

Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

Lukas Prokop

March to July 2016

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This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n \in \mathbb{N}}$ be a complex series with $\lim_{n \rightarrow \infty} z_n = z$ and $\lim_{n \rightarrow \infty} (1 + \frac{z_n}{n})^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire \mathbb{C} .

$$\exp(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\exp(z + w) = \exp(z) \cdot \exp(w)$$

$$\lim_{z \rightarrow 0} \frac{\exp(z) - 1}{z} = 1$$

$$\exp(1) = e \in \mathbb{R}$$

$$z = \frac{m}{n} \in \mathbb{Q} \wedge n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z \quad \text{for } z \in \mathbb{C}$$

It holds that

$$\exp(z) \neq 0 \quad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$

$$e^x > 0 \quad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in \mathbb{R} .

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^0 = 1$$

Let $n \in \mathbb{N}$

$$\lim_{x \rightarrow +\infty} \frac{e^x}{x^n} = \infty$$

$$\lim_{x \rightarrow -\infty} e^x \cdot x^n = 0$$

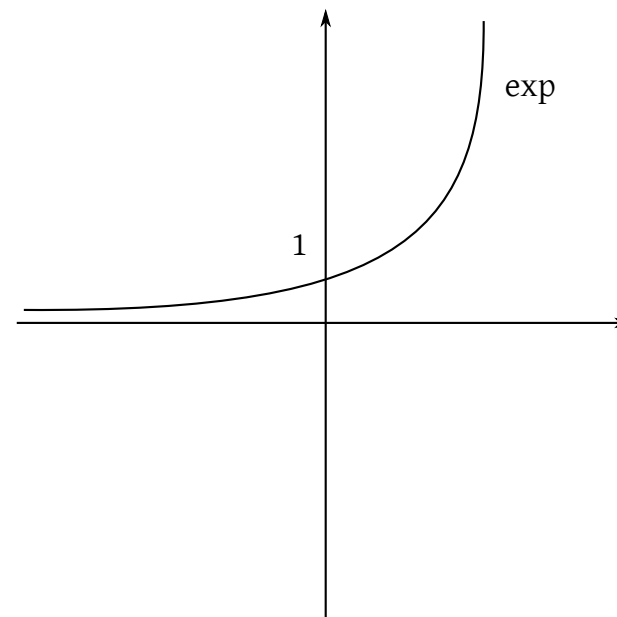


Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp : \mathbb{R} \rightarrow (0, \infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. $\exp : \mathbb{R} \rightarrow (0, \infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course “Analysis 1”.

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y] : \quad e^\xi = y$$

Case 2 Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w] : \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp : \mathbb{R} \rightarrow (0, \infty)$ is bijective. \square

Definition 1. We call the inverse function *natural logarithm*¹.

$$\exp^{-1} : (0, \infty) \rightarrow \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R} : \ln(e^x) = x$ and $\forall y \in (0, \infty) : e^{\ln(y)} = y$.
- $\ln : (0, \infty) \rightarrow \mathbb{R}$ is strictly monotonically increasing

Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xrightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! \square

¹In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Limes:

$$\lim_{x \rightarrow 1} \frac{\ln(x)}{x - 1} = 1$$

Proof. •

$$x \cdot y = e^{\ln(x \cdot y)}$$

$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$

Injectivity of exp:

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \rightarrow \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \rightarrow \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.

$$\lim_{n \rightarrow \infty} y_n = \ln(1) = 0$$

$$\lim_{n \rightarrow \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \rightarrow \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$

where

$$e^0 = 1 \Rightarrow \ln(1) = 0$$

\square

Theorem 1 (Logarithmic growth). $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \rightarrow \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \rightarrow \infty \Leftrightarrow \xi \rightarrow \infty$$

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \rightarrow \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \rightarrow \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \rightarrow \infty} \frac{\xi^2}{e^\xi} = 0$. \square

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

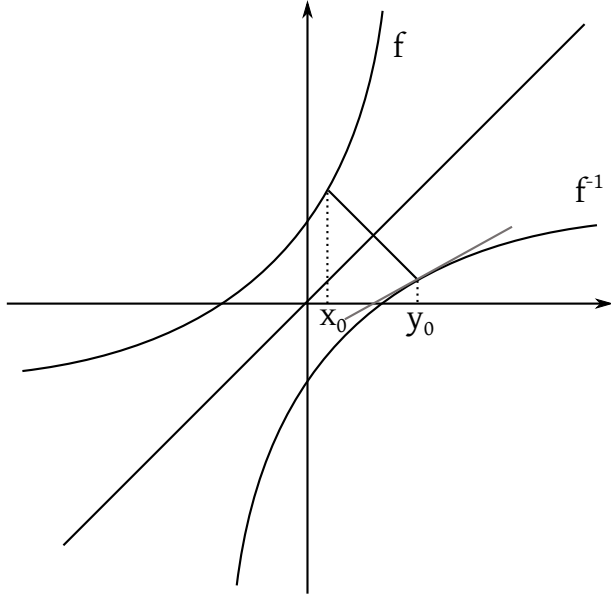


Figure 2: A geometric proof of differentiability

Proof. First approach Let $x > 0$, $x_n \rightarrow x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \quad e^{\xi} = x \quad \xi_n \rightarrow \xi$$

Then it holds that

$$\lim_{n \rightarrow \infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n \rightarrow \infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n \rightarrow \infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$

$$f'(f^{-1}(y)) \cdot (f^{-1})'$$

$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$\begin{aligned} [\ln(x)]' &= \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}} \\ &= \frac{1}{x} \cdot \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x} \rightarrow 0 \end{aligned}$$

$1 + \frac{h}{x} = w$ then it holds that $h \rightarrow 0 \Rightarrow w \rightarrow 1$.

$$\begin{aligned} \frac{h}{x} &= w - 1 \\ \lim_{h \rightarrow 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} &= \lim_{h \rightarrow 0} \frac{\ln(w)}{w - 1} = 1 \end{aligned}$$

□

Remark 1. The exponential function can be defined from \mathbb{C} to \mathbb{C} .

$$\exp : \mathbb{C} \rightarrow \mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire \mathbb{C} (or $\mathbb{C} \setminus \{0\}$). We can only define a continuous inverse function of \exp in $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I \rightarrow \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I) \rightarrow \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

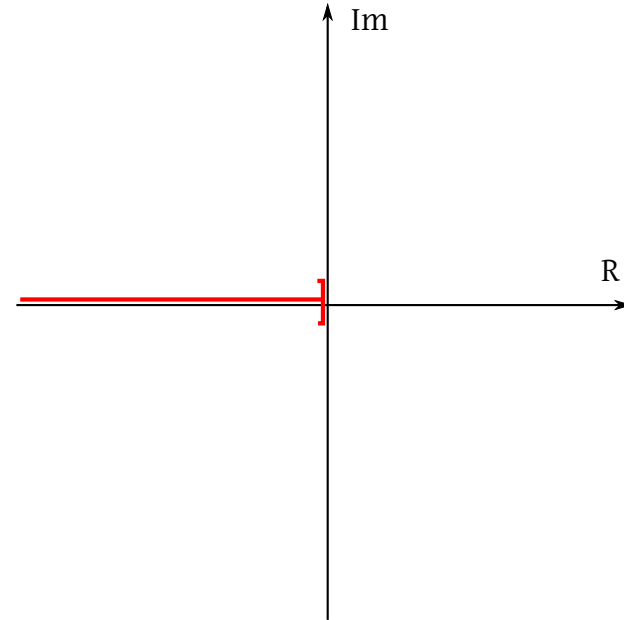


Figure 3: Continuous exponential function in \mathbb{C}

Proof. Let $y_n \rightarrow y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \Rightarrow x_n \neq x_0$.

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} \\ &= \lim_{n \rightarrow \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{\lim_{n \rightarrow \infty} \underbrace{\frac{f(x_n) - f(x_0)}{x_n - x_0}}_{\text{ex} = f'(x_0)}} = \frac{1}{f'(x_0)} \end{aligned}$$

□

Lemma 2. Let $f : I \rightarrow \mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \forall x \in I$$

Proof. \Rightarrow Immediate.

\Leftarrow Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

□

Definition 2. Let I be an interval, $f : I \rightarrow \mathbb{R}$. A function $F : I \rightarrow \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \rightarrow \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f . Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xrightarrow{\text{Lemma 2}} F_1 - F_2 = \text{const}$$

□

Theorem 4. Let I be an interval. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable functions in I .

$$f_n : I \rightarrow \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \rightarrow \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ($f_n \rightarrow f$ pointwise)
2. for every $x \in I$ let $(f'_n(x))_{n \in \mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n \rightarrow \infty} f'_n(x)$ exists for every x)

3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon |u - v| \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n \rightarrow \infty} f'_n(x)$.

$$f'(x) = \left[\lim_{n \rightarrow \infty} f \right]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\begin{aligned} & \left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right| \\ &= \left| \frac{f(x) - f(x_0)}{x - x_0} - \lim_{n \rightarrow \infty} f'_N(x_0) \right| \\ &= \left| \frac{f(x) - f(x_0)}{x - x_0} - f'_N(x_0) \right| + \left| f'_N(x_0) - \lim_{n \rightarrow \infty} f'_n(x_0) \right| \forall N \in \mathbb{N} \\ &\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| \\ &\quad + \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) \right| + |f'_N(x_0) - \varphi(x_0)| \end{aligned}$$

1st term

$$\begin{aligned} & \left| \frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0} \right| = \left| \frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0} \right| \\ & \leq \frac{\varepsilon |x - x_0|}{3 |x - x_0|} \stackrel{\text{condition 3}}{=} \frac{\varepsilon}{3} \end{aligned}$$

for sufficiently large N .

3rd term $|f'_N(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N .

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left| \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| - f'_N(x_0)$$

Differentiability of f_N : Therefore for $|x - x_0| < \delta$.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$. \square

Theorem 5. Let $f_n : I \rightarrow \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ ($n \in \mathbb{N}$) and f_n is differentiable in I .

Assumption:

1. $f_n \rightarrow f$ converges pointwise in I (like the first statement in the previous Theorem)
2. There exists $g : I \rightarrow \mathbb{R}$ such that $f'_n \rightarrow g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem). Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \rightarrow f$ pointwise
- $f'_n(x) \rightarrow \varphi(x)$ for every x
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon |u - v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \rightarrow \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ and it holds that

- $f_n \rightarrow f$ pointwise in I for $n \rightarrow \infty$
- $\exists g : I \rightarrow \mathbb{R}$ such that $f'_n \rightarrow g$ is *uniform* in I , hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f'_n(x) - g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6.

We look at the conditions of Theorem 6:

2. Uniform convergences of $f'_n \rightarrow g$ implies pointwise convergence

$$\forall x \in I : f'_n(x) \rightarrow g(x)$$

3. From uniform convergence of $f'_n \rightarrow g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f'_n(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$\begin{aligned} |f'_n(x) - f'_m(x)| &= |f'_n(x) - g(x) + g(x) - f'_m(x)| \\ &\leq |f'_n(x) - g(x)| + |g(x) - f'_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $(f'_n)_{n \in \mathbb{N}}$ is a uniform Cauchy sequence.

Let $\varepsilon > 0$ be arbitrary and N such that $n, m \geq N$ and $x \in I$:

$$|f'_n(x) - f'_m(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m \rightarrow \infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$\begin{aligned} &= \lim_{m \rightarrow \infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)| \\ &= \lim_{m \rightarrow \infty} |f'_m(\xi_{m,n}) - f'_n(\xi_{m,n})| \cdot |u - v| \end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

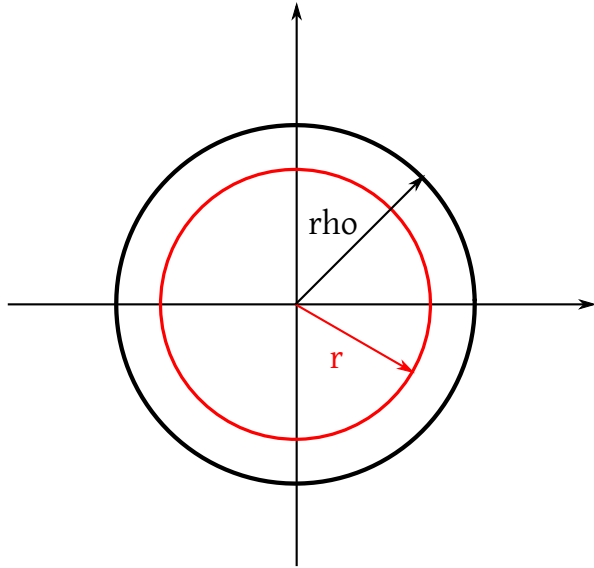


Figure 4: Convergence radius

Remark 2 (An application of Theorem 7). Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \quad L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^n a_k z^k \quad \dots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \rightarrow P(z)$ uniform in $\overline{B(0, r)}$ ².

$$P_n(x) \rightarrow P(x) \forall x \in [-r, r]$$

□ Compare with Figure 4.

$$P'_n(x) = \sum_{k=0}^n a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1} (j+1) x^j$$

is the $n - 1$ -th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1} (j+1) z^j$$

Convergence radius of Q ?

$$\begin{aligned} \tilde{L} &= \limsup_{j \rightarrow \infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j \rightarrow \infty} |a_{j+1}|^{\frac{j+1}{j}} \cdot \frac{1}{j+1} \cdot (j+1)^{\frac{j+1}{j} \cdot \frac{1}{j+1}} \\ &= \limsup_{j \rightarrow \infty} \underbrace{\left(|a_{j+1}|^{\frac{j+1}{j}} \cdot \frac{1}{j+1} \right)}_{L^1=L} \cdot \underbrace{\lim_{j \rightarrow \infty} \left[(j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L \end{aligned}$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P'_n(z) = \sum_{k=1}^n k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0, r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

²Where overline means “closed”

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k \rightarrow \infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k \rightarrow \infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$\begin{aligned} [-\ln(1-x)]' &= -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x} \\ \Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) &= \text{constant} \end{aligned}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$\begin{aligned} 0 + 0 &= 0 = \text{constant} \\ \Rightarrow \ln(1-x) &= -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } |x| < 1 \end{aligned}$$

Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\begin{aligned} \Rightarrow \ln(1 - (-x)) &= \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2 \sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \quad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{in } (-1, 1)$$

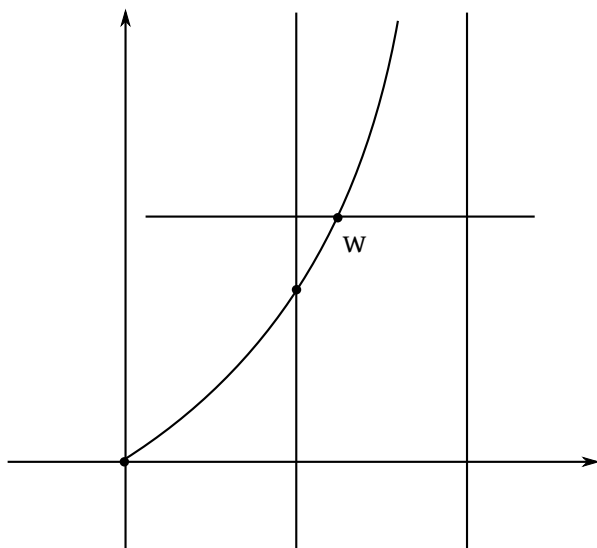
Solve $\frac{1+x}{1-x} = w$ for x .

$$\Rightarrow 1+x = w - wx$$

$$x(1+w) = w-1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2 \sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$


 Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometric functions

We define trigonometric functions using the exponential function in \mathbb{C} .

