# Introduction to Functional Analysis

Lecture notes, University of Technology, Graz
based on the lecture by Martin Holler

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## Contents



# 0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces $U, V$
  matrices $A \in \mathcal{M}^{n\times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
  functions $f : \mathbb{R}^n \to \mathbb{R} \mapsto$ functionals $f : U \to \mathbb{R}$

- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
  - basis of Applied Mathematics
  - partial differential equations
  - physical modelling
  - inverse problems (operator $A$ models some physical measurement process)
  - Optimization and optimal control

A motivating example was presented with slides.

## 0.1 Application examples

Let $K : U \to \mathbb{R}^m$ with $U$ as vector space describe a physical model. For example, $K$ is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0,1] \to \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find $u$. Typically inversion of $K$ is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$ where $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$ and $\|\cdot\|_u$ is a norm on $U$. Or alternatively, let $U = C^1([0,1]^2)$ and solve $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 dx}$.

Other examples are JPEG compression and upsampling of images.

## 0.2 Our first problem

Let $U := C^1([0,1]^2)$ be a normed space, $K : U \to \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \|Ku - d\| + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \implies u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \to \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ linear.

*Proof.* **Case 1 (trivial model):** Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \tag{1}$$

Take $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim_{n\to\infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u\in\mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u\in\mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \, \forall n \in \mathbb{N}$

$$\implies \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i\in\mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \to u$ as $i \to \infty$.

Now: Show that $u$ solves Problem (1). $\nabla$ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u\in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i\to\infty} \left\|\nabla u_{n_i}\right\| + \lambda \left\|u_{n_i} - d\right\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that $\hat{u}$ is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of $\nabla$, $\|\cdot\|_2$

**Case 2 ($K$ arbitrary):** 1. $K$ arbitrary does not provide boundedness anymore. Define $X := \operatorname{kernel}(\nabla) \cap \operatorname{kernel}(k)$ and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^{n} x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \qquad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that $X^\perp$ is called *orthogonal complement.*

**Claim 0.1.** *If $\hat{u}$ solves $\min_{u\in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. Then $\hat{u}$ solves Problem* (1).

*Proof.* Let $\hat{u}$ be a solution on $X^\perp$. Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^\perp$. Now we have:

$$\begin{aligned}
\|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \left\|\nabla(u_1 + u_2)\right\|_2 + \lambda \left\|k(u_1 + u_2) - d\right\|_2 \\
&= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\
&\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2
\end{aligned}$$

Thus $\hat{u}$ solves our problem (1). □

Take again $(u_n)_{n\in\mathbb{N}}$ be such that $u_n \in X^\perp \forall n$ and

$$\lim_{n\to\infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u\in X^\perp} \|\nabla_u\|_2 + \lambda \|ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \operatorname{kernel}(\nabla) + \operatorname{kernel}(\nabla)^\perp$. $\nabla : \operatorname{kernel}(\nabla)^\perp \to \operatorname{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \operatorname{kernel}(\nabla)^\perp \implies v \in$

kernel$(\nabla)$ $\implies$ $\|v_2\| = (v, v) = 0$. Thus, $\nabla^{-1}$ : image$(\nabla) \to$ kernel$(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^2\right\|_2 &= \left\|\nabla^{-1}\nabla u_n^2\right\|_2 = \left\|\nabla^{-1}\right\| \cdot \left\|\nabla u_n^2\right\|_2 \leq \left\|\nabla^{-1}\right\| \\
&\leq \left\|\nabla^{-1}\right\| \left(\left\|\nabla u_n^2\right\|_2 + \lambda \|Ku_n - d\|_2\right) \\
&= \left\|\nabla^{-1}\right\| \left( \underbrace{\|\nabla u_n\|_2}_{=\|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\
&< C \text{ for some } C > 0
\end{aligned}$$

Than $\left\|u_n^2\right\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \to$ image$(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap$ kernel$(\nabla)$ $\implies$ $v \in$ kernel$(K)$. Hence $v \in$ kernel$(K) \cap$ kernel$(\nabla) = X$ $\implies$ $v \in X \cap X^\perp$ $\implies$ $v = 0$. Hence $K^{-1}$ : image$(K) \to X^\perp \cap$ kernel$(\nabla)$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^n\right\|_2 &= \left\|K^{-1}Ku_n^n\right\|_2 \leq \left\|K^{-1}\right\| \left\|Ku_n^n\right\|_2 \\
&= \frac{\|K\|}{\lambda} \left(\lambda \left\|K(u_1^n + u_2^n) - Ku_n^n\right\|_2 + \|\nabla u_n\|_2\right) \\
&\leq \frac{\|K\|}{\lambda} \left( \underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \left\|d - Ku_1^2\right\|}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\
&< D \text{ for some } D > 0
\end{aligned}$$

$\implies$ $(u_n^n)_n$bounded $\implies$ $(u_n) = (u_n^n + u_n^n)_n$ is bounded

$\implies$ $(u_n)_n$ admits a subsequence converging to some $\hat{u}$. As in Case 1, $\hat{u}$ is a solution to Problem (1).

In summary,

1. $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 \, dx}$ with $U = C^1([0,1]^2)$ relevant for application.

2. Discrete version: $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:

   (a) complementary subspaces $X^\perp$

   (b) boundedness and compactness

   (c) continuity

   (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

$\square$

↓ *This lecture took place on 2019/03/07.*

Define $U = C^1([0,1]^2)$. Let $Y$ is some Banach space and $K : U \to Y$ is linear and continuous.

Consider the problem $(P_\infty)$ given by $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2}$.

**Proposition 0.2.** *There exists a solution of $(P_\infty)$.*

*Proof.* Take $(u_n)_{n\in\mathbb{N}}$ as a sequence in $U$ such that $\lim_{n\to\infty} \|\nabla u_1\|_2 + \lambda \|Ku_n - d\|_n \to \inf_{u\in U}(\dots)$. Now we want to show that $(u_n)_{n\in\mathbb{N}}$ is bounded.

**Case 1:** Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\implies \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \le \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\implies (u_n)_{n\in\mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n\in\mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change $U$ to $U = \left\{u : [0,1]^2 \to \mathbb{R} \mid \sqrt{\int_{[0,1]^2}} < \infty\right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } v \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u\in U} R(u) + \lambda \|K_{u-d}\|_2$ instead.

In this setting, $(u_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i\in\mathbb{N}}$).

Our next step is to use continuity to show that $\hat{u}$ is a solution.

Problem: $u \mapsto \|u - d\|_2$ is, in general, not continuous with respect to weak convergence.

*But* it is always true that $\|\hat{u} - d\|_2 \le \liminf_{i\to\infty} \left\|u_{n_i} - d\right\|_L$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \le \liminf_{i\to\infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u\varphi = -\int_0^1 u\partial_x\varphi \forall\varphi \in C^\infty([0,1]^2)$$

$\varphi = 0$ in $K \setminus [0,1]^2$ for some $K \Subset (0,1)^2$.

$$\implies \int_{[0,1]^2} \nabla u\varphi = -\int_{[0,1]^2} u \cdot (\partial_{x_i}\varphi_1 + \partial_{x_2}\varphi_2)$$

$$\forall\varphi : (\varphi_1, \varphi_2) = C^\infty([0,1]^2, \mathbb{R}^2) + \text{ zero on boundary}$$

We define $w : [0,1]^2 \to \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w\varphi = -\int_{[0,1]^2} u(\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2) \text{ holds } \forall \varphi$$

Then $w$ is called *weak gradient* of $u$. We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i\to\infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i\to\infty} R(u_{n_i}) + \liminf_{i\to\infty} \lambda \left\|u_{n_i} - d\right\|_2 \\ &\leq \liminf_{i\to\infty} \Big(R(u_{n_i}) + \lambda \left\|u_{n_i} - d\right\|_2\Big) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

**Case 2:** Works as in the finite-dimensional setting using

- $X := \text{kernel}(A) \cap \text{kernel}(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \; \forall u \in \text{kernel}(\nabla)^\perp$ is called *Poincare inequality.*

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normal spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contemplementary subspaces
8. Hilbert spaces

↓ *This lecture took place on 2019/03/12.*

**Remark.** *1. Literature: UGU, in particular: Biezis, Werner*

*2. In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified*

# 1 Topological and metric spaces

**Remark** (Motivation)**.** *Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies*

**Definition 1.1** (Topology)**.** *Let* $X$ *be a set and* $\tau \subset \mathcal{P}(X) = \{$*"set of subsets of X"*$\}$*. We say that* $\tau$ *is a* topology *on* $X$ *if*

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. *For any collection of sets* $(U_i)_{i \in I}$ *with* $I$ *as some index set. We have* $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$.

$(X, \tau)$ *is called* topological space.

*A set* $U \subset X$ *is called* open *if* $U \in \tau$ *and is called closed if* $U^C \in \tau$.

**Remark.** *By the third property of topologies,* $\bigcap_{i \in I} V_i$ *is closed for any collection* $(V_i)_{i \in I}$ *of closed sets.*

**Definition 1.2** (Metric)**.** *Let* $X$ *be a set,* $d : X \times X \to \mathbb{R}$ *be such that* $\forall x, y, z \in X$

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

*Then* $d$ *is called a* metric on $X$ *and* $(X, d)$ *is called* metric space.

**Definition 1.3** (Norm)**.** *Let* $X$ *be a vector space. A function* $\|\cdot\| : X \to \mathbb{R}$ *is called* norm *if* $\forall x, y \in X, \lambda \in \mathbb{K}$

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\left\|x + y\right\| \leq \|x\| + \left\|y\right\|$

*Then* $(X, \|\cdot\|)$ *is called* normed space.

