Linear Algebra 2 – Practicals

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Contents

Exercises, I did on the board:

Exercise 1

Exercise 1. Determine the matrix representation of the linear map

$$f : \mathbb{R}_2[x] \to \mathbb{R}_3[x]$$

$$p(x) \mapsto x \cdot p(x)$$

in terms of the bases $B = \left\{1, x, x^2 - 1\right\} \subseteq \mathbb{R}_2[x]$ and $C = \left\{1, x, x^2 - 1, x^3 - 2x\right\} \subseteq \mathbb{R}_3[x]$

0.1 Blackboard solution

$$\mathcal{L}(\left\{\underbrace{1, x, x^2 - 1}_{b_1, b_2, b_3}\right\}) \to \mathcal{L}(\left\{\underbrace{1, x, x^2 - 1, x^3 - 2x}_{c_1, c_2, c_3, c_4}\right\})$$

$$f : \alpha \mapsto x \cdot \alpha$$

$$f(1) = x = 1c_2$$

$$f(x) = x = x^2 = 1c_3 + 1c_1$$

$$f(x^2 - 1) = x^3 - x = 1c_4 + 1c_2$$

	b_1	b_2	b_3
c_1	0	1	0
c_2	1	0	1
c_3	0	1	0
c_4		0	1

0.2 My solution

$$\begin{aligned} B &= \left\{1, x, x^2 - 1\right\} =: \{b_1, b_2, b_3\} \\ C &= \left\{1, x, x^2 - 1, x^3 - 2x\right\} =: \{c_1, c_2, c_3, c_4\}\ f(b_1) &&= x \cdot (1) = x \\ f(b_2) = x \cdot (x) = x^2 \\ f(b_3) = x \cdot (x^2 - 1) = x^3 - x \end{aligned}$$

$$\begin{aligned} x &= \lambda_1 \cdot 1 + \lambda_2 \cdot x + \lambda_3 \cdot (x^2 - 1) + \lambda_4 \cdot (x^3 - 2x) \\ &= \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3 \end{aligned}$$

By coefficient comparison, we get $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 - 2\lambda_4 = 1$ where $\lambda_4 \overset{!}{=} 0$. Hence $\lambda_2 = 1$.

$$\implies \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$x^2 = \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3$$

By coefficient comparison, we get $\lambda_3 = 1$ and $\lambda_1 = \lambda_2 = \lambda_4 = 0$.

$$\implies \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x^3 - x = \lambda_1 - \lambda_3 + (\lambda_2 - 2\lambda_4)x + \lambda_3 x^2 + \lambda_4 x^3$$

By coefficient comparison, we get $\lambda_1 = \lambda_3 = 0$ and $\lambda_2 - 2\lambda_4 = -1$ with $\lambda_4 \stackrel{!}{=} 1$, hence $\lambda_2 = 1$.

$$\implies \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

So our solution is,

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Exercise 2

Exercise 2. A chain complex C is a sequence of linear maps

$$0 = V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} V_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} V_0 \xrightarrow{f_0} 0$$

with the property such that $\operatorname{im} f_{k+1} \subseteq \ker f_k$ for all $0 \leq k \leq n-1$, hence, $f_k \circ f_{k+1} = 0$. The quotient space $H_k(C) = \ker f_k / \operatorname{im} f_{k+1}$ is called k-th ***homology*** of the complex. Show that for finite-dimensional chain complexs (hence, $\dim V_k < \infty$ for all k) the following formula holds:

$$\sum_{k=0}^{n-1} (-1)^k \dim V_k = \sum_{k=0}^{n-1} (-1)^k \dim H_k(C)$$

0.3 Blackboard solution

$V \subset W$ vector spaces.

$$V = \mathcal{L}\{v_1, \ldots, v_n\} \qquad W = \mathcal{L}\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$$

${}^{W}\!/\!_{V} = \{[x]_V : x \in W\}$

$$[x]_n := \{x + v \mid v \in V\}$$

$[w_1]_v, \ldots, [w_n]_v$ is a basis of vector space ${}^{W}\!/\!_{V}$.

for $x, y \in W$,

$$x \sim_V y := x - y \in V$$

$$y + v_2 \in [y]_V$$

$$[x]_V \odot [y]_V = [x + v_1 + y + v_2]_V$$

$$[x]_V \odot [y]_V = [x + y]_V$$
$$\alpha[x]_V = [\alpha x]_V$$

$$\sum_{k=0}^{n-1}(-1)^k \dim V_k = \sum_{k=0}^{n-1}(-1)^k \dim H_k(C).$$

where $\dim(V_k) = \dim \operatorname{kernel}(f_k) + \dim \operatorname{image}(f_k)$ and $\dim(H_k) = \dim \operatorname{kernel}(f_k) - \dim \operatorname{image}(f_k) = \dim \operatorname{kernel}(f_k) - \dim \operatorname{image}(f_{k+1})$.

Exercise 3

Exercise 3. Let $A \in \mathbb{K}^{n\times n}$ be a nilpotent matrix, hence, there exists $k \in \mathbb{N}$ such that $A^k = 0$.

- Show that $I - A$ is invertible with $(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$.
- Use the previous result to derive the inverse of the matrix:

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

0.4 Blackboard solution

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} = \frac{x^n - 1[= (x-1)(1 + x + x^2 + \cdots + x^{n-1}])}{x - 1}$$

Just verify:

$$(I - A)(I - A + A^2 + \cdots + A^{n-1})$$

Exercise 4

Exercise 4. 1. Let A be an invertible $n\times n$ matrix over the field $\mathbb{K}$ and u, v are column vectors (hence, $n \times 1$ matrices), such that $\sigma = 1 + v^t A^{-1} u \neq 0$. Show that $(A + uv^t)$ is invertible and that

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} uv^t A^{-1}$$

2. Apply this formula, to determine the inverse of matrix

$$\begin{pmatrix} 5 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

efficiently.

0.5 Blackboard solution

$$(A+uv^t)^{-1} = A^{-1} - \frac{1}{\sigma}A^{-1}uv^tA^{-1} \qquad \text{(Sherman-Morrison-Formula)}$$
$$\sigma = 1 + v^tA^{-1}u \neq 0$$

$$\begin{aligned}(A+uv^t)(A^{-1} - \frac{1}{\sigma}A^{-1}uv^tA^{-1}) &= AA^{-1} + uv^tA^{-1} - \frac{1}{\sigma}(AA^{-1}uv^tA^{-1} + uv^tA^{-1}uv^tA^{-1}) \\ &= I + uv^tA^{-1} - \frac{1}{\sigma}(uv^tA^{-1} + (v^tA^{-1}u)uv^tA^{-1}) \\ &= I + uv^tA^{-1} - \frac{1}{\sigma}(1 + v^tA^{-1}u)uv^tA^{-1} \\ &= I + uv^tA^{-1} - \frac{\sigma}{\sigma}uv^tA^{-1} = I\end{aligned}$$

These practicals took place on 2018/03/14.

Exercise 5

Exercise 5. a. Determine the dual basis of $(\mathbb{R})^4$ to B

$$B := \left\{ \begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\\1\end{pmatrix} \right\}$$

b. Determine the matrix of the distinct (why distinct?) projection map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ with

$$\operatorname{image}\varphi = \mathcal{L}\left\{ \begin{pmatrix}1\\2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\\1\end{pmatrix} \right\} \text{ and } \operatorname{kernel}\varphi = \mathcal{L}\left\{ \begin{pmatrix}-1\\-2\\2\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\\1\end{pmatrix} \right\}$$

0.6 Blackboard solution

It must hold that

$$\langle b_1, b_1^* \rangle = 1$$
$$\langle b_2, b_2^* \rangle = 0$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & -2 & 2 & -1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 & -4 & -5 & 4 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 0 & 2 & 5 & 3 & -2 \\ 0 & 0 & 0 & 1 & 5 & 15 & 8 & -6 \end{pmatrix}$$

Pay attention! We transposed the matrix initially. Now we can read the solution vectors in columns. You can also transpose it only in the end.

$$B^* = \left\{ b_1^*, b_2^*, b_3^*, b_4^* \right\}$$

where e.g. $b_1^* = (3, 1, 2, 5)^T$.

Exercise b: $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$.

$$\operatorname{image}\varphi = L((b_1, b_2))$$

$$\text{kernel}\, \varphi = L((b_3, b_4))$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot B^{*^T} = P$$

$$P = \begin{pmatrix} -12 & 3 & 7 & 20 \\ -6 & 2 & 4 & 10 \\ 6 & -1 & -3 & -10 \\ -4 & 2 & 5 & 15 \end{pmatrix}$$

Why distinct? The projection matrix is given with

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where row i is b_i and column j is d_j where b and d are the bases of the two vector spaces.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_B = 1 \cdot b_1 + 2 \cdot b_2 + 3 \cdot b_3 + 4 \cdot b_4$$

$$P_{E,E} = \Phi_B^E \cdot P_{B,B} \cdot \overbrace{\underbrace{\Phi_E^B}_{(\Phi_B^E)^{-1}} v_E}^{v_B}$$

How to compute the inverse efficiently?

Let $A, B, C \in \mathbb{R}^{2 \times 2}$.

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} A\alpha & A\beta + B\gamma \\ 0 & C\gamma \end{pmatrix} \stackrel{!}{=} \infty$$

$$\alpha = A^{-1} \qquad \gamma = C^{-1}$$

$$\beta = -A^{-1} B \gamma$$

Exercise 6

Exercise 6. Let $V = \mathbb{R}[x]_2$.

$$\xi_1 < \xi_2 < \xi_3 \in \mathbb{R}$$

0.7 Whiteboard solution

Exercise a:

$$\beta_i : V \to \mathbb{R}$$

$$p(x) \mapsto p(\xi_i)$$

$$\dim(V) = \dim(V^*) = 3$$

$$\sum a_i \beta_i = 0 \iff a_i = 0 \forall i$$

$$\forall p \in \mathbb{R}[x]_2 : \sum a_i \beta_i(p(x)) \stackrel{!}{=} 0$$

$$\forall p \in \mathbb{R}[x]_2 : \sum a_i \beta_i(\xi_i) \stackrel{!}{=} 0$$

$$\implies p_1(\xi_1) = p_1(x_2) = 0 \implies a_3 = 0 \dots a_i = 0 \forall i$$

hence linear independent.

Exercise b:

$$\gamma : p(x) \mapsto p'(\xi_2)$$
$$\gamma(p(x)) = \sum a_i \beta_i(p(x)) = \sum a_i p(\xi_i) = p'(\xi_2)$$
$$p(x) = \alpha + \beta x + \delta x^2$$
$$\implies p'(\xi_2) = \beta + 2\delta\xi_2$$
$$p(x) = \alpha + \beta x + \delta x^2$$

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \end{pmatrix}}_{=A} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2\xi_2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{\xi_2\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_1)} & \dots \\ -\frac{\xi_3\xi_1}{(\xi_2-\xi_1)(\xi_3-\xi_2)} & \dots \\ \frac{\xi_1\xi_2}{(\xi_3-\xi_1)(\xi_3-\xi_2)} & \dots \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 1 \\ 2\xi_2 \end{pmatrix} = \begin{pmatrix} \frac{\xi_2-\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_1)} \\ \frac{\xi_1-2\xi_2+\xi_3}{(\xi_2-\xi_1)(\xi_3-\xi_2)} \\ \frac{\xi_2-\xi_1}{(\xi_3-\xi_1)(\xi_3-\xi_2)} \end{pmatrix}$$

Exercise c:

$$B = \{b_1(x), b_2(x), b_3(x)\}$$
$$l_i = \sum_{j=1}^{2} a_{ji} x^j$$
$$\beta_l(l_i(x)) = \delta_{li}$$

$$\begin{pmatrix} 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \\ 1 & \xi_3 & \xi_3^2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In essence, we look for $p(x) = \frac{(x_1-x)(x_2-x)}{(\xi_1-\xi_3)(\xi_2-\xi_3)}$. This is a Lagrange polynomial with $l_3 = p$.

Exercise 7

Exercise 7. Let V be a vector space with $\dim V = n < \infty$ and $U \subseteq V$ is a subspace with $\dim U = m$.

a. Show that $U^\perp = \{v^* \in V^*\}\, U \subseteq \operatorname{kernel} v^*$ is a subspace of dual space V^* and give $\dim U^\perp$.

b. Is $\{v^* \in V^* \mid U = \operatorname{kernel} v^*\}$ also a subspace?

Exercise a:

$$\left(U^\perp \quad = \{v^* \in V^* \mid U \subseteq \operatorname{kernel} v^*\} \quad = \{v^* \in V \mid \forall u \in U : v^*(u) = 0\}\right)$$

We prove subspace criteria:

1. $U^\perp \neq \emptyset$. Let $v^* : V \to \mathbb{K}$ with $v \mapsto 0$.

2. $$\forall u_1^\perp, u_2^\perp \in U^\perp \forall \lambda, \mu \in \mathbb{K} : \lambda u_1^\perp + \mu u_2^\perp \in U^\perp$$
$$\lambda \underbrace{v_1^*(u)}_{0} + \mu v_2^*(u) = 0 \qquad \text{for } \forall v_1^*, v_2^* \in U^\perp, u \in U$$

Now, we need to determine the dimension $\dim U^\perp$.

Let $B_U = \{v_1, v_2, \ldots, v_m\}$.

$$B_V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$
$$B_{V^*} = \left\{v_1^*, v_2^*, \ldots, v_n^*\right\}$$
$$B_{U^\perp} = \left\{v_{m+1}^*, v_{m+2}^*, \ldots, v_n^*\right\} \text{ is basis of } U^\perp$$

$$\forall u \in U : v_j^*(u) = 0^C \forall j \in \{m+1, \ldots, n\}$$
$$B_U = \{v_1, v_2, \ldots, v_m\}$$
$$B_V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$
$$B_{V^*} = \left\{v_1^*, v_2^*, \ldots, v_n^*\right\}$$
$$B_{U^\perp} = \left\{v_{m+1}^*, v_{m+2}^*, \ldots, v_n^*\right\} \text{ is basis of } U^\perp$$
$$\implies \dim(U^\perp) = n - m$$

Exercise b:

$$W^\perp = \{v^* \in V^* \mid U = \mathrm{kernel}(v^*)\}$$

The reason was given orally.

Exercise 8

Exercise 8. Let $f \in \mathrm{Hom}(V, W)$ be a linear map between two finite-dimensional vector space with bases $B \subseteq V$ and $C \subseteq W$. We define the transposed map

$$f^T : W^* \to V^*$$
$$w^* \mapsto w^* \circ f$$

Hence $f^T(w^*)$ is a linear functional and $(f^T(w^*))(v) = w^*(f(v))$

a. Show that f^T is linear.

b. Show that the matrix representation, in regards of dual bases C^* and B^*, has the following matrix representation: $\Phi^{C^*}_{B^*}(f^T) = \Phi^B_C(f)^T$

Exercise a: Let $v \in V$ and $\lambda \in \mathbb{K}$, $w_1^*, w_2^* \in W^*$.

$$(f^T(w_1^* + w_2^*))(v) = (w_1^* + w_2^*)f(v) = w_1^*(f(v)) + w_2^*(f(w_1^*))(v) + (f^T(w_2^*))(v)$$
$$(f^T(\lambda w_1^*))(v) = (\lambda w_1^*)(f(v)) = \lambda w_1^*(f(v)) = \lambda(f^T(w_1^*))(v)$$

We proved $g(w_1 + \lambda w_2) = g(w_1) + \lambda g(w_2)$. Hence f^* is linear.