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n \rightarrow \infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

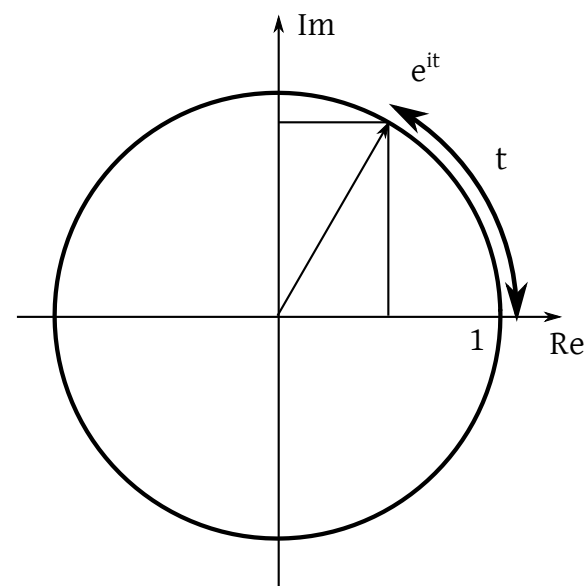
$$e^{-it} = \lim_{n \rightarrow \infty} \left(1 - \frac{it}{n} \right)^n = \lim_{n \rightarrow \infty} \left[\overline{\left(1 + \frac{it}{n} \right)} \right]^n$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \overline{\left(1 + \frac{it}{n} \right)^n} = \overline{\lim_{n \rightarrow \infty} \left(1 + \frac{it}{n} \right)^n} = \overline{e^{it}} \\ &|e^{it}|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it} \\ &e^{it-it} = e^0 = 1 \end{aligned}$$

So it holds that $\forall t \in \mathbb{R}$:

$$|e^{it}| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.


 Figure 6: Unit circle in \mathbb{C} with t

We define the cosine function $\cos : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

$$1. \quad e^{it} = \cos(t) + i \cdot \sin(t) \text{ (Euler's identity)}$$

$$2. \quad |e^{it}|^2 = 1 = (\cos t)^2 + (\sin t)^2$$

3.

$$\begin{aligned} \Re(z) &= \frac{1}{2}(z + \bar{z}) \\ \Rightarrow \cos(t) &= \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{-it}) \end{aligned}$$

$$\begin{aligned} \Im(z) &= \frac{1}{2i}[z - \bar{z}] \\ \sin(t) &= \Im(e^{it}) = \frac{1}{2i}[e^{it} - e^{-it}] \end{aligned}$$

4.

$$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \rightarrow \mathbb{C}$ and $\cos : \mathbb{C} \rightarrow \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}[e^{iz} + e^{-iz}]$$

$$\sin(z) = \frac{1}{2i}[e^{iz} - e^{-iz}]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$\begin{aligned} t \in \mathbb{R} : \cos t &= \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{-it}) \\ \sin t &= \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it}) \end{aligned}$$

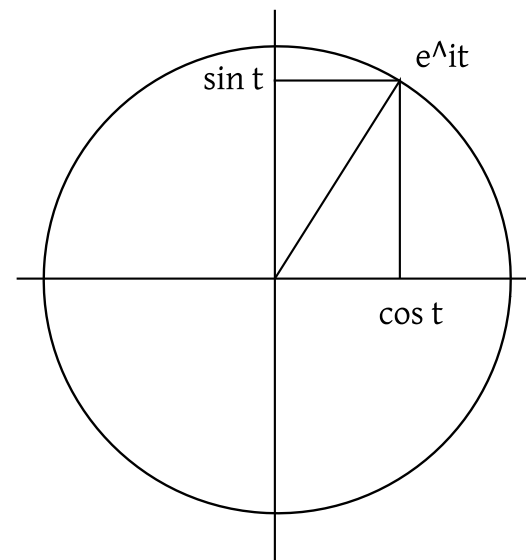


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i(-z)}) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence). Let $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty} (a_n + b_n)$ is absolute convergent with sum $s + s'$.

series sum. Absolute convergence. Show that $\sum_{k=0}^n |a_k + b_k| = t_n$ and $(t_n)_{n \in \mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^n |a_k + b_k| \leq \underbrace{\sum_{k=0}^n |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^n |b_k|}_{\text{bounded}}$$

□

Example 1 (Application). Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0, \delta)$. Then also $\sum_{k=0}^{\infty} (a_k + b_k) z^k$ is convergent in $B(0, \delta)$ and it holds that $\sum_{k=0}^{\infty} (a_k + b_k) z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \quad i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 = i^0 \quad i^5 = i \quad \dots$$

$$\Rightarrow = 1 + i \frac{z}{1!} - \frac{z^2}{2!} - i \frac{z^3}{3!} + \frac{z^4}{4!} + i \frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \quad (-i)^1 = -i \quad (-i)^2 = -1 \quad (-i)^3 = i \quad (-i)^4 = 1 \quad \dots$$

$$\Rightarrow = 1 - i \frac{z}{1!} - \frac{z^2}{2!} + i \frac{z^3}{3!} + \frac{z^4}{4!} - i \frac{z^5}{5!} - \frac{z^6}{6!} + \dots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \dots$$

Followingly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C}$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \dots$$

$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and subtraction theorems). We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i \sin(z+w)) \\ \Rightarrow &= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z \cdot \cos w) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i(-\cos z \sin w - \cos w \sin z) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i \sin(-(z+w))) \\ \Rightarrow &= \cos(z+w) - i \sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2 \cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\begin{aligned} \cos(z-w) &= \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \cdot \underbrace{\sin w}_{=-\sin(-w)} \\ \sin(z-w) &= -\cos z \cdot \sin(w) + \sin(z) \cos(w) \end{aligned}$$

Corollary 1.

$$\begin{aligned} z &= \frac{1}{2}(z+w) + \frac{1}{2}(z-w) \\ \Rightarrow \cos z &= \cos \frac{z+w}{2} \cos \frac{z-w}{2} - \sin \frac{z+w}{2} \sin \frac{z-w}{2} \\ w &= \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w) \\ \cos w &= \cos \frac{z+w}{2} \cdot \cos \frac{z-w}{2} + \sin \frac{z+w}{2} \cdot \sin \frac{z-w}{2} \\ \cos z - \cos w &= -2 \sin \frac{z+w}{2} \sin \frac{z-w}{2} \end{aligned}$$

Analogously,

$$\sin z - \sin w = 2 \cos \frac{z+w}{2} \cdot \cos \frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z \rightarrow 0 \\ z \neq 0}} \frac{\sin z}{z} &= \lim_{z \rightarrow 0} \frac{1}{2i} \left(\frac{e^{iz} - e^{-iz}}{z} \right) \\ &= \lim_{z \rightarrow 0} e^{-iz} \left(\frac{e^{2iz} - 1}{2iz} \right) \\ &= \underbrace{\lim_{z \rightarrow 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z \rightarrow 0} \frac{e^{2iz} - 1}{2iz}}_{\substack{e=2iz; z \rightarrow 0 \Leftrightarrow w=0 \\ \lim_{w \rightarrow 0} \frac{e^w - 1}{w} = 1}} \end{aligned}$$

So it holds that

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled “definition of π ” and “periodicity”.

Let $x \in \mathbb{R}$.

$$\cos x = \underbrace{1}_{=c_0} - \underbrace{\frac{x^2}{2}}_{=c_1} + \underbrace{\frac{x^4}{24}}_{=c_2} - \underbrace{\frac{x^6}{720}}_{=c_3} + \underbrace{\frac{x^8}{40320}}_{=c_4} - \dots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \quad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0, 2]$ and $k \geq 1$ it holds that

$$\left| \frac{c_{k+1}}{c_k} \right| = \left| \frac{x^2}{(2k+2)(2k+1)} \right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k \geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0, 2]$.

Similarly for $x \in (0, 2]$:

$$\left| \frac{s_{k+1}}{s_k} \right| = \left| \frac{x^2}{(2k+2)(2k+3)} \right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{in } [0, 2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0, 2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$

Let $x \in (0, 2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1 - \frac{4}{6} = \frac{1}{3} > 0} > 0$$

So it holds that $\sin(x) > 0$ in $(0, 2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = -2 \cdot \underbrace{\sin \frac{z+w}{2}}_{\in (0,2]} \cdot \underbrace{\sin \frac{z-w}{2}}_{\in (0,2]} = \underbrace{\phantom{-2 \cdot \sin \frac{z+w}{2} \cdot \sin \frac{z-w}{2}}}_{<0} > 0$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that \cos is a strictly monotonically decreasing function in $[0, 2]$. Hence \cos has only one root because it is continuous in $(0, 2]$.

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in $(0, 2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos \frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left| \sin \frac{\pi}{2} \right| = 1$$

We know that $\sin x > 0$ for $x \in (0, 2]$.

$$\Rightarrow \sin \frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

TODO: table missing

$$e^{i\pi} = e^{i\frac{\pi}{2} + i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$

$$e^{i\frac{3}{2}\pi} = e^{i\pi + i\frac{1}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$

$$e^{z+2i\pi} = e^z \cdot (e^{i\pi})^2 = e^z$$

So the exponential function is periodic in \mathbb{C} with period $2i\pi$.

$$\begin{aligned} \cos(z + 2\pi) &= \frac{1}{2} (e^{iz+2\pi i} + e^{-iz-2\pi i}) \\ &= \frac{1}{2} \left(e^{iz} + e^{-iz} \cdot \underbrace{\frac{1}{e^{2\pi i}}}_{=1} \right) = \cos z \end{aligned}$$

Therefore the cosine is periodic in \mathbb{C} with period 2π . Analogously, sine is periodic in \mathbb{C} with period 2π .

This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

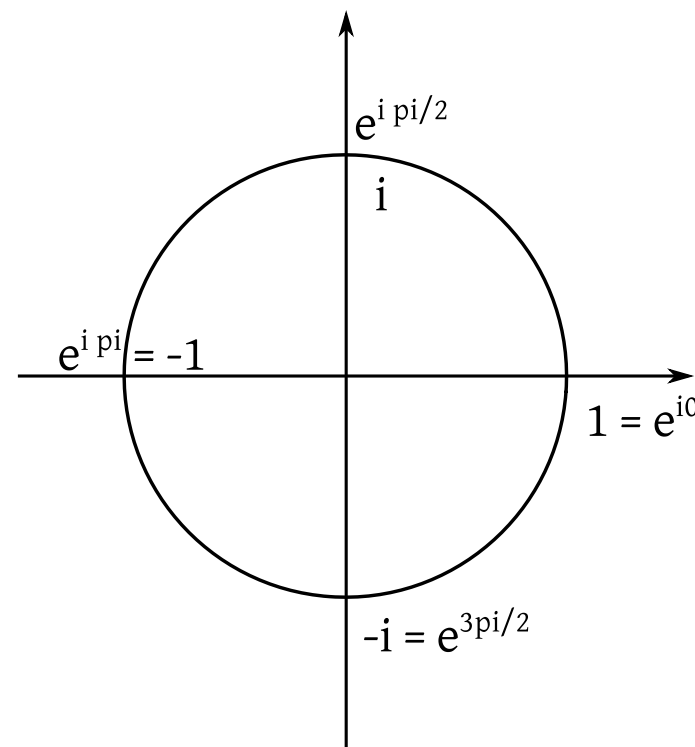
3.5 Periodicity and roots of trigonometric functions

TODO: equations missing

$$\cos(z + 2\pi) = \cos(z)$$

$$\sin(z + 2\pi) = \sin(z)$$

Remark 3. We will show: $\forall c \in (0, 2\pi)$, \cos and \sin are non-periodic with period c , hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x + c)$.



Definition 5.

$$f : \mathbb{C} \rightarrow \mathbb{C} \quad (f : \mathbb{R} \rightarrow \mathbb{R})$$

is called *periodic* with period $c \in \mathbb{C}$ ($c \in \mathbb{R}$) if $\forall z \in \mathbb{C}$ it holds that

$$f(z + c) = f(z)$$

$$(\forall x \in \mathbb{R} : f(x + c) = f(x))$$

c is called *period of f* .

Remark 4. If f is periodic with period $c \in \mathbb{C}$, then f is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark 5.

$$\begin{aligned} z &= u + iv \\ \Re(i \cdot z) &= \Re(iu - v) = -v = -\Im(z) \\ \Im(i \cdot z) &= \Im(iu - v) = u = \Re(z) \end{aligned}$$

Remark 6. Let $x \in \mathbb{R}$.

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \Re(e^{i(x+\frac{\pi}{2})}) \\ &= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}}) \\ &= \Re(ie^{ix}) \\ &= -\Im(e^{ix}) \\ &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \Im(e^{i(x+\frac{\pi}{2})}) \\ &= \Im(ie^{ix}) \\ &= \Re(e^{ix}) \\ &= \cos(x) \end{aligned}$$

$$\begin{aligned} \cos\left(x - \frac{\pi}{2}\right) &= \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x - \frac{\pi}{2}\right) &= -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\cos(x) \end{aligned}$$

Summary:

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x - \frac{\pi}{2}\right) &= \sin(x) \\ \sin\left(x - \frac{\pi}{2}\right) &= -\cos(x) \end{aligned}$$

Remark 7 (A remark on the name “cosine”).

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of x (Compare with Figure 8).

Remark 8.

$$\begin{aligned} \cos(x + \pi) &= \Re(e^{i(x+\pi)}) \\ &= \Re(-e^{ix}) \\ &= -\cos(x) \\ \sin(x + \pi) &= -\sin(x) \end{aligned}$$

Remark 9. Let $0 < c < 2\pi$. Assume \cos is periodic with period c . We know that \cos has exactly one root in $[0, 2]$,

$$\cos(x) = \cos(-x)$$

\cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

Therefore \cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.

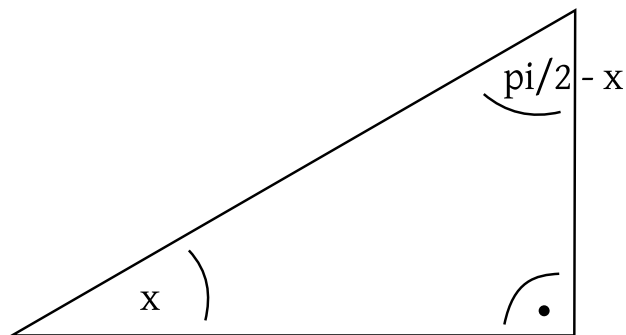


Figure 8: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$

c would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of } \cos \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore \cos is not periodic with period c . Hence 2π is indeed the smallest period of \cos .

Analogously it holds for \sin .

Remark 10 (Roots of \cos).

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \quad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \quad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$

$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \quad \forall l \in \mathbb{Z}$$

3.6 Derivatives of trigonometric functions

It holds that

$$\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$$

Furthermore it holds that

$$\lim_{z \rightarrow 0} \frac{1 - \cos z}{z} = 0$$

Proof.

$$\begin{aligned} \frac{1 - \cos z}{z} &= \frac{1}{z} \left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots \right) \\ &= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots \end{aligned}$$

is convergent in \mathbb{C} and (especially) continuous in 0

$$\lim_{z \rightarrow 0} \left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots \right) = 0$$

□

$$\lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h}$$

This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{\sin z}{z} &= 1 \\ \lim_{z \rightarrow 0} \frac{1 - \cos z}{z} &= 0 \end{aligned}$$

Lemma 6. The trigonometric functions \sin and \cos are differentiable in \mathbb{R} (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x) \quad \sin'(x) = \cos(x)$$

Proof.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(h)}{h} &= \lim_{h \rightarrow 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cos(h) - 1}{h} - \lim_{h \rightarrow 0} \frac{\sin x \cdot \sin h}{h} \\ &= \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h}}_{=0} - \sin x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin(h)}{h}}_{=1} \\ &= -\sin(x) \end{aligned}$$

Analogously:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(h)}{h} &= \lim_{h \rightarrow 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} \\ &= \sin(x) \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h}}_{=0} + \cos(x) \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_{=1} \\ &= \cos(x) \end{aligned}$$

□

TODO: incomplete graphics, verify text

Figure 9. We now use tools of integral calculus:

Let $I = [a, b]$ and $\gamma : I \rightarrow \mathbb{R} (\mathbb{R}^2)$.

