

# Mathematical analysis 1 – Lecture notes

course by Wolfgang Ring

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## 1 Propositional logic

This lecture took place on 1st of October 2015 with lecturer Wolfgang Ring.

- Discussion about motivation for visiting university
- Kurt Gödel: Gödel's incompleteness theorem
- propositional logic (and/or/implication/equivalence operation)
  - $p \implies q$ : “p implies q” (“notwendig”), “q requires p” (“hinreichend”)
  - Indirect proof:  $(\neg q \implies \neg p) \Leftrightarrow (p \implies q)$
  - Proof by contradiction: claim  $p$ , claim  $\neg q$ , show that  $p \wedge \neg q$  is not possible
  - commutative law:  $a \wedge b \Leftrightarrow b \wedge a$
  - associative law:  $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  - distributive law:  $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
  - DeMorgan's law:  $\neg(a \wedge b) \Leftrightarrow (\neg a) \vee (\neg b)$
- First-order logic
  - $\forall x \in \mathbb{N} : x \in \mathbb{R}$
  - $\forall x \in M : P(x)$
  - $\neg[(\forall x \in M)P(x)] \Leftrightarrow \exists x \in M : \neg P(x)$
- Peano's axioms: rationale for induction proofs

The lecture on 8th of October 2015 got cancelled spontaneously.

## 2 First-Order Logic

This lecture took place on 12th of October 2015 with lecturer Wolfgang Ring.

Literature recommendation:

- “Analysis 1 (Mathematik für das Lehramt)”, Oliver Deiser

Let  $A$  and  $B$  be statements.

- Logical equivalence is given iff the truth table of both expressions is the same.
- $\neg(\neg A) \Leftrightarrow A$
- $(A \vee B) \Leftrightarrow (B \vee A)$
- $(A \wedge B) \Leftrightarrow (B \wedge A)$
- $a \implies b$ : implication

Boolean Laws:

$$\neg(A \implies B) \Leftrightarrow A \wedge \neg B \quad (1)$$

$$A \Leftrightarrow B \implies (A \implies B) \wedge (B \implies A) \quad (2)$$

“contraposition” or “indirect proof”

$$\neg B \implies \neg A \quad (3)$$

$$A \implies B \Leftrightarrow (\neg B \implies \neg A) \quad (4)$$

$$(A \Leftrightarrow B) \Leftrightarrow (\neg A \Leftrightarrow \neg B) \quad (5)$$

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B \quad (6)$$

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B \quad (7)$$

$$\neg(A \implies B) \Leftrightarrow (A \wedge \neg B) \quad (8)$$

$$A \wedge (B \vee C) \Leftrightarrow ((A \wedge B) \vee (A \wedge C)) \quad (9)$$

$$A \vee (B \wedge C) \Leftrightarrow ((A \vee B) \wedge (A \vee C)) \quad (10)$$

$$(A \implies B) \Leftrightarrow (\neg A \vee B) \quad (11)$$

“proof by contradiction”

$$((A \implies B) \wedge (A \implies \neg B)) \implies \neg A \quad (12)$$

“conclusion”

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C) \quad (13)$$

$$\begin{aligned} A \vee B &\Leftrightarrow \neg(\neg A) \vee \neg(\neg B) \Leftrightarrow \neg(\neg A \wedge \neg B) \\ \neg(A \vee B) &\Leftrightarrow \neg(\neg(\neg A) \vee (\neg B)) \end{aligned}$$

Distributive laws:

- $(A \vee B) \wedge C \Leftrightarrow (A \wedge C) \vee (B \wedge C)$
- $(A \wedge B) \vee C \Leftrightarrow (A \vee C) \wedge (B \vee C)$

## 2.1 Tautologies

A *tautology* is the composition of statements, which always yields the truth value true, independent of the truth value of its subexpressions.

Examples of tautologies:

“**Law of excluded middle**”  $A \vee \neg A$

**equivalences are always tautologies**  $A \Leftrightarrow \neg(\neg A)$

**implication of itself**  $A \rightarrow A$

Tautology with multiple statements:

**implication with or and not**  $(A \rightarrow B) \Leftrightarrow (\neg A \vee B)$

**proof by contradiction**  $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$

**chain inference**  $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$

This lecture took place on 14th of Oct 2015 with lecturer Wolfgang Ring.

*Proof.* We prove,  $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$ .

$$\begin{aligned} (A \rightarrow B) \wedge (A \rightarrow \neg B) &\Leftrightarrow (\neg A \vee B) \wedge (\neg A \vee \neg B) \\ &\Leftrightarrow \underbrace{(B \wedge \neg B)}_{\perp} \vee \neg A \\ &\Leftrightarrow \neg A \end{aligned}$$

Special case:  $A = B$ .

$$(A \rightarrow A) \wedge (A \rightarrow \neg A) \rightarrow \neg A$$

$$(A \rightarrow \neg A) \rightarrow \neg A$$

□

## 2.2 Negation of a tautology

- is called *contradiction*.
- has always truth value false.

*Proof.*

$$\begin{aligned} (A \vee B) \rightarrow C &\Leftrightarrow \neg(A \vee B) \vee C \Leftrightarrow (\neg A \wedge \neg B) \vee C \\ (\neg A \vee C) \wedge (\neg B \vee C) &\Leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C) \end{aligned}$$

$$(A \vee B) \rightarrow C \Leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C)$$

$$(A \wedge B) \rightarrow C \Leftrightarrow (A \rightarrow C) \vee (B \rightarrow C)$$

$$A \rightarrow (B \wedge C) \Leftrightarrow (A \rightarrow B) \wedge (A \rightarrow C)$$

$$A \rightarrow (B \vee C) \Leftrightarrow (A \rightarrow B) \vee (A \rightarrow C)$$

□

*Example proof by contradiction: Number of prime numbers.* We prove a statement by Euklid of Alexandria, 300 BC:

The number of prime numbers is infinite.

Assume the number of prime numbers is finite. Then there exists some  $N \in \mathbb{N}$  such that  $\mathbb{P} = \{P_1, P_2, \dots, P_n\}$  is the set of all prime numbers.

Every integer can be represented as product of prime numbers. Therefore for every integer there exists at least one prime number that divides this number (without remainder).

Let  $m = p_1 \cdot p_2 \cdot \dots \cdot p_N + 1$ . Let  $a$  be a prime number that divides  $m$ .

It holds that: Every  $p_i \in \mathbb{P}$  is not a divisor of  $m$ . Because when dividing  $\frac{m}{p_i}$ , the remainder is always one.

So  $q \in \mathbb{P}$ , so there exists more than  $N$  prime numbers (at least  $N + 1$ ). This contradicts with our assumption, that only  $N$  prime numbers exist.

Therefore always one more prime number exists. So the number of prime numbers is infinite.  $\square$

### 2.3 Quantifiers

Quantified statements are statements, in which objects of a set occur.

*Example:* Let  $P(x) = (x > 0)$ . Its truth value cannot be determined if the set  $X$  is not defined.

**Definition 1.** Let  $M$  be a set,  $x \in M$  and  $P(x)$  a predicate.

The composed statement: for every  $x \in M$ , it holds that  $P(x)$  is true, if the truth value of  $P(x)$  is always true independent of the selection of  $x \in M$ .

**Example 1.** Let  $M = \mathbb{R}$  and  $P(x) = (x^2 + 1 > 0)$ .

This is true for all  $x \in M$ . We denote:  $\forall x \in M : P(x)$ .

**Example 2.** Let  $M = \mathbb{R}$  and  $P(x) = (x^2 - 1 > 0)$ .

This is *not* true for all  $x \in M$ . We denote:  $\exists x \in M : \neg P(x)$ .

**Definition 2.**  $\forall x \in M : P(x)$  does not hold if and only if  $\exists x \in M : \neg P(x)$ .

$\forall$  is called *all quantifier*.  $\exists$  is called *existence quantifier*.

Negation works as follows:

$$\neg(\forall x \in M : P(x)) \Leftrightarrow \exists x \in M : \neg P(x)$$

$$\neg(\exists x \in M : P(x)) \Leftrightarrow \forall x \in M : \neg P(x)$$

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This lecture took place on 15th of Oct 2015 with lecturer Wolfgang Ring.

$$\forall x \in M : (P(x) \wedge Q(x)) \iff (\forall x \in X : P(x)) \wedge (\forall y \in M : Q(y))$$

$$\forall x \in M : (P(x) \vee Q(x)) \leftrightarrow (\forall x \in M : P(x)) \wedge (\forall x \in M : Q(x))$$

Counterexample:

$$M = \mathbb{R} \quad P(x) := (x > 0)$$

A statement  $B$  is stronger than  $C$  if  $C$

### 2.4 Composition of several quantifiers

1. The order of quantifiers matters.
2. For every real number  $x$ , there exists an integer  $n \in \mathbb{N}$  with the property  $n > x$ :

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$$

The statement does not hold if the order is changed.

$$\exists n \in \mathbb{N} \forall x \in \mathbb{R} : n > x$$

## 3 Sets

We consider objects, which we call *sets*. For every set  $M$  and every element  $x$ , it holds that

$$x \in M \vee \neg(x \in M)$$

Consider the set  $L = \{M : M \text{ is a set and } M \notin M\}$ . Does  $L \notin L$  or  $L \in L$  hold?

If  $L \notin L$ , then  $L$  satisfies the definition and therefore  $L \in L$ . If  $L \in L$ , then elements of  $L$  satisfy the property; therefore  $L \notin L$ .

Set operations:

- union
- intersection
- subsets

- $\forall S : \emptyset \subseteq S$
- complete induction

**Theorem 1.** (Pythagoreans, 450 BC)

$$\forall n \in \mathbb{N}_+ : \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

*Proof.* **Induction base  $n = 1$**

$$P(1) : 1 = \frac{1(1+1)}{2} \quad \checkmark$$

**Induction step  $n \rightarrow n+1$**

Assume  $P(n)$  is true. So  $(1 + 2 + \dots + n) = \frac{n(n+1)}{2}$ .

$$\begin{aligned} [(1 + 2 + \dots + n) + (n+1)] &= \frac{n(n+1)}{2} + (n+1) = (n+1) \left( \frac{n}{2} + 1 \right) \\ &= (n+1) \cdot \frac{(n+2)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark \end{aligned}$$

So, it simply holds that:

$$\begin{aligned} s &= 1 + 2 + 3 + \dots + n \\ 2 \cdot s &= \underbrace{n}_{\text{number of items}} \cdot \underbrace{(n+1)}_{\text{sum}} \Rightarrow s = \frac{n \cdot (n+1)}{2} \end{aligned}$$

□

This lecture took place on 21st of October 2015 with lecturer Ring Wolfgang.

- Let  $X$  be a set.  $M = \{x \in X : P(x)\}$ .
- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$  ... “enumerating set representation”
- $M = \{x \in X \mid P(x)\}$ ,  $N = \{x \in X \mid Q(x)\}$
- $M \cup N = \{x \in X \mid P(x) \vee Q(x)\}$

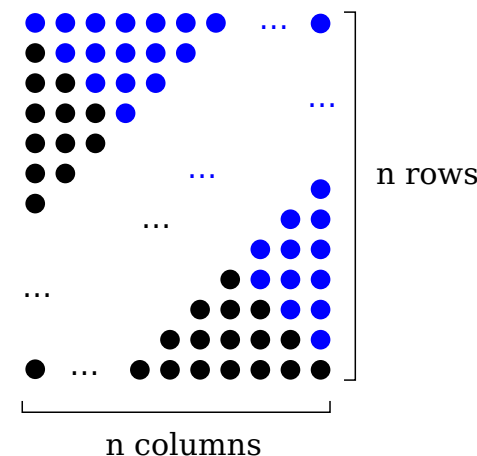


Figure 1: Illustration of the triangular number (illustrative proof)

- Let  $X$  be a set.  $A_0 \subseteq X$ ,  $A_1 \subseteq X$ ,  $A_2 \subseteq X$ , etc
- $\forall n \in \mathbb{N} : A_n \subseteq X$
- $A_0 \cup A_1 \cup A_2 \cup \dots = \bigcup_{n=1}^{\infty} A_n = \{x \in X \mid (x \in A_0) \vee (x \in A_1) \vee \dots\} = \{x \in X \mid \exists n \in \mathbb{N} : x \in A_n\}$
- $A_0 \cap A_1 \cap A_2 \cap \dots = \bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall n \in \mathbb{N} : x \in A_n\}$

### 3.1 Cartesian product

**Definition 3.** Let  $A$  and  $B$  sets. The *cartesian product* of  $A$  and  $B$  is given as:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

This operation is not commutative!

**Definition 4.** We denote  $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ .

**Example 3.**

$$\begin{aligned} A &= \{a, b, c, d, e, f, g, h\} \\ B &= \{1, 2, 3, 4, 5, 6, 7, 8\} \\ A \times B &= \{(a, 1), (a, 2), (a, 3), \dots, (a, 8), (b, 1), (b, 2), \dots\} \end{aligned}$$

**Example 4.**

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

e.g.  $(1, \frac{9}{8}) \in \mathbb{R} \times \mathbb{R}$ .

**Definition 5.** Let  $A_1, A_2, \dots, A_n$  be sets.

$$A_n = A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

instead of  $\underbrace{A \times A \times \dots \times A}_{n \text{ times}} = A^n$ .

### 3.2 Power set

**Definition 6.** Let  $X$  be a set. Then  $\mathcal{P}(X)$  is the *power set* of  $x$ .

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}$$

## 4 Mappings and functions

**Definition 7.** Let  $A$  and  $B$  be sets. A *mapping*  $f$  from  $A$  to  $B$  (denoted  $f : A \rightarrow B$ ) is an assignment, such that for every  $x \in A$  one  $y \in B$  is assigned. We denote the corresponding  $y \in B$  for some  $x \in A$  with  $y = f(x)$ .  $A$  is called *domain*,  $B$  is called *co-domain*.

**Definition 8** (Alternative definition of mappings). A mapping  $f$  is a subset of  $A \times B$  which fulfills the following properties:

- $\forall x \in A : (\exists y \in B : (x, y) \in f)$
- $\forall x \in A \wedge (y_1, y_2 \in B) : [(x, y_1) \in f \wedge (x, y_2) \in f] \implies y_1 = y_2$

Notation:

$$\begin{aligned} (x, y) \notin f &\Leftrightarrow y \neq f(x) \\ \{(x, f(x)) \in \mid x \in A\} &\Rightarrow \text{graph from } f \end{aligned}$$

**Definition 9.** Let  $f : A \rightarrow B$  be a mapping.

- The mapping  $f$  is called *surjective*, if  $\forall y \in B : \exists x \in A : y = f(x)$ .
- The mapping  $f$  is called *injective*, if

$$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$

- Let  $B' \subseteq B$ . Then we denote  $f^{-1}(B') = \{x \in A \mid f(x) \in B'\}$  as the *preimage of  $f$* .

Attention! The preimage distinguishes itself from the domain (it is a subset) and the inverse function  $f^{-1}$  (a function must not be invertible to have a preimage)!

- Let  $A' \subseteq A$ . Then we call  $f(A') = \{f(x) \mid x \in A'\} \subseteq B$  the *image of  $A'$  under  $f$* .

Special case:  $A' = A$ , then  $f(A) \subseteq B$  is the image of  $A$  under  $f$ .

Let  $f : A \rightarrow B$  be a mapping. We define  $\tilde{f} : A \rightarrow f(A) \subseteq B$  with  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . The mapping  $\tilde{f}$  is surjective  $\forall y \in f(A)$  there exists one  $x \in A$  such that  $y = \tilde{f}(x)$ .

- A mapping is called *bijective* iff the mapping is surjective and injective.

### 4.1 Bernoulli's inequality

**Definition 10** (Bernoulli's inequality). Let  $x \in \mathbb{R}$  with  $x > -1$  and  $x \neq 0$ . Let  $n \in \mathbb{N}$  with  $n > 1$ . Then it holds that

$$(1 + x)^n > 1 + nx$$

*Proof.* Proof by complete induction.

**Induction base  $n = 2$**

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x \quad \checkmark$$

because  $x^2 > 0$  for  $x \neq 0$ .

**Induction step  $n \rightarrow n+1$**

Assume  $(1+x)^2 > 1+n$ , then  $x > -1$  and  $x \neq 0$ .

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n \cdot \underbrace{(1+x)}_{>0} > (1+nx) \cdot (1+x) \\ &= (1+nx+x+nx^2) = (1+(n+1) \cdot x + \underbrace{nx^2}_{>0}) > 1+(n+1) \cdot x \end{aligned}$$

□

Back to sets and functions (notes missing):

- injective, surjective, bijective function
- composition of functions: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .  $g \circ f : X \rightarrow Z$  is defined as  $g(f(x))$  (“g after f”).
- Let  $f$  and  $g$  be mappings. If  $f$  and  $g$  are injective,  $f \circ g$  is injective. If  $f$  and  $g$  are surjective,  $f \circ g$  is surjective. If  $f$  and  $g$  are bijective,  $f \circ g$  is bijective.
- Identity function,  $f \circ \text{id} = \text{id} \circ f = f$
- properties of an inverse function,  $f \circ f^{-1} : X \rightarrow X$ ,  $f^{-1} \circ f : X \rightarrow X$

## 5 About sums of integers

This lecture took place on 21st of Oct 2015 with lecturer Wolfgang Ring.

**Definition 11.** The summation notation is defined as,

$$\sum_{k=h}^l a_k$$

Iteration over all values from  $l$  to  $h$  (inclusive) and evaluation of the enclosed expression with  $k$  as iteration value. The resulting terms are added up and the sum gives the result of the summation expression.

Laws:

$$\sum_{k=l}^h a_k = \sum_{i=l}^h a_i \quad (14)$$

$$\sum_{k=l}^h (a_k + b_k) = \left( \sum_{k=l}^h a_k \right) + \left( \sum_{k=l}^h b_k \right) \quad (15)$$

$$\sum_{k=0}^h a_k = a_0 + \sum_{k=1}^h a_k \quad \text{“Extraction of the initial value”} \quad (16)$$

$$\sum_{k=0}^h a_k = a_h + \sum_{k=0}^{h-1} a_k \quad \text{“Extraction of the final value”} \quad (17)$$

$$\sum_{k=u+n}^{h+n} a_k = \sum_{k=u}^h a_{k+n} \quad \text{“index shifting”} \quad (18)$$

$$\sum_{k=l}^h \lambda \cdot a_k = \lambda \cdot \sum_{k=l}^h a_k \quad \text{“extraction of a constant  $\lambda$ ”} \quad (19)$$

$$\sum_{k=0}^n n = \frac{n(n+1)}{2} \quad \text{“triangular sum”} \quad (20)$$

We consider  $S_n = \{(a_1, a_2, \dots, a_n) : a_i \in M_n \forall i = 1, \dots, n \text{ with } a_i \neq a_j\} \subseteq M_n \times M_n \times \dots \times M_n$ .  $S_n$  is the set of all arrangements of the numbers  $1, \dots, n$ .

Example:  $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$

### 5.1 Factorials

**Theorem 2.** It holds that  $|S_n| = n!$  for all  $n \in \mathbb{N}$

*Proof.* Proof by induction over  $n$ .



**Induction base**  $n = 1$ :  $M_1 = \{1\}, S_1 = \{(1)\} \Rightarrow |S_1| = 1 = 1! \quad \checkmark$

**Induction step**  $n \rightarrow n + 1$ :

$$S_{n+1} = \{(a_1, a_2, \dots, a_n) : a_i \in M_{n+1} \forall i \in M_{n+1}, a_i \neq a_j \text{ for } i \neq j\}$$

For  $l \in M_{n+1}$ :

$$W_l = \{(a_1, \dots, a_{n+1}) \in S_{n+1} : a_l = n + 1\}$$

It holds that  $W_l \cap W_j = \emptyset$  for  $l \neq j$  and  $S_{n+1} = W_1 \cup W_2 \cup \dots \cup W_{n+1}$ .

Then it holds that  $|S_{n+1}| = |W_1| + |W_2| + \dots + |W_{n+1}| = \sum_{l=1}^{n+1} |W_l|$

**Theorem 3.** Claim: For every  $l \in M_{n+1}$  it holds that  $|W_l| = |S_n| = n!$ .

*Proof.* We build a bijective map  $\phi_l : W_l \rightarrow S_n$ .

$$W_l = \{(a_1, a_2, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1})\}$$

$$: a_i \in M_n, \forall i \neq l, a_i \neq a_j \forall i \neq j$$

$$\phi((a_1, a_2, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$= (a_1, a_2, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1}) \in S_n$$

$S_n$  is surjective. Let  $(b_1, \dots, b_n) \in S_n$ , then it holds that  $(b_1, \dots, b_{l-1}, n + 1, b_l, \dots, b_n) \in W_l$

$$\phi_l((b_1, \dots, b_{l-1}, n + 1, b_l, \dots, b_n)) = (b_1, \dots, b_n)$$

$S_n$  is injective.

$$\phi_l((a_1, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$= \phi_l((a_1, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$\Rightarrow (a_1, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1}) = (a_1, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1})$$

$\phi$  is bijective.

Therefore  $|W_l| = |S_n| = n!$ . Therefore  $|S_{n+1}| = \sum_{l=1}^{n+1} |S_n| = \sum_{l=1}^{n+1} n! = (n + 1)n! = (n + 1)!$

**Remark 1.** Let  $f : M_n \rightarrow M_n$ .  $f$  is represented as

$$(1, 2, 3, 4, \dots, n - 1, n) \rightarrow (f(1), f(2), f(3), f(4), \dots, f(n - 1), f(n))$$

Therefore  $(f(1), f(2), \dots, f(n)) \in S_n$ . Analogously every  $(a_1, \dots, a_n) \in S_n$  defined by  $f(k) = a_k$  for  $k = 1, \dots, n$  is a bijective mapping  $f : M_n \rightarrow M_n$ . Therefore we set  $S_n = \{f : M_n \rightarrow M_n : f \text{ is bijective}\}$ .  $S_n$  is called symmetric group of  $n$  elements.

## 5.2 Binomial coefficients

**Definition 12.** Let  $n \in \mathbb{N}$ ,  $k \in \mathbb{N}$  with  $k \leq n$ . We define

$$\binom{n}{k} = \frac{n!}{k!(n - k)!} \quad \text{“binomial coefficient } n \text{ choose } k\text{”}$$

It holds that

$$\begin{aligned} \binom{n}{k} &= \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{(1 \cdot 2 \cdot \dots \cdot k)(1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - k))} \\ &= \frac{n(n - 1) \cdot \dots \cdot (k + 1)}{(1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - k))} \end{aligned}$$

Factorial laws:

$$\binom{1}{0} = \frac{n!}{0!(n - 0)!} = 1 \quad \forall n \in \mathbb{N}$$

$$\binom{n}{n} = \frac{n!}{n!(n - n)!} = \frac{n!}{n! \cdot 1} = 1$$

$$\binom{n}{n - k} = \frac{n!}{(n - k)!(n - (n - k))!} = \frac{n!}{k!(n - k)!} = \binom{n}{k} \quad \text{“symmetrical”}$$

□ A recursive definition is given by

$$\binom{n}{k} = \binom{n - 1}{k - 1} + \binom{n - 1}{k} \quad n \geq 1, 1 \leq k \leq n - 1$$

*Proof.*

$$\begin{aligned}
 \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(n-1)!(n-1-(k-1))!} \\
 &= \frac{(n-1)!}{k!(n-1-k)!} \\
 &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\
 &= \frac{k \cdot (n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
 &= \frac{n(n-1)!}{k!(n-1)!} = \frac{n!}{k!(n-k)!} \\
 &= \binom{n}{k}
 \end{aligned}$$

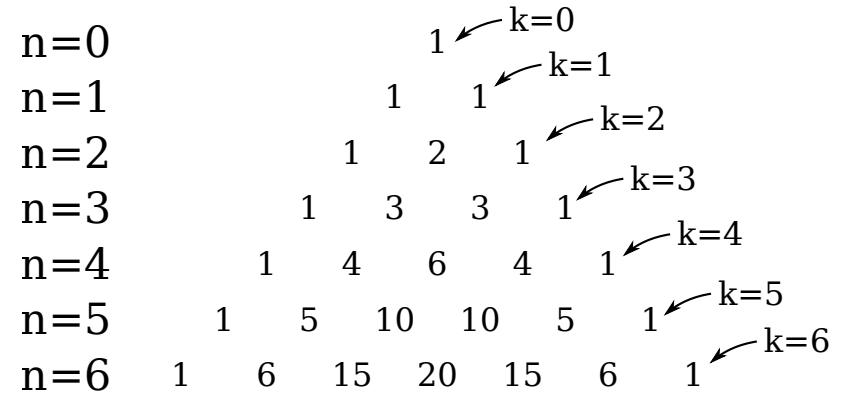


Figure 2: Pascal's triangle describes binomial coefficients. For every element of the triangle it holds that, it is adding up the two numbers above a number. The margins are defined by 1. For example 5 is given by  $\binom{5}{4}$ .

□

### 5.3 Arrangement in Pascal's triangle

**Theorem 4.** Let  $T_n^k = \{A \subseteq M_n : |A| = k\}$ . Then it holds that  $|T_n^k| = \binom{n}{k}$ .

Example:  $T_3^2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ .

$$|T_3^2| = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$$

*Proof.* Let  $n$  be fixed. Induction for  $k$ .

**Induction base**  $k = 0$

$$\begin{aligned}
 T_n^0 &= \{\emptyset\} \\
 |T_n^0| &= 1 = \binom{n}{0}
 \end{aligned}$$

**Induction step**  $k \rightarrow k + 1$

$$\begin{aligned}
 T_n^k &= \underbrace{\{\{a_1, \dots, a_k\} : a_i \in M_n, (i = 1, \dots, k), a_i \neq a_j \text{ for } i \neq j\}}_{A_1} \\
 &\cup \underbrace{\{\{a_1, \dots, a_{k-1}\} \cup [n] : [n] \in M_{n-1}\}}_{A_2} \\
 |T_n^k| &= |A_1| + |A_2|
 \end{aligned}$$

□

This lecture took place on 28th of October 2015 with lecturer Ring Wolfgang.

Let  $A, B$  be sets and define

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

Then the domain of  $A \setminus B$  is "A without B".

**Theorem 5.**

$$T_n^x = \{x \subseteq M_x : |X| = x\}$$

Let  $k \in \mathbb{N}$  and  $0 \leq k \leq 1$ .

$$|T_n^x| = \binom{1}{k}$$

There are exactly  $\binom{n}{k}$   $k$ -ary subsets of  $M_n$ .

*Proof.*

$$M_0 = \emptyset \quad T_0^0 = \{\emptyset\} \quad |T_n^0| = 1 = \binom{0}{n}$$

Proof by complete induction over  $n$  of the following statement:

$$\forall n \in \mathbb{N} : \forall k \in \mathbb{N} \text{ with } 0 \leq k \leq n : |T_n^k| = \binom{n}{k}$$

**Induction base**  $n = 0$  is fine. For  $n = 1$  there are two cases:  $k = 0$  or  $k = 1$ .

$$M_1 = \{1\}$$

$$T_1^0 = \{\emptyset\} \quad |T_1^0| = 1 = \binom{1}{0}$$

$$T_1^1 = \{\{1\}\} \quad |T_1^1| = 1 = \binom{1}{1}$$

Is also fine.

**Induction step** The hypothesis is our assumption:

$$\forall 0 \leq k \leq 1 : |T_n^k| = \binom{n}{k}$$

Consider  $M_{n+1}$ . Special case  $k = 0$ :

$$T_{n+1}^0 = \{\emptyset\} \quad |T_{n+1}^0| = 1 = \binom{n+1}{0}$$

Special case  $k = n + 1$ :

$$T_n = \{M_{n+1}\} \quad |T_{Nn+1}^{n+1}| = 1 = \binom{n+1}{n+1}$$

Let  $1 \leq k \leq n$ .

$$T_{n+1}^x \text{ TODO}$$

Union is disjoint  $\Rightarrow |T_{n+1}^k| = |R_{n+1}^k| + |S_{n+1}^k|$

$$R_{n+1}^k = \{A \subseteq M_n : |A| = k\} = T_n^k$$

$$|R_{n+1}^k| = |T_n^k| = \binom{n}{k}$$

by induction hypothesis.

$$S_{n+1}^k = \{A \subseteq M_{n+1} : A = A' \cup \{n+1\} : A' \subseteq M_n : |A'| = k-1\}$$

We prove  $|S_{n+1}^k| = |T_n^{k-1}|$ .

$$f : S_{n+1}^k \rightarrow T_n^{k-1}$$

$$f(A) = f(A' \cup \{n+1\}) = A'$$

$f$  is bijective.  $f$  is surjective: Let  $A' \in T_n^{k-1}$  define  $A = A' \cup \{n+1\} \in S_{n+1}^k$  and  $f(A) = A'$ .  $f$  is injective: Let  $f(A) = f(B)$  and  $A = A' \cup \{n+1\} \in S_{n+1}^k$ .

$$B = B' \cup \{n+1\} \in S_{n+1}^k. \quad A', B' \in T_n^{k-1}.$$

$$f(A) = f(B) \Rightarrow A' = B' \Rightarrow A' \cup \{n+1\} = B' \cup \{n+1\} \Rightarrow A = B$$

$$|S_{n+1}^k| = |T_n^{k-1}| \stackrel{\text{ind. hypo.}}{=} \binom{n}{k-1}$$

Therefore  $|T_{n+1}^k| = \binom{n}{n} = \binom{n}{k-1} = \binom{n+1}{k}$ . The last equation follows from the recursive definition of binomial coefficients.

□

## 5.4 Binomial theorem

**Theorem 6** (Binomial theorem). Let  $a, b \in \mathbb{R}$  (or  $a, b \in \mathbb{C}$ ). Then it holds that

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

*Proof.* 1. Proof by induction over  $n$ .

**Induction step**  $n = 0$ :  $(a + b)^0 = 1$

$$\sum_{k=0}^0 \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^0 = 1$$

**Induction step**  $n \rightarrow n + 1$

$$\begin{aligned} (a + b)^{n+1} &= (a + b)^n \cdot (a + b) = \left( \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \right) (a + b) \\ &= \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n-k+1} \\ &= \underbrace{\sum_{n=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\substack{\text{index shift} \\ h+1=j, h=0 \\ \Rightarrow j=1, h=j-1, h=n-1 \\ \Rightarrow j=n}} + \underbrace{\binom{n}{n} a^{n+1} \cdot b^0}_{a^{n+1}} \\ &\quad + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1} \\ &= \sum_{j=1}^n \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} \\ &\quad + \binom{n+1}{n+1} a^{n+1} + \binom{n+1}{0} b^{n+1} \end{aligned}$$

Renaming  $j$  to  $k$ :

$$\begin{aligned} &= \sum_{k=1}^n \underbrace{\left[ \binom{n}{k-1} + \binom{n}{k} \right]}_{\binom{n+1}{k} \text{ by recursive definition}} a^k b^{n+1-k} \\ &\quad + \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a^0 b^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \end{aligned}$$

Therefore the binomial theorem holds for  $n + 1$ .

□

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This lecture took place on 29th of October 2015 with lecturer Ring Wolfgang.

$$\forall a, b \in \mathbb{R}, n \in \mathbb{N} : (a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

**Induction base**  $n = 0, n = 1$  follows immediately

**Induction step**

$$(a + b)^n = \underbrace{(a + b)(a + b)(a + b)(a + b) \dots (a + b)}_{n \text{ times}}$$

When multiplying the products  $a^n b^{n-k}$  are created ( $0 \leq k \leq n$ ).  $a^n b^{n-k}$  are created iff  $a$  is the factor resulting from  $k$  parenthesis groups and  $b$  originates from the remaining  $(n - k)$  groups. There are exactly  $\binom{n}{k}$  possibilities to select from  $n$  groups.  $a^k b^{n-k}$  occurs  $\binom{n}{k}$  times. Therefore

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

This is a rather informal proof, but suffices at this point.

## 6 Arithmetics of numbers

We consider two fundamental arithmetic operators and determine fundamental properties.

**Definition 13.** Let  $K$  be a set where two arithmetic operators are defined: Therefore  $\forall a, b \in K$  let  $a + b \in K$  and  $a \cdot b \in K$ .

We require the following properties:

$$\mathbf{A1} \quad \forall a, b \in K : a + b = b + a$$

$$\mathbf{A2} \quad \forall a, b, c \in K : (a + b) + c = a + (b + c)$$

$$\mathbf{A3} \quad \exists 0 \in K \forall a \in K : a + 0 = a$$

$$\mathbf{A4} \quad \forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$$

Then  $(K, +)$  is a commutative group (“abelian group”). In general we denote  $\tilde{a}$  as  $-a$ . We define  $a - b = a + (-b)$  (“subtraction”).

$$\mathbf{M1} \quad \forall a, b \in K : a \cdot b = b \cdot a$$

$$\mathbf{M2} \quad \forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$\mathbf{M3} \quad \exists 1 \in K : a \cdot 1 = a \forall a \in K \text{ (neutral element)}$$

$$\mathbf{M4} \quad \forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$$

In general we denote  $\hat{a}$  as  $a^{-1}$ .

