Linear Algebra 2 – Lecture Notes

Lukas Prokop

summer term 2016

Contents

This lecture took place on 29th of Feb 2016 (Prof. Franz Lehner).

Exam: written and orally

Tutorial session:

- Every Monday, 18:30-20:00, SR 11.34
- Contact: gernot.holler@edu.uni-graz.at

Konversatorium:

- Every Monday, 10:00–10:45, SR 11.33

Topics, wie already discussed:

- Vector spaces
- Linear maps and their equivalence with matrices
- We introduced equivalence of matrices ($PAQ = B$)
- We defined the following techniques:
  - Rank
  - Linear equation system
  - Inverse matrices
  - Basis transformation

In this semester, we will discuss:

- PAP^{-1}, which is related to eigenvalues and diagonalization, hence $\bigvee_P^? PAP^{-1} = D$.

1 Linear maps (cont.)

1.1 Addition to chapter 5.2.4

$\mathrm{Hom}(V,W)$ in special case $W=\mathbb{K}$. We define,

$$V^* := \mathrm{Hom}(V,\mathbb{K})$$

also denoted V' is called *dual space* of vector space V. The elements $v* \in V*$ is called *linear forms* or *linear functionals*.

We denote,

$$v^*(v) =: \langle v*, v\rangle$$

1.2 Example

$$V = \mathbb{K}^n$$

$v^*: V \to \mathbb{K}$ is uniquely defined with values $v^*(e_i) =: a_i$.

$$\langle v^*, v\rangle = \left\langle v^*, \sum_{i=1}^n v_i e_i \right\rangle = \sum_{i=1}^n v_i \langle v^*, e_i\rangle$$

$$\left(v^*\left(\sum_{i=1}^n v_i e_i\right) = \sum_{i=1}^n v_i v^*(e_i) = \sum_{i=1}^n a_i v_i \right)$$

1.3 More general

We know, $\dim \mathrm{Hom}(V,W) = \dim V \cdot \dim W$.

Theorem 1. *Let V be a vector space over $\mathbb{K}$.*

- $\dim V =: n < \infty \Rightarrow \dim V^* = n$
  More precisely: Let $(b_1,\dots,b_n)$ be a basis of V. Then

  $$b_k^* : b_i \mapsto \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & \textit{else} \end{cases}$$

  is a basis of V^* and is called dual basis.

- *For $v^* \in V^*$ it holds that $v^* = \sum_{k=1}^n \langle v^*, b_k\rangle \cdot b_k^*$.*
- *If $\dim V = \infty, (b_i)_{i\in I}$ bass, then it holds that*

  $$(b_k^*)_{k\in I}, \langle b_k^*, b_i\rangle = \delta_{ik}$$

  is not *a basis of V^*.*

Proof.
- Special case of 5.18
  (b_k^*) is linear independent, hence in $\sum_{i=1}^n \lambda_i b_i^* = 0$ all $\lambda_i = 0$.

  $$0 = \left\langle \sum_{i=1}^n \lambda_i b_i^*, b_k \right\rangle = \sum_{i=1}^n \lambda_i \left\langle \underbrace{b_i^*, b_k}_{\delta_{ik}} \right\rangle = \lambda_k \forall k$$

- Let $v \in V$ with $v = \sum_{i=1}^n v_i b_i$. We need to show

  $$\langle v^*, v\rangle \overset{!}{=} \left\langle \sum_{k=1}^n \langle v^*, b_w\rangle b_n^*, v \right\rangle$$

  $$\begin{aligned}
  \left\langle \sum_{k=1}^n \langle v^*, b_k\rangle b_k^*, v \right\rangle &= \sum_{k=1}^n \langle v^*, b_k\rangle \langle b_k^*, v\rangle \\
  &= \sum_{k=1}^n \langle v^*, b_k\rangle \left\langle b_k^*, \sum_{i=1}^n v_i b_i \right\rangle \\
  &= \sum_{k=1}^n \sum_{i=1}^n \langle v^*, b_k\rangle \underbrace{\langle b_k^*, b_i\rangle}_{\delta_{ki}} \cdot v_i \\
  &= \sum_{k=1}^n \langle v^*, b_k\rangle \langle v^*, b_k\rangle \cdot v_k \\
  &= \left\langle v^*, \sum_{k=1}^n v_k b_k \right\rangle \\
  &= \langle v^*, v\rangle
  \end{aligned}$$

- (To be done in the practicals) Consider the functional

  $$\langle v^*, b_i\rangle = 1 \Rightarrow v^* \notin L((v_i^*)_{i\in I})$$

$\square$

1.4 Remark and a definition for bilinearity

The mapping $V^* \times V \to \mathbb{K}$ is linear in v (with fixed v^*) with $(v^*, v) \mapsto \langle v^*, v\rangle$ is linear in v^* (with fixed v). Such a mapping is called *bilinear*.