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

Assumption: $\gamma_1 : [a, b] \rightarrow \mathbb{R}^n$.

$$\gamma'(t) = \begin{bmatrix} \gamma'_1(t) \\ \vdots \\ \gamma'_n(t) \end{bmatrix}$$

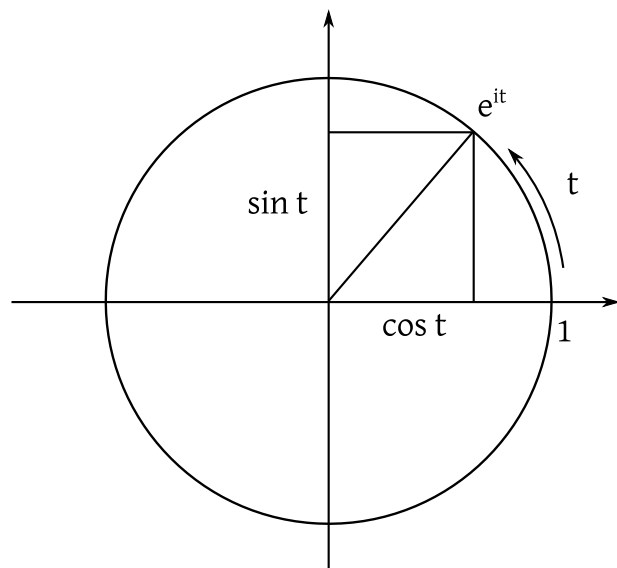


Figure 9: The arc length is related to sin and cos

TODO: graphics missing

Let $t \in [a, b]$. Then the arc length of γ between a and t is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \, d\tau$$

We identify \mathbb{C} with \mathbb{R}^2 :

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

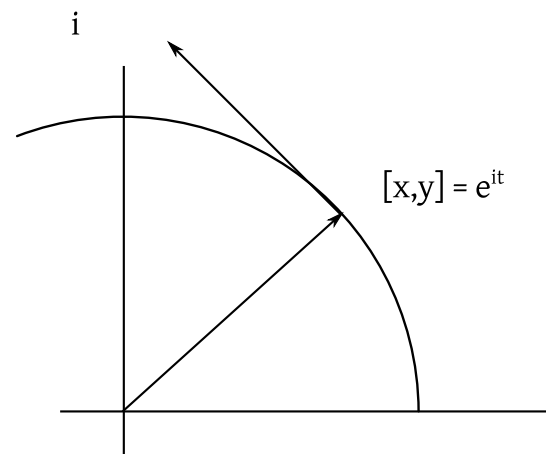
$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \rightarrow \mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$


 Figure 10: Derivative in \mathbb{R}^2

Compare with Figure 10.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \, d\tau = \int_0^t 1 \, d\tau = t$$

4 Integration calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will be discussed in further details in the courses “Functional analysis” and “Measure and integration theory”. For now, we look at the basis (as discussed by Königsberger).

Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \rightarrow \mathbb{R}$. We call ϕ a *step function*, if $n \in \mathbb{N}$ and x_0, \dots, x_n exist such that

$$x_0 = a < x_1 < x_2 < \dots < x_n = b$$

and $\phi|_{(x_{j-1}, x_j)} = c_j$ is constant. The points x_j define a partition of the interval $[a, b]$.

$\tau[a, b]$ defines the set of step functions of interval $[a, b]$. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 11).

Definition 6. Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a step function and $x_0 = a < x_1 < \dots < x_n = b$ as partition of $[a, b]$ and let $\phi|_{(x_{j-1}, x_j)} = c_j$ for $j = 1, \dots, n$. Then we define

$$\int_a^b \phi dx = \sum_{j=1}^n c_j \Delta x_j$$

where $\Delta x_j = x_j - x_{j-1}$ (for $j = 1, \dots, n$).

$$\int_a^b \phi dx \text{ is called } \textit{integral} \text{ of } \phi \text{ over } [a, b]$$

ϕ is the step function in terms of the partition $\{x_0, x_1, \dots, x_n\}$.

It remains to show that if ϕ satisfies the definition of a step function in terms of partition $\{x_0, \dots, x_n\}$ and $\phi|_{(x_{j-1}, x_j)} = c_j$ (TODO: text missing: “but ...”) and ϕ is a step function in terms of $\{w_0, w_1, \dots, w_m\}$ and $\phi|_{(w_{l-1}, w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^n c_j \Delta x_j = \sum_{l=1}^m c'_l \Delta w_l$$

Compare with Figure 12.

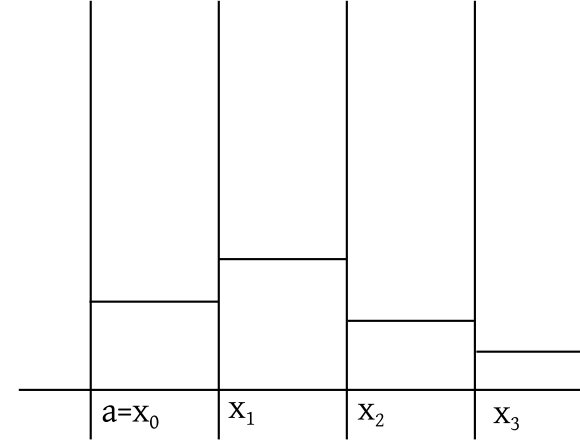


Figure 11: Partition of an area into rectangles

Proof. Let $Z = \{x_0, \dots, x_n\}$ and $Z' = \{w_0, \dots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \dots < \alpha_L = b$$

Because $Z \subseteq Z''$,

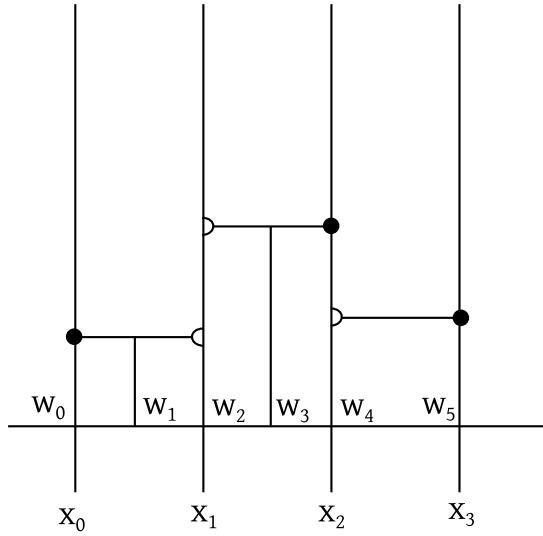
$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly,

$$k_{j-1} < k_j$$

Let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$


 Figure 12: Step function φ

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \dots, L\}$ there exists $j \in \{1, \dots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$\Rightarrow \varphi|_{(\alpha_{l-1}, \alpha_l)}$ is constant

Hence φ is a step function in terms of the partition $\{\alpha_0, \dots, \alpha_L\}$.

Let $l \in \{0, 1, \dots, L\}$ and j such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c_l'' = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c_l'' = c_j$.

$$\begin{aligned} \sum_{l=1}^L c_l'' \cdot \Delta \alpha_l &= \sum_{j=1}^n \sum_{l=k_{j-1}+1}^{k_j} c_l'' \Delta \alpha_l \\ &= \sum_{j=1}^n c_j \sum_{l=k_{j-1}}^{k_j} \Delta \alpha_l \end{aligned}$$

$$\begin{aligned} \sum_{l=k_{j-1}+1}^{k_j} \Delta \alpha_l &= (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1}) + (\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2}) \\ &\quad + \dots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1}) \end{aligned}$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_j - x_{j-1} = \Delta x_j$$

Analogously,

$$\sum_{l=1}^L c_l'' \cdot \Delta \alpha_l = \sum_{k=1}^m c_k' \Delta w_k$$

So it holds that

$$\sum_{j=1}^n c_j \Delta x_j = \sum_{k=1}^m c_k' \Delta w_k$$

□

This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

Lemma 7. Let $\varphi \in \tau[a, b]$ be a step function in terms of partition $a = x_0 < x_1 < \dots < x_n = b$. Let $a = \alpha_0 < \alpha_1 < \dots < \alpha_L = b$ with $Z = \{x_0, \dots, x_n\} \subseteq \{\alpha_0, \alpha_1, \dots, \alpha_L\} = z'$ (z' has more intervals than Z').

Then also φ is step function in terms of partition z' .

Proof. see above

□

Lemma 8. Let $\varphi_1, \varphi_2 \in \tau[a, b]$ and $\alpha, \beta \in \mathbb{C}$.

Then it holds that

- $\alpha\varphi + \beta\psi \in \tau[a, b]$ and

$$\int_a^b (\alpha\varphi + \beta\psi) dx = \alpha \int_a^b \varphi dx + \beta \int_a^b \psi dx$$

Hence (“linearity”),

$$\int_a^b : \tau[a, b] \rightarrow \mathbb{R} \text{ is linear}$$

- $|\varphi| \in \tau[a, b]$ and it holds that

$$\left| \int_a^b \varphi dx \right| \leq \int_a^b |\varphi| dx \leq \|\varphi\|_\infty (b - a)$$

Reminder: $\|\varphi\|_\infty = \max \{|\varphi(x)| : x \in [a, b]\}$

This gives “boundedness”.

- Let φ and ψ be real values and it holds that

$$\forall x \in [a, b] : \varphi(x) \leq \psi(x)$$

Then TODO Monotonicity

Proof. • Let $\varphi|_{(x_{k-1}, x_k)} = c_k$ $\psi|_{(w_{j-1}, w_j)} = d_k$

$$z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\} = \{x_0, \dots, x_n\} \cup \{w_0, \dots, w_m\}$$

where α_i is sorted ascendingly. φ and ψ are step functions in terms of z'' , hence

$$\begin{aligned} \varphi|_{(\alpha_{i-1}, \alpha_i)} &= c'_i \text{ and } \psi|_{(\alpha_{i-1}, \alpha_i)} = d'_i \\ \Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1}, \alpha_i)} &= \alpha c'_i + \beta d'_i \text{ constant} \end{aligned}$$

$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a, b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi) dx = \sum_{i=1}^L (\alpha c'_i + \beta d'_i) \cdot \Delta\alpha_i$$

$$\begin{aligned} &= \alpha \sum_{i=1}^L c'_i \Delta\alpha_i + \beta \sum_{i=1}^L d'_i \Delta\alpha_i \\ &= \alpha \int_a^b \varphi dx + \beta \int_a^b \psi dx \end{aligned}$$

- Let $\varphi|_{(x_{i-1}, x_i)} = c_i$ ($i = 1, \dots, n$). Then,

$$|\varphi| |_{(x_{i-1}, x_i)} = |c_i|$$

$$\left| \sum_{i=1}^n c_i \Delta x_i \right| \leq \sum_{i=1}^n |c_i| \cdot \underbrace{|\Delta x_i|}_{x_i - x_{i-1} > 0} = \sum_{i=1}^n |c_i| \cdot \Delta x_i = \int_a^b |\varphi| dx$$

$$\begin{aligned} &\leq \sum_{i=1}^n \|\varphi\|_\infty \Delta x_i = \|\varphi\|_\infty \sum_{i=1}^n \Delta x_i \\ &= \|\varphi\|_\infty ((x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})) \\ &= \|\varphi\|_\infty (x_n - x_0) = \|\varphi\|_\infty (b - a) \end{aligned}$$

- Let φ , ψ and z'' as in the linearity statement.

$$\left. \begin{aligned} \varphi|_{(\alpha_{i-1}, \alpha_i)} &= c'_i \in \mathbb{R} \\ \psi|_{(\alpha_{i-1}, \alpha_i)} &= d'_i \in \mathbb{R} \end{aligned} \right\}$$

$$\begin{aligned} \int_a^b \varphi dx &= \sum_{i=1}^L c'_i \underbrace{\Delta\alpha_i}_{>0} \leq \sum_{i=1}^L d'_i \Delta\alpha_i \\ &= \int_a^b \psi dx \end{aligned}$$

□

Definition 7. Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\chi_A) : \mathbb{R} \rightarrow \mathbb{R}$ as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

the characteristic function of A . Hence $\chi_A(x)$ is 1 if and only if x is inside interval A .

Remark 11. Let $a \leq a' < b' \leq b$. Then

$$\chi_{(a',b')} \in \tau[a, b] \quad \int_a^b \chi_{(a',b')} dx = 1 \cdot (b' - a')$$

Every linear combination of characteristic functions is also in $\tau[a, b]$.

On the opposite side, let $\varphi \in \tau[a, b]$ with $\varphi|_{(x_{i-1}, x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \leq i \leq n$ and $0 \leq j \leq n$.

$$\Rightarrow \varphi = \sum_{i=1}^n c_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^n r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_a^b \varphi dx = \sum_{i=1}^n c_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^n c_j \int_a^b \chi_{(x_{i-1}, x_i)} dx$$

5 Regulated functions

Definition 8. Let $D \subseteq \mathbb{R}$. Let x_0 be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n \in \mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n \rightarrow \infty} z_n = x_0$. Let $f : D \rightarrow \mathbb{C}$ be given.

We state that f has left-sided limit y_0 in x_0 if

$$\begin{aligned} \forall \varepsilon > 0 \exists \delta > 0 : [x \in D \cap (-\infty, x_0) \wedge |x - x_0| < \delta] \\ \Rightarrow |f(x) - y_0| < \varepsilon \end{aligned}$$

Equivalently $\forall (z_n)_{n \in \mathbb{N}}$ with $z_n \in D$ and $z_n < x_0$ and $\lim_{n \rightarrow \infty} z_n = x_0 \quad \forall n \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} f(z_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by (x_0, ∞) .

We denote: y_0 is left-sided limit of f in x_0 :

$$y_0 = \lim_{x \rightarrow x_0^-} f(x)$$

and right-sided limit of f in x_0 :

$$y_0 = \lim_{x \rightarrow x_0^+} f(x)$$

Definition 9. Let $a, b \in \mathbb{R}$ and $a < b$. A function $f : [a, b] \rightarrow \mathbb{C}$ is called *regulated functions* if

- $\forall x \in (a, b)$ f has a left-sided and a right-sided limes in x
- f has a right-sided limes in a
- f has a left-sided limes in b

Examples for regulated functions:

- Every continuous function in $[a, b]$ is a regulated function.
- Every step function is a regulated function.
Why? Consider $x \in (x_{i-1}, x_i)$. Then

$$\lim_{\xi \rightarrow x^+} \varphi(\xi) = c_i = \lim_{\xi \rightarrow x^-} \varphi(\xi)$$

Let $x = x_i$ be a partitioning point.

$$\lim_{x \rightarrow a^+} f(x) \text{ and } \lim_{x \rightarrow b^-} f(x)$$

So $\tau[a, b] \subseteq R[a, b]$. Compare with Figure 13.