Exercise b:

$$\Phi^{C^*}_{B^*}(f^T) = \Phi^B_C(f)^T$$
$$\{v_1 \ldots, v_n\} = B \qquad \{w_1, \ldots, w_m\} = C$$

$$
\begin{aligned}
f(v_j) &= \sum_{i=1}^{m} m_{ij} w_i \\
(f^t(w_i^*))(v_k) = w_j^*(f(v_k)) = w_j^* &\left(\sum_{l=1}^{m} lkw_l\right) = m_{jk} \\
m_{jk} &= \sum_{l=1}^{n} m_{jl} \underbrace{v_l^*(v_k)}_{\delta_{lk}} \\
&= \sum_{l=1}^{n} m_{jl} v_l^*(v_k) \\
\implies f^T(w_j^*) &= \sum_{l=1}^{n} A_{l,j} v_l^* \\
\implies A &= \Phi_C^B(f)^T \\
A &= \Phi_{B^*}^{C^*}(f^T)
\end{aligned}
$$

These practicals took place on 2018/03/21.

Exercise 10

Exercise 9. A permutation $\pi \in \sigma_n$ is called cyclic, if there exists some $k \geq 1$ and a sequence $i_1, i_2, \ldots, i_k$ such that $\pi(i_j) = i_{j+1}$ for $1 \leq j \leq k-1$, $\pi(i_k) = i_1$ and $\pi(i) = i$ for $i \notin \{i_1, i_2, \ldots, i_k\}$, hence

$$i_1 \to i_2 \to \cdots \to i_1.$$

and all other i are fixed. Common notation: $\pi = (i_1, i_2, \ldots, i_k)$.

- Show, that two cyclic permutations $\pi = (i_1, i_2, \ldots, i_k)$ and $\rho = (j_1, j_2, \ldots, j_l)$ commutate ($\pi \circ \rho = \rho \circ \pi$), if $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, j_2, \ldots, j_l\} = \emptyset$.
- Decompose the cycle into a product of transpositions and show that for a cyclic permutation, it holds that $\text{sign}(\pi) = (-1)^{k-1}$.

For the first part,

Let $\text{supp}(\pi) \cap \text{supp}(\rho) = \emptyset$ where $\text{supp}(\pi)$ defines the elements in the cycle of permutation π.

$i \notin \text{supp}(\pi) \cup \text{supp}(\rho)$

$$
\begin{aligned}
&\implies \rho(i) = i = \pi(i) = i \\
&\implies \pi(\rho(i)) = \rho(\pi(i)) = i
\end{aligned}
$$

$i \in \text{supp}(\pi)$ $\;i \in \text{supp}(\pi) \implies \pi(i) \in \text{supp}(\pi)$

$$\rho(\pi(i)) = \pi(i) \implies \rho(\pi(i)) = \pi(i) = \pi(\rho(i))$$

For the second part,

$$\pi = \tau_1 \cdot \tau_2 \cdot \cdots = (i_1, i_2)(i_2, i_3) \ldots (i_{k-1}, i_k)(i_k, i_1)$$

giving $k-1$ transposition.

$$\implies \text{sign}(\pi) = (-1)^{k-1}$$

$$
\begin{gathered}
\tau_{24} = 1432 \\
T_{34}^{2341} T_{23}^{2314} T_{42}^{2134}
\end{gathered}
$$

Exercise 11

Exercise 10. Let $\pi \in \sigma_n$ be a permutation and $i \in \{1, 2, \dots, n\}$.

1. Show that the sequence $i, \pi(i), \pi^2(i), \dots$ is periodic and that the first number occuring twice is i.
2. The sequence $(i, \pi(i), \pi^2(i), \dots, \pi^{k-1}(i))$, where k is the smallest exponent such that $\pi^k(i) = i$, is called *cycle of i*. Show that the relation $i \sim j :\iff$ (j is in inside the cycle) defines an equivalence relation in $\{1, 2, \dots, n\}$.
3. Show that every permutation can be written as product of commutative cycles.
4. Apply this decomposition to permutation π in Exercise 9.

Exercise (a).

k is certainly finite, because of the pidgeonhole principle. Furthermore smaller than n, because there are at most n numbers it can be mapped to. We have n distinct elements. i is the first element, which is not mapped to any number. So i is the first number which will occur for the second time. This implies that the map is bijective, which is given for any permutation.

Exercise (b).

Reflexivity is trivial. Symmetry: Let $\pi^l(i) = j$, then $\pi^{k-l}(j) = i$. This shows that both are in the same cycle and symmetry is given. If $i \sim j \land j \sim m \implies i \sim m$.

$$\pi(i) = j \qquad \pi^p(j) = m \iff \pi^p(\pi^l(i)) = m \iff \pi^{p+l}(i) = m$$

$$\pi^p \circ \pi^l(i) = m$$

Exercise (c).

$$1 \quad \pi(1) \quad \pi(\pi(1)) \quad \pi(\pi(\pi(1))) \quad \dots$$

$$\pi = (\quad)(1, \dots, \pi^{k-1})a_2 \pi(a_2) \neq a_2$$

Exercise (d).

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix} = (1253)(467)$$

Exercise 12

Exercise 11. Show that every permutation $\pi \in \sigma_n$ can be written as composition of permutations $\gamma = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & n-1 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 1 & \dots & n-1 & n \end{pmatrix}$

From the lecture:

> Every permutation $\sigma \in \sigma_n$ with $\sigma \neq \mathrm{id}$ can be denoted as a product of transpositions.

1. Consider the theorem from the lecture.
2. Every transposition can be represented as composition of swapping two neighbors.

$$\tau_{ij} = (i, i+1)(i+1, i+2) \dots (j-1, j)(j-2, j-1) \dots (i, i+1)$$

3.

$$\tau_{i,i+1} = \gamma^{i-1} \cdot \tau \cdot \gamma^{-(i-1)}$$

Exercise 13

Exercise 12. In the sliding 6-puzzle, which permutations can be reached?

We begin with the initial position (right-bottom shows the vacant field) and need to end with the initial position as well. We can only do transpositions with the vacant field.

1. even number of transpositions
2. signature $\pi = (-1)^{\#\text{ transpositions}}$
3. no permutation with $\operatorname{sign} -1$

The second item is wrong.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$$

Any permutation is a product of π_1 and π_2.

We can permute in a shape of the infinity symbol.

Exercise 14

Exercise 13. Determine the determinant using three different methods (Leibniz, Laplace, Gauss-Jordan) of the matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

TODO

Exercise 15

Exercise 14. The numbers 18270, 16128, 63042, 17304 and 17934 are divisible by 42. Show that the determinant

$$\det(A) = \begin{vmatrix} 1 & 8 & 2 & 7 & 0 \\ 1 & 6 & 1 & 2 & 8 \\ 6 & 3 & 0 & 4 & 2 \\ 1 & 7 & 3 & 0 & 4 \\ 1 & 7 & 9 & 3 & 4 \end{vmatrix}$$

is divisible by 42 without explicit evaluation.

$$\begin{vmatrix} 1 & 8 & 2 & 7 & 0 \\ 1 & 6 & 1 & 2 & 8 \\ 6 & 3 & 0 & 4 & 2 \\ 1 & 7 & 3 & 0 & 4 \\ 1 & 7 & 9 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 2 & 7 & 18270 \\ 1 & 6 & 1 & 2 & 16128 \\ 6 & 3 & 0 & 4 & 63042 \\ 1 & 7 & 3 & 0 & 17304 \\ 1 & 7 & 9 & 3 & 17934 \end{vmatrix}$$

$$\det(A) = \sum_{k=1}^{5} a_{k,5} \underbrace{(-1)^{k+5} \det A_{k,5}}_{\in \mathbb{Z}}$$

$\det(A)$ consists of 5 summands, which are divisible by 42 each, hence the sum is divisible

These practicals took place on 2018/04/11.

Exercise 17

Exercise 15. Evaluate the determinants:

0.8 Exercise 17a

Exercise 16.

$$\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1-y \end{vmatrix}$$

$$\begin{vmatrix} 0 & -x & -x & y+xy-x \\ 0 & -x & 0 & y \\ 0 & 0 & y & y \\ 1 & 1 & 1 & 1-y \end{vmatrix} = -1 \cdot \begin{vmatrix} -x & -x & y+xy-x \\ -x & 0 & y \\ 0 & y & y \end{vmatrix}$$
$$= (-1)(-xy^2 - (xy)^2 + x^2y - x^2y + xy^2) = (xy)^2$$

0.8.1 A simpler solution

Assume $C \in \mathrm{GL}(\mathbb{R})$ and $\vec{V}, \vec{W} \in \mathbb{R}^n$ where GL is the set of invertible matrices. Then it holds that

$$\det(C + \vec{v}\vec{w}^t) = \det C \left(1 + \langle C^{-1}\vec{v}, \vec{w}\rangle\right)$$

where $\langle \cdot, \cdot \rangle$ is an inner product with $\langle \vec{v}, \vec{w} \rangle = v_1 \cdot w_1 + \ldots + v_n \cdot w_n$.

$$A\vec{x} = \vec{b}$$
$$x_i = \frac{\det(A_j)}{\det A}$$

0.9 Exercise 17b

Exercise 17.

$$\begin{vmatrix} x & 0 & \ldots & a_0 \\ -1 & x & \ldots & a_1 \\ & -1 & \ddots & \\ & \ddots & \ddots & \\ 0 & & -1 & x + a_{n-1} \end{vmatrix}$$

Alternative approach: Use Laplace expansion theorem along the last column.

Always consider: A division by x requires a case distinction!

Case 1: $x \neq 0$:

$$\begin{vmatrix} x & & \ldots & a_0 \\ 0 & x & \ldots & a_1 + \frac{a_0}{x} \\ & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ 0 & & -1 & x + a_{n-1} \end{vmatrix} = \begin{vmatrix} x & & a_0 \\ & \ddots & a_1 + \frac{a_0}{x} \\ & & x + a_{n-1} + \frac{a_{n-2}}{x} + \ldots + \frac{a_0}{x^{n-1}} \end{vmatrix}$$

$$= x^{n-1}(x + a_{n-1} + \frac{a_{n-2}}{x} + \ldots) = x^n + x^{n-1}a_{n-1} + \ldots + a_0 = x^n + \sum_{i=1}^{n} a_{n-i}x^{n-i}$$

Case 2: $x = 0$.

$$\begin{vmatrix} 0 & & a_0 \\ -1 & \ddots & \\ & & -1 \cdot a_{n-1} \end{vmatrix} = (-1)^{n+1} \cdot a_0 \cdot \begin{vmatrix} -1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & -1 \end{vmatrix} = (-1)^{n+1} \cdot a_0 \cdot (-1)^{n-1} = (-1)^{2n} \cdot a_0 = a_0$$

0.10 Exercise 17c

Exercise 18.

$$\begin{vmatrix} 0 & 0 & \dots & & a_n & & \\ 0 & 0 & \dots & a_{n-1} & * & & \\ \vdots & & & & \vdots & & \\ 0 & a_2 & & & \vdots a_1 & * & * \end{vmatrix}$$

Case distinction: n is even.

$$= (-1)^{\frac{n}{2}} \begin{vmatrix} a_1 & * & & * \\ & a_2 & * & \vdots \\ & & \ddots & \\ 0 & & & a_n \end{vmatrix} = (-1)^{\frac{n-1}{2}} \prod_{i=1}^{n} a_i$$

You can skip the case distinction if you use the Gaussian bracket: $(-1)^{\lfloor \frac{n}{2} \rfloor}$

Exercise 18

Exercise 19. Show: There exists some matrix $A \in \mathbb{R}^{n \times n}$ with entries $a_{ij} = \pm 1$ such that $\det(A) = n!$ if and only if $n < 3$.

Hint: For $n = 2$, it is easy. For $n = 3$, consider why no all summands in Leibniz' formula for determinants have the same sign. The case $n > 3$ can be reduced to the case $n = 3$.

For $n = 2$,

$$2! = 2 \qquad \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 - (-1) = 2$$

For $n = 3$, we consider the Rule of Sarrus and assume such a matrix A exists. Because $n! = 6$, we need all summands of the Rule of Sarrus to be positive. We consider the diagonals given in the Rule of Sarrus and recognize, that both diagonals use the same elements. Consider the diagonals with positive sign. All of them must either use zero or two -1. At the same time, all diagonals with negative sign must either use three or one -1. This contradicts assuming they use the same elements. The proof by contradiction has been completed.

Now we look for the generalization of $n \to n + 1$ for $n \geq 3$.

This will be proven by complete induction.

Induction hypothesis $A \in \mathbb{R}^{n \times n}$ with $a_{ij} = \pm 1$

Induction base $n = 3$ has been proven

Induction step We apply Laplace expansion along one row. Let $\varepsilon^{(i)}$ be the value of $\det(A_n^{(i)})$ where A_n is a square matrix of dimension $n \times n$.

$$\det(A_{n+1}) = +\underbrace{\det(A_n^{(1)})}_{<n!} - \underbrace{\det(A_n^{(2)})}_{<n!} + \underbrace{\det(A_n^{(3)})}_{<n!} - \dots$$

$$= \sum_{i=1}^{n+1} \det(A_n^{(i)}) = \sum_{i=1}^{n+1} \varepsilon^{(i)} < (n+1)n! = (n+1)!$$

Hence $\det(A_{n+1}) < (n+1)n!$.

Exercise 19

Exercise 20. (a) Let $\mathbb{K}$ be a field and $a_1, a_2, \dots, a_n \in \mathbb{K}$. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \prod_{i<j} (a_j - a_i)$$

(b) Conclude from this, that for given pairwise different numbers $x_0, x_1, \dots, x_n \in \mathbb{K}$ and arbitrary $y_0, y_1, \dots, y_n \in \mathbb{K}$ there exists exactly one polynomial $p(x) \in \mathbb{K}[x]$ with degree n, such that $p(x_i) = y_i$ for all i.

(c) Extra point to be solved on a computer: Determine for each different n, one polynomial $p(x) \in \mathbb{R}[x]$, such that $p(x_k) = |\, x_k \,|$, $k = -n, \dots, n$, with $x_k = \frac{k}{n}$.

0.11 Exercise 19a

Induction base: $n = 2$.

$$\begin{vmatrix} 1 & a_1 \\ 1 & a_2 \end{vmatrix} = (a_2 - a_1)$$

Induction step: $n - 1 \to n$.

