

# Mathematical analysis 1 – Lecture notes

course by Wolfgang Ring

Lukas Prokop

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## 1 Propositional logic

This lecture took place on 1st of October 2015 with lecturer Wolfgang Ring.

- Discussion about motivation for visiting university
- Kurt Gödel: Gödel's incompleteness theorem
- propositional logic (and/or/implication/equivalence operation)
  - $p \implies q$ : “p implies q” (“notwendig”), “q requires p” (“hinreichend”)
  - Indirect proof:  $(\neg q \implies \neg p) \Leftrightarrow (p \implies q)$
  - Proof by contradiction: claim  $p$ , claim  $\neg q$ , show that  $p \wedge \neg q$  is not possible
  - commutative law:  $a \wedge b \Leftrightarrow b \wedge a$
  - associative law:  $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  - distributive law:  $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
  - DeMorgan's law:  $\neg(a \wedge b) \Leftrightarrow (\neg a) \vee (\neg b)$
- First-order logic
  - $\forall x \in \mathbb{N} : x \in \mathbb{R}$
  - $\forall x \in M : P(x)$
  - $\neg[(\forall x \in M)P(x)] \Leftrightarrow \exists x \in M : \neg P(x)$
- Peano's axioms: rationale for induction proofs

The lecture on 8th of October 2015 got cancelled spontaneously.

## 2 First-Order Logic

This lecture took place on 12th of October 2015 with lecturer Wolfgang Ring.

Literature recommendation:

- “Analysis 1 (Mathematik für das Lehramt)”, Oliver Deiser

Let  $A$  and  $B$  be statements.

- Logical equivalence is given iff the truth table of both expressions is the same.
- $\neg(\neg A) \Leftrightarrow A$
- $(A \vee B) \Leftrightarrow (B \vee A)$
- $(A \wedge B) \Leftrightarrow (B \wedge A)$
- $a \implies b$ : implication

Boolean Laws:

$$\neg(A \implies B) \Leftrightarrow A \wedge \neg B \quad (1)$$

$$A \Leftrightarrow B \implies (A \implies B) \wedge (B \implies A) \quad (2)$$

“contraposition” or “indirect proof”

$$\neg B \implies \neg A \quad (3)$$

$$A \implies B \Leftrightarrow (\neg B \implies \neg A) \quad (4)$$

$$(A \Leftrightarrow B) \Leftrightarrow (\neg A \Leftrightarrow \neg B) \quad (5)$$

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B \quad (6)$$

$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B \quad (7)$$

$$\neg(A \implies B) \Leftrightarrow (A \wedge \neg B) \quad (8)$$

$$A \wedge (B \vee C) \Leftrightarrow ((A \wedge B) \vee (A \wedge C)) \quad (9)$$

$$A \vee (B \wedge C) \Leftrightarrow ((A \vee B) \wedge (A \vee C)) \quad (10)$$

$$(A \implies B) \Leftrightarrow (\neg A \vee B) \quad (11)$$

“proof by contradiction”

$$((A \implies B) \wedge (A \implies \neg B)) \implies \neg A \quad (12)$$

“conclusion”

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C) \quad (13)$$

$$\begin{aligned} A \vee B &\Leftrightarrow \neg(\neg A) \vee \neg(\neg B) \Leftrightarrow \neg(\neg A \wedge \neg B) \\ \neg(A \vee B) &\Leftrightarrow \neg(\neg(\neg A) \vee (\neg B)) \end{aligned}$$

Distributive laws:

- $(A \vee B) \wedge C \Leftrightarrow (A \wedge C) \vee (B \wedge C)$
- $(A \wedge B) \vee C \Leftrightarrow (A \vee C) \wedge (B \vee C)$

## 2.1 Tautologies

A *tautology* is the composition of statements, which always yields the truth value true, independent of the truth value of its subexpressions.

Examples of tautologies:

**“Law of excluded middle”**  $A \vee \neg A$

**equivalences with itself are always tautologies**  $A \Leftrightarrow \neg(\neg A)$

**implication of itself**  $A \rightarrow A$

Tautology with multiple statements:

**implication with or and not**  $(A \rightarrow B) \Leftrightarrow (\neg A \vee B)$

**proof by contradiction**  $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$

**chain inference**  $[(A \rightarrow B) \wedge (B \rightarrow C)] \rightarrow (A \rightarrow C)$

This lecture took place on 14th of Oct 2015 with lecturer Wolfgang Ring.

*Proof.* We prove,  $[(A \rightarrow B) \wedge (A \rightarrow \neg B)] \rightarrow \neg A$ .

$$\begin{aligned} (A \rightarrow B) \wedge (A \rightarrow \neg B) &\Leftrightarrow (\neg A \vee B) \wedge (\neg A \vee \neg B) \\ &\Leftrightarrow \underbrace{(B \wedge \neg B)}_{\perp} \vee \neg A \\ &\Leftrightarrow \neg A \end{aligned}$$

*special case*  $A = B$ .

$$\begin{aligned} (A \rightarrow A) \wedge (A \rightarrow \neg A) &\rightarrow \neg A \\ (A \rightarrow \neg A) &\rightarrow \neg A \end{aligned}$$

## 2.2 Negation of a tautology

- is called *contradiction*.
- has always truth value false.

*Proof.*

$$\begin{aligned} (A \vee B) \rightarrow C &\Leftrightarrow \neg(A \vee B) \vee C \Leftrightarrow (\neg A \wedge \neg B) \vee C \\ \Leftrightarrow (\neg A \vee C) \wedge (\neg B \vee C) &\Leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C) \end{aligned}$$

Laws:

$$\begin{aligned} (A \vee B) \rightarrow C &\Leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C) \\ (A \wedge B) \rightarrow C &\Leftrightarrow (A \rightarrow C) \vee (B \rightarrow C) \\ A \rightarrow (B \wedge C) &\Leftrightarrow (A \rightarrow B) \wedge (A \rightarrow C) \\ A \rightarrow (B \vee C) &\Leftrightarrow (A \rightarrow B) \vee (A \rightarrow C) \end{aligned}$$

*Example proof by contradiction: Number of prime numbers.* We prove a statement by Euklid of Alexandria, 300 BC:

The number of prime numbers is infinite.

Assume the number of prime numbers is finite. Then there exists some  $N \in \mathbb{N}$  such that  $\mathbb{P} = \{p_1, p_2, \dots, p_N\}$  is the set of all prime numbers.

Every integer can be represented as product of prime numbers. Therefore for every integer there exists at least one prime number that divides this number (without remainder).

Let  $m = p_1 \cdot p_2 \cdot \dots \cdot p_N + 1$ . Let  $a$  be a prime number that divides  $m$ .

It holds that: Every  $p_i \in \mathbb{P}$  is not a divisor of  $m$ . Because when dividing  $\frac{m}{p_i}$ , the remainder is always one.

So  $a \in \mathbb{P}$ , so there exists more than  $N$  prime numbers (at least  $N + 1$ ). This contradicts with our assumption, that only  $N$  prime numbers exist.

Therefore always one more prime number exists. So the number of prime numbers is infinite.  $\square$

### 2.3 Quantifiers

Quantified statements are statements, in which objects of a set occur.

*Example:* Let  $P(x) = (x > 0)$ . Its truth value cannot be determined if the set  $X$  is not defined.

**Definition 1.** Let  $M$  be a set,  $x \in M$  and  $P(x)$  a predicate.

For every  $x \in M$ , it holds that  $P(x)$  is true, iff the truth value of  $P(x)$  is always true independent of the selection of  $x \in M$ .

**Example 1.** Let  $M = \mathbb{R}$  and  $P(x) = (x^2 + 1 > 0)$ .

This is true for all  $x \in M$ . We denote:  $\forall x \in M : P(x)$ .

**Example 2.** Let  $M = \mathbb{R}$  and  $P(x) = (x^2 - 1 > 0)$ .

This is *not* true for all  $x \in M$ . We denote:  $\exists x \in M : \neg P(x)$ .

**Definition 2.**  $\forall x \in M : P(x)$  does not hold if and only if  $\exists x \in M : \neg P(x)$ .

$\forall$  is called *all quantifier*.  $\exists$  is called *existence quantifier*.

Negation works as follows:

$$\neg (\forall x \in M : P(x)) \iff (\exists x \in M : \neg P(x))$$

$$\neg (\exists x \in M : P(x)) \iff (\forall x \in M : \neg P(x))$$

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This lecture took place on 15th of Oct 2015 with lecturer Wolfgang Ring.

$$\forall x \in M : (P(x) \wedge Q(x)) \iff (\forall x \in X : P(x)) \wedge (\forall y \in M : Q(y))$$

Counterexample:

$$M = \mathbb{R} \quad P(x) := (x > 0)$$

A statement  $B$  is stronger than  $C$  if  $B$  implies at least the same propositions that  $C$  imply. “ $B$  is stronger than  $C$ ” means “ $\{D \mid C \rightarrow D\} \subsetneq \{D \mid B \rightarrow D\}$ ”. In that sense the stronger statement covers more cases.

### 2.4 Composition of several quantifiers

**Theorem 1.** The order of quantifiers matters.

*Proof.* For every real number  $x$ , there exists  $n \in \mathbb{N}$  with the property  $n > x$ :

$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$$

The statement does not hold if the order is changed.

$$\exists n \in \mathbb{N} \forall x \in \mathbb{R} : n > x$$

$\square$

## 3 Sets

We consider objects, which we call *sets*. For every set  $M$  and every element  $x$ , it holds that

$$x \in M \vee \neg(x \in M)$$

### 3.1 Russell’s paradox

Consider the set  $L = \{M : M \text{ is a set and } M \notin M\}$ . Does  $L \notin L$  or  $L \in L$  hold?

If  $L \notin L$ , then  $L$  satisfies the definition and therefore  $L \in L$ . If  $L \in L$ , then elements of  $L$  satisfy the property; therefore  $L \notin L$ .

Set operations:

- union:  $x \in (L \cup M) \iff x \in L \vee x \in M$
- intersection:  $x \in (L \cap M) \iff x \in L \wedge x \in M$
- subsets:  $L \subseteq M \iff \forall x : x \in L \implies x \in M$
- $\forall S : \emptyset \subseteq S$

### 3.2 Complete induction

**Theorem 2.** (Pythagoreans, 450 BC)

$$\forall n \in \mathbb{N}_+ : \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

*Proof.* **Induction base  $n = 1$**

$$P(1) : 1 = \frac{1(1+1)}{2} \quad \checkmark$$

**Induction step  $n \rightarrow n+1$**

Assume  $P(n)$  is true. So  $(1 + 2 + \dots + n) = \frac{n(n+1)}{2}$ .

$$\begin{aligned} [(1 + 2 + \dots + n) + (n+1)] &= \frac{n(n+1)}{2} + (n+1) = (n+1) \left( \frac{n}{2} + 1 \right) \\ &= (n+1) \cdot \frac{(n+2)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark \end{aligned}$$

So, it simply holds that:

$$s = 1 + 2 + 3 + \dots + n$$

$$2 \cdot s = \underbrace{n}_{\text{number of items}} \cdot \underbrace{(n+1)}_{\text{sum}} \Rightarrow s = \frac{n \cdot (n+1)}{2}$$

□

This lecture took place on 21st of October 2015 with lecturer Wolfgang Ring.

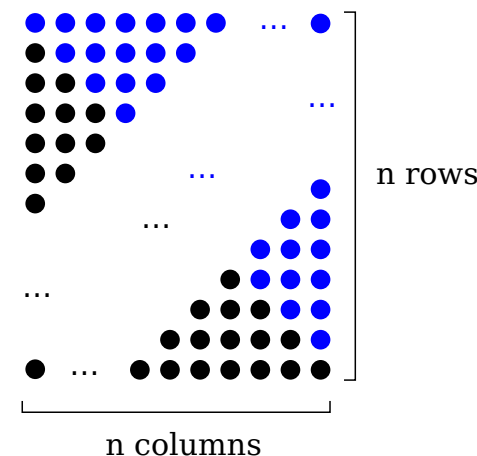


Figure 1: Illustration of the triangular number  $\frac{n(n+1)}{2}$  (illustrative proof)

### 3.3 Notations to describe sets

- Let  $X$  be a set.  $M = \{x \in X : P(x)\}$ .
- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$  ... “enumerating set representation”
- $M = \{x \in X \mid P(x)\}$ ,  $N = \{x \in X \mid Q(x)\}$
- $M \cup N = \{x \in X \mid P(x) \vee Q(x)\}$
- Sets as union or intersection of sets:
  - Let  $X$  be a set.  $A_0 \subseteq X$ ,  $A_1 \subseteq X$ ,  $A_2 \subseteq X$ , etc
  - $\forall n \in \mathbb{N} : A_n \subseteq X$
  - $A_0 \cup A_1 \cup A_2 \cup \dots = \bigcup_{n=1}^{\infty} A_n = \{x \in X \mid (x \in A_0) \vee (x \in A_1) \vee \dots\} = \{x \in X \mid \exists n \in \mathbb{N} : x \in A_n\}$
  - $A_0 \cap A_1 \cap A_2 \cap \dots = \bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall n \in \mathbb{N} : x \in A_n\}$

### 3.4 Cartesian product

**Definition 3.** Let  $A$  and  $B$  sets. The *cartesian product* of  $A$  and  $B$  is given as:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

This operation is *not* commutative!

**Definition 4.** We denote  $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ .

**Example 3.**

$$\begin{aligned} A &= \{a, b, c, d, e, f, g, h\} \\ B &= \{1, 2, 3, 4, 5, 6, 7, 8\} \\ A \times B &= \{(a, 1), (a, 2), (a, 3), \dots, (a, 8), (b, 1), (b, 2), \dots\} \end{aligned}$$

**Example 4.**

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

e.g.  $(1, \frac{9}{8}) \in \mathbb{R} \times \mathbb{R}$ .

**Definition 5.** Let  $A_1, A_2, \dots, A_n$  be sets.

$$A_n = A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

instead of  $\underbrace{A \times A \times \dots \times A}_{n \text{ times}} = A^n$ .

### 3.5 Power set

**Definition 6.** Let  $X$  be a set. Then  $\mathcal{P}(X)$  is the *power set* of  $X$ , i.e. containing all subsets of  $X$ :

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}$$

## 4 Mappings and functions

**Definition 7.** Let  $A$  and  $B$  be sets. A *mapping*  $f$  from  $A$  to  $B$  (denoted  $f : A \rightarrow B$ ) is an assignment, such that for every  $x \in A$  one  $y \in B$  is assigned. We denote the corresponding  $y \in B$  for some  $x \in A$  with  $y = f(x)$ .  $A$  is called *domain*,  $B$  is called *co-domain*.

**Definition 8** (Alternative definition of mappings). A mapping  $f$  is a subset of  $A \times B$  which fulfills the following properties:

- $\forall x \in A : (\exists y \in B : (x, y) \in f)$
- $\forall x \in A \wedge (y_1, y_2 \in B) : [(x, y_1) \in f \wedge (x, y_2) \in f] \implies y_1 = y_2$

Notation:

$$(x, y) \in f \Leftrightarrow y = f(x)$$

$$\{(x, f(x)) \mid x \in A\} =: \text{graph of } f$$

**Definition 9.** Let  $f : A \rightarrow B$  be a mapping.

- The mapping  $f$  is called *surjective*, if  $\forall y \in B : \exists x \in A : y = f(x)$ .
- The mapping  $f$  is called *injective*, if

$$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \implies x_1 = x_2).$$

- Let  $B' \subseteq B$ . We denote  $f^{-1}(B') = \{x \in A \mid f(x) \in B'\}$  as the *preimage* of  $f$ .

**Attention!** The preimage distinguishes itself from the domain (it is a subset) and the inverse function  $f^{-1}$  (a function must not be invertible to have a preimage)!

- Let  $A' \subseteq A$ . Then we call  $f(A') = \{f(x) \mid x \in A'\} \subseteq B$  the *image* of  $A'$  under  $f$ .

Special case:  $A' = A$ , then  $f(A) \subseteq B$  is the image of  $A$  under  $f$ .

Let  $f : A \rightarrow B$  be a mapping. We define  $\tilde{f} : A \rightarrow f(A) \subseteq B$  with  $\tilde{f}(x) = f(x)$  for all  $x \in A$ . The mapping  $\tilde{f}$  is surjective, i.e.  $\forall y \in f(A)$  there exists one  $x \in A$  such that  $y = \tilde{f}(x)$ .

- A mapping is called *bijective* iff the mapping is surjective and injective.

## 4.1 Bernoulli's inequality

**Definition 10** (Bernoulli's inequality). Let  $x \in \mathbb{R}$  with  $x > -1$  and  $x \neq 0$ . Let  $n \in \mathbb{N}$  with  $n > 1$ . Then it holds that

$$(1 + x)^n > 1 + nx$$

*Proof.* Proof by complete induction.

**Induction base  $n = 2$**

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x \quad \checkmark$$

because  $x^2 > 0$  for  $x \neq 0$ .

**Induction step  $n \rightarrow n + 1$**

Assume  $(1 + x)^2 > 1 + n$ , then  $x > -1$  and  $x \neq 0$ .

$$\begin{aligned} (1 + x)^{n+1} &= (1 + x)^n \cdot \underbrace{(1 + x)}_{>0} \underset{\text{by ind. hypo.}}{>} (1 + nx) \cdot (1 + x) \\ &= (1 + nx + x + nx^2) = (1 + (n + 1) \cdot x + \underbrace{nx^2}_{>0}) > 1 + (n + 1) \cdot x \end{aligned}$$

□

Back to sets and functions (notes are missing, topics covered as follows):

- injective, surjective, bijective function
- composition of functions: Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .  $g \circ f : X \rightarrow Z$  is defined as  $g(f(x))$  (“g after f”).
- Let  $f$  and  $g$  be mappings. If  $f$  and  $g$  are injective,  $f \circ g$  is injective. If  $f$  and  $g$  are surjective,  $f \circ g$  is surjective. If  $f$  and  $g$  are bijective,  $f \circ g$  is bijective.
- Identity function,  $f \circ \text{id} = \text{id} \circ f = f$
- properties of an inverse function,  $f \circ f^{-1} : X \rightarrow X$ ,  $f^{-1} \circ f : X \rightarrow X$

## 5 About sums of integers

This lecture took place on 21st of Oct 2015 with lecturer Wolfgang Ring.

**Definition 11.** The summation notation is defined as,

$$\sum_{k=h}^l a_k$$

Iteration over all values from  $l$  to  $h$  (inclusive) and evaluation of the enclosed expression with  $k$  as iteration value. The resulting terms are added up and the sum gives the result of the summation expression.

Laws:

$$\sum_{k=l}^h a_k = \sum_{i=l}^h a_i \quad (14)$$

$$\sum_{k=l}^h (a_k + b_k) = \left( \sum_{k=l}^h a_k \right) + \left( \sum_{k=l}^h b_k \right) \quad (15)$$

$$\sum_{k=0}^h a_k = a_0 + \sum_{k=1}^h a_k \quad \text{“Extraction of the initial value”} \quad (16)$$

$$\sum_{k=0}^h a_k = a_h + \sum_{k=0}^{h-1} a_k \quad \text{“Extraction of the final value”} \quad (17)$$

$$\sum_{k=u+n}^{h+n} a_k = \sum_{k=u}^h a_{k+n} \quad \text{“index shifting”} \quad (18)$$

$$\sum_{k=l}^h \lambda \cdot a_k = \lambda \cdot \sum_{k=l}^h a_k \quad \text{“extraction of a constant } \lambda \text{”} \quad (19)$$

$$\sum_{k=0}^n n = \frac{n(n+1)}{2} \quad \text{“triangular sum”} \quad (20)$$



## 5.1 Factorials

We consider  $S_n = \{(a_1, a_2, \dots, a_n) : a_i \in M_n \forall i = 1, \dots, n \text{ with } a_i \neq a_j\} \subseteq M_n \times M_n \times \dots \times M_n$ .  $S_n$  is the set of all arrangements of the numbers  $1, \dots, n$ .

Example:  $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$

**Theorem 3.** It holds that  $|S_n| = n!$  for all  $n \in \mathbb{N}$

*Proof.* Proof by induction over  $n$ .

**Induction base**  $n = 1$ :  $M_1 = \{1\}, S_1 = \{(1)\} \Rightarrow |S_1| = 1 = 1! \quad \checkmark$

**Induction step**  $n \rightarrow n + 1$ :

$$S_{n+1} = \{(a_1, a_2, \dots, a_n) : a_i \in M_{n+1} \forall i \in M_{n+1}, a_i \neq a_j \text{ for } i \neq j\}$$

For  $l \in M_{n+1}$ :

$$W_l = \{(a_1, \dots, a_{n+1}) \in S_{n+1} : a_l = n + 1\}$$

It holds that  $W_l \cap W_j = \emptyset$  for  $l \neq j$  and  $S_{n+1} = W_1 \cup W_2 \cup W_3 \cup \dots \cup W_{n+1}$ . Then it holds that  $|S_{n+1}| = |W_1| + |W_2| + \dots + |W_{n+1}| = \sum_{l=1}^{n+1} |W_l|$

**Theorem 4.** Claim: For every  $l \in M_{n+1}$  it holds that  $|W_l| = |S_n| = n!$ .

*Proof.* We build a bijective map  $\phi_l : W_l \rightarrow S_n$ .

$$W_l = \{(a_1, a_2, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1})\}$$

$$: a_i \in M_n, \forall i \neq l, a_i \neq a_j \forall i \neq j$$

$$\phi_l((a_1, a_2, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$= (a_1, a_2, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1}) \in S_n$$

$S_n$  is surjective: Let  $(b_1, \dots, b_n) \in S_n$ , then it holds that  $(b_1, \dots, b_{l-1}, n + 1, b_l, \dots, b_n) \in W_l$

$$\phi_l((b_1, \dots, b_{l-1}, n + 1, b_l, \dots, b_n)) = (b_1, \dots, b_n)$$

$S_n$  is injective.

$$\phi_l((a_1, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$= \phi_l((a_1, \dots, a_{l-1}, n + 1, a_{l+1}, \dots, a_{n+1}))$$

$$\Rightarrow (a_1, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1}) = (a_1, \dots, a_{l-1}, a_{l+1}, \dots, a_{n+1})$$

Hence,  $\phi$  is bijective.  $\square$

Therefore  $|W_l| = |S_n| = n!$ . Therefore  $|S_{n+1}| = \sum_{l=1}^{n+1} |S_n| = \sum_{l=1}^{n+1} n! = (n + 1)n! = (n + 1)!$

$\square$

**Remark 1.** Let  $f : M_n \rightarrow M_n$ .  $f$  is represented as

$$(1, 2, 3, 4, \dots, n - 1, n) \rightarrow (f(1), f(2), f(3), f(4), \dots, f(n - 1), f(n))$$

Therefore  $(f(1), f(2), \dots, f(n)) \in S_n$ . Analogously every  $(a_1, \dots, a_n) \in S_n$  defined by  $f(k) = a_k$  for  $k = 1, \dots, n$  is a bijective mapping  $f : M_n \rightarrow M_n$ . Therefore we set  $S_n = \{f : M_n \rightarrow M_n : f \text{ is bijective}\}$ .  $S_n$  is called *symmetric group of  $n$  elements*.

## 5.2 Binomial coefficients

**Definition 12.** Let  $n \in \mathbb{N}, k \in \mathbb{N}$  with  $k \leq n$ . We define

$$\binom{n}{k} = \frac{n!}{k!(n - k)!} \quad \text{“binomial coefficient } n \text{ choose } k\text{”}$$

It holds that

$$\begin{aligned} \binom{n}{k} &= \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{(1 \cdot 2 \cdot \dots \cdot k)(1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - k))} \\ &= \frac{n(n - 1) \cdot \dots \cdot (k + 1)}{(1 \cdot 2 \cdot 3 \cdot \dots \cdot (n - k))} \end{aligned}$$

Factorial laws:

$$\begin{aligned} \binom{1}{0} &= \frac{n!}{0!(n-0)!} = 1 \quad \forall n \in \mathbb{N} \\ \binom{n}{n} &= \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = 1 \\ \binom{n}{n-k} &= \frac{n!}{(n-k)!(n-n+k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \quad \text{“symmetrical”} \end{aligned}$$

A recursive definition is given by Pascal’s Rule:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad n \geq 1, 1 \leq k \leq n-1$$

*Proof.*

$$\begin{aligned} \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \\ &\quad + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{(n-1)! \cdot k}{k! \cdot (n-k)!} + \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} \\ &= \frac{k \cdot (n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{(n-k+k)(n-1)!}{k!(n-k)!} \\ &= \frac{n(n-1)!}{k!(n-1)!} = \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k} \end{aligned}$$

□

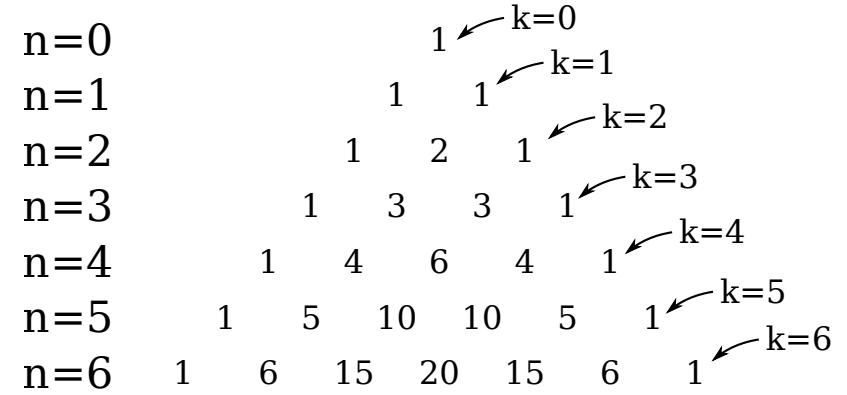


Figure 2: Pascal’s triangle describes binomial coefficients. For every element of the triangle it holds that, it is adding up the two numbers above a number. The margins are defined by 1. For example 5 is given by  $\binom{5}{4}$ .

### 5.3 Arrangement in Pascal’s triangle

**Theorem 5.** Let  $T_n^k = \{A \subseteq M_n : |A| = k\}$ . Then it holds that  $|T_n^k| = \binom{n}{k}$ .

Example:  $T_3^2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ .

$$|T_3^2| = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$$

*Proof.* Let  $n$  be fixed. Complete induction over  $k$ .

**Induction base**  $k = 0$

$$\begin{aligned} T_n^0 &= \{\emptyset\} \\ |T_n^0| &= 1 = \binom{n}{0} \end{aligned}$$

**Induction step**  $k \rightarrow k + 1$

$$T_n^k = \underbrace{\{\{a_1, \dots, a_k\} : a_i \in M_n, (i = 1, \dots, k), a_i \neq a_j \text{ for } i \neq j\}}_{A_1} \cup \underbrace{\{\{a_1, \dots, a_{k-1}\} \cup [n] \in M_{n-1}\}}_{A_2}$$

$$|T_n^k| = |A_1| + |A_2|$$

□

This lecture took place on 28th of October 2015 with lecturer Wolfgang Ring.

Let  $A, B$  be sets and define

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

Then  $A \setminus B$  is called “A without B”.

**Theorem 6.**

$$T_n^x = \{x \subseteq M_x : |X| = x\}$$

Let  $k \in \mathbb{N}$  and  $0 \leq k \leq 1$ .

$$|T_n^x| = \binom{1}{k}$$

There are exactly  $\binom{n}{k}$   $k$ -ary subsets of  $M_n$ .

*Proof.*

$$M_0 = \emptyset \quad T_0^0 = \{\emptyset\} \quad |T_0^0| = 1 = \binom{0}{n}$$

Proof by complete induction over  $n$  of the following statement:

$$\forall n \in \mathbb{N} : \forall k \in \mathbb{N} \text{ with } 0 \leq k \leq n : |T_n^k| = \binom{n}{k}$$

**Induction base**  $n = 0$  is fine. For  $n = 1$  there are two cases:  $k = 0$  or  $k = 1$ .

$$M_1 = \{1\}$$

$$T_1^0 = \{\emptyset\} \quad |T_1^0| = 1 = \binom{1}{0}$$

$$T_1^1 = \{\{1\}\} \quad |T_1^1| = 1 = \binom{1}{1}$$

Is also fine.

**Induction step** The hypothesis is our assumption:

$$\forall 0 \leq k \leq 1 : |T_n^k| = \binom{n}{k}$$

Consider  $M_{n+1}$ . Special case  $k = 0$ :

$$T_{n+1}^0 = \{\emptyset\} \quad |T_{n+1}^0| = 1 = \binom{n+1}{0}$$

Special case  $k = n + 1$ :

$$T_n = \{M_{n+1}\} \quad |T_{n+1}^{n+1}| = 1 = \binom{n+1}{n+1}$$

Let  $1 \leq k \leq n$ .

$$T_{n+1}^x := \{(a_1, \dots, a_x) : a_i \in M_{n+1}, \forall i \in \{1, \dots, x\}, a_i \neq a_j \forall (i, j) \in \{1, \dots, x\}\}$$

$$R_{n+1}^k := \{(a_1, \dots, a_{k-1}, n+1) : (a_1, \dots, a_{k-1}) \in T_n^k\}$$

$$S_{n+1}^k := \{(a_1, \dots, a_{l-1}, n+1, a_{l+1}, \dots, a_{k-1}) : (a_1, \dots, a_l, \dots, a_k) \in T_n^k\}$$

$$\text{Union is disjoint} \Rightarrow |T_{n+1}^k| = |R_{n+1}^k| + |S_{n+1}^k|$$

$$R_{n+1}^k = \{A \subseteq M_n : |A| = k\} = T_n^k$$

$$|R_{n+1}^k| = |T_n^k| = \binom{n}{k}$$

by induction hypothesis.

$$S_{n+1}^k = \{A \subseteq M_{n+1} : A = A' \cup \{n+1\} : A' \subseteq M_n : |A'| = k-1\}$$

$$\text{We prove } |S_{n+1}^k| = |T_n^{k-1}|.$$

$$f : S_{n+1}^k \rightarrow T_n^{k-1}$$

$$f(A) = f(A' \cup \{n+1\}) = A'$$

$f$  is bijective.  $f$  is surjective: Let  $A' \in T_n^k$  define  $A = A' \cup \{n+1\} \in S_{n+1}^n$  and  $f(A) = A'$ .  $f$  is injective: Let  $f(A) = f(B)$  and  $A = A' \cup \{n+1\} \in S_{n+1}^k$ .

$$B = B' \cup \{n+1\} \in S_{n+1}^k. \quad A', B' \in T_n^{k-1}.$$

$$f(A) = f(B) \Rightarrow A' = B' \Rightarrow A' \cup \{n+1\} = B' \cup \{n+1\} \Rightarrow A = B$$

$$|S_{n+1}^k| = |T_n^{k-1}| \stackrel{\text{ind. hypo.}}{=} \binom{n}{k-1}$$

Therefore  $|T_{n+1}^k| = \binom{n}{n} = \binom{n}{k-1} = \binom{n+1}{k}$ . The last equality follows from the recursive definition of binomial coefficients.

□

## 5.4 Binomial theorem

**Theorem 7** (Binomial theorem). Let  $a, b \in \mathbb{R}$  (or  $a, b \in \mathbb{C}$ ). Then it holds that

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

*Proof.* 1. Proof by induction over  $n$ .

**Induction step**  $n=0$ :  $(a+b)^0 = 1$

$$\sum_{k=0}^0 \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^0 = 1$$

**Induction step**  $n \rightarrow n+1$

$$(a+b)^{n+1} = (a+b)^n \cdot (a+b) = \left( \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \right) (a+b)$$

$$\begin{aligned} &= \sum_{k=0}^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^n \binom{n}{k} a^k b^{n-k+1} \\ &= \underbrace{\sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\substack{\text{index shift} \\ k+1=j, k=j-1 \\ k=0 \Rightarrow j=1 \\ k=n-1 \Rightarrow j=n}} + \underbrace{\binom{n}{n} a^{n+1} \cdot b^0}_{=a^{n+1}} \\ &\quad + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1} \\ &= \sum_{j=1}^n \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{k=1}^n \binom{n}{k} a^k b^{n+1-k} \\ &\quad + \binom{n+1}{n+1} a^{n+1} + \binom{n+1}{0} b^{n+1} \end{aligned}$$

Renaming  $j$  to  $k$ . Then it holds that:

$$\begin{aligned} &= \sum_{k=1}^n \underbrace{\left[ \binom{n}{k-1} + \binom{n}{k} \right]}_{\substack{\binom{n+1}{k} \text{ by recursive definition}}} a^k b^{n+1-k} \\ &\quad + \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a^0 b^{n+1} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k} \end{aligned}$$

Therefore the binomial theorem holds for  $n+1$ .

□

This lecture took place on 29th of October 2015 with lecturer Wolfgang Ring.

$$\forall a, b \in \mathbb{R}, n \in \mathbb{N} : (a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

**Induction base**  $n = 0, n = 1$  follows immediately

**Induction step**

$$(a + b)^n = \underbrace{(a + b)(a + b)(a + b)(a + b) \dots (a + b)}_{n \text{ times}}$$

When multiplying the products  $a^n b^{n-k}$  are created ( $0 \leq k \leq n$ ).  $a^n b^{n-k}$  are created iff  $a$  is the factor resulting from  $k$  parenthesis groups and  $b$  originates from the remaining  $(n - k)$  groups. There are exactly  $\binom{n}{k}$  possibilities to select from  $n$  groups.  $a^k b^{n-k}$  occurs  $\binom{n}{k}$  times. Therefore

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

This is a rather informal proof, but suffices at this point.

## 6 Arithmetics of numbers

We consider two fundamental arithmetic operators and determine fundamental properties.

**Definition 13.** Let  $K$  be a set where two arithmetic operators are defined: Therefore  $\forall a, b \in K$  let  $a + b \in K$  and  $a \cdot b \in K$ .

We require the following properties:

**A1**  $\forall a, b \in K : a + b = b + a$

**A2**  $\forall a, b, c \in K : (a + b) + c = a + (b + c)$

**A3**  $\exists 0 \in K \forall a \in K : a + 0 = a$

**A4**  $\forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$

Then  $(K, +)$  is a commutative group (“abelian group”). In general we denote  $\tilde{a}$  as  $-a$ . We define  $a - b = a + (-b)$  (“subtraction”).

**M1**  $\forall a, b \in K : a \cdot b = b \cdot a$

**M2**  $\forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$

**M3**  $\exists 1 \in K : a \cdot 1 = a \forall a \in K$  (neutral element)

**M4**  $\forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$

In general we denote  $\hat{a}$  as  $a^{-1}$ .

We set  $\frac{a}{b} = a \cdot b^{-1}$ .

$$\frac{1}{b} = 1 \cdot b^{-1} \text{ for } b \neq 0$$

**Definition 14** (Composition). Compatibility of  $+$  and  $\cdot$ :

**D**  $\forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$

Under these conditions  $K$  is called a *field*.

**Example 5.** Examples for fields:  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

In every field it holds that

- the inverse element of  $a$  is unique ( $\tilde{a}$  is unique). Let  $-a$  be the inverse element of  $a$  and  $a + b = 0 \Rightarrow b = -a$

*Proof.* Let  $\tilde{a}$  be the inverse of  $a$ . Let  $\tilde{b}$  the inverse of  $a$ . Show  $\tilde{a} = \tilde{b}$ .

$$\Rightarrow \tilde{a} + a = 0 \wedge \tilde{b} + a = 0$$

$$\Rightarrow \tilde{a} + a = \tilde{b} + a$$

$$\Rightarrow \tilde{a} = \tilde{b}$$

□

- $0 \cdot a = 0$

*Proof.*

$$0 = 0 + 0$$

follows from **D**.

$$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$$

$$0 \cdot a + (-0 \cdot a) = 0 \cdot a + [0 \cdot a + (-0 \cdot a)]$$

$$0 = 0 \cdot a$$

$$\bullet -a = (-1) \cdot a$$

*Proof.*

$$a + (-1) \cdot a = (1 + (-1))a = 0$$

$$a + (-1) \cdot a = 0$$

$$-a = (-1) \cdot a$$

## 6.1 Integers and the field of rational numbers $\mathbb{Q}$

For  $\mathbb{N}$ , **A1**, **A2** and **A3** hold. If  $n \geq m$ , then also  $n - m \in \mathbb{N}$ .  $n - m = k \in \mathbb{N}$  is defined in such a way that  $n = m + k$ .

**Corollary 1.** Extension:

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \dots\} = \mathbb{N}_+ \cup \{0\} \cup \{-n : n \in \mathbb{N}_0\}$$

We define  $-0 := 0$  and  $\forall n \in \mathbb{N}_+$  let  $n + (-n) := 0$ .

Therefore for every  $z \in \mathbb{Z}$  exists some  $\tilde{z}$  such that  $z + \tilde{z} = 0$ .

$$\bullet z \in \mathbb{Z}_+ \Rightarrow \tilde{z} = -z$$

$$\bullet z = 0 \Rightarrow \tilde{z} = 0$$

$$\bullet z = -n \text{ for } n \in \mathbb{N}_+$$

$$\bullet \tilde{z} = n$$

$$\forall z \in \mathbb{Z} \exists \tilde{z} \in \mathbb{Z} : z + \tilde{z} = 0$$

In general we denote  $\tilde{z} = (-z)$ . Also  $-(-z) = z$ .

□

For  $z, w \in \mathbb{Z}$ :

$$z + w = \begin{cases} z + w & z, w \in \mathbb{N} \\ (-z) + (-w) & -z, -w \in \mathbb{N} \\ z - (-w) & z, -w \in \mathbb{N} \text{ and } z > (-w) \\ -((-w) - z) & z, -w \in \mathbb{N} \text{ and } (-w) > z \end{cases}$$

$$z \cdot w = \begin{cases} z \cdot w & z, w \in \mathbb{N} \\ (-z)(-w) & -z, -w \in \mathbb{N} \\ -((-z) \cdot w) & -z \in \mathbb{N}, w \in \mathbb{N} \end{cases}$$

In  $\mathbb{Z}$  the properties **A1**, **A2**, **A3**, **A4**, **M1**, **M2**, **M3** and **D** hold.

□

**Definition 15.**

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

where  $\frac{m}{n} = \frac{m'}{n'} \Leftrightarrow m \cdot n' = n \cdot m'$ .  $\mathbb{Q}$  is called the set of rational numbers.

We define

$$\frac{m}{n} + \frac{k}{l} := \frac{ml + nk}{nl}$$

$$\frac{m}{n} \cdot \frac{k}{l} = \frac{mk}{nl}$$

Show that

$$\begin{aligned} \frac{m}{n} &= \frac{m'}{n'} \text{ and } \frac{k}{l} = \frac{k'}{l'} \\ \Rightarrow \frac{ml + nk}{nl} &= \frac{m'l' + n'k'}{n'l'} \\ \Rightarrow (ml + nk)(n'l') &= (m'l' + n'k') \\ \Leftrightarrow mn' \cdot ll' + nn' \cdot kl &= m'n \cdot ll' + nn' \cdot k'l \end{aligned}$$

Analogously for  $\frac{m}{n} \cdot \frac{k}{l}$ .

**A1–A4**, **M1–M4** and **D** hold for  $\mathbb{Q}$ .

For  $z \in \mathbb{Z}$  we set  $z = \frac{z}{1}$ . Therefore it holds that  $\mathbb{Z} \subseteq \mathbb{Q}$ .  $0 = \frac{0}{1}$  and  $\frac{m}{n} + 0 = \frac{m}{n} + \frac{0}{1} = \frac{m \cdot 1 + n \cdot 0}{n \cdot 1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$ .  $0$  is neutral in regards of addition in  $\mathbb{Q}$ .

Inverse element in regards of addition:

$$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{n^2} = \frac{(m + (-m))n}{n \cdot n} = \frac{0n}{n^2} = \frac{0}{1}$$

because  $0 \cdot 1 = 0 \cdot n^2$ .

Concerning multiplication:

$$1 = \frac{1}{1} \quad \frac{m}{n} \cdot \frac{1}{1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$$

$1$  is a neutral element in regards of multiplication in  $\mathbb{Q}$ .

Let  $\frac{m}{n} \in \mathbb{Q} \setminus \{0\} \implies m \neq 0 \implies \frac{n}{m} \in \mathbb{Q}$  and  $\frac{m}{n} \cdot \frac{n}{m} = \frac{mn}{mn} = \frac{1}{1}$ , because  $m \cdot n \cdot 1 = 1 \cdot m \cdot n$ .

**Corollary 2.**

$$\forall \frac{m}{n} \in \mathbb{Q} : -\frac{m}{n} = \frac{-m}{n}$$

$$\forall \frac{m}{n} \in \mathbb{Q} \setminus \{0\} : \left(\frac{m}{n}\right)^{-1} = \left(\frac{n}{m}\right)$$

Therefore  $\mathbb{Q}$  is a field.

This lecture took place on 30th of October 2015 with lecturer Wolfgang Ring.

Literature:

- Ebbinghaus et al., “Zahlen”, Springer Verlag
- E. Landau: “Grundlagen der Analysis”, uses Peano axioms to build calculus

## 6.2 Ordered fields

**Definition 16.** Let  $K$  be a field. We assume that  $K$  is taken from two sets:  $K = K_+ \cup \{0\} \cup K_-$  with  $0 \notin K_+, 0 \notin K_-$ . It holds that

- $\forall a \in K$  it holds that either  $a \in K_-$  or  $a = 0$  or  $a \in K_+$   
 $a \in K_+ \Leftrightarrow -a \in K_-$

- $\forall a, b \in K_+ : a + b \in K_+ \wedge a \cdot b \in K_+$

If those properties are satisfied, such a field is called an *ordered field*. Instead of  $a \in K_+$  we write  $a > 0$  (namely “positive numbers”) and  $a < 0$  for  $a \in K_-$  correspondingly (namely “negative numbers”).

For arbitrary  $a, b \in K$  we define

$$a > b \Leftrightarrow a - b > 0$$

It holds that  $a > b \Leftrightarrow b < a$ .

$$a \geq b \Leftrightarrow a > b \vee a = b$$

**Lemma 1.** Let  $K$  be an ordered field. Then it holds that

1.  $a \in K_+ \wedge b \in K_- \Rightarrow a \cdot b \in K_-$   
 $a \in K_- \wedge b \in K_- \Rightarrow a \cdot b \in K_+$
2.  $\forall a, b \in K$  one of the following relations hold:

$$a > b \vee a = b \vee a < b$$

Therefore  $<$  defines a total order on  $K$ .

3.  $\forall a, b, c \in K : [(a < b) \wedge (b < c) \implies a < c]$   
 Therefore  $<$  is transitive.
4. If  $a > b > 0$  then  $\frac{1}{a} < \frac{1}{b}$  If  $a > 0$  holds, then also  $a^{-1} = \frac{1}{a} > 0$ .
5.  $\forall a, b, c \in K : a < b \implies a + c < b + c$
6.  $\forall a, b \in K : \forall c > 0 : [a > b \implies ac > bc]$   
 $\forall a, b \in K : \forall c < 0 : [a > b \implies ac < bc]$
7.  $\forall a \in K \setminus \{0\} : a^2 = a \cdot a > 0$

*Proof.* 1. We know from the practicals:  $\forall a, b \in K : (-a)(-b) = ab$

$$(-a)b = -(ab)$$

Let  $a \in K_+, b \in K_-$ , therefore  $a \in K_+, (-b) \in K_-$ , then it holds that  $ab = (-a)(-b) = -(a(-b)) \in K_-$ . Let  $a \in K_-$  and  $b \in K_-$  therefore  $(-a) \in K_+ \wedge (-b) \in K_+ \implies ab = (-a)(-b) \in K_+$ .

