}}_{\text{ex}=f'(x_0)}} = \frac{1}{f'(x_0)}$$

□

**Lemma 2.** Let $f : I\to\mathbb{R}$ where $I$ is some interval. Then it holds that

$$f = \text{ const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0\forall x\in I$$

*Proof.* $\Rightarrow$ Immediate.

$\Leftarrow$ Let $f$ be differentiable and $f' \equiv 0$. Assume $f$ is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

$\square$

**Definition 2.** Let $I$ be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of $f$ if $F$ is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

**Lemma 3.** Let $f : I \to \mathbb{R}$. Let $F_1$ and $F_2$ be two primitive functions of $f$. Then it holds that $F_1 - F_2 = \text{const}$.

*Proof.* $F_1$, $F_2$ are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

$\square$

**Theorem 4.** Let $I$ be an interval. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable functions in $I$.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n \in \mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every $x$)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon\, |u - v|\, \forall u, v \in I$$

Then $f$ is differentiable in $I$ and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

*Proof.* Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0) \right| + \left| f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0) \right| \forall N \in \mathbb{N}$$

$$\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right|$$

$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right| + |f_N'(x_0) - \varphi(x_0)|$$

**1st term**

$$\left| \frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0} \right| = \left| \frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0} \right|$$

$$\leq \frac{\varepsilon}{3} \frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large $N$.

**3rd term** $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large $N$.

Now let $N$ be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

**2nd term**

$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0}\right| - f_N'(x_0)$$

Differentiability of $f_N$: Therefore for $|x - x_0| < \delta$.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$f$ is differentiable in $x_0$ and $f'(x_0) = \varphi(x_0)$. □

**Theorem 5.** Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ $(n \in \mathbb{N})$ and $f_n$ is differentiable in $I$.

Assumption:

1. $f_n \to f$ converges pointwise in $I$ (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in $I$

Then $f$ is differentiable in $I$ and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

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This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

**Theorem 6** (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $I$ and let $f_n$ be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f_n'(x) \to \varphi(x)$ for every $x$
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon\,|u - v|$

Then it holds that $f$ is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

**Theorem 7.** Let $f_n$ and $f$ be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in $I$ for $n \to \infty$
- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in $I$, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f_n'(x) - g(x)| < \varepsilon$

Then $f$ is differentiable in $I$ and $f'(x) = g(x) \forall x \in I$.

*Proof.* We check whether the two conditions lead to the conditions of Theorem 6. We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence

$$\forall x \in I : f_n'(x) \to g(x)$$

3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and $N$ is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$\begin{aligned} |f_n'(x) - f_m'(x)| &= |f_n'(x) - g(x) + g(x) - f_m'(x)| \\ &\leq |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.
Let $\varepsilon > 0$ be arbitrary and $N$ such that $n, m \geq N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m\to\infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$\begin{aligned} &= \lim_{m\to\infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)| \\ &= \lim_{m\to\infty} |f_m'(\xi_{m,n}) - f_n'(\xi_{m,n})| \cdot |u - v| \end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

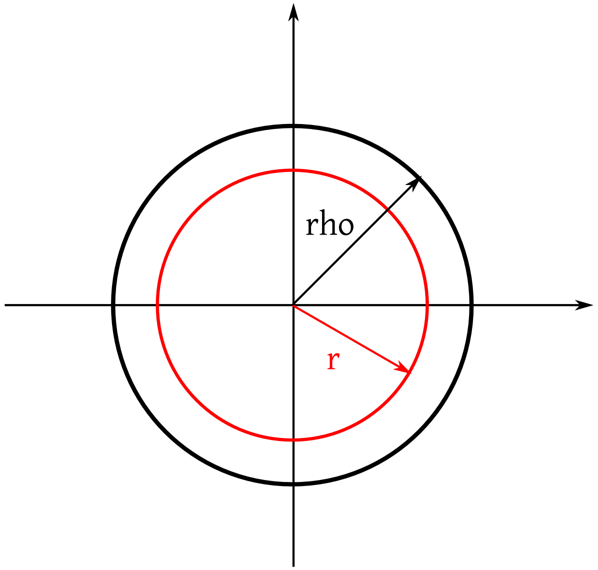


Figure 4: Convergence radius

**Remark 2** (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$



Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

$\square$

Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of $Q$?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \underbrace{\limsup_{j\to\infty} \left( |a_{j+1}|^{\overbrace{\frac{1}{j+1}}^{\to 1}} \right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty} \left[ (j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and $P$ is differentiable in point $x$ with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

[2]Where overline means "closed"

**Lemma 4.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then $P$ is differentiable in $x$ and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

## 2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly $g$ is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$



Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for $x$.

$$\Rightarrow 1 + x = w - wx$$

$$x(1+w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

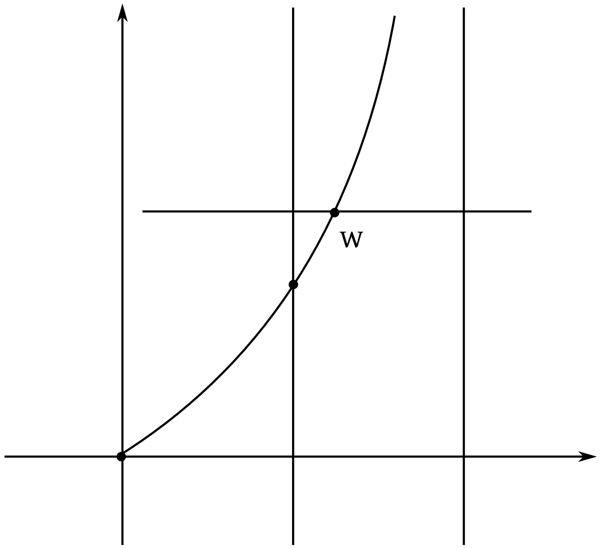


Figure 5: Plot of $\frac{1+x}{1-x}$

## 3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left( \underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[ \overline{\left(1 + \frac{it}{n}\right)} \right]^n$$



$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So $e^{it}$ lies inside the complex unit circle. Compare with Figure 6.

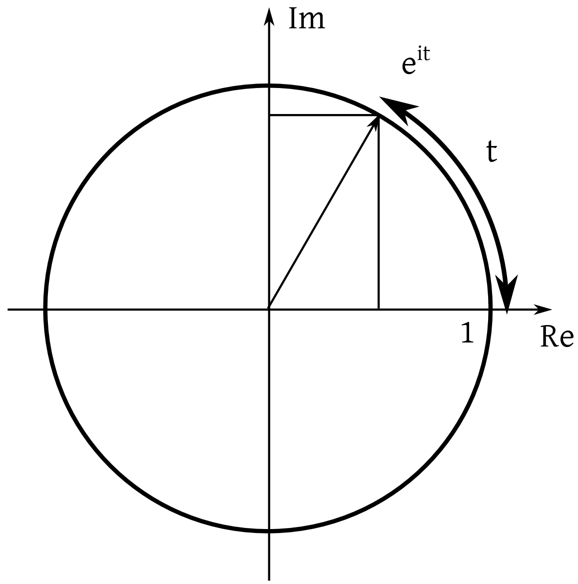


Figure 6: Unit circle in $C$ with $t$

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$ $$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$ $$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$ $$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

**Definition 3.** Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

---

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$
$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$



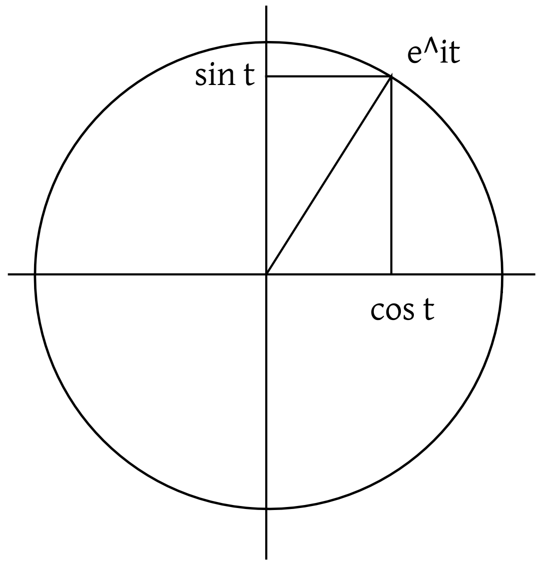


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

## 3.1 Series representation of trigonometric functions

**Lemma 5** (Addition of series of absolute convergence)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

*series sum.* Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

□



**Example 1** (Application)**.** Let $P(z) \coloneqq \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) \coloneqq \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0,\delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0,\delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

## 3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \ldots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \ldots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \ldots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots$$

Followingly,

$$\begin{aligned}\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C} \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \ldots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!}\end{aligned}$$

## 3.3 Functional equations of trigonometric functions

**Theorem 8** (Addition and substraction theorems)**.** We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i\sin(z+w)) \\ &\Rightarrow= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z \cdot \cos w) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i\,(-\cos z \sin w - \cos w \sin z) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i\sin(-(z+w))) \\ &\Rightarrow= \cos(z+w) - i\sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2\cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\cos(z-w) = \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)}$$

$$\sin(z-w) = -\cos z \cdot \sin(w) + \sin(z)\cos(w)$$



**Corollary 1.**

$$z = \frac{1}{2}(z+w) + \frac{1}{2}(z-w)$$

$$\Rightarrow \cos z = \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2}$$

$$w = \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w)$$

$$\cos w = \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2}$$

$$\cos z - \cos w = -2\sin\frac{z+w}{2}\sin\frac{z-w}{2}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z\to 0 \\ z\neq 0}} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right) \\ &= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz} - 1}{2iz}\right) \\ &= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z\to 0} \frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz; z\to 0 \Leftrightarrow w=0 \\ \lim_{w\to 0} \frac{e^w-1}{w}=1}} \end{aligned}$$

So it holds that

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

## 3.4 Trigonometric functions for real arguments

Subtitled "definition of $\pi$" and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \dots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{x^2}{(2k+2)(2k+1)}\right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k\geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left|\frac{s_{k+1}}{s_k}\right| = \left|\frac{x^2}{(2k+2)(2k+3)}\right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{ in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$



Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1-\frac{4}{6}=\frac{1}{3}>0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = \underbrace{-2 \cdot \underbrace{\sin \underbrace{\frac{z+w}{2}}_{\in(0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in(0,2]}}_{>0}}_{<0}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that cos is a strictly monotonically decreasing functionin $[0,2)$. Hence cos has only one root because it is continuous in $(0,2]$.

**Definition 4.** The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where $\xi$ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos\frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin\frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$$

TODO: table missing

$$e^{i\pi} = e^{i\frac{\pi}{2}+i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi+\frac{i}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$
$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{iz+2\pi i} + e^{-iz-2\pi i}\right) \\ &= \frac{1}{2}\left(e^{iz} + e^{-iz} \cdot \frac{1}{\underbrace{e^{2\pi i}}_{=1}}\right) = \cos z\end{aligned}$$

Therefore the cosine is periodic in $\mathbb{C}$ with period $2\pi$. Analogously, sine is periodic in $\mathbb{C}$ with period $2\pi$.

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This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

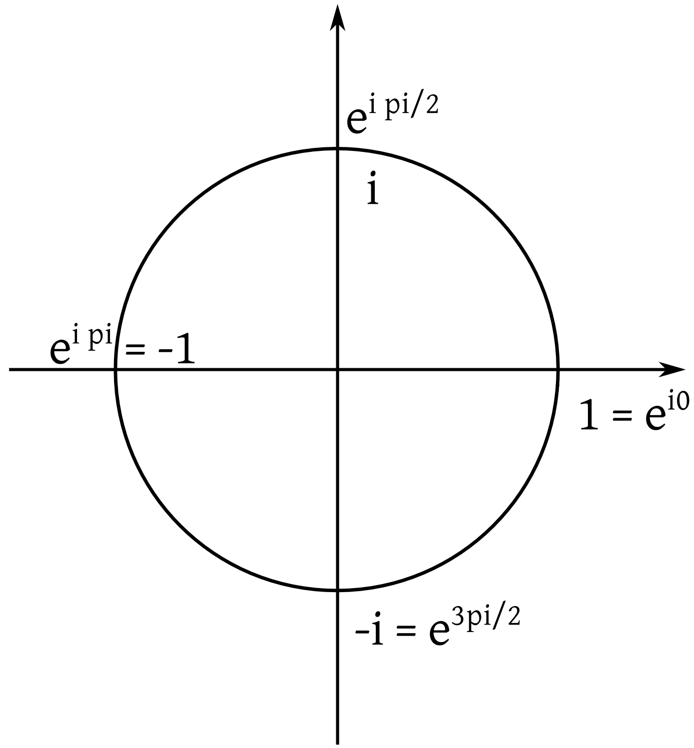
### 3.5 Periodicity and roots of trigonometric functions

TODO: equations missing

$$\cos(z+2\pi) = \cos(z)$$
$$\sin(z+2\pi) = \sin(z)$$

**Remark 3.** We will show: $\forall c \in (0, 2\pi)$, cos and sin are non-periodic with period $c$, hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x+c)$.






**Definition 5.**

$$f : \mathbb{C} \to \mathbb{C} \qquad (f : \mathbb{R} \to \mathbb{R})$$

is called *periodic* with period $c \in \mathbb{C}$ ($c \in \mathbb{R}$) if $\forall z \in \mathbb{C}$ it holds that

$$f(z+c) = f(z)$$
$$(\forall x \in \mathbb{R} : f(x+c) = f(x))$$

$c$ is called *period of $f$*.

**Remark 4.** If $f$ is periodic with period $c \in \mathbb{C}$, then $f$ is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

**Remark 5.**

$$z = u + iv$$
$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$
$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

**Remark 6.** Let $x \in \mathbb{R}$.

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \Re(e^{i(x+\frac{\pi}{2})}) \\ &= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}}) \\ &= \Re(ie^{ix}) \\ &= -\Im(e^{ix}) \\ &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \Im\left(e^{i(x+\frac{\pi}{2})}\right) \\ &= \Im(ie^{ix}) \\ &= \Re(e^{ix}) \\ &= \cos(x) \end{aligned}$$

$$\begin{aligned} \cos\left(x - \frac{\pi}{2}\right) &= \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x - \frac{\pi}{2}\right) &= -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\cos(x) \end{aligned}$$



Summary:

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x - \frac{\pi}{2}\right) &= \sin(x) \\ \sin\left(x - \frac{\pi}{2}\right) &= -\cos(x) \end{aligned}$$

**Remark 7** (A remark on the name "cosine")**.**

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of $x$ (Compare with Figure 8).

**Remark 8.**

$$\begin{aligned} \cos(x + \pi) &= \Re(e^{i(x+\pi)}) \\ &= \Re(-e^{ix}) \\ &= -\cos(x) \\ \sin(x + \pi) &= -\sin(x) \end{aligned}$$

**Remark 9.** Let $0 < c < 2\pi$. Assume cos is periodic with period $c$. We know that cos has exactly one root in $[0, 2]$,

$$\cos(x) = \cos(-x)$$

cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

   Therefore cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.

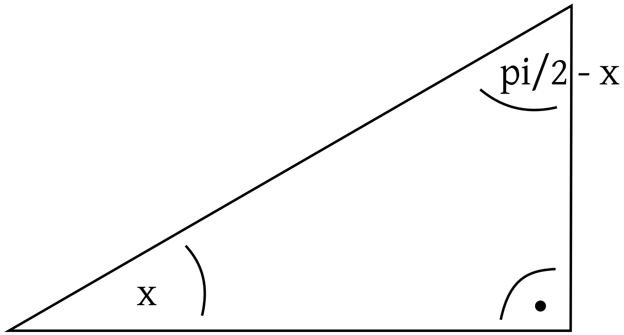



Figure 8: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$



$c$ would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of cos in } (-\frac{\pi}{2}, \frac{\pi}{2})$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period $c$. Hence $2\pi$ is indeed the smallest period of cos.

Analogously it holds for sin.

