Linear Algebra 2 – Lecture Notes

Lukas Prokop

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Contents

This lecture took place on 29th of Feb 2016 (Prof. Franz Lehner).

Exam: written and orally

Tutorial session:

- Every Monday, 18:30-20:00, SR 11.34
- Contact: gernot.holler@edu.uni-graz.at

Konversatorium:

- Every Monday, 10:00–10:45, SR 11.33

Topics, wie already discussed:

- Vector spaces
- Linear maps and their equivalence with matrices
- We introduced equivalence of matrices ($PAQ = B$)
- We defined the following techniques:
  - Rank
  - Linear equation system
  - Inverse matrices
  - Basis transformation

In this semester, we will discuss:

- PAP^{-1}, which is related to eigenvalues and diagonalization, hence $\bigvee_P^? PAP^{-1} = D$.

1 Linear maps (cont.)

1.1 Addition to chapter 5.2.4

$\operatorname{Hom}(V,W)$ in special case $W=\mathbb{K}$. We define,

$$V^* := \operatorname{Hom}(V,\mathbb{K})$$

also denoted V' is called *dual space* of vector space V. The elements $v* \in V*$ are called *linear forms* or *linear functionals.*

We denote,

$$v^*(v) =: \langle v*, v\rangle$$

1.2 Example

$$V = \mathbb{K}^n$$

$v^*: V \to \mathbb{K}$ is uniquely defined with values $v^*(e_i) =: a_i$.

$$\langle v^*, v\rangle = \left\langle v^*, \sum_{i=1}^n v_i e_i \right\rangle = \sum_{i=1}^n v_i \langle v^*, e_i\rangle$$

$$v^*\left(\sum_{i=1}^n v_i e_i\right) = \sum_{i=1}^n v_i v^*(e_i) = \sum_{i=1}^n a_i v_i$$

1.3 More general

We know, $\dim \operatorname{Hom}(V,W) = \dim V \cdot \dim W$.

Theorem 1. *Let V be a vector space over $\mathbb{K}$.*

- $\dim V =: n < \infty \Rightarrow \dim V^* = n$
  More precisely: Let $(b_1,\dots,b_n)$ be a basis of V. Then

  $$b_k^* : b_i \mapsto \delta_{ik} = \begin{cases} 1 & i=k \\ 0 & \textit{else} \end{cases}$$

  is a basis of V^* and is called dual basis.

- *For $v^* \in V^*$ it holds that $v^* = \sum_{k=1}^n \langle v^*, b_k\rangle \cdot b_k^*$.*

- *If $\dim V = \infty, (b_i)_{i\in I}$ is a basis, then it holds that $(b_k^*)_{k\in I}$ with*

  $$\langle b_k^*, b_i\rangle = \delta_{ik}$$

  is not *a basis of V^*.*

Proof.

- Special case of 5.18
  (b_k^*) is linear independent, hence in $\sum_{i=1}^n \lambda_i b_i^* = 0$ all $\lambda_i = 0$.

  $$0 = \left\langle \sum_{i=1}^n \lambda_i b_i^*, b_k \right\rangle = \sum_{i=1}^n \lambda_i \left\langle \underbrace{b_i^*, b_k}_{\delta_{ik}} \right\rangle = \lambda_k \forall k$$

- Let $v \in V$ with $v = \sum_{i=1}^n v_i b_i$. We need to show

  $$\langle v^*, v\rangle \overset{!}{=} \left\langle \sum_{k=1}^n \langle v^*, b_w\rangle b_n^*, v \right\rangle$$

  $$\begin{aligned}
  \left\langle \sum_{k=1}^n \langle v^*, b_k\rangle b_k^*, v \right\rangle &= \sum_{k=1}^n \langle v^*, b_k\rangle \langle b_k^*, v\rangle \\
  &= \sum_{k=1}^n \langle v^*, b_k\rangle \left\langle b_k^*, \sum_{i=1}^n v_i b_i \right\rangle \\
  &= \sum_{k=1}^n \sum_{i=1}^n \langle v^*, b_k\rangle \underbrace{\langle b_k^*, b_i\rangle}_{\delta_{ki}} \cdot v_i \\
  &= \sum_{k=1}^n \langle v^*, b_k\rangle \langle v^*, b_k\rangle \cdot v_k \\
  &= \left\langle v^*, \sum_{k=1}^n v_k b_k \right\rangle \\
  &= \langle v^*, v\rangle
  \end{aligned}$$

- (To be done in the practicals) Consider the functional

  $$\langle v^*, b_i\rangle = 1 \Rightarrow v^* \notin L((v_i^*)_{i\in I})$$

□

1.4 Remark and a definition for bilinearity

The mapping $V^* \times V \to \mathbb{K}$ is linear in v (with fixed v^*) with $(v^*, v) \mapsto \langle v^*, v \rangle$ is linear in v^* (with fixed v). Such a mapping is called *bilinear*.

A mapping $F : V_1 \times \ldots \times V_n \to W$ is called *multilinear* (n-linear) if it is linear in every component. Formally:

$$F(v_1, \ldots, v_{k-1}, \lambda v'_k + \mu v''_k, v_{k+1}, \ldots, v_n)$$

$$= \lambda F(v_1, \ldots, v_{k-1}, v'_k, v_{k+1}, \ldots, v_n) + \mu F(v_1, \ldots, v''_k, v_{k+1}, \ldots, v_n)$$

1.5 Example

$$V = \mathbb{K}[x] \text{ polynomials}$$

Basis: $\left\{x^k \,\middle|\, k \in \mathbb{N}_0\right\}$ and $\dim V = \aleph_0$

Every $v^* \in V^*$ is uniquely defined by $a_k \coloneqq \left\langle v^*, x^k \right\rangle$

$$(a_k)_{k \in \mathbb{N}_0}$$

$V^* \cong \mathbb{K}[[t]]$ are the formal power series

$$= \left\{ \sum_{k=0}^{\infty} a_k t^k \,\middle|\, a_k \in \mathbb{K} \right\}$$

$$\lambda \sum_{k=0}^{\infty} a_k t^k + \mu \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (\lambda a_k + \mu b_k) t^k$$

(Compare with Taylor series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$))

$$\left\langle \sum_{k=0}^{\infty} a_k t^k, \sum_{k=0}^{n} b_k x^k \right\rangle \eqqcolon \sum_{k=0}^{n} a_k b_k \text{ is well-defined}$$

$$\to \mathbb{K}[x]^* \cong \mathbb{K}[[t]]$$

1.6 Example

$$C[0,1] \text{ continuous functions}$$

Example:

Example 1.

$$x \in [0,1] \qquad \delta_x : C[0,1] \to \mathbb{R}$$

$$f \mapsto f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$I(f) = \int_0^1 f(x)\, dx \textit{ is linear}$$

$$\langle I_g, f \rangle = \int_0^1 f(x) g(x) \,\mathrm{d}x$$

$$g \in C[0,1] \text{ is fixed}$$

$$\Rightarrow I_g \in C[0,1]$$

$$\langle I_g, \lambda f_1 + \mu f_2 \rangle' = \int_0^1 (\lambda f_1(x) + \mu f_2(x)) g(x) \,\mathrm{d}x$$

$$= \lambda \int_0^1 f_1(x) g(x) \,\mathrm{d}x + \mu \int_0^1 f_2(x) g(x) \,\mathrm{d}x$$

This also works with non-continuous g (it suffices to have g integratable). (Compare with measure theory and Riesz' theorem)

Does there exist some g such that $f(x) = \langle \delta_x, f \rangle = \int_0^1 f(t) g(t) \,\mathrm{d}t$. (Compare with Dirac's δ function and Schwartz/Sobder theory)

$$V^{**} = (V^*)^* \cong V \text{ if } \dim V < \infty$$

Lemma 1. *Let V be a vector space over $\mathbb{K}$. It requires that $\dim V < \infty$ and the Axiom of Choice holds.*

- $v \in V \setminus \{0\} \Leftrightarrow \bigvee_{v^* \in V^*} \langle v^*, v \rangle \neq 0$

- $\bigwedge_{v\in V} v = 0 \Leftrightarrow \bigwedge_{v^*\in V^*} \langle v^*, v\rangle = 0$

Proof. Addition v to a basis B of V: Define $v^* \in V^*$ by

$$\langle v^*, b\rangle = \begin{cases} 1 & b = v \\ 0 & b \neq v \end{cases} \text{ for } b \in B$$

□

Theorem 2. *Let V be a vector space over $\mathbb{K}$.*

- *The map $\iota : V \to V^{**} := (V^*)^*$ is called* bidual space.

$$\langle \iota(v), v^*\rangle := \langle v^*, v\rangle$$

  is linear and injective.

- *if $\dim V < \infty$, then isomorphism.*

Proof.
- Linearity

$$\iota(\lambda v + \mu w) \stackrel{!}{=} \lambda\iota(v) + \mu\iota(w)$$

  must hold in every point $v^* \in V^*$:

$$\begin{aligned} \langle \iota(\lambda v + \mu w), v^*\rangle &= \langle v^*, \lambda v + \mu w\rangle \\ &= \lambda \langle v^*, v\rangle + \mu \langle v^*, w\rangle \\ &= \lambda \langle \iota(v), v^*\rangle + \mu \langle \iota(w), v^*\rangle \\ &= \langle \lambda\iota(v) + \mu\iota(w), v^*\rangle \end{aligned}$$

  Is it injective? Let $v \in \ker\iota$.

$$\langle \iota(v), v^*\rangle = 0 \quad \forall v^* \in V^*$$

$$\Rightarrow \langle v^*, v\rangle = 0 \quad \forall v^* \in V^*$$

$$\xRightarrow{\text{Lemma 1}} v = 0$$

- Follows immediately, because the dimension is equal.

□

Definition 1. *Let V, W be vector spaces over $\mathbb{K}$. $f \in \mathrm{Hom}(V, W)$. We define $f^T \in \mathrm{Hom}(W^*, V^*)$ using $f^T(w^*) \in V^*$ via*

$$\left\langle f^T(w^*), v\right\rangle = \langle w^*, f(v)\rangle = w^*(f(v)) = w^* \circ f(v)$$

$$f^T(w^*) = w^* \circ f \text{ is linear} \Rightarrow f^T(w^*) \in V^*$$

V to W (with f) and W to $\mathbb{K}$ (with w^*).

f^T is called transposed map.

Example 2. *(See practicals) Let $\dim V = n$ and $\dim W = m$ with $B \subseteq V$ and $C \subseteq W$ as bases and dual bases $B^* \subseteq V^*$ and $C^* \subseteq W^*$*

$$\Phi^{C^*}_{B^*}(f^T) = \Phi^B_C(f)^T \quad \textit{transposition of matrices}$$

This lecture took place on 2nd of March 2016 (Franz Lehner).