**Remark.** *If* $\dim(x) < \infty$*, all norms on* $X$ *are equivalent.*

**Example.**

1. *Let* $X$ *be a set then* $\tau = \{\emptyset, X\}$ *is a topology.*
2. $(X, \mathcal{P}(X))$ *is a topological space.*
3. *Define* $S^{d-1} := \left\{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\right\}$ *and* $d(x, y) := r$ *where* $r$ *is the length of the shortest connection between* $x$ *and* $y$ *on* $S^{d-1}$*. Then* $d$ *is a metric on* $S^{d-1}$

4. $X := \{u : [0,1] \to \mathbb{R} \mid u$ *is continuous*$\}$ *then* $\|u\|_\infty := \sup_{x\in[0,1]} \left|u(x)\right|$ *is a norm on* $X$

5. $l^p := \{(X_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{K}\forall u$ *and* $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ *with* $p \in [1,\infty)$ *and* $\left\|(x_i)_{i\in\mathbb{N}}\right\|_p := (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. *Then* $(l^p, \|\cdot\|_p)$ *is a normed space (the proof will be done later).*

**Remark.**

$$L^\infty := \left\{(X_i)_{i\in\mathbb{N}} \mid \sup_{i\in\mathbb{N}} |x_i| < \infty\right\}$$

$$\left\|(X_i)_{i\in\mathbb{N}}\right\| = \sup_i |X_i|$$

**Proposition 1.4.** *Let* $X$ *be a set.*

1. *If* $(X,d)$ *is a metric space, define for* $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ *and* $\tau = \{U \in \mathcal{P}(x) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \in U\}$. *Then* $(X,\tau)$ *is a* topological space. *We say that* $\tau$ *is the topology induced by* $d$ *and we have that* $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$

2. *If* $(X, \|\cdot\|)$ *is a normed space, define* $d : X \times X \to \mathbb{R}$ *with* $(x,y) \mapsto \left\|x-y\right\|$. *Then* $(X,d)$ *is a metric space and* $d$ *is called the metric induced by* $\|\cdot\|$.

**Remark** (Consequence)**.** *Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.*

**Definition 1.5.** *Let* $(X,\tau)$ *be a topological space.* $U \subset X$. $x \in X$.

1. $U$ *is called a neighborhood of* $x$ *if* $\exists V \in \tau - x \in X \subset U : \mathcal{U}(x)$ *is defined as the set of all neighborhoods of* $x$

2. 
   - $x$ *is called* interior point *of* $U$ *if* $U \in \mathcal{U}$
   - $x$ *is called* adjacent point *of* $U$ *if* $\forall V \in \tau$ *such that* $x \in V : V \cap U \neq \emptyset$
   - $x$ *is called* cluster point *of* $U$ *if it is an adjacent point of* $U \setminus \{x\}$.

   *The third property is stronger.*

3. *Notational conventions:*

$$\mathring{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \mathring{U}$$

**Proposition 1.6.** *Let* $(X,\tau)$ *be a topological space,* $U \in X$. *Then*

1. $U$ *is open* $\iff \mathring{U} = U$

2. $U$ *is closed* $\iff \overline{U} = U$

3. $\mathring{U} = \bigcup_{\substack{V\in\tau \\ V\subset U}} V$ *and* $\mathring{U}$ *is open [“*$\mathring{U}$ *is the largest open set in* $U$*”]*

4. $\overline{U} = \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$ *and* $\overline{U}$ *is closed ["*$\overline{U}$ *is the smallest closed set containing* $U$*"]*

*Proof.* 3. $\subset$ Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V$

$\supset$ Let $x \in \bigcup_{\substack{V \in \tau \\ V \in U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \in U \implies x \in \mathring{U}$

$\mathring{U}$ is open because it is the union of open sets.

1. $\implies$ $\mathring{U} \subset U$ by definition. $U$ is open, so $U \subset \bigcup_{\substack{V \subset \tau \\ V \subset U}} V \overset{(3)}{=} \mathring{U}$

2. $\implies$ $V \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \implies x \in U^C \in \tau$ and $U \cap U^C = \emptyset$. This contradicts to $x \in \overline{U}$.

$\impliedby$ Take $x \in U^C = \overline{U}^C$.

$\overset{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$

$\implies V \cap U = \emptyset \implies V \subset U^C$

$\implies U^C$ open $\implies U$ closed

4. We prove the fourth property without the second.

$\subset$ Take $x \in \overline{U}$. Take closed $V$ such that $U \subset V$ if $x \notin V \implies x \in V^C$ which is open and $V^C \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.

$\supset$ Take $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}}$. Suppose $x \notin \overline{U}$.

$\implies \exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$

$\implies U \subset Z^C$, $Z^C$ closed, $x \notin Z^C$. This contradicts to $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$

$\overline{U}$ closed follows since the intersection of closed sets is closed.

$\square$

**Definition 1.7** (Limit)**.** *Let* $(X, \tau)$ *be a topological space,* $(X_n)_{n \in \mathbb{N}}$ *be a sequence in* $X$*. Henceforth, we write* $(X_n)_n$ *for* $(X_n)_{n \in \mathbb{N}}$ *and* $\hat{x} \in X$*. We say* $x_n \to x$ *in* $\tau$ *as* $n \to \infty$ *("*$x_n$ *converges to* $x$*", "*$x$ *is limit of* $x_n$*") if*

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

**Definition 1.8** (Proposition and definition)**.** *Let* $(X, \tau)$ *be a topological space. We say that* $(X, \tau)$ *is* $T_2$ *(or Hausdorff) if*

$$\forall x, y \in X \text{ with } x \neq y \ \exists U, V \in \tau : x \in U, v \in V \text{ and } U \cap V = \emptyset$$

- *In a* $T_2$*-sphere, the limit of any sequence is unique.*
- *If* $\tau$ *is induced by a metric, then* $(X, \tau)$ *is* $T_2$*.*

*Proof.* 1. Take $(x_n)_n$ to be a sequence and assume $x_n$ converges to $\hat{x}$ and $\hat{y}$ with $\hat{x} \neq \hat{y}$. By $T_2$, $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergenc, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\left\{n_x, n_y\right\} : x_i \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x,y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology $\tau$. Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x,y) \leq d(x,z) + d(z,y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

**Definition 1.9.** *Let $(X, \tau)$ be a topological space, $U \subset V \subset X$. We say that $U$ is* dense *in $V$, if $V \subset \overline{U}$. We say that $X$ is* separable *if there exists a countable, dense subset.*

**Definition 1.10.** *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \to Y$ a function. We say $f$ is* continuous *at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. $f$ is called* continuous *if it is continous at any $x \in X$.*

**Proposition 1.11.** *With $(X, \tau_X), (Y, \tau_Y)$ and $f$ as above, $f$ is continuous $\iff$ $f^{-1}(V) \in \tau_X \forall V \in \tau_Y$*

*Proof.* Left as an exercise to the reader. □

**Definition 1.12.** *Let $(X, \tau)$ be a $T_2$ topological space, $M \subset X$ called* compact *if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ ("$(U_i)_{i \in I}$ is an open covering of $M$"), there exists $U_{i_1}, \dots, U_{i_n}$ such that $M \subset \bigcup_{k=1}^{n} U_{i_k}$ ("there exists a finite subcover").*

**Remark.** *Compactness can also be defined without $T_2$, this is also referred to as* quasi-compact*.*

**Remark** (Exercise)**.** *Reconsider the previous results for metric and normed spaces.*

↓ *This lecture took place on 2019/03/14.*

**Definition 1.13.** *Let $(X, d)$ be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in $X$. Then we say,*

1. *$V$ is* bounded *if $\exists x \in X, r > 0$ such that $U \in B_r(x)$*
2. *$(x_n)_n$ is a* Cauchy sequence *if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$*
3. *$X$ is* complete *if any Cauchy sequence in $X$ admits a limit point*
4. *$X$ is a* Banach space *if it is a normed space and complete*

**Proposition 1.14.** *Let $(X, d)$ be a metric space. $(x_n)_n$ be a sequence in $X$. Then*

1. *$x_n \to x$ in the induced topology $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$*

2. *If $x_n \to x$, then $(x_n)_n$ is bounded as subset of $X$ and $(x_n)_n$ is Cauchy.*

3. *If $U \subset X$ is closed and $X$ is complete. Then $(U, d)$ is a complete metric space.*

*Proof.* 1. We prove both directions:

$\implies$ True, since $B_\varepsilon(x)$ is open $\forall \varepsilon 0$

$\impliedby$ Let $x \in V$ with $V$ open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$
$V$ open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$
$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1,\ldots,n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in $U$, then $(y_n)_n$ is a Cauchy sequence in $X \implies \exists x \in X : y_n \to x$ as $n \to x$ if $x \notin U \implies x \in U^C \implies \exists n_0 \in N$ such that $y_{n_0} \in U^C$ due to $U^C$ open. This is a contradiction to $(y_n)_n$ in $U$

□

**Proposition 1.15.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. $f : X \to Y$. The following are equivalent (TFAE):*

- *$f$ is continuous (with respect to the induced topology)*
- *$\forall (\mathrm{X}_n)_n$ such that $x_n \to x \implies f(x_n) \to f(x)$*

*Proof.* Firstly, we prove that the first statement implies the second statement.

Take $(x_n)_n$ converging to $x$. Take $V \in \tau_y$ such that $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X$ such that $x \in \hat{U} \subset U$

$\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_0) \to f(x)$

**Remark.** 1. $\implies$ 2. *holds true in any topological space*

2. $\implies$ 1. *Not.*

Secondly, we prove that the second statement implies the first statement.

Suppose $f$ is not continuous, find $x_n \to x$ such that $f(x_n) \to f(x)$ is wrong. If $f$ is not contiuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(u) \not\subset V \forall U \in \mathcal{U}(x)$

$$\implies \exists \hat{V} \in \tau_Y \text{ such that } f(u) \not\subset \hat{V} \forall U \in U(x), f(x) \in \hat{V}$$

$$\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\implies (x_n)_n$ converges to $x$ but $f(x_n) \notin \hat{V} \implies f(x_n) \not\to f(x)$. This gives a contradiction. □

**Definition 1.16.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces. Let* $f : X \to Y$.

$f$ *is* uniformly continuous *iff*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

**Proposition 1.17.** *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces.* $M \subset X$, $f : M \to Y$. *If* $M$ *is dense in* $X$, $Y$ *is complete and* $f$ *is uniformly continuous.*

$$\implies \exists! \hat{f} : X \to Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

*Proof.* Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \to x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(y_1), f(y_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(y_n), f(x_n)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in $M, \hat{y} \in Y$ such that $f(\hat{x}_n) \to \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X, (x_n)_n$ is a sequence in $M$ such that $x_n \to x, f(x_n) \to y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$:

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d(f(x_n), f(x)) < 3\varepsilon$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.
2. $\hat{f}$ is uniformly continuous.