2. Let  $a, b \in K$ . Then one of the following properties hold:

$$a - b > 0 \vee a - b = 0 \vee a - b < 0$$

Equivalently,

$$a > b \vee a = b \vee a < b$$

3. Let  $a > b$  and  $b > c$ . Therefore  $a - b > 0$  and  $b - c > 0$ .

$$\implies (a - b) + (b - c) > 0$$

$$a(-b + b) - c > 0$$

$$a - c > 0 \iff a > c$$

4. Let  $a > 0 \implies a^{-1} \neq 0$ . Assume  $\frac{1}{a} = a^{-1} < 0 \implies a^{-1} \cdot a = 1 < 0$ .  
Otherwise it holds that  $1 = 1 \cdot 1 = 1^2 > 0$ .

5. Let  $a > b > 0$ . Then it holds that

$$a^{-1}b^{-1}(b - a) = a^{-1}b^{-1}b - a^{-1}b^{-1}a = -a^{-1} \cdot b^{-1} = \frac{1}{a} \cdot \frac{1}{b} \Rightarrow a^{-1} < b^{-1}$$

6.  $a < b$  therefore  $a - b < 0 \implies a + c - c - b < 0 \implies (a + c) - (b + c) < 0$

$$\iff a + c < b + c$$

7. Let  $a > b, c > 0 \implies (a - b) > 0 \implies (a - b) \cdot c > 0 \implies ac - bc > 0 \implies ac > bc$ . For the second statement, it holds analogously:  $a < b, c < 0 \implies (a - b) < 0 \implies (a - b) \cdot c < 0 \implies ac - bc < 0 \implies ac < bc$

8.  $a > 0 \implies a \cdot a > 0$ . Let  $a < 0 \implies (-a) > 0$ . It holds  $a \cdot a = (-a)(-a) > 0$ . Therefore the square of two numbers is always positive.

□

### 6.3 Remarks about some common fields

**Remark 2.**  $\mathbb{C}$  is not an ordered field.  $\mathbb{N}, \mathbb{Z}$  and  $\mathbb{Q}$  are ordered.

**Remark 3.** Let  $q \in \mathbb{Q}$ .

a) Let  $m, n \in \mathbb{N}_+$  such that  $q = \frac{m}{n}$  then  $q > 0$ .

b) Let  $m, n \in \mathbb{N}_+$  such that  $q = -\frac{m}{n}$  then  $q < 0$ .

We show that  $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ . Every  $q \in \mathbb{Q}$  has a representation of either a) or b), but not both.  $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$ .

$$q \neq 0 \Rightarrow q = \begin{cases} \frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{-n} & m, n \in \mathbb{N}_+ \end{cases}$$

$$q = \frac{n}{-m} = \frac{-n}{m}$$

because  $nm = (-n)(-m)$ .

$$q = \frac{-m}{-n} = \frac{m}{n}$$

because  $(-m) \cdot n = m \cdot (-n)$ .

**Remark 4.** We want to show that  $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$ . Let  $q \in \mathbb{Q}_+ \cap \mathbb{Q}_-$ .

$$q = \frac{m}{n} = -\frac{m'}{n'} \quad m, n, m', n' \in \mathbb{N}_+$$

$$\Rightarrow n \cdot n' = (-m')n$$

$$\Rightarrow \underbrace{mn'}_{\in \mathbb{N}_+} + \underbrace{m'n}_{\in \mathbb{N}_+} = 0 \quad \text{!}$$

Furthermore  $p \in \mathbb{Q}_+ \wedge q \in \mathbb{Q}_+$

$$\Rightarrow p + q \in \mathbb{Q}_+ \wedge pq \in \mathbb{Q}_+$$

$$\Rightarrow p = \frac{k}{l} \quad q = \frac{m}{n} \quad k, l, m, n \in \mathbb{N}_+$$

$$p + q = \frac{\overbrace{kn + ml}^{\in \mathbb{N}_+}}{nm} \in \mathbb{Q}_+$$



$$pq = \frac{k}{l} \cdot \frac{m}{n} = \frac{\overbrace{km}^{\in \mathbb{N}_+}}{\underbrace{ln}_{\in \mathbb{N}_+}} \in \mathbb{Q}_+$$

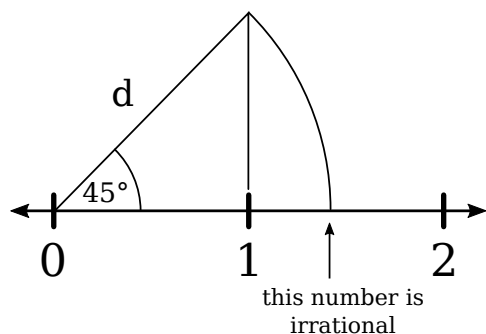


Figure 3: Illustration of an irrational number

**Definition 17.** Let  $K$  be an ordered field  $a \in K$ . The absolute value of  $a$  is defined as

$$|a| = \begin{cases} a & \text{if } a \in K_+ \\ 0 & \text{if } a = 0 \\ -a & \text{if } a \in K_- \end{cases}$$

**Remark 5.** Let  $K$  be an ordered field. Then it holds that

$$\mathbb{Q} \subseteq K \subseteq \mathbb{R}$$

except for isomorphism.

## 6.4 Triangle inequality

**Theorem 8.**

$$\forall a, b \in K : |a + b| \leq |a| + |b| \quad \text{“Triangle inequality”}$$

*Proof.* **Case 1** Let  $a \cdot b > 0$

$$\implies a \cdot b > 0 : |ab| = ab \quad |a| \cdot |b| = ab$$

**Case 2**

$$a > 0, b < 0 : a \cdot b < 0 : |ab| = -ab \quad |a| \cdot |b| = a \cdot (-b)$$

$$b < 0 \implies -b > 0 \implies b < -b \implies \underbrace{a+b}_{|a+b|} < \underbrace{a-b}_{|a|+|b|}$$

**Case 3**

$$a < 0, b < 0 : a \cdot b > 0 : |ab| = ab \quad |a| = -a \quad |b| = -b$$

$$|a| \cdot |b| = -a \cdot -b = ab$$

$$|a + b| = |(-1)(a + b)| = |-a - b| = \left| \underbrace{|a|}_{\geq 0} + \underbrace{|b|}_{\geq 0} \right| = |a| + |b|$$

**Case 4**

$$a > 0, b < 0 : a + b < 0$$

$$|a| = a \quad |b| = b \quad |a + b| = -(a + b) = -a - b$$

$$a > 0 \implies -a < 0 \quad -a - b < a - b$$

$$-(a + b) = |a + b|$$

□

This lecture took place on 4th of November 2015 with lecturer Wolfgang Ring.

## 6.5 Laws for absolute values

**Theorem 9.** Let  $y \geq 0$ . Then it holds that  $|x| \leq y \Leftrightarrow -y \leq x \wedge x \leq y$

*Proof.* First direction  $\Rightarrow$ :

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

**Case 1** Let  $x \geq 0$ . Then

$$|x| \leq y \Rightarrow x \leq y \Rightarrow -y \leq x$$

because  $-y \leq 0 \wedge x \geq 0$  anyways.

**Case 2** Let  $x < 0$ , therefore  $|x| = -x$ . Because

$$-x \leq y \Rightarrow x \geq -y$$

$x \leq y$  holds anyways because  $x < 0$  and  $y \geq 0$ .

Second direction  $\Leftarrow$ :

Let  $-y \leq x \leq y$ .

**Case 1**  $x \geq 0$ :  $|x| = x \leq y$  because of the second inequality.

**Case 2**  $x < 0$ :  $|x| = -x$

$$-(-1) \Rightarrow -(-y) \geq -x \text{ or equivalently } y \geq -x = |x|$$

□

**Theorem 10.**

$$|x| = 0 \Leftrightarrow x = 0$$

$$\forall a \in K : |a| = |-a|$$

$$\forall \varepsilon > 0 : |x - y| \leq \varepsilon \Leftrightarrow x = y$$

*Proof.* **First direction**  $\Rightarrow$  Without loss of generality:  $x \geq y$ .

$$x \neq y \Rightarrow \exists \varepsilon > 0 : |x - y| > \varepsilon$$

Let  $x \neq y$ . Because  $x \geq y$  holds, so does  $x > y$ . Therefore  $x - y > 0$ . We define  $\varepsilon = \frac{x-y}{2} < x - y$

$$2 = 1 + 1 > 1$$

$$2^{-1} = \frac{1}{2} < 1 = 1^{-1}$$

Therefore it holds that  $\varepsilon : |x - y| = x - y > \frac{1}{2}(x - y) = \varepsilon > 0$ .

**Second direction**  $\Leftarrow$   $x = y \Rightarrow |x - y| = 0 \leq \varepsilon \forall \varepsilon > 0$

□

**Theorem 11** (Inversed triangle inequality). Let  $a, b \in K$ . Then it holds that

$$||a| - |b|| \leq |a - b|$$

*Proof.* Show that  $-|a - b| \leq |a| - |b| \leq |a - b|$ .

**First inequality**

$$|b| = |b - a + a| \leq |b - a| + |a| \Rightarrow -|a - b| \leq |a| - |b|$$

**Second inequality**

$$|a| = |a - b + b| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$$

□

## 6.6 Irrational numbers approximated by rational numbers

Additional remark from 14th of January 2016.

$\mathbb{Q}$  is dense in  $\mathbb{R}$ .

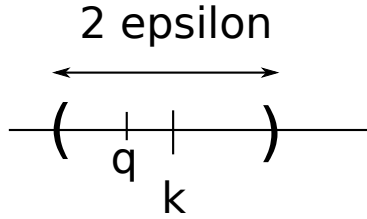
**Theorem 12.** For all  $x \in \mathbb{R}$  and for every  $\varepsilon > 0$  there exists  $q \in \mathbb{Q}$  with  $|x - q| < \varepsilon$ .

**Lemma 2.** Let  $A \subseteq \mathbb{N}$  and  $A \neq \emptyset$ . Then a minimum of  $A$  exists.

*Proof.* Proof by complete induction.

We show: Let  $A \subseteq \mathbb{N}$  such that no minimum exists. Then it holds that  $A = \emptyset$ .

Let  $C = \{k \in \mathbb{N} \mid \forall n \in A : k < n\}$ .  $C$  is the set of all lower bounds of  $A$  with operator  $<$ . We show:  $0 \in C$  and  $\forall k \in C \Rightarrow k + 1 \in C$ .


 Figure 4:  $k$  and an  $\varepsilon$  environment

**Induction base** Assume  $0 \notin C$ , hence for  $k = 0$  it holds that

$$\exists n \in A \subset \mathbb{N} : n \leq 0$$

$\Rightarrow n \geq 0$  anyways and hence  $n = 0$  and  $0 \in A$ .  $\Rightarrow 0 = \min A$  and  $A$  has a minimum.

This is a contradiction. So  $0 \notin C$  does not hold. So  $0 \in C$ .

**Induction step** Let  $k \in C$ , hence  $\forall n \in A : k < n$ .

$$\Rightarrow \forall n \in A : k + 1 \leq n$$

Even  $<$  holds. Assume  $\exists n \in A : k + 1 = n$  and  $\forall n' \in A : k + 1 \leq n'$ .

$$\Rightarrow k + 1 \in A \wedge k + 1 \text{ is lower bound of } A$$

Therefore  $k + 1 = \min A$ .

This is a contradiction to the assumption that  $\min A$  does not exist. Therefore  $\forall n \in A : k + 1 < n \Rightarrow k + 1 \in C$ .

Due to the workings of induction:  $\forall k \in \mathbb{N} : k \in C$  equivalently means  $C = \mathbb{N}$ .

Therefore  $A = \emptyset$  holds. Assume  $m \in A$ , so it holds that  $m \notin C$ , because  $\neg(m < m)$ .

□

*Proof.* Case distinction:

$x > 0$  Let  $\varepsilon > 0$  be arbitrary. Choose  $n \in \mathbb{N}_+$  such that  $\frac{1}{n} < \varepsilon$  and define  $A = \{k \in \mathbb{N} \mid k > n \cdot x\}$ .

We know that  $A \neq \emptyset$  (Archimedean axiom). Let  $M = \min A$ , then it holds that  $m > n \cdot x$  and  $m - 1 \leq n \cdot x$ . Therefore  $x < \frac{m}{n}$  and  $x \geq \frac{m-1}{n}$ . Therefore it holds that

$$\left| x - \frac{m}{n} \right| = \frac{m}{n} - x \leq \frac{m}{n} - \frac{m-1}{n} = \frac{m - m + 1}{n} = \frac{1}{n} < \varepsilon$$

and  $\frac{m}{n} = q \in \mathbb{Q}$ .

$x < 0$  Therefore  $-x > 0$  and  $\exists q \in \mathbb{Q}$ :

$$|q - (-x)| < \varepsilon$$

$$|x + q| = \left| x - \underbrace{(-q)}_{\in \mathbb{Q}} \right|$$

$x = 0$  Let  $q = 0 \in \mathbb{Q}$ .

□

**Corollary 3.**  $\forall x \in \mathbb{R}$  and  $\forall \varepsilon > 0$  it holds that

$$\mathbb{Q} \cap B(x, \varepsilon) = \mathbb{Q} \cap (x - \varepsilon, x + \varepsilon) \neq \emptyset$$

Therefore  $x$  is a contact point of  $\mathbb{Q}$ .

**Remark.** It even holds that  $x$  is limit point of  $\mathbb{Q}$ .

$$\overline{\mathbb{Q}} = \{x \in \mathbb{R} \mid x \text{ is contact point of } \mathbb{Q}\} = \mathbb{R}$$

We say  $\mathbb{Q}$  is *dense* (or: lies in) in  $\mathbb{R}$ .

Alternative characterization of contact points:

$$\forall x \in \mathbb{R} \exists (q_n)_{n \in \mathbb{N}} \text{ with } q_n \in \mathbb{Q} \text{ with } \lim_{n \rightarrow \infty} q_n = x$$

## 6.7 Intervals

This lecture took place on 5th of November 2015 with lecturer Wolfgang Ring.

**Definition 18** (Intervals). Let  $a, b \in K$ .

$$(a, b) = \{x \in K \mid (x > a) \wedge (x < b)\}$$

$$[a, b) = \{x \in K \mid (x \geq a) \wedge (x < b)\}$$

$$(a, b] = \{x \in K \mid (x > a) \wedge (x \leq b)\}$$

$$[a, b] = \{x \in K \mid (x \geq a) \wedge (x \leq b)\}$$

**Theorem 13** (Laws for intervals).

$$(a, b) = \emptyset \text{ if } b \leq a \quad (21)$$

$$[a, b) = \emptyset \text{ if } b < a \quad (22)$$

$$[a, a] = \{a\} \quad (23)$$

If  $I$  is an non-empty interval (hence  $I \neq \emptyset$ ), then  $|I| = b - a$  is called *length of the interval*. Furthermore

$$(a, \infty) = \{x \in K \mid x > a\} \quad (24)$$

$$[a, \infty) = \{x \in K \mid x \geq a\} \quad (25)$$

$$(-\infty, a) = \{x \in K \mid x < a\} \quad (26)$$

$$(-\infty, a] = \{x \in K \mid x \leq a\} \quad (27)$$

**Theorem 14.**  $\mathbb{Q}$  is arithmetically incomplete.

*Proof.* We define a mapping from  $\mathbb{N}_+$  to  $\mathbb{N}$ : Let  $n \in \mathbb{N}_+$  then we know that  $n$  can be represented distinctly as product of prime numbers. Let  $Z(n)$  be the number of twos in the prime product representation.

Examples:

$$Z(14) = Z(2 \cdot 7) = 1$$

$$Z(15) = Z(3 \cdot 5) = 0$$

$$Z(24) = Z(2 \cdot 2 \cdot 2 \cdot 3) = 3$$

It holds that  $Z(2n) = Z(n) + 1 \forall n \in \mathbb{N}_+$  and  $Z(n^2) = Z(n) \cdot 2 \forall n \in \mathbb{N}_+$ .

We claim,

$$\nexists q : q = \frac{m}{n} \text{ with } q^2 = 2$$

Proof by contradiction:

$$1. \text{ Assume } \left(\frac{m}{n}\right)^2 = 2.$$

$$2. \text{ Then } \frac{m^2}{n^2} = 2.$$

$$3. \text{ Then } m^2 = 2 \cdot n^2.$$

$$4. \text{ With } Z(m^2) = 2 \cdot Z(n).$$

$$5. \text{ With } Z(2 \cdot n^2) = Z(n^2) + 1 = 2 \cdot Z(n) + 1.$$

$$6. \text{ If } m^2 = 2n^2, \text{ then } Z(m^2) \text{ must be even and } Z(2 \cdot n^2) \text{ must be odd.}$$

$$7. \text{ Then equality cannot be satisfied } \nexists$$

□

## 6.8 Archimedean property and Completeness axiom

**Theorem 15.**  $\mathbb{Q}$  is geometrically incomplete.

We consider an infinite straight number line. We define  $\mathbb{R}$  as ordered field with properties:

**Archimedean property**  $\mathbb{N} \subseteq \mathbb{R}$  with  $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : x < n$

$$\mathbb{N} \subseteq \mathbb{R} \forall n \in \mathbb{N} : -n \in \mathbb{R}$$

$$\Rightarrow \forall n \in \mathbb{N}_+ : n^{-1} \in \mathbb{R}$$

$$\Rightarrow \mathbb{Z} \subseteq \mathbb{R}$$

$$\text{Therefore } \forall m \in \mathbb{N} : m \cdot \frac{1}{n} = \frac{m}{n} \in \mathbb{R} \Rightarrow \mathbb{Q} \subseteq \mathbb{R}.$$

**Definition 19.** Let  $I_0, I_1, \dots, I_z$ .  $(I_n)_{n \in \mathbb{N}}$  is a sequence of closed intervals with

1.  $\forall a \in \mathbb{N} : I_{n+1} \subseteq I_n$
2.  $\forall \varepsilon > 0 \exists n \in \mathbb{N} : n \geq N \Rightarrow |I_n| < \varepsilon$

**Completeness axiom** Let  $(I_n)_{n \in \mathbb{N}}$  be nested intervals in  $\mathbb{R}$ . Then there exists some  $x \in \mathbb{R} : x \in I_n : \forall n \in \mathbb{N}_+$ .

Be aware, there exists only *one*  $x \in \mathbb{R}$  with the property:  $x \in I_n \forall n \in \mathbb{N}$ .

Assume  $x \in I_n$  and  $y \in I_n \forall n \in \mathbb{N}$  and  $x \neq y$ .

$$|\beta - \alpha| \leq b - a = |I|$$

*Proof.* Without loss of generality:  $\alpha \leq \beta$ . Then it holds that  $|\beta - \alpha| = \beta - \alpha \leq \beta + (-\alpha) \leq b + (-a) = b - a = |I|$ .

$$a \leq \alpha \Rightarrow -a \geq -\alpha$$

Consider arbitrary small  $\varepsilon > 0$  and  $N \in \mathbb{N}$  sufficiently large, such that  $|I_n| < \varepsilon$ . Because  $x, y \in I_n \Rightarrow |x - y| < \varepsilon \Rightarrow x = y$ .  $\square$

**Corollary 4.** From the Archimedean property it follows that,

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

*Proof.* Let  $x > \frac{1}{\varepsilon} \in \mathbb{R}$ . Archimedean property:  $\exists N \in \mathbb{N} : N > x$ .

For  $n \geq N$  it holds that  $n > x > 0 \Rightarrow \frac{1}{n} < \frac{1}{x} = \varepsilon$ .

**Corollary 5.** Let  $p \in \mathbb{R}, p > 1 \forall x \in \mathbb{R} : n \geq N \Rightarrow p^n > x$ .

*Proof.*  $p > 1 + u$  with  $u = p - 1$

$$p^n = (1 + u)^n \underset{\text{Bernoulli}}{>} 1 - nu = 1 + n(p - 1)$$

Let  $x \in \mathbb{R}$  arbitrary, select  $N \in \mathbb{N} : \frac{x-1}{p-1} < N$ .

Then it holds for  $n \geq N$  :

$$\underbrace{\frac{x-1}{p-1}}_{>0} \Leftrightarrow x - 1 < n \cdot (p - 1) \Leftrightarrow x < 1 + n(p - 1) < p^n$$

**Theorem 16.** Let  $q \in \mathbb{R}$  with  $|q| < 1$ . Then it holds that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |q^n| = |q|^n < \varepsilon$$

*Proof.* Let  $s = |q| \geq 0$ . Consider  $q > 0$ . Then

$$\begin{aligned} q^n &= 0 \\ |q^n| &= 0 \\ |q|^n &< \varepsilon \forall \varepsilon > 0 \forall n \in \mathbb{N} \end{aligned}$$

Let  $q \neq 0$ , then  $0 < s < 1$ . Let  $p = \frac{1}{s} \Rightarrow p > 1$ . Choose arbitrary  $\varepsilon > 0$  and  $x = \frac{1}{\varepsilon}$ . Because of the Completeness axiom

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow p^n > X$$

So it holds that

$$\begin{aligned} \frac{1}{p^n} &= s^n < \frac{1}{x} = \varepsilon \forall n \geq N \\ &\Rightarrow (|q|)^n = |q^n| \end{aligned}$$

$\square$

**Theorem 17.** Let  $x \in \mathbb{R}, x > 0$  and let  $k \in \mathbb{N}_+$ . Then there exists a distinct  $y \in \mathbb{R}$  with  $y \geq 0$  such that

$$y^k = x$$

We denote  $y = \sqrt[k]{x}$  and conclude there exists  $k$ -th root numbers.

*Proof.* Idea: Construct nested intervals.

$(I_n)_{n \in \mathbb{N}}$  such that  $y \in \bigcap_{n \in \mathbb{N}} I_n$  satisfies the property that  $y^k = x$ .

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k$$

We define  $J_0 = [a_0, b_0]$  with  $a_0 = 0$  and  $b_0 = 1 + x$ . Then it holds that

$$a_0^k = 0^k = 0 \leq x$$

$$b_0^k = (1 + x)^k = 1 + kx + \binom{k}{2}x^2 + \dots + x^k \geq 1 + kx > 0$$

$\square$

$\square$

This lecture took place on 6th of November 2015 with lecturer Wolfgang Ring.

**Theorem 18.** We prove:

$$0 \leq y_1 < y_2 \Rightarrow y_1^k \leq y_2^k$$

*Proof.* A short proof by a student:

**k = 2**

$$y^{k+1} = y^k \cdot y < y_2^k x < y_2^k y_2 = y^{k+1}$$

**k → k + 1**

$$y_1^2 < y_2^2$$

**Theorem 19.** Let  $a, b \in K$  and  $k \in \mathbb{N}$ . Then it holds that

$$a^k - b^k = (a - b) \left( \sum_{j=0}^{k-1} a^{k-1-j} b^j \right)$$

$$a^2 - b^2 = (a - b)(a + b)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

*Proof.*

$$\begin{aligned} (a - b) \left( \sum_{j=0}^{k-1} a^{k-j-1} b^j \right) &= \sum_{j=0}^{j-1} a^{k-j} b^j - \sum_{j=0}^{k-1} a^{k-j-1} b^{j+1} \\ &= a^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \underbrace{b^{k-1}}_{j=k-1} - \sum_{j=0}^{k-2} a^{k-j-1} b^{j+1} \\ &= a^k - b^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \sum_{l=1}^{k-1} a^{k-l} b^l \\ &= a^k \end{aligned}$$

**Theorem 20.** Let  $y_2 > y_1$  then

$$\begin{aligned} y_2^k - y_1^k &= \underbrace{(y_2 - y_1)}_{>0} \underbrace{\left( \sum_{j=0}^{k-1} y_2^{k-j-1} y_1^j \right)}_{>0} \\ &\Rightarrow y_2^k - y_1^k > 0 \end{aligned}$$

*Proof.*

$$\forall x \geq 0 \in \mathbb{R} : \exists y \geq 0 \in \mathbb{R} : y^k = x \text{ with } k \in \mathbb{N}_+$$

Special case  $x = 0$  and  $y = 0$  is the solution.

Let  $x > 0$ : We construct  $y$  with  $y \in \bigcap_{k=0}^{\infty} I_n$  where  $I_n$  are nested intervals. Specifically  $I_n$  must have the properties:

□

- $I_n = [a_n, b_n]$  with  $a^k \leq x, b_n^k \geq x \quad \forall n \in \mathbb{N}$
- $I_{n+1} \subseteq I_n : |I_n| = \frac{1}{2} |I_{n+1}| = \left(\frac{1}{2}\right)^n |I_0|$

$$n = 0 \quad I_0 = [0, x - 1]$$

$$a_0 = b \quad b_0 = x + 1$$

$$a_0^k = 0 < x \quad \checkmark$$

$$b_0^k = (1 + x)^k = 1 + kx + \binom{k}{2} x^2 + \dots + x^k > 1 + kx > x \text{ for } k \geq 1$$

Let  $I_n$  be given:  $I_n = [a_n, b_n]$ . Define  $m_n = \frac{1}{2}(a_n + b_n)$

**Case 1**

$$m_n^k \geq x \Rightarrow \text{let } a_{n+1} = a_n, b_{n+1} = m$$

$$I_{n+1} = [a_n, m_n] \subseteq [a_n, b_n] = I_n$$

$$|I_{n+1}| = m_n - a_n = \frac{1}{2} a_n + \frac{1}{2} b_n - a_n$$

$$\frac{1}{2} (b_n - a_n) = \frac{1}{2} |I_n|$$

$$a_{n+1}^k = a^k \leq x \quad \checkmark$$

□

All conditions are satisfied.

**Case 2**  $m_n^k < x$  : Let  $a_{n+1} = m_1, b_{n+1} = b_n$ . It holds that  $a_{n+1} = m_n < x, b_{n+1} = b_n \geq x$  ✓. Furthermore it holds that  $I_{n+1} \subseteq I$  and  $|I_{n+1}| = \frac{1}{2} |I_n|$ .

$I_n$  is set of nested intervals. Let  $\varepsilon > 0$  be arbitrary. Then

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{1+x}$$

For those  $n \geq N$  it holds that

$$|I_n| = \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^n (x+1) < \frac{\varepsilon}{1+x} \cdot (1+x)$$

Let  $y \in I_n \forall n \in \mathbb{N}$ . Further nesting of intervals:

$$(I_n)_{n \in \mathbb{N}} \text{ with } I_n = [a_n^k, b_n^k]$$

It holds that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \text{ because } I_{n+1} \subseteq I_n \Rightarrow a_n^b \leq a_{n+1}^k < b_{n+1}^k \leq b_n^k$$

Length of  $I_n$ :

$$I_n = b_n^k - a_n^k = (b_n - a_n) \sum_{j=0}^{k-1} a_n^{k-1-j} b_n^j$$

Because  $I_n \leq I_0 \Rightarrow a_n < b_0 \Rightarrow b_n \leq b_0$ ,

$$< (b_n - b_0) \sum_{j=0}^{k-1} b_0^{k-1-j} b_0^j$$

$$= (b_n - a_n) k b_0^k = (b_n - a_n) k (1+x)^k$$

Let  $\varepsilon > 0$  be arbitrary. Find some  $N \in \mathbb{N}$  with  $n \geq N$ :

$$|I_n| = (b_n - a_n) < \frac{\varepsilon}{k(1+x)^k}$$

For those  $n$  it holds that

$$|I_n| < |I_n| \cdot k(1-x)^k < \frac{\varepsilon}{k(1+x)^k} k(1+x)^k = \varepsilon$$

Therefore  $(I_n)_{n \in \mathbb{N}}$  a set of nested intervals.

$\exists z \in \mathbb{R}$  with  $z \in [a_n^k, b_n^k] : \forall n \in \mathbb{N}$  and  $z$  is unique. By construction of  $I_n$  it holds that  $a_n^k \leq x \leq b_n^k$

$$\Rightarrow x \in I_n \forall n \in \mathbb{N} \Rightarrow x = z \in \bigcap_{n \in \mathbb{N}} I_n.$$

On the opposite side it holds that  $y \in I_n$  (hence  $a_n \leq y \leq b_n \Rightarrow a_n^k \leq y^k \leq b_n^k$ ). So  $y^k \in I_n \forall n \in \mathbb{N} \Rightarrow y^k = z = x$ . So we have found some  $y^k$  which is  $x$ . But is  $y \geq 0$  with  $y^k = x$  unique?

Let  $y_1 \neq y_2$  with  $y_1^k = y_2^k = x$  and without loss of generality,

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k \quad \text{`}$$

So,  $y$  is unique.

□

## 7 Supremum property of $\mathbb{R}$

### 7.1 Boundedness in $\mathbb{R}$

**Definition 20.** Let  $A \subseteq \mathbb{R}$ .

- We call  $A$  to be *bounded above* if there exists some  $u \in \mathbb{R}$  such that  $\forall a \in A : a \leq u$ .
- A number  $u$  with that property is called *upper bound* of  $A$ .
- We call  $A$  to be *bounded below* if there exists some  $l \in \mathbb{R}$  such that  $\forall a \in A : a \geq l$ .
- A number  $l$  with that property is called *lower bound* of  $A$ .
- $A$  is called *bounded* if there exists a lower and upper bound of  $A$ .

**Corollary 6.** Let  $(a, b)$  be bounded. Let  $u$  be its upper bound and let  $v \geq u$ . Then  $v$  is also an upper bound of  $(a, b)$ .

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This lecture took place on 11th of November 2015 with lecturer Wolfgang Ring.

## 7.2 Supremum and infimum in $\mathbb{R}$

**Definition 21.** Let  $A$  be bounded above. Assume  $s \in \mathbb{R}$  has the properties

1.  $s$  is an upper bound for  $A$
2.  $\forall \sigma \in \mathbb{R} : \sigma < s$ :  $\sigma$  is not an upper bound for  $A$ .

If those properties are satisfied, we call  $s$  *supremum of  $A$* . A supremum  $s$  is always the smallest upper bound of  $A$ . We denote  $s = \sup A$ .

There exists at most one supremum for  $A$ . Let  $s_1$  and  $s_2$  be two suprema, then  $s_1 \neq s_2$ . So wlog.  $\sigma_1 < \sigma_2$ . This invalidates the supremum property of  $s_2 \Rightarrow s_1$  is not a supremum of  $A$   $\nabla$ .

Analogously an *infimum of  $A$*  is the greatest lower bound of  $A$ . Let  $A$  be bounded below.  $t \in \mathbb{R}$  is called *infimum of  $A$*  if

1.  $\forall a \in A : t \leq a$  ( $t$  is a lower bound of  $A$ )
2.  $\forall x > t$  so  $x$  is no lower bound of  $A$

$$\Leftrightarrow \exists a \in A : a < x$$

We denote  $t = \inf A$ .

**Definition 22.** Let  $A \subseteq \mathbb{R}$ . We denote  $u = \max A$  for the *maximum of  $A$*  if

1.  $u \in A$  (is element of  $A$ )
2.  $\forall a \in A : a \leq u$  (is an upper bound)

$l \in \mathbb{R}$  denoted  $l = \min A$  is called minimum of  $A$  if

1.  $l \in A$  (is element of  $A$ )
2.  $\forall a \in A : l \leq a$  ( $l$  is a lower bound)

**Theorem 21.** Let  $A \subseteq \mathbb{R}$  and  $u$  be the maximum of  $A$ . Then it holds that  $u = \sup A$ . If  $l = \min A \Rightarrow l = \inf A$ .

*Proof.* We need to show, that  $l$  is an upper bound of  $A$ . This follows by definition. For  $x < u$  it holds that  $x$  not an upper bound.

Let  $x < u$ , because  $u \in A$  there exists some element  $y$  in  $A$  with  $y > x$ . Therefore  $x$  is not an upper bound of  $A$ .  $\square$

**Example 6.**

$$A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} = \left\{ \frac{1}{n} : n \in \mathbb{N}_+ \right\}$$

Then it holds that  $1 \in A$  and  $1 \geq \frac{1}{n} \forall n \in \mathbb{N}_+$ . Therefore  $1 = \max A = \sup A$ .

$0 = \inf A$ , because  $0$  is a lower bound of  $A$  ( $\frac{1}{n} > 0 \forall n \in \mathbb{N}_+$ ). Let  $\varepsilon > 0$ , then  $\exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} \leq \varepsilon$ . Therefore  $\varepsilon$  is not a lower bound of  $A$ .

So  $A$  does not have a minimum, because otherwise  $l = \max A = \inf A = 0$ .

**Theorem 22.** Let  $A \neq \emptyset$  and  $A \subseteq \mathbb{R}$  be bounded above. So some  $s = \sup A \in \mathbb{R}$  exists (therefore  $\mathbb{R}$  has a supremum property).

*Proof.* We construct nested intervals  $(I_n)_{n \in \mathbb{N}}$  such that for  $s \in \bigcap_{n \in \mathbb{N}} I_n$  gilt  $s = \sup A$ . We construct  $I_{n+1}$  inductively using  $I_n$

**Case  $n = 0$**

Because  $A \neq \emptyset$ , we select  $a_0 \in A$ . Because  $A$  is bounded above,  $\exists b_0 \in \mathbb{R}$  such that  $b_0$  is an upper bound of  $A$ . We define  $I_0 = [a_0, b_0]$ .

**Case  $n \rightarrow n + 1$**

Let  $a_0 = b_0$ , then it holds that  $b_0$  is upper bound and  $b_0 \in A$ . We call that terminating condition. Therefore  $b_0 = \max A = \sup A$  and the supremum was found. Instead of  $n$  we use  $n + 1$ . Let  $I_0 = [a_n, b_n]$  with  $a_n \neq b_n$  and  $a_n \in A$ ,  $b_n$  is an upper bound of  $A$ . Furthermore it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider  $I_{n+1}$  such that the same properties are satisfied. Let  $m_1 = \frac{1}{2}(a_1 + b_1)$ . It holds that  $a_n < m_n < b_n$ .



**Case  $m_n$  is an upper bound of  $A$**  Then we set  $a_{n+1} = a_n \in A$  and  $b_{n+1} = m_n$  is an upper bound of  $A$ .

$$\begin{aligned} |I_{n+1}| &= b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \\ &= \frac{1}{2}b_1 - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^{n+1} |I_n| \quad \checkmark \end{aligned}$$

**Case  $m_n$  is not an upper bound of  $A$**  Therefore  $\exists x \in A$  with  $x > m_n$ .

**Subcase  $x = b_1$**  So  $b_1$  is an upper bound. Therefore  $x \in A$  and  $x$  is upper bound.

$$x = \max A = \sup A$$

We found the supremum.

**Subcase  $m_n < x < b_n$**  Let  $a_{n+1} = x \in A$  and  $b_{n+1} = b_n$  is an upper bound and

$$\begin{aligned} I_{n+1} &= b_{n+1} - a_{n+1} = b_n - x < b_n - m_n - b_n - \frac{1}{2}(b_n - a_n) + \frac{1}{2}(b_n - a_n) \\ &= \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0| \end{aligned}$$

We have found supremum  $s = \sup A$ .

If in any case the terminating condition holds, then we have found the supremum.

The remaining case is  $\forall n \in \mathbb{N} : a_n < b_n, a_n \in A, b_n$  is upper bound of  $A$ .

$$|I_n| = b_n - a_n \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider  $\varepsilon > 0$  and  $N$  such that  $n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{|I_0|}$ . For those  $n$  it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| < \frac{\varepsilon}{|I_0|} |I_0| = \varepsilon$$

Therefore  $(I_n)_{n \in \mathbb{N}}$  are nested intervals.

□

What remains for completeness:  $s \in \mathbb{R}, s \in I_n : \forall n \in \mathbb{N}$ . We need to show that  $s = \sup A$ .

This lecture took place on 12th of November 2015 with lecturer Wolfgang Ring.

**Theorem 23.** Completeness of  $\mathbb{R}$ :

$$\exists s \in \mathbb{R} : s \in I_n \forall n \in \mathbb{N}$$

**Proof cont.** Every set with an upper bound has a supremum.

We construct  $(I_n)_{n \in \mathbb{N}}$  with  $I_n = [a_n, b_n]$  and  $I_{n+1} \subseteq I_n$ .  $\forall n \in \mathbb{N} : a_n \in A, b_n$  is the upper bound of  $A$ .

$$|I_{n+1}| \leq \frac{1}{2} |I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

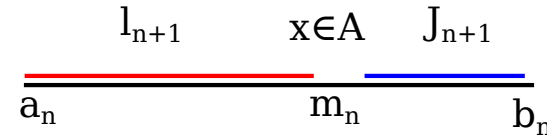


Figure 5: Relation of  $a_n$  and  $b_n$  and  $J_{n+1}$

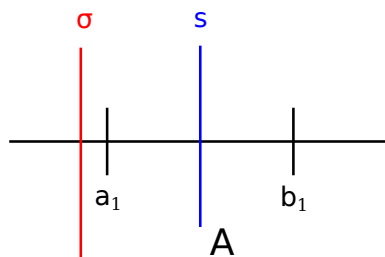
Consider  $I_{n+1} \subseteq I_n$  with  $a_n < b_n \forall n \in \mathbb{N}$ .

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

1. Claim:  $s$  is  $\sup A$ .

We need to show (by contradiction):  $S$  is upper bound of  $A$ . Assume  $a \in A$  and  $a > s$ . Let  $\varepsilon = a - s > 0$  and choose  $N$  sufficiently large such that

$$|I_n| < \varepsilon = a - s$$


 Figure 6: Illustration of  $s$  between  $a_n$  and  $b_n$ 

Then it holds that

$$b_N = \underbrace{b_n - a_n}_{< \varepsilon} + \underbrace{a_N}_{< s} < s + \varepsilon = a$$

$$\Rightarrow b_N < a \in A$$

Because  $b_n$  is an upper bound.

2.  $\forall \sigma < s$  it holds that  $\sigma$  is not an upper bound of  $A$ . Let  $\sigma < s$  and  $\varepsilon = s - \sigma > 0$  and choose  $n \in \mathbb{N}$  large enough such that  $b_n - a_n < \varepsilon$ . Then it holds that

$$\begin{aligned} a_N &= a_n - b_n + b_N \\ &> -\varepsilon + s \\ &= -s + \sigma + s = \sigma \quad \checkmark \end{aligned}$$

Therefore it holds that  $s$  is smallest upper bound of  $A$  and therefore supremum.

**Theorem 24.** Every set with a lower bound in  $\mathbb{R}$  has an infimum. Every set with an upper bound in  $\mathbb{R}$  has a supremum.

**Theorem 25.** Remember that  $M$  has the same cardinality like  $A$  if  $\varphi : M \rightarrow A$ .  $\varphi$  is bijective,  $M$  is called countably infinite if  $M$  has the same cardinality like  $\mathbb{N}$ .

Let  $\varphi : \mathbb{N} \rightarrow M$  be bijective therefore  $M = \{\varphi(1), \varphi(2), \varphi(3), \dots\} = \{\varphi(n) \mid n \in \mathbb{N}\}$  and  $\varphi(i) \neq \varphi(j)$  for  $i \neq j$ .

**Notation.**  $\varphi(n) = m_n$ .

$M = \{m_0, m_1, m_2, \dots\}$  with  $m_i \neq m_j$  for  $i \neq j$ .  $\varphi$  is a complete enumeration of all elements of  $M$ .

Therefore every element of  $M$  has the structure:  $m_n$  with  $i \in \mathbb{N}$ .

**Theorem 26.**

$$\mathbb{Q}^+ = \left\{ \frac{m}{n}, m \in \mathbb{N}, n \in \mathbb{N}_+ \right\}$$

The set  $\mathbb{Q}^+$  is countably infinite.

*Proof.* We enumerate the elements of  $\mathbb{Q}^+$ .

$$\mathbb{Q}_+ = \{q_0, q_1, q_2, \dots\}$$

$$\mathbb{Q}_- = \{-q_0, -q_1, -q_2, \dots\}$$

$$\mathbb{Q} = \{0, q_0, -q_0, q_1, -q_1, \dots\}$$

An enumeration exists. So  $\mathbb{Q}$  is countably infinite.  $\square$

**Theorem 27.** There is no bijective relation  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$ . Therefore we call  $\mathbb{R}$  *uncountable*.

*Proof.* We provide a proof by contradiction. Assume  $\mathbb{R} = \{x_0, x_1, x_2, x_3, \dots\}$  is countable.

We construct nested intervals.

$\square$  **Case  $n = 0$**

$$I_0 = [x_0 + 1, x_0 + 2]$$

Let  $|I_0| = 1$  and  $x_0 \notin I_0$ .

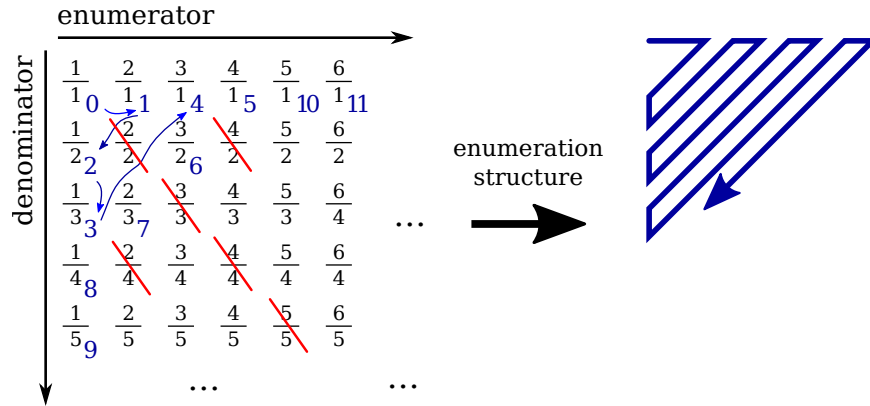


Figure 7: A complete enumeration of  $\mathbb{Q}^+$  (diagonalization argument). We traverse the whole matrix diagonally. The blue numbers indicate the enumeration and red lines cross out values already enumerated. On the right-hand side the general order of the enumeration is illustrated.

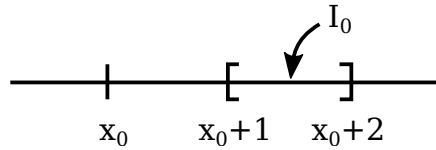


Figure 8: Construction of a nested interval and its  $I_0$

**Case  $n \rightarrow n+1$**  Assume  $I_0 \dots I_n$  were already defined with  $x_k \notin I_k$  for  $0 \leq k \leq n$ .

$$I_{k+1} \leq I_k \text{ for } k = 0, \dots, n-1$$

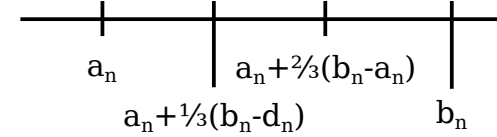


Figure 9: Construction of a nested interval and its  $I_n$

$$|I_k| = \left(\frac{1}{3}\right)^k$$

We construct  $I_{n+1}$ . Let  $I_n = [a_n, b_n]$ .

$$\begin{aligned} I_n^1 &= \left[ a_n, \frac{2}{3}a_n + \frac{1}{3}b_n \right] \\ I_n^2 &= \left[ \frac{2}{3}a_n + \frac{1}{3}b_n, \frac{1}{3}a_n + \frac{2}{3}b_n \right] \\ I_n^3 &= \left[ \frac{1}{3}a_n + \frac{2}{3}b_n, b_n \right] \end{aligned}$$

So  $x_n$  certainly is not contained in all three intervals  $I_n^1$ ,  $I_n^2$  and  $I_n^3$  because  $I_n^1 \cap I_n^2 \cap I_n^3 = \emptyset$ . Choose  $I_{n+1}$  as one of the three intervals  $I_n^l$  with  $x_{n+1} \notin I_n^l = I_{n+1}$ .  $I_{n+1} < I_n$ .

$$|I_{n+1}| = \frac{1}{3}I_n = \left(\frac{1}{3}\right)^{n+1}$$

For  $\varepsilon > 0$  it holds that there exists some  $N \in \mathbb{N}$  such that  $n \geq N \Rightarrow |I_1| = \left(\frac{1}{3}\right)^n < \varepsilon$ . Therefore nested intervals  $I_n$  are given.

