

Linear Algebra 2 – Lecture Notes

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This lecture took place on 29th of Feb 2016 (Prof. Franz Lehner).

Exam: written and orally

Tutorial session:

- Every Monday, 18:30-20:00, SR 11.34
- Contact: gernot.holler@edu.uni-graz.at

Konversatorium:

- Every Monday, 10:00–10:45, SR 11.33

Topics, wie already discussed:

- Vector spaces
- Linear maps and their equivalence with matrices
- We introduced equivalence of matrices ($PAQ = B$)
- We defined the following techniques:
 - Rank
 - Linear equation system
 - Inverse matrices
 - Basis transformation

In this semester, we will discuss:

- PAP^{-1} , which is related to eigenvalues and diagonalization, hence $\bigvee_P PAP^{-1} = D$.

1 Linear maps (cont.)

1.1 Addition to chapter 5.2.4

$\text{Hom}(V, W)$ in special case $W = \mathbb{K}$. We define,

$$V^* := \text{Hom}(V, \mathbb{K})$$

also denoted V' is called *dual space* of vector space V . The elements $v^* \in V^*$ is called *linear forms* or *linear functionals*.

We denote,

$$v^*(v) =: \langle v^*, v \rangle$$

1.2 Example

$$V = \mathbb{K}^n$$

$v^* : V \rightarrow \mathbb{K}$ is uniquely defined with values $v^*(e_i) =: a_i$.

$$\begin{aligned} \langle v^*, v \rangle &= \left\langle v^*, \sum_{i=1}^n v_i e_i \right\rangle = \sum_{i=1}^n v_i \langle v^*, e_i \rangle \\ \left(v^* \left(\sum_{i=1}^n v_i e_i \right) \right) &= \sum_{i=1}^n v_i v^*(e_i) = \sum_{i=1}^n a_i v_i \end{aligned}$$

1.3 More general

We know, $\dim \text{Hom}(V, W) = \dim V \cdot \dim W$.

Theorem 1. *Let V be a vector space over \mathbb{K} .*

- $\dim V =: n < \infty \Rightarrow \dim V^* = n$
More precisely: Let (b_1, \dots, b_n) be a basis of V . Then

$$b_k^* : b_i \mapsto \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & \text{else} \end{cases}$$

is a basis of V^* and is called dual basis.

- For $v^* \in V^*$ it holds that $v^* = \sum_{k=1}^n \langle v^*, b_k \rangle \cdot b_k^*$.
- If $\dim V = \infty, (b_i)_{i \in I}$ bass, then it holds that

$$(b_k^*)_{k \in I}, \langle b_k^*, b_i \rangle = \delta_{ik}$$

is not a basis of V^* .

Proof. • Special case of 5.18

(b_k^*) is linear independent, hence in $\sum_{i=1}^n \lambda_i b_i^* = 0$ all $\lambda_i = 0$.

$$0 = \left\langle \sum_{i=1}^n \lambda_i b_i^*, b_k \right\rangle = \sum_{i=1}^n \lambda_i \underbrace{\langle b_i^*, b_k \rangle}_{\delta_{ik}} = \lambda_k \forall k$$

- Let $v \in V$ with $v = \sum_{i=1}^n v_i b_i$. We need to show

$$\begin{aligned} \langle v^*, v \rangle &\stackrel{!}{=} \left\langle \sum_{k=1}^n \langle v^*, b_k \rangle b_k^*, v \right\rangle \\ \left\langle \sum_{k=1}^n \langle v^*, b_k \rangle b_k^*, v \right\rangle &= \sum_{k=1}^n \langle v^*, b_k \rangle \langle b_k^*, v \rangle \\ &= \sum_{k=1}^n \langle v^*, b_k \rangle \left\langle b_k^*, \sum_{i=1}^n v_i b_i \right\rangle \\ &= \sum_{k=1}^n \sum_{i=1}^n \langle v^*, b_k \rangle \underbrace{\langle b_k^*, b_i \rangle}_{\delta_{ki}} \cdot v_i \\ &= \sum_{k=1}^n \langle v^*, b_k \rangle \langle v^*, b_k \rangle \cdot v_k \\ &= \left\langle v^*, \sum_{k=1}^n v_k b_k \right\rangle \\ &= \langle v^*, v \rangle \end{aligned}$$

- (To be done in the practicals) Consider the functional

$$\langle v^*, b_i \rangle = 1 \Rightarrow v^* \notin L((v_i^*)_{i \in I})$$

□

1.4 Remark and a definition for bilinearity

The mapping $V^* \times V \rightarrow \mathbb{K}$ is linear in v (with fixed v^*) with $(v^*, v) \mapsto \langle v^*, v \rangle$ is linear in v^* (with fixed v). Such a mapping is called *bilinear*.