**Remark 10** (Roots of cos)**.**

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$
$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \qquad \forall l \in \mathbb{Z}$$

## 3.6 Derivatives of trigonometric functions

It holds that

$$\boxed{\lim_{z\to 0}\frac{\sin z}{z}=1}$$

Furthermore it holds that

$$\boxed{\lim_{z\to 0}\frac{1-\cos z}{z}=0}$$

*Proof.*

$$\begin{aligned}\frac{1-\cos z}{z}&=\frac{1}{z}\left(1-1+\frac{z^2}{2}-\frac{z^4}{4!}+\frac{z^6}{6!}-\frac{z^8}{8!}+\dots\right)\\&=\frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}-\frac{z^7}{8!}+\dots\end{aligned}$$

is convergent in $\mathbb{C}$ and (especially) continuous in 0

$$\lim_{z\to 0}\left(\frac{z}{2!}-\frac{z^3}{4!}+\frac{z^5}{6!}-\dots\right)=0$$

□

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

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This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z\to 0}\frac{\sin z}{z}=1$$
$$\lim_{z\to 0}\frac{1-\cos z}{z}=0$$

**Lemma 6.** The trigonometric functions sin and cos are differentiable in $\mathbb{R}$ (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x)=-\sin(x)\qquad\sin'(x)=\cos(x)$$

*Proof.*

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos(h)}{h}&=\lim_{h\to 0}\frac{\cos x\cdot\cos h-\sin x\cdot\sin h-\cos x}{h}\\&=\lim_{h\to 0}\cos x\cdot\frac{\cos(h)-1}{h}-\lim_{h\to 0}\frac{\sin x\cdot\sin h}{h}\\&=\cos x\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0}-\sin x\cdot\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1}\\&=-\sin(x)\end{aligned}$$

Analogously:

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin(h)}{h}&=\lim_{h\to 0}\frac{\sin x\cdot\cos h+\sin h\cdot\cos x-\sin x}{h}\\&=\sin(x)\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0}+\cos(x)\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{n}}_{=1}\\&=\cos(x)\end{aligned}$$

□

TODO: incomplete graphics, verify text

Figure 9. We now use tools of integral calculus:

Let $I=[a,b]$ and $\gamma: I\to\mathbb{R}$ ($\mathbb{R}^2$).

$$\gamma(t)=\begin{bmatrix}\gamma_1(t)\\\vdots\\\gamma_n(t)\end{bmatrix}$$

Assumption: $\gamma_1:[a,b]\to\mathbb{R}^n$.

$$\gamma'(t)=\begin{bmatrix}\gamma_1'(t)\\\vdots\\\gamma_n'(t)\end{bmatrix}$$

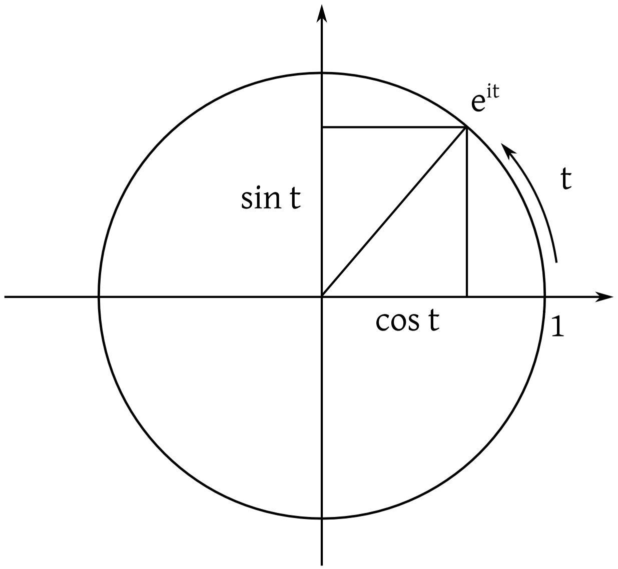


Figure 9: The arc length is related to sin and cos

TODO: graphics missing

Let $t \in [a,b]$. Then the arc length of $\gamma$ between $a$ and $t$ is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \, d\tau$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \to \mathbb{C}$$



$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

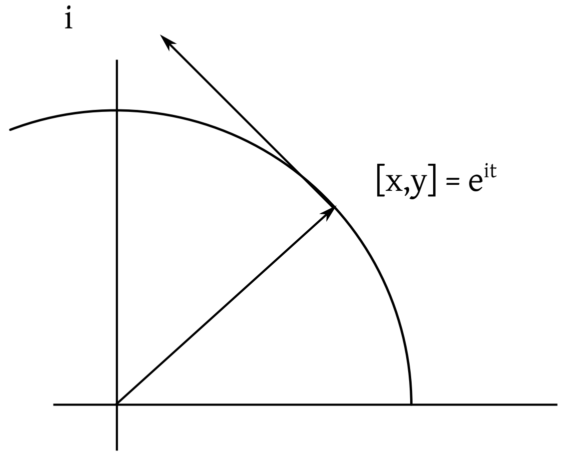


Figure 10: Derivative in $\mathbb{R}^2$

Compare with Figure 10.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \, d\tau = \int_0^t 1 \, d\tau = t$$

# 4 Integration calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses "Functional analysis" and "Measure and integration theory". For now, we look at the basis (as discussed by Königsberger).

Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \to \mathbb{R}$. We call $\varphi$ a *step function*, if $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1},x_j)} = c_j$ is constant. The points $x_j$ define a partition of the interval $[a, b]$.

$\tau[a, b]$ defines the set of step functions of interval $[a, b]$. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 11).

**Definition 6.** Let $\varphi : [a, b] \to \mathbb{R}$ be a step function and $x_0 = a < x_1 < \ldots < x_n = b$ as partition of $[a, b]$ and let $\varphi|_{(x_{j-1},x_j)} = c_j$ for $j = 1, \ldots, n$. Then we define

$$\int_a^b \varphi \, dx = \sum_{j=1}^{n} c_j \triangle x_j$$

where $\triangle x_j = x_j - x_{j-1}$ (for $j = 1, \ldots, n$).

$$\int_a^b \varphi \, dx \text{ is called } \textit{integral} \text{ of } \varphi \text{ over } [a, b]$$

$\varphi$ is the step function in terms of the partition $\{x_0, x_1, \ldots, x_5\}$.

It remains to show that if $\varphi$ satisfies the definition of a step function in terms of partition $\{x_0, \ldots, x_n\}$ and $\varphi|_{(x_{j-1},x_j)} = c_j$ (TODO: text missing: "but …") and $\varphi$ is a step function in terms of $\{w_0, w_1, \ldots, w_m\}$ and $\varphi|_{(w_{l-1},w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{l=1}^{m} c'_l \triangle w_l$$

Compare with Figure 12.



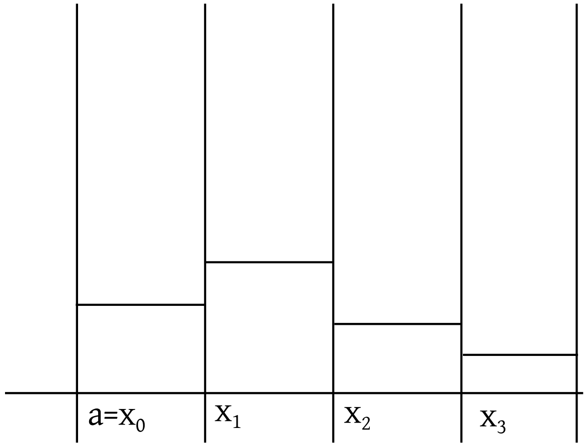


Figure 11: Partition of an area into rectangles

*Proof.* Let $Z = \{x_0, \ldots, x_n\}$ and $Z' = \{w_0, \ldots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly,

$$k_{j-1} < k_j$$

Let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$

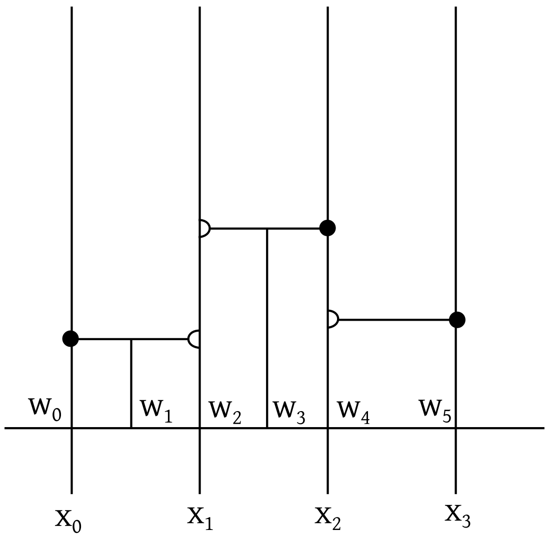


Figure 12: Step function $\varphi$

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \ldots, L\}$ there exists $j \in \{1, \ldots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1},\alpha_l)} \text{ is constant}$$

Hence $\varphi$ is a step function in terms of the partition $\{\alpha_0, \ldots, \alpha_L\}$.

Let $l \in \{0, 1, \ldots, L\}$ and $j$ such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c_l'' = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c_l'' = c_j$.



$$\sum_{l=1}^{L} c_l'' \cdot \triangle\alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c_l'' \triangle\alpha_l$$

$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}}^{k_j} \triangle\alpha_l$$

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1}) + (\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2})$$

$$+ \ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_j - x_{j-1} = \triangle x_j$$

Analogously,

$$\sum_{l=1}^{L} c_l'' \cdot \triangle\alpha_l = \sum_{k=1}^{m} c_k' \triangle w_k$$

So it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{k=1}^{m} c_k' \triangle w_k$$

□

---

This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

**Lemma 7.** Let $\varphi \in \tau[a, b]$ be a step function in terms of partition $a = x_0 < x_1 < \ldots < x_n = b$. Let $a = \alpha_0 < \alpha_1 < \ldots < \alpha_L = b$ with $Z = \{x_0, \ldots, x_n\} \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = z'$ ($z'$ has more intervals than $Z'$).

Then also $\varphi$ is step function in terms of partition $z'$.

*Proof.* see above □

**Lemma 8.** Let $\varphi_1, \varphi_2 \in \tau[a,b]$ and $\alpha, \beta \in \mathbb{C}$.
Then it holds that

- $\alpha\varphi + \beta\psi \in \tau[a,b]$ and
$$\int_a^b (\alpha\varphi + \beta\psi)\,dx = \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$
Hence ("linearity"),
$$\int_a^b : \tau[a,b] \to \mathbb{R} \text{ is linear}$$
- $|\varphi| \in \tau[a,b]$ and it holds that
$$\left|\int_a^b \varphi\,dx\right| \leq \int_a^b |\varphi|\ dx \leq \|\varphi\|_\infty (b-a)$$
Reminder: $\|\varphi\|_\infty = \max\{|\varphi(x)| : x \in [a,b]\}$
This gives "boundedness".
- Let $\varphi$ and $\psi$ be real values and it holds that
$$\forall x \in [a,b] : \varphi(x) \leq \psi(x)$$
Then TODO Monotonicity

*Proof.*

- Let $\varphi|_{(x_{k-1},x_k)} = c_k$ $\psi|_{(w_{j-1},w_j)} = d_k$
$$z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\} = \{x_0, \dots, x_n\} \cup \{w_0, \dots, w_m\}$$
where $\alpha_i$ is sorted ascendingly. $\varphi$ and $\psi$ are step functions in terms of $z''$, hence
$$\varphi|_{(\alpha_{i-1},\alpha_i)} = c_i' \text{ and } \psi|_{(\alpha_{i-1},\alpha_i)} = d_i'$$
$$\Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1},\alpha_i)} = \alpha c_i' + \beta d_i' \text{ constant}$$
$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a,b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi)\,dx = \sum_{i=1}^{L} (\alpha c_i' + \beta d_i') \cdot \triangle\alpha_i$$
$$= \alpha \sum_{i=1}^{L} c_i' \triangle\alpha_i + \beta \sum_{i=1}^{L} d_i' \triangle\alpha_i$$
$$= \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$
- Let $\varphi|_{(x_{i-1},x_i)} = c_i$ $(i = 1, \dots, n)$. Then,
$$|\varphi|\,|_{(x_{i-1},x_i)} = |c_i|$$
$$\left|\sum_{i=1}^{n} c_i \triangle x_i\right| \leq \sum_{i=1}^{n} |c_i| \cdot \underbrace{|\triangle x_i|}_{x_i - x_{i-1} > 0} = \sum_{i=1}^{n} |c_i| \cdot \triangle x_i = \int_a^b |\varphi|\ dx$$
$$\leq \sum_{i=1}^{n} \|\varphi\|_\infty \triangle x_i = \|\varphi\|_\infty \sum_{i=1}^{n} \triangle x_i$$
$$= \|\varphi\|_\infty ((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}))$$
$$= \|\varphi\|_\infty (x_n - x_0) = \|\varphi\|_\infty (b-a)$$
- Let $\varphi$, $\psi$ and $z''$ as in the linearity statement.
$$\left.\begin{aligned} \varphi|_{(\alpha_{i-1},\alpha_i)} &= c_i' \in \mathbb{R} \\ \psi|_{(\alpha_{i-1},\alpha_i)} &= d_i' \in \mathbb{R} \end{aligned}\right\}$$
$$\int_a^b \varphi\,dx = \sum_{i=1}^{L} c_i' \underbrace{\triangle\alpha_i}_{>0} \leq \sum_{i=1}^{L} d_i'\,dx$$
$$\int_a^b \varphi\,dx$$

□

**Definition 7.** Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\infty_A) : \mathbb{R} \to \mathbb{R}$ as
$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$
the characteristic function of $A$. Hence $\chi_A(x)$ is 1 if and only if $x$ is inside interval $A$.

**Remark 11.** Let $a \leq a' < b' \leq b$. Then

$$\chi_{(a',b')} \in \tau[a,b] \qquad \int_a^b \chi_{(a',b')}\, dx = 1 \cdot (b' - a')$$

Every linear combination of characteristic functions is also in $\tau[a,b]$.

On the opposite side, let $\varphi \in \tau[a,b]$ with $\varphi|_{(x_{i-1},x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \leq i \leq n$ and $0 \leq j \leq n$.

$$\Rightarrow \varphi = \sum_{i=1}^n c_i \chi_{(x_{i-1},x_i)} + \sum_{j=0}^n r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_a^b \varphi\, dx = \sum_{i=1}^n c_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^n c_j \int_a^b \chi_{(x_{i-1},x_i)}\, dx$$

# 5 Regulated functions

**Definition 8.** Let $D \subseteq \mathbb{R}$. Let $x_0$ be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n\in\mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n\to\infty} z_n = x_0$. Let $f : D \to \mathbb{C}$ be given.

We state that $f$ has left-sided limit $y_0$ in $x_0$ if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \cap (-\infty, x_0) \land |x - x_0| < \delta]$$
$$\Rightarrow |f(x) - y_0| < \varepsilon$$

Equivalently $\forall (z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $z_n < x_0$ and $\lim_{n\to\infty} z_n = x_0$ $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} f(x_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by $(x_0, \infty)$.

We denote: $y_0$ is left-sided limit of $f$ in $x_0$:

$$y_0 = \lim_{x\to x_0^-} f(x)$$

and right-sided limit of $f$ in $x_0$:

$$y_0 = \lim_{x\to x_0^+} f(x)$$

**Definition 9.** Let $a, b \in \mathbb{R}$ and $a < b$. A function $f : [a,b] \to \mathbb{C}$ is called *regulated functions* if

- $\forall x \in (a,b)$ $f$ has a left-sided and a right-sided limes in $x$
- $f$ has a right-sided limes in $a$
- $f$ has a left-sided limes in $b$

Examples for regulated functions:

- Every continuous function in $[a,b]$ is a regulated function.
- Every step function is a regulated function.
  Why? Consider $x \in (x_{i-1}, x_i)$. Then

  $$\lim_{\xi\to x^+} \varphi(\xi) = c_i = \lim_{\xi\to x^-} \varphi(\xi)$$

  Let $x = x_i$ be a partitioning point.

  $$\lim TODO \text{ and } \lim TODO$$

  So $\tau[a,b] \subseteq R[a,b]$. Compare with Figure 13.
- Let $f : [a,b] \to \mathbb{R}$ be monotonically. Then it holds that

  $$f \in R[a,b]$$

## 5.1 Approximation theorem for regulated functions

Let $f : [a,b] \in \mathbb{C}$. Then it holds that $f \in R[a,b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a,b]$ such that $\|f - \varphi\|_\infty < \varepsilon$. Hence $\forall x \in [a,b] : |f(x)|$ TODO

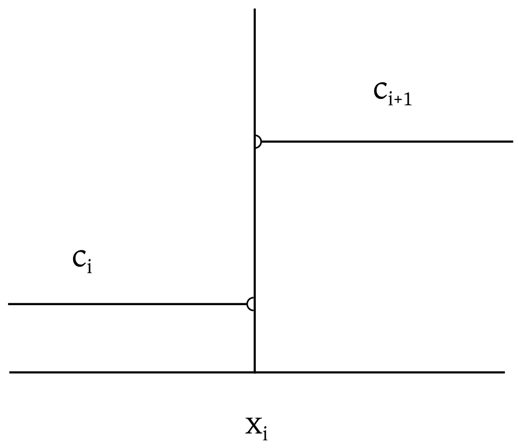


Figure 13: Step functions are also regulated functions

$$\Leftrightarrow \underbrace{\sup\{|f(x)-\varphi(x)| : x\in[a,b]\}}_{\|f-\varphi\|_\infty} < \varepsilon$$

$S\ \varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a,b]$ such that

$$|\varphi_n(x) - f(x)| < \varepsilon \forall x \in [a,b]$$

hence $f$ is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges continuously towards $f$.

*Proof.* $\Rightarrow$ Let $f\in R[a,b]$. Assume $\exists\varepsilon>0$ fixed such that $\forall\varphi\in\tau[a,b]$

$$\exists x\in[a,b] : |\varphi(x)-f(x)| \geq \varepsilon$$



We build nested intervals such that the desired property $|\varphi(x)-f(x)|\geq\varepsilon$ holds on every subinterval $[a_n,b_n]$.

Induction:

$n=0$ Let $a_0=a$ and $b_0=b$, hence the property holds in $[a_0,b_0]$.