We set  $\frac{a}{b} = a \cdot b^{-1}$ .

$$\frac{1}{b} = 1 \cdot b^{-1} \text{ for } b \neq 0$$

**Definition 14** (Composition). Compatibility of  $+$  and  $\cdot$ :

$$\mathbf{D} \quad \forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$$

Under these conditions  $K$  is called a *field*.

**Example 5.** Examples for fields:  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

In every field it holds that

- the inverse element of  $a$  is unique ( $\tilde{a}$  is unique). Let  $-a$  be the inverse element of  $a$  and  $a + b = 0 \Rightarrow b = -a$

*Proof.* TODO

$$(a + (-a)) + (b + 0) = a + b =$$

□

- $0 \cdot a = 0$

*Proof.*

$$0 = 0 + 0$$

follows from **D**.

$$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$$

$$0 \cdot a + (-0 \cdot a) = 0 \cdot a + [0 \cdot a + (-0 \cdot a)]$$

$$0 = 0 \cdot a$$

□

- $-a = (-1) \cdot a$

*Proof.*

$$a + (-1) \cdot a = (1 + (-1))a = 0$$

$$a + (-1) \cdot a = 0$$

$$-a = (-1) \cdot a$$

□

## 6.1 Integers and the field of rational numbers $\mathbb{Q}$

For  $\mathbb{N}$ , **A1**, **A2** and **A3**. If  $n \geq m$ , then also  $n - m \in \mathbb{N}$ .  $n - m = k \in \mathbb{N}$  is defined in such a way that  $n = m + k$ .

**Corollary 1.** Extension:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \dots\} = \mathbb{N}_+ \cup \{0\} \cup \{-n : n \in \mathbb{N}_0\}$$

We define  $-0 := 0$  and  $\forall n \in \mathbb{N}_+$  let  $n + (-n) := 0$ .

Therefore for every  $z \in \mathbb{Z}$  exists some  $\tilde{z}$  such that  $z + \tilde{z} = 0$ .

- $z \in \mathbb{Z}_+ \Rightarrow \tilde{z} = -z$
- $z = 0 \Rightarrow \tilde{z} = 0$
- $z = -n$  for  $n \in \mathbb{N}_+$
- $\tilde{z} = n$

$$\forall z \in \mathbb{Z} \exists \tilde{z} \in \mathbb{Z} : z + \tilde{z} = 0$$

In general we denote  $\tilde{z} = (-z)$ . Also  $-(-z) = z$ .

For  $z, w \in \mathbb{Z}$ :

$$z + w = \begin{cases} z + w & z, w \in \mathbb{N} \\ (-z) + (-w) & -z, -w \in \mathbb{N} \\ z - (-w) & z, -w \in \mathbb{N} \text{ and } z > (-w) \\ -((-w) - z) & z, -w \in \mathbb{N} \text{ and } (-w) > z \end{cases}$$

$$z \cdot w = \begin{cases} z \cdot w & z, w \in \mathbb{N} \\ (-z)(-w) & -z, -w \in \mathbb{N} \\ -((-z) \cdot w) & -z \in \mathbb{N}, w \in \mathbb{N} \end{cases}$$

In  $\mathbb{Z}$  the properties **A1**, **A2**, **A3**, **A4**, **M1**, **M2**, **M3** and **D** hold.

**Definition 15.**

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

where  $\frac{m}{n} = \frac{m'}{n'} \Leftrightarrow m \cdot n' = n \cdot m'$ .  $\mathbb{Q}$  is called the set of rational numbers.

We define

$$\frac{m}{n} + \frac{k}{l} := \frac{ml + nk}{nl}$$

$$\frac{m}{n} \cdot \frac{k}{l} = \frac{mk}{nl}$$

Show that

$$\frac{m}{n} = \frac{m'}{n'} \text{ and } \frac{k}{l} = \frac{k'}{l'}$$

$$\Rightarrow \frac{ml + nk}{nl} = \frac{m'l' + n'k'}{n'l'}$$

$$\Rightarrow (ml + nk)(n'l') = (m'l' + n'k')nl$$

$$\Leftrightarrow mn'l' + nn' \cdot kl = m'n \cdot ll' + nn' \cdot k'l$$

Analogously for  $\frac{m}{n} \cdot \frac{k}{l}$ .

**A1–A4**, **M1–M4** and **D** hold for  $\mathbb{Q}$ .

For  $z \in \mathbb{Z}$  we set  $z = \frac{z}{1}$ . Therefore it holds that  $\mathbb{Z} \subseteq \mathbb{Q}$ .  $0 = \frac{0}{1}$  and  $\frac{m}{n} + 0 = \frac{m}{n} + \frac{0}{1} = \frac{m \cdot 1 + n \cdot 0}{n \cdot 1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$ . 0 is neutral in regards of addition in  $\mathbb{Q}$ .

Inverse element in regards of addition:

$$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{n^2} = \frac{(m + (-m))n}{n \cdot n} = \frac{0n}{n^2} = \frac{0}{1}$$

because  $0 \cdot 1 = 0 \cdot n^2$ .

Concerning multiplication:

$$1 = \frac{1}{1} \quad \frac{m}{n} \cdot \frac{1}{1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$$

1 is a neutral element in regards of multiplication in  $\mathbb{Q}$ .

Let  $\frac{m}{n} \in \mathbb{Q} \setminus \{0\} \Rightarrow m \neq 0 \Rightarrow \frac{n}{m} \in \mathbb{Q}$  and  $\frac{m}{n} \cdot \frac{n}{m} = \frac{mn}{mn} = \frac{1}{1}$ . TODO: verify because  $m \cdot n \cdot 1 = 1 \cdot m \cdot n$ .

**Corollary 2.**

$$\forall \frac{m}{n} \in \mathbb{Q} : -\frac{m}{n} = \frac{-m}{n}$$

$$\forall \frac{m}{n} \in \mathbb{Q} \setminus \{0\} : \left(\frac{m}{n}\right)^{-1} = \left(\frac{n}{m}\right)$$

Therefore  $\mathbb{Q}$  is a field.

This lecture took place on 30th of October 2015 with lecturer Ring Wolfgang.

Literature:

- Ebbinghaus et al., “Zahlen”, Springer Verlag
- E. Landau: “Grundlagen der Analysis”, uses Peano axioms to build calculus

## 6.2 Ordered fields

**Definition 16.** Let  $K$  be a field. We assume that  $K$  is taken from two sets:  $K = K_+ \cup \{0\} \cup K_-$  with  $0 \notin K_+, 0 \notin K_-$ . It holds that

- $\forall a \in K$  it holds that either  $a \in K_+$  or  $a = 0$  or  $a \in K_-$   
 $a \in K_+ \Leftrightarrow -a \in K_-$
- $\forall a, b \in K_+ : a + b \in K \wedge a \cdot b \in K$

If those properties are satisfied, such a field is called an *ordered field*. Instead of  $a \in K_+$  we write  $a > 0$  (namely “positive numbers”) and  $a < 0$  for  $a \in K_-$  correspondingly (namely “negative numbers”).

For arbitrary  $a, b \in K$  we define

$$a > b \Leftrightarrow a - b > 0$$

It holds that  $a > b \Leftrightarrow b < a$ .

$$a \geq b \Leftrightarrow a > b \vee a = b$$

**Lemma 1.** Let  $K$  be an ordered field. Then it holds that

1.  $a \in K_+ \wedge b \in K_- \Rightarrow a \cdot b \in K_-$   
 $a \in K_- \wedge b \in K_- \Rightarrow a \cdot b \in K_+$

2.  $\forall a, b \in K$  one of the following relations hold:

$$a > b \vee a = b \vee a < b$$

Therefore  $<$  defines a total order on  $K$ .

3.  $\forall a, b, c \in K : [(a < b) \wedge (b < c) \Rightarrow a < c]$

Therefore  $<$  is transitive.

4. If  $a > b > 0$  then  $\frac{1}{a} < \frac{1}{b}$  If  $a > 0$  holds, then also  $a^{-1} = \frac{1}{a} > 0$ .

5.  $\forall a, b, c \in K : a < b \Rightarrow a + c < b + c$

6.  $\forall a, b \in K : \forall c > 0 : [a > b \Rightarrow ac > bc]$   
 $\forall a, b \in K : \forall c < 0 : [a > b \Rightarrow ac < bc]$

7.  $\forall a \in K \setminus \{0\} : a^2 = a \cdot a > 0$

*Proof.* 1. We know from the practicals:  $\forall a, b \in K : (-a)(-b) = ab$

$$(-a)b = -(ab)$$

Let  $a \in K_+, b \in K_-$ , therefore  $a \in K_+, (-b) \in K_-$ , then it holds that  $ab = (-a)(-b) = -(a(-b)) \in K_-$ . Let  $a \in K_-$  and  $b \in K_-$  therefore  $(-a) \in K_+ \wedge (-b) \in K_+ \Rightarrow ab = (-a)(-b) \in K_+$ .

2. Let  $a, b \in K$ . Then one of the following properties hold:

$$a - b > 0 \vee a - b = 0 \vee a - b < 0$$

Equivalently,

$$a > b \vee a = b \vee a < b$$

3. Let  $a > b$  and  $b > c$ . Therefore  $a - b > 0$  and  $b - c > 0$ .

$$\Rightarrow (a - b) + (b - c) > 0$$

$$a(-b + b) - c > 0$$

$$a - c > 0 \Leftrightarrow a > c$$

4. Let  $a > 0 \Rightarrow a^{-1} \neq 0$ . Assume  $\frac{1}{a} = a^{-1} < 0 \Rightarrow a^{-1} \cdot a = 1 < 0$ . Otherwise it holds that  $1 = 1 \cdot 1 = 1^2 > 0$ .

5. Let  $a > b > 0$ . Then it holds that

$$a^{-1}b^{-1}(b-a) = a^{-1}b^{-1}b - a^{-1}b^{-1}a = -a^{-1} \cdot b^{-1} = \frac{1}{a} \cdot \frac{1}{b} \Rightarrow a^{-1} < b^{-1}$$

6.  $a < b$  therefore  $a - b < 0 \Rightarrow a + c - c - b < 0 \Rightarrow (a + c) - (b + c) < 0$

$$\Leftrightarrow a + c < b + c$$

7. Let  $a > b, c > 0 \Rightarrow (a - b) > 0 \Rightarrow (a - b) \cdot c > 0 \Rightarrow ac - bc > 0 \Rightarrow ac > bc$ .

For the second statement, it holds analogously:  $a < b, c < 0 \Rightarrow (a - b) < 0 \Rightarrow (a - b) \cdot c < 0 \Rightarrow ac - bc < 0 \Rightarrow ac < bc$

8.  $a > 0 \Rightarrow a \cdot a > 0$ . Let  $a < 0 \Rightarrow (-a) > 0$ . It holds  $a \cdot a = (-a)(-a) > 0$ . Therefore the square of two numbers is always positive.

□

### 6.3 Remarks about some common fields

**Remark 2.**  $\mathbb{C}$  is not an ordered field.  $\mathbb{N}, \mathbb{Z}$  and  $\mathbb{Q}$  are ordered.

**Remark 3.** Let  $q \in \mathbb{Q}$ .

a) Let  $m, n \in \mathbb{N}_+$  such that  $q = \frac{m}{n}$  then  $q > 0$ .

b) Let  $m, n \in \mathbb{N}_+$  such that  $q = -\frac{m}{n}$  then  $q < 0$ .

We show that  $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ . Every  $q \in \mathbb{Q}$  has a representation of either a) or b), but not both.  $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$ .

$$q \neq 0 \Rightarrow q = \begin{cases} \frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{-n} & m, n \in \mathbb{N}_+ \end{cases}$$

$$q = \frac{n}{-m} = \frac{-n}{m}$$

because  $nm = (-n)(-m)$ .

$$q = \frac{-m}{-n} = \frac{m}{n}$$

because  $(-m) \cdot n = m \cdot (-n)$ .

**Remark 4.** We want to show that  $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$ . Let  $q \in \mathbb{Q}_+ \cap \mathbb{Q}_-$ .

$$\begin{aligned} q = \frac{m}{n} = -\frac{m'}{n'} \quad m, n, m', n' \in \mathbb{N}_+ \\ \Rightarrow n \cdot n' = (-m')n \\ \Rightarrow \underbrace{mn'}_{\in \mathbb{N}_+} + \underbrace{m'n}_{\in \mathbb{N}_+} = 0 \quad \text{!} \end{aligned}$$

Furthermore  $p \in \mathbb{Q}_+ \wedge q \in \mathbb{Q}_+$

$$\Rightarrow p + q \in \mathbb{Q}_+ \wedge pq \in \mathbb{Q}_+$$

$$\Rightarrow p = \frac{k}{l} \quad q = \frac{m}{n} \quad k, l, m, n \in \mathbb{N}_+$$

$$p + q = \frac{\overbrace{kn + ml}^{\in \mathbb{N}_+}}{nm} \in \mathbb{Q}_+$$

$$pq = \frac{k}{l} \cdot \frac{m}{n} = \frac{\overbrace{km}^{\in \mathbb{N}_+}}{\underbrace{ln}_{\in \mathbb{N}_+}} \in \mathbb{Q}_+$$

**Definition 17.** Let  $K$  be an ordered field  $a \in K$ . The absolute value of  $a$  is defined as

$$|a| = \begin{cases} a & \text{if } a \in K_+ \\ 0 & \text{if } a = 0 \\ -a & \text{if } a \in K_- \end{cases}$$

**Remark 5.** Let  $K$  be an ordered field. Then it holds that

$$\mathbb{Q} \subseteq K \subseteq \mathbb{R}$$

except for isomorphism.



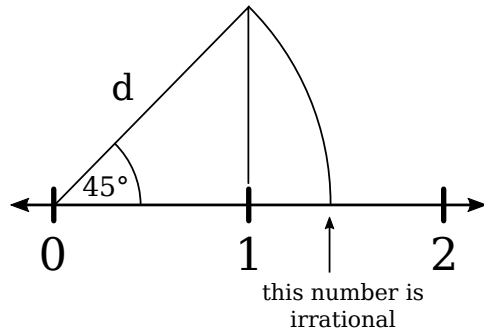


Figure 3: Illustration of an irrational number

## 6.4 Triangle inequality

**Theorem 7.**

$$\forall a, b \in K : |a + b| \leq |a| + |b| \quad \text{“Triangle inequality”}$$

*Proof.* **Case 1**

$$a \cdot b > 0 \Rightarrow a \cdot b > 0 : |ab| = ab \quad |a| \cdot |b| = ab$$

**Case 2**

$$a > 0, b < 0 : a \cdot b < 0 : |ab| = -ab \quad |a| \cdot |b| = a \cdot (-b)$$

$$b < 0 \Rightarrow -b > 0 \Rightarrow b < -b \Rightarrow \underbrace{a+b}_{|a+b|} < \underbrace{a-b}_{|a|+|b|}$$

**Case 3**

$$a < 0, b < 0 : a \cdot b > 0 : |ab| = ab \quad |a| = -a \quad |b| = -b$$

$$|a| \cdot |b| = -a \cdot -b = ab$$

**Case 4**

$$a > 0, b < 0 : a + b < 0$$

$$|a| = a \quad |b| = b \quad |a + b| = -(a + b) = -a - b$$

$$a > 0 \Rightarrow -a < 0 \quad -a - b < a - b$$

$$-(a + b) = |a + b|$$

□

This lecture took place on 4th of November 2015 with lecturer Wolfgang Ring.

## 6.5 Laws for absolute values

**Theorem 8.** Let  $y \geq 0$ . Then it holds that  $|x| \leq y \Leftrightarrow -y \leq x \wedge x \leq y$

*Proof.* First direction  $\Rightarrow$ :

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

**Case 1** Let  $x \geq 0$ . Then

$$|x| \leq y \Rightarrow x \leq y \Rightarrow -y \leq x$$

because  $-y \leq 0 \wedge x \geq 0$  anyways.

**Case 2** Let  $x < 0$ , therefore  $|x| = -x$ . Because

$$-x \leq y \Rightarrow x \geq -y$$

$x \leq y$  holds anyways because  $x < 0$  and  $y \geq 0$ .

Second direction  $\Leftarrow$ :

Let  $-y \leq x \leq y$ .

**Case 1**  $x \geq 0 : |x| = x \leq y$  because of the second inequality.

**Case 2**  $x < 0 : |x| = -x$

$$-(-1) \Rightarrow -(-y) \geq -x \text{ or equivalently } y \geq -x = |x|$$

**Theorem 9.**

$$\begin{aligned} |x| = 0 &\Leftrightarrow x = 0 \\ \forall a \in K : |a| &= |-a| \\ \forall \varepsilon > 0 : |x - y| &\leq \varepsilon \Leftrightarrow x = y \end{aligned}$$

*Proof.* **First direction**  $\Rightarrow$  Without loss of generality:  $x \geq y$ .

$$x \neq y \Rightarrow \exists \varepsilon > 0 : |x - y| > \varepsilon$$

Let  $x \neq y$ . Because  $x \geq y$  holds, so does  $x > y$ . Therefore  $x - y > 0$ . We define  $\varepsilon = \frac{x-y}{2} < x - y$

$$\begin{aligned} 2 &= 1 + 1 > 1 \\ 2^{-1} &= \frac{1}{2} < 1 = 1^{-1} \end{aligned}$$

Therefore it holds that  $\varepsilon : |x - y| = x - y > \frac{1}{2}(x - y) = \varepsilon > 0$ .

**Second direction**  $\Leftarrow x = y \Rightarrow |x - y| = 0 \leq \varepsilon \forall \varepsilon > 0$

□

**Theorem 10** (Inversed triangle inequality). Let  $a, b \in K$ . Then it holds that

$$||a| - |b|| \leq |a - b|$$

*Proof.* Show that  $-|a - b| \leq |a| - |b| \leq |a - b|$ .

**First inequality**

$$|b| = |b - a + a| \leq |b - a| + |a| \Rightarrow -|a - b| \leq |a| - |b|$$

**Second inequality**

$$|a| = |a - b + b| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$$

□

## 6.6 Irrational numbers approximated by rational numbers

Additional remark from 14th of January 2016.

□  $Q$  is dense in  $\mathbb{R}$ .

**Theorem 11.** For all  $x \in \mathbb{R}$  and for every  $\varepsilon > 0$  there exists  $q \in Q$  with  $|x - q| < \varepsilon$ .

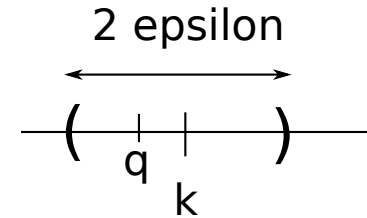


Figure 4:  $k$  and an  $\varepsilon$  environment

**Lemma 2.** Let  $A \subseteq \mathbb{N}$  and  $A \neq \emptyset$ . Then a minimum of  $A$  exists.

*Proof.* Proof by complete induction.

We show: Let  $A \subseteq \mathbb{N}$  such that no minimum exists. Then it holds that  $A = \emptyset$ .

Let  $C = \{k \in \mathbb{N} \mid \forall n \in A : k < n\}$ .  $C$  is the set of all lower bounds of  $A$  with operator  $<$ . We show:  $0 \in C$  and  $\forall k \in C \Rightarrow k + 1 \in C$ .

**Induction base** Assume  $0 \notin C$ , hence for  $k = 0$  it holds that

$$\exists n \in A \subset \mathbb{N} : n \leq 0$$

$\Rightarrow n \geq 0$  anyways and hence  $n = 0$  and  $0 \in A$ .  $\Rightarrow 0 = \min A$  and  $A$  has a minimum.

□

This is a contradiction. So  $0 \notin C$  does not hold. So  $0 \in C$ .

**Induction step** Let  $k \in C$ , hence  $\forall n \in A : k < n$ .

$$\Rightarrow \forall n \in A : k + 1 \leq n$$

Even  $<$  holds. Assume  $\exists n \in A : k + 1 = n$  and  $\forall n' \in A : k + 1 \leq n'$ .

$$\Rightarrow k + 1 \in A \wedge k + 1 \text{ is lower bound of } A$$

Therefore  $k + 1 = \min A$ .

This is a contradiction to the assumption that  $\min A$  does not exist. Therefore  $\forall n \in A : k + 1 < n \Rightarrow k + 1 \in C$ .

Due to the workings of induction:  $\forall k \in \mathbb{N} : k \in C$  equivalently means  $C = \mathbb{N}$ .

Therefore  $A = \emptyset$  holds. Assume  $m \in A$ , so it holds that  $m \notin C$ , because  $\neg(m < m)$ .

□

*Proof.* Case distinction:

$x > 0$  Let  $\varepsilon > 0$  be arbitrary. Choose  $n \in \mathbb{N}_+$  such that  $\frac{1}{n} < \varepsilon$  and define  $A = \{k \in \mathbb{N} \mid k > n \cdot x\}$ .

We know that  $A \neq \emptyset$  (Archimedean axiom). Let  $M = \min A$ , then it holds that  $m > n \cdot x$  and  $m - 1 \leq n \cdot x$ . Therefore  $x < \frac{m}{n}$  and  $x \geq \frac{m-1}{n}$ . Therefore it holds that

$$\left| x - \frac{m}{n} \right| = \frac{m}{n} - x \leq \frac{m}{n} - \frac{m-1}{n} = \frac{m-m+1}{n} = \frac{1}{n} < \varepsilon$$

and  $\frac{m}{n} = q \in \mathbb{Q}$ .

$x < 0$  Therefore  $-x > 0$  and  $\exists q \in \mathbb{Q}$ :

$$|q - (-x)| < \varepsilon$$

$$|x + q| = \left| x - \underbrace{(-q)}_{\in \mathbb{Q}} \right|$$

$x = 0$  Let  $q = 0 \in \mathbb{Q}$ .

□

**Corollary 3.**  $\forall x \in \mathbb{R}$  and  $\forall \varepsilon > 0$  it holds that

$$\mathbb{Q} \cap B(x, \varepsilon) = \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon) \neq \emptyset$$

Therefore  $x$  is a contact point of  $\mathbb{Q}$ .

**Remark.** It even holds that  $x$  is limit point of  $\mathbb{Q}$ .

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} \mid x \text{ is contact point of } \mathbb{Q}\} = \mathbb{R}$$

We say  $\mathbb{Q}$  is *dense* (or: lies in) in  $\mathbb{R}$ .

Alternative characterization of contact points:

$$\forall x \in \mathbb{R} \exists (q_n)_{n \in \mathbb{N}} \text{ with } q_n \in \mathbb{Q} \text{ with } \lim_{n \rightarrow \infty} q_n = x$$

## 6.7 Intervals

This lecture took place on 5th of November 2015 with lecturer Wolfgang Ring.

**Definition 18** (Intervals). Let  $a, b \in K$ .

$$(a, b) = \{x \in K \mid (x > a) \wedge (x < b)\}$$

$$[a, b) = \{x \in K \mid (x \geq a) \wedge (x < b)\}$$

$$(a, b] = \{x \in K \mid (x > a) \wedge (x \leq b)\}$$

$$[a, b] = \{x \in K \mid (x \geq a) \wedge (x \leq b)\}$$

**Theorem 12** (Laws for intervals).

$$(a, b) = \emptyset \text{ if } b \leq a \tag{21}$$

$$[a, b) = \emptyset \text{ if } b < a \tag{22}$$

$$[a, a] = \{a\} \tag{23}$$

If  $I$  is an non-empty interval (hence  $I \neq \emptyset$ ), then  $|I| = b - a$  is called *length of the interval*. Furthermore

$$(a, \infty) = \{x \in K \mid x > a\} \quad (24)$$

$$[a, \infty) = \{x \in K \mid x \geq a\} \quad (25)$$

$$(-\infty, a) = \{x \in K \mid x < a\} \quad (26)$$

$$(-\infty, a] = \{x \in K \mid x \leq a\} \quad (27)$$

**Theorem 13.**  $\mathbb{Q}$  is arithmetically incomplete.

*Proof.* We define a mapping from  $\mathbb{N}_+$  to  $\mathbb{N}$ : Let  $n \in \mathbb{N}_+$  then we know that  $n$  can be represented distinctly as product of prime numbers. Let  $Z(n)$  be the number of twos in the prime product representation.

Examples:

$$Z(14) = Z(2 \cdot 7) = 1$$

$$Z(15) = Z(3 \cdot 5) = 0$$

$$Z(24) = Z(2 \cdot 2 \cdot 2 \cdot 3) = 3$$

It holds that  $Z(2n) = Z(n) + 1 \forall n \in \mathbb{N}_+$  and  $Z(n^2) = Z(n) \cdot 2 \forall n \in \mathbb{N}_+$ .

We claim,

$$\nexists q : q = \frac{m}{n} \text{ with } q^2 = 2$$

Proof by contradiction:

1. Assume  $\left(\frac{m}{n}\right)^2 = 2$ .
2. Then  $\frac{m^2}{n^2} = 2$ .
3. Then  $m^2 = 2 \cdot n^2$ .
4. With  $Z(m^2) = 2 \cdot Z(n)$ .
5. With  $Z(2 \cdot n^2) = Z(n^2) + 1 = 2 \cdot Z(n) + 1$ .
6. If  $m^2 = 2n^2$ , then  $Z(m^2)$  must be even and  $Z(2 \cdot n^2)$  must be odd.
7. Then equality cannot be satisfied  $\nexists$

□

## 6.8 Archimedean property and Completeness axiom

**Theorem 14.**  $\mathbb{Q}$  is geometrically incomplete.

We consider an infinite straight number line. We define  $\mathbb{R}$  as ordered field with properties:

**Archimedean property**  $\mathbb{N} \subseteq \mathbb{R}$  with  $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : x < n$

$$\mathbb{N} \subseteq \mathbb{R} \forall n \in \mathbb{N} : -n \in \mathbb{N}$$

$$\Rightarrow \forall n \in \mathbb{N}_+ : n^{-1} \in \mathbb{R}$$

$$\Rightarrow \mathbb{Z} \subseteq \mathbb{R}$$

Therefore  $\forall m \in \mathbb{N} : m \cdot \frac{1}{n} = \frac{m}{n} \in \mathbb{R} \Rightarrow \mathbb{Q} \subseteq \mathbb{R}$ .

**Definition 19.** Let  $I_0, I_1, \dots, I_z$ .  $(I_n)_{n \in \mathbb{N}}$  is a sequence of closed intervals with

1.  $\forall a \in \mathbb{N} : I_{n+1} \subseteq I_n$
2.  $\forall \varepsilon > 0 \exists n \in \mathbb{N} : n \geq N \Rightarrow |I_n| < \varepsilon$

**Completeness axiom** Let  $(I_n)_{n \in \mathbb{N}}$  be nested intervals in  $\mathbb{R}$ . Then there exists some  $x \in \mathbb{R} : x \in I_n : \forall n \in \mathbb{N}_+$ .

Be aware, there exists only *one*  $x \in \mathbb{R}$  with the property:  $x \in I_n \forall n \in \mathbb{N}$ .

Assume  $x \in I_n$  and  $y \in I_n \forall n \in \mathbb{N}$  and  $x \neq y$ .

$$|\beta - \alpha| \leq b - a = |I|$$

*Proof.* Without loss of generality:  $\alpha \leq \beta$ . Then it holds that  $|\beta - \alpha| = \beta - \alpha \leq \beta + (-\alpha) \leq b + (-\alpha) = b - \alpha = |I|$ .

$$a \leq \alpha \Rightarrow -a \geq -\alpha$$

Consider arbitrary small  $\varepsilon > 0$  and  $N \in \mathbb{N}$  sufficiently large, such that  $|I_n| < \varepsilon$ . Because  $x, y \in I_n \Rightarrow |x - y| < \varepsilon \Rightarrow x = y$ . □

**Corollary 4.** From the Archimedean property it follows that,

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

*Proof.* Let  $x > \frac{1}{\varepsilon} \in \mathbb{R}$ . Archimedean property:  $\exists N \in \mathbb{N} : N > x$ .

For  $n \geq N$  it holds that  $n > x > 0 \Rightarrow \frac{1}{n} < \frac{1}{x} = \varepsilon$ .

**Corollary 5.** Let  $p \in \mathbb{R}, p > 1 \forall x \in \mathbb{R} : n \geq N \Rightarrow p^n > x$ .

*Proof.*  $p > 1 + u$  with  $u = p - 1$

$$p^n = (1 + u)^n \underset{\text{Bernoulli}}{>} 1 - nu = 1 + n(p - 1)$$

Let  $x \in \mathbb{R}$  arbitrary, select  $N \in \mathbb{N} : \frac{x-1}{p-1} < N$ .

Then it holds for  $n \geq N$  :

$$\underbrace{\frac{x-1}{p-1}}_{>0} \Leftrightarrow x - 1 < n \cdot (p - 1) \Leftrightarrow x < 1 + n(p - 1) < p^n$$

**Theorem 15.** Let  $q \in \mathbb{R}$  with  $|q| < 1$ . Then it holds that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |q^n| = |q|^n < \varepsilon$$

*Proof.* Let  $s = |q| \geq 0$ . Consider  $q > 0$ . Then

$$\begin{aligned} q^n &= 0 \\ |q^n| &= 0 \\ |q|^n &< \varepsilon \forall \varepsilon > 0 \forall n \in \mathbb{N} \end{aligned}$$

Let  $q \neq 0$ , then  $0 < s < 1$ . Let  $p = \frac{1}{s} \Rightarrow p > 1$ . Choose arbitrary  $\varepsilon > 0$  and  $x = \frac{1}{\varepsilon}$ . Because of the Completeness axiom

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow p^n > X$$

So it holds that

$$\begin{aligned} \frac{1}{p^n} &= s^n < \frac{1}{x} = \varepsilon \forall n \geq N \\ &\Rightarrow (|q|)^n = |q^n| \end{aligned}$$

□

□ **Theorem 16.** Let  $x \in \mathbb{R}, x > 0$  and let  $k \in \mathbb{N}_+$ . Then there exists a distinct  $y \in \mathbb{R}$  with  $y \geq 0$  such that

$$y^k = x$$

We denote  $y = \sqrt[k]{x}$  and conclude there exists  $k$ -th root numbers.

*Proof.* Idea: Construct nested intervals.

$(I_n)_{n \in \mathbb{N}}$  such that  $y \in \bigcap_{n \in \mathbb{N}} I_n$  satisfies the property that  $y^k = x$ .

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k$$

We define  $J_0 = [a_0, b_0]$  with  $a_0 = 0$  and  $b_0 = 1 + x$ . Then it holds that

$$a_0^k = 0^k = 0 \leq x$$

$$\square \quad b_0^k = (1 + x)^k = 1 + kx + \binom{k}{2}x^2 + \dots + x^k \geq 1 + kx > 0$$

□

This lecture took place on 6th of November 2015 with lecturer Wolfgang Ring.

**Theorem 17.** We prove:

$$0 \leq y_1 < y_2 \Rightarrow y_1^k \leq y_2^k$$

*Proof.* A short proof by a student:

**k = 2**

$$y^{k+1} = y^k \cdot y < y_2^k x < y_2^k y_2 = y^{k+1}$$

**k → k + 1**

$$y_1^2 < y_2^2$$

**Theorem 18.** Let  $a, b \in K$  and  $k \in \mathbb{N}$ . Then it holds that

$$a^k - b^k = (a - b) \left( \sum_{j=0}^{k-1} a^{k-1-j} b^j \right)$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

*Proof.*

$$\begin{aligned} (a - b) \left( \sum_{j=0}^{k-1} a^{k-j-1} b^j \right) &= \sum_{j=0}^{j-1} a^{k-j} b^j - \sum_{j=0}^{k-1} a^{k-j-1} b^{j+1} \\ &= a^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \underbrace{b^{k-1}}_{j=k-1} - \sum_{j=0}^{k-2} a^{k-j-1} b^{j+1} \\ &= a^k - b^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \sum_{l=1}^{k-1} a^{k-l} b^l \\ &= a^k \end{aligned}$$

**Theorem 19.** Let  $y_2 > y_1$  then

$$\begin{aligned} y_2^k - y_1^k &= \underbrace{(y_2 - y_1)}_{>0} \underbrace{\left( \sum_{j=0}^{k-1} y_2^{k-j-1} y_1^j \right)}_{>0} \\ &\Rightarrow y_2^k - y_1^k > 0 \end{aligned}$$

*Proof.*

$$\forall x \geq 0 \in \mathbb{R} : \exists y \geq 0 \in \mathbb{R} : y^k = x \text{ with } k \in \mathbb{N}_+$$

Special case  $x = 0$  and  $y = 0$  is the solution.