A mapping $F : V_1 \times \dots \times V_n \rightarrow W$ is called *multilinear* (n -linear) if it is linear in every component. Formally:

$$\begin{aligned} & F(v_1, \dots, v_{k-1}, \lambda v'_k + \mu v''_k, v_{k+1}, \dots, v_n) \\ &= \lambda F(v_1, \dots, v_{k-1}, v'_k, v_{k+1}, \dots, v_n) + \mu F(v_1, \dots, v_{k-1}, v''_k, v_{k+1}, \dots, v_n) \end{aligned}$$

1.5 Example

$V = \mathbb{K}[x]$ polynomials

Basis: $\{x^k \mid k \in \mathbb{N}_0\}$ and $\dim V = \aleph_0$

Every $v^* \in V^*$ is uniquely defined by $a_k := \langle v^*, x^k \rangle$

$$(a_k)_{k \in \mathbb{N}_0}$$

$V^* \cong \mathbb{K}[[t]]$ are the formal power series

$$= \left\{ \sum_{k=0}^{\infty} a_k t^k \mid a_k \in \mathbb{K} \right\}$$

$$\lambda \sum_{k=0}^{\infty} a_k t^k + \mu \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (\lambda a_k + \mu b_k) t^k$$

(Compare with Taylor series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$)

$$\left\langle \sum_{k=0}^{\infty} a_k t^k, \sum_{k=0}^n b_k x^k \right\rangle = \sum_{k=0}^n a_k b_k \text{ is well-defined}$$

$$\rightarrow \mathbb{K}[x]^* \cong \mathbb{K}[[t]]$$

1.6 Example

$C[0, 1]$ continuous functions

Example:

Example 1.

$$x \in [0, 1] \quad \delta_x : C[0, 1] \rightarrow \mathbb{R}$$

$$f \mapsto f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$I(f) = \int_0^1 f(x) dx \text{ is linear}$$

$$\langle I_g, f \rangle = \int_0^1 f(x)g(x) dx$$

$g \in C[0, 1]$ is fixed

$$\Rightarrow I_g \in C[0, 1]$$

$$\langle I_g, \lambda f_1 + \mu f_2 \rangle' = \int_0^1 (\lambda f_1(x) + \mu f_2(x))g(x) dx$$

$$= \lambda \int_0^1 f_1(x)g(x) dx + \mu \int_0^1 f_2(x)g(x) dx$$

This also works with non-continuous g (it suffices to have g integrable). (Compare with measure theory and Riesz' theorem)

Does there exist some g such that $f(x) = \langle \delta_x, f \rangle = \int_0^1 f(t)g(t) dt$. (Compare with Dirac's δ function and Schwartz/Sobder theory)

$$V^{**} = (V^*)^* \cong V \text{ if } \dim V < \infty$$

Lemma 1. Let V be a vector space over \mathbb{K} . It requires that $\dim V < \infty$ and the Axiom of Choice holds.

$$\bullet v \in V \setminus \{0\} \Leftrightarrow \bigvee_{v^* \in V^*} \langle v^*, v \rangle \neq 0$$

- $\bigwedge_{v \in V} v = 0 \Leftrightarrow \bigwedge_{v^* \in V^*} \langle v^*, v \rangle = 0$

Proof. Addition v to a basis B of V : Define $v^* \in V^*$ by

$$\langle v^*, b \rangle = \begin{cases} 1 & b = v \\ 0 & b \neq v \end{cases} \text{ for } b \in B$$

Theorem 2. Let V be a vector space over \mathbb{K} .

- The map $\iota : V \rightarrow V^{**} := (V^*)^*$ is called *bidual space*.

$$\langle \iota(v), v^* \rangle := \langle v^*, v \rangle$$

is linear and injective.

- if $\dim V < \infty$, then isomorphism.