Let  $x \in \mathbb{R}$  such that  $\forall n \in \mathbb{N} : x \in I_n$  (because of completeness law). Then it holds that  $\forall x_n : x \neq x_n$ .  $x \in I_n$  and  $x_n \notin I_n$ . Therefore  $x \in \{x_0, x_1, x_2, \dots\} = \mathbb{R}$ .

This contradicts with the assumption that  $\mathbb{R}$  is countable.

## 8 Complex numbers $\mathbb{C}$

We introduce a new arithmetic unit denoted  $i$ , which extends the field  $\mathbb{R}$ . Elements of  $\mathbb{C}$  are represented as  $a + bi$  with  $a, b \in \mathbb{R}$ .

$$\forall a, b \in \mathbb{R} : a + bi = 0 \Leftrightarrow a = 0 \wedge b = 0 \quad (28)$$

$$i^2 = -1 \quad (29)$$

$$\text{associativity, commutativity etc holds} \quad (30)$$

This lecture took place on 13th of November 2015 with lecturer Wolfgang Ring.

**Definition 23.** We consider an “arithmetic element”  $i$  extending  $\mathbb{R}$  (“conjugate”, dt. “adjungiert”). Arithmetic operations are well-defined for  $i$ . Associativity and commutativity holds. It holds that

- $a + ib = 0$  with  $a, b \in \mathbb{R} \Leftrightarrow a = 0 \wedge b = 0$
- $i^2 = -1$  i.e.  $i^2 + 1 = 0$ .
- Arithmetic operations still hold.

By the first law,

$$a + ib = a' + ib' \Leftrightarrow (a - a') + i(b - b') = 0 \Leftrightarrow a - a' = 0 \wedge b - b' = 0 \text{ therefore } a = a' \wedge b = b' \text{ Therefore } w = z^{-1} = \frac{1}{z}.$$

By the second law,  $i$  is the solution of the quadratic equation  $i^2 + 1 = 0$ .

Let  $z = a + ib$  a complex number. We call  $i$  the “imaginary unit”.

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$

$\mathbb{C}$  is the field of complex numbers with the following properties:

- For addition, it holds that

$$(a + ib) + (c + id) = (a + b) + i(b + d) \subseteq \mathbb{C}$$

and

$$(a + ib) + (-a - ib) = (a - a) + i(b - b) = 0 + i \cdot 0 = 0$$

- • For multiplication, it holds that

$$(a + ib) \cdot (c + id) = (ac + \underbrace{(i)^2}_{=-1} bd) + i(bc + ad)$$

$$(ac - bd) + i(bc + ad)$$

- Laws **A<sub>n</sub>** to **A<sub>4</sub>**, **M<sub>1</sub>** to **M<sub>3</sub>** and **D** hold.

- The one element exists:

$$1 = 1 + 0 \cdot i$$

$$(a + i \cdot b)(1 + i \cdot 0) = (a + (i)^2 \cdot 0) + i(b + 0) = a + ib$$

- **M4** holds: Let  $z \in \mathbb{C} \setminus \{0\}$ . Let  $z = a + ib$  and  $\neg(a = 0 \wedge b = 0) \Leftrightarrow a^2 + b^2 > 0$ .

We define

$$\begin{aligned} w &= \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \\ z \cdot w &= (a + ib) \left( \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \right) \\ &= \left( \underbrace{\frac{a^2}{a^2 + b^2} - \frac{b \cdot (-b)}{a^2 + b^2}}_{=1} \right) + i \cdot \left( \underbrace{\frac{ba}{a^2 + b^2} - \frac{a \cdot b}{a^2 + b^2}}_{=0} \right) \\ &= 1 + i \cdot 0 = 1 \end{aligned}$$

Therefore  $\mathbb{C}$  is a field.

We denote

$$a = \Re(z)$$

$$b = \Im(z)$$

$$\bar{z} = a - ib$$

$$|z| = \sqrt{a^2 + b^2}$$

$a$  is called *real part of  $z$* .  $b$  is called *imaginary part of  $z$* .  $z$  is called complex conjugate.  $|z|$  is called absolute value of  $z$ .

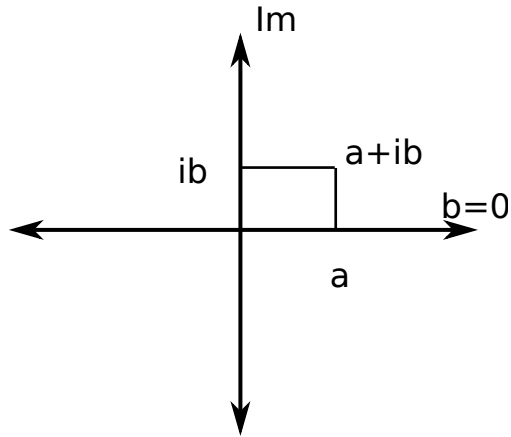


Figure 10: Illustration of complex numbers

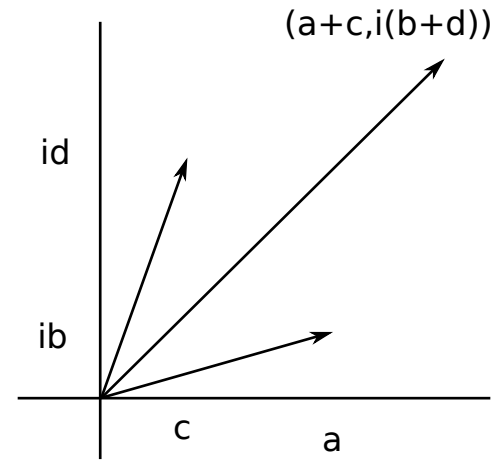


Figure 11: Illustration of complex number addition

**Theorem 28.**

$$\overline{(\bar{z})} = z$$

*Proof.*

$$\overline{(\bar{z})} = \overline{(a - ib)} = (a - (-ib)) = a + ib = z$$

**Theorem 29.**

$$\Re(z) = \frac{1}{2}(z + \bar{z})$$

**Theorem 30.**

$$\frac{1}{2}(z + \bar{z}) = \frac{1}{2}(a + ib + a - ib) = \frac{1}{2}(2a) = a \checkmark$$

**Theorem 31.**

$$\Im(z) = \frac{1}{2i}(z - \bar{z})$$

*Proof.*

$$\frac{1}{2i}(a + ib - (a - ib)) = \frac{1}{2i}(2ib) = b \checkmark$$

□

□ **Theorem 32.**

$$z \in \mathbb{R} \Leftrightarrow z = \bar{z}$$

*Proof.*

$$z = a \in \mathbb{R} \Rightarrow \bar{z} = a = z$$

On the opposite, let  $z = \bar{z}$  therefore

$$a + ib = a - ib \Rightarrow 2ib = 0 \Rightarrow b = 0$$

Therefore  $z = a \in \mathbb{R}$ .

□

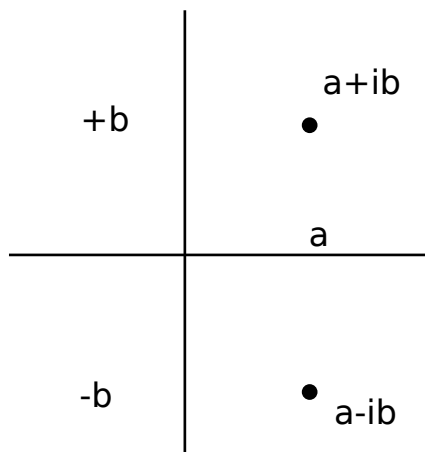


Figure 12: Illustration of the complex conjugate

**Theorem 33.**

$$z \in i\mathbb{R} = \{ib : b \in \mathbb{R}\} \Leftrightarrow z = -\bar{z}$$

Proof follows analogously.

**Theorem 34.** It holds that  $|z| = \sqrt{z \cdot \bar{z}}$ .

*Proof.*

$$\begin{aligned} \sqrt{z \cdot \bar{z}} &= ((a + ib)(a - ib))^{\frac{1}{2}} \\ &= (a^2 - (ib)^2)^{\frac{1}{2}} = (a^2 - i^2 b^2)^{\frac{1}{2}} \\ &= (a^2 + b^2)^{\frac{1}{2}} = |z| \quad \checkmark \end{aligned}$$

**Theorem 35.** Let  $z, w \in \mathbb{C}$ :

$$\overline{(zw)} = \bar{z} \cdot \bar{w}$$

*Proof.*

$$z = a + ib \quad w = c + id$$

$$zw = (ac - bd) + i(bc + ad)$$

$$\overline{zw} = (ac - bd) - i(bc + ad)$$

$$\bar{z}\bar{w} = (a - ib)(c - id) = (ac - bd) - i(bc + ad)$$

$$\bar{z} \cdot \bar{w} = (ac - (-b)(-d)) + i(-bc + a(-d)) = (ac - bd) - i(bc + ad)$$

□

**Corollary 7.**

$$\overline{z + w} = \bar{z} + \bar{w}$$

**Theorem 36.**

$$|zw| = |z| \cdot |w|$$

*Proof.*

$$\begin{aligned} |z \cdot w| &= (zw) \cdot (\overline{z \cdot w})^{\frac{1}{2}} \\ &= (z \cdot \bar{z} \cdot w \cdot \bar{w})^{\frac{1}{2}} = (z \cdot \bar{z})^{\frac{1}{2}} \cdot (w \cdot \bar{w})^{\frac{1}{2}} = |z| \cdot |w| \end{aligned}$$

□

**Theorem 37.**

$$z = 0 \Leftrightarrow |z| = 0 \in \mathbb{R}$$

*Proof.*

$$z = 0 = 0 + i0 \Rightarrow |z| = \sqrt{0^2 + 0^2} = 0$$

$$\text{Let } |z| = \sqrt{a^2 + b^2} = 0 \Rightarrow a^2 + b^2 = 0.$$

$$\Rightarrow a = 0 \wedge b = 0$$

□

□ **Theorem 38.**

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

$$|\Im(z)| = |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| =$$

**Theorem 39.** The triangle inequality holds:

$$\forall z, w \in \mathbb{C} : |z + w| \leq |z| + |w|$$

**Remark 6.** Let  $0 \leq y_1 < y_2$  with  $y_1, y_2 \in \mathbb{R}$ . Let  $k \in \mathbb{N}_+$ . Then it holds that

$$\sqrt[k]{y_1} < \sqrt[k]{y_2}$$

*Proof.* Indirect proof: Let  $\sqrt[k]{y_1} \geq \sqrt[k]{y_2} \geq 0$ .

$$\Rightarrow (\sqrt[k]{y_1})^k \geq (\sqrt[k]{y_2})^k$$

therefore  $y_1 \geq y_2$ . This is the negation of our assumption.  $\square$

**Proof of the triangle inequality.** We show that  $|z + w|^2 \leq (|z| + |w|)^2$ .

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = \underbrace{z\bar{z}}_{|z|^2} + w\bar{z} + z\bar{w} + \underbrace{w\bar{w}}_{|w|^2} \\ &= 2\Re(w\bar{z}) \\ &= (w\bar{z} + \overline{(w\bar{z})}) \\ &\quad \underbrace{\bar{w} \cdot \bar{z} = \bar{w} \cdot z}_{\bar{w} \cdot \bar{z} = \bar{w} \cdot z} \\ &= |z|^2 + 2\Re(w\bar{z}) + |w|^2 \\ &\leq |z|^2 + 2|\Re(w\bar{z})| + |w|^2 \\ &\leq |z|^2 + 2 \cdot |w\bar{z}| + |w|^2 \\ &= |z|^2 + 2 \cdot |w| \cdot |\bar{z}| + |w|^2 \\ &= |z|^2 + 2 \cdot |w| \cdot |z| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

**Theorem 40.** In our previous proof there was a small loop hole: We need to show that

$$|z| = |\bar{z}|$$

*Proof.*

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

## 8.1 Interpretation of multiplication

Multiplication with  $i$ . Let  $z = a + ib$ .

$$iz = i \cdot a + i^2 \cdot b = (-b) + ia$$

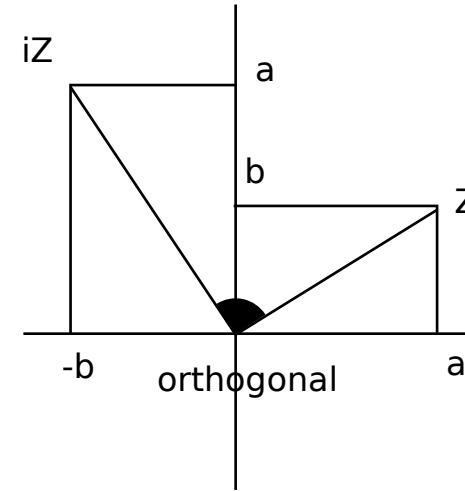


Figure 13: Multiplication corresponds to a rotation by  $90^\circ$

Multiplication with  $i$  rotates  $z$  counter-clockwise by  $90^\circ$  in the plane.

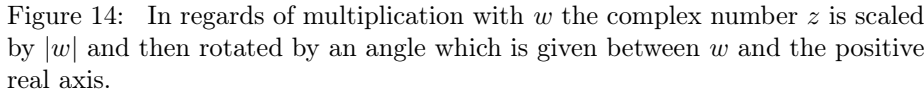
Let  $z \in \mathbb{C}$  and  $w = c + id$ .

This lecture took place on 18th of November 2015 with lecturer Wolfgang Ring.  $\square$

## 8.2 Taking roots

$$\forall a \in \mathbb{R} : a \geq 0 \forall n \in \mathbb{N}_+ : \exists x \geq 0 \in \mathbb{R} : x^n = a$$

Taking the  $n$ -th root only works for positive integers, because  $\forall x \geq 0 : x^2 \geq 0$  and no solution in  $\mathbb{R}$  exists for the equation  $x^2 = -1$ .  $\square$



## 9 Sequences of real and complex elements

$$\forall n \in \mathbb{N} : a(n) \in \mathbb{R}$$

Analogously for the complex numbers  $\mathbb{C}$  and general sets  $X$ .

$$\left(\sqrt[n]{2}\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$

**Definition 25.** Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence.  $(a_n)_{n \in \mathbb{N}}$  is called *bounded above* if  $o \in \mathbb{R}$  exists such that  $\forall n \in \mathbb{N} : a_n \leq o$ .  $(a_n)_{n \in \mathbb{N}}$  is called *bounded below* if  $u \in \mathbb{R}$  exists such that  $\forall n \in \mathbb{N} : a_n > u$ .

**Example 9.**  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = \frac{n}{n+1}$  is bounded below by 0 and bounded above by 1:  $n \leq n+1 \Rightarrow n \frac{1}{n+1} < \frac{n+1}{n+1} = 1\checkmark$ .

**Definition 26.**

- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically increasing* if  $\forall n \in \mathbb{N} : a_{n+1} \geq a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically decreasing* if  $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically strictly increasing* if  $\forall n \in \mathbb{N} : a_{n+1} > a_n$ .
- $(a_n)_{n \in \mathbb{N}}$  is called *monotonically strictly decreasing* if  $\forall n \in \mathbb{N} : a_{n+1} < a_n$ .

- $(a_n)_{n \in \mathbb{N}}$  is called bounded if  $(|a_n|)_{n \in \mathbb{N}}$  is a bounded real sequence. Hence  $\exists o \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| \leq o$ .

- The lower bound is implicitly given by 0.



**Example 10.**  $a_n := i^n$  and  $(a_n)_{n \in \mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, i, -1, \dots)$

$$|1| = 1 \quad |-1| = 1 \quad |i| = \sqrt{0^2 + 1^2} = 1 \quad |-i| = \sqrt{0^2 + (-1)^2} = 1$$

So  $(|a_n|)_{n \in \mathbb{N}} = (1, 1, 1, 1, 1, \dots)$ . It holds that

$$|z| = |-z| = |\bar{z}|$$

**Definition 27.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathbb{C}$  and let  $a \in \mathbb{C}$ . We state:  $(a_n)_{n \in \mathbb{N}}$  has a limit (lat. limes)  $a$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$$

We denote

$$\lim_{n \rightarrow \infty} a_n = a$$

The distance  $|a_n - a|$  becomes arbitrary small, if  $n$  is sufficiently large.

A sequence, which has a limit, is called *convergent*. A sequence, which does not have a limit, is called *divergent*.

**Remark 7.** Sometimes we consider mappings  $a : \mathbb{N}_+ \rightarrow \mathbb{C}$ , which we also call sequences:

$$a \leftrightarrow (a_1, a_2, \dots)$$

**Example 11.**

$$a_n = \frac{1}{n}$$

We know:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \rightarrow \frac{1}{n} < \varepsilon$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Let  $q \in \mathbb{C}$ ,  $|q| < 1$ .

We know  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \rightarrow |q^n - 0| < \varepsilon$ .

$$\lim_{n \rightarrow \infty} q^n = 0$$

---

This lecture took place on 19th of November 2015 with lecturer Wolfgang Ring.

**Remark 8.** Consider  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$  as a circle with radius  $\varepsilon$ . So if  $n$  is sufficiently large, all new sequence elements are located inside the circle.

**Lemma 3.** A sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in \mathbb{C}$  can have at most one limit.

*Proof.* Assume  $a$  and  $b$  are limes of  $(a_n)_{n \in \mathbb{N}}$ . Then we prove:

$$\forall \varepsilon > 0 : |a - b| < \varepsilon$$

$$\implies a = b$$

Let  $\varepsilon > 0$  arbitrary: Because  $a = \lim_{n \rightarrow \infty} a_n$  there exists

$$N_1 \in \mathbb{N} : [n \geq N_1 \implies |a_n - a| < \frac{\varepsilon}{2}]$$

Because  $b = \lim_{n \rightarrow \infty} b_n$  there exists

$$N_2 \in \mathbb{N} : [n \geq N_2 \implies |b_n - b| < \frac{\varepsilon}{2}]$$

Let  $N = \max(N_1, N_2)$ , hence  $N \geq N_1 \wedge N \geq N_2$ .

$$\implies |a_N - a| < \frac{\varepsilon}{2} \wedge |a_N - b| < \frac{\varepsilon}{2}$$

$$|a - b| = |a - \underbrace{a_N + a_N}_{0} - b| \leq \underbrace{|a - a_N|}_{< \frac{\varepsilon}{2}} + \underbrace{|a_N - b|}_{< \frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

**Theorem 41** (Well-known convergent sequences.).

1. Let  $s = \frac{p}{q} \in \mathbb{Q}_+$  and  $n \in \mathbb{N}_+$ . Consider  $(\frac{1}{n^s})_{n \in \mathbb{N}}$ .

$$n^s = n^{\frac{p}{q}} := \sqrt[q]{n^p}$$

It holds that

$$\lim_{n \rightarrow \infty} \frac{1}{n^s} = 0$$

2. Let  $q \in \mathbb{C}, |q| < 1$ . Then it holds that

$$\lim_{n \rightarrow \infty} q^n = 0$$

3. Let  $a \in \mathbb{R}, a > 0, n \in \mathbb{N}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \sqrt[n]{a} = 1$$

4. It holds that ( $n \in \mathbb{N}_+$ )

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

5. Let  $z \in \mathbb{C} : |z| > 1$ . Let  $k \in \mathbb{N}$ . Then it holds that

$$\lim_{n \rightarrow \infty} \frac{n^k}{z^n} = 0$$

**Remark 9** (Remark to sequence 5).  $|z^n|$  grows faster than  $n^k$ .

**Proof of sequence 1.** Let  $0 \leq x_n < x_2$ .

$$\Rightarrow 0 \leq x_1^p < x_2^p \Rightarrow \sqrt[p]{x_1^p} < \sqrt[p]{x_2^p}$$

Therefore  $f(x) = x^s$  is strongly monotonic rising for  $x \in (0, \infty)$ . Let  $\varepsilon > 0$  arbitrary and  $N > \frac{1}{\varepsilon^{\frac{1}{s}}} = \varepsilon^{\frac{1}{s}} = \varepsilon^{-\frac{s}{1}}$ . Then it holds that  $n \geq N$ :

$$\begin{aligned} \left| \frac{1}{n^s} - 0 \right| &= \frac{1}{n^s} \leq \frac{1}{N^s} \\ \frac{1}{n^s} < \frac{1}{N^s} &\Rightarrow n^s \geq N^s \\ \frac{1}{n^s} \leq \frac{1}{N^s} &< \frac{1}{\left(\frac{1}{\varepsilon^{\frac{1}{s}}}\right)^s} = \frac{1}{\varepsilon} = \varepsilon \end{aligned}$$

**Proof of sequence 2.** Already done. □

**Proof of sequence 3. Case  $a > 1$**  Let  $a > 1$ . Consider  $\varepsilon > 0$ . Show that  $|\sqrt[n]{a} - 1| < \varepsilon$  for sufficiently large  $n$ .

$$x_n = \sqrt[n]{a} - 1 = |\sqrt[n]{a} - 1|$$

$$a > 1 \Rightarrow \sqrt[n]{a} > \sqrt[n]{1} = 1 \Rightarrow \sqrt[n]{a} - 1 > 0$$

It holds that  $x_n + 1 = \sqrt[n]{a}$ , i.e.  $(x_n + 1)^n = a$ .

$$a = \underbrace{(x_1 + 1)^n}_{>0} \underbrace{>}_{\text{Bernoulli}} 1 + n \cdot x_n$$

$$\Rightarrow x_n < \frac{a - 1}{n}$$

$$\begin{aligned} N &> \frac{a - 1}{\varepsilon} \xrightarrow{\text{for } x \geq N} |\sqrt[n]{a} - 1| = x_n \\ &< \frac{a - 1}{n} \leq \frac{a - 1}{N} < \frac{a - 1}{\frac{a - 1}{\varepsilon}} = \varepsilon \end{aligned}$$

**Case  $a = 1$**

$$\sqrt[n]{a} = \sqrt[n]{1} = 1$$

$$(\sqrt[n]{a})_{n \in \mathbb{N}} = (1, 1, 1, 1, \dots)$$

has the limit 1.

**Case  $0 < a < 1$**  Let  $0 < a < 1 \Rightarrow 0 < \sqrt[n]{a} < \sqrt[n]{1} = 1$ .

$$x_n = 1 - \sqrt[n]{a} > 0$$

Show that  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow x_n < \varepsilon]$ .

$$x_n = 1 - \sqrt[n]{a} = \sqrt[n]{a} \left( \frac{1}{\sqrt[n]{a}} - 1 \right) = \sqrt[n]{a} \left( \sqrt[n]{\frac{1}{a}} - 1 \right) < \left( \sqrt[n]{a'} - 1 \right)$$

with  $a' = \frac{1}{a} > 1$ . From case  $a > 1$  we already know

$$\begin{aligned} \exists N \in \mathbb{N} : [n \geq N \Rightarrow |\sqrt[n]{a'} - 1| = \sqrt[n]{a'} - 1 < \varepsilon] \\ \Rightarrow x_n < \varepsilon \end{aligned}$$

□

□

**Proof of sequence 4.** This proof works similar to the proof of sequence 3.

$$x_n = \sqrt[n]{n} - 1 > 0 \text{ for } n \geq 2$$

Therefore  $|x_n| = x_n$ . Let  $\varepsilon > 0$  be arbitrary.

$$x_n + 1 = \sqrt[n]{n} \quad \text{i.e.} \quad (x_n + 1)^n = n$$

$$n = (1 + x_n)^n = 1 + \underbrace{nx_n}_{>0} + \underbrace{\binom{n}{2}x_n^2}_{>0} + \underbrace{\binom{n}{3}x_n^3}_{>0} + \underbrace{\dots + x_n^n}_{>0} > 1 + \binom{n}{2}x_n^2$$

All expressions we remove are positive (but we don't remove all positive expressions).

$$x_n^2 < \frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2 \cdot 1}} = \frac{2}{n}$$

$$x_n < \sqrt{\frac{2}{n}}$$

Choose  $N > \frac{2}{\varepsilon^2}$ . Then it holds for  $n \geq N$  that

$$x_n < \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} < \sqrt{\frac{2}{\frac{2}{\varepsilon^2}}} = \varepsilon$$

Consider  $\sqrt{\frac{2}{n}} < \varepsilon$  hence  $\frac{2}{n} < \varepsilon^2$  hence  $n > \frac{2}{\varepsilon^2}$ .

**Proof of sequence 5.**

$$|z| > 1 \text{ thus } x = |z| - 1 > 0 \text{ it holds that } |z| = 1 + x$$

We show that for  $\varepsilon > 0$  arbitrary, there exists  $N \in \mathbb{N}$ :

$$n \geq N \implies \left| \frac{n^k}{z^n} - 0 \right| = \left| \frac{n^k}{z^n} \right| = \frac{n^k}{|z|^n} < \varepsilon$$

Let  $\varepsilon > 0$  be given,

• For  $n > 2k$  it holds that  $n - k > n - \frac{n}{2} = \frac{n}{2}$ .

$$|z|^n = (1 + x)^n = \sum_{j=0}^n \binom{n}{j} x^j > \underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1}$$

$$n > 2k \geq k + 1$$

$$\underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1} = \frac{\overbrace{n}^{>\frac{n}{2}} \overbrace{(n-1)}^{>\frac{n}{2}} \overbrace{(n-2)}^{>\frac{n}{2}} \dots \overbrace{(n-k)}^{>\frac{n}{2}}}{(k+1)!} x^{k+1} > \frac{\frac{n^{k+1}}{2^{k+1}}}{(k+1)!} x^{k+1}$$

Therefore  $|z|^n > \frac{n^{k+1}}{2^{k+1}(k+1)!} x^{k+1}$ . So,

$$\frac{n^k}{|z|^n} < \frac{n^k \cdot 2^{k+1}(k+1)!}{n^{k+1} \cdot x^{k+1}} = \frac{2^{k+1}(k+1)!}{\underbrace{x^{k+1}}_{= \text{constant} \wedge >0}} \cdot \frac{1}{n} = M \cdot \frac{1}{n}$$

$$\frac{n^k}{|z|^n} < M \cdot \frac{1}{n} \text{ for } n > 2k$$

Consider  $N$  such that  $N > \frac{M}{\varepsilon}$  and  $N > 2k$ . Then it holds that

$$\frac{n^k}{|z|^n} < M \frac{1}{n} \leq \frac{M}{N} < \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$$

□

□

**Lemma 4.** Every convergent sequence is bounded (in  $\mathbb{C}$ ).

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent. This means especially e.g.  $\varepsilon = 13$ .

$$\exists N \in \mathbb{N} \text{ s.t. } [n \geq N \implies |a_n - a| < 13]$$

Consider  $O > 0$  such that

$$O = \max\{|a_0|, |a_1|, |a_2|, \dots, |a_{N-1}|, |a| + 13\}$$

So  $O \geq |a_n|$  for  $n \in \{0, \dots, N\}$ . Then for  $0 \leq n < N$  it holds that  $|a_n| < O$ . ✓

For  $n \geq N$  it holds that

$$|a_n| = |a_n - a + a| \leq \underbrace{|a_n - a|}_{< 13} + |a| < \underbrace{13 + |a|}_{\leq O}$$

Therefore  $(|a_n|)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  and followingly  $(|a_n|)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{C}$ . □

**Theorem 42.** Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then the following laws hold:

1.  $\lim_{n \rightarrow \infty} (a_n + b_n)$  is convergent with limes  $a + b$
2.  $\lim_{n \rightarrow \infty} (a_n \cdot b_n)$  is convergent with limes  $a \cdot b$
3.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  is convergent with limes  $\frac{a}{b}$  if  $\forall n \in \mathbb{N} : b_n \neq 0 \wedge b \neq 0$ .

*Proof.* 1. Let  $\varepsilon > 0$  arbitrary. Because  $(a_n)_{n \in \mathbb{N}}$  is convergent,

$$\exists N_1 : [n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}]$$

$(b_n)$  is convergent hence

$$\exists N_2 : [n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}]$$

$N = \max\{N_1, N_2\}$ , hence for  $n \geq N$  both statements above hold. Let  $n \geq N$ , then the triangle inequality holds:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{< \frac{\varepsilon}{2}} < \varepsilon$$

2.  $(a_n)_{n \in \mathbb{N}}$  is convergent and therefore also bounded. Therefore,

$$\exists m \geq 0 : \forall n \in \mathbb{N} : |a_n| \leq m$$

$(b_n)_{n \in \mathbb{N}}$  is convergent, hence

$$\exists N_1 : n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2} \cdot \frac{1}{m + 1}$$

$(a_n)_{n \in \mathbb{N}}$  is convergent, hence

$$\exists N_2 \leq N : n \geq N_2 \Rightarrow |a_n - a| < \frac{\varepsilon}{2} \frac{1}{|b| + 1}$$

$N = \max\{N_1, N_2\}$ . For  $n \geq N$  both relations above hold. Let  $n \geq N$ :

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| = |a_n| |b_n - b| + |b| |a_n - a| \\ &\leq m \frac{\varepsilon}{2} \frac{1}{m + 1} + |b| \frac{\varepsilon}{2} \frac{1}{|b| + 1} < \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} \cdot 1 = \varepsilon \end{aligned}$$

3. Left for the practicals. □

## 9.2 Laws for convergent complex sequences

**Theorem 43.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$ ,  $(a_n \rightarrow a)$ . Then it holds that

- $(\Re(a_n))_{n \in \mathbb{N}}$  is convergent.

$$\lim_{n \rightarrow \infty} (\Re(a_n)) = \Re(a)$$

- $(\Im(a_n))_{n \in \mathbb{N}}$  is convergent.

$$\lim_{n \rightarrow \infty} (\Im(a_n)) = \Im(a)$$

- $(|a_n|)_{n \in \mathbb{N}}$  is a convergent real sequence.

$$\lim_{n \rightarrow \infty} |a_n| = |a|$$

- $(\overline{a_n})_{n \in \mathbb{N}}$  is convergent with

$$\lim_{n \rightarrow \infty} \overline{a_n} = \overline{a}$$

On the opposite, let  $(a_n)_{n \in \mathbb{N}}$  with  $a_n = \alpha_n + i\beta_n$  a sequence of complex numbers. Let  $a_n = \alpha_n + i\beta_n$  is convergent with limes  $\alpha + i\beta$  which is  $a$ . Let  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  be convergent with limes  $\alpha$  i.e.  $\beta$ . Then  $(a_n)_{n \in \mathbb{N}}$  is a convergent complex sequence with limes  $a = \alpha + \beta i$ .

*Proof.* Let  $\varepsilon > 0$ . Consider  $N$  such that  $n \geq N \Rightarrow |a_n - a| < \varepsilon$ .

$$\underbrace{|a_n - a|}_{(\alpha_n - \alpha) + (\beta_n - \beta)i} = \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2}$$

TODO

Therefore  $(\alpha_n) = (\Re(a_n))_{n \in \mathbb{N}}$  is convergent.  $(\beta_n) = (\Im(a_n))_{n \in \mathbb{N}}$  is convergent.

Let  $\varepsilon > 0$ . Consider  $N$  such that  $n \geq N \Rightarrow |a_n - a| < \varepsilon$ .

$$||a_n| - |a|| \leq |a_n - a| < \varepsilon \text{ for } n \geq N$$

inverse triangular inequality

Now we need to show  $\alpha_n \rightarrow \alpha$  and  $\beta_n \rightarrow \beta$

$$\Rightarrow a_n \rightarrow a$$

Let  $\varepsilon > 0$  be arbitrary. Because  $(\alpha_n)_{n \in \mathbb{N}}$  be convergent, there exists  $N_1 \in \mathbb{N}$ :

$$n \geq N_1 \Rightarrow |\alpha_n - \alpha| < \frac{\varepsilon}{\sqrt{2}}$$

$(\beta_n)_{n \in \mathbb{N}}$  is convergent. So,

$$\exists N_2 \in \mathbb{N} : n \geq N_2$$

$$|\beta_n - \beta| < \frac{\varepsilon}{\sqrt{2}}$$

For  $N = \max\{N_1, N_2\}$  and  $n \geq N$  both relations hold.

Let  $n \geq N$ :

$$\begin{aligned} |a_n - a| &= |(\alpha_n - \alpha) + i(\beta_n - \beta)| \\ &= \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon \end{aligned}$$

$$\begin{aligned} &\Rightarrow \lim_{n \rightarrow \infty} \alpha_n = \alpha \wedge \lim_{n \rightarrow \infty} \beta_n = \beta \\ &\Rightarrow \lim_{n \rightarrow \infty} (-\beta_n) = -\beta \quad \text{“multiplication rule”} \\ &\Rightarrow (\overline{a_n})_{n \in \mathbb{N}} = (\underbrace{\alpha_n}_{\text{convergent}} - \underbrace{i\beta_n}_{\text{convergent}})_{n \in \mathbb{N}} \\ &\Rightarrow \lim_{n \rightarrow \infty} \overline{a_n} = \alpha - i\beta = \overline{a} \end{aligned}$$

□

### 9.3 Further laws for sequences

**Theorem 44.** Let  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  be convergent in  $\mathbb{R}$  with limes  $a$  (i.e.  $b$ ) and it must hold that  $\forall n \in \mathbb{N} : a_n \leq b_n$ . Then also  $a \leq b$ .

*Proof.* Consider  $a - b = \varepsilon > 0$ .

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$$

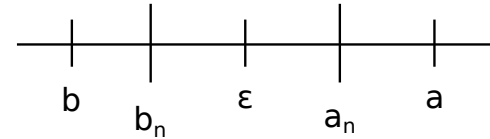


Figure 15: the sequences  $a_n$ ,  $b_n$  and limes  $a$ ,  $b$  and  $\varepsilon$  in relation

For  $N = \max\{N_1, N_2\}$ :

$$b_N = b_N - b + b \leq b + |b_N - b| < b + \frac{\varepsilon}{2} = b + \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$a_N = \underbrace{a_N - a}_{\geq -|a_n - a|} + a \geq a - |a_n - a| > a - \frac{\varepsilon}{2} = a - \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$b_N < \frac{1}{2}(a+b) < d_N$$

Attention:

$$a_n < b_n \not\Rightarrow a < b$$

Example:  $a_n = 0, b_n = \frac{1}{n}$ .

## 9.4 Convergence criteria

Are there criteria such that if the sequences have a specific structure, they are obviously convergent?

### 9.4.1 Squeeze theorem

**Theorem 45.** Let  $(A_n)_{n \in \mathbb{N}}$  and  $(B_n)_{n \in \mathbb{N}}$  be convergent real sequences with  $\lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} B_n = A$ . Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence and  $M \in \mathbb{N}$  such that

$$\forall n \geq M : A_n \leq a_n \leq B_n$$

Then it holds that  $(a_n)_{n \in \mathbb{N}}$  is also convergent and  $\lim a_n = A$ .

*Proof.* Let  $\varepsilon > 0$  be arbitrary. Consider  $N$  such that,

- $N \geq M$
- $n \geq N \Rightarrow |A_n - A| < \varepsilon$
- $n \geq N \Rightarrow |B_n - A| < \varepsilon$

Then it holds that for  $n \geq N$ :

$$\left. \begin{array}{l} A - a_n \leq A - A_n \leq |A - A_n| < \varepsilon \\ a_n - A \leq B_n - A \leq |B_n - A| < \varepsilon \end{array} \right\} = 1$$

$$\Rightarrow |a_n - A| < \varepsilon$$

$$\lim_{n \rightarrow \infty} a_n = A$$

□

**Example 12.** Let  $s \in \mathbb{Q}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \left( \sqrt[n]{n^s} \right) = 1$$

□ We apply the squeeze theorem:

$$n^2 \geq 1 \forall n \in \mathbb{N}$$

$$\Rightarrow \sqrt[n]{n^s} \geq 1$$

Let  $k \in \mathbb{N}_+$ . Then it holds that

$$\lim_{n \rightarrow \infty} \sqrt[n]{n^k} = \lim_{n \rightarrow \infty} \underbrace{\sqrt[n]{n} \sqrt[n]{n} \dots \sqrt[n]{n}}_{k \text{ times}}$$

$$= 1 \cdot 1 \cdot 1 \dots = 1$$

For the last two lines we actually need to read them from right to left.

Let  $s = \frac{p}{q}$ .

$$\Rightarrow n^s = n^{\frac{p}{q}} \leq q \cdot \left( n^{\frac{p}{q}} \right)^q = n^p$$

$$q \geq 1 \Rightarrow \sqrt[n]{n^s} \leq \underbrace{\sqrt[n]{n^p}}_{\text{convergent with limes 1}} \quad p \in \mathbb{N}$$

Then it holds that  $\lim_{n \rightarrow \infty} \sqrt[n]{n^s} = 1$  with the squeezing theorem.

**Remark 10.** Let  $A \subseteq \mathbb{R}$  be bounded above. Then it holds that

$$S = \sup A \Leftrightarrow s \text{ is upper bound of } A \wedge \forall \varepsilon > 0 \exists a \in A : a > s - \varepsilon$$

*Proof.* Implication from left to right: Let  $s = \sup A$ . Then it holds that  $s$  is upper bound of  $A$  and  $s - \varepsilon < s$  is not an upper bound. Therefore  $\exists a \in A : a > s - \varepsilon$ .

Implication from right to left: Consider that both statements on the RHS hold. So  $s$  is an upper bound. We need to show that any  $t$  is not an upper bound with  $t > s$ . Let  $t < s, s - t = \varepsilon > 0$ . Therefore  $t = s - \varepsilon$ . Because of the right statement  $\exists a \in A : a > s - \varepsilon = t$  therefore  $t$  is not an upper bound.  $\square$

**Remark 11.** Analogously:

$$\sigma = \inf A \Leftrightarrow \sigma \text{ is lower bound} \wedge \forall \varepsilon > 0 \exists a \in A : a < \sigma + \varepsilon$$

**Theorem 46.** Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded monotonic sequence. Then  $(a_n)_{n \in \mathbb{N}}$  has a limes  $a$  with

- $a = \sup \{a_n : n \in \mathbb{N}\}$  if  $(a_n)_{n \in \mathbb{N}}$  is monotonically increasing.
- $a = \inf \{a_n : n \in \mathbb{N}\}$  if  $(a_n)_{n \in \mathbb{N}}$  is monotonically decreasing.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be monotonically increasing. Let  $a = \sup \{a_n : n \in \mathbb{N}\}$ . Let  $\varepsilon > 0$  be arbitrary. Because  $a$  is a supremum, there exists  $a_N \in \{a_n : n \in \mathbb{N}\}$  such that  $a_N > a - \varepsilon$ .

$$\Rightarrow \underbrace{a - a_N}_{\geq 0} < \varepsilon$$

because  $a$  is an upper bound. Therefore

$$|a - a_N| < \varepsilon$$

Let  $n \geq N$  then it holds that

$$|a - a_n| \underbrace{=}_{a \text{ is upper bound}} a - a_n \leq a - a_N$$

because  $a_N \leq a_n$  is increasing:

$$a - a_N < \varepsilon$$

Therefore  $\lim_{n \rightarrow \infty} a_n = a$ .  $\square$

This lecture took place on 25th of November 2015 with lecturer Wolfgang Ring.

Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence. If  $(a_n)_{n \in \mathbb{N}}$  is bounded and monotonous. Then  $(a_n)_{n \in \mathbb{N}} \in \mathbb{N}$  is convergent.

**Example: Wallis product**

John Wallis (1616–1703)

$$p_n = \frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)} = \prod_{k=1}^n \frac{2k}{2k-1}$$

Consider

$$\alpha_n = \frac{p_n}{\sqrt{n}} \quad \beta_n = \frac{p_n}{\sqrt{n+1}}$$

We need to show that

- $(\alpha_n)$  is monotonously decreasing
- $(\beta_n)$  is monotonously increasing

$$\forall n \in \mathbb{N} : n \geq 1 : \alpha_n > \beta_n$$

Both are convergent.

1. Show that,

$$\alpha_{n+1} < \alpha_n \Leftrightarrow \frac{\alpha_{n+1}}{\alpha_n} < 1 \Leftrightarrow \frac{(\alpha_{n+1})^2}{(\alpha_n)^2} < 1$$

$$\begin{aligned} \left( \frac{\alpha_{n+1}}{\alpha_n} \right)^2 &= \left( \frac{\frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n+2)}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1) \cdot (2n+1)} \cdot \frac{1}{\sqrt{n+1}}}{\frac{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n+1)} \cdot \frac{1}{\sqrt{n}}} \right)^2 \\ &= \frac{(2n+2)^2 \cdot n}{(2n+1)^2 (n+1)} = \frac{4n^3 + 8n^2 + 4n}{(4n^2 + 4n + 1) \cdot (n+1)} = \frac{4n^3 + 8n^2 + 4n}{4n^3 + 8n^2 + 5n + 1} < 1 \end{aligned}$$

2. We show,

$$\begin{aligned} \left( \frac{\beta_{n+1}}{\beta_n} \right)^2 &= \frac{(2n+2)^2 \cdot (n+1)}{(2n+1)^2 \cdot (n+2)} = \frac{(4n^2 + 8n + 4)(n+1)}{(4n^2 + 2n + 1)(n+2)} \\ &= \frac{4n^3 + 12n^2 + 12n + 4}{4n^3 + 12n^2 + 9n + 2} > 1 \Rightarrow \beta_{n+1} > \beta_n \Rightarrow \beta_n \text{ is monotonically increasing} \end{aligned}$$

Let  $p = \lim_{n \rightarrow \infty} a_n$  and  $p' = \lim_{n \rightarrow \infty} b_n$ .

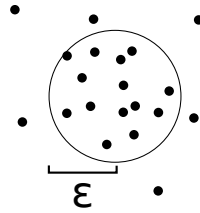


Figure 16: Illustration of a limit point in the Euclidean plane. The point is represented as circle with radius  $\varepsilon$ . Finitely many points lie outside the limit point; infinitely many inside.

$$\begin{aligned}\beta_n &= \frac{p_n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \alpha_n \cdot \sqrt{\frac{n}{n+1}} \\ \lim_{n \rightarrow \infty} \beta_n &= \lim_{n \rightarrow \infty} \alpha_n \sqrt{\frac{n}{n+1}} = \lim_{n \rightarrow \infty} \alpha_n \cdot \underbrace{\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}}}_{=1} \\ &\Rightarrow \lim_{n \rightarrow \infty} \beta_n = \lim_{n \rightarrow \infty} a_n \Rightarrow p = p'\end{aligned}$$

It holds that  $p = \lim_{n \rightarrow \infty} \frac{p_n}{\sqrt{n}} = \sqrt{n}$ .

