Linear Algebra 2 – Practicals

Lukas Prokop

summer term 2016

Contents

1 Solution of the last lecture exam of Analysis 1

1.1 Exam: Exercise 1

Exercise 1. Determine the limes of

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

does not help us. What about this representation?

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1} = \frac{a(n-1)+b(n+1)}{(n+1)(n-1)}$$

$$a(n-1)+b(n+1) = 1$$

$$(a+b)n + (b-a) = 1$$

$$\Rightarrow a+b = 0 \land b-a = 1$$

$$\Rightarrow a = -\frac{1}{2} \quad b = \frac{1}{2}$$

Followingly,

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right)$$

Okay, how to proceed? Let's build a pre-factor:

$$\frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$$
$$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m := \frac{1}{2} \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n+1}$$

Please be aware that we explicitly define S_m because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$= \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=4}^{m+2} \frac{1}{n-1}$$
$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2}\right) - \frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1}\right)$$

We already know $\frac{1}{m} \xrightarrow{m\to\infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m\to\infty} 0$. Followingly also $\frac{1}{2}\left(\frac{1}{m} + \frac{1}{m+1}\right) \xrightarrow{m\to\infty} 0$.

1.2 Exam: Exercise 2

Exercise 2. A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n \in \mathbb{N}}$$

$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \qquad a_1 = \frac{3}{2} \qquad a_2 = \frac{5}{4} \qquad a_3 \frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \qquad a_1 = 4 \qquad a_2 = \frac{5}{2} \qquad a_3 \frac{7}{4}$$

Exercise 3. a) Show that $1 \overset{!}{<} a_n \overset{!}{\leq} a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over n to prove this statement:

induction base $1 < a_0 \leq a_0$ holds trivially.

induction step We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &\leq \frac{1}{2}(a_0 + a_0) && \text{[induction hypothesis and } 1 < a_0] \end{aligned}$$

$$
\begin{aligned}
a_{n+1} &= \frac{1}{2}(a_n + 1) \\
&> \frac{1}{2}(1+1) && \text{[induction hypothesis]} \\
&= 1
\end{aligned}
$$

Exercise 4. b) Prove that $a_{n+1} \overset{!}{<} a_n \quad \forall n \in \mathbb{N}$

$$
\begin{aligned}
a_{n+1} &= \frac{1}{2}(a_n + 1) \\
&< \frac{1}{2}(a_n + a_n) && \text{[we have proven: } a_n > 1\text{]}
\end{aligned}
$$

Exercise 5. c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$
\begin{gathered}
b_n := a_n - 1 \qquad \forall n \in \mathbb{N} \\
b_0 := a_0 - 1 \\
b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2} b_n
\end{gathered}
$$

$$
\begin{gathered}
b_n = \frac{1}{2^n} b_0 \to 0 \cdot b_0 = 0 \\
\Rightarrow b_n \to 0 \\
\Rightarrow a_n = b_n + 1 \to 1
\end{gathered}
$$

Does it work to just show: $1 = \frac{1}{2}(1+1)$? Nope, because in points of continuity this might be true even though 1 is not its limes.

Let $a_n \to a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$
a_{n+1} \to a \qquad \frac{1}{2}(a_n + 1) \to \frac{1}{2}(a+1) \qquad a = \frac{1}{2}(a+1)
$$

1.3 Exam: Exercise 3

Exercise 6. $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an ε-δ-proof.

If we don't need an ε-δ-proof, we would argue with the Algebraic Continuity Theorem: The function f is a composition of continuous functions, hence a continuous function itself.

ε-δ-definition:

$$
\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon
$$

If $|x - x_0| < \delta$,

$$
\begin{aligned}
|f(x) - f(x_0)| &= |2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)| \\
&= |2x^2 + 5x - 2x_0^2 - 5x_0| \\
&\leq 2|x^2 - x_0^2| + 5|x - x_0| \\
&= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\
&= 2|x + x_0||x - x_0| + 5|x - x_0| \\
&\leq 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\
&\leq 2(|x_0| + \delta + |x_0|) + 5\delta
\end{aligned}
$$

Our goal: we are able to claim $\overset{!}{<} \varepsilon$

$$
\begin{aligned}
&= 4|x_0|\delta + 2\delta^2 + 5\delta \\
&= 2\delta^2 + (4|x_0| + 5)\delta
\end{aligned}
$$

In general (here it does not apply), that x_0 might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$2\delta^2 < \frac{\varepsilon}{2}$$

$$\delta < \frac{\sqrt{\varepsilon}}{2}$$

$$(4|x_0| + 5)\delta < \varepsilon$$

$$\delta < \frac{\varepsilon}{4|x_0| + 5}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{5}, \frac{\varepsilon}{4|x_0|+6}\right)$. Then the ε-δ definition shows that f is continuous.

2 Exam: Exercise 4

Exercise 7. Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$
\begin{aligned}
h(0) &= f(0) - f\left(\frac{1}{2}\right) \\
h(1) &= f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0)
\end{aligned}
$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x + \frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore h is continuous, because it is a composition of continuous functions.

Case 1: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\Rightarrow f(\xi) = f(\xi + \frac{1}{2})$$

Case 2: $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

Case 3: $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

3 Exercise 1

Exercise 8. Investigate the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(x|x| + x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

4 Exercise 2

Exercise 9. Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x\to 1} \frac{\ln x}{x-1}$
2. $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1-\sin x)}$
4. $\lim_{x\to 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$