$n\mapsto n+1$ Let $m=\frac{1}{2}(a_n+b_n)$. In $[a_n,b_n]$ the property holds.
Then the property either holds in $[a_n,m]$ or $[m,b_n]$. If the property does not hold in $[a_n,m]$:

$$\exists\varphi_1\in\tau[a_n,m] \text{ with } |\varphi_1(\xi)-f(\xi)|<\varepsilon \qquad \forall\xi\in[a_n,m]$$

If the property does not hold in $[m,b_n]$:

$$\exists\varphi_2\in\tau[m,b_n] \text{ with } |\varphi_2(\xi)-f(\xi)|<\varepsilon \qquad \forall\xi\in[m,b_n]$$

Let

$$\varphi(x)=\begin{cases}\varphi_1(x) & \text{for } x\in[a_n,m]\\ \varphi_2(x) & \text{for } x\in[m,b_n]\end{cases}$$

$$\Rightarrow \varphi\in\tau[a,b] \text{ and } |\varphi(\xi)-f(\xi)|<\varepsilon \qquad \forall\xi\in[a_n,b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1},b_{n+1}]$.
$([a_n,b_n])_{n\in\mathbb{N}}$ are nested intervals. Let $\varphi\in\bigcap_{n\in\mathbb{N}}[a_n,b_n]$.

**Case** $\xi\in(\mathbf{a},\mathbf{b})$ Let $\varepsilon$ satisfy the desired property. $f\in R[a,b]$, hence $f$ has left-sided limit $c_-$ in $\xi$ and right-sided limit $c_+$. Hence $\exists\delta>0$ such that

- $|x-\xi|<\delta\wedge a\leq x<\xi\Rightarrow|f(x)-c_-|<\varepsilon$
- $|x-\xi|<\delta\wedge\delta<x\leq b\Rightarrow|f(x)-c_+|<\varepsilon$

Choose $\delta$ sufficiently small such that

$$a<\xi-\delta<\xi+\delta<b$$

Let

$$\varphi(x)=\begin{cases}c_- & \text{for } x\in(\xi-\delta,\xi)\\ f(\xi) & \text{for } x=\xi\\ c_+ & \text{for } x\in(\xi,\xi+\delta)\end{cases}$$

$\varphi$ is necessarily a step function in $(\xi-\delta,\xi+\delta)$ and it holds that $\forall x \in (\xi-\delta,\xi+\delta) : |\varphi(x) - f(x)| < \varepsilon$.
Let $n$ be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi-\delta,\xi+\delta)$$

then

$$\varphi|_{[a_n,b_n]} \in \tau[a_n,b_n] \text{ and } |\varphi(x) - f(x)| < \varepsilon \quad \forall x \in [a_n,b_n]$$

This is a contradiction to our desired property.

For $\xi = a$ or $\xi = b$ only with one-sided limit.

□

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This lecture took place on 17th of March 2016 with lecturer Wolfgang Ring.

We learned: All regulated functions can be approximated with step functions.

$f \in R[a,b]$ in the proof $\Leftrightarrow$ $f$ is uniform limit of step functions. We have prove direction $\Rightarrow$.

**Lemma 9** (Cauchy criterion for limits of functions)**.** Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and $z_0$ is a limit point of $D$. Then $f$ has a limit in $z_0$ if and only if $\forall \varepsilon > 0 \exists \delta > 0 :$ $v, w \in D \setminus \{z_0\} \wedge |v - z_0| < \delta \wedge |w - z_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$.

If $D \subseteq \mathbb{R}$ and $x_0$ is limit point of $D \cap (x_0, \infty)$, then $f$ has a *right-sided limit* in $x_0$ if and only if $\forall \varepsilon > 0 \exists \delta > 0 : [v, w \in D \cap (x_0, \infty) \wedge |v - x_0| < \delta \wedge |w - x_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon]$.

Analogously for left-sided limit.

*Proof.* This proof is done only for the first point.

$\Rightarrow$

Assume $f$ has a limit $\eta$ in $z_0$. Choose $\delta$ such that $v, w \in D$ with $|v - z_0| < \delta$ and $|w - z_0| < \delta$ implies that $|f(v) - \eta| < \frac{\varepsilon}{2}$ and $|f(w) - \eta| < \frac{\varepsilon}{2}$. Then $|f(v) - f(w)| \leq |f(v) - \eta| + |\eta - f(w)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Leftarrow$

Assume the Cauchy criterion holds. Show: There exists $\eta \in \mathbb{C}$ such that for every sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D \setminus \{z_0\}$ with $\lim_{n\to\infty} w_n = z_0$ it holds that $\lim_{n\to\infty} f(w_n) = \eta$.

Let $(w_n)_{n\in\mathbb{N}}$ be as above. Show: $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and $\delta$ as above. Choose $N \in \mathbb{N}$ such that $n, m \geq N$

$$\Rightarrow |w_n - z_0| < \delta \wedge |w_m - z_0| < \delta$$

The Cauchy criterion holds for $n, m \geq N$:

$$|f(w_n) - f(w_m)| < \varepsilon$$

So $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence and (because $\mathbb{C}$ is complete) is also convergent. So $\exists \eta' \in \mathbb{C} : \lim_{n\to\infty} f(w_n) = \eta'$.

It remains to show: $\eta'$ is unique.

Let $(v_n)_{n\in\mathbb{N}}$ be another sequence with $\lim_{n\to\infty} v_n = z_0$ and $v_n \in D \setminus \{z_0\}$. As above: $\exists \eta'' \in \mathbb{C}$ such that $\lim_{n\to\infty} f(v_n) = \eta''$.

We construct:

$$(\xi_n)_{n\in\mathbb{N}} = (w_0, v_0, w_1, v_1, w_2, v_2, \ldots)$$

Then it holds that $\lim_{n\to\infty} \xi_n = z_0$.

We use the argument from above: $(f(\xi_n))_{n\in\mathbb{N}}$ is convergent, hence $\lim_{n\to\infty} f(\xi_n) = \eta$. Both subsequences $(f(w_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ must have the same limit, hence $\eta' = \eta = \eta''$.

□

*Proof of approximation theorem.* $\Leftarrow$

Let $f = \lim_{n\to\infty} \varphi_n$ be uniform on $[a,b]$. Let $\varphi_n \in \tau[a,b]$ and let $x_0 \in [a,b)$. Show: $f$ has a right-sided limit in $x_0$. Let $\varepsilon > 0$ arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that

$$|f(x) - \varphi_N(x)| < \frac{\varepsilon}{2} \forall x \in [a,b]$$

$\varphi_N$ is a step function (hence interval-wise constant). Choose $\delta > 0$ such that $\varphi_N|_{(x_0,x_0+\delta)} = c$ constant. Let $v, w \in (x_0, x_0 + \delta)$. Then it holds that

$$|f(v) - f(w)| \leq |f(v) - c| + |c - f(w)|$$

$$= |f(v) - \varphi_N(v)| + |f(w) - \varphi_N(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The Cauchy criterion implies that $f$ has a right-sided limit in $x_0$.

□

**Corollary 2.** $f \in R[a, b]$ if and only if $f(x) = \sum_{j=0}^{\infty} \psi_j(x)$ with $\psi_j \in \tau[a, b]$ and the series converges uniformly in $[a, b]$.

*Proof.* $\Leftarrow$

Let $\varphi_n = \sum_{j=0}^{n} \psi_j \in \tau[a, b]$ and $\varphi_n \to f$ continuously in $[a, b]$. From the approximation theorem it follows that $f \in R[a, b]$.

$\Rightarrow$

Let $f \in R[a, b]$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to f$ uniform in $[a, b]$. Let $\psi_0 = \varphi_0$.

$$\psi_j := \varphi_j - \varphi_{j-1} \text{ for } j \geq 1$$

Then it holds that

$$\sum_{j=0}^{n} \psi_j = \varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \ldots + (\varphi_{n-1} - \varphi_{n-2}) + (\varphi_n - \varphi_{n-1}) = \varphi_n$$

and $(\varphi_n)_{n\in\mathbb{N}}$ converges uniform if and only if the series is uniformly convergent.

□

**Lemma 10** (Sidenote)**.** Let $(f_n)_{n\in\mathbb{N}}$ with $f_n : D \to \mathbb{C}$ a sequence of functions in $D$, let $z_0 \in D$ and $\forall n \in \mathbb{N}$ $f_n$ is continuous in $z_0$. Furthermore let $f : D \to \mathbb{C}$ and $f_n \to f$ is uniform in $D$. Then $f$ is continuous in $z_0$.

*Proof.* Let $\varepsilon > 0$ arbitrary. Choose $N$ sufficiently large such that $|f(z) - f_w(z)| < \frac{\varepsilon}{3}$ $\forall z \in D$ (uniform convergence). Because $f_N$ is continuous in $z_0$, $\exists \delta > 0$ such that $z \in D$ and $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$.

Then for $|z - z_0| < \delta$ (with $z \in D$)

$$\underbrace{|f(z) - f(z_0)|}_{<\frac{\varepsilon}{3}} \leq \underbrace{|f(z) - f_N(z)| + |f_N(z) - f_N(z_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{<\frac{\varepsilon}{3}}$$

□



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This lecture took place on 18th of March 2016 with lecturer Wolfgang Ring.

**Theorem 9.** Let $f$ be a regulated function in $[a, b]$. Then $f$ is in at most countable infinite points of $[a, b]$ non-continuous.

*Proof.*

$$f = \sum_{k=0}^{\infty} \psi_k$$

where $\psi_k$ is a sequence of step functions and and the series is uniformly convergent. $\psi_k \in \tau[a, b]$.

Let $\{x_0^k, \ldots, x_{n(k)}^k\}$ be the partition points of $\psi_k$. Then $\psi_k$ is continuous in $[a, b] \setminus Z_k$. Let $Z = \bigcup_{k=0}^{\infty} Z_k$ be countable. Let $x \in [a, b] \setminus Z$ and $\varphi_n = \sum_{k=0}^{n} \psi_k$. Then it holds that $\varphi_n \to f$ is uniform in $[a, b]$ and $\varphi_n$ is continuous in $x$, because $x \notin Z$.

From Lemma 10 it follows that $f$ is continuous in $x$. □

## 5.2 Norms and vector spaces

**Definition 10** (Normed vector spaces)**.** Let $V$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). A map $n : V \mapsto [0, \infty)$ is called *norm* in $V$, if

1. $n(V) = 0 \Leftrightarrow V = 0$ ($V$ is null vector)
   "definity"
2. $\forall \lambda \in \mathbb{C}$ ($\mathbb{R}$) $\forall v \in V : n(\lambda v) = |\lambda| \cdot n(v)$ "positive homogeneity"
3. $\forall v, w \in V : n(v + w) \leq n(v) + n(w)$
   "triangle inequality"

Common notation: $\|v\|$ for $n(v)$ ("norm of $v$")
A vector space satisfying the norm properties is called *Normed vector space*

**Example 2.**
- $|x|$ is a norm in $\mathbb{R}$.
- $|z|$ is a norm in $\mathbb{C}$.

$\|\vec{x}\|$ is norm in $\mathbb{R}^n$.

Let $D \subseteq \mathbb{C}$.

$$B(D) = \{f : D \to \mathbb{C} : f \text{ limited to } D\}$$

$B(D)$ is a vector space. For $f \in B(D)$ we define:

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\}$$

"supremum norm" of $\infty$-norm of $f$ in $D$.

It holds that $\|\cdot\|_\infty$ is a norm in $B(D)$.

$$\|f\|_\infty = 0 \Leftrightarrow \sup\left\{\underbrace{|f(z)|}_{\geq 0} : z \in D\right\} = 0$$

$$\Leftrightarrow |f(z)| = 0 \quad \forall z \in D$$
$$\Rightarrow f = 0 \text{ in } B(D)$$

Homogeneity:

$$\begin{aligned}|\lambda \cdot f|_\infty &= \sup\{|\lambda f(z)| : z \in D\}\\ &= \sup\{|\lambda|\,|f(z)| : z \in D\}\\ &= \sup\{|f(z)| : z \in D\} \cdot |\lambda|\\ &= |\lambda| \cdot \|f\|_\infty\end{aligned}$$

Triangle inequality: Let $f, g \in B(D)$.

$$\begin{aligned}\|f+g\|_\infty &= \sup\{|f(z) + g(z)| : z \in D\}\\ &= \sup\left\{\underbrace{|f(z)|}_{\leq\|f\|_\infty} + \underbrace{|g(z)|}_{\leq\|g\|_\infty} :\right\}\\ &\leq TODO \qquad\qquad = \|f\|_\infty + \|g\|_\infty\end{aligned}$$

**Remark 12.** Let $V \subseteq B(D)$ be an arbitrary subvectorspace of $B(D)$. So $\|\cdot\|_\infty$ is also a norm in $V$.



Important example:

$$V = \mathcal{C}_b(D) = \{f : D \to \mathbb{C} : f \text{ is continuous and bounded in } D\}$$

Special case: $D = K$ compact in $\mathbb{C}$. Then every continuous function is also bounded.

$$\begin{aligned}\mathcal{C}(K) &= \{f : K \to \mathbb{C} : f \text{ is continuous}\}\\ &\subseteq B(K) \qquad \text{(sub vector space)}\end{aligned}$$

Another special case: $D = [a, b] \subseteq \mathbb{C}$

$$\tau[a, b] \subseteq B([a, b]) \text{ and}$$

$$R[a, b] \subseteq B([a, b])$$

**Remark 13** (Further properties of the norm)**.** The inverse triangle inequality holds:

$$\forall v, w \in V : |\|v\| - \|w\|| \leq \|v - w\|$$

*Proof.*

$$v = (v - w) + w$$

From triangle inequality it follows that

$$\|v\| \leq \|v - w\| + \|w\|$$

$$w = (w - v) + w$$
$$\|w\| \leq \|w - v\| + \|w\|$$

$$\begin{aligned}&= \|(-1)\cdot(v - w)\| + \|v\|\\ &= |(-1)| \cdot \|v - w\| + \|v\|\\ &= \|v - w\| + \|v\|\end{aligned}$$

$$\begin{aligned}\text{requirement 1} &\Rightarrow \|v\| - \|w\| \leq \|v - w\|\\ \text{requirement 2} &\Rightarrow \|w\| - \|v\| \leq \|v - w\|\\ \text{requirements} &\Rightarrow TODO\end{aligned}$$

□

**Definition 11.** Let $V$ be a normed vector space, $(v_n)_{n\in\mathbb{N}}$ be a sequence of elements in $V$ and $v \in V$. We define $(v_n)_{n\in\mathbb{N}}$ is convergent with limit $V$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq \mathbb{N} \Rightarrow \|V_n - V\| \leq \varepsilon]$$

**Remark 14** (Metric on $V$)**.**

$$d(v, w) = \|v - w\|$$

defines a metric on $V$. Properties of a metric:

1. $d(v, w) \geq 0$
2. $d(v, w) = 0 \Leftrightarrow v = w$
3. $\|v - w\| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$

Triangle inequality of metrics: Let $v, w, u \in V$.

$$d(v, u) = \|v - u\| = \|v - w + w - u\|$$

$$\leq \|v - w\| + \|w - u\| = d(v, w) + d(w, u)$$

Works only if $d(v, w) = d(w, v)$ and can be simply proven:

$$d(v, w) = \|v - w\| = \|w - v\| = d(w, v)$$

**Remark 15.** $(V_n)_{n\in\mathbb{N}}$ is called *Cauchy sequence in* $V$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|v_n - v_m\| < \varepsilon]$$

$V$ is called *complete normed vector space* if every Cauchy sequence in $V$ is also a convergent sequence in $V$.

A complete normed vector space is called *Banach space.*

## 5.3 Integration of regulated functions

**Theorem 10.** Let $f \in \mathcal{B}[a, b]$ and $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n \in \tau[a, b]$ and $\varphi_n \to_{n\to\infty} f$ uniform in $[a, b]$ ($\Leftrightarrow \|\varphi_n - f\| \to 0$ for $n \to \infty$).



Then we define

$$\int_a^b f\, dx = \lim_{n\to\infty} \int_a^b \varphi_n\, dx$$

for the integral of $f$ in $[a, b]$. The right-sided limit exists for every sequence $(\varphi)_{n\in\mathbb{N}}$ with the property above and is independent of the choice of the sequence $(\varphi_n)_{n\in\mathbb{N}}$.

*Proof.* Let $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [\underbrace{n \geq N \Rightarrow |\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]}_{\sup\{|\varphi_n(x) - f(x)| : x \in [a,b] \leq \varepsilon\}}]$$

$$\Rightarrow \|\varphi_n - f\|_\infty \leq \varepsilon$$

So $\varphi_n$ converges towards $f$ in terms of $\|\cdot\|_\infty$ in $\mathcal{B}[a, b]$.

