Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

Lukas Prokop

March to July 2016

Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$
$$\exp(z+w) = \exp(z)\cdot\exp(w)$$
$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n} \in \mathbb{Q} \land n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote
$$\exp(z) = e^z \qquad \text{for } z \in \mathbb{C}$$

It holds that
$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$
$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.
$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,
$$e^0 = 1$$

Let $n \in \mathbb{N}$
$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$
$$\lim_{x\to-\infty} e^x \cdot x^n = 0$$

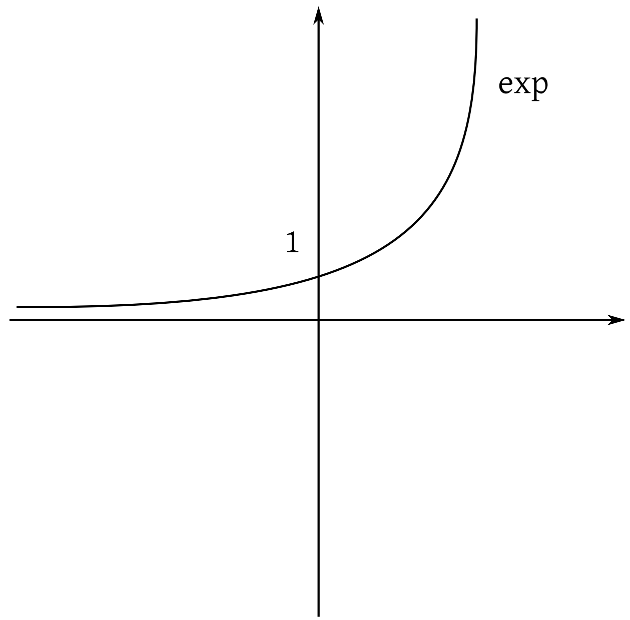

Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp : \mathbb{R} \to (0,\infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. $\exp : \mathbb{R} \to (0,\infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course “Analysis 1”.

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y]: \quad e^\xi = y$$

Case 2 Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w]: \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp : \mathbb{R} \to (0, \infty)$ is bijective. $\square$

Definition 1. We call the inverse function *natural logarithm*[1].

$$\exp^{-1} : (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R} : \ln(e^x) = x$ and $\forall y \in (0, \infty) : e^{\ln(y)} = y$.
- $\ln : (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

  Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xRightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! $\square$

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Limes:
$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Proof.

- $$x \cdot y = e^{\ln(x \cdot y)}$$
$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$
Injectivity of exp:
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.
$$\lim_{n \to \infty} y_n = \ln(1) = 0$$
$$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$
where
$$e^0 = 1 \Rightarrow \ln(1) = 0$$

$\square$

Theorem 1 (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. $\square$

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

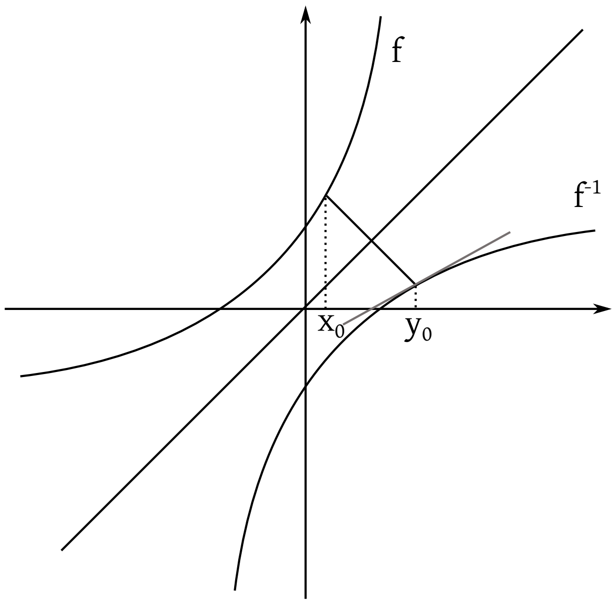

Figure 2: A geometric proof of differentiability

Proof. **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$= \frac{1}{x}\cdot\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x}\to 0$$

$1+\frac{h}{x} = w$ then it holds that $h\to 0 \Rightarrow w\to 1$.

$$\frac{h}{x} = w-1$$

$$\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{=}\lim_{h\to 0}\frac{\ln(w)}{w-1} = 1$$

□

Remark 1. The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C}\to\mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C}\setminus\{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C}\setminus\{x\in\mathbb{R} : x\le 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I\to\mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I)\to\mathbb{R}$ is defined and the inverse function.

