Linear Algebra 2
List of potential exam questions

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7 Determinants

- Give the three determinant properties
- Give the definition of determinant forms and multilinearity
- Prove $\triangle(a_1, \ldots, a_k + \lambda a_i, \ldots, a_n) = \triangle(a_1, \ldots, a_k, \ldots, a_n) \forall \lambda \in \mathbb{K}, \forall i \neq k$
- Prove $\triangle(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = -\triangle(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n)$
- Define permutations
- Define transpositions
- Prove: Every permutation can be represented as product of transpositions
- Is the decomposition of a permutation as transpositions unique?
- Define and determine the signature of a permutation
- Prove $\text{sign}(\pi) = \prod_{\substack{i,j \\ i<j}} \frac{\pi(j) - \pi(i)}{j - i}$
- Prove: every transposition has sign -1
- Prove: $\text{sign}(\pi \circ \sigma) = \text{sign}(\pi) \cdot \text{sign}(\sigma)$
- Prove: $\forall \sigma \in \sigma_n : \triangle(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \text{sign}(a) \cdot \triangle(a_1, \ldots, a_n)$
- Define Leibniz' formula for determinants
- Prove Leibniz' formula
- Prove: Let B and C be two bases of a vector space. The determinant of B is non-trivial iff the determinant of C is non-trivial.
- Prove: $\triangle$ non-trivial $\iff \triangle(b_1, \ldots, b_n) \neq 0$ for every basis
- Prove: Let $\triangle$ be a non-trivial determinant form $\triangle(v_1, \ldots, v_n) \neq 0 \iff v_1, \ldots, v_n$ is linearly independent.
- Prove: Two determinant forms are different only by some factor
- Prove: $f : V \to V$ is invertible $\iff \det(f) \neq 0$.
- Prove: For a matrix $A \in \mathbb{K}^{n \times n}$ it holds that $\det A \neq 0 \iff A$ has full rank.
- Prove: $f, g : V \to V$ linear. $\implies \det(f \circ g) = \det(f) \cdot \det(g)$
- Prove $\det(A \cdot B) = \det(A) \cdot \det(B)$ directly
- Prove: $\det(A^{-1}) = \frac{1}{\det(A)}$ if invertible
- Prove: $\det(A) = 0 \iff \text{rank}(A) < n$
- Prove: $\det(A^t) = \det(A)$

- Define $\mathrm{perm}(A)$

- Prove: Let A be an upper triangular matrix, hence $a_{ij} = 0$ if $i > j$. $\implies \det(A) = a_{11}a_{22}\ldots a_{nn}$.

- Prove: $\begin{vmatrix} & & & 0 \\ & B & & 0 \\ & & & 0 \\ * & * & * & a_{nn} \end{vmatrix} = \det(B) \cdot a_{nn}$

- Define Laplace Expansion

- Define the cofactor of a matrix

- Define the complementary matrix

- Prove: $A^{-1} = \frac{1}{\det A}\hat{A}$

- Give and prove Cramer's rule

8   Inner products

- Give a geometrical proof of the Pythagorean theorem

- Define the scalar product in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Prove $\langle a, b \rangle = \langle b, a \rangle$, $\langle \lambda a, b \rangle = \lambda \langle a, b \rangle = \langle a, \lambda b \rangle$ and $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$ in $\mathbb{R}^2$

- Prove that the scalar product in $\mathbb{R}^3$ is the dot product.

- Give the law of Cosines

- Define an outer product (in $\mathbb{R}^3$)

- Prove $b \times a = -a \times b$, $(\lambda a) \times b = \lambda(a \times b) = a \times (\lambda b)$ and $(a + b) \times c = a \times c + b \times c$

- Define bilinearity

- Define antisymmetry

- Define an inner product

- The function value of an inner product is an element of which domain?

- Define positive/negative-(semi)definite and indefinite inner products (in terms of inner products and in terms of matrices).

- Define a scalar product in terms of inner products.

- What is a positive definite inner product in Hermitian form?

- What is a unitary product?

- When does the scalar product satisfy $\langle a, b \rangle = \langle b, a \rangle$?