A mapping $F : V_1 \times \ldots \times V_n \to W$ is called *multilinear* (n-linear) if it is linear in every component. Formally:

$$F(v_1, \ldots, v_{k-1}, \lambda v_k' + \mu v_k'', v_{k+1}, \ldots, v_n)$$

$$= \lambda F(v_1, \ldots, v_{k-1}, v_k', v_{k+1}, \ldots, v_n) + \mu F(v_1, \ldots, v_k'', v_{k+1}, \ldots, v_n)$$

1.5 Example

$$V = \mathbb{K}[x] \text{ polynomials}$$

Basis: $\left\{x^k \,\middle|\, k \in \mathbb{N}_0\right\}$ and $\dim V = \aleph_0$

Every $v^* \in V^*$ is uniquely defined by $a_k \coloneqq \left\langle v^*, x^k\right\rangle$

$$(a_k)_{k\in\mathbb{N}_0}$$

$V^* \cong \mathbb{K}[[t]]$ are the formal power series

$$= \left\{\sum_{k=0}^{\infty} a_k t^k \,\middle|\, a_k \in \mathbb{K}\right\}$$

$$\lambda \sum_{k=0}^{\infty} a_k t^k + \mu \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (\lambda a_k + \mu b_k) t^k$$

(Compare with Taylor series $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$)

$$\left\langle \sum_{k=0}^{\infty} a_k t^k, \sum_{k=0}^{n} b_k x^k \right\rangle \eqqcolon \sum_{k=0}^{n} a_k b_k \text{ is well-defined}$$

$$\to \mathbb{K}[x]^* \cong \mathbb{K}[[t]]$$

1.6 Example

$$C[0,1] \text{ continuous functions}$$

Example:

Example 1.

$$x \in [0,1] \qquad \delta_x : C[0,1] \to \mathbb{R}$$
$$f \mapsto f(x)$$
$$\langle \delta_x, f\rangle = f(x)$$
$$\langle \delta_x, f\rangle = f(x)$$
$$I(f) = \int_0^1 f(x)\,dx \textit{ is linear}$$

$$\langle I_g, f\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$$
$$g \in C[0,1] \text{ is fixed}$$
$$\Rightarrow I_g \in C[0,1]$$
$$\langle I_g, \lambda f_1 + \mu f_2\rangle' = \int_0^1 (\lambda f_1(x) + \mu f_2(x))g(x)\,\mathrm{d}x$$
$$= \lambda \int_0^1 f_1(x)g(x)\,\mathrm{d}x + \mu \int_0^1 f_2(x)g(x)\,\mathrm{d}x$$

This also works with non-continuous g (it suffices to have g integratable). (Compare with measure theory and Riesz' theorem)

Does there exist some g such that $f(x) = \langle \delta_x, f\rangle = \int_0^1 f(t)g(t)\,\mathrm{d}t$. (Compare with Dirac's δ function and Schwartz/Sobder theory)

$$V^{**} = (V^*)^* \cong V \text{ if } \dim V < \infty$$

Lemma 1. *Let V be a vector space over $\mathbb{K}$. It requires that* $\dim V < \infty$ *and the Axiom of Choice holds.*

- $v \in V \setminus \{0\} \Leftrightarrow \bigvee_{v^* \in V^*} \langle v^*, v\rangle \neq 0$

- $\bigwedge_{v\in V} v = 0 \Leftrightarrow \bigwedge_{v^*\in V^*} \langle v^*, v\rangle = 0$

Proof. Addition v to a basis B of V: Define $v^* \in V^*$ by

$$\langle v^*, b\rangle = \begin{cases} 1 & b = v \\ 0 & b \neq v \end{cases} \text{ for } b \in B$$

□

Theorem 2. *Let V be a vector space over $\mathbb{K}$.*

- *The map $\iota : V \to V^{**} := (V^*)^*$ is called* bidual space*.*

$$\langle \iota(v), v^*\rangle := \langle v^*, v\rangle$$

  is linear and injective.

- *if $\dim V < \infty$, then isomorphism.*

Proof.
- Linearity

$$\iota(\lambda v + \mu w) \stackrel{!}{=} \lambda\iota(v) + \mu\iota(w)$$

  must hold in every point $v^* \in V^*$:

$$\begin{aligned} \langle \iota(\lambda v + \mu w), v^*\rangle &= \langle v^*, \lambda v + \mu w\rangle \\ &= \lambda \langle v^*, v\rangle + \mu \langle v^*, w\rangle \\ &= \lambda \langle \iota(v), v^*\rangle + \mu \langle \iota(w), v^*\rangle \\ &= \langle \lambda\iota(v) + \mu\iota(w), v^*\rangle \end{aligned}$$

  Is it injective? Let $v \in \ker \iota$.

$$\langle \iota(v), v^*\rangle = 0 \quad \forall v^* \in V^*$$

$$\Rightarrow \langle v^*, v\rangle = 0 \quad \forall v^* \in V^*$$

$$\xRightarrow{\text{Lemma 1}} v = 0$$

- Follows immediately, because the dimension is equal.

□

Definition 1. *Let V, W be vector spaces over $\mathbb{K}$. $f \in \mathrm{Hom}(V, W)$. We define $f^T \in \mathrm{Hom}(W^*, V^*)$ using $f^T(w^*) \in V^*$ via*

$$\left\langle f^T(w^*), v\right\rangle = \langle w^*, f(v)\rangle = w^*(f(v)) = w^* \circ f(v)$$

$$f^T(w^*) = w^* \circ f \textit{ is linear} \Rightarrow f^T(w^*) \in V^*$$

V to W (with f) and W to $\mathbb{K}$ (with w^*).

f^T is called transposed map.

Example 2. *(See practicals) Let $\dim V = n$ and $\dim W = m$ with $B \subseteq V$ and $C \subseteq W$ as bases and dual bases $B^* \subseteq V^*$ and $C^* \subseteq W^*$*

$$\Phi^{C^*}_{B^*}(f^T) = \Phi^B_C(f)^T \quad \textit{transposition of matrices}$$

This lecture took place on 2nd of March 2016 (Franz Lehner).