## 9.5 On limit points and subsequences

**Definition 28.** Let  $(a_n)_{n \in \mathbb{N}}$  be a complex sequence. The complex value  $a$  is called *limit point* (german “Häufungspunkt”) of  $(a_n)_{n \in \mathbb{N}}$  if  $\forall \varepsilon > 0 : |a_n - a| < \varepsilon$  for infinitely many indices  $n \in \mathbb{N}$ . Hence infinitely many values of the sequence lie within a circle with center  $a$  and radius  $\varepsilon$ .

**Remark 12.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limit  $a$ . Then it holds that  $a$  is the only limit point of the sequence  $(a_n)_{n \in \mathbb{N}}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent. Let

$$\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a|$$

Therefore  $\forall n \in \{N, N+1, N+2, \dots\}$  it holds that  $|a_n - a| < \varepsilon$ . Assume  $a' \in \mathbb{C}$  is another limit point with  $a \neq a'$ . Let

$$\varepsilon = \frac{|a - a'|}{2} > 0$$

Let  $N \in \mathbb{N}$  such that  $\forall n \geq N : |a_n - a| < \varepsilon$ .

$$\begin{aligned}\Rightarrow n \in \mathbb{N} : |a' - a_n| &= |a' - a + a - a_n| = |a' - a - (a_n - a)| \geq |a' - a| - |a_n - a| \\ &= 2\varepsilon - |a_n - a| > 2\varepsilon - \varepsilon = \varepsilon\end{aligned}$$

At most for  $n \in \{1, \dots, N-1\}$  it is possible that  $|a_n - a'| < \varepsilon$ .  $\square$

**Remark 13.**  $a_n = (-1)^n$  has the limit points  $+1$  and  $-1$ .

The lecture on 26th of November 2015 got cancelled.

This lecture took place on 27th of November 2015 with lecturer Wolfgang Ring.

**Definition 29.** Let  $a \in \mathbb{C}$  and  $r > 0$  and

$$B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

and we call  $B(a, r)$  an *open* circle with center  $a$  and radius  $r$ . So the circle itself is not part of the set, unlike the following set:

$$B'(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}$$

Let  $a$  be a limit point of  $(a_n)_{n \in \mathbb{N}} \Leftrightarrow \forall \varepsilon > 0. B(a, \varepsilon)$  contains infinitely many sequence values.

**Example 13.**

$$\begin{aligned}a_n &= \frac{1}{2} \left[ 1 + (-1)^n \left( \frac{1-n}{n} \right) \right] \quad n \geq 1 \\ &\Rightarrow a_1 = \frac{1}{2} \quad a_2 = \frac{1}{4} \quad a_3 = \frac{5}{6} \\ a_4 &= \frac{1}{8} \quad a_5 = \frac{9}{10} \quad a_6 = \frac{1}{12} \quad a_7 = \frac{13}{14}\end{aligned}$$

“ $\frac{5}{6}$ ? Ah, passt ma eh besa.” (Wolfgang Ring)



Estimated limit points:  $a = 0, b = 1$ . □

*Proof.* Let  $\varepsilon > 0$  and  $a = 0$ . We consider sequence values with even index. So for indices it holds that  $n = 2k$ .

$$\begin{aligned}
 |a_{2k} - 0| &= \left| \frac{1}{2} \left( 1 + \underbrace{(-1)^{2k}}_{+1} \left( \frac{1-2k}{2k} \right) \right) \right| \\
 &= \frac{1}{2} \left| 1 + \frac{1-2k}{2k} \right| \\
 &= \frac{1}{2} \left| \frac{2k+1-2k}{2k} \right| \\
 &= \frac{1}{4k} < \varepsilon \text{ if } \underbrace{k > \frac{1}{4\varepsilon}}_{\substack{\text{infinitely many ks} \\ \text{satisfy the relation}}}
 \end{aligned}$$

Let  $\varepsilon > 0$  and  $b = 1$ . We consider sequence values of structure  $n = 2k + 1$ .

$$\begin{aligned}
 |a_{2k+1} - 1| &= \left| \frac{1}{2} \left[ 1 + \underbrace{(-1)^{2k+1}}_{=-1} \left[ \frac{1-(2k+1)}{2k+1} \right] \right] - 1 \right| \\
 &= \left| \frac{1}{2} \left[ 1 - \frac{-2k}{2k+1} \right] - 1 \right| \\
 &= \left| \frac{1}{2} \frac{2k+1+2k}{2k+1} - 1 \right| \\
 &= \left| \frac{4k+1}{4k+2} - 1 \right| \\
 &= \left| \frac{4k+1-4k-2}{4k+2} \right| \\
 &= \frac{1}{4k+2} \\
 &< \varepsilon
 \end{aligned}$$

$$\text{if } 4k+2 > \frac{1}{\varepsilon} \Rightarrow \underbrace{k}_{\substack{\text{infinitely many} \\ \text{indices}}} > \frac{1}{4} \left( \frac{1}{\varepsilon} - 2 \right)$$

**Example 14.**  $(c_n)_{n \in \mathbb{N}}$  is defined with  $c_n = i^n$ .

$$(c_n)_{n \in \mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, \dots)$$

What are its limit points?

**Definition 30.** Let  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in \mathbb{C}$ . For example,

$$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$$

We remove some elements

$$(1, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \dots)$$

A *subsequence* is created. We also reenumerate the numbers:

$$(\underbrace{1}_{n_0}, \underbrace{\frac{1}{3}}_{n_1}, \underbrace{\frac{1}{4}}_{n_2}, \underbrace{\frac{1}{6}}_{n_3}, \dots)$$

Let  $n : \mathbb{N} \rightarrow \mathbb{N}$  be strictly monotonically increasing. Therefore

$$\forall k \in \mathbb{N} : n(k+1) > n(k) \Rightarrow n_{k+1} > n_k$$

We call  $(n_k)_{k \in \mathbb{N}}$  an *index subsequence* and  $(a_{n_k})_{k \in \mathbb{N}}$  is called subsequence of  $(a_n)_{n \in \mathbb{N}}$ .

**Lemma 5.** Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$  and  $(a_{n_k})_{k \in \mathbb{N}}$  a subsequence of  $(a_n)_{n \in \mathbb{N}}$ . Then also the subsequence is convergent and has the same limes  $a$ .

*Proof.* For every subsequence index  $n_k$  with  $k \in \mathbb{N}$  it holds that  $n_k \geq k$ .

Proof by induction:  $k = 0$

$n_0 \in \mathbb{N}$

$$n_0 \geq 0 = k \quad \checkmark$$

$n_k \geq k$  Because  $\underbrace{n_{k+1}}_{\in \mathbb{N}} > n_k$  (strictly monotonic). Therefore,

$$n_{k+1} \geq n_k + 1 > k + 1$$

Proof of limes:  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . Let  $\varepsilon > 0$ . Because  $(a_n)_{n \in \mathbb{N}}$  is convergent, it holds that  $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \varepsilon$ . Let  $k \geq N$ . This holds because  $n_k \geq k \geq N : |a_{n_k} - a| < \varepsilon$ . Therefore  $(a_{n_k})_{k \in \mathbb{N}}$  has limes  $a$ .  $\square$

**Lemma 6.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$ . Then it holds that  $a \in \mathbb{C}$  is limit point if and only if there exists some subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} a_{n_k} = a$ .

*Proof.* We first prove direction  $\Leftarrow$ .

Assume  $(a_{n_k})_{k \in \mathbb{N}}$  is a convergent subsequence of  $(a_n)_{n \in \mathbb{N}}$  with limes  $a$ . Let  $\varepsilon > 0$ .

$$\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_{n_k} - a| < \varepsilon$$

Therefore  $B(a, \varepsilon)$  has infinitely many sequence elements of  $(a_{n_k})_{k \in \mathbb{N}}$  and therefore also infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ .

We prove direction  $\Rightarrow$ .

We build a convergent subsequence. Consider  $k \in \mathbb{N}$  with  $k \geq 1$ .

$$\varepsilon_k = \frac{1}{k}$$

We define  $n_0 = 0$  and  $a_{n_0} = a_0$ . Assume  $a_{n_0}, a_{n_1}, \dots, a_{n_{k-1}}$  are already defined.

Definition of  $a_{n_k}$ : In  $B(a, \varepsilon_k)$  there are infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ . We consider  $n_k > n_{k-1}$  and  $a_{n_k} \in B(a, \varepsilon_k)$ .

Then it holds that  $\lim_{k \rightarrow \infty} a_{n_k} = a$ . Let  $\varepsilon > 0$  be arbitrary. Consider  $K > \frac{1}{\varepsilon}$ . Hence  $\varepsilon > \frac{1}{K} = \varepsilon_K$  for all  $k \geq K$  it holds that  $n_k \geq n_K$  and  $|a_{n_k} - a| < \varepsilon_k = \frac{1}{k} \leq \frac{1}{K} < \varepsilon$ .  $\square$

## 9.6 Bolzano-Weierstrass theorem

Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897)

**Theorem 47.** Every bounded sequence of real numbers has a limit point in  $\mathbb{R}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathbb{R}$ , hence  $\exists M > 0$  such that all sequence elements  $a_n$  in  $I_0 = [-M, M]$  and let  $F_0 = \{n \in \mathbb{N} \mid a_n \in I_0\} = \mathbb{N}$  (index set).  $F_0$  is infinite. We build nested intervals with the properties:

- $I_{n+1} \subseteq I_n$
- $|I_{n+1}| = \frac{1}{2} |I_n|$
- $F_n = \{k \in \mathbb{N} \mid a_k \in I_n\}$  is infinite.

This construction is inductive:

**induction base**  $I_0$   $\checkmark$

**induction step** Let  $I_n = [A_n, B_n]$  be given and  $M_n = \frac{1}{2}(A_n + B_n)$ . Let  $J_n = [A_n, M_n]$  and  $L_n = [M_n, B_n]$ . It holds that  $J_n \subseteq I_n \wedge L_n \subseteq I_n$  and  $|J_n| = \frac{1}{2} |I_n| \wedge |L_n| = \frac{1}{2} |I_n|$ . Because there are infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$  in  $I_n$  and  $I_n = J_n \cup L_n$ , in at least one subinterval there have to be infinitely many sequence elements.

Therefore select  $I_{n+1} = J_n$  if  $J_n$  contains infinitely many sequence elements and consider  $I_{n+1} = L_n$  if  $J_n$  contains only finitely many sequence elements. Therefore  $I_{n+1}$  contains infinitely many sequence elements.

$$F_{n+1} = \{k \in \mathbb{N} \mid a_k \in I_{n+1}\}$$

is infinite. So  $(I_n)_{n \in \mathbb{N}}$  is a nested interval.

Let  $a \in \bigcap_{n \in \mathbb{N}} I_n$  (completeness of  $\mathbb{R}$ ).

Claim:  $a$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Let  $\varepsilon > 0$  be given and  $n$  sufficiently large, such that  $|I_n| = B_n - A_n < \varepsilon$ . Then it holds that for every  $x \in I_n$  that  $|x - a| \leq B_n - A_n < \varepsilon$  (with  $x \in I_n, a \in I_n$ ). Because  $I_n$  contains infinitely many sequence elements of  $(a_n)_{n \in \mathbb{N}}$ , it holds that infinitely many sequence elements  $a_k$  satisfy the relation  $|a_n - a| < \varepsilon$ . Therefore  $a$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .  $\square$

**Corollary 8** (typical definition of the Bolzano-Weierstrass theorem). Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

**Theorem 48** (Bolzano-Weierstrass theorem in  $\mathbb{C}$ ). Let  $(a_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathbb{C}$ . Then  $(a_n)_{n \in \mathbb{N}}$  has a convergent subsequence and therefore also at least one limit point in  $\mathbb{C}$ .

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be bounded.  $a_n = \alpha_n + i\beta_n$ . So  $(\alpha_n)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$  as well as  $(\beta_n)_{n \in \mathbb{N}}$  is bounded in  $\mathbb{R}$ .

Consider a convergent subsequence of  $(\alpha_n)_{n \in \mathbb{N}}$ ,  $(\alpha_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} \alpha_{n_k} = \alpha$ . Now consider bounded  $(\beta_{n_k})_{k \in \mathbb{N}}$ . From the Bolzano-Weierstrass theorem it follows that there exists a convergent subsequence  $(\beta_{n_{k_l}})_{l \in \mathbb{N}}$  with  $\beta = \lim_{l \rightarrow \infty} \beta_{n_{k_l}}$ .

$(\alpha_{n_{k_l}})_{l \in \mathbb{N}}$  is subsequence of  $(\alpha_{n_k})_{k \in \mathbb{N}}$  convergent with limit point  $\alpha$ .

Let  $a_{n_{k_l}} = \alpha_{n_{k_l}} + i\beta_{n_{k_l}}$  be a subsequence of  $(a_n)_{n \in \mathbb{N}}$ .

Real and imaginary parts are convergent, therefore  $\lim_{l \rightarrow \infty} a_{n_{k_l}} = a = \alpha + i\beta$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  contains a convergent subsequence.  $\square$

This lecture took place on 2nd of December 2015 with lecturer Wolfgang Ring.

**Theorem 49** (Weierstrass-Bolzano theorem). Every bounded sequence in  $\mathbb{C}$  has a convergent subsequence.

**Theorem 50** (Convergence). Let  $(x_n)_{n \in \mathbb{N}}$  be convergent in  $\mathbb{C}$  with limes  $x$ .

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : |x_n - x| < \varepsilon$$

**Definition 31** (Metric space). Let  $X$  be a set. We call  $d : X \times X \rightarrow \mathbb{R}$  a *distance function* (or *metric*) on  $X$  if,

- $\forall x \in X : d(x, x) = 0$
- $\forall x, y \in X : d(x, y) = d(y, x)$  (symmetry)
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called *metric space*.

**Example 15.**  $X = \mathbb{C}$ ,  $d(x, y) = |x - y|$ .

**Definition 32** (Convergence with metric spaces). Let  $X$  be a metric space.  $(x_n)_{n \in \mathbb{N}}$  is a sequence of elements in  $X$ . Let  $x \in X$ . We call  $(x_n)_{n \in \mathbb{N}}$  convergent with limes  $x$  if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : d(x_n, x) < \varepsilon$$

**Definition 33.** Let  $K \subseteq X$  be a subset of the metrical space  $X$ . We call  $K$  *pre-compact* if every sequence  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in K$  has a convergent subsequence.  $K$  is called *compact* if the limes  $a$  of the convergent subsequence is also in  $K$ .

**Definition 34.** In  $\mathbb{C}$  it holds that every bounded set is pre-compact.

## 9.7 Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$

Augustin-Louis Cauchy (1789–1857)

**Definition 35.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{C}$ . We call  $(a_n)_{n \in \mathbb{N}}$  a *Cauchy sequence* (fundamental sequence) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow |a_n - a_m| < \varepsilon$$

**Definition 36** (Cauchy sequence in a metric space). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . We call  $(a_n)_{n \in \mathbb{N}}$  a *Cauchy sequence* (fundamental sequence) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow d(a_n, a_m) < \varepsilon$$

**Lemma 7.** Every convergent sequence  $(a_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  is a Cauchy sequence.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be convergent with limes  $a$ . Let  $\varepsilon > 0$  be arbitrary.

Convergence implies that  $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$ . For  $m, n \geq N$  it holds that

$$|a_n - a_m| = |a_n - a + a - a_m| \leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2} \text{ because } n \geq N} + \underbrace{|a - a_m|}_{< \frac{\varepsilon}{2} \text{ because } m \geq N}$$

$\square$

**Lemma 8.** Every Cauchy sequence  $(a_n)_{n \in \mathbb{N}}$  in  $\mathbb{C}$  is bounded.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ . The Cauchy condition for  $\varepsilon = 1$  states:

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |a_n - a_m| < 1$$

specifically  $m = N : \forall n \geq N$

$$|a_n - a_N| < 1$$

Therefore  $|a_n| = |a_n - a_N + a_N| \leq \underbrace{|a_n - a_N|}_{< 1} + |a_N| < |a_N| + 1$ .

Let  $m = \max\{|a_0|, |a_1|, \dots, |a_{N-1}|\}$  and  $M = \max\{m, |a_N| + 1\}$ .

Then for  $n \leq N - 1$  it holds that

$$|a_n| \leq m \leq M$$

and for  $n \geq N$  it holds that

$$|a_n| \leq |a_N| + 1 \leq M$$

Therefore  $\forall n \in \mathbb{N} : |a_n| \leq M$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  is bounded. □

## 9.8 Is $\mathbb{C}$ , $\mathbb{R}$ and $\mathbb{Q}$ complete?

**Theorem 51** (Cauchy sequences and limes). Every Cauchy sequence in  $\mathbb{C}$  has a limes and is therefore convergent. Followingly we call  $\mathbb{C}$  to be *complete*.

*Proof.* Let  $(a_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ . We know that  $(a_n)_{n \in \mathbb{N}}$  is bounded. From the Bolzano-Weierstrass theorem it follows that a limit point  $a$  of  $(a_n)_{n \in \mathbb{N}}$  exists. Let  $\varepsilon > 0$  be arbitrary.

1. We choose  $N \in \mathbb{N}$  sufficiently large such that

$$n, m \geq N \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2}$$

2. Because  $B(a, \frac{\varepsilon}{2})$  contains infinitely many sequence elements ( $a$  is limit point),  $K \geq N$  exists with  $|a - a_K| < \frac{\varepsilon}{2}$ .

Let  $n \geq N$ . Then

$$|a_n - a| = |a_n - a_K + a_K - a| \leq \underbrace{|a_n - a_K|}_{< \frac{\varepsilon}{2} \text{ (Cauchy seq.)}} + \underbrace{|a_K - a|}_{< \frac{\varepsilon}{2} \text{ (limit point } a)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore  $(a_n)_{n \in \mathbb{N}}$  is convergent with limes  $a$ .

We have proven that if  $(a_n)_{n \in \mathbb{N}}$  has a limit point, this limit point is also its limes.

We concluded: nested intervals  $\Rightarrow$  compactness / Bolzano-Weierstrass theorem  $\Rightarrow$  completeness.

Actually nested intervals are equivalent to completeness. □

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This lecture took place on 3rd of December 2015 with lecturer Wolfgang Ring.

**Corollary 9.**  $\mathbb{C}$  is complete.

*Proof.* Let  $(z_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathbb{C}$ .

$$z_n = a_n + ib_n$$

Then  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are Cauchy sequences in  $\mathbb{R}$ .

Show that this property: Let  $\varepsilon > 0$ . Because  $(z_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, it holds that

$$\exists N \in \mathbb{N} : n, m \geq N \Rightarrow |z_n - z_m| < \varepsilon$$

Because  $|a_n - a_m| \leq |z_n - z_m|$  and  $|b_n - b_m| \leq |z_n - z_m|$  hold, it follows that for  $n, m \geq N : |a_n - a_m| < \varepsilon \wedge |b_n - b_m| < \varepsilon$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are Cauchy sequences.

Because  $\mathbb{R}$  is complete, it follows that  $\exists a \in \mathbb{R}$  such that

$$a = \lim_{n \rightarrow \infty} a_n \text{ and } \exists b \in \mathbb{R}$$

with  $b = \lim_{n \rightarrow \infty} b_n$ . Because  $\lim_{n \rightarrow \infty} z_n = z = a + ib$ ,

$$\Leftrightarrow a = \lim_{n \rightarrow \infty} a_n \wedge b = \lim_{n \rightarrow \infty} b_n$$

□

**Example 16.** We show a counterexample for the completeness of  $\mathbb{Q}$ . So we have Cauchy sequences with limes, which lies outside  $\mathbb{Q}$ .

We define a recursion:

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) & \text{if } n > 0 \end{cases}$$

We observe,  $\forall n \in \mathbb{N} : a_n > 0 \wedge a_n \in \mathbb{Q}$ .

Proof by complete induction:

**Induction base:**  $n = 0$

$$a_0 = 2 > 0 \wedge 2 \in \mathbb{Q} \quad \checkmark$$

**Induction step:**  $n \rightarrow n + 1$  Let  $a_n > 0$  and  $a_n \in \mathbb{Q}$ .

$$a_{n+1} = \frac{1}{2} \left( \underbrace{a_n}_{>0} + \underbrace{\frac{2}{a_n}}_{>0} \right) > 0$$

and  $a_{n+1} \in \mathbb{Q}$ .

We prove by induction:  $\forall n \in \mathbb{N} : a_n^2 > 2$ .

**Induction base:**  $n = 0$

$$a_0 = 2 \quad a_0^2 = 4 > 2 \quad \checkmark$$

**Induction step:**  $n \rightarrow n + 1$  It holds that  $a_n^2 - 2 > 0$ .

$$\begin{aligned} a_{n+1}^2 - 2 &= \frac{1}{4} \left( a_n^2 + 4 + \frac{4}{a_n^2} \right) - 2 = \frac{1}{4a_n^2} (a_n^4 + 4a_n^2 + 4 - 8a_n^2) \\ &= \frac{1}{4a_n^2} (a_n^4 - 4a_n^2 + 4) = \frac{1}{4a_n^2} \underbrace{(a_n^2 - 2)^2}_{>0} > 0 \end{aligned}$$

Furthermore it holds that  $a_{n+1} < a_n$ .

$$\begin{aligned} 2a_{n+1} &= a_n + \frac{2}{a_n} \Rightarrow 2(a_{n+1} - a_n) = -a_n + \frac{2}{a_n} = \frac{\overbrace{2 - a_n^2}^{<0}}{a_n} < 0 \\ \Rightarrow a_{n+1} - a_n &< 0 \Rightarrow a_{n+1} < a_n \end{aligned}$$

Therefore the sequence  $(a_n)_{n \in \mathbb{N}}$  is strictly monotonically decreasing and is bound by below. Therefore some  $a \in \mathbb{R}$  exists with  $a = \lim_{n \rightarrow \infty} a_n$ .

Monotonicity really depends on the completeness of  $\mathbb{R}$ . We cannot argue equivalently to Theorem 46 with the supremum.

For this example we know that  $(a_n)_{n \in \mathbb{N}}$  is convergent in  $\mathbb{R}$ .  $(a_n)_{n \in \mathbb{N}}$  is Cauchy sequence in  $\mathbb{R}$ . So  $(a_n)_{n \in \mathbb{N}}$  is Cauchy sequence in  $\mathbb{Q}$ .

For the limes  $a$  it holds that,

$$\begin{aligned} a &= \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2} \left( a_n + \frac{2}{a_n} \right) \\ &= \frac{1}{2} \lim_{n \rightarrow \infty} a_n + \frac{1}{2} \frac{2}{\lim_{n \rightarrow \infty} a_n} = \frac{1}{2}a + \frac{1}{a} \end{aligned}$$

$$a = \frac{1}{2}a + \frac{1}{a} \Rightarrow \frac{1}{2}a = \frac{1}{a}$$

$$a^2 = 2 \Rightarrow a = +\sqrt{2} \notin \mathbb{Q}$$

Therefore  $(a_n)_{n \in \mathbb{N}}$  is *not* convergent in  $\mathbb{Q}$ . We found a convergent Cauchy sequence whose limes is not in  $\mathbb{Q}$  which immediately means that  $\mathbb{Q}$  is incomplete.

**Definition 37** (Tending towards infinity). Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of real numbers.

- We state  $(a_n)_{n \in \mathbb{N}}$  *tends to infinity* with limes  $+\infty$ :

$$\lim_{n \rightarrow \infty} a_n = +\infty$$

$$\text{if } \forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n > M$$

- We state  $(a_n)_{n \in \mathbb{N}}$  *tends to negative infinity* with limes  $-\infty$ :

$$\lim_{n \rightarrow \infty} a_n = -\infty$$

$$\forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n < -M$$

**Example 17.**

$$a_n = \frac{n^2 + 2}{n + 1}$$

has limes  $+\infty$ . The proof is given in the practicals. We show that ...

$$\frac{n^2 + 2}{n + 1} > M \Leftrightarrow \dots$$

**Definition 38** (Limes superior, Limes inferior). Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence which is bounded above and

$$H = \{\xi \in \mathbb{R} \mid \xi \text{ is limit point of } (a_n)_{n \in \mathbb{N}}\} \neq \emptyset$$

Then  $H$  is also bounded by above and we call  $S^* = \sup H$  a *limes superior* of the sequence  $(a_n)_{n \in \mathbb{N}}$ . We denote:

$$S^* = \limsup_{n \rightarrow \infty} a_n$$

Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence which is bounded below and

$$H = \{\xi \in \mathbb{R} \mid \xi \text{ is limit point of } (a_n)_{n \in \mathbb{N}}\} \neq \emptyset$$

Then  $H$  is also bounded by below and we call  $S_* = \inf H$  a *limes inferior* of the sequence  $(a_n)_{n \in \mathbb{N}}$ . We denote:

$$S_* = \liminf_{n \rightarrow \infty} a_n$$

**Theorem 52.** If  $(a_n)_{n \in \mathbb{N}}$  is bounded by above by  $M$ ,  $H \neq \emptyset$ , then  $M$  is also an upper bound of  $H$ .

*Proof.* Assume  $\exists s \in H$  with  $s > M$ . Choose  $\varepsilon = s - M > 0$ . Because  $s$  is a limit point of  $(a_n)_{n \in \mathbb{N}}$  it holds that  $(s - \varepsilon, s + \varepsilon)$  contains infinitely many sequence elements. So for infinitely many indices  $n$  it holds that,

$$a_n > s - \varepsilon = s - (s - M) = M$$

This contradicts with  $M$  being the upper bound of the sequence.  $\square$

**Lemma 9.** Let  $(a_n)_{n \in \mathbb{N}}$  be bounded by above.  $a_n \in \mathbb{R}$ . Let  $H \neq \emptyset$  be defined as above. Then it holds that

$$S^* = \limsup_{n \rightarrow \infty} (a_n) = \max H$$

ie.  $S^*$  is a limit point itself of the sequence.

*Proof.* Show that  $S^*$  itself is a limit point of the sequence. Let  $\varepsilon > 0$ : Choose  $\xi \in H$  such that

$$\xi > S^* - \frac{\varepsilon}{2} \Rightarrow S^* - \xi = |S^* - \xi| < \frac{\varepsilon}{2}$$

Because  $\xi$  is a limit point of the sequence, in  $(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2})$  there are infinitely many sequence elements.

Let  $x \in (\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}) \Leftrightarrow |x - \xi| < \frac{\varepsilon}{2}$ . Then it holds that

$$|x - S^*| = |x - \xi + \xi - S^*| \leq \underbrace{|x - \xi|}_{< \frac{\varepsilon}{2}} + \underbrace{|\xi - S^*|}_{= S^* - \xi < \frac{\varepsilon}{2}}$$

$$\Rightarrow x \in (S^* - \varepsilon, S^* + \varepsilon)$$

Followingly,

$$\underbrace{\left(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}\right)}_{\text{contains infinitely many sequence elements}} \subseteq \underbrace{(S^* - \varepsilon, S^* + \varepsilon)}_{\text{contains infinitely many sequence elements}}.$$

$\square$

**Remark 14.** The analogous statement holds for the limes inferior.

$$S^* = \limsup_{n \rightarrow \infty} a_n \Leftrightarrow$$

1.  $S^* \in H$ , therefore  $S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .
2.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

*Proof.* Let  $S^* = \limsup_{n \rightarrow \infty} a_n$ .

1. The first property holds immediately.
2. We use an indirect proof.

$$\Rightarrow \exists \varepsilon > 0 : \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Therefore infinitely many sequence elements  $a_n$  exist with  $a_n \geq S^* + \varepsilon$ . We sort the sequence elements in a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$ . It holds that

$$S^* + \varepsilon \leq a_{n_k} \leq M$$

$(a_{n_k})_{k \in \mathbb{N}}$  is bounded and has a limit point  $S$  with  $S^* + \varepsilon < S \Rightarrow S > S^*$ .  $S$  is also a limit point of the original sequence  $(a_n)_{n \in \mathbb{N}}$  with  $S > S^* = \max H$ . This is a contradiction.

□

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This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

**Theorem 53** (Repetition of the theorem). Let  $(a_n)_{n \in \mathbb{N}}$  be bounded above and let  $(a_n)_{n \in \mathbb{N}}$  has a limit point. Then it holds that  $S^* = \limsup_{n \rightarrow \infty} a_n \Leftrightarrow$

1.  $S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .
2.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

Therefore above  $S^* + \varepsilon$  there are only finitely many sequence elements.

*Proof.* We prove the first direction  $\Rightarrow$ .

Let  $S^* = \limsup_{n \rightarrow \infty} a_n$ . Let  $\varepsilon > 0$  be arbitrary. The first property follows immediately. The second property needs to be shown.

*Proof by contradiction for the second property.*

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Then we build a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  from  $(a_n)_{n \in \mathbb{N}}$  with  $a_{n_k} \geq S^* + \varepsilon$ .

The subsequence is built inductively:

$n = 0$  then (because the second property holds negated) there exists  $x_n \geq 0 : a_{n_0} \geq S^* + \varepsilon$ .

$k \rightarrow k+1$  Let  $a_{n_0}, a_{n_1}, \dots, a_{n_k}$  be found with  $a_{n_l} \geq S^* + \varepsilon$  with  $l = 0, \dots, k$  and  $n_l < n_{l+1}$ . Let  $N = n_k + 1$ . Because the second property holds negated,  $n_{k+1} \geq N > n_k$  such that  $a_{n_{k+1}} \geq S^* + \varepsilon$ .

The subsequence's elements have the properties:

- $a_{n_k} \geq S^* + \varepsilon \quad \forall k \in \mathbb{N}$

- Because  $(a_n)_{n \in \mathbb{N}}$  is bounded above, also  $(a_{n_k})_{k \in \mathbb{N}}$  is bounded above

From the Bolzano-Weierstrass theorem it follows that  $(a_{n_k})_{k \in \mathbb{N}}$  has a limit point  $S \geq S^* + \varepsilon$ . Because every limit point of  $(a_{n_k})_{k \in \mathbb{N}}$  is a limit point of  $(a_n)_{n \in \mathbb{N}}$ , it holds that  $S$  is limit point of  $(a_n)_{n \in \mathbb{N}}$  and  $S > S^* + \varepsilon > S^*$ . This is a contradiction. □

We prove the second direction  $\Leftarrow$ .

Assume properties 1 and 2 hold. It remains to show that  $S^*$  is the largest limit point. Assume  $S > S^*$ . We need to show that  $S$  cannot be a limit point.

$$\varepsilon = \frac{S - S^*}{2} > 0 \Rightarrow 2\varepsilon = S - S^* \Rightarrow S^* + \varepsilon = S - \varepsilon$$

Because the second property holds, there exists some  $N \in \mathbb{N}$  such that  $\forall n \geq N \Rightarrow a_n < S^* + \varepsilon$ . Therefore only finitely many sequence elements are larger than  $S^* + \varepsilon = S - \varepsilon$ . Therefore at most finitely many sequence elements  $(S - \varepsilon, S + \varepsilon)$ . Followingly  $S$  is not a limit point. □

**Theorem 54** (Analogous result for limes inferior).

$$S_* = \liminf_{n \rightarrow \infty} a_n \Leftrightarrow$$

1.  $S_*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ .
2.  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n > S_* - \varepsilon$

**Theorem 55.** Let  $(a_n)_{n \in \mathbb{N}}$  be bounded above and  $(a_n)_{n \in \mathbb{N}}$  has a limit point.

- Let  $k \in \mathbb{N}$ . We define

$$A_k = \{a_k, a_{k+1}, a_{k+2}, \dots\} = \{a_j : j \geq k\}$$

- It holds that  $A_{k+1} \subseteq A_k$  and  $A_k$  is bounded above<sup>1</sup>.

---

<sup>1</sup>Obviously.

We define  $S_k = \sup A_k$ . Then  $(S_k)_{k \in \mathbb{N}}$  is a monotonically decreasing sequence in  $\mathbb{R}$  and  $(S_k)_{k \in \mathbb{N}}$  is bounded below. Therefore  $(S_k)_{k \in \mathbb{N}}$  is convergent and it holds that

$$\lim_{n \rightarrow \infty} S_k = \inf \{S_k : k \in \mathbb{N}\} = S^*$$

It turns out that

$$S^* = \limsup_{n \rightarrow \infty} a_n$$

We denote

$$\lim_{k \rightarrow \infty} \sup A_k = \lim_{k \rightarrow \infty} \sup \{a_j : j \geq k\} = \inf \{\sup A_k : k \in \mathbb{N}\} = \limsup_{n \rightarrow \infty} a_n$$

*Proof.*

$$A_{k+1} \subseteq A_k \Rightarrow \sup A_{k+1} \leq \sup A_k \Rightarrow S_{k+1} \leq S_k$$

$(S_k)_{k \in \mathbb{N}}$  is bounded below. Choose  $\xi \in H$  and  $\xi$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Then  $\xi - 1$  is a lower bound for  $(S_k)_{k \in \mathbb{N}}$  because infinitely many sequence elements are in  $(\xi - 1, \xi + 1)$ . Therefore,

$$\forall k \in \mathbb{N} : \exists n \geq k : a_n > \xi - 1 \Rightarrow S_k = \sup A_k > \xi - 1 \quad \checkmark$$

We know that  $(S_k)_{k \in \mathbb{N}}$  is convergent. Let  $S^* = \lim_{n \rightarrow \infty} S_k$ . We show the first property:

$S^*$  is limit point of  $(a_n)_{n \in \mathbb{N}}$ . Let  $\varepsilon > 0$  be given. We need to show that infinitely many sequence elements are in  $(S^* - \varepsilon, S^* + \varepsilon)$ .

Because  $\lim_{k \rightarrow \infty} S_k = S^*$  there exists some

$$N \in \mathbb{N} : k \geq N \Rightarrow \underbrace{|S_k - S^*|}_{-S^*} < \frac{\varepsilon}{2}.$$

We build a subsequence of  $(a_n)_{n \in \mathbb{N}}$  inductively, which is entirely inside  $(S^* - \varepsilon, S^* + \varepsilon)$ . Because  $S_N = \sup \{a_N, a_{N+1}, a_{N+2}, \dots\}$  exists, there exists  $a_j \geq S_N - \frac{\varepsilon}{2}$  with  $j \geq N$ .

$$\Rightarrow \underbrace{S_N - a_j}_{=|S_N - a_j|} \leq \frac{\varepsilon}{2}$$

$k = 0$  Choose  $n_0 = j \geq N$  ( $j$  from above), therefore it holds that

$$\begin{aligned} |S^* - a_{n_0}| &= |S^* - S_N + S_N - a_{n_0}| \leq |S^* - S_N| + |S_N - a_j| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore  $a_{n_0} \in (S^* - \varepsilon, S^* + \varepsilon)$ .

$k \rightarrow k + 1$  Consider  $a_{n_0}, a_{n_1}, \dots, a_{n_k}$  such that  $n_k > n_{k-1} > \dots > n_0 \geq N$  holds and  $|a_{n_l} - S^*| < \varepsilon$ . Because  $n_k + 1 > N$  holds

$$|S^* - S_{n_k+1}| < \frac{\varepsilon}{2}$$

because  $S_{n_k+1} = \sup \{a_{n_k+1}, a_{n_k+2}, \dots\}$ , exists  $j' \geq n_k + 1 > n_k$  such that

$$|S_{n_k+1} - a_{j'}| = S_{n_k+1} - a_{j'} < \frac{\varepsilon}{2}$$

Choose  $n_{k+1} = j'$  from above.

$$\begin{aligned} n_{k+1} \geq n_k + 1 > n_k \text{ and } |S^* - a_{n_{k+1}}| &= |S^* - S_{n_k+1} + S_{n_k+1} - a_{j'}| \\ &\leq |S^* - S_{n_k+1}| + |S_{n_k+1} - a_{j'}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Therefore we have found a subsequence  $(a_n)_{n \in \mathbb{N}}$  such that

$$\forall k \in \mathbb{N} : a_{n_k} \in (S^* - \varepsilon, S^* + \varepsilon)$$

$\Rightarrow S^*$  is limit point of the sequence.

We show that  $S^*$  is the largest limit point. Let  $S < S^*$ . We show that  $S$  is not a limit point.

Let  $\varepsilon = \frac{1}{2}(S - S^*) > 0$  such that  $S^* + \varepsilon = S - \varepsilon$ . Choose  $k \in \mathbb{N}$  such that  $S_k - S^* = |S_k - S^*| < \varepsilon$ .  $\forall n \geq K$  it holds that  $a_n \leq S_k < S^* + \varepsilon = S - \varepsilon$ . Therefore there are at most finitely many sequence elements in  $(S - \varepsilon, S + \varepsilon)$ . Therefore  $S$  is not a limit point.  $\square$

The analogous result for the limes inferior also holds and is given in the practicals.



## 10 Infinite series

**Definition 39.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of complex values. We define

- $S_0 = a_0$
- $S_1 = a_0 + a_1$
- $S_2 = a_0 + a_1 + a_2$
- $\dots$
- $S_n = a_0 + a_1 + \dots + a_n = \sum_{k=0}^n a_k$

We call  $(S_n)_{n \in \mathbb{N}}$  an *infinite series* with  $a_k$  sequence elements. We call  $S_n$  the *n-th partial sum* of the series. The series is called *convergent* if  $(S_n)_{n \in \mathbb{N}}$  is a convergent series in  $\mathbb{C}$ . For a convergent series instead of

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \underbrace{\sum_{k=0}^n a_k}_{=S_n}$$

we denote

$$S = \sum_{k=0}^{\infty} a_k$$

Actually a series must be denoted like a sequence with  $(S_n)_{n \in \mathbb{N}}$ . But we also say “let  $\sum_{k=0}^{\infty} a_k$  be a series” (but actually the sum of partial sums is meant). So this an ambiguous definition (per default always assume that the sum of partial sums is considered).

### 10.1 The geometric series

**Theorem 56.** Let  $q \in \mathbb{C}$  with  $q \neq 1$ . Consider  $\sum_{k=0}^{\infty} q^k$  hence  $S_n = \sum_{k=0}^n q^k$ . The limes of this series is given with  $\frac{1-q^{n+1}}{1-q}$  for  $|q| < 1$ .

*Proof.* We find a simple equation for  $S_n$ :

$$S_n - q \cdot S_n = (1 - q)S_n$$

$$\begin{aligned} & (1 + q + q^2 + \dots + q^n) - q(1 + q + q^2 + \dots + q^n) \\ &= (1 + q + q^2 + \dots + q^n) - (q + q^2 + \dots + q^n + q^{n+1}) \\ &= (1 - q^{n+1}) \end{aligned}$$

Therefore  $(1 - q) \cdot S_n = 1 - q^{n+1}$ . That is,

$$S_n = \frac{1 - q^{n+1}}{1 - q}$$

If  $|q| < 1$  it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} q^{n+1} &= q \lim_{n \rightarrow \infty} q^n = q \cdot 0 = 0 \\ \lim_{n \rightarrow \infty} S_n &= \frac{1 - \lim_{n \rightarrow \infty} q^{n+1}}{1 - q} = \frac{1}{1 - q} \\ \sum_{k=0}^{\infty} q^k &= \frac{1}{1 - q} \end{aligned}$$

If  $|q| > 1$  it holds that

$$|S_n| = \frac{1}{|1 - q|} \cdot |1 - q^{n+1}| \geq \frac{1}{|1 - q|} (|q^{n+1}| - 1)$$

This is the inversed triangle inequality.

$$= \frac{1}{|1 - q|} \left( \underbrace{|q|^{n+1}}_{\rightarrow \infty} - 1 \right)$$

Hence  $(S_n)_{n \in \mathbb{N}}$  is unbounded and therefore not convergent. □

**Theorem 57.** Let  $a_n = \frac{1}{n}$  hence  $\sum_{k=1}^{\infty} \frac{1}{k}$ .

$$\sum_{k=1}^{\infty} \frac{1}{n} \text{ is divergent}$$

*Proof.* Consider

$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots \\
 &> \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \dots \\
 &= \frac{1}{2} + 2 \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \dots \right) \\
 &= \frac{1}{2} + 2 \frac{1}{2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) \\
 &= \frac{1}{2} + \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \right) \\
 &= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{n}
 \end{aligned}$$

So we have,

$$\sum_{k=1}^{\infty} \frac{1}{n} > \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{n}$$

Let  $\sum_{k=1}^{\infty} \frac{1}{n} = H$ , then  $H > \frac{1}{2} + H$  must hold for some real value. This is impossible, so  $H$  cannot exist. Hence  $\sum_{k=1}^{\infty} \frac{1}{n}$  diverges.  $\square$

This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

## 10.2 Remark about notation of convergence

Let  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in \mathbb{C}$  be convergent with limes  $a$ .

Notation:  $a_n = \lim_{n \rightarrow \infty} a$

or even shorter:  $a_n \rightarrow a$  for  $n \rightarrow \infty$

$$a_n \rightarrow_{n \rightarrow \infty} a$$

We call  $(a_n)_{n \in \mathbb{N}}$  a zero sequence if  $(a_n)_{n \in \mathbb{N}}$  is convergent with  $\lim_{n \rightarrow \infty} a_n = 0$ .

## 10.3 Convergence tests

**Theorem 58.** Let  $(a_k)_{k \in \mathbb{N}}$  with  $a_k \in \mathbb{R}$  and  $a_k > 0$  be a *real* sequence. Then  $\sum_{k=0}^{\infty} a_k$  is convergent if and only if  $s_n = \sum_{k=0}^n a_k$  is a *bounded* sequence in  $\mathbb{R}$ .

*Proof.*  $\Rightarrow$  Let  $(s_n)_{n \in \mathbb{N}}$  be convergent in  $\mathbb{R}$ , then it holds that  $(s_n)_{n \in \mathbb{N}}$  is also bounded.

$\Leftarrow$   $(s_n)_{n \in \mathbb{N}}$  is bounded.

$$\begin{aligned}
 s_n - s_{n-1} &= (a_0 + \dots + a_{n-1} + a_n) - \\
 &\quad (a_0 + \dots + a_{n-1}) \\
 &= a_n \geq 0
 \end{aligned}$$

Hence,  $s_n \geq s_{n-1}$  so  $(s_n)_{n \in \mathbb{N}}$  is monotonically increasing and therefore also convergent.  $\square$

**Theorem 59.** Let  $\alpha \in \mathbb{Q}_+$ . Then it holds that: The series  $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$  is

**convergent** if  $\alpha > 1$

**divergent** if  $\alpha \leq 1$

**Case 1:  $\alpha > 1$**  We know: Map  $f(x) = x^\alpha$  is monotonically increasing.

$$x < y \Rightarrow x^\alpha < y^\alpha$$

Let  $S_{\alpha,n} = \sum_{k=1}^n \frac{1}{k^\alpha}$  be the  $n$ -th partial sum.  $n = 2^k - 1$ .

$$\begin{aligned}
 S_{\alpha,2^k-1} &= \underbrace{1}_{2^0 \text{ terms}} + \underbrace{\frac{1}{2^\alpha} + \frac{1}{3^\alpha}}_{2^1 \text{ terms}} + \underbrace{\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}}_{2^2 \text{ terms}} \\
 &\quad + \underbrace{\frac{1}{8^\alpha} + \dots + \frac{1}{15^\alpha}}_{2^3 \text{ terms}} + \dots + \underbrace{\frac{1}{(2^{k-1})^\alpha} + \dots + \frac{1}{(2^k-1)^\alpha}}_{2^{k-1} \text{ terms}} \\
 &< 1 + 2\frac{1}{2^\alpha} + 4\frac{1}{4^\alpha} + 8\frac{1}{8^\alpha} + \dots + 2^{k-1}\frac{1}{(2^{k-1})^\alpha} \\
 &= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \frac{1}{8^{\alpha-1}} + \dots + \frac{1}{(2^{k-1})^{\alpha-1}} \\
 &= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \left(\frac{1}{2^{\alpha-1}}\right)^3 + \dots \\
 &= \underbrace{\sum_{j=0}^{k-1} \left(\frac{1}{2^{\alpha-1}}\right)^j}_{\text{geometric series}} \\
 &= \frac{1 - \left(\frac{1}{2^{\alpha-1}}\right)^k}{1 - \frac{1}{2^{\alpha-1}}} \\
 &< \frac{1}{1 - \frac{1}{2^{\alpha-1}}} = \frac{2^{\alpha-1}}{2^{\alpha-1} - 1}
 \end{aligned}$$

Therefore  $(S_{\alpha,2^k-1})$  is bounded. Let  $n \in \mathbb{N}$  be arbitrary and choose a sufficiently large  $K$  such that  $2^K > n + 1$ . Therefore  $2^k - 1 > n$ . Because  $\frac{1}{j^\alpha} > 0$  for all  $j \geq 1$ , it holds that  $S_{2^k-1} > S_n$ . At the same time  $S_{2^k-1} < \frac{2^{\alpha-1}}{2^{\alpha-1}-1}$ . So  $(S_n)_{n \in \mathbb{N}}$  is bounded. Hence  $\sum_{k=1}^\infty \frac{1}{k^\alpha}$  is convergent.