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & a_1 & & \dots & & a_1^{n-1} \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 & & \dots & a_2^{n-1} - a_1^{n-1} \\ \dots & & & & & \\ 0 & a_n - a_1 & a_2^2 - a_1^2 & & & a_n^{n-1} - a_1^{n-1} \end{vmatrix}$$

The following equation holds:

$$(x^n - y^n) = (x - y) \sum_{i=0}^{n-1} x^{n-1-i} y^i$$

$$= \begin{vmatrix} (a_2 - a_1) & (a_2^2 - a_1^2) & (a_2^{n-1} - a_1^{n-1}) \\ \vdots & \vdots & \vdots \\ (a_n - a_1) & (a_n^2 - a_n^2) & (a_n^{n-1} - a_1^{n-1}) \end{vmatrix} = \prod_{i=2}^{n} (a_j - a_1) \cdot \begin{vmatrix} 1 & (a_2 + a_1) & (a_2^{n-2} + a_2^{n-3} a_1 + \dots + a_1^{n-2}) \\ 1 & (a_3 + a_1) & \vdots \\ \vdots & \vdots & \vdots \\ 1 & (a_n + a_1) & (a_n^{n-2} + \dots + a_1^{n-2}) \end{vmatrix}$$

$$= \prod_{j=2}^{n} (a_j - a_1) \cdot \begin{vmatrix} 1 & a_2 & a_2^2 & \dots & a_2^{n-2} \\ 1 & a_3 & a_3^2 & \dots & a_3^{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-2} \end{vmatrix}$$

$$= \prod_{j=2}^{n} (a_j - a_1) \prod_{\substack{i<j \\ i,j \neq 1}}^{n} (a_j - a_i) = \prod_{i<j}^{n} (a_j - a_i)$$

0.12 Exercise 19b

Show: there exists exactly one polynomial $p \in \mathbb{K}_n[x](\forall i \in \{0,\dots,n\}) : p(x_i) = y_i$.

$$p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\begin{array}{ccccccccc} a_0 & + & a_1 x_0 & + & \dots & a_n x_0^n & = & y_0 \\ a_0 & + & a_1 x_1 & + & \dots & a_n x_1^n & = & y_1 \\ \vdots \\ a_0 & + & a_1 x_n & + & \dots & a_n x_n^n & = & y_n \end{array} \iff \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$\det(M) = \prod_{i<j} (x_j - x_i)$$

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & \dots & x_0^1 \\ \vdots & & \vdots \\ 1 & \dots & x_n^n \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

0.13 Exercise 20

Exercise 21. Let $A, B \in \mathbb{K}^{n \times n}$. Show by elementary row- and column transformations, that the following identity for block matrices holds:

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$$

Derive an alternative proof for the multiplication law of determinants ($\det(AB) = \det(A) \cdot \det(B)$).

1. We consider the left-hand side.
2. We add the $n+1$-th row to the first row multiplied by a_{11} and use the result as row $n+1$. As a result, the value in $a_{n+1,1}$ becomes 0.
3. We add the $n+2$-th row to the first row multiplied by a_{21} and use the result as row $n+2$. As a result, the value in $a_{n+2,1}$ becomes 0.
4. We also do this process for columns and the second row.
5. As a result we get $\begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$.

$$\det(AB) = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix} = \begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = (-1)^n \begin{vmatrix} I & B \\ A & 0 \end{vmatrix} = (-1)^n (-1)^n \begin{vmatrix} A & 0 \\ I & B \end{vmatrix} = (-1)^{2n} \det(A) \det(B)$$

Exercise 21

Exercise 22. Prove by induction:

$$A := \begin{vmatrix} \alpha & \beta & \beta & \dots & \beta \\ \beta & \alpha & \beta & \dots & \beta \\ \vdots & & & \ddots & \vdots \\ \beta & \beta & \beta & \dots & \alpha \end{vmatrix} = (\alpha - \beta)^{n-1}(\alpha + (n-1)\beta)$$

Induction base For $n = 1$, it holds that $|\alpha| = \alpha$. Induction base satisfied.

Induction step

$$\frac{1}{\alpha^n}\begin{vmatrix}\alpha & \alpha\beta & \alpha\beta & \dots & \\ \beta & \alpha^2 & & & \\ \vdots & & \ddots & & \\ & & & & a^2\end{vmatrix}$$

$$= \frac{1}{\alpha^n}\begin{vmatrix}\alpha & \alpha\beta & \alpha\beta & \dots & \\ \beta & \alpha^2 & & & \\ \vdots & & \ddots & & \\ & & & & a^2\end{vmatrix}$$

$$= \frac{1}{\alpha^n}\begin{vmatrix}\alpha & 0 & 0 & \dots & \\ \beta & \alpha^2-\beta^2 & & & \\ \vdots & \beta(\alpha-\beta) & \ddots & & \\ & & & & \end{vmatrix}$$

$$= \alpha\frac{1}{alpha^n}\begin{vmatrix}\alpha^2-\beta^2 & & \beta(\alpha-\beta) \\ & \ddots & \\ \beta(\alpha-\beta) & & \alpha^2-\beta^2\end{vmatrix} =: d$$

$$\begin{aligned} d &= \frac{1}{\alpha^{n-1}}(\alpha^2-\beta^2-\alpha\beta+\beta^2)^{n-1}\left(\alpha^2-\beta^2+(n-1)(\alpha\beta-\beta^2)\right) \\ &= \frac{1}{\alpha^{n-1}}(\alpha(\alpha-\beta))^{n-1}(\alpha+\beta)(\alpha-\beta)+(n-1)\beta(\alpha-\beta) \\ &= \frac{1}{\alpha^{n-1}}\alpha^{n-1}(\alpha-\beta)^{n-1}\cdot(\alpha-\beta)(\alpha+\beta+(n-1)\beta) \\ &= (\alpha-\beta)^n(\alpha+n\beta) \end{aligned}$$

Again: the division by α implies that $\alpha \neq 0$. It is important to consider $\alpha = 0$. It is easy to show this case, but if you skip it, points are lost.

Exercise 22

Exercise 23. Let $P_i = (x_i, y_i)$ are pairwise different points in $\mathbb{R}^2$.

1. Show that the uniquely determined line g crossing points P_1 and P_2 can be described by the following equation:

$$g = \left\{(x,y)\in\mathbb{R}^2 : \begin{vmatrix}1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y\end{vmatrix} = 0\right\}$$

2. Show that the uniquely determined circle k crossing points P_1, P_2 and P_3, can be described by:

$$k = \left\{(x,y)\in\mathbb{R}^2 : \begin{vmatrix}1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x & y & x^2+y^2\end{vmatrix}\right\} = 0$$

   What is the result, if the points are colinear?

3. Determine the center of the circle crossing points $(-4, 1)$, $(-2, -3)$ and $(4, 5)$.

0.14 Exercise 22a

$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

Again, consider: $x_2 = x_1$ separately!

Laplace expansion along the last row:

$$1 \cdot (x_1 y_2 - x_2 y_1) - x(y_2 - y_1) + y(x_2 - x_1) \stackrel{!}{=} 0$$

$$\underbrace{\frac{(x_1 y_2 - x_2 y_1)}{x_2 - x_1}}_{d} - x \underbrace{\frac{(y_2 - y_1)}{x_2 - x_1}}_{k}$$

$$y_0 = \frac{y_2 - y_1}{x_2 - x_1} x_1 + d$$

$$d = y_1 - \frac{(y_2 - y_1)x_1}{(x_2 - x_1)x_1} = \frac{y_1 x_2 - y_1 x_1 - y_2 x_1 + y_2 x_1}{x_2 - x_1}$$

This corresponds to the slope of the line. Hence, our model matches the formula (the one involving the determinant).

What about $x_2 = x_1$? Then the second column is a linear combination of the others. Hence, determinant equals 0.

0.15 Exercise 22b

Consider 3 points P_1, P_2 and P_3. Consider point A half-way of $\overline{P_1 P_2}$. Consider point B half-way of $\overline{P_1 P_3}$. If the line g_1, orthogonal to $P_1 P_2$ and crossing A, crosses with the line g_2, orthogonal to $P_1 P_3$ and crossing B, meet this crosspoint M is the center of the circumference circle of P_1, P_2 and P_3.

$$v_1 = P_2 - P_1 = (2, -4) \to A = P_1 + \frac{v_1}{2} = (-3, -1)$$
$$v_2 = P_3 - P_1 = (8, 4) \to B = P_1 + \frac{v_2}{2} = (0, 3)$$

$$n_1 = \perp v_1 = (4, 2)$$
$$n_2 = \perp v_2 = (4, -8)$$

$$g_1 = A + t \cdot n_1$$
$$g_2 = B + s \cdot n_2$$

0.16 Exercise 22c

$$\begin{pmatrix}-3\\-1\end{pmatrix} + t\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix} + s\begin{pmatrix}4\\-8\end{pmatrix}$$

Gives $t = 1$ and

$$\begin{pmatrix}-3\\-1\end{pmatrix} + 1\begin{pmatrix}4\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = M$$

0.17 Exercise 22b: What if all points are colinear?

A generic circle equation is given by

$$(x - \overline{x})^2 + (y - \overline{y})^2 = r^2$$

$$x^2 - 2x\overline{x} + \overline{x}^2 + y^2 - 2y\overline{y} + \overline{y}^2 = r^2$$

$$x^2 + y^2 = \underbrace{r^2 - \overline{x}^2 - \overline{y}^2}_{K} + 2\overline{y}y + 2\overline{x}x$$

$$M \cdot \begin{pmatrix} K \\ 2\overline{x} \\ 2\overline{y} \end{pmatrix} = V$$

where M are the first three columns and V is the last column.

Exercise 23

Exercise 24. Let $A, B, C, D \in \mathbb{K}_{n\times n}$ be matrices. D is invertible and M is a $2n \times 2n$ block matrix.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

1. Show: M is invertible iff $A - BD^{-1}C$ is invertible
2. Show: $\det(M) = \det(A - BD^{-1}C)\det(D)$

0.18 Exercise 23a

$$\det(M) = \underbrace{\det(A - BD^{-1}C)}_{\neq 0 \text{ if invertible}} \underbrace{\det(D)}_{\neq 0 \text{ if invertible}}$$

$\det(D)$ is invertible by the exercise specification.

$$\det(A - BD^{-1}C) \neq 0 \implies A - BD^{-1}C = \text{ invertible}$$

0.19 Exercise 23b

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & B \\ 0 & D \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{bmatrix}$$

$$\begin{vmatrix} I & B \\ 0 & D \end{vmatrix} \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & I \end{vmatrix} = \det(D) \cdot \det(A - BD^{-1}C)\det(I)$$

Exercise 25

Exercise 25. Let A be a $m \times n$ matrix. Show that $\operatorname{rank}(A)$ is identical with the largest number $k \in \{1, 2, \ldots, \min(m, n)\}$ for which a non-vanishing subdeterminant of order k exists, hence index sets $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$, such that

$$\left| A_{i_k, j_k} \right| := \begin{vmatrix} a_{i_1,j_1} & a_{i_1,j_2} & \ldots & a_{i_1,j_k} \\ a_{i_2,j_1} & a_{i_2,j_2} & \ldots & a_{i_2,j_k} \\ \ldots & \ldots & \ddots & \vdots \\ a_{i_k,j_1} & a_{i_k,j_2} & \ldots & a_{i_k,j_k} \end{vmatrix} \neq 0$$

Assume $k \geq \operatorname{rank}(A)$.

$$A \to \tilde{A}$$

$m - \operatorname{rank}(A)$ rows and $n - \operatorname{rank}(A)$ columns. $\operatorname{rank}(A)$ is the number linear independent rows (or equivalently, columns)

$$\implies k \leq \operatorname{rank}(A) \implies k = \operatorname{rank}(A)$$

Exercise 26

Exercise 26. Let $A \in \mathbb{K}^{m\times n}$, $B \in \mathbb{K}^{n\times m}$. Show that

$$\det(AB) = \sum_{i_1<i_2<\ldots<i_m} \begin{vmatrix} a_{1i_1} & a_{1i_2} & \ldots & a_{1i_m} \\ a_{2i_1} & a_{2i_2} & \ldots & a_{2i_m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{mi_1} & a_{mi_2} & \ldots & a_{mi_m} \end{vmatrix} \begin{vmatrix} a_{i_1 1} & a_{i_2 1} & \ldots & a_{i_m 1} \\ a_{i_1 2} & a_{i_2 2} & \ldots & a_{i_m 2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1 m} & a_{i_2 m} & \ldots & a_{i_m m} \end{vmatrix}$$

Hint: Use Leibniz formula.

$$\det(AB) = \sum \det(A_{i_m\ldots}) \det(B^{i_1\ldots i_m})$$

$$A, B \in \mathbb{K}^{m\times m}$$

$$\det(AB) = \sum_{\sigma\in S_m} \operatorname{sign}(\sigma) \prod_{i=1}^{m} (AB)_{i\,\operatorname{sign}(i)} = \sum_{\sigma\in S_m} \operatorname{sign}(\sigma) \prod_{i=1}^{n} \left(\sum_{k=1}^{n} A_{i,k} B_{k\sigma(i)}\right)$$

Let $N = \{1, \ldots, n\}$. Let $M = \{1, \ldots, m\}$. Let N^M be the functions mapping M to N.

$$\sum_{\sigma\in\sigma_n} \operatorname{sign}(\sigma) \sum_{k\in N^M} \prod_{i=1}^{m} A_{ik(i)} B_{k(i)\sigma(i)}$$

$$= \sum_{k\in N^M} \sum_{\sigma\in S_m} \operatorname{sign}(\sigma) \prod_{i=1}^{m} A_{ik(i)} \prod_{i=1}^{m} B_{k(i)\sigma(i)} = \sum_{k\in N^M} \prod_{i=1}^{m} A_{ik(i)} \underbrace{\sum_{\sigma\in S_m} \operatorname{sign}(\sigma) \prod_{i=1}^{m} B_{k(i)\sigma(i)}}_{\det(B^{k(1)\ldots k(m)})}$$

Let $k, \tilde{k} \in N^M$. $k \sim \tilde{k} :\iff \operatorname{image}(k) = \operatorname{image}(\tilde{k})$.

$$= \sum_{k\in N^M \text{injective}/\sim} \sum_{\tilde{k}\sim k} \prod_{i=1}^{m} A_{ik(i)} \qquad \underbrace{\det(B^{(\tilde{k}(1)\ldots\tilde{k}(m))})}_{\operatorname{sign}(\delta)\det(B^{k(1)\ldots k(m)} \text{ with } k(1)<k(2)<\ldots<k(n)} \qquad)$$

where $\cdot/\sim$ denotes the set of equivalence classes. $\tilde{k} \sim k \implies \exists \delta \in \delta_m : \tilde{k} = k \circ \delta$.

$$= \sum_{k\in N^M \ \text{injective}/\sim} \left(\sum_{\delta\in\delta_m} \operatorname{sign}(\delta) \prod A_{ik(\delta_i)}\right) \det(B^*)$$

Exercise 28

Exercise 27. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show

1. $A \geq 0 \iff \exists B \in \mathbb{C}^{n\times n} : A = B^* \cdot B$
2. $A > 0 \implies A$ regular and $A^{-1} > 0$
3. Let $A \geq 0 \implies a_{ii} \geq 0 \forall i$ and if $\exists i : a_{ii} = 0 \implies a_{ij} = 0$
4. Does the following generalized first-minors criterion apply? "A $n \times n$ matrix A is positive semidefinite iff $\det A_r \geq 0 \forall r = 1, 2, \ldots, n$"

0.20 Exercise 28a

Direction $\Leftarrow$.

Let B be given such that $B^* \cdot B = A$.

$$z^* \cdot B^* \cdot B \cdot z = (Bz)^* \cdot B \cdot z$$
$$(Bz)^* = z^* B^*$$
$$(Bz)^* Bz = [v_1, \dots, v_n] \cdot \begin{bmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{bmatrix} := \sum_{i=1}^{n} \overline{v_1} \cdot v_1 = \sum_{i=1}^{n} \mid v_i \mid^2 \geq 0$$

Direction $\Rightarrow$.