2 Determinants

Leibnitz 1693 (3×3 matrices)
Seki Takukazu 1685 (most general version)
Gauß 1801 ("determinant")
Cayley 1845 (on matrices)

$n = 2$

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

$$\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}$$

1. Case 1: $a \neq 0$ (multiply first row $-\frac{a}{b}$ times second row)

$$\begin{array}{cc} a & b \\ c & d \\ \hline a & b \\ 0 & d - \frac{bc}{a} \end{array}$$

Unique solution:

$$d - \frac{bc}{a} \neq 0$$

2. Case 2: $c \neq 0$ (multiple second row $-\frac{a}{c}$ times first row)

$$\begin{array}{cc} a & b \\ c & d \\ \hline 0 & b - \frac{ad}{c} \\ c & d \end{array}$$

Unique solution:

$$b - \frac{ad}{c} \neq 0$$

This gives us

$$ad - bc \neq 0$$

Definition 2.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called determinant *of* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2.1 Properties of determinants

- The determinant is bilinear in the columns and rows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v, w)$$

where v and w are column vectors of A.

$$\det(\lambda v_1 + \mu v_2, w) = \lambda \det(v_1, w) + \mu \det(v_2, w)$$

$$\det(v, \lambda w + \mu w_2) = \lambda \det(v, w_1) + \mu \det(v, w_2)$$

$$\det(\lambda v_1 + \mu v_2, w) = \begin{vmatrix} \lambda a_1 + \mu a_2 & b \\ \lambda c_1 + \mu c_2 & d \end{vmatrix}$$

$$\begin{aligned} &= (\lambda a_1 + \mu a_2)d - (\lambda c_1 + \mu c_2)b \\ &= \lambda(a_1 d - c_1 b) + \mu(a_2 d - c_2 b) \\ &= \lambda \begin{vmatrix} a_1 & b \\ c_1 & d \end{vmatrix} + \mu \begin{vmatrix} a_2 & b \\ c_2 & d \end{vmatrix} \end{aligned}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

- $\det(v, v) = 0$.

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ac = 0$$

- $$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(e_1, e_2) = 1$$

Theorem 3. *The properties 1–3 of determinants (see above) characterize the determinant.*

Let $\varphi : \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}$

- *bilinear*
- $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$
- $\varphi(e_1, e_2) = 1$. *Then it holds that* $\varphi = \det$.

Proof. To show: $\varphi(v, w) = \det(v, w) \forall v, w \in \mathbb{K}^2$

$$v = \underbrace{ae_1 + ce_2}_{\begin{pmatrix} a \\ c \end{pmatrix}} \qquad w = \underbrace{be_1 + de_2}_{\begin{pmatrix} b \\ d \end{pmatrix}}$$

$$\begin{aligned} \varphi(v, w) &= \varphi(ae_1 + ce_2, be_1 + de_2) \\ &= a\varphi(e_1, be_1 + de_2) + c \cdot \varphi(e_2, be_1 + de_2) \\ &= ad\underbrace{\varphi(e_1, e_2)}_{=1} + \underbrace{ab\varphi(e_1, e_1)}_{=0} + cb\varphi(e_2, e_1) + cd\underbrace{\varphi(e_2, e_2)}_{=0} \end{aligned}$$

□

Lemma 2. *From (i) bilinearity and (ii) $\bigwedge_{v\in\mathbb{K}^2}\varphi(v,v)=0$ it follows that*

$$\bigwedge_{v,w\in\mathbb{K}^2}\varphi(v,w)=-\varphi(w,v)$$

$$\begin{aligned}0 &\overset{(ii)}{=} \varphi(v+w,v+w) \overset{(i)}{=} \varphi(v,v)+\varphi(v,w)+\varphi(w,v)+\varphi(w,w)\\ &\overset{(ii)}{=} \varphi(v,w)+\varphi(w,v)\end{aligned}$$

2.2 Geometric interpretation of the determinant

Consider an area with w defining its breath and v its depth (hence the area spanning vectors). Let e_1 and e_2 be the spanning vectors of a rectangle corresponding to the parallelogram. $\det(v,w)$ is the surface of the spanned parallelogram. The sign defines the orientation of the pair (v,w).

$$\det(e_1,e_2)=1 \qquad \det(e_2,e_1)=-1$$

There are surfaces where the surface is infinite if you follow a vector in some direction:

- Möbius strip
- Klein's bottle (named after Felix Klein)

$$A=|\,v\,|\cdot h$$

Consider Figure 1. h is the length of the projection of w to $v^\perp$.

$$v=\begin{pmatrix}a\\b\end{pmatrix}\rightarrow\vec{n}=\begin{pmatrix}-b\\a\end{pmatrix}$$

$$\langle\begin{pmatrix}c\\d\end{pmatrix},\begin{pmatrix}-b\\a\end{pmatrix}\rangle=ad-bc$$

Second proof. $A(v,w)$ satisfies properties (i)—(iii).

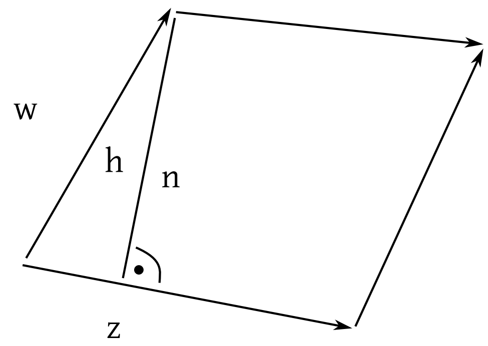

Figure 1: Parallelogram

- Property (iii) follows immediately (the area of unit vectors in two dimensions is 1).
- Property (ii) follows immediately (the area of two vectors in the same direction is 0).