**Case 2:**  $\alpha \leq 1$  Then it holds that  $k^\alpha \leq k$  and therefore  $\frac{1}{k^\alpha} \geq \frac{1}{k}$ . Because  $S_{\alpha,n} \geq S_{1,n}$  and because  $S_{1,n}$  is unbounded, it holds that  $(S_{\alpha,n})_{n \in \mathbb{N}}$  is unbounded and followingly  $\sum_{k=0}^\infty \frac{1}{k^\alpha}$  is divergent.

**Remark 15.**  $\alpha \in \mathbb{Q}_+$  can be replaced by  $\alpha \in \mathbb{R}_+$ . It is even possible to choose  $\alpha \in \mathbb{C}$ . Then we can define  $\zeta : M \subseteq \mathbb{C} \rightarrow \mathbb{C}$  with  $\xi(z) = \sum_{k=1}^\infty \frac{1}{k^z}$ . This is Riemann's Zeta function.

**Definition 40.** Let  $(a_n)_{n \in \mathbb{N}}$  be a real sequence with  $a_n \geq 0$ . Then we call  $(\alpha_n)_{n \in \mathbb{N}}$  with  $\alpha_n = (-1)^n a_n$ , or equivalently  $\alpha_n = (-1)^{n+1} a_n$ , an .

A series of structure  $\sum_{k=0}^\infty (-1)^k a_k$  with  $a_k \geq 0$  is called *alternating series*.

## 10.4 Leibniz convergence criterion

Gottfried Wilhelm Leibniz (1646–1716)

**Theorem 60** (Leibniz convergence criterion). Let  $(a_n)_{n \in \mathbb{N}}$  be a real, monotonically zero sequence with  $a_n \geq a_{n+1} \geq 0 \quad \forall n \in \mathbb{N}$ . Then  $\sum_{k=0}^\infty (-1)^k a_k$  is convergent.

*Proof.*

$$\begin{aligned}
 S_{2n-1} &= \sum_{k=0}^{2n-1} (-1)^k a_k \\
 S_{2n} &= \sum_{k=0}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} \\
 &= S_{2n-1} + a_{2n} \\
 S_{2n+1} &= S_{2n-1} + \underbrace{a_{2n} - a_{2n-1}}_{\geq 0} \\
 S_{2n+2} &= \underbrace{S_{2n-1} + a_{2n}}_{S_{2n}} - \underbrace{a_{2n+1} + a_{2n+2}}_{=-(a_{2n+1}-a_{2n+2}) \geq 0}
 \end{aligned}$$

Therefore it holds that  $S_{2n+1} \geq S_{2n-1}$ ,  $S_{2n+2} \leq S_{2n}$  and  $S_{2n} \geq S_{2n-1}$ .

$(S_{2n})_{n \in \mathbb{N}}$  is monotonically decreasing.  $(S_{2n+1})_{n \in \mathbb{N}}$  is monotonically increasing.

It holds that:  $\forall m, n \in \mathbb{N} : S_{2n} \geq S_{2m-1}$ .

*Proof. Case 1: m > n*

$$S_{2m+1} \leq S_{2n} \leq S_{2n} \quad \checkmark$$

**Case 2:  $m \leq n$**

$$S_{2m+1} \leq S_{2n+1} \underbrace{\leq}_{\alpha < 1} S_{2n}$$

So  $(S_{2n})_{n \in \mathbb{N}}$  is monotonically decreasing and bounded by below (for example by  $S_1$ ). Therefore  $S_{2n} \rightarrow S^*$  for  $n \rightarrow \infty$ .  $(S_{2n+1})$  is monotonically increasing and bounded by above by  $S_*$ :

$$S_{2n+1} \rightarrow S_* \text{ for } n \rightarrow \infty$$

It holds that  $S_* \leq S^*$  because  $S_{2n+1} \leq S_{2n}$ .  $\square$

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This lecture took place on 10th of December 2015 with lecturer Wolfgang Ring.

Given  $S_* \leq S^*$ , we show that  $S^* = S_*$  and we prove that  $\forall \varepsilon > 0 : S^* - S_* < \varepsilon$ .

Let  $\varepsilon > 0$  and choose  $N$  sufficiently large, such that  $a_{2N} < \varepsilon$ .

$$a_{2N} = S_{2N} - S_{2N-1} > S^* - S_*$$

$$a_{2N} < \varepsilon$$

So  $\forall \varepsilon > 0$ , it holds that

$$S^* - S_* = |S^* - S_*| < \varepsilon$$

$$\Rightarrow S^* = S_* = S$$

So it holds that,

$$\lim_{n \rightarrow \infty} S_n = S^* = S_* = S$$

and the series converges.  $\square$

**Example 18.**

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{k} \text{ is convergent}$$

## 10.5 Series in $\mathbb{C}$ and absolute convergence

**Theorem 61** (Cauchy convergence criterion). The complex series  $\sum_{k=0}^{\infty} a_k$  is convergent if and only if the partial sums  $(s_n)_{n \in \mathbb{N}}$  are a Cauchy sequence in  $\mathbb{C}$ .

**Remark 16.** Therefore

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n, m > N$$

$$\Rightarrow |S_n - S_m| < \varepsilon$$

Therefore without loss of generality,  $n \geq m$ .

$$S_n - S_m = \sum_{k=0}^n a_k - \sum_{k=0}^m a_k = \sum_{k=m+1}^n a_k$$

Hence  $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N$ .

$$\left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

Equivalently, with  $m+1 = n$  and  $n - m = l$ .

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \text{ and } l \in \mathbb{N}$$

$$\left| \sum_{k=0}^l a_{n+k} \right| < \varepsilon$$

*Proof by  $(S_n)_{n \in \mathbb{N}}$  being convergent.*

$$(S_n)_{n \in \mathbb{N}} \Leftrightarrow \text{Cauchy sequence}$$

**Lemma 10.** If  $\sum_{k=0}^{\infty} a_n$  is convergent in  $\mathbb{C}$ , then  $(a_n)_{n \in \mathbb{N}}$  is a zero sequence.

*Proof.* Follows directly from the Cauchy criterion for  $l = 0$ .  $\square$

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N : \underbrace{\left| \sum_{k=0}^0 a_{n+k} \right|}_{|a_n|} < \varepsilon \quad \text{hence } a_n \rightarrow 0$$

$\square$

□ *Proof.* 1. We need to show that  $\sum_{n=0}^{\infty} \underbrace{|a_k|}_{\geq 0}$  is convergent. It suffices to show that

**Definition 41.** The complex series  $\sum_{k=0}^{\infty} a_k$  is called *absolute convergent* if the real series  $\sum_{k=0}^{\infty} |a_k|$  is convergent.

**Example 19.**

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n^2} \quad \text{absolute convergent}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{n} \quad \text{absolute convergent (Leibniz)}$$

$$\sum_{k=0}^n |a_k| = \sigma_n$$

$(\sigma_n)_{n \in \mathbb{N}}$  is bounded. Let  $n \geq N$ .

**Lemma 11.** Let  $\sum_{k=0}^{\infty} a_k$  be absolute convergent. Then  $\sum_{k=0}^{\infty} a_k$  is also convergent.

*Proof.* Let  $\sum_{k=0}^{\infty} |a_k|$  be convergent. From the Cauchy criterion it follows that,

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N :$$

$$\left| \sum_{k=m+1}^n |a_k| \right| = \sum_{k=m+1}^n |a_k| \geq \left| \sum_{k=m+1}^n a_k \right| < \varepsilon$$

$\Rightarrow \sum_{k=0}^{\infty} a_k$  is convergent according to Cauchy criterion. □

$$\begin{aligned} \sigma_n &= \sum_{k=0}^n |a_k| \\ &= |a_0| + |a_1| + \cdots + |a_{N-1}| + \sum_{k=N}^n |a_k| \\ &\leq |a_0| + \cdots + |a_{N-1}| + \underbrace{\sum_{k=N}^{\infty} |b_k|}_{s \geq 0} \\ &\quad \underbrace{\hspace{10em}}_M \end{aligned}$$

## 10.6 Direct comparison test

**Theorem 62** (Direct comparison test (dt. Majorantenkriterium)).

1. Let  $\sum_{k=0}^{\infty} a_k$  and  $\sum_{k=0}^{\infty} b_k$  be complex series. Let  $\sum_{k=0}^{\infty} b_k$  be absolute convergent and  $\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_k| \leq |b_k|$ .

Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.  $\sum_{k=0}^{\infty} b_k$  is called *majorant* of  $\sum_{k=0}^{\infty} a_k$ .

2. (dt. Minorantenkriterium) Let  $\sum_{k=0}^{\infty} a_k$  and  $\sum_{k=0}^{\infty} b_k$  be complex series. Let  $\sum_{k=0}^{\infty} a_k$  be divergent. Assume  $\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_k| \leq |b_k|$ . Then also  $\sum_{k=0}^{\infty} b_k$  is divergent.  $\sum_{k=0}^{\infty} a_k$  is *minorant* of  $\sum_{k=0}^{\infty} b_k$ .

Therefore  $(\sigma_n)_{n \in \mathbb{N}}$  is bounded and therefore  $\sum_{n=0}^{\infty} a_n$  is absolute convergent.

2. Let  $\sum_{k=0}^{\infty} a_k$  be divergent. Then also  $\sum_{k=0}^{\infty} |a_k|$  is divergent. Otherwise  $\sum_{k=0}^{\infty} a_k$  is absolute convergent and therefore convergent.

$$\Rightarrow \sigma_n = \sum_{k=0}^n |a_k|$$

$(\sigma_n)_{n \in \mathbb{N}}$  is unbounded. Because

$$\begin{aligned} \sum_{k=0}^n |b_k| &= |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^n |b_k| \\ &\geq |b_0| + \cdots + |b_{N-1}| + \sum_{k=N}^N |a_k| \\ &= \underbrace{|b_0| + \cdots + |b_{N-1}| - (|a_0| + \cdots + |N-1|)}_z + \sum_{k=0}^n |a_k| \\ &= z + \sigma_n \end{aligned}$$

$z + \sigma_n$  is unbounded. Therefore  $\sum_{k=0}^{\infty} |b_k|$  is not convergent. Therefore  $\sum_{k=0}^{\infty} b_k$  is not absolute convergent.

□

## 10.7 Ratio test

**Theorem 63** (Ratio test (dt. Quotientenkriterium)). 1. Let  $\sum_{k=0}^{\infty} a_k$  be a complex series. Assume  $\exists q \in [0, 1)$  with  $(0 \leq q < 1)$  and  $N \in \mathbb{N}$  such that

- $\frac{|a_{n+1}|}{|a_n|} < q \quad \forall n \geq N$  with  $|a_n| \neq 0$ , or “Ratio test”
- $\sqrt[n]{|a_n|} < q \quad \forall n \geq N$  “Root test”

Then the series  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

2. Assume there exists  $q > 1$  and  $N \in \mathbb{N}$  such that

- $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$
- $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$

Then  $\sum_{k=0}^{\infty} a_k$  is divergent.

*Proof.* This follows from the direct comparison criterion. Compare with geometric series  $\sum_{k=0}^{\infty} q^k$ .

1. Assume the second statement of the ratio test holds. Therefore  $\forall n \geq N$  it holds that  $\sqrt[n]{|a_n|} \leq q \Leftrightarrow |a_n| \leq q^n$ .  $|a_n| \leq q^n$ . Due to the direct comparison test,  $\sum_{k=0}^{\infty} q^k \checkmark$ .

Assume the first statement of the ratio test does not hold.

$$\frac{|a_{n+1}|}{|a_n|} \leq q (< 1)$$

Then it holds that  $\forall k \in \mathbb{N}$ :

$$|a_{k+N}| \leq |a_N| \cdot q^k$$

Proof by induction over  $k$ :

**k = 0**

$$|a_N| \leq |a_N| \cdot q^0 \quad \checkmark$$

**k → k + 1** Assume  $|a_{N+k}| \leq |a_N| \cdot q^k$ . Because

$$\frac{|a_{N+k+1}|}{|a_{N+k}|} \leq q \Rightarrow |a_{N+k+1}| \leq q |a_{N+k}| \leq q \cdot |a_N| \cdot q^k = |a_N| q^{k+1} \quad \checkmark$$

We set

$$b_k = \begin{cases} 0 & \text{for } k = 0, 1, 2, \dots, N-1 \\ |a_N| \cdot q^{k-N} & \text{for } k \geq N \end{cases}$$

$$\sum_{k=0}^{\infty} b_k = 0 + 0 + 0 + \cdots + 0 + |a_N| \cdot q^0 + |a_N| \cdot q^1 + |a_N| q^2 + \dots$$

$$= |a_N| \sum_{j=0}^{\infty} q^j \text{ is absolute convergent}$$

$\sum_{k=0}^{\infty} b_k$  is an absolute convergent majorant for  $\sum_{k=0}^{\infty} a_k$ .

$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ is convergent}$$

2. Assume the second statement (square root test) holds:  $\sqrt[n]{|a_n|} \geq q$  or equivalently  $\underbrace{|a_n|}_{\text{unbounded}} \geq \underbrace{q^n}_{\text{unbounded}}$ . Therefore  $(a_n)_{n \in \mathbb{N}}$  is no zero sequence. Therefore  $\sum_{k=1}^{\infty} a_k$  is divergent.

Assume the first statement holds.

$$\Rightarrow |a_{N+k}| \geq |a_N| \cdot q^k$$

Because  $|a_N| \cdot q^k$  is unbounded,  $|a_{N+k}|$  is unbounded.  $(a_k)_{k \in \mathbb{N}}$  are not zero sequences.

**Remark 17.** Assume  $\frac{|a_{n+1}|}{|a_n|}$  is bounded and  $q = \limsup_{n \rightarrow \infty} \left( \frac{|a_{n+1}|}{|a_n|} \right) < 1$ . Let  $2\varepsilon = 1 - q > 0$ .

$$\begin{aligned} \Rightarrow \exists N \in \mathbb{N} : n \geq N : \frac{|a_{n+1}|}{|a_n|} &< q + \varepsilon \\ &= q + \frac{1}{2}(1 - q) = \frac{1}{2}(1 + q) = 1 - \varepsilon < 1 \end{aligned}$$

Due to the ratio test, the series  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

**Lemma 12.** Let  $\sum_{k=0}^{\infty} a_k$  be a complex series with  $a_k \neq 0 \forall k \in \mathbb{N}$  and if it holds that

$$q = \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = q$$

This lecture took place on 11th of December 2015 with lecturer Wolfgang Ring.

## 10.8 Revision

So  $\sum_{k=0}^{\infty}$  is absolute convergent if  $\exists q \in [0, 1) \exists N \in \mathbb{N}$ .

$$\bullet \frac{|a_{n+1}|}{|a_n|} \leq q \quad \forall n \geq N$$

$$\bullet \sqrt[n]{|a_n|} \leq q \quad \forall n \geq N$$

If  $q > 1$  and either  $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$  or  $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$ , then this series is convergent.

**Corollary 10.** Let  $q = \limsup_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$ , then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent. Let  $q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ . Then  $\sum_{k=0}^{\infty} a_k$  is absolute convergent.

Let  $q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ . Then  $\sum_{k=0}^{\infty} a_k$  is divergent.

*Proof.* Let  $q = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ .

$$2\varepsilon = 1 - q > 0$$

Then there exists some  $N \in \mathbb{N} : n \geq N$

$$\Rightarrow \sqrt[n]{|a_n|} \leq q + \varepsilon = 1 - \varepsilon < 1$$

Is absolute convergent according to the square root theorem.

We also need to show divergence: Let  $q > 1$  be limit point of  $\sqrt[n]{|a_n|}$ . So there exists some subsequence  $\left( \sqrt[n_k]{|a_{n_k}|} \right)_{k \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \sqrt[n_k]{|a_{n_k}|} = q > 1 \Rightarrow \varepsilon = \frac{1}{2}(q - 1) > 0$ .

$$\sqrt[n_k]{|a_{n_k}|} > q - \varepsilon \quad \forall k \geq K$$

$$\Rightarrow |a_{n_k}| > (q - \varepsilon)^{n_k} = (1 + \varepsilon)^{n_k} > 1$$

$$\Rightarrow (|a_{n_k}|)_{k \in \mathbb{N}} \text{ is not a zero sequence}$$

$$\Rightarrow (|a_n|)_{n \in \mathbb{N}} \text{ is also not a zero sequence}$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ is divergent}$$

□

**Example 20** (Binomial series). Let  $n \in \mathbb{N}$  and  $k \in \{0, 1, 2, \dots, n\}$ .

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1 \cdot 2 \cdot \dots \cdot (n-k)(n-k+1) \cdot \dots \cdot n}{k! \cdot 1 \cdot 2 \cdot \dots \cdot (n-k)}$$

$$= \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!}$$

Let  $s \in \mathbb{C}$ . We define the binomial coefficient  $\binom{s}{k} = \frac{s \cdot (s-1) \cdot (s-2) \cdot \dots \cdot (s-k+1)}{k!}$ . Also let  $\binom{s}{0} = 1$  and  $\binom{s}{1} = s$ . Let  $k > n$  and  $n \in \mathbb{N}$ , then

$$\binom{n}{k} = \frac{n(n-1) \cdot \dots \cdot \overbrace{(n-n)}^0 \cdot \dots \cdot (n-k+1)}{k!} = 0$$

**Example 21.** We define the binomial series for  $s, z \in \mathbb{C}$  with

$$B_S(z) = \sum_{k=0}^{\infty} \underbrace{\binom{s}{k}}_{:=a_k} z^k$$

What about convergence? Well,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left| \frac{s \cdot (s-1) \cdot \dots \cdot (s-(k+1)+1)}{(k+1)!} z^{k+1} \right|}{\left| \frac{s(s-1)(s-2) \cdot \dots \cdot (s-k+1)}{k!} z^k \right|}$$

$$\frac{|a_{k+1}|}{|a_k|} = \left| \frac{(s-k)z}{k+1} \right| = \left| \frac{\left( \overbrace{\frac{s}{k}}^{\rightarrow 0} - 1 \right) \cdot z}{1 + \underbrace{\frac{1}{k}}_{\rightarrow 0}} \right| \rightarrow |z|$$

Therefore  $B_S(z)$  is convergent for  $|z| < 1$  and divergent for  $|z| > 1$ . So geometrically, it is convergent within a circle of radius 1 or  $i$  (at center  $(0,0)$ ) and divergent outside.

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

We know, for  $s \in \mathbb{N}$ :

$$B_S(z) = \sum_{k=0}^{\infty} \binom{n}{k} z^k = \sum_{k=0}^n \binom{n}{k} z^k = (1+z)^n$$

Remind that  $\binom{n}{k} = 0$  for  $k > n$ .

Therefore

$$(1+z)^s := \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

This is the definition of a power function i.e.

$$z = \xi - 1 \quad 1+z = \xi$$

$$\xi^S = \sum_{k=0}^{\infty} \binom{s}{k} (\xi-1)^k$$

is convergent for  $|\xi-1| < 1$ .

Geometrically, this is a circle of radius 1 or  $i$  (at center  $(1,0)$ ).

## 11 Power series

**Definition 42.** A power series (in one variable) is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

So we have one free variable. Coefficients of the series contain a variable.

- In  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  all summands are fixed.
- However  $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$  with  $|z| < 1$  is variable with the variable  $z$ .

**Example 22.**

$$f : B(0,1) \rightarrow \mathbb{C}$$

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

Mapping:

$$B_S : B(0,1) \rightarrow \mathbb{C}$$



$$\varepsilon(z) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$$

Let  $z \in \mathbb{C}$  arbitrary.

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left| \frac{z^{k+1}}{(k+1)!} \right|}{\left| \frac{z^k}{k!} \right|} = \left| \frac{z}{k+1} \right| \rightarrow 0$$

$\Rightarrow \varepsilon(z)$  is convergent for all  $z \in \mathbb{C}$ .

$$\varepsilon : \mathbb{C} \rightarrow \mathbb{C}$$

**Corollary 11.** Using series sum we can define mappings (functions).

**Definition 43.** Let  $(a_n)_{n \in \mathbb{N}}$  be a complex sequence and let  $z \in \mathbb{C}$ . Then  $\sum_{k=0}^{\infty} a_k \cdot z^k$  is called *power series with coefficient sequence*  $(a_k)_{k \in \mathbb{N}}$ .

Its convergence property depends on  $z$ . For  $z = 0$  every power series is convergent.

$$\sum_{k=0}^{\infty} a_k \cdot 0^k$$

Because we define  $0^0 := 1$  here, the constant series  $a_0$  is given.

**Lemma 13.** Let  $\sum_{k=0}^{\infty} a_k z^k$  is a power series in  $\mathbb{C}$  and  $z_0 \in \mathbb{C} \setminus \{0\}$  such that  $\sum_{k=0}^{\infty} a_k z_0^k$  is convergent. Then the power series is absolute convergent for all  $z$  with  $|z| < |z_0|$ .

Geometrically, if the series is convergent at one point  $z_0$  at the circle, it is convergent in all points of the circle.

*Proof.* Direct comparison test: Because  $\sum_{k=0}^{\infty} a_k z_0^k$  is convergent, it holds that  $\lim_{k \rightarrow \infty} a_k z_0^k = 0$ . Therefore  $(a_k z_0^k)_{n \in \mathbb{N}}$  is also bounded and there exists some  $m \geq 0$  such that  $|a_k z_0^k| \leq m \quad \forall k \in \mathbb{N}$ .

Let  $|z| < |z_0|$ . Then,

$$|a_k z^k| = \left| a_k \frac{z^k}{z_0^k} \cdot z_0^k \right| = |a_k z_0^k| = \underbrace{|a_k z_0^k|}_{\leq m} \underbrace{\left| \frac{z}{z_0} \right|^k}_{:=q} \leq m \cdot q^k$$

with  $0 \leq q < 1$ . Therefore  $\sum_{k=0}^{\infty} a_k z^k$  is convergent because of the direct comparison test with  $\sum_{k=0}^{\infty} m \cdot q^k = m \cdot \sum_{k=0}^{\infty} q^k$ .  $\square$

**Definition 44.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series in  $\mathbb{C}$ . We define

$$\rho(P) = \sup \{r \geq 0, r \in \mathbb{R} : P(r) \text{ is convergent}\}$$

$\rho(P)$  is called convergence radius of  $P$ . If  $\{r \geq 0 : P(r) \text{ is convergent}\}$  is unbounded, then we define  $\rho(P) = \infty$ .

**Lemma 14.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series in  $\mathbb{C}$  and let  $\rho(P)$  be its convergence radius of  $P$ . Then  $P(z)$  is absolute convergent for all  $z \in \mathbb{C}$  with  $|z| < \rho(P)$ .

*Proof.* For  $\rho(P) = 0$ , nothing has to be shown.

Let  $\rho(P) > 0$  and  $|z| < \rho(P)$ , then  $\varepsilon := \rho(P) - |z|$ . Because  $\rho(P) = \sup \{r \geq 0 : P(r) \text{ is convergent}\}$ , there exists some  $r \in \mathbb{R}$  such that  $\rho(P) - \varepsilon < r \leq \rho(P)$  and  $P(r)$  is convergent.  $\rho(P) - \varepsilon = |z| < r$ . So  $P(z)$  is absolute convergent according to Lemma 13.

Geometrically,  $\rho(P)$  is a circle and its interior is convergent. On the outside the power series is divergent. The convergence property at the circle itself is unknown (not generally uniform).  $\square$

**Lemma 15.** Let  $z \in \mathbb{C}$ ,  $P$  is a power series and  $|z| > \rho(P)$ . Then  $\sum_{k=0}^{\infty} a_k z^k$  is divergent for this point.

*Proof.* Proof by contradiction. Assume  $P(z)$  is convergent and  $|z| > \rho(P)$ . Let  $\varepsilon = 2(|z| - \rho(P))$ . Then  $\rho(P) + \varepsilon < |z|$  with  $\rho(P) + \varepsilon > \rho(P)$ . From the previous lemma it follows that  $P(\rho(P) + \varepsilon)$  is convergent. But this contradicts with  $\rho(P) = \sup \{r \geq 0 : P(r) \text{ is convergent}\}$ .  $\square$

**Remark 18.**  $B(0, \rho(P))$  is called *convergence circle* of  $P$ .

**Theorem 64** (Formulas to compute  $\rho(P)$ ). Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series. Then it holds in every case that,

- $\rho(P) = \frac{1}{L}$  with  $L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$  (for  $L = \infty$  if  $\left(\sqrt[n]{|a_n|}\right)_{n \in \mathbb{N}}$  is unbounded and  $\frac{1}{\infty} := 0$ ) (Cauchy & Hadamard)

- If  $q := \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then the convergence disk of this power series is  $\frac{1}{q}$ :

$$\rho(P) = \frac{1}{q}$$

with  $\frac{1}{0} := \infty$  and  $\frac{1}{\infty} := 0$ .

This lecture took place on 16th of December 2015 with lecturer Wolfgang Ring.

## 11.1 Equations for $\rho(P)$

**Theorem 65.**

$$P(z) = \sum_{k=0}^{\infty} a_k z^k$$

$$L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$\rho(P) = \frac{1}{L} \quad \text{“Cauchy-Hadamard theorem”}$$

If  $q = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$  exists, then it holds that  $\rho(P) = \frac{1}{q}$  (Euler).

*Proof.* 1. Let  $z \neq 0$  and let  $L^* = \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k z^k|} = \limsup_{k \rightarrow \infty} |z| \sqrt[k]{|a_k|} = |z| \cdot k$ . Due to the square root criterion it holds that:

- If  $|z| L < 1$ , then  $\sum_{k=0}^{\infty} a_k z^k$  is absolute convergent.
- If  $|z| L > 1$ , then  $\sum_{k=0}^{\infty} a_k z^k$  is absolute divergent.

Therefore for  $|z| < \frac{1}{L}$ ,  $P$  is convergent. For  $|z| > \frac{1}{L}$ ,  $P$  is divergent.

$$\Rightarrow \rho(P) = \frac{1}{L}$$

2. Ratio test: Assume  $q = \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$  exists. The ratio test for  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  gives us

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1} \cdot z^{k+1}}{a_k \cdot z^k} \right| = |z| \lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = |z| \cdot q$$

Therefore  $P$  is convergent, if  $|z| \cdot q < 1 \Leftrightarrow |z| < \frac{1}{q}$ . And  $P$  is divergent, if  $|z| \cdot q > 1 \Leftrightarrow |z| > \frac{1}{q}$ . □

**Remark 19.** What happens for  $|z| = \rho(P)$ ? We need a different approach for convergence/divergence.

1.

$$G(z) = \sum_{k=0}^{\infty} z^k \quad L = \limsup_{k \rightarrow \infty} \sqrt[k]{|1|}$$

$$\rho(G) = 1$$

2.

$$H(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^k$$

$$q = \lim_{k \rightarrow \infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{k}{k+1} \right| = \lim_{k \rightarrow \infty} \left| \frac{1}{1 + \underbrace{\frac{1}{k}}_{\rightarrow 0}} \right|$$

3.

$$Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$$

$$q = \lim_{n \rightarrow \infty} \left( \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right) = 1$$

$$\rho(Q) = 1$$

**Case 1** Let  $z \in \mathbb{C}$  with  $|z| = 1$ . Then  $G(z)$  is not convergent because  $(z^k)_{k \in \mathbb{Z}}$  is not a zero sequence because  $|z^k| = |z|^k = 1$ . So geometrically, the circle itself of the convergence circle is divergent.

**Case 2** Consider  $H(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$ .  $H$  is divergent for  $z = 1$ . For  $z = -1$ ,  $H(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$  is convergent according to the Leibniz criterion.

**Case 3** For  $Q(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$  and let  $|z| = 1$ . Then it holds that  $\left| \frac{z^k}{k^2} \right| \leq \frac{1}{k^2}$ .  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  is absolute convergent. The direct comparison test tells us that  $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$  is absolute convergent.

## 12 Functions and their regularity properties

Recall: Let  $D \subseteq \mathbb{C}$  (or  $\subseteq \mathbb{R}$ ). A mapping  $f : D \Rightarrow \mathbb{C}$  (or  $f : D \rightarrow \mathbb{R}$ ) is a function. Depending on the domain, we call the function *complex* or *real*.

### 12.1 Fundamental topological terminology

Recall:  $B(z, r) = \{\zeta \in \mathbb{C} : |z - \zeta| < r\}$ . Geometrically this corresponds to an open circular disk with center  $z$  and radius  $r$ .

Analogously,  $B(x, r) = \{y \in \mathbb{R} : |y - x| < r\} = (x - r, x + r)$  in  $\mathbb{R}$ .

**Definition 45.** Let  $U \subseteq \mathbb{C}$  ( $U \subseteq \mathbb{R}$ ) and  $z_0 \in U$ . Then  $U$  is called surrounding of  $z_0$  in  $\mathbb{C}$ , if  $\exists r > 0 : B(z_0, r) \subseteq U$ .

- $O \subseteq \mathbb{C}$  if called *open set* if  $\forall z \in O$ :  $O$  is surrounding of  $z$ .

$$\Leftrightarrow \forall z \in O : \exists r = r(z) : B(z, r) \subseteq O$$

- $A \subseteq \mathbb{C}$  is called *closed set*, if  $\mathbb{C} \setminus A$  is an open set.

**Theorem 66.** 1. Let  $I$  be a set and  $\forall i \in I$  let  $O_i$  be an open set in  $\mathbb{C}$ . Then  $\bigcup_{i \in I} O_i = \{z \in \mathbb{C} : \exists i \in I : z \in O_i\}$  is an open set.

2. Let  $O_1, O_2, \dots, O_n$  be open sets. Then  $\bigcap_{k=1}^n O_k = O_1 \cap O_2 \cap \dots \cap O_n$  is open.
3. If  $\emptyset$  is open, then  $\mathbb{C}$  is open.
4.  $I$  is a set  $\forall i \in I$ . Let  $A_i$  be closed. Then  $\bigcap_{i \in I} A_i$  is closed.
5. Let  $A_1, A_2, \dots, A_n$  be closed, then  $A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{k=1}^n A_k$  is closed.

*Proof.* 1. Let  $z \in \bigcup_{i \in I} O_i$ . Show that  $\exists r > 0 : B(z, r) \subseteq \bigcup_{i \in I} O_i$ .

Let  $z \in \bigcup_{i \in I} O_i$ , therefore  $\exists j \in I : z \in O_j$ . Because  $O_i$  is open,  $\exists r > 0 : B(z, r) \subseteq O_j \subseteq \bigcup_{j \in I} O_j$ .

2. Let  $O_1, \dots, O_n$  and let  $z \in O_k$ . Hence  $\forall k \in \{1, \dots, n\} : z \in O_k$  with  $O_k$  as open set.  $\exists r_k > 0 : B(z, r_k) \subseteq O_k$ . Let  $r = \min \{r_1, r_2, \dots, r_n\} > 0$ . Then it holds that  $B(z, r) = \{\zeta \in \mathbb{C} : |\zeta - z| < r\} \subseteq \{\zeta \in \mathbb{C} : |\zeta - z| < r_k\} = B(z, r_k) \subseteq O_k$  because  $r \leq r_k$ .

So  $\forall k \in \{1, \dots, n\} : B(z, r) \subseteq O_k$ . Otherwise  $B(z, r) \subseteq \bigcap_{k=1}^n O_k \Rightarrow \bigcap_{k=1}^n O_k$  is open.

3. Let  $O = \emptyset$ . Then it holds that  $\forall z \in \emptyset : B(z, 1) \subseteq \emptyset$ . So  $\emptyset$  is open. For  $O = \mathbb{C}$  it holds that  $\forall z \in \mathbb{C} : B(z, 1) \subseteq \mathbb{C}$ , therefore  $\mathbb{C}$  is open.
4. Let  $A_i$  be closed and  $A = \bigcap_{i \in I} A_i$  and  $O = \mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin \bigcap_{i \in I} A_i\}$ .  $O = \mathbb{C} \setminus A = \{z \in \mathbb{C} : z \notin \bigcap_{i \in I} A_i\} = \bigcup_{j \in I} \{z \in \mathbb{C} : z \notin A_j\} = \bigcup_{j \in I} (\mathbb{C} \setminus A_j) \rightarrow$  open. So  $\mathbb{C} \setminus A$  is open, therefore  $A$  is closed.

$$\mathbb{C} \setminus \bigcap_{j \in I} A_j = \bigcup_{j \in I} (\mathbb{C} \setminus A_j)$$

The last statement was proven by DeMorgan.

5. Let  $A = \bigcup_{k=1}^n A_k$ .

$$\mathbb{C} \setminus A = \mathbb{C} \setminus \bigcup_{k=1}^n A_k = \bigcap_{k=1}^n (\mathbb{C} \setminus A_k)$$

where  $\mathbb{C} \setminus A_n$  is an open set. So  $A$  is closed. □

**Theorem 67.**  $A \subseteq \mathbb{C}$  is closed  $\Leftrightarrow \forall (a_n)_{n \in \mathbb{N}}$  with  $a_n \in A$  and  $(a_n)_{n \in \mathbb{N}}$  is convergent with limes  $a \in \mathbb{C}$ , then  $a \in A$ .

*Proof.*  $\Rightarrow$  Let  $A$  be closed ( $\mathbb{C} \setminus A$  is open) and  $(a_n)_{n \in \mathbb{N}}$  is a convergent sequence with  $\lim_{n \rightarrow \infty} a_n = a$ . Show that  $a \in A$ .

Proove by contradiction: Assume  $a \notin A$ , so  $a \in \mathbb{C} \setminus A$ .

Because  $\mathbb{C} \setminus A$  is an open set,  $\exists r > 0 : B(a, r) \subseteq \mathbb{C} \setminus A$ . And  $B(a, r) \cap A = \emptyset$  so it holds that  $\forall n \in \mathbb{N} : a_n \notin B(a, r)$  with  $a_n \in A$ . So it holds that

$\forall n \in \mathbb{N} : |a_n - a| \geq r > 0$ . This is contradiction to the assumption that  $a_n$  converges to  $a$  for  $n \rightarrow \infty$ .

$\Leftarrow$  Assume the limes of every convergent sequence with sequence elements in  $A$ , is again in  $A$ . We show that for  $z \notin A$  ( $z \in \mathbb{C} \setminus A$ ) there exists  $\varepsilon > 0$  :  $B(z, \varepsilon) \cap A = \emptyset \Leftrightarrow B(z, \varepsilon) \subseteq \mathbb{C} \setminus A$ .

We prove the existence of such an  $\varepsilon$  by contradiction: So we assume such a  $\varepsilon$  does not exist:

$$\forall \varepsilon > 0 : B(z, \varepsilon) \cap A \neq \emptyset$$

Especially:  $\varepsilon = \frac{1}{n}$  with  $n \in \mathbb{N}_+$ .

$$B(z, \frac{1}{n}) \cap A \neq \emptyset \text{ therefore } \exists a_n \in A \cap B(z, \frac{1}{n})$$

therefore  $a_n \in A \wedge |a_n - z| < \frac{1}{n}$ . So this constructed sequence  $(a_n)_{n \in \mathbb{N}}$  satisfies:

$$a_n \in A : |a_n - z| < \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} a_n = z$$

By hypothesis, it holds that  $z \in A$ , but this is a contradiction to  $z \in \mathbb{C} \setminus A$ . So it is shown that  $\mathbb{C} \setminus A$  is an open set. So  $A$  is closed.  $\square$

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TODO

**Definition 46.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . A point  $z \in \mathbb{C}(\mathbb{R})$  is called *contact point* of a set  $M$ , if  $\forall r > 0 : B(z, r) \cap M \neq \emptyset$ . A point  $z \in \mathbb{C}(\mathbb{R})$  is called *limit point* of a set  $M$  if  $\forall r > 0$  it holds that  $B(z, r)$  contains a point  $w \in M$  with  $w \neq z$ .

Every limit point is also contact point.

**Lemma 16.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . It holds that

1.  $z \in \mathbb{C}$  is a contact point of  $M$  if and only if  $\exists (z_n)_{n \in \mathbb{N}} : z_n \in M$  and  $\lim_{n \rightarrow \infty} z_n = z$ .
2.  $z \in \mathbb{C}$  is a limit point of  $M$  if and only if  $\exists (z_n)_{n \in \mathbb{N}} : z_n \in M$  with  $z_n \neq z \forall n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} z_n = z$ .

*Proof.* 1. Let  $z$  be a contact point of  $M$ . Choose  $r_n = \frac{1}{n}$ , due to contact point property there exists  $z_n \in M$  to  $r_n$  with  $z_n \in B(z, \frac{1}{n})$  hence  $|z_n - z| < \frac{1}{n}$ . Then it holds  $\lim_{n \rightarrow \infty} z_n = z$ , then for  $\varepsilon > 0$  arbitrary let  $N$  be arbitrary large, such that  $\frac{1}{N} < \varepsilon$ . Then it holds that for  $n \geq N$ :

$$|z_n - z| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

$\Leftarrow$  Assume  $\exists (z_n)_{n \in \mathbb{N}}$  with limes  $z$ .  $z_n \in M$ . Choose  $r > 0$  arbitrary. Due to convergenze of  $(z_n)_{n \in \mathbb{N}}$  there exists some  $N \in \mathbb{N} : n \geq N$  such that  $|z_n - z| < r$ .

$$\Rightarrow z_n \in M \wedge z_n \in B(z, r) \Rightarrow z \text{ is contact point of } M$$

Also,

$$\Rightarrow z_n (\neq z) \in M \wedge z_n \in B(z, r) \Rightarrow z \text{ is limit point of } M$$

$\square$

**Theorem 68.**  $A \subseteq \mathbb{C}(\mathbb{R}, \mathbb{R}^n)$  is closed if and only if for every contact point  $z$  of  $A$  it holds that  $z \in A$ .

*Proof. Direction  $\Rightarrow$*  Let  $A$  be closed and  $z$  is a contact point of  $A$ . Due to Lemma 16 there exists  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in A$  and  $\lim_{n \rightarrow \infty} z_n = z$ . By the Lemma before the last, it holds that  $z \in A$ .

**Direction  $\Leftarrow$**  Assume for all contact points  $z$  of  $A$  it holds that  $z \in A$ . By the Lemma before the last: Let  $(z_n)_{n \in \mathbb{N}}$  be a convergent sequence with  $z_n \in A$  and  $\lim_{n \rightarrow \infty} z_n = z$ .

Show that  $z \in A$ .

This follows immediately because by the previous Lemma, it holds that  $z = \lim_{n \rightarrow \infty} z_n$  is a contact point TODO and by assumption  $z \in A$ .  $\square$

**Remark 20.** In general it holds that  $z \in M$ , then  $z$  is a contact point of  $M$ . Because  $\{z\} \subseteq B(z, r) \cap M$  with  $B(z, r) \cap M \neq \emptyset$ .

**Definition 47.** Let  $M \subseteq \mathbb{C}(\mathbb{R})$ . We define  $\overline{M} = \{z \in \mathbb{C} : z \text{ is contact point of } M\}$ .  $\overline{M}$  is called *closed hull*. It holds that  $M \subseteq \overline{M}$  and  $M$  is closed  $\Leftrightarrow M = \overline{M}$ .

**Definition 48.** A set  $K \subseteq \mathbb{C}$  ( $\mathbb{R}$ ,  $\mathbb{R}^n$ ) is called *compact*, if for each sequence  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in K$ , a subsequence  $(z_{n_l})_{l \in \mathbb{N}}$  exists which is convergent and its limit is inside  $K$ .

**Remark 21.** There are equivalent definitions which do not use sequences (e.g. using open covers).

**Theorem 69** (Bolzano-Weierstrass theorem for sets).  $K \subseteq \mathbb{C}$  is compact if and only if  $K$  is bounded and closed.

*Proof. Direction  $\Leftarrow$*  Let  $K$  be bounded and closed and let  $(z_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $K$ . Then  $(z_n)_{n \in \mathbb{N}}$  is a bounded sequence. Due to the Bolzano-Weierstrass Theorem for sequences, there exists some convergent subsequence  $(z_{n_l})_{l \in \mathbb{N}}$  with  $\lim_{l \rightarrow \infty} z_{n_l} = z$  where  $z_{n_l} \in K$ . Followingly  $z$  is contact point in  $K$ . Because  $K$  is closed, it holds that  $z \in K$ .

*Direction  $\Rightarrow$*  Let  $K$  be compact. Assume  $K$  is not bounded. Therefore for  $m = 1, 2, \dots, 5$ , there exists  $z_m \in K$  with  $|z_m| > m$ .  $(z_m)_{m \in \mathbb{N}}$  has certainly no convergent subsequence, because every subsequence  $(z_m)_{m \in \mathbb{N}}$  is also unbounded and therefore not convergent. This is a contradiction.

It remains to show that  $K$  is bounded. Let  $z \in \overline{K}$  ( $z$  is a contact point of  $K$ ). There exists a sequence  $(z_n)_{n \in \mathbb{N}}$  with  $z_n \in K$  and  $z = \lim_{n \rightarrow \infty} z_n$ . Because  $K$  is compact, there exists a subsequence  $(z_{n_k})_{k \in \mathbb{N}}$  of  $(z_n)_{n \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} z_{n_k} = w$  and  $w \in K$ . Because  $(z_n)_{n \in \mathbb{N}}$  is already convergent, every subsequence is convergent with limit  $z$ . It follows that  $w \in K$  and  $w = z$ , so  $z \in K$ . So  $K$  is bounded.

## 13 Continuous functions

**Definition 49.** Let  $D \subseteq \mathbb{C}$  ( $D \subseteq \mathbb{R}$ ) and  $f : D \rightarrow \mathbb{C}$  be a function. We say “ $f$  is continuous” (dt. “stetig”) iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in D \text{ with } |z - z_0| < \delta : |f(z) - f(z_0)| < \varepsilon$$

Intuitively, the difference of function values are arbitrary close to each other if the difference of the arguments is sufficiently small.