- Let $f : [a, b] \rightarrow \mathbb{R}$ be monotonically. Then it holds that

$$f \in R[a, b]$$

5.1 Approximation theorem for regulated functions

Let $f : [a, b] \in \mathbb{C}$. Then it holds that $f \in R[a, b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a, b]$ such that $\|f - \varphi\|_\infty < \varepsilon$. Hence $\forall x \in [a, b] : |f(x) - \varphi(x)| < \varepsilon$

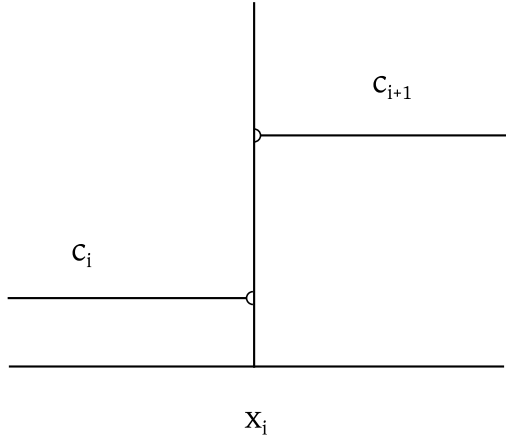


Figure 13: Step functions are also regulated functions

$$\Leftrightarrow \underbrace{\sup \{|f(x) - \varphi(x)| : x \in [a, b]\}}_{\|f - \varphi\|_\infty} < \varepsilon$$

$S \varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a, b]$ such that

$$|\varphi_n(x) - f(x)| < \varepsilon \forall x \in [a, b]$$

hence f is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n \in \mathbb{N}}$ converges continuously towards f .

Proof. \Rightarrow Let $f \in R[a, b]$. Assume $\exists \varepsilon > 0$ fixed such that $\forall \varphi \in \tau[a, b]$

$$\exists x \in [a, b] : |\varphi(x) - f(x)| \geq \varepsilon$$

We build nested intervals such that the desired property $|\varphi(x) - f(x)| \geq \varepsilon$ holds on every subinterval $[a_n, b_n]$.

Induction:

$n = 0$ Let $a_0 = a$ and $b_0 = b$, hence the property holds in $[a_0, b_0]$.

$n \mapsto n + 1$ Let $m = \frac{1}{2}(a_n + b_n)$. In $[a_n, b_n]$ the property holds.

Then the property either holds in $[a_n, m]$ or $[m, b_n]$. If the property does not hold in $[a_n, m]$:

$$\exists \varphi_1 \in \tau[a_n, m] \text{ with } |\varphi_1(\xi) - f(\xi)| < \varepsilon \quad \forall \xi \in [a_n, m]$$

If the property does not hold in $[m, b_n]$:

$$\exists \varphi_2 \in \tau[m, b_n] \text{ with } |\varphi_2(\xi) - f(\xi)| < \varepsilon \quad \forall \xi \in [m, b_n]$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in [a_n, m] \\ \varphi_2(x) & \text{for } x \in [m, b_n] \end{cases}$$

$$\Rightarrow \varphi \in \tau[a, b] \text{ and } |\varphi(\xi) - f(\xi)| < \varepsilon \quad \forall \xi \in [a_n, b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1}, b_{n+1}]$.

$([a_n, b_n])_{n \in \mathbb{N}}$ are nested intervals. Let $\varphi \in \bigcap_{n \in \mathbb{N}} \tau[a_n, b_n]$.

Case $\xi \in (a, b)$ Let ε satisfy the desired property. $f \in R[a, b]$, hence f has left-sided limit c_- in ξ and right-sided limit c_+ . Hence $\exists \delta > 0$ such that

- $|x - \xi| < \delta \wedge a \leq x < \xi \Rightarrow |f(x) - c_-| < \varepsilon$
- $|x - \xi| < \delta \wedge \delta < x \leq b \Rightarrow |f(x) - c_+| < \varepsilon$

Choose δ sufficiently small such that

$$a < \xi - \delta < \xi + \delta < b$$

Let

$$\varphi(x) = \begin{cases} c_- & \text{for } x \in (\xi - \delta, \xi) \\ f(\xi) & \text{for } x = \xi \\ c_+ & \text{for } x \in (\xi, \xi + \delta) \end{cases}$$

φ is necessarily a step function in $(\xi - \delta, \xi + \delta)$ and it holds that $\forall x \in (\xi - \delta, \xi + \delta) : |\varphi(x) - f(x)| < \varepsilon$.

Let n be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi - \delta, \xi + \delta)$$

then

$$\varphi|_{[a_n, b_n]} \in \tau[a_n, b_n] \text{ and } |\varphi(x) - f(x)| < \varepsilon \quad \forall x \in [a_n, b_n]$$

This is a contradiction to our desired property.

For $\xi = a$ or $\xi = b$ only with one-sided limit.

□

This lecture took place on 17th of March 2016 with lecturer Wolfgang Ring.

We learned: All regulated functions can be approximated with step functions.

$f \in R[a, b]$ in the proof $\Leftrightarrow f$ is uniform limit of step functions. We have prove direction \Rightarrow .

Lemma 9 (Cauchy criterion for limits of functions). Let $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$ and z_0 is a limit point of D . Then f has a limit in z_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : v, w \in D \setminus \{z_0\} \wedge |v - z_0| < \delta \wedge |w - z_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$.

If $D \subseteq \mathbb{R}$ and x_0 is limit point of $D \cap (x_0, \infty)$, then f has a *right-sided limit* in x_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : [v, w \in D \cap (x_0, \infty) \wedge |v - x_0| < \delta \wedge |w - x_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon]$.

Analogously for left-sided limit.

Proof. This proof is done only for the first point.

\Rightarrow

Assume f has a limit η in z_0 . Choose δ such that $v, w \in D$ with $|v - z_0| < \delta$ and $|w - z_0| < \delta$ implies that $|f(v) - \eta| < \frac{\varepsilon}{2}$ and $|f(w) - \eta| < \frac{\varepsilon}{2}$. Then $|f(v) - f(w)| \leq |f(v) - \eta| + |\eta - f(w)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

\Leftarrow

Assume the Cauchy criterion holds. Show: There exists $\eta \in \mathbb{C}$ such that

for every sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in D \setminus \{z_0\}$ with $\lim_{n \rightarrow \infty} w_n = z_0$ it holds that $\lim_{n \rightarrow \infty} f(w_n) = \eta$.

Let $(w_n)_{n \in \mathbb{N}}$ be as above. Show: $(f(w_n))_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and δ as above. Choose $N \in \mathbb{N}$ such that $n, m \geq N$

$$\Rightarrow |w_n - z_0| < \delta \wedge |w_m - z_0| < \delta$$

The Cauchy criterion holds for $n, m \geq N$:

$$|f(w_n) - f(w_m)| < \varepsilon$$

So $(f(w_n))_{n \in \mathbb{N}}$ is a Cauchy sequence and (because \mathbb{C} is complete) is also convergent. So $\exists \eta' \in \mathbb{C} : \lim_{n \rightarrow \infty} f(w_n) = \eta'$.

It remains to show: η' is unique.

Let $(v_n)_{n \in \mathbb{N}}$ be another sequence with $\lim_{n \rightarrow \infty} v_n = z_0$ and $v_n \in D \setminus \{z_0\}$. As above: $\exists \eta'' \in \mathbb{C}$ such that $\lim_{n \rightarrow \infty} f(v_n) = \eta''$.

We construct:

$$(\xi_n)_{n \in \mathbb{N}} = (w_0, v_0, w_1, v_1, w_2, v_2, \dots)$$

Then it holds that $\lim_{n \rightarrow \infty} \xi_n = z_0$.

We use the argument from above: $(f(\xi_n))_{n \in \mathbb{N}}$ is convergent, hence $\lim_{n \rightarrow \infty} f(\xi_n) = \eta$. Both subsequences $(f(w_n))_{n \in \mathbb{N}}$ and $(f(v_n))_{n \in \mathbb{N}}$ must have the same limit, hence $\eta' = \eta = \eta''$.

□

Proof of approximation theorem. \Leftarrow

Let $f = \lim_{n \rightarrow \infty} \varphi_n$ be uniform on $[a, b]$. Let $\varphi_n \in \tau[a, b]$ and let $x_0 \in [a, b]$. Show: f has a right-sided limit in x_0 . Let $\varepsilon > 0$ arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that

$$|f(x) - \varphi_N(x)| < \frac{\varepsilon}{2} \forall x \in [a, b]$$

φ_N is a step function (hence interval-wise constant). Choose $\delta > 0$ such that $\varphi_N|_{(x_0, x_0 + \delta)} = c$ constant. Let $v, w \in (x_0, x_0 + \delta)$. Then it holds that

$$|f(v) - f(w)| \leq |f(v) - c| + |c - f(w)|$$

$$= |f(v) - \varphi_N(v)| + |f(w) - \varphi_N(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The Cauchy criterion implies that f has a right-sided limit in x_0 .

□

Corollary 2. $f \in R[a, b]$ if and only if $f(x) = \sum_{j=0}^{\infty} \psi_j(x)$ with $\psi_j \in \tau[a, b]$ and the series converges uniformly in $[a, b]$.

Proof. \Leftarrow

Let $\varphi_n = \sum_{j=0}^n \psi_j \in \tau[a, b]$ and $\varphi_n \rightarrow f$ continuously in $[a, b]$. From the approximation theorem it follows that $f \in R[a, b]$.

\Rightarrow

Let $f \in R[a, b]$. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of step functions with $\varphi_n \rightarrow f$ uniform in $[a, b]$. Let $\psi_0 = \varphi_0$.

$$\psi_j := \varphi_j - \varphi_{j-1} \text{ for } j \geq 1$$

Then it holds that

$$\sum_{j=0}^n \psi_j = \varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \dots + (\varphi_{n-1} - \varphi_{n-2}) + (\varphi_n - \varphi_{n-1}) = \varphi_n$$

and $(\varphi_n)_{n \in \mathbb{N}}$ converges uniform if and only if the series is uniformly convergent.

□

Lemma 10 (Sidenote). Let $(f_n)_{n \in \mathbb{N}}$ with $f_n : D \rightarrow \mathbb{C}$ a sequence of functions in D , let $z_0 \in D$ and $\forall n \in \mathbb{N}$ f_n is continuous in z_0 . Furthermore let $f : D \rightarrow \mathbb{C}$ and $f_n \rightarrow f$ is uniform in D . Then f is continuous in z_0 .

Proof. Let $\varepsilon > 0$ arbitrary. Choose N sufficiently large such that $|f(z) - f_N(z)| < \frac{\varepsilon}{3} \quad \forall z \in D$ (uniform convergence). Because f_N is continuous in z_0 , $\exists \delta > 0$ such that $z \in D$ and $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$.

Then for $|z - z_0| < \delta$ (with $z \in D$)

$$\underbrace{|f(z) - f(z_0)|}_{< \frac{\varepsilon}{3}} \leq \underbrace{|f(z) - f_N(z)| + |f_N(z) - f_N(z_0)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{< \frac{\varepsilon}{3}}$$

□

This lecture took place on 18th of March 2016 with lecturer Wolfgang Ring.

Theorem 9. Let f be a regulated function in $[a, b]$. Then f is in at most countable infinite points of $[a, b]$ non-continuous.

Proof.

$$f = \sum_{k=0}^{\infty} \psi_k$$

where ψ_k is a sequence of step functions and the series is uniformly convergent. $\psi_k \in \tau[a, b]$.

Let $\{x_0^k, \dots, x_{n(k)}^k\}$ be the partition points of ψ_k . Then ψ_k is continuous in $[a, b] \setminus Z_k$. Let $Z = \bigcup_{k=0}^{\infty} Z_k$ be countable. Let $x \in [a, b] \setminus Z$ and $\varphi_n = \sum_{k=0}^n \psi_k$. Then it holds that $\varphi_n \rightarrow f$ is uniform in $[a, b]$ and φ_n is continuous in x , because $x \notin Z$.

From Lemma 10 it follows that f is continuous in x .

□

5.2 Norms and vector spaces

Definition 10 (Normed vector spaces). Let V be a vector space over \mathbb{C} (or \mathbb{R}). A map $n : V \mapsto [0, \infty)$ is called *norm* in V , if

1. $n(V) = 0 \Leftrightarrow V = 0$ (V is null vector)
“definiteness”
2. $\forall \lambda \in \mathbb{C} \ (\mathbb{R}) \ \forall v \in V : n(\lambda v) = |\lambda| \cdot n(v)$ “positive homogeneity”
3. $\forall v, w \in V : n(v + w) \leq n(v) + n(w)$
“triangle inequality”

Common notation: $\|v\|$ for $n(v)$ (“norm of v ”)

A vector space satisfying the norm properties is called *Normed vector space*

Example 2. • $|x|$ is a norm in \mathbb{R} .

□

• $|z|$ is a norm in \mathbb{C} .

$\|\vec{x}\|$ is norm in \mathbb{R}^n .

Let $D \subseteq \mathbb{C}$.

$$B(D) = \{f : D \rightarrow \mathbb{C} : f \text{ limited to } D\}$$

$B(D)$ is a vector space. For $f \in B(D)$ we define:

$$\|f\|_\infty = \sup \{|f(z)| : z \in D\}$$

“supremum norm” of ∞ -norm of f in D .

It holds that $\|\cdot\|_\infty$ is a norm in $B(D)$.

$$\begin{aligned} \|f\|_\infty = 0 &\Leftrightarrow \sup \left\{ \underbrace{|f(z)|}_{\geq 0} : z \in D \right\} = 0 \\ &\Leftrightarrow |f(z)| = 0 \quad \forall z \in D \\ &\Rightarrow f = 0 \text{ in } B(D) \end{aligned}$$

Homogeneity:

$$\begin{aligned} |\lambda \cdot f|_\infty &= \sup \{|\lambda f(z)| : z \in D\} \\ &= \sup \{|\lambda| |f(z)| : z \in D\} \\ &= \sup \{|f(z)| : z \in D\} \cdot |\lambda| \\ &= |\lambda| \cdot \|f\|_\infty \end{aligned}$$

Triangle inequality: Let $f, g \in B(D)$.

$$\begin{aligned} \|f + g\|_\infty &= \sup \{|f(z) + g(z)| : z \in D\} \\ &= \sup \left\{ \underbrace{|f(z)|}_{\leq \|f\|_\infty} + \underbrace{|g(z)|}_{\leq \|g\|_\infty} : \right\} \\ &\leq \text{TODO} \qquad \qquad \qquad = \|f\|_\infty + \|g\|_\infty \end{aligned}$$

Remark 12. Let $V \subseteq B(D)$ be an arbitrary subvectorspace of $B(D)$. So $\|\cdot\|_\infty$ is also a norm in V .

Important example:

$$V = \mathcal{C}_b(D) = \{f : D \rightarrow \mathbb{C} : f \text{ is continuous and bounded in } D\}$$

Special case: $D = K$ compact in \mathbb{C} . Then every continuous function is also bounded.

$$\begin{aligned} \mathcal{C}(K) &= \{f : K \rightarrow \mathbb{C} : f \text{ is continuous}\} \\ &\subseteq B(K) \quad (\text{sub vector space}) \end{aligned}$$

Another special case: $D = [a, b] \subseteq \mathbb{C}$

$$\tau[a, b] \subseteq B([a, b]) \text{ and}$$

$$R[a, b] \subseteq B([a, b])$$

Remark 13 (Further properties of the norm). The inverse triangle inequality holds:

$$\forall v, w \in V : ||v| - |w|| \leq \|v - w\|$$

Proof.

$$v = (v - w) + w$$

From triangle inequality it follows that

$$\|v\| \leq \|v - w\| + \|w\|$$

$$w = (w - v) + v$$

$$\|w\| \leq \|w - v\| + \|v\|$$

$$= \|(-1) \cdot (v - w)\| + \|v\|$$

$$= |(-1)| \cdot \|v - w\| + \|v\|$$

$$= \|v - w\| + \|v\|$$

$$\text{requirement 1} \Rightarrow \|v\| - \|w\| \leq \|v - w\|$$

$$\text{requirement 2} \Rightarrow \|w\| - \|v\| \leq \|v - w\|$$

$$\text{requirements} \Rightarrow \text{TODO}$$

□

Definition 11. Let V be a normed vector space, $(v_n)_{n \in \mathbb{N}}$ be a sequence of elements in V and $v \in V$. We define $(v_n)_{n \in \mathbb{N}}$ is convergent with limit V if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow \|v_n - v\| \leq \varepsilon]$$

Remark 14 (Metric on V).

$$d(v, w) = \|v - w\|$$

defines a metric on V . Properties of a metric:

1. $d(v, w) \geq 0$
2. $d(v, w) = 0 \Leftrightarrow v = w$
3. $\|v - w\| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$

Triangle inequality of metrics: Let $v, w, u \in V$.

$$\begin{aligned} d(v, u) &= \|v - u\| = \|v - w + w - u\| \\ &\leq \|v - w\| + \|w - u\| = d(v, w) + d(w, u) \end{aligned}$$

Works only if $d(v, w) = d(w, v)$ and can be simply proven:

$$d(v, w) = \|v - w\| = \|w - v\| = d(w, v)$$

Remark 15. $(v_n)_{n \in \mathbb{N}}$ is called *Cauchy sequence* in V if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|v_n - v_m\| < \varepsilon]$$

V is called *complete normed vector space* if every Cauchy sequence in V is also a convergent sequence in V .