Property (i) defines the linearity in v

1. If v,w are linear dependent, then $A(v,w)=0$ (one is a multiple of the other)
2. $n\in\mathbb{N}$ with $A(nv,w)=nA(v,w)$

3. For $\tilde{v} = n \cdot v$:

$$A(\tilde{v}, w) = n \cdot A(\frac{\tilde{v}}{n}, w)$$

$$\Rightarrow A(\frac{\tilde{v}}{n}, w) = \frac{1}{n} A(\tilde{v}, w)$$

$$\begin{aligned} A(nv, w) &= nA(v, w) \\ A(\frac{1}{n} v, w) &= \frac{1}{n} A(v, w) \\ A(\frac{m}{n} v, w) &= \frac{m}{n} A(v, w) \\ A(-v, w) &= -A(v, w) \end{aligned}$$

From continuity it follows that $A(\lambda u, w) = \lambda A(v, w)$ for $\lambda \in \mathbb{R}$. Analogously $A(v, \lambda w) = \lambda A(v, w)$.

4. The sum is given with

$$A(v + w, w) = A(v, w)$$

Compare with Figure 2, where area(2) + area(3) = area(2) + area(1).

$$\begin{aligned} A(\lambda v + \mu w, w) &= A(\lambda v + \mu w, \frac{1}{\mu} \mu w) \\ &= \frac{1}{\mu} A(\lambda v + \mu w, \mu w) \\ &= \frac{1}{\mu} A(\lambda v, \mu w) \\ &= A(\lambda v, w) \end{aligned}$$

General case: v, w are linear independent and therefore basis of $\mathbb{R}^2$. Besides that, v_1 and v_2 are arbitrary.

$$\begin{aligned} v_1 &= \lambda_1 v + \mu_1 w \\ v_2 &= \lambda_2 v + \mu_2 w \end{aligned}$$

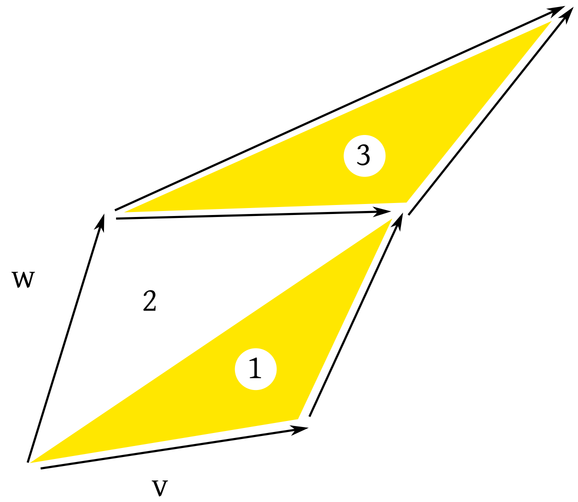

Figure 2: Translation of area 1 to area 3.

$$\begin{aligned} A(v_1 + v_2, w) &= A(\lambda_1 v + \mu_1 w + \lambda_2 v + \mu_2 w, w) \\ &= A((\lambda_1 + \lambda_2) v + (\mu_1 + \mu_2) w, w) \\ &= A((\lambda_1 + \lambda_2) v, w) \\ &= (\lambda_1 + \lambda_2) A(v, w) \\ &= A(\lambda_1 v, w) + A(\lambda_2 v, w) \end{aligned}$$

$$A(\lambda_1 v + \mu_1 w, w) + A(\lambda_2 v + \mu_2 w, w) = A(v_1, w) + A(v_2, w)$$

Additivity follows.

$\square$

Definition 3. *Let* $\dim V = n$*. A* determinant form *is a map*

$$\triangle : V^n \to \mathbb{K}$$

with properties:

1.

$$\bigwedge_{\lambda} \bigwedge_{k} \bigwedge_{a_1,\dots,a_n \in V} \triangle(a_1,\dots,a_{k-1},\lambda a_k, a_{k+1},\dots,a_n) = \lambda \triangle(a_1,\dots,a_k,\dots,a_n)$$

2.

$$\bigwedge_{k} \bigwedge_{\substack{a_1,\dots,a_n \\ a_k', a_k''}} \triangle(a_1,\dots,a_{k-1},a_k'+a_k'',a_{k+1},\dots,a_n)$$

$$:= \triangle(a_1,\dots,a_{k-1},a_k'+a_k'',a_{k+1},\dots,a_n)$$

3.

$$\triangle(a_1,\dots,a_n) = 0$$

if $\bigvee_{k \neq l} a_k = e_l$ *if* $\triangle \neq 0$*, i.e.* $\triangle$ *is non-trivial.*

Multilinearity is defined by the first two properties. Multilinearity means linearity in a_k *if* $a_1,\dots,a_{k-1},a_{k+1},\dots,a_n$ *get fixed.*

Theorem 4.

$$\dim V = n$$

$$\triangle : V^n \to \mathbb{K} \text{ is determinant form}$$

Then,

4.

$$\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{i \neq j} \triangle(a_1,\dots,a_{i-1},a_i+\lambda a_j,a_{i+1},\dots,a_n) = \triangle(a_1,\dots,a_i,\dots,a_n)$$