Proof. • Linearity

$$\iota(\lambda v + \mu w) \stackrel{!}{=} \lambda \iota(v) + \mu \iota(w)$$

must hold in every point $v^* \in V^*$:

$$\begin{aligned} \langle \iota(\lambda v + \mu w), v^* \rangle &= \langle v^*, \lambda v + \mu w \rangle \\ &= \lambda \langle v^*, v \rangle + \mu \langle v^*, w \rangle \\ &= \lambda \langle \iota(v), v^* \rangle + \mu \langle \iota(w), v^* \rangle \\ &= \langle \lambda \iota(v) + \mu \iota(w), v^* \rangle \end{aligned}$$

Is it injective? Let $v \in \ker \iota$.

$$\langle \iota(v), v^* \rangle = 0 \quad \forall v^* \in V^*$$

$$\Rightarrow \langle v^*, v \rangle = 0 \quad \forall v^* \in V^*$$

$$\xrightarrow{\text{Lemma 1}} v = 0$$

- Follows immediately, because the dimension is equal.

Definition 1. Let V, W be vector spaces over \mathbb{K} . $f \in \text{Hom}(V, W)$. We define $f^T \in \text{Hom}(W^*, V^*)$ using $f^T(w^*) \in V^*$ via

$$\langle f^T(w^*), v \rangle = \langle w^*, f(v) \rangle = w^*(f(v)) = w^* \circ f(v)$$

$$f^T(w^*) = w^* \circ f \text{ is linear} \Rightarrow f^T(w^*) \in V^*$$

V to W (with f) and W to \mathbb{K} (with w^*).

□ f^T is called *transposed map*.

Example 2. (See practicals) Let $\dim V = n$ and $\dim W = m$ with $B \subseteq V$ and $C \subseteq W$ as bases and dual bases $B^* \subseteq V^*$ and $C^* \subseteq W^*$

$$\Phi_{B^*}^{C^*}(f^T) = \Phi_C^B(f)^T \quad \text{transposition of matrices}$$

This lecture took place on 2nd of March 2016 (Franz Lehner).

2 Determinants

Leibnitz 1693 (3×3 matrices)

Seki Takukazu 1685 (most general version)

Gauß 1801 (“determinant”)

Cayley 1845 (on matrices)

$$n = 2$$

$$ax + by = e$$

$$cx + dy = f$$

$$\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}$$

1. Case 1: $a \neq 0$ (multiply first row $-\frac{a}{b}$ times second row)

$$\begin{array}{cc|c} a & b & \\ c & d & \\ \hline a & b & \\ 0 & d - \frac{bc}{a} & \end{array}$$

□

Unique solution:

$$d - \frac{bc}{a} \neq 0$$

2. Case 2: $c \neq 0$ (multiple second row $-\frac{a}{c}$ times first row)

$$\begin{array}{cc} a & b \\ c & d \\ 0 & b - \frac{ad}{c} \\ c & d \end{array}$$

Unique solution:

$$b - \frac{ad}{c} \neq 0$$

This gives us

$$ad - bc \neq 0$$

Definition 2.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2.1 Properties of determinants

- The determinant is bilinear in the columns and rows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v, w)$$

where v and w are column vectors of A .

$$\det(\lambda v_1 + \mu v_2, w) = \lambda \det(v_1, w) + \mu \det(v_2, w)$$

$$\det(v, \lambda w + \mu w_2) = \lambda \det(v, w_1) + \mu \det(v, w_2)$$

$$\det(\lambda v_1 + \mu v_2, w) = \begin{vmatrix} \lambda a_1 + \mu a_2 & b \\ \lambda c_1 + \mu c_2 & d \end{vmatrix}$$

$$= (\lambda a_1 + \mu a_2)d - (\lambda c_1 + \mu c_2)b$$

$$= \lambda(a_1d - c_1b) + \mu(a_2d - c_2b)$$

$$= \lambda \begin{vmatrix} a_1 & b \\ c_1 & d \end{vmatrix} + \mu \begin{vmatrix} a_2 & b \\ c_2 & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

- $\det(v, v) = 0$.

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ac = 0$$

-

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(e_1, e_2) = 1$$

Theorem 3. The properties 1–3 of determinants (see above) characterize the determinant.

Let $\varphi : \mathbb{K}^2 \times \mathbb{K}^2 \rightarrow \mathbb{K}$

- bilinear
- $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$
- $\varphi(e_1, e_2) = 1$. Then it holds that $\varphi = \det$.