□

↓ *This lecture took place on 2019/03/19.*

**Proposition 1.18.** *Let* $(X, d)$ *be a metric space,* $M \subset X$.

1. *M is compact, so* $\forall (X_i)_{i \in I}$ *with* $X_i$ *a closed set* $\forall i$ *such that* $(\bigcap_{i \in I} X_i) \cap M = \emptyset$.

$$\implies \exists X_{i_1}, \ldots, X_{i_n} \text{ such that } \left( \bigcap_{i=1}^{n} X_{ij} \right) \cap M = \emptyset$$

2. *M is compact, so M is closed and bounded.*

*Proof.* 1. We note that $\forall (X_i)_{i \in I}$ is a family of closed sets. $(X_i^C)_{i \in I}$ is a family of open sets and $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

**Proposition 1.19.** *Let $(X, d)$ be a metric space, $M \subset X$. TFAE:*

1. *$M$ is compact.*
2. *Every infinite subset of $M$ admits a cluster point.*
3. *Every sequence of $M$ admits a convergent subsequence.*
4. *$M$ is complete and totally bounded, where totally bounded is defined as*

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

**Remark.** *1. totally bounded $\implies$ bounded (proof is left as an exercise)*

2. *If $\dim(x) < \infty$, then compact $\iff$ complete and bounded (see course Analysis I)*
3. *$\dim(x) < \infty \iff \overline{B_1}(0)$ is compact*

*where the last two items imply that X is a normed space.*

*Proof.* $1 \to 2$ If $M$ is finite, (2) always holds true. So assume that $M$ is infinite. Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists xinC$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(X_n)_n$ is a sequence of distinct points and $x_n \to x$. This implies that $x$ is a cluster point of $C$. This gives a contradiction.

Now $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$. If $M$ is compact, then

$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$

$\implies C \subset M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$

$\implies C$ is finite

This is a contradiction.

$2 \to 3$ Let $(x_n)_n$ be a sequence in $M$.

**Case 1:** $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies (x_n)_n$ admits a convergent sequence.

**Case 2:** $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid x \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

$3 \to 4$ Suppose that $M$ is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^{n} B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in $M$ as follows: Given $x_1, \dots, x_n$, choose $x_1 \in M$ arbitrary and $x_{i+1} \in M \setminus \bigcup_{j=1}^{i} B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in $M$ and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convenient subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in $M$, then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \to x$ as $i \to \infty$. Since $(x_n)_n$ is Cauchy, $x_n \to x$ as $n \to \infty$ [left as an exercise]. Thus $M$ is complete.

$4 \to 1$ Let $(U_i)_{i\in I}$ be an open covering of $M$ and assume that $(U_i)_{i\in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a\in E_n} B_{\frac{1}{2^n}(a)}$. Define $\Omega \coloneqq \left\{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } (U_i)_i\right\}$. We recursively define a sequence $(a_n)_n$ in $M$ such that

$$\forall n \in \mathbb{N} : a_n \subset E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n} \cap B_{\frac{1}{2^{n-1}}}}(a_{n-1}) \neq \emptyset$$

**Goal:** Show $(a_n)_n \to a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

**Step 1** $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a\in C_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \to n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let $\tilde{E}_{n+1} = \left\{a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset\right\}$.

Since $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a\in\tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$. [Take $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$, but if $B_{\frac{1}{2^{n-1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a\in\tilde{E}_{n+1}} B_{\frac{1}{a^{n+1}}}(a)$$

Hence there exists $a_{n+1}$ such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$ and $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

**Step 2** Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) \leq d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. $M$ is complete, so $\exists a \in M : a_n \xrightarrow{n\to\infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} and \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for $n$ sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

**Proposition 1.20.** *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces.* $M \subset X$ *compact. Let* $f : X \to Y$ *be continuous. Then*

1. $f(M)$ *is compact*
2. $f|_M : M \to Y$ *is uniformly continuous.*

*Proof.* 1. Let $(U_i)_{i \in I}$ be an open covering of $f(M)$

$\implies$ $(f^{-1}(U_i))_{i \in I}$ is an open covering of $M$ [why!]

$\implies$ $\exists c_1, \ldots, c_n$ such that $M \subset \bigcup_{i=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{i=1}^n U_{i_j}$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n, (y_n)_n$ sequences in $M$ satisfying condition (*). $M$ is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \to \infty} 0$$

□

↓ *This lecture took place on 2019/03/21.*

**Proposition 1.21** (Proposition and definition)**.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces.* $g : X \to Y$ *is a function.* $g$ *is called* Lipschitz continuous *if* $\exists L > 0$ *such that* $d_Y(\varphi(x), \varphi(y)) \leq L d_X(x, y) \forall x, y \in X$*. Any Lipschitz continuous function is uniformly continuous.*

*Proof.* Left as an exercise to the reader. □

**Theorem 1.22** (Arzelà-Ascoli theorem)**.** *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces and assume that* $X$ *is compact. Define* $C(X, Y) := \{f : X \to Y \mid f \text{ continuous}\}$ *and* $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$*. Then*

1. $d_C$ *is well-defined and* $(C(X, Y), d_C)$ *is a complete metric space*
2. *A set* $M \subset C(X, Y)$ *is compact iff*
   (a) $\forall x \in X$ *the set* $M_X := \{f(x) \mid f \in M\}$ *is compact*
   (b) $M$ *is* equicontinuous, *i.e.* $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

*Proof.* 1. Show that: $d_C(f,g) < \infty$.

Pick $f, g \in C(X,Y)$. Because $X$ is compact, $f(X), g(X)$ compact $\implies$ $f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\le d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\le D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x\in X} d(f(x), g(x)) \end{aligned}$$

Showing that $d_C$ is a metric is left as an exercise.

Show that $(C(X,Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X,Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because $Y$ is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n\to\infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to $d_C$: Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \ge n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \ge n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \ge n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x)) \le \lim_{m\to\infty} d(f_m, f_n) < \infty$ (the proof follows below)

$$\implies \sup_{x\in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X,Y) \implies f_n \to f$ with respect to $d_C$. Show that $f \in C(X,Y)$. Take $\varepsilon > 0$. Let $n_0$ such that $\sup_{x\in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x,y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x,y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\le d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \ge n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x))$.

In general, we have $\forall x, y, z \in (Z, d_Z)$ with $d_Z$ as a metric.

$$\left| d(x,z) - d(y,z) \right| \le d(x,y)$$

*Proof.*

$$d(x,z) \le d(x,y) + d(y,z) \implies d(x,z) - d(y,z) \le d(x,y) \tag{2}$$
$$d(y,z) \le d(y,x) + d(x,z) \implies d(y,z) - d(x,z) \le d(x,y) \tag{3}$$
$$(2) \text{ and } (3) \implies \left| d(x,z) - d(y,z) \right| \le d(x,y) \tag{4}$$

□

Consequently, $\forall z \in Z$, $x_n \to x$ in Z: $d(x_n, z) \to d(x,z)$ since $\left| d(x_n,z) - d(x,z) \right| \le d(x_n, x) \to 0$.

2. We need to prove both directions.

$\implies$ (a) For $x \in X$ fixed, define $g_X : M \to Y$ with $f \mapsto f(x)$. Then $d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \le d_C(f_1, f_2)$

$\implies$ $g_X$ is Lipschitz continuous, in particular continuous

$\implies$ $M_X = g_X(M)$ compact

(b) Take $\varepsilon > 0$. $M$ is totally bounded, so $\exists f_1, \ldots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$. $\forall i \in \{1, \ldots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$. Define $\delta := \min_i \delta_i > 0$. Then $\forall x, y \in X : d(x, y) < \delta$ and $\forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0})$

$$\implies d(f(x), f(y)) \le \underbrace{d(f(x), f_{i_0}(x))}_{\le d_C(f, f_{i_0}) \le \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\le \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\le d_C(f_{i_0}, f) \le \frac{\varepsilon}{3}} < \varepsilon$$

$\impliedby$ We prove the other direction.

> ↓ *This lecture took place on 2019/03/26.*

$B$ is complete since it is a closed subset of a Banach space.
Show: $M$ is totally bounded.
Consider $\varepsilon > 0$. Show: $\exists f_1, \ldots, f_n$ such that $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$.

- Because $M$ is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X, $\exists x_1, \ldots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \ldots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{i=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{i_i})$

Compare with Figure 1.
Now, for each tuple of indices $(y_{1,j_1}, \ldots, y_{n,j_n})$ define $f_{y_{1,j_1}, \ldots, y_{n,j_n}} \in C(x, y)$ to be such that $f_{y_{1,j_1}, \ldots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an $f$ exists. The set $F$ of all such functions is finite. We show that $M \subset \bigcup_{q \in F} B_\varepsilon(q)$.
Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \ldots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_\alpha \in F$ accordingly.
Take $x \in X$ arbitrary and $x_i$ such that $x \in B_\delta(x_i)$

$$\begin{aligned} \implies d(f(x), f_\alpha(x)) &\le d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_\alpha) &= \sup_{x \in X} d(f(y), f_\alpha(x)) < \varepsilon \end{aligned}$$

□

**Remark.** *Compare this to the fact that $B_1(0)$ in $C(X, Y)$ is* not *compact.*

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

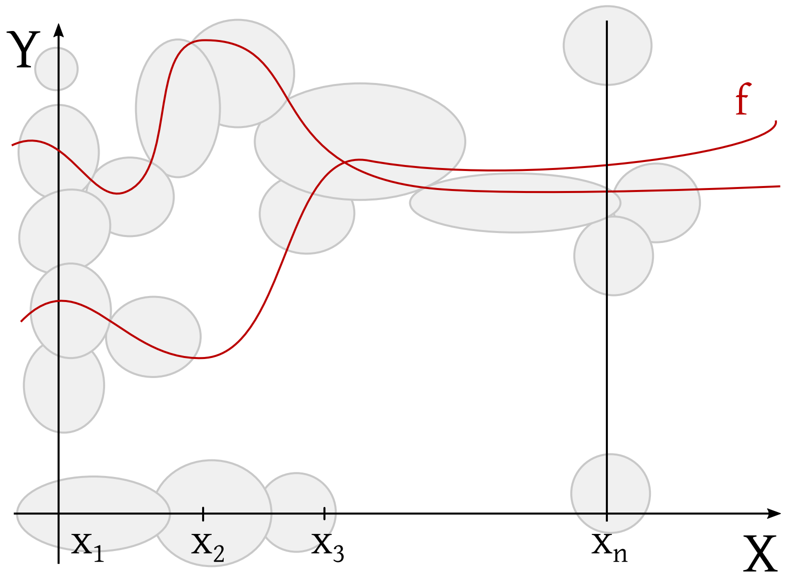


Figure 1: Covering of a function graph

**Remark** (Motivation)**.** *In general, let $(X, d)$ be a metric space. Let $A$ and $B$ be open and dense, then also $A \cap B$ is dense.*

*Proof.* Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] = \emptyset$. Take $x \in Y, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. $A$ is dense. $A$ is open, so $\exists \varepsilon > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. $B$ is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_\varepsilon(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally, $\forall A_1, \ldots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many $A_i$?

