Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

Lukas Prokop

March to July 2016

Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$
$$\exp(z+w) = \exp(z)\cdot\exp(w)$$
$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n} \in \mathbb{Q} \wedge n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote
$$\exp(z) = e^z \qquad \text{for } z \in \mathbb{C}$$

It holds that
$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$
$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.
$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,
$$e^0 = 1$$

Let $n \in \mathbb{N}$
$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$
$$\lim_{x\to-\infty} e^x \cdot x^n = 0$$

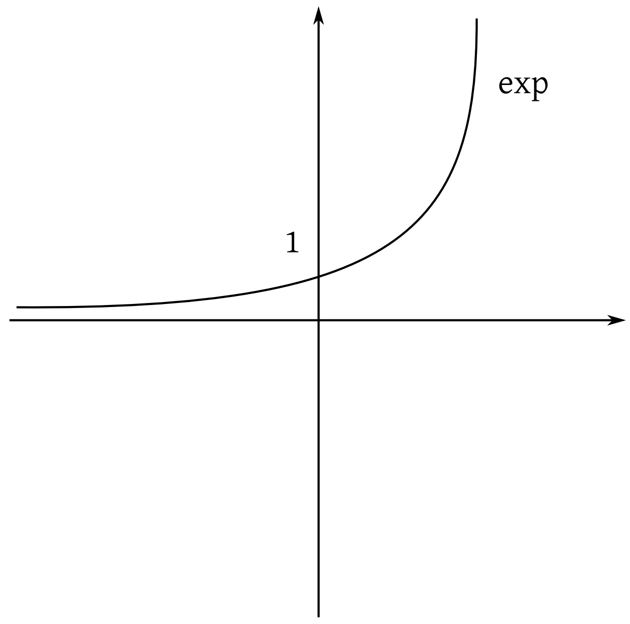

Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp : \mathbb{R} \to (0,\infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. $\exp : \mathbb{R} \to (0,\infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y]: \quad e^\xi = y$$

Case 2 Let $y \in (0,1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w]: \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp: \mathbb{R} \to (0, \infty)$ is bijective. □

Definition 1. We call the inverse function *natural logarithm*[1].

$$\exp^{-1}: (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R}: \ln(e^x) = x$ and $\forall y \in (0, \infty): e^{\ln(y)} = y$.
- $\ln: (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xRightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! □

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Limes:
$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Proof.

- $$x \cdot y = e^{\ln(x \cdot y)}$$
$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$
Injectivity of exp:
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.
$$\lim_{n \to \infty} y_n = \ln(1) = 0$$
$$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$
where
$$e^0 = 1 \Rightarrow \ln(1) = 0$$

□

Theorem 1 (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. □

Theorem 2. The logarithm function is differentiable in $(0,\infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

Proof. **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

Second approach using chain rule

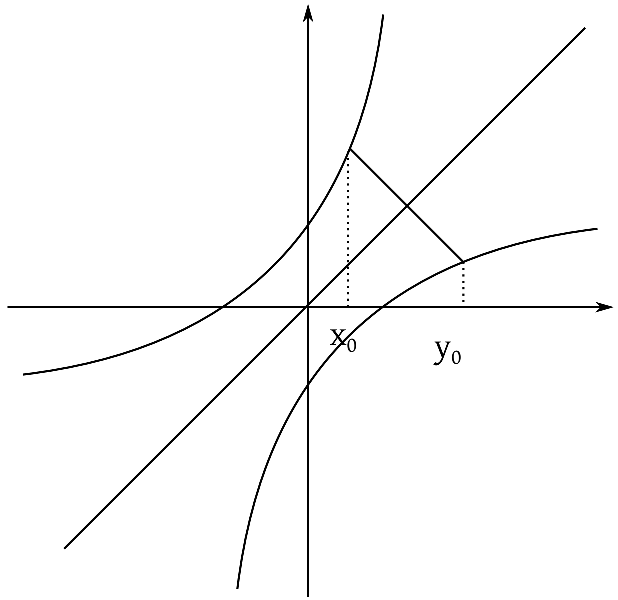

Figure 2: A geometric proof of differentiability

□

2.2 Extension of the functional equation of logarithm

Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$
$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln\frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln\frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}}$$

$$= \frac{1}{x} \cdot \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x} \to 0$$