"Addition of λa_j *to* a_i *does not change* $\triangle$*"*

5.

$$\bigwedge_{i>j} \triangle(a_1,\dots,a_{j-1},a_i,a_{j+1},\dots,a_{i-1},a_j,a_{i+1},\dots,a_n)$$

$$= -\triangle(a_1,\dots,a_j,\dots,a_i,\dots,a_n)$$

"Exchanging a_i *with* a_j *inverts the sign"*

Proof. 4.

$$\triangle(a_1,\dots,a_i+\lambda a_j,\dots,a_n)$$

Without loss of generality: $i < j$. From properties 1 and 2 it follows that:

$$= \triangle(a_1,\dots,a_i,a_j,a_n) + \lambda \triangle(a_1,\dots,a_j,a_j,\dots,a_k)$$

Oh, a_j occurs twice! Once at index i and once at index j.

$$= 0$$

due to property 3.

5.

$$\begin{aligned}
0 &\overset{\text{property 3}}{=} \triangle(a_1,\dots,a_{i-1},a_i+a_j,\dots,a_{j-1},a_i+a_j,\dots,a_n) \\
&= \triangle(a_1,\dots,a_{i-1},\mathbf{a_i},\dots,a_{j-1},\mathbf{a_i},\dots,a_n) = \mathbf{0} \\
&+ \triangle(a_1,\dots,a_{i-1},\mathbf{a_i},\dots,a_{j-1},\mathbf{a_j},\dots,a_n) \\
&+ \triangle(a_1,\dots,a_{i-1},\mathbf{a_j},\dots,a_{j-1},\mathbf{a_i},\dots,a_n) \\
&+ \triangle(a_1,\dots,a_{i-1},\mathbf{a_j},\dots,a_{j-1},\mathbf{a_j},\dots,a_n) = \mathbf{0} \\
&\Rightarrow \delta
\end{aligned}$$

□

Definition 4. *A permutation of order* n *is a bijective mapping* $\pi : \{1,\dots,n\} \to \{1,\dots,n\}$.

$$\sigma_n = \text{ set of all permutations}$$

Remark 1. *Notation: We write the elements in the first row and their images in the second row.*

Definition 5. σ_n *constitutes (in terms of composition) a group with neutral element id, the so-called symmetric group.*

In the previous course (Theorem 1.40) we hvae proven: Compositions of bijective functions are bijective.

Remark 2. *For $n \geq 3$, σ_n is non-commutative*

Theorem 5.

$$|\,\sigma_n\,| = n!$$

Remark 3. *These are "a lot"!*

Example 3.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Definition 6. *A* transposition *is a permutation of the structure*

$$\tau = \tau_{ij} : \begin{array}{c} i \mapsto j \\ j \mapsto i \\ k \mapsto h \end{array} \quad \textit{if } k \notin \{i, j\}$$

Then $\tau_{ij}^{-1} = \tau_{ij}$, hence $\tau_{ij}^2 = id$.

Theorem 6. *σ_n is generated by transpositions. With other words, every permutation π can be represented as composition of transpositions*

$$\pi = \tau_1 \circ \ldots \circ \tau_k$$

Proof.

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

If $\pi = \mathrm{id}$,

$$\pi = \pi \quad \tau := \mathrm{id}$$

If $\pi \neq \mathrm{id}$,

$$k_1 = \min\{k \,|\, k \neq \pi(k)\}$$

1.

$$\tau_1 = \tau_{k_1 \pi(k_1)}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & \ldots & \substack{k-1 \\ 1} & k_1 & \ldots \\ 1 & \ldots & \substack{k-1 \\ 1} & k_1 & \ldots \end{pmatrix}$$

Example: Consider $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$.

$$k_1 = 2$$

$$\tau_1 = \tau_{23}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}$$

2.

$$k_2 = \min\{k \,|\, k \neq \pi_1(k)\} > k_1$$

$$\tau_2 = \tau_{k_2, \pi(k_2)}$$

And so on and so forth. $k_j > k_{j-1}$ ends after $\leq n$ steps.

$$\tau_k \circ \tau_{k-1} \circ \ldots \circ \tau_1 \circ \pi = \mathrm{id}$$

$$\Rightarrow \pi = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k$$

Regarding the example:

$$k_2 = 3$$

$$\tau_2 = \tau_{35}$$

$$\pi_2 = \tau_2 \circ \pi_1 = \tau_2 \circ \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}$$

$$k_3 = 5 \quad \tau_3 = \tau_{57}$$

$$\Rightarrow \pi = \tau_{23} \circ \tau_{35} \circ \tau_{57}$$

□

Definition 7. *A* malposition *of π is a pair (i,j) such that $i<j$ with $\pi(i)>\pi(j)$. Let F_π be the set of malpositions of π.*

$$f_\pi := |\, F_\pi \,|$$

$$\operatorname{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

Example 4.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$$

$$F_\pi = \{(2,7),(3,4),(3,7),(4,7),(5,6),(5,7),(6,7)\}$$

$$f_\pi = 7 \qquad \operatorname{sign}(\pi) = -1$$

This lecture took place on 7th of March 2016 (Franz Lehner).