Let f in $x_0\in I$ be differentiable and $f'(x_0)\neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

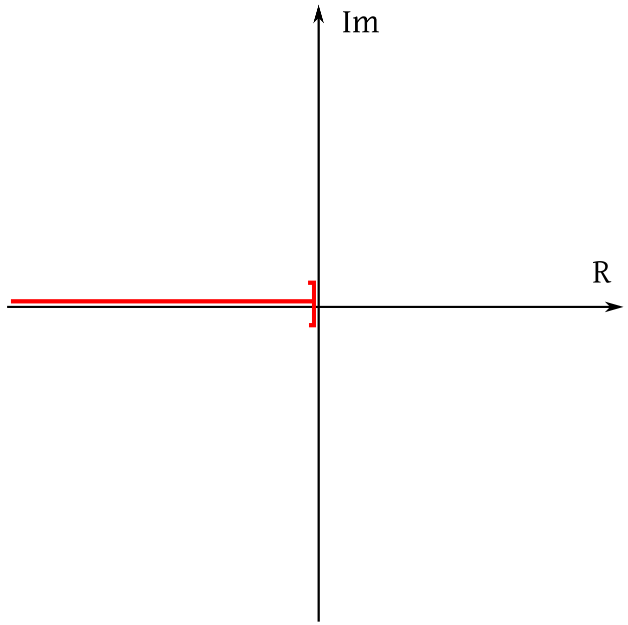

Figure 3: Continuous exponential function in $\mathbb{C}$

Proof. Let $y_n\to y_0$ and $y_n\in f(I)$; $y_0 = f(x_0)$; $y_0\in f(I)$; $y_n = f(x_n)$. $y_n\neq y_0\Rightarrow x_n\neq x_0$.

$$\lim_{n\to\infty}\frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}$$

$$= \lim_{n\to\infty}\frac{x_n-x_0}{f(x_n)-f(x_0)} = \frac{1}{\lim_{n\to\infty}\underbrace{\frac{f(x_n)-f(x_0)}{x_n-x_0}}_{\text{ex}=f'(x_0)}} = \frac{1}{f'(x_0)}$$

□

Lemma 2. Let $f : I\to\mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{ const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0\forall x\in I$$

Proof. $\Rightarrow$ Immediate.

$\Leftarrow$ Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

$\square$

Definition 2. Let I be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

$\square$

Theorem 4. Let I be an interval. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n\in\mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every x)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon\, |u - v|\, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\begin{aligned}
&\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| \\
&= \left|\frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0)\right| \\
&= \left|\frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0)\right| + \left|f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0)\right| \forall N \in \mathbb{N} \\
&\leq \left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right| \\
&+ \left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0)\right| + |f_N'(x_0) - \varphi(x_0)|
\end{aligned}$$

1st term

$$\left|\frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0}\right| = \left|\frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0}\right|$$

$$\leq \frac{\varepsilon}{3}\frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left|\frac{f_N(x)-f_N(x_0)}{x-x_0}\right| - f_N'(x_0)$$

Differentiability of f_N: Therefore for $|x-x_0|<\delta$.

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0)=\varphi(x_0)$. □

Theorem 5. Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ $(n \in \mathbb{N})$ and f_n is differentiable in I.

Assumption:

1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f_n'(x) \to \varphi(x)$ for every x
- $\forall\varepsilon>0 \forall u,v\in I \exists N : n \geq N \Rightarrow |(f_n-f)(u)-(f_n-f)(v)| < \varepsilon\,|u-v|$

Then it holds that f is differentiable and $f'(x)=\varphi(x)\forall x\in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n\to\infty$
- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in I, hence $\forall\varepsilon>0\exists N\in\mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f_n'(x)-g(x)|<\varepsilon$