- Why is the scalar product defined with sesquilinearity and not bilinearity?

- Define norms

- Every $\square$ induces a $\square$. How?

- Define l-norms

- Prove: $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm on V

- Give and prove: CBS inequality

- Prove: Let V be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $B = \{b_1, \ldots, b_n\}$ be a basis. $\langle, \rangle$ is an inner product. There exists a unique matrix A in Hermitian form such that $\forall x, y \in V : \langle x, y \rangle = \Phi_B(x)^T \cdot A \cdot \overline{\Phi_B(y)}$. Additionally show: If $\langle, \rangle$ is positive definite, A is invertible.

- Define the conjugate transpose of a matrix.

- Define self-adjoint matrices.

- Define symmetrical matrices.

- Define Hermitian matrices.

- Give one simple example each for a (positive/negative) (semi)definite matrix and indefinite matrix.

- Define congruent matrices.

- Prove: Every Hermitian matrix is congruent to a diagonal matrix D of form $\text{diag}(D) = (1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$.

- Give and prove Sylvester's law of inertia.

- Define the index and signature of a matrix.

- Prove: $A > 0 \iff A \hat{=} I \iff \text{ind}(A) = n$

- Prove: $A \geq 0 \iff \text{ind}(A) = \text{sign}(A) = \text{rank}(A)$

- Prove: $A \hat{=} B \iff \text{ind}(A) = \text{ind}(B) \land \text{sign}(A) = \text{sign}(B)$

- Prove: $\det(C^*) = \overline{\det(C)}$

- Prove: $A = A^* \implies \det(A) \in \mathbb{R}$

- Prove: $A = A^*, B = B^*, A \hat{B} \implies \text{sign} \det(A) = \text{sign} \det(B)$

- Prove: $A > 0 \implies \det(A) > 0$

- Define the minor of a matrix.

- Let $A = A^*$. Prove: $A > 0 \iff$ all first minors A_r satisfy $\det(A_r) > 0$

- All submatrices of a positive definite matrix are □

- Let $A = A^*$. Prove: $A < 0 \iff (-1)^r \det(A_r) > 0 \forall r \in \{1, \ldots, n\}$

- What is an Euclidean space? What is a unitary space?

- What is a Hilbert space?

- Give the parallelogram law.

- Define orthogonal and orthonormal families of vectors.

- Define orthonormal bases of vectors.

- Let $(v_i)_{i \in I} \subseteq V$, $v_i \neq 0 \forall i$. Prove: $(v_i)_{i \in I}$ is orthogonal, then $(v_i)_{i \in I}$ is linear independent.

- Let $B = (b_1, \ldots, b_n)$ is an orthonormal basis of an finite-dimensional vector space over $\mathbb{K}$. For $v \in V$, let $\Phi_B(v) = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix}^T$. For $w \in V$, let $\Phi_B(w) = \begin{pmatrix} \mu_1 & \cdots & \mu_n \end{pmatrix}^T$. Prove:
  1. $\lambda_i = \langle v, b_i \rangle$
  2. $\langle v, w \rangle = \sum_{i=1}^n \lambda_i \overline{\mu_i}$

- Let V be a vector space with scalar product. $M, N \subseteq V$ are partitions.
  1. $M^\perp$ is a subspace.
  2. $M \subseteq N \implies N^\perp \subseteq M^\perp$
     $(M_1 \cup M_2)^\perp = M_1^\perp \cap M_2^\perp$
  3. $\{0\}^\perp = V$
  4. $V^\perp = \{0\}$
  5. $M \cap M^\perp \subseteq \{0\}$
  6. $M^\perp = \mathcal{L}(M)^\perp$

7. $M \subseteq (M^\perp)^\perp$

- Prove: Let $U \subseteq V$ be a subspace. $U + U^\perp$ is a direct sum in $\mathbb{R}^n$ such that $U + U^\perp = \mathbb{R}^n$.

- Define convexity of functions and sets.