Let $N$ be sufficiently large such that

$$\forall n \geq N : \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(b - a)}$$

Then it holds for $i_n = \inf_a^b \varphi_n\, dx$ and $n, m \geq N$,

$$\begin{aligned}
|i_n - i_m| &= \left| \int_a^b \varphi_n\, dx - \int_a^b \varphi_m\, dx \right| \\
&= \left| \int_a^b (\varphi_n - \varphi_m)\, dx \right| \\
&\leq \|\varphi_n - \varphi_m\|_\infty (b - a) \\
&= \|\varphi_n - f + f - \varphi_m\|_\infty (b - a) \\
&\leq (\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty)(b - a) \\
&< \left( \frac{\varepsilon}{2(b - a)} + \frac{\varepsilon}{2(b - a)} \right) (b - a) \\
&= \varepsilon
\end{aligned}$$

So $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and therefore convergent. So there exists

$$\lim_{n\to\infty} \int_a^b \varphi_n\, dx$$

Let $i = \lim_{n\to\infty} \int_a^b \varphi_n \, dx$. Let $(\psi_n)_{n\in\mathbb{N}}$ be another sequence of step functions with $\psi_n \to_{n\to\infty} f$ is uniform in $[a,b]$. Analogously as above:

$$j_n = \int_a^b \psi_n \, dx$$

$(j_n)_{n\in\mathbb{N}}$ is convergent and has limes $j$.

Show that $i = j$. We again use a zip-like construction:

$$F = (\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \ldots)$$

$F$ is a sequence of step functions, which converge towards $f$ uniformly. Let $l$ be the limit of integrals of this sequence of step functions. Then it holds that TODO (subsequences have the same limit)

$$i = l = j$$

□

**Theorem 11** (Elementary properties of the integral)**.** Let $f, g \in \mathcal{B}[a,b]$ and $\alpha, \beta \in \mathbb{C}$. Then it holds that

**linearity**

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx$$

**boundedness**

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx \leq \|f\|_\infty (b-a)$$

**monotonicity** Let $f, g \in \mathcal{B}[a,b]$ with values in $\mathbb{R}$ and it holds that

$$f(x) \leq g(x) \qquad \forall x \in [a,b]$$

Then it holds that

$$\int_a^b f \, dx \leq \int_a^b g \, dx$$



*Proof.*
- Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences of step functions with $\varphi_n \to f$ and $\psi_n \to g$ uniform in $[a,b]$. Then it holds that

$$\alpha\varphi_n + \beta\psi_n \to_{n\to\infty} \alpha f + \beta g$$

(proof left as exercise to the reader)
uniform in $[a,b]$. So it holds that

$$\begin{aligned}\int_a^b (\alpha f + \beta g) \, dx &= \lim_{n\to\infty} \int_a^b (\alpha\varphi_n + \beta\psi_n) \, dx \\ &= \alpha \lim_{n\to\infty} \int_a^b \varphi_n \, dx + \beta \lim_{n\to\infty} \int_a^b \varphi_n \, dx \\ &= \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx\end{aligned}$$

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to_{n\to\infty} f$ continuous in $[a,b]$. Then also $(|\varphi_n|)_{n\in\mathbb{N}}$ is a sequence of step functions and it holds that

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a,b]$$

*Proof.* Let $N$ be sufficiently large such that $\forall n \geq N \forall x \in [a,b]$ :

$$|\varphi_n(x) - f(x)| < \varepsilon \Rightarrow ||\varphi_n(x)| - |f(x)|| \leq |\varphi_n(x) - f(x)| < \varepsilon$$

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a,b]$$

So it holds that

$$\left|\int_a^b f \, dx\right| = \left|\lim_{n\to\infty} \int_a^b \varphi_n \, dx\right| = \lim_{n\to\infty} \left|\int_a^b \varphi_n \, dx\right| \leq \lim_{n\to\infty} \int_a^b |\varphi_n| \, dx = \int_a^b |f| \, dx$$

Because $|f - \varphi_n|_\infty \to_{n\to\infty} 0$ it follows that

$$|\|f\|_\infty - \|\varphi_n\|_\infty| \leq \|f - \varphi_n\|_\infty \to 0$$

hence $\|f\|_\infty = \lim_{n\to\infty} \|\varphi_n\|_\infty$. □

Hence,

$$\begin{aligned}\int_a^b |f| \, dx &= \lim_{n\to\infty} \int_a^b |\varphi_n| \, dx \\ &\leq \lim_{n\to\infty} \|\varphi_n\|_\infty (b-a) \\ &= \|f\|_\infty (b-a)\end{aligned}$$

**Remark 16.** We have proven that $\|\cdot\| : V \to [0,\infty)$ is a continuous map, hence $v_n \to v \Rightarrow \|v_n\| \to \|v\|$.

□

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This lecture took place on 12th of April 2016 with lecturer Wolfgang Ring.

**Definition 12.** Let $f : [a,b] \to \mathbb{R}$ be given. Let $x_0 \in [a,b)$. We claim that $f$ has a right-sided derivative $f'_+(x_0)$ in $x_0$ if the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

has a right-sided limit in $x_0$. Then $f$ is denoted with $f'_+(x_0)$.

$$f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$$

Analogously for the left-sided derivative: Let $x_0 \in (a,b]$. $f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ if the limit exists.

**Theorem 12** (Mean value theorem of calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and $p : [a,b] \to \mathbb{R}$ is a regulated function with $p(x) \geq 0 \quad \forall x \in [a,b]$.

Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x) \cdot p(x) \, dx = f(\xi) \cdot \int_a^b p(x) \, dx$$

*Proof.* Let $M = \max\{f(x) : x \in [a,b]\}$ and $m = \min\{f(x) : x \in [a,b]\}$

$$mp(x) \leq f(x) \underbrace{p(x)}_{\geq 0} \leq Mp(x) \qquad \forall x \in [a,b]$$

Due to monotonicity of the integral it holds that

$$m \int_a^b p(x) \, dx \leq \int_a^b f(x)p(x) \, dx \leq M \int_a^b p(x) \, dx$$

hence $\exists \eta \in [m,M]$ such that $\eta \cdot \int_a^b p(x) \, dx = \int_a^b f(x)p(x) \, dx$. From the Intermediate Value Theorem it follows that $\exists \xi \in [a,b] : \eta = f(\xi)$.

$$\Rightarrow f(\xi) : \int_a^b p(x) \, dx = \int_a^b f(x)p(x) \, dx$$

□

**Remark 17.** Consider $p \equiv 1$.

$$\exists \xi \in [a,b] : \int_a^b f(x) \cdot 1 \, dx = f(\xi) \cdot \int_a^b 1 \, dx = f(\xi) \cdot (b-a)$$

**Lemma 11.** Let $I = [a,b]$ and $f \in R[a,b]$ and $a \leq \alpha < \beta < \gamma \leq b$ (compare with Figure 15). Then $f|_{[\alpha,\gamma]} \in R[\alpha,\gamma]$.

Furthermore it holds that

$$\int_\alpha^\beta f(x) \, dx = \int_\alpha^\beta f(x) \, dx + \int_\beta^\gamma f(x) \, dx$$

*Proof.* Let $\varphi$ be a step function in $[\alpha,\gamma]$. Then $\varphi|_{[\alpha,\beta]} \in \tau[\alpha,\beta]$ and $\varphi|_{[\beta,\gamma]} \in \tau[\beta,\gamma]$. Furthermore it holds (proof not given here)

$$\int_\alpha^\gamma \varphi \, dx = \int_\alpha^\beta \varphi \, dx = \int_\beta^\gamma \varphi \, dx$$

For $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of subsequences with $\varphi_n \to f$ continuous in $[\alpha,\gamma]$.

$$\Rightarrow \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]} \text{ uniform in } [\alpha,\beta]$$

analogously for $[\beta,\gamma]$.

$$\int_\alpha^\gamma f dx = \lim_{n\to\infty} \int_\alpha^\gamma \varphi_n \, dx = \lim_{n\to\infty} \left[ \int_\alpha^\beta \varphi_n \, dx + \int_\beta^\gamma \varphi_n \, dx \right]$$

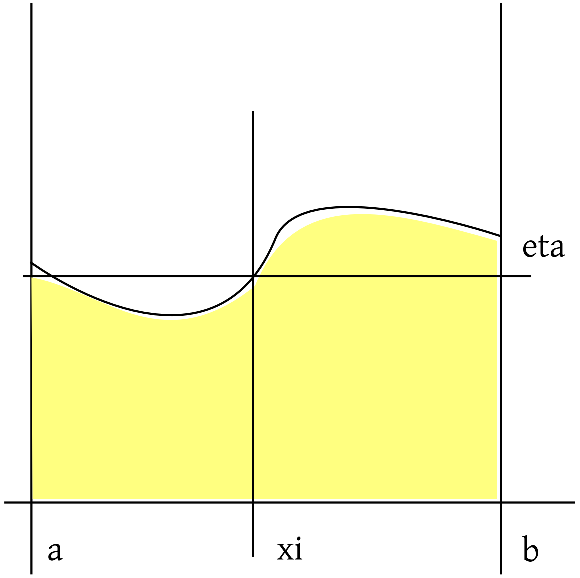


Figure 14: Mean value Theorem

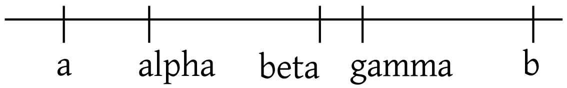


Figure 15: Relation of $a \leq \alpha < \beta < \gamma \leq b$

$$= \underbrace{\lim_{n\to\infty} \int_\alpha^\beta \varphi_n \, dx}_{=\int_\alpha^\beta f \, dx} + \underbrace{\lim_{n\to\infty} \int_\beta^\gamma \varphi_n \, dx}_{=\int_\beta^\gamma f \, dx}$$



$\square$

**Remark 18.** Notation ($\alpha, \beta \in [a, b]$):

$$\int_\beta^\alpha f(x) \, dx = -\int_\alpha^\beta f(x) \, dx$$

So it follows that

$$\int_\alpha^\alpha f(x) \, dx = -\int_\alpha^\alpha f(x) \, dx = 0$$

With this notation it holds that $\forall \alpha, \beta, \gamma \in I$:

$$\int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f(x) \, dx$$

independent of the relation of $\alpha, \beta, \gamma$ towards each other. For $\alpha < \beta < \gamma$ everything is fine.

Let's also look at $\beta < \gamma < \alpha$ as an exercise.

Then it holds that

$$\int_\beta^\alpha f \, dx = \int_\beta^\gamma f \, dx + \int_\gamma^\alpha f \, dx$$

$$-\int_\alpha^\beta f \, dx = \int_\beta^\gamma f \, dx - \int_\alpha^\gamma f \, dx$$

$$\Rightarrow \int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx$$

Case $\alpha = \beta$ or $\beta = \gamma$ is trivial.

**Theorem 13** (Fundamental theorem of Calculus)**.** Originally formulated by Isaac Barrow (1630–1677). Followingly popularized by Newton (1642–1727) and Leibniz (1646–1716).

Let $f : I \to \mathbb{R}$ be a regulated function. $I$ is an interval and $a \in I$ is fixed. For $x \in I$ we define

$$F(x) = \int_a^x f(\xi) \, d\xi$$

Then it holds that (two variants/characterizations)

1. $F$ is right-sided derivable and also left-sided derivable for every $x_0 \in I$ and it holds that
$$F'_+(x) = f_+(x_0) = \lim_{x \to x_0^+} f(x) \qquad \text{and}$$
$$F'_-(x) = f_-(x_0) = \lim_{x \to x_0^-} f(x)$$
Especially if $f$ is continuous in $x_0$, then $F$ is differentiable in $x_0$ with derivative $F'(x_0) = f(x_0)$.

We call a function with the properties of $F$ above a *primitive function* of the regulated function $f$.

2. Let $\Phi : I \to \mathbb{R}$ be an arbitrary primitive function of $f$ and let $a, b \in I$. Then it holds that
$$\int_a^b f(\xi)\, d\xi = \Phi(b) - \Phi(a)$$

The first characterization claims that (informally speaking) the derivative for the upper limit of the integral of $f$ gives $f$.

Let $f = \Phi'$ ($\Phi$ is our primitive function of $f$). The second characterization claims that the integral of a derivative of $\Phi$ gives $\Phi$.
$$\int_a^b \Phi'\, dx = \Phi(b) - \Phi(a)$$

*Proof.* 1. Let $x_1, x_2 \in I$ and wlog $x_1 \leq x_2$.
$$\begin{aligned}|F(x_1) - F(x_2)| &= \left|\int_a^{x_1} f(\xi)\, d\xi - \int_a^{x_2} f(\xi)\, d\xi\right| \\ &= \left|\int_a^{x_1} f(\xi)\, d\xi + \int_{x_2}^{a} f(\xi)\, d\xi\right| \\ &= \left|\int_{x_2}^{x_1} f(\xi)\, d\xi\right| = \left|\int_{x_1}^{x_2} f(\xi)\, d\xi\right| \\ &\leq \int_{x_1}^{x_2} |f(\xi)|\, d\xi \leq |x_2 - x_1| \cdot \|f\|_\infty\end{aligned}$$
hence $F$ is Lipschitz continuous in $I$. So $F$ is continuous in $I$.

One-sided limits:

Let $\varepsilon > 0$ arbitrary and $x_0 \in I$ and $\delta$ such that $\forall x \in (x_0, x_0 + \delta)$ it holds that:
$$|f(x) - f_+(x_0)| < \varepsilon$$
$$\begin{aligned}&\left|\frac{F(x) - F(x_0)}{x - x_0} - f_+(x_0)\right| \\ &= \frac{1}{|x - x_0|}\left|\int_a^x f(\xi)\, d\xi - \int_a^{x_0} f(\xi)\, d\xi - f_+(x_0) \cdot (x - x_0)\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\, d\xi - f_+(x_0)\int_{x_0}^x 1 d\xi\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\, d\xi - \int_{x_0}^x f_+(x_0)\, d\xi\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi) - f_+(x_0)\, d\xi\right| \\ &\leq \frac{1}{|x - x_0|}\int_{x_0}^x |f(\xi) - f_+(x_0)|\, d\xi\end{aligned}$$
$\xi \in (x_0, x) \subseteq (x_0, x_0 + \delta)$
$$\begin{aligned}&< \frac{1}{|x - x_0|} \cdot \varepsilon \underbrace{\int_{x_0}^x 1\, d\xi}_{|x - x_0|} \\ &= \varepsilon \\ &\Rightarrow F'_+(x_0) = f_+(x_0)\end{aligned}$$
Analogously $F'_-(x_0) = f_-(x_0)$.

□

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This lecture took place on 14th of April 2016 with lecturer Wolfgang Ring.

**Theorem 14** (Addition: Lipschitz continuity of differentiable functions)**.** Let $I = [a, b]$, $f : I \to \mathbb{R}$ and $f$ is continuous in $I$. Let $A \subseteq I$. Let $A$ be countable and $f$ is differentiable in $I \setminus A$ and $\exists L > 0 : |f'(x)| \leq L \quad \forall x \in I \setminus A$.

Then it holds that $\forall x_1, x_2 \in I$:
$$|f(x_1) - f(x_2)| \leq L\, |x_1 - x_2|$$

*Proof.* Without loss of generality, $x_1 < x_2$. Let $\varepsilon > 0$, define $F_\varepsilon : I \to \mathbb{R}$

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L+\varepsilon)(x - x_1)$$

Show $F_\varepsilon(x_2) \le 0$.

Assume there is some $\varepsilon' > 0$ with $F_{\varepsilon'}(x_2) > 0$. It holds that

- $F_{\varepsilon'}(A) \subseteq \mathbb{R}$ is countable
- $0 = F_{\varepsilon'}(x_1) < F_{\varepsilon'}(x_2)$. Because $F_{\varepsilon'}$ is continuous (by Intermediate Value Theorem, $[0, F_{\varepsilon'}(x_2)] \subseteq F_{\varepsilon'}([x_1, x_2])$) and $[0, F_{\varepsilon'}(x_2)]$ contains overcountably many points, $F_{\varepsilon'}(A)$ is countable.

$$\Rightarrow \exists \gamma : 0 < \gamma < F_{\varepsilon'}(x_2)$$

and

$$\gamma \in F_{\varepsilon'}([x_1, x_2] \setminus A)$$

Let $\underbrace{F_{\varepsilon'}^{-1}(\{y\})}_{B} \cap [x_1, x_2] = \{x \in [x_1, x_2] \,|\, F_{\varepsilon'}(x) = y\}$.

$B$ is bounded. Let $c = \sup B$. Let $(\xi_n)_{n\in\mathbb{N}}$, $\xi_n \in B$ with $\lim_{n\to\infty} \xi_n = c$. Then it holds that $c \in [x_1, x_2]$ and $F_{\varepsilon'}(\xi_n) = y \xRightarrow{\text{continuity of } F_{\varepsilon'}} \underbrace{\lim_{n\to\infty} F_{\varepsilon'}(\xi_n)}_{y} = F_{\varepsilon'}(c)$.

Therefore $c = \max B = \max\{x \in [x_1, x_2] : F_{\varepsilon'}(x) = y\}$. Because $F_{\varepsilon'}(x_2) > y$ and $F_{\varepsilon'}(x_1) = 0 < \gamma$, it holds that $x_1 < c < x_2$.