Let  $x > 0$ : We construct  $y$  with  $y \in \bigcap_{k=0}^{\infty} I_n$  where  $I_n$  are nested intervals. Specifically  $I_n$  must have the properties:

$$\square \quad \bullet \quad I_n = [a_1, b_n] \text{ with } a^k \leq x, b_n^k \geq x \quad \forall n \in \mathbb{N}$$

$$\bullet \quad I_{n+1} \subseteq I_n : |I_n| = \frac{1}{2} |I_{n+1}| = \left(\frac{1}{2}\right)^n |I_0|$$

$$n = 0 \quad I_0 = [0, x - 1]$$

$$a_0 = b \quad b_0 = x + 1$$

$$a_0^k = 0 < x \quad \checkmark$$

$$b_0^k = (1 + x)^k = 1 + kx + \binom{k}{2} x^2 + \dots + x^k > 1 + kx > x \text{ for } k \geq 1$$

Let  $I_n$  be given:  $I_n = [a_n, b_n]$ . Define  $m_n = \frac{1}{2}(a_n + b_n)$

**Case 1**

$$m_n^k \geq x \Rightarrow \text{let } a_{n+1} = a_n, b_{n+1} = m$$

$$I_{n+1} = [a_n, m_n] \subseteq [a_n, b_n] = I_n$$

$$|I_{n+1}| = m_n - a_n = \frac{1}{2} a_n + \frac{1}{2} b_n - a_n$$

$$\frac{1}{2} (b_n - a_n) = \frac{1}{2} |I_n|$$

$$a_{n+1}^k = a^k \leq x \quad \checkmark$$

$\square$

All conditions are satisfied.

**Case 2**  $m_n^k < x$ : Let  $a_{n+1} = m_1, b_{n+1} = b_n$ . It holds that  $a_{n+1} = m_n < x, b_{n+1} = b_n \geq x \quad \checkmark$ . Furthermore it holds that  $I_{n+1} \subseteq I$  and  $|I_{n+1}| = \frac{1}{2} |I_n|$ .

$I_n$  is set of nested intervals. Let  $\varepsilon > 0$  be arbitrary. Then

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{1+x}$$

For those  $n \geq N$  it holds that

$$|I_n| = \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^n (x+1) < \frac{\varepsilon}{1+x} \cdot (1+x)$$

Let  $y \in I_n \forall n \in \mathbb{N}$ . Further nesting of intervals:

$$(I_n)_{n \in \mathbb{N}} \text{ with } I_n = [a_n^k, b_n^k]$$

It holds that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \text{ because } I_{n+1} \subseteq I_n \Rightarrow a_n^b \leq a_{n+1}^k < b_{n+1}^k \leq b_n^k$$

Length of  $I_n$ :

$$I_n = b_n^k - a_n^k = (b_n - a_n) \sum_{j=0}^{k-1} a_n^{k-1-j} b_n^j$$

Because  $I_n \subseteq I_0 \Rightarrow a_n < b_0 \Rightarrow b_n \leq b_0$ ,

$$< (b_n - b_0) \sum_{j=0}^{k-1} b_0^{k-1-j} b_0^j$$

$$= (b_n - a_n) k b_0^k = (b_n - a_n) k (1+x)^k$$

Let  $\varepsilon > 0$  be arbitrary. Find some  $N \in \mathbb{N}$  with  $n \geq N$ :

$$|I_n| = (b_n - a_n) < \frac{\varepsilon}{k(1+x)^k}$$

For those  $n$  it holds that

$$|I_n| < |I_n| \cdot k(1-x)^k < \frac{\varepsilon}{k(1+x)^k} k(1+x)^k = \varepsilon$$

Therefore  $(I_n)_{n \in \mathbb{N}}$  a set of nested intervals.

$\exists z \in \mathbb{R}$  with  $z \in [a_n^k, b_n^k] : \forall n \in \mathbb{N}$  and  $z$  is unique. By construction of  $I_n$  it holds that  $a_n^k \leq x \leq b_n^k$

$$\Rightarrow x \in I_n \forall n \in \mathbb{N} \Rightarrow x = z \in \bigcap_{n \in \mathbb{N}} I_n.$$

On the opposite side it holds that  $y \in I_n$  (hence  $a_n \leq y \leq b_n \Rightarrow a_n^k \leq y^k \leq b_n^k$ ). So  $y^k \in I_n \forall n \in \mathbb{N} \Rightarrow y^k = z = x$ . So we have found some  $y^k$  which is  $x$ . But is  $y \geq 0$  with  $y^k = x$  unique?

Let  $y_1 \neq y_2$  with  $y_1^k = y_2^k = x$  and without loss of generality,

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k \quad \text{`}$$

So,  $y$  is unique.

## 7 Supremum property of $\mathbb{R}$

### 7.1 Boundedness in $\mathbb{R}$

**Definition 20.** Let  $A \subseteq \mathbb{R}$ .

- We call  $A$  to be *bounded above* if there exists some  $u \in \mathbb{R}$  such that  $\forall a \in A : a \leq u$ .
- A number  $u$  with that property is called *upper bound of  $A$* .
- We call  $A$  to be *bounded below* if there exists some  $l \in \mathbb{R}$  such that  $\forall a \in A : a \geq l$ .
- A number  $l$  with that property is called *lower bound of  $A$* .
- $A$  is called *bounded* if there exists a lower and upper bound of  $A$ .

**Corollary 6.** Let  $(a, b)$  be bounded. Let  $u$  be its upper bound and let  $v \geq u$ . Then  $v$  is also an upper bound of  $(a, b)$ .

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This lecture took place on 11th of November 2015 with lecturer Wolfgang Ring.

### 7.2 Supremum and infimum in $\mathbb{R}$

**Definition 21.** Let  $A$  be bounded above. Assume  $s \in \mathbb{R}$  has the properties

1.  $s$  is an upper bound for  $A$
2.  $\forall \sigma \in \mathbb{R} : \sigma < s : \sigma$  is not an upper bound for  $A$ .

If those properties are satisfied, we call  $s$  *supremum of  $A$* . A supremum  $s$  is always the smallest upper bound of  $A$ . We denote  $s = \sup A$ .

There exists at most one supremum for  $A$ . Let  $s_1$  and  $s_2$  be two suprema, then  $s_1 \neq s_2$ . So wlog.  $s_1 < s_2$ . This invalidates the supremum property of  $s_2 \Rightarrow s_1$  is not a supremum of  $A$   $\nmid$ .

Analogously an *infimum* of  $A$  is the greatest lower bound of  $A$ . Let  $A$  be bounded below.  $t \in \mathbb{R}$  is called *infimum* of  $A$  if

1.  $\forall a \in A : t \leq a$  ( $t$  is a lower bound of  $A$ )
2.  $\forall x > t$  so  $x$  is no lower bound of  $A$

$$\Leftrightarrow \exists a \in A : a < x$$

We denote  $t = \inf A$ .

**Definition 22.** Let  $A \subseteq \mathbb{R}$ . We denote  $u = \max A$  for the *maximum* of  $A$  if

1.  $u \in A$  (is element of  $A$ )
2.  $\forall a \in A : a \leq u$  (is an upper bound)

$l \in \mathbb{R}$  denoted  $l = \min A$  is called minimum of  $A$  if

1.  $l \in A$  (is element of  $A$ )
2.  $\forall a \in A : l \leq a$  ( $l$  is a lower bound)

**Theorem 20.** Let  $A \subseteq \mathbb{R}$  and  $u$  be the maximum of  $A$ . Then it holds that  $u = \sup A$ . If  $l = \min A \Rightarrow l = \inf A$ .

*Proof.* We need to show, that  $l$  is an upper bound of  $A$ . This follows by definition. For  $x < u$  it holds that  $x$  not an upper bound.

Let  $x < u$ , because  $u \in A$  there exists some element  $y$  in  $A$  with  $y > x$ . Therefore  $x$  is not an upper bound of  $A$ .  $\square$

**Example 6.**

$$A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} = \left\{ \frac{1}{n} : n \in \mathbb{N}_+ \right\}$$

Then it holds that  $1 \in A$  and  $1 \geq \frac{1}{n} \forall n \in \mathbb{N}_+$ . Therefore  $1 = \max A = \sup A$ .

$0 = \inf A$ , because  $0$  is a lower bound of  $A$  ( $\frac{1}{n} > 0 \forall n \in \mathbb{N}_+$ ). Let  $\varepsilon > 0$ , then  $\exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} \leq \varepsilon$ . Therefore  $\varepsilon$  is not a lower bound of  $A$ .

So  $A$  does not have a minimum, because otherwise  $l = \max A = \inf A = 0$ .

**Theorem 21.** Let  $A \neq \emptyset$  and  $A \subseteq \mathbb{R}$  be bounded above. So some  $s = \sup A \in \mathbb{R}$  exists (therefore  $\mathbb{R}$  has a supremum property).

*Proof.* We construct nested intervals  $(I_n)_{n \in \mathbb{N}}$  such that for  $s \in \bigcap_{n \in \mathbb{N}} I_n$  gilt  $s = \sup A$ . We construct  $I_{n+1}$  inductively using  $I_n$

**Case  $n = 0$**

Because  $A \neq \emptyset$ , we select  $a_0 \in A$ . Because  $A$  is bounded above,  $\exists b_0 \in \mathbb{R}$  such that  $b_0$  is an upper bound of  $A$ . We define  $I_0 = [a_0, b_0]$ .

**Case  $n \rightarrow n + 1$**

Let  $a_0 = b_0$ , then it holds that  $b_0$  is upper bound and  $b_0 \in A$ . We call that terminating condition. Therefore  $b_0 = \max A = \sup A$  and the supremum was found. Instead of  $n$  we use  $n + 1$ . Let  $I_0 = [a_n, b_n]$  with  $a_n \neq b_n$  and  $a_n \in A$ ,  $b_n$  is an upper bound of  $A$ . Furthermore it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider  $I_{n+1}$  such that the same properties are satisfied. Let  $m_1 = \frac{1}{2}(a_1 + b_1)$ . It holds that  $a_n < m_n < b_n$ .

**Case  $m_n$  is an upper bound of  $A$**  Then we set  $a_{n+1} = a_n \in A$  and  $b_{n+1} = m_n$  is an upper bound of  $A$ .

$$|I_{n+1}| = b_{n+1} - a_{n+1} = \frac{1}{2}(b_n + a_n) - a_n$$

$$= \frac{1}{2}b_1 - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^{n+1} |I_n| \quad \checkmark$$

**Case  $m_n$  is not an upper bound of  $A$**  Therefore  $\exists x \in A$  with  $x > m_n$ .

**Subcase  $x = b_1$**  So  $b_1$  is an upper bound. Therefore  $x \in A$  and  $x$  is upper bound.

$$x = \max A = \sup A$$

We found the supremum.



**Subcase  $m_n < x < b_n$**  Let  $a_{n+1} = x \in A$  and  $b_{n+1} = b_n$  is an upper bound and

$$\begin{aligned} I_{n+1} &= b_{n+1} - a_{n+1} - b_n - x < b_n - m_n - b_n - \frac{1}{2}(b_n + a_n) + \frac{1}{2}(b_n - a_n) \\ &= \frac{1}{2} |I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0| \end{aligned}$$

We have found supremum  $s = \sup A$ .

If in any case the terminating condition holds, then we have found the supremum.

The remaining case is  $\forall n \in \mathbb{N} : a_n < b_n, a_n \in A, b_n$  is upper bound of  $A$ .

$$|I_n| = b_n - a_n \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider  $\varepsilon > 0$  and  $N$  such that  $n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{|I_0|}$ . For those  $n$  it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| < \frac{\varepsilon}{|I_0|} |I_0| = \varepsilon$$

Therefore  $(I_n)_{n \in \mathbb{N}}$  are nested intervals.

□

What remains for completeness:  $s \in \mathbb{R}, s \in I_n : \forall n \in \mathbb{N}$ . We need to show that  $s = \sup A$ .

This lecture took place on 12th of November 2015 with lecturer Wolfgang Ring.

**Theorem 22.** Completeness of  $\mathbb{R}$ :

$$\exists s \in \mathbb{R} : s \in I_n \forall n \in \mathbb{N}$$

**Proof cont.** Every set with an upper bound has a supremum.

We construct  $(I_n)_{n \in \mathbb{N}}$  with  $I_n = [a_n, b_n]$  and  $I_{n+1} \subseteq I_n$ .  $\forall n \in \mathbb{N} : a_n \in A, b_n$  is the upper bound of  $A$ .

$$|I_{n+1}| \leq \frac{1}{2} |I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

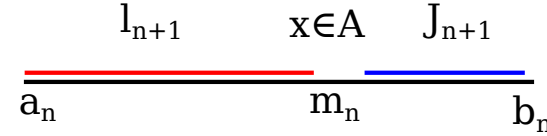


Figure 5: Relation of  $a_n$  and  $b_n$  and  $J_{n+1}$

Consider  $I_{n+1} \subseteq I_n$  with  $a_n < b_n \forall n \in \mathbb{N}$ .

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

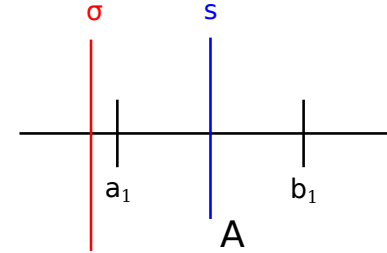


Figure 6: Illustration of  $s$  between  $a_n$  and  $b_n$

1. Claim:  $s$  is  $\sup A$ .

We need to show (by contradiction):  $S$  is upper bound of  $A$ . Assume  $a \in A$  and  $a > s$ . Let  $\varepsilon = a - s > 0$  and choose  $N$  sufficiently large such that

$$|I_n| < \varepsilon = a - s$$

Then it holds that

$$b_N = \underbrace{b_n - a_n}_{\varepsilon} \not\leq \underbrace{a_N}_{< s} < s + \varepsilon = a$$

$$\Rightarrow b_N < a \in A$$

Because  $b_n$  is an upper bound.

2.  $\forall \sigma < s$  it holds that  $\sigma$  is not an upper bound of  $A$ . Let  $\sigma < s$  and  $\varepsilon = s - \sigma > 0$  and choose  $n \in \mathbb{N}$  large enough such that  $b_N - a_N < \varepsilon$ . Then it holds that

$$\begin{aligned} a_N &= a_N - b_N + b_N \\ &> -\varepsilon + s \\ &= -s + \sigma + s = \sigma \quad \checkmark \end{aligned}$$

Therefore it holds that  $s$  is smallest upper bound of  $A$  and therefore supremum.

**Theorem 23.** Every set with a lower bound in  $\mathbb{R}$  has an infimum. Every set with an upper bound in  $\mathbb{R}$  has a supremum.

**Theorem 24.** Remember that  $M$  has the same cardinality like  $\mathbb{N}$  if  $\varphi : M \rightarrow \mathbb{N}$  is bijective,  $M$  is called countably infinite if  $M$  has the same cardinality like  $\mathbb{N}$ .

Let  $\varphi : \mathbb{N} \rightarrow M$  be bijective therefore  $M = \{\varphi(1), \varphi(2), \varphi(3), \dots\} = \{\varphi(n) \mid n \in \mathbb{N}\}$  and  $\varphi(i) \neq \varphi(j)$  for  $i \neq j$ .

**Notation.**  $\varphi(n) = m_n$ .

$M = \{m_0, m_1, m_2, \dots\}$  with  $m_i \neq m_j$  for  $i \neq j$ .  $\varphi$  is a complete enumeration of all elements of  $M$ .

Therefore every element of  $M$  has the structure:  $m_n$  with  $n \in \mathbb{N}$ .

**Theorem 25.**

$$\mathbb{Q}^+ = \left\{ \frac{m}{n}, m \in \mathbb{N}, n \in \mathbb{N}_+ \right\}$$

The set  $\mathbb{Q}^+$  is countably infinite.

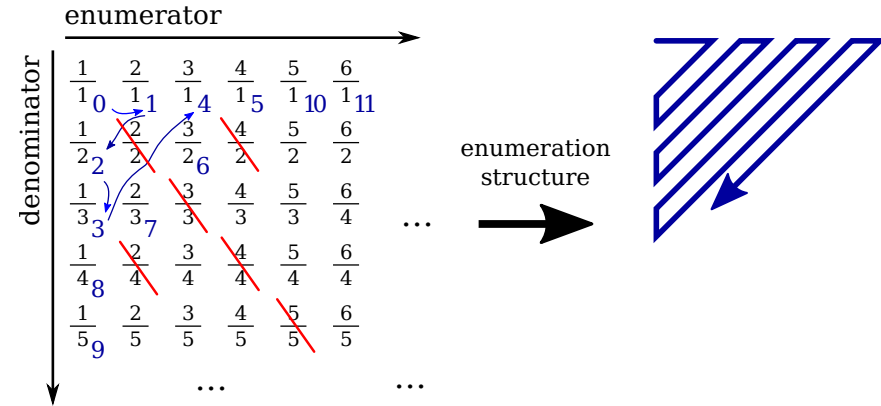


Figure 7: A complete enumeration of  $\mathbb{Q}^+$  (diagonalization argument). We traverse the whole matrix diagonally. The blue numbers indicate the enumeration and red lines cross out values already enumerated. On the right-hand side the general order of the enumeration is illustrated.

*Proof.* We enumerate the elements of  $\mathbb{Q}^+$ .

$$\mathbb{Q}_+ = \{q_0, q_1, q_2, \dots\}$$

$$\mathbb{Q}_- = \{-q_0, -q_1, -q_2, \dots\}$$

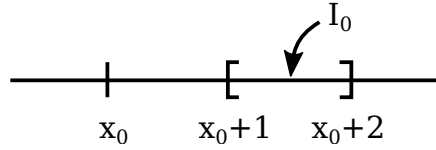
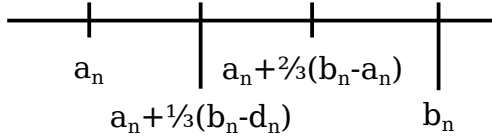
$$\mathbb{Q} = \{0, q_0, -q_0, q_1, -q_1, \dots\}$$

An enumeration exists. So  $\mathbb{Q}$  is countably infinite.  $\square$

**Theorem 26.** There is no bijective relation  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$ . Therefore we call  $\mathbb{R}$  uncountable.

*Proof.* We provide a proof by contradiction. Assume  $\mathbb{R} = \{x_0, x_1, x_2, x_3, \dots\}$  is countable.

We construct nested intervals.


 Figure 8: Construction of a nested interval and its  $I_0$ 

 Figure 9: Construction of a nested interval and its  $I_n$ 

**Case  $n = 0$**

$$I_0 = [x_0 + 1, x_0 + 2]$$

Let  $|I_0| = 1$  and  $x_0 \notin I_0$ .

**Case  $n \rightarrow n + 1$**  Assume  $I_0 \dots I_n$  were already defined with  $x_k \notin I_k$  for  $0 \leq k \leq n$ .

$$I_{k+1} \leq I_k \text{ for } k = 0, \dots, n - 1$$

$$|I_k| = \left(\frac{1}{3}\right)^k$$

We construct  $I_{n+1}$ . Let  $I_n = [a_n, b_n]$ .

$$I_n^1 = \left[a_n, \frac{2}{3}a_n + \frac{1}{3}b_n\right]$$

$$I_n^2 = \left[\frac{2}{3}a_n + \frac{1}{3}b_n, \frac{1}{3}a_n + \frac{2}{3}b_n\right]$$

$$I_n^3 = \left[\frac{1}{3}a_n + \frac{2}{3}b_n, b_n\right]$$

So  $x_n$  certainly is not contained in all three intervals  $I_n^1, I_n^2$  and  $I_n^3$  because  $I_n^1 \cap I_n^2 \cap I_n^3 = \emptyset$ . Choose  $I_{n+1}$  as one of the three intervals  $I_n^l$  with  $x_{n+1} \notin I_n^l = I_{n+1}$ .  $I_{n+1} < I_n$ .

$$|I_{n+1}| = \frac{1}{3}I_n = \left(\frac{1}{3}\right)^{n+1}$$

For  $\varepsilon > 0$  it holds that there exists some  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |I_n| = \left(\frac{1}{3}\right)^n < \varepsilon$ . Therefore nested intervals  $I_n$  are given.

Let  $x \in \mathbb{R}$  such that  $\forall n \in \mathbb{N} : x \in I_n$  (because of completeness law). Then it holds that  $\forall x_n : x \neq x_n$ .  $x \in I_n$  and  $x_n \notin I_n$ . Therefore  $x \in \{x_0, x_1, x_2, \dots\} = \mathbb{R}$ .

This contradicts with the assumption that  $\mathbb{R}$  is countable.

□

## 8 Complex numbers $\mathbb{C}$

We introduce a new arithmetic unit denoted  $i$ , which extends the field  $\mathbb{R}$ . Elements of  $\mathbb{C}$  are represented as  $a + bi$  with  $a, b \in \mathbb{R}$ .

$$\forall a, b \in \mathbb{R} : a + bi = 0 \Leftrightarrow a = 0 \wedge b = 0 \quad (28)$$

$$i^2 = -1 \quad (29)$$

$$\text{associativity, commutativity etc holds} \quad (30)$$

This lecture took place on 13th of November 2015 with lecturer Wolfgang Ring.

**Definition 23.** We consider an “arithmetic element”  $i$  extending  $\mathbb{R}$  (“adjungiert”). Arithmetic operations are well-defined for  $i$ . Associativity and commutativity holds. It holds that

- $a + ib = 0$  with  $a, b \in \mathbb{R} \Leftrightarrow a = 0 \wedge b = 0$
- $i^2 = -1$  i.e.  $i^2 + 1 = 0$ .
- Arithmetic operations still hold.

By the first law,

$$a + ib = a' + ib' \Leftrightarrow (a - a') + i(b - b') = 0 \Leftrightarrow a - a' = 0 \wedge b - b' = 0 \text{ therefore } a = a' \wedge b = b'$$

By the second law,  $i$  is the solution of the quadratic equation  $i^2 + 1 = 0$ .

Let  $z = a + ib$  a complex number. We call  $i$  the “imaginary unit”.

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$

$\mathbb{C}$  is the field of complex numbers with the following properties:

- For addition, it holds that

$$(a + ib) + (c + id) = (a + b) + i(b + d) \subseteq \mathbb{C}$$

and

$$(a + ib) + (-a - ib) = (a - a) + i(b - b) = 0 + i \cdot 0 = 0$$

- For multiplication, it holds that

$$(a + ib) \cdot (c + id) = (ac + \underbrace{(i)^2}_{=-1} bd) + i(bc + ad)$$

$$(ac - bd) + i(bc + ad)$$

- Laws **A<sub>n</sub>** to **A<sub>4</sub>**, **M<sub>1</sub>** to **M<sub>3</sub>** and **D** hold.

- The one element exists:

$$1 = 1 + 0 \cdot i$$

$$(a + i \cdot b)(1 + i \cdot 0) = (a + (i)^2 \cdot 0) + i(b + 0) = a + ib$$

- **M4** holds: Let  $z \in \mathbb{C} \setminus \{0\}$ . Let  $z = a + ib$  and  $\neg(a = 0 \wedge b = 0) \Leftrightarrow a^2 + b^2 > 0$ .

We define

$$\begin{aligned} w &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \\ z \cdot w &= (a + ib) \left( \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \right) \\ &= \left( \underbrace{\frac{a^2}{a^2 + b^2} - \frac{b \cdot (-b)}{a^2 + b^2}}_{=1} \right) + i \cdot \left( \underbrace{\frac{ba}{a^2 + b^2} - \frac{a \cdot b}{a^2 + b^2}}_{=0} \right) \\ &= 1 + i \cdot 0 = 1 \end{aligned}$$

Therefore  $w = z^{-1} = \frac{1}{z}$ .

Therefore  $\mathbb{C}$  is a field.

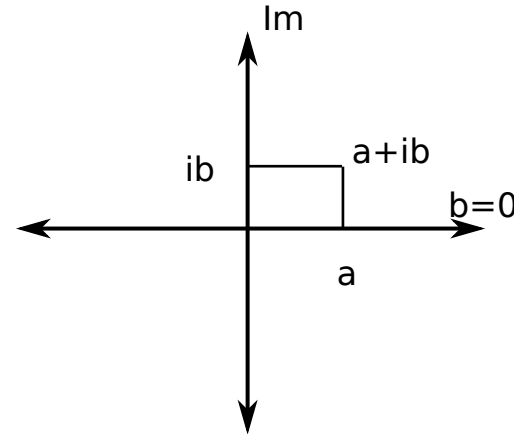


Figure 10: Illustration of complex numbers

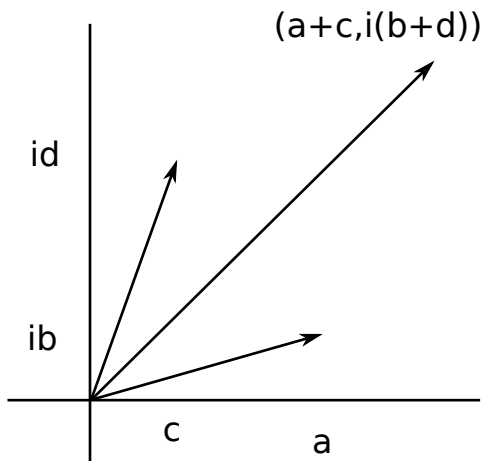


Figure 11: Illustration of complex number addition

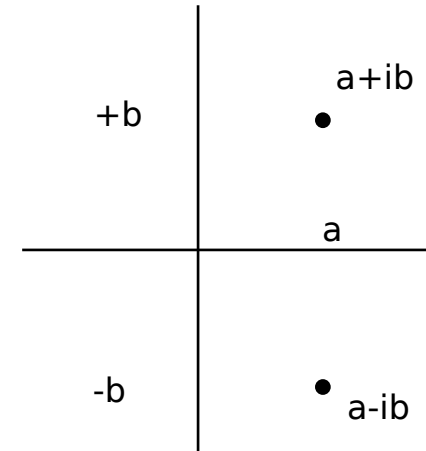


Figure 12: Illustration of the complex conjugate

We denote

$$\begin{aligned} a &= \Re(z) \\ b &= \Im(z) \\ \bar{z} &= a - ib \\ |z| &= \sqrt{a^2 + b^2} \end{aligned}$$

$a$  is called *real part of  $z$* .  $b$  is called *imaginary part of  $z$* .  $z$  is called complex conjugate.  $|z|$  is called absolute value of  $z$ .

**Theorem 27.**

$$\overline{(\bar{z})} = z$$

*Proof.*

$$\overline{(\bar{z})} = \overline{(a - ib)} = (a - (-ib)) = a + ib = z$$

**Theorem 28.**

$$\Re(z) = \frac{1}{2}(z + \bar{z})$$

**Theorem 29.**

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + ib + a - ib) = \frac{1}{2}(2a) = a$$

**Theorem 30.**

$$\Im(z) = \frac{1}{2i}(z - \bar{z})$$

*Proof.*

$$\frac{1}{2i}(a + ib - (a - ib)) = \frac{1}{2i}(2ib) = b$$

□

**Theorem 31.**

□

$$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$$

*Proof.*

$$z = a \in \mathbb{R} \Rightarrow \bar{z} = a = z$$

On the opposite, let  $z = \bar{z}$  therefore

$$a = ib = a - ib \Rightarrow 2ib = 0 \Rightarrow b = 0$$

Therefore  $z = a \in \mathbb{R}$ .

**Theorem 32.**

$$z \in i\mathbb{R} = \{ib : b \in \mathbb{R}\} \Leftrightarrow z = -\bar{z}$$

Proof follows analogously.

**Theorem 33.** It holds that  $|z| = \sqrt{z \cdot \bar{z}}$ .

*Proof.*

$$\begin{aligned} \sqrt{z \cdot \bar{z}} &= ((a + ib)(a - ib))^{\frac{1}{2}} \\ &= (a^2 - (ib)^2)^{\frac{1}{2}} = (a^2 - i^2 b^2)^{\frac{1}{2}} \\ &= (a^2 + b^2)^{\frac{1}{2}} = |z| \quad \checkmark \end{aligned}$$

**Theorem 34.** Let  $z, w \in \mathbb{C}$ :

$$\overline{(zw)} = \bar{z} \cdot \bar{w}$$

*Proof.*

$$\begin{aligned} z &= a + ib & w &= c + id \\ zw &= (ac - bd) + i(bc + ad) \\ \overline{zw} &= (ac - bd) - i(bc + ad) \\ \overline{zw} &= a - ib & \bar{w} &= c - id \end{aligned}$$

$$\bar{z} \cdot \bar{w} = (a - (-b))(-d) + i(-bc + a(-d)) = (ac - bd) - i(bc + ad)$$

**Corollary 7.**

$$\overline{z + w} = \bar{z} + \bar{w}$$

**Theorem 35.**

$$|zw| = |z| \cdot |w|$$

*Proof.*

$$\begin{aligned} |z \cdot w| &= (zw) \cdot (\overline{z \cdot w})^{\frac{1}{2}} \\ &= (z \cdot \bar{z} \cdot w \cdot \bar{w})^{\frac{1}{2}} = (z \cdot \bar{z})^{\frac{1}{2}} \cdot (w \cdot \bar{w})^{\frac{1}{2}} = |z| \cdot |w| \end{aligned}$$

□

□

**Theorem 36.**

$$z = 0 \Leftrightarrow |z| = 0 \in \mathbb{R}$$

*Proof.*

$$z = 0 = 0 + i0 \Rightarrow |z| = \sqrt{0^2 + 0^2} = 0$$

$$\text{Let } |z| = \sqrt{a^2 + b^2} = 0 \Rightarrow a^2 + b^2 = 0.$$

$$\Rightarrow a = 0 \wedge b = 0$$

□

**Theorem 37.**

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

$$|\Im(z)| = |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| =$$

**Theorem 38.** The triangle inequality holds:

$$\forall z, w \in \mathbb{C} : |z + w| \leq |z| + |w|$$

**Remark 6.** Let  $0 \leq y_1 < y_2$  with  $y_1, y_2 \in \mathbb{R}$ . Let  $k \in \mathbb{N}_+$ . Then it holds that

$$\sqrt[k]{y_1} < \sqrt[k]{y_2}$$

*Proof.* Indirect proof: Let  $\sqrt[k]{y_1} \geq \sqrt[k]{y_2} \geq 0$ .

□

$$\Rightarrow (\sqrt[k]{y_1})^k \geq (\sqrt[k]{y_2})^k$$

therefore  $y_1 \geq y_2$ . This is the negation of our assumption.

□

**Proof of the triangle inequality.** We show that  $|z + w|^2 \leq (|z| + |w|)^2$ .

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = \underbrace{z\bar{z}}_{|z|^2} + w\bar{z} + z\bar{w} + \underbrace{w\bar{w}}_{|w|^2} \\
 &= 2\Re(w\bar{z}) \\
 &= (w\bar{z} + \overline{(w\bar{z})}) \\
 &\quad \overline{w\bar{z}} = \bar{w} \cdot z \\
 &= |z|^2 + 2\Re(w\bar{z}) + |w|^2 \\
 &\leq |z|^2 + 2|\Re(w\bar{z})| + |w|^2 \\
 &\leq |z|^2 + 2 \cdot |w\bar{z}| + |w|^2 \\
 &= |z|^2 + 2 \cdot |w| \cdot |\bar{z}| + |w|^2 \\
 &= |z|^2 + 2 \cdot |w| \cdot |z| + |w|^2 \\
 &= (|z| + |w|)^2
 \end{aligned}$$

□

**Theorem 39.** In our previous proof there was a small loop hole: We need to show that

$$|z| = |\bar{z}|$$

*Proof.*

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

## 8.1 Interpretation of multiplication

Multiplication with  $i$ . Let  $z = a + ib$ .

$$iz = i \cdot a + i^2 \cdot b = (-b) + ia$$

Multiplication with  $i$  rotates  $z$  counter-clockwise by  $90^\circ$  in the plane.

Let  $z \in \mathbb{C}$  and  $w = c + id$ .

This lecture took place on 18th of November 2015 with lecturer Wolfgang Ring.

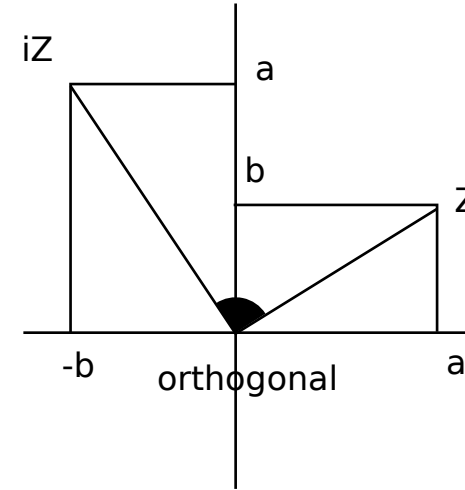


Figure 13: Multiplication corresponds to a rotation by  $90^\circ$

## 8.2 Taking roots

$$\forall a \in \mathbb{R} : a \geq 0 \forall n \in \mathbb{N}_+ : \exists x \geq 0 \in \mathbb{R} : x^n = a$$

Taking the  $n$ -th root only works for positive integers, because  $\forall x \geq 0 : x^2 \geq 0$  and no solution in  $\mathbb{R}$  exists for the equation  $x^2 = -1$ . □

In  $\mathbb{C}$  it holds that  $\forall w \in \mathbb{C} \setminus \{0\}$ .  $\forall n \in \mathbb{N}$  there exist exactly  $n$  different solutions of the equation  $z^n = w$ .