- Let $V = U \dot{+} U^\perp$. Prove: $\forall x, y \in V : \langle x, \pi_U(y) \rangle = \langle \pi_U(x), y \rangle = \langle \pi_U(x), \pi_U(y) \rangle$

- Let $V = U \dot{+} U^\perp$. Prove: $\left\| \pi_U(x) \right\| \leq \|x\| \wedge \left\| \pi_U(x) \right\| = \|x\| \iff x \in U$

- Define the Gram matrix.

- Let $v_1, \ldots, v_m \in V$. $G = \text{Gram}(v_1, \ldots, v_m)$. Prove: $G = G^*$ is Hermitian, positive *semi*definite.

- Let $v_1, \ldots, v_m \in V$. $G = \text{Gram}(v_1, \ldots, v_m)$. Prove: $\xi \in \ker G \iff \sum_{i=1}^{m} \overline{\xi_i} v_i = 0$

- Let $v_1, \ldots, v_m \in V$. $G = \text{Gram}(v_1, \ldots, v_m)$. Prove: G is positive definite iff $(v_1, \ldots, v_m)$ are linear independent.

- Give Bessel's inequality. Give an intuition what the inequality says/when it can be useful.

- Give Parseval's identity. Give an intuition what the inequality says/when it can be useful.

- Give and prove the Gram–Schmidt process for orthogonalization

- Define Laguerre polynomials

- Define Hermite polynomials

- How are Laguerre and Hermite polynomials related to the Gram–Schmidt process?

- Give Riesz representation theorem

- Prove Riesz representation theorem

- Does Riesz representation theorem hold for infinite-dimensional spaces?

- Prove: $v = 0 \iff \forall w \in V : \langle v, w \rangle = 0$

- Prove: $\langle x, y \rangle = \langle x, z \rangle \ \forall x \implies y = z$

- Prove: $\|v\| = \sup \{ |\langle v, w \rangle| \mid \|w\| \leq 1 \}$

- Define adjoint maps

- Prove: Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: For every $v \in V$ the map $w \mapsto \langle T(w), v \rangle_V$ is linear.

- Prove: Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: $\forall v \in V \exists! u \in W \forall w \in W : \langle T(w), v \rangle_V = \langle w, u \rangle_W$ and $T^*(v) = u$.

- Prove: Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Show: $T^* \in \text{Hom}(V, W)$

- Prove: Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be spaces with a scalar product. $\dim V, \dim W < \infty$. $T : W \to V$ linear. Prove: the map $\text{Hom}(W, V) \mapsto \text{Hom}(V, W)$ with $T \mapsto T^*$ is antilinear and $T^{**} = T$.

- Does every linear map have an adjoint map?

- Define involutions

- Prove: Let $B \subseteq V, C \subseteq W$ be orthonormal bases. $f \in \text{Hom}(V, W)$.

$$\Phi_B^C(f^*) = \Phi_C^B(f)^* = \overline{\Phi_C^B(f)^T}$$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $(g \circ f)^* = f^* \circ g^*$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $f^{**} = f$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $\ker f = (\operatorname{im} f^*)^\perp$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: $\operatorname{im} f = (\ker f^*)^\perp$

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: f injective $\iff f^*$ surjective

- Let U, V, W be finite-dimensional. $U \xrightarrow{f} V \xrightarrow{g} W$. Prove: f surjective $\iff f^*$ injective

- Define self-adjoint matrices

- Define linear isometries (= unitary transformations)

- Prove: unitary transformations are injective

- If $\dim V = \dim W < \infty$ and $f : V \to W$ is linear and unitary, then f is invertible and $f^{-1} = f^*$.

- If $\dim V = \infty$, $f : V \to V$ is isometry, it does not imply that f is invertible.

- Define unitary matrices

- Let U be a unitary matrix. Prove: $UU^* = I \iff U^*U = I$.

- Define orthogonal matrices

- Eigenspaces are always $\square$

- Prove: $T \in \mathbb{C}^{n \times n}$ is unitary $\implies T$ is unitary

- Prove: $T \in \mathbb{C}^{n \times n}$. $\forall x, y \in \mathbb{C}^n : \langle Tx, Ty \rangle = \langle x, y \rangle \iff$ the columns of T define an orthonormal basis of $\mathbb{C}^n$

- Define isometries in metric spaces.