Consider $x \in (c, x_2]$ and let $\varphi(x) := \frac{F_{\varepsilon'}(x) - F_{\varepsilon'}(c)}{x-c}$. Furthermore $F_{\varepsilon'}(x) > \gamma = F_{\varepsilon'}(c)$ for $x \in (c, x_2]$. Because if we define $F_{\varepsilon'}(x) < \gamma$, then (due to Intermediate Value Theorem) $\exists \xi \in (x, x_2)$ with $F_{\varepsilon'}(\xi) = \gamma$, so $\exists \xi \in B$ which would be a contradiction to $c = \max B$.

$$\begin{aligned}
\varphi(x) &= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L+\varepsilon')(x - x_1 - c + x_1)}{x-c} \\
&= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L+\varepsilon')(x-c)}{x-c} \\
&\overset{\text{inv. triangle ineq.}}{\le} \frac{|f(x) - f(c)|}{x-c} - (L+\varepsilon')
\end{aligned}$$

Now as far as $c \notin A$ holds, $f$ is differentiable in $c$ and it holds that $|f'(c)| \le L$, hence there exists an interval $(c, d)$, $d < x_2$ and $d > c$, such that

$$\frac{|f(x) - f(c)|}{x-c} < L + \varepsilon'$$

Because $F_{\varepsilon'}(x) > \gamma$,

$$\begin{aligned}
&\Rightarrow \varphi(x) > 0 \qquad \forall x \in (c, x_2] \\
&\Rightarrow 0 < \varphi(x) \le |f(x) - f(c)|\, x - c - (L+\varepsilon') \\
&\Rightarrow \left|\frac{f(x) - f(c)}{x-c}\right| > L + \varepsilon'
\end{aligned}$$

This is a contradiction to the assumption that $F_{\varepsilon'}(x_2) > 0$. So $F_\varepsilon(x_2) \le 0 \quad \forall \varepsilon > 0$

$$\Rightarrow F_0(x_2) \le 0 \Rightarrow |f(x_2) - f(x_1)| \le L - |x_2 - x_1|$$

$\square$

**Remark 19.** Let $f$ be differentiable in $[a,b]$ and $|f'(x)| < L \quad \forall x \in [a,b]$. Let $x_1, x_2 \in [a,b]$

$$|f(x_L) - f(x_1)| = |f'(\xi) \cdot (x_2 - x_1)| \le L\,|x_2 - x_1|$$

by Mean Value Theorem of differential calculus.

**Corollary 3.** Let $f, g : I \to \mathbb{R}$. $I$ as above and $f, g$ are differentiable in $I \setminus A$, $A$ countable and it holds that $f'(x) = g'(x) \quad \forall x \in I \setminus A$. There exists a constant $k$ such that

$$f(x) = g(x) + k \qquad \forall x \in I$$

*Proof.* We use the previous Theorem for

$$h(x) = f(x) - g(x)$$

Then it holds that $|h'(x)| = 0 = L \quad \forall x \in I \setminus A$.

$$\Rightarrow |h(x_1) - h(x_2)| \le 0 \cdot |x_1 - x_2| \quad \forall x_1, x_2 \in I$$

$$\Rightarrow h(x_1) = h(x_2) \quad \forall x_1, x_2 \in I$$

$$\begin{aligned} f(x_1) - g(x_1) &= f(x_2) - g(x_2) \quad \forall x_1, x_2 \in I \\ &= k \ldots \text{ constant} \end{aligned}$$

$\forall x_1 \in I$ it holds that $f(x_1) = g(x_1) + k$. □

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This lecture took place on 15th of April 2016 with lecturer Wolfgang Ring.

*cont, 2nd part.* We need to show: Let $f$ be a regulated function and $\Phi$ is a primitive function of $f$ with the following properties

$$\Phi'(x) = f(x) \quad \forall x \in I \text{ where f is continuous}$$

$$\Phi'_+(x) = \lim_{\xi \to x_+} f(x)$$

$$\Phi'_-(x) = \lim_{\xi \to x_-} f(x) \quad \forall x \in I$$

Then it holds that

$$\int_\alpha^\beta f(x)\,dx = \Phi(\beta) - \Phi(\alpha)$$

*Proof.* For $\Phi(x) = \int_\alpha^x f(\xi)\,d\xi = F(x)$ (where $F$ is also a primitive function) it holds that

$$\int_\alpha^\beta f(\xi)\,d\xi = F(\beta) - \underbrace{F(\alpha)}_{=0}$$

Because $\Phi$ and $F$ are both primitive functions of $f$, $\Phi'$ and $F'$ correspond in all continuous points, hence everywhere, but one countable set.

By the uniqueness theorem, it holds that

$$\Phi(x) = F(x) + c$$

$$F(x) = \Phi(x) - c$$

$$\int_a^b f(\xi)\,d\xi = F(b) - F(a) = \Phi(b) - c - \Phi(a) + c = \Phi(b) - \Phi(a)$$

□



□

**Remark 20** (Notational remark)**.** Let $f$ be a regulated function. Then we denote

$$\int f(x)\,dx = \begin{cases} \text{the set of all primitive function of } f \\ \text{an arbitrary primitive function of } f \end{cases}$$

$\int f(x)\,dx$ is called *indefinite integral.*

**Remark 21.**

$$\int x^n\,dx = \frac{1}{n+1} x^{n+1} \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

If you consider all primitive functions of the indefinite integral, you consider a constant $c \in \mathbb{R}$.

$$\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

Let $x > 0$: $(\ln x)' = \frac{1}{x}$.
Let $x < 0$: $(\ln -x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases} = \ln|x| \qquad \text{for } x \neq 0$$

$$\int \cos x\,dx = \sin x$$

$$\int \sin x\,dx = -\cos x$$

$$\int e^{cx}\,dx = \frac{1}{c} \cdot e^{cx} \quad (c \neq 0)$$

**Lemma 12.** Let $f_1$ and $f_2$ be regulated functions in $I = [a, b]$ and there exists some countable set $A$ such that

$$f_1(x) = f_2(x) \qquad \forall x \in I \setminus A$$

Then it holds that

$$\int f_1(x)\,dx = \int f_2(x)\,dx \text{ and } \int_a^b f_1(x)\,dx = \int_a^b f_2(x)\,dx \qquad \forall a, b \in I$$

*Proof.* Let $F_1$ be a primitive function on $f_1$, $F_2$ be a primitive function of $f_2$. Then it holds that $F_1' = F_2'$ in $I \setminus A$. Due to identity theorem:

$$\Rightarrow F_1 = F_2 + c \Rightarrow \int f_1\,dx = \int f_2\,dx$$

□

**Remark 22.** Example of a function, which is differentiable everywhere. Its derivative is not a regulated function.

Let $I = [-1,1]$ and

$$f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$ it holds that

$$f'(x) = 2x\cos\sin\frac{1}{x} - \frac{x^2}{x^2}\cdot\cos\frac{1}{x}$$
$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$f'(0) = \lim_{h\to 0}\frac{1}{h}\left[h^2\cdot\sin\frac{1}{h} - 0\right] = \lim_{h\to 0}\underbrace{h}_{\to 0}\cdot\underbrace{\sin\frac{1}{h}}_{\in[-1,1]} = 0$$

$$f'(x) = \begin{cases} 0 & \text{for } x = 0 \\ \underbrace{2x\sin\frac{1}{x} - \cos\frac{1}{x}}_{\text{has no one-sided limit in } x=0} & \text{for } x \neq 0 \end{cases}$$
$$f'_+(0) \neq \lim_{x\to 0^+} f'(x)$$

## 5.4 Integration techniques

**Theorem 15** (Integration by parts (dt. "partielle Integration")). Let $u, v : I \to \mathbb{R}$ be both primitive functions of regulated functions. Then also $u \cdot v$ is a primitive function of a regulated function and it holds that

$$\int u'v\,dx = u\cdot v - \int u\cdot v'\,dx$$

and

$$\int_a^b u'v\,dx = \underbrace{u(b)\cdot v(b) - u(a)\cdot v(a)}_{=:u\cdot v|_a^b} - \int_a^b u\cdot v'\,dx$$

*Proof.* $u$ is continuous and therefore a regulated function.
$v$ is continuous and therefore a regulated function.

$u'$ and $v'$ are regulated function by assumption.

$$\Rightarrow (u'\cdot v + u\cdot v') \in \mathcal{R}(I)$$

$u\cdot v$ is differentiable in every point in which $u$ and $v$ is differentiable. Let $u$ be differentiable in $I\setminus A$, $v$ is differentiable in $I \setminus B$.

$$\Rightarrow u\cdot v \text{ is differentiable in } I\setminus\underbrace{(A\cup B)}_{\text{countable}}$$

In $I\setminus(A\cup B)$ it holds that

$$(u\cdot v)'(x) = u'(x)\cdot v(x) + u(x)v'(x)$$

Hence the function $u\cdot v$ is primitive function of the regulated function $(u'v+uv')$.

$$\Rightarrow \int (u'v + uv')\,dx = u\cdot v$$

$$\Rightarrow \int_a^b (u'v+uv')\,dx = u(b)v(b) - u(a)v(a)$$

□

**Example 3.** Let $a \neq -1$ and $x > 0$.

$$\int x^a \ln x\,dx = \left| \begin{matrix} u' = x^a & u = \frac{1}{1+a}\cdot x^{a+1} \\ v = \ln x & v' = \frac{1}{x} \end{matrix} \right|$$

$$\overset{\text{int. by parts}}{=} \frac{1}{1+a}x^{1+a}\cdot\ln x - \frac{1}{1+a}\int x^a\,dx$$

$$= \frac{1}{1+a}x^{1+a}\ln x - \frac{1}{(1+a)^2}x^{1+a} = \frac{1}{1+a}x^{1+a}\left[\ln x - \frac{1}{1+a}\right]$$

**Example 4.**

$$\int \cos^k(x)\,dx \text{ for } k = 2,3,4,\dots$$

$$\left|\begin{array}{ll} u' = \cos x & \Rightarrow u = \sin x \\ v = \cos^{k-1}(x) & v' = -(k-1)\cdot\cos^{k-2}(x)\cdot\sin(x) \end{array}\right|$$

$$\int \cos^k(x)\,dx = \cos^{k-1}(x)\cdot\sin(x) + \int (k-1)\cdot\cos^{k-2}(x)\cdot\underbrace{\sin^2(x)}_{1-\cos^2(x)}\,dx$$

$$= \cos^{k-1}(x)\cdot\sin(x) + (k-1)\cdot\int\cos^{k-2}(x)\,dx - (k-1)\cdot\int\cos^k(x)\,dx$$

Recognize that we have $\int\cos^k(x)\,dx$ twice in the equation (LHS and RHS, RHS with a sign).

$$k\cdot\int\cos^k(x)\,dx = \cos^{k-1}(x)\cdot\sin(x) + (k-1)\int\cos^{k-2}(x)\,dx$$

$$\int\cos^k(x)\,dx = \frac{1}{k}\cos^{k-1}(x)\sin(x) + \frac{k-1}{k}\int\cos^{k-2}(x)\,dx$$

Recursion formula.

Analogously,

$$\int\sin^k(x)\,dx = -\frac{1}{k}\sin^{k-1}(x)\cos(x) + \frac{k-1}{k}\int\sin^{k-2}(x)\,dx$$

Let $c_m = \int_0^{\frac{\pi}{2}}\cos^m(x)\,dx$. Then it holds that

$$c_{2n} = \frac{(2n-1)}{2n}\cdot\frac{(2(n-1)-1)}{2(n-1)}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \left(\prod_{k=1}^{n}\frac{2k-1}{2k}\right)\cdot\frac{\pi}{2}$$

$$c_{2n+1} = \left(\prod_{k=1}^{n}\frac{2k}{2k+1}\right)$$

Proof by complete induction:



**Case** $n = 0$

$$\int_0^{\frac{\pi}{2}}\cos^{2\cdot 0}x\,dx = \int_0^{\frac{\pi}{2}}1\,dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}}\cos^{2\cdot 0+1}x\,dx = \int_0^{\frac{\pi}{2}}\cos x\,dx = \sin(x)|_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}}\cos^{2(n+1)}\,dx = \frac{1}{2(n+1)}\cdot\cos^{2(n+1)-1}(x)\cdot\sin(x)|_0^{\frac{\pi}{2}} + \frac{2(n+1)-1}{2(n+1)}\cdot\int_0^{\frac{\pi}{2}}\cos^{2n}(x)\,dx$$

$$= \frac{2n+1}{2n+2}\cdot\left(\underbrace{\prod_{k=1}^{n}\frac{2k-1}{2k}}_{\text{induction hypothesis}}\right)\cdot\frac{\pi}{2} = \left(\prod_{k=1}^{n+1}\frac{2k-1}{2k}\right)\cdot\frac{\pi}{2}$$

**Theorem 16** (Wallis product)**.** (John Wallis, 1616–1703)

$$\frac{\pi}{2} = \lim_{n\to\infty} w_n \quad\text{with}\quad w_n = \prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots$$

*Proof.*

$$\frac{\pi}{2}\cdot\frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2}\cdot\frac{\prod_{k=1}^{n}\frac{2k}{2k+1}}{\prod_{k=1}^{n}\frac{2k-1}{2k}\cdot\frac{\pi}{2}} = \prod_{k=1}^{n}\frac{(2k)^2}{(2k+1)(2k-1)} = w_n$$

It remains to show: $\lim_{n\to\infty}\frac{c_{2n+1}}{c_{2n}} = 1$.

In $\left[0,\frac{\pi}{2}\right]$ it holds that $0\le\cos x\le 1$.

$$\Rightarrow \int_0^{\frac{\pi}{2}}\cos^{2n}(x)\,dx \ge \int_0^{\frac{\pi}{2}}\cos^{2n+1}(x)\,dx \ge \int_0^{\frac{\pi}{2}}\cos^{2n+2}(x)\,dx$$

$$c_{2n}\ge c_{2n+1}\ge c_{2n+2}$$

$$1 \ge \frac{c_{2n+1}}{c_{2n}} \ge \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1}\frac{2k-1}{2k}}{\prod_{k=1}^{n}\frac{2k-1}{2k}} = \underbrace{\frac{2n+1}{2n+2}}_{\to 1\text{ for }n\to\infty}$$

$\Rightarrow \frac{c_{2n+1}}{c_{2n}}$ converges and limit is 1.

$$\lim_{n\to\infty} \frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} = \lim_{n\to\infty} w_n$$

□

**Theorem 17** (Substitution law)**.** Let $f : I \to \mathbb{R}$ be a regulated function with primitive function $F$. Furthermore $t : [\alpha, \beta] \to I$ is continuously differentiable. Then $F \circ t$ is a primitive function for function $(f \circ t) \cdot t'$ and it holds that

$$\int_\alpha^\beta f(t(x)) \cdot t'(x)\, dx = \int_{t(\alpha)}^{t(\beta)} f(t)\, dt$$

*Proof.* The right-side integral is given (according to the Fundamental Theorem) by

$$F(t(\beta)) - F(t(\alpha))$$

The left-side integral, because of

$$F(t(x))' = F'(t(x)) \cdot t(x)$$

Hence $F = t$ is primitive function of the left-side integral. So it holds that

$$\int_a^b f(t(x)) \cdot t'(x)\, dx = F \circ t(b) - F \circ t(a) = F(t(b)) - F(t(a))$$

□

**Example 5.**

$$\int_0^1 x\sqrt{1+x^2}\, dx = \frac{1}{2} \int_0^1 2x\sqrt{1+x^2}\, dx$$

$$\left| \begin{matrix} t(x) = 1 + x^2 & t'(x) = 2x \\ f(y) = \sqrt{y} & \end{matrix} \right|$$

$$= \frac{1}{2} \int_1^2 \sqrt{x}\, dx = \frac{1}{2} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^2 = \frac{2}{3}^{\frac{3}{2}} - \frac{1}{3}^{\frac{3}{2}} = \frac{1}{3}(\sqrt{8} - 1)$$

$$\int_0^1 x \cdot \sqrt{1+x^2}\, dx = \left| \begin{matrix} \overbrace{\begin{matrix} y = x^2 + 1 \\ \frac{dy}{dx} = 2x \end{matrix}}^{\text{transform variables}} & \begin{matrix} \underbrace{dy = 2x\, dx}_{\text{transformation of differences}} \\ x\, dx = \frac{1}{2}\, dy \end{matrix} \end{matrix} \right|$$

Transformation of limits:

$$x = 0 \Leftrightarrow y = 1 \qquad x = 1 \Leftrightarrow y = 2$$

$$= \frac{1}{2} \int_1^2 \sqrt{y}\, dy = \frac{1}{2} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \Bigg|_1^2 = \frac{(x^2+1)^{\frac{3}{2}}}{3} \Bigg|_0^1$$

Hence it is also necessary to transform the limits.

**Example 6** (Integration by parts)**.**

$$\int \ln x\, dx = \left| \begin{matrix} v' = 1 & v = x \\ u = \ln x & u' = \frac{1}{x} \end{matrix} \right| = x \ln x - \int x \frac{1}{x}\, dx = x \ln x - x$$

**Theorem 18.** Ivan M. Niven (published in 1947, 1915–1999)
It holds: $\pi^2$ is an irrational number. So $\pi$ is irrational.