Proof. To show: $\varphi(v, w) = \det(v, w) \forall v, w \in \mathbb{K}^2$

$$v = \underbrace{ae_1 + ce_2}_{\begin{pmatrix} a \\ c \end{pmatrix}} \quad w = \underbrace{be_1 + de_2}_{\begin{pmatrix} b \\ d \end{pmatrix}}$$

$$\varphi(v, w) = \varphi(ae_1 + ce_2, be_1 + de_2)$$

$$= a\varphi(e_1, be_1 + de_2) + c \cdot \varphi(e_2, be_1 + de_2)$$

$$= ad \underbrace{\varphi(e_1, e_2)}_{=1} + \underbrace{ab\varphi(e_1, e_1)}_{=0} + cb\varphi(e_2, e_1) + cd \underbrace{\varphi(e_2, e_2)}_{=0}$$

□

Lemma 2. From (i) bilinearity and (ii) $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$ it follows that

$$\bigwedge_{v, w \in \mathbb{K}^2} \varphi(v, w) = -\varphi(w, v)$$

$$\begin{aligned} 0 &\stackrel{(ii)}{=} \varphi(v+w, v+w) \stackrel{(i)}{=} \varphi(v, v) + \varphi(v, w) + \varphi(w, v) + \varphi(w, w) \\ &\stackrel{(ii)}{=} \varphi(v, w) + \varphi(w, v) \end{aligned}$$

2.2 Geometric interpretation of the determinant

Consider an area with w defining its breath and v its depth (hence the area spanning vectors). Let e_1 and e_2 be the spanning vectors of a rectangle corresponding to the parallelogram. $\det(v, w)$ is the surface of the spanned parallelogram. The sign defines the orientation of the pair (v, w) .

$$\det(e_1, e_2) = 1 \quad \det(e_2, e_1) = -1$$

There are surfaces where the surface is infinite if you follow a vector in some direction:

- Möbius strip
- Klein's bottle (named after Felix Klein)

$$A = |v| \cdot h$$

Consider Figure 1. h is the length of the projection of w to v^\perp .

$$\begin{aligned} v = \begin{pmatrix} a \\ b \end{pmatrix} &\rightarrow \vec{n} = \begin{pmatrix} -b \\ a \end{pmatrix} \\ \left\langle \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle &= ad - bc \end{aligned}$$

Second proof. $A(v, w)$ satisfies properties (i)–(iii).

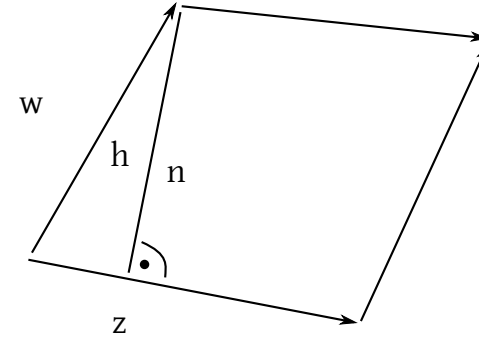


Figure 1: Parallelogram

- Property (iii) follows immediately (the area of unit vectors in two dimensions is 1).
- Property (ii) follows immediately (the area of two vectors in the same direction is 0).

Property (i) defines the linearity in v

1. If v, w are linear dependent, then $A(v, w) = 0$ (one is a multiple of the other)
2. $n \in \mathbb{N}$ with $A(nv, w) = nA(v, w)$

3. For $\tilde{v} = n \cdot v$:

$$A(\tilde{v}, w) = n \cdot A\left(\frac{\tilde{v}}{n}, w\right)$$

$$\Rightarrow A\left(\frac{\tilde{v}}{n}, w\right) = \frac{1}{n} A(\tilde{v}, w)$$

$$A(nv, w) = nA(v, w)$$

$$A\left(\frac{1}{n}v, w\right) = \frac{1}{n}A(v, w)$$

$$A\left(\frac{m}{n}v, w\right) = \frac{m}{n}A(v, w)$$

$$A(-v, w) = -A(v, w)$$

From continuity it follows that $A(\lambda v, w) = \lambda A(v, w)$ for $\lambda \in \mathbb{R}$. Analogously $A(v, \lambda w) = \lambda A(v, w)$.