- Is translation an isometry?

- Is translation unitary?

- What does the counterclockwise transformation matrix in $\mathbb{R}^2$ look like?

- Define the orthogonal group

- Define the unitary group

- Define the special orthogonal group

- Define the special unitary group

- Define the general linear group

- Define the special linear group

- Prove: $U \in \mathcal{U}(n) \implies |\det(U)| = 1$

- Give the general layout of a quaternion

- What is the multiplicative identity element of quaternions?

9 Polynomials

- Define algebras

- Does an algebra satisfy associativity?

- Does an algebra satisfy commutativity?

- Define Jordan algebras

- Define the polynomial algebra

- Define the algebra of formal power series

- Define the degree of polynomials. Give the degree of polynomials $0, 1, x$ and x^2.

- Prove: Let $p(x)$ and $q(x)$ be polynomials. $\deg(p(x)) + \deg(q(x)) = \deg(g(x) \cdot q(x))$

- Define zero division freedom for polynomials

- Prove: Every polynomial induces a polynomial function and the map $p(x) \mapsto p_f(x)$ is linear and multiplicative.

- Define algebra homomorphisms

- Give and prove the insertion theorem for polynomials

- Define the root of a polynomial

- There is a formula to find roots of any quadratic equation. What is the highest polynomial degree such that a formula exists?

- What is the idea of Cardanos cubic formula?

- Prove polynomial division with remainder: $p(x), q(x) \in \mathbb{K}[x]$, $q(x) \neq 0$. Then there exists exactly one polynomial $s(x), r(x) \in \mathbb{K}[x]$, $p(x) = s(x) \cdot q(x) + r(x)$. with $\deg r(x) < \deg q(x)$.

- Give Ruffini-Horner's method

- Define reducibility of polynomials

- Give the Fundamental Theorem of Algebra

- Define the greatest common divisor for polynomials

- Prove Bezout's identity for polynomials

10 Eigenvalues and eigenvectors

- Define eigenvalues and eigenvectors

- Give an intuitive notion of eigenvalues and eigenvectors

- Can 0 be an eigenvalue? Can $\vec{0}$ be an eigenvector?

- Does every matrix have eigenvalues and eigenvectors?

- Give simple examples in $\mathbb{R}^{2 \times 2}$ and $\mathbb{Q}^{2 \times 2}$ without eigenvectors and eigenvalues.

- Define spectrums of linear maps

- Let $A, B \in \mathbb{C}^{n \times n}$. Prove: $\operatorname{spec}(AB) = \operatorname{spec}(BA)$

- What kind of linear map do you need to consider eigenspaces (necessary, but not sufficient condition)?

- Define eigenspaces of linear maps

- What is the eigenvalue of the matrix $\lambda \cdot \operatorname{id}$?

- Let $b_1, \ldots, b_n$ be a basis of V. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$. Then there exists a linear map f such that $f(b_i) = \lambda_i \cdot b_i$ and this map is $\square$.

- Prove: Left-sided eigenvalue $\iff$ right-sided eigenvalue
- Write it down formally: The spectrum does not depend on the choice of the basis
- Prove: The spectrum does not depend on the choice of the basis
- Define characteristic polynomials of matrices
- Let $A \in \mathbb{K}^{n \times n}$. What is the degree of the characteristic polynomial of A?
- Complete and prove: λ is eigenvalue $\iff \chi_A(\lambda) = \square$
- Prove: A square matrix A is invertible if and only if 0 is not an eigenvalue of A
- Define symmetrical minors
- Prove: $\chi_{T^{-1}AT}(x) = \chi_A(x)$
- Define diagonalizable matrices
- Define equivalent matrices
- Define similar matrices
- Prove: A is diagonalizable $\iff \exists$ basis of eigenvectors.
- Define exponentiation of matrices
- Define Fibonacci sequences
- Give an explanation of the relationship of rabbit populations and the Fibonacci sequence
- Give the iteration matrix of the Fibonacci sequence
- The golden ratio is the root of which polynomial?
- Give the golden ratio
- Prove: eigenvectors corresponding to different eigenvalues are linear independent.
- Prove: an $n \times n$ matrix with n different eigenvalues is diagonalizable.
- Define the geometric and algebraic multiplicity of an eigenvalue
- Prove: A matrix is diagonalizable iff for different eigenvalues $\lambda_1, \ldots, \lambda_r$ it holds that $\sum_{i=1}^{r} d(\lambda_i) = n$
- Give an argument why there are at most n eigenvalues for a matrix in $\mathbb{K}^{n \times n}$
- Prove: For every eigenvalue λ, it holds that $d(\lambda) \leq k(\lambda)$
- Define nilpotent matrices