## 9 Sequences of real and complex elements

**Definition 24.** Let  $a$  be a mapping  $\mathbb{N} \rightarrow \mathbb{R}$  is called *sequence* of real numbers.

$$\forall n \in \mathbb{N} : a(n) \in \mathbb{R}$$

We denote  $a_n := a(n)$ . Instead of  $a : \mathbb{N} \rightarrow \mathbb{C}$  we write  $(a_n)_{n \in \mathbb{N}} = (a_0, a_1, \dots)$ .

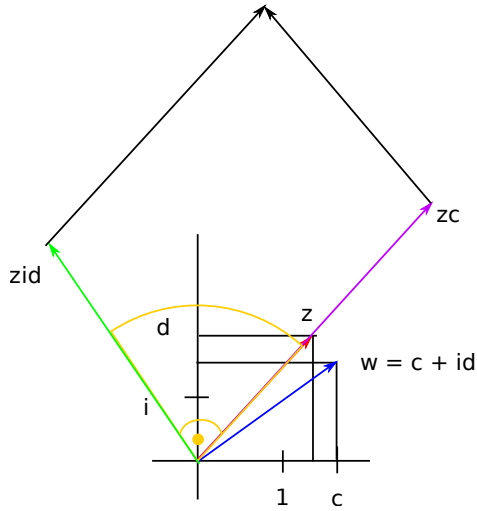


Figure 14: In regards of multiplication with  $w$  the complex number  $z$  is scaled by  $|w|$  and then rotated by an angle which is given between  $w$  and the positive real axis.

Analogously for the complex numbers  $\mathbb{C}$  and general sets  $X$ .

**Example 7.**  $a_n = \sqrt[n]{2} \frac{1}{n+1}$  with  $(a_n)_{n \in \mathbb{N}}$ . Or simply:

$$\left( \sqrt[n]{2} \frac{1}{n+1} \right)_{n \in \mathbb{N}}$$

**Example 8.** Let  $(I_n)_{n \in \mathbb{N}}$  be nested intervals. Therefore  $(I_n)_{n \in \mathbb{N}}$  is a sequence of elements in  $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ .

**Definition 25.** Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence.  $(a_n)_{n \in \mathbb{N}}$  is called *bounded above* if  $o \in \mathbb{R}$  exists such that  $\forall n \in \mathbb{N} : a_n \leq o$ .  $(a_n)_{n \in \mathbb{N}}$  is called *bounded below* if  $u \in \mathbb{R}$  exists such that  $\forall n \in \mathbb{N} : a_n \geq u$ .

$(a_n)_{n \in \mathbb{N}}$  is called *bounded*, if  $(a_n)_{n \in \mathbb{N}}$  is bounded above and below.

**Example 9.**  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = \frac{n}{n+1}$  is bounded below by 0 and bounded above by 1:  $n \leq n+1 \Rightarrow n \frac{1}{n+1} < \frac{n+1}{n+1} = 1 \checkmark$ .

## 9.1 Monotonicity

**Definition 26.**

- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically increasing* if  $\forall n \in \mathbb{N} : a_{n+1} \geq a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically decreasing* if  $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically strictly increasing* if  $\forall n \in \mathbb{N} : a_{n+1} > a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically strictly decreasing* if  $\forall n \in \mathbb{N} : a_{n+1} < a_n$ .

In  $\mathbb{C}$ , elements are not ordered, so we need to define an order explicitly. Let  $(a_n)_{n \in \mathbb{N}}$  a complex sequence. We define:

- $(a_n)_{n \in \mathbb{N}}$  is called *bounded* if  $(|a_n|)_{n \in \mathbb{N}}$  is a bounded real sequence. Hence  $\exists o \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| \leq o$ .
- The lower bound is implicitly given by 0.

**Example 10.**  $a_n := i^n$  and  $(a_n)_{n \in \mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, i, -1, \dots)$

$$|1| = 1 \quad |-1| = 1 \quad |i| = \sqrt{0^2 + 1^2} = 1 \quad |-i| = \sqrt{0^2 + (-1)^2} = 1$$

So  $(|a_n|)_{n \in \mathbb{N}} = (1, 1, 1, 1, 1, \dots)$ . It holds that

$$|z| = |-z| = |\bar{z}|$$

**Definition 27.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathbb{C}$  and let  $a \in \mathbb{C}$ . We state:  $(a_n)_{n \in \mathbb{N}}$  has a limit (lat. *limes*)  $a$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$$

We denote

$$\lim_{n \rightarrow \infty} a_n = a$$

The distance  $|a_n - a|$  becomes arbitrary small, if  $n$  is sufficiently large.

A sequence, which has a limit, is called *convergent*. A sequence, which does not have a limit, is called *divergent*.



**Remark 7.** Sometimes we consider mappings  $a : \mathbb{N}_+ \rightarrow \mathbb{C}$ , which we also call sequences: Let  $N = \max(N_1, N_2)$ , hence  $N \geq N_1 \wedge N \geq N \geq N_2$ .

$$a \leftrightarrow (a_1, a_2, \dots) \quad \Rightarrow |a_N - a| < \frac{\varepsilon}{2} \wedge |a_N - b| < \frac{\varepsilon}{2}$$

**Example 11.**

$$a_n = \frac{1}{n}$$

We know:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \rightarrow \frac{1}{n} < \varepsilon$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Let  $q \in \mathbb{C}$ ,  $|q| < 1$ .

We know  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \rightarrow |q^n - 0| < \varepsilon$ .

$$\lim_{n \rightarrow \infty} q^n = 0$$

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This lecture took place on 19th of November 2015 with lecturer Wolfgang Ring.

**Remark 8.** Consider  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$  as a circle with radius  $\varepsilon$ . So if  $n$  is sufficiently large, all new sequence elements are located inside the circle.

**Lemma 3.** A sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in \mathbb{C}$  can have at most one limit.

*Proof.* Assume  $a$  and  $b$  are limes of  $(a_n)_{n \in \mathbb{N}}$ . Then we prove:

$$\forall \varepsilon > 0 : |a - b| < \varepsilon$$

$$\Rightarrow a = b$$

Let  $\varepsilon > 0$  arbitrary: Because  $a = \lim_{n \rightarrow \infty} a_n$  there exists

$$N_1 \in \mathbb{N} : [n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}]$$

Because  $b = \lim_{n \rightarrow \infty} b_n$  there exists

$$N_1 \in \mathbb{N} : [n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}]$$

**Theorem 40** (Well-known convergent sequences.).

1. Let  $s = \frac{p}{q} \in \mathbb{Q}_+$  and  $n \in \mathbb{N}_+$ . Consider  $(\frac{1}{n^s})_{n \in \mathbb{N}}$ .

$$n^s = n^{\frac{p}{q}} := \sqrt[q]{n^p}$$

It holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n^s} = 0$$

2. Let  $q \in \mathbb{C}$ ,  $|q| < 1$ . Then it holds that

$$\lim_{n \rightarrow \infty} q^n = 0$$

3. Let  $a \in \mathbb{R}$ ,  $a > 0$ ,  $n \in \mathbb{N}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

4. It holds that ( $n \in \mathbb{N}_+$ )

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

5. Let  $z \in \mathbb{C} : |z| > 1$ . Let  $k \in \mathbb{N}$ . Then it holds that

$$\lim_{n \rightarrow \infty} \frac{n^k}{z^n} = 0$$

**Remark 9** (Remark to sequence 5).  $|z^n|$  grows faster than  $n^k$ .

**Proof of sequence 1.** Let  $0 \leq x_n < x_2$ .

$$\Rightarrow 0 \leq x_1^p < x_2^p \Rightarrow \sqrt[p]{x_1^p} < \sqrt[p]{x_2^p}$$

Therefore  $f(x) = x^s$  is strongly monotonic rising for  $x \in (0, \infty)$ . Let  $\varepsilon > 0$  arbitrary and  $N > \frac{1}{\varepsilon^{\frac{1}{s}}} = \varepsilon^{\frac{1}{s}} = \varepsilon^{-\frac{s}{1}}$ . Then it holds that  $n \geq N$ :

$$\begin{aligned} \left| \frac{1}{n^s} - 0 \right| &= \frac{1}{n^s} \leq \frac{1}{N^s} \\ \frac{1}{n^s} < \frac{1}{N^s} &\Rightarrow n^s \geq N^s \\ \frac{1}{n^s} \leq \frac{1}{N^s} &< \frac{1}{\left(\frac{1}{\varepsilon^{\frac{1}{s}}}\right)^s} = \frac{1}{\varepsilon} = \varepsilon \end{aligned}$$

**Proof of sequence 2.** Already done.

**Proof of sequence 3. Case  $a > 1$**  Let  $a > 1$ . Consider  $\varepsilon > 0$ . Show that  $|\sqrt[a]{a} - 1| < \varepsilon$  for sufficiently large  $n$ .

$$x_n = \sqrt[a]{a} - 1 = |\sqrt[a]{a} - 1|$$

$$a > 1 \Rightarrow \sqrt[a]{a} > \sqrt[a]{1} = 1 \Rightarrow \sqrt[a]{a} - 1 > 0$$

It holds that  $x_n + 1 = \sqrt[a]{a}$ , i.e.  $(x_n + 1)^n = a$ .

$$\begin{aligned} a &= \underbrace{(x_1 + 1)^n}_{>0} \underbrace{>}_{\text{Bernoulli}} 1 + n \cdot x_n \\ &\Rightarrow x_n < \frac{a - 1}{n} \end{aligned}$$

$$\begin{aligned} N &> \frac{a - 1}{\varepsilon} \xrightarrow{\text{for } x \geq N} |\sqrt[a]{a} - 1| = x_n \\ &< \frac{a - 1}{n} \leq \frac{a - 1}{N} < \frac{a - 1}{\frac{a - 1}{\varepsilon}} = \varepsilon \end{aligned}$$

**Case  $a = 1$**

$$\begin{aligned} \sqrt[a]{a} &= \sqrt[a]{1} = 1 \\ (\sqrt[a]{a})_{n \in \mathbb{N}} &= (1, 1, 1, 1, \dots) \end{aligned}$$

has the limit 1.

**Case  $0 < a < 1$**  Let  $0 < a < 1 \Rightarrow 0 < \sqrt[a]{a} < \sqrt[a]{1} = 1$ .

$$x_n = 1 - \sqrt[a]{a} > 0$$

Show that  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow x_n < \varepsilon]$ .

$$x_n = 1 - \sqrt[a]{a} = \sqrt[a]{a} \left( \frac{1}{\sqrt[a]{a}} - 1 \right) = \sqrt[a]{a} \left( \sqrt[n]{\frac{1}{a}} - 1 \right) < \left( \sqrt[a]{a'} - 1 \right)$$

with  $a' = \frac{1}{a} > 1$ . From case  $a > 1$  we already know

$$\begin{aligned} \exists N \in \mathbb{N} : [n \geq N \Rightarrow |\sqrt[a']{a'} - 1| &= \sqrt[a']{a'} - 1 < \varepsilon] \\ &\Rightarrow x_n < \varepsilon \end{aligned}$$

**Proof of sequence 4.** This proof works similar to the proof of sequence 3.

$$x_n = \sqrt[n]{n} - 1 > 0 \text{ for } n \geq 2$$

Therefore  $|x_n| = x_n$ . Let  $\varepsilon > 0$  be arbitrary.

$$x_n + 1 = \sqrt[n]{n} \quad \text{i.e.} \quad (x_n + 1)^n = n$$

$$n = (1 + x_n)^n = 1 + \underbrace{nx_n}_{>0} + \underbrace{\binom{n}{2}x_n^2}_{>0} + \underbrace{\binom{n}{3}x_n^3}_{>0} + \underbrace{\dots + x_n^n}_{>0} > 1 + \binom{n}{2}x_n^2$$

All expressions we remove are positive (but we don't remove all positive expressions).

$$x_n^2 < \frac{n - 1}{\binom{n}{2}} = \frac{n - 1}{\frac{n(n-1)}{2 \cdot 1}} = \frac{2}{n}$$

$$x_n < \sqrt{\frac{2}{n}}$$

Choose  $N > \frac{2}{\varepsilon^2}$ . Then it holds for  $n \geq N$  that

$$x_n < \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} < \sqrt{\frac{2}{\varepsilon^2}} = \varepsilon$$

Consider  $\sqrt{\frac{2}{n}} < \varepsilon$  hence  $\frac{2}{n} < \varepsilon^2$  hence  $n > \frac{2}{\varepsilon^2}$ .

**Proof of sequence 5.**

$|z| > 1$  thus  $x = |z| - 1 > 0$  it holds that  $|z| = 1 + x$

We show that for  $\varepsilon > 0$  arbitrary, there exists  $N \in \mathbb{N}$ :

$$n \geq N \implies \left| \frac{n^k}{z^n} - 0 \right| = \left| \frac{n^k}{z^n} \right| = \frac{n^k}{|z|^n} < \varepsilon$$

Let  $\varepsilon > 0$  be given,

- For  $n > 2k$  it holds that  $n - k > n - \frac{n}{2} = \frac{n}{2}$ .

$$|z|^n = (1+x)^n = \sum_{j=0}^n \binom{n}{j} x^j > \underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1}$$

$$n > 2k \geq k+1$$

$$\underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1} = \frac{\overbrace{n}^{>\frac{n}{2}} \overbrace{(n-1)}^{>\frac{n}{2}} \overbrace{(n-2)}^{>\frac{n}{2}} \dots \overbrace{(n-k)}^{>\frac{n}{2}}}{(k+1)!} x^{k+1} > \frac{\frac{n^{k+1}}{2^{k+1}}}{(k+1)!} x^{k+1}$$

Therefore  $|z|^n > \frac{n^{k+1}}{2^{k+1}(k+1)!} x^{k+1}$ . So,

$$\frac{n^k}{|z|^n} < \frac{n^k \cdot 2^{k+1}(k+1)!}{n^{k+1} \cdot x^{k+1}} = \frac{2^{k+1}(k+1)!}{\underbrace{x^{n+1}}_{\text{constant} \wedge > 0}} \cdot \frac{1}{n} = M \cdot \frac{1}{n}$$

$\underbrace{\quad}_{=: M}$

$$\frac{n^k}{|z|^n} < M \cdot \frac{1}{n} \text{ for } n > 2k$$

Consider  $N$  such that  $N > \frac{M}{\varepsilon}$  and  $N > 2k$ . Then it holds that

$$\frac{n^k}{|z|^n} < M \frac{1}{n} \leq \frac{M}{N} < \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$$

□

□

**Lemma 4.** Every convergent sequence is bounded (in  $\mathbb{C}$ ).

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent. This means especially e.g.  $\varepsilon = 13$ .

$$\exists N \in \mathbb{N} \text{ s.t. } [n \geq N \implies |a_n - a| < 13]$$

Consider  $O > 0$  such that

$$O = \max \{|a_0|, |a_1|, |a_2|, \dots, |a_{N-1}|, |a| + 13\}$$

So  $O \geq |a_n|$  for  $n \in \{0, \dots, N\}$ . Then for  $0 \leq n < N$  it holds that  $|a_n| < O$ . ✓

For  $n \geq N$  it holds that

$$|a_n| = |a_n - a + a| \leq \underbrace{|a_n - a|}_{<13} + |a| < \underbrace{13 + |a|}_{\leq O}$$

Therefore  $(|a_n|)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  and followingly  $(|a_n|)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{C}$ . □

**Theorem 41.** Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then the following laws hold:

1.  $\lim_{n \rightarrow \infty} (a_n + b_n)$  is convergent with limes  $a + b$
2.  $\lim_{n \rightarrow \infty} (a_n \cdot b_n)$  is convergent with limes  $a \cdot b$
3.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  is convergent with limes  $\frac{a}{b}$  if  $\forall n \in \mathbb{N} : b_n \neq 0 \wedge b \neq 0$ .

*Proof.* 1. Let  $\varepsilon > 0$  arbitrary. Because  $(a_n)_{n \in \mathbb{N}}$  is convergent,

$$\exists N_1 : \left[ n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2} \right]$$

$(b_n)$  is convergent hence

$$\exists N_2 : \left[ n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \right]$$

$N = \max\{N_1, N_2\}$ , hence for  $n \geq N$  both statements above hold. Let  $n \geq N$ , then the triangle inequality holds:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{< \frac{\varepsilon}{2}} < \varepsilon$$

2.  $(a_n)_{n \in \mathbb{N}}$  is convergent and therefore also bounded. Therefore,

$$\exists m \geq 0 : \forall n \in \mathbb{N} : |a_n| \leq m$$

$(b_n)_{n \in \mathbb{N}}$  is convergent, hence

$$\exists N_1 : n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \cdot \frac{1}{m+1}$$

$(a_n)_{n \in \mathbb{N}}$  is convergent, hence

$$\exists N_2 \leq N : n \geq N_2 \Rightarrow |a_n - a| < \frac{\varepsilon}{2} \frac{1}{|b|+1}$$

$N = \max\{N_1, N_2\}$ . For  $n \geq N$  both relations above hold. Let  $n \geq N$ :

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| = |a_n| |b_n - b| + |b| |a_n - a| \\ &\leq m \frac{\varepsilon}{2} \frac{1}{m+1} + |b| \frac{\varepsilon}{2} \frac{1}{|b|+1} < \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} \cdot 1 = \varepsilon \end{aligned}$$

3. Left for the practicals.

□

## 9.2 Laws for convergent complex sequences

**Theorem 42.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$ ,  $(a_n \rightarrow a)$ . Then it holds that

- $(\Re(a_n))_{n \in \mathbb{N}}$  is convergent.

$$\lim_{n \rightarrow \infty} (\Re(a_n)) = \Re(a)$$

- $(\Im(a_n))_{n \in \mathbb{N}}$  is convergent.

$$\lim_{n \rightarrow \infty} (\Im(a_n)) = \Im(a)$$

- $(|a_n|)_{n \in \mathbb{N}}$  is a convergent real sequence.

$$\lim_{n \rightarrow \infty} |a_n| = |a|$$

- $(\overline{a_n})_{n \in \mathbb{N}}$  is convergent with

$$\lim_{n \rightarrow \infty} \overline{a_n} = \overline{a}$$

On the opposite, let  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = \alpha_n + i\beta_n$  a sequence of complex numbers. Let  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  be convergent with limes  $\alpha$  i.e.  $\beta$ . Then  $(a_n)_{n \in \mathbb{N}}$  is a convergent complex sequence with limes  $a = \alpha + \beta i$ .

*Proof.* Let  $\varepsilon > 0$ . Consider  $N$  such that  $n \geq N \Rightarrow |a_n - a| < \varepsilon$ .

$$\underbrace{|a_n - a|}_{(\alpha_n - \alpha) + (\beta_n - \beta)i} = \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2}$$

TODO

Therefore  $(\alpha_n) = (\Re(a_n))_{n \in \mathbb{N}}$  is convergent.  $(\beta_n) = (\Im(a_n))_{n \in \mathbb{N}}$  is convergent.

Let  $\varepsilon > 0$ . Consider  $N$  such that  $n \geq N \Rightarrow |a_n - a| < \varepsilon$ .

$$||a_n| - |a|| \leq |a_n - a| < \varepsilon \text{ for } n \geq N$$

inverse triangular inequality

Now we need to show  $\alpha_n \rightarrow \alpha$  and  $\beta_n \rightarrow \beta$

$$\Rightarrow a_n \rightarrow a$$

Let  $\varepsilon > 0$  be arbitrary. Because  $(\alpha_n)_{n \in \mathbb{N}}$  be convergent, there exists  $N_1 \in \mathbb{N}$ :

$$n \geq N_1 \Rightarrow |\alpha_n - \alpha| < \frac{\varepsilon}{\sqrt{2}}$$

$(\beta_n)_{n \in \mathbb{N}}$  is convergent. So,

$$\exists N_2 \in \mathbb{N} : n \geq N_2$$

$$|\beta_n - \beta| < \frac{\varepsilon}{\sqrt{2}}$$

For  $N = \max\{N_1, N_2\}$  and  $n \geq N$  both relations hold.

Let  $n \geq N$ :

$$\begin{aligned} |a_n - a| &= |(\alpha_n - \alpha) + i(\beta_n - \beta)| \\ &= \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon \end{aligned}$$

Let  $a_n = \alpha_n + i\beta_n$  is convergent with limes  $\alpha + i\beta$  which is  $a$ .

$$\begin{aligned} &\Rightarrow \lim_{n \rightarrow \infty} \alpha_n = \alpha \wedge \lim_{n \rightarrow \infty} \beta_n = \beta \\ &\Rightarrow \lim_{n \rightarrow \infty} (-\beta_n) = -\beta \quad \text{“multiplication rule”} \\ &\Rightarrow (\overline{a_n})_{n \in \mathbb{N}} = \left( \underbrace{\alpha_n}_{\text{convergent}} - \underbrace{i\beta_n}_{\text{convergent}} \right)_{n \in \mathbb{N}} \\ &\Rightarrow \lim_{n \rightarrow \infty} \overline{a_n} = \alpha - i\beta = \overline{a} \end{aligned}$$

*Proof.* Consider  $a - b = \varepsilon > 0$ .

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$$

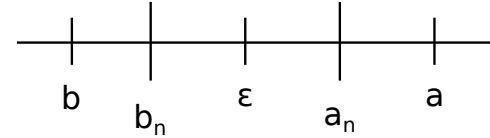


Figure 15: the sequences  $a_n, b_n$  and limes  $a, b$  and  $\varepsilon$  in relation

For  $N = \max\{N_1, N_2\}$ :

$$\begin{aligned} b_N &= b_N - b + b \leq b + |b_N - b| < b + \frac{\varepsilon}{2} = b + \frac{a - b}{2} = \frac{1}{2}(a + b) \\ a_N &= \underbrace{a_N - a}_{\geq -|a_n - a|} + a \geq a - |a_n - a| > a - \frac{\varepsilon}{2} = a - \frac{a - b}{2} = \frac{1}{2}(a + b) \\ b_N &< \frac{1}{2}(a + b) < a_N \end{aligned}$$

Attention:

$$a_n < b_n \not\Rightarrow a < b$$

□ Example:  $a_n = 0, b_n = \frac{1}{n}$ . □

### 9.3 Further laws for sequences

**Theorem 43.** Let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be convergent in  $\mathbb{R}$  with limes  $a$  (i.e.  $b$ ) and it must hold that  $\forall n \in \mathbb{N} : a_n \leq b_n$ . Then also  $a \leq b$ .

### 9.4 Convergence criteria

Are there criteria such that if the sequences have a specific structure, they are obviously convergent?

## 9.4.1 Squeeze theorem

**Theorem 44.** Let  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$  be convergent real sequences with  $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n = A$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence and  $M \in \mathbb{N}$  such that

$$\forall n \geq M : A_n \leq a_n \leq B_n$$

Then it holds that  $(a_n)_{n \in \mathbb{N}}$  is also convergent and  $\lim a_n = A$ .

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Consider  $N$  such that,

- $N \geq M$
- $n \geq N \Rightarrow |A_n - A| < \varepsilon$
- $n \geq N \Rightarrow |B_n - A| < \varepsilon$

Then it holds that for  $n \geq N$ :

$$\left. \begin{array}{l} A - a_n \leq A - A_n \leq |A - A_n| < \varepsilon \\ a_n - A \leq B_n - A \leq |B_n - A| < \varepsilon \end{array} \right\} = 1$$

$$\Rightarrow |a_n - A| < \varepsilon$$

$$\lim_{n \rightarrow \infty} a_n = A$$

**Example 12.** Let  $s \in \mathbb{Q}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \left( \sqrt[n]{n^s} \right) = 1$$

We apply the squeeze theorem:

$$n^2 \geq 1 \forall n \in \mathbb{N}$$

$$\Rightarrow \sqrt[n]{n^s} \geq 1$$

Let  $k \in \mathbb{N}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \sqrt[n]{n^k} = \lim_{n \rightarrow \infty} \underbrace{\sqrt[n]{n} \sqrt[n]{n} \dots \sqrt[n]{n}}_{k \text{ times}}$$

$$= 1 \cdot 1 \cdot 1 \dots = 1$$

For the last two lines we actually need to read them from right to left.

Let  $s = \frac{p}{q}$ .

$$\Rightarrow n^s = n^{\frac{p}{q}} \leq q \cdot \left( n^{\frac{p}{q}} \right)^q = n^p$$

$$q \geq 1 \Rightarrow \sqrt[q]{n^s} \leq \underbrace{\sqrt[q]{n^p}}_{\text{convergent with limes 1}} \quad p \in \mathbb{N}$$

Then it holds that  $\lim_{n \rightarrow \infty} \sqrt[q]{n^s} = 1$  with the squeezing theorem.

**Remark 10.** Let  $A \subseteq \mathbb{R}$  be bounded above. Then it holds that

$$S = \sup A \Leftrightarrow s \text{ is upper bound of } A \wedge \forall \varepsilon > 0 \exists a \in A : a > s - \varepsilon$$

*Proof.* Implication from left to right: Let  $s = \sup A$ . Then it holds that  $s$  is upper bound of  $A$  and  $s - \varepsilon < s$  is not an upper bound. Therefore  $\exists a \in A : a > s - \varepsilon$ .

Implication from right to left: Consider that both statements on the RHS hold. So  $s$  is an upper bound. We need to show that any  $t$  is not an upper bound with  $t > s$ . Let  $t < s, s - t = \varepsilon > 0$ . Therefore  $t = s - \varepsilon$ . Because of the right statement  $\exists a \in A : a > s - \varepsilon = t$  therefore  $t$  is not an upper bound.  $\square$

$\square$  **Remark 11.** Analogously:

$$\sigma = \inf A \Leftrightarrow \sigma \text{ is lower bound} \wedge \forall \varepsilon > 0 \exists a \in A : a < \sigma + \varepsilon$$

**Theorem 45.** Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded monotonic sequence. Then  $(a_n)_{n \in \mathbb{N}}$  has a limes  $a$  with

- $a = \sup \{a_n : n \in \mathbb{N}\}$  if  $(a_n)_{n \in \mathbb{N}}$  is monotonically increasing.
- $a = \inf \{a_n : n \in \mathbb{N}\}$  if  $(a_n)_{n \in \mathbb{N}}$  is monotonically decreasing.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be monotonically increasing. Let  $a = \sup \{a_n : n \in \mathbb{N}\}$ . Let  $\varepsilon > 0$  be arbitrary. Because  $a$  is a supremum, there exists  $a_N \in \{a_n : n \in \mathbb{N}\}$  such that  $a_N > a - \varepsilon$ .

$$\Rightarrow \underbrace{a - a_N}_{\geq 0} < \varepsilon$$

because  $a$  is an upper bound. Therefore

$$|a - a_N| < \varepsilon$$

Let  $n \geq N$  then it holds that

$$|a - a_n| \underbrace{=}_{a \text{ is upper bound}} a - a_n \leq a - a_N$$

because  $a_N \leq a_n$  is increasing:

$$a - a_N < \varepsilon$$

Therefore  $\lim_{n \rightarrow \infty} a_n = a$ .  $\square$

This lecture took place on 25th of November 2015 with lecturer Wolfgang Ring.

Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence. If  $(a_n)_{n \in \mathbb{N}}$  is bounded and monotonous. Then  $(a_n)_{n \in \mathbb{N}} \in \mathbb{N}$  is convergent.

**Example: Wallis product** John Wallis (1616–1703)

$$p_n = \frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} = \prod_{k=1}^n \frac{2k}{2k-1}$$

Consider

$$\alpha_n = \frac{p_n}{\sqrt{n}} \quad \beta_n = \frac{p_n}{\sqrt{n+1}}$$

We need to show that

- $(\alpha_n)$  is monotonously decreasing
- $(\beta_n)$  is monotonously increasing

$$\forall n \in \mathbb{N} : n \geq 1 : \alpha_n > \beta_n$$

Both are convergent.

1. Show that,

$$\alpha_{n+1} < \alpha_n \Leftrightarrow \frac{\alpha_{n+1}}{\alpha_n} < 1 \Leftrightarrow \frac{(\alpha_{n+1})^2}{(\alpha_n)^2} < 1$$

$$\begin{aligned} \left( \frac{\alpha_{n+1}}{\alpha_n} \right)^2 &= \left( \frac{\frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n+2)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \cdot \frac{1}{\sqrt{n+1}}}{\frac{2 \cdot 4 \cdot \dots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} \cdot \frac{1}{\sqrt{n+1}}} \right)^2 \\ &= \frac{(2n+2)^2 \cdot n}{(2n+1)^2 (n+1)} = \frac{4n^3 + 8n^2 + 4n}{(4n^2 + 4n + 1) \cdot (n+1)} = \frac{4n^3 + 8n^2 + 4n}{4n^3 + 8n^2 + 5n + 1} < 1 \end{aligned}$$

2. We show,

$$\begin{aligned} \left( \frac{\beta_{n+1}}{\beta_n} \right)^2 &= \frac{(2n+2)^2 \cdot (n+1)}{(2n+1)^2 \cdot (n+2)} = \frac{(4n^2 + 8n + 4)(n+1)}{(4n^2 + 2n + 1)(n+2)} \\ &= \frac{4n^3 + 12n^2 + 12n + 4}{4n^3 + 12n^2 + 9n + 2} > 1 \Rightarrow \beta_{n+1} > \beta_n \Rightarrow \beta_n \text{ is monotonically increasing} \end{aligned}$$

Let  $p = \lim_{n \rightarrow \infty} a_n$  and  $p' = \lim_{n \rightarrow \infty} b_n$ .

$$\begin{aligned} \beta_n &= \frac{p_n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \alpha_n \cdot \sqrt{\frac{n}{n+1}} \\ \lim_{n \rightarrow \infty} \beta_n &= \lim_{n \rightarrow \infty} \alpha_n \sqrt{\frac{n}{n+1}} = \lim_{n \rightarrow \infty} \alpha_n \cdot \underbrace{\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}}}_{=1} \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} a_n \Rightarrow p = p'$$

It holds that  $p = \lim_{n \rightarrow \infty} \frac{p_n}{\sqrt{n}} = \sqrt{n}$ .

## 9.5 On limit points and subsequences

**Definition 28.** Let  $(a_n)_{n \in \mathbb{N}}$  be a complex sequence. The complex value  $a$  is called *limit point* (german “Häufungspunkt”) of  $(a_n)_{n \in \mathbb{N}}$  if  $\forall \varepsilon > 0 : |a_n - a| < \varepsilon$  for infinitely many indices  $n \in \mathbb{N}$ . Hence infinitely many values of the sequence lie within a circle with center  $a$  and radius  $\varepsilon$ .

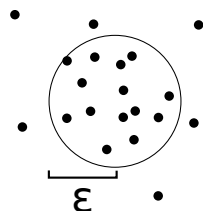


Figure 16: Illustration of a limit point in the Euclidean plane. The point is represented as circle with radius  $\varepsilon$ . Finitely many points lie outside the limit point; infinitely many inside.

**Remark 12.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limit  $a$ . Then it holds that  $a$  is the only limit point of the sequence  $(a_n)_{n \in \mathbb{N}}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent. Let

$$\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a|$$

Therefore  $\forall n \in \{N, N+1, N+2, \dots\}$  it holds that  $|a_n - a| < \varepsilon$ . Assume  $a' \in \mathbb{C}$  is another limit point with  $a \neq a'$ . Let

$$\varepsilon = \frac{|a - a'|}{2} > 0$$

Let  $N \in \mathbb{N}$  such that  $\forall n \geq N : |a_n - a| < \varepsilon$ .

$$\Rightarrow n \in \mathbb{N} : |a' - a_n| = |a' - a + a - a_n| = |a' - a - (a_n - a)| \geq |a' - a| - |a_n - a|$$

$$= 2\varepsilon - |a_n - a| > 2\varepsilon - \varepsilon = \varepsilon$$

At most for  $n \in \{1, \dots, N-1\}$  it is possible that  $|a_n - a'| < \varepsilon$ . □

**Remark 13.**  $a_n = (-1)^n$  has the limit points  $+1$  and  $-1$ .

The lecture on 26th of November 2015 got cancelled.

This lecture took place on 27th of November 2015 with lecturer Wolfgang Ring.

**Definition 29.** Let  $a \in \mathbb{C}$  and  $r > 0$  and

$$B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

and we call  $B(a, r)$  an *open* circle with center  $a$  and radius  $r$ . So the circle itself is not part of the set, unlike the following set:

$$B'(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}$$

Let  $a$  be a limit point of  $(a_n)_{n \in \mathbb{N}} \Leftrightarrow \forall \varepsilon > 0. B(a, \varepsilon)$  contains infinitely many sequence values.

**Example 13.**

$$a_n = \frac{1}{2} \left[ 1 + (-1)^n \left( \frac{1-n}{n} \right) \right] \quad n \geq 1$$

$$\Rightarrow a_1 = \frac{1}{2} \quad a_2 = \frac{1}{4} \quad a_3 = \frac{5}{6}$$

$$a_4 = \frac{1}{8} \quad a_5 = \frac{9}{10} \quad a_6 = \frac{1}{12} \quad a_7 = \frac{13}{14}$$

“ $\frac{5}{6}$ ? Ah, passt ma eh besa.” (Wolfgang Ring)

Estimated limit points:  $a = 0, b = 1$ .