Side remark:

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Let $A \geq 0$.

$$\implies \exists C \in \mathbb{C}^{n \times n} : A = C^* \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} C$$

$$A = C^* \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} C = C^* \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix}^* \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} C = (C')^* \cdot C' \iff A = (C')^* \cdot C'$$

0.21 Exercise 28b

$A > 0$, iff $A \hat{=} I_n$.

$$B^* AB = I_n \iff B^* AB = I_n \iff AB = (B^*)^{-1} \iff ABB^* = I_n$$
$$B^* A = B^{-1} \qquad \underbrace{BB^* A}_{A^{-1}} = I_n$$

$A^{-1} > 0$.

$$A^{-1} \hat{=} I_n \iff \exists C \in \mathrm{GL}(n, \mathbb{C}) : C^* \cdot A^{-1} \cdot C = I_n \iff A^{-1} = (C^*)^{-1} \cdot C^{-1}$$
$$A^{-1} = B \cdot B^* \qquad (B^{-1})^* = C$$

0.22 Exercise 28c

Show: $A \geq 0 \implies a_{ii} = 0$ and $a_{ii} = 0 \implies$ without loss of generality $a_{11} = 0 \qquad a_{1i} \neq 0 \quad a_{ij} = 0 \forall j$.

$$A = B^* B \implies a_{11} = \sum_{j=1}^{n} \overline{b_{j1}} \cdot b_{j1} = \sum_{j=1}^{n} \left| b_{j1} \right|^2 \stackrel{!}{=} 0$$
$$\implies b_{j1} = 0 \forall j$$
$$a_{1i} = 0 \qquad \text{gives a contradiction}$$

0.23 Exercise 28d

$$A = 0 \iff \det(A_r) \geq 0 \forall r \in \{1, \ldots, n\}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

Exercise 29

Exercise 28. Show

1. $A \leq B :\iff B - A \geq 0$ (hence $B - A$ is semidefinite) defines an order relation on the set of self-adjoint matrices.

2. If $B > 0$ and $A \geq B$, then $A > 0$

0.24 Exercise 29a

An order relation is a partial order. We show:

reflexivity xRx

anti symmetry $xRy \wedge yRx \implies x = y$

transitivity $xRy \wedge yRz \implies xRz$

We show antisymmetry.

$$\forall A \in M \text{ with } B - A \geq 0 \text{ and } A - B = 0$$

it holds that $\forall x \in V$:

$$x^T(B - A)\overline{x} \geq 0 \wedge x^T(A - B)\overline{x} \geq 0$$
$$x^T(B - A)\overline{x} = 0 \implies x^T B\overline{x} = x^T A\overline{x}$$

$$B - A = C^* DC$$

$$D = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$A - B = C^*(-D)C$$
$$D = -D = 0$$
$$\implies B = A$$

We show reflexivity.

$$\forall A \in M \forall x \in V : 0 = x^T \cdot 0 \cdot \overline{x} = x^T(A - A)\overline{x} = 0 \implies A - A \geq 0$$

We show transitivity.

$$\forall A, B, C \in M : B - A \geq 0 \wedge A - B > 0$$

It holds that

$$\forall x \in V : x^T(B-A)\overline{x} \geq 1 \qquad x^T(C-B)\overline{x} \geq 0$$

$$\implies 0 \leq x^T(B-A)\overline{x} + x^T(C-B)\overline{x}$$

$$= x^T\left((B-A)\overline{x} + (C-B)\overline{x}\right) = x^T(B-A, C-B)\overline{x} = \underbrace{x^T(C-A)\overline{x}}_{0\leq}$$

$$\implies C - A \geq 0$$

0.25 Exercise 29b

Let $B > 0$ and $A \geq B$ then it holds that $A > 0$.

$$\forall x \in V : x^T B\overline{x} > 0 : x^T(A-B)\overline{x} - x^T A\overline{x} - x^T B\overline{x} \geq 0$$

$$\implies x^T A\overline{x} \geq x^T B\overline{x} > 0$$

$$\langle x, x\rangle_B = x^T Bx = x^T Ax = \langle x, x\rangle_A$$

$$x = e_j \implies B_{jj} = A_{jj}$$

$$A = 0 \qquad B = \operatorname{rot}(\frac{\pi}{2}) \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \pi$$

Exercise 30

Exercise 29. Let $Tr(A) = \sum_{i=1}^n a_{ii}$ be the trace of an $n \times n$ matrix over $\mathbb{R}$ or $\mathbb{C}$. Show:

1. $Tr : \mathbb{K}^{n\times n} \to \mathbb{K}$ is linear and for $A \in \mathbb{K}^{n\times m}, B \in \mathbb{K}^{m\times n}$ it holds that $Tr(AB) = Tr(BA)$ but in general $Tr(ABC) = Tr(ACB)$ does not hold.
2. Let A, B be $n \times n$ matrices. B is invertible. Show $Tr(B^{-1}AB) = Tr(A)$.
3. Show: $\nexists A, B : AB - BA = I$
4. Show that $\langle A, B\rangle = Tr(B^*A)$ defines a positive definite scalar product over $\mathbb{C}^{n\times n}$.
5. Find a real matrix A such that $Tr(A^2) < 0$
6. For a fixed positive definite matrix A, $\langle A, B\rangle_Q = Tr(B^*QA)$ defines a positive definite scalar product.

Hint: Exercise 28 can be helpful.

0.26 Exercise 30a

Show linearity.

$$\forall A, B \in \mathbb{K}^{n\times n} : \underbrace{Tr(A+B)}_{\sum_{i=1}^n (a_{ii}+b_{ii})} = \underbrace{Tr(A)}_{\sum_{i=1}^n a_{ii}} + \underbrace{Tr(B)}_{\sum_{i=1}^n b_{ii}}$$

$$\lambda \in K : \lambda Tr(A) + Tr(\lambda A)$$

$$\lambda \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda a_{ii}$$

Show that multiplication is commutative for two traces. Let $A \in \mathbb{K}^{n \times m}, B \in \mathbb{K}^{m \times n}$.

$$Tr(AB) = Tr(BA)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} a_{ji} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ji} b_{ij}$$

Show that multiplication is not commutative in general. Does not hold unless $B = C$.

$$Tr(ABC) \neq Tr(ACB)$$

0.27 Exercise 30b

Show that $Tr(B^{-1}(AB)) = Tr(A) \iff Tr(ABB^{-1}) = Tr(A)$.

0.28 Exercise 30c

Let $A, B \in \mathbb{K}^{n \times n}$.

$$Tr(I_n) = n$$
$$Tr(AB - BA) = Tr(AB) - Tr(BA) = 0$$
$$0 \neq n$$

This gives a contradiction.

0.29 Exercise 30d

1. Sesquilinearity:

$$\langle A + \lambda B, C \rangle \stackrel{!}{=} \langle A, C \rangle + \lambda \langle B, C \rangle$$
$$\langle A, C + \lambda B \rangle = Tr((C + \lambda B)^* A) = Tr((C^* + \overline{\lambda} B^*)A) = Tr(C^* A + \lambda B)$$

2. Positive definiteness:

$$\langle A, A \rangle > 0 \qquad A \neq 0$$
$$\mathrm{Tr}(A^* A) = \sum_{j=1}^{n} \sum_{l=1}^{n} \overline{a}_{l_j} a_{l_j} = \sum_{j=1}^{n} \sum_{l=1}^{n} |a_{lj}|^2$$

0.30 Exercise 30e

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

0.31 Exercise 30f

Q is positive definite.

$$\begin{aligned}
\langle A, B \rangle_Q &= Tr(B^* Q A) \\
\langle A, A \rangle_Q &= Tr(A^* M^* M A) \\
&= Tr((MA)^* MA) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \overline{(ma)}_j (ma)_j = \sum_{i=1}^{n} \sum_{j=1}^{n} |ma_j|^2
\end{aligned}$$

$$Q = C^* DC$$
$$\exists M : (Q)^{-1} = (M^* M)^{-1} = M^{-1}(M^*)^{-1}$$

Show $MA \neq 0$ if $A \neq 0$ and $\exists M^{-1} \iff A = M^{-1}0$ gives a contradiction. Thus, we are finished.

Exercise 31

Exercise 30. Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Show:

1. $A \geq 0 \iff \exists x_1, x_2, \ldots, x_n \in \mathbb{C}^{n\times 1} : A = \sum_{i=1}^{n} x_i x_i^*$.
2. Let C be the matrix with entries $c_{ij} = a_{ij} b_{ij}$. If $A \geq 0$ and $B \geq 0$, then also $C \geq 0$.

0.32 Exercise 31a

By Exercise 28, we know: $A \geq 0 \implies \exists B : A \cdot B^* \cdot B$.

$$x_i \ldots (B^*)_i \qquad x_i^* \ldots (B)_i$$

$$(x_i \cdot x_i^*)_W = x_i^k \cdot x_i^{*j}$$
$$\sum_i (x_i x_i^*)_{k,j} = (B^*)_k \cdot (B)_j$$

$$a_{kj} = \sum_{i=1}^{n} b_k^* b_{ij}$$

0.33 Exercise 31b

Direction $\Leftarrow$.

$$A = \sum_{i=-1}^{n} x_i x_i^*$$
$$y^T A \overline{y} = y^T \sum_{i=1}^{n} x_i x_i^* \cdot \overline{y} = \sum (y^T x_i)_{1\times 1} (x_i^* \overline{y})_{1\times 1} = \sum \left\| y^T x_i \right\|^2 \geq 0$$

Direction $\Rightarrow$.

$$A = \sum x_i x_i^* \qquad B = \sum y_i y_i^*$$
$$c_{ij} = a_{ij} \cdot b_{ij} = \sum_{k=1}^{n} x_k^i \cdot \overline{x_k^j} \cdot \sum_{l=1}^{n} y_l^i \overline{y_l^j} = \sum_{k,l=1}^{n} \underbrace{\left(x_k^i y_l^j\right)}_{z_{k,l^i}} \underbrace{\left(\overline{x_k^j y_l^j}\right)}_{\overline{z_{k,l^j}}}$$
$$\implies C = \sum_{k,l=1}^{n} z_{k,l} \cdot z_{k,l}^*$$

Exercise 32

Exercise 31. Let $(V, \langle .,. \rangle)$ be a vector space with scalar product and $U \subseteq V$ is a subspace. Show:

1. $U^\perp = U^{\perp\perp\perp}$;
2. $V = U \dot{+} U^\perp \implies U = U^{\perp\perp}$.
3. Show that the following construction is a counterexample for inversion of the previous statement: $V = C[-1, 1]$ with scalar product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$ and subspace $U = \{f \in C[-1, 1] | f(t) = 0 \forall t < 0\}$.

0.34 Exercise 32a

$$U^\perp = \{v \in V : \forall u \in U : \langle u, v \rangle = 0\}$$

We prove:

1. $U^\perp \subseteq U^{\perp\perp\perp}$
2. $U^\perp \supseteq U^{\perp\perp\perp}$

We begin with (1.)

Let $v \in U^\perp \implies v \in U^{\perp\perp\perp}$

$$U^{\perp\perp\perp} = \left\{v \in V \middle| \langle v, u'' \rangle = 0 \forall u'' \in U^{\perp\perp}\right\}$$

By definition, this satisfies the claim.

In other words: we know $U \subseteq U^{\perp\perp}$. Consider $W = U^\perp$. Then $W \subset W^{\perp\perp}$.

We prove (2.)

Let $x \in U^{\perp\perp\perp} \implies \forall u \in U^{\perp\perp} : \langle x, u \rangle = 0$. Because $U \subseteq U^{\perp\perp}$, $\implies \forall u' \in U : \langle x, u' \rangle = 0 \implies x \in U^\perp$. Hence $U^\perp \in U^{\perp\perp\perp}$.

0.35 Exercise 32b

$$V = U + U^\perp \implies U = U^{\perp\perp}$$

Show that $U^{\perp\perp} \subseteq U$. Let $x \in U^{\perp\perp}$. $x = U + W$. $u \in U, w \in U^\perp$.

$$\implies \forall y \in U^\perp : \langle x, y \rangle = 0 = \langle u + w, y \rangle = \langle u, y \rangle + \langle w, y \rangle = 0 \implies w = 0$$

$$\implies x = u \in U$$

0.36 Exercise 32c

Example for $U = U^{\perp\perp}$ but $V \neq U + U^\perp$.

$$V = [-1, 1] \qquad \langle f, g \rangle = \int_{-1}^{1} f(x) \cdot g(x)\,dx$$
$$U = \{f \in C[-1, 1] : f(t) = 0 \forall t < 0\}$$

Claim:

$$U^\perp = \{f \in C[-1, 1], f(t) = 0 \forall t \geq 0\}$$

Assume $f \in U^\perp$. Choose $g \in U$. We build a triangle below the point $f(g)$ and function f. The area of the triangle is non-negative and therefore non-zero.

Claim:

$$U^{\perp\perp} = \{f \in C[-1, 1], f(t) = 0 \forall t < 0\} \implies U = U^{\perp\perp}$$

Exercise 33

Exercise 32. Let $V = \mathbb{R}^{n\times n}$ and $\langle A, B\rangle = \mathrm{Tr}(B^T A)$ the scalar product of Exercise 30. Determine the orthogonal complement.

$$\left\{A \in \mathbb{R}^{n\times n} \middle| A = A^T\right\}^\perp$$

$$U = \left\{A \in \mathbb{R}^{n\times 1} : A = A^T\right\}, \qquad V = \mathbb{R}^{n\times n}$$

$$A_{ii} = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}$$

positive at (i, i).

$$A_{ij} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}$$

positive at (j, i) with $i \neq j$.

$$\mathrm{Tr}(B^T A) = \sum_{k,i=1}^{n} B_{ik} A_{ki} \stackrel{!}{=} 0$$

For $A = A_{ii} \implies B_{ii} = 0$. For $A = A_{ij} \implies B_{ij} + B_{ji} = 0$. Skew-symmetric.

Exercise 34

Exercise 33. Let

$$U = \left\{x \in \mathbb{R}^5 \middle| {}^{x_1 - x_2 + x_3 - x_4 + x_5 = 0}_{x_1 + x_3 + x_5 = 0}\right\}$$

be a subspace of $\mathbb{R}^5$ and $v = (1, -1, 1, -1, 1)^T$.

1. Determine the orthogonal projection $\pi_U(v)$ using the Gramian matrix.
2. Determine the orthonormal basis of U
3. Determine $\pi_U(v)$ using the orthonormal basis.
4. Determine the matrix representation of π_U in terms of the canonical basis.

0.37 Exercise 34b

$$\tilde{a}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\|\tilde{a}_1\| = \sqrt{2}$$

$$a_1 = \frac{1}{\sqrt{2}} \tilde{a}_1$$

$$\tilde{a}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\|\tilde{a}_2\| = \sqrt{2} \qquad a_2 = \frac{1}{\sqrt{2}}\tilde{a}_2$$

$$\tilde{a}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\left\langle \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$\|\tilde{a}_3\| = \frac{\sqrt{6}}{2} \qquad a_3 = \frac{2}{\sqrt{6}}\tilde{a}_3$$

0.38 Exercise 34d

$$P = \sum_{i=1}^{3} a_i a_i^*$$

$$P = \begin{pmatrix} \frac{2}{3} & 0 & -\frac{2}{3} & 0 & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{2}{3} & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{pmatrix}$$

Exercise 35

Exercise 34. Given the data $\vec{x} = (-2, -1, 1, 2)$ and $y = (1, 1, -1, 1)$. Determine the coefficients a_0, a_1, a_2 of the quadratic polynomial function f using an orthogonal projection.

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto a_0 + a_1 x + a_2 x^2$$

such that the value

$$\sum_{i=1}^{4} (f(x_i) - y_i)^2$$

is minimal. Reason that the solution is unique.