11   Jordan normal form

- Define invariant subspaces
- Show: Eigenspaces are invariant subspaces
- Prove: Let $A \in \mathbb{K}^{n \times n}, V = \mathbb{K}^n$. If $U \subseteq V$ is invariant and $p(x) \in \mathbb{K}[x]$, then U is invariant under $p(A)$.
- Prove: Let $A \in \mathbb{K}^{n \times n}, V = \mathbb{K}^n$. $U_1, \ldots, U_k$ are invariant subspaces. Then $\bigcap_{i=1}^{k} U_i$ and $\boldsymbol{+}_{i=1}^{k} U_i$ are invariant with respect to A.
- Prove: Let $f : V \to V$ and let $U \subseteq V$ be an invariant subspace. Then $f|_U : U \to U$ is a homomorphism
- Prove: Let $f : V \to V$. If V can be decomposed into a direct sum of invariant subspaces, then A can be transformed into block diagonal form.
- Let $\dim V = n, f \in \text{End}(V)$. Prove: $\{0\} \subseteq \ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \ldots$ and $\text{im } f \supseteq \text{im } f^2 \supseteq \text{im } f^3 \supseteq \ldots$

- Let $\dim V = n, f \in \mathrm{End}(V)$. Prove: $\exists m \leq n : \ker f^m = \ker f^{m+1}$

- Let $\dim V = n, f \in \mathrm{End}(V)$. Prove:

$$\ker f^m = \ker f^{m+1} \iff \operatorname{im} f^m = \operatorname{im} f^{m+1} \iff \ker f^m = \ker f^{m+k} \forall k \geq 1 \iff$$

$$\operatorname{im} f^m = \operatorname{im} f^{m+k} \forall k \geq 1 \iff \ker f^m \cap \operatorname{im} f^m = \{0\} \iff V = \ker f^m \dotplus \operatorname{im} f^m$$

- Explain Fitting's Lemma in a few words.

- Define generalized spaces. Define generalized eigenvectors. How do generalized eigenvectors differ from eigenvectors?

- Prove: Let $\lambda_1, \ldots, \lambda_k$ be different eigenvalues of A and $\ker(\lambda I - A)^{r_i}$ the corresponding generalized spaces where

$$\ker(\lambda_i I - A)^{r_i-1} \subsetneq \ker(\lambda_i I - A)^{r_i} = \ker(\lambda_i \cdot I - A)^{r_i+1}$$

$$\implies \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i - A)^{r_i} \cap \ker((\lambda_1 I - A)^{r_1}(\lambda_2 I - A)^{r_2} \ldots (\lambda_k I - A)^{r_k}) = \{0\}$$

- Prove: $\forall \lambda \neq \mu \in \mathrm{spec}(A) \forall k, l \geq 1 : \ker(\lambda I - A)^k \cap \ker(\mu I - A)^l = \{0\}$

- Prove: The sum $\sum_{i=1}^{k} \ker(\lambda_i I - A)^{r_i}$ is direct for arbitrary pairwise different $\lambda_1, \ldots, \lambda_k$.

