1 Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

Ω is the set of outcomes (n-tuples, 1 of 365 days for n people)

$$\Omega = \{(d_1, d_2, \ldots, d_n) : 1 \leq d_i \leq 365 \,\forall i \in \{1, \ldots, n\}\}$$

$\mathcal{A}$ is the set of events (power set of Ω)

$$\mathcal{A} = \mathcal{P}(\Omega)$$

$$\begin{aligned}
\{\} &\in \mathcal{A} \\
\Omega &\in \mathcal{A} \\
\{(d_1, d_2, \ldots, d_n) : d_i = d_j \text{ for any } i \neq j\} &\in \mathcal{A} \\
\{(d_1, d_2, \ldots, d_n) : d_i \neq d_j \forall i \neq j\} &\in \mathcal{A}
\end{aligned}$$

$\mathbb{P}$ is the probability measure for given $A \in \mathcal{A}$

$$\mathbb{P}(\{\}) = 0$$

$$\mathbb{P}(\Omega) = 1$$

$$\mathbb{P}(\{(d_1, d_2, \ldots, d_n) : d_i \neq d_j \forall i \neq j\}) = \frac{n! \cdot \binom{365}{n}}{365^n}$$

$$\mathbb{P}(\{(d_1, d_2, \ldots, d_n) : d_i = d_j \text{ for any } i \neq j\}) = 1 - \frac{n! \cdot \binom{365}{n}}{365^n}$$

where $\binom{365}{n}$ is the number of possibilities to assign n people to 365 dates without collision. Once you have chosen n slots, there are $n!$ ways to permute the specific assignment for person 1 to n. As usual we divide the number of desired outcomes by the number of possible outcomes. The possible outcomes are left. We have 365^n ways to assign one of 365 days to n people.

Whiteboard solution:

$$\mathbb{P}((x_1, \ldots, x_n)) = 365^n$$

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 1)$$

Event $A = \{\, (x_1, \ldots, x_n) \in \Omega \mid \text{at least two } x_i \text{ are equal}\}$.

$$A^C = \{\, (x_1, \ldots, x_n) \in \Omega \mid x_i \neq x_j, j \neq i\}$$

$$\mathbb{P}(A^C) = \frac{|A^C|}{|\Omega|} = \frac{365 \cdot 364 \cdot \cdots \cdot (365 - n + 1)}{365^n}$$

2 Find n such that $\mathbb{P}(A) > 0.5$ with $|A| = 1$

Using R:

```
# via https://stackoverflow.com/a/40527881
ramanujan <- function(n){
  n*log(n) - n + log(n*(1 + 4*n*(1+2*n)))/6 + log(pi)/2
}
bignchoosek <- function(n,k){
  exp(ramanujan(n) - ramanujan(k) - ramanujan(n-k))
}
f <- function (n) { 1 - factorial(n) * bignchoosek(365, n)/365^n }
f(5)
# [1] 0.02713187
for (i in 1:365) {
  if (f(i) > 0.5) {
    print(i)
    break
  }
}
# [1] 23
```

Using Python:

```
>>> import math
>>> fac = math.factorial
>>> f = lambda n: 1 - fac(n) * (fac(365) / (fac(n) * fac(365 - n))) / 365.0**n
>>> for i in range(1,365):
...     if f(i) > 0.5:
...         print(i)
...         break
...
23
```

Answer: 23

3 Find n such that $\mathbb{P}(A) > 0.99$ with $|A| = 1$

Using R:

```
for (i in 1:365) {
  if (f(i) > 0.99) {
    print(i)
    break
  }
}
# [1] 57
```

Using Python:

```
>>> for i in range(1,365):
...     if f(i) > 0.99:
...         print(i)
...         break
...
57
```