Partial exam, Exercise 4

$$A = \{a_{ij}\} \qquad C = \{(-1)^{i+j} a_{ij}\}$$

$$\det(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i \in \sigma(i)} \operatorname{sign}(\sigma)$$

$$\det(C) = \sum_{\sigma \in S_n} \underbrace{\prod_{i=1}^{n} \underbrace{c_{i,\sigma(i)}}_{i-\sigma(i)}}_{\prod_{i=1}^{n} a_{i,\sigma(i)} (-1)^{\sum i - \sigma(i)}} \operatorname{sign}(\sigma)$$

Partial exam, Exercise 5

$$U_i \perp U_j \qquad i \neq j$$

$$\forall u_i, w_i \in U_i : \sum w_i = \sum u_i \iff w_i = u_i$$

Consider $\sum(w_i - u_i)$. $\sum(w_i - u_i) = 0$.

$$0 = \|w_i - u_i\|^2 = \langle \sum w_i - u_i, \sum w_i - u_i \rangle = \sum \|w_i - u_i\|^2$$

Exercise 39

Exercise 35. $\langle u, v\rangle = u^T A v$ is the scalar product with $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Determine the adjugate map to the linear map $f(x, y) = (2x - y, x + y)$.

$$V = \mathbb{R}^2 \qquad \langle u, v\rangle = u^T A v \qquad f(x, y) = 2(x - y, x + y)$$

$$f^* : \forall x, y \in \mathbb{R} : \langle f^*(x), y, = \rangle \langle x, f(y)\rangle$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad f^*\begin{pmatrix} x \\ y \end{pmatrix} = C\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\implies C^T A = AB \implies C^T = ABA^{-1}$$

$$\langle f^*(x), y\rangle = \langle Cx, y\rangle = (Cx)^T Ay = x^T C^T Ay$$

$$\langle x, f(y)\rangle = x^T A(By)$$

$$C^T = \begin{pmatrix} 6 & 7 \\ 3 & -3 \end{pmatrix} \implies C = \begin{pmatrix} 6 & 3 \\ -7 & -3 \end{pmatrix}$$

Exercise 40

Exercise 36. 1. Determine the matrix representation of the orthogonal reflection σ_U on the plane $U = \{x \in \mathbb{R}^3 : x_1 + x_2 - x_3 = 0\}$ in regards of an appropriate orthonormal basis and in regards of a standard basis.

2. Let σ_V be an orthogonal reflection on the plane

$$V = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

Determine the matrix of the composition $\rho = \sigma_V \circ \sigma_U$ in regards of the standard basis and give a reason, why ρ is a rotation. Determine rotation axis and rotation angle of ρ.

Exercise (a).

$$V = \{x \in \mathbb{R}^3 | x_1 + x_2 - x_3 = 0\} \qquad \vec{n} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Points on the plane U:

$$(0, 0, 0)$$
$$(1, -1, 0)$$
$$(1, 1, 2)$$

$$P_0(x_{10}, x_{20}, x_{30})$$

Solve equation system.

$$\begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} + \lambda_n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\lambda_n = \frac{1}{3}(x_{30} - x_{20} - x_{10})$$

$$\delta_{UB} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\sigma_U(p_0) = \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \end{pmatrix} + 2\frac{1}{3}(x_{30} - x_{20} - x_{10}) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} x_{10} - 2x_{20} + 2x_{30} \\ -2x_{10} + x_{20} + 2x_{30} \\ 2x_{10} + 2x_{20} + x_{30} \end{pmatrix} \delta_{U,B} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Exercise (b).

$$p \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{15}{9} \\ -\frac{15}{9} \\ \frac{1}{9} \end{pmatrix}$$

$$\begin{aligned} \cos\varphi &= \frac{\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{9} \end{pmatrix} \rangle}{\sqrt{6} \cdot \sqrt{\frac{151}{81}}} \\ &= \frac{-\frac{28}{9}}{\sqrt{6} \cdot \sqrt{\frac{451}{81}}} \end{aligned}$$

$$\varphi = 122.5°$$

But these calculations contain an error. $\approx 141°$ should be correct.

Exercise 41

Exercise 37. Show that every matrix $U \in \mathrm{SU}_2(\mathbb{C})$ has structure $U = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$ with $|z|^2 + |w|^2 = 1$.

$$U \in \mathrm{SU}_2(\mathbb{C}) \iff U = \begin{bmatrix} z & -\overline{w} \\ w & \overline{w} \end{bmatrix} \land |z|^2 + |w|^2 = 1$$

Direction $\Leftarrow$.

Is easy. $U^*U = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Direction $\Rightarrow$.

$$\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = U^{-1}$$

$$U^{-1} = \frac{1}{\det(U)} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

$$\implies d = \overline{a} \qquad b = -\overline{c}$$

Exercise 42

Exercise 38. Quaternions are elements of a 4-dimensional vector space

$$\mathbb{H} = \{a_0 + a_1 i + a_2 j + a_3 k : a_i \in \mathbb{R}\}$$

over $\mathbb{R}$ with formal basis $\{1, i, j, k\}$ and multiplication laws:

$$ij = k = -ji \qquad jk = i = -kj \qquad ki = j = -ik \qquad i^2 = j^2 = k^2 = -1$$

Show that

1. Quaterions give an associative algebra.
2. Every quaternion has a multiplicative inverse.
3. The map $\Phi : \mathbb{H} \to M_2(\mathbb{C})$

$$a_0 + a_1 i + a_2 j + a_3 k \mapsto a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + a_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

4. Show that $\mathrm{SU}_2(\mathbb{C}) \simeq \{q \in \mathbb{H} \mid q\overline{q} = 1\}$. Hint: compare with 41.

Exercise (a).

Simply long calculations.

Exercise (b).

Let $q \in \mathbb{H} \setminus \{0\}$.

$$(a_0 + a_1 i + a_2 j + a_3 k)(a_0 - a_1 i - a_2 j - a_3 k) = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

$$q^{-1} = \frac{a_0 - a_1 i - a_2 j - a_3 k}{a_0^2 + a_1^2 + a_2^2 + a_3^2}$$

0 is a quaternion, but just like in the real numbers, a multiplicative inverse only exists for the group except for 0.

Exercise (c).

$$(a_1 i + a_2 j) \mapsto a_1 \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{=:A} + a_2 \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=:B}$$

$AB = C$. $C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. And so on and so forth.

Exercise (d).

$$\mathbb{H}_1 := \{q \in \mathbb{H} \mid g\overline{g} = 1\}$$

$$\Phi : \mathbb{H} \to M_2(\mathbb{C})$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \mapsto \begin{bmatrix} \alpha + i\beta & \gamma + i\delta \\ -(\gamma + i\delta) & \overline{\alpha + i\beta} \end{bmatrix}$$

Prove injectivity:

$$p, q \in \mathbb{H}_1$$

Show: $\Phi(p) = \Phi(q) \implies p = q$.

$$\begin{bmatrix}\alpha_1 + i\beta_1 & \gamma_1 + i\delta_1 \\ \dots & \dots\end{bmatrix} - \begin{bmatrix}\alpha_1 + i\beta_1 & \gamma_1 + i\delta_1 \\ \dots & \dots\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}$$
$$(\alpha_1 + i\beta_1) - (\alpha_2 + i\beta_2) = 0 \implies \alpha_1 = \alpha_2 \wedge \beta_1 = \beta_2$$

Prove surjectivity:

Immediate if you look at the matrix representation above.

Remark: Rationale for quaternions

$$v - (v_x, v_y, v_z) \in S^2 = \left\{|\, x\,|^2 + |\, y\,|^2 + |\, z\,|^2 = 1\right\}$$
$$v \in [0, 2\pi]$$

Rotation with axis v and angle $\theta = R^v_\theta$.

$$q^V_v := \cos(\frac{v}{2}) - (v_x i + v_y j + v_z k)\sin(\frac{v}{2})$$
$$R^V_v w = q^v_v w \overline{q^v}_w$$

with $w = (w_x i + w_y j + w_z k)$.

Every rotation matrix can be represented as quaternion.

Exercise 38

Exercise b.

Let $\{p_n\}_{n\geq 0}$ be orthogonal polynomials in $\mathbb{R}[x]$ in regards of $\int fgw\,dx$ (from Exercise a) $\det(P_n) = n$ with leading coefficients $(p_n) = 1$. What is $xp_n(x)$? $\sum_{j=0}^{n+i} \alpha_j p_j$. The claim is $\alpha_j = 0 \forall j \in \{0, \dots, n-2\}$. $\alpha_{n+1} = 1$.

How about $\langle xp_n, p_0\rangle$?

$$\langle xp_n, p_0\rangle = \int_a^b xp_2 1w\,dt + 0 = \int_a^b xp_2 1w\,dt + \int_a^b p_n \cdot c \cdot w\,dt = \int_a^b xp_2 1w\,dt + \langle p_n, c\cdot p_0\rangle$$
$$\int_a^b p_n \underbrace{(x+c)}_{=:p_1} w\,dt = \underbrace{\langle p_n, p_n\rangle}_{n>1} = 0$$

How about $\langle xp_n, p_1\rangle$?

$$\langle xp_1, p_1\rangle = \int_a^b \overbrace{xp_n(x+c)}^{p_n(x^2+cx)} w\,dt + \underbrace{0}_{=\int_a^b p_n(x+c)\lambda w\,dt = \langle p_n, \lambda p_1\rangle}$$
$$= \int_a^b p_1(x^2 + (c+\lambda)x + \lambda c)w\,dt + \underbrace{0}_{\langle p_n, \gamma\rangle}$$
$$\gamma = \gamma \cdot p_0$$
$$\int_a^b p_n \underbrace{\left(x^2 + (c+\lambda)x + c\lambda + \gamma\right)}_{p_2} w\,dt$$
$$\langle p_n, p_2\rangle = 0$$

These practicals took place on 2018/05/23.

Exercise 43

Exercise 39. The derivative of the polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{K}[x]$ is defined (over an arbitrary field!) as

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.$$

Show:

1. The map $p(x) \mapsto p'(x)$ is linear and the Leibniz rule holds
$$(pq)'(x) = p'(x)q(x) + p(x)q'(x)$$
as well as the chain rule
$$(p \circ q)'(x) = p'(q(x))q'(x)$$
2. If $q(x)$ is an irreducible factor of $p(x)$ with multiplicity ≥ 2, then also $q(x)$ is a divisor of $\gcd(p(x), p'(x))$
3. For $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ the inverse of the previous item holds as well. What can go wrong in finite fields?

0.39 Exercise 43a

$$(x^n x^m)' = (x^{n+m})' = (n+m)x^{n+m-1}$$
$$\stackrel{?}{=} n x^{n-1} x^m + m x^{m-1} x^n$$

0.40 Exercise 43b

$$((p(x))^n)' = n p(x)^{n-1} p'(x)$$
$$p(x)' = (q(x)q(x)p_R(x))' = q(x)'q(x)p_R(x)xq'(x)q(x)p_R(x) + p_R'(x)q(x)g(x)$$

0.41 Exercise 43c

Remark: $q(x)$ is irreducible.

$$q(x) | p(x)$$
$$q(x) | p'(x)$$

$$q(x)^2 = p(x)$$

Exercise 44

Exercise 40.

$$p(x) := 8 + 12x + 14x^2 + 13x^3 + 6x^4 + x^5$$

Decompose polynomial $p(x)$ into irreducible factors over

$(a)\ \mathbb{Q} \qquad (b)\ \mathbb{R} \qquad (c)\ \mathbb{C} \qquad (d)\ \mathbb{Z}_{11} \qquad (e)\ \mathbb{Z}_{13}$

0.42 Exercise 44c

$$(x+2)^3(x^2+1)$$

$$\begin{aligned} x^2+1 &= 0 \\ x^2 &= -1 \\ &= \pm 1 \end{aligned}$$

Factors are:

$$(x+2)^3(x+i)(x-i)$$

0.43 Exercise 44a,b

Factors are:

$$(x^2)^3(x^2+1)$$

0.44 Exercise 44d

Considering (x^2+1) in $\mathbb{Z}_{11}$.

$$\begin{aligned} x^2 &\equiv -1 \\ x^2 &\equiv -1 \pmod{11} \\ x^2 &\equiv 10 \pmod{11} \\ x^2 &\equiv -1 \pmod{11} \end{aligned}$$

0.45 Exercise 44e

Considering (x^2+1) in $\mathbb{Z}_{13}$.

$$x^2 \equiv -1 \pmod{13}$$

holds for $x = 5 \wedge x = 8$.

$$(x^2+1) = (x-5)(x-8) = x^2 - \underbrace{13x}_{\equiv 0} + \underbrace{40}_{\equiv 1}$$

Exercise 45

Exercise 41. Let $p(x) := x^7 - x^5 + x^4 - x^3 + x - 1$ and $q(x) := x^8 - x^5 - x^4 + x^3 - 2x^2 + 2x - 2$. Determine $\gcd(p(x), q(x))$ with Euclidean algorithm over $\mathbb{Q}[x]$ and polynomials $a(x)$ and $b(x)$ such that $a(x)p(x) + b(x)q(x) = \gcd(p(x), q(x))$

$$x^8 - x^5 - x^4 + x^3 - 2x^2 + 2x - 2 = (x^3 - 2) \cdot \gcd$$

$$\gcd = x^5 + x^2 - x + 1 \stackrel{!}{=} (x^2+1)(x^3-x+1)$$

$$a(x)p(x) + b(x)q(x) = \gcd$$

Evaluation:

$$q(x) : p(x) = x$$

$$p(x) : R_1 = x + 2$$

$$
\begin{aligned}
3x^5 + 3x^2 - 3x + 3 &= R_2 \\
R_1 : R_2 &= \frac{1}{3}x - \frac{2}{3} \\
0 &= R_3
\end{aligned}
$$

$$
\begin{aligned}
q(x) &= p(x)x + R_1 \\
&= [(x+2)\cdot R_1 + R_2]\cdot x + R_1 \\
&= \left[(x+2)\cdot\left(\frac{1}{3}x - \frac{2}{3}\right)R_2 + R_2\right]\cdot x + \left[\frac{1}{3}x - \frac{2}{3}\right]R_2 \\
R_2 &\left[\left[(x+2)\left(\frac{1}{3}x^2 - \frac{2}{3}\right) + 1\right]\cdot x + \left(\frac{1}{3}x - \frac{2}{3}\right)\right] \\
&= \frac{1}{3}R_2\,[[(x+2)(1x-2)+3]\cdot x + (1x-2)] \\
&\frac{1}{3}R_2[(x^2-4)+3]\cdot x + (1x-2) \\
&x^3 - 4x + 3x + 1x - 2 \\
&x^3 - x + x - 2 \\
&\frac{1}{3}R_2(x^3-2)
\end{aligned}
$$

i	b_i	a_i
1	0	1
2	1	$0-(x+2)$
3	$-x-2$	$1-[x(x+2)]$

$$1-[x(x+2)] = x^2+2x+1$$

Exercise 46

Exercise 42. Let $p(x)$ and $q(x)$ be non-disappearing polynomials of degree m and n over some field $\mathbb{K}$. Show:

- $\gcd(p(x), q(x)) = 1 \iff$ polynomials $a(x)$ and $b(x)$ exist such that $a(x)p(x) + b(x)q(x) = 1$.
- $\gcd(p(x), q(x))$ is non-trivial iff polynomials $A(x)$ and $B(x)$ exist with $\deg A(x) < n$ and $\deg B(x) < m$ such that $A(x)p(x) + B(x)q(x) = 0$.
- Let $p(x) = p_0 + p_1x + \cdots + p_mx^m$ and $q(x) = q_0 + q_1x + \cdots + q_nx^n$ polynomials of degree m and n with $p_m, q_n \neq 0$. Show that $p(x)$ and $q(x)$ have a non-trivial common divisor iff the determinant $R(p,q)$ disappears.

$$
R(p,q) := \begin{bmatrix}
p_m & 0 & \dots & 0 & q_n & 0 & \dots & 0 \\
p_{m-1} & p_m & \ddots & \vdots & q_{n-1} & q_n & \ddots & \vdots \\
\vdots & p_{m-1} & \ddots & 0 & \vdots & q_{n-1} & \ddots & 0 \\
p_0 & \vdots & \ddots & p_m & q_0 & \vdots & \ddots & q_n \\
0 & p_0 & \ddots & p_{m-1} & 0 & q_0 & \ddots & q_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \dots & p_0 & 0 & 0 & \dots & q_0
\end{bmatrix}
$$

with $n+m$ columns.