2 Determinants

Leibnitz 1693 (3×3 matrices)
Seki Takukazu 1685 (most general version)
Gauß 1801 (“determinant”)
Cayley 1845 (on matrices)

$n = 2$

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

$$\left.\begin{array}{cc} a & b \\ c & d \end{array}\right| \begin{array}{c} e \\ f \end{array}$$

1. Case 1: $a \neq 0$ (multiply first row $-\frac{a}{b}$ times second row)

$$\begin{array}{cc} a & b \\ c & d \\ \hline a & b \\ 0 & d - \frac{bc}{a} \end{array}$$

Unique solution:

$$d - \frac{bc}{a} \neq 0$$

2. Case 2: $c \neq 0$ (multiple second row $-\frac{a}{c}$ times first row)

$$\begin{array}{cc} a & b \\ c & d \\ \hline 0 & b - \frac{ad}{c} \\ c & d \end{array}$$

Unique solution:

$$b - \frac{ad}{c} \neq 0$$

This gives us

$$ad - bc \neq 0$$

Definition 2.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called determinant *of* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2.1 Properties of determinants

- The determinant is bilinear in the columns and rows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v, w)$$

where v and w are column vectors of A.

$$\det(\lambda v_1 + \mu v_2, w) = \lambda \det(v_1, w) + \mu \det(v_2, w)$$

$$\det(v, \lambda w_1 + \mu w_2) = \lambda \det(v, w_1) + \mu \det(v, w_2)$$

$$\det(\lambda v_1 + \mu v_2, w) = \begin{vmatrix} \lambda a_1 + \mu a_2 & b \\ \lambda c_1 + \mu c_2 & d \end{vmatrix}$$

$$\begin{aligned} &= (\lambda a_1 + \mu a_2)d - (\lambda c_1 + \mu c_2)b \\ &= \lambda(a_1 d - c_1 b) + \mu(a_2 d - c_2 b) \\ &= \lambda \begin{vmatrix} a_1 & b \\ c_1 & d \end{vmatrix} + \mu \begin{vmatrix} a_2 & b \\ c_2 & d \end{vmatrix} \end{aligned}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

- $\det(v, v) = 0$.

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ac = 0$$

- $$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(e_1, e_2) = 1$$

Theorem 3. *The properties 1–3 of determinants (see above) characterize the determinant.*

Let $\varphi : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}$

- *bilinear*
- $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$
- $\varphi(e_1, e_2) = 1$. *Then it holds that* $\varphi = \det$.

Proof. To show: $\varphi(v, w) = \det(v, w) \forall v, w \in \mathbb{K}^2$

$$v = \underbrace{ae_1 + ce_2}_{\begin{pmatrix} a \\ c \end{pmatrix}} \qquad w = \underbrace{be_1 + de_2}_{\begin{pmatrix} b \\ d \end{pmatrix}}$$

$$\begin{aligned} \varphi(v, w) &= \varphi(ae_1 + ce_2, be_1 + de_2) \\ &= a\varphi(e_1, be_1 + de_2) + c \cdot \varphi(e_2, be_1 + de_2) \\ &= ad\,\underbrace{\varphi(e_1, e_2)}_{=1} + \underbrace{ab\varphi(e_1, e_1)}_{=0} + cb\varphi(e_2, e_1) + cd\,\underbrace{\varphi(e_2, e_2)}_{=0} \end{aligned}$$

□

Lemma 2. *From (i) bilinearity and (ii)* $\bigwedge_{v\in\mathbb{K}^2}\varphi(v,v)=0$ *it follows that*

$$\bigwedge_{v,w\in\mathbb{K}^2}\varphi(v,w)=-\varphi(w,v)$$

$$\begin{aligned}0 &\overset{(ii)}{=} \varphi(v+w,v+w) \overset{(i)}{=} \varphi(v,v)+\varphi(v,w)+\varphi(w,v)+\varphi(w,w)\\ &\overset{(ii)}{=} \varphi(v,w)+\varphi(w,v)\end{aligned}$$

2.2 Geometric interpretation of the determinant

Consider an area with w defining its breath and v its depth (hence the area spanning vectors). Let e_1 and e_2 be the spanning vectors of a rectangle corresponding to the parallelogram. $\det(v,w)$ is the surface of the spanned parallelogram. The sign defines the orientation of the pair (v,w).

$$\det(e_1,e_2)=1 \qquad \det(e_2,e_1)=-1$$

There are surfaces where the surface is infinite if you follow a vector in some direction:

- Möbius strip
- Klein's bottle (named after Felix Klein)

$$A=|\,v\,|\cdot h$$

Consider Figure 1. h is the length of the projection of w to $v^{\perp}$.

$$v=\begin{pmatrix}a\\b\end{pmatrix}\to\vec{n}=\begin{pmatrix}-b\\a\end{pmatrix}$$

$$\langle\begin{pmatrix}c\\d\end{pmatrix},\begin{pmatrix}-b\\a\end{pmatrix}\rangle=ad-bc$$

Second proof. $A(v,w)$ satisfies properties (i)—(iii).

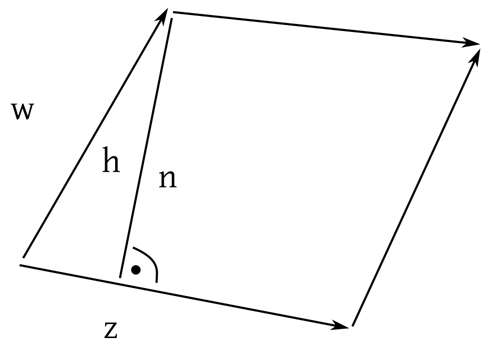

Figure 1: Parallelogram

- Property (iii) follows immediately (the area of unit vectors in two dimensions is 1).
- Property (ii) follows immediately (the area of two vectors in the same direction is 0).

Property (i) defines the linearity in v

1. If v,w are linear dependent, then $A(v,w)=0$ (one is a multiple of the other)
2. $n\in\mathbb{N}$ with $A(nv,w)=nA(v,w)$

3. For $\tilde{v} = n \cdot v$:

$$A(\tilde{v}, w) = n \cdot A(\frac{\tilde{v}}{n}, w)$$

$$\Rightarrow A(\frac{\tilde{v}}{n}, w) = \frac{1}{n} A(\tilde{v}, w)$$

$$\begin{aligned} A(nv, w) &= nA(v, w) \\ A(\frac{1}{n}v, w) &= \frac{1}{n}A(v, w) \\ A(\frac{m}{n}v, w) &= \frac{m}{n}A(v, w) \\ A(-v, w) &= -A(v, w) \end{aligned}$$

From continuity it follows that $A(\lambda u, w) = \lambda A(v, w)$ for $\lambda \in \mathbb{R}$. Analogously $A(v, \lambda w) = \lambda A(v, w)$.

4. The sum is given with

$$A(v + w, w) = A(v, w)$$

Compare with Figure 2, where area(2) + area(3) = area(2) + area(1).

$$\begin{aligned} A(\lambda v + \mu w, w) &= A(\lambda v + \mu w, \frac{1}{\mu}\mu w) \\ &= \frac{1}{\mu} A(\lambda v + \mu w, \mu w) \\ &= \frac{1}{\mu} A(\lambda v, \mu w) \\ &= A(\lambda v, w) \end{aligned}$$

General case: v, w are linear independent and therefore basis of $\mathbb{R}^2$. Besides that, v_1 and v_2 are arbitrary.

$$\begin{aligned} v_1 &= \lambda_1 v + \mu_1 w \\ v_2 &= \lambda_2 v + \mu_2 w \end{aligned}$$

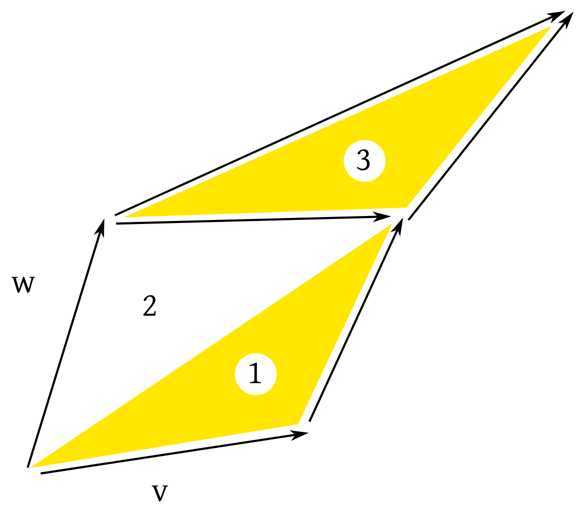

Figure 2: Translation of area 1 to area 3.

$$\begin{aligned} A(v_1 + v_2, w) &= A(\lambda_1 v + \mu_1 w + \lambda_2 v + \mu_2 w, w) \\ &= A((\lambda_1 + \lambda_2)v + (\mu_1 + \mu_2)w, w) \\ &= A((\lambda_1 + \lambda_2)v, w) \\ &= (\lambda_1 + \lambda_2)A(v, w) \\ &= A(\lambda_1 v, w) + A(\lambda_2 v, w) \end{aligned}$$

$$A(\lambda_1 v + \mu_1 w, w) + A(\lambda_2 v + \mu_2 w, w) = A(v_1, w) + A(v_2, w)$$

Additivity follows.

$\square$

Definition 3. *Let* $\dim V = n$*. A* determinant form *is a map*

$$\triangle : V^n \to \mathbb{K}$$

with properties:

1.
$$\bigwedge_{\lambda} \bigwedge_{k} \bigwedge_{a_1,\dots,a_n \in V} \triangle(a_1, \dots, a_{k-1}, \lambda a_k, a_{k+1}, \dots, a_n) = \lambda \triangle(a_1, \dots, a_k, \dots, a_n)$$

2.
$$\bigwedge_{k} \bigwedge_{\substack{a_1,\dots,a_n \\ a_k', a_k''}} \triangle(a_1, \dots, a_{k-1}, a_k' + a_k'', a_{k+1}, \dots, a_n)$$
$$:= \triangle(a_1, \dots, a_{k-1}, a_k' + a_k'', a_{k+1}, \dots, a_n)$$

3.
$$\triangle(a_1, \dots, a_n) = 0$$
if $\bigvee_{k \neq l} a_k = e_l$ *if* $\triangle \neq 0$*, i.e.* $\triangle$ *is non-trivial.*

Multilinearity is defined by the first two properties. Multilinearity means linearity in a_k *if* $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$ *gets fixed.*

Theorem 4.
$$\dim V = n$$
$$\triangle : V^n \to \mathbb{K} \text{ is determinant form}$$

Then,

4.
$$\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{i \neq j} \triangle(a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, a_n) = \triangle(a_1, \dots, a_i, \dots, a_n)$$
"Addition of λa_j *to* a_i *does not change* $\triangle$*"*

5.
$$\bigwedge_{i>j} \triangle(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n)$$
$$= -\triangle(a_1, \dots, a_j, \dots, a_i, \dots, a_n)$$
"Exchanging a_i *with* a_j *inverts the sign"*

Proof. 4.
$$\triangle(a_1, \dots, a_i + \lambda a_j, \dots, a_n)$$
Without loss of generality: $i < j$. From properties 1 and 2 it follows that:
$$= \triangle(a_1, \dots, a_i, a_j, a_n) + \lambda \triangle(a_1, \dots, a_j, a_j, \dots, a_k)$$
Oh, a_j occurs twice! Once at index i and once at index j.
$$= 0$$
due to property 3.