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Recall: Determinant form:

1. $\triangle(a_1,\dots,\lambda a_k,\dots,a_n) = \lambda\triangle(a_1,\dots,a_n)$
2. $\triangle(a_1,\dots,a_k'+a_k'',\dots,a_n) = \triangle(a,\dots,a_k',\dots,a_n) + \triangle(a_1,\dots,a_k'',\dots,a_n)$
3. $\triangle(a_1,\dots,a_k,\dots,a_l,\dots,a_n) = 0$ if $a_k = a_l$

Conclusions:

4. $\triangle(a_1,\dots,a_k+\lambda a_l,\dots,a_n) = \triangle(a_1,\dots,a_n)$ if $k \neq l$
5. $\triangle(a_1,\dots,a_k,\dots,a_l,\dots a_n) = -\triangle(a_1,\dots,a_l,\dots,a_k,\dots a_n)$

$$\triangle(a_{\pi(1)},\dots,a_{\pi(n)}) = (-1)^k \triangle(a_1,\dots,a_n)$$

Decompose $\pi = \tau_1 \circ \ldots \circ \tau_k \circ \tau_{12} \circ \tau_{12}$. This decomposition is not distinct (k is distinct mod2)

$$\pi \in \sigma_n \qquad \text{permutation}$$

$$F_\pi = \{(i,j) \,|\, i<j, \pi(i)>\pi(j), \text{ malpositions }\}$$

$$f_\pi = |\, F_\pi \,|$$

$$\operatorname{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

Theorem 7.
- $\bigwedge_{\pi\in\sigma_n} \operatorname{sign}(\pi) = \prod_{1\le i<j\le n} \frac{\pi(j)-\pi(i)}{j-i}$
- *For transposition τ it holds that* $\operatorname{sign}(\tau) = -1$

Proof.
- Every pair $\{i,j\}$ occurs in the enumerator exactly once.

  $$\frac{\prod_{i<j} \pi(j)-\pi(i)}{\prod_{i<j}(j-i)}$$

  Denominator: $j>i$, positive. Enumerator: positive if $\pi(j)>\pi(i)$, negative if $\pi(i)>\pi(j)$.
-
  $$\tau = \begin{pmatrix} 1 & \dots & k & \dots & l & \dots & n \\ 1 & \dots & l & \dots & k & \dots & n \end{pmatrix}$$

  $$F_\tau(\underbrace{(k,k+1),(k,k+2),\dots,(k,l-1)}_{\text{malpositions with } k,\ l-k \text{ times}},(k,l),\underbrace{(k+1,l),\dots,(l-1,l)}_{l-k-1 \text{ times}})$$

  Example:

  $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 8 & 5 & 6 & 7 & 4 & 9 & 10 \end{pmatrix}$$

  Yields 7 malpositions (8 needs to be repositioned with 3 transpositions, 4 needs to be repositions with 4 transpositions).

□

$$\operatorname{sign}(\pi) = \prod_{i<j} \frac{\pi(j)-\pi(i)}{j-i} \qquad \binom{n}{2} \text{ factors}$$

$$\operatorname{sign}(\tau) = -1$$

Theorem 8. *1.* $\operatorname{sign}(id) = 1$

2. $\operatorname{sign}(\pi \circ \sigma) = \operatorname{sign}(\pi) \cdot \operatorname{sign}(\sigma)$, *hence*

$$\operatorname{sign} \sigma_n \to (\{+1, -1\}, \cdot)$$

is a group homomorphism. (In general: A group homomorphism $h : G \to (\mathcal{T}, \cdot)$ *is called* character*)*

3. $\operatorname{sign}(\pi^{-1}) = \operatorname{sign}(\pi)$

Remark 4.

$$\mathcal{T} = \{z \in \mathbb{C} \,|\, |\, z\,| = 1\}$$

Torus with multiplication is a group.

$$|\, z_1 \cdot z_2 \,| = |\, z_1 \,| \cdot |\, z_2 \,| = 1$$

Proof. 1. trivial

2.

$$\begin{aligned} \operatorname{sign}(\pi \cdot \sigma) &= \prod_{i<j} \frac{\pi \circ \sigma(j) - \pi \circ \sigma(i)}{j-i} \\ &= \underbrace{\prod_{i<j} \frac{\pi(\sigma(j)) - \pi(\sigma(i))}{\sigma(j) - \sigma(i)}}_{=\operatorname{sign}(\pi)} \cdot \underbrace{\prod_{i<j} \frac{\sigma(j) - \sigma(i)}{j-i}}_{\operatorname{sign}(\sigma)} \end{aligned}$$

3. Group homomorphism!

□

Corollary 1. • *If* $\pi = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k$, *product of transpositions*

$$\Rightarrow \operatorname{sign}(\pi) = (-1)^k$$

• $\mathfrak{a}_n := \ker(\operatorname{sign}) = \{\pi \in \sigma_n \,|\, \operatorname{sign}(\pi) = 1\}$

"even permutations", "alternating group"

$$|\, \mathfrak{a}_n \,| = \frac{n!}{2}$$

Corollary 2.

$$\triangle : V^k \to \mathbb{K} \textit{ determinant form}$$

then it holds that

$$\bigwedge_{\pi \in \sigma_n} \bigwedge_{a_1, \ldots, a_n \in V} \triangle(a_{\pi(1)}, \ldots, a_{\pi(n)}) = \operatorname{sign}(\pi) \cdot \triangle(a_1, \ldots, a_n)$$

Proof. • If $\pi = \tau_{kl}$ transposition $\xrightarrow{\text{Theorem 4}}$ $\triangle(a_{\tau(1)}, \ldots, a_{\pi(n)}) = -\triangle(a_1, \ldots, a_n) = \operatorname{sign}(\tau_{kl}) \cdot \triangle(a_1, \ldots, a_n)$