*Proof.* Let  $\varepsilon > 0$  and  $a = 0$ . We consider sequence values with even index. So for indices it holds that  $n = 2k$ .

$$\begin{aligned} |a_{2k} - 0| &= \left| \frac{1}{2} \left( 1 + \underbrace{(-1)^{2k}}_{+1} \left( \frac{1-2k}{2k} \right) \right) \right| \\ &= \frac{1}{2} \left| 1 + \frac{1-2k}{2k} \right| \\ &= \frac{1}{2} \left| \frac{2k+1-2k}{2k} \right| \\ &= \frac{1}{4k} < \varepsilon \text{ if } \underbrace{k > \frac{1}{4\varepsilon}}_{\text{infinitely many ks satisfy the relation}} \end{aligned}$$



Let  $\varepsilon > 0$  and  $b = 1$ . We consider sequence values of structure  $n = 2k + 1$ .

$$\begin{aligned}
 |a_{2k+1} - 1| &= \left| \frac{1}{2} \left[ 1 + \underbrace{(-1)^{2k+1}}_{=-1} \left[ \frac{1 - (2k+1)}{2k+1} \right] \right] - 1 \right| \\
 &= \left| \frac{1}{2} \left[ 1 - \frac{2k}{2k+1} \right] - 1 \right| \\
 &= \left| \frac{1}{2} \frac{2k+1+2k}{2k+1} - 1 \right| \\
 &= \left| \frac{4k+1}{4k+2} - 1 \right| \\
 &= \left| \frac{4k+1-4k-2}{4k+2} \right| \\
 &= \frac{1}{4k+2} \\
 &< \varepsilon
 \end{aligned}$$

$$\text{if } 4k+2 > \frac{1}{\varepsilon} \Rightarrow \underbrace{k}_{\text{infinitely many indices}} > \frac{1}{4} \left( \frac{1}{\varepsilon} - 2 \right)$$

**Example 14.**  $(c_n)_{n \in \mathbb{N}}$  is defined with  $c_n = i^n$ .

$$(c_n)_{n \in \mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, \dots)$$

What are its limit points?

**Definition 30.** Let  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in \mathbb{C}$ . For example,

$$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$$

We remove some elements

$$(1, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \dots)$$

A *subsequence* is created. We also reenumerate the numbers:

$$(\underbrace{1}_{n_0}, \underbrace{\frac{1}{3}}_{n_1}, \underbrace{\frac{1}{4}}_{n_2}, \underbrace{\frac{1}{6}}_{n_3}, \dots)$$

Let  $n : \mathbb{N} \rightarrow \mathbb{N}$  be strictly monotonically increasing. Therefore

$$\forall k \in \mathbb{N} : n(k+1) > n(k) \Rightarrow n_{k+1} > n_k$$

We call  $(n_k)_{k \in \mathbb{N}}$  an *index subsequence* and  $(a_{n_k})_{k \in \mathbb{N}}$  is called subsequence of  $(a_n)_{n \in \mathbb{N}}$ .

**Lemma 5.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$  and  $(a_{n_k})_{k \in \mathbb{N}}$  a subsequence of  $(a_n)_{n \in \mathbb{N}}$ . Then also the subsequence is convergent and has the same limes  $a$ .

*Proof.* For every subsequence index  $n_k$  with  $k \in \mathbb{N}$  it holds that  $n_k \geq k$ .

Proof by induction:  $k = 0$

$$n_0 \in \mathbb{N}$$

$$n_0 \geq 0 = k \quad \checkmark$$

$n_k \geq k$  Because  $\underbrace{n_{k+1}}_{\in \mathbb{N}} > n_k$  (strictly monotonic). Therefore,

□

$$n_{k+1} \geq n_k + 1 > k + 1$$

Proof of limes:  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . Let  $\varepsilon > 0$ . Because  $(a_n)_{n \in \mathbb{N}}$  is convergent, it holds that  $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \varepsilon$ . Let  $k \geq N$ . This holds because  $n_k \geq k \geq N : |a_{n_k} - a| < \varepsilon$ . Therefore  $(a_{n_k})_{k \in \mathbb{N}}$  has limes  $a$ . □

**Lemma 6.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$ . Then it holds that  $a \in \mathbb{C}$  is limit point if and only if there exists some subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} a_{n_k} = a$ .

*Proof.* We first prove direction  $\Leftarrow$ .

Assume  $(a_{n_k})_{k \in \mathbb{N}}$  is a convergent subsequence of  $(a_n)_{n \in \mathbb{N}}$  with limes  $a$ . Let  $\varepsilon > 0$ .

$$\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_{n_k} - a| < \varepsilon$$

Therefore  $B(a, \varepsilon)$  has infinitely many sequence elements of  $(a_{n_k})_{k \in \mathbb{N}}$  and therefore also infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ .

We prove direction  $\Rightarrow$ .

We build a convergent subsequence. Consider  $k \in \mathbb{N}$  with  $k \geq 1$ .

$$\varepsilon_k = \frac{1}{k}$$

We define  $n_0 = 0$  and  $a_{n_0} = a_0$ . Assume  $a_{n_0}, a_{n_1}, \dots, a_{n_{k-1}}$  are already defined.

Definition of  $a_{n_k}$ : In  $B(a, \varepsilon_k)$  there are infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ . We consider  $n_k > n_{k-1}$  and  $a_{n_k} \in B(a, \varepsilon_k)$ .

Then it holds that  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . Let  $\varepsilon > 0$  be arbitrary. Consider  $K > \frac{1}{\varepsilon}$ . Hence  $\varepsilon > \frac{1}{K} = \varepsilon_K$  for all  $k \geq K$  it holds that  $n_k \geq n_K$  and  $|a_{n_k} - a| < \varepsilon_k = \frac{1}{k} \leq \frac{1}{K} < \varepsilon$ .  $\square$

## 9.6 Bolzano-Weierstrass theorem

Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897)

**Theorem 46.** Every bounded sequence of real numbers has a limit point in  $\mathbb{R}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathbb{R}$ , hence  $\exists M > 0$  such that all sequence elements  $a_n$  in  $I_0 = [-M, M]$  and let  $F_0 = \{n \in \mathbb{N} \mid a_n \in I_0\} = \mathbb{N}$  (index set).  $F_0$  is infinite. We build nested intervals with the properties:

- $I_{n+1} \subseteq I_n$
- $|I_{n+1}| = \frac{1}{2} |I_n|$
- $F_n = \{k \in \mathbb{N} \mid a_k \in I_n\}$  is infinite.

This construction is inductive:

**induction base**  $I_0$  ✓

**induction step** Let  $I_n = [A_n, B_n]$  be given and  $M_n = \frac{1}{2}(A_n + B_n)$ . Let  $J_n = [A_n, M_n]$  and  $L_n = [M_n, B_n]$ . It holds that  $J_n \subseteq I_n \wedge L_n \subseteq I_n$  and

$|J_n| = \frac{1}{2} |I_n| \wedge |L_n| = \frac{1}{2} |I_n|$ . Because there are infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$  in  $I_n$  and  $I_n = J_n \cup L_n$ , in at least one subinterval there have to be infinitely many sequence elements.

Therefore select  $I_{n+1} = J_n$  if  $J_n$  contains infinitely many sequence elements and consider  $I_{n+1} = L_n$  if  $J_n$  contains only finitely many sequence elements. Therefore  $I_{n+1}$  contains infinitely many sequence elements.

$$F_{n+1} = \{k \in \mathbb{N} \mid a_k \in I_{n+1}\}$$

is infinite. So  $(I_n)_{n \in \mathbb{N}}$  is a nested interval.

Let  $a \in \bigcap_{n \in \mathbb{N}} I_n$  (completeness of  $\mathbb{R}$ ).

Claim:  $a$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Let  $\varepsilon > 0$  be given and  $n$  sufficiently large, such that  $|I_n| = B_n - A_n < \varepsilon$ . Then it holds that for every  $x \in I_n$  that  $|x - a| \leq B_n - A_n < \varepsilon$  (with  $x \in I_n, a \in I_n$ ). Because  $I_n$  contains infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ , it holds that infinitely many sequence elements  $a_k$  satisfy the relation  $|a_n - a| < \varepsilon$ . Therefore  $a$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .  $\square$

**Corollary 8** (typical definition of the Bolzano-Weierstrass theorem). Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

**Theorem 47** (Bolzano-Weierstrass theorem in  $\mathbb{C}$ ). Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathbb{C}$ . Then  $(a_n)_{n \in \mathbb{N}}$  has a convergent subsequence and therefore also at least one limit point in  $\mathbb{C}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be bounded.  $a_n = \alpha_n + i\beta_n$ . So  $(\alpha_n)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  as well as  $(\beta_n)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$ .

Consider a convergent subsequence of  $(\alpha_n)_{n \in \mathbb{N}}$ ,  $(\alpha_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} \alpha_{n_k} = \alpha$ . Now consider bounded  $(\beta_{n_k})_{k \in \mathbb{N}}$ . From the Bolzano-Weierstrass theorem it follows that there exists a convergent subsequence  $(\beta_{n_{k_l}})_{l \in \mathbb{N}}$  with  $\beta = \lim_{l \rightarrow \infty} \beta_{n_{k_l}}$ .

$(\alpha_{n_{k_l}})_{l \in \mathbb{N}}$  is subsequence of  $(\alpha_{n_k})_{k \in \mathbb{N}}$  convergent with limit point  $\alpha$ .

Let  $a_{n_{k_l}} = \alpha_{n_{k_l}} + i\beta_{n_{k_l}}$  be a subsequence of  $(a_n)_{n \in \mathbb{N}}$ .

Real and imaginary parts are convergent, therefore  $\lim_{l \rightarrow \infty} a_{n_{k_l}} = a = \alpha + i\beta$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  contains a convergent subsequence.  $\square$

This lecture took place on 2nd of December 2015 with lecturer Wolfgang Ring.

**Theorem 48** (Weierstrass-Bolzano theorem). Every bounded sequence in  $\mathbb{C}$  has a convergent subsequence.

**Theorem 49** (Convergence). Let  $(x_n)_{n \in \mathbb{N}}$  be convergent in  $\mathbb{C}$  with limes  $x$ .

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : |x_n - x| < \varepsilon$$

**Definition 31** (Metric space). Let  $X$  be a set. We call  $d : X \times X \rightarrow \mathbb{R}$  a *distance function* (or *metric*) on  $X$  if,

- $\forall x \in X : d(x, x) = 0$
- $\forall x, y \in X : d(x, y) = d(y, x)$  (symmetry)
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called *metric space*.

**Example 15.**  $X = \mathbb{C}$ ,  $d(x, y) = |x - y|$ .

**Definition 32** (Convergence with metric spaces). Let  $X$  be a metric space.  $(x_n)_{n \in \mathbb{N}}$  is a sequence of elements in  $X$ . Let  $x \in X$ . We call  $(x_n)_{n \in \mathbb{N}}$  convergent with limes  $x$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : d(x_n, x) < \varepsilon$$

**Definition 33.** Let  $K \subseteq X$  be a subset of the metrical space  $X$ . We call  $K$  *pre-compact* if every sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in K$  has a convergent subsequence.  $K$  is called *compact* if the limes  $a$  of the convergent subsequence is also in  $K$ .

**Definition 34.** In  $\mathbb{C}$  it holds that every bounded set is pre-compact.

## 9.7 Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$

Augustin-Louis Cauchy (1789–1857)

**Definition 35.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$ . We call  $(a_n)_{n \in \mathbb{N}}$  a *Cauchy sequence* (fundamental sequence) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow |a_n - a_m| < \varepsilon$$

**Definition 36** (Cauchy sequence in a metric space). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . We call  $(a_n)_{n \in \mathbb{N}}$  a *Cauchy sequence* (fundamental sequence) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow d(a_n, a_m) < \varepsilon$$

**Lemma 7.** Every convergent sequence  $(a_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  is a Cauchy sequence.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$ . Let  $\varepsilon > 0$  be arbitrary.

Convergence implies that  $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$ . For  $m, n \geq N$  it holds that

$$|a_n - a_m| = |a_n - a + a - a_m| \leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2} \text{ because } n \geq N} + \underbrace{|a - a_m|}_{< \frac{\varepsilon}{2} \text{ because } m \geq N}$$

□

**Lemma 8.** Every Cauchy sequence  $(a_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  is bounded.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ . The Cauchy condition for  $\varepsilon = 1$  states:

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |a_n - a_m| < 1$$

specifically  $m = N : \forall n \geq N$

$$|a_n - a_N| < 1$$

Therefore  $|a_n| = |a_n - a_N + a_N| \leq \underbrace{|a_n - a_N|}_{< 1} + |a_N| < |a_N| + 1$ .

Let  $m = \max\{|a_0|, |a_1|, \dots, |a_{N-1}|\}$  and  $M = \max\{m, |a_N| + 1\}$ .

Then for  $n \leq N - 1$  it holds that

$$|a_n| \leq m \leq M$$

and for  $n \geq N$  it holds that

$$|a_n| \leq |a_N| + 1 \leq M$$

Therefore  $\forall n \in \mathbb{N} : |a_n| \leq M$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  is bounded.

□

## 9.8 Is $\mathbb{C}$ , $\mathbb{R}$ and $\mathbb{Q}$ complete?

**Theorem 50** (Cauchy sequences and limes). Every Cauchy sequence in  $\mathbb{C}$  has a limes and is therefore convergent. Followingly we call  $\mathbb{C}$  to be *complete*.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ . We know that  $(a_n)_{n \in \mathbb{N}}$  is bounded. From the Bolzano-Weierstrass theorem it follows that a limit point  $a$  of  $(a_n)_{n \in \mathbb{N}}$  exists. Let  $\varepsilon > 0$  be arbitrary.

1. We choose  $N \leq \mathbb{N}$  sufficiently large such that

$$n, m \geq N \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2}$$

2. Because  $B(a, \frac{\varepsilon}{2})$  contains infinitely many sequence elements ( $a$  is limit point),  $K \geq N$  exists with  $|a - a_K| < \frac{\varepsilon}{2}$ .

Let  $n \geq N$ . Then

$$|a_n - a| = |a_n - a_K + a_K - a| \leq \underbrace{|a_n - a_K|}_{< \frac{\varepsilon}{2} \text{ (Cauchy seq.)}} + \underbrace{|a_K - a|}_{< \frac{\varepsilon}{2} \text{ (limit point } a)}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore  $(a_n)_{n \in \mathbb{N}}$  is convergent with limes  $a$ .

We have proven that if  $(a_n)_{n \in \mathbb{N}}$  has a limit point, this limit point is also its limes.

We concluded: nested intervals  $\Rightarrow$  compactness / Bolzano-Weierstrass theorem  $\Rightarrow$  completeness.

Actually nested intervals are equivalent to completeness.  $\square$

This lecture took place on 3rd of December 2015 with lecturer Wolfgang Ring.

**Corollary 9.**  $\mathbb{C}$  is complete.

*Proof.* Let  $(z_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ .

$$z_n = a_n + ib_n$$

Then  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are Cauchy sequences in  $\mathbb{R}$ .

Show that this property: Let  $\varepsilon > 0$ . Because  $(z_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, it holds that

$$\exists N \in \mathbb{N} : n, m \geq N \Rightarrow |z_n - z_m| < \varepsilon$$

Because  $|a_n - a_m| \leq |z_n - z_m|$  and  $|b_n - b_m| \leq |z_n - z_m|$  hold, it follows that for  $n, m \geq N : |a_n - a_m| < \varepsilon \wedge |b_n - b_m| < \varepsilon$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are Cauchy sequences.

Because  $\mathbb{R}$  is complete, it follows that  $\exists a \in \mathbb{R}$  such that

$$a = \lim_{n \rightarrow \infty} a_n \text{ and } \exists b \in \mathbb{R}$$

with  $b = \lim_{n \rightarrow \infty} b_n$ . Because  $\lim_{n \rightarrow \infty} z_n = z = a + ib$ ,

$$\Leftrightarrow a = \lim_{n \rightarrow \infty} a_n \wedge b = \lim_{n \rightarrow \infty} b_n$$

$\square$

**Example 16.** We show a counterexample for the completeness of  $\mathbb{Q}$ . So we have Cauchy sequences with limes, which lies outside  $\mathbb{Q}$ .

We define a recursion:

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) & \text{if } n > 0 \end{cases}$$

We observe,  $\forall n \in \mathbb{N} : a_n > 0 \wedge a_n \in \mathbb{Q}$ .

Proof by complete induction:

**Induction base:**  $n = 0$

$$a_0 = 2 > 0 \wedge 2 \in \mathbb{Q} \quad \checkmark$$

**Induction step:**  $n \rightarrow n + 1$  Let  $a_n > 0$  and  $a_n \in \mathbb{Q}$ .

$$a_{n+1} = \frac{1}{2} \left( \underbrace{a_n}_{>0} + \underbrace{\frac{2}{a_n}}_{>0} \right) > 0$$

and  $a_{n+1} \in \mathbb{Q}$ .

We prove by induction:  $\forall n \in \mathbb{N} : a_n^2 > 2$ .

**Induction base:**  $n = 0$

$$a_0 = 2 \quad a_0^2 = 4 > 2 \quad \checkmark$$

**Induction step:**  $n \rightarrow n + 1$  It holds that  $a_n^2 - 2 > 0$ .

$$\begin{aligned} a_{n+1}^2 - 2 &= \frac{1}{4} \left( a_n^2 + 4 + \frac{4}{a_n^2} \right) - 2 = \frac{1}{4a_n^2} (a_n^4 + 4a_n^2 + 4 - 8a_n^2) \\ &= \frac{1}{4a_n^2} (a_n^4 - 4a_n^2 + 4) = \frac{1}{4a_n^2} \underbrace{(a_n^2 - 2)^2}_{>0} > 0 \end{aligned}$$

Furthermore it holds that  $a_{n+1} < a_n$ .

$$\begin{aligned} 2a_{n+1} &= a_n + \frac{2}{a_n} \Rightarrow 2(a_{n+1} - a_n) = -a_n + \frac{2}{a_n} = \frac{2 - a_n^2}{a_n} \stackrel{<0}{<} 0 \\ &\Rightarrow a_{n+1} - a_n < 0 \Rightarrow a_{n+1} < a_n \end{aligned}$$

Therefore the sequence  $(a_n)_{n \in \mathbb{N}}$  is strictly monotonically decreasing and is bound by below. Therefore some  $a \in \mathbb{R}$  exists with  $a = \lim_{n \rightarrow \infty} a_n$ .

Monotonicity really depends on the completeness of  $\mathbb{R}$ . We cannot argue equivalently to Theorem 45 with the supremum.

For this example we know that  $(a_n)_{n \in \mathbb{N}}$  is convergent in  $\mathbb{R}$ .  $(a_n)_{n \in \mathbb{N}}$  is Cauchy sequence in  $\mathbb{R}$ . So  $(a_n)_{n \in \mathbb{N}}$  is Cauchy sequence in  $\mathbb{Q}$ .

For the limes  $a$  it holds that,

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} a_n + \frac{1}{2} \frac{2}{\lim_{n \rightarrow \infty} a_n} = \frac{1}{2} a + \frac{1}{a} \\ a &= \frac{1}{2} a + \frac{1}{a} \Rightarrow \frac{1}{2} a = \frac{1}{a} \end{aligned}$$

$$a^2 = 2 \Rightarrow a = +\sqrt{2} \notin \mathbb{Q}$$

Therefore  $(a_n)_{n \in \mathbb{N}}$  is *not* convergent in  $\mathbb{Q}$ . We found a convergent Cauchy sequence whose limes is not in  $\mathbb{Q}$  which immediately means that  $\mathbb{Q}$  is incomplete.

**Definition 37** (Tending towards infinity). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers.

- We state  $(a_n)_{n \in \mathbb{N}}$  *tends to infinity* with limes  $+\infty$ :

$$\lim_{n \rightarrow \infty} a_n = +\infty$$

$$\text{if } \forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n > M$$

- We state  $(a_n)_{n \in \mathbb{N}}$  *tends to negative infinity* with limes  $-\infty$ :

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

$$\forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n < -M$$

**Example 17.**

$$a_n = \frac{n^2 + 2}{n + 1}$$

has limes  $+\infty$ . The proof is given in the practicals. We show that ...

$$\frac{n^2 + 2}{n + 1} > M \Leftrightarrow \dots$$

**Definition 38** (Limes superior, Limes inferior). Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence which is bounded above and

$$H = \{ \xi \in \mathbb{R} \mid \xi \text{ is limit point of } (a_n)_{n \in \mathbb{N}} \} \neq \emptyset$$

Then  $H$  is also bounded by above and we call  $S^* = \sup H$  a *limes superior* of the sequence  $(a_n)_{n \in \mathbb{N}}$ . We denote:

$$S^* = \limsup_{n \rightarrow \infty} a_n$$

Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence which is bounded below and

$$H = \{\xi \in \mathbb{R} \mid \xi \text{ is limit point of } (a_n)_{n \in \mathbb{N}}\} \neq \emptyset$$

Then  $H$  is also bounded by below and we call  $S^* = \inf H$  a *limes inferior* of the sequence  $(a_n)_{n \in \mathbb{N}}$ . We denote:

$$S_* = \liminf_{n \rightarrow \infty} a_n$$

**Theorem 51.** If  $(a_n)_{n \in \mathbb{N}}$  is bounded by above by  $M$ ,  $H \neq \emptyset$ , then  $M$  is also an upper bound of  $H$ .

*Proof.* Assume  $\exists s \in H$  with  $s > M$ . Choose  $\varepsilon = s - M > 0$ . Because  $S$  is a limit point of  $(a_n)_{n \in \mathbb{N}}$  it holds that  $(s - \varepsilon, s + \varepsilon)$  contains infinitely many sequence elements. So for infinitely many indices  $n$  it holds that,

$$a_n > s - \varepsilon = s - (s - M) = M$$

This contradicts with  $M$  being the upper bound of the sequence. □

**Lemma 9.** Let  $(a_n)_{n \in \mathbb{N}}$  be bounded by above.  $a_n \in \mathbb{R}$ . Let  $H \neq \emptyset$  be defined as above. Then it holds that

$$S^* = \limsup_{n \rightarrow \infty} (a_n) = \max H$$

ie.  $S^*$  is a limit point itself of the sequence.

*Proof.* Show that  $S^*$  itself is a limit point of the sequence. Let  $\varepsilon > 0$ : Choose  $\xi \in H$  such that

$$\xi > S^* - \frac{\varepsilon}{2} \Rightarrow S^* - \xi = |S^* - \xi| < \frac{\varepsilon}{2}$$

Because  $\xi$  is a limit point of the sequence, in  $(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2})$  there are infinitely many sequence elements.

Let  $x \in (\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}) \Leftrightarrow |x - \xi| < \frac{\varepsilon}{2}$ . Then it holds that

$$\begin{aligned} |x - S^*| &= |x - \xi + \xi - S^*| \leq \underbrace{|x - \xi|}_{< \frac{\varepsilon}{2}} + \underbrace{|\xi - S^*|}_{= S^* - \xi < \frac{\varepsilon}{2}} \\ &\Rightarrow x \in (S^* - \varepsilon, S^* + \varepsilon) \end{aligned}$$

Followingly,

$$\underbrace{\left(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}\right)}_{\text{contains infinitely many sequence elements}} \subseteq \underbrace{(S^* - \varepsilon, S^* + \varepsilon)}_{\text{contains infinitely many sequence elements}}.$$

□

**Remark 14.** The analogous statement holds for the limes inferior.

$$S^* = \limsup_{n \rightarrow \infty} a_n \Leftrightarrow$$

1.  $S^* \in H$ , therefore  $S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .
2.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

*Proof.* Let  $S^* = \limsup_{n \rightarrow \infty} a_n$ .

1. The first property holds immediately.
2. We use an indirect proof.

$$\Rightarrow \exists \varepsilon > 0 : \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Therefore infinitely many sequence elements  $a_n$  exist with  $a_n \geq S^* + \varepsilon$ . We sort the sequence elements in a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$ . It holds that

$$S^* + \varepsilon \leq a_{n_k} \leq M$$

$(a_{n_k})_{k \in \mathbb{N}}$  is bounded and has a limit point  $S$  with  $S^* + \varepsilon < S \Rightarrow S > S^*$ .  $S$  is also a limit point of the original sequence  $(a_n)_{n \in \mathbb{N}}$  with  $S > S^* = \max H$ . This is a contradiction. □

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This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

**Theorem 52** (Repetition of the theorem). Let  $(a_n)_{n \in \mathbb{N}}$  be bounded above and let  $(a_n)_{n \in \mathbb{N}}$  has a limit point. Then it holds that  $S^* = \limsup_{n \rightarrow \infty} a_n \Leftrightarrow$

1.  $S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .

$$2. \forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$$

Therefore above  $S^* + \varepsilon$  there are only finitely many sequence elements.

*Proof.* We prove the first direction  $\Rightarrow$ .

Let  $S^* = \limsup_{n \rightarrow \infty} a_n$ . Let  $\varepsilon > 0$  be arbitrary. The first property follows immediately. The second property needs to be shown.

*Proof by contradiction for the second property.*

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Then we build a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  from  $(a_n)_{n \in \mathbb{N}}$  with  $a_{n_k} \geq S^* + \varepsilon$ .

The subsequence is built inductively:

$n = 0$  then (because the second property holds negated) there exists  $x_n \geq 0$  :  
 $a_{n_0} \geq S^* + \varepsilon$ .

$k \rightarrow k+1$  Let  $a_{n_0}, a_{n_1}, \dots, a_{n_k}$  be found with  $a_{n_l} \geq S^* + \varepsilon$  with  $l = 0, \dots, k$  and  $n_l < n_{l+1}$ . Let  $N = n_k + 1$ . Because the second property holds negated,  $n_{k+1} \geq N > n_k$  such that  $a_{n_{k+1}} \geq S^* + \varepsilon$ .

The subsequence's elements have the properties:

- $a_{n_k} \geq S^* + \varepsilon \quad \forall k \in \mathbb{N}$
- Because  $(a_n)_{n \in \mathbb{N}}$  is bounded above, also  $(a_{n_k})_{k \in \mathbb{N}}$  is bounded above

From the Bolzano-Weierstrass theorem it follows that  $(a_{n_k})_{k \in \mathbb{N}}$  has a limit point  $S \geq S^* + \varepsilon$ . Because every limit point of  $(a_{n_k})_{k \in \mathbb{N}}$  is a limit point of  $(a_n)_{n \in \mathbb{N}}$ , it holds that  $S$  is limit point of  $(a_n)_{n \in \mathbb{N}}$  and  $S > S^* + \varepsilon > S^*$ . This is a contradiction.  $\square$

We prove the second direction  $\Leftarrow$ .

Assume properties 1 and 2 hold. It remains to show that  $S^*$  is the largest limit point. Assume  $S > S^*$ . We need to show that  $S$  cannot be a limit point.

$$\varepsilon = \frac{S - S^*}{2} > 0 \Rightarrow 2\varepsilon = S - S^* \Rightarrow S^* + \varepsilon = S - \varepsilon$$

Because the second property holds, there exists some  $N \in \mathbb{N}$  such that  $\forall n \geq N \Rightarrow a_n < S^* + \varepsilon$ . Therefore only finitely many sequence elements are larger than  $S^* + \varepsilon = S - \varepsilon$ . Therefore at most finitely many sequence elements  $(S - \varepsilon, S + \varepsilon)$ . Followingly  $S$  is not a limit point.  $\square$

**Theorem 53** (Analogous result for limes inferior).

$$S_* = \liminf_{n \rightarrow \infty} a_n \Leftrightarrow$$

1.  $S_*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .
2.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n > S_* - \varepsilon$

**Theorem 54.** Let  $(a_n)_{n \in \mathbb{N}}$  be bounded above and  $(a_n)_{n \in \mathbb{N}}$  has a limit point.

- Let  $k \in \mathbb{N}$ . We define

$$A_k = \{a_k, a_{k+1}, a_{k+2}, \dots\} = \{a_j : j \geq k\}$$

- It holds that  $A_{k+1} \subseteq A_k$  and  $A_k$  is bounded above<sup>1</sup>.

We define  $S_k = \sup A_k$ . Then  $(S_k)_{k \in \mathbb{N}}$  is a monotonically decreasing sequence in  $\mathbb{R}$  and  $(S_k)_{k \in \mathbb{N}}$  is bounded below. Therefore  $(S_k)_{k \in \mathbb{N}}$  is convergent and it holds that

$$\lim_{n \rightarrow \infty} S_k = \inf \{S_k : k \in \mathbb{N}\} = S^*$$

It turns out that

$$S^* = \limsup_{n \rightarrow \infty} a_n$$

We denote

$$\lim_{k \rightarrow \infty} \sup A_k = \lim_{k \rightarrow \infty} \sup \{a_j : j \geq k\} = \inf \{\sup A_k : k \in \mathbb{N}\} = \limsup_{n \rightarrow \infty} a_n$$

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<sup>1</sup>Obviously.

*Proof.*

$$A_{k+1} \subseteq A_k \Rightarrow \sup A_{k+1} \leq \sup A_k \Rightarrow S_{k+1} \leq S_k$$

$(S_k)_{k \in \mathbb{N}}$  is bounded below. Choose  $\xi \in H$  and  $\xi$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Then  $\xi - 1$  is a lower bound for  $(S_k)_{k \in \mathbb{N}}$  because infinitely many sequence elements are in  $(\xi - 1, \xi + 1)$ . Therefore,

$$\forall k \in \mathbb{N} : \exists n \geq k : a_n > \xi - 1 \Rightarrow S_k = \sup A_k > \xi - 1 \quad \checkmark$$

We know that  $(S_k)_{k \in \mathbb{N}}$  is convergent. Let  $S^* = \lim_{n \rightarrow \infty} S_k$ . We show the first property:

$S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Let  $\varepsilon > 0$  be given. We need to show that infinitely many sequence elements are in  $(S^* - \varepsilon, S^* + \varepsilon)$ .

Because  $\lim_{k \rightarrow \infty} S_k = S^*$  there exists some

$$N \in \mathbb{N} : k \geq N \Rightarrow \underbrace{|S_k - S^*|}_{-S^*} < \frac{\varepsilon}{2}.$$

We build a subsequence of  $(a_n)_{n \in \mathbb{N}}$  inductively, which is entirely inside  $(S^* - \varepsilon, S^* + \varepsilon)$ . Because  $S_N = \sup \{a_N, a_{N+1}, a_{N+2}, \dots\}$  exists, there exists  $a_j \geq S_N - \frac{\varepsilon}{2}$  with  $j \geq N$ .

$$\Rightarrow \underbrace{S_N - a_j}_{=|S_N - a_j|} \leq \frac{\varepsilon}{2}$$

$k = 0$  Choose  $n_0 = j \geq N$  ( $j$  from above), therefore it holds that

$$\begin{aligned} |S^* - a_{n_0}| &= |S^* - S_N + S_N - a_{n_0}| \leq |S^* - S_N| + |S_N - a_j| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore  $a_{n_0} \in (S^* - \varepsilon, S^* + \varepsilon)$ .

$k \rightarrow k + 1$  Consider  $a_{n_0}, a_{n_1}, \dots, a_{n_k}$  such that  $n_k > n_{k-1} > \dots > n_0 \geq N$  holds and  $|a_{n_l} - S^*| < \varepsilon$ . Because  $n_k + 1 > N$  holds

$$|S^* - S_{n_k+1}| < \frac{\varepsilon}{2}$$

because  $S_{n_k+1} = \sup \{a_{n_k+1}, a_{n_k+2}, \dots\}$ , exists  $j' \geq n_k + 1 > n_k$  such that

$$|S_{n_k+1} - a_{j'}| = S_{n_k+1} - a_{j'} < \frac{\varepsilon}{2}$$

Choose  $n_{k+1} = j'$  from above.

$$\begin{aligned} n_{k+1} &\geq n_k + 1 > n_k \text{ and } |S^* - a_{n_{k+1}}| = |S^* - S_{n_k+1} + S_{n_k+1} - a_{j'}| \\ &\leq |S^* - S_{n_k+1}| + |S_{n_k+1} - a_{j'}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore we have found a subsequence  $(a_n)_{n \in \mathbb{N}}$  such that

$$\forall k \in \mathbb{N} : a_{n_k} \in (S^* - \varepsilon, S^* + \varepsilon)$$

$\Rightarrow S^*$  is limit point of the sequence.

We show that  $S^*$  is the largest limit point. Let  $S < S^*$ . We show that  $S$  is not a limit point.

Let  $\varepsilon = \frac{1}{2}(S^* - S) > 0$  such that  $S^* + \varepsilon = S - \varepsilon$ . Choose  $k \in \mathbb{N}$  such that  $S_k - S^* = |S_k - S^*| < \varepsilon$ .  $\forall n \geq K$  it holds that  $a_n \leq S_k < S^* + \varepsilon = S - \varepsilon$ . Therefore there are at most finitely many sequence elements in  $(S - \varepsilon, S + \varepsilon)$ . Therefore  $S$  is not a limit point.  $\square$

The analogous result for the limes inferior also holds and is given in the practicals.