5.
$$\begin{aligned} 0 &\overset{\text{property 3}}{=} \triangle(a_1, \dots, a_{i-1}, a_i + a_j, \dots, a_{j-1}, a_i + a_j, \dots, a_n) \\ &= \triangle(a_1, \dots, a_{i-1}, \mathbf{a_i}, \dots, a_{j-1}, \mathbf{a_i}, \dots, a_n) = \mathbf{0} \\ &+ \triangle(a_1, \dots, a_{i-1}, \mathbf{a_i}, \dots, a_{j-1}, \mathbf{a_j}, \dots, a_n) \\ &+ \triangle(a_1, \dots, a_{i-1}, \mathbf{a_j}, \dots, a_{j-1}, \mathbf{a_i}, \dots, a_n) \\ &+ \triangle(a_1, \dots, a_{i-1}, \mathbf{a_j}, \dots, a_{j-1}, \mathbf{a_j}, \dots, a_n) = \mathbf{0} \\ &\Rightarrow \delta \end{aligned}$$

□

Definition 4. *A permutation of order* n *is a bijective mapping* $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$*.*
$$\sigma_n = \text{ set of all permutations}$$

Remark 1. *Notation: We write the elements in the first row and their images in the second row.*

Definition 5. σ_n *constitutes (in terms of composition) a group with neutral element id, the so-called symmetric group.*

In the previous course (Theorem 1.40) we have proven: Compositions of bijective functions are bijective.

Remark 2. *For $n \geq 3$, σ_n is non-commutative*

Theorem 5.

$$|\,\sigma_n\,| = n!$$

Remark 3. *These are "a lot"!*

Example 3.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Definition 6. *A* transposition *is a permutation of the structure*

$$\tau = \tau_{ij} : \begin{array}{c} i \mapsto j \\ j \mapsto i \\ k \mapsto h \end{array} \quad \textit{if } k \notin \{i, j\}$$

Then $\tau_{ij}^{-1} = \tau_{ij}$, hence $\tau_{ij}^2 = id$.

Theorem 6. *σ_n is generated by transpositions. With other words, every permutation π can be represented as composition of transpositions*

$$\pi = \tau_1 \circ \ldots \circ \tau_k$$

Proof.

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

If $\pi = \mathrm{id}$,

$$\pi = \pi \quad \tau := \mathrm{id}$$

If $\pi \neq \mathrm{id}$,

$$k_1 = \min\{k \,|\, k \neq \pi(k)\}$$

1.

$$\tau_1 = \tau_{k_1 \pi(k_1)}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & \ldots & k_1-1 & k_1 & \ldots \\ 1 & \ldots & k_1-1 & k_1 & \ldots \end{pmatrix}$$

Example: Consider $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$.

$$k_1 = 2$$

$$\tau_1 = \tau_{23}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}$$

2.

$$k_2 = \min\{k \,|\, k \neq \pi_1(k)\} > k_1$$

$$\tau_2 = \tau_{k_2, \pi(k_2)}$$

And so on and so forth. $k_j > k_{j-1}$ ends after $\leq n$ steps.

$$\tau_k \circ \tau_{k-1} \circ \ldots \circ \tau_1 \circ \pi = \mathrm{id}$$

$$\Rightarrow \pi = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k$$

Regarding the example:

$$k_2 = 3$$

$$\tau_2 = \tau_{35}$$

$$\pi_2 = \tau_2 \circ \pi_1 = \tau_2 \circ \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}$$

$$k_3 = 5 \quad \tau_3 = \tau_{57}$$

$$\Rightarrow \pi = \tau_{23} \circ \tau_{35} \circ \tau_{57}$$

□

Definition 7. *An* inversion *of* π *is a pair* (i,j) *such that* $i<j$ *with* $\pi(i)>\pi(j)$. *Let* F_π *be the set of inversions of* π.

$$f_\pi := |\,F_\pi\,|$$

$$\operatorname{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

Example 4.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$$

$$F_\pi = \{(2,7),(3,4),(3,7),(4,7),(5,6),(5,7),(6,7)\}$$

$$f_\pi = 7 \qquad \operatorname{sign}(\pi) = -1$$

This lecture took place on 7th of March 2016 (Franz Lehner).

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Recall: Determinant form:

1. $\triangle(a_1,\dots,\lambda a_k,\dots,a_n) = \lambda\triangle(a_1,\dots,a_n)$
2. $\triangle(a_1,\dots,a_k'+a_k'',\dots,a_n) = \triangle(a,\dots,a_k',\dots,a_n) + \triangle(a_1,\dots,a_k'',\dots,a_n)$
3. $\triangle(a_1,\dots,a_k,\dots,a_l,\dots,a_n) = 0$ if $a_k = a_l$

Conclusions:

4. $\triangle(a_1,\dots,a_k+\lambda a_l,\dots,a_n) = \triangle(a_1,\dots,a_n)$ if $k \neq l$
5. $\triangle(a_1,\dots,a_k,\dots,a_l,\dots a_n) = -\triangle(a_1,\dots,a_l,\dots,a_k,\dots a_n)$

$$\triangle(a_{\pi(1)},\dots,a_{\pi(n)}) = (-1)^k\triangle(a_1,\dots,a_n)$$

Decompose $\pi = \tau_1 \circ \ldots \circ \tau_k \circ \tau_{12} \circ \tau_{12}$. This decomposition is not distinct (k is distinct mod2)

$$\pi \in \sigma_n \qquad \text{permutation}$$

$$F_\pi = \{(i,j)\,|\, i<j, \pi(i)>\pi(j),\ \text{inversions}\ \}$$

$$f_\pi = |\,F_\pi\,|$$

$$\operatorname{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

Theorem 7.

- $\bigwedge_{\pi\in\sigma_n} \operatorname{sign}(\pi) = \prod_{1\le i<j\le n} \frac{\pi(j)-\pi(i)}{j-i}$
- *For transposition* τ *it holds that* $\operatorname{sign}(\tau) = -1$

Proof.

- Every pair $\{i,j\}$ occurs in the enumerator exactly once.

$$\frac{\prod_{i<j}\pi(j)-\pi(i)}{\prod_{i<j}(j-i)}$$

Denominator: $j>i$, positive. Enumerator: positive if $\pi(j)>\pi(i)$, negative if $\pi(i)>\pi(j)$.

- $$\tau = \begin{pmatrix} 1 & \dots & k & \dots & l & \dots & n \\ 1 & \dots & l & \dots & k & \dots & n \end{pmatrix}$$

$$F_\tau(\underbrace{(k,k+1),(k,k+2),\dots,(k,l-1)}_{\text{inversions with } k,\ l-k \text{ times}},(k,l),\underbrace{(k+1,l),\dots,(l-1,l)}_{l-k-1 \text{ times}})$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 8 & 5 & 6 & 7 & 4 & 9 & 10 \end{pmatrix}$$

Yields 7 inversions (8 needs to be repositioned with 3 transpositions, 4 needs to be repositions with 4 transpositions).

□

$$\operatorname{sign}(\pi) = \prod_{i<j}\frac{\pi(j)-\pi(i)}{j-i} \qquad \binom{n}{2} \text{ factors}$$

$$\operatorname{sign}(\tau) = -1$$

Theorem 8. *1.* $\operatorname{sign}(id) = 1$

2. $\operatorname{sign}(\pi \circ \sigma) = \operatorname{sign}(\pi) \cdot \operatorname{sign}(\sigma)$, *hence*

$$\operatorname{sign} \sigma_n \to (\{+1, -1\}, \cdot)$$

is a group homomorphism. (In general: A group homomorphism $h : G \to (\mathcal{T}, \cdot)$ is called character*)*

3. $\operatorname{sign}(\pi^{-1}) = \operatorname{sign}(\pi)$

Remark 4.

$$\mathcal{T} = \{z \in \mathbb{C} \mid |\, z \,| = 1\}$$

Torus with multiplication is a group.

$$|\, z_1 \cdot z_2 \,| = |\, z_1 \,| \cdot |\, z_2 \,| = 1$$

Proof. 1. trivial

2.

$$\begin{aligned}\operatorname{sign}(\pi \cdot \sigma) &= \prod_{i<j} \frac{\pi \circ \sigma(j) - \pi \circ \sigma(i)}{j - i} \\ &= \underbrace{\prod_{i<j} \frac{\pi(\sigma(j)) - \pi(\sigma(i))}{\sigma(j) - \sigma(i)}}_{=\operatorname{sign}(\pi)} \cdot \underbrace{\prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j - i}}_{\operatorname{sign}(\sigma)}\end{aligned}$$

3. Group homomorphism!

□

Corollary 1. • *If $\pi = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k$, product of transpositions*

$$\Rightarrow \operatorname{sign}(\pi) = (-1)^k$$

• $\mathfrak{a}_n := \ker(\operatorname{sign}) = \{\pi \in \sigma_n \mid \operatorname{sign}(\pi) = 1\}$

"even permutations", "alternating group"

$$|\, \mathfrak{a}_n \,| = \frac{n!}{2}$$

Corollary 2.

$$\triangle : V^k \to \mathbb{K} \text{ determinant form}$$

then it holds that

$$\bigwedge_{\pi \in \sigma_n} \bigwedge_{a_1, \ldots, a_n \in V} \triangle(a_{\pi(1)}, \ldots, a_{\pi(n)}) = \operatorname{sign}(\pi) \cdot \triangle(a_1, \ldots, a_n)$$

Proof. • If $\pi = \tau_{kl}$ transposition $\xrightarrow{\text{Theorem 4}}$ $\triangle(a_{\tau(1)}, \ldots, a_{\pi(n)}) = -\triangle(a_1, \ldots, a_n) = \operatorname{sign}(\tau_{kl}) \cdot \triangle(a_1, \ldots, a_n)$