0.46 Exercise 46a

$\gcd(p(x), q(x)) = 1 \iff \exists a(x), b(x) : a(x)p(x) + b(x)q(x) = 1.$

Personal notes:

$$\begin{aligned} a(x)\cdot p(x) + b(x)\cdot q(x) &= 1 \\ 1\cdot gcd(p(x), q(x)) &= d(x) \qquad \deg(d(x)) > 1 \\ a(x)\cdot d(x)\cdot p_1(x) + b(x)\cdot d(x)\cdot q_1(x) &= 1 \\ d(x)\cdot [a(x)\cdot p_1(x) + b(x)\cdot q_1(x)] &= 1 \\ a(x)\cdot p_1(x) + b(x)\cdot q_1(x) &= \frac{1}{d(x)} \end{aligned}$$

1 has a lower polynomial degree than $d(x)$. Thus $\frac{1}{d(x)}$ represents a hyperbola. As $x \to \infty$, a polynomial goes to infinity or minus infinity. A hyperbola goes to zero. Thus, $\frac{1}{d(x)}$ cannot be a polynomial unless $d(x) = 1$.

0.47 Exercise 46b

$$\deg(\gcd(p(x), q(x))) \geq 1 \iff \exists A(x), B(x)$$

s.t. $\deg(A(x)) < n = \deg(q(x))$ and $\deg(B(x)) < m = \deg(p(x))$ and $A(x)p(x) + B(x)q(x) = 0$.

Direction $\implies$.

$$p(x) = \gcd\cdot \underbrace{p_R(x)}_{\deg<m}$$

$$q(x) = \gcd\cdot \underbrace{q_R(x)}_{\deg<n}$$

$$A(x) = q_R(x)(-1) \qquad B(x) = p_R(x)$$

$$\begin{gathered} a(x)p_R(x)\cdot\gcd(p,q) + b(x)q_R\cdot\gcd(p,q) = 1 \\ \deg(\text{polynom}(x)\cdot\gcd) = \deg(1) = 0 \\ \underbrace{\deg(\text{polynom}(x))}_{\leq 0} + \underbrace{\deg(\gcd)}_{\geq 0} = 0 \end{gathered}$$

Direction $\impliedby$.

Without loss of generality, A, B = 1.

$$A(x)p(x) = -B(x)q(x)$$

$$\underbrace{\deg(A(x)p(x))}_{<m+n} = \deg(B\cdot q)$$

$$\implies p(x)|B(x)q(x)$$

$\implies \exists$ factor/irreducible polynomial.

$$p(x) = \prod_{i=1}^{m'} \underbrace{p_i(x)}_{\text{irreducible}}$$

$$\deg(B) < m$$

$p(x)$ does not divide $B(x)$. Thus, $\exists i \in \{1, \ldots, m\} : p_i | q(x)$.

Personal notes: “nichtverschwindend” = “nicht konstant”

$$\deg(p) = m \qquad \deg(g) = n$$

$$\gcd(p(x), g(x)) \neq 1 \iff \exists A(x), B(x) : A(x)p(x) + B(x)q(x) = 0 \iff \exists A(x), B(x) : \frac{A(x)}{B(x)} = -\frac{g(x)}{p(x)}$$

Direction $\implies$:

$$\gcd(p(x), g(x)) = d(x) \qquad \deg d \geq 1$$
$$p(x) = d(x) \cdot p_1(x) \qquad g(x) = d(x) \cdot g_1(x)$$
$$\implies \frac{g(x)}{p(x)} = \frac{d(x)}{d(x)} \cdot \frac{g_1(x)}{p_1(x)} = -\left(-\frac{g_1(x)}{p_1(x)}\right)$$
$$\implies \frac{g(x)}{p(x)} = -\frac{A(x)}{B(x)}$$

0.48 Exercise 46c

$$L : \mathbb{R}^{n+m} \to \mathbb{R}$$

$$\begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \\ b_m \\ b_{m-1} \\ \vdots \\ b_0 \end{pmatrix} \mapsto p(x) \overbrace{(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0^0)}^{=:A(x)} + q(x) \overbrace{(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0)}^{=:B(x)} \to \mathbb{R}^{n+m}$$

$\angle(\vec{v}) = R(q, p) \cdot \vec{v}$

Exercise 47

> **Exercise 43**. Let $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and $p(x) \in \mathbb{K}[x]$ is a polynomial. Determine $p(A)$. Hint: Consider the polynomials $p(x) = x^n$, $n = 1, 2, \ldots$ and prove the result.

$$p(x) = \alpha_m x^m + \alpha_{m-1}x^{m-1} + \cdots + \alpha_1 x^1 + \alpha_0 x^0$$
$$A^2 = A \cdot A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2ab \\ 0 & a^2 \end{pmatrix}$$
$$A^3 = \begin{pmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{pmatrix}$$
$$A^n = \begin{pmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{pmatrix}$$
$$\forall A > 0 : p(A) = \begin{pmatrix} p(a) & p'(a)b \\ 0 & p(a) \end{pmatrix} \qquad a \neq 0$$

Exercise 48

Exercise 44. Determine the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 9 & 6 & -2 & 3 \\ -15 & -9 & 4 & -5 \\ 15 & 9 & -4 & 5 \\ 12 & 6 & -4 & 4 \end{bmatrix}$$

over $\mathbb{R}$ and $\mathbb{C}$ and if possible, a matrix B such that $B^{-1}AB = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

$$\det(A - \lambda I) = \lambda^4 - \lambda^2$$

$$\begin{aligned} v_1 &= \begin{pmatrix} 0 & -2 & 3 & 6 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 2 & -2 & 4 & 0 \end{pmatrix} \\ v_3 &= \begin{pmatrix} -i-3 & 5 & -5 & 2i-4 \end{pmatrix} \\ v_4 &= \begin{pmatrix} i-3 & 5 & -5 & -2i-4 \end{pmatrix} \end{aligned}$$

Exercise 49

Exercise 45. Let A be a $\mathbb{K}^{n\times n}$ matrix. Show:

1. If $p(x) \in \mathbb{K}[x]$ is a polynomial such that $p(A) = 0$, then all eigenvalues $\lambda \in \operatorname{spec}(A)$ satisfy $p(\lambda) = 0$.
2. If A is regular, then the eigenvalues are given by

   $$\operatorname{spec}(A^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \operatorname{spec}(A)\right\}$$

   and the associated eigenspaces are the same.

Exercise 50

Exercise 46. Let $A, B \in \mathbb{K}^{n\times n}$. Show that $\operatorname{spec}(AB) = \operatorname{spec}(BA)$.

We use a case distinction. Consider $\lambda \neq 0$.

There exists $v \neq 0$, such that

$$ABv = \lambda v \iff (BA)(Bv) = \lambda(Bv) \iff \lambda \in \operatorname{spec}(BA) \text{ with } Bv \neq 0$$

We chose some $v \neq 0$. The opposite direction works analogously, but we cannot simply claim the proof works for both directions (because we chose some specific v).

Consider $\lambda = 0$.

$$\det(0I - AB) = 0 \iff \det(-A)\det(B) = 0 \iff \det(-BA) = 0 \iff 0 \in \operatorname{spec}(BA)$$

Exercise 51

Exercise 47. Let A be a $\mathbb{K}^{n\times n}$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

1. Show there exist matrices $M_1, M_2, \ldots, M_n \in \mathbb{K}^{n\times n}$ with properties:

(a) idempotent, $M_i^2 = M_i$

(b) $M_i M_j = 0$ if $i \neq j$

(c) $\operatorname{rank}(M_i) = 1$

such that $A = \sum_{i=1}^n \lambda_i M_i$. Furthermore $A^k = \sum_{i=1}^n \lambda_i^k M_i \forall k \in \mathbb{N}$.

2. Let $\mathbb{K} = \mathbb{C}$. Determine the matrices $M_1, M_2, \ldots, M_n$ for the $n \times n$ matrix

$$\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \vdots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

$$\exists B \in \mathbb{K}^{n \times n} : B^{-1}AB = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \iff A = BDB^{-1}$$

$$\begin{aligned} a_{ij} &= (BDB^{-1})_{ij} \\ &= \sum_{k=1}^n (BD)_{ik}(B^{-1})_{kj} \\ &= \sum_{k=1}^n \left((\sum_{i=1}^n B)_{il} \cdot (D)_k \right) \cdot (B^{-1})_{kj} \\ &= \sum_{k=1}^n (B_{ik}) \cdot \lambda_k \cdot (B^{-1})_{kj} \end{aligned}$$

$$B = (b_{ij})_{1 \leq i,j \leq n} \qquad B^{-1} = (b'_{ij})_{1 \leq i,j \leq n}$$

$$a_{ij} = \sum_{k=1}^n \lambda_k \cdot b_{ik} \cdot b'_{kj}$$

$$M_i = (b_{ir} \cdot b'_{rj})_{1 \leq i,j \leq n} \implies A = \sum_{i=1}^n \lambda_i \cdot M_i$$

The three properties:

1.

$$\begin{aligned} (M_r^2) &= \sum_{k=1}^n (M_r)_{ik} \cdot (M_r)_{kj} \\ &= \sum_{k=1}^n b_{ir} \cdot b'_{rk} \cdot b_{kr} \cdot b'_{ij} \\ &= b_{ir} \cdot b_{ij} \cdot \sum_{k=1}^n b'_{rk} \cdot b_{kl} \\ &= (M_r)_{ij}(B^{-1}B)_{rr} = (M_r)_{ij} \end{aligned}$$

2. Let $r \neq s$.

$$\begin{aligned} (M_r M_s)_{ij} &= \sum_{k=1}^n (M_i)_{ik}(M_j)_{kj} \\ &= \sum_{k=1}^n b_{ii} \cdot b'_{ik} \cdot b_{ks} \cdot b'_{si} \\ &= b_{ir} \cdot b'_{sj} \cdot (B^{-1}B)_{is} = 0 \end{aligned}$$

3. $\operatorname{rank}(M_i) = 1$. Let $r \in \{1, \ldots, n\}$ is arbitrarily fixed and let v_k be the k-th column vector. 1

0.49 Exercise 51b

$$\mathbb{K} = \mathbb{C}$$

$$T = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & \ddots & 0 & 0 \\ \vdots & \ddots & 0 & \vdots \end{pmatrix}$$

$$\det(\lambda - T) = \begin{vmatrix} \lambda & 0 & & -1 \\ -1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & -1 & \lambda \end{vmatrix} = \lambda \begin{vmatrix} \lambda & & 0 \\ -1 & \ddots & \\ \vdots & \ddots & \\ 0 & & -i\lambda \end{vmatrix} + (-1)(-1)^{n+1} \begin{vmatrix} -1 & \lambda & \\ & \ddots & \ddots \\ & & -1 \end{vmatrix}$$

$$= \lambda^n - 1 \overset{!}{=} 0 \implies \lambda = \sqrt[n]{n}$$

$$\lambda_k = e^{i\frac{2\pi k}{n}} : k \in \{1, \dots, n\}$$

$$Tv_k = T \begin{pmatrix} v_{k,1} \\ \vdots \\ v_{k,n} \end{pmatrix} = \begin{pmatrix} v_{k,1} \\ v_{k,2} \\ \vdots \\ v_{k,n-1} \end{pmatrix} = \lambda_k \begin{pmatrix} v_{k,1} \\ \vdots \\ v_{k,n} \end{pmatrix} \implies v_{k,l} = \frac{e^{-i\frac{2\pi k l}{n}}}{\sqrt{n}}$$

$$v_{k,n} = \lambda_k v_{k,n} \qquad v_{k,l} = \lambda_k v_{k,l+1} \qquad l = 1, \dots, n-2$$

$$\langle v_k, v_m \rangle = \sum_{l=1}^{n} e^{-i\frac{2\pi(k-m)l}{n}} = \frac{e^{-i\frac{2\pi(k-m)}{n}} \left(1 - e^{-i2\pi(k-m)\frac{n}{n}}\right)}{1 - e^{-i\frac{2\pi(k-m)}{n}}} = 0$$

$$\implies B = \begin{pmatrix} \vdots v_1 \vdots & \dots & \vdots v_n \vdots \end{pmatrix}$$

$$B^{-1} = B^*$$

$$M_k = B \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix} \cdot B^{-1} = v_k \cdot v_k^* = M_k$$

These practicals took place on 2018/06/06.

Exercise 52

Exercise 48. Let $A \in \mathbb{K}^{n\times n}$. Show equivalence of the following statements:

1. $\operatorname{rank}(A) = 1$
2. 0 is an eigenvalue of geometric multiplicity $n - 1$
3. There are vectors $x, y \in \mathbb{K}^n$ with $x, y \neq 0$ such that $A = xy^T$

Trivial for $n = 1$. Let $n > 1$.

Let $A \in \mathbb{K}^{n\times n}$.

$1 \to 2.$

$$\operatorname{rank}(A) = 1 \iff \operatorname{rowrank}(A) = 1 \implies A = \begin{bmatrix} a_1 & \dots & a_n \\ \lambda_1 a_1 & \dots & \lambda_n a_n \\ \vdots & \ddots & \vdots \\ \lambda_n a_1 & \dots & \lambda_n a_n \end{bmatrix}$$

$$\det(0 \cdot I - A) = 0 = \det(-A) = 0$$

$$d(0) = \dim\ker(0 \cdot I - A) = \dim\ker(-A)$$

$$\dim\operatorname{image}(A) = \operatorname{rank}(A) = 1$$

$$\underbrace{\dim(A)}_{=n} = 1 + d(0)$$

$1 \iff 3.$

$$A = \begin{bmatrix} \lambda_1 a_1 & \dots & \lambda_n a_n \\ \vdots & \ddots & \vdots \\ \lambda_n a_1 & \dots & \lambda_n a_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} [a_1, \dots, a_n] = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}^T$$

$2 \to 1.$ $0 \in \operatorname{spec}(A)$. $d(0) = n - 1 = \dim\ker(A - 0 \cdot I) = \dim\ker(A)$.

$$\dim(A) = \dim\operatorname{image}(A) + \dim\ker(A) = \operatorname{rank}(A) + n - 1$$

$$n = \operatorname{rank}(A) + n - 1 \implies \operatorname{rank}(A) = 1$$

Exercise 53

Exercise 49.

1. Let $f : V \to V$ be a diagonalizable linear map and $W \subseteq V$ an invariant subspace. Show that $f|_W : W \to W$ is diagonalizable as well.
2. Let $A, B \in \mathbb{K}^{n \times n}$ be diagonalizable matrices. Show that $AB = BA$ holds iff there exists a basis whose entries are eigenvectors of A and B simultaneously.

Hint: Show that the eigenspaces of A and B are invariant.