- Prove: Let $\lambda_1, \ldots, \lambda_k$ be pairwise different eigenvalues of $A \in \mathbb{K}^{n \times n}$. Let $W := \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i I - A)^n$. $V = \ker(\lambda_1 I - A)^n \oplus \cdots \oplus \ker(\lambda_k I - A)^n \oplus \bigcap_{i=1}^{k} \operatorname{im}(\lambda_i I - A)^n$

- Prove: W is invariant under A and $\lambda_i \notin \mathrm{spec}(A|_W) \forall i \in \{1, \ldots, k\}$

- Prove: Let $\mathbb{K}$ be algebraically closed and let $\lambda_1, \ldots, \lambda_k$ be all eigenvalues of a matrix $A \in \mathbb{K}^{n \times n}$. Then $\mathbb{K}^n = \ker(\lambda_1 I - A)^n \oplus \cdots \oplus \ker(\lambda_k I - A)^n$.

- What is the index of a linear map?

- Define nilpotent matrices.

- The sum of nilpotent matrices is □

- The product of nilpotent matrices is □

- Assume all generalized spaces are eigenspaces. What about diagonizability?

- Prove: Let $\ker(f^m) \subseteq \ker(f^{m+1}) \subseteq \ker(f^{m+2})$

$$u_1 \ldots u_p \ldots \text{ basis of } \ker f^n$$
$$u_1 \ldots u_p v_1 \ldots v_k \ldots \text{ basis of } \ker f^{m+1}$$
$$u_1 \ldots u_p v_1 \ldots v_k w_1 \ldots w_r \ldots \text{ basis of } \ker f^{m+2}$$

Then $(u_1 \ldots u_p, f(w_1), \ldots, f(w_r))$ is linear independent.

- What kind of matrix is the Jordan normal form of a matrix?

- Prove: Let $\dim V = n$. $f : V \to V$ is nilpotent of index p ($f^p = 0$). $d = \dim \ker f$. Then there exists a basis B of V such that

$$\mathrm{diag}\left(\Phi_B^B(f)\right) = ([N_1], [N_2], \ldots, [N_d])$$

where

$$N_i = \begin{bmatrix} 0 & 1 & \ddots & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & \ddots & 0 \end{bmatrix}_{n_i \times n_i} \qquad p = n_1 \geq n_2 \geq \cdots \geq n_d \geq 1 \qquad n_1 + \cdots + n_d = n$$

- Define Jordan blocks of length k of a given eigenvalue λ.

- Prove: Let $\mathbb{K}$ be an algebraically closed field. Then every matrix $A \in \mathbb{K}^{n+m}$ is similar to a matrix of Jordan normal form.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: $\sum_{i=1}^{q} k_i = n$

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: Geometric multiplicity of λ equals the number of corresponding Jordan blocks. Algebraic multiplicity of λ equals the num of sizes of corresponding Jordan blocks.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Show: The smallest exponent r such that $\ker((\lambda I - A)^r) = \ker((\lambda I - A)^{r+1})$ is the largest length of a corresponding Jordan block.

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: Let $k \in \mathbb{N}$. $\#\{i : \lambda_i = \lambda \wedge k_i \geq k + 1\} = \text{rank}(\lambda I - A)^k - \text{rank}(\lambda I - A)^{k+1}$

- Let $B^{-1}AB = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_q \end{bmatrix} \in \mathbb{K}^{n+m}$ be a Jordan normal form with $J_i = J_{k_i}(\lambda_i)$. Prove: The Jordan blocks are uniquely determined (except for the order)

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : \begin{smallmatrix} \mathbb{K}[x] \to \mathbb{K}^{n+n} \\ p(x) \mapsto p(A) \end{smallmatrix}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right)$.

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : \begin{smallmatrix} \mathbb{K}[x] \to \mathbb{K}^{n+n} \\ p(x) \mapsto p(A) \end{smallmatrix}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix}$.