*Proof by contradiction.* Let $\pi^2 = \frac{a}{b} \in \mathbb{Q}$.

Because $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ (practicals!) there exists $n \in \mathbb{N}$ such that $\pi \frac{a^n}{n!} < 1$.

$$f(x) = \frac{1}{n!} x^n (1-x)^n$$

is symmetrical along axis $x = \frac{1}{2}$

$$= \frac{1}{n!}\sum_{k=n}^{2n} c_k x^k \qquad \text{with } c_k = (-1)^{k-n}\binom{n}{k-n} = \pm\binom{n}{k-n} \in \mathbb{Z}$$

$f^{(\mu)}(0) = 0$ for $\mu = 0, 1, \dots, n-1 \in \mathbb{Z}$ and also:

$f^{(\mu)}(1) \in \mathbb{Z}$ for $\mu = n, n+1, \dots, 2n$

$$f^{(\mu)}(x) = \frac{1}{n!}\sum_{k=0}^{2n} \underbrace{k(k-1)\dots(k-\mu+1)}_{=\mu!} \cdot c_k \cdot x^{k-\mu}$$

$$\begin{aligned}
f^{(\mu)}(0) &= \frac{1}{n!}\mu!\left(\pm\binom{n}{\mu-n}\right) \cdot 1 \\
&= \frac{1}{n!}\mu!\frac{n!}{(\mu-n)!(n-\mu+n)!} \\
&= \frac{\mu!}{(\mu-n)!(2n-\mu)!} \\
&= \frac{(\mu-n+1)(\mu-n+2)\dots\mu}{1\cdot 2\cdot 3\dots(2n-\mu)} \\
&\in \mathbb{Z}
\end{aligned}$$

Why does $\in \mathbb{Z}$ hold?

$$\underbrace{\frac{\mu!}{n!}\binom{n}{\mu-n}}_{\in\mathbb{Z}} \in \mathbb{Z} \qquad n \le \mu \le 2n$$

$$(n+1)(n+2)\dots\nu \in \mathbb{Z}$$

$$n \le \mu \le 2n$$

$f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$, analogously $f^{(\mu)}(1) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$.

$$F(x) = b^n\left(\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) + (-1)^n f^{2n}(x)\pi^0\right)$$



$F(0) \in \mathbb{Z}$ because $f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu = 0, 2, 4, 6, \dots, 2n$

$$\pi^2 = \frac{a}{b} \qquad \pi^{2n-2l} = \frac{a^{k-l}}{b^{n-l}}$$

$$b^n \cdot \pi^{2n-2l} = a^{n-l} \cdot b^l \in \mathbb{Z}$$

Analogously for $F(1) \in \mathbb{Z}$.

$$\begin{aligned}
&(F'(x)\cdot\sin(\pi x) - \pi F(x)\cdot\cos(\pi x))' \\
&= F''(x)\cdot\sin(\pi x) + \pi^2\cdot F(x)\cdot\sin\pi x + F'(x)(\cos(\pi x) - \pi\cos\pi x) \\
&= (F''(x) + \pi^2 F(x))\cdot\sin(\pi x)
\end{aligned}$$

$$F''(x) = b^n\cdot\left(\pi^{2n}\cdot f''(x) + \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) - \dots + (-1)^n f^{(2n+2)}(x)\right)$$

$$\Rightarrow F''(x) + \pi^2\cdot F(x)$$

$$= b^n\left(\pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n f^{(2n+2)}(x)\right)$$

$$+ b^n\left(\pi^{2n+2} f(x) - \pi^{2n} f''(x) + \pi^{2n-2} f^{(4)}(x) - \pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n \pi^2\cdot f^{(2n)}(x)\right)$$

Almost all expressions cancel each other out. So it holds that

$$\begin{aligned}
&(F'(x)\cdot\sin(\pi x) - \pi F(x)\cos(\pi x))' \\
&= \pi^{2n+2}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&= \frac{a^{n+1}}{b^{n+1}}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&= \frac{a^{n+1}}{b}\cdot f(x)\cdot\sin(\pi x) \\
&= \pi^2\cdot a^n f(x)\cdot\sin(\pi x) \\
&= \pi\left(\pi a^n f(x)\sin(\pi x)\right)
\end{aligned}$$

$$\begin{aligned}
I &= \pi\int_0^1 a^n f(x)\cdot\sin(\pi x)\,dx \\
&= \frac{1}{\pi}\cdot\left[F'(x)\cdot\sin(\pi x) - \pi\cdot F(x)\cos(\pi x)\right]\Big|_0^1 \\
&= F(1) + F(0) \in \mathbb{Z}
\end{aligned}$$

On the other hand it holds that

$$f(x) = \frac{1}{n!} \underbrace{x^n}_{\leq 1} \underbrace{(1-x)^n}_{\leq 1}$$

So $0 \leq f(x) \leq \frac{1}{n!}$. Hence,

$$0 \leq a^n f(x) \cdot \sin(\pi x) \leq \frac{a^n}{n!} < \frac{1}{\pi}$$

So $0 < I < 1 \Rightarrow I \in \mathbb{Z}$. This is a contradiction to our assumption that $I \in \mathbb{Z}$. □

**Remark 23.** Hence $\pi$ is not rational. So there exists no linear affine function $g(x) = ax + b$ with $a, b \in \mathbb{Z}$ such that $\pi$ is root of $g$.

**Remark 24.** We state, $\xi \in \mathbb{R}$ is an *algebraic* number if polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_n$$

exists with $a_i \in \mathbb{Z}$ for $i = 0, \ldots, n$ and $P(\xi) = 0$.

Algebraic numbers are a generalization of rational numbers.

$\eta \in \mathbb{R}$ is called *transcendental*, if $\eta$ is not algebraic.

**Remark 25.** $\pi$ is transcendental.

**Theorem 19** (Integration of non-compact intervals)**.**

$$\int_0^\infty e^{-x}\,dx = \lim_{c \to \infty} \int_0^c e^{-x}\,dx$$

**Definition 13** (Definition of indefinite integrals)**.** Let $I$ be an interval with boundary values $a$ and $b$ with $-\infty \leq a < b \leq \infty$.

Let $f$ be a regulated function in $I$. Then we define

1. if $I = [a, b)$, $\int_a^b f(x)\,dx = \lim_{\beta \to b_-} \int_a^\beta f(x)\,dx$
2. if $I = (a, b]$, $\int_a^b f(x)\,dx = \lim_{\alpha \to a_+} \int_\alpha^a f(x)\,dx$
3. if $I = (a, b)$, we choose $c \in I$ and $\int_a^b f(x)\,dx = \lim_{\alpha \to a_+} \int_\alpha^c f(x)\,dx + \lim_{\beta \to b_-} \int_c^\beta f(x)\,dx$.



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This lecture took place on 21st of April 2016 with lecturer Wolfgang Ring.

$$f : [a, b) \to \mathbb{R} \qquad b \in (-\infty, \infty]$$

$$\int_a^b f(x)\,dx = \lim_{\beta \to b_-} \int_a^b f(x)\,dx$$

**Example 7** (Classic examples)**.** 1. Let $s > 1$.

$$\begin{aligned}\int_1^\infty \frac{1}{x^s}\,dx &= \lim \int_1^\beta x^{-s}\,dx \\ &= \frac{1}{-s+1} \cdot x^{-s+1}\Big|_1^\beta \\ &= \lim_{\beta \to \infty} \frac{1}{1-s} \cdot \frac{1}{\beta^{s-1}} - \frac{1}{1-s}\end{aligned}$$

$s - 1 > 0$ and $\frac{1}{1-s} \to 1$

$$= \frac{1}{s-1} \qquad \text{so indefinite integral exists}$$

2. Let $s < 1$.

$$\begin{aligned}\int_0^1 x^{-s}\,dx &= \lim_{\alpha \to 0_+} \int_\alpha^1 x^{-s}\,dx \\ &= \lim_{\alpha \to 0_+} \frac{1}{-s+1} x^{-s+1}\Big|_\alpha^1 \\ &= \frac{1}{1-s} - \underbrace{\lim_{\alpha \to 0_+} \frac{1}{1-s} \alpha^{1-s}}_{=0} \\ &= \frac{1}{1-s}\end{aligned}$$

Compare with Figure 16.

3.

$$\begin{aligned}\int_0^\infty e^{-cx}\,dx &= \lim_{\beta\to\infty}\int_0^\beta e^{-cx}\,dx\\ &= \lim_{\beta\to\infty}\frac{1}{-c}\cdot e^{-cx}\Big|_0^\beta\\ &= \lim_{\beta\to\infty}\left(-\frac{1}{c}\cdot e^{-c\beta}\right)+\frac{1}{c}\\ &= \frac{1}{c}\end{aligned}$$

4.

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{\alpha\to-\infty}\int_\alpha^0\frac{1}{1+x^2}\,dx+\lim_{\beta\to0}\int_0^\beta\frac{1}{1+x^2}\,dx\\ &= \arctan(0)-\underbrace{\lim_{\alpha\to-\infty}\arctan(\alpha)}_{-\frac{\pi}{2}}+\underbrace{\lim_{\beta\to\infty}\arctan(\beta)}_{\frac{\pi}{2}}-\arctan(0)\\ &= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}\\ &= \pi\end{aligned}$$

**Remark 26.** "Integral converges" means "an (indefinite) integral exists"

**Remark 27.**

$$\arctan'(x)=\frac{1}{1+x^2}$$

$$\tan'(x)=\frac{\cos x\cdot\cos x-(\sin x)(-\sin x)}{\cos^2 x}=\frac{1}{\cos^2(x)}$$

$$\tan(x)=\frac{\sin x}{\cos x}$$



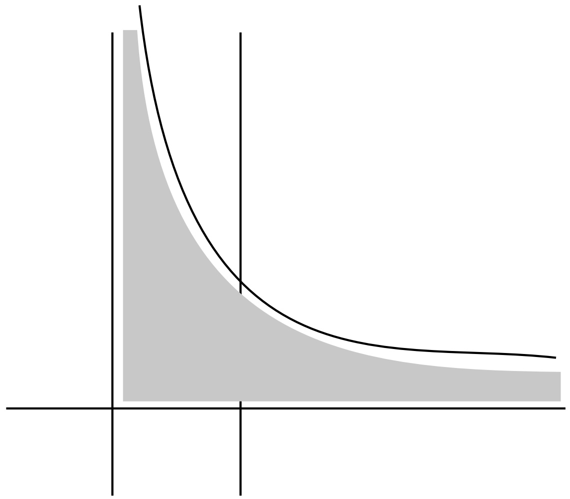

Figure 16: $\frac{1}{1-s}$

$$\begin{aligned}\arctan'(x) &= \frac{1}{\tan'(\arctan(x))}\\ &= |\arctan x = s|\\ &= \left(\frac{1}{\cos^2(s)}\right)^{-1}\\ &= \left(\frac{\cos^2(s)+\sin^2(s)}{\cos^2(s)}\right)^{-1}\\ &= \left(1+\left(\frac{\sin s}{\cos s}\right)^2\right)^{-1}\\ &= \left(1+[\tan(\arctan x)]^2\right)^{-1}\\ &= (1+x^2)^{-1}\\ &= \frac{1}{1+x^2}\end{aligned}$$

**Theorem 20** (Direct comparison test for indefinite integrals)**.** (dt. "Majorantenkriterium für uneigentliche Integrals") Let $f, g$ be regulated functions in $[a, b]$ and $|f(x)| \leq g(x) \quad \forall x \in [a, b)$. Assume $\int_a^b g(x)\,dx$ exists. Then also $\int_a^b |f(x)|\,dx$ exists and also $\int_a^b f(x)\,dx$.

*Proof.*

$$G(\beta) = \int_a^\beta g(x)\,dx$$

We know that $\lim_{\beta\to b_-} G(\beta)$ exists.

Cauchy criterion: $\forall \varepsilon > 0$ there exists a left-sided environment of $b$ such that for all $u, v$ in this environment it holds that

$$\underbrace{|G(u) - G(v)|}_{\int_u^v g(x)\,dx} < \varepsilon$$

Because $|f| \leq g$ it holds that

$$F(\beta) = \int_a^\beta |f(x)|\,dx$$

and also that

$$\left|\int_u^v |f(x)|\,dx\right| = |F(v) - F(u)| \overset{\text{monotonicity}}{\leq} \left|\int_u^v g(x)\,dx\right| < \varepsilon$$

Hence $\lim_{\beta\to b} F(\beta)$ exists because of the Cauchy criterion. So $\int_a^b |f(x)|\,dx$ exists. Analogously for $f$ instead of $|f|$. $\square$

**Example 8.**

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ exists}$$

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \quad \text{continuous in } 0$$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 f(x)\,dx \text{ exists because } f \text{ is continuous}$$



$$\begin{aligned}
\lim_{\beta\to\infty} \int_1^\beta \frac{\sin x}{x}\,dx &= \begin{vmatrix} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{vmatrix} \\
&= \lim_{\beta\to\infty} \frac{1}{x} \cdot (-\cos x)\Big|_1^\beta - \int_1^\beta \frac{\cos x}{x^2}\,dx \\
&= \lim_{\beta\to\infty} \left[\underbrace{-\frac{1}{\beta} \cdot \cos\beta}_{\to 0} + \cos 1 - \int_1^\beta \frac{\cos x}{x^2}\,dx\right] \\
&= \lim_{\beta\to\infty} \int_1^\beta \frac{\cos x}{x^2}\,dx
\end{aligned}$$

The last expression exists, because $\frac{1}{x^2}$ is a majorant for $\frac{\cos(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ exists.

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This lecture took place on 22nd of April 2016 with lecturer Wolfgang Ring.

$$\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx \text{ does not exist}$$

$$\begin{aligned}
\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right|\,dx &\geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|\,dx \\
= \frac{1}{(k+1)\pi}(\pm 1) \cdot (-\cos x)\Big|_{k\pi}^{(k+1)\pi} &= \frac{1}{(k+1)\pi}(\pm 1)(\pm 2) \\
&= \frac{2}{(k+1)\pi}
\end{aligned}$$

$$\underbrace{\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right|\,dx}_{\text{unbounded}\Leftarrow} \geq \frac{2}{\pi} \cdot \underbrace{\sum_{k=0}^n \frac{1}{k+1}}_{\substack{\text{harmonic series,} \\ \text{divergent}}}$$

In terms of the Lebesgue integral, $\int_0^\infty \frac{\sin x}{x}\,dx$ does not exist.

We can define new types of integration which yield new types of function which are not representable with techniques discussed so far.

**Example 9** (The Eulerian Γ-function)**.**

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \text{ for } x > 0$$

The function variable of the Γ-function is a parameter of the integrand.

The indefinite integral from above exists,

$$\lim_{\alpha\to 0_+}\int_\alpha^1 \underbrace{t^{x-1}e^{-t}}_{>0}\,dt \text{ exists}$$

of $\int_\alpha^1 t^{x-1}e^{-t}\,dt$ is bounded in terms of $\alpha$.

$$\int_\alpha^1 t^{x-1}\underbrace{e^{-t}}_{<1}\,dt < \underbrace{\int_\alpha^1 t^{x-1}\,dt}_{\text{converges for } x-1>-1}$$

hence for $x > 0$.

Right-side integral boundary:

$$\int_1^\infty t^{x-1}e^{-t}\,dt \text{ converges?}$$

**Example 10** (Claim)**.** There exists $c > 0$ such that

$$t^{x-1}e^{-t} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$t^{x-1}\cdot e^{-\frac{t}{2}} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$\begin{aligned}\lim_{t\to\infty}\left(t^{x-1}\cdot e^{-\frac{t}{2}}\right) &= \begin{vmatrix}\frac{t}{2} = s\\ t = 2s\end{vmatrix}\\ &= \lim_{s\to\infty}(2s)^x - 1e^{-s}\\ &\leq \lim_{s\to\infty}(2s)^{\lfloor x\rfloor+1-1}\cdot e^{-s}\end{aligned}$$



with $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$

$$\begin{aligned}&= \lim_{s\to\infty}(2s)^{\lfloor x\rfloor}\cdot e^{-s}\\ &\leq \lim_{s\to\infty}s^{\lfloor x\rfloor+1}\cdot e^{-s}\end{aligned}$$

because $s^{n+1} > (2s)^n$ for $s > 2^n$.

Hence for $\varepsilon > 0$, $\exists t$ such that

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| < \varepsilon \text{ if } t > L$$

and

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| \leq M \text{ for } t \in \underbrace{[1, L]}_{\text{compact}}$$

$\Rightarrow$ for $t \in [1,\infty)$ it holds that

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| \leq \max\{M,\varepsilon\} =: c$$

$$t^{x-1}e^{-\frac{t}{2}} \leq c$$

$$\int_0^\infty t^{x-1}e^{-t}\,dt \leq \int_0^\infty c\cdot e^{-\frac{t}{2}}\,dt = c\cdot\left(-2\cdot e^{-\frac{t}{2}}\right)\Big|_0^\infty = 2c$$

hence $\int_0^\infty t^{x-1}e^{-t}\,dt$ exists.