0.50 Exercise 53a

Diagonalizable $\iff$ $\exists$ basis B of V of eigenvectors V of f.

$$V = \oplus \gamma_{\lambda_i}$$

Let n be the number of eigenvalues.

$$\forall v \in V : \exists! v_i \in \gamma_{\lambda_i} : v = \sum_{i=1}^{n} v_i$$

$$\forall w \in W : \exists! v_i \in \gamma_{\lambda_i} : w = \sum_{i=1}^{n} w_i \implies \exists! B = \{b_1, \dots, b_m\} : b_i \in \gamma_{\lambda_{j_i}}$$

Recall that

$$V \cap W = W = \oplus \qquad \underbrace{(\gamma_{\lambda_i} \cap W)}_{\neq 0 \text{ thus gives a vector of the basis}}$$

0.51 Exercise 53b

Let C be the basis of eigenvectors of A and B.

Direction $\Longleftarrow$.

$$\forall c \in C : ABc = A\lambda_C^B c = \lambda_C^A \lambda_C^B c = B\lambda_C^A C = BAc \implies AB = BA$$

Direction $\Longrightarrow$. Let $x \in \gamma_\lambda^A$ and let $AB = BA$.

$$BAx = B\lambda x = \lambda Bx = ABx \implies Bx \in \gamma_\lambda^A$$

By (1), $B|_{\ker(\lambda - A)}$ diagonal $\forall \lambda \in \operatorname{spec}(A)$.

$$\implies \exists b_1', \dots, b_n' =: B \text{ for } \ker(\lambda_i - A) \implies B = \bigcup_i B^i \text{ basis of } V$$

$$V = \oplus \ker(\lambda_i - A)$$

Exercise 54

Exercise 50. Let $A \in \mathbb{K}^{n \times n}$ be a matrix. For a given vector $v \in \mathbb{K}^n$ consider the sequence $v, Av, A^2v, \dots$.

$$m := \min\left\{k \,\middle|\, \exists c_0, c_1, \dots, c_{k-1} : A^k v = c_0 v + c_1 Av + \dots + c_{k-1} A^{k-1} v\right\}$$

Show:

1. $v, Av, A^2v, \dots, A^{m-1}v$ are linear independent.
2. $U_v = \mathcal{L}\left\{v, Av, \dots, A^{m-1}v\right\}$ is the smallest A-invariant subspace containing v.
3. Let $A^m v = \sum_{i=0}^{m-1} c_i A^i v$. Determine the matrix representation of restriction $C = \Phi_B^B(f_A|_{U_v})$ in regards of basis $B = (v, Av, A^2v, \dots, A^{m-1}v)$.

0.52 Exercise 54a

$$c_0 v + c_1 Av + \dots + c_{m-1} A^{m-1} v = 0 \implies c_i = 0 \forall i = 0, \dots, n-1$$
$$k = \max\{i : c_i \neq 0\} \implies k \leq m-1$$

It holds that $c_0 v + c_1 Av + \dots + c_{k+1} A^{k-1} v = -c_k A^k v \implies m \leq k$.

$m \leq k$ contradicts with $k \leq m-1$.

0.53 Exercise 54b

$$\forall w \in U_v : Aw \in U_v$$

Assume $\exists W : v \in W, W$ invariant.

$$B = \text{ basis of } W$$
$$\implies B = \{v, b_2, \dots, b_m\} = \{v, Av, \dots\}$$

0.54 Exercise 54c

$$B = \left\{w, Av, \dots, A^{m-1}v\right\}$$
$$A^n v = \sum_{i=0}^{n-1} c_i A^i v$$
$$\Phi_B^B(f|_{U_v})$$

b_1	b_2	$\dots$	b_{n-1}	b_{n-2}
b_1	0	0	$\ddots$	c_0
b_2	1	0	$\ddots$	c_1
$\vdots$	$\vdots$			$\vdots$
b_n	0	0	$\ddots$	c_n

Exercise 55

Exercise 51. 1. Let $A \in \mathbb{C}^{n\times n}$ and $p(x), q(x) \in \mathbb{C}[x]$ polynomials with $p(A)q(A) = 0$ and $\gcd(p(x), q(x)) =$
1. Show that image $q(A) = \ker p(A)$.

2. Let A be an idempotent matrix ($A^2 = A$). Show that A is diagonalizable.

0.55 Exercise 55a

$$A \in \mathbb{C}^{n\times n} \qquad p(x)q(x) \in C[x] \qquad p(A)q(A) = 0 \qquad \gcd(p, q) = 1$$

1. $\forall v \in V : p(A)q(A)v = 0 \implies \operatorname{image}(q(A)) \subseteq \ker(p(A))$
2. By exercise 46, $\gcd(p, q) = 1 \iff \exists a, b \in \mathbb{C}[x] : p(x)a(x) + q(x)b(x) = 1$

$$\implies \forall v \in V : p(A)a(A)v + q(A)b(A)v = v \implies V = \operatorname{image} p(A) + \operatorname{image} q(A)$$
$$\implies \dim \operatorname{image} p(A) + \dim \operatorname{image} q(A) \geq n \qquad n = \dim(V)$$
$$\dim \operatorname{image} p(A) + \dim \ker p(A) = n$$
$$\implies \dim \operatorname{image} q(A) \geq \dim \ker p(A) \iff \operatorname{image}(q(A)) \subseteq \ker(p(A))$$

0.56 Exercise 55b

$$\implies \lambda_i = \{0, 1\}$$
$$A^2 v = A\lambda_i v_i = \lambda_i^2 v_i = \lambda_i v_i$$

Exercise 56

Exercise 52. Determine all invariant subspaces of matrices (over $\mathbb{C}$)

$$(a) \quad A = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \qquad (b) \quad B = \begin{bmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{bmatrix}$$

$$\chi_A(\lambda) = (\lambda + 1)(\lambda - 1)^2 \lambda$$

Consider $\lambda = 0$.

$$\ker(A) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & -1 & \\ & 0 & & 0 \end{pmatrix} \implies v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad U_3 = \mathcal{L}\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Consider $\lambda = 1$.

$$v_2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad a_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \implies \mathcal{L}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} = U_4$$

Consider $\lambda = -1$.

$$v_4 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \mathcal{L}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\} = U_5$$

$$\text{image}(A) = \left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\} = U_6$$

Is this set complete? Difficult to prove.

Alternative approach (a constructive approach):

$$W \subseteq \mathbb{C}^4, \text{ invariant in regards of } A$$

W is of structure on the left side.

$$v = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$\mathcal{L}(v)$ is invariant.

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \in W$$

$$\begin{pmatrix}0\\0\\0\\d\end{pmatrix} = \begin{pmatrix}a\\b\\c\\d\end{pmatrix} - \begin{pmatrix}a\\b\\c\\0\end{pmatrix} \in W$$

$$A\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}a\\-b\\c\\0\end{pmatrix}$$

$$A^2\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}a\\-b\\c\\0\end{pmatrix}$$

If $d = 0 \implies \underbrace{W}_{=W^{-1}} \subset \mathbb{C}^3 \times \{0\}$. If $d \neq 0 \implies e_k \in W \implies W = W^{-1} \oplus \mathcal{L}(e_4)$ with $W^{-1} \subseteq \mathbb{C}^3$.

$$\begin{pmatrix}a\\b\\c\\0\end{pmatrix} \in W^{-1}$$

$$A\begin{pmatrix}a\\b\\c\\0\end{pmatrix}=\begin{pmatrix}a\\-b\\c\\0\end{pmatrix}\in W^{-1}$$

$$\begin{pmatrix}a\\b\\c\\0\end{pmatrix}-\begin{pmatrix}a\\-b\\c\\0\end{pmatrix}\in W^{(-1)}$$

$$b=0 \implies W^{-1}\in\mathcal{L}(e_1,e_3)$$

$$b\neq 0 \implies e_2\in W^{-1}\subseteq W$$

$$W=W^{-2}\oplus\mathcal{L}(e_2,e_4)$$

$$\underbrace{\begin{pmatrix}a\\b\\c\\0\end{pmatrix}-\begin{pmatrix}a\\-b\\c\\0\end{pmatrix}}_{\begin{pmatrix}0\\b\\0\\0\end{pmatrix}}\in W^{(-1)}$$

$$A\begin{pmatrix}a\\b\\c\\0\end{pmatrix}=\begin{pmatrix}a\\-b\\c\\0\end{pmatrix}\in W^{-1}$$

$$A\begin{pmatrix}a\\0\\c\\0\end{pmatrix}=\begin{pmatrix}a\\0\\c\\0\end{pmatrix}$$

Exercise 57

Exercise 53. We call $J_k(\lambda)$ the Jordan block of length k of eigenvalue λ, hence

$$J_k(\lambda)=\begin{bmatrix}\lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1\\ & & & & \lambda\end{bmatrix}$$

Given a block diagonal matrix with the following Jordan blocks $J_k(\lambda)$:

$$A=\begin{bmatrix}J_2(0) &&&&&&&\\ & J_3(0) &&&&&&\\ && J_3(0) &&&&&\\ &&& J_1(1) &&&&\\ &&&& J_1(1) &&&\\ &&&&& J_2(1) &&\\ &&&&&& J_3(1) &\\ &&&&&&& J_5(1)\end{bmatrix}$$

Determine $\dim\ker(\lambda I - A)^k$ for $\lambda\in\{0,1\}$ and $0\le k\le 20$ and bases of all eigenspaces and main spaces.

(e_1, e_3, e_6).

Let $k = 2$. $(e_1, e_2, e_3, e_4, e_6, e_7)$.

k \ λ	0	1
1	3	5
2	6	8
3	8	10
4	8	11
5	8	12
6	8	12
$\vdots$	$\vdots$	$\vdots$

$$\dim \ker(\lambda 1 - A)^2$$

$$(e_9 e_{10} e_{11} e_{13} e_{16}) + (e_{12} e_{14} e_{12}) + (e_{15} e_{18}) + (e_{19})$$

Exercise 58

Exercise 54. For a matrix $A \in \mathbb{C}^{20\times 20}$ the following kernel dimensions are known:

k	1	2	3	4	5	6	7
$\ker(A-2I)^k$	1	2	3	4	5	6	6
$\ker(A-I)^k$	0	0	0	0	0	0	0
$\ker(A)^k$	3	4	5	6	7	7	7
$\ker(A+I)^k$	3	6	6	6	6	6	6
$\ker(A+2I)^k$	0	1	1	1	1	1	1

1. Two numbers in this table are wrong. Find and fix them.
2. Determine a Jordan normal form and the minimal polynomial of A.

0 in the last row is wrong. If $\ker(A + 2I) = \{0\}$, then $(A + 2I)$ is regular. The product of regular matrices is regular. Hence the kernel must be trivial ($= \{0\}$).

$$\begin{array}{lll} \lambda = -1 & k = 7 & \text{not 7, but 6} \\ \lambda = -2 & k = 1 & \text{not 0, but 1} \end{array}$$

$$J = \operatorname{diag}(J_6(2), J_5(0), J_1(0), J_2(-1), J_2(-1), J_2(-1), J_1(-2))$$

$$a_S(\lambda) = 2a_J - a_{J-1} - a_{J+1}$$

$$(\lambda - 2)^6 \lambda^5 (\lambda + 1)^2 (\lambda + 2)$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

Exercise 61

Exercise 55. Determine and interpret the evaluation of the exponential function of the cross product:

$$e^{\vec{\varphi}x}]\vec{v} = \vec{v} + \vec{\varphi} \times \vec{v} + \frac{1}{2!}\vec{\varphi} \times (\vec{\varphi} \times \vec{v}) + \frac{1}{3!}\vec{\varphi} \times (\vec{\varphi}(\vec{\varphi} \times \vec{v})) + \dots$$

where $\vec{\varphi} := (q, 0, 0)^T$. Hint: matrix representation of linear map $\vec{\varphi}\times : \vec{x} \mapsto \vec{\varphi} \times \vec{x}$.

$$\vec{\varphi} = \begin{pmatrix} \varphi \\ 0 \\ 0 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \varphi$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right) \cdot \vec{v} = \sum$$

Exercise 59

Exercise 56. The matrix A has the characteristic polynomial

$$A = \begin{bmatrix} -5 & -1 & -5 & -7 & -7 & -4 \\ 1 & -3 & 0 & -1 & -1 & 0 \\ 7 & 5 & 8 & 12 & 12 & 6 \\ 3 & -3 & 2 & 2 & 1 & 1 \\ -8 & 6 & -6 & -6 & -5 & -4 \\ 7 & -10 & 4 & 1 & 1 & 3 \end{bmatrix}$$

1. Determine for every main space $\ker(\lambda_i - A)^{r_i}$ a basis $(u_1^{(i)}, u_2^{(i)}, \dots, u_{n_i}^{(i)})$ in such a way that every $(u_1^{(i)}, u_2^{(i)}, \dots, u_{m_{i,k}}^{(i)})$ is a basis of $\ker(\lambda_i - A)^k$.
2. Determine a Jordan normal form J and the minimal polynomial of A and a regular matrix B such that $B^{-1}AB = J$.

$$\chi_A = (x-1)^3(x+1)^3$$

$$\ker(A-1) = \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, x_1, x_3 \in \mathbb{R} \right\}$$

$$\ker(A+1) = \left\{ x_4 \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \\ 1 \\ -2 \\ 3 \end{pmatrix}, x_4 \in \mathbb{R} \right\}$$

$$\dim\ker(A-1) = 2 \qquad \dim\ker(A+1) = 1$$

$$J = \begin{pmatrix} 1 & & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ & & & & & 1 \end{pmatrix}$$

$$B : B^{-1}AB = J$$
$$\implies AB = BJ$$
$$B = (b_1, b_2, \dots, b_6)$$

where b_i are column vectors.

$$AB = \begin{pmatrix} b_1 & b_2 & b_2 + b_3 & -b_4 & b_4 - b_5 & b_5 - b_6 \end{pmatrix}$$

$$\begin{aligned} (A-1)b_1 &= 0 \qquad & (A+1)b_4 &= 0 \\ (A-1)b_2 &= 0 \qquad & (A+1)b_5 &= b_4 \\ (A-1)b_3 &= b_2 \qquad & (A+1)b_0 &= b_5 \end{aligned}$$

$$B = \begin{pmatrix} 0 & 1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & -1 & 2 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -2 & 0 & -1 \\ 0 & -2 & 3 & 3 & 0 & 0 \end{pmatrix}$$

Exercise 63

Exercise 57. Show that a matrix is nilpotent if and only if 0 is the only eigenvalue.

A is nilpotent $\iff \exists k \in \mathbb{N} : A^k = 0$. We assume an algebraically closed field, because otherwise a counterexample can be found.

Direction $\implies$.

Assume there exists a eigenvalue $\lambda \neq 0$: $A^k = 0 \iff \ker(0 \cdot I - A)^k = V$.

$$A^k x = \lambda^k x$$

This gives a contradiction. If the field is algebraically closed, then 0 is the only eigenvalue.

Direction $\impliedby$.