• If $\pi = \tau_1 \circ \ldots \circ \tau_k = \tau_1 \circ \tilde{\pi}, \tilde{\pi} = \tau_2 \circ \ldots \circ \tau_k$

$$\triangle(a_{\tau_1 \circ \tilde{\pi}(1)}, \ldots, a_{\tau_1 \circ \tilde{\pi}(n)}) = -\triangle(a_{\tilde{\pi}(1)}, \ldots, a_{\tilde{\pi}(n)}) = (-1)^2 \cdot \triangle(a_{\tilde{\pi}(1)}, a_{\tilde{\pi}(n)}) \to (-1)^k \cdot \triangle(a_1, \ldots$$

□

Theorem 9 (Leibnitz' definition of $\det(A)$)**.** *Let* $B = (b_1, \ldots, b_n)$ *be the basis of* V. $a_1, \ldots, a_n \in V$ *with coordinates*

$$\Phi_B(a_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

$$A := [a_{ij}]_{i,j=1,\ldots,n} = [\Phi_B(a_1), \Phi_B(a_2), \ldots, \Phi_B(a_n)]$$

Then it holds that for every determinant form $\triangle : V^k \to \mathbb{K}$*:*

$$\triangle(a_1, \ldots, a_n) = \det(A) \cdot \triangle(b_1, \ldots, b_n)$$

where

$$\det(A) := \sum_{\pi \in \sigma_n} \operatorname{sign}_{\mathbb{K}} \pi a_{\pi(1),1} a_{\pi(2),2} \ldots a_{\pi(n),n}$$

is the determinant of A

Example 5. *Example* $(n = 2)$*:*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$\operatorname{sign}\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = 1$$

$$\operatorname{sign}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} = -1$$

Proof.

$$a_j = \sum_{j=1}^{n} a_{ij} b_i$$

$$\begin{aligned}\triangle(a_1,\dots,a_n) &= \triangle\left(\sum_{i_=1}^{n} a_{i,1} b_i, \sum_{i_2=1}^{n} a_{i_2,2} b_i, \dots, \sum_{i_n=1}^{n} a_{i_n,n} b_i\right)\\ &= \sum_{i_1=1}^{n} a_{i,1} \sum_{i_2=1}^{n} a_{i_2,2} \dots \sum_{i_n=1}^{n} a_{i_n,n} \underbrace{\triangle(b_i, b_{i_2}, \dots, b_{i_n})}_{=0 \text{ if some } i_k = i_l}\end{aligned}$$

So summands with equal indices disappear. It holds that $\sum_{i_1,\dots,i_n}$ such that $i_1,\dots,i_n$ are different. Hence every value from $\{1,\dots,n\}$ occurs exactly once. This is the set of all permutations π $(i_j = \pi(j))$

$$= \sum_{\pi\in\sigma_n} a_{\pi(1)1} a_{\pi(2)2} \dots a_{\pi(n)n} \underbrace{\triangle(b_{\pi(1)},\dots,b_{\pi(n)})}_{\operatorname{sign}(\pi)\cdot\triangle(b_1,\dots,b_n)}$$

□

Corollary 3. *A determinant form is* uniquely *defined on a basis* $(b_1,\dots,b_n)$ *by the value* $\triangle(b_1,\dots,b_n)$*. Especially* $\triangle$ *is nontrivial,*

$\Leftrightarrow$ $\triangle(b_1,\dots,b_n) \neq 0$ *on some basis.*

$\Leftrightarrow$ $\triangle(b_1,\dots,b_n) \neq 0$ *in every basis* $b_1,\dots,b_n$*.*

Let $\triangle(b'_1,\dots,b'_n) = 0$ *for some other basis, represent* $b_1,\dots,b_n$ *in basis* $b'_1,\dots,b'_n$

$$b_j = \sum a_{ij} b'_i \Rightarrow \triangle(b_1,\dots,b_n) = \det(A)\cdot\triangle(b'_1,\dots,b'_n) = 0$$

$$\triangle(a_1,\dots,a_n) = \det(A)\cdot\triangle(b_1,\dots,b_n)$$

Theorem 10. *Let* $B = (b_1,\dots,b_n)$ *be a basis of* V *over* $\mathbb{K}$*.* $c \in \mathbb{K}$*. For* $a_1,\dots,a_n \in V$*, let* $A = [\Phi_B(a_1),\dots,\Phi_B(a_n)]$*. Then*

$$\triangle(a_1,\dots,a_n) = c\cdot\det(A)$$

defines a determinant form, specifically the unique determinant form with value

$$\triangle(b_1,\dots,b_n) = c$$

Proof. The 3 properties of a determinant form:

1.

$$\begin{aligned}\triangle(a_1,\dots,\lambda a_k,\dots,a_n) &= c\cdot\det\left[\Phi_B(a_1),\dots,\lambda\cdot\Phi_B(a_k),\dots,\Phi_B(a_n)\right]\\ &= c\cdot\sum_{\pi\in\sigma_n}\operatorname{sign}\pi\cdot a_{\pi(1),1} a_{\pi(2),2}\dots - \lambda a_{\pi(k),k}\dots a_{\pi(n),n}\\ &= \lambda\cdot c\cdot\sum_{\pi\in\sigma_n}\operatorname{sign}\pi\cdot a_{\pi(1),1} a_{\pi(2),2}\dots a_{\pi(n),n}\\ &= \lambda\cdot\triangle(a_1,\dots,a_n)\end{aligned}$$