## 10 Infinite series

**Definition 39.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of complex values. We define

- $S_0 = a_0$
- $S_1 = a_0 + a_1$
- $S_2 = a_0 + a_1 + a_2$
- $\dots$
- $S_n = a_0 + a_1 + \dots + a_n = \sum_{k=0}^n a_k$



We call  $(S_n)_{n \in \mathbb{N}}$  an *infinite series* with  $a_k$  sequence elements. We call  $S_n$  the *n-th partial sum* of the series. The series is called *convergent* if  $(S_n)_{n \in \mathbb{N}}$  is a convergent series in  $\mathbb{C}$ . For a convergent series instead of

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \underbrace{\sum_{k=0}^n a_k}_{=S_n}$$

we denote

$$S = \sum_{k=0}^{\infty} a_k$$

Actually a series must be denoted like a sequence with  $(S_n)_{n \in \mathbb{N}}$ . But we also say “let  $\sum_{k=0}^{\infty} a_k$  be a series” (but actually the sum of partial sums is meant). So this an ambiguous definition (per default always assume that the sum of partial sums is considered).

## 10.1 The geometric series

**Theorem 55.** Let  $q \in \mathbb{C}$  with  $q \neq 1$ . Consider  $\sum_{k=0}^{\infty} q^k$  hence  $S_n = \sum_{k=0}^n q^k$ . The limes of this series is given with  $\frac{1-q^{n+1}}{1-q}$  for  $|q| < 1$ .

*Proof.* We find a simple equation for  $S_n$ :

$$\begin{aligned} S_n - q \cdot S_n &= (1 - q)S_n \\ (1 + q + q^2 + \dots + q^n) - q(1 + q + q^2 + \dots + q^n) \\ &= (1 + q + q^2 + \dots + q^n) - (q + q^2 + \dots + q^n + q^{n+1}) \\ &= (1 - q^{n+1}) \end{aligned}$$

Therefore  $(1 - q) \cdot S_n = 1 - q^{n+1}$ . That is,

$$S_n = \frac{1 - q^{n+1}}{1 - q}$$

If  $|q| < 1$  it holds that

$$\lim_{n \rightarrow \infty} q^{n+1} = q \lim_{n \rightarrow \infty} q^n = q \cdot 0 = 0$$

$$\lim_{n \rightarrow \infty} S_n = \frac{1 - \lim_{n \rightarrow \infty} q^{n+1}}{1 - q} = \frac{1}{1 - q}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1 - q}$$

If  $|q| > 1$  it holds that

$$|S_n| = \frac{1}{|1 - q|} \cdot |1 - q^{n+1}| \geq \frac{1}{|1 - q|} (|q^{n+1}| - 1)$$

This is the inversed triangle inequality.

$$= \frac{1}{|1 - q|} \left( \underbrace{|q|^{n+1}}_{\rightarrow \infty} - 1 \right)$$

Hence  $(S_n)_{n \in \mathbb{N}}$  is unbounded and therefore not convergent. □

**Theorem 56.** Let  $a_n = \frac{1}{n}$  hence  $\sum_{k=1}^{\infty} \frac{1}{k}$ .

$$\sum_{k=1}^{\infty} \frac{1}{n} \text{ is divergent}$$

*Proof.* Consider

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\ &> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \dots \\ &= \frac{1}{2} + 2 \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \dots \right) \\ &= \frac{1}{2} + 2 \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) \\ &= \frac{1}{2} + \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) \end{aligned}$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{n}$$

So we have,

$$\sum_{k=1}^{\infty} \frac{1}{n} > \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{n}$$

Let  $\sum_{k=1}^{\infty} \frac{1}{n} = H$ , then  $H > \frac{1}{2} + H$  must hold for some real value. This is impossible, so  $H$  cannot exist. Hence  $\sum_{k=1}^{\infty} \frac{1}{n}$  diverges.  $\square$

This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

TODO

**Case 1**

$$\begin{aligned} n &= 2^k - 1 \\ S_{\alpha, 2^k - 1} &= \underbrace{\frac{1}{2^\alpha}}_{< \frac{1}{2^\alpha}} + \underbrace{\frac{1}{2^\alpha} + \frac{1}{3^\alpha}}_{2^1 \text{ terms}} + \underbrace{\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}}_{2^2 \text{ terms}} \\ &\quad + \underbrace{\frac{1}{8^\alpha} + \dots}_{2^3 \text{ terms}} \text{ TODO} \\ &< 1 + 2 \frac{1}{2^\alpha} + 4 \frac{1}{4^\alpha} + 8 \frac{1}{8^\alpha} + \dots + 2^{k-1} \frac{1}{(2^{k-1})^\alpha} \\ &= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \frac{1}{8^{\alpha-1}} + \dots + \frac{1}{(2^{n-1})^{\alpha-1}} \\ &= 1 + \frac{1}{2^{\alpha-1}} + \left( \frac{1}{2^{\alpha-1}} \right)^2 + \left( \frac{1}{3^{\alpha-1}} \right)^3 + \dots \\ &= \underbrace{\sum_{j=0}^{k-1} \left( \frac{1}{2^{\alpha-1}} \right)^j}_{\text{geometric series}} = \frac{1 - \left( \frac{1}{2^{\alpha-1}} \right)^k}{1 - \frac{1}{2^{\alpha-1}}} \end{aligned}$$

Therefore  $(S_{\alpha, 2^k - 1})$  is bounded. Let  $n \in \mathbb{N}$  be arbitrary and choose a sufficiently large  $K$  such that  $2^K > n + 1$ . Therefore  $2^k - 1 > n$ . Because

$\frac{1}{j^\alpha} > 0$  for all  $j \geq 1$ , it holds that  $S_{2^k - 1} > S_n$ . At the same time  $S_{2^k - 1} < \frac{2^{\alpha-1}}{2^{\alpha-1}-1}$ . So  $(S_n)_{n \in \mathbb{N}}$  is bounded. Hence  $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$  is convergent.

**Case 2:**  $\alpha \leq 1$  Then it holds that  $k^\alpha \leq k$  and therefore  $\frac{1}{k^\alpha} \geq \frac{1}{k}$ . Because  $S_{\alpha, n} \geq S_{1, n}$  and because  $S_{1, n}$  is unbounded, it holds that  $(S_{\alpha, n})_{n \in \mathbb{N}}$  is unbounded and followingly  $\sum_{k=0}^{\infty} \frac{1}{k^\alpha}$  is divergent.

**Remark 15.**  $\alpha \in \mathbb{Q}_+$  can be replaced by  $\alpha \in \mathbb{R}_+$ . It is even possible to choose  $\alpha \in \mathbb{C}$ . Then we can define  $\zeta : M \subseteq \mathbb{C} \rightarrow \mathbb{C}$  with  $\xi(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$ . This is Riemann's Zeta function.

**Definition 40.** Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence with  $a_n \geq 0$ . Then we call  $(\alpha_n)_{n \in \mathbb{N}}$  with  $\alpha_n = (-1)^n a_n$ , or equivalently  $\alpha_n = (-1)^{n+1} a_n$ , an .

A series of structure  $\sum_{k=0}^{\infty} (-1)^k a_k$  with  $a_k \geq 0$  is called *alternating series*.

## 10.2 Leibniz convergence criterion

Gottfried Wilhelm Leibniz (1646–1716)

**Theorem 57** (Leibniz convergence criterion). Let  $(a_n)_{n \in \mathbb{N}}$  be a real, monotonically zero sequence with  $a_n \geq a_{n+1} \geq 0 \quad \forall n \in \mathbb{N}$ . Then  $\sum_{k=0}^{\infty} (-1)^k a_k$  is convergent.

*Proof.*

$$\begin{aligned} S_{2n-1} &= \sum_{k=0}^{2n-1} (-1)^k a_k \\ S_{2n} &= \sum_{k=0}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} \\ &= S_{2n-1} + a_{2n} \\ S_{2n+1} &= S_{2n-1} + \underbrace{a_{2n} - a_{2n-1}}_{\geq 0} \\ S_{2n+2} &= \underbrace{S_{2n-1} + a_{2n}}_{S_{2n}} - \underbrace{a_{2n+1} + a_{2n+2}}_{= -(a_{2n+1} - a_{2n+2}) \geq 0} \end{aligned}$$

Therefore it holds that  $S_{2n+1} \geq S_{2n-1}$ ,  $S_{2n+2} \leq S_{2n}$  and  $S_{2n} \geq S_{2n-1}$ .

$(S_{2n})_{n \in \mathbb{N}}$  is monotonically decreasing.  $(S_{2n+1})_{n \in \mathbb{N}}$  is monotonically increasing.

It holds that:  $\forall m, n \in \mathbb{N} : S_{2n} \geq S_{2m-1}$ .

*Proof. Case 1:  $m > n$*

$$S_{2m+1} \leq S_{2n} \leq S_{2n} \quad \checkmark$$

**Case 2:  $m \leq n$**

$$S_{2m+1} \leq S_{2n+1} \underbrace{\leq}_{\alpha < 1} S_{2n}$$

So  $(S_{2n})_{n \in \mathbb{N}}$  is monotonically decreasing and bounded by below (for example by  $S_1$ ). Therefore  $S_{2n} \rightarrow S^*$  for  $n \rightarrow \infty$ .  $(S_{2n+1})_{n \in \mathbb{N}}$  is monotonically increasing and bounded by above by  $S_*$ :

$$S_{2n+1} \rightarrow S_* \text{ for } n \rightarrow \infty$$

It holds that  $S_* \leq S^*$  because  $S_{2n+1} \leq S_{2n}$ .  $\square$

This lecture took place on 10th of December 2015 with lecturer Wolfgang Ring.

Given  $S_* \leq S^*$ , we show that  $S^* = S_*$  and we prove that  $\forall \varepsilon > 0 : S^* - S_* < \varepsilon$ .

Let  $\varepsilon > 0$  and choose  $N$  sufficiently large, such that  $a_{2N} < \varepsilon$ .

$$a_{2N} = S_{2N} - S_{2N-1} > S^* - S_*$$

$$a_{2N} < \varepsilon$$

So  $\forall \varepsilon > 0$ , it holds that

$$S^* - S_* = |S^* - S_*| < \varepsilon$$

$$\Rightarrow S^* = S_* = S$$

So it holds that,

$$\lim_{n \rightarrow \infty} S_n = S^* = S_* = S$$

and the series converges.  $\square$

**Example 18.**

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k} \text{ is convergent}$$

### 10.3 Series in $\mathbb{C}$ and absolute convergence

**Theorem 58** (Cauchy convergence criterion). The complex series  $\sum_{k=0}^{\infty} a_k$  is convergent if and only if the partial sums  $(s_n)_{n \in \mathbb{N}}$  are a Cauchy sequence in  $\mathbb{C}$ .

**Remark 16.** Therefore

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n, m > N$$

$$\Rightarrow |S_n - S_m| < \varepsilon$$

Therefore without loss of generality,  $n \geq m$ .

$$S_n - S_m = \sum_{k=0}^n a_k - \sum_{k=0}^m a_k = \sum_{k=m+1}^n a_k$$

Hence  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N$ .

$$\left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

Equivalently, with  $m+1 = n$  and  $n - m = l$ .

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \text{ and } l \in \mathbb{N}$$

$$\left| \sum_{k=0}^l a_{n+k} \right| < \varepsilon$$

*Proof by  $(S_n)_{n \in \mathbb{N}}$  being convergent.*

$$(S_n)_{n \in \mathbb{N}} \Leftrightarrow \text{Cauchy sequence}$$

**Lemma 10.** If  $\sum_{k=0}^{\infty} a_n$  is convergent in  $\mathbb{C}$ , then  $(a_n)_{n \in \mathbb{N}}$  is a zero sequence.

*Proof.* Follows directly from the Cauchy criterion for  $l = 0$ .

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N : \underbrace{\left| \sum_{k=0}^0 a_{n+k} \right|}_{|a_n|} < \varepsilon \quad \text{hence } a_n \rightarrow 0$$

$\square$

□ *Proof.* 1. We need to show that  $\sum_{n=0}^{\infty} \underbrace{|a_k|}_{\geq 0}$  is convergent. It suffices to show that

**Definition 41.** The complex series  $\sum_{k=0}^{\infty} a_k$  is called *absolute convergent* if the real series  $\sum_{k=0}^{\infty} |a_k|$  is convergent.

**Example 19.**

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n^2} \quad \text{absolute convergent}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n} \quad \text{absolute convergent (Leibniz)}$$

**Lemma 11.** Let  $\sum_{k=0}^{\infty} a_k$  be absolute convergent. Then  $\sum_{k=0}^{\infty} a_k$  is also convergent.

*Proof.* Let  $\sum_{k=0}^{\infty} |a_k|$  be convergent. From the Cauchy criterion it follows that,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N :$$

$$\left| \sum_{k=m+1}^n |a_k| \right| = \sum_{k=m+1}^n |a_k| \geq \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

$\Rightarrow \sum_{k=0}^{\infty} a_k$  is convergent according to Cauchy criterion. □

## 10.4 Direct comparison test

**Theorem 59** (Direct comparison test (dt. Majorantenkriterium)).

1. Let  $\sum_{k=0}^{\infty} a_k$  and  $\sum_{k=0}^{\infty} b_k$  be complex series. Let  $\sum_{k=0}^{\infty} b_k$  be absolute convergent and  $\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_k| \leq |b_k|$ .

Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.  $\sum_{k=0}^{\infty} b_k$  is called *majorant* of  $\sum_{k=0}^{\infty} a_k$ .

2. (dt. Minorantenkriterium) Let  $\sum_{k=0}^{\infty} a_k$  and  $\sum_{k=0}^{\infty} b_k$  be complex series. Let  $\sum_{k=0}^{\infty} a_k$  be divergent. Assume  $\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_k| \leq |b_k|$ . Then also  $\sum_{k=0}^{\infty} b_k$  is divergent.  $\sum_{k=0}^{\infty} a_k$  is *minorant* of  $\sum_{k=0}^{\infty} b_k$ .

$$\sum_{k=0}^n |a_k| = \sigma_n$$

$(\sigma_n)_{n \in \mathbb{N}}$  is bounded. Let  $n \geq N$ .

$$\begin{aligned} \sigma_n &= \sum_{k=0}^n |a_k| \\ &= |a_0| + |a_1| + \cdots + |a_{N-1}| + \sum_{k=N}^n |a_k| \\ &\leq |a_0| + \cdots + |a_{N-1}| + \underbrace{\sum_{k=N}^{\infty} |b_k|}_{s \geq 0} \\ &\quad \underbrace{\hspace{10em}}_M \end{aligned}$$

Therefore  $(\sigma_n)_{n \in \mathbb{N}}$  is bounded and therefore  $\sum_{n=0}^{\infty} a_n$  is absolute convergent.

2. Let  $\sum_{k=0}^{\infty} a_k$  be divergent. Then also  $\sum_{k=0}^{\infty} |a_k|$  is divergent. Otherwise  $\sum_{k=0}^{\infty} a_k$  is absolute convergent and therefore convergent.

$$\Rightarrow \sigma_n = \sum_{k=0}^n |a_k|$$

$(\sigma_n)_{n \in \mathbb{N}}$  is unbounded. Because

$$\begin{aligned} \sum_{k=0}^n |b_k| &= |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^n |b_k| \\ &\geq |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^N |a_k| \\ &= |b_0| + \cdots + |b_{N-1}| - (|a_0| + \cdots + |a_{N-1}|) + \underbrace{\sum_{k=0}^n |a_k|}_z \\ &= z + \sigma_n \end{aligned}$$

$z + \sigma_n$  is unbounded. Therefore  $\sum_{k=0}^{\infty} |b_k|$  is not convergent. Therefore  $\sum_{k=0}^{\infty} b_k$  is not absolute convergent.  $\square$

## 10.5 Ratio test

**Theorem 60** (Ratio test (dt. Quotientenkriterium)). 1. Let  $\sum_{k=0}^{\infty} a_k$  be a complex series. Assume  $\exists q \in [0, 1)$  with  $(0 \leq q < 1)$  and  $N \in \mathbb{N}$  such that

- $\frac{|a_{n+1}|}{|a_n|} < q \quad \forall n \geq N$  with  $|a_n| \neq 0$ , or
- $\sqrt[n]{|a_n|} < q \quad \forall n \geq N$

Then the series  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

“Ratio test”

2. Assume there exists  $q > 1$  and  $N \in \mathbb{N}$  such that

- $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$
- $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$

Then  $\sum_{k=0}^{\infty} a_k$  is divergent.

“Square root test”

*Proof.* This follows from the direct comparison criterion. Compare with geometric series  $\sum_{k=0}^{\infty} q^k$ .

1. Assume the two statement of the ratio test holds. Therefore  $\forall n \geq N$  it holds that  $\sqrt[n]{|a_n|} \leq q \Leftrightarrow |a_n| \leq q^n$ . Due to the direct comparison test,  $\sum_{k=0}^{\infty} q^k$  ✓.

Assume the first statement of the ratio test does not hold.

$$\frac{|a_{n+1}|}{|a_n|} \leq q (< 1)$$

Then it holds that  $\forall k \in \mathbb{N}$ :

$$|a_{k+N}| \leq |a_N| \cdot q^k$$

Proof by induction over  $k$ :

$\mathbf{k} = 0$

$$|a_N| \leq |a_N| \cdot q^0 \quad \checkmark$$

$\mathbf{k} \rightarrow \mathbf{k} + 1$  Assume  $|a_{N+k}| \leq |a_N| \cdot q^k$ . Because

$$\frac{|a_{N+k+1}|}{|a_{N+k}|} \leq q \Rightarrow |a_{N+k+1}| \leq q |a_{N+k}| \leq q \cdot |a_N| \cdot q^k = |a_N| q^{k+1} \quad \checkmark$$

We set

$$b_k = \begin{cases} 0 & \text{for } k = 0, 1, 2, \dots, N-1 \\ |a_N| \cdot q^{K-n} & \text{for } n \geq N \end{cases}$$

$$\sum_{k=0}^{\infty} b_k = 0 + 0 + 0 + \cdots + 0 + |a_N| \text{ TODO}$$

$$= |a_N| \sum_{j=0}^{\infty} q_j \text{ is absolute convergent}$$

$\sum_{k=0}^{\infty} b_k$  is an absolute convergent majorant. for  $\sum_{k=0}^{\infty} a_k \Rightarrow \sum_{k=0}^{\infty} a_k$  is convergent.

2. Assume the second statement (square root test) holds. **TODO** Therefore  $(a_n)_{n \in \mathbb{N}}$  is not a zero sequence. Therefore  $\sum_{k=1}^{\infty} a_k$  is divergent.

Assume the first statement holds.

$$\Rightarrow |a_{N+k}| \geq |a_N| \cdot q^k$$

Because  $|a_N| \cdot q^k$  is unbounded,  $|a_{N+k}|$  is unbounded.  $(a_k)_{k \in \mathbb{N}}$  are not zero sequences. □

**Remark 17.** Assume  $\frac{|a_{n+1}|}{|a_n|}$  is bounded and  $q = \limsup_{n \rightarrow \infty} \left( \frac{|a_{n+1}|}{|a_n|} \right) < 1$ . Let  $2\varepsilon = 1 - q > 0$ .

$$\begin{aligned} \Rightarrow \exists N \in \mathbb{N} : n \geq N : \frac{|a_{n+1}|}{|a_n|} &< q + \varepsilon \\ &= q + \frac{1}{2}(1 - q) = \frac{1}{2}(1 + q) = 1 - \varepsilon < 1 \end{aligned}$$

Due to the ratio test, the series  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

**Lemma 12.** Let  $\sum_{k=0}^{\infty} a_k$  be a complex series with  $a_k \neq 0 \forall k \in \mathbb{N}$  and if it holds that

$$q = \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = q$$

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This lecture took place on 11th of December 2015 with lecturer Wolfgang Ring.

## 10.6 Revision

So  $\sum_{k=0}^{\infty}$  is absolute convergent if  $\exists q \in [0, 1) \exists N \in \mathbb{N}$ .

- $\frac{|a_{n+1}|}{|a_n|} \leq q \quad \forall n \geq N$
- $\sqrt[n]{|a_n|} \leq q \quad \forall n \geq N$

If  $q > 1$  and either  $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$  or  $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$ , then this series is convergent.

**Corollary 10.** Let  $q = \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$ , then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent. Let  $q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ . Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

Let  $q = \limsup \sqrt[n]{|a_n|} > 1$ . Then  $\sum_{k=0}^{\infty} a_k$  is divergent.

*Proof.* Let  $q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ .

$$2\varepsilon = 1 - q > 0$$

Then there exists some  $N \in \mathbb{N} : n \geq N$

$$\Rightarrow \sqrt[n]{|a_n|} \leq q + \varepsilon = 1 - \varepsilon < 1$$

Is absolute convergent according to the square root theorem.

We also need to show divergence: Let  $q > 1$  be limit point of  $\sqrt[n]{|a_n|}$ . So there exists some subsequence  $\left( \sqrt[n_k]{|a_{n_k}|} \right)_{k \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \sqrt[n_k]{|a_{n_k}|} = q > 1 \Rightarrow \varepsilon = \frac{1}{2}(q - 1) > 0$ .

$$\begin{aligned} \sqrt[n_k]{|a_{n_k}|} &> q - \varepsilon \quad \forall k \geq K \\ \Rightarrow |a_{n_k}| &> (q - \varepsilon)^{n_k} = (1 + \varepsilon)^{n_k} > 1 \\ \Rightarrow (|a_{n_k}|)_{k \in \mathbb{N}} &\text{ is not a zero sequence} \\ \Rightarrow (|a_n|)_{n \in \mathbb{N}} &\text{ is also not a zero sequence} \end{aligned}$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ is divergent}$$

□

**Example 20** (Binomial series). Let  $n \in \mathbb{N}$  and  $k \in \{0, 1, 2, \dots, n\}$ .

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} = \frac{1 \cdot 2 \cdot \dots \cdot (n-k)(n-k+1) \cdot \dots \cdot n}{k! \cdot 1 \cdot 2 \cdot \dots \cdot (n-k)} \\ &= \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!} \end{aligned}$$

Let  $s \in \mathbb{C}$ . We define the binomial coefficient  $\binom{s}{k} = \frac{s \cdot (s-1) \cdot (s-2) \cdot \dots \cdot (s-k+1)}{k!}$ . Also let  $\binom{s}{0} = 1$  and  $\binom{s}{1} = s$ . Let  $k > n$  and  $n \in \mathbb{N}$ , then

$$\binom{n}{k} = \frac{n(n-1) \cdot \dots \cdot \overbrace{(n-n)}^0 \cdot \dots \cdot (n-k+1)}{k!} = 0$$

**Example 21.** We define the binomial series for  $s, z \in \mathbb{C}$  with

$$B_S(z) = \sum_{k=0}^{\infty} \underbrace{\binom{s}{k}}_{:=a_k} z^k$$

What about convergence? Well,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left| \frac{s \cdot (s-1) \cdot \dots \cdot (s-(k+1)+1)}{(k+1)!} z^{k+1} \right|}{\left| \frac{s \cdot (s-1) \cdot (s-2) \cdot \dots \cdot (s-k+1)}{k!} z^k \right|}$$

$$\frac{|a_{k+1}|}{|a_k|} = \left| \frac{\left( \overbrace{\frac{s}{k}}^{\rightarrow 0} - 1 \right) \cdot z}{1 + \underbrace{\frac{1}{k}}_{\rightarrow 0}} \right| \rightarrow |z|$$

Therefore  $B_S(z)$  is convergent for  $|z| < 1$  and divergent for  $|z| > 1$ . So geometrically, it is convergent within a circle of radius 1 or  $i$  (at center  $(0, 0)$ ) and divergent outside.

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

We know, for  $s \in \mathbb{N}$ :

$$B_S(z) = \sum_{k=0}^{\infty} \binom{n}{k} z^k = \sum_{k=0}^n \binom{n}{k} z^k = (1+z)^n$$

Remind that  $\binom{n}{k} = 0$  for  $k > n$ .

Therefore

$$(1+z)^s := \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

This is the definition of a power function i.e.

$$z = \xi - 1 \quad 1 + z = \xi$$

$$\xi^S = \sum_{k=0}^{\infty} \binom{s}{k} (\xi - 1)^k$$

is convergent for  $|\xi - 1| < 1$ .

Geometrically, this is a circle of radius 1 or  $i$  (at center  $(1, 0)$ ).

## 11 Power series

**Definition 42.** A power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

So we have one free variable. Coefficients of the series contain a variable.

- In  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  all summands are fixed.
- However  $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$  with  $|z| < 1$  is variable with the variable  $z$ .

**Example 22.**

$$f : B(0, 1) \rightarrow \mathbb{C}$$

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

Mapping:

$$B_S : B(0, 1) \rightarrow \mathbb{C}$$

$$\varepsilon(z) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$$

Let  $z \in \mathbb{C}$  arbitrary.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left| \frac{z^{k+1}}{(k+1)!} \right|}{\left| \frac{z^k}{k!} \right|} = \left| \frac{z}{k+1} \right| \rightarrow 0$$

$\Rightarrow \varepsilon(z)$  is convergent for all  $z \in \mathbb{C}$ .

$$\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$$

**Corollary 11.** Using series sum we can define mappings (functions).

**Definition 43.** Let  $(a_n)_{n \in \mathbb{N}}$  be a complex sequence and let  $z \in \mathbb{C}$ . Then  $\sum_{k=0}^{\infty} a_k \cdot z^k$  is called *power series with coefficient sequence*  $(a_k)_{k \in \mathbb{N}}$ .

Its convergence property depends on  $z$ . For  $z = 0$  every power series is convergent.

$$\sum_{k=0}^{\infty} a_k \cdot 0^k$$

Because we define  $0^0 := 1$  here, the constant series  $a_0$  is given.

**Lemma 13.** Let  $\sum_{k=0}^{\infty} a_k z^k$  is a power series in  $\mathbb{C}$  and  $z_0 \in \mathbb{C} \setminus \{0\}$  such that  $\sum_{k=0}^{\infty} a_k z_0^k$  is convergent. Then the power series is absolute convergent for all  $z$  with  $|z| < |z_0|$ .

Geometrically, if the series is convergent at one point  $z_0$  at the circle, it is convergent in all points of the circle.

*Proof.* Direct comparison test: Because  $\sum_{k=0}^{\infty} a_k z_0^k$  is convergent, it holds that  $\lim_{k \rightarrow \infty} a_k z_0^k = 0$ . Therefore  $(a_k z_0^k)_{n \in \mathbb{N}}$  is also bounded and there exists some  $m \geq 0$  such that  $|a_k z_0^k| \leq m \quad \forall k \in \mathbb{N}$ .

Let  $|z| < |z_0|$ . Then,

$$|a_k z^k| = \left| a_k \frac{z^k}{z_0^k} \cdot z_0^k \right| = |a_k z_0^k| = \underbrace{|a_k z_0^k|}_{\leq m} \underbrace{\left| \frac{z}{z_0} \right|^k}_{:=q} \leq m \cdot q^k$$

with  $0 \leq q < 1$ . Therefore  $\sum_{k=0}^{\infty} a_k z^k$  is convergent because of the direct comparison test with  $\sum_{k=0}^{\infty} m \cdot q^k = m \cdot \sum_{k=0}^{\infty} q^k$ .  $\square$

**Definition 44.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series in  $\mathbb{C}$ . We define

$$\rho(P) = \sup \{r \geq 0, r \in \mathbb{R} : P(r) \text{ is convergent}\}$$

$\rho(P)$  is called convergence radius of  $P$ . If  $\{r \geq 0 : P(r) \text{ is convergent}\}$  is unbounded, then we define  $\rho(P) = \infty$ .

**Lemma 14.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series in  $\mathbb{C}$  and let  $\rho(P)$  be its convergence radius of  $P$ . Then  $P(z)$  is absolute convergent for all  $z \in \mathbb{C}$  with  $|z| < \rho(P)$ .

*Proof.* For  $\rho(P) = 0$ , nothing has to be shown.

Let  $\rho(P) > 0$  and  $|z| < \rho(P)$ , then  $\varepsilon := \rho(P) - |z|$ . Because  $\rho(P) = \sup \{r \geq 0 : P(r) \text{ is convergent}\}$ , there exists some  $r \in \mathbb{R}$  such that  $\rho(P) - \varepsilon < r \leq \rho(P)$  and  $P(r)$  is convergent.  $\rho(P) - \varepsilon = |z| < r$ . So  $P(z)$  is absolute convergent according to Lemma 13.

Geometrically,  $\rho(P)$  is a circle and its interior is convergent. On the outside the power series is divergent. The convergence property at the circle itself is unknown (not generally uniform).  $\square$

**Lemma 15.** Let  $z \in \mathbb{C}$ ,  $P$  is a power series and  $|z| > \rho(P)$ . Then  $\sum_{k=0}^{\infty} a_k z^k$  is divergent for this point.

*Proof.* Proof by contradiction. Assume  $P(z)$  is convergent and  $|z| > \rho(P)$ . Let  $\varepsilon = 2(|z| - \rho(P))$ . Then  $\rho(P) + \varepsilon < |z|$  with  $\rho(P) + \varepsilon > \rho(P)$ . From the previous lemma it follows that  $P(\rho(P) + \varepsilon)$  is convergent. But this contradicts with  $\rho(P) = \sup \{r \geq 0 : P(r) \text{ is convergent}\}$ .  $\square$

**Remark 18.**  $B(0, \rho(P))$  is called *convergence circle* of  $P$ .

**Theorem 61** (Formulas to compute  $\rho(P)$ ). Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series. Then it holds in every case that,

- $\rho(P) = \frac{1}{L}$  with  $L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$  (for  $L = \infty$  if  $\left(\sqrt[n]{|a_n|}\right)_{n \in \mathbb{N}}$  is unbounded and  $\frac{1}{\infty} := 0$ ) (Cauchy & Hadamard)



- If  $q := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then the convergence disk of this power series is  $\frac{1}{q}$ :

$$\rho(P) = \frac{1}{q}$$

with  $\frac{1}{0} := \infty$  and  $\frac{1}{\infty} := 0$ .

This lecture took place on 16th of December 2015 with lecturer Wolfgang Ring.

## 11.1 Equations for $\rho(P)$

**Theorem 62.**

$$P(z) = \sum_{k=0}^{\infty} a_k z^k$$

$$L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$\rho(P) = \frac{1}{L} \quad \text{“Cauchy-Hadamard theorem”}$$

If  $q = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then it holds that  $\rho(P) = \frac{1}{q}$  (Euler).

*Proof.* 1. Let  $z \neq 0$  and let  $L^* = \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k z^k|} = \limsup_{k \rightarrow \infty} |z| \sqrt[k]{|a_k|} = |z| \cdot L$ . Due to the square root criterion it holds that:

- If  $|z| L < 1$ , then  $\sum_{k=0}^{\infty} a_k z^k$  is absolute convergent.
- If  $|z| L > 1$ , then  $\sum_{k=0}^{\infty} a_k z^k$  is absolute divergent.

Therefore for  $|z| < \frac{1}{L}$ ,  $P$  is convergent. For  $|z| > \frac{1}{L}$ ,  $P$  is divergent.

$$\Rightarrow \rho(P) = \frac{1}{L}$$

2. Ratio test: Assume  $q = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$  exists. The ratio test for  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  gives us

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1} \cdot z^{k+1}}{a_k \cdot z^k} \right| = |z| \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = |z| \cdot q$$

Therefore  $P$  is convergent, if  $|z| \cdot q < 1 \Leftrightarrow |z| < \frac{1}{q}$ . And  $P$  is divergent, if  $|z| \cdot q > 1 \Leftrightarrow |z| > \frac{1}{q}$ . □

**Remark 19.** What happens for  $|z| = \rho(P)$ ? We need a different approach for convergence/divergence.

1.

$$G(z) = \sum_{k=0}^{\infty} z^k \quad L = \limsup_{k \rightarrow \infty} \sqrt[k]{|1|}$$

$$\rho(G) = 1$$

2.

$$H(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$$

$$q = \lim_{k \rightarrow \infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{k}{k+1} \right| = \lim_{k \rightarrow \infty} \left| \frac{1}{1 + \underbrace{\frac{1}{k}}_{\rightarrow 0}} \right|$$

3.

$$Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

$$q = \lim_{n \rightarrow \infty} \left( \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right) = 1$$

$$\rho(Q) = 1$$

**Case 1** Let  $z \in \mathbb{C}$  with  $|z| = 1$ . Then  $G(z)$  is not convergent because  $(z^k)_{k \in \mathbb{Z}}$  is not a zero sequence because  $|z^k| = |z|^k = 1$ . So geometrically, the circle itself of the convergence circle is divergent.

**Case 2** Consider  $H(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$ .  $H$  is divergent for  $z = 1$ . For  $z = -1$ ,  $H(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$  is convergent according to the Leibniz criterion.

**Case 3** For  $Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$  and let  $|z| = 1$ . Then it holds that  $\left| \frac{z^k}{k^2} \right| \leq \frac{1}{k^2}$ .  
 $\sum_{k=1}^{\infty} \frac{1}{k^2}$  is absolute convergent. The direct comparison test tells us that  
 $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$  is absolute convergent.