$1 + \frac{h}{x} = w$ then it holds that $h \to 0 \Rightarrow w \to 1$.

$$\frac{h}{x} = w - 1$$

$$\lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{=} \lim_{h\to 0} \frac{\ln(w)}{w-1} = 1$$

□

Remark 1. The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C} \to \mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C} \setminus \{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I \to \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I) \to \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Proof. Let $y_n \to y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \Rightarrow x_n \neq x_0$.

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}$$

$$= \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{\lim_{n\to\infty} \underbrace{\frac{f(x_n) - f(x_0)}{x_n - x_0}}_{\text{ex}=f'(x_0)}} = \frac{1}{f'(x_0)}$$

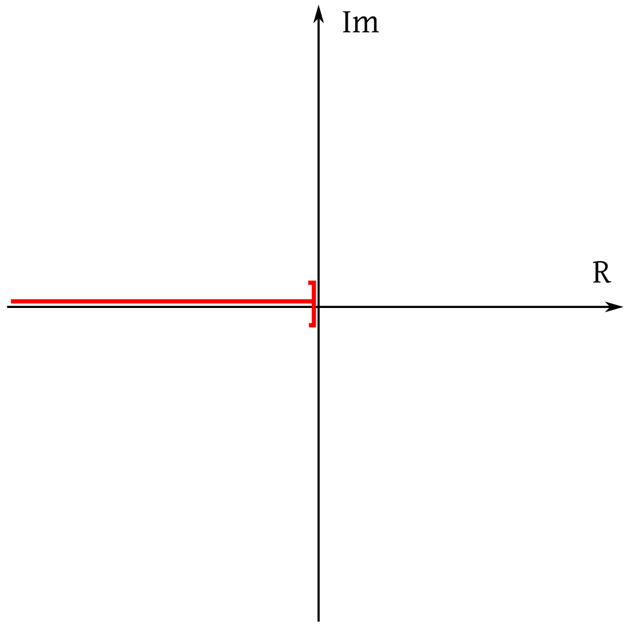

Figure 3: Continuous exponential function in $\mathbb{C}$

□

Lemma 2. Let $f : I \to \mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{ const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \forall x \in I$$

Proof. $\Rightarrow$ Immediate.

$\Leftarrow$ Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

$\square$

Definition 2. Let I be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

$\square$

Theorem 4. Let I be an interval. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n\in\mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every x)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon \, |u - v| \, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right|$$

$$= \left|\frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0)\right|$$

$$= \left|\frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0)\right| + \left|f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0)\right| \forall N \in \mathbb{N}$$

$$\leq \left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right|$$

$$+ \left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0)\right| + |f_N'(x_0) - \varphi(x_0)|$$

1st term

$$\left|\frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0}\right| = \left|\frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0}\right|$$

$$\leq \frac{\varepsilon}{3} \frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0}\right| - f_N'(x_0)$$

Differentiability of f_N: Therefore for $|x - x_0| < \delta$.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$. $\square$

Theorem 5. Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ ($n \in \mathbb{N}$) and f_n is differentiable in I.

Assumption:

1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f_n'(x) \to \varphi(x)$ for every x
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon\,|u - v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n \to \infty$
- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in I, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f_n'(x) - g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6.

We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence

$$\forall x \in I : f_n'(x) \to g(x)$$

3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$|f_n'(x) - f_m'(x)| = |f_n'(x) - g(x) + g(x) - f_m'(x)|$$

$$\leq |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.