Then f is differentiable in I and $f'(x)=g(x)\forall x\in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6. We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence
$$\forall x \in I : f_n'(x) \to g(x)$$
3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon>0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:
$$|f_n'(x)-g(x)| < \frac{\varepsilon}{2}$$
Choose $n,m \geq N$ and $x\in I$ arbitrary. Then it holds that
$$\begin{aligned}|f_n'(x)-f_m'(x)| &= |f_n'(x)-g(x)+g(x)-f_m'(x)|\\ &\leq |f_n'(x)-g(x)|+|g(x)-f_m'(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$
So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.
Let $\varepsilon>0$ be arbitrary and N such that $n,m\geq N$ and $x\in I$:
$$|f_n'(x)-f_m'(x)|<\varepsilon$$
Consider the third condition of Theorem 6. Let $u,v\in I$
$$|(f-f_n)(u)-(f-f_n)(v)| = \lim_{m\to\infty} |(f_m-f_n)(u)-(f_m-f_n)(v)|$$
where (f_m-f_n) and (f_m-f_n) is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)
$$\begin{aligned}&= \lim_{m\to\infty} |(f_m-f_n)'(\xi_{m,n})\cdot(u-v)|\\ &= \lim_{m\to\infty} |f_m'(\xi_{m,n})-f_n'(\xi_{m,n})|\cdot|u-v|\end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

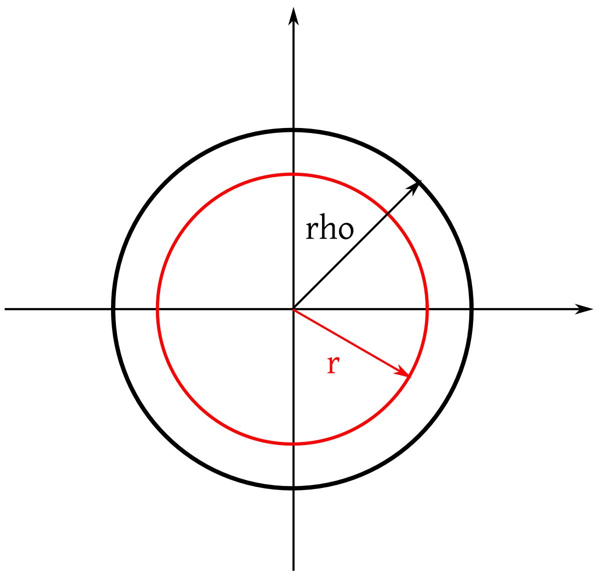

Figure 4: Convergence radius

Remark 2 (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

□

Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \limsup_{j\to\infty} \underbrace{\left(|a_{j+1}|^{\overbrace{\frac{1}{j+1}}^{\to 1}} \right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty} \left[(j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

[2]Where overline means “closed”

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$

Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\Rightarrow 1 + x = w - wx$$

$$x(1+w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

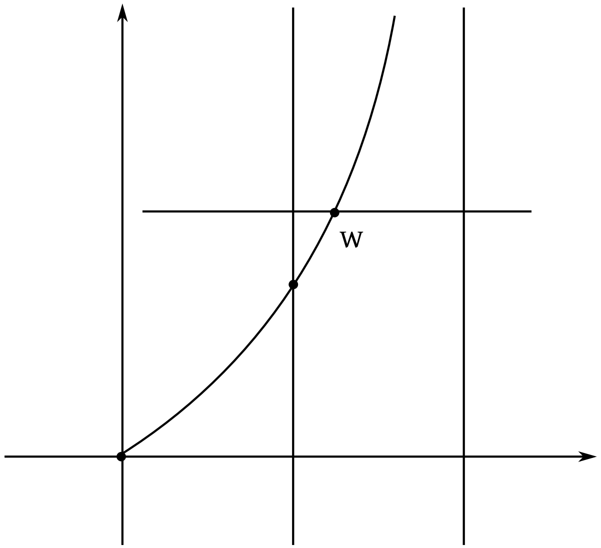

Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[\overline{\left(1 + \frac{it}{n}\right)} \right]^n$$

$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

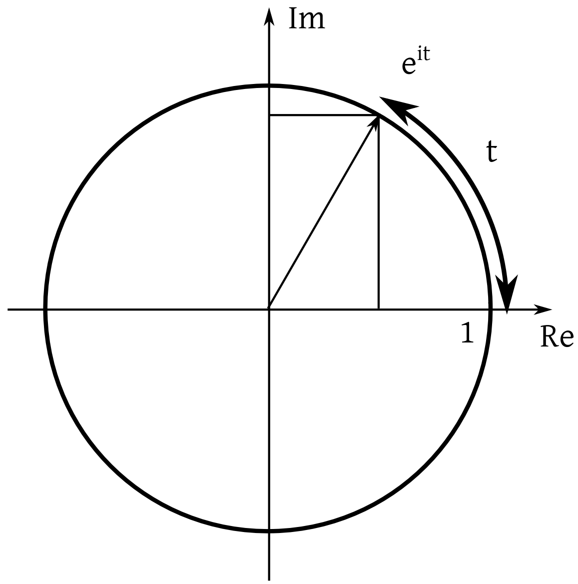

Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$
$$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$
$$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$
$$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$
$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$