## 12 Functions and their regularity properties

Recall: Let  $D \subseteq \mathbb{C}$  (or  $\subseteq \mathbb{R}$ ). A mapping  $f : D \Rightarrow \mathbb{C}$  (or  $f : D \rightarrow \mathbb{R}$ ) is a function. Depending on the domain, we call the function *complex* or *real*.

### 12.1 Fundamental topological terminology

Recall:  $B(z, r) = \{\zeta \in \mathbb{C} : |z - \zeta| < r\}$ . Geometrically this corresponds to an open circular disk with center  $z$  and radius  $r$ .

Analogously,  $B(x, r) = \{y \in \mathbb{R} : |y - x| < r\} = (x - r, x + r)$  in  $\mathbb{R}$ .

**Definition 45.** Let  $U \subseteq \mathbb{C}$  ( $U \subseteq \mathbb{R}$ ) and  $z_0 \in U$ . Then  $U$  is called *surrounding* of  $z_0$  in  $\mathbb{C}$ , if  $\exists r > 0 : B(z_0, r) \subseteq U$ .

- $O \subseteq \mathbb{C}$  if called *open set* if  $\forall z \in O : O$  is surrounding of  $z$ .

$$\Leftrightarrow \forall z \in O : \exists r = r(z) : B(z, r) \subseteq O$$

- $A \subseteq \mathbb{C}$  is called *closed set*, if  $\mathbb{C} \setminus A$  is an open set.

**Theorem 63.** 1. Let  $I$  be a set and  $\forall i \in I$  let  $O_i$  be an open set in  $\mathbb{C}$ . Then  $\bigcup_{i \in I} O_i = \{z \in \mathbb{C} : \exists i \in I : z \in O_i\}$  is an open set.

2. Let  $O_1, O_2, \dots, O_n$  be open sets. Then  $\bigcap_{k=1}^n O_k = O_1 \cap O_2 \cap \dots \cap O_n$  is open.

3. If  $\emptyset$  is open, then  $\mathbb{C}$  is open.

4.  $I$  is a set  $\forall i \in I$ . Let  $A_i$  be closed. Then  $\bigcap_{i \in I} A_i$  is closed.

5. Let  $A_1, A_2, \dots, A_n$  be closed, then  $A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{k=1}^n A_k$  is closed.

*Proof.* 1. Let  $z \in \bigcup_{i \in I} O_i$ . Show that  $\exists r > 0 : B(z, r) \subseteq \bigcup_{i \in I} O_i$ .

Let  $z \in \bigcup_{i \in I} O_i$ , therefore  $\exists j \in I : z \in O_j$ . Because  $O_i$  is open,  $\exists r > 0 : B(z, r) \subseteq O_j \subseteq \bigcup_{j \in I} O_j$ .

2. Let  $O_1, \dots, O_n$  and let  $z \in O_k$ . Hence  $\forall k \in \{1, \dots, n\} : z \in O_k$  with  $O_k$  as open set.  $\exists r_k > 0 : B(z, r_k) \subseteq O_k$ . Let  $r = \min \{r_1, r_2, \dots, r_n\} > 0$ . Then it holds that  $B(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\} \subseteq \{\zeta \in \mathbb{C} : |\zeta - z| < r_k\} = B(z, r_k) \subseteq O_k$  because  $r \leq r_k$ .

So  $\forall k \in \{1, \dots, n\} : B(z, r) \subseteq O_k$ . Otherwise  $B(z, r) \subseteq \bigcap_{k=1}^n O_k \Rightarrow \bigcap_{k=1}^n O_k$  is open.

3. Let  $O = \emptyset$ . Then it holds that  $\forall z \in \emptyset : B(z, 1) \subseteq \emptyset$ . So  $\emptyset$  is open. For  $O = \mathbb{C}$  it holds that  $\forall z \in \mathbb{C} : B(z, 1) \subseteq \mathbb{C}$ , therefore  $\mathbb{C}$  is open.

4. Let  $A_i$  be closed and  $A = \bigcap_{i \in I} A_i$  and  $O = \mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin \bigcap_{i \in I} A_i\}$ .  
 $O = \mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin \bigcap_{i \in I} A_i\} = \{z \in \mathbb{C} : \exists j \in I : z \notin A_j\} = \bigcup_{j \in I} \{z \in \mathbb{C} : z \notin A_j\} = \bigcup_{j \in I} (\mathbb{C} \setminus A_j) \rightarrow$  open. So  $\mathbb{C} \setminus A$  is open, therefore  $A$  is closed.

$$\mathbb{C} \setminus \bigcap_{j \in I} A_j = \bigcup_{j \in I} (\mathbb{C} \setminus A_j)$$

The last statement was proven by DeMorgan.

5. Let  $A = \bigcup_{k=1}^n A_k$ .

$$\mathbb{C} \setminus A = \mathbb{C} \setminus \bigcup_{k=1}^n A_k = \bigcap_{k=1}^n (\mathbb{C} \setminus A_k)$$

where  $\mathbb{C} \setminus A_n$  is an open set. So  $A$  is closed. □

**Theorem 64.**  $A \subseteq \mathbb{C}$  is closed  $\Leftrightarrow \forall (a_n)_{n \in \mathbb{N}}$  with  $a_n \in A$  and  $(a_n)_{n \in \mathbb{N}}$  is convergent with limes  $a \in \mathbb{C}$ , then  $a \in A$ .

*Proof.*  $\Rightarrow$  Let  $A$  be closed ( $\mathbb{C} \setminus A$  is open) and  $(a_n)_{n \in \mathbb{N}}$  is a convergent sequence with  $\lim_{n \rightarrow \infty} a_n = a$ . Show that  $a \in A$ .

Proove by contradiction: Assume  $a \notin A$ , so  $a \in \mathbb{C} \setminus A$ .

Because  $\mathbb{C} \setminus A$  is an open set,  $\exists r > 0 : B(a, r) \subseteq \mathbb{C} \setminus A$ . And  $B(a, r) \cap A = \emptyset$  so it holds that  $\forall n \in \mathbb{N} : a_n \notin B(a, r)$  with  $a_n \in A$ . So it holds that

$\forall n \in \mathbb{N} : |a_n - a| \geq r > 0$ . This is contradiction to the assumption that  $a_n$  converges to  $a$  for  $n \rightarrow \infty$ .

$\Leftarrow$  Assume the limes of every convergent sequence with sequence elements in  $A$ , is again in  $A$ . We show that for  $z \notin A$  ( $z \in \mathbb{C} \setminus A$ ) there exists  $\varepsilon > 0$  :  $B(z, \varepsilon) \cap A = \emptyset \Leftrightarrow B(z, \varepsilon) \subseteq \mathbb{C} \setminus A$ .

We prove the existence of such an  $\varepsilon$  by contradiction: So we assume such a  $\varepsilon$  does not exist:

$$\forall \varepsilon > 0 : B(z, \varepsilon) \cap A \neq \emptyset$$

Especially:  $\varepsilon = \frac{1}{n}$  with  $n \in \mathbb{N}_+$ .

$$B(z, \frac{1}{n}) \cap A \neq \emptyset \text{ therefore } \exists a_n \in A \cap B(z, \frac{1}{n})$$

therefore  $a_n \in A \wedge |a_n - z| < \frac{1}{n}$ . So this constructed sequence  $(a_n)_{n \in \mathbb{N}}$  satisfies:

$$a_n \in A : |a_n - z| < \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} a_n = z$$

By hypothesis, it holds that  $z \in A$ , but this is a contradiction to  $z \in \mathbb{C} \setminus A$ . So it is shown that  $\mathbb{C} \setminus A$  is an open set. So  $A$  is closed.  $\square$

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This lecture took place on 17th of December 2015 with lecturer Wolfgang Ring.

TODO

**Definition 46.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . A point  $z \in \mathbb{C}(\mathbb{R})$  is called *contact point* of a set  $M$ , if  $\forall r > 0 : B(z, r) \cap M \neq \emptyset$ . A point  $z \in \mathbb{C}(\mathbb{R})$  is called *limit point* of a set  $M$  if  $\forall r > 0$  it holds that  $B(z, r)$  contains a point  $w \in M$  with  $w \neq z$ .

Every limit point is also contact point.

**Lemma 16.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . It holds that

1.  $z \in \mathbb{C}$  is a contact point of  $M$  if and only if  $\exists (z_n)_{n \in \mathbb{N}} : z_n \in M$  and  $\lim_{n \rightarrow \infty} z_n = z$ .
2.  $z \in \mathbb{C}$  is a limit point of  $M$  if and only if  $\exists (z_n)_{n \in \mathbb{N}} : z_n \in M$  with  $z_n \neq z \forall n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} z_n = z$ .

*Proof.* 1. Let  $z$  be a contact point of  $M$ . Choose  $r_n = \frac{1}{n}$ , due to contact point property there exists  $z_n \in M$  to  $r_n$  with  $z_n \in B(z, \frac{1}{n})$  hence  $|z_n - z| < \frac{1}{n}$ . Then it holds  $\lim_{n \rightarrow \infty} z_n = z$ , then for  $\varepsilon > 0$  arbitrary let  $N$  be arbitrary large, such that  $\frac{1}{N} < \varepsilon$ . Then it holds that for  $n \geq N$ :

$$|z_n - z| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$\Leftarrow$  Assume  $\exists (z_n)_{n \in \mathbb{N}}$  with limes  $z$ .  $z_n \in M$ . Choose  $r > 0$  arbitrary. Due to convergenze of  $(z_n)_{n \in \mathbb{N}}$  there exists some  $N \in \mathbb{N} : n \geq N$  such that  $|z_n - z| < r$ .

$$\Rightarrow z_n \in M \wedge z_n \in B(z, r) \Rightarrow z \text{ is contact point of } M$$

Also,

$$\Rightarrow z_n (\neq z) \in M \wedge z_n \in B(z, r) \Rightarrow z \text{ is limit point of } M$$

$\square$

**Theorem 65.**  $A \subseteq \mathbb{C}(\mathbb{R}, \mathbb{R}^n)$  is closed if and only if for every contact point  $z$  of  $A$  it holds that  $z \in A$ .

*Proof. Direction  $\Rightarrow$*  Let  $A$  be closed and  $z$  is a contact point of  $A$ . Due to Lemma 16 there exists  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in A$  and  $\lim_{n \rightarrow \infty} z_n = z$ . By the Lemma before the last, it holds that  $z \in A$ .

**Direction  $\Leftarrow$**  Assume for all contact points  $z$  of  $A$  it holds that  $z \in A$ . By the Lemma before the last: Let  $(z_n)_{n \in \mathbb{N}}$  be a convergent sequence with  $z_n \in A$  and  $\lim_{n \rightarrow \infty} z_n = z$ .

Show that  $z \in A$ .

This follows immediately because by the previous Lemma, it holds that  $z = \lim_{n \rightarrow \infty} z_n$  is a contact point TODO and by assumption  $z \in A$ .  $\square$

**Remark 20.** In general it holds that  $z \in M$ , then  $z$  is a contact point of  $M$ . Because  $\{z\} \subseteq B(z, r) \cap M$  with  $B(z, r) \cap M \neq \emptyset$ .

**Definition 47.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . We define  $\overline{M} = \{z \in \mathbb{C} : z \text{ is contact point of } M\}$ .  $\overline{M}$  is called *closed hull*. It holds that  $M \subseteq \overline{M}$  and  $M$  is closed  $\Leftrightarrow M = \overline{M}$ .

**Definition 48.** A set  $K \subseteq \mathbb{C}$  ( $\mathbb{R}$ ,  $\mathbb{R}^n$ ) is called *compact*, if for each sequence  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in K$ , a subsequence  $(z_{n_l})_{l \in \mathbb{N}}$  exists which is convergent and its limit is inside  $K$ .

**Remark 21.** There are equivalent definitions which do not use sequences (e.g. using open covers).

**Theorem 66** (Bolzano-Weierstrass theorem for sets).  $K \subseteq \mathbb{C}$  is compact if and only if  $K$  is bounded and closed.

*Proof. Direction  $\Leftarrow$*  Let  $K$  be bounded and closed and let  $(z_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $K$ . Then  $(z_n)_{n \in \mathbb{N}}$  is a bounded sequence. Due to the Bolzano-Weierstrass Theorem for sequences, there exists some convergent subsequence  $(z_{n_l})_{l \in \mathbb{N}}$  with  $\lim_{l \rightarrow \infty} z_{n_l} = z$  where  $z_{n_l} \in K$ . Followingly  $z$  is contact point in  $K$ . Because  $K$  is closed, it holds that  $z \in K$ .

*Direction  $\Rightarrow$*  Let  $K$  be compact. Assume  $K$  is not bounded. Therefore for  $m = 1, 2, \dots, 5$ , there exists  $z_m \in K$  with  $|z_m| > m$ . TODO  $(z_n)_{n \in \mathbb{N}}$  is also unbounded and therefore not convergent. This is a contradiction.

It remains to show that  $K$  is bounded. Let  $z \in \overline{K}$  ( $z$  is a contact point of  $K$ ). There exists a sequence  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in K$  and  $z = \lim_{n \rightarrow \infty} z_n$ . Because  $K$  is compact, there exists a subsequence  $(z_{n_k})_{k \in \mathbb{N}}$  of  $(z_n)_{n \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} z_{n_k} = w$  and  $w \in K$ . Because  $(z_n)_{n \in \mathbb{N}}$  is already convergent, every subsequence is convergent with limit  $z$ . It follows that  $w \in K$  and  $w = z$ , so  $z \in K$ . So  $K$  is bounded.

## 13 Continous functions

**Definition 49.** Let  $D \subseteq \mathbb{C}$  ( $D \subseteq \mathbb{R}$ ) and  $f : D \rightarrow \mathbb{C}$  be a function. We say “ $f$  is continous” (dt. “stetig”) iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in D \text{ with } |z - z_0| < \delta : |f(z) - f(z_0)| < \varepsilon$$

Intuitively, the difference of function values are arbitrary close to each other if the difference of the arguments is sufficiently small.

**Example 23.** 1.  $D$  is “strange”. Specifying the codomain and discussion of continuity in regards of this codomain is very important!

2. A non-continous function  $f$  has a non-continuity in  $z_0$ . So  $\varepsilon$  cannot be arbitrary small.

3.  $f : \mathbb{C} \rightarrow \mathbb{C}$ . TODO Let  $z_0 \in \mathbb{C}$  arbitrary. Then  $f$  is continuous in  $z_0$ .

**Example 24.** Let  $\varepsilon > 0$  be arbitrary. Find  $\delta > 0$  such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| = |z^2 - z_0^2| < \varepsilon.$$

Define  $\delta = \min \left( 1, \frac{\varepsilon}{1+2|z_0|} \right)$ . For  $|z - z_0| < \delta$  it holds that

$$\begin{aligned} |f(z) - f(z_0)| &= |z^2 - z_0^2| = |(z - z_0)(z + z_0)| \\ &= |z - z_0| \cdot |z + z_0| \\ &= |z - z_0 + z_0 + z_0| \cdot |z - z_0| \\ &= \underbrace{|z - z_0 + z_0|}_{=0} + z_0 \cdot |z - z_0| \\ &\leq (\underbrace{|z - z_0|}_{<1} + 2|z_0|) \underbrace{|z - z_0|}_{<\varepsilon} \\ &= (1 + 2|z_0|) \text{ TODO} \end{aligned}$$

**Example 25.** Let  $D = [0, \infty) \subseteq \mathbb{R}$ . Let  $f(x) = \sqrt[k]{x}$  be continuous in every point  $x_0 \in D$ .

Let  $\varepsilon > 0$  be given. Claim: It holds that  $|\sqrt[k]{x} - \sqrt[k]{x_0}| \leq \sqrt[k]{|x - x_0|}$ .

Proof: Show that for  $a, b \geq 0$ , it holds that  $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$ .

Assume  $\sqrt[k]{a+b} > \sqrt[k]{a} + \sqrt[k]{b}$ . Taking the  $k$ -th power keeps monotonicity:

$$\begin{aligned} (\sqrt[k]{a+b})^k &= a+b > \left( \sqrt[k]{a} + \sqrt[k]{b} \right)^k \\ &= a + \underbrace{\sum_{j=1}^{k-1} \binom{k}{j} a^{\frac{k-j}{k}} b^{\frac{j}{k}}}_{>0} + b \geq a+b \end{aligned}$$

This lecture took place on 18th of December 2015 with lecturer Wolfgang Ring.

We prove  $|\sqrt[k]{x} - \sqrt[k]{x_0}| \leq \sqrt[k]{|x - x_0|}$  using  $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$ .

$$\begin{aligned} |x| &= \left| \underbrace{x - x_0}_a + \underbrace{x_0}_b \right| \leq \underbrace{|x - x_0|}_a + \underbrace{|x_0|}_b \\ \sqrt[k]{|x|} &\leq \sqrt[k]{|x - x_0| + |x_0|} \leq \sqrt[k]{|x - x_0|} + \sqrt[k]{|x_0|} \\ \sqrt[k]{|x|} - \sqrt[k]{|x_0|} &\leq \sqrt[k]{|x - x_0|} \end{aligned}$$

Analogously:

$$\begin{aligned} |x_0| &= |x_0 - x + x| \leq \underbrace{|x_0 - x|}_a + \underbrace{|x|}_b \\ \Rightarrow \sqrt[k]{|x_0|} - \sqrt[k]{|x|} &\leq \sqrt[k]{|x - x_0|} \\ \left| \underbrace{\sqrt[k]{|x|}}_{f(x)} - \underbrace{\sqrt[k]{|x_0|}}_{f(x_0)} \right| &\leq \sqrt[k]{|x - x_0|} \end{aligned}$$

Let  $\varepsilon > 0$  arbitrary. Let  $\delta := \varepsilon^k$ . For  $|x - x_0| < \delta = \varepsilon^k$  it holds that

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \sqrt[k]{|x|} - \sqrt[k]{|x_0|} \right| \\ &\leq \sqrt[k]{|x - x_0|} < \sqrt[k]{\delta} = \sqrt[k]{\varepsilon^k} = \varepsilon \quad \checkmark \end{aligned}$$

**Theorem 67** (Sequence criterion for continuity). Let  $f : D \subset \mathbb{C} \Rightarrow \mathbb{C}$  ( $D \subseteq \mathbb{R}$ ). Then it holds that  $f$  is continuous in  $z_0 \in D$  if and only if for every convergent sequence  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D \forall n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  it holds that  $(f(w_n))_{n \in \mathbb{N}}$  is convergent and  $\lim_{n \rightarrow \infty} f(w_n) = f(z_0)$ .

In a different way, this theorem states:

$$w_n \rightarrow_{n \rightarrow \infty} z_0 \Rightarrow f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$$

*Proof.* **Direction  $\Rightarrow$**  Let  $f$  be continuous in  $z_0$  and  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D$  with  $\lim_{n \rightarrow \infty} w_n = z_0$ . Show that  $f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$ .

Let  $\varepsilon > 0$  arbitrary. Because  $f$  is continuous, there exists some  $\delta > 0$  such that  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$  ( $z \in D$ ). So  $(w_n)_{n \in \mathbb{N}}$  converges to  $z_0$ . So there exists  $N \in \mathbb{N} : n \geq N \Rightarrow |w_n - z_0| < \delta$ . For those indices it holds that:  $|f(w_n) - f(z_0)| < \varepsilon$ . Hence  $\lim_{n \rightarrow \infty} f(w_n) = f(z_0)$ .

**Direction  $\Leftarrow$**  Proof by contradiction: For every sequence  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D$  and  $w_n \rightarrow z_0$  it holds that:  $f(w_n) \rightarrow f(z_0)$  for  $n \rightarrow \infty$ . Assume  $f$  is not continuous in  $z_0$ .

So  $\exists \tilde{\varepsilon} > 0 : \forall \delta > 0 \exists z_\delta \in D$  with

$$|z_\delta - z_0| < \delta \wedge |f(z_\delta) - f(z_0)| \geq \varepsilon$$

We choose  $\delta_n = \frac{1}{n}$  for  $n = 1, 2, 3, \dots$

$$w_n := z_{\delta_n}$$

So it holds that

$$\forall n \in \mathbb{N} : |w_n - z_0| < \frac{1}{n} \wedge |f(w_n) - f(z_0)| \geq \tilde{\varepsilon}$$

Hence  $w_n \in D$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  and for  $\varepsilon > 0$  we choose  $N$  such that  $\frac{1}{N} < \varepsilon$ . Then it holds for  $n \geq N$ :  $\frac{1}{n} < \frac{1}{N} < \varepsilon$  and therefore  $|w_n - z_0| < \frac{1}{n} < \varepsilon$ , but  $f(w_n)$  does not converge to  $f(z_0)$ , because  $|f(w_n) - f(z_0)| \geq \tilde{\varepsilon} > 0$ . This is a contradiction to our assumption. □

**Definition 50.** Let  $f : D \rightarrow \mathbb{C}$  ( $D \subseteq \mathbb{C}$  or  $D \subseteq \mathbb{R}$ ). We call  $f$  “continuous on  $D$ ” if  $f$  is continuous in every point  $z \in D$ .

### 13.1 Laws for continuous functions

**Theorem 68.** Let  $f : D \rightarrow \mathbb{C}$  and  $g : D \rightarrow \mathbb{C}$  be functions and  $f$  and  $g$  are continuous in  $z_0 \in D$ . Then it holds that

1.  $(f + g) : D \rightarrow \mathbb{C}$  and  $(f + g)(z) = f(z) + g(z)$ .  
So the sum function  $(f + g)$  is continuous in  $z_0$ .
2.  $(f \cdot g) : D \rightarrow \mathbb{C}$  and  $(f \cdot g)(z) = f(z) \cdot g(z)$ .  
The product function is continuous in  $z_0$ .
3. Let  $g(z) \neq 0 \forall z \in D$ . Then  $\left(\frac{f}{g}\right) : D \rightarrow \mathbb{C}$  with  $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$ .  
The quotient function  $\left(\frac{f}{g}\right)$  is continuous in  $z_0$ .

*Proof.* Let  $(w_n)_{n \in \mathbb{N}}$  be an arbitrary sequence with  $w_n \in D$  and  $\lim_{n \rightarrow \infty} w_n = z_0$ . Due to the sequence criterion it holds that  $f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$  and  $g(w_n) \rightarrow_{n \rightarrow \infty} g(z_0)$ . The laws for convergent sequences state that,

$$\begin{aligned} f(w_n) \cdot g(w_n) &\rightarrow_{n \rightarrow \infty} f(z_0) \cdot g(z_0) \\ f(w_n) + g(w_n) &\rightarrow_{n \rightarrow \infty} f(z_0) + g(z_0) \\ \frac{f(w_n)}{g(w_n)} &\rightarrow_{n \rightarrow \infty} \frac{f(z_0)}{g(z_0)} \end{aligned}$$

Hence  $(f + g)$ ,  $(f \cdot g)$  and  $\left(\frac{f}{g}\right)$  is continuous in  $z_0$ .

**Corollary 12.**

- $k : \mathbb{C} \rightarrow \mathbb{C}$ ,  $k(z) = c \in \mathbb{C}$  is a constant function.  $k$  is continuous in  $\mathbb{C}$ .
- The function  $f(z) = z$  is continuous in  $\mathbb{C}$ , because we can choose  $\delta = \varepsilon$ .

$$|z - z_0| < \varepsilon \Rightarrow |f(z) - f(z_0)| = |z - z_0| < \varepsilon$$

- The functions  $p_n(z) = z^n$  for  $n = 0, 1, 2, \dots$  are continuous in  $\mathbb{C}$  as products of continuous functions.
- All polynomials  $P(z) = \sum_{k=0}^n a_k z^k$  with  $a_k \in \mathbb{C}$  are continuous in  $\mathbb{C}$ .
- Let  $D = B(0, \rho(P))$  with  $\rho(P)$  is convergence radius of the power series

$$P(z) = \sum_{k=0}^{\infty} a_k z^k$$

Then  $P(z)$  is continuous in  $B(0, \rho(P))$ .

- Let  $P(z) = \sum_{k=0}^n a_k z^k$  and  $Q(z) = \sum_{l=0}^m b_l z^l$  be polynomials. And let  $D = \{z \in \mathbb{C} : Q(z) \neq 0\}$ . Then  $\left(\frac{P}{Q}\right) : D \rightarrow \mathbb{C}$  is continuous in  $D$ . Therefore all rational functions are continuous in all points except for the roots of the denominator:

$$\frac{a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \dots + b_m z^m}$$

**Theorem 69.** Let  $f : D \rightarrow U \subseteq \mathbb{C}$  and  $g : U \rightarrow \mathbb{C}$  be two functions. Let  $f$  be continuous in  $z_0 \in D$  and let  $g$  be continuous in  $y_0 = f(z_0) \in U$ . Then  $g \circ f : D \rightarrow \mathbb{C}$  is continuous in  $z_0$ .

*Proof.* Due to the sequence criterion: Let  $(w_n)_{n \in \mathbb{N}}$  ( $w_n \in D$ ) with  $\lim_{n \rightarrow \infty} w_n = z_0$ . The sequence criterion for  $f$  yields

$$\lim_{n \rightarrow \infty} \underbrace{f(w_n)}_{\in U} = f(z_0) = y_0$$

□ The sequence criterion for  $g$  states that

$$\lim_{n \rightarrow \infty} \underbrace{g(f(w_n))}_{g \circ f(w_n)} = g(y_0) = \underbrace{g(f(z_0))}_{g \circ f(z_0)}$$

So  $g \circ f$  is continuous in  $z_0$ . □

We know  $w_k(x) = \sqrt[k]{x}$  is continuous in  $[0, \infty)$ .

$$P_l(x) = x^l \text{ is continuous in } \mathbb{C}$$

$$\Rightarrow P_l \circ w_k \text{ is continuous in } [0, \infty)$$

$$p_0 \circ w_k(x) = p_l(\sqrt[k]{x}) = (\sqrt[k]{x})^l = x^{\frac{l}{k}} \text{ is continuous.}$$

- $n(z) = |z|$  is continuous in  $\mathbb{C}$ .

Let  $\varepsilon > 0$  be arbitrary. It holds that

$$|n(z) - n(z_0)| = ||z| - |z_0|| \leq |z - z_0|$$

Choose  $\delta = \varepsilon$ . Then for  $|z - z_0| < \delta = \varepsilon$  it holds that  $|n(z) - n(z_0)| < \varepsilon$ .

- $\Re : \mathbb{C} \rightarrow \mathbb{C}$  and  $\Im : \mathbb{C} \rightarrow \mathbb{C}$  are continuous in  $\mathbb{C}$ . Because  $|\Re(z) - \Re(z_0)| \leq |z - z_0|$  ✓.
- Let  $f, g : D \rightarrow \mathbb{R}$ . Then  $\max(f, g) : D \rightarrow \mathbb{R}$   $(\max(f, g))(z) = \max\{f(z), g(z)\}$  is continuous in  $D$ . because  $\max f(z), g(z) = \frac{1}{2}(|f(z) - g(z)| + f(z) + g(z))$ .

This lecture took place on 7th of January 2016 with lecturer Wolfgang Ring.

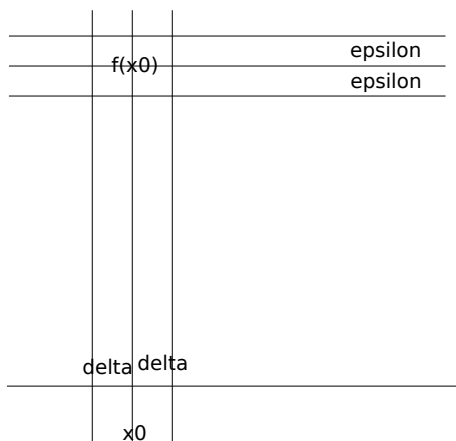
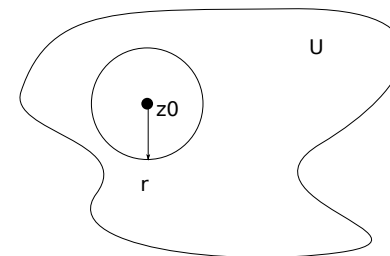


Figure 17: The notion of continuity


 Figure 18: Environment with radius  $r$ 

### 13.2 Revision of the continuity definition

$f$  is continuous in  $x_0$  if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \wedge |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon]$$

Reminder: Let  $z_0 \in U \subseteq \mathbb{C}$ .  $U$  is called environment of  $z_0$  if  $r > 0$  exists such that  $B(z_0, r) \subset U$ .

**Definition 51.** Let  $D \subseteq \mathbb{C}$  and  $z_0 \in U \subseteq D$ . We call  $U$  *environment* of  $z_0$  in  $D$  if  $\exists r > 0$  such that  $B(z_0, r) \cap D \subseteq U$ .

**Theorem 70.** Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$ . Let  $z_0 \in D$ . Then  $f$  is continuous in  $z_0$  if and only if for every environment  $U$  of  $y_0 = f(z_0)$  it holds that  $V = f^{-1}(U)$  is an environment of  $z_0 \in D$  (where  $f^{-1}$  denotes the preimage).

*Proof.*  $\Rightarrow$

Let  $f$  be continuous in  $z_0$  and let  $U$  be an environment of  $y_0 = f(z_0)$ , hence

$\exists \varepsilon > 0 : B(y_0, \varepsilon) \subseteq U$  with  $y_0 = f(z_0)$ . Because  $f$  is continuous in  $z_0$ , it holds that

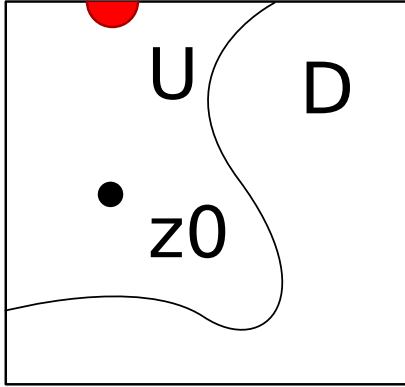
$$\exists \delta > 0 : |z - z_0| < \delta \wedge z \in D \Rightarrow \underbrace{|f(z) - f(z_0)|}_{\substack{f(z) \in B(f(z_0), \varepsilon) \\ B(y_0, \varepsilon) \subseteq U}} < \varepsilon.$$

This requires:

$$\begin{aligned} |z - z_0| < \delta \wedge z \in D &\Leftrightarrow z \in B(z_0, \delta) \wedge z \in D \\ &\Rightarrow z \in B(z_0, \delta) \cap D \end{aligned}$$

Therefore we can redefine continuity as:

$$z \in B(z_0, \delta) \cap D \Rightarrow f(z) \in B(y_0, \varepsilon)$$


 Figure 19: Environment  $U$ 

So it holds that

$$\forall z \in B(z_0, \delta) \cap D \Rightarrow z \in f^{-1}(B(y_0, \varepsilon)) \subseteq f^{-1}(U)$$

So it holds that  $B(z_0, \delta) \cap D \subseteq f^{-1}(U)$ .

⇐

Let the preimage of every environment in  $y_0$  be an environment of  $z_0$  in  $D$ . Let  $\varepsilon > 0$  arbitrary. Then it holds that  $B(y_0, \varepsilon)$  is an environment of  $y_0$ . By assumption it holds that  $V = f^{-1}(B(y_0, \varepsilon))$  is an environment of  $z_0$  in  $D$ , hence

$$\exists \delta > 0 : B(z_0, \delta) \cap D \subseteq f^{-1}(B(y_0, \varepsilon)).$$

Therefore for  $z \in B(z_0, \delta) \cap D$  it holds that  $f(z) \in B(y_0, \varepsilon)$ .

In other words:

$$|z - z_0| < \delta \wedge z \in D \Rightarrow |f(z) - \underbrace{f(z_0)}_{=y_0}| < \varepsilon$$

So  $f$  is continuous in  $z_0$ .

This notion of continuity is the most general one accepted by the mathematical community. It can be used in all topological spaces.  $\square$

### 13.3 Variants of continuity

**Definition 52.** Let  $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function  $f$  called uniformly continuous in  $D$  if

$$\forall \varepsilon > 0 \exists \delta > 0 : [\forall z_0, z_1 \in D \text{ with } |z_1 - z_0| < \delta \Rightarrow |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that  $\delta$  only depends on  $\varepsilon$ , meaning that it can be arbitrarily shifted on the  $x$ -axis ( $\delta = \delta(\varepsilon)$ ).

Reminder:  $f$  is continuous in  $D$

$$\Leftrightarrow \forall z_0 \in D \forall \varepsilon > 0 \exists \delta > 0 : [\forall z_1 \in D \wedge |z_1 - z_0| < \delta \Rightarrow |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that  $\delta$  depends on  $z_0$  and  $\varepsilon$  ( $\delta = \delta(\varepsilon, z_0)$ ). Therefore this second definition provides more freedom to parameter  $\delta$ . So uniform continuity implies continuity in  $D$ .