• If $\pi = \tau_1 \circ \ldots \circ \tau_k = \tau_1 \circ \tilde{\pi}, \tilde{\pi} = \tau_2 \circ \ldots \circ \tau_k$

$$\begin{aligned}&\triangle(a_{\tau_1 \circ \tilde{\pi}(1)}, \ldots, a_{\tau_1 \circ \tilde{\pi}(n)}) \\ &= -\triangle(a_{\tilde{\pi}(1)}, \ldots, a_{\tilde{\pi}(n)}) \\ &= (-1)^2 \cdot \triangle(a_{\tilde{\pi}(1)}, a_{\tilde{\pi}(n)}) \\ &\to (-1)^k \cdot \triangle(a_1, \ldots, a_n)\end{aligned}$$

□

Theorem 9 (Leibnitz' definition of $\det(A)$)**.** *Let $B = (b_1, \ldots, b_n)$ be the basis of V. $a_1, \ldots, a_n \in V$ with coordinates*

$$\Phi_B(a_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

$$A := [a_{ij}]_{i,j=1,\ldots,n} = [\Phi_B(a_1), \Phi_B(a_2), \ldots, \Phi_B(a_n)]$$

Then it holds that for every determinant form $\triangle : V^k \to \mathbb{K}$:

$$\triangle(a_1, \ldots, a_n) = \det(A) \cdot \triangle(b_1, \ldots, b_n)$$

where

$$\det(A) := \sum_{\pi \in \sigma_n} \operatorname{sign}_{\mathbb{K}} \pi a_{\pi(1),1} a_{\pi(2),2} \ldots a_{\pi(n),n}$$

is the determinant of A

Example 5. *Example ($n = 2$):*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$\operatorname{sign}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 1$$

$$\operatorname{sign}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -1$$

Proof.

$$a_j = \sum_{j=1}^{n} a_{ij} b_i$$

$$\begin{aligned} \triangle(a_1, \dots, a_n) &= \triangle\left(\sum_{i_=1}^{n} a_{i,1} b_i, \sum_{i_2=1}^{n} a_{i_2,2} b_i, \dots, \sum_{i_n=1}^{n} a_{i_n,n} b_i\right) \\ &= \sum_{i_1=1}^{n} a_{i,1} \sum_{i_2=1}^{n} a_{i_2,2} \dots \sum_{i_n=1}^{n} a_{i_n,n} \underbrace{\triangle(b_i, b_{i_2}, \dots, b_{i_n})}_{=0 \text{ if some } i_k = i_l} \end{aligned}$$

So summands with equal indices disappear. It holds that $\sum_{i_1,\dots,i_n}$ such that $i_1, \dots, i_n$ are different. Hence every value from $\{1, \dots, n\}$ occurs exactly once. This is the set of all permutations π ($i_j = \pi(j)$)

$$= \sum_{\pi \in \sigma_n} a_{\pi(1)1} a_{\pi(2)2} \dots a_{\pi(n)n} \underbrace{\triangle(b_{\pi(1)}, \dots, b_{\pi(n)})}_{\operatorname{sign}(\pi) \cdot \triangle(b_1, \dots, b_n)}$$

□

Corollary 3. *A determinant form is* uniquely *defined on a basis $(b_1, \dots, b_n)$ by the value $\triangle(b_1, \dots, b_n)$. Especially $\triangle$ is nontrivial,*

$\Leftrightarrow$ $\triangle(b_1, \dots, b_n) \neq 0$ *on some basis.*

$\Leftrightarrow$ $\triangle(b_1, \dots, b_n) \neq 0$ *in every basis $b_1, \dots, b_n$.*

Let $\triangle(b'_1, \dots, b'_n) = 0$ for some other basis, represent $b_1, \dots, b_n$ in basis $b'_1, \dots, b'_n$

$$b_j = \sum a_{ij} b'_i \Rightarrow \triangle(b_1, \dots, b_n) = \det(A) \cdot \triangle(b'_1, \dots, b'_n) = 0$$

$$\triangle(a_1, \dots, a_n) = \det(A) \cdot \triangle(b_1, \dots, b_n)$$

Theorem 10. *Let $B = (b_1, \dots, b_n)$ be a basis of V over $\mathbb{K}$. $c \in \mathbb{K}$. For $a_1, \dots, a_n \in V$, let $A = [\Phi_B(a_1), \dots, \Phi_B(a_n)]$. Then*

$$\triangle(a_1, \dots, a_n) = c \cdot \det(A)$$

defines a determinant form, specifically the unique determinant form with value

$$\triangle(b_1, \dots, b_n) = c$$

Proof. The 3 properties of a determinant form:

1.

$$\begin{aligned} \triangle(a_1, \dots, \lambda a_k, \dots, a_n) &= c \cdot \det\left[\Phi_B(a_1), \dots, \lambda \cdot \Phi_B(a_k), \dots, \Phi_B(a_n)\right] \\ &= c \cdot \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot a_{\pi(1),1} a_{\pi(2),2} \dots \lambda a_{\pi(k),k} \dots a_{\pi(n),n} \\ &= \lambda \cdot c \cdot \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot a_{\pi(1),1} a_{\pi(2),2} \dots a_{\pi(n),n} \\ &= \lambda \cdot \triangle(a_1, \dots, a_n) \end{aligned}$$

2.

$$\begin{aligned} &\triangle(a_1, \dots, a'_k + a''_k, \dots, a_n) \\ &= c \cdot \det\left[\Phi_B(a_1), \dots, \Phi_B(a'_k) + \Phi_B(a''_k), \dots, \Phi_B(a_n)\right] \\ &= c \cdot \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot a_{\pi(1),1} \cdot a_{\pi(2),2} \cdot \dots \left(a'_{\pi(k),k} + a''_{\pi(k),k}\right) \cdot \dots \cdot a_{\pi(n),n} \\ &= c \cdot \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot a_{\pi(1),1} \cdot \dots \cdot a'_{\pi(k),k} \dots a_{\pi(n),n} \\ &+ c \cdot \sum_{\pi \in \sigma_n} \operatorname{sign}(\pi) a_{\pi(1),1} \dots a''_{\pi(k),k} \dots a_{\pi(n),n} \\ &= \triangle(a_1, \dots, a'_k, \dots, a_n) + \triangle(a_1, \dots, a''_k, \dots, a_n) \end{aligned}$$

3. Let $a_k = a_l$ for $k < l$. Show that $\triangle(a_1, \dots, a_n) = 0$

$$\tau_{kl} = \text{ transposition exchanging } k \text{ and } l$$

$$\sigma_n = \mathfrak{a}_n \dot\cup (\mathfrak{a}_n \cdot \tau_{kl})$$

Claim: $\{\pi \,|\, \operatorname{sign} \pi = -1\} = \{\pi \circ \tau_{kl} \,|\, \operatorname{sign}\pi = +1\}$

$\supseteq$ If $\operatorname{sign}\pi = +1 \Rightarrow \operatorname{sign}(\pi \circ \tau_{kl}) = \underbrace{\operatorname{sign}\pi}_{+1} \cdot \underbrace{\operatorname{sign}\tau_{kl}}_{-1} = -1$

$\subseteq$ If $\operatorname{sign}\pi = -1 \Rightarrow \operatorname{sign}(\pi \circ \tau_{kl}) = +1 \Rightarrow \pi = \underbrace{(\pi \circ \tau_{kl})}_{\in \mathfrak{a}_n} \circ \tau_{kl} \in \mathfrak{a}_n \cdot \tau_{kl}$

$$\begin{aligned}
\triangle(a_1, \dots, a_n) &= c \cdot \sum_{\pi \in \sigma_n = \mathfrak{a}_n \cup \mathfrak{a}_n \cdot \tau_{kl}} \operatorname{sign}(\pi) a_{\pi(1),1} \cdots a_{\pi(n),n} \\
&= \underbrace{c \cdot \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \cdots a_{\pi(n),n}}_{\text{even}} \\
&\quad \underbrace{- \sum_{\pi \in \mathfrak{a}_n} a_{\pi\circ\tau_{kl}(1),1} \cdots a_{\pi\circ\tau_{kl}(k),k} \cdots a_{\pi\circ\tau_{ul}(l),l} \cdots a_{\pi\circ\tau_{kl}(n),n}}_{\text{odd}} \\
&= c \cdot \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \cdots, a_{\pi(n),n} \\
&\quad - \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \cdots \underbrace{a_{\pi(l),k}}_{a_{\pi(l),l}} \cdots \quad \underset{a_{\pi(k),k}}{} \underbrace{a_{\pi(k)l}}_{\text{because } a_k = a_l} \quad \cdots a_{\pi(n),n}
\end{aligned}$$

What we did:

(a) $a_{\pi(l),k} = a_{\pi(l),l}$ and $a_{\pi(k),l} = a_{\pi(k),k}$ because $a_k = a_l$
(b) exchange factors

$$\begin{aligned}
&= c \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \cdots a_{\pi(k),k} \cdots a_{\pi(l),l} \cdots a_{\pi(n),n} \\
&- c \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \cdots a_{\pi(k),k} \cdots a_{\pi(l),l} \cdots a_{\pi(n),n} \\
&= 0
\end{aligned}$$

Value for $(b_1, \dots, b_n)$

$$a_{ij} = \delta_{ij} \Rightarrow A = I$$

$$\det(I) = \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot \delta_{\pi(1),1} \cdots \delta_{\pi(n),n} = +1$$

for all $\pi(j) = j$ otherwise 0.

$$\Rightarrow \pi = \text{id is the only summand}$$

$$\triangle(b_1, \dots, b_n) = \det(I) \cdot c = c$$

$\square$

Remark 5. *"$\mathfrak{a}_n$ is the subgroup of index 2" denoted* $[\sigma_n : \mathfrak{a}_n] = 2$

You might be familiar with:

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$$
$$[\mathbb{Z} : n\mathbb{Z}] = n$$

Theorem 11 (Summary).
- *The set of determinant forms $\triangle : V^n \to \mathbb{K}$ constructs a one-dimensional vector space, $\Lambda^n V$*
- *There exists a non-trivial determinant form with $\triangle(b_1, \dots, b_n) = 1$*

This lecture took place on 9th of March 2016 (Franz Lehner).

Revision:

$$\triangle : V^n \to \mathbb{K}$$
$$\triangle(a_1, \dots, a_n) = \det A \cdot \triangle(b_1, \dots, b_n)$$

$$\phi_B(a_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$\det A = \sum_{\pi \in \sigma_n} \operatorname{sign}\pi \cdot a_{\pi(1),1} \cdots a_{\pi(n),n}$$

$$\triangle(v_1, \dots, v_n) \neq 0 \Leftrightarrow v_1, \dots, v_n \text{ linear independent } (\Leftrightarrow \text{basis})$$

Theorem 12.