Let $\varepsilon > 0$ be arbitrary and N such that $n, m \geq N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m\to\infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the Mittelwertsatz der Differentialrechnung

$$= \lim_{m\to\infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)|$$
$$= \lim_{m\to\infty} |f_m'(\xi_{m,n}) - f_n'(\xi_{m,n})| \cdot |u - v|$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

□

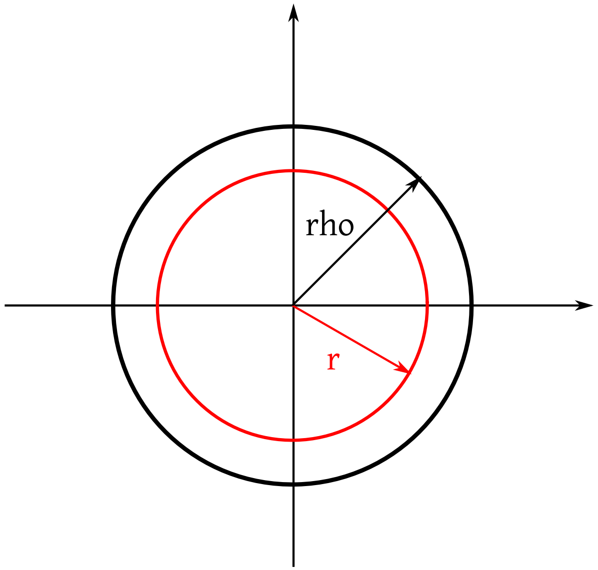

Figure 4: Convergence radius

Remark 2 (An application of Theorem 7). Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

[2] Where overline means "closed"

Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \underbrace{\limsup_{j\to\infty} \left(|a_{j+1}|^{\frac{\overbrace{1}^{\to 1}}{j+1}} \right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty} \left[(j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$

Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\Rightarrow 1 + x = w - wx$$

$$x(1+w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

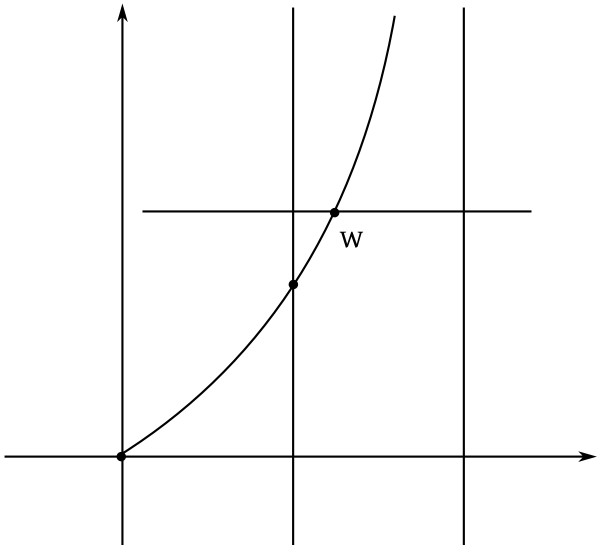

Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[\overline{\left(1 + \frac{it}{n}\right)}\right]^n$$

$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

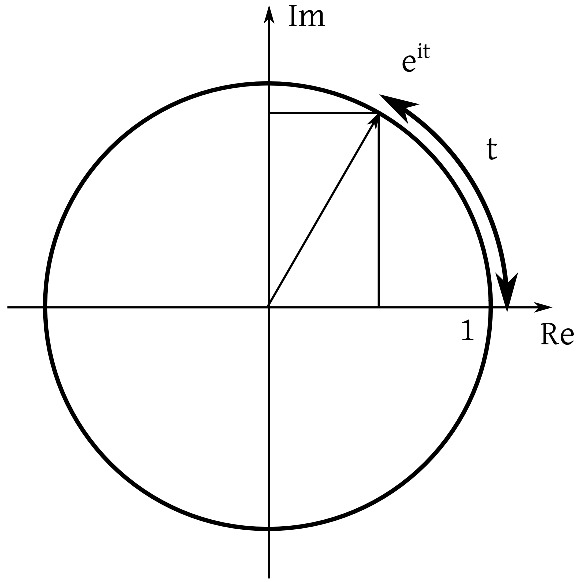

Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$ $$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$ $$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$ $$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

German keywords

English keywords