**Example 26.** Let  $f : (0, 1] \rightarrow \mathbb{C}$  and  $f(x) = \frac{1}{x}$ .  $f$  is continuous in every point  $x_0 \in (0, 1]$ . However,  $f$  is not uniformly continuous.

$$\forall \varepsilon > 0 \exists \delta > 0 : \left[ \forall x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \Rightarrow \left| \frac{1}{x_0} - \frac{1}{x_1} \right| < \varepsilon \right]$$

The negation is given with:

$$\exists \varepsilon > 0 \forall \delta > 0 : \left[ \exists x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \wedge \left| \frac{1}{x_0} - \frac{1}{x_1} \right| \geq \varepsilon \right]$$

We look at  $\varepsilon = 1$ . Let  $\delta > 0$  arbitrary. We choose  $x_0 = \frac{1}{n}$  and  $x_1 = \frac{1}{n+1}$  for appropriate  $n \in \mathbb{N}_+$ . Then it holds that

$$|x_0 - x_1| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \underbrace{\leq}_{\text{for } n \in \mathbb{N}_+} \frac{1}{n} < \delta$$



if  $n > \frac{1}{\delta}$

$$\left| \frac{1}{x_0} - \frac{1}{x_1} \right| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = |n - (n+1)| = |-1| = 1$$

Therefore  $f(x) = \frac{1}{x}$  is not uniformly continuous in  $(0, 1]$ .

Remark:  $f(x) = \frac{1}{x}$  is uniformly continuous in  $D = [\frac{1}{100}, 1]$ , but not in  $\mathbb{R}$ .

**Definition 53** (Lipschitz continuity). Another notion of continuity is given by Rudolf Lipschitz (1832–1903).

$f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  is called Lipschitz continuous if  $k \geq 0$  exists such that

$$\forall z_1, z_2 \in D : |f(z_1) - f(z_2)| \leq k |z_1 - z_2|$$

The value  $k$  is called Lipschitz constant for  $f$ .

**Definition 54** (Hölder continuity). Yet another notion of continuity is given by Otto Hölder (1859–1937).

$f$  is called Hölder continuous with exponent  $H \in (0, 1]$  if there exists  $k > 0$  such that

$$\forall z_1, z_2 \in D : |f(z_1) - f(z_2)| \leq k |z_1 - z_2|^H$$

**Corollary 13.** A hierarchy for those continuity notion is given:

Lipschitz continuous  $\subseteq$  uniformly continuous  $\subseteq$  continuous in  $D$ .

**Theorem 71.** Let  $K \subseteq \mathbb{C}$  be compact. Let  $f : K \rightarrow \mathbb{C}$  be continuous in  $K$ . Then  $f(K) = \{y = f(z) : z \in K\} \subset \mathbb{C}$  is compact in  $\mathbb{C}$ .

*Proof.* Every sequence  $(y_n)_{n \in \mathbb{N}}$ , with  $y_n = f(z_n)$  and  $z_n \in K$  where  $y_n \in f(K)$ , has a convergent subsequence. The sequence of preimage values  $(z_n)_{n \in \mathbb{N}}$  is a sequence in  $K$  which, followingly, has a convergent subsequence. Let  $(z_{n_k})_{k \in \mathbb{N}}$   $\lim_{k \rightarrow \infty} z_{n_k} = z \in K$ . Because of the sequence criterion for continuity it holds that

$$\lim_{k \rightarrow \infty} y_{n_k} = \lim_{k \rightarrow \infty} f(z_{n_k}) = f(z) \in f(K)$$

with  $y = f(k)$ . So  $(y_n)_{n \in \mathbb{N}}$  has a convergent subsequence with limes  $y \in f(K)$ . Therefore  $f(K)$  is compact.  $\square$

**Definition 55.** Let  $f : D \rightarrow \mathbb{R}$  and  $D \subseteq \mathbb{C}$ . A point  $z_{\max} \in D$  is called global maximum of  $f$  if  $f(z_{\max}) \geq f(z) \quad \forall z \in D$ .

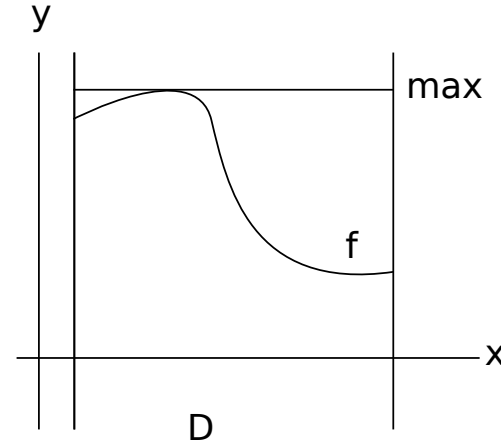


Figure 20: Illustration of a global maximum

**Definition 56.** Let  $f : K \rightarrow \mathbb{R}$  ( $K \subseteq \mathbb{C}$ ) is continuous in  $K$  and  $K$  is compact in  $\mathbb{C}$ . Then  $f$  has a global maximum and a global minimum.

**Remark 22.** For non-compact definition sets this statement is not generally true. For example,  $f(x) = \frac{1}{x}$  in  $D = (0, 1)$  has neither a global maximum nor a global minimum.

*Proof.*

$$f(K) \subseteq \mathbb{R}$$

is compact (because of the previous theorem) and therefore bounded and closed in  $\mathbb{R}$  (by Theorem by Bolzano-Weierstrass). Because  $f(K)$  is bounded,  $f(K)$  has a supremum  $\zeta^*$  and an infimum  $\zeta_*$  (supremum property). Supremum and infimum are contact points of  $f(K)$ . Because  $f(K)$  is closed it holds that

$$\zeta^* \in f(K) \text{ and } \zeta_* \in f(K)$$

Therefore there exists  $z_{\min} \in K$  with  $f(z_{\min}) = \zeta_*$  and  $z_{\max} \in K$  with  $f(z_{\max}) = \zeta^*$ . Because  $f(K)$  is closed, it holds that  $\zeta^* \in f(K)$  and  $\zeta_* \in f(K)$ . Therefore there exists  $z_{\min} \in K$  with  $f(z_{\min}) = \zeta_*$  and  $z_{\max} \in K$  with  $f(z_{\max}) \geq y$ , therefore  $\forall z \in K : f(z_{\max}) \geq f(z)$ .

Therefore  $z_{\max}$  is a global maximum. The analogous statement holds for  $\zeta_*$  and a global minimum.  $\square$

**Theorem 72** (A very universal theorem about maxima). A continuous function has a global maximum in a compact domain.

Using this method to show existence of a value is called “direct method of variation computations”.

This lecture took place on 8th of January 2016 with lecturer Wolfgang Ring.

Continuity and compactness implies existence of a maximum and minimum.

“Direct method of calculus of variations” (dt. “direkte Methode der Variationsrechnung”).

**Theorem 73** (Intermediate value theorem for continuous functions). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous with  $a \leq b$ . Let

$$m^* = \max \{f(x) : x \in [a, b]\}$$

$$m_* = \min \{f(x) : x \in [a, b]\}$$

$m^*$  and  $m_*$  exist because  $[a, b]$  is compact (bounded and closed).

Let  $m_* \leq \eta \leq m^*$ .

Then there exists  $\xi \in [a, b]$  with  $f(\xi) = \eta$ . The function  $f$  takes any value for some  $x$  in  $m_*$  and  $m^*$ . Compare with Figure 21.

*Proof.* Let  $a_0 \in [a, b]$  such that  $f(a_0) = m_*$  and  $b_0 \in [a, b]$  such that  $f(b_0) = m^*$ . Without loss of generality:  $a_0 = b_0$ . If  $a_0 > b_0$  it holds that  $\max \{f(x) : x \in [a, b]\} = f(b_0) = f(a_0) = \min \{f(x) : x \in [a, b]\}$ . If  $\max = \min$ , then  $f$  is constant, hence  $f(x) = m_* = m^* \quad \forall x \in [a, b]$ .

$$m_* = \eta \leq m^* \Rightarrow \eta = m_* = m^* \wedge f(x) = \eta \quad \forall x \in [a, b]$$

Consider  $a_0 \leq b_0$ . We know,  $f(a_0) = m_* \leq \eta \leq m^* = f(b_0)$ . We use nested intervals:

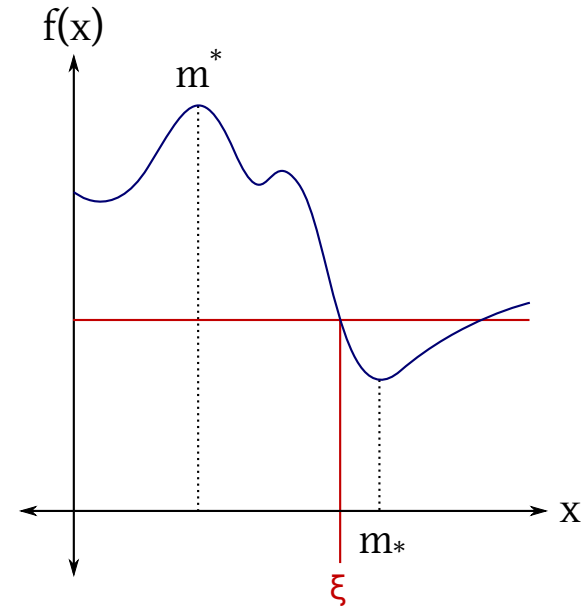


Figure 21:  $\zeta$  in  $[a, b]$

Assume  $I_n = [a_n, b_n]$  for  $n \in \mathbb{N}$  was already found with the property  $f(a_n) \leq \eta \leq f(b_n)$ . Let  $m_n = \frac{1}{2}(a_n + b_n)$  be the midpoint of  $I_n$ .

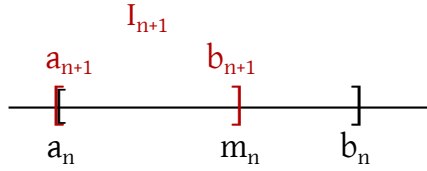
**Case  $f(m_n) \geq \eta$**  If  $f(m_n) \geq \eta$  we set  $b_{n+1} = m_n$  and  $a_{n+1} = a_n$  (compare Figure ??) and it holds that  $f(a_{n+1}) = f(a_n) \leq \eta$  and  $f(b_{n+1}) = f(m_n) \geq \eta$ . Furthermore  $|I_{n+1}| = \frac{1}{2} |I_n|$ .

**Case  $f(m_n) < \eta$**  Let  $a_{n+1} = m_n$  and  $b_{n+1} = b_n$ .

$$I_{n+1} = [a_{n+1}, b_{n+1}]$$

$$f(a_{n+1}) = f(m_n) < \eta$$

$$f(b_{n+1}) = f(b_n) \geq \eta$$


 Figure 22: Interval  $I_{n+1}$ 

$$|I_{n+1}| = \frac{1}{2} |I_n|$$

Nested interval  $(I_n)_{n \in \mathbb{N}}$  has the property:

$$I_{n+1} \leq I_n \quad |I_n| = \left(\frac{1}{2}\right)^n \cdot |I_0| = \left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)$$

and  $f(a_n) \leq \eta \leq f(b_n)$ .  $(I_n)_{n \in \mathbb{N}}$  is are nested intervals. Let  $\xi \in \bigcap_{n \in \mathbb{N}} I_n$  and it holds that  $|\xi - a_n| \leq |b_n - a_n| = \underbrace{\left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)}_{\rightarrow 0 \text{ for } n \rightarrow \infty}$ . Therefore  $\lim_{n \rightarrow \infty} a_n = \xi$

and

$$|b_n - \xi| \leq |b_n - a_n| = \underbrace{\left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)}_{\rightarrow 0 \text{ for } n \rightarrow \infty}.$$

So  $\lim_{n \rightarrow \infty} b_n = \xi$ .

Because  $f$  is continuous on  $[a, b]$ , it holds that

$$\begin{aligned} \eta \leq f(b_n) \quad \forall n \in \mathbb{N} &\Rightarrow \eta \leq \lim_{n \rightarrow \infty} f(b_n) \\ \text{continuity} &\Rightarrow \lim_{n \rightarrow \infty} f(b_n) = f(\xi) \end{aligned}$$

So,

$$\eta \leq \lim_{n \rightarrow \infty} f(b_n) = f(\xi) = \lim_{n \rightarrow \infty} f(a_n) \leq \eta.$$

Therefore  $\eta = f(\xi)$ .

**Remark 23.** From this we can derive continuity for a numerical algorithm for solving  $f(x) = \eta$ . It's called *bisection method*.

**Remark 24.** Often the intermediate value theorem is defined as:

Let  $\eta$  be between  $f(a)$  and  $f(b)$ . Then there exists  $\xi \in [a, b]$  such that  $f(\xi) = \eta$ . Obviously because  $m_* \leq f(a)$  and  $f(b) \leq m^*$ .

**Definition 57** (Limes of a function). Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$ . Let  $z$  be a limit point of  $D$ . We say, that  $f$  in  $z$  has the limes  $w$  if the function

$$\hat{f} : D \cup \{z\} \rightarrow \mathbb{C}$$

$$\hat{f}(\xi) = \begin{cases} f(\xi) & \text{if } \xi \neq z \\ w & \text{if } \xi = z \end{cases}$$

is continuous. We denote  $\lim_{\xi \rightarrow z} f(\xi)$ .

**Example 27.** See Figures 23, 24 and 25.

**Lemma 17.** Let  $f : D \rightarrow \mathbb{C}$  given and  $z$  is a limit point of  $D \subseteq \mathbb{C}$ . Then  $f$  has a limes  $w \in \mathbb{C}$  if and only if one of the equivalent conditions hold.

$$\bullet \quad \forall \varepsilon > 0 \exists \delta > 0 \forall \xi \in D : |z - \xi| < \delta \Rightarrow \underbrace{|f(\xi) - w|}_{\hat{f}(\xi) - \hat{f}(z)} < \varepsilon$$

“Continuity of  $\hat{f}$ ”

$$\bullet \quad \forall (\xi_n)_{n \in \mathbb{N}} \text{ with } \xi_n \in D \setminus \{z\} \text{ and } \lim_{n \rightarrow \infty} \xi_n = z \text{ holds.}$$

$$\lim_{n \rightarrow \infty} f(\xi_n) = w$$

“Sequence criterion for  $\hat{f}$ ”

**Example 28.**  $f : \mathbb{C} \setminus \{1\} \rightarrow \mathbb{C}$  with

$$f(z) = \frac{z^2 - 1}{z - 1}$$

For  $z \neq 1$  it holds that:

$$f(z) = \frac{(z - 1)(z + 1)}{(z - 1)} = (z + 1)$$

□

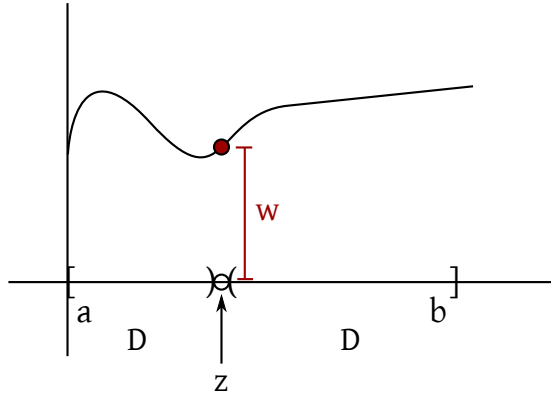


Figure 23: Example 1 with  $D = [a, b] \setminus \{z\}$  and  $w = \lim_{\xi \rightarrow z} f(\xi)$

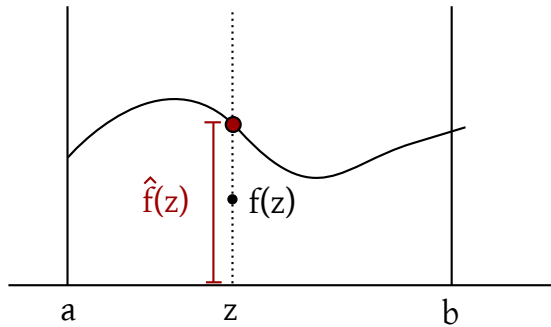


Figure 24: Example 2 which defines function new in point  $z$  with  $D = [a, b]$  and  $\lim_{\xi \rightarrow z} f(\xi)$ .  $f$  is not continuous in  $z$ , but  $\hat{f}$  is continuous in  $z$ .

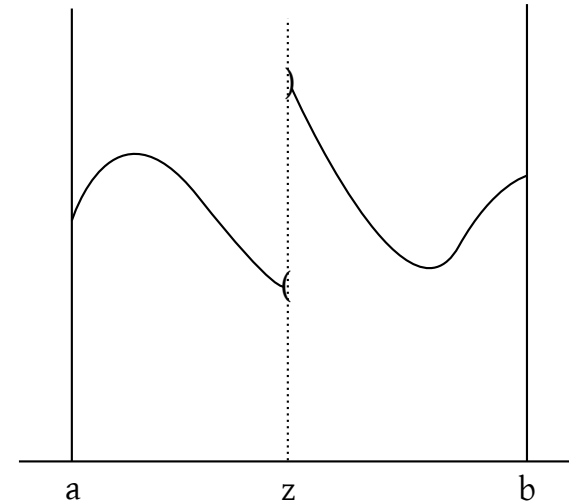


Figure 25: Example 3 with  $D = [a, b] \setminus \{z\}$ .  $f$  does not have a limit in  $z$ . Due to the jumping point, it is not a continuous function. Therefore we cannot find  $\varepsilon$ . We say  $\hat{f}$  is a continuous continuation of  $f$  in point  $z$ .

Let

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq 1 \\ 2 & \text{if } z = 1 \end{cases}$$

$\hat{f}(z) = z + 1$  in  $\mathbb{C}$  is continuous.  $f$  has limes  $w = 2$  in point  $z = 1$ .

**Example 29.** Let  $s \in \mathbb{Q} \setminus \{0\}$  and  $D = (-1, \infty) \setminus \{0\}$

$$f(x) = \frac{(1+x)^s - 1}{x}$$

It holds that  $\lim_{x \rightarrow 0} f(x) = s$ .

for  $|x| < 1$ .

$$\begin{aligned} (1+x)^s &= \sum_{k=0}^{\infty} \binom{s}{k} x^k \Rightarrow \frac{(1+x)^s - 1}{x} \\ &\Rightarrow \frac{(1+x)^s - 1}{x} = \frac{\sum_{k=1}^{\infty} \binom{s}{k} x^k}{x} = \sum_{k=1}^{\infty} \binom{s}{k} \cdot x^{k-1} \\ \lim_{x \rightarrow 0} \underbrace{\left( \sum_{k=1}^{\infty} \binom{s}{k} x^{k-1} \right)}_{f(x)} &= \sum_{k=1}^{\infty} \binom{s}{k} 0^{k-1} = \binom{s}{1} = s \end{aligned}$$

We need the following theorem: A power series is in its convergence radius a continuous function.  $\square$

## 14 Differential calculus

Let  $f : (a, b) \rightarrow \mathbb{R}$  be given. with  $a < b$ .

Idea: We want  $f$  close to point  $x_0 \in (a, b)$  be approximated by a linear-affine function  $a(x) = k(x - x_0) + d$ .

$$a(x) = k(x - x_0) + d = kx + \underbrace{(-kx_0 + d)}_{\tilde{d}} = kx + \tilde{d}$$

$\tilde{a}(x) = kx$  is linear. Linear and constant functions are linear affine.  $a$  should (at least) cross point  $x_0$ , ie.  $f(x_0)$ . Compare with Figure 26.

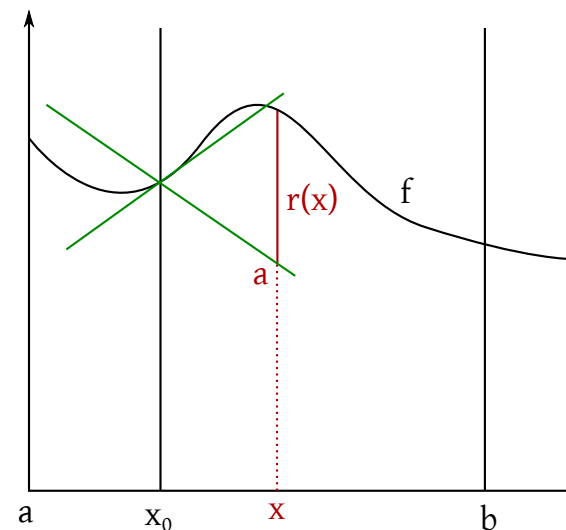


Figure 26: Differential  $f$  in  $x_0$

$$\Rightarrow a(x_0) = k(\underbrace{x_0 - x_0}_0) + a \stackrel{!}{=} f(x_0) \Rightarrow d = f(x_0)$$

$$\Rightarrow a(x) = k(x - x_0) + f(x_0)$$

How should we select  $k$  such that the approximation of  $f$  is best possible by selection of  $a$ . We consider the deviation.

$$f(x) = f(x) - a(x)$$

$r(x)$  should be as small as possible in  $x_0$ . Therefore  $\lim_{x \rightarrow x_0} r(x) = 0$ .

$$\lim_{x \rightarrow x_0} r(x) = \lim_{x \rightarrow x_0} [f(x) - f(x_0) - k \cdot (x - x_0)] = 0 \quad \forall k$$

We need:  $r(x)$  should converge to 0 very quickly for  $x \rightarrow x_0$ .

Idea: Require that  $\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0$ .  $\frac{1}{x - x_0}$  is unbounded close to  $x_0$ .  
 $\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0$  means  $\lim_{x \rightarrow x_0} \left| \frac{r(x)}{x - x_0} - 0 \right| = 0$

$$\Rightarrow \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0) - k \cdot (x - x_0)}{x - x_0} \right| = \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - k \right|$$

Hence,

$$\Rightarrow k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

with

$$\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0,$$

$k$  is uniquely identified with

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

---

This lecture took place on 13th of January 2016 with lecturer Wolfgang Ring.

TODO: radius.pdf

TODO: another image missing

$$y = kx + d$$

$$d = k \cdot (x_0) - k \cdot x_0$$

TODO: missing a few lines

**Definition 58** (Landau's symbols). Let  $g : D \rightarrow \mathbb{C}$ ,  $D \subseteq \mathbb{C}$ . Let  $z_0$  be a limit point of  $g$  and assume  $g$  has a limit point 0 for  $z \rightarrow z_0$ . Therefore,

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall z \in D \wedge |z - z_0| < \delta$$

where  $z \neq z_0$ .

$$\Rightarrow |g(z) - 0| < \varepsilon$$

We say that  $y$  is of order  $\mathcal{O}(n)$  in point  $z_0$ , if  $k \geq 0$  and some  $r > 0$  such that

$$|g(z)| \leq K |z - z_0|^n \quad \forall z \in D \text{ with } |z - z_0| < r \wedge z \neq z_0$$

We denote it with  $g(z) = \mathcal{O}(|z - z_0|^n)$ .

We say that  $g$  is of order  $o(n)$  if  $r > 0$  and some function  $k : (0, r) \rightarrow \mathbb{R}^+$  with  $\lim_{x \rightarrow 0} k(x) = 0$  exists, such that

$$|g(z)| \leq k(|z - z_0|) \cdot |z - z_0|^n \quad \forall z \in D \text{ with } |z - z_0| < r \wedge z \neq z_0$$

We denote,

$$g(z) = o(|z - z_0|^n)$$

**Corollary 14.** It holds that,  $g : \mathcal{O}(|z - z_0|^n) \Leftrightarrow \exists r > 0$  such that

$$\frac{|g(z)|}{|z - z_0|^n}$$

is bounded in  $B(z_0, r) \setminus \{z_0\}$  and  $g = o(|z - z_0|^n)$ , if  $\exists r > 0$  such that  $\frac{|g(z)|}{|z - z_0|^n}$  in point  $z_0$  has limit point 0.

**Corollary 15.** For determination of the slope  $k$  for the best-achievable linear-affine approximation of  $f$  it must hold that

$$f(x) - (f(x_0) - k(x - x_0)) = o(|x - x_0|)$$

**Definition 59.** Let  $f : (a, b) \rightarrow \mathbb{R}$  and  $x_0 \in (a, b)$ . We claim that  $f$  in  $x_0$  is *differentiable*, if the limit point of the function  $\frac{f(x) - f(x_0)}{x - x_0}$  exists. The corresponding limit point  $k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$  is called *derivative of  $f$  in  $x_0$* .

We can compute  $k$  using  $k = f'(x_0)$ .

Alternatively:  $f$  is differentiable in  $x_0$  if  $x \in \mathbb{R}$  exists, such that  $r : (a, b) \setminus \{0\} \rightarrow \mathbb{R}$  with  $r(x) = f(x) - f(x_0) - k(x - x_0)$  is of order  $o(1)$  in  $x_0$ .

$$f(x) - f(x_0) - k(x - x_0) = \mathcal{O}(|x - x_0|)$$

The second definition is more general and can also be applied for functions  $f : \mathcal{O} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**Corollary 16.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ . Then the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \in (a, b) \setminus \{x_0\} \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

$\varphi : (a, b) \rightarrow \mathbb{R}$  and  $\varphi$  is continuous in  $x_0$ .

Show that  $\lim_{x \rightarrow x_0} \varphi(x) = \varphi(x_0)$ .

$$f(x) = f(x_0) + \varphi(x)(x - x_0)$$

because  $\varphi(x) = \frac{f(x)-f(x_0)}{x-x_0}$  for  $x \neq x_0$ .  $f(x)$  is constant,  $\varphi(x)$  is continuous in  $x_0$  and  $(x - x_0)$  is continuous in  $(a, b)$ . For  $x = x_0$ ,  $f(x) = f(x_0) + \varphi(x)(x - x_0)$  holds as well.

Therefore all expressions of  $f(x_0) + \varphi(x)(x - x_0)$  are continuous in  $x_0$ , followingly  $f$  is continuous in  $x_0$ .

**Lemma 18.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ .

$$k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is slope of affine function, which approximates  $f$  in  $x_0$ .

Plot of this function:

$$y(x) = f'(x_0)(x - x_0) + f(x_0)$$

is called *tangent* of  $f$  in  $x_0$ .

---

This lecture took place on 14th of Jan 2016 with lecturer Wolfgang Ring.

**Theorem 74** (Convergence, limes and differentiable functions). Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ . Therefore the equivalent defining properties hold.

1.  $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in (a, b)$  with  $|x - x_0| < \delta$  and  $x \neq x_0$  it holds that  $\left| \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \right| < \varepsilon$ . This constitutes a definition of the limes.

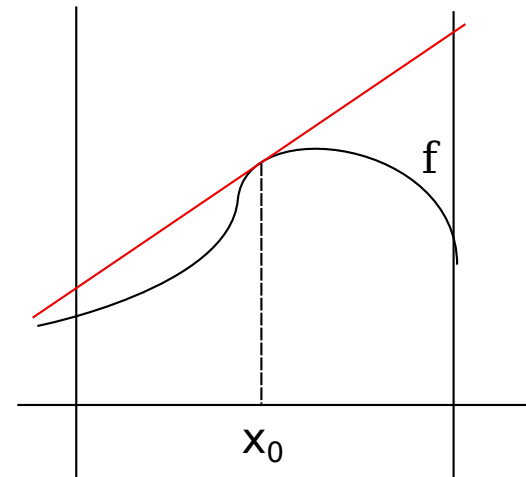


Figure 27: Tangent of  $f$  in  $x_0$

2. For all  $(\xi_n)_{n \in \mathbb{N}}$  with  $\xi_n \in (a, b)$  and  $\xi_n \neq x_0$  and  $\lim_{n \rightarrow \infty} \xi_n = x_0$ , it holds that

$$\left( \frac{f(\xi_n) - f(x_0)}{\xi_n - x_0} \right)_{n \in \mathbb{N}} \text{ is convergent towards } f'(x_0)$$

This is the sequence criterion for the limes.

3. For all  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that  $\forall x \in (a, b)$  with  $|x - x_0| < \delta$  it holds that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \varepsilon |x - x_0|$$

holds also for  $x = x_0$ .

The (3) implies the (1): Assume (3) holds and choose  $\delta$  such that  $\forall |x - x_0| < \delta$

it holds that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{\varepsilon}{2} |x - x_0| \underbrace{\leq}_{\text{for } x \neq x_0} \varepsilon |x - x_0| \underbrace{\Rightarrow}_{\text{divide by } x - x_0} (1)$$

### 14.1 Derivation of common functions

Let  $p_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $p_n(x) = x^n$ . Let  $x_0 \in \mathbb{R}$  and  $x \neq x_0$  and  $n \in \mathbb{N}$ . Then it holds that

$$\begin{aligned} \frac{p_n(x) - p_n(x_0)}{x - x_0} &= \frac{x^n - x_0^n}{x - x_0} \\ &= \frac{(x - x_0) \cdot \sum_{k=0}^{n-1} x^k x_0^{n-1-k}}{x - x_0} \\ &= \sum_{k=0}^{n-1} x^k x_0^{n-1-k} \\ &\xrightarrow{x \rightarrow x_0} \sum_{k=0}^{n-1} x_0^k x_0^{n-1-k} \\ &= \sum_{k=0}^{n-1} x_0^{n-1} \\ &= n x_0^{n-1} \end{aligned}$$

Therefore  $p_n$  is differentiable in  $x_0$  and  $p'_n(x_0) = n x_0^{n-1}$ .

$$(x^n)' = n x^{n-1} \quad \forall n \in \mathbb{N}$$

1. Let  $f(x) = a^x$  with  $a > 0$ . This function is called *exponential function* with

basis  $a$ . It holds that:

$$\begin{aligned} \frac{a^x - a^{x_0}}{x - x_0} &= \frac{a^{x_0} \cdot a^{x-x_0} - a^{x_0}}{x - x_0} \\ &= a^{x_0} \cdot \frac{a^{x-x_0} - 1}{x - x_0} \\ &\xrightarrow{x \rightarrow x_0} a^{x_0} \cdot \lim_{x \rightarrow x_0} \frac{a^{x-x_0} - 1}{x - x_0} \end{aligned}$$

$$\left| \begin{array}{c} x - x_0 = h \\ x \rightarrow x_0 \Leftrightarrow h \rightarrow 0 \end{array} \right| = a^{x_0} \lim_{h \rightarrow 0} \underbrace{\frac{a^h - 1}{h}}_{=c \in \mathbb{R}}$$

Therefore  $|a^x|' = c \cdot a^x$  with  $c = \lim_{h \rightarrow 0}$ . TODO content missing

In the special case that this constant  $c$  is the Eulerian number  $e$ , it holds that:

$$(e^x)' = e^x$$

2.  $\log : (0, \infty) \rightarrow \mathbb{R}$  with  $e^{\log x} = x \forall x > 0$  or equivalently  $\log(e^y) = y \forall y \in \mathbb{R}$ .

$$\frac{\log x - \log x_0}{x - x_0} = \frac{\log \frac{x}{x_0}}{\frac{x}{x_0} - 1} = \frac{1}{x_0} \frac{\log \frac{x}{x_0}}{\frac{x}{x_0} - 1} \rightarrow \frac{1}{x_0} \cdot \lim_{h \rightarrow 1} \underbrace{\frac{\log h}{h-1}}_{=1} = \frac{1}{x_0}$$

Therefore  $(\log x)' = \frac{1}{x}$  for  $x > 0$ .

### 14.2 Derivation laws

**Theorem 75.** Let  $f, g : (a, b) \rightarrow \mathbb{R}$ . Let  $x_0 \in (a, b)$  and let  $f, g$  be differentiable in  $x_0$ . Then it holds that

- $f + g : (a, b) \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and the derivative is given by  $(f + g)'(x_0) = f'(x_0) + g'(x_0)$ .
- Let  $\lambda \in \mathbb{R}$ . Then it holds that  $\lambda \cdot f : (a, b) \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and it holds that  $(\lambda f)'(x_0) = \lambda \cdot (f'(x_0))$ .
- Let  $f \cdot g : (a, b) \rightarrow \mathbb{R}$  be differentiable and it holds that

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + g'(x_0) \cdot f(x_0)$$



This is the so-called *product law for derivatives*.

*Proof.* • Addition holds:

$$\begin{aligned} f'(x_0) + g'(x_0) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} = (f + g)'(x_0) \end{aligned}$$

• Multiplication with a scalar holds:

$$\lambda f'(x_0) = \lambda \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\lambda f(x) - \lambda f(x_0)}{x - x_0} = (\lambda f)'(x_0)$$

• The product law holds:

$$\begin{aligned} &f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ &= g(x_0) \cdot \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \underbrace{f(x_0)}_{=\lim_{x \rightarrow x_0} f(x)} \cdot \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \end{aligned}$$

because  $f$  is differentiable and therefore continuous in  $x_0$ .

$$\begin{aligned} &= \lim_{x \rightarrow x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{f(x) \cdot g(x) - f(x) \cdot g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0) + g(x)f(x) - g(x_0)f(x)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x) \cdot g(x) - f(x_0)g(x_0)}{x - x_0} = (f \cdot g)'(x_0) \end{aligned}$$

□

**Definition 60.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be given. Assume  $f$  is differentiable in *every* point  $x_0 \in (a, b)$ , then we call  $f$  is *differentiable on interval*  $(a, b)$ . The mapping  $f' : (a, b) \rightarrow \mathbb{R}$  which assigns  $x \in (a, b)$  its  $f'(x)$ , is called *derivative function*.

$f$  is called *continuously differentiable* if  $f'$  is a continuous function on  $(a, b)$ .

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