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

Lemma 3. *Let V, W be vector spaces over $\mathbb{K}$ with $\dim V = \dim W = n$.*

$$\triangle : W^n \to \mathbb{K}$$

$$f : V \to W$$

$$\Rightarrow f^n : V^n \to W^n \xrightarrow{\triangle} \mathbb{K}$$

$$(v_1, \ldots, v_n) \mapsto (f(v_1), \ldots, f(v_n))$$

Then $\triangle^f : V^n \to \mathbb{K}$

$$\triangle^f(v_1, \ldots, v_n) = \triangle(f(v_1), \ldots, f(v_n))$$

is a determinant form in V.

Proof. 1.

$$\begin{aligned}\triangle f(v_1, \ldots, \lambda v_k, \ldots, v_n) &= \triangle(f(v_1), \ldots, f(\lambda v_k), \ldots, f(v_n)) \\ &= \lambda \triangle(f(v_1), \ldots, f(v_n)) \\ &= \lambda \cdot \triangle^f(v_1, \ldots, v_n)\end{aligned}$$

2.

$$\begin{aligned}&= \triangle^f(v_1, \ldots, v_k', +v_k'', \ldots, v_n) \\ &= \triangle(f(v_1), \ldots, f(v_k' + v_k''), \ldots, f(v_n)) \\ &= \triangle(f(v_1), \ldots, f(v_k') + f(v_k''), \ldots, f(v_n)) \\ &= \triangle(f(v_1), \ldots, f(v_k'), \ldots, f(v_n)) + \triangle(f(v_1), \ldots, f(v_k''), \ldots, f(v_n)) \\ &= \triangle^f(v_1, \ldots, v_k', \ldots, v_n) + \triangle^f(v_1, \ldots, v_k'', \ldots, v_n)\end{aligned}$$

3.

$$\begin{aligned}\triangle^f(v_1, \ldots, v_k, \ldots, v_l, \ldots, v_n) \qquad v_k = v_l \Rightarrow f(v_k) = f(v_l) \\ = \triangle(f(v_1), \ldots, f(v_k), \ldots, f(v_l), \ldots, f(v_n)) \\ = 0\end{aligned}$$

□

Corollary 4 (Conclusions for $V = W$)**.**

$$\triangle : V^n \to \mathbb{K}$$

non-trivial determinant form

$$f : V \to V$$

$$\Rightarrow \triangle^f \textit{ is a determinant form}$$

$$\dim \bigwedge^n V = 1 \Rightarrow \bigvee_{c_f \in \mathbb{K}} \triangle^k = c_f \cdot \triangle$$

$$c_f =: \det f \textit{ is called determinant of } f$$

Corollary 5. *Let V, $\triangle$ and f be like above.*

1. *For every basis $B = (b_1, \ldots, b_n)$ it holds that*

$$\triangle^f(b_1, \ldots b_n) = \triangle(f(b_1), \ldots, f(b_n)) = \det(f) \cdot \triangle(b_1, \ldots, b_n)$$

$$\det(f) = \frac{\triangle(f(b_1), \ldots, f(b_n))}{\triangle(b_1, \ldots, b_n)}$$

2. *with $a_j = f(b_j)$ it holds that*

$$\det \Phi_B^B(f) = \det(f)$$

$$A = \Phi_B^B(f)$$

$$a_{ij} = i\textit{-th coordinate of } f(b_j) \textit{ and } s_j(A) = \Phi_B(f(b_j)).$$

Theorem 13. *Let $f : V \to V$ be an isomorphism $\Leftrightarrow \det(f) \neq 0$.*

Proof. Let f be an isomorphism.

$$\begin{aligned}&\Leftrightarrow (f(b_1), \ldots, f(b_n)) \text{ is basis} \\ &\Leftrightarrow \triangle(f(b_1), \ldots, f(b_n)) \neq 0 \\ &\Leftrightarrow \det(f) \cdot \triangle(b_1, \ldots, b_n) \\ &\Leftrightarrow \det(f) \neq 0\end{aligned}$$

□

Theorem 14. *Let $f, g : V \to V$ be linear.*

$$\Rightarrow \det(f \circ g) = \det(f) \cdot \det(g)$$

Remark 6. *We show: $f \circ g$ is isomorphism $\Leftrightarrow$ f and g are isomorphisms.*

If f, g are invertible, then $f \circ g$ are invertible.

1. $$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

2. *Attention! This is wrong, if $\dim = \infty$! For example: $\delta : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ over $\mathbb{K}^\infty$ is injective, but not surjective!*

$$S_L : (x_1, x_2, \dots) = (x_2, x_3, \dots)$$

is not injective, but surjective.

$$S_L \circ S_R = I$$
$$S_R \circ S_L - I - P_1$$

If $f \circ g$ is bijective, then g is injective and f surjective.

$$\xRightarrow{\dim < \infty} g \textit{ bijective}, f \textit{ bijective}$$

Proof. Case distinction:

$\det(f \circ g) = 0$

$$\begin{aligned}
&\xLeftrightarrow{\text{Theorem 13}} f \circ g \text{ is not bijective} \\
&\Leftrightarrow f \text{ is not bijective or g not bijective} \\
&\Leftrightarrow \det(f) = 0 \vee \det(g) = 0 \\
&\Leftrightarrow \det(f) \cdot \det(g) = 0
\end{aligned}$$

$\det(f \circ g) \neq 0$

$$\begin{aligned}
&\Leftrightarrow f \circ g \text{ is bijective} \\
&\Rightarrow g \text{ bijective} \\
&\Rightarrow \triangle^g \text{ non-trivial}
\end{aligned}$$

Let $(b_1, \dots, b_n)$ be a basis of V, then $\triangle$ is non-trivial determinant.

$$\begin{aligned}
\det(f \circ g) &= \frac{\triangle(f \circ g(b_1) \dots, f \circ g(b_n))}{\triangle(b_1, \dots, b_n)} \\
&= \frac{\triangle(f(g(b_1)), \dots, f(g(b_n)))}{\triangle(g(b_1), \dots, g(b_n))} \cdot \frac{\triangle(g(b_1), \dots, g(b_n))}{\triangle(b_1, \dots, b_n)} \\
&= \frac{\triangle((b_1'), \dots, f(b_n'))}{\triangle(b_1', \dots, b_n')} \cdot \frac{\triangle(g(b_1), \dots, g(b_n))}{\triangle(b_1, \dots, b_n)} \\
&= \det(f) \cdot \det(g)
\end{aligned}$$

$b_i' = g(b_i)$ are also a basis, because g is bijective.

□

Corollary 6. *Let $A, B \in \mathbb{K}^{n \times n}$.*

1. $\det(A \cdot B) = \det(A) \cdot \det(B)$
2. *A is regular* $\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$
3. $\det(A) = 0 \Leftrightarrow \operatorname{rank}(A) < n$
4. $\det(A^t) = \det(A)$

Proof. 1. A first proof follows from Theorem 14.
A second proof approach is:

$$A = [s_1, \dots, s_n] \qquad \text{column vectors}$$

$$A \cdot B = \left[\sum_{i_1=1}^{n} s_{i_1} \cdot b_{i_1,1}, \sum_{i_2=1}^{n} s_{i_2} b_{i_2,2}, \dots, \sum_{i_n=1}^{n} s_{i_n} b_{i_n,n} \right]$$

Select determinent form such that $\triangle(e_1, \dots, e_n) = 1$.

$$\det(A \cdot B) = \triangle\left(\sum_{i_1=1}^{n} s_{i_1} b_i, \dots, \sum_{i_n=1}^{n} s_{i_n} b_{i_n,n} \right)$$

From multilinearity it follows that

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_n=1}^{n} b_{i_1,1}b_{i_2,2}\cdots b_{i_n,n}\triangle(s_{i_1},\ldots,s_{i_n})$$

If two indices satisfy $i_k = i_l \Rightarrow \triangle = 0$.

$$\Rightarrow \sum_{\text{different indices}} = \sum_{\text{permutations}}$$

$$= \underbrace{\sum_{\pi\in\sigma_n} b_{\pi(1),1}b_{\pi(2),2}\cdots b_{\pi(n),n}}_{\det(B)} \underbrace{\triangle(s_{\pi(1)},\ldots,s_{\pi(n)})}_{\operatorname{sign}(\pi)\underbrace{\triangle(s_1,\ldots,s_n)}_{\det(A)}}$$

$$= \det A \cdot \det B$$

Be aware that $\det(B)$ also includes $\operatorname{sign}(\pi)$ from the right-hand side.

2.

$$A\cdot A^{-1} = I \Leftrightarrow \det(A\cdot A^{-1}) = \det I = 1$$

$$\det(A\cdot A^{-1}) \overset{1.}{=} \det(A)\cdot\det(A^{-1})$$

3. $\det(A) = 0$ and $\det(A) = \det(f_A)$.

$$\Leftrightarrow f_A \text{ is not bijective } \Leftrightarrow \operatorname{rank}(A) < n$$

4.

$$\begin{aligned}\det(A^T) &= \sum_{\pi\in\sigma_n}\operatorname{sign}(\pi)a^T_{\pi(1),1}\cdots a^T_{\pi(n),n}\\ &= \sum_{\pi\in\sigma_n}\operatorname{sign}(\pi)a_{1,\pi(1)}\cdots a_{n,\pi(n)}\\ &= \sum_{\pi\in\sigma_n}\operatorname{sign}\pi a_{\pi^{-1}(1),1}\cdots a_{\pi^{-1}(n),1}\\ &= \sum_{\rho}\operatorname{sign}\rho^{-1}a \qquad\qquad \rho = \pi^1\end{aligned}$$

For fixed π:

$$\prod_{j=1}^{n} a_{j,\pi(j)} = \prod_{k=1}^{n} a_{\pi^{-1}(k),k}$$

$$\pi(j) = 1 \Leftrightarrow j = \pi'(1)$$

$$\pi(j) = k \Leftrightarrow j = \pi'(k)$$

$$\sum_{\pi}\operatorname{sign}\pi a_{\pi^{-1}(1),1}\cdots a_{\pi^{-1}(n),n}$$

$$= \sum\operatorname{sign}(\rho^{-1})a_{\rho(1),1}\cdots a_{\rho(n),n} = \sum_{\rho}\operatorname{sign}(\rho)a_{\rho(1),1}\cdots a_{\rho(n),n} = \det A$$

Remark:

$$\sigma_n \to \sigma_n \text{ is bijective}$$

$$\pi \mapsto \pi^{-1}$$

$$\operatorname{sign}(\rho) = (-1)^k \text{ where } \rho = \tau_1,\ldots,\tau_k$$

$$\Rightarrow \rho^{-1} = \tau_k\circ\ldots\circ\tau_n$$

$$\operatorname{sign}\rho^{-1} = (-1)^k$$

□

Remark 7 (Determination of determinants)**.** dim ≤ 3

For $n = 2$:

$$\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For $n = 3$:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} = \sum_{\pi\in\sigma_3}\operatorname{sign}(\pi)a_{\pi(1),1}a_{\pi(2),2}a_{\pi(3),3}$$

General linear group:

$$\begin{aligned}
\mathrm{GL}(n,\mathbb{K}) &= \textit{ group of invertible matrices}\\
&= \left\{A \in \mathbb{K}^{n\times n} \,\middle|\, \det(A) \neq 0\right\}\\
\mathrm{SL}(n,\mathbb{K}) &= \textit{ special linear group}\\
&= \left\{A \in \mathbb{K}^{n\times n} \,\middle|\, \det(A) = 1\right\}
\end{aligned}$$

σ_3 is a coxeter group.

$$\sigma_3 = \langle \tau_{12}, \tau_{23} \rangle$$

Is created by two transpositions.