**Example 23.** 1.  $D$  is “strange”. Specifying the codomain and discussion of continuity in regards of this codomain is very important!

2. A non-continuous function  $f$  has a non-continuity in  $z_0$ . So  $\varepsilon$  cannot be arbitrary small.

3.  $f : \mathbb{C} \rightarrow \mathbb{C}$ .  $f(z) = z^2$ . Let  $z_0 \in \mathbb{C}$  arbitrary. Then  $f$  is continuous in  $z_0$ .

**Example 24.** Let  $\varepsilon > 0$  be arbitrary. Find  $\delta > 0$  such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| = |z^2 - z_0^2| < \varepsilon.$$

Define  $\delta = \min \left( 1, \frac{\varepsilon}{1+2|z_0|} \right)$ . For  $|z - z_0| < \delta$  it holds that

$$\begin{aligned} |f(z) - f(z_0)| &= |z^2 - z_0^2| = |(z - z_0)(z + z_0)| \\ &= |z - z_0| \cdot |z + z_0| \\ &= |z - z_0 + z_0 + z_0| \cdot |z - z_0| \\ &= \underbrace{|z - z_0|}_{<1} \cdot \underbrace{|z_0 + z_0|}_{<2|z_0| + \varepsilon} \\ &\leq (1 + 2|z_0|) \varepsilon \end{aligned}$$

**Example 25.** Let  $D = [0, \infty) \subseteq \mathbb{R}$ . Let  $f(x) = \sqrt[k]{x}$  be continuous in every point  $x_0 \in D$ .

Let  $\varepsilon > 0$  be given. Claim: It holds that  $|\sqrt[k]{x} - \sqrt[k]{x_0}| \leq \sqrt[k]{|x - x_0|}$ .

□

Proof: Show that for  $a, b \geq 0$ , it holds that  $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$ .

Assume  $\sqrt[k]{a+b} > \sqrt[k]{a} + \sqrt[k]{b}$ . Taking the  $k$ -th power keeps monotonicity:

$$\begin{aligned} (\sqrt[k]{a+b})^k &= a+b > \left( \sqrt[k]{a} + \sqrt[k]{b} \right)^k \\ &= a + \underbrace{\sum_{j=1}^{k-1} \binom{k}{j} a^{\frac{k-j}{k}} b^{\frac{j}{k}}}_{>0} + b \geq a+b \end{aligned}$$

This lecture took place on 18th of December 2015 with lecturer Wolfgang Ring.

We prove  $|\sqrt[k]{x} - \sqrt[k]{x_0}| \leq \sqrt[k]{|x - x_0|}$  using  $\sqrt[k]{a+b} \leq \sqrt[k]{a} + \sqrt[k]{b}$ .

$$\begin{aligned} |x| &= \left| \underbrace{x - x_0}_a + \underbrace{x_0}_b \right| \leq \underbrace{|x - x_0|}_a + \underbrace{|x_0|}_b \\ \sqrt[k]{|x|} &\leq \sqrt[k]{|x - x_0| + |x_0|} \leq \sqrt[k]{|x - x_0|} + \sqrt[k]{|x_0|} \\ \sqrt[k]{|x|} - \sqrt[k]{|x_0|} &\leq \sqrt[k]{|x - x_0|} \end{aligned}$$

Analogously:

$$\begin{aligned} |x_0| &= |x_0 - x + x| \leq \underbrace{|x_0 - x|}_a + \underbrace{|x|}_b \\ \Rightarrow \sqrt[k]{|x_0|} - \sqrt[k]{|x|} &\leq \sqrt[k]{|x - x_0|} \\ \left| \underbrace{\sqrt[k]{|x|}}_{f(x)} - \underbrace{\sqrt[k]{|x_0|}}_{f(x_0)} \right| &\leq \sqrt[k]{|x - x_0|} \end{aligned}$$

Let  $\varepsilon > 0$  arbitrary. Let  $\delta := \varepsilon^k$ . For  $|x - x_0| < \delta = \varepsilon^k$  it holds that

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \sqrt[k]{|x|} - \sqrt[k]{|x_0|} \right| \\ &\leq \sqrt[k]{|x - x_0|} < \sqrt[k]{\delta} = \sqrt[k]{\varepsilon^k} = \varepsilon \quad \checkmark \end{aligned}$$

**Theorem 70** (Sequence criterion for continuity). Let  $f : D \subset \mathbb{C} \Rightarrow \mathbb{C}$  ( $D \subseteq \mathbb{R}$ ). Then it holds that  $f$  is continuous in  $z_0 \in D$  if and only if for every convergent sequence  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D \forall n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  it holds that  $(f(w_n))_{n \in \mathbb{N}}$  is convergent and  $\lim_{n \rightarrow \infty} f(w_n) = f(z_0)$ .

In a different way, this theorem states:

$$w_n \rightarrow_{n \rightarrow \infty} z_0 \Rightarrow f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$$

*Proof.* **Direction**  $\Rightarrow$  Let  $f$  be continuous in  $z_0$  and  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D$  with  $\lim_{n \rightarrow \infty} w_n = z_0$ . Show that  $f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$ .

Let  $\varepsilon > 0$  arbitrary. Because  $f$  is continuous, there exists some  $\delta > 0$  such that  $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$  ( $z \in D$ ). So  $(w_n)_{n \in \mathbb{N}}$  converges to  $z_0$ . So there exists  $N \in \mathbb{N} : n \geq N \Rightarrow |w_n - z_0| < \delta$ . For those indices it holds that:  $|f(w_n) - f(z_0)| < \varepsilon$ . Hence  $\lim_{n \rightarrow \infty} f(w_n) = f(z_0)$ .

**Direction**  $\Leftarrow$  Proof by contradiction: For every sequence  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in D$  and  $w_n \rightarrow z_0$  it holds that:  $f(w_n) \rightarrow f(z_0)$  for  $n \rightarrow \infty$ . Assume  $f$  is not continuous in  $z_0$ .

So  $\exists \varepsilon > 0 : \forall \delta > 0 \exists z_\delta \in D$  with

$$|z_\delta - z_0| < \delta \wedge |f(z_\delta) - f(z_0)| \geq \varepsilon$$

We choose  $\delta_n = \frac{1}{n}$  for  $n = 1, 2, 3, \dots$

$$w_n := z_{\delta_n}$$

So it holds that

$$\forall n \in \mathbb{N} : |w_n - z_0| < \frac{1}{n} \wedge |f(w_n) - f(z_0)| \geq \varepsilon$$

Hence  $w_n \in D$  and  $\lim_{n \rightarrow \infty} w_n = z_0$  and for  $\varepsilon > 0$  we choose  $N$  such that  $\frac{1}{N} < \varepsilon$ . Then it holds for  $n \geq N$ :  $\frac{1}{n} < \frac{1}{N} < \varepsilon$  and therefore  $|w_n - z_0| < \frac{1}{n} < \varepsilon$ , but  $|f(w_n) - f(z_0)| \geq \varepsilon > 0$ . This is a contradiction to our assumption.  $\square$

**Definition 50.** Let  $f : D \rightarrow \mathbb{C}$  ( $D \subseteq \mathbb{C}$  or  $D \subseteq \mathbb{R}$ ). We call  $f$  “continuous on  $D$ ” if  $f$  is continuous in every point  $z \in D$ .

### 13.1 Laws for continuous functions

**Theorem 71.** Let  $f : D \rightarrow \mathbb{C}$  and  $g : D \rightarrow \mathbb{C}$  be functions and  $f$  and  $g$  are continuous in  $z_0 \in D$ . Then it holds that

1.  $(f + g) : D \rightarrow \mathbb{C}$  and  $(f + g)(z) = f(z) + g(z)$ .  
So the sum function  $(f + g)$  is continuous in  $z_0$ .
2.  $(f \cdot g) : D \rightarrow \mathbb{C}$  and  $(f \cdot g)(z) = f(z) \cdot g(z)$ .  
The product function is continuous in  $z_0$ .
3. Let  $g(z) \neq 0 \forall z \in D$ . Then  $\left(\frac{f}{g}\right) : D \rightarrow \mathbb{C}$  with  $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$ .  
The quotient function  $\left(\frac{f}{g}\right)$  is continuous in  $z_0$ .

*Proof.* Let  $(w_n)_{n \in \mathbb{N}}$  be an arbitrary sequence with  $w_n \in D$  and  $\lim_{n \rightarrow \infty} w_n = z_0$ . Due to the sequence criterion it holds that  $f(w_n) \rightarrow_{n \rightarrow \infty} f(z_0)$  and  $g(w_n) \rightarrow_{n \rightarrow \infty} g(z_0)$ . The laws for convergent sequences state that,

$$\begin{aligned} f(w_n) \cdot g(w_n) &\rightarrow_{n \rightarrow \infty} f(z_0) \cdot g(z_0) \\ f(w_n) + g(w_n) &\rightarrow_{n \rightarrow \infty} f(z_0) + g(z_0) \\ \frac{f(w_n)}{g(w_n)} &\rightarrow_{n \rightarrow \infty} \frac{f(z_0)}{g(z_0)} \end{aligned}$$

Hence  $(f + g)$ ,  $(f \cdot g)$  and  $\left(\frac{f}{g}\right)$  is continuous in  $z_0$ .

**Corollary 12.**

- $k : \mathbb{C} \rightarrow \mathbb{C}$ ,  $k(z) = c \in \mathbb{C}$  is a constant function.  $k$  is continuous in  $\mathbb{C}$ .
- The function  $f(z) = z$  is continuous in  $\mathbb{C}$ , because we can choose  $\delta = \varepsilon$ .

$$|z - z_0| < \varepsilon \Rightarrow |f(z) - f(z_0)| = |z - z_0| < \varepsilon$$

- The functions  $p_n(z) = z^n$  for  $n = 0, 1, 2, \dots$  are continuous in  $\mathbb{C}$  as products of continuous functions.
- All polynomials  $P(z) = \sum_{k=0}^n a_k z^k$  with  $a_k \in \mathbb{C}$  are continuous in  $\mathbb{C}$ .
- Let  $D = B(0, \rho(P))$  with  $\rho(P)$  is convergence radius of the power series

$$P(z) = \sum_{k=0}^{\infty} a_k z^k$$

Then  $P(z)$  is continuous in  $B(0, \rho(P))$ .

- Let  $P(z) = \sum_{k=0}^n a_k z^k$  and  $Q(z) = \sum_{l=0}^m b_l z^l$  be polynomials. And let  $D = \{z \in \mathbb{C} : Q(z) \neq 0\}$ . Then  $\left(\frac{P}{Q}\right) : D \rightarrow \mathbb{C}$  is continuous in  $D$ . Therefore all rational functions are continuous in all points except for the roots of the denominator:

$$\frac{a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n}{b_0 + b_1 z + b_2 z^2 + \dots + b_m z^m}$$

**Theorem 72.** Let  $f : D \rightarrow U \subseteq \mathbb{C}$  and  $g : U \rightarrow \mathbb{C}$  be two functions. Let  $f$  be continuous in  $z_0 \in D$  and let  $g$  be continuous in  $y_0 = f(z_0) \in U$ . Then  $g \circ f : D \rightarrow \mathbb{C}$  is continuous in  $z_0$ .

*Proof.* Due to the sequence criterion: Let  $(w_n)_{n \in \mathbb{N}}$  ( $w_n \in D$ ) with  $\lim_{n \rightarrow \infty} w_n = z_0$ . The sequence criterion for  $f$  yields

$$\lim_{n \rightarrow \infty} \underbrace{f(w_n)}_{\in U} = f(z_0) = y_0$$

□ The sequence criterion for  $g$  states that

$$\lim_{n \rightarrow \infty} \underbrace{g(f(w_n))}_{g \circ f(w_n)} = g(y_0) = \underbrace{g(f(z_0))}_{g \circ f(z_0)}$$

So  $g \circ f$  is continuous in  $z_0$ . □

We know  $w_k(x) = \sqrt[k]{x}$  is continuous in  $[0, \infty)$ .

$$P_l(x) = x^l \text{ is continuous in } \mathbb{C}$$

$$\Rightarrow P_l \circ w_k \text{ is continuous in } [0, \infty)$$

$$p_0 \circ w_k(x) = p_l(\sqrt[k]{x}) = (\sqrt[k]{x})^l = x^{\frac{l}{k}} \text{ is continuous.}$$

- $n(z) = |z|$  is continuous in  $\mathbb{C}$ .

Let  $\varepsilon > 0$  be arbitrary. It holds that

$$|n(z) - n(z_0)| = ||z| - |z_0|| \leq |z - z_0|$$

Choose  $\delta = \varepsilon$ . Then for  $|z - z_0| < \delta = \varepsilon$  it holds that  $|n(z) - n(z_0)| < \varepsilon$ .

- $\Re : \mathbb{C} \rightarrow \mathbb{C}$  and  $\Im : \mathbb{C} \rightarrow \mathbb{C}$  are continuous in  $\mathbb{C}$ . Because  $|\Re(z) - \Re(z_0)| \leq |z - z_0|$  ✓.
- Let  $f, g : D \rightarrow \mathbb{R}$ . Then  $\max(f, g) : D \rightarrow \mathbb{R}$   $(\max(f, g))(z) = \max\{f(z), g(z)\}$  is continuous in  $D$ . because  $\max f(z), g(z) = \frac{1}{2}(|f(z) - g(z)| + f(z) + g(z))$ .

This lecture took place on 7th of January 2016 with lecturer Wolfgang Ring.

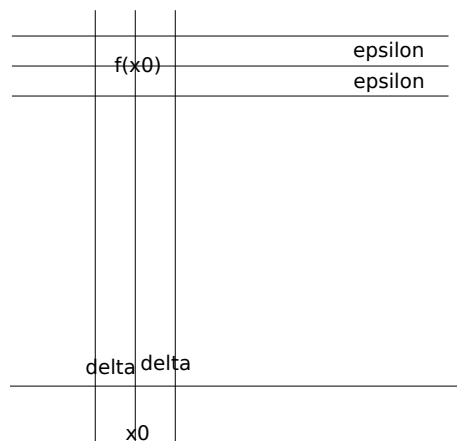
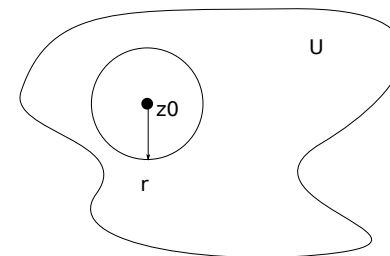


Figure 17: The notion of continuity


 Figure 18: Environment with radius  $r$ 

### 13.2 Revision of the continuity definition

$f$  is continuous in  $x_0$  if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \wedge |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon]$$

Reminder: Let  $z_0 \in U \subseteq \mathbb{C}$ .  $U$  is called environment of  $z_0$  if  $r > 0$  exists such that  $B(z_0, r) \subset U$ .

**Definition 51.** Let  $D \subseteq \mathbb{C}$  and  $z_0 \in U \subseteq D$ . We call  $U$  *environment* of  $z_0$  in  $D$  if  $\exists r > 0$  such that  $B(z_0, r) \cap D \subseteq U$ .

**Theorem 73.** Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$ . Let  $z_0 \in D$ . Then  $f$  is continuous in  $z_0$  if and only if for every environment  $U$  of  $y_0 = f(z_0)$  it holds that  $V = f^{-1}(U)$  is an environment of  $z_0 \in D$  (where  $f^{-1}$  denotes the preimage).

*Proof.*  $\Rightarrow$

Let  $f$  be continuous in  $z_0$  and let  $U$  be an environment of  $y_0 = f(z_0)$ , hence

$\exists \varepsilon > 0 : B(y_0, \varepsilon) \subseteq U$  with  $y_0 = f(z_0)$ . Because  $f$  is continuous in  $z_0$ , it holds that

$$\exists \delta > 0 : |z - z_0| < \delta \wedge z \in D \Rightarrow \underbrace{|f(z) - f(z_0)| < \varepsilon}_{\substack{f(z) \in B(f(z_0), \varepsilon) \\ B(y_0, \varepsilon) \subseteq U}}$$

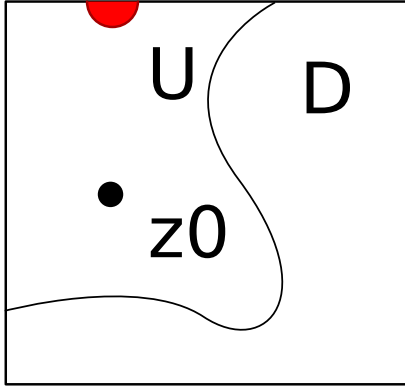
This requires:

$$\begin{aligned} |z - z_0| < \delta \wedge z \in D &\Leftrightarrow z \in B(z_0, \delta) \wedge z \in D \\ &\Rightarrow z \in B(z_0, \delta) \cap D \end{aligned}$$

Therefore we can redefine continuity as:

$$z \in B(z_0, \delta) \cap D \Rightarrow f(z) \in B(y_0, \varepsilon)$$




 Figure 19: Environment  $U$ 

So it holds that

$$\forall z \in B(z_0, \delta) \cap D \Rightarrow z \in f^{-1}(B(y_0, \varepsilon)) \subseteq f^{-1}(U)$$

So it holds that  $B(z_0, \delta) \cap D \subseteq f^{-1}(U)$ .

⇐

Let the preimage of every environment in  $y_0$  be an environment of  $z_0$  in  $D$ . Let  $\varepsilon > 0$  arbitrary. Then it holds that  $B(y_0, \varepsilon)$  is an environment of  $y_0$ . By assumption it holds that  $V = f^{-1}(B(y_0, \varepsilon))$  is an environment of  $z_0$  in  $D$ , hence

$$\exists \delta > 0 : B(z_0, \delta) \cap D \subseteq f^{-1}(B(y_0, \varepsilon)).$$

Therefore for  $z \in B(z_0, \delta) \cap D$  it holds that  $f(z) \in B(y_0, \varepsilon)$ .

In other words:

$$|z - z_0| < \delta \wedge z \in D \Rightarrow |f(z) - \underbrace{f(z_0)}_{=y_0}| < \varepsilon$$

So  $f$  is continuous in  $z_0$ .

This notion of continuity is the most general one accepted by the mathematical community. It can be used in all topological spaces.  $\square$

### 13.3 Variants of continuity

**Definition 52.** Let  $f : D \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a function  $f$  called uniformly continuous in  $D$  if

$$\forall \varepsilon > 0 \exists \delta > 0 : [\forall z_0, z_1 \in D \text{ with } |z_1 - z_0| < \delta \Rightarrow |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that  $\delta$  only depends on  $\varepsilon$ , meaning that it can be arbitrarily shifted on the  $x$ -axis ( $\delta = \delta(\varepsilon)$ ).

Reminder:  $f$  is continuous in  $D$

$$\Leftrightarrow \forall z_0 \in D \forall \varepsilon > 0 \exists \delta > 0 : [\forall z_1 \in D \wedge |z_1 - z_0| < \delta \Rightarrow |f(z_1) - f(z_0)| < \varepsilon]$$

Recognize that  $\delta$  depends on  $z_0$  and  $\varepsilon$  ( $\delta = \delta(\varepsilon, z_0)$ ). Therefore this second definition provides more freedom to parameter  $\delta$ . So uniform continuity implies continuity in  $D$ .

**Example 26.** Let  $f : (0, 1] \rightarrow \mathbb{C}$  and  $f(x) = \frac{1}{x}$ .  $f$  is continuous in every point  $x_0 \in (0, 1]$ . However,  $f$  is not uniformly continuous.

$$\forall \varepsilon > 0 \exists \delta > 0 : \left[ \forall x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \Rightarrow \left| \frac{1}{x_0} - \frac{1}{x_1} \right| < \varepsilon \right]$$

The negation is given with:

$$\exists \varepsilon > 0 \forall \delta > 0 : \left[ \exists x_0, x_1 \in D \text{ with } |x_0 - x_1| < \delta \wedge \left| \frac{1}{x_0} - \frac{1}{x_1} \right| \geq \varepsilon \right]$$

We look at  $\varepsilon = 1$ . Let  $\delta > 0$  arbitrary. We choose  $x_0 = \frac{1}{n}$  and  $x_1 = \frac{1}{n+1}$  for appropriate  $n \in \mathbb{N}_+$ . Then it holds that

$$|x_0 - x_1| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{n+1-n}{n(n+1)} = \frac{1}{n(n+1)} \underset{\text{for } n \in \mathbb{N}_+}{\leq} \frac{1}{n} < \delta$$

if  $n > \frac{1}{\delta}$

$$\left| \frac{1}{x_0} - \frac{1}{x_1} \right| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = |n - (n+1)| = |-1| = 1$$

Therefore  $f(x) = \frac{1}{x}$  is not uniformly continuous in  $(0, 1]$ .

Remark:  $f(x) = \frac{1}{x}$  is uniformly continuous in  $D = [\frac{1}{100}, 1]$ , but not in  $\mathbb{R}$ .

**Definition 53** (Lipschitz continuity). Another notion of continuity is given by Rudolf Lipschitz (1832–1903).

$f : D \subset \mathbb{C} \rightarrow \mathbb{C}$  is called Lipschitz continuous if  $k \geq 0$  exists such that

$$\forall z_1, z_2 \in D : |f(z_1) - f(z_2)| \leq k |z_1 - z_2|$$

The value  $k$  is called Lipschitz constant for  $f$ .

**Definition 54** (Hölder continuity). Yet another notion of continuity is given by Otto Hölder (1859–1937).

$f$  is called Hölder continuous with exponent  $H \in (0, 1]$  if there exists  $k > 0$  such that

$$\forall z_1, z_2 \in D : |f(z_1) - f(z_2)| \leq k |z_1 - z_2|^H$$

**Corollary 13.** A hierarchy for those continuity notion is given:

Lipschitz continuous  $\subseteq$  uniformly continuous  $\subseteq$  continuous in  $D$ .

**Theorem 74.** Let  $K \subseteq \mathbb{C}$  be compact. Let  $f : K \rightarrow \mathbb{C}$  be continuous in  $K$ . Then  $f(K) = \{y = f(z) : z \in K\} \subset \mathbb{C}$  is compact in  $\mathbb{C}$ .

*Proof.* Every sequence  $(y_n)_{n \in \mathbb{N}}$ , with  $y_n = f(z_n)$  and  $z_n \in K$  where  $y_n \in f(K)$ , has a convergent subsequence. The sequence of preimage values  $(z_n)_{n \in \mathbb{N}}$  is a sequence in  $K$  which, followingly, has a convergent subsequence. Let  $(z_{n_k})_{k \in \mathbb{N}}$   $\lim_{k \rightarrow \infty} z_{n_k} = z \in K$ . Because of the sequence criterion for continuity it holds that

$$\lim_{k \rightarrow \infty} y_{n_k} = \lim_{k \rightarrow \infty} f(z_{n_k}) = f(z) \in f(K)$$

with  $y = f(k)$ . So  $(y_n)_{n \in \mathbb{N}}$  has a convergent subsequence with limes  $y \in f(K)$ . Therefore  $f(K)$  is compact.  $\square$

**Definition 55.** Let  $f : D \rightarrow \mathbb{R}$  and  $D \subseteq \mathbb{C}$ . A point  $z_{\max} \in D$  is called global maximum of  $f$  if  $f(z_{\max}) \geq f(z) \quad \forall z \in D$ .

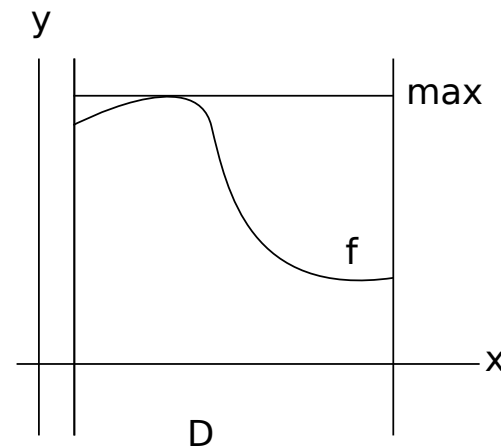


Figure 20: Illustration of a global maximum

**Definition 56.** Let  $f : K \rightarrow \mathbb{R}$  ( $K \subseteq \mathbb{C}$ ) is continuous in  $K$  and  $K$  is compact in  $\mathbb{C}$ . Then  $f$  has a global maximum and a global minimum.

**Remark 22.** For non-compact definition sets this statement is not generally true. For example,  $f(x) = \frac{1}{x}$  in  $D = (0, 1)$  has neither a global maximum nor a global minimum.

*Proof.*

$$f(K) \subseteq \mathbb{R}$$

is compact (because of the previous theorem) and therefore bounded and closed in  $\mathbb{R}$  (by Theorem by Bolzano-Weierstrass). Because  $f(K)$  is bounded,  $f(K)$  has a supremum  $\zeta^*$  and an infimum  $\zeta_*$  (supremum property). Supremum and infimum are contact points of  $f(K)$ . Because  $f(K)$  is closed it holds that

$$\zeta^* \in f(K) \text{ and } \zeta_* \in f(K)$$

Therefore there exists  $z_{\min} \in K$  with  $f(z_{\min}) = \zeta_*$  and  $z_{\max} \in K$  with  $f(z_{\max}) = \zeta^*$ . Because  $f(K)$  is closed, it holds that  $\zeta^* \in f(K)$  and  $\zeta_* \in f(K)$ . Therefore there exists  $z_{\min} \in K$  with  $f(z_{\min}) = \zeta_*$  and  $f(z_{\max}) \in K$  with  $f(z_{\max}) \geq y$ , therefore  $\forall z \in K : f(z_{\max}) \geq f(z)$ .

Therefore  $z_{\max}$  is a global maximum. The analogous statement holds for  $\zeta_*$  and a global minimum.  $\square$

**Theorem 75** (A very universal theorem about maxima). A continuous function has a global maximum in a compact domain.

Using this method to show existence of a value is called “direct method of variation computations”.

This lecture took place on 8th of January 2016 with lecturer Wolfgang Ring.

Continuity and compactness implies existence of a maximum and minimum.

“Direct method of calculus of variations” (dt. “direkte Methode der Variationsrechnung”).

**Theorem 76** (Intermediate value theorem for continuous functions). Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous with  $a \leq b$ . Let

$$m^* = \max \{f(x) : x \in [a, b]\}$$

$$m_* = \min \{f(x) : x \in [a, b]\}$$

$m^*$  and  $m_*$  exist because  $[a, b]$  is compact (bounded and closed).

Let  $m_* \leq \eta \leq m^*$ .

Then there exists  $\xi \in [a, b]$  with  $f(\xi) = \eta$ . The function  $f$  takes any value for some  $x$  in  $m_*$  and  $m^*$ . Compare with Figure 21.

*Proof.* Let  $a_0 \in [a, b]$  such that  $f(a_0) = m_*$  and  $b_0 \in [a, b]$  such that  $f(b_0) = m^*$ . Without loss of generality:  $a_0 = b_0$ . If  $a_0 > b_0$  it holds that  $\max \{f(x) : x \in [a, b]\} = f(b_0) = f(a_0) = \min \{f(x) : x \in [a, b]\}$ . If  $\max = \min$ , then  $f$  is constant, hence  $f(x) = m_* = m^* \quad \forall x \in [a, b]$ .

$$m_* = \eta \leq m^* \Rightarrow \eta = m_* = m^* \wedge f(x) = \eta \quad \forall x \in [a, b]$$

Consider  $a_0 \leq b_0$ . We know,  $f(a_0) = m_* \leq \eta \leq m^* = f(b_0)$ . We use nested intervals:

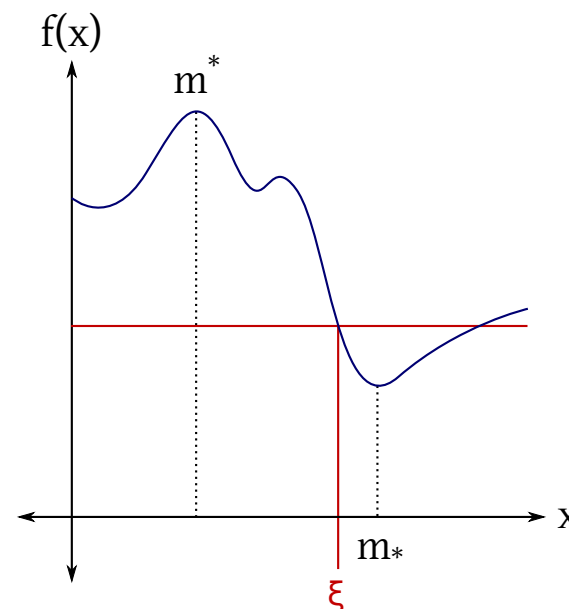


Figure 21:  $\zeta$  in  $[a, b]$

Assume  $I_n = [a_n, b_n]$  for  $n \in \mathbb{N}$  was already found with the property  $f(a_n) \leq \eta \leq f(b_n)$ . Let  $m_n = \frac{1}{2}(a_n + b_n)$  be the midpoint of  $I_n$ .

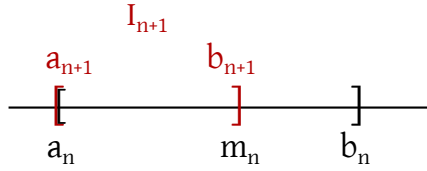
**Case  $f(m_n) \geq \eta$**  If  $f(m_n) \geq \eta$  we set  $b_{n+1} = m_n$  and  $a_{n+1} = a_n$  (compare Figure ??) and it holds that  $f(a_{n+1}) = f(a_n) \leq \eta$  and  $f(b_{n+1}) = f(m_n) \geq \eta$ . Furthermore  $|I_{n+1}| = \frac{1}{2}|I_n|$ .

**Case  $f(m_n) < \eta$**  Let  $a_{n+1} = m_n$  and  $b_{n+1} = b_n$ .

$$I_{n+1} = [a_{n+1}, b_{n+1}]$$

$$f(a_{n+1}) = f(m_n) < \eta$$

$$f(b_{n+1}) = f(b_n) \geq \eta$$


 Figure 22: Interval  $I_{n+1}$ 

$$|I_{n+1}| = \frac{1}{2} |I_n|$$

Nested interval  $(I_n)_{n \in \mathbb{N}}$  has the property:

$$I_{n+1} \leq I_n \quad |I_n| = \left(\frac{1}{2}\right)^n \cdot |I_0| = \left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)$$

and  $f(a_n) \leq \eta \leq f(b_n)$ .  $(I_n)_{n \in \mathbb{N}}$  is are nested intervals. Let  $\xi \in \bigcap_{n \in \mathbb{N}} I_n$  and it holds that  $|\xi - a_n| \leq |b_n - a_n| = \underbrace{\left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)}_{\rightarrow 0 \text{ for } n \rightarrow \infty}$ . Therefore  $\lim_{n \rightarrow \infty} a_n = \xi$

and

$$|b_n - \xi| \leq |b_n - a_n| = \underbrace{\left(\frac{1}{2}\right)^n \cdot (b_0 - a_0)}_{\rightarrow 0 \text{ for } n \rightarrow \infty}.$$

So  $\lim_{n \rightarrow \infty} b_n = \xi$ .

Because  $f$  is continuous on  $[a, b]$ , it holds that

$$\eta \leq f(b_n) \quad \forall n \in \mathbb{N} \Rightarrow \eta \leq \lim_{n \rightarrow \infty} f(b_n)$$

$$\text{continuity} \Rightarrow \lim_{n \rightarrow \infty} f(b_n) = f(\xi)$$

So,

$$\eta \leq \lim_{n \rightarrow \infty} f(b_n) = f(\xi) = \lim_{n \rightarrow \infty} f(a_n) \leq \eta.$$

Therefore  $\eta = f(\xi)$ .

**Remark 23.** From this we can derive continuity for a numerical algorithm for solving  $f(x) = \eta$ . It's called *bisection method*.

**Remark 24.** Often the intermediate value theorem is defined as:

Let  $\eta$  be between  $f(a)$  and  $f(b)$ . Then there exists  $\xi \in [a, b]$  such that  $f(\xi) = \eta$ . Obviously because  $m_* \leq f(a)$  and  $f(b) \leq m^*$ .

**Definition 57** (Limes of a function). Let  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$ . Let  $z$  be a limit point of  $D$ . We say, that  $f$  in  $z$  has the limes  $w$  if the function

$$\hat{f} : D \cup \{z\} \rightarrow \mathbb{C}$$

$$\hat{f}(\xi) = \begin{cases} f(\xi) & \text{if } \xi \neq z \\ w & \text{if } \xi = z \end{cases}$$

is continuous. We denote  $\lim_{\xi \rightarrow z} f(\xi)$ .

**Example 27.** See Figures 23, 24 and 25.

**Lemma 17.** Let  $f : D \rightarrow \mathbb{C}$  given and  $z$  is a limit point of  $D \subseteq \mathbb{C}$ . Then  $f$  has a limes  $w \in \mathbb{C}$  if and only if one of the equivalent conditions hold.

$$\bullet \quad \forall \varepsilon > 0 \exists \delta > 0 \forall \xi \in D : |z - \xi| < \delta \Rightarrow \underbrace{|f(\xi) - w|}_{\hat{f}(\xi) - \hat{f}(z)} < \varepsilon$$

“Continuity of  $\hat{f}$ ”

$$\bullet \quad \forall (\xi_n)_{n \in \mathbb{N}} \text{ with } \xi_n \in D \setminus \{z\} \text{ and } \lim_{n \rightarrow \infty} \xi_n = z \text{ holds.}$$

$$\lim_{n \rightarrow \infty} f(\xi_n) = w$$

“Sequence criterion for  $\hat{f}$ ”

**Example 28.**  $f : \mathbb{C} \setminus \{1\} \rightarrow \mathbb{C}$  with

$$f(z) = \frac{z^2 - 1}{z - 1}$$

For  $z \neq 1$  it holds that:

$$f(z) = \frac{(z - 1)(z + 1)}{(z - 1)} = (z + 1)$$

□

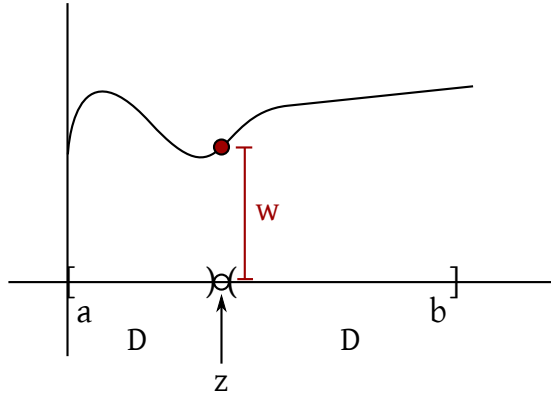


Figure 23: Example 1 with  $D = [a, b] \setminus \{z\}$  and  $w = \lim_{\xi \rightarrow z} f(\xi)$

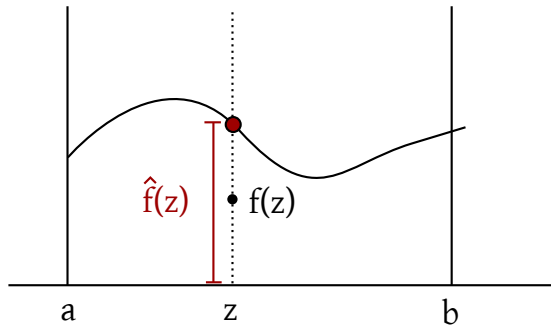


Figure 24: Example 2 which defines function new in point  $z$  with  $D = [a, b]$  and  $\lim_{\xi \rightarrow z} f(\xi)$ .  $f$  is not continuous in  $z$ , but  $\hat{f}$  is continuous in  $z$ .

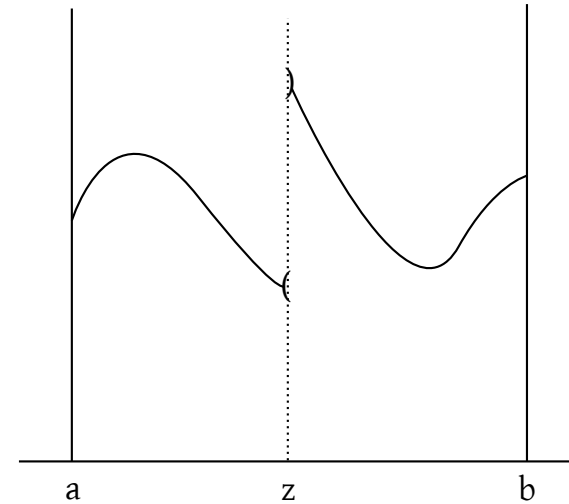


Figure 25: Example 3 with  $D = [a, b] \setminus \{z\}$ .  $f$  does not have a limit in  $z$ . Due to the jumping point, it is not a continuous function. Therefore we cannot find  $\varepsilon$ . We say  $\hat{f}$  is a continuous continuation of  $f$  in point  $z$ .

Let

$$\hat{f}(z) = \begin{cases} f(z) & \text{if } z \neq 1 \\ 2 & \text{if } z = 1 \end{cases}$$

$\hat{f}(z) = z + 1$  in  $\mathbb{C}$  is continuous.  $f$  has limes  $w = 2$  in point  $z = 1$ .

**Example 29.** Let  $s \in \mathbb{Q} \setminus \{0\}$  and  $D = (-1, \infty) \setminus \{0\}$

$$f(x) = \frac{(1+x)^s - 1}{x}$$

It holds that  $\lim_{x \rightarrow 0} f(x) = s$ .

for  $|x| < 1$ .

$$\begin{aligned} (1+x)^s &= \sum_{k=0}^{\infty} \binom{s}{k} x^k \Rightarrow \frac{(1+x)^s - 1}{x} \\ &\Rightarrow \frac{(1+x)^s - 1}{x} = \frac{\sum_{k=1}^{\infty} \binom{s}{k} x^k}{x} = \sum_{k=1}^{\infty} \binom{s}{k} \cdot x^{k-1} \\ \lim_{x \rightarrow 0} \underbrace{\left( \sum_{k=1}^{\infty} \binom{s}{k} x^{k-1} \right)}_{f(x)} &= \sum_{k=1}^{\infty} \binom{s}{k} 0^{k-1} = \binom{s}{1} = s \end{aligned}$$

We need the following theorem: A power series is in its convergence radius a continuous function.  $\square$

## 14 Differential calculus

Let  $f : (a, b) \rightarrow \mathbb{R}$  be given. with  $a < b$ .

Idea: We want  $f$  close to point  $x_0 \in (a, b)$  be approximated by a linear-affine function  $a(x) = k(x - x_0) + d$ .

$$a(x) = k(x - x_0) + d = kx + \underbrace{(-kx_0 + d)}_{\tilde{d}} = kx + \tilde{d}$$

$\tilde{a}(x) = kx$  is linear. Linear and constant functions are linear affine.  $a$  should (at least) cross point  $x_0$ , ie.  $f(x_0)$ . Compare with Figure 26.

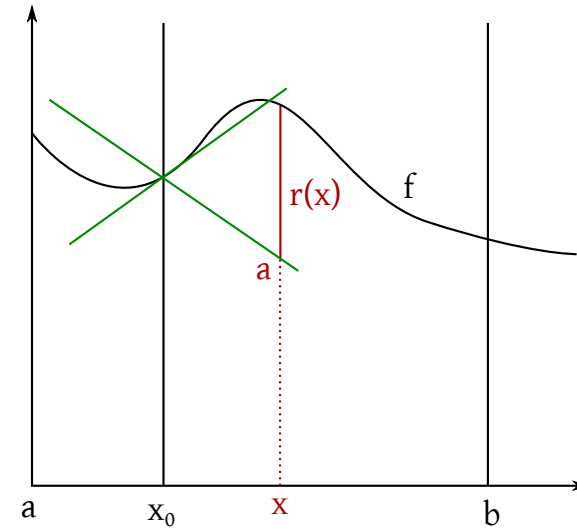


Figure 26: Differential  $f$  in  $x_0$

$$\Rightarrow a(x_0) = k(\underbrace{x_0 - x_0}_0) + a \stackrel{!}{=} f(x_0) \Rightarrow d = f(x_0)$$

$$\Rightarrow a(x) = k(x - x_0) + f(x_0)$$

How should we select  $k$  such that the approximation of  $f$  is best possible by selection of  $a$ . We consider the deviation.

$$f(x) = f(x) - a(x)$$

$r(x)$  should be as small as possible in  $x_0$ . Therefore  $\lim_{x \rightarrow x_0} r(x) = 0$ .

$$\lim_{x \rightarrow x_0} r(x) = \lim_{x \rightarrow x_0} [f(x) - f(x_0) - k \cdot (x - x_0)] = 0 \quad \forall k$$

We need:  $r(x)$  should converge to 0 very quickly for  $x \rightarrow x_0$ .

Idea: Require that  $\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0$ .  $\frac{1}{x - x_0}$  is unbounded close to  $x_0$ .

$$\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0 \text{ means } \lim_{x \rightarrow x_0} \left| \frac{r(x)}{x - x_0} - 0 \right| = 0$$

$$\Rightarrow \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0) - k \cdot (x - x_0)}{x - x_0} \right| = \lim_{x \rightarrow x_0} \left| \frac{f(x) - f(x_0)}{x - x_0} - k \right|$$

Hence,

$$\Rightarrow k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

with

$$\lim_{x \rightarrow x_0} \frac{r(x)}{x - x_0} = 0,$$

$k$  is uniquely identified with

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

This lecture took place on 13th of January 2016 with lecturer Wolfgang Ring.

TODO: another figure missing

$$y = kx + d$$

$$d = k \cdot (x_0) - k \cdot x_0$$

TODO: missing a few lines

**Definition 58** (Landau's symbols). Let  $g : D \rightarrow \mathbb{C}$ ,  $D \subseteq \mathbb{C}$ . Let  $z_0$  be a limit point of  $g$  and assume  $g$  has a limit point 0 for  $z \rightarrow z_0$ . Therefore,

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall z \in D \wedge |z - z_0| < \delta$$

where  $z \neq z_0$ .

$$\Rightarrow |g(z) - 0| < \varepsilon$$

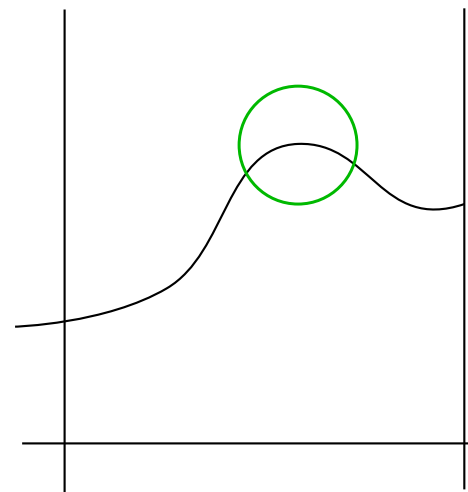


Figure 27: Derivative

We say that  $y$  is of order  $\mathcal{O}(n)$  in point  $z_0$ , if  $k \geq 0$  and some  $r > 0$  such that

$$|g(z)| \leq K |z - z_0|^n \quad \forall z \in D \text{ with } |z - z_0| < r \wedge z \neq z_0$$

We denote it with  $g(z) = \mathcal{O}(|z - z_0|^n)$ .