Let $A \in \mathbb{K}^{n \times n} \implies \chi_A(\lambda) = \lambda^n$.

$$\chi_A(A) = 0 \xRightarrow{\text{Cayleigh-Hamilton}} A^r = 0 \implies \exists k \in \mathbb{N} : A^k = 0$$

Exercise 64

Exercise 58. Determine a unitary matrix U, that diagonalizes the matrix

$$\begin{pmatrix} i & 1 & i & -1 \\ -1 & i & 1 & i \\ i & -1 & i & 1 \\ 1 & i & -1 & i \end{pmatrix}$$

$$U = \begin{pmatrix} -i/\sqrt{2} & 1/\sqrt{6} & i/\sqrt{12} & -i/2 \\ 1/\sqrt{2} & -i/\sqrt{6} & 1/\sqrt{12} & -1/2 \\ 0 & 2/\sqrt{6} & -i/\sqrt{12} & i/2 \\ 0 & 0 & 3/\sqrt{12} & 1/2 \end{pmatrix}$$

Exercise 65

Exercise 59. Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Show that for an arbitrary vector $x \in \mathbb{C}^n$ it holds that

$$\|Ax\| \leq \max_{1\leq i\leq n} |\lambda_i| \|x\|$$

$A \in \mathbb{C}^{n\times n}$ is normal and $\lambda_1, \dots, \lambda_n$ are eigenvalues. $x \in \mathbb{C}^n$. Show that $\|Ax\| \leq \max_{1\leq i\leq n} |\lambda_i| \|x\|$. $B = \{u_1, \dots, u_n\} \subseteq \mathbb{C}^n \implies \exists$ linear combination : $x = \sum_i \mu_i u_i$.

$$\begin{aligned}
\|Ax\| &= \left\| A \cdot \sum_{i=1}^{n} \mu_i u_i \right\| \\
&= \left\| \sum_{j=1}^{n} \mu_i \cdot \lambda_i \cdot u_i \right\| \\
&= \sqrt{\langle \sum_{i=1}^{n} \mu_i \lambda_i u_i \rangle} \\
&= \sqrt{\sum_{i=1}^{n} \langle \mu_i \lambda_i u_i, \mu_i \lambda_i u_i \rangle} \\
&= \sqrt{\sum_{i=1}^{n} |\lambda_i|^2 \underbrace{\langle \mu_i u_i, \mu_i u_i \rangle}_{>0}} \\
&\leq \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} \langle \mu_i u_i, \mu_i u_i \rangle} \\
&= \max_{1\leq j\leq n} \left|\lambda_j\right| \left\| \sum_{i=1}^{n} \mu_i u_i \right\| \\
&= \max_{1\leq j\leq n} \left|\lambda_j\right| \|x\|
\end{aligned}$$

Exercise 66

Exercise 60. Determine the Schur normal form of matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ -1 & -3 & -1 \end{bmatrix}$$

Determine the eigenvalues:

$$\det(A - \lambda I) = 0 \qquad \chi_A(\lambda) = (\lambda + 1)(\lambda - 1)^2$$

Determine the eigenvectors:

$$\lambda = -1 \implies \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$W_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad W_1^* = W_1$$

$$W^* A W = \begin{pmatrix} -1 & -3 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Consider the bottom-right 2×2 matrix as A_2.

$$A_2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \qquad \lambda_{2_1} = 1 \quad \lambda_{2_2} = 1$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$W_2^* \cdot W_1^* \cdot A \cdot W_1 \cdot W_2 = \begin{pmatrix} -1 & -\sqrt{2} & -2\sqrt{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

The diagonal elements are the eigenvalues and it is an upper triangular matrix.

Exercise 67

Exercise 61. Let $A \in \mathbb{C}^{n \times n}$ be a normal matrix. Show that for an arbitrary matrix $B \in \mathbb{C}^{n \times n}$:

$$AB = BA \implies A^*B = BA^*$$

$A \in \mathbb{C}^{n \times n}$ is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$ and $B \in \mathbb{C}^{n \times n}$ arbitrary. Let k be the number of different eigenvalues. $\Lambda_j := \lambda_{n_j} \cdot I$.

$\exists U \in U_n(\mathbb{C}) : U^*AU = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_k)$.

$$AB = BA \iff U^*AUU^*BU = U^*BUU^*AU$$

$C := U^*BU$.

$$\iff DC = CD \iff \sum_{\gamma=1}^{n} d_{i\gamma} c_{\gamma j} \iff \lambda_i \cdot c_{ij} = c_{ij} \cdot \lambda_j \iff c_{ij} = 0 \text{ if } \lambda_i \neq \lambda_j$$

$C = \operatorname{diag}(C_1, \ldots, C_k)$ with $\dim C_i = \dim \Lambda_i$. $D^* = \overline{D} = \operatorname{diag}(\overline{\Lambda_1}, \overline{\Lambda_k})$.

$$D^*C = \operatorname{diag}(\overline{\Lambda}_1 C_1, \ldots, \overline{\Lambda}_k C_k) = \operatorname{diag}(\overline{\lambda}_{n_1} C_1, \ldots, \overline{\lambda}_{n_k} C_k) = \operatorname{diag}(C_1 \overline{\Lambda}_1, \ldots, C_k \overline{\Lambda}_k)$$

$$= CD^* \implies D^*C = CD^* \iff (U^*AU)^* \cdot U^*BU = U^*BU \cdot (U^*AU)^*$$

$$\iff U^*A^*UU^*BU = U^*BUU^*A^*U \iff A^*B = BA^*$$

Also, $Av = \lambda v$.

$$ABv = BAv = \lambda Bv$$

$$B(E_j) \subseteq E_\lambda$$

$$\text{eigenspace}(A, \lambda) = \text{eigenspace}(A^*, \overline{\lambda})$$

Exercise 68

Exercise 62. Determine the translation and a rotation, that transforms the quadric Q into normal form.

$$Q = \left\{ x \,\middle|\, 16x_1^2 - 24x_1x_2 + 9x_2^2 + 30x_1 + 40x_2 + 50 = 0 \right\} \subseteq \mathbb{R}^2$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad x^T A x + b^T x + \gamma = 0$$

$$\implies A = \begin{pmatrix} 16 & -12 \\ -12 & 9 \end{pmatrix} \implies b = \begin{pmatrix} 30 \\ 40 \end{pmatrix} \qquad \gamma = 50$$

Task 1: rotation.

$$A = QDQ^*$$

$$\chi_A(\lambda) = (\lambda - 16)(\lambda - 9) - 144 = \lambda(\lambda - 25)$$

$$\implies \lambda_1 = 0 \qquad \lambda_2 = 25$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}$$

$$\ker(0 - A) = \ker \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} = \mathcal{L} \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \implies u_1 = \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\ker(25 - 1) = \mathcal{L} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \implies u_2 = \frac{1}{5} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\implies Q = \frac{1}{5} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \qquad \text{“rotation matrix”}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(Qy)^T A (Qy) + b^T (Qy) + \gamma = 0$$

$$y^T \cdot \underbrace{Q^T A Q}_{D} \cdot y + b^T Q y + \gamma = 0$$

Insert: $25y_2^2 + \xi D y_1 + \xi 0 = 0$.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

$$y_2^2 + 2y_1 + 2 = 0$$

$$(z_2 + t_2)^2 + 2(z_1 + t_1) + 2 = 0$$

$$z_2^2 + 2z_2 t_2 + t_2^2 + 2z_1 + 2t_1 + 20$$

$$2t_2 \overset{!}{=} 0 \implies t_2 = 0$$

$$2t_2 + t_2^2 + 2 \overset{!}{=} 0 \implies t_1 = -1$$

$$t = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$z_2^2 + 2z_1 = 0$$

$$b^T \cdot Q = \frac{1}{3} \begin{pmatrix} 30 & 40 \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 50 & 0 \end{pmatrix}$$

Translation is not a linear map. Thus, it is described in vector multiplication, not as a matrix.

(rotation matrix $\implies$ determinant is one)

1 Exercise 71

$$\ker A \cap \ker B = \{0\} \qquad AB = BA$$

1.1 Exercise 71a

$$\ker A^2 \cap \ker B = \{0\}$$

Let $x \in \ker A^2 \cap \ker B : A^2 x = 0 \implies Ax \in \ker A$. $Bx = 0$. $BAx = ABx = 0 \implies Ax \in \ker B$.

Thus, $Ax \in \ker A \cap \ker B = \{0\} \implies Ax = 0 \implies x \in \ker A$. By assumption $x \in \ker B$. Thus, $x \in \ker A \cap \ker B = \{0\} \implies x = 0$.

1.2 Exercise 71b

$$\ker A^2 \cap \ker B^2 \overset{!}{=} 0$$
$$A' = B \qquad B' = A^2$$
$$\ker A' \cap \ker B' = \{0\}$$
$$A'B' = B'A'$$

$$BA^2 = BAA = ABA = AAB = A^2 B$$

We apply (a) to A' and B':

$$\underbrace{\ker(A')^2 \cap \ker(B')}_{=\ker B^2 \cap \ker A^2} = \{0\}$$

1.3 Exercise 71c

Inductive:

$$A^{(2^k)} \cdot B^{(2^k)} = B^{(2^k)} \cdot A^{2^k}$$

Induction hypothesis: $\ker A^{2^k} \cap \ker B^{2^k} = \{0\}$.

We apply (b) to A^{2^k} and B^{2^k}.

$$\ker(A^{2^k})^2 \cap \ker(B^{2^k})^2 = \{0\}$$

Recall that $(A^{2^k})^2 = A^{2^k} \cdot A^{2^k} = A^{2^k} + A^{2^k} = A^{2 \cdot 2^k} = A^{2^{k+1}}$.

1.4 Exercise 71d

$$\ker A^r \cap \ker B^s = \{0\}$$

choose k such that $2^k \geq \max(r, s)$.

$$\ker A^r \leq \ker A^{2^k} \wedge \ker B^s \leq \ker B^{2^k} \implies \ker A^r \cap \ker B^s \subseteq \ker A^{2^k} \cap \ker B^{2^k} = \{0\}$$

Main spaces $\lambda - A$ and $\mu - A$

1. commute
2. $\ker(\lambda - A) \cap \ker(\mu - A) = \{0\}$

$$\implies \ker(\lambda - A)^r \cap \ker(\mu - A)^s = \{0\}$$

1.5 Exercise 72

1. Determine eigenvalues.
2. Determine eigenvectors.
3. Determine eigenspaces and *main* spaces.

   For eigenvalue 8, we get:

$$\mathcal{L}\left(\begin{pmatrix}1\\0\\0\\1\\0\end{pmatrix}\right) = \ker(8 - A)^k \qquad \forall k$$

Eigenspace = main space, thus no further computations are required.

$$B = \begin{bmatrix}1 & \dots\\0 & \dots\\0 & \dots\\1 & \dots\\0 & \dots\end{bmatrix} \qquad J = \begin{bmatrix}8 & \dots\\ \vdots & \ddots\end{bmatrix}$$

For eigenvalue 0, we get:

$$\ker A = \mathcal{L}\left(\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\2\end{pmatrix}\right)$$

$$\ker A^2 = \mathcal{L}\left(\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\2\end{pmatrix}, \begin{pmatrix}3\\2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}\right) =: \{u_1, u_2, u_3, u_4\}$$

$\dim = 4 \implies$ this is the main space.

$$v_1^{(2)} - v_3 = \begin{pmatrix}3\\2\\0\\0\\0\end{pmatrix} \in \ker A^2$$

$$\implies A \cdot u_3 \in \ker A$$

are linear independent. $Au_4 \in \ker A$.

$$A = \begin{pmatrix}4 & -2 & -4 & 4 & -2\\0 & 0 & 0 & 0 & 0\\2 & 1 & -2 & -2 & -1\\4 & -6 & -4 & 4 & -2\\-4 & 6 & 4 & 4 & 2\end{pmatrix}$$

$$v_1^{(1)} = Av_1^{(2)} = \begin{pmatrix}8\\0\\8\\0\\0\end{pmatrix} = 8u_1$$

$$v_2^{(1)} = Av_2^{(2)} = A \cdot \begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-4\\0\\8\end{pmatrix} = 4 \cdot u_2 - 4 \cdot u_1$$

$$\mathcal{L}(u_1, u_2) = \mathcal{L}(v_1^{(1)}, v_2^{(1)})$$

$$v_1^{(1)} \quad v_2^{(1)} \quad | \quad v_1^{(2)} \quad v_2^{(2)}$$

$$\text{order: } v_1^{(1)} \to v_1^{(2)} \to v_2^{(1)} \to v_2^{(2)}$$

$$B = \begin{bmatrix} 1 & 8 & 3 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 8 & 0 & -4 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 8 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 8 & & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

where the second column is $v_1^{(1)}$, the third column is $v_1^{(2)}$, the third column is $v_2^{(1)}$ and the fourth column is $v_2^{(2)}$.

$$(A - \lambda I)v_i^{(k)} = v_i^{(k-1)} \implies A \cdot v_i^{(k)} = \lambda v_i^{(k)} + v_i^{(k-1)}$$

2 Exercise 73

2.1 Exercise 73a

$$\text{JNF}(A) = \text{JNF}(A^T)$$

Eigenvalues of A and A^T are the same.

$$\dim\ker(\lambda - A) = \dim\ker(\lambda - A^T)$$

$$n - \text{rank}(\lambda - A)^k = n - \text{rank}(\lambda - A^T)^k$$

with $\dim\ker(\lambda - A) = n - \text{rank}(\lambda - A)^k$ and $\dim\ker(\lambda - A^T) = n - \text{rank}(\lambda - A^T)^k$.

This defines the size of the blocks uniquely.

2.2 Exercise 73b

$$A = TJT^{-1} \qquad A^T = (T^t)^{-t} \cdot J^t \cdot T^t$$

$$J^t = \begin{bmatrix} -2 & & & & \\ 1 & -2 & & & \\ & & -2 & & \\ & & 1 & -2 & \\ & & & 1 & -2 \end{bmatrix}$$

Find V such that

$$V^{-1}J^TV = J$$

$$\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} -2 & \\ 1 & -2 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} -2 & & \\ 1 & -2 & \\ & 1 & -2 \end{bmatrix}\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$VJ^TV = J \implies J^T = VJV$$

$$A^T = (T^T)^1 J^T T^T = (T^T)^{-1} VJVT^t = UJU^{-1}$$

$$U = (T^t)^{-1}V$$

3 Exercise 74

Given $x_1, \dots, x_n \in \mathbb{C}$ and $y_1, \dots, y_n \in \mathbb{C}$. Then there exists $g(x) \in \mathbb{C}[x]_n$ such that $p(x_i) = y_i$.

Direction $\Longleftarrow$.

Immediate. $A^* = p(A) \implies AA^* = Ap(A) = (xp(x))(A) = (p(x)x)(A) = p(A)A = A^*A$.

Direction $\implies$.

$$AA^* = A^*A \implies A = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^*$$

$$A^* = U \begin{bmatrix} \overline{\lambda}_1 & & \\ & \ddots & \\ & & \overline{\lambda}_n \end{bmatrix} U^*$$

$$p(A) = U \cdot \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_n) \end{bmatrix} \cdot U^*$$

$p(x)$ is polynomial of degree $\leq n$ such that $p(\lambda_i) = \overline{\lambda_i} \implies p(A) = A^*$.

4 Exercise 75

A is normal $\implies \exists$ ONB: $u_1, \dots, u_n$ of eigenvectors.

$x \in \mathbb{C}^n$ with $\|x\| = 1$.

$$x = \sum_1^n \alpha_i u_i$$

$$\|x\|^2 = \sum |\alpha_i|^2 = 1$$

$$\beta_i = |\alpha_i|^2 \implies W(A) = \left\{\sum \lambda_i \beta_i\right\} \sum \beta_i = 1$$

$$\langle Ax, x\rangle = \langle A \sum \alpha_i u_i, \sum \alpha_j u_j\rangle = \langle \sum \alpha_i \lambda_i u_i, \sum \alpha_j u_j\rangle = \sum_{i,j\gamma} \alpha_i \lambda_i \overline{\alpha}_j \underbrace{\langle u_i, u_j\rangle}_{\delta_{ij}} = \sum |\alpha_i|^2 \lambda_i$$