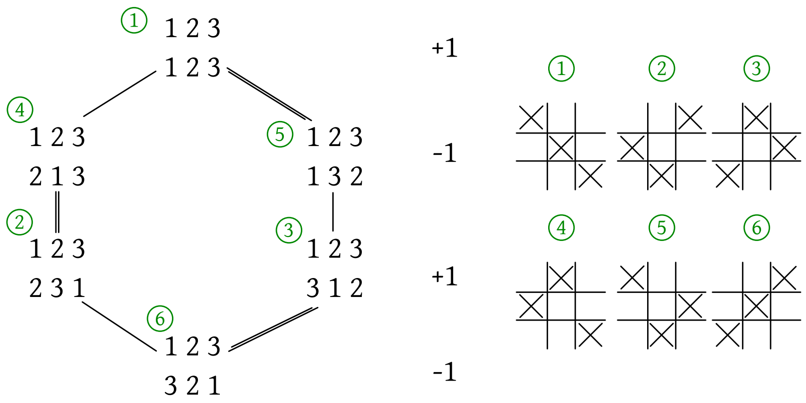

Figure 3: Sign of a permutation

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13}$$

corresponding to (1) + (2) + (3) + (4) + (5) + (6) in Figure 3.

Remark 8 (Rule of Sarrus)**.** *Compare with Figure 4.*

You write the first two columns next to right side of the matrix. You add up all 3 diagonals (the product of their values) from top left diagonally to the right bottom and subtract all 3 diagonals from left bottom to the top right.

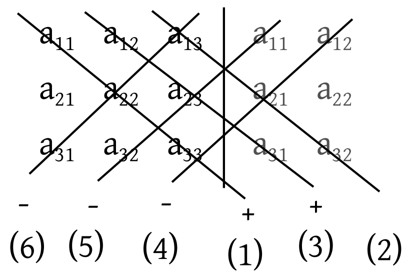

Figure 4: Rule of Sarrus visualized

The rule of Sarrus does not hold for $n = 4$!

Example 6.

$$\det \begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = 1 \cdot 5 \cdot 42 + 2 \cdot 14 \cdot 5 + 5 \cdot 2 \cdot 14 - 5 \cdot 5 \cdot 5 - 14 \cdot 14 \cdot 1 - 2 \cdot 2 \cdot 42$$

$$= 14(1 \cdot 5 \cdot 3 + 2 \cdot 5 + 5 \cdot 2 - 14 - 2 \cdot 2 \cdot 3) - 125 = 14 \cdot 9 - 125 = 1$$

It turns out, if we use Catalan numbers, we always end up with determinant 1.

Lemma 4. *Let A be an upper triangular matrix, hence $a_{ij} = 0 \forall i > j$. Then it holds that $\det A = a_{11}a_{22} \dots a_{nn}$.*

Proof.
$$\det A = \sum_{\pi \in \sigma_n} \operatorname{sign} \pi a_{\pi(1),1} \dots a_{\pi(n),n}$$
it must hold that
$$\pi(j) \leq j \qquad \forall j$$
$$\Rightarrow \pi(1) = 1, \pi(2) = 2, \dots, \pi(n) = n$$
The only permutation which contributes something is the identity. And sign id $=$ 1, hence
$$= 1 \cdot a_{11} a_{22} \dots a_{nn}$$
□

Lemma 5 (Elementary row and column transformations)**.**
$$A = [a_{ij}] \in \mathbb{K}^{n \times n}$$

1.
$$s_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} \textit{ column vectors}$$
$$\Rightarrow \det[as_1, \dots, s_i + \lambda s_j, \dots, s_n] = \det(A) \qquad i \neq j$$

2. Let $z_i = [a_{i_1}, \dots, a_{i_n}]$ *rows of A.*
$$\det \begin{bmatrix} z_1 \\ \vdots \\ z_i + \lambda z_j \\ \vdots \\ z_n = \end{bmatrix} = \det A \qquad \textit{for } i \neq j$$

Proof. 1. compare with determinant form

2. $\det A = \det A^T$

□

Example 7.
$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 4 & 17 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

This lecture took place on 14th of March 2016 (Franz Lehner).

Lemma 6. *Recall: The following operations do not change the determinant:*

- $\triangle(s_1, \dots, s_i + \lambda s_j, \dots, s_n) = \triangle(s_1, \dots, s_n)$
  Addition of a multiple of a column (or row) to another
- *Gauss-Jordan operations (elementary row/column transformations)*

Example 8.
$$\begin{vmatrix} 1 & 0 & 3 & -2 \\ 2 & 6 & 4 & 1 \\ 3 & 3 & -1 & -1 \\ -1 & 2 & 4 & 1 \end{vmatrix} \rightsquigarrow \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 3 & -10 & 5 \\ 0 & 2 & 7 & -1 \end{vmatrix} \rightsquigarrow \frac{1}{3}\frac{1}{2} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 6 & -20 & 10 \\ 0 & 6 & 21 & -3 \end{vmatrix}$$

We multiplied the third row times 2 and the fourth row times 3. Be aware that this way we avoided fractions in the matrix.

$$\rightsquigarrow \frac{1}{6} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 0 & -18 & 5 \\ 0 & 0 & 23 & -8 \end{vmatrix} \cdot \frac{23}{18} = \frac{1}{6} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 0 & -8 & 5 \\ 0 & 0 & 0 & -8 + 5\frac{23}{18} \end{vmatrix}$$

Even though we have a fraction $\frac{1}{6}$ *at the front, our result will remain to be integral (i.e. without decimal points).*

Triangular matrix:
$$\frac{1}{6} \cdot 1 \cdot 6 \cdot (-18) \cdot \left(-8 + \frac{5 \cdot 23}{18}\right)$$
$$= -(-18 \cdot 8 + 5 \cdot 23) = -(-144 + 115) = 29$$

Lemma 7. *1.*

$$\begin{vmatrix} a_{11} & * \;\dots\; * \\ 0 & \\ \vdots & B \\ 0 & \end{vmatrix} = a_{11} \cdot \det B$$

2.

$$\begin{vmatrix} B & \begin{matrix} 0 \\ 0 \\ \vdots \end{matrix} \\ * \;\dots\; * & a_{nn} \end{vmatrix} = \det B \cdot a_{nn}$$

Proof.

$$\det A = \sum_{\pi \in \sigma_n} (-1)^\pi a_{\pi(1),1} \dots a_{\pi(2),2}$$

2.

$$\begin{aligned} a_{\pi(n),n} &= 0 \text{ except when } \pi(n) = n \\ &= \sum_{\pi \in \sigma_n} (-1)^\pi a_{\pi(1),1} \dots a_{\pi(n),n} \\ &= \sum_{\rho \in \sigma_{n-1}} (-1)^\rho a_{\rho(1),1} \dots a_{\rho(n-1),n-1} a_{\rho(n),n} \qquad = \det B \cdot a_{nn} \end{aligned}$$

□

Definition 8. *Let $A \in \mathbb{K}^{n \times n}$.*

$$1 \le k, l \le n$$

$A_{k,l}$ (dimension $(n-1) \times (n-1)$) which is generated by A if you cancel out row k and column l.

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,l-1} & a_{1,l+1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,l-1} & a_{2,l+1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & \dots & a_{k-1,l-1} & a_{k-1,l+1} & \dots & a_{k-1,n} \\ a_{k+1,1} & \dots & a_{k+1,l-1} & a_{k+1,l+1} & \dots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,l-1} & a_{n,l+1} & \dots & a_{n,n} \end{vmatrix}$$

Theorem 15 (Generative theorem of Laplace (dt. Entwicklungssatz von Laplace))**.** *Let $A \in K^{n \times n}$, then it holds that*

$$\det(A) = \sum_{k=1}^{n} a_{k,l} \cdot (-1)^{k+l} \cdot \det A_{k,l}$$

Generation to l-th column.

$$\det A = \sum_{l=1}^{n} a_{k,l} \cdot (-1)^{k+l} \cdot \det A_{k,l}$$

Generation to k-th row.

Proof. l-th column is

$$a_l = \sum_{k=1}^{n} a_{kl} e_k$$

$$\begin{aligned}\det(A) &= \triangle(a_1,\dots,a_n)\\ &= \triangle(a_1,\dots,a_{l-1},\sum_{k=1}^n a_{kl}e_k,\dots,a_n)\\ &= \sum_{k=1}^n a_{kl}\triangle(a_1,\dots,a_{l-1},e_k,\dots a_n)\\ &= \sum_{k=1}^n a_{kl}\begin{vmatrix} a_{11} & \dots & a_{1,l-1} & 0 & a_{1,l+1} & \dots & a_{1,n}\\ & & & \vdots & & & \\ & & & 0 & & & \\ \vdots & & & 1 & & & \vdots\\ & & & 0 & & & \\ & & & \vdots & & & \\ a_{n1} & \dots & a_{n,l-1} & 0 & a_{n,l+1} & \dots & a_{n,n}\end{vmatrix}\end{aligned}$$

where 1 is given on the k-th row and the l-th column which is e_k.

We exchange the l-th column with the $(l-1)$-th, then $(l-2)$-th and so on and so forth ... This requires $(l-1)$ transpositions.

$$\sum_{k=1}^n a_{kl}(-1)^{l-1}\begin{vmatrix} 0 & a_{11} & \dots & a_{1,l-1} & a_{1,l-1} & \dots & a_{1,n}\\ \vdots & a_{21} & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & \dots & \dots & \dots & \dots\\ 1 & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & \dots & \dots & \dots & \dots & \dots\\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & a_{n1} & \dots & a_{n,l-1} & a_{n,l-1} & \dots & a_{n,n}\end{vmatrix}$$

where 1 is given on the k-th row.