A complete normed vector space is called *Banach space*.

5.3 Integration of regulated functions

Theorem 10. Let $f \in \mathcal{B}[a, b]$ and $(\varphi_n)_{n \in \mathbb{N}}$ with $\varphi_n \in \mathcal{T}[a, b]$ and $\varphi_n \rightarrow_{n \rightarrow \infty} f$ uniform in $[a, b]$ ($\Leftrightarrow \|\varphi_n - f\| \rightarrow 0$ for $n \rightarrow \infty$).

Then we define

$$\int_a^b f \, dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n \, dx$$

for the integral of f in $[a, b]$. The right-sided limit exists for every sequence $(\varphi_n)_{n \in \mathbb{N}}$ with the property above and is independent of the choice of the sequence $(\varphi_n)_{n \in \mathbb{N}}$.

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\begin{aligned} \forall \varepsilon > 0 \exists N \in \mathbb{N} : &\underbrace{[n \geq N \Rightarrow |\varphi_n(x) - f(x)| < \varepsilon \forall x \in [a, b]]}_{\sup\{|\varphi_n(x) - f(x)| : x \in [a, b] \leq \varepsilon\}} \\ &\Rightarrow \|\varphi_n - f\|_\infty \leq \varepsilon \end{aligned}$$

So φ_n converges towards f in terms of $\|\cdot\|_\infty$ in $\mathcal{B}[a, b]$.

Let N be sufficiently large such that

$$\forall n \geq N : \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(b-a)}$$

Then it holds for $i_n = \int_a^b \varphi_n \, dx$ and $n, m \geq N$,

$$\begin{aligned} |i_n - i_m| &= \left| \int_a^b \varphi_n \, dx - \int_a^b \varphi_m \, dx \right| \\ &= \left| \int_a^b (\varphi_n - \varphi_m) \, dx \right| \\ &\leq \|\varphi_n - \varphi_m\|_\infty (b-a) \\ &= \|\varphi_n - f + f - \varphi_m\|_\infty (b-a) \\ &\leq (\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty)(b-a) \\ &< \left(\frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} \right) (b-a) \\ &= \varepsilon \end{aligned}$$

So $(i_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{C} and therefore convergent. So there exists

$$\lim_{n \rightarrow \infty} \int_a^b \varphi_n \, dx$$

Let $i = \lim_{n \rightarrow \infty} \int_a^b \varphi_n dx$. Let $(\psi_n)_{n \in \mathbb{N}}$ be another sequence of step functions with $\psi_n \rightarrow_{n \rightarrow \infty} f$ is uniform in $[a, b]$. Analogously as above:

$$j_n = \int_a^b \psi_n dx$$

$(j_n)_{n \in \mathbb{N}}$ is convergent and has limes j .

Show that $i = j$. We again use a zip-like construction:

$$F = (\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \dots)$$

F is a sequence of step functions, which converge towards f uniformly. Let l be the limit of integrals of this sequence of step functions. Then it holds that
TODO (subsequences have the same limit)

$$i = l = j$$

□

Theorem 11 (Elementary properties of the integral). Let $f, g \in \mathcal{B}[a, b]$ and $\alpha, \beta \in \mathbb{C}$. Then it holds that

linearity

$$\int_a^b (\alpha f + \beta g) dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

boundedness

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx \leq \|f\|_\infty (b - a)$$

monotonicity Let $f, g \in \mathcal{B}[a, b]$ with values in \mathbb{R} and it holds that

$$f(x) \leq g(x) \quad \forall x \in [a, b]$$

Then it holds that

$$\int_a^b f dx \leq \int_a^b g dx$$

Proof. • Let $(\varphi_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ be sequences of step functions with $\varphi_n \rightarrow f$ and $\psi_n \rightarrow g$ uniform in $[a, b]$. Then it holds that

$$\alpha \varphi_n + \beta \psi_n \rightarrow_{n \rightarrow \infty} \alpha f + \beta g$$

(proof left as exercise to the reader)

uniform in $[a, b]$. So it holds that

$$\begin{aligned} \int_a^b (\alpha f + \beta g) dx &= \lim_{n \rightarrow \infty} \int_a^b (\alpha \varphi_n + \beta \psi_n) dx \\ &= \alpha \lim_{n \rightarrow \infty} \int_a^b \varphi_n dx + \beta \lim_{n \rightarrow \infty} \int_a^b \psi_n dx \\ &= \alpha \int_a^b f dx + \beta \int_a^b g dx \end{aligned}$$

• Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of step functions with $\varphi_n \rightarrow_{n \rightarrow \infty} f$ continuous in $[a, b]$. Then also $(|\varphi_n|)_{n \in \mathbb{N}}$ is a sequence of step functions and it holds that

$$|\varphi_n| \rightarrow_{n \rightarrow \infty} |f| \text{ uniform in } [a, b]$$

Proof. Let N be sufficiently large such that $\forall n \geq N \forall x \in [a, b]$:

$$|\varphi_n(x) - f(x)| < \varepsilon \Rightarrow ||\varphi_n(x)| - |f(x)|| \leq |\varphi_n(x) - f(x)| < \varepsilon$$

$$|\varphi_n| \rightarrow_{n \rightarrow \infty} |f| \text{ uniform in } [a, b]$$

So it holds that

$$\left| \int_a^b f dx \right| = \left| \lim_{n \rightarrow \infty} \int_a^b \varphi_n dx \right| = \lim_{n \rightarrow \infty} \left| \int_a^b \varphi_n dx \right| \leq \lim_{n \rightarrow \infty} \int_a^b |\varphi_n| dx = \int_a^b |f| dx$$

Because $|f - \varphi_n|_\infty \rightarrow_{n \rightarrow \infty} 0$ it follows that

$$||f| - |\varphi_n||_\infty \leq \|f - \varphi_n\|_\infty \rightarrow 0$$

hence $\|f\|_\infty = \lim_{n \rightarrow \infty} \|\varphi_n\|_\infty$.

□

Hence,

$$\begin{aligned} \int_a^b |f| \, dx &= \lim_{n \rightarrow \infty} \int_a^b |\varphi_n| \, dx \\ &\leq \lim_{n \rightarrow \infty} \|\varphi_n\|_\infty (b-a) \\ &= \|f\|_\infty (b-a) \end{aligned}$$

Remark 16. We have proven that $\|\cdot\| : V \rightarrow [0, \infty)$ is a continuous map, hence $v_n \rightarrow v \Rightarrow \|v_n\| \rightarrow \|v\|$.

□

This lecture took place on 12th of April 2016 with lecturer Wolfgang Ring.

Definition 12. Let $f : [a, b] \rightarrow \mathbb{R}$ be given. Let $x_0 \in [a, b)$. We claim that f has a right-sided derivative $f'_+(x_0)$ in x_0 if the function

$$\varphi(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

has a right-sided limit in x_0 . Then f is denoted with $f'_+(x_0)$.

$$f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0}$$

Analogously for the left-sided derivative: Let $x_0 \in (a, b]$. $f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$ if the limit exists.

Theorem 12 (Mean value theorem of calculus). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous in $[a, b]$ and $p : [a, b] \rightarrow \mathbb{R}$ is a regulated function with $p(x) \geq 0 \quad \forall x \in [a, b]$.

Then there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x) \cdot p(x) \, dx = f(\xi) \cdot \int_a^b p(x) \, dx$$

Proof. Let $M = \max \{f(x) : x \in [a, b]\}$ and $m = \min \{f(x) : x \in [a, b]\}$

$$mp(x) \leq f(x) \underbrace{p(x)}_{\geq 0} \leq Mp(x) \quad \forall x \in [a, b]$$

Due to monotonicity of the integral it holds that

$$m \int_a^b p(x) \, dx \leq \int_a^b f(x)p(x) \, dx \leq M \int_a^b p(x) \, dx$$

hence $\exists \eta \in [m, M]$ such that $\eta \cdot \int_a^b p(x) \, dx = \int_a^b f(x)p(x) \, dx$. From the Intermediate Value Theorem it follows that $\exists \xi \in [a, b] : \eta = f(\xi)$.

$$\Rightarrow f(\xi) : \int_a^b p(x) \, dx = \int_a^b f(x)p(x) \, dx$$

□

Remark 17. Consider $p \equiv 1$.

$$\exists \xi \in [a, b] : \int_a^b f(x) \cdot 1 \, dx = f(\xi) \cdot \int_a^b 1 \, dx = f(\xi) \cdot (b-a)$$

Lemma 11. Let $I = [a, b]$ and $f \in R[a, b]$ and $a \leq \alpha < \beta < \gamma \leq b$ (compare with Figure 15). Then $f|_{[\alpha, \gamma]} \in R[\alpha, \gamma]$.

Furthermore it holds that

$$\int_\alpha^\beta f(x) \, dx = \int_\alpha^\beta f(x) \, dx + \int_\beta^\gamma f(x) \, dx$$

Proof. Let φ be a step function in $[\alpha, \gamma]$. Then $\varphi|_{[\alpha, \beta]} \in \tau[\alpha, \beta]$ and $\varphi|_{[\beta, \gamma]} \in \tau[\beta, \gamma]$. Furthermore it holds (proof not given here)

$$\int_\alpha^\gamma \varphi \, dx = \int_\alpha^\beta \varphi \, dx + \int_\beta^\gamma \varphi \, dx$$

For $(\varphi_n)_{n \in \mathbb{N}}$ a sequence of subsequences with $\varphi_n \rightarrow f$ continuous in $[\alpha, \gamma]$.

$$\Rightarrow \varphi_n|_{[\alpha, \beta]} \rightarrow f|_{[\alpha, \beta]} \text{ uniform in } [\alpha, \beta]$$

analogously for $[\beta, \gamma]$.

$$\int_\alpha^\gamma f \, dx = \lim_{n \rightarrow \infty} \int_\alpha^\gamma \varphi_n \, dx = \lim_{n \rightarrow \infty} \left[\int_\alpha^\beta \varphi_n \, dx + \int_\beta^\gamma \varphi_n \, dx \right]$$

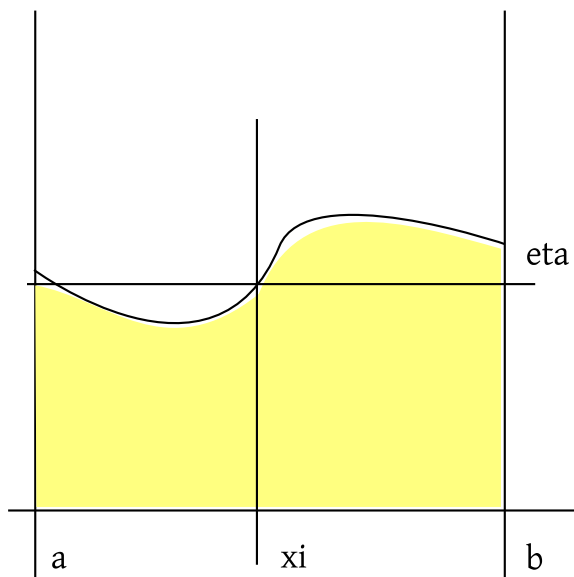
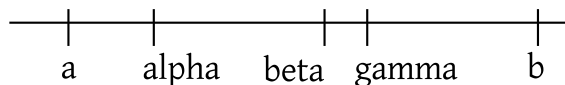


Figure 14: Mean value Theorem


 Figure 15: Relation of $a \leq \alpha < \beta < \gamma \leq b$

$$= \underbrace{\lim_{n \rightarrow \infty} \int_{\alpha}^{\beta} \varphi_n dx}_{= \int_{\alpha}^{\beta} f dx} + \underbrace{\lim_{n \rightarrow \infty} \int_{\beta}^{\gamma} \varphi_n dx}_{= \int_{\beta}^{\gamma} f dx}$$

□

Remark 18. Notation $(\alpha, \beta \in [a, b])$:

$$\int_{\beta}^{\alpha} f(x) dx = - \int_{\alpha}^{\beta} f(x) dx$$

So it follows that

$$\int_{\alpha}^{\alpha} f(x) dx = - \int_{\alpha}^{\alpha} f(x) dx = 0$$

With this notation it holds that $\forall \alpha, \beta, \gamma \in I$:

$$\int_{\alpha}^{\gamma} f dx = \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f(x) dx$$

independent of the relation of α, β, γ towards each other. For $\alpha < \beta < \gamma$ everything is fine.

Let's also look at $\beta < \gamma < \alpha$ as an exercise.

Then it holds that

$$\begin{aligned} \int_{\beta}^{\alpha} f dx &= \int_{\beta}^{\gamma} f dx + \int_{\gamma}^{\alpha} f dx \\ - \int_{\alpha}^{\beta} f dx &= \int_{\beta}^{\gamma} f dx - \int_{\alpha}^{\gamma} f dx \\ \Rightarrow \int_{\alpha}^{\gamma} f dx &= \int_{\alpha}^{\beta} f dx + \int_{\beta}^{\gamma} f dx \end{aligned}$$

Case $\alpha = \beta$ or $\beta = \gamma$ is trivial.

Theorem 13 (Fundamental theorem of Calculus). Originally formulated by Isaac Barrow (1630–1677). Followingly popularized by Newton (1642–1727) and Leibniz (1646–1716).

Let $f : I \rightarrow \mathbb{R}$ be a regulated function. I is an interval and $a \in I$ is fixed. For $x \in I$ we define

$$F(x) = \int_a^x f(\xi) d\xi$$

Then it holds that (two variants/characterizations)

1. F is right-sided derivable and also left-sided derivable for every $x_0 \in I$ and it holds that

$$F'_+(x) = f_+(x_0) = \lim_{x \rightarrow x_0^+} f(x) \quad \text{and} \\ F'_-(x) = f_-(x_0) = \lim_{x \rightarrow x_0^-} f(x)$$

Especially if f is continuous in x_0 , then F is differentiable in x_0 with derivative $F'(x_0) = f(x_0)$.

We call a function with the properties of F above a *primitive function* of the regulated function f .

2. Let $\Phi : I \rightarrow \mathbb{R}$ be an arbitrary primitive function of f and let $a, b \in I$. Then it holds that

$$\int_a^b f(\xi) d\xi = \Phi(b) - \Phi(a)$$

The first characterization claims that (informally speaking) the derivative for the upper limit of the integral of f gives f .