2.

$$\begin{aligned}&= \triangle(a_1,\dots,a'_k + a''_k,\dots,a_n)\\ &= c\cdot\det\left[\Phi_B(a_1),\dots,\Phi_B(a'_k)+\Phi_B(a''_k),\dots,\Phi_B(a_n)\right]\\ &= c\cdot\sum_{\pi\in\sigma_n}\operatorname{sign}\pi\cdot a_{\pi(1),1}\cdot a_{\pi(2),2}\cdot\dots\left(a'_{\pi(k),k} + a''_{\pi(k),k}\right),\dots,a_{\pi(n),n}\\ &= c\cdot\sum_{\pi\in\sigma_n}\operatorname{sign}\pi\cdot a_{\pi(1),1}\cdot a'_{\pi(k),k}\dots a_{\pi(n),n} + c\cdot\sum_{\pi\in\sigma_n}\operatorname{sign}(\pi) a_{\pi(1),1}\dots a''_{\pi(k),k}\dots a_{\pi(n),n}\\ &= \triangle(a_1,\dots,a'_k,\dots,a_n) + \triangle(a_1,\dots,a''_k,\dots,a_n)\end{aligned}$$

3. Let $a_k = a_l$ for $k <$. Show that $\triangle(a_1,\dots,a_n) = 0$

$$\tau_{kl} = \text{ transposition exchanging } k \text{ and } l$$

$$\sigma_n = \mathfrak{a}_n \dot\cup (\mathfrak{a}_n\cdot\tau_{kl})$$

Claim: $\{\pi \mid \operatorname{sign}\pi = -1\} = \{\pi\circ\tau_{kl} \mid \operatorname{sign}\pi = +1\}$

$\supseteq$ If $\operatorname{sign}\pi = +1 \Rightarrow \operatorname{sign}(\pi\circ\tau_{kl}) = \underbrace{\operatorname{sign}\pi}_{+1}\cdot\underbrace{\operatorname{sign}\tau_{kl}}_{-1} = -1$

$\subseteq$ If $\operatorname{sign}\pi = -1 \Rightarrow \operatorname{sign}(\pi\circ\tau_{kl}) = +1 \Rightarrow \pi = \underbrace{(\pi\circ\tau_{kl})}_{\in\mathfrak{a}_n}\circ\tau_{kl} \in \mathfrak{a}_n\cdot\tau_{kl}$

$$\begin{aligned}
\triangle(a_1,\dots,a_n) &= c\cdot\sum_{\pi\in\sigma_n=\mathfrak{a}_n\cup\mathfrak{a}_n\cdot\tau_{kl}} \operatorname{sign}(\pi)a_{\pi(1),1}\dots a_{\pi(n),n}\\
&= c\cdot\sum_{\pi\in\mathfrak{a}_n} a_{\pi(1),1}\dots a_{\pi(n),n}\\
&- \sum_{\pi\in\mathfrak{a}_n} a_{\pi\circ\tau_{kl}(1),1}\dots a_{\pi\circ\tau_{kl}(k),k}\dots a_{\pi\circ\tau_{ul}(l),l}\dots a_{\pi\circ\tau_{kl}(n),n}\\
&= c\cdot\sum_{\pi\in\mathfrak{a}_n} a_{\pi(1),1}\dots, a_{\pi(n),n} - \sum_{\pi\in\mathfrak{a}_n} a_{\pi(1),1}\dots a_{\pi(l),k}\dots a_{\pi(k)l}\dots a_{\pi(n),n}
\end{aligned}$$

What we did:

(a) $a_{\pi(l),k} = a_{\pi(l),l}$ and $a_{\pi(k),l} = a_{\pi(k),k}$ because $a_k = a_l$
(b) exchange factors

$$\begin{aligned}
&= c\sum_{\pi\in\mathfrak{a}_n} a_{\pi(1),1}\dots a_{\pi(k),k}\dots a_{\pi(l),l}\dots a_{\pi(n),n}\\
&- c\sum_{\pi\in\mathfrak{a}_n} a_{\pi(1),1}\dots a_{\pi(k),k}\dots a_{\pi(l),l}\dots a_{\pi(n),n}\\
&= 0
\end{aligned}$$

Value for $(b_1,\dots,b_n)$

$$a_{ij} = \delta_{ij} \Rightarrow A = I$$

$$\det(I) = \sum_{\pi\in\sigma_n} \operatorname{sign}\pi\cdot\delta_{\pi(1),1}\dots\delta_{\pi(n),n} = +1$$

for all $\pi(j) = j$ otherwise 0.

$$\Rightarrow \pi = \text{id is the only summand}$$

$$\triangle(b_1,\dots,b_n) = \det(I)\cdot c = c$$

$\square$

Remark 5. *"$\mathfrak{a}_n$ is the subgroup of index 2"* $[\sigma_n : \mathfrak{a}_n] = 2$

You might be familiar with:

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$$
$$[\mathbb{Z} : n\mathbb{Z}] = n$$

Theorem 11 (Summary).
- *The set of determinant forms $\triangle : V^n \to \mathbb{K}$ constructs a one-dimensional vector space, $\Lambda^n V$*
- *There exists a non-trivial determinant form with $\triangle(b_1,\dots,b_n) = 1$*

German keywords

English keywords