Exchange k-th and $(k-1)$-th row, then $(k-2)$-th and so on and so forth ... This requires $k-1$ transpositions.

$$= \sum_{k=1}^n a_{kl}(-1)^{k-1+l-1}\begin{array}{|c|c|}\hline 1 & \\ \hline 0 & \\ \vdots & \\ \dots & A_{k,l}\\ \vdots & \\ 0 & \\ \hline\end{array} = \sum_{l=1}^n a_{k,l}(-1)^{k+l}\det A_{k,l}$$

□

Example 9.

$$\begin{vmatrix}1 & 2 & 5\\ 2 & 5 & 14\\ 5 & 14 & 42\end{vmatrix} = 1\cdot\begin{vmatrix}5 & 14\\ 14 & 42\end{vmatrix} - 2\cdot\begin{vmatrix}2 & 14\\ 5 & 42\end{vmatrix} + 5\cdot\begin{vmatrix}2 & 5\\ 5 & 4\end{vmatrix}$$

+	−	+
−	+	−
+	−	+

where the top right + refers to the third summand (submatrix) and the top middle − refers to the second summand (submatrix).

$$= (5\cdot 42 - 14\cdot 14) - 2\cdot(2\cdot 42 - 5\cdot 14) + 5\cdot(2\cdot 14 - 5\cdot 5) = 14 - 2\cdot 14 + 5\cdot 3 = 1$$

Theorem 16. *A is invertible iff $\det A \neq 0$.*

Let $A \in K^{n\times n}$, $\hat{A} := [\hat{a}_{kl}]_{k,l=1,\dots,n}$ is the complementary matrix *or* adjoint matrix.

$$\hat{a}_{kl} = (-1)^{k+l}\det A_{lk}$$

Then

$$A^{-1} = \frac{1}{\det A}\cdot\hat{A}$$

Proof. Show that $B := \hat{A}\cdot A = \det A\cdot I$.

$$b_{k,l} = \sum_{j=1}^n \hat{a}_{kj}a_{jl} = \sum_{j=1}^n(-1)^{k+j}\det A_{jk}a_{jl}$$

Case $k = l$

$$b_{kk} = \sum_{j=1}^{n} (-1)^{k+j} a_{jk} \det A_{jk} = \det A \text{ (Laplace generation to } k\text{-th column)}$$

Case $k \neq l$ Without loss of generality $k < l$.

$$0 = \det \begin{bmatrix} a_{11} & \dots & a_{1l} & \dots & a_{1l} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nl} & \dots & a_{nl} & \dots & a_{nn} \end{bmatrix}$$

We replace the k-th column (left column with a_{1l} in the middle) by the l-th column (right column with a_{1l} in the middle).

Laplace generation by k-th column:

$$= \sum_{j=1}^{n} a_{jl} \det \begin{bmatrix} a_{11} & \dots & 0 & \dots & a_{1l} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & 0 & \dots & a_{nl} & \dots & a_{nn} \end{bmatrix}$$

Similar to Laplace:

$$= \sum_{j=1}^{n} a_{jl} (-1)^{j+l} \det A_{jk} = \sum_{j=1}^{n} a_{jl} \hat{a}_{kj} = b_{kl}$$

□

Example 10 (Cayley 1855). *Cayley considered it as partial derivations:*

$$\frac{1}{\triangledown} \begin{vmatrix} \partial_a \triangledown & \partial_c \triangledown \\ \partial_b \triangledown & \partial_d \triangledown \end{vmatrix}$$

Consider $n = 2$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Consider $n = 3$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} a_{22} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

Example 11.

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 5 & 14 \\ 14 & 42 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 14 & 42 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 5 & 14 \end{vmatrix} \\ -\begin{vmatrix} 2 & 14 \\ 5 & 42 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 5 & 42 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 2 & 14 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 5 & 14 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 14 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 14 & -14 & 3 \\ -14 & 17 & -4 \\ 3 & -4 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 5 & 14 \\ 14 & 42 \end{vmatrix} = 5 \cdot 3 \cdot 14 - 14 \cdot 14 = 14$$

$$\begin{vmatrix} 2 & 5 \\ 14 & 42 \end{vmatrix} = 2 \cdot 3 \cdot 14 - 5 \cdot 14 = 14$$

Theorem 17 (Arnold's hypothesis)**.** *"No theorem in mathematics is named after it's original author"*

Proof. No proof provided here. □

Theorem 18 (Cramer's rule)**.** *Originally by McLansin (1748) based on work by Leibniz (1678) and reformulated by G. Cramer (1750).*

A regular $n \times n$ matrix with column vectors $a_1, \dots, a_n \in \mathbb{K}^n$.

Then the unique solution to the equation system $Ax = b$ is given by

$$x_i := \frac{\triangle(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)}{\triangle(a_1, \dots, a_n)} = \frac{\det(a_1, \dots, b, \dots, a_n)}{\det A}$$

Its complexity is given by $n + 1$ determinants!

Proof.

$$b = \sum_{j=1}^{n} b_j e_j$$

$$x = A^{-1}b - \frac{1}{\det A}\hat{A} \cdot b$$

$$x_i = \frac{1}{\det A}\sum_{j=1}^{n} \hat{a}_{ij} b_j = \frac{1}{\det A}\sum_{j=1}^{n} (-1)^{i+j} \det(A_j) b_j$$

$$= \frac{1}{\det A}\sum_{j=1}^{n} \triangle(a_1, \dots, a_{i-1}, \dots, a_{j-1}, e_j, a_{j+1}, \dots, a_n) \cdot b_j$$

$$= \frac{1}{\det A}\triangle(a_1, \dots, a_{i-1}, \underbrace{\sum_{j=1}^{n} b_j e_j}_{=b}, \dots, a_n)$$

□

Example 12.

$$2x_1 + 2x_2 = 7$$

$$x_1 - 3x_2 = 0$$

$$A = \begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

$$\det A = -8 \qquad x_1 = \frac{\begin{vmatrix} 7 & 2 \\ 0 & -3 \end{vmatrix}}{-8} = \frac{21}{8} \qquad x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 0 \end{vmatrix}}{-8} = \frac{7}{8}$$

Remark 9.

- *in higher dimensions ($n \geq 4$) Cramer's rule is disallowed.*
  1. *too computationally intense*
  2. *numerically unstable (small errors have large effects)*
- *Anyways, still useful for theoretical considerations*
  1. *the map $A \mapsto \det A$ is C^∞ (polynomial!) (this denotes infinite differentiability)*
  2. *The set of invertible matrices in $\mathbb{R}^{n\times n}$ is open, because if $\det A \neq 0$, then also $\det \tilde{A} \neq 0$ as long as $|\, a_{ij} - \tilde{a}_{ij} \,| < \delta$.*
  3. *The solution of the equation system $Ax = b$, for invertible A, depends continuously and differentiable on A and b:*

$$x_i = \underbrace{\frac{1}{\det A}}_{\text{continuous as long as } \det A \neq 0} \underbrace{\hat{A}b}_{\text{polynomial}}$$

  4. *The map $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$*

$$A \mapsto A^{-1}$$

is continuous.

$$A^{-1} = \frac{1}{\det A} \cdot \hat{A}$$

So $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

This lecture took place on 16th of March 2016 (Franz Lehner).

3 Inner products

Descartes introduced "La Géometrie" (1637).

Definition 9. *The length of a vector in $\mathbb{R}^2/\mathbb{R}^3$ is:*

$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

Definition 10 (Scalar product)**.**

$$\cos\theta = \cos(2\pi - \theta)$$

The scalar product is defined as

$$\langle a, b \rangle = \|a\| \cdot \|b\| \cdot \cos\theta$$

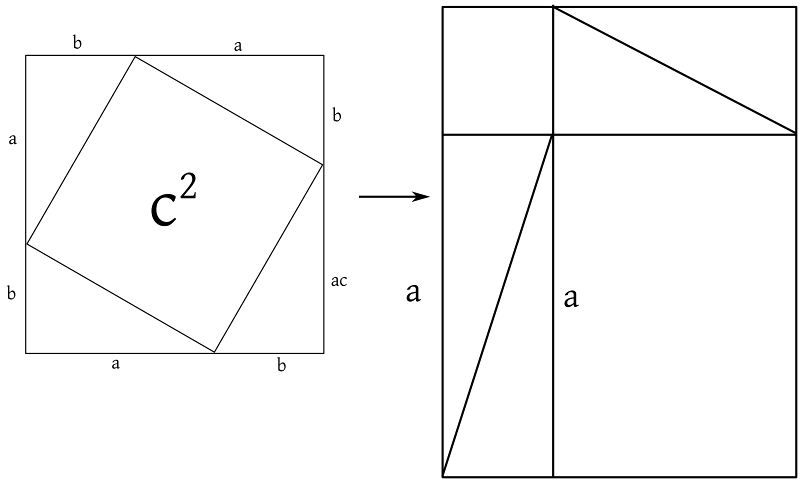

Figure 5: Pythagorian proof of $c^2 = a^2 + b^2$

Theorem 19. *The following properties hold:*

- $\|\lambda \cdot a\| = |\,\lambda\,| \cdot \|a\|$
- $\|a + b\| \leq \|a\| + \|b\|$ *(triangle inequality)*
- $\langle a, a\rangle = \|a\|^2 \geq 0$
- $\langle a, a\rangle = 0 \Leftrightarrow a = 0$
- $\langle a, b\rangle = 0 \Leftrightarrow a = 0 \vee b = 0$

$$\langle a, b\rangle > 0 \Leftrightarrow \textit{ acute angle}$$

$$\langle a, b\rangle < 0 \Leftrightarrow \textit{ obtuse angle}$$

Theorem 20.

$$\langle a, b\rangle = \langle b, a\rangle \tag{1}$$
$$\langle \lambda a, b\rangle = \lambda \langle a, b\rangle \tag{2}$$
$$\langle a + b, c\rangle = \langle a, c\rangle + \langle b, c\rangle \tag{3}$$

So it actually describes a bilinear map.

Proof.
- immediate
- $\lambda > 0$ immediate

  $\lambda < 0$ Angle θ becomes $\pi - \theta$.

$$\cos(\pi - \theta) = -\cos\theta$$

$$\langle \lambda a, b\rangle = |\,\lambda\,| \cdot \|a\| \cdot \|b\| \cos(\pi - \theta) = -|\,\lambda\,| \cdot \|a\| \cdot \|b\| \cdot \cos\theta = \lambda \langle a, b\rangle$$

- Let $b = e$, $\|e\| = 1$.

$$\langle a, e\rangle = \|a\| \cdot \cos\theta$$

$$\langle a + b, c\rangle = \|c\| \left\langle a + b, \frac{c}{\|c\|} \right\rangle - \|c\| \left(\left\langle a, \frac{c}{\|c\|} \right\rangle + \left\langle b, \frac{c}{\|c\|} \right\rangle \right) = \langle a, c\rangle + \langle b, c\rangle$$

  Compare with Figure 6.