Let $f = \Phi'$ (Φ is our primitive function of f). The second characterization claims that the integral of a derivative of Φ gives Φ .

$$\int_a^b \Phi' dx = \Phi(b) - \Phi(a)$$

Proof. 1. Let $x_1, x_2 \in I$ and wlog $x_1 \leq x_2$.

$$\begin{aligned} |F(x_1) - F(x_2)| &= \left| \int_a^{x_1} f(\xi) d\xi - \int_a^{x_2} f(\xi) d\xi \right| \\ &= \left| \int_a^{x_1} f(\xi) d\xi + \int_{x_2}^a f(\xi) d\xi \right| \\ &= \left| \int_{x_2}^{x_1} f(\xi) d\xi \right| = \left| \int_{x_1}^{x_2} f(\xi) d\xi \right| \\ &\leq \int_{x_1}^{x_2} |f(\xi)| d\xi \leq |x_2 - x_1| \cdot \|f\|_\infty \end{aligned}$$

hence F is Lipschitz continuous in I . So F is continuous in I .

One-sided limits:

Let $\varepsilon > 0$ arbitrary and $x_0 \in I$ and δ such that $\forall x \in (x_0, x_0 + \delta)$ it holds that:

$$|f(x) - f_+(x_0)| < \varepsilon$$

$$\begin{aligned} &\left| \frac{F(x) - F(x_0)}{x - x_0} - f_+(x_0) \right| \\ &= \frac{1}{|x - x_0|} \left| \int_a^x f(\xi) d\xi - \int_a^{x_0} f(\xi) d\xi - f_+(x_0) \cdot (x - x_0) \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(\xi) d\xi - f_+(x_0) \int_{x_0}^x 1 d\xi \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x f(\xi) d\xi - \int_{x_0}^x f_+(x_0) d\xi \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x (f(\xi) - f_+(x_0)) d\xi \right| \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(\xi) - f_+(x_0)| d\xi \end{aligned}$$

$$\xi \in (x_0, x) \subseteq (x_0, x_0 + \delta)$$

$$< \frac{1}{|x - x_0|} \cdot \varepsilon \underbrace{\int_{x_0}^x 1 d\xi}_{|x - x_0|}$$

$$= \varepsilon$$

$$\Rightarrow F'_+(x_0) = f_+(x_0)$$

Analogously $F'_-(x_0) = f_-(x_0)$.

□

This lecture took place on 14th of April 2016 with lecturer Wolfgang Ring.

Theorem 14 (Addition: Lipschitz continuity of differentiable functions). Let $I = [a, b]$, $f : I \rightarrow \mathbb{R}$ and f is continuous in I . Let $A \subseteq I$. Let A be countable and f is differentiable in $I \setminus A$ and $\exists L > 0 : |f'(x)| \leq L \quad \forall x \in I \setminus A$.

Then it holds that $\forall x_1, x_2 \in I$:

$$|f(x_1) - f(x_2)| \leq L |x_1 - x_2|$$

Proof. Without loss of generality, $x_1 < x_2$. Let $\varepsilon > 0$, define $F_\varepsilon : I \rightarrow \mathbb{R}$

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L + \varepsilon)(x - x_1)$$

Show $F_\varepsilon(x_2) \leq 0$.

Assume there is some $\varepsilon' > 0$ with $F_{\varepsilon'}(x_2) > 0$. It holds that

- $F_{\varepsilon'}(A) \subseteq \mathbb{R}$ is countable
- $0 = F_{\varepsilon'}(x_1) < F_{\varepsilon'}(x_2)$. Because $F_{\varepsilon'}$ is continuous (by Intermediate Value Theorem, $[0, F_{\varepsilon'}(x_2)] \subseteq F_{\varepsilon'}([x_1, x_2])$) and $[0, F_{\varepsilon'}(x_2)]$ contains overcountably many points, $F_{\varepsilon'}(A)$ is countable.

$$\Rightarrow \exists \gamma : 0 < \gamma < F_{\varepsilon'}(x_2)$$

and

$$\gamma \in F_{\varepsilon'}([x_1, x_2] \setminus A)$$

Let $\underbrace{F_{\varepsilon'}^{-1}(\{\gamma\})}_B \cap [x_1, x_2] = \{x \in [x_1, x_2] \mid F_{\varepsilon'}(x) = \gamma\}$.

B is bounded. Let $c = \sup B$. Let $(\xi_n)_{n \in \mathbb{N}}$, $\xi_n \in B$ with $\lim_{n \rightarrow \infty} \xi_n = c$. Then it holds that $c \in [x_1, x_2]$ and $F_{\varepsilon'}(\xi_n) = \gamma \xrightarrow{\text{continuity of } F_{\varepsilon'}} \lim_{n \rightarrow \infty} F_{\varepsilon'}(\xi_n) = \underbrace{F_{\varepsilon'}(c)}_y$.

Therefore $c = \max B = \max \{x \in [x_1, x_2] : F_{\varepsilon'}(x) = \gamma\}$. Because $F_{\varepsilon'}(x_2) > \gamma$ and $F_{\varepsilon'}(x_1) = 0 < \gamma$, it holds that $x_1 < c < x_2$.

Consider $x \in (c, x_2]$ and let $\varphi(x) := \frac{F_{\varepsilon'}(x) - F_{\varepsilon'}(c)}{x - c}$. Furthermore $F_{\varepsilon'}(x) > \gamma = F_{\varepsilon'}(c)$ for $x \in (c, x_2]$. Because if we define $F_{\varepsilon'}(x) < \gamma$, then (due to Intermediate Value Theorem) $\exists \xi \in (x, x_2)$ with $F_{\varepsilon'}(\xi) = \gamma$, so $\exists \xi \in B$ which would be a contradiction to $c = \max B$.

$$\begin{aligned} \varphi(x) &= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - x_1 - c + x_1)}{x - c} \\ &= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - c)}{x - c} \\ &\stackrel{\text{inv. triangle ineq.}}{\leq} \frac{|f(x) - f(c)|}{x - c} - (L + \varepsilon') \end{aligned}$$

Now as far as $c \notin A$ holds, f is differentiable in c and it holds that $|f'(c)| \leq L$, hence there exists an interval (c, d) , $d < x_2$ and $d > c$, such that

$$\frac{|f(x) - f(c)|}{x - c} < L + \varepsilon'$$

Because $F_{\varepsilon'}(x) > \gamma$,

$$\Rightarrow \varphi(x) > 0 \quad \forall x \in (c, x_2]$$

$$\Rightarrow 0 < \varphi(x) \leq |f(x) - f(c)| x - c - (L + \varepsilon')$$

$$\Rightarrow \left| \frac{f(x) - f(c)}{x - c} \right| > L + \varepsilon'$$

This is a contradiction to the assumption that $F_{\varepsilon'}(x_2) > 0$. So $F_\varepsilon(x_2) \leq 0 \quad \forall \varepsilon > 0$

$$\Rightarrow F_0(x_2) \leq 0 \Rightarrow |f(x_2) - f(x_1)| \leq L|x_2 - x_1|$$

□

Remark 19. Let f be differentiable in $[a, b]$ and $|f'(x)| < L \quad \forall x \in [a, b]$. Let $x_1, x_2 \in [a, b]$

$$|f(x_2) - f(x_1)| = |f'(\xi) \cdot (x_2 - x_1)| \leq L|x_2 - x_1|$$

by Mean Value Theorem of differential calculus.

Corollary 3. Let $f, g : I \rightarrow \mathbb{R}$. I as above and f, g are differentiable in $I \setminus A$, A countable and it holds that $f'(x) = g'(x) \quad \forall x \in I \setminus A$. There exists a constant k such that

$$f(x) = g(x) + k \quad \forall x \in I$$

Proof. We use the previous Theorem for

$$h(x) = f(x) - g(x)$$

Then it holds that $|h'(x)| = 0 = L \quad \forall x \in I \setminus A$.

$$\Rightarrow |h(x_1) - h(x_2)| \leq 0 \cdot |x_1 - x_2| \quad \forall x_1, x_2 \in I$$

$$\Rightarrow h(x_1) = h(x_2) \quad \forall x_1, x_2 \in I$$

□

$$\begin{aligned} f(x_1) - g(x_1) &= f(x_2) - g(x_2) \quad \forall x_1, x_2 \in I \\ &= k \dots \text{constant} \end{aligned}$$

$\forall x_1 \in I$ it holds that $f(x_1) = g(x_1) + k$.

□

This lecture took place on 15th of April 2016 with lecturer Wolfgang Ring.

cont, 2nd part. We need to show: Let f be a regulated function and Φ is a primitive function of f with the following properties

$$\Phi'(x) = f(x) \quad \forall x \in I \text{ where } f \text{ is continuous}$$

$$\Phi'_+(x) = \lim_{\xi \rightarrow x_+} f(\xi)$$

$$\Phi'_-(x) = \lim_{\xi \rightarrow x_-} f(\xi) \quad \forall x \in I$$

Then it holds that

$$\int_{\alpha}^{\beta} f(x) dx = \Phi(\beta) - \Phi(\alpha)$$

Proof. For $\Phi(x) = \int_{\alpha}^x f(\xi) d\xi = F(x)$ (where F is also a primitive function) it holds that

$$\int_{\alpha}^{\beta} f(\xi) d\xi = F(\beta) - \underbrace{F(\alpha)}_{=0}$$

Because Φ and F are both primitive functions of f , Φ' and F' correspond in all continuous points, hence everywhere, but one countable set.

By the uniqueness theorem, it holds that

$$\Phi(x) = F(x) + c$$

$$F(x) = \Phi(x) - c$$

$$\int_a^b f(\xi) d\xi = F(b) - F(a) = \Phi(b) - c - \Phi(a) + c = \Phi(b) - \Phi(a)$$

□

Remark 20 (Notational remark). Let f be a regulated function. Then we denote

$$\int f(x) dx = \begin{cases} \text{the set of all primitive function of } f \\ \text{an arbitrary primitive function of } f \end{cases}$$

$\int f(x) dx$ is called *indefinite integral*.

Remark 21.

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad \forall n \in \mathbb{R} \setminus \{-1\} \quad \forall x > 0$$

If you consider all primitive functions of the indefinite integral, you consider a constant $c \in \mathbb{R}$.

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c \quad \forall n \in \mathbb{R} \setminus \{-1\} \quad \forall x > 0$$

Let $x > 0$: $(\ln x)' = \frac{1}{x}$.

Let $x < 0$: $(\ln -x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

$$\int \frac{1}{x} dx = \begin{cases} \ln(x) & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases} = \ln|x| \quad \text{for } x \neq 0$$

$$\int \cos x dx = \sin x$$

$$\int \sin x dx = -\cos x$$

$$\int e^{cx} dx = \frac{1}{c} \cdot e^{cx} \quad (c \neq 0)$$

Lemma 12. Let f_1 and f_2 be regulated functions in $I = [a, b]$ and there exists some countable set A such that

$$f_1(x) = f_2(x) \quad \forall x \in I \setminus A$$

Then it holds that

$$\int f_1(x) dx = \int f_2(x) dx \text{ and } \int_a^b f_1(x) dx = \int_a^b f_2(x) dx \quad \forall a, b \in I$$

Proof. Let F_1 be a primitive function on f_1 , F_2 be a primitive function of f_2 . and Then it holds that $F_1' = F_2'$ in $I \setminus A$. Due to identity theorem:

$$\Rightarrow F_1 = F_2 + c \Rightarrow \int f_1 dx = \int f_2 dx$$

Remark 22. Example of a function, which is differentiable everywhere. Its derivative is not a regulated function.

Let $I = [-1, 1]$ and

$$f(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$ it holds that

$$f'(x) = 2x \cos \sin \frac{1}{x} - \frac{x^2}{x^2} \cdot \cos \frac{1}{x}$$

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

$$f'(0) = \lim_{h \rightarrow 0} \frac{1}{h} \left[h^2 \cdot \sin \frac{1}{h} - 0 \right] = \lim_{h \rightarrow 0} \underbrace{h}_{\rightarrow 0} \cdot \underbrace{\sin \frac{1}{h}}_{\in [-1, 1]} = 0$$

$$f'(x) = \begin{cases} 0 & \text{for } x = 0 \\ \underbrace{2x \sin \frac{1}{x} - \cos \frac{1}{x}}_{\text{has no one-sided limit in } x=0} & \text{for } x \neq 0 \end{cases}$$

$$f'_+(0) \neq \lim_{x \rightarrow 0^+} f'(x)$$

5.4 Integration techniques

Theorem 15 (Integration by parts (dt. “partielle Integration”). Let $u, v : I \rightarrow \mathbb{R}$ be both primitive functions of regulated functions. Then also $u \cdot v$ is a primitive function of a regulated function and it holds that

$$\int u'v dx = u \cdot v - \int u \cdot v' dx$$

$$\int_a^b u'v dx = \underbrace{u(b) \cdot v(b) - u(a) \cdot v(a)}_{=: u \cdot v|_a^b} - \int_a^b u \cdot v' dx$$

□ *Proof.* u is continuous and therefore a regulated function. v is continuous and therefore a regulated function. u' and v' are regulated function by assumption.

$$\Rightarrow (u' \cdot v + u \cdot v') \in \mathcal{R}(I)$$

$u \cdot v$ is differentiable in every point in which u and v is differentiable. Let u be differentiable in $I \setminus A$, v is differentiable in $I \setminus B$.

$$\Rightarrow u \cdot v \text{ is differentiable in } I \setminus \underbrace{(A \cup B)}_{\text{countable}}$$

In $I \setminus (A \cup B)$ it holds that

$$(u \cdot v)'(x) = u'(x) \cdot v(x) + u(x)v'(x)$$

Hence the function $u \cdot v$ is primitive function of the regulated function $(u'v + uv')$.

$$\Rightarrow \int (u'v + uv') dx = u \cdot v$$

$$\Rightarrow \int_a^b (u'v + uv') dx = u(b)v(b) - u(a)v(a)$$

□

Example 3. Let $a \neq -1$ and $x > 0$.

$$\int x^a \ln x dx = \left| \begin{array}{ll} u' = x^a & u = \frac{1}{1+a} \cdot x^{a+1} \\ v = \ln x & v' = \frac{1}{x} \end{array} \right|$$

$$\stackrel{\text{int. by parts}}{=} \frac{1}{1+a} x^{1+a} \cdot \ln x - \frac{1}{1+a} \int x^a dx$$

$$= \frac{1}{1+a} x^{1+a} \ln x - \frac{1}{(1+a)^2} x^{1+a} = \frac{1}{1+a} x^{1+a} \left[\ln x - \frac{1}{1+a} \right]$$

Example 4.

$$\int \cos^k(x) dx \text{ for } k = 2, 3, 4, \dots$$

$$\left| \begin{array}{l} u' = \cos x \\ v = \cos^{k-1}(x) \end{array} \right. \Rightarrow u = \sin x \quad v' = -(k-1) \cdot \cos^{k-2}(x) \cdot \sin(x)$$

$$\begin{aligned} \int \cos^k(x) dx &= \cos^{k-1}(x) \cdot \sin(x) + \int (k-1) \cdot \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{1-\cos^2(x)} dx \\ &= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \cdot \int \cos^{k-2}(x) dx - (k-1) \cdot \int \cos^k(x) dx \end{aligned}$$

Recognize that we have $\int \cos^k(x) dx$ twice in the equation (LHS and RHS, RHS with a sign).

$$\begin{aligned} k \cdot \int \cos^k(x) dx &= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x) dx \\ \int \cos^k(x) dx &= \frac{1}{k} \cos^{k-1}(x) \sin(x) + \frac{k-1}{k} \int \cos^{k-2}(x) dx \end{aligned}$$

Recursion formula.