We say that  $g$  is of order  $o(n)$  if  $r > 0$  and some function  $k : (0, r) \rightarrow \mathbb{R}^+$  with  $\lim_{x \rightarrow 0} k(x) = 0$  exists, such that

$$|g(z)| \leq k(|z - z_0|) \cdot |z - z_0|^n \quad \forall z \in D \text{ with } |z - z_0| < r \wedge z \neq z_0$$

We denote,

$$g(z) = o(|z - z_0|^n)$$

**Corollary 14.** It holds that,  $g : \mathcal{O}(|z - z_0|^n) \Leftrightarrow \exists r > 0$  such that

$$\frac{|g(z)|}{|z - z_0|^n}$$

is bounded in  $B(z_0, r) \setminus \{z_0\}$  and  $g = o(|z - z_0|^n)$ , if  $\exists r > 0$  such that  $\frac{|g(z)|}{|z - z_0|^n}$  in point  $z_0$  has limit point 0.

**Corollary 15.** For determination of the slope  $k$  for the best-achievable linear-affine approximation of  $f$  it must hold that

$$f(x) - (f(x_0) + k(x - x_0)) = o(|x - x_0|)$$

**Definition 59.** Let  $f : (a, b) \rightarrow \mathbb{R}$  and  $x_0 \in (a, b)$ . We claim that  $f$  in  $x_0$  is *differentiable*, if the limit point of the function  $\frac{f(x) - f(x_0)}{x - x_0}$  exists. The corresponding limit point  $k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$  is called *derivative of  $f$  in  $x_0$* .

We can compute  $k$  using  $k = f'(x_0)$ .

Alternatively:  $f$  is differentiable in  $x_0$  if  $x \in \mathbb{R}$  exists, such that  $r : (a, b) \setminus \{0\} \rightarrow \mathbb{R}$  with  $r(x) = f(x) - f(x_0) - k(x - x_0)$  is of order  $o(1)$  in  $x_0$ .

$$f(x) - f(x_0) - k(x - x_0) = \mathcal{O}(|x - x_0|)$$

The second definition is more general and can also be applied for functions  $f : \mathcal{O} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$ .

**Corollary 16.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ . Then the function

$$\varphi(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x \in (a, b) \setminus \{x_0\} \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

$\varphi : (a, b) \rightarrow \mathbb{R}$  and  $\varphi$  is continuous in  $x_0$ .

Show that  $\lim_{x \rightarrow x_0} \varphi(x) = \varphi(x_0)$ .

$$f(x) = f(x_0) + \varphi(x)(x - x_0)$$

because  $\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0}$  for  $x \neq x_0$ .  $f(x)$  is constant,  $\varphi(x)$  is continuous in  $x_0$  and  $(x - x_0)$  is continuous in  $(a, b)$ . For  $x = x_0$ ,  $f(x) = f(x_0) + \varphi(x)(x - x_0)$  holds as well.

Therefore all expressions of  $f(x_0) + \varphi(x)(x - x_0)$  are continuous in  $x_0$ , followingly  $f$  is continuous in  $x_0$ .

**Lemma 18.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ .

$$k = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is slope of affine function, which approximates  $f$  in  $x_0$ .

Plot of this function:

$$y(x) = f'(x_0)(x - x_0) + f(x_0)$$

is called *tangent* of  $f$  in  $x_0$ .

This lecture took place on 14th of Jan 2016 with lecturer Wolfgang Ring.

**Theorem 77** (Convergence, limes and differentiable functions). Let  $f : (a, b) \rightarrow \mathbb{R}$  be differentiable in  $x_0 \in (a, b)$ . Therefore the equivalent defining properties hold.

1.  $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in (a, b)$  with  $|x - x_0| < \delta$  and  $x \neq x_0$  it holds that  $\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon$ . This constitutes a definition of the limes.

2. For all  $(\xi_n)_{n \in \mathbb{N}}$  with  $\xi_n \in (a, b)$  and  $\xi_n \neq x_0$  and  $\lim_{n \rightarrow \infty} \xi_n = x_0$ , it holds that

$$\left( \frac{f(\xi_n) - f(x_0)}{\xi_n - x_0} \right)_{n \in \mathbb{N}} \text{ is convergent towards } f'(x_0)$$

This is the sequence criterion for the limes.

3. For all  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that  $\forall x \in (a, b)$  with  $|x - x_0| < \delta$  it holds that

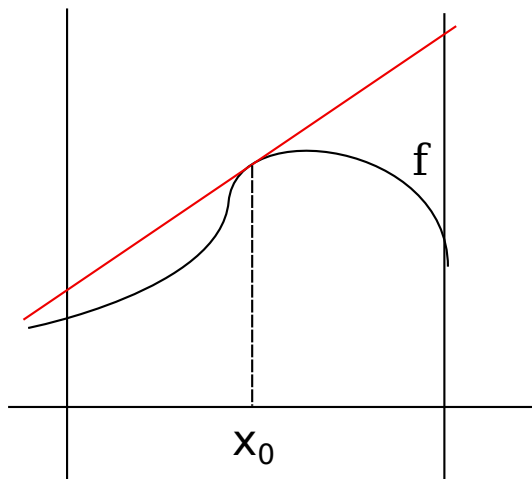
$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \varepsilon |x - x_0|$$

holds also for  $x = x_0$ .

The (3) implies the (1): Assume (3) holds and choose  $\delta$  such that  $\forall |x - x_0| < \delta$  it holds that

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{\varepsilon}{2} |x - x_0| \underbrace{\leq}_{\text{for } x \neq x_0} \varepsilon |x - x_0| \underbrace{\Rightarrow}_{\text{divide by } x - x_0} (1)$$




 Figure 28: Tangent of  $f$  in  $x_0$ 

### 14.1 Derivation of common functions

Let  $p_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $p_n(x) = x^n$ . Let  $x_0 \in \mathbb{R}$  and  $x \neq x_0$  and  $n \in \mathbb{N}$ . Then it holds that

$$\begin{aligned} \frac{p_n(x) - p_n(x_0)}{x - x_0} &= \frac{x^n - x_0^n}{x - x_0} \\ &= \frac{(x - x_0) \cdot \sum_{k=0}^{n-1} x^k x_0^{n-1-k}}{x - x_0} \\ &= \sum_{k=0}^{n-1} x^k x_0^{n-1-k} \\ &\rightarrow_{x \rightarrow x_0} \sum_{k=0}^{n-1} x_0^k x_0^{n-1-k} \\ &= \sum_{k=0}^{n-1} x_0^{n-1} \\ &= n x_0^{n-1} \end{aligned}$$

Therefore  $p_n$  is differentiable in  $x_0$  and  $p'_n(x_0) = n x_0^{n-1}$ .

$$(x^n)' = n x^{n-1} \quad \forall n \in \mathbb{N}$$

1. Let  $f(x) = a^x$  with  $a > 0$ . This function is called *exponential function* with basis  $a$ . It holds that:

$$\begin{aligned} \frac{a^x - a^{x_0}}{x - x_0} &= \frac{a^{x_0} \cdot a^{x-x_0} - a^{x_0}}{x - x_0} \\ &= a^{x_0} \cdot \frac{a^{x-x_0} - 1}{x - x_0} \\ &\rightarrow_{x \rightarrow x_0} a^{x_0} \cdot \lim_{x \rightarrow x_0} \frac{a^{x-x_0} - 1}{x - x_0} \end{aligned}$$

$$\left| \begin{array}{l} x - x_0 = h \\ x \rightarrow x_0 \Leftrightarrow h \rightarrow 0 \end{array} \right| = a^{x_0} \lim_{\substack{h \rightarrow 0 \\ = c \in \mathbb{R}}} \frac{a^h - 1}{h}$$

Therefore  $|a^x|' = c \cdot a^k$  with  $c = \lim_{h \rightarrow 0}$ . TODO content missing

In the special case that this constant  $h$  is the Eulerian number  $e$ , it holds that:

$$(e^x)' = e^x$$

2.  $\log : (0, \infty) \rightarrow \mathbb{R}$  with  $e^{\log x} = x \forall x > 0$  or equivalently  $\log(e^y) = y \forall y \in \mathbb{R}$ .

$$\frac{\log x - \log x_0}{x - x_0} = \frac{\log \frac{x}{x_0}}{x - x_0} = \frac{1}{x_0} \frac{\log \frac{x}{x_0}}{\frac{x}{x_0} - 1} \rightarrow \frac{1}{x_0} \cdot \lim_{h \rightarrow 1} \frac{\log h}{h - 1} = \frac{1}{x_0}$$

Therefore  $(\log x)' = \frac{1}{x}$  for  $x > 0$ .

## 14.2 Derivation laws

**Theorem 78.** Let  $f, g : (a, b) \rightarrow \mathbb{R}$ . Let  $x_0 \in (a, b)$  and let  $f, g$  be differentiable in  $x_0$ . Then it holds that

- $f + g : (a, b) \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and the derivative is given by  $(f + g)'(x_0) = f'(x_0) + g'(x_0)$ .
- Let  $\lambda \in \mathbb{R}$ . Then it holds that  $\lambda \cdot f : (a, b) \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and it holds that  $(\lambda f)'(x_0) = \lambda \cdot (f'(x_0))$ .
- Let  $f \cdot g : (a, b) \rightarrow \mathbb{R}$  be differentiable and it holds that

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + g'(x_0) \cdot f(x_0)$$

This is the so-called *product law for derivatives*.

*Proof.* • Addition holds:

$$\begin{aligned} f'(x_0) + g'(x_0) &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) + g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{(f(x) + g(x)) - (f(x_0) + g(x_0))}{x - x_0} = (f + g)'(x_0) \end{aligned}$$

- Multiplication with a scalar holds:

$$\lambda f'(x_0) = \lambda \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\lambda f(x) - \lambda f(x_0)}{x - x_0} = (\lambda f)'(x_0)$$

- The product law holds:

$$\begin{aligned} &f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ &= g(x_0) \cdot \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \underbrace{f(x_0)}_{=\lim_{x \rightarrow x_0} f(x)} \cdot \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \end{aligned}$$

because  $f$  is differentiable and therefore continuous in  $x_0$ .

$$\begin{aligned} &= \lim_{x \rightarrow x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{f(x) \cdot g(x) - f(x) \cdot g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{g(x_0)f(x) - g(x_0)f(x_0) + g(x)f(x) - g(x_0)f(x)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x) \cdot g(x) - f(x_0)g(x_0)}{x - x_0} = (f \cdot g)'(x_0) \end{aligned}$$

□

**Definition 60.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be given. Assume  $f$  is differentiable in *every* point  $x_0 \in (a, b)$ , then we call  $f$  is *differentiable on interval*  $(a, b)$ . The mapping  $f' : (a, b) \rightarrow \mathbb{R}$  which assigns  $x \in (a, b)$  its  $f'(x)$ , is called *derivative function*.

$f$  is called *continuously* differentiable if  $f'$  is a continuous function on  $(a, b)$ .

This lecture took place on 15th of Jan 2015 with lecturer Wolfgang Ring.

Exam date: 4th February 2016 14:00.

**Remark 25.** Let  $D \subseteq \mathbb{R}$  and let  $x_0 \in D$  be limit point of  $D$ . Then the function

$$\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0} \text{ in } D \setminus \{x_0\}$$

can be investigated and the question of existence of a limes of  $\varphi$  (theoretically) answered.

Therefore the function  $f : [a, b] \rightarrow \mathbb{R}$  can be discussed in term of convergence and  $f'(a)$  and  $f'(b)$  can be defined (under the assumption that the limes exists)

$$k = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \Leftrightarrow \forall (\xi)_{n \in \mathbb{N}}, \min \xi_n \geq a, \lim_{n \rightarrow \infty} \xi_n = a \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{f(\xi_n) - f(a)}{\xi_n - a} = k$$

The derivative in  $a$  is *right-sided*. The derivative in  $b$  is *left-sided*.

**Remark 26.** Functions that are not differentiable:

- $f(x) = x$  is not differentiable in  $x = 0$ .

*Proof.* Let  $\varepsilon_1 = \frac{1}{n}$ .

$$\lim_{n \rightarrow \infty} \frac{f(\xi_n) - f(0)}{\xi_n - 0} = \lim_{n \rightarrow \infty} \frac{\left| \frac{1}{n} \right| - |0|}{\frac{1}{n} - 0} = 1 \xrightarrow{n \rightarrow \infty} 1$$

“right-sided limes”

Let  $\eta_n = -\frac{1}{n}$ .

$$\lim_{n \rightarrow \infty} \frac{f(\eta_n) - f(0)}{\eta_n - 0} = \frac{\left| -\frac{1}{n} \right| - 0}{-\frac{1}{n} - 0} = \frac{\frac{1}{n}}{-\frac{1}{n}} = -1 \xrightarrow{n \rightarrow \infty} -1$$

“left-sided limes”

Therefore limes of  $f(\xi_n)$  and  $f(\eta_n)$  are different even though both sequences  $(\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$  have the same limes. Therefore it is not differentiable in  $x = 0$ .  $\square$

- Consider  $g : [a, b] \rightarrow \mathbb{R}$  with  $g(x) = \sqrt{x}$ . Claim:  $g$  is not differentiable in  $x = 0$ .

*Proof.* Let  $(\xi)_{n \in \mathbb{N}}$  and  $\xi_n = \frac{1}{n} \Rightarrow \lim_{n \rightarrow \infty} \xi_n = 0$ .

$$\frac{g(\xi_n) - g(0)}{\xi_n - 0} = \frac{\sqrt{\frac{1}{n}} - \sqrt{0}}{\frac{1}{n} - 0} = \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}$$

$(\sqrt{n})_{n \in \mathbb{N}}$  is unbounded, therefore not convergent.

### 14.3 Computing with the limes of functions

We actually used that already (for example, when proving the product law for derivatives).

**Theorem 79.** Let  $f, g : D \rightarrow \mathbb{C}$  with  $d \subseteq \mathbb{C}$ . Let  $z_0 \in \mathbb{C}$  be limit point of  $D$  and  $f$  has limes  $a \in \mathbb{C}$  in  $z_0$  and  $g$  has limes  $b$  in  $z_0$ . Then

- $(f + g)$  has limes  $a + b$  in  $z_0$ .
- $(f \cdot g)$  has limes  $a \cdot b$  in  $z_0$
- If  $g(z) \neq 0 \quad \forall z \in D$  and  $b \neq 0$ , then  $\frac{f}{g}$  has the limes  $\frac{a}{b}$  in  $z_0$ .

*Proof.* Sequence criterion and laws for convergent sequences. Let  $(\xi)_{n \in \mathbb{N}}$  and  $\xi_n \in D$  and  $\lim_{n \rightarrow \infty} \xi_n = z_0$ . Because  $f$  has limes  $a$  and  $g$  has limes  $b$ , it holds that

$$\lim_{n \rightarrow \infty} f(\xi_n) = a \wedge \lim_{n \rightarrow \infty} g(\xi_n) = b$$

Due to the laws for convergent sequences:

$$\underbrace{\lim_{n \rightarrow \infty} f(\xi_n) + \lim_{n \rightarrow \infty} g(\xi_n)}_{a+b} \\ = \lim_{n \rightarrow \infty} (f(\xi_n) + g(\xi_n)) = \lim_{n \rightarrow \infty} (f + g)(\xi_n)$$

Therefore  $\lim_{\xi \rightarrow z_0} (f + g)(\xi) = a + b$ .

The proofs work analogously for  $\cdot$  and  $/$ .  $\square$

### 14.4 Other equivalent definitions of differential calculus

**Theorem 80.**

$$f : [a, b] \rightarrow \mathbb{R} \text{ or } f : (a, b) \rightarrow \mathbb{R}$$

In general, let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$  and  $x_0 \in I$ . Then  $f$  is differentiable in  $x_0$  if and only if there exists  $\varphi : I \rightarrow \mathbb{R}$  such that  $\varphi$  is continuous in  $x_0$  and  $f(x) = f(x_0) + \varphi(x)(x - x_0)$ .

$\square$  If  $\varphi$  exists with such properties,  $f'(x_0) = \varphi(x_0)$ .

*Proof.*  $\Leftarrow$  Let  $x \neq x_0$ ,  $x \in I$  and it holds that  $f(x) = f(x_0) + \varphi(x)(x - x_0)$ , then

$$\varphi(x) = \frac{f(x) - f(x_0)}{x - x_0}$$

because  $\varphi$  is continuous, there exists some limes

$$\lim_{x \rightarrow x_0} \varphi(x) = \varphi(x_0)$$

Hence  $f$  is differentiable and  $f'(x_0)$ .

$\Rightarrow$  Let  $f$  be differentiable. Then we define

$$\varphi(x) = \begin{cases} \frac{f(x) - f(x_0)}{x - x_0} & \text{if } x \neq x_0 \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

then  $\varphi$  is continuous in  $x_0$  and

$$f(x) = f(x_0) + \varphi(x)(x - x_0) \text{ for } x \neq x_0$$

$$f(x_0) = f(x_0) + \varphi(x_0) \underbrace{(x_0 - x_0)}_0 \text{ for } x = x_0$$

□

**Theorem 81.** Let  $J, I$  be intervals.

$$f : I \rightarrow J$$

$$g : J \rightarrow \mathbb{R}$$

$f$  is differentiable in  $x_0 \in I$  and let  $g$  be differentiable in  $y_0 = f(x_0)$ . Then  $g \circ f : I \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and it holds that

$$(g \circ f)'(x_0) = g'(y_0) \cdot f'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

*Proof.*  $f$  is differentiable implies  $\exists \varphi : I \rightarrow \mathbb{R}$  is continuous in  $x_0$  with  $f(x) = f(x_0) + \varphi(x)(x - x_0)$ .

$g$  is differentiable implies  $\exists \psi : J \rightarrow \mathbb{R}$  with  $g(y) = g(y_0) + \psi(y)(y - y_0)$  is continuous.

Let  $y \in f(I)$ , hence  $y = f(x)$  and  $y_0 = f(x_0)$ . It follows (due to the previous theorems) that

$$g(f(x)) = g(f(x_0)) + \psi(f(x)) \underbrace{(f(x) - f(x_0))}_{\varphi(x)(x - x_0)}$$

$$= g(f(x_0)) + \psi(f(x))\varphi(x)(x - x_0)$$

$$g \circ f(x) = g \circ f(x_0) + (\psi \cdot f)(x) \cdot \varphi(x) \cdot (x - x_0)$$

$$\vartheta(x) = \psi \circ f(x) \cdot \varphi(x)$$

with  $\vartheta : I \rightarrow \mathbb{R}$  and  $f$  is continuous in  $x_0$ , because it is differentiable,  $\psi$  is continuous in  $y_0 = f(x_0)$  and  $\varphi$  is continuous in  $x_0$ . Therefore  $\vartheta$  is continuous in  $x_0$  and  $g \circ f(x) = g \circ f(x_0) + \vartheta(x)(x - x_0)$ . Therefore  $g \circ f$  is differentiable in  $x_0$  and

$$(g \circ f)'(x_0) = \vartheta(x_0) = \underbrace{\psi(f(x_0))}_{g'(f(x_0))} \cdot \underbrace{\varphi(x_0)}_{f'(x_0)}.$$

□

**Example 30.**

$$f : \mathbb{R} \rightarrow \mathbb{R}^+, f(x) = x^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}, g(x) = e^x$$

$$g \circ f : \mathbb{R} \rightarrow \mathbb{R}$$

$$g \circ f(x) = e^{f(x)} = e^{x^2}$$

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

$$g'(y) = e^y, g'(f(x_0)) = e^{f(x_0)} = e^{x_0^2}$$

$$f'(x_0) = 2x_0$$

$$(e^{x^2})' = \underbrace{e^{x^2}}_{\text{outer derivative}} \cdot \underbrace{2x}_{\text{inner derivative}}$$

$$f \circ g : \mathbb{R} \rightarrow \mathbb{R}$$

$$(f \circ g)(x) = (e^x)^2$$

$$(f \circ g)'(x) = \underbrace{f'(g(x))}_{2(y(x))=2e^x} \circ \underbrace{g'(x)}_{=e^x}$$

$$\Rightarrow 2 \cdot e^x \cdot e^x = 2e^{2x}$$

**Example 31.** We decompose this function  $h$ .

$$h(x) = \cos(\sqrt{x^2 + 1})$$

$$h(x) = g \circ f(x)$$

So we either get

$$g(y) = \cos(\sqrt{y})$$

$$f(x) = x^2 + 1$$

or

$$g(y) = \cos(y)$$

$$f(x) = \sqrt{x^2 + 1}$$

Both are correct. Not the second decomposition is way more useful.

**Theorem 82.** Consider  $r : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  and  $r(x) = \frac{1}{x}$ . Then it holds that  $r$  is differentiable for all  $x_0 \neq 0$  and  $r'(x_0) = -\frac{1}{x_0^2}$ .

*Proof.*

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{\frac{x_0 - x}{x x_0}}{x - x_0} \\ &= - \lim_{x \rightarrow x_0} \frac{1}{x x_0} \\ &= -\frac{1}{x_0^2} \end{aligned}$$

□

**Theorem 83.** Let  $g : I \rightarrow \mathbb{R}$  with  $g(x) \neq 0 \quad \forall x \in I$  where  $I$  is an interval. Let  $g$  be differentiable in  $x_0 \in I$ . Then  $\frac{1}{g} : I \rightarrow \mathbb{R}$  is differentiable in  $x_0$  and it holds

$$\text{that } \left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}.$$

Furthermore let  $f : I \rightarrow \mathbb{R}$  differentiable in  $x_0$ . Then the quotient  $\left(\frac{f}{g}\right)$  is differentiable in  $x_0$  and it holds that

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{(g(x_0))^2}$$

“Quotient law”

*Proof.* To be done rigorously next Wednesday.

Idea:  $\frac{1}{g} = r \circ g$  and quotient law

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

□

This lecture took place on 20th of January 2016 with lecturer Wolfgang Ring.

*Proof.*

$$\frac{1}{g} = r \circ g \quad r(y) = \frac{1}{y}$$

Chain rule:  $x_n \in I$  and  $g$  differentiable in  $x_0$ ,  $y_0 = g(x_0) \neq 0$  and  $r(y) = \frac{1}{y}$  in  $y_0$ . Therefore  $g \circ y$  is in  $x_0$  and

$$(r \circ g)'x_0 = r'(g(x_0)) \cdot g'(x_0) = -\frac{1}{g(x_0)^2} \cdot r'(x_0)$$

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

Product law:

$$\left(\frac{f}{g}\right)'(x_0) = f'(x_0) \cdot \frac{1}{g(x_0)} + f(x_0) \cdot \left(-\frac{g'(x_0)}{(g(x_0))^2}\right) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$$

□

**Remark 27.** What is differential calculus good for?

Geometrical investigation of functions.

**Definition 61.** Let  $f : I \rightarrow \mathbb{R}$  be a function.  $I$  is an interval. We call  $x_0 \in I$  a *local maximum* of  $f$ , if  $\varepsilon > 0$  exists such that

$$[x \in I \wedge |x - x_0| < \varepsilon] \Rightarrow f(x) \leq f(x_0)$$

We call  $x_0 \in I$  a *local minimum* of  $f$ , if  $\varepsilon > 0$  exists such that

$$[x \in I \wedge |x - x_0| < \varepsilon] \Rightarrow f(x) \geq f(x_0)$$

**Theorem 84** (Necessary optimality criterion). Let  $f : I \rightarrow \mathbb{R}$  be differentiable and  $I$  is an interval. Let  $x_0 \in I$  be a local maximum of  $f$ . Then there exists  $\varepsilon > 0$  such that for all  $x \in I$  with  $|x - x_0| < \varepsilon$  the following relation holds:

$$f'(x_0)(x - x_0) \leq 0.$$

**Remark 28.** This is a more general statement than  $f'(x_0) = 0$ .

*Proof.* Let  $x_0$  be a local maximum. Assume

$$\forall \varepsilon > 0 \exists x_\varepsilon : |x_\varepsilon - x_0| < \varepsilon \wedge f'(x_0)(x_\varepsilon - x_0) > 0$$

Especially:  $\varepsilon = \frac{1}{n}$ ,  $x_\varepsilon = x_n$ . Therefore it holds that  $\lim_{n \rightarrow \infty} x_n = x_0$  and  $f'(x_0)(x_n - x_0) > 0$ . Followingly both factors must be non-zero, hence  $f'(x_0) \neq 0$ .  $\square$

**Theorem 85** (Differentiability of  $f$  in  $x_0$ ).

$$f(x_0) = f(x_0) - f'(x_0)(x_n - x_0) + \underbrace{r(x_0)(x_n - x_0)}_{\mathcal{O}(|x_n - x_0|)}$$

$$\lim_{x \rightarrow x_0} r(x) = 0$$

Let  $n$  sufficiently large such that

$$|f(x_n)| \leq \frac{1}{2} \underbrace{|f'(x_0)|}_{>0} \quad \forall n \geq N$$

Then it holds that

$$\begin{aligned} f(x_n) - f(x_0) &= \underbrace{f'(x_0)(x_n - x_0)}_{>0} + r(x_n)(x_n - x_0) \\ &= |f'(x_0)(x_n - x_0)| + r(x_n)(x_n - x_0) \\ &\geq |f'(x_0)| |x_n - x_0| - |r(x_n)| |x_n - x_0| \\ &= \left( |f'(x_0)| - \underbrace{|r(x_n)|}_{\leq \frac{1}{2}|f'(x_0)|} \right) \cdot |x_n - x_0| \geq \frac{1}{2} \end{aligned}$$

$$= \frac{1}{2} \underbrace{f'(x_0)(x_n - x_0)}_{>0} > 0$$

and therefore  $f(x_n) > f(x_0) \quad \forall n \geq N$ . This is a contradiction to the assumption that  $x_0$  is a local maximum.

**Remark 29.**  $x_0$  is a local minimum. Therefore

$$f'(x_0)(x - x_0) \geq 0 \quad \forall |x - x_0| < \varepsilon \text{ where } x \in I$$

**Corollary 17.** Let  $I$  be an interval and  $x_0$  an inner point of  $I$  (therefore  $\exists \varepsilon > 0 : (x_0 - \varepsilon, x_0 + \varepsilon) \subset I$ ). Assume  $f : I \rightarrow \mathbb{R}$  has a local maximum (or minimum) in  $x_0$  and let  $f$  be differentiable. Then it holds that

$$f'(x_0) = 0$$

*Proof.* Let  $\varepsilon > 0$  such that  $(x_0 - \varepsilon, x_0 + \varepsilon) \subset I$  and let  $x = x_0 + \frac{\varepsilon}{2} \in I$ .

The optimality criterion is given with:

$$f'(x_0) \cdot (x - x_0) = f'(x_0) \left( x_0 + \frac{\varepsilon}{2} - x_0 \right) = \frac{\varepsilon}{2} f'(x_0) \leq 0$$

$$w = x_0 - \frac{\varepsilon}{2} \in I$$

Necessary optimality criterion:

$$f'(x_0)(w - x_0) = f'(x_0) \left( x_0 - \frac{\varepsilon}{2} - x_0 \right) = -\frac{\varepsilon}{2} f'(x_0) \leq 0$$

$$f'(x_0) \leq 0 \text{ and } f'(x_0) \geq 0 \Rightarrow f'(x_0) = 0$$

$\square$

This lecture took place on 21st of January 2016 with lecturer Wolfgang Ring.

**Theorem 86** (Consideration of optimal points at the borders of  $I$ ). Let  $I = [a, b]$  and  $x_0 = a$  is a local maximum. Then the necessary optimality criterion (NOC) yields:

$$\text{NOC:} \quad f'(a)(x - a) \leq 0 \quad x \in [a, b]$$

and  $x$  is sufficiently close to  $a$ . Choose  $\varepsilon$  small enough such that for  $x = a + \varepsilon$  (necessary optimality criterion)

$$\Rightarrow f'(a)(a' - \varepsilon - a') = \varepsilon f'(a) \leq 0$$

$$\Rightarrow f'(a) \geq 0$$

Analogously:

$x_0 = a$  is a local minimum. So  $f'(a) \geq 0$ .

$x_0 = b$  is a local maximum. So  $f'(b) \leq 0$ .

Michel Rolle (1652–1719)

**Theorem 87** (Rolle's theorem). Let  $I = [a, b]$  and  $f : I \rightarrow \mathbb{R}$  is differentiable in  $I$ . Furthermore it holds that  $f(a) = f(b)$ . Then there exists some  $\xi \in [a, b]$  with  $f'(\xi) = 0$ .

*Proof. Case 1:  $f$  constant* Therefore  $f(x) = f(a) = f(b) \forall x \in [a, b]$

$$\Rightarrow f'(x) = 0 \forall x \in [a, b]$$

**Case 2:  $f$  is non-constant** Therefore  $\exists x \in (a, b)$  with  $f(x) \neq f(a)$ . Without loss of generality:  $f(x) > f(a) = f(b)$ .  $[a, b]$  is a compact interval.  $f$  is continuous in  $[a, b]$  (because it's differentiable). The theorem about the existence of a global maximum tells us:

$$\exists \xi \in [a, b] : f(\xi) \geq f(z) \quad \forall z \in [a, b]$$

$$f(\xi) \geq f(x) > f(a) = f(b)$$

$$\Rightarrow \xi \neq a \wedge \xi \neq b$$

So  $\xi$  is an inner point of  $[a, b]$ , hence  $\xi \in (a, b)$ .

Analogously the same holds for a minimum: Without loss of generality:  $f(x) < f(a) = f(b)$ . And the same proof works for a global minimum.

□

**Theorem 88** (Mean value theorem). Let  $I = [a, b]$  be a compact interval with  $a < b$  and let  $f : I \rightarrow \mathbb{R}$  be differentiable in  $[a, b]$ . Then there exists some  $\xi \in [a, b]$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

(Sogan,  $\xi \in [a, b]$ )

Equivalently,

$$f(b) = f(a) + f'(\xi)(b - a)$$

$$f(a) = f(b) + f'(\xi)(a - b)$$

*Proof.* Let  $g(x) = f(x) - s(x)$ .

$$= f(x) - \underbrace{\left[ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]}_{\text{linear, hence differentiable}}$$

$$\Rightarrow g(a) = f(a) - [f(a) - 0] = 0$$

$$g(b) = f(b) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(b - a) \right] = 0$$

By the Rolle's Theorem it follows that

$$\exists \xi \in [a, b] \text{ with } g'(\xi) = 0$$

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

$$g'(\xi) = 0 \Rightarrow f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

□

**Definition 62** (Monotonicity for functions). Let  $I$  be an interval,  $f : I \rightarrow \mathbb{R}$ . Then  $f$  is called *monotonically increasing* in  $I$  if

$$x_1, x_2 \in I \wedge x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

$f$  is called *monotonically decreasing* in  $I$  if

$$x_1, x_2 \in I \wedge x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

$f$  is called *strictly monotonically increasing* in  $I$

$$x_1, x_2 \in I \wedge x_1 \leq x_2 \Rightarrow f(x_1) < f(x_2)$$

$f$  is called *strictly monotonically decreasing* in  $I$

$$x_1, x_2 \in I \wedge x_1 \leq x_2 \Rightarrow f(x_1) > f(x_2)$$

**Theorem 89.** Let  $f : I \rightarrow \mathbb{R}$  be differentiable in  $I$  where  $I$  is some interval. Then

- $f$  is monotonically increasing in  $I \Leftrightarrow f'(x) \geq 0 \quad \forall x \in I$
- $f$  is monotonically decreasing in  $I \Leftrightarrow f'(x) \leq 0 \quad \forall x \in I$

*Proof.* We only show the proof for monotonically increasing functions. It follows analogously for monotonically decreasing functions.

$\Rightarrow$  Let  $f$  be monotonically increasing and  $x_0 \in I$ . Let  $(w_n)_{n \in \mathbb{N}}$  and  $w_n \in I$  with  $\lim_{n \rightarrow \infty} w_n = x_0, w_1 \neq x_0 \quad \forall n \in \mathbb{N}$ . Then it holds that

$$f'(x_0) = \lim_{n \rightarrow \infty} \underbrace{\frac{f(w_n) - f(x_0)}{w_n - x_0}}_{S_n}$$

- If  $w_n > x_0$ , then  $f(w_n) \geq f(x_0)$  due to monotonicity.

$$\Rightarrow S_n \geq 0$$

- If  $w_n < x_0$  (hence  $w_n - x_0 < 0$ ), then  $f(w_n) \leq f(x_0)$  hence  $f(w_n) - f(x_0) \leq 0$ , due to monotonicity.

$$\Rightarrow S_n \geq 0$$

$$\Rightarrow f'(x_0) = \lim_{n \rightarrow \infty} S_n \geq 0$$

$\Leftarrow$  Let  $f'(x) \geq 0 \forall x \in I$ . Show that  $f$  is monotonically increasing.

Proof by contradiction: Assume the opposite.  $f$  is not monotonically increasing, so there exist  $x_1, x_2 \in I$  with  $x_1 < x_2$  and  $f(x_1) > f(x_2)$ .  $f$  is differentiable in  $[x_1, x_2] \subseteq I$ . The Intermediate Value Theorem tells us that  $\exists \xi \in (x_1, x_2)$  with

$$f'(\xi) = \frac{\overbrace{f(x_2) - f(x_1)}^{<0}}{\underbrace{x_2 - x_1}_{>0}} \\ \Rightarrow f'(\xi) < 0$$

This contradicts with our assumption that  $f'(x) \geq 0 \forall x \in I$ .

**Lemma 19.** Let  $f : I \rightarrow \mathbb{R}$  where  $I$  is an interval. Let  $f$  be differentiable in  $I$ . Assume

$$f'(x) > 0 \quad \forall x \in I$$

Then it follows that  $f$  is strictly monotonically increasing.

Assume

$$f'(x) > 0 \quad \forall x \in I$$

Then it follows that  $f$  is strictly monotonically decreasing.

**Attention!** This is a necessary, but not sufficient condition!  $f(x) = x^3$  is strictly monotonically increasing in  $\mathbb{R}$ , but  $f'(x) = 3x^2$  and therefore  $f'(0) = 0$ .

*Proof.* See the previous proof, part  $\Leftarrow$ , and use  $f(x_1) \geq f(x_2)$  and  $f'(\xi) \leq 0$  in contradiction to  $f'(x) > 0 \quad \forall x \in I$ .  $\square$

**Theorem 90** (Generalization of the IVT). Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be differentiable in  $[a, b]$  and  $g'(x) \neq 0$  for all  $x \in [a, b]$ . Then it holds that

$$g(a) \neq g(b)$$

and there exists  $\xi \in (a, b)$  with

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(If  $g(x) = x$ , the IVT is given as special case)

*Proof.*

$$F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a))$$

It holds that  $g(a) \neq g(b)$ , because  $g(a) = g(b)$ . Rolle's Theorem implies that  $g'(\xi) = 0$  for some  $\xi \in (a, b)$ . This is a contradiction to our assumption.

$F$  is well-defined and differentiable in  $[a, b]$ .

$$F(a) = f(a) - 0$$



$$\begin{aligned} F(b) &= f(b) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(b) - g(a)) \\ &= f(b) - f(b) + f(a) \\ &= f(a) \end{aligned}$$

By Rolle's Theorem it follows that

$$\begin{aligned} \exists \xi \notin (a, b) \text{ with } F'(\xi) &= 0 \\ F'(x) &= f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(x) \\ F'(\xi) = 0 &\Rightarrow \frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)} \end{aligned}$$

Guillaume Francois Antoine Marquis de l'Hôpital (1661–1704)

**Example 32** (Application of this generalization). Assume  $f, g$  are differentiable in  $I$ . Let  $x_0 \in I$  with  $f(x_0) = g(x_0)$ . Therefore  $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = y_0$ .

If  $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)}$  exists, then

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{\xi \rightarrow x_0} \frac{f'(\xi)}{g'(\xi)}$$

“L'Hôpital's rule”

*Proof.* Assuming the generalization of the IVT, we have:

$$\exists \xi \in [x, x_0] \text{ wlog. } x < x_0 : \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}$$

and for  $|x - x_0| < \varepsilon$  it holds that

$$\begin{aligned} |\xi - x_0| &< \varepsilon \\ \Rightarrow x \rightarrow x_0 &\Rightarrow \xi \rightarrow x_0 \end{aligned}$$

**Example 33.**

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{\xi \rightarrow 0} \frac{e^\xi}{1} = 1$$

This holds only if the limit actually exists.

**Corollary 18.** Corollaries following this monotonicity criterion:

- Let  $f : I \rightarrow \mathbb{R}$  differentiable in  $I$  and let  $x_0 \in I$  be a local maximum. Then there exists  $\varepsilon > 0$  such that for all  $x \in I$  with  $x \in (x_0 - \varepsilon, x_0]$  it holds that

$$f(x) \leq f(x_0) \wedge \forall w \in I \text{ with } w \in [x_0, x_0 + \varepsilon) : f(w) \leq f(x_0)$$

- Assume  $f$  is monotonically increasing in  $(x_0 - \varepsilon, x_0]$  and  $f$  is monotonically decreasing in  $[x_0, x_0 + \varepsilon)$

□

$$\Rightarrow \exists x, \tilde{x} \in (x_0 - \varepsilon, x_0] : f(x) \leq f(\tilde{x}) \wedge \forall w, \tilde{w} \in [x_0, x_0 + \varepsilon)$$

with  $\tilde{w} \leq w$  it holds that  $f(\tilde{w}) \geq f(w)$ . Especially for  $\tilde{x} = x_0$  and  $\tilde{w} = x_0$  it holds that

$$f(x) \leq f(x_0) \wedge f(x_0) \geq f(w)$$

Condition for local maximum: Therefore if  $\varepsilon > 0$  exists, such that  $f$  in  $I \cap (x_0 - \varepsilon, x_0]$  monotonically increasing and  $f$  in  $I \cap [x_0, x_0 + \varepsilon)$  is monotonically decreasing, then  $f$  has a local maximum in  $x_0$ .

This is a sufficient condition for a maximum. So if this condition holds, a maximum is given.

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This lecture took place on 22nd of Jan 2015 with lecturer

**Theorem 91.** Let  $(w_n)_{n \in \mathbb{N}}$  with  $w_n \in I$  such that  $\lim_{n \rightarrow \infty} w_n = x_0$  and

$$\xi_n \in \begin{cases} [w_n, x_0] & \text{if } w_n < x_0 \\ [x_0, w_n] & \text{if } x_0 < w_n \end{cases}$$

with

$$\frac{f(w_n) - f(x_0)}{g(w_n) - g(x_0)} = \frac{f'(\xi_n)}{g'(\xi_n)}$$

□

Because  $|\xi_n - x_0| < \underbrace{|w_n - x_0|}_{\rightarrow 0}$  it holds that

$$\lim_{n \rightarrow \infty} \xi_n = x_0.$$

If  $\lim_{n \rightarrow \infty} \frac{f'(\xi_n)}{g'(\xi_n)} = d$ .

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{f(w_n) - f(x_0)}{g(w_n) - g(x_0)} = d$$

## 14.5 Sufficient optimality criteria

**Theorem 92.** Let  $f : I \rightarrow \mathbb{R}$ ,  $x_0 \in I$ . If  $\varepsilon > 0$  exists, such that  $f$  is monotonically increasing in  $(x_0 - \varepsilon, x_0] \cap I$  and  $f$  is monotonically decreasing in  $[x_0, x_0 + \varepsilon) \cap I$ , then  $f$  has a local maximum in  $f$ .

**Remark 30.** Informal: Increasing to the right, decreasing to the left? So it must be a local maximum.

**Remark 31.** This is a sufficient, but not necessary condition. Compare with Figure 33.

**Remark 32.** TODO: content missing

that  $f'(x) \geq 0$  and  $\forall x \in [x_0, x_0 + \varepsilon)$  hold, that  $f'(x) \leq 0$ , then  $f$  is monotonically increasing to the left of  $x_0$  and monotonically decreasing to the right, hence  $x_0$  is a local maximum.

Hence, the point when  $f'$  changes its sign, a maximum (or minimum) is given.

All statements hold analogously for the minimum (negate the operators).

## 14.6 Behavior of curvatures in functions

**Remark 33.** Assume the line on the graph defines our road. Do we need to drive to the left or right in a curvature?

**Definition 63.** Let  $I$  be an interval  $f : I \rightarrow \mathbb{R}$ . Then  $f$  is called *convex* in  $I$  if  $\forall a, b \in I$  with  $a < b$  and for all  $\lambda \in [0, 1]$  it holds that

$$f((1 - \lambda) \cdot a + \lambda \cdot b) \leq (1 - \lambda)f(a) + \lambda f(b)$$

$f$  is called *concave* if the following holds:

$$f((1 - \lambda) \cdot a + \lambda \cdot b) \geq (1 - \lambda)f(a) + \lambda f(b)$$

$f$  is called *strictly convex* if the following holds:

$$f((1 - \lambda) \cdot a + \lambda \cdot b) < (1 - \lambda)f(a) + \lambda f(b)$$

$f$  is called *strictly concave* if the following holds:

$$f((1 - \lambda) \cdot a + \lambda \cdot b) > (1 - \lambda)f(a) + \lambda f(b)$$

**Remark 34.** Let  $\lambda \in [0, 1]$ .

$$(1 - \lambda) \cdot a + \lambda \cdot b \leq (1 - \lambda) \cdot b + \lambda \cdot b = b$$

$$(1 - \lambda) \cdot a + \lambda \cdot a = a$$

$(1 - \lambda) \cdot a + \lambda \cdot b$  defines an arbitrary point in  $[a, b]$ . It's called *convex combination* of  $a$  and  $b$ .

**Remark 35.** In case of convexness, the function graph lies underneath the function. Compare with Figure 34.

**Theorem 93.** Let  $f : I \rightarrow \mathbb{R}$  be differentiable and  $I$  an interval. Then it holds that  $f$  is convex in  $I$

$$\Leftrightarrow f' : I \rightarrow \mathbb{R}$$

is monotonically increasing. Analogously for concave and monotonically decreasing.

*Proof.*  $\Leftarrow$  Let  $f' : I \rightarrow \mathbb{R}$  be monotonically increasing. Let  $a, b \in I$ ,  $a < b$  and let  $\lambda \in (0, 1]$ .

Let  $\lambda = 0$ . Then it holds that

$$f((1 - 0) \cdot a + 0 \cdot b) = f(a) \leq (1 - 0) \cdot f(a) + 0 \cdot f(b)$$

Hence convexity condition is satisfied. Analogously it holds for  $\lambda = 1$ .

$$f((1 - 1) \cdot a + 1 \cdot b) = f(b) = (1 - 1) \cdot f(a) + 1 \cdot f(b)$$

$$\begin{aligned} & (1-\lambda)f(a) + \lambda f(b) - \underbrace{1}_{((1-\lambda)+\lambda)} \cdot f((1-\lambda) \cdot a + \lambda \cdot b) \\ &= (1-\lambda)f(a) - (1-\lambda)f((1-\lambda)a + \lambda b) + \lambda f(b) - \lambda f((1-\lambda) \cdot a + \lambda b) \\ &= (1-\lambda) (f(a) - f((1-\lambda)a + \lambda b)) + \lambda [f(b) - f((1-\lambda)a + \lambda b)] \end{aligned}$$
$$= \lambda[f(b) - f(x_\lambda)] - (1 - \lambda)[f(x_\lambda) - f(a)]$$

$$\in (x_\lambda, b) \text{ such that } f(b) - f(x_\lambda) = f'(\xi_2)(b - x_\lambda)$$
$$f(b) - f(x_\lambda) = f'(\xi_2)(b - x_\lambda)$$
$$\exists \xi_1 \in (a, x_\lambda) \text{ such that } f(x_\lambda) - f(a) = f'(\xi_\lambda)(x_\lambda - a) \text{ and } f'(\xi_1)\lambda(b - a)$$

because  $f'$  is monotonically increasing and  $\xi_1 < x_\lambda < \xi_2$  holds.