It holds that $\Gamma(1) = 1$ because,

$$\int_0^\infty e^{-t}\,dt = 1$$

Furthermore it holds that for all $x > 0$,

$$\Gamma(x+1) = x\cdot\Gamma(x)$$

$$\begin{aligned}\Gamma(x+1) = \int_0^\infty t^{x+1-1}e^{-t}\,dt &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}}\int_\varepsilon^R t^x e^{-t}\,dt\\ &= \begin{vmatrix}u = t^x & u' = x\cdot t^{x-1}\\ v' = e^{-t} & v = -e^{-t}\end{vmatrix}\end{aligned}$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left[ -t^x e^{-t}\Big|_{t=\varepsilon}^{R} + \int_\varepsilon^R x \cdot t^{x-1} \cdot e^{-t}\,dt \right]$$

$$= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left( \underbrace{-R^x \cdot e^{-R}}_{\to 0 \text{ for } R\to\infty} + \underbrace{\varepsilon^x \cdot e^{-\varepsilon}}_{\to 0 \text{ for } \varepsilon\to 0} \right) + x \cdot \int_0^\infty t^{x-1} e^{-t}\,dt = x \cdot \Gamma(x)$$

So it holds that

$$\begin{aligned} T(2) &= 1 \cdot T(1) = 1 \\ T(3) &= 2 \cdot T(2) = 2 \cdot 1 \\ T(4) &= 4 \cdot T(3) = 3 \cdot 2 \cdot 1 \\ T(5) &= 4 \cdot T(4) = 4 \cdot 3 \cdot 2 \cdot 1 \end{aligned}$$

By complete induction we can show that

$$\Gamma(n+1) = n! \qquad \forall n \in \mathbb{N}$$

## 5.5 Some important inequalities

**Theorem 21** (Young's inequality)**.** Let $f : [0,\infty) \to [0,\infty)$ be continously differentiable, $f(0) = 0$; $f$ is strictly monotonically increasing and unbounded (hence $f$ is injective because of strong monotonicity and surjective because of unboundedness).

So there exists $f^{-1} : [0,\infty) \to [0,\infty)$.

Let $a, b \geq 0$. Then it holds that

$$a \cdot b \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy$$

Equality holds if and only if,

$$b = f(a) \text{ i.e. } a = f^{-1}(b)$$

Compare with Figure 17



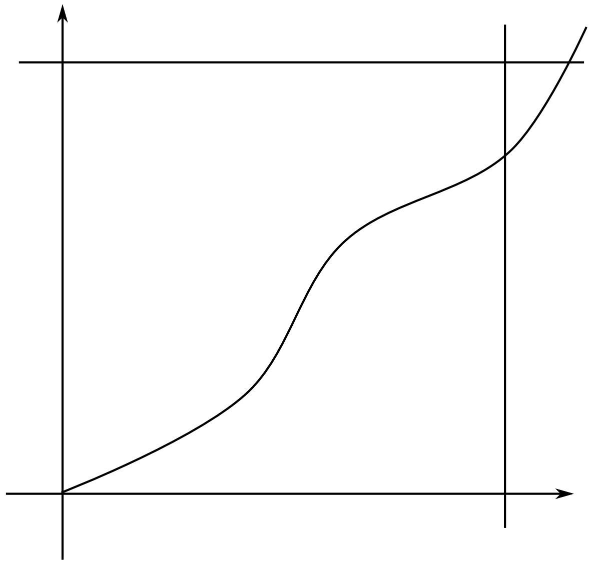

Figure 17: Young's inequality: the blue and red areas are larger than the green area

*Proof.*

$$\int_0^b f^{-1}(y)\,dy \overset{\text{substitution}}{=} \left| \begin{array}{c} y = f(x) \\ dy = f'(x)\,dx \\ y = 0 \Leftrightarrow x = f^{-1}(0) = 0 \\ y = b \Leftrightarrow x = f^{-1}(b) \end{array} \right|$$

$$= TODO \qquad = f(x)x\big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} 1 \cdot f(x)\,dx$$

$$= f(f^{-1}(b)) \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,dx$$

$$= b \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\,dx$$

Therefore,

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy = \int_0^a f(x)\,dx + \int_{f^{-1}(b)}^0 f(x)\,dx + b \cdot f^{-1}(b)$$

**Case 1:** $f^{-1}(b) = a \;\; (f(a) = b)$

$$\Rightarrow I = \underbrace{\int_a^b f(x)\,dx}_{=0} + b \cdot a = ab$$

Proven.

**Case 2:** $b < f(a) \Leftrightarrow f^{-1}(b) < a$ $f$ is strictly monotonically increasing, hence $f(x) > f(f^{-1}(b)) = b$ for all $x \in (f^{-1}(b), a]$.

$$\int_{f^{-1}(b)}^a f(x)\,dx > b \cdot \int_{f^{-1}(b)}^a 1\,dx$$

$$= b \cdot \big(a - f^{-1}(b)\big)$$

$$I > b\big(a - f^{-1}(b)\big) + b \cdot f^{-1}(b) = a \cdot b$$

Proven.

**Case 3:** $b > f(a)$

$$I = \underbrace{-\int_a^{f^{-1}(b)} f(x)\,dx}_{\text{strictly mon. decreasing}} + bf^{-1}(b)$$

For $(-f(x))$ it holds that:

$$> (-f(f^{-1}(b)) = -b)$$

$$> (-b)\big(f^{-1}(b) - a\big) + b \cdot f^{-1}(b) = a \cdot b$$

Proven.

$\square$

**Remark 28.** Young's inequality also holds if $f$ has all the properties above but is not necessarily differentiable.

**Theorem 22** (Young's inequality, special case)**.** Let $A, B \geq 0$. $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (hence $p$ and $q$ are "conjugate exponents"). Then it holds that

$$A \cdot B \leq \frac{A^p}{p} + \frac{B^q}{q}$$

*Proof.* Let $f(x) = x^{p-1}$ satisfy the requirements for Young's inequality.

$$f^{-1}(y) = y^{\frac{1}{p-1}}$$

$$\left(\frac{1}{q} = 1 - \frac{1}{q} \qquad q = \left(1 - \frac{1}{q}\right)^{-1}\right)$$

$$q - 1 = \left(1 - \frac{1}{p}\right)^{-1} - 1 = \left(\frac{p-1}{p}\right)^{-1} - 1$$

$$= \frac{p}{p-1} - 1 = \frac{p - p + 1}{p-1} = \frac{1}{p-1}$$

$$f^{-1}(y) = y^{q-1}$$

Therefore

$$A \cdot B \leq \int_0^A x^{p-1}\,dx + \int_0^B y^{q-1}\,dy = \frac{x^p}{p}\bigg|_0^A + \frac{y^q}{q}\bigg|_0^B = \frac{A^p}{p} + \frac{B^q}{q}$$

$\square$

**Remark 29.** Equality holds if $A^p = B^q$. The proof is left as an exercise to the reader.

**Theorem 23** (Hölder's inequality)**.** Let $I$ be an interval, $a, b$ are boundary values of $I$ ($a, b \in [-\infty, \infty]$). Let $p, q$ be conjugate exponents, hence $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Let $f_1$ and $f_2$ be regulated functions in $I$ and

$$\int_a^b |f_1(x)|^p \, dx \text{ exists and } \int_a^b |f_2(x)|^q \, dx \text{ exists}$$

Let

$$\|f_1\|_p = \left(\int_a^b |f_1(x)|^p \, dx\right)^{\frac{1}{p}}$$

and

$$\|f_2\|_q = \left(\int_a^b |f_2(x)|^q \, dx\right)^{\frac{1}{q}}$$

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)| \, dx \text{ exists and } \int_a^b |f_1(x) f_2(x)| \, dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

*Proof.* Let $A = \frac{|f_1(x)|}{\|f_1\|_p}$ and $B = \frac{|f_2(x)|}{\|f_2\|_q}$.

$$A \cdot b \leq \frac{A^p}{p} + \frac{B^q}{q}$$

$$\overset{\text{integration}}{\Rightarrow} \int_a^b \frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q} \, dx \leq \frac{1}{p} \int_a^b \frac{|f_1(x)|^p}{\|f_1\|_p^p} \, dx + \frac{1}{q} \int_a^b \frac{|f_2(x)|^q}{\|f_2\|_q^q} \, dx$$

$$\Rightarrow \frac{1}{\|f_1\|_p \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| \, dx$$

$$\leq \frac{1}{p} \frac{1}{\|f_1\|_p^p} \underbrace{\int_a^b |f_1(x)|^p \, dx}_{\|f_1\|_p^q} + \frac{1}{q} \frac{1}{\|f_2\|_q^q} \underbrace{\int_a^b |f_2(x)|^q \, dx}_{\|f_2\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$= \underbrace{\int_a^b |f_1(x) \cdot f_2(x)| \, dx}_{\text{exists}} \leq \|f_1\|_p \cdot \|f_2\|_q$$

$\square$

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This lecture took place on 28th of April 2016 with lecturer Wolfgang Ring.

**Example 11** (Special case $p = q = 2$)**.** Let $p = q = 2$. $\frac{1}{2} + \frac{1}{2} = 1$ holds.

$$\int_a^b |f_1(x) \cdot f_2(x)| \, dx \leq \left(\int_a^b |f_1(x)|^2 \, dx\right)^{\frac{1}{2}} \cdot \left(\int_a^b |f_2(x)|^2 \, dx\right)^{\frac{1}{2}}$$

$$\int_a^b |f_1(x) \cdot f_2(x)| \, dx \geq \left|\int_a^b f_1(x) \cdot f_2(x) \, dx\right|$$

$f_1$ and $f_2$ such that $\|f_i\|_2 < \infty$ for $i = 1, 2$, then

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) \cdot f_2(x) \, dx$$

is a scalar (= inner) product in the vector space of functions with norm:

$$\|f\| = (\langle f, f \rangle)^{\frac{1}{2}} = \|f\|_2$$

The resulting inequality is named "Cauchy-Schwarz inequality"

$$|\langle f_1, f_2 \rangle| \leq \|f_1\|_2 \cdot \|f_2\|_2$$

## 5.6 Elementwise integration of series

**Lemma 13.** Let $f_n \in R(I)$ with $I$ as interval, $f_n$ converges uniformly to $f$ in $I$. Then also $f$ is a regulated function and

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \int_a^b f_n(x) \, dx$$

*Proof.* We know $f$ is a regulated function if and only if $f$ can be uniformly approximated using a step function.

Let $\varepsilon > 0$ be arbitrary. Because $f$ is the uniform limit of $f_n$, there exists $n \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$. Because $f_N$ is a regulated function, there exists $\varphi \in \tau(I)$ with

$$\|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} \Rightarrow \|f - \varphi\|_\infty \leq \|f - f_N\|_\infty + \|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $f$ is a regulated function. Choose $N$ such that $\forall n \geq N$:

$$\|f - f_n\|_\infty < \frac{\varepsilon}{b-a}$$

Then it holds that

$$\begin{aligned}\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &\leq \int_a^b |f_n(x) - f(x)|\,dx \\ &\leq \int_a^b \underbrace{\|f_n - f\|_\infty}_{<\frac{\varepsilon}{b-a}}\,dx \\ &< \frac{\varepsilon}{b-a} \cdot (b-a) \\ &= \varepsilon\end{aligned}$$

□

**Example 12** (Application)**.** Let $f(x) = \sum_{n=0}^\infty a_k x^k$ is a power series. Let $\rho_f$ be a convergence radius of $f$ and $0 < r < \rho_f$. Then it holds that

$$f_n(x) = \sum_{k=0}^n a_k x^k \text{ converges uniformly to } f \text{ in } [-r, r]$$

$$f_n \in R([-r, r])$$

$$\Rightarrow \int_{-r}^r f(x)\,dx = \lim_{n\to\infty} \int_{-r}^r f_n(x)\,dx$$

The integral is determined by elementwise integration

$$\int_{-r}^r a_k x^k\,dx = a_k \frac{x^{k+1}}{k+1}\Big|_{-r}^r$$



Analogously for integration over any compact interval $[a, b] \subset (-\rho_f, \rho_f)$ i.e. for the indefinite integration. Hence,

$$\sum_{k=0}^\infty a_k \frac{x^{k+1}}{n+1} + c$$

is primitive function of $f$ uniformly convergent on every interval $[-r, r] \subseteq (-\rho_f, \rho_f)$.

**Example 13.**

$$F : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad F(x) = \arctan(x)$$

$$F'(x) = f(x) = \frac{1}{1+x^2} = \sum_{k=0}^\infty \left(-(x^2)\right)^k = \sum_{k=0}^\infty (-1)^k x^{2k} \qquad \forall x \in (-1, 1)$$

Elementwise integration:

$$F(x) = \arctan(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1} + c$$

$$\arctan(0) = 0 = c$$

Hence,

$$\arctan(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{2k+1} \quad \text{in } (-1, 1)$$

Compare with Figure 18

# 6 Taylor polynomials and Taylor series

**Theorem 24.** Approximation of a function with polynomials or representation of a function using a power series.

$$\mathcal{C}^n((a, b)) = \{f : (a, b) \to \mathbb{R} \,|\, f \text{ differentiable } n \text{ times in } (a, b)\}$$

Hence $f^{(k)} : (a, b) \to \mathbb{R}$ is continuous for $k = 0, 1, \ldots, n$. Choose $x_0 \in (a, b)$. Find a polynomial $T_f^a(x)$ of degree $n$ such that

$$\left(T_f^a\right)^{(k)}(x_0) = f^{(k)}(x_0)$$

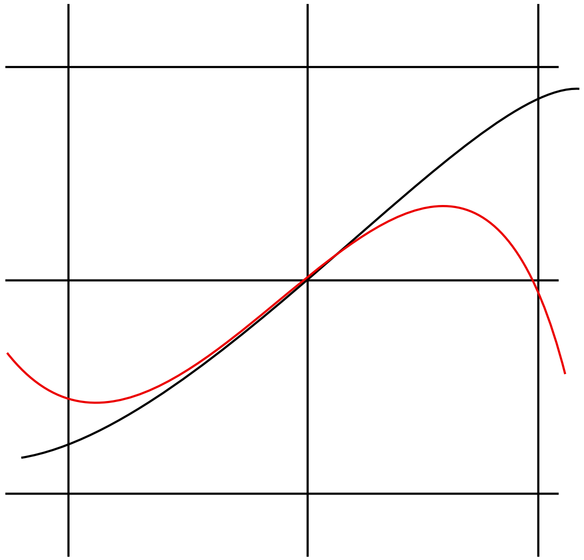

Figure 18: Approximation of $\arctan(x)$

It holds that $T_f^a$ can be determined uniquely as

$$T_f^a(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Taylor polynomial of $n$-th order of $f$ in point $x_0$.

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This lecture took place on 29th of April 2016 with lecturer Kniely Michael.

**Definition 14** (Additional remark to Taylor polynomials)**.** Let $P(x) := \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$. Let $k \in \{1, \dots, n\}$.

1. $x_0$ is called $k$-th root of $P$ iff $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$.



2. It holds that $x_0$ is a $k$-th root of $P$ iff

$$\forall j \in \{0, \dots, k - 1\} : P^{(j)}(x_0) = 0 \land P^{(k)}(x_0) \neq 0$$

*Complete induction over $k$.* $\mathbf{k = 1}$

$\Rightarrow$: Let $x_0$ be 1st root of $P$.

$$P(x) = (x - x_0)Q(x) \Rightarrow P^{(0)}(x_0) = 0 \land P^{(1)}(x_0) = Q(x_0) \neq 0.$$

$\Leftarrow$: Let $P^{(0)}(x_0) = 0$.

$$P^{(1)}(y_0) \neq 0$$

Division with remainder $\Rightarrow$

$$P(x) = (x - x_0)Q(x) + R(x) \text{ with } \deg(R) < \deg(x - x_0) = 1$$

with $R$ constant.

$$0 = P(y_0) = R \Rightarrow P(x) = (x - x_0)Q(x)$$

$$x \neq x_0 \Rightarrow Q(x) = \frac{P(x)}{x - x_0} = \frac{P(x) - P(x_0)}{x - x_0} \Rightarrow Q(x_0)$$

$$\overset{\text{Q continuous}}{=} \lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{P(x) - P(x_0)}{x - x_0} = P^{(1)}(x_0) \neq 0$$

$\mathbf{k \geq 2, k - 1 \to k}$ $\Rightarrow$. Let $x_0$ be the k-th root of $P$. Hence $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$. Let $\tilde{P}(x) := (x - x_0)^{k-1} Q(x)$. $x_0$ is $(k - 1)$-th root of $\tilde{P}$.

$$\xRightarrow{\text{ind. hypo.}} \tilde{P}^{(j)}(x_0) = 0 \land \tilde{P}^{(k-1)}(x_0) \neq 0 \quad \forall j \in \{0, \dots, k - 2\}$$

$$P(x) = (x - x_0)\tilde{P}(x) \Rightarrow P^{(j)}(x) = (x - x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x)$$

We prove the last statement using complete induction:

*Proof.* $j = 0$ Follows immediately.