Analogously,

$$\int \sin^k(x) dx = -\frac{1}{k} \sin^{k-1}(x) \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x) dx$$

Let $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x) dx$. Then it holds that

$$c_{2n} = \frac{(2n-1)}{2n} \cdot \frac{(2(n-1)-1)}{2(n-1)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \left(\prod_{k=1}^n \frac{2k-1}{2k} \right) \cdot \frac{\pi}{2}$$

$$c_{2n+1} = \left(\prod_{k=1}^n \frac{2k}{2k+1} \right)$$

Proof by complete induction:

Case $n = 0$

$$\int_0^{\frac{\pi}{2}} \cos^{2 \cdot 0} x dx = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^{2 \cdot 0 + 1} x dx = \int_0^{\frac{\pi}{2}} \cos x dx = \sin(x) \Big|_0^{\frac{\pi}{2}} = 1$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^{2(n+1)} dx &= \frac{1}{2(n+1)} \cdot \cos^{2(n+1)-1}(x) \cdot \sin(x) \Big|_0^{\frac{\pi}{2}} + \frac{2(n+1)-1}{2(n+1)} \cdot \int_0^{\frac{\pi}{2}} \cos^{2n}(x) dx \\ &= \frac{2n+1}{2n+2} \cdot \left(\underbrace{\prod_{k=1}^n \frac{2k-1}{2k}}_{\text{induction hypothesis}} \right) \cdot \frac{\pi}{2} = \left(\prod_{k=1}^{n+1} \frac{2k-1}{2k} \right) \cdot \frac{\pi}{2} \end{aligned}$$

Theorem 16 (Wallis product). (John Wallis, 1616–1703)

$$\frac{\pi}{2} = \lim_{n \rightarrow \infty} w_n \quad \text{with} \quad w_n = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

Proof.

$$\frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} \cdot \frac{\prod_{k=1}^n \frac{2k}{2k+1}}{\prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2}} = \prod_{k=1}^n \frac{(2k)^2}{(2k+1)(2k-1)} = w_n$$

It remains to show: $\lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = 1$.

In $[0, \frac{\pi}{2}]$ it holds that $0 \leq \cos x \leq 1$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^{2n}(x) dx \geq \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x) dx \geq \int_0^{\frac{\pi}{2}} \cos^{2n+2}(x) dx$$

$$c_{2n} \geq c_{2n+1} \geq c_{2n+2}$$

$$1 \geq \frac{c_{2n+1}}{c_{2n}} \geq \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1} \frac{2k-1}{2k}}{\prod_{k=1}^n \frac{2k-1}{2k}} = \underbrace{\frac{2n+1}{2n+2}}_{\rightarrow 1 \text{ for } n \rightarrow \infty}$$

$\Rightarrow \frac{c_{2n+1}}{c_{2n}}$ converges and limit is 1.

$$\lim_{n \rightarrow \infty} \frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} = \lim_{n \rightarrow \infty} w_n$$

□

Theorem 17 (Substitution law). Let $f : I \rightarrow \mathbb{R}$ be a regulated function with primitive function F . Furthermore $t : [\alpha, \beta] \rightarrow I$ is continuously differentiable. Then $F \circ t$ is a primitive function for function $(f \circ t) \cdot t'$ and it holds that

$$\int_{\alpha}^{\beta} f(t(x)) \cdot t'(x) dx = \int_{t(\alpha)}^{t(\beta)} f(t) dt$$

Proof. The right-side integral is given (according to the Fundamental Theorem) by

$$F(t(\beta)) - F(t(\alpha))$$

The left-side integral, because of

$$F(t(x))' = F'(t(x)) \cdot t'(x)$$

Hence $F \circ t$ is primitive function of the left-side integral. So it holds that

$$\int_a^b f(t(x)) \cdot t'(x) dx = F \circ t(b) - F \circ t(a) = F(t(b)) - F(t(a))$$

Example 5.

$$\begin{aligned} \int_0^1 x \sqrt{1+x^2} dx &= \frac{1}{2} \int_0^1 2x \sqrt{1+x^2} dx \\ &\left| \begin{array}{l} t(x) = 1+x^2 \quad t'(x) = 2x \\ f(y) = \sqrt{y} \end{array} \right| \\ &= \frac{1}{2} \int_1^2 \sqrt{y} dy = \frac{1}{2} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^2 = \frac{2^{\frac{3}{2}}}{3} - \frac{1^{\frac{3}{2}}}{3} = \frac{1}{3}(\sqrt{8} - 1) \end{aligned}$$

$$\int_0^1 x \cdot \sqrt{1+x^2} dx = \left| \begin{array}{l} \text{transform variables} \\ y = x^2 + 1 \\ \frac{dy}{dx} = 2x \\ \text{transformation of differences} \\ x dx = \frac{1}{2} dy \end{array} \right|$$

Transformation of limits:

$$x = 0 \Leftrightarrow y = 1 \quad x = 1 \Leftrightarrow y = 2$$

$$= \frac{1}{2} \int_1^2 \sqrt{y} dy = \frac{1}{2} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^2 = \frac{(x^2 + 1)^{\frac{3}{2}}}{3} \Big|_0^1$$

Hence it is also necessary to transform the limits.

Example 6 (Integration by parts).

$$\int \ln x dx = \left| \begin{array}{ll} v' = 1 & v = x \\ u = \ln x & u' = \frac{1}{x} \end{array} \right| = x \ln x - \int x \frac{1}{x} dx = x \ln x - x$$

Theorem 18. Ivan M. Niven (published in 1947, 1915–1999)

□

It holds: π^2 is an irrational number. So π is irrational.

Proof by contradiction. Let $\pi^2 = \frac{a}{b} \in \mathbb{Q}$.

Because $\lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$ (practicals!) there exists $n \in \mathbb{N}$ such that $\pi \frac{a^n}{n!} < 1$.

$$f(x) = \frac{1}{n!} x^n (1-x)^n$$

is symmetrical along axis $x = \frac{1}{2}$

$$= \frac{1}{n!} \sum_{k=n}^{2n} c_k x^k \quad \text{with } c_k = (-1)^{k-n} \binom{n}{k-n} = \pm \binom{n}{k-n} \in \mathbb{Z}$$

$$f^{(\mu)}(0) = 0 \text{ for } \mu = 0, 1, \dots, n-1 \in \mathbb{Z} \quad \text{and also:}$$

$$f^{(\mu)}(1) \in \mathbb{Z} \text{ for } \mu = n, n+1, \dots, 2n$$

$$f^{(\mu)}(x) = \frac{1}{n!} \sum_{k=0}^{2n} \underbrace{k(k-1) \dots (k-\mu+1)}_{=\mu!} \cdot c_k \cdot x^{k-\mu}$$

$$\begin{aligned} f^{(\mu)}(0) &= \frac{1}{n!} \mu! \left(\pm \binom{n}{\mu-n} \right) \cdot 1 \\ &= \frac{1}{n!} \mu! \frac{n!}{(\mu-n)!(n-\mu+n)!} \\ &= \frac{\mu!}{(\mu-n)!(2n-\mu)!} \\ &= \frac{(\mu-n+1)(\mu-n+2) \dots \mu}{1 \cdot 2 \cdot 3 \dots (2n-\mu)} \\ &\in \mathbb{Z} \end{aligned}$$

Why does $\in \mathbb{Z}$ hold?

$$\begin{aligned} \frac{\mu!}{n!} \underbrace{\binom{n}{\mu-n}}_{\in \mathbb{Z}} &\in \mathbb{Z} \quad n \leq \mu \leq 2n \\ (n+1)(n+2) \dots \nu &\in \mathbb{Z} \end{aligned}$$

$$n \leq \mu \leq 2n$$

$f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$, analogously $f^{(\mu)}(1) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$.

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) + (-1)^n f^{(2n)}(x) \pi^0 \right)$$

$F(0) \in \mathbb{Z}$ because $f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu = 0, 2, 4, 6, \dots, 2n$

$$\begin{aligned} \pi^2 &= \frac{a}{b} & \pi^{2n-2l} &= \frac{a^{k-l}}{b^{n-l}} \\ b^n \cdot \pi^{2n-2l} &= a^{n-l} \cdot b^l \in \mathbb{Z} \end{aligned}$$

Analogously for $F(1) \in \mathbb{Z}$.

$$\begin{aligned} &(F'(x) \cdot \sin(\pi x) - \pi F(x) \cdot \cos(\pi x))' \\ &= F''(x) \cdot \sin(\pi x) + \pi^2 \cdot F(x) \cdot \sin \pi x + F'(x) (\cos(\pi x) - \pi \cos \pi x) \\ &= (F''(x) + \pi^2 F(x)) \cdot \sin(\pi x) \\ F''(x) &= b^n \cdot \left(\pi^{2n} \cdot f''(x) + \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) - \dots + (-1)^n f^{(2n+2)}(x) \right) \\ &\Rightarrow F''(x) + \pi^2 \cdot F(x) \\ &= b^n \left(\pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n f^{(2n+2)}(x) \right) \\ &+ b^n \left(\pi^{2n+2} f(x) - \pi^{2n} f''(x) + \pi^{2n-2} f^{(4)}(x) - \pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n \pi^2 \cdot f^{(2n)}(x) \right) \end{aligned}$$

Almost all expressions cancel each other out. So it holds that

$$\begin{aligned} &(F'(x) \cdot \sin(\pi x) - \pi F(x) \cos(\pi x))' \\ &= \pi^{2n+2} \cdot b^n \cdot f(x) \cdot \sin(\pi x) \\ &= \frac{a^{n+1}}{b^{n+1}} \cdot b^n \cdot f(x) \cdot \sin(\pi x) \\ &= \frac{a^{n+1}}{b} \cdot f(x) \cdot \sin(\pi x) \\ &= \pi^2 \cdot a^n f(x) \cdot \sin(\pi x) \\ &= \pi (\pi a^n f(x) \sin(\pi x)) \end{aligned}$$

$$\begin{aligned} I &= \pi \int_0^1 a^n f(x) \cdot \sin(\pi x) dx \\ &= \frac{1}{\pi} \cdot [F'(x) \cdot \sin(\pi x) - \pi \cdot F(x) \cos(\pi x)] \Big|_0^1 \\ &= F(1) + F(0) \in \mathbb{Z} \end{aligned}$$

On the other hand it holds that

$$f(x) = \frac{1}{n!} \underbrace{x^n}_{\leq 1} \underbrace{(1-x)^n}_{\leq 1}$$

So $0 \leq f(x) \leq \frac{1}{n!}$. Hence,

$$0 \leq a^n f(x) \cdot \sin(\pi x) \leq \frac{a^n}{n!} < \frac{1}{\pi}$$

So $0 < I < 1 \Rightarrow I \in \mathbb{Z}$. This is a contradiction to our assumption that $I \in \mathbb{Z}$. \square **Example 7** (Classic examples). 1. Let $s > 1$.

Remark 23. Hence π is not rational. So there exists no linear affine function $g(x) = ax + b$ with $a, b \in \mathbb{Z}$ such that π is root of g .

Remark 24. We state, $\xi \in \mathbb{R}$ is an *algebraic* number if polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

exists with $a_i \in \mathbb{Z}$ for $i = 0, \dots, n$ and $P(\xi) = 0$.

Algebraic numbers are a generalization of rational numbers.

$\eta \in \mathbb{R}$ is called *transcendental*, if η is not algebraic.

Remark 25. π is transcendental.

Theorem 19 (Integration of non-compact intervals).

$$\int_0^\infty e^{-x} dx = \lim_{c \rightarrow \infty} \int_0^c e^{-x} dx$$

Definition 13 (Definition of indefinite integrals). Let I be an interval with boundary values a and b with $-\infty \leq a < b \leq \infty$.

Let f be a regulated function in I . Then we define

1. if $I = [a, b]$, $\int_a^b f(x) dx = \lim_{\beta \rightarrow b-} \int_a^\beta f(x) dx$
2. if $I = (a, b]$, $\int_a^b f(x) dx = \lim_{\alpha \rightarrow a+} \int_\alpha^b f(x) dx$
3. if $I = (a, b)$, we choose $c \in I$ and $\int_a^b f(x) dx = \lim_{\alpha \rightarrow a+} \int_\alpha^c f(x) dx + \lim_{\beta \rightarrow b-} \int_c^\beta f(x) dx$.

This lecture took place on 21st of April 2016 with lecturer Wolfgang Ring.

$$f : [a, b] \rightarrow \mathbb{R} \quad b \in (-\infty, \infty]$$

$$\int_a^b f(x) dx = \lim_{\beta \rightarrow b-} \int_a^\beta f(x) dx$$

$$\begin{aligned} \int_1^\infty \frac{1}{x^s} dx &= \lim \int_1^\beta x^{-s} dx \\ &= \frac{1}{-s+1} \cdot x^{-s+1} \Big|_1^\beta \\ &= \lim_{\beta \rightarrow \infty} \frac{1}{1-s} \cdot \frac{1}{\beta^{s-1}} - \frac{1}{1-s} \end{aligned}$$

$$s-1 > 0 \text{ and } \frac{1}{1-s} \rightarrow 1$$

$$= \frac{1}{s-1} \quad \text{so indefinite integral exists}$$

2. Let $s < 1$.

$$\begin{aligned} \int_0^1 x^{-s} dx &= \lim_{\alpha \rightarrow 0+} \int_\alpha^1 x^{-s} dx \\ &= \lim_{\alpha \rightarrow 0+} \frac{1}{-s+1} x^{-s+1} \Big|_\alpha^1 \\ &= \frac{1}{1-s} - \underbrace{\lim_{\alpha \rightarrow 0+} \frac{1}{1-s} \alpha^{1-s}}_{=0} \\ &= \frac{1}{1-s} \end{aligned}$$

Compare with Figure 16.

3.

$$\begin{aligned}
 \int_0^\infty e^{-cx} dx &= \lim_{\beta \rightarrow \infty} \int_0^\beta e^{-cx} dx \\
 &= \lim_{\beta \rightarrow \infty} \frac{1}{-c} \cdot e^{-cx} \Big|_0^\beta \\
 &= \lim_{\beta \rightarrow \infty} \left(-\frac{1}{c} \cdot e^{-c\beta} \right) + \frac{1}{c} \\
 &= \frac{1}{c}
 \end{aligned}$$

4.

$$\begin{aligned}
 \int_{-\infty}^\infty \frac{1}{1+x^2} dx &= \lim_{\alpha \rightarrow -\infty} \int_\alpha^0 \frac{1}{1+x^2} dx + \lim_{\beta \rightarrow \infty} \int_0^\beta \frac{1}{1+x^2} dx \\
 &= \arctan(0) - \underbrace{\lim_{\alpha \rightarrow -\infty} \arctan(\alpha)}_{-\frac{\pi}{2}} + \underbrace{\lim_{\beta \rightarrow \infty} \arctan(\beta)}_{\frac{\pi}{2}} - \arctan(0) \\
 &= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} \\
 &= \pi
 \end{aligned}$$

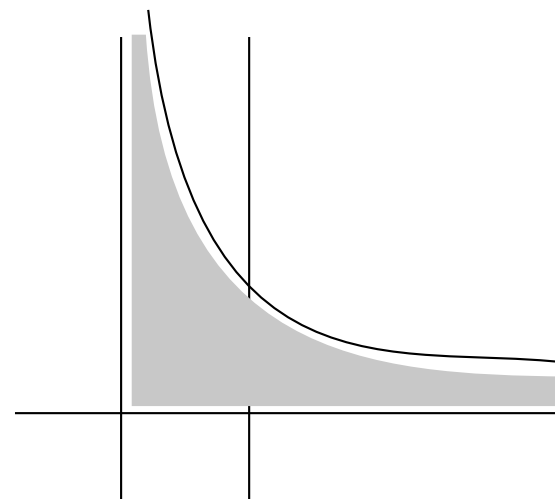
Remark 26. “Integral converges” means “an (indefinite) integral exists”

Remark 27.

$$\arctan'(x) = \frac{1}{1+x^2}$$

$$\tan'(x) = \frac{\cos x \cdot \cos x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2(x)}$$

$$\tan(x) = \frac{\sin x}{\cos x}$$


 Figure 16: $\frac{1}{1-s}$

$$\begin{aligned}
 \arctan'(x) &= \frac{1}{\tan'(\arctan(x))} \\
 &= \left| \arctan x = s \right| \\
 &= \left(\frac{1}{\cos^2(s)} \right)^{-1} \\
 &= \left(\frac{\cos^2(s) + \sin^2(s)}{\cos^2(s)} \right)^{-1} \\
 &= \left(1 + \left(\frac{\sin s}{\cos s} \right)^2 \right)^{-1} \\
 &= \left(1 + [\tan(\arctan x)]^2 \right)^{-1} \\
 &= (1+x^2)^{-1} \\
 &= \frac{1}{1+x^2}
 \end{aligned}$$

Theorem 20 (Direct comparison test for indefinite integrals). (dt. “Majorantenkriterium für uneigentliche Integrals”) Let f, g be regulated functions in $[a, b]$ and $|f(x)| \leq g(x) \quad \forall x \in [a, b]$. Assume $\int_a^b g(x) dx$ exists. Then also $\int_a^b |f(x)| dx$ exists and also $\int_a^b f(x) dx$.