Therefore it holds that  $(1 - \lambda)f(a) + \lambda f(b) \geq f(x_\lambda)$

$$f'(x_1) \leq f'(x_2)$$
$$\lim_{n \rightarrow \infty} w_n = x_1 \text{ and } \lim_{n \rightarrow \infty} z_n = x_2$$
$$\frac{f(x_2) - f(z_n)}{x_2 - z_n} - \frac{f(w_n) - f(x_1)}{w_n - x_1}$$

## Convexity: From

It follows that

So

for  $n \rightarrow \infty$ . So  $f'(x_2) \geq f'(x_0)$ .

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**Definition 64.** Let  $f : I \rightarrow \mathbb{R}$  and  $x_0 \in I$ . Assume  $x_0$  is an inner point of  $I$  and  $\exists \varepsilon > 0$  such that  $(x_0 - \varepsilon, x_0 + \varepsilon) \subseteq I$  and  $f$  in  $(x_0 - \varepsilon, x_0]$  is convex and  $f$  in  $[x_0, x_0 + \varepsilon)$  is concave, then  $x_0$  is called *inflection point*.

If  $f$  is concave in  $(x_0 - \varepsilon, x_0]$  and convex in  $[x_0, x_0 + \varepsilon)$ , then  $x_0$  is also an inflection point.

**Definition 65** (Higher derivatives). Assume  $f : I \rightarrow \mathbb{R}$  is differentiable in  $I$  and the derivative  $f' : I \rightarrow \mathbb{R}$  in a point  $x_0 \in I$  itself is differentiable. Then  $f''(x_0) = (f')'(x_0)$  is called *second derivative* of  $f$  in  $x_0$ .

Analogously for higher derivatives: Let the derivative function of order  $n$  ( $n \in \mathbb{N}$ ) be already defined and let itself be differentiable in  $x_0$ , then

$$f^{(n-1)} : I \rightarrow \mathbb{R}$$

is called derivative function of  $(n-1)$ -th order where

$$f^{(0)} = f, f^{(1)} = f'$$

Then we let

$$f^{(n)}(x_0) = (f^{(n-1)})'(x_0)$$

**Remark 36.** We can use the second derivative to check the monotonicity of the first derivative.

$$f^{(2)} : I \rightarrow \mathbb{R}, \quad f^{(2)}(x) \geq 0 \quad \forall x \in I$$

$$\Rightarrow f^{(1)} = f' \text{ is monotonical in } I$$

$$\Rightarrow f \text{ is convex in } I$$

**Remark 37.** Let  $f$  be convex in  $I$  and differentiable in  $x_0$ . Then it holds with  $t : I \rightarrow \mathbb{R}$  and  $t(f) = f(x_0) + f'(x_0)(x - x_0)$ , which is the tangent of  $f$  in  $x_0$ , that

$$\forall x \in I : t(x) \leq f(x)$$

This lecture took place on 27th of January 2016 with lecturer Wolfgang Ring.

TODO: something missing here?

$$P(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$L = \limsup_{k \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$\delta = \frac{1}{L} \quad P(z) \text{ is convergent}$$

## 14.7 Function sequences and uniform convergence

**Sequences, we know:**

$$(z_n)_{n \in \mathbb{N}} \quad z_n \in \mathbb{C} \quad \text{sequence of complex numbers}$$

$$(I_n)_{n \in \mathbb{N}} \quad I_{n+1} \subseteq I_n \quad \text{sequence of intervals}$$

**Function sequences:** Consider  $(f_n)_{n \in \mathbb{N}}$  with  $f : D \rightarrow \mathbb{C}$  with  $D \subseteq \mathbb{C}$ . Then  $(f_n)_{n \in \mathbb{N}}$  is called *function sequence*. It is important to recognize that all functions have the same co-domain.

**Definition 66.** Let  $D \subseteq \mathbb{C}$  and  $f_n : D \rightarrow \mathbb{C}$  for  $n \in \mathbb{N}$  and  $f : D \rightarrow \mathbb{C}$ . We say the function sequence  $(f_n)_{n \in \mathbb{N}}$  is *uniformly convergent* with  $f$  if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} : [n \geq N_\varepsilon \Rightarrow |f_n(z) - f(z)| < \varepsilon \forall z \in D]$$

**Lemma 20.** Let  $(f_n)_{n \in \mathbb{N}}$  be a function sequence in  $D \subseteq \mathbb{C}$  and  $f : D \rightarrow \mathbb{C}$ . Then it holds  $(f_n)_{n \in \mathbb{N}}$  is uniformly convergent in  $D$  towards  $f$  if and only if

$$\lim_{n \rightarrow \infty} \sup \{|f_n(z) - f(z)| : z \in D\} = 0$$

*Proof.*  $\Rightarrow$  Let  $f$  be a uniform limit of  $(f_n)_{n \in \mathbb{N}}$ . Then  $\forall \varepsilon > 0 \exists N_\varepsilon : [n \geq N_\varepsilon \Rightarrow |f_n(z) - f(z)| < \varepsilon \forall z \in D]$

$$\text{for } n \geq N_\varepsilon \text{ it holds that } \sup \{|f_n(z) - f(z)| : z \in D\}$$

So it holds that

$$\sup \{|f_n(z) - f(z)| : z \in D\} \rightarrow_{n \rightarrow \infty} 0$$

$\Leftarrow$  Let  $\varepsilon > 0$ . Convergence of supremum sequence implies that

$$\exists N_\varepsilon \in \mathbb{N} : [n \geq N_\varepsilon \Rightarrow \sup |f_n(z) - f(z)| : z \in D < \varepsilon]$$

for those  $n$  and for every  $z \in D$  it holds that

$$|f_n(z) - f(z)| < \varepsilon$$

□

**Remark 38.** Let  $B(D) = \{f : D \rightarrow \mathbb{C} \text{ with } f \text{ is bound to } D\}$  and

$$\|f\|_\infty = \sup \{|f(z)| : z \in D\}$$

Then it holds that  $(f_n)_{n \in \mathbb{N}}$  converges uniformly towards  $f$  (with  $f_n \in B(D)$  and  $f \in B(D)$ )

$$\Leftrightarrow \|f_n - f\|_\infty \rightarrow 0 \text{ for } n \rightarrow \infty$$

**Remark 39.** It can be shown that  $B(D)$  is a vector space and  $\|\cdot\|_\infty$  is a *norm* in  $B(D)$ , hence

$$\|f\|_\infty =$$

TODO

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty = \forall f, g \in B(D), \alpha \in \mathbb{C}$$

$\|\cdot\|_\infty$  is called *supremum norm* in  $D$ .

$$C_b(D) := \{f : D \rightarrow \mathbb{C}, f \in B(D) \text{ and } f \text{ is continuous in } D\} \subseteq B(D)$$

The supremum norm can also be defined on  $C_b(D)$ .

If  $D = K \subseteq \mathbb{C}$  is compact in  $\mathbb{C}$ , it follows immediately that every continuous function is bounded.

Show that  $\{|f(z)| : z \in K\}$  is a bounded set in  $\mathbb{R}$ .

$$|f| : D \rightarrow \mathbb{R}$$

$|f|$  is the composition of two functions, namely  $f$  and the absolute value function. Both are continuous.  $|f|$  has a maximum, hence  $\exists z_0 \in K : |f(z)| \leq |f(z_0)| \forall z \in K$ . So  $|f(z_0)|$  is upper bound of  $\{|f(z)| : z \in K\}$ .

$$C(K) = \{f : K \rightarrow \mathbb{C} : f \text{ is continuous}\} \subseteq B(K)$$

and for  $f \in C(K)$  it holds that

$$\|f\|_\infty = \sup \{|f(z)| : z \in K\} = \max \{|f(z)| : z \in K\}$$

**Theorem 94.** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of functions in  $D$ . TODO Assume  $f$  TODO  $(f_n)_{n \in \mathbb{N}}$  is uniformly convergent towards  $f$  in  $D$ . Then  $f$  is continuous in  $D$ .

*Proof.* Let  $\varepsilon > 0$  be given and  $z_0 \in D$ . Show that  $\exists \delta > 0$  such that for all  $z \in D$  with

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

1. Because  $(f_n)_{n \in \mathbb{N}}$  converges uniformly towards  $f$ , there exists some

$$N \in \mathbb{N} : |f_N(w) - f(w)| < \frac{\varepsilon}{3} \forall w \in D$$

2. If  $f_N$  is continuous on its own, then

$$\exists \delta > 0 \text{ such that } z \in D \text{ and } |z - z_0| < \delta \Rightarrow |f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$$

Let  $z \in D$  and  $|z - z_0| < \delta$  (with  $\delta$  properties as above). Then it holds that

$$|f(z) - f(z_0)| = \left| f(z) - \underbrace{f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0)}_{=0} - f(z_0) \right|$$

$$\underbrace{\leq}_{\text{triangle inequality}} \underbrace{|f(z) - f_N(z)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(z) - f_N(z_0)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{< \frac{\varepsilon}{3}}$$

The middle term is  $< \frac{\varepsilon}{3}$  because  $f$  is continuous. The other terms are  $< \frac{\varepsilon}{3}$  because of convergence and selection of  $N$ .

So overall  $< \varepsilon$ . So  $f$  is continuous in  $z_0$ . Because  $z_0 \in D$  is arbitrary, it holds for all  $z_0$ . So  $f$  is continuous in  $D$ .

□

This lecture took place on 28th of January 2016 with lecturer Wolfgang Ring.

“The continuous limit of a sequence of continuous functions is continuous”

## 15 Power series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{absolute convergent } \forall z \in B(0, \rho)$$

where  $\rho$  is the convergence radius.  $\rho = \frac{1}{L}$  with

$$L = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

**Lemma 21** (Remaining term estimation). Let  $P(z) = \sum_{n=0}^{\infty} a_n z^n$  be a power series with convergence radius  $\rho > 0$  and let

$$R_n(z) = \sum_{k=n}^{\infty} a_k z^k \quad (k \in \mathbb{N})$$

Assume  $0 \leq |z| \leq r < \rho$ . Then there exists a constant  $c = c(r)$  such that

$$|R_n(z)| \leq c \left( \frac{|z|}{r} \right)^n$$

*Proof.*

$$\begin{aligned} |R_n(z)| &= \left| \sum_{k=n}^{\infty} a_k z^k \right| \leq \sum_{k=n}^{\infty} |a_k| |z|^k = \sum_{k=n}^{\infty} |a_k| \cdot r^k \cdot \underbrace{\frac{|z|^k}{r^k}}_{\leq \frac{|z|^n}{r^n}} \\ &\leq \frac{|z|^n}{r^n} \sum_{k=n}^{\infty} |a_k| r^k \leq \frac{|z|^n}{r^n} \frac{|z|^n}{r^n} \underbrace{\sum_{k=0}^{\infty} |a_k| r^k}_{=c(r)} \end{aligned}$$

Is  $c(r)$ , because the series is absolute convergent and so the series has some value we call  $c$ .  $r \in B(0, \rho)$ .  $\square$

**Theorem 95.** Let  $P(z) = \sum_{k=0}^{\infty} a_k z^k$  be a power series with convergence radius  $\rho > 0$  and let  $0 \leq r < \rho$ . We define

$$P_n(z) = \sum_{k=0}^n a_k z^k$$

( $n$ -th partial sum of the series)

Then  $(P_n)_{n \in \mathbb{N}}$  converges uniformly towards  $P$  in  $B(0, r)$ .

*Proof.* Let  $\hat{r} = \frac{1}{2}(r + \rho)$ , hence  $r < \hat{r} < \rho$ . Then it holds that  $P(\hat{r})$  is convergent (because  $\hat{r} \in B(0, \rho)$ )

So  $\forall z \in B(0, r)$ , the remaining term estimation theorem holds.

$$\begin{aligned} \exists c(\hat{r}) : \left| \sum_{k=n+1}^{\infty} a_k z^k \right| &\leq \frac{|z|^{n+1}}{\hat{r}^{n+1}} \cdot c(\hat{r}) \\ &\leq c(\hat{r}) \cdot \frac{r^{n+1}}{\hat{r}^{n+1}} = c(\hat{r}) \left( \frac{r}{\hat{r}} \right)^{n+1} \end{aligned}$$

Let  $\varepsilon > 0$  be arbitrary and  $N$  sufficiently large such that

$$\left( \underbrace{\frac{r}{\hat{r}}}_{< 1} \right)^{N+1} < \frac{\varepsilon}{c(\hat{r})}$$

Then for all  $n \geq N$  and for all  $z \in B(0, r)$  it holds that

$$\begin{aligned} |P(z) - P_n(z)| &= \left| \sum_{k=0}^{\infty} a_k z^k - \sum_{k=0}^n a_k z^k \right| \\ &= \left| \sum_{k=n+1}^{\infty} a_k z^k \right| \leq \left( \frac{r}{\hat{r}} \right)^{n+1} \cdot c(\hat{r}) \\ &\leq \left( \frac{r}{\hat{r}} \right)^{N+1} \cdot c(\hat{r}) < \frac{\varepsilon}{c(\hat{r})} \cdot c(\hat{r}) = \varepsilon \end{aligned}$$

So it holds that  $P_n \rightarrow P$  is uniform on  $B(0, r)$ .  $\square$

**Corollary 19.**  $P_n(z)$  is continuous in  $\overline{B(0, r)}$

$$\Rightarrow P : \overline{B(0, r)} \rightarrow \mathbb{C} \text{ is continuous}$$

Let  $z \in B(0, \rho)$ , hence  $|z| < \rho$ . Let  $r = \frac{1}{2}(|z| + \rho)$ .

$P$  is continuous in  $\overline{B(0, r)}$  and  $z \in B(0, r)$ . Hence it holds that  $P$  is continuous in  $z$ . So it holds that  $P$  is continuous in  $B(0, \rho)$ . Compare with Figure 36.

### 15.1 The exponential function and its relatives

We want to define the function  $f_{\text{ex}} : \mathbb{C} \rightarrow \mathbb{C}$ , which behaves like  $z \mapsto b^z$ . We want to achieve the power laws in  $f_{\text{ex}}$  as well. We require:

$$(F) \quad f_{\text{ex}}(z_1) \cdot f_{\text{ex}}(z_2) = f_{\text{ex}}(z_1 + z_2) \quad \forall z_1, z_2 \in \mathbb{C}$$

“Functional equation of the exponential function”

**Corollary 20.**

$$f_{\text{ex}}(z) = f_{\text{ex}}(z + 0) = f_{\text{ex}}(z) \cdot f(0)$$

Let  $z \in \mathbb{C}$  such that  $f_{\text{ex}}(z) \neq 0$ . We divide, followingly,

$$f_{\text{ex}}(0) = 1$$

**Corollary 21.** Let  $z \in \mathbb{C}$  be arbitrary and  $k \in \mathbb{N}_+$ . Then

$$z = \underbrace{\frac{z}{k} + \frac{z}{k} + \dots + \frac{z}{k}}_{k \text{ times}}$$

$$f_{\text{ex}}(z) = f_{\text{ex}}\left(\frac{z}{k} + \dots + \frac{z}{k}\right) = \left(f_{\text{ex}}\left(\frac{z}{k}\right)\right)^k$$

**Corollary 22.** Assume:  $f_{\text{ex}}$  is continuous in 0. Let  $z \in \mathbb{C}$  fixed,  $k \in \mathbb{N}$ , then it holds that

$$\frac{z}{k} \rightarrow_{k \rightarrow \infty} 0$$

So it holds that

$$f_{\text{ex}}\left(\frac{z}{k}\right) \rightarrow f_{\text{ex}}(0) = 1$$

**Remark 40.** Approach: Consider  $f_{\text{ex}}(\frac{z}{k}) = 1 + \frac{w_k}{k}$  where  $w_k$  as enumerator is undefined, small and not really important.

**Corollary 23.**

$$w_k = K \cdot \left(f_{\text{ex}}\left(\frac{z}{k}\right) - 1\right)$$

$$f_{\text{ex}}(z) = \left(1 + \frac{w_k}{k}\right)^k$$

Desired:

$$w = \lim_{k \rightarrow \infty} w_k$$

$$f_{\text{ex}}(z) = \lim_{k \rightarrow \infty} \left(1 + \frac{w_k}{k}\right)^k = \lim_{k \rightarrow \infty} \left(1 + \frac{w}{k}\right)^k$$

If the limit of  $w_k$  actually exists, then  $w_k$  depends on  $z$

$$\lim_{k \rightarrow \infty} w_k = \lim_{k \rightarrow \infty} \frac{f_{\text{ex}}\left(\frac{z}{k}\right) - 1}{\frac{1}{k}} = \lim_{k \rightarrow \infty} z \cdot \frac{f_{\text{ex}}\left(\frac{z}{k}\right) - 1}{\frac{z}{k}} = z \cdot \underbrace{\lim_{\substack{w \rightarrow 0 \\ w \in \mathbb{C}}} \frac{f(w) - 1}{w}}$$

With the assumption that this limit actually exists. Then it follows that,

$$w = \lim_{k \rightarrow \infty} w_k = c \cdot z$$

$$f_{\text{ex}}(z) = \lim_{k \rightarrow \infty} \left(1 + \frac{c \cdot z}{k}\right)^k$$

As a general toolbox to define exponential functions.

**Corollary 24.** For  $c = 1$  we get the definition of  $e^z$ .

### 15.2 Fundamental lemma of exponential function

For every convergent complex sequence  $(w_k)_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} w_k = w$  it holds that

$$\lim_{k \rightarrow \infty} \left(1 + \frac{w_k}{k}\right)^k = \sum_{n=0}^{\infty} \frac{1}{n!} w^n$$

**Remark 41.** The constant sequence  $z_n = w \quad \forall k \in \mathbb{N}$  has limit  $w$  and therefore it holds that

$$\lim_{k \rightarrow \infty} \left(1 + \frac{z_k}{k}\right)^k = \underbrace{\lim_{k \rightarrow \infty} \left(1 + \frac{w}{k}\right)^k}_{\text{with } w} = \sum_{n=0}^{\infty} \frac{1}{n!} w^n = \underbrace{\lim_{k \rightarrow \infty} \left(1 + \frac{w_k}{k}\right)^k}_{\text{with } w_k}$$

*Proof of the fundamental lemma.* Let  $\varepsilon > 0$  arbitrary. We choose  $K \in \mathbb{N}$ , such that  $n \geq K \Rightarrow |w_k| \leq |w| + 1$  (this theorem holds because  $|w_k| \rightarrow_{k \rightarrow \infty} |w|$ ). At the same time let  $K$  be sufficiently large such that

$$\sum_{k=K}^{\infty} \frac{(|w| + 1)^k}{k!} < \frac{\varepsilon}{3}$$

This is possible, because the series  $\sum_{n=0}^{\infty} \frac{z^n}{n!}$  converges in  $\mathbb{C}$ .

Let  $n \geq K$ . Then

$$\left| \left(1 + \frac{w_n}{n}\right)^n - \sum_{k=0}^{\infty} \frac{w_n^k}{k!} \right| \stackrel{\text{triangle inequality}}{\leq} \underbrace{\left| \left(1 + \frac{w_n}{n}\right)^n - \sum_{k=0}^{K-1} \frac{w_n^k}{k!} \right|}_{\text{apply binomial theorem}} + \left| \sum_{k=K}^{\infty} \frac{w_n^k}{k!} \right|$$

So it holds for  $n \geq N$ :

$$\left| \sum_{k=0}^n \binom{n}{k} \frac{w_n^k}{n^k} - \sum_{n=0}^{k-1} \frac{w_n^k}{k!} \right| + \left| \sum_{k=K}^{\infty} \frac{w_n^k}{k!} \right|$$

$$\leq \left| \sum_{k=0}^{k-1} \left( \binom{n}{k} \frac{w_n^k}{n^k} - \frac{w_n^k}{k!} \right) \right| + \left| \sum_{k=K}^n \binom{n}{k} \frac{w_n^k}{n^k} \right| + \underbrace{\sum_{n=K}^{\infty} \frac{(|w|+1)^k}{k!}}_{< \frac{\varepsilon}{3}}$$

Second expression:

$$\binom{n}{k} \cdot \frac{1}{n^k} = \frac{1}{k!} \underbrace{\frac{n}{n} \frac{n-1}{n} \frac{n-2}{n} \dots \frac{n-k+1}{n}}_{k \text{ times}} < \frac{1}{k!}$$

$$= \left| \sum_{k=K}^n \binom{n}{k} \frac{w_n^k}{k!} \right| \leq \sum_{k=K}^n \binom{n}{k} \frac{|w_n|^k}{n^k} < \sum_{k=K}^{\infty} \frac{1}{k!} (|w|+1)^k < \frac{\varepsilon}{3}$$

First expression:

$$\lim_{n \rightarrow \infty} \binom{n}{k} \frac{1}{n^k} = \lim_{n \rightarrow \infty} \frac{1}{k!} \cdot \frac{n}{n} \frac{n-1}{n} \dots \frac{n-k+1}{n}$$

$$= \frac{1}{k!} \lim_{n \rightarrow \infty} \underbrace{\frac{n-1}{n}}_{=1} \cdot \lim_{n \rightarrow \infty} \underbrace{\frac{n-2}{n}}_{=1} \dots \lim_{n \rightarrow \infty} \underbrace{\frac{n-k+1}{n}}_{=1} = \frac{1}{k!}$$

Therefore it holds that,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{K-1} \underbrace{\binom{n}{k} \frac{1}{n^k}}_{\rightarrow \frac{1}{k!}} \underbrace{w_n^k}_{\rightarrow w} = \sum_{k=0}^K \frac{1}{k!} w^k$$

Therefore some  $N \in \mathbb{N}$  exists such that for  $n \geq N$  it holds that,

$$\left| \sum_{k=0}^{K-1} \binom{n}{k} \frac{1}{n^k} w_n^k - \sum_{k=0}^{K-1} \frac{1}{k!} w^k \right| < \frac{\varepsilon}{3}$$

$$\left| \left(1 + \frac{w_n}{n}\right)^n - \sum_{k=0}^n \frac{w_n^k}{k!} \right| < \varepsilon$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{w_n}{n}\right)^n = \sum_{k=0}^{\infty} \frac{w^k}{k!}$$

□

**Definition 67** (Exponential function). We define for some  $z \in \mathbb{C}$

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

For every sequence  $z_n \in \mathbb{C}$  with  $\lim_{n \rightarrow \infty} z_n = z$  it holds that

$$\exp(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z_n}{n}\right)^n$$

Especially for  $z_n = z$  it holds that

$$\exp(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n$$

---

This lecture took place on 29th of Jan 2016 with lecturer Wolfgang Ring.

$$w_n \rightarrow w \in \mathbb{C}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(1 + \frac{w_n}{n}\right)^n = \sum_{k=0}^{\infty} \frac{w^k}{k!} \quad \text{Fundamental lemma}$$

$$\exp(z) = \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$



We desire an exponential function satisfying:

$$f_{\text{ex}}(z) \cdot f_{\text{ex}}(w) = f_{\text{ex}}(z + w)$$

**Theorem 96.** The exponential function  $\exp : \mathbb{C} \rightarrow \mathbb{C}$  is defined on entire  $\mathbb{C}$  and it holds that

$$(F) \quad \forall z, w \in \mathbb{C} : \exp(z) \cdot \exp(w) = \exp(z + w)$$

$$(A) \quad \lim_{\zeta \rightarrow 0} \frac{\exp(\zeta) - 1}{\zeta} = 1$$

Furthermore the exponential function is the *only* function satisfying properties (A) and (F).

*Proof.* The power series  $\sum_{k=0}^{\infty} \frac{z^k}{k!}$  has convergence radius  $\rho = \infty$ , hence the exponential function is defined on entire  $\mathbb{C}$ .

What about property (F)?

$$\begin{aligned} \exp(z) \exp(w) &= \lim_{n \rightarrow \infty} \left(1 + \frac{z}{n}\right)^n \cdot \lim_{n \rightarrow \infty} \left(1 + \frac{w}{n}\right)^n \\ &= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{z}{n}\right) \left(1 + \frac{w}{n}\right)\right]^n = \lim_{n \rightarrow \infty} \left(1 + \frac{z + w + \frac{zw}{n}}{n}\right)^n \end{aligned}$$

It holds that  $\zeta_n = z + w + \frac{zw}{n} \rightarrow z + w$ , hence  $\lim_{n \rightarrow \infty} \zeta_n = z + w$ . So,

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{\zeta_n}{n}\right) \underset{\text{fundamental theorem}}{=} = \sum_{k=0}^{\infty} \frac{(z + w)^k}{k!} = \exp(z + w)$$

What about property (A)?

$$\exp(\zeta) - 1 = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} - 1 = \sum_{k=1}^{\infty} \frac{\zeta^k}{k!} = \zeta \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!}$$

for  $\zeta \neq 0$  it is,

$$\frac{\exp(\zeta) - 1}{\zeta} = \sum_{k=1}^{\infty} \frac{\zeta^{k-1}}{k!} = \underbrace{\sum_{l=0}^{\infty} \frac{\zeta^l}{(l+1)!}}_{Q(\zeta)} \quad \text{power series converging in } \mathbb{C}$$

So  $\rho = \infty$ . Theorem about continuity of power series:

$$\lim_{\zeta \rightarrow 0} Q(\zeta) = Q(0) = \frac{1}{1!} = 1$$

So it holds that

$$\lim_{\zeta \rightarrow 0} \frac{\exp(\zeta) - 1}{\zeta} = 1$$

Proof for uniqueness: Let  $f_{\text{ex}}$  be a function which satisfies (A) and (F). Let  $z \in \mathbb{C}$  arbitrary.

Approach:

$$f_{\text{ex}}\left(\frac{z}{n}\right) = 1 + \frac{w_n}{n}$$

Then it holds that

$$\lim_{n \rightarrow \infty} f_{\text{ex}}\left(\frac{z}{n}\right) = f_{\text{ex}}(0) = 1$$

$$f_w = \frac{f_{\text{ex}}\left(\frac{z}{n} - 1\right)}{\frac{1}{n}}$$

Because of (F) it holds that

$$f(z) = \left(f\left(\frac{z}{n}\right)\right)^n = \left(1 + \frac{w_n}{n}\right)^n$$

$$w_n = z \cdot \frac{f_{\text{ex}}\left(\frac{z}{n}\right) - 1}{\frac{z}{n}}$$

and

$$\lim_{n \rightarrow \infty} w_n = z \underbrace{\lim_{n \rightarrow \infty} \frac{f_{\text{ex}}\left(\frac{z}{n}\right) - 1}{\frac{z}{n}}}_{=1} = z$$

$$f_{\text{ex}}(z) = \left(1 + \frac{w_n}{n}\right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{w_n}{n}\right)^n \underset{\text{fundamental theorem}}{=} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z)$$

□

Let  $n \in \mathbb{N}$ .

$$\exp(n) = \exp(\underbrace{1 + 1 + \dots + 1}_{n \text{ times}}) = \exp(1)$$

We let

$$\exp(1) = e = \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \in \mathbb{R}$$

$e$  is the Eulerian number (irrational,  $\approx 2.718281828459045$ ).

Leonard Euler (1707–1783)

Let  $m \in \mathbb{N}^+$ , then it holds that

$$\underbrace{\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}}_{m \text{ times}} = 1$$

Therefore

$$\exp\left(\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}\right) = \exp\left(\frac{1}{m}\right)^m = \underbrace{e}_{\exp(1)}$$

$$\exp\left(\frac{1}{m}\right) = \sqrt[m]{e} = e^{\frac{1}{m}}$$

$$\exp\left(\frac{n}{m}\right) = \exp\left(\underbrace{\frac{1}{m} + \frac{1}{m} + \dots + \frac{1}{m}}_{n \text{ times}}\right) = \exp\left(\frac{1}{m}\right)^n = \left(e^{\frac{1}{m}}\right)^n = e^{\frac{n}{m}}$$

Let  $z \in \mathbb{C}$ , then it holds that  $z - z = 0$ .

$$1 = \exp(0) = \exp(z + (-z)) = \exp(z) \cdot \exp(-z)$$

$$\Rightarrow \forall z \in \mathbb{C} : \exp(z) \neq 0$$

and

$$\exp(-z) = \frac{1}{\exp(z)} = \exp(z)^{-1}$$

The exponential function does not have roots (i.e.  $x$  such that  $f(x) = 0$ ).

So for  $\frac{n}{m} \in \mathbb{Q}_-$ ,  $n < 0$ ,  $m > 0$  it holds that

$$\exp\left(\frac{n}{m}\right) = \frac{1}{\underbrace{\exp\left(-\frac{n}{m}\right)}_{\in \mathbb{Q}_+}} = \frac{1}{e^{-\frac{n}{m}}} = e^{\frac{n}{m}}$$

So it holds that

$$\forall q \in \mathbb{Q} : \exp(q) = e^q$$

We denote for  $z \in \mathbb{C}$ :

$$\exp(z) = e^z$$

### 15.3 The exponential function for real arguments

**Theorem 97.**  $\exp : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable in  $\mathbb{R}$  and it holds that  $\exp' = \exp$ .

*Proof.* Let  $x_0 \in \mathbb{R}$  and consider

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{\exp(x) - \exp(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{\exp(x - x_0 + x_0) - \exp(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{\exp(x - x_0) \cdot \exp(x_0) - \exp(x_0)}{x - x_0} \\ &= \exp(x_0) \cdot \lim_{x \rightarrow x_0} \frac{\exp(x - x_0) - 1}{x - x_0} \\ &= \exp(x_0) \cdot \underbrace{\lim_{x \rightarrow x_0 \rightarrow 0} \frac{\exp(x - x_0) - 1}{x - x_0}}_{=1 \text{ because of (A)}} = \exp(x_0) \end{aligned}$$

So it has been proved that

$$\exp'(x_0) = \exp(x_0)$$

□

**Corollary 25.** •  $e^x > 0 \quad \forall x \in \mathbb{R}$

- $\exp$  is strictly monotonically increasing in  $\mathbb{R}$
- $\exp$  is strictly convex in  $\mathbb{R}$

*Proof.* • We already know that  $e^x \neq 0 \quad \forall x \in \mathbb{R}$ .

$$e^x = e^{\frac{x}{2} + \frac{x}{2}} = \underbrace{\left(e^{\frac{x}{2}}\right)^2}_{\geq 0 \text{ as square}}$$

$$e^x \neq 0 \Rightarrow e^x > 0$$

- So it holds that  $\forall x \in \mathbb{R} : \exp'(x) > 0$

$\underbrace{\Rightarrow}_{\text{monotonic property}}$   $\exp$  is strictly monotonically increasing

- The derivative  $\exp'$  of  $\exp$  is strictly monotonically increasing. Hence  $\exp$  is strictly convex (Convexity criterion)

□

**Definition 68** (Reminder of tendency towards infinity for functions). Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . We say  $f$  tends to infinity  $a \in \mathbb{R}$  for  $x$  to infinity if

$$\forall \varepsilon > 0 \exists M \in \mathbb{R} : x > M \Rightarrow |f(x) - a| < \varepsilon$$

$$\lim_{x \rightarrow \infty} f(x) = a$$

We say  $f$  for  $x$  to  $\infty$  tends to infinity if TODO

**Theorem 98** (exponential growth). Let  $n \in \mathbb{N}$ . Then it holds that

- $\lim_{n \rightarrow \infty} \frac{e^x}{x^n} = +\infty$   
exp with  $x \rightarrow \infty$  grows stronger than any  $x^n$
- $\lim_{x \rightarrow -\infty} e^x \cdot x^n = 0$   
exp with  $x \rightarrow -\infty$  drops stronger towards zero than any  $x^n$  grows

*Proof.* • Let  $L > 0$  arbitrary,  $n \in \mathbb{N}$  is fixed. For  $x > 0$  it holds that

$$e^x = \sum_{k=0}^{\infty} \underbrace{\frac{x^k}{k!}}_{>0} > \frac{x^{n+1}}{(n+1)!}$$

Hence

$$\frac{e^x}{x^n} > \frac{\frac{x^{n+1}}{(n+1)!}}{x^n} = \frac{x}{(n+1)!} > L \text{ if } x > \underbrace{L \cdot (n+1)!}_{=M}$$

- Let  $\xi = -x$ .

$$\lim_{x \rightarrow -\infty} e^x \cdot x^n = \lim_{\xi \rightarrow +\infty} e^{-\xi} \cdot (-\xi)^n = - \lim_{\xi \rightarrow +\infty} \frac{\xi^n}{e^\xi} = 0$$

□

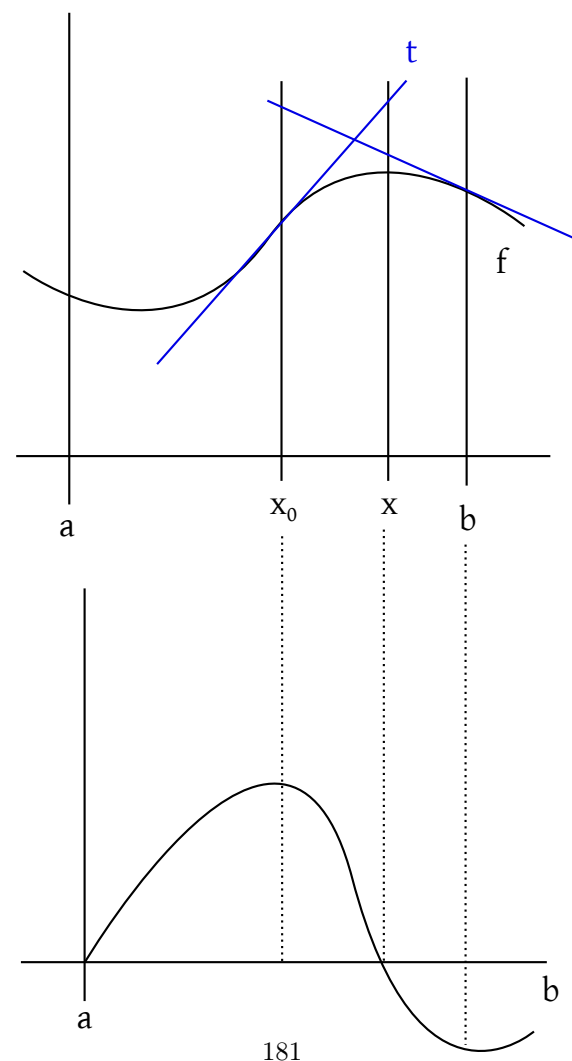


Figure 29: Slopes and tangents of two functions

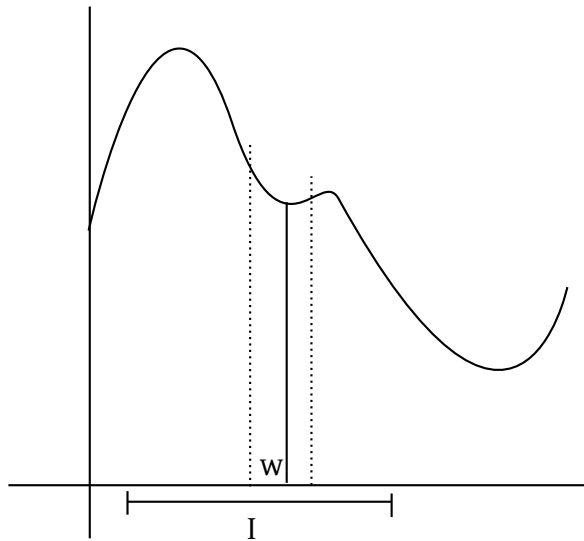


Figure 30: Local minimum  $w$

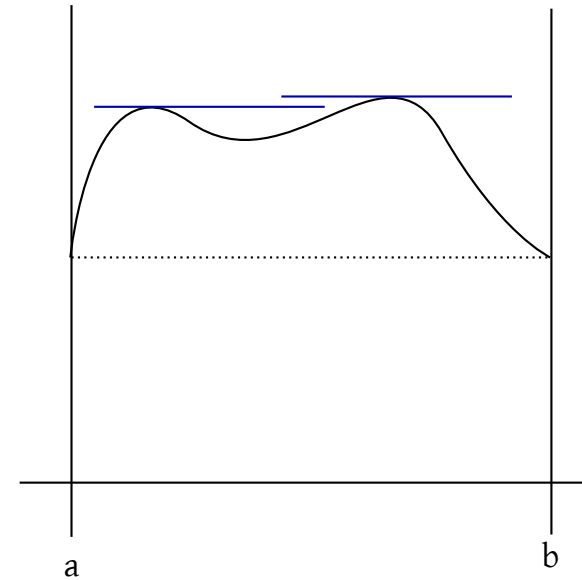


Figure 31: Rolle's theorem says that one  $x$  with  $f'(x) = 0$  must exist between two points  $x_1$  and  $x_2$  with  $f(x_1) = f(x_2)$  and  $x_1 \neq x_2$

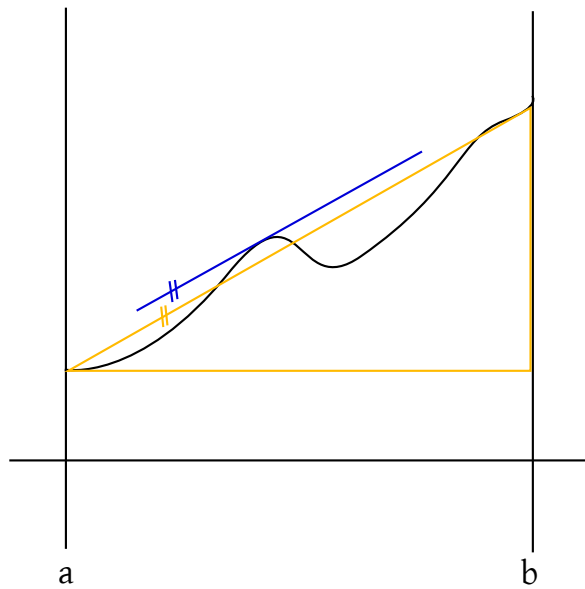


Figure 32: The Intermediate Value Theorem (IVT) claims that some tangent exists which is parallel to the line connecting  $f(a)$  and  $f(b)$

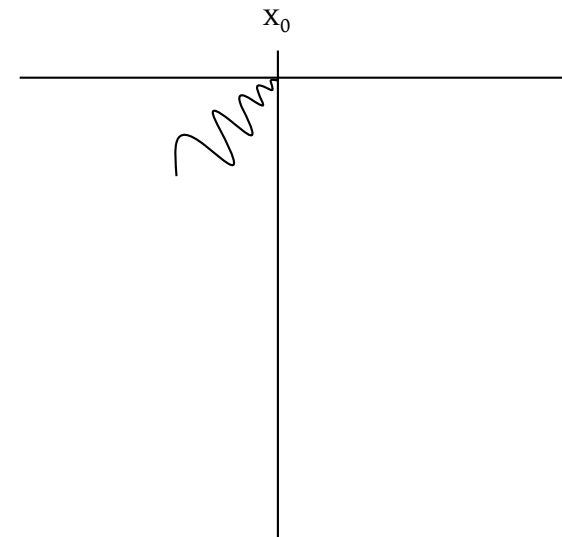


Figure 33: This is not a local maximum, but Theorem 92 holds

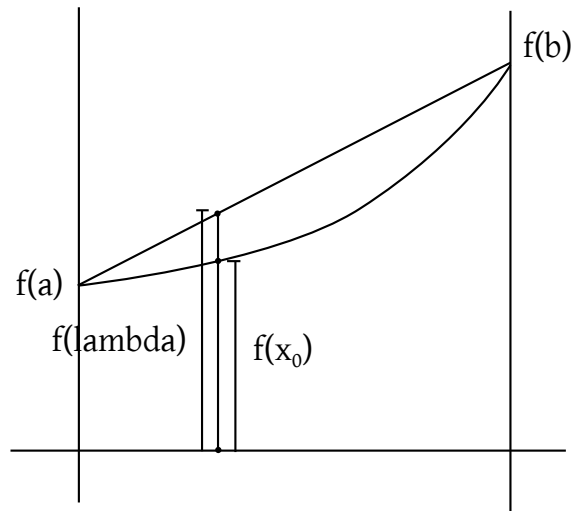


Figure 34: Convex combination

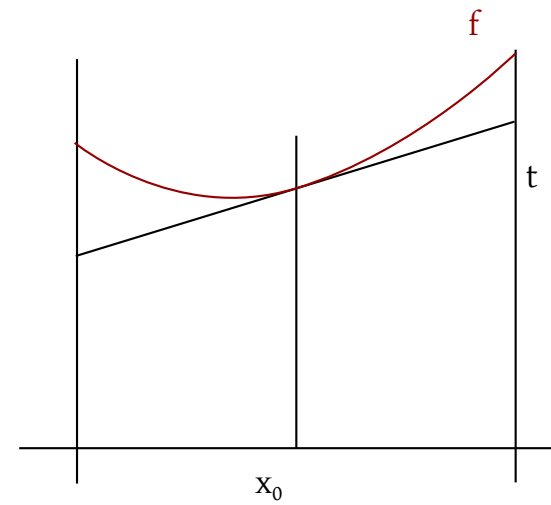


Figure 35: Tangent in  $x_0$

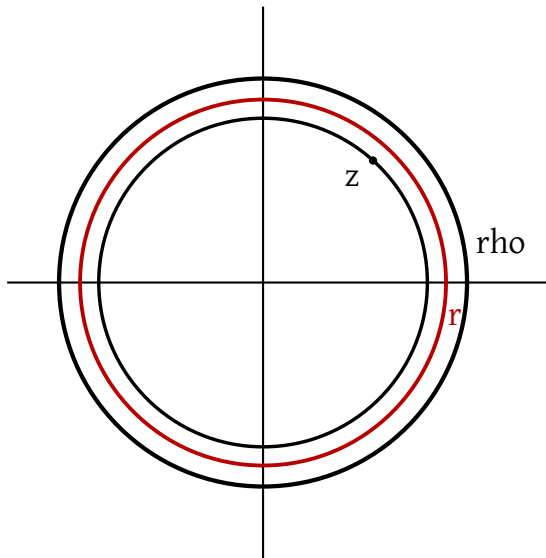


Figure 36: Convergence radius of power series

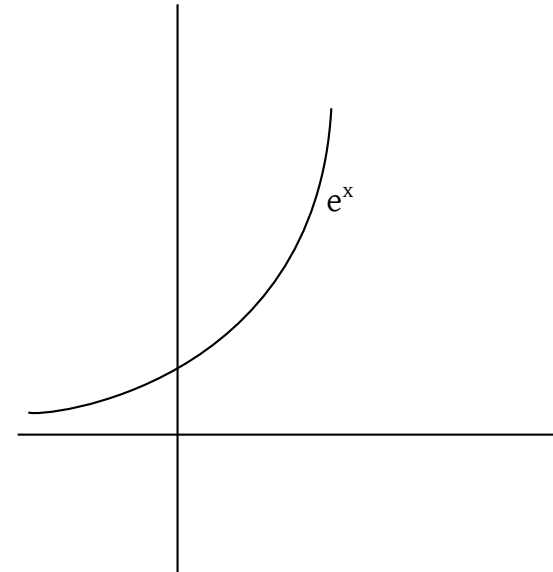


Figure 37: Plot of the general exponential function  $e^x$

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