**Theorem 1.23** (Baire theorem)**.** *Let $(X, d)$ be a complete metric space. Let $(O_n)_{n \in \mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.*

*Proof.* Let $D := \bigcap_{n \in \mathbb{N}} O_n$. Show that for $x \in X$, $\varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n \in \mathbb{N}}$.

**$\mathbf{n = 1}$** Take $x_1, \varepsilon_1$ such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

**$\mathbf{n - 1 \to n}$** Given $x_{n-1}, \varepsilon_{n-1}$, take $x_n, \varepsilon_n$ such that

$$\overline{B_{C_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_N}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, X complete } \implies \exists x \in X : x_n \to x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_N}}(x_N) \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_n)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

**Definition 1.24.** *Let $(X, d)$ be a metric space, $M \subset X$. We say*

- *$M$ is* nowhere dense*(dt. "Nirgends dicht"), if $\overset{\circ}{\overline{M}} = \emptyset$*
- *$M$ is of* first category $\iff$ *$M$ is the countable union of nowhere dense sets*
- *$M$ is of* second category $\iff$ *$M$ is not of first category*

**Theorem 1.25** (Baire category theorem (weaker version))**.** *Let $(X, d)$ be a complete metric space. Then $(X, d)$ is of second category.*

*In otheor words (which is a useful formulation): If $X = \bigcup_{n \in \mathbb{N}} C_n \implies \exists n_0 :$ $\overset{\circ}{\overline{C}} \neq \emptyset$. In particular, if*

$$X = \bigcup_{n \in \mathbb{N}} C_n \textit{ with } C_n \textit{ closed } \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

*Proof.* Suppose that $X = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O_n}$ with $\overset{\circ}{\overline{O}}_n = \emptyset \forall n$

$$\overset{\circ}{\overline{O}}_n = \emptyset \implies \overline{\overline{O}_n^C} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overline{O}_n^C = \emptyset \implies B_\varepsilon(x) \subset \overline{O}_n \implies \overset{\circ}{\overline{O}}_n \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O}_n^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O}_n^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n \in \mathbb{N}} \overline{O}_n^C \text{ is dense}$$

$$\bigcap_{n \in \mathbb{N}} \overline{O}_n^C = \left( \bigcup_{n \in \mathbb{N}} \overline{O}_n \right)^C = X^C = \emptyset$$

gives a contradiction □

**Remark.** 1. *This is a fundamental theorem in Functional Analysis*

2. *This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise)*

# 2 Normed space

## 2.1 Fundamentals

**Definition 2.1.** *Let $X$ be a vector space. A function $\|\cdot\| : X \to [0, \infty)$ is called* seminorm *if*

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \, \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \, \forall x, yinX$

*The first property differs between a norm and a seminorm.*

*The tuple $(X, \|\cdot\|)$ is called a* semi-normed space. *We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.*

**Definition 2.2** (Definition and proposition)**.** *Let $(X, \|\cdot\|)$ be a semi-normed space and $(x_n)_n$ be a sequence in $X$. We say that*

- $\sum_{n=1}^{\infty} x_n$ converges *to $x \in X$ and write $x = \sum_{n=1}^{\infty} x_n$ if $\lim_{m\to\infty} \sum_{n=1}^{m} x_n = x$*
- $\sum_{n=1}^{\infty} x_n$ *is* absolutely convergent *if $\sum_{n=1}^{\infty} \|x_n\|$ converges [ $\iff$ $(\sum_{n=1}^{m} \|x_n\|)_m$ is bounded]*

*It holds that $X$ is complete iff any absolutely converging series converges.*

*Proof.* $\implies$ Take $m_1 < m_2$ arbitrary, then

$$\left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| \leq \sum_{n=m_1+1}^{m_2} \|x_1\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\| \leq \left\| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\| \right\|$$

$$\implies \left( \sum_{n=1}^{m} x_n \right)_m \text{ is Cauchy } \implies \text{ convergent}$$

$\impliedby$ Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick $N_k$ such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \left\| x_{n_{k+1}} - x_{n_k} \right\| \leq 2^{-k}$$

Define $y_k := x_{n_{k-1}} - x_{n_k} \implies \sum_k \left\| y_{n_w} \right\| \leq \sum_k 2^{-k} < \infty$

$$\implies \exists y \in X : \sum_{k=1}^{n} y_k \to y \text{ as } n \to \infty$$

$$\sum_{k=1}^{n} y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \to y - x_{n_1} \text{ as } n \to \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is convergent.

□

**Remark.** *In $\mathbb{R}^n$, $\sum_n x_n$ is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction $\implies$ is true.*

> ↓ *This lecture took place on 2019/03/28.*

**Proposition 2.3** (Proposition and definition)**.** *Let $X$ be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on $X$. We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if*

$$\exists m, M > 0 \forall x \in X : m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

*TFAE:*

1. *$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*
2. *For any sequence $(x_n)_n$ and $x \in X, x_n \to x$ with respect to $\|\cdot\|_1 \iff x_n \to x$ with respect to $\|\cdot\|_2$*
3. *For any sequence $(x_n)_n$ we have,*

$$x_n \to 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \to 0 \text{ with respect to } \|\cdot\|_2$$

*Proof.* (1) $\implies$ (2) $\implies$ (3) is immediate.

It remains to show that:

(3) $\implies$ (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \; \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n\|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\|y_n\|_1 = \frac{1}{n} \to 0$ hence $y_n \to 0$, but $\|y_n\|_2 > n\|y_n\|_1 = 1$.

$$\implies y_n \not\to 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

**Remark.** *If $\dim(X) < \infty$, then any two norms on $X$ are equivalent.*

**Definition 2.4** (Quotient spaces)**.** *Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation "$\sim$" on $X$ with $x \sim y :\iff x - y \in Y$.*

*Then $\sim$ defines an equivalence relation on $X$. We define*

- $[X]_\sim = \{y \in X \mid x \sim y\}$ *...the equivalence class of $x \in X$*

- $X/Y := \{[x]_\sim \mid x \in X\}$ *...the quotient space*
- $\pi : \begin{cases} X \to X/y \\ x \mapsto [x]_\sim \end{cases}$

*Defining* $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \qquad \hat{0} := [0]$$

*We get that:*

1. *$X/Y$ is a vector space*
2. *$\left\|[x]\right\|_{X/Y} := \inf_{y\in[x]} \left\|y\right\|_X$ is a semi-norm.*
3. *If $Y$ is closed, then $\|\cdot\|_{X/Y}$ is a norm.*
4. *If $X$ is complete and $Y$ closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.*

*Proof.*
- Equivalence relation
- Vector space with "$+$" and "$\lambda[x]$" are well-defined

This is left as an exercise to the reader.

2. – First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\left\|[0]\right\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y\in Y} \|Y\| \leq \|0\| = 0$$

– Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$.
Show that: $\left\|\lambda[x]\right\|_{X/Y} = |\lambda| \left\|[x]\right\|_{X/Y}$.
Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\left\|\lambda[x]\right\|_{X/Y} = \left\|[\lambda x]\right\|_{X/Y} = \inf_{y\in[\lambda x]} \left\|y\right\| = \inf_{y\in X, \frac{y}{\lambda}\in[x]} \left\|y\right\| = \inf_{w\in[x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u\in[x]} \|u\|}^{\|[x]\|_{X/Y}}$$

– Take $[x_1], [x_2] \in X/Y, \varepsilon > 0$. We note that

$$\left\|[x]\right\|_{X/Y} = \inf_{\substack{y\in X \\ w\in Y \\ w:=x\cdot y}} \left\|y\right\| = \inf_{w\in Y} \|x - w\|$$

Hence we can take $y_1, y_2 \in Y$ such that $\left\|x_1 - y_i\right\| < \left\|[x_i]\right\|_{X/Y} + \varepsilon \qquad \varepsilon \in [1, 2)$.

$$\implies \left\|[x_1] + [x_2]\right\|_{X/Y} = \left\|[x_1 + x_2]\right\|_{X/Y} \leq \left\|x_1 + x_2 - (y_1 + y_2)\right\|$$
$$\leq \left\|x_1 - y_1\right\| + \left\|x_2 - y_2\right\| \leq \left\|[x_1]\right\|_{X/Y} + \left\|[x_2]\right\|_{X/Y} + 2\varepsilon$$

Since $\varepsilon$ was arbitrary, the assertion follows.

3. Suppose $Y$ is closed if $\left\|[x]\right\|_{X/Y} = 0$, then

$$\inf_{y\in Y} \left\|x - y\right\| = 0 \implies \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n\to\infty} \left\|x - y_n\right\| = 0$$

$$Y \text{ closed } \implies x \in Y \implies [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in $X/Y$ and suppose that $\sum_{i=1}^{\infty} \left\|[x_n]\right\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, $X/Y$ is complete.

   Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \left\|[x_n]\right\|_{X/Y} + 2^{-n}$

$$\implies \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} \left( \left\|[x_n]\right\|_{X/Y} + 2^{-n} \right) < c < \infty$$

$$X \text{ complete } \implies \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \qquad \left\| [x] - \sum_{n=1}^{m} \underbrace{[x_n]}_{[\tilde{x}_n]} \right\|_{X/Y} \leq \left\| \underbrace{x - \sum_{k=0}^{n} \tilde{x}_k}_{\to 0} \right\|$$

□

↓ *This lecture took place on 2019/04/02.*

**Corollary 2.5.** *Let $X$ be a vector space with semi-norm $\|\cdot\|_X : X \to [0, \infty)$. Then*

- $N = \{x \in X \mid \|x\|_X = 0\}$ *is a subspace at* $X$
- $\left\|[X]\right\| := \|X\|_p$ *is a norm on* $X/N$
- *If* $X$ *is complete, then* $(X/N, \|\cdot\|)$ *is a Banach space.*

*Proof.* The proof is left as an exercise. □

**Proposition 2.6.** *Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then*

- $\overline{U}$ *is also a subspace.*
- $X$ *is separable iff* $\exists A \subset X$ *complete such that* $X = \overline{\mathcal{L}(A)}$ *where* $L(A) = \{\sum_{i=1}^{n} \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathcal{K}, n \in \mathbb{N}\}$

*Proof.*
- Left as an exercise
- $\implies$ True since $\exists A \subset X$ countable such that $\overline{A} = X \implies \underline{X} = \overline{A} \subset \overline{L(A)} \subset X$

$\Longleftarrow$ Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^{n} (\lambda_i + i\mu_i)x_i \mid \lambda_i, \mu_i \in \mathbb{X}, x \in A, n \in \mathbb{N} \right\}$$

where $i$ is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then $B$ is countable.
Show: $\forall x \in X \forall \varepsilon \exists x \in B : \left\| x - y \right\| < \varepsilon$.
Take $x \in X, \varepsilon > 0 \implies \exists x_0 \in \mathcal{L}(A) : \|x - x_i\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=0}^{n} (\lambda_i + i\mu_i)x_i$ with $\lambda_i, \mu_i \in \mathbb{R}, x_i \in A$. Choose $\lambda', \mu_i' \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda_i')^2 + (\mu_i - \mu_i')^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^{n} \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let $y := \sum_{j=1}^{n} (\lambda_i' + \mu_i')x_i \subset B$.

$$\begin{aligned} \implies \left\| x - y \right\| \leq \|x - x_0\| + \left\| x_0 - y \right\| & \leq \frac{\varepsilon}{2} \\ \leq \sum_{i=1}^{n} \left| (\lambda_i + i\varepsilon_i) - (\lambda_i' + i\mu_i') \right| \|x_i\| & \\ \leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^{n} \|x_i\|} = \varepsilon & \end{aligned}$$

□

**Proposition 2.7** (Proposition and definition). *Let $(X, \|\cdot\|_{x_i})$ for $i = 1, \dots, n$ be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \ldots \otimes X_n = \bigotimes_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

*For $p \in [1, \infty]$, define*

$$\left\| (x_1, \dots, x_n) \right\|_{\otimes_i X_i, p} = \begin{cases} \left( \sum_{i=1}^{n} \|x_i\|_{x_i}^{p} \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1,\dots,n} \|x_i\|_{x_i} & \text{if } p = \infty \end{cases}$$

*Then*

- *$(\bigotimes_i X_i, \|\cdot\|_{\bigotimes_i X_i, p})$ is a normed space with respect to componentwise addition and multiplication.*
- *If all $X_i$ are complete, then $\bigotimes_{i=1}^{n} X_i$ is complete.*
- *All norms $\|\cdot\|_{\bigotimes_i X_i, p}$ are equivalent.*

*Proof.*

- Vector space properties: Left as an exercise
- Norm: $\|x\|_{\bigotimes_i X_i, n} = 0 \iff x = 0$
  $\|\lambda x\|_{\bigotimes_i X_i, p} = |\lambda| \|x\|_{\bigotimes_i X_i, p}$

- Triangle inequality: $p = 1$, $p = \infty$
  $p \in (1, \infty)$. Take $x, y \in \bigotimes_{i=1}^n X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i, p}$.

$$
\begin{aligned}
\implies \left\|x+y\right\|_p^p &= \sum_{i=1}^n \left\|x_i+y_i\right\|_{X_i} \left\|x_i+y_i\right\|_{x_i}^{p-1} \\
&\leq \sum_{i=1}^n \|x_i\|_{X_i} \left\|x_i+y_i\right\|_{X_i}^{p-1} + \sum_{i=1}^n \left\|y_i\right\|_{X_i} \left\|x_i+y_i\right\|_{X_i}^{p-1} \\
&\underbrace{\leq}_{\text{Hölder ineq.}} \left(\sum_{i=1}^n \|X_i\|_{X_i}^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n \left\|x_i+y_i\right\|_{X_i}^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^n \left\|y_i\right\|_{x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n \left\|x_i+y_i\right\|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \|x\|_p \left\|x+y\right\|_p^{p-1} + \left\|y\right\|_p \left\|x+y\right\|_p^{p-1} \\
&= \left(\|x\|_p + \left\|y\right\|_p\right) \cdot \left\|x+y\right\|_p^{p-1}
\end{aligned}
$$

$$\implies \left\|x+y\right\|_p \leq \|x\|_p + \left\|y\right\|_p \text{ if } x+y \neq 0 \text{ (trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in $\mathbb{R}^n$)

□

**Definition 2.8.** *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \to Y$ is linear such that $\left\|j(x)\right\|_Y = \|x\|_X$ (hence $j$ is injective) then $j$ is called isometric embedding from $X$ to $Y$. If $j$ is bijective, then $j$ is called* isometric isomorphism *and we say $X = Y$ up to isomorphism.*

**Proposition 2.9.** *Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_X)$ a Banach space such that*

1. *$\exists$ isometric embedding, $i : X \to \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [$\hat{X}$ can be regarded as completion of $X$]*
2. *If $j_1 : X \to Y$ is an isometric embedding on $Y$, a Banach space*

$$\implies \exists i_2 : \hat{X} \to Y$$

*an isometric embedding such that $j_2 \circ j = j_1$ and if $\overline{j_1(x)} = Y$ then $j_2$ is an isometric isomorphism. Thus "the completion is essentially unique".*

*Proof.* 1. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is } Cauchy\}$. $\hat{X}$ is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \qquad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define $\left\|(x_n)_n\right\|_{\tilde{X}} := \lim_{n\to\infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in $\mathbb{R}$]. Then $\|\cdot\|_{\tilde{X}}$ is a semi-norm (proof is left as an exercise). Setting $N =$

$\left\{(X_n)_n \mid \left\|(X_n)_n\right\|_{\hat{X}} = 0\right\}$. By Corollary 2.5, $\hat{X} \coloneqq \hat{X} \setminus N$ with $\left\|[(X_n)_n]\right\|_{\hat{X}} = \left\|(X_n)_n\right\|_{\hat{X}}$ is a normed space. Define

$$j : X \to \hat{X} \qquad x \mapsto [(x)_n]$$

then $j$ is linear and $\left\|j(x)\right\|_{\hat{x}} = \left\|[(x)_n]\right\|_{\hat{x}} = \lim_{n\to\infty} \|x\| = \|x\|$. So $j$ is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n \coloneqq j(x_n) \in \hat{X}$.

$$\begin{aligned} \implies \left\|y_m - [(x_n)_n]\right\|_{\hat{X}} &= \left\|(x_m)_n - (x_n)_n\right\|_{\hat{X}} = \lim_{n\to\infty} \|x_m - x_n\| \\ &= \lim_{n \geq n_0} \|x_m - x_n\| < \varepsilon \end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: $\hat{X}$ is complete.

Let $(y_n)_n$ be Cauchy in $\hat{X}$. Pick $X_n \in X$ such that $\left\|j(x_n) - y_n\right\|_{\hat{X}} \leq \frac{1}{n}$ $(\overline{j(x)} = \hat{x})$

$$\implies \|x_n - x_m\|_X = \left\|j(x_n) - j(x_m)\right\|_{\hat{X}} \leq \left\|j(x_n) - y_n\right\|_{\hat{X}} + \left\|y_n - y_m\right\|_{\hat{X}} + \left\|y_n - j(x_n)\right\|_{\hat{X}}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \left\|y_n - y_m\right\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick $n_1$ such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\implies \forall n, m > \max(n_0, n_0) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\implies$ $(x_n)_n$ is Cauchy. Let $y \coloneqq (X_n)_n \in \tilde{X}$. Then

$$\left\|y_n - [y]\right\|_{\hat{X}} \leq \left\|y_n - j(x_n)\right\|_{\hat{X}} + \left\|j(x_n) - [y]\right\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n\to\infty} \|x_n - x_m\|_X \xrightarrow{n\to\infty} 0$$

2. ↓ *This lecture took place on 2019/04/04.*

Let $\hat{x} \in \hat{X} \implies \exists (x_n)_n \in X$ such that $j(x_n) \to \hat{x} \implies \|x_n - x_m\|_X = \left\|j(x_n) - j(x_m)\right\|_{\hat{X}}$.

$\implies$ $(x_n)_n$ is a Cauchy sequence.

$\implies$ $j_1(x_n)$ is a Cauchy sequence in $Y$.

$\implies$ $\exists \lim_{n\to\infty} j_1(x_n) \coloneqq y$

Using this, we define $j_2 : \hat{X} \to Y$ with $\hat{x} \mapsto \lim_{n\to\infty} j_1(x_1)$ where $j(x_1) \to \hat{x}$.

Well-defined? Take $\hat{x} \in \hat{X}$ and $j(x_n) \to \hat{x}$, $j(y_n) \to \hat{x}$.

$$\implies \left\|i_1(x_n) - j_1(y_n)\right\| = \left\|x_n - y_n\right\| = \left\|j(x_n) - j(y_n)\right\| \to 0 \text{ as } n \to \infty$$

$$\implies \lim_{n\to\infty} j_1(x_n) = \lim_{n\to\infty} j_1(y_n) \implies j_1 \text{ well-defined}$$

Show linearity is left as an exercise. By isometry, take $\hat{x} \in \hat{X}$,

$$\left|i_2(\hat{x})\right| \underbrace{=}_{j(x_n)\to\hat{x}} \lim_{n\to\infty} \left\|j_1(x_1)\right\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \left\|i(x_n)\right\| = \|\hat{x}\|$$

Show: $j_2 \circ j = j_1$. Take $x \in X \implies (x_n)$ is such that $j(x) \to j(x) \implies j_2(j(x)) = \lim_{n\to\infty} j_1(x) = j_1(x)$.