□

Theorem 21.

$$\left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Proof.

$$\begin{aligned}
\langle a, b\rangle &= \langle a_1 e_1 + a_2 e_2 + a_3 e_3, b\rangle \\
&= a_1 \langle e_1, b\rangle + a_2 \langle e_2, b\rangle + a_3 \langle e_3, b\rangle \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
\langle e_i, b\rangle &= \langle e_i, b_1 e_1 + b_2 e_2 + b_3 e_3\rangle \\
&= b_1 \langle e_i, e_1\rangle + b_2 \langle e_i, e_2\rangle + b_3 \langle e_i, e_3\rangle \\
&= b_1 \delta_{i1} + b_2 \delta_{i2} + b_3 \delta_{i3} \\
&= b_i
\end{aligned}$$

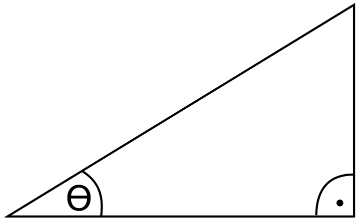

Figure 6: $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$

with $\dim \langle e_i, e_j\rangle = \delta_{ij}$.

Example 13 (Law of cosines)**.**

$$a^2 + b^2 = c^2 + 2ab\cos\gamma$$

Compare with Figure 7.

$$\begin{aligned}\|c\|^2 &= \langle a-b, a-b\rangle \\ &= \langle a, a\rangle - \langle a, b\rangle - \langle b, a\rangle + \langle b, b\rangle \\ &= \|a\|^2 - 2\cdot\|a\|\|b\|\cos\gamma + \|b\|^2\end{aligned}$$

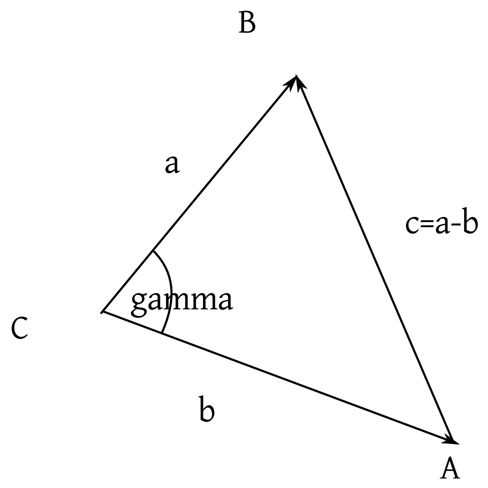

Figure 7: Law of cosines

□

Theorem 22. *Theorem by Thales TODO: image*

$$\begin{aligned}\langle a-b, -a-b\rangle &= \|a-b\|\|a+b\|\cos\theta \\ \langle a-b, -a-b\rangle &= -\langle a-b, a+b\rangle \\ &= -\left(\langle a, a\rangle - \langle b, a\rangle + \langle a, b\rangle - \langle b, b\rangle\right) \\ &= -\left(\|a\|^2 - \|b\|^2\right) \\ &= 0 \\ &\Rightarrow \theta = \frac{\pi}{2}\end{aligned}$$

Remark 10. *How do we find the normal vector?*

$$\vec{n} = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_2 & -a_1 \end{pmatrix} \right\rangle = a_1 a_2 - a_2 a_1 = 0$$

Definition 11 (Outer product)**.** *"Outer product", "cross product" or "vector product"*

TODO: image missing

This is only available in $\mathbb{R}^3$.

Let $a, b \in \mathbb{R}^3$*, then* $a \times b$ *is the vector with properties:*

- $\|a \times b\| = \|a\| \cdot \|b\| \cdot \sin\theta$
  This corresponds to the are of a parallelogram.

  $$\|b\| \cdot \sin\theta = \text{ heigh of a parallelogram}$$

- $a \times b \perp a, b$

  $$\langle a \times b, a \rangle = 0$$
  $$\langle a \times b, b \rangle = 0$$

- $(a, b, a \times b)$ *are clockwise (consider a screw coming out of Figure)*

$$a \times b = 0 \Leftrightarrow a = 0 \vee b = 0 \vee a, b \text{ are linear dependent}$$

Theorem 23. *1.* $b \times a = -a \times b$ *(counter-clockwise)*

2. $(\lambda a) \times b = \lambda \cdot a \times b = a \times (\lambda b)$

3. $(a + b) \times c = a \times c + b \times c$

So it is bilinear in $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

Proof.

$$a \times c, b \times c, (a + b) \times c \in E$$

Let $a', b', (a + b)'$ be the projection of a, b and $a + b$ in the plane.

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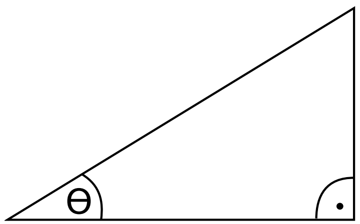

Figure 8: Theorem 23, third statement

1.
$$(a+b)' = a' + b'$$
Projection of the sum = sum of projections.

2.
$$a \times c = a' \times c$$
$$\|a' \times c\| = \|a'\| \cdot \|c\|$$
$$\begin{aligned}\|a \times c\| &= \|a\| \cdot \|c\| \cdot \sin\theta \\ &= \|a'\| \cdot \|c\|\end{aligned}$$
$$\|a'\| = \|c\| \cdot \sin\theta$$
and they have the same direction.

TODO: image missing

3.
$$(a' + b') \times c = c' \times c + b' \times c$$
From above:

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$$\|a' \times c\| = \|c\| \cdot \|a'\|$$

So this operation is linear.

$$\begin{aligned}(a+b) \times c &\overset{2}{=} (a+b)' \times c \\ &\overset{1}{=} (a' + b') \times c \\ &\overset{3}{=} (a' \times c + b' \times c) \\ &\overset{2}{=} a \times c + b \times c\end{aligned}$$

□

Corollary 7. *The cross product is a map $x : \mathbb{R}^3 \to \mathbb{R}^3 \to \mathbb{R}^3$ with properties:*

- *bilinear*
- *anti-symmetric*
- *"chiral", namely*
$$e_1 \times e_2 = e_3$$
$$e_2 \times e_3 = e_1$$
$$e_3 \times e_1 = e_2$$

Corollary 8.

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_2 & b_2 \end{vmatrix} \end{bmatrix}$$

$$\overset{Laplace}{=} \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

Formally, matrices in a vector of values are disallowed, but as far as it boils down to addition, this is fine.

Proof.

$$\begin{aligned}&(a_1e_1 + a_2e_2 + a_3e_3) \times (b_1e_1 + b_2e_2 + b_3e_3) \\ &= a_1b_1e_1 \times e_1 + a_1b_2e_1 \times e_2 + a_1b_3e_1 \times e_3 \\ &+ a_2b_1e_2 \times e_1 + a_2b_2e_2 \times e_2 + a_2b_3e_3 \times e_3 \\ &+ a_3b_1e_3 \times e_1 + a_3b_2e_3 \times e_2 + a_3b_3e_3 \times e_3 \\ &= a_1b_2e_3 + a_1b_3(-e_2) + a_2b_1(-e_3) + a_2b_3e_1 + a_3b_1e_2 + a_3b_2(-e_1) \\ &= (a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3\end{aligned}$$

□

Theorem 24 (Scalar triple product)**.** *The three-dimensional parallelogram is called "Spat" in German (compare with Figure 9).*

$$\langle a \times b, c\rangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \textit{volume of spanned 3-dimensional parallelogram}$$

$\|a \times b\|$ *is the area of the parallelogram.* $\langle a \times b, c\rangle = \|a \times b\| \cdot \|c\| \cdot \cos\theta$ *where*

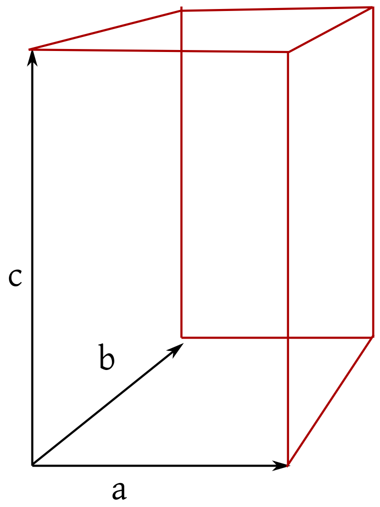

Figure 9: Three-dimensional parallelogram

$\|c\| \cdot \cos\theta$ *is the height of the 3-dimensional parallelogram.*

$$\langle a \times b, c\rangle = \left\langle \begin{pmatrix} \begin{vmatrix} a_1 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\rangle$$

$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \cdot c_1 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \cdot c_2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot c_3 =$ *Laplace generated by third column*

Example 14. *Given a plane in parameter representation:*

$$E = \{v_0 + \lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\}$$

Find $\alpha_1, \alpha_2, \alpha_3$ *and* β *with ("implicit representation")*

$$E = \{x \mid \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta\}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = a \times b$$

TODO: image missing

$$\beta = \langle v_0, a \times b\rangle$$

In the following chapters we always consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 12. *An inner product over a vector space in* $\mathbb{R}$ *or* $\mathbb{C}$ *is a map* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ *with properties:*

- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \quad \forall x, y, z \in V$
- $\langle \lambda x, y\rangle = \lambda \langle x, y\rangle \quad \forall x, y \in V \forall \lambda \in \mathbb{K}$
- $\langle y, x\rangle = \overline{\langle x, y\rangle} \quad \forall x, y \in V$

where $\overline{\langle x, y\rangle}$ *denotes the complex conjugate. Especially* $\langle x, x\rangle \in \mathbb{R} \forall x \in V$.

An inner product is called

positive semidefinite *if* $\langle x, x\rangle \geq 0 \quad \forall x$

positive definite *if* $\langle x, x \rangle > 0 \quad \forall x \neq 0$

negative semidefinite *if* $\langle x, x \rangle \leq 0 \quad \forall x$

negative definite *if* $\langle x, x \rangle < 0 \quad \forall x \neq 0$

indefinite *if* $\exists x : \langle x, x \rangle > 0 \wedge \exists y : \langle y, y \rangle < 0$

Definition 13. *Scalar product if* $\mathbb{K} = \mathbb{R}$
Hermitian product (or unitary product) if $\mathbb{K} = \mathbb{C}$

Quadratic form if $\mathbb{K} = \mathbb{R}$
Hermitian form if $\mathbb{K} = \mathbb{C}$

Lemma 8.

- $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
- $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$
- $\langle x, 0 \rangle = 0$

Linear in x and anti-linear in y!

Swsquilinear

German keywords

English keywords