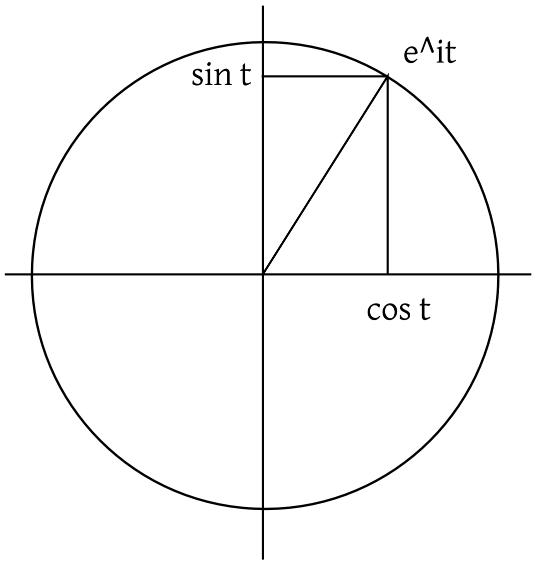

Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$\begin{aligned} z \in \mathbb{C} : \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \end{aligned}$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

series sum. Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

□

Example 1 (Application)**.** Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0, \delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0, \delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \ldots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \ldots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \ldots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots$$

Followingly,

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C} \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \ldots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!} \end{aligned}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and substraction theorems)**.** We derive them directly:
Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i\sin(z+w)) \\ &\Rightarrow= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z c\dot{c}os w) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i\left(-\cos z \sin w - \cos w \sin z\right) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i\sin(-(z+w))) \\ &\Rightarrow= \cos\left(z+w\right) - i\sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2\cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\begin{aligned} \cos(z-w) &= \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)} \\ \sin(z-w) &= -\cos z \cdot \sin(w) + \sin(z)\cos(w) \end{aligned}$$

Corollary 1.

$$\begin{aligned} z &= \frac{1}{2}(z+w) + \frac{1}{2}(z-w) \\ \Rightarrow \cos z &= \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2} \\ w &= \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w) \\ \cos w &= \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2} \\ \cos z - \cos w &= -2\sin\frac{z+w}{2}\sin\frac{z-w}{2} \end{aligned}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z\to 0\\ z\neq 0}} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right) \\ &= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz} - 1}{2iz}\right) \\ &= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z\to 0}\frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz; z\to 0 \Leftrightarrow w=0\\ \lim_{w\to 0}\frac{e^w-1}{w}=1}} \end{aligned}$$

So it holds that

$$\lim_{z\to 0}\frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled "definition of π" and "periodicity".
Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \dots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{x^2}{(2k+2)(2k+1)}\right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k\geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left|\frac{s_{k+1}}{s_k}\right| = \left|\frac{x^2}{(2k+2)(2k+3)}\right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{ in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$

Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1-\frac{4}{6}=\frac{1}{3}>0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = \underbrace{-2 \cdot \underbrace{\sin \underbrace{\frac{z+w}{2}}_{\in(0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in(0,2]}}_{>0}}_{<0}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that cos is a strictly monotonically decreasing functionin $[0,2)$. Hence cos has only one root because it is continuous in $(0,2]$.

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos\frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin\frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$e^{i\pi} = e^{i\frac{\pi}{2}+i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi+\frac{i}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$
$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{iz+2\pi i} + e^{-iz-2\pi i}\right) \\ &= \frac{1}{2}\left(e^{iz} + e^{-iz} \cdot \frac{1}{\underbrace{e^{2\pi i}}_{=1}}\right) = \cos z\end{aligned}$$

Therefore the cosine is periodic in $\mathbb{C}$ with period 2π. Analogously, sine is periodic in $\mathbb{C}$ with period 2π.

German keywords

English keywords