Assume that $\overline{j_1(x)} = Y$. Take $y \in Y$. Find $\hat{x} \in \hat{X}$ such that $i_2(\hat{x}) = y$. By $\overline{j_1(x)} = Y \implies \exists (x_n)_n$ in X such that $i_1(x_n) \to Y \implies (j_1(x_n))_n$ is Cauchy.

$$\implies (x_n)_n \text{ Cauchy } \implies (j(x_n))_n \text{ is Cauchy}$$

$$\xRightarrow{\hat{X}\text{ complete}} \exists\hat{x} \text{ such that } \lim_{n\to\infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n\to\infty} j_2(x_n) = Y$$

□

## 2.2 Important examples of normed spaces

**Definition 2.10** (Basic notation)**.** *Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \to \mathbb{K}^M$ with $N, M \subset \mathbb{N}$.*

- *We call $\Omega$ a* domain*(dt. "Gebiet") if $\Omega$ is open and connected, where connected means that $\forall x, y \in \Omega$ there is a curve in $\Omega$ connecting $X$ and $Y$.*
- *For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$. If $f$ is r-times continuously differentiable, we set for $\alpha \in \mathbb{N}_0^N$, $\{\alpha\} \leq r$.*

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial^{\alpha_1}_{x_1} \dots \partial^{\alpha_n}_{x_n} x_n} f$$

*where $\frac{\partial^{\alpha_1}}{\partial^{\alpha_i} x_i}$ is the partial derivative of $f$ with respect to $x_i$ of order $\alpha_i$.*

**Example 2.11.** *Let $N = 2$ and $\alpha = (1, 1)$.*

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

*Let $\alpha = (2, 0)$.*

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- *For $z \in \mathbb{K}^N$ we denote $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$.*[1]
- *We say $E \subset \Omega$ is compact in $\Omega$ and we write $E \Subset \Omega$ if $E$ is compact.*

**Remark.** *If $E \Subset \Omega$, then $\exists \delta > 0 : \inf\left\{\left\|x - y\right\| \mid x \in E, y \in \partial\Omega\right\} > 0$.*

*Proof.* Left as an exercise (use compactness) □

[1]This is an abuse of notation with $|\alpha|$ for $\alpha \in \mathbb{N}_0^N$

- $f$ *is* compactly supported *in* $\Omega$ *if* $\operatorname{supp}(f) \ll \Omega$.
- $\operatorname{supp}(f) := \overline{\left\{x \in \Omega \mid \left\|f(x)\right\| > 0\right\}}$

↓ *This lecture took place on 2019/04/09.*

**Definition 2.12** (Definition and proposition, Spaces of continuous functions)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be a domair. We define*

$$C_b(\Omega, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \varphi \textit{ bounded}\right\} \textit{ with } \left\|\varphi\right\|_{C_b} = \left\|\varphi\right\|_\infty = \sup_{x \in \Omega} \left|\varphi(x)\right|$$

$$C(\overline{\Omega}, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \varphi \textit{ can be continuously extended to } \overline{\Omega}\right\}, \left\|\varphi\right\|_C := \left\|\varphi\right\|_\infty$$

$$C^r(\overline{\Omega}, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid D^\alpha\varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \leq r\right\} \textit{ and} \left\|\varphi\right\|_{C^r} = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq r}} \left\|D^\infty \varphi\right\|_\infty$$

$$C_C^r(\Omega, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \operatorname{supp}(\varphi) \ll \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M)\right\} \textit{ and } \left\|\varphi\right\|_{C_C^r} = \left\|\varphi\right\|_{C^r}$$

$$C^\infty(\overline{\Omega}, \mathbb{K}^M) = \bigcap_{r \in \mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M)$$

$$D(\Omega, \mathbb{K}^M) = C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r \in \mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \textit{ in } C^r(\overline{\Omega}, \mathbb{K}^M)$$

*Then for any bounded* $\Omega$, $C^r, C_0^r, C_b$ *are Banach spaces and* $C_C^r$ *is a normed space.*

*Recall:* $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

*Proof.* The functions $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$ are norms (proof is left as an exercise).

Show that $C_b$ is complete: Take $(\varphi_n)_n$ in $C_b$ to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because $\left|\varphi_n(x) - \varphi_m(x)\right| \leq \left\|\varphi_n - \varphi_m\right\|_\infty$. Hence we can define $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$.

Show: $\varphi_n \to \varphi$ in $\|\cdot\|_\infty$. Take $\varepsilon > 0$. Show that $\exists n_0 \forall n \geq n_0 : \left\|\varphi - \varphi_n\right\|_\infty < \varepsilon$. Take $n_0$ such that $\forall n, m \geq n_0 : \left\|\varphi_n - \varphi_m\right\|_\infty < \varepsilon$. Take $m \geq n_0$.

$$\implies \forall x \in \Omega : \left|\varphi(x) - \varphi_m(x)\right| = \lim_{\substack{n\to\infty \\ n \geq n_0}} \left|\varphi_n(x) - \varphi_m(x)\right| < \left\|\varphi_n - \varphi_m\right\|_\infty$$

Show: $\varphi$ is bounded, i.e. $\exists C > 0 : \left|\varphi(x)\right| \leq C < \left\|\varphi_n - \varphi_m\right\|_\varepsilon < \infty$. Take $n$ suchthat $\left\|\varphi - \varphi_n\right\|_\infty < 1$

$$\implies \forall x \in \Omega : \left|\varphi(x)\right| > \left|\varphi(x) - \varphi_n(x)\right| + \left|\varphi_n(x)\right| \leq 1 + \underbrace{\left\|\varphi_n\right\|}_{=C}$$

Now $C^r(\overline{\Omega}, \mathbb{K}^n)$ is a subspace of $C^b(\Omega, \mathbb{K}^n)$. Also $C^r(\overline{\Omega}, \mathbb{K}^n)$ is closed, since the uniform limit of $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$ with respect to $\|\cdot\|_{C^r}$ is again in $C^r(\overline{\Omega}, \mathbb{K}^M)$ [a result from Analysis].

$$\implies C^r(\overline{\Omega}, \mathbb{K}^M) \text{ is a Banach space}$$

$C^r_C(\overline{\Omega}, \mathbb{K}^M)$ is closed by definition, hence Banach.

$C^r_C(\Omega, \mathbb{K}^M)$ is a vector space, since $\forall \lambda \in \mathbb{K} : \varphi \in C^r_0(\Omega, \mathbb{K}^M) : \operatorname{supp}(\lambda\varphi) = \operatorname{supp}(\varphi)$ and for $\varphi, \Psi \in C^r_0(\Omega, \mathbb{K}^M) : \operatorname{supp}(\varphi + \Psi) \ll \Omega$. $\square$

**Definition 2.13** (Definition and proposition). *Let $(\Omega, \Sigma, \mu)$ with $\Omega \subset \mathbb{R}^N$ be a measure space (i.e. $\Sigma$ is a sigma algebra and $\mu$ is a measure). For $p \in [1, \infty)$, we define*

$$\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) = \left\{ f : \Omega \to \mathbb{K}^M \mid f\ \mu-\ \textit{measurable and}\ \int_\Omega \left|f(x)\right|^p d\mu(x) < \infty \right\}$$

$$\left\|f\right\|_p^* = \left( \int_\Omega \left\|f(x)\right\|^p d\mu(x) \right)^{\frac{1}{p}}$$

$$\mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) := \left\{ f : \Omega \to \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} \left|f(x)\right| < \infty \right\}$$

$$\left\|f\right\|_\infty^* = \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} \left|f(x)\right|$$

*Our proposition is that these are semi-norms.*

*Proof.* Show that $\|\cdot\|_p^*$ for $p \in [1, \infty]$ are seminorms.

They cannot be norms since $\left\|f\right\|_p^* = 0$ for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0.$ $\square$

**Proposition 2.14** (Hölder inequality). *Let $p \in [1, \infty]$ and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \textit{if } \in (1, \infty) \\ 1 & \textit{if } p = \infty \\ \infty & \textit{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

*If $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $q \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$ then for both*

$$f \cdot g : \Omega \to \mathbb{K} \textit{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)}$$

$$f \otimes g : \Omega \to \mathbb{K}^M \textit{ with } x \mapsto (f_i(x), \varphi_i(x))_{i=1}^M$$

*we have that* $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$ *and* $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$ *and* $\left\|f \otimes g\right\|_1^* \leq \left\|fg\right\|_1^* \leq \left\|f\right\|_p^* \cdot \left\|q\right\|_q^*$.

*Proof.* **Case** $p \in (1, \infty)$**:** Intermediate result: $\forall \sigma, \tau \geq 0, r \in (0,1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$ [AGM-inequality].

*Proof.*

Case $\sigma = 0$ or $\tau = 0$: immediate

Case $\sigma, \tau > 0$:

$$\begin{aligned} \log(\sigma^r \tau^{1-r}) &= r\log(\sigma) + (1-r)\log(\tau) \leq \log(r\sigma + (1-r)\tau) \\ &\quad \text{since } \log''(x) \leq 0 \text{ implies that log is concave} \\ \text{log is monotonic} \implies \sigma^r\tau^{1-r} &\leq r\sigma + (1-r)\tau \end{aligned}$$

□

Let $A := \left(\left\|f\right\|_p^*\right)^p$ and $B := \left(\left\|\varphi\right\|_1^*\right)^q$ with $r = \frac{1}{p} \in (0,1]$ we get

$$\forall x \in \Omega : \left(\frac{\left|f(x)\right|^p}{A}\right)^{\frac{1}{p}} \left(\frac{\varphi(x)}{B}\right)^{\frac{1}{q}} = \frac{1}{p}\frac{\left|f(x)\right|^p}{A} + \frac{1}{q}\frac{\left|q(x)\right|^q}{B}$$

$$\begin{aligned} \implies \frac{\int_\Omega \left|f(x)\right|\left|q(x)\right| \, d\mu(x)}{A^{\frac{1}{p}}B^{\frac{1}{q}}} &\leq \frac{1}{p}\frac{\int_\Omega \left|f(x)\right|^p \, d\mu(x)}{A} + \frac{1}{q}\frac{\int_\Omega \left|q(x)\right|^q \, d\mu(x)}{B} \\ \implies \int_\Omega \left|f(x)\right|\left|g(x)\right| \, d\mu(x) &\leq \left\|f\right\|_p^* \left\|g\right\|_q^* = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Now: $\left\|f \cdot g\right\|_x^* \leq \left\|f\right\|_p^* \cdot \left\|g\right\|_q^*$ follows since $\left|\langle x, y\rangle\right| \leq |x|\left|y\right| \forall x, y \in \mathbb{K}^M$.