Proof.

$$G(\beta) = \int_a^\beta g(x) dx$$

We know that $\lim_{\beta \rightarrow b-} G(\beta)$ exists.

Cauchy criterion: $\forall \varepsilon > 0$ there exists a left-sided environment of b such that for all u, v in this environment it holds that

$$\underbrace{|G(u) - G(v)|}_{\int_u^v g(x) dx} < \varepsilon$$

Because $|f| \leq g$ it holds that

$$F(\beta) = \int_a^\beta |f(x)| dx$$

and also that

$$\left| \int_u^v |f(x)| dx \right| = |F(v) - F(u)| \stackrel{\text{monotonicity}}{\leq} \left| \int_u^v g(x) dx \right| < \varepsilon$$

Hence $\lim_{\beta \rightarrow b} F(\beta)$ exists because of the Cauchy criterion. So $\int_a^b |f(x)| dx$ exists. Analogously for f instead of $|f|$. \square

Example 8.

$$\int_0^\infty \frac{\sin x}{x} dx \text{ exists}$$

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \text{ continuous in } 0$$

$$\int_0^1 \frac{\sin x}{x} dx = \int_0^1 f(x) dx \text{ exists because } f \text{ is continuous}$$

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\sin x}{x} dx &= \left| \begin{array}{ll} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{array} \right| \\ &= \lim_{\beta \rightarrow \infty} \frac{1}{x} \cdot (-\cos x) \Big|_1^\beta - \int_1^\beta \frac{\cos x}{x^2} dx \\ &= \lim_{\beta \rightarrow \infty} \left[\underbrace{-\frac{1}{\beta} \cdot \cos \beta + \cos 1}_{\rightarrow 0} - \int_1^\beta \frac{\cos x}{x^2} dx \right] \\ &= \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\cos x}{x^2} dx \end{aligned}$$

The last expression exists, because $\frac{1}{x^2}$ is a majorant for $\frac{\cos(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2} dx$ exists.

This lecture took place on 22nd of April 2016 with lecturer Wolfgang Ring.

$$\begin{aligned} \int_0^\infty \left| \frac{\sin x}{x} \right| dx &\text{ does not exist} \\ \int_{k\pi}^{(k+1)\pi} \left| \frac{\sin x}{x} \right| dx &\geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx \\ &= \frac{1}{(k+1)\pi} (\pm 1) \cdot (-\cos x) \Big|_{k\pi}^{(k+1)\pi} = \frac{1}{(k+1)\pi} (\pm 1)(\pm 2) \\ &= \frac{2}{(k+1)\pi} \end{aligned}$$

$$\underbrace{\int_0^{(n+1)\pi} \left| \frac{\sin x}{x} \right| dx}_{\text{unbounded} \leftarrow} \geq \frac{2}{\pi} \cdot \underbrace{\sum_{k=0}^n \frac{1}{k+1}}_{\text{harmonic series, divergent}}$$

In terms of the Lebesgue integral, $\int_0^\infty \frac{\sin x}{x} dx$ does not exist.

We can define new types of integration which yield new types of function which are not representable with techniques discussed so far.

Example 9 (The Eulerian Γ -function).

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \text{ for } x > 0$$

The function variable of the Γ -function is a parameter of the integrand.

The indefinite integral from above exists,

$$\lim_{\alpha \rightarrow 0^+} \int_\alpha^1 \underbrace{t^{x-1} e^{-t}}_{>0} dt \text{ exists}$$

of $\int_\alpha^1 t^{x-1} e^{-t} dt$ is bounded in terms of α .

$$\int_\alpha^1 t^{x-1} \underbrace{e^{-t}}_{<1} dt < \underbrace{\int_\alpha^1 t^{x-1} dt}_{\text{converges for } x-1 > -1}$$

hence for $x > 0$.

Right-side integral boundary:

$$\int_1^\infty t^{x-1} e^{-t} dt \text{ converges?}$$

Example 10 (Claim). There exists $c > 0$ such that

$$t^{x-1} e^{-t} < c \cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$t^{x-1} \cdot e^{-\frac{t}{2}} < c \cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(t^{x-1} \cdot e^{-\frac{t}{2}} \right) &= \left| \frac{\frac{t}{2} = s}{t = 2s} \right| \\ &= \lim_{s \rightarrow \infty} (2s)^x - 1e^{-s} \\ &\leq \lim_{s \rightarrow \infty} (2s)^{\lfloor x \rfloor + 1 - 1} \cdot e^{-s} \end{aligned}$$

with $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$

$$\begin{aligned} &= \lim_{s \rightarrow \infty} (2s)^{\lfloor x \rfloor} \cdot e^{-s} \\ &\leq \lim_{s \rightarrow \infty} s^{\lfloor x \rfloor + 1} \cdot e^{-s} \end{aligned}$$

because $s^{n+1} > (2s)^n$ for $s > 2^n$.

Hence for $\varepsilon > 0$, $\exists t$ such that

$$\left| t^{x-1} e^{-\frac{t}{2}} \right| < \varepsilon \text{ if } t > L$$

and

$$\left| t^{x-1} e^{-\frac{t}{2}} \right| \leq M \text{ for } t \in \underbrace{[1, L]}_{\text{compact}}$$

\Rightarrow for $t \in [1, \infty)$ it holds that

$$\left| t^{x-1} e^{-\frac{t}{2}} \right| \leq \max \{M, \varepsilon\} =: c$$

$$\begin{aligned} t^{x-1} e^{-\frac{t}{2}} &\leq c \\ \int_0^\infty t^{x-1} e^{-t} dt &\leq \int_0^\infty c \cdot e^{-\frac{t}{2}} dt = c \cdot \left(-2 \cdot e^{-\frac{t}{2}} \right) \Big|_0^\infty = 2c \end{aligned}$$

hence $\int_0^\infty t^{x-1} e^{-t} dt$ exists.

It holds that $\Gamma(1) = 1$ because,

$$\int_0^\infty e^{-t} dt = 1$$

Furthermore it holds that for all $x > 0$,

$$\begin{aligned} \Gamma(x+1) &= x \cdot \Gamma(x) \\ \Gamma(x+1) &= \int_0^\infty t^{x+1-1} e^{-t} dt = \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\varepsilon^R t^x e^{-t} dt \\ &= \left| \begin{array}{ll} u = t^x & u' = x \cdot t^{x-1} \\ v' = e^{-t} & v = -e^{-t} \end{array} \right| \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \left[-t^x e^{-t} \Big|_{t=\varepsilon}^R + \int_{\varepsilon}^R x \cdot t^{x-1} \cdot e^{-t} dt \right] \\
 &= \lim_{\substack{\varepsilon \rightarrow 0 \\ R \rightarrow \infty}} \left(\underbrace{-R^x \cdot e^{-R}}_{\rightarrow 0 \text{ for } R \rightarrow \infty} + \underbrace{\varepsilon^x \cdot e^{-\varepsilon}}_{\rightarrow 0 \text{ for } \varepsilon \rightarrow 0} \right) + x \cdot \int_0^{\infty} t^{x-1} e^{-t} dt = x \cdot \Gamma(x)
 \end{aligned}$$

So it holds that

$$\begin{aligned}
 T(2) &= 1 \cdot T(1) = 1 \\
 T(3) &= 2 \cdot T(2) = 2 \cdot 1 \\
 T(4) &= 4 \cdot T(3) = 3 \cdot 2 \cdot 1 \\
 T(5) &= 4 \cdot T(4) = 4 \cdot 3 \cdot 2 \cdot 1
 \end{aligned}$$

By complete induction we can show that

$$\Gamma(n+1) = n! \quad \forall n \in \mathbb{N}$$

5.5 Some important inequalities

Theorem 21 (Young's inequality). Let $f : [0, \infty) \rightarrow [0, \infty)$ be continuously differentiable, $f(0) = 0$; f is strictly monotonically increasing and unbounded (hence f is injective because of strong monotonicity and surjective because of unboundedness).

So there exists $f^{-1} : [0, \infty) \rightarrow [0, \infty)$.

Let $a, b \geq 0$. Then it holds that

$$a \cdot b \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy$$

Equality holds if and only if,

$$b = f(a) \text{ i.e. } a = f^{-1}(b)$$

Compare with Figure 17

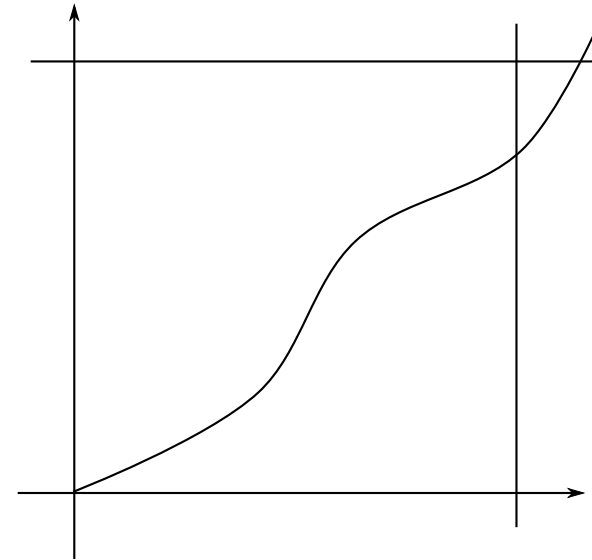


Figure 17: Young's inequality: the blue and red areas are larger than the green area

Proof.

$$\int_0^b f^{-1}(y) dy \stackrel{\text{substitution}}{=} \left| \begin{array}{l} y = f(x) \\ dy = f'(x) dx \\ y = 0 \Leftrightarrow x = f^{-1}(0) = 0 \\ y = b \Leftrightarrow x = f^{-1}(b) \end{array} \right|$$

$$= \text{TODO} \quad = f(x)x|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} 1 \cdot f(x) dx$$

$$= f(f^{-1}(b)) \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x) dx$$

$$= b \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x) dx$$

Therefore,

$$\int_0^a f(x) dx + \int_0^b f^{-1}(y) dy = \int_0^a f(x) dx + \int_{f^{-1}(b)}^0 f(x) dx + b \cdot f^{-1}(b)$$

Case 1: $f^{-1}(b) = a$ ($f(a) = b$)

$$\Rightarrow I = \underbrace{\int_a^b f(x) dx}_{=0} + b \cdot a = ab$$

Proven.

Case 2: $b < f(a) \Leftrightarrow f^{-1}(b) < a$ f is strictly monotonically increasing, hence $f(x) > f(f^{-1}(b)) = b$ for all $x \in (f^{-1}(b), a]$.

$$\int_{f^{-1}(b)}^a f(x) dx > b \cdot \int_{f^{-1}(b)}^a 1 dx$$

$$= b \cdot (a - f^{-1}(b))$$

$$I > b(a - f^{-1}(b)) + b \cdot f^{-1}(b) = a \cdot b$$

Proven.

Case 3: $b > f(a)$

$$I = - \underbrace{\int_a^{f^{-1}(b)} f(x) dx}_{\text{strictly mon. decreasing}} + b f^{-1}(b)$$

For $(-f(x))$ it holds that:

$$> (-f(f^{-1}(b))) = -b$$

Proven.

$$> (-b)(f^{-1}(b) - a) + b \cdot f^{-1}(b) = a \cdot b$$

□

Remark 28. Young's inequality also holds if f has all the properties above but is not necessarily differentiable.

Theorem 22 (Young's inequality, special case). Let $A, B \geq 0$. $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (hence p and q are “conjugate exponents”). Then it holds that

$$A \cdot B \leq \frac{A^p}{p} + \frac{B^q}{q}$$

Proof. Let $f(x) = x^{p-1}$ satisfy the requirements for Young's inequality.

$$f^{-1}(y) = y^{\frac{1}{p-1}}$$

$$\left(\frac{1}{q} = 1 - \frac{1}{p} \quad q = \left(1 - \frac{1}{p} \right)^{-1} \right)$$

$$q - 1 = \left(1 - \frac{1}{p} \right)^{-1} - 1 = \left(\frac{p-1}{p} \right)^{-1} - 1$$

$$= \frac{p}{p-1} - 1 = \frac{p-p+1}{p-1} = \frac{1}{p-1}$$

$$f^{-1}(y) = y^{q-1}$$

Therefore

$$A \cdot B \leq \int_0^A x^{p-1} dx + \int_0^B y^{q-1} dy = \frac{x^p}{p} \Big|_0^A + \frac{y^q}{q} \Big|_0^B = \frac{A^p}{p} + \frac{B^q}{q}$$

□

Remark 29. Equality holds if $A^p = B^q$. The proof is left as an exercise to the reader.

Theorem 23 (Hölder's inequality). Let I be an interval, a, b are boundary values of I ($a, b \in [-\infty, \infty]$). Let p, q be conjugate exponents, hence $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let f_1 and f_2 be regulated functions in I and

$$\int_a^b |f_1(x)|^p dx \text{ exists and } \int_a^b |f_2(x)|^q dx \text{ exists}$$

Let

$$\|f_1\|_p = \left(\int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}}$$

and

$$\|f_2\|_q = \left(\int_a^b |f_2(x)|^q dx \right)^{\frac{1}{q}}$$

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)| dx \text{ exists and } \int_a^b |f_1(x)f_2(x)| dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

Proof. Let $A = \frac{|f_1(x)|}{\|f_1\|_p}$ and $B = \frac{|f_2(x)|}{\|f_2\|_q}$.

$$A \cdot B \leq \frac{A^p}{p} + \frac{B^q}{q} \xrightarrow{\text{integration}} \int_a^b \frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q} dx \leq \frac{1}{p} \int_a^b \frac{|f_1(x)|^p}{\|f_1\|_p^p} dx + \frac{1}{q} \int_a^b \frac{|f_2(x)|^q}{\|f_2\|_q^q} dx$$

$$\Rightarrow \frac{1}{\|f_1\|_p \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| dx \leq \frac{1}{p} \frac{1}{\|f_1\|_p^p} \underbrace{\int_a^b |f_1(x)|^p dx}_{\|f_1\|_p^p} + \frac{1}{q} \frac{1}{\|f_2\|_q^q} \underbrace{\int_a^b |f_2(x)|^q dx}_{\|f_2\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$= \underbrace{\int_a^b |f_1(x) \cdot f_2(x)| dx}_{\text{exists}} \leq \|f_1\|_p \cdot \|f_2\|_q$$

□

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