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : \begin{smallmatrix} \mathbb{K}[x] \to \mathbb{K}^{n+n} \\ p(x) \mapsto p(A) \end{smallmatrix}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $p\left(\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{bmatrix}\right) = \begin{bmatrix} p(A_1) & & \\ & \ddots & \\ & & p(A_n) \end{bmatrix}$

- Let $A \in \mathbb{K}^{n+n}$ matrix. $\Psi_A : \begin{smallmatrix} \mathbb{K}[x] \to \mathbb{K}^{n+n} \\ p(x) \mapsto p(A) \end{smallmatrix}$ and $a_0 + a_1 x + \cdots + a_k x^k \mapsto a_0 \cdot I + a_1 A + \cdots + a_k A^k$. Prove: $A = T^{-1}BT \implies p(A) = T^{-1}p(B)T$

- For some Jordan block $J_k(\lambda)$ it holds that,

$$p(J_k(\lambda))_{i,j} = \begin{cases} \frac{p^{(j-i)}(\lambda)}{(j-i)} & j > i \\ p(\lambda) & j = i \\ 0 & j < i \text{ (below the diagonal)} \end{cases}$$

- Let $A \in \mathbb{K}^{N \times N}$. Prove: $\exists p(x) \in \mathbb{K}[x] : p(A) = 0$

- Define the annihilator of a matrix.

- Let $A \in \mathbb{K}^{N \times N}$. Prove: $\exists$ a unique polynomial $m_A(x) \in \mathbb{K}[x]$ with minimal degree and leading coefficients 1 and $p(x) \in$ Ann$(A) \iff m_A(x)\,\big|\,p(x)$

- Let $A \in \mathbb{K}^{N \times N}$. Prove: $m_A(\lambda) = 0\, \forall \lambda \in \operatorname{spec}(A)$

- Give and prove the Cayley-Hamilton Theorem

- Prove: $m_A(x)\,\big|\,\chi_A(x)$

- Conclude: the roots of $m_A(x)$ are the eigenvalues of A

- Recognize: The minimal polynomial has the structure $m_A(x) = \prod (\lambda - \lambda_i)^{m_i}$ where m_i is the smallest exponent for $\ker(\lambda_i - A)^m = \ker(\lambda_i - A)^{m+1}$, hence this equals the largest length of a Jordan block for eigenvalue λ_i.

- Recognize: A is diagonalizable $\iff$ all $m_i = 1 \iff m_A(x) = \prod_{i=1}^{k}(\lambda - \lambda_i) \iff m_A(x)$ has only simple roots.

- $p(A) \cdot x = p(\lambda) \cdot x \qquad$ if $\lambda \in \operatorname{spec}(A) \implies p(\lambda) \in \operatorname{spec}(p(A))$

- What is an important precondition for the spectral mapping theorem?

- Give and prove the spectral mapping theorem

12 Normal matrices

- Define normal matrices

- If a matrix is self-adjoint, then is it necessarily $\square$

- A real-valued self-adjoint matrix is called $\square$

- Are unitary matrices normal?

- Prove: A is normal $\iff$ A is unitarily diagonalizable.

- The sum of normal matrices is $\square$

- The product of normal matrices is $\square$

- $A \in \mathbb{C}^{n \times n}$ is normal. Prove: $\ker A = \ker A^*$

- $A \in \mathbb{C}^{n \times n}$ is normal. Prove: $\ker A = \ker A^2$

- Let $A \in \mathbb{C}^{n \times n}$ is normal. Prove: $\ker(\lambda I - A) = \ker(\overline{\lambda} I - A^*)$

- Let $A \in \mathbb{C}^{n \times n}$ is normal. Prove: $\ker(\lambda I - A)^2 = \ker(\lambda I - A)$

- Let A be normal. Prove: $\lambda \neq \mu \in \operatorname{spec} A \implies \ker(\lambda I - A) \perp \ker(\mu I - A)$

- Matrix A is normal. What can you say about generalized eigenspaces?

- Let $A \in \mathbb{C}^{n \times n}$ be normal. Prove: Eigenvectors of different eigenvalues are orthogonal to each other.

- Let $A \in \mathbb{C}^{n \times n}$. Prove: A is normal $\implies$ $\exists$ orthonormal basis of eigenvectors

- Let $A \in \mathbb{C}^{n \times n}$. Prove: $\exists$ orthonormal basis of eigenvectors $\implies$ $\exists$ unitary matrix U such that $U^*AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

- Let $A \in \mathbb{C}^{n \times n}$. Prove: $\exists$ unitary matrix U such that $U^*AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \implies A$ is normal.