Also:

$$\forall x \in \Omega : \left|f \otimes g(x)\right| = \sum_{i=1}^M \left|f_i(x)\right|\left|g_i(x)\right| = \begin{pmatrix} \left|f_1(x)\right| & \left|g_1(x)\right| \\ \vdots & \vdots \\ \left|f_n(x)\right| & \left|g_n(x)\right| \end{pmatrix} \leq \left|f(x)\right|\left|g(x)\right|$$

$$\implies \int_\Omega \left|f \otimes g(x)\right| \, d\mu(x) \leq \left\|f\right\|_p^* \cdot \left\|g\right\|_q^*$$

**Case** $p \in \{1, \infty\}$**:** Without loss of generality assume that $p = 1, q = \infty$. $\forall N \in \Sigma$ with $\mu(N) = 0$ we get

$$\begin{aligned} \int_\Omega \left|f(x)\right|\left|g(x)\right| \, d\mu(x) &= \int_{\Omega \setminus N} \left|f(x)\right|\left|q(x)\right| \, \mu(x) \\ &\leq \int_{\Omega \setminus N} \left|f(x)\right| \, d\mu(x) \cdot \sup_{x \in \Omega \setminus N} \left|q(x)\right| \quad = \int_\Omega \left|f(x)\right| \, d\mu(x) \cdot \sup_{x \in \Omega \setminus N} \left|q(x)\right| \end{aligned}$$

Taking the infimum over all such $N$, then

$$\int_\Omega |f(x)| |g(x)| \, d\mu(x) \leq \|f\|_1^* \cdot \|g\|_\infty^*$$

And the result follows again from $|\langle x, y\rangle| \leq |x| \cdot |y|$ and componentwise $|\langle x_i, y_i\rangle_i| \leq |x| |y| \, \forall x, y \in \mathbb{K}^M$

□

**Proposition 2.15** (Minkowski inequality)**.** *For $p \in [1, \infty], f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$, we have that $\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*$ with $\|f\|_\infty := \inf_{\mu(N)\to 0} \sup_{x\in\Omega\setminus N} |f(x)|$.*

*Proof.* **Case $p = 1$:** trivial

**Case $p \in (1, \infty)$:**

$$\begin{aligned}
\left(\|f + g\|_p^*\right)^p &= \int_\Omega |f(x) + g(x)| \, d\mu(x) \\
&= \int_\Omega |f(x)| \cdot |f(x) + g(x)|^{p-1} \, d\mu(x) \\
&+ \int_\Omega |g(x)| |f(x) + g(x)|^{p-1} \, d\mu(x) \\
&\leq \|f\|_p^* \cdot \left\| |f + g|^{p-1} \right\|_q^* + \|g\|_p^* \cdot \left\| |f + g|^{p-1} \right\|_q^*
\end{aligned}$$

Recognize that $\left(\int |f + g|^p\right)^{\frac{1}{q}} = \left(\int |f + g|^{(p-1)q}\right)^{\frac{1}{q}}$ because $p = q \cdot (p - 1)$

$$\begin{aligned}
&= \left(\|f\|_p^* + \|q\|_p^*\right) \||f + g|\|_p^* \\
\implies \quad &\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*
\end{aligned}$$

↓ *This lecture took place on 2019/04/11.*

**Case $p = \infty$:** First, note that $\forall f \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) \exists N \in \Sigma$ such that $\mu(N) = 0$ and $\|f\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty := \sup_{x\in\Omega\setminus N} |f(x)|$.

**Claim 2.16.**

$$\|f\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty := \sup_{x\in\Omega\setminus N} |f(x)| = \sup_{x\in\Omega\setminus\hat{N}} |f(x)| \text{ for } \mu(\hat{N}) = 0$$

*Proof.* For all $n \in \mathbb{N}$, define $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $\left\|f|_{\Omega\setminus N_n}\right\|_\infty \leq \|f\|_\infty^* + \frac{1}{n}$. Thus with $N := \bigcup_{n\in\mathbb{N}^n} N_n \implies \mu(N) = 0$ and $\|f\|_\infty^* \leq \left\|f|_{\Omega\setminus N}\right\|_\infty \leq \|f\|_\infty^* + \frac{1}{n}$. $n \to \infty \implies \|f\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty$. □

For $f, g \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu)$, pick $N_f, N_G$ such that $\mu(N_C) = \mu(N_g) = 0$ and $\|f\|_\infty^* = \left\|f|_{\Omega\setminus N_\varepsilon}\right\|_\infty$ and $\|g\|_\infty^* = \left\|g|_{\Omega\setminus N_g}\right\|_\infty$.

$$\begin{aligned}
\implies \|f+g\|_\infty^* &\leq \left\|(f+g)_{\Omega\setminus(N_f\cup N_g)}\right\|_\infty \\
&\leq \left\|f|_{f\setminus(N_f\cup N_g)}\right\|_\infty + \left\|g|_{\Omega\setminus(N_f\cup N_g)}\right\|_\infty \\
&\leq \left\|f|_{\Omega\setminus N_f}\right\|_\infty + \left\|g|_{\Omega\setminus N_g}\right\|_\infty = \|f\|_\infty^* + \|g\|_\infty^*
\end{aligned}$$

□

**Proposition 2.17.** *Let $p \in [1, \infty]$. Then $\|\cdot\|_p^*$ is a seminorm on $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$ and $\mathcal{L}^n(\Omega, \mathcal{K}^M, \mu)$ is complete with the seminorm. With $M := \left\{f \in \mathcal{L}^\infty \mid \|f\|_p^* = 0\right\}$, we get that $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)/M$ is a Banach space with respect to $\|[f]\|_p := \|f\|_p^*$.*

*Proof.* Seminorm is clear by Minkowski's inequality. Give completeness of $f^p(\cdot)$, the rest follows from Corollary 2.5.

Hence, show that $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ is complete.

Assume $p < \infty$. By Proposition 2.2, it suffices to show that for $f_n(t_n)_n$ in $\mathcal{L}^p(\cdot)$ such that $a := \sum_{n=1}^\infty \|f_n\|_p^* < \infty$.

$$\implies \exists f \in \mathcal{L}^p(\cdot) : f = \sum_{n=1}^\infty f_n$$

Define $\hat{q}(x) := \sum_{n=1}^\infty |f_n(x)| \in [0, \infty]$. Define $\hat{q}_n(x) := \sum_{i=1}^n |f_i(x)|$. Then $q_n$ is measurable and by Minkowski's inequality,

$$\|q_n\|_p^* \leq \sum_{i=1}^n \|f_i\|_p^* \leq \sum_{i=1}^\infty \|f_i\|_p^* = a < \infty$$

Also $\hat{q}_n^p : x \to \hat{q}_n(x)^n$ is a sequence of positive functions and it is monotonically increasing and converging to $\hat{g}^p$.

By Beppo-Levi (from measure theory):

$$\infty_\Omega \hat{g}^p = \lim_{n\to\infty} \int_\Omega \hat{g}_n = \lim_{n\to\infty} (\|q^n\|_p^*)^p = a^p < \infty$$

$\implies$ $\hat{g}^p < \infty$ almost everywhere (except for a $\mu$ zero-set). Define $g : \Omega \to \mathbb{R}$,

$$x \mapsto \begin{cases} \hat{g}(x) & \text{if } \hat{g}(x) < \infty \\ 0 & \text{else} \end{cases}$$

We get that $g \in \mathcal{L}^n(\Omega, \mathbb{R}, \mu)$ and $g(x) = \lim_{n\to\infty} \sum_{i=1}^n |f_i(x)|$ $\mu$-almost everywhere. Furthermore, by completeness of $\mathbb{K}^M$, $f(x) := \sum_{i=1}^\infty f_i(x)$ exists for $\mu$-almost everywhere. $x \in \Omega$.

Show: $f = \sum_{i=1}^{\infty} f_i$ in $\mathcal{L}^n(\cdot)$, i.e. show that $\lim_{n\to\infty} \int_\Omega \left|\sum_{i=1}^{\infty} f_i\right|_{d_N}^p = \sigma$.

$$\left\|\sum_{i=1}^{n-1} f_i - \sum_{i=1}^{\infty} f_i\right\|_p^* = \left\|\sum_{i=n}^{\infty} f_i\right\|_p^* \xrightarrow{!} 0$$

By contruction, $\left|f\right| \leq q$ almost everywhere $\implies \int_\Omega \left|f\right|^p \leq \int_\Omega g^p < \infty$. Set $h_n(x) = \left|\sum_{i=n}^{\infty} f_i(x)\right|^p$. Then $h_n(x) \to 0$ for $\mu$-almost everywhere $x \in \Omega$ and $h_n(x) \geq 0$ and

$$0 \leq h_n(x) \leq \left(\sum_{i=n}^{\infty} \left|f_i(x)\right|\right)^p \leq q(x)^p$$

Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega h_n(x) = \int_\Omega \lim_{n\to\infty} h_n(x) = 0$$

This completes the assertion since

$$\int_\Omega h_n(x) = \int_\Omega \left|\sum_{i=n}^{\infty} f_i(x)\right|^p = \int_\Omega \left|\sum_{i=1}^{n-1} f_i(x) - f(x)\right|^p = \left(\left\|\sum_{i=1}^{n-1} f_i - f\right\|_p^*\right)^p$$

□

↓ *This lecture took place on 2019/04/30.*

**Proposition** (Proposition 2.15 again)**.** *Let $p \in [1, \infty]$. Then $\|\cdot\|_{L^p}$ is a seminorm, $\mathcal{L}^p(\Omega, \mathbb{K}^n, \mu)$ is complete and $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\cdot)/N$ where $N = \left\{f \mid \left\|f\right\|_{L^p} = 0\right\}$ is a Banach space.*

*Proof.* Assume $p \in [1, \infty]$, then the proof of the last lecture is given.

Assume $p = \infty$. Let $(f_n)_n$ be Cauchy in $\mathcal{L}^\infty$. Remember: $\left\|f\right\|_{L^\infty} := \inf_{\mu(N)=0} \sup_{x\in\Omega\setminus N} \left|f(x)\right|$. Pick $N_{n,m}$ such that $\mu(N_{n,m}) = 0$ and $\left|f_n - f_m\right|_\infty = \left\|(f_n - f_m)\big|_{\Omega\setminus N_{n,m}}\right\|_\infty$. Set $N = \bigcup_{n,m} N_{n,m} \implies \mu(N) = 0$.