4. The sum is given with

$$A(v + w, w) = A(v, w)$$

Compare with Figure 2, where $\text{area}(2) + \text{area}(3) = \text{area}(2) + \text{area}(1)$.

$$\begin{aligned} A(\lambda v + \mu w, w) &= A\left(\lambda v + \mu w, \frac{1}{\mu} \mu w\right) \\ &= \frac{1}{\mu} A(\lambda v + \mu w, \mu w) \\ &= \frac{1}{\mu} A(\lambda v, \mu w) \\ &= A(\lambda v, w) \end{aligned}$$

General case: v, w are linear independent and therefore basis of \mathbb{R}^2 . Besides that, v_1 and v_2 are arbitrary.

$$v_1 = \lambda_1 v + \mu_1 w$$

$$v_2 = \lambda_2 v + \mu_2 w$$

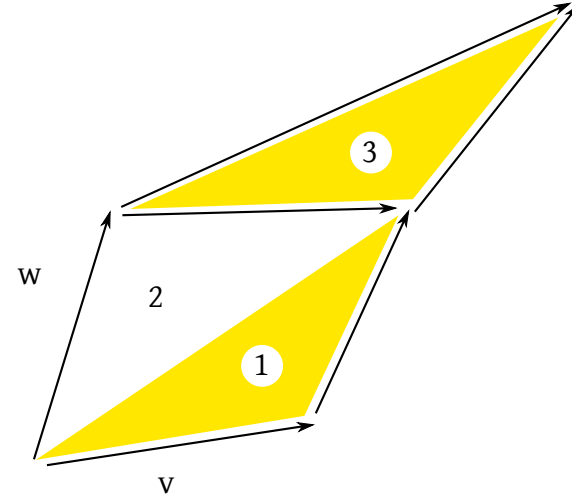


Figure 2: Translation of area 1 to area 3.

$$\begin{aligned} A(v_1 + v_2, w) &= A(\lambda_1 v + \mu_1 w + \lambda_2 v + \mu_2 w, w) \\ &= A((\lambda_1 + \lambda_2)v + (\mu_1 + \mu_2)w, w) \\ &= A((\lambda_1 + \lambda_2)v, w) \\ &= (\lambda_1 + \lambda_2)A(v, w) \\ &= A(\lambda_1 v, w) + A(\lambda_2 v, w) \end{aligned}$$

$$A(\lambda_1 v + \mu_1 w, w) + A(\lambda_2 v + \mu_2 w, w) = A(v_1, w) + A(v_2, w)$$

Additivity follows.

□

Definition 3. Let $\dim V = n$. A determinant form is a map

$$\Delta : V^n \rightarrow \mathbb{K}$$

with properties:

1.

$$\bigwedge_{\lambda} \bigwedge_k \bigwedge_{a_1, \dots, a_n \in V} \Delta(a_1, \dots, a_{k-1}, \lambda a_k, a_{k+1}, \dots, a_n) = \lambda \Delta(a_1, \dots, a_k, \dots, a_n)$$

2.

$$\begin{aligned} \bigwedge_k \bigwedge_{\substack{a_1, \dots, a_n \\ a'_k, a''_k}} \Delta(a_1, \dots, a_{k-1}, a'_k + a''_k, a_{k+1}, \dots, a_n) \\ := \Delta(a_1, \dots, a_{k-1}, a'_k + a''_k, a_{k+1}, \dots, a_n) \end{aligned}$$

3.

$$\Delta(a_1, \dots, a_n) = 0$$

if $\bigvee_{k \neq l} a_k = e_l$ if $\Delta \neq 0$, i.e. Δ is non-trivial.

Multilinearity is defined by the first two properties. Multilinearity means linearity in a_k if $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ get fixed.

Theorem 4.

$$\dim V = n$$

$$\Delta : V^n \rightarrow \mathbb{K} \text{ is determinant form}$$

Then,

4.

$$\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{i \neq j} \Delta(a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, a_n) = \Delta(a_1, \dots, a_i, \dots, a_n)$$

“Addition of λa_j to a_i does not change Δ ”

5.

$$\begin{aligned} \bigwedge_{i > j} \Delta(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n) \\ = -\Delta(a_1, \dots, a_j, \dots, a_i, \dots, a_n) \end{aligned}$$

“Exchanging a_i with a_j inverts the sign”

Proof. 4.

$$\Delta(a_1, \dots, a_i + \lambda a_j, \dots, a_n)$$

Without loss of generality: $i < j$. From properties 1 and 2 it follows that:

$$= \Delta(a_1, \dots, a_i, a_j, a_n) + \lambda \Delta(a_1, \dots, a_j, a_j, \dots, a_k)$$

Oh, a_j occurs twice! Once at index i and once at index j .

$$= 0$$

due to property 3.