- Define Schur's decomposition

- If $A \in \mathbb{R}^{n \times n}$ and $\chi_A(\lambda)$ decomposes into linear factors. $\exists U \in \mathcal{U}(n) : U^*AU = R$ is an upper triangular matrix. Then $U\,\square$

- Prove: A matrix is normal $\iff$ Schur normal form = diagonal matrix.

- Prove: Let $A \in \mathbb{C}^{n \times n}, A = A^* \implies \operatorname{spec}(A) \subseteq \mathbb{R}$.

- Let $A \in \mathbb{C}^{n \times n}$ self-adjoint. Prove: $A \geq 0 \iff \operatorname{spec}(A) \subseteq [0, \infty)$.

- For which matrices is the square root of a matrix defined?

- Prove: The square root of a positive definite matrix is exists and is unique.
- Define Cholesky decompositions
- Define Schur complements of D in M. Give the dimensions of D and M.
- Let $A \in \mathbb{C}^{n \times n}$. $A > 0$, $b \in \mathbb{C}^n$, $\gamma > 0$.

$$\det \left[\begin{array}{c|c} A & B \\ \hline b^* & \gamma \end{array} \right] = \det A \cdot (\gamma - b^* A^{-1} b)$$

- Cholesky decomposition is a $\square$ decomposition
- Prove: The lower triangular matrix in the Cholesky decomposition is unique
- Prove: If $A > 0$, then $\det A \leq a_{11} a_{22} \ldots a_{nn}$.
- Give and prove Hadamard's inequality
- Which matrices have polar decompositions?
- Define the polar decomposition of a matrix A.
- Prove: Let $A \in \mathbb{C}^{n \times n}$. $|A| := (A^* A)^{\frac{1}{2}}$. Then $\exists U \in \mathcal{U}(n)$ such that $A = U \cdot |A|$.
- Which matrices have singular values?
- Define singular value decompositions of matrices.
- Define the numerical range of a matrix A
- Define the numerical radius of matrix A
- Prove: $\operatorname{spec}(A) \subseteq W(A)$
- Give and prove the Theorem by Toeplitz–Hausdorff
- Define Rayleigh quotients.
- Which requirements are given for matrix A in the Rayleigh-Ritz Theorem and the Courant–Fischer–Weyl min–max principle?
- Give and prove the Rayleigh-Ritz Theorem.
- Give and prove the Courant–Fischer–Weyl min–max principle.
- Which one is more generic: Rayleigh-Ritz Theorem or Courant–Fischer–Weyl min–max principle?
- Give and prove the Cauchy interlacing theorem
- Prove: A, B are self-adjoint $\in \mathbb{C}^{n \times n}$. $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B)$
- Define Geršgorin discs
- Give and prove the Geršgorin theorem
- What is the spectral gap of a matrix?

13 Matrix norms

- Give one scalar product for matrices in $\mathbb{C}^{m \times n}$.
- Define Frobenius norms (= Schur norms) (= Hilbert-Schmidt norms)
- Let V, W be normed vectorspaces. $f \in \operatorname{Hom}(V, W)$. Defined the so-called induced norm.
- Define optimal norms
- Define the spectral radius of a matrix
- What does every compatible matrix norm satisfy with respect to the spectral radius.
- Prove: The spectral radius is not a norm.
- Prove: $\forall A \in \mathbb{C}^{n \times n} \, \forall \varepsilon > 0 \, \exists$ norm on $\mathbb{C}^n$: the induced matrix norm satisfies $\|A\| \leq \rho(A) + \varepsilon$
- Define condition numbers of matrices

14 Non-negative matrices

- Define non-negative matrices
- Give the Perron–Frobenius theorem

15 Generic/recall question

- Which matrices satisfy $AB = BA$?
- Which matrices satisfy $A^{-1} = A$?
- Which matrices satisfy $A^T = A$?
- Which matrices satisfy $AA^* = I$?
- Which matrices satisfy $AA^* = A^*A$?
- Which matrices satisfy $A = A^*$?

Orthogonal complements