Then $\tilde{f}$ is the uniform limit of $f_n \cdot \mathbf{1}_{\Omega\setminus N}$. Hence $\tilde{f}$ is measurable. Also $\left\|\hat{f}\right\|_{L^\infty} := \inf_{\mu(M)=0} \sup_{X\subset\Omega\setminus M} \left|f(x)\right| \leq \left\|f\right\|_\infty \implies \hat{f} \in L^\infty(\Omega, \mathbb{K}^n, \mu)$. Also $\left\|f_n - \hat{f}\right\|_{L^\infty} = \left\|(f_n - f)\big|_{\Omega\setminus N}\right\|_\infty = \left\|f_n\big|_{\Omega\setminus N} - f\right\|_\infty \to 0$ as $n \to \infty$. □

Now $(f_n\big|_N)_n$ is Cauchy with respect to $\|\cdot\|_\infty$. Since $\forall n, m$:

$$\begin{aligned}\left\|f_n\big|_{N^C} - f_M\big|_{N^C}\right\|_\infty &= \left\|(f_n - f_M)\big|_{N^C}\right\|_\infty \\ &\leq \left\|(f_n - f_m)_{N_{m,n}^C}\right\|_\infty \\ &= \left\|f_n - f_m\right\|_{L^\infty}\end{aligned}$$

As in the proof of $C_b$ being a Banach space:

$$\implies \exists f : \Omega \setminus N \to \mathbb{K}^M : \left\|f\right\|_\infty < \infty \text{ and } f_n\big|_{N^C} \to f \text{ w.r.t. } \|\cdot\|_\infty$$

**Remark** (Important special cases)**. Case 1** *$\mu = \mathcal{L}^N$ is the Lebesgue measure on $\Omega \subset \mathbb{R}^N$ (a domain). In this case we write $L^p(\Omega, \mathbb{K}^M) := L^p(\Omega, \mathbb{K}^M, \lambda^M)$ and $L^p(\Omega) := L^p(\Omega, \mathbb{K})$. Here the space $L^p(\Omega, \mathbb{K})$ is considered as functions which are defined almost everywhere.*

**Case 2** *Set $\Omega = \mathbb{N}, \sigma = \mathbb{P}(\mathbb{N}), \mu_c(A) = |A|$.*

*Then*

- *$f : \Omega \to \mathbb{K}^m$ is identified with a sequence $(x_n)_n$ with $x_n \in \mathbb{K}^M$.*
- *$\int_\Omega f(x)\, d\mu(x) \sim \sum_{i\in\mathbb{N}} x_i \in \mathbb{K}^M$*
- *$\mu_c(A) = 0 \iff A = \emptyset$ and the equivalence class construction becomes obsolete.*

*And we denote,*

$$l^p(\mathbb{N}, \mathbb{K}^M) = \mathcal{L}^p(\mathbb{N}, \mathbb{K}^M, \mu_c) \qquad l^p := l^p(\mathbb{N}) = l^p(\mathbb{N}, \mathbb{K})$$

### 2.2.1 Basic properties of Lebesgue spaces

**Proposition 2.18.** *The space $l^p(\mathbb{N}, \mathbb{K}^M)$ is separable for $p \in [1, \infty]$ and* not *separable for $p = \infty$.*

*Proof.* $p < \infty$ Define $l_{i,j} \in l^p(\mathbb{N}, \mathbb{K}^M)$ as

$$(l_{ij})_k := \begin{cases} 0 & \text{if } i \neq k \\ \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^T & \text{if } i = k \end{cases}$$

Then $A := \left\{e_{ij} \mid i \in \mathbb{N}, j \in \{1, \dots, M\}\right\}$ is countable.

It suffices to show that $\overline{\operatorname{span}(A)} = l^p(\mathbb{N}, \mathbb{K}^M)$.

This is true since $\forall x \in l^p(\mathbb{N}, \mathbb{K}^M) : \forall \varepsilon > 0 \exists n_0 : \sum_{n_0=1}^\infty |x_i|^p < \varepsilon$ and hence $\left\|x - \sum_{i=1}^{n_0} \sum_{j=1}^M x_{ij} e_{ij}\right\|^p = \left(\sum_{i=n_0+1} |x_i|^n\right)^{\frac{1}{n}} < \varepsilon^{\frac{1}{p}}$

$p = \infty$ It suffices to show that $L^\infty(\mathbb{N})$ is not separable (why?). For $M \subset \mathbb{N}$ define $\mathbf{1}_M \in L^\infty$. Then $\triangle := \{\mathbf{1}_M \mid M \subset \mathbb{N}\}$ is uncountable.

For $A \subset L^\infty$ countable and $x \in A$ set $M_X = \left\{y \in L^\infty \mid \left\|x - y\right\|_\infty < \frac{1}{3}\right\} = B_{\frac{1}{3}}(x)$. Then each $M_X$ contains at most one element of $\triangle$ since if $\mathbf{1}_M \neq \mathbf{1}_{M'}$ are such that $\mathbf{1}_M, \mathbf{1}_{M'} \in M_X$.

$$\implies 1 = \|\mathbf{1}_M - \mathbf{1}_{M'}\|_\infty \leq \|\mathbf{1}_M - x\| + \|\mathbf{1}_{X'} - x\| < \frac{2}{3}$$

This gives a contradiction.

$\triangle$ is uncountable, $\{M_X \mid x \in A\}$ is countable.

$$\implies \exists \hat{M} \in \mathbb{N} : \mathbf{1}_{\hat{M}} \notin M_x \forall x \in A$$

$$\implies \left\|\mathbf{1}_{\hat{M}} - x\right\|_\infty \geq \frac{1}{3} \forall x \in A$$

Hence, $A$ is not dense. Since $A$ was arbitrary countable. Thus $L^\infty$ is not separable.

□

### 2.2.2 Separability of $L^p$ requires a density result

**Proposition 2.19.** *Let $f \in L^p(\mathbb{R}^N, \mathbb{K}^M)$. Let $p < \infty$. Then $\exists (f_n)_n \in \ldots$ $C_C(\mathbb{R}^N, \mathbb{R}^M)$ such that $\left\|f_n - f\right\|_{L^n} \to 0$ as $n \to \infty$.*

*Proof.* **Step 1** Reduction to step functions with $E \in \Sigma$.

$$\xi_E(x) := \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$$

Take $f \in L^p(\ldots)$. For $\varepsilon > 0$, define

$$E_C = \lceil x : \varepsilon \leq \left|f\right| \leq \frac{1}{\varepsilon} \rceil$$

Then $E_\varepsilon \in \Sigma$ and $\int_{\mathbb{R}^N} \left|f\right|^p \geq \varepsilon^p \left|E_\varepsilon\right|$ where $\left|E_\varepsilon\right| := L^N(E_\varepsilon)$.

$$\left|E_\varepsilon\right| < \infty \text{ and } \int_{\mathbb{R}^N} \left|\mathbf{1}_{E_\varepsilon} f\right| \leq \frac{1}{\varepsilon} \cdot \left|E_\varepsilon\right| < \infty$$

$$\implies \mathbf{1}_{E_\varepsilon} f \text{ is integrable } \implies \exists (q_{n,\varepsilon})_n \text{ step functions}$$

such that $\int_{\mathbb{R}^N} \left|\mathbf{1}_{E_\varepsilon} f - q_{n,\varepsilon}\right| \to 0$ as $n \to \infty$. Define

$$f_{n,\varepsilon}(x) := \begin{cases} q_{N,\varepsilon}(x) & \text{if } x \in E_\varepsilon, \left|q_{n,\varepsilon}(x)\right| \leq \frac{2}{\varepsilon} \\ \frac{2}{\varepsilon} \frac{q_{n,\varepsilon}(x)}{\left|q_{n,\varepsilon}(x)\right|} & \text{if } x \in E_\varepsilon, \left|q_{n,\varepsilon}(x)\right| > \frac{2}{\varepsilon} \\ 0 & \text{else} \end{cases}$$

Hence $(f_{n,\varepsilon})_n$ is a sequence of step functions. For $x \in E_\varepsilon$ such that $\left|q_{n,\varepsilon}(x)\right| > \frac{2}{\varepsilon}$.

$$\implies \left|f_{n,\varepsilon}(x) - f(x)\right| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} = \frac{3}{\varepsilon} \leq 3 \underbrace{\left(\left|q_{n,\varepsilon}(x)\right| - \left|f(x)\right|\right)}_{\geq \frac{1}{\varepsilon}} \leq 3 \left|q_{n,\varepsilon}(x) - f(x)\right|$$

$$\int_{\mathbb{R}^n} \left|f_{n,\varepsilon}(x) - X_{E_\varepsilon}(x) f(x)\right| dx \leq 3 \int_{\mathbb{R}^N} \left|g_{n,\varepsilon}(x) - \mathbf{1}_{E_\varepsilon}(x) f(x)\right| dx \to 0 \text{ as } n \to \infty$$

$$\int_{\mathbb{R}^N} \left|f - f_{n,\varepsilon}\right|^p \leq \int_{\mathbb{R}^N \setminus E_\varepsilon} \left|f\right|^p + \left(\frac{3}{\varepsilon}\right)^{p-1} \underbrace{\int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right|}_{(*)} =: (X)$$

$$(*) = \int_{E_\varepsilon} \left|R - f_{n,\varepsilon}\right|^p = \int_{\mathbb{R}^N} \left|f \cdot \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right|^p = \int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right| \left( \underbrace{\left|f \mathbf{1}_{E_\varepsilon}\right|}_{\leq \frac{1}{3}} + \underbrace{\left|f_{n,\varepsilon}\right|^{p-1}}_{\leq \frac{2}{3}} \right)$$

Now given $\delta > 0$, we first fix $\varepsilon > 0$ such that $\int_{\mathbb{R}^N \setminus E_\varepsilon} \left|f\right|^p < \frac{\delta}{2}$. Then we find $n_0$ such that $\left(\frac{3}{\varepsilon}\right)^{n-1} \int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right| < \frac{\delta}{2}$. This is possible since $\mathbb{R}^N = \bigcup_{\varepsilon > 0} E_\varepsilon$ and $\int_{\mathbb{R}^N} \left|f\right|^n < \infty$.

$$\Longrightarrow (X) < \delta$$

Now suppose $\forall \varepsilon > 0 \forall E \in \Sigma : \exists \varphi \in C_C(\mathbb{R}^N, \mathbb{K}^M)$ such that $\left\|\mathbf{1}_E - \varphi\right\| < \varepsilon$. We need to show that this is true. Then for $f \in L^p(\mathbb{R}^N, \mathbb{K}), \varepsilon > 0$, we pick

$$g = \sum_{i=1}^n \underbrace{c_i}_{\in \mathbb{K}^M} \cdot \underbrace{\mathbf{1}_{E_i}}_{\in \Sigma}$$

such that $\left\|f - g\right\|_p < \frac{\varepsilon}{2}$ (possible by what we just showed). For $i \in \mathbb{N}$, pick $\varphi_i \in C_C(\mathbb{R}^N, \mathbb{R})$ such that $\left\|\mathbf{1}_{E_i} - \varphi_i\right\|_p \leq \frac{2^{-i}\varepsilon}{|C_i| 2}$

$$\begin{aligned} \Longrightarrow \left\|f - \underbrace{\sum_{i=1}^n c_i \cdot \varphi_i}_{\in C_C(\mathbb{R}^n, \mathbb{K}^n)}\right\|_p &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \left\|c_i \mathbf{1}_{E_i} - c_i \varphi_i\right\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n |c_i| \cdot \left\|\mathbf{1}_{E_i} - \varphi_i\right\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n 2^{-i} \cdot \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

$\square$

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