5.

$$\begin{aligned} 0 &\stackrel{\text{property 3}}{=} \Delta(a_1, \dots, a_{i-1}, a_i + a_j, \dots, a_{j-1}, a_i + a_j, \dots, a_n) \\ &= \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_i, \dots, a_{j-1}, \mathbf{a}_i, \dots, a_n) = \mathbf{0} \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_i, \dots, a_{j-1}, \mathbf{a}_j, \dots, a_n) \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_j, \dots, a_{j-1}, \mathbf{a}_i, \dots, a_n) \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_j, \dots, a_{j-1}, \mathbf{a}_j, \dots, a_n) = \mathbf{0} \\ &\Rightarrow \delta \end{aligned}$$

□

Definition 4. A permutation of order n is a bijective mapping $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$.

$\sigma_n =$ set of all permutations

Remark 1. Notation: We write the elements in the first row and their images in the second row.

Definition 5. σ_n constitutes (in terms of composition) a group with neutral element id , the so-called symmetric group.

In the previous course (Theorem 1.40) we have proven: Compositions of bijective functions are bijective. 1.

Remark 2. For $n \geq 3$, σ_n is non-commutative

Theorem 5.

$$|\sigma_n| = n!$$

Remark 3. These are “a lot”!

Example 3.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Definition 6. A transposition is a permutation of the structure

$$\tau = \tau_{ij} : \begin{array}{l} i \mapsto j \\ j \mapsto i \text{ if } k \notin \{i, j\} \\ k \mapsto k \end{array}$$

Then $\tau_{ij}^{-1} = \tau_{ij}$, hence $\tau_{ij}^2 = \text{id}$.

Theorem 6. σ_n is generated by transpositions. With other words, every permutation π can be represented as composition of transpositions

$$\pi = \tau_1 \circ \dots \circ \tau_k$$

Proof.

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

If $\pi = \text{id}$,

$$\pi = \pi \circ \tau := \text{id}$$

If $\pi \neq \text{id}$,

$$k_1 = \min \{k \mid k \neq \pi(k)\}$$

$$\tau_1 = \tau_{k_1 \pi(k_1)}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & \dots & k-1 & k_1 & \dots \\ 1 & \dots & k-1 & k_1 & \dots \end{pmatrix}$$

Example: Consider $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$.

$$k_1 = 2$$

$$\tau_1 = \tau_{23}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}$$

2.

$$k_2 = \min \{k \mid k \neq \pi_1(k)\} > k_1$$

$$\tau_2 = \tau_{k_2, \pi(k_2)}$$

And so on and so forth. $k_j > k_{j-1}$ ends after $\leq n$ steps.

$$\tau_k \circ \tau_{k-1} \circ \dots \circ \tau_1 \circ \pi = \text{id}$$

$$\Rightarrow \pi = \tau_1 \circ \tau_2 \circ \dots \circ \tau_k$$

Regarding the example:

$$k_2 = 3$$

$$\tau_2 = \tau_{35}$$

$$\pi_2 = \tau_2 \circ \pi_1 = \tau_2 \circ \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}$$

$$k_3 = 5 \quad \tau_3 = \tau_{57}$$

$$\Rightarrow \pi = \tau_{23} \circ \tau_{35} \circ \tau_{57}$$

□

Definition 7. A malposition of π is a pair (i, j) such that $i < j$ with $\pi(i) > \pi(j)$. Let F_π be the set of malpositions of π .

$$f_\pi := |F_\pi|$$

$$\text{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

Example 4.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$$

$$F_\pi = \{(2, 7), (3, 4), (3, 7), (4, 7), (5, 6), (5, 7), (6, 7)\}$$

$$f_\pi = 7 \quad \text{sign}(\pi) = -1$$

German keywords

Bidualraum, 7
Bilineare Abbildung, 5
Determinantenform, 13
Determinante, 9
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