$j \geq 0, j \to j + 1$

$$P^{(j+1)}(x) = \left(P^{(j)}\right)'(x)$$

$$= \tilde{P}^{(j)}(x) + \tilde{P}^{(j+1)}(x)(x - x_0)$$

$$+j\tilde{P}^{(j)}(x) = (x-x_0)\tilde{P}^{(j+1)}(x) + (j+1)P^j(x).$$

$$P^{(j)}(x_0) = j\tilde{P}^{(j-1)}(x_0)$$

$$\begin{cases} = 0 & j = 0,\dots,k-1 \\ \neq 0 & j = k \end{cases}$$

□

We then prove the second part: $\Leftarrow$.

Let $P^{(j)}(x_0) = 0$ for $j \in \{0,\dots,k-1\}$, $P^{(k)}(x_0) \neq 0$. It holds that $P(y_0) = 0$ because of $P^{(0)}(x_0) = 0$. Like above: $P(x) = (x-x_0)\tilde{P}(x)$ and

$$P^{(j)}(x) = (x-x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x).$$

$$j \in \{1,\dots,k-1\} \Rightarrow 0 = P^{(j)}(x_0) = TODO$$

$$\Rightarrow \forall l \in \{0,\dots,k-2\} : \tilde{P}^{(l)}(x_0) = 0$$

TODO

$$0 \neq P^{(k)}(x_0) = k\tilde{P}^{(k-1)}(x_0) \Rightarrow \tilde{P}^{)k-1}(x_0) \neq 0$$

induction hypothesis $\Rightarrow$

$$\tilde{P}(x) = (x-x_0)^{k-1}Q(x) \text{ with } Q(x_0) \neq 0$$

$$\Rightarrow P(x) = (x-x_0)\tilde{P}(x) = (x-x_0)^k Q(x).$$

□

**Theorem 25.** Let $f$ in $\mathbb{C}^n((a,b))$ with $n \in \mathbb{N}$. Let $a,b \in [-\infty,\infty]$, $x_0 \in (a,b)$. Find a polynomial $T$ of degree $n$ such property

$$\forall k \in \{0,\dots,n\} : T^{(k)}(x_0) = f^{(k)}(x_0). \tag{1}$$

Claim:

$$T_f^n(x) \equiv T_f^n(x;x_0) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

where $x_0$ is the base point, is the only polynmial of degree $n$, which satisfies property 1.

$T_f^n$ is called *Taylor polynomial* of $n$-th degree of $f$ in $x_0$.

*Proof.* Let $k \in \{0,\dots,n\}$.

$$(T_g^n)^{(k)}(x) = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} j(j-1)\cdot\ldots\cdot(j-(k-1))(x-x_0)^{j-k}$$

$$\left(T_f^n\right)^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!}\underbrace{(k\cdot\ldots\cdot(k-(k-1)))}_{=k!} = f^{(k)}(x_0).$$

Let $T(x) = \sum_{j=0}^{n} a_j x^j$ be a polynomial, which satisfies 1. For $P := T_g^n - T$ it holds that $P^{(k)}(x_0) = 0$ for all $k \in \{0,\dots,n\}$. And $P$ is a polynomial of degree at most $n$. $x_0$ is at least an $(n+1)$-th root of $P \Rightarrow P \equiv 0$. □

**Definition 15** (Deviation, error, remainder)**.**

$$R_g^{n+1}(x;x_0) \equiv R_g^{n+1}(x) := f(x) - T_g^n(x;x_0)$$

**Theorem 26** (Integration form of the remainder)**.** Let $f \in C^{n+1}((a,b),\mathbb{C})$, $n \in \mathbb{N}$, $a,b \in [-\infty,\infty]$, $x_0, x \in (a,b)$. Then it holds that

$$R_g^{n+1}(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt$$

*Complete induction over $n$.* Let $n = 0$.

$$R_g^1(x) = f(x) - T_g^0(x) = f(x) - f(x_0)$$

$$\frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt = \int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0).$$

Consider $n \geq 1, n-1 \to n$. From induction hypothesis we consider

$$\begin{aligned}\Rightarrow f(x) - T_g^{n-1}(x) = R_g^n(x) &= \frac{1}{(n-1)!}\int_{x_0}^{x}(x-1)^{n-1}f^{(n)}(t)\,dt \\ &= -\frac{(x-t)^n}{n(n-1)!}f^{(n)}(t)\Big|_{x_0}^{x} + \int_{x_0}^{x}\frac{(x-t)^n}{n(n-1)!}f^{(n+1)}(t)\,dt \\ &= \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) + \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\,dt\end{aligned}$$

$$\Rightarrow R_f^{n+1}(x) = f(x) - T_g^n(x) = f(x) - T_g^{n-1}(x) - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
$$= \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{n+1}(t)\,dt$$

Recognize that we consider $f$ over $\mathbb{C}$. In the next theorem we will only consider it in $\mathbb{R}$. □

**Theorem 27** (Lagrange representation of remainder)**.** Let $f \in C^{n+1}((a,b),\mathbb{R}), n \in \mathbb{N}$, $a,b \in [-\infty,\infty]$, $x_0, x \in (a,b)$. Then there exists some $\xi$ between $x_0$ and $x$ such that

$$R_g^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

*Proof.*

$$R_f^{n+1}(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)\,dt$$

**Case 1:** $x \geq x_0$**:**

$$\forall t \in [x_0, x] : (x-t)^n \geq 0$$

$f \mapsto (x-1)^n$ regulated function. $t \mapsto f^{(n+1)}(t)$ continuous. Hence,

$$\exists \xi \in [x_0, x] : \int_{x_0}^{x} (x-1)^n f^{n+1}(t)\,dt = f^{n+1}(\xi)\int_{x_0}^{x}(x-t)^n\,dt$$
$$= f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{n+1}$$
$$\Rightarrow R_f^{n+1}(x) = \frac{f^{n+1}}{(n+1)!}(x-x_0)^{n+1}.$$

**Case 2:** $x < x_0$**:**

$$\forall t \in [x, x_0] : (t-x)^n \geq 0 \qquad \text{analogously}$$

$$\exists \xi \in [x, x_0] : \int_{x}^{x_0} (t-x)^n f^{(n+1)}(t)\,dt$$
$$= f^{(n+1)}(\xi)\int_{x}^{x_0}(1-x)^n\,dt$$
$$= \frac{f^{(n+1)}(\xi)}{n+1}(x_0-x)^{n+1}$$
$$\Rightarrow R_g^{n+1}(x) = \frac{(-1)^{n+1}}{n!}\int_{x}^{x_0}(t-x)^n f^{(n+1)}(t)\,dt$$
$$= (-1)^{n+1}\frac{f^{(n+1)}(\xi)}{(n+1)!}(x_0-x)^{n+1}$$
$$= \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

□

**Corollary 4** (Sufficient criterion for local extremes)**.** Let $f \in C^{n+1}((a,b),\mathbb{R}), x_0 \in (a,b)$ with $f^{(n)}(x_0) = \ldots = f^{(n)}(x_0) = 0, f^{(n+1)}(x_0) \neq 0$. Then $f$ has the following in $x_0$:

- a strict local minimum, if $n$ is odd and $f^{(n+1)}(x_0) > 0$.
- a strict local maximum, if $n$ is odd and $f^{(n+1)}(x_0) < 0$.
- no extreme, if $n$ is even.

*Proof.* **Case 1:** $f^{(n+1)}(x_0) > 0$**:**
$f^{(n+1)}$ is continuous $\Rightarrow$

$$\exists \varepsilon > 0 : f^{(n+1)} > 0 \text{ in } (x_0-\varepsilon, x_0+\varepsilon) =: I$$

by Induction hypothesis it holds that

$$\forall x \in (a,b) : f(x) = T_g^n(x) + R_g^{n+1}(x) = f(x_0) + R_f^{n+1}(x).$$

If $n$ is even, then $n+1$ is odd, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} > 0.$$

So,

$$\forall x \in I \setminus \{x_0\} : f(x) > f(x_0)$$

If $n$ is odd, $n+1$ is even, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$$\begin{cases} > 0 & x > x_0 \\ < 0 & x < x_0 \end{cases}$$

$\Rightarrow f$ has no extremum in $x_0$.

**Case 2:** $f^{(n+1)}(x_0) < 0$ follows analogously like Case 1.

□

**Theorem 28** (Qualitative Taylor formula)**.** Let $f \in C^n((a,b),\mathbb{C}), x, x_0 \in (a,b)$. There exists some $r \in C((a,b),\mathbb{C})$ with $r(x_0) = 0$ and

$$f(x) = T_f^n(x) + (x-x_0)^n r(x) \tag{2}$$

*Proof.* Equation 2 only has to be shown for $f : (a,b) \to \mathbb{R}$, because for $f : (a,b) \to \mathbb{C}$, $f = f_R + i f_I$ with $f_R, f_I : (a,b) \to \mathbb{R}$. Representations for $f_R$ and $f_I$ provide corresponding representations for $f$. Hence let $f : (a,b) \to \mathbb{R}$. Let $r : (a,b) \to \mathbb{R}$.

$$x \mapsto \frac{f(x) - T_f^n(x)}{(x-x_0)^n}, x \neq x_0 \text{ and } r(x_0) := 0$$

We only need to show:
$r$ is continuous in $x_0$, hence $\lim_{x\to x_0} r(x) = r(x_0) = 0$.

$$x \in (a,b) \setminus \{x_0\} \Rightarrow r(x) = \frac{1}{(x-x_0)^n}\left(f(x) - T_f^{n-1}(x) - \frac{f^{(n)}}{n!}(x-x_0)^n\right)$$

$$= \frac{1}{(x-x_0)^n}\left(R_g^n(x) - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\right)$$

$$= \frac{1}{(x-x_0)^n}\left(\frac{f^{(n)}(\xi)}{n!}(x-x_0)^n - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\right)$$

$$= \frac{1}{n!}\left(f^{(n)}(\xi) - f^{(n)}(x_0)\right)$$

$\xi$ is between $x_0$ and $x$. $f^{(n)}$ is continuous and $\xi \to x_0$ for $x \to x_0$

$$\Rightarrow r(x) = \frac{1}{n!}(f^{(n)}(\xi) - f^{(n)}(x_0)) \overset{x\to x_0}{\to} 0$$

□

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This lecture took place on 3rd of May 2016 with lecturer Wolfgang Ring.

**Theorem 29.** Assumption: Let $f : I \to \mathbb{R}$ be arbitrarily often continuously derivable. Hence,

$$T_f^n(x;x_0) \text{ exists for } \forall n \in \mathbb{N}$$

Therefore we can consider a power series

$$T_f(x;x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$T_f(x;x_0)$ is called Taylor series of $f$ in $x_0$. Is a power sereis in $\xi = (x-x_0)$. Converges for $|\xi| = |x-x_0| < \rho(T_f)$.

If $\rho(T_f) > 0$, it holds that $T_f(x;x_0) = f(x)$?

$$\lim_{n\to\infty} T_f^n(x;x_0) = T_f(x;x_0) = f(x) \text{ for } |x-x_0| < \rho(T_f)$$

is *not* always satisfied.

**Example 14.**

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Compare with Figure 19.

$$f_-^{(n)}(0) = 0$$

$$f_+^{(n)}(0) = \lim_{x\to 0^+} f^{(n)}(x)$$

$$f^{(n)}(x) = R(x) \cdot e^{-\frac{1}{x}}$$

with $R(x) = \frac{P(x)}{Q(x)}$ with $P$ and $Q$ as polynomials. $R$ is a rational function (i.e. division of two polynomials).

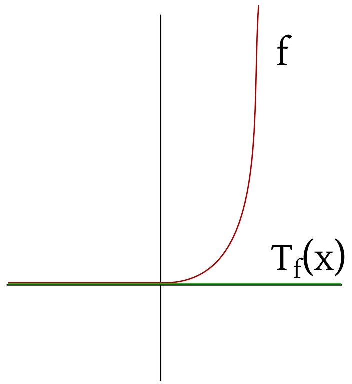


Figure 19: Plot of $f$

$$\lim_{x\to 0_+} R(x)\cdot e^{-\frac{1}{x}} = 0$$

Hence $f^{(n)}(0) = 0$ and therefore Taylor series $T_f(x;0) = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0$.

**Remark 30.** Taylor:

$$R_f(x) = T_f(x;0) - f(x)$$

It holds that

$$|R_f(x)| \leq c_n \cdot |x|^n \qquad \forall n \in \mathbb{N}$$

**Theorem 30.** Let $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series in $\xi = x-x_0$. Let $\rho(f) > 0$. We already know that $f$ is differentiable for all $|\xi| = |x-x_0| < \rho(f)$ (differentiable by $x$) and $f'$ is a power series with convergence radius $\rho(f') = \rho(f)$.

$$f'(x) = \sum_{k=1}^{\infty} a_k \cdot kx^{k-1}$$

By complete induction it follows that:

- For all $n \in \mathbb{N}$ there exists $f^{(n)}(x)$ as power series of form

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k \cdot k \cdot (k-1)\cdot(k-2)\cdot\ldots(k-n+1)\cdot x^{k-n}$$

- $f^{(n)}$ as convergent power series is a continuous function. Hence,

$$f^{(n)}(x_0) = a_n \cdot n \cdot (n-1)\cdot(n-2)\ldots(n-n+1) = a_n \cdot n!$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Backsubstitution in the power series yields

$$f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = T_f(x;x_0)$$

Hence $f$ has a power series representation, then the power series is the Taylor series in $f$.

**Remark 31.** A function representable with a power series is called *analytical*. In the complex space, once differentiable means arbitrary often differentiable.

# 7 Curves in $\mathbb{R}^n$

**Definition 16.** A parametric curve is a map $\gamma : I \to \mathbb{R}^n$ where $I$ is an interval.

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

where every function $\gamma_i : I \to \mathbb{R}$ $(i = 1,\ldots,n)$ is continuous. Often we write $\gamma_i(t) = x_i(t)$. If every $\gamma_i$ is differentiable in $I$, a differentiable, parameterized curve is given. $t$ is the curve parameter.

We call $\Gamma = \{\gamma(t)\,|\,t \in I\} = \gamma(I) \subseteq \mathbb{R}$ the trace of the curve $\gamma$.

**Example 15.**

$$\gamma : [0, 4\pi] \to \mathbb{R}^2$$

$$\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

In this example, every point on the curve is hit twice by the function.

$$\Gamma = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R} \,\middle|\, x_1^2 + x_2^2 - 1 = 0 \right\}$$

$F(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$ is called trace equation of the curve

$$\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \text{ in } I = [0, 4\pi]$$

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, then

$$x_2^1 + x_2^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 1 - 1 = 0$$

On the inverse, let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1$. Then there exists $t \in [0, 2\pi)$ such that $x_1 = \cos t$ and $x_2 = \sin t$.

In this example it holds that $\tilde{\gamma} \neq \gamma$, but $T = \tilde{T}$.

**Example 16.** Let $\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$.

$$\forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \tilde{T} : T(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$

but

$$\tilde{T} \neq \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, F(x_1, x_2) = 0 \right\}$$

**Definition 17.** Let $\gamma : I \to \mathbb{R}^n$ be a differentiable, parameterized curve. We define

$$\dot{\gamma}(t) = \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$



and we call $\dot{\gamma}(t)$ the derivation vector of $\gamma$ in $t$. If $\gamma$ is considered as motion curve, then $\dot{\gamma}(t)$ is considered as speed vector of $\gamma$ in $t$.

Consider

$$\dot{\gamma}(t) = \lim_{h \to 0} \frac{1}{R} [\gamma(t + h) - \gamma(t)]$$

as illustrated in Figure 20.

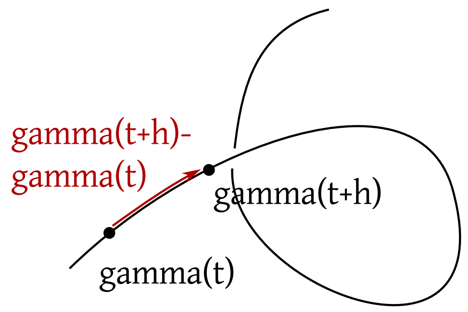


Figure 20: Curve example

If $\dot{\gamma}(t) \neq 0$, then $\dot{\gamma}$ is tangential into $\Gamma$ and we denote $\dot{\gamma}(t)$ as tangential vector of $\gamma$ in $t$.

If $\dot{\gamma}(t) \neq 0$, we set

$$T_\gamma(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|_2}$$

and we call $T_\gamma(t)$ the tangential unit vector of $\gamma$ in $t$.

**Example 17.**

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

$$\gamma(t) = \begin{bmatrix} t^2 - 1 \\ t^3 - 1 \end{bmatrix} \text{ differentiable}$$

$$\gamma(1) = \begin{bmatrix} 1 - 1 \\ 1 - 1 \end{bmatrix} = 0$$

$$\gamma(-1) = \begin{bmatrix} 1 - 1 \\ -1 + 1 \end{bmatrix} = 0$$

## German keywords

## English keywords