# Linear Algebra 2 – Practicals

Lukas Prokop

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## Contents

# 1 Solution of the last lecture exam of Analysis 1

## 1.1 Exam: Exercise 1

**Exercise 1.** Determine the limes of

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

does not help us. What about this representation?

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1} = \frac{a(n-1)+b(n+1)}{(n+1)(n-1)}$$

$$a(n-1) + b(n+1) = 1$$

$$(a+b)n + (b-a) = 1$$

$$\Rightarrow a+b = 0 \wedge b-a = 1$$

$$\Rightarrow a = -\frac{1}{2} \quad b = \frac{1}{2}$$

Followingly,

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left( \frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right)$$

Okay, how to proceed? Let's build a pre-factor:

$$\frac{1}{2} \sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots$$

$$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m := \frac{1}{2}\sum_{n=2}^{m}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n-1} - \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n+1}$$

Please be aware that we explicitly define $S_m$ because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$\begin{aligned} &= \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n-1} - \frac{1}{2}\sum_{n=4}^{m+2}\frac{1}{n-1} \\ &= \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{m}+\frac{1}{m+1}\right) \end{aligned}$$

We already know $\frac{1}{m} \xrightarrow{m\to\infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m\to\infty} 0$. Followingly also $\frac{1}{2}\left(\frac{1}{m}+\frac{1}{m+1}\right) \xrightarrow{m\to\infty} 0$.

## 1.2 Exam: Exercise 2

**Exercise 2.** A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n\in\mathbb{N}}$$

$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \qquad a_1 = \frac{3}{2} \qquad a_2 = \frac{5}{4} \qquad a_3\frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \qquad a_1 = 4 \qquad a_2 = \frac{5}{2} \qquad a_3\frac{7}{4}$$

**Exercise 3.** a) Show that $1 \overset{!}{<} a_n \overset{!}{\leq} a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over $n$ to prove this statement:

**induction base** $1 < a_0 \leq a_0$ holds trivially.

**induction step** We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n+1) \\ &\leq \frac{1}{2}(a_0+a_0) && \text{[induction hypothesis and } 1 < a_0\text{]} \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n+1) \\ &> \frac{1}{2}(1+1) && \text{[induction hypothesis]} \\ &= 1 \end{aligned}$$

**Exercise 4.** b) Prove that $a_{n+1} \overset{!}{<} a_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &< \frac{1}{2}(a_n + a_n) && \text{[we have proven: } a_n > 1\text{]} \end{aligned}$$

**Exercise 5.** c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$b_n := a_n - 1 \qquad \forall n \in \mathbb{N}$$
$$b_0 := a_0 - 1$$
$$b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2} b_n$$

$$b_n = \frac{1}{2^n} b_0 \to 0 \cdot b_0 = 0$$
$$\Rightarrow b_n \to 0$$
$$\Rightarrow a_n = b_n + 1 \to 1$$

Does it work to just show: $1 = \frac{1}{2}(1+1)$? Nope, because in points of continuity this might be true even though 1 is not its limes.

Let $a_n \to a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$a_{n+1} \to a \qquad \frac{1}{2}(a_n + 1) \to \frac{1}{2}(a + 1) \qquad a = \frac{1}{2}(a+1)$$

## 1.3 Exam: Exercise 3

**Exercise 6.** $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an $\varepsilon$-$\delta$-proof.

If we don't need an $\varepsilon$-$\delta$-proof, we would argue with the Algebraic Continuity Theorem: The function $f$ is a composition of continuous functions, hence a continuous function itself.

$\varepsilon$-$\delta$-definition:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If $|x - x_0| < \delta$,

$$\begin{aligned} |f(x) - f(x_0)| &= \left|2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)\right| \\ &= \left|2x^2 + 5x - 2x_0^2 - 5x_0\right| \\ &\leq 2\left|x^2 - x_0^2\right| + 5|x - x_0| \\ &= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\ &= 2|x + x_0||x - x_0| + 5|x - x_0| \\ &\leq 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\ &\leq 2(|x_0| + \delta + |x_0|) + 5\delta \end{aligned}$$

Our goal: we are able to claim $\overset{!}{<} \varepsilon$

$$\begin{aligned} &= 4|x_0|\delta + 2\delta^2 + 5\delta \\ &= 2\delta^2 + (4|x_0| + 5)\delta \end{aligned}$$

In general (here it does not apply), that $x_0$ might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$\begin{aligned} 2\delta^2 &< \frac{\varepsilon}{2} \\ \delta &< \frac{\sqrt{\varepsilon}}{2} \\ (4|x_0| + 5)\delta &< \varepsilon \\ \delta &< \frac{\varepsilon}{4|x_0| + 5} \end{aligned}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{5}, \frac{\varepsilon}{4|x_0|+6}\right)$. Then the $\varepsilon$-$\delta$ definition shows that $f$ is continuous.

## 2 Exam: Exercise 4

**Exercise 7.** Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\begin{aligned} h(0) &= f(0) - f\left(\frac{1}{2}\right) \\ h(1) &= f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0) \end{aligned}$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x + \frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore $h$ is continuous, because it is a composition of continuous functions.

**Case 1**: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\Rightarrow f(\xi) = f(\xi + \frac{1}{2})$$

**Case 2:** $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

**Case 3:** $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

# 3 Exercise 1

**Exercise 8.** Investigate the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(x\,|\,x\,|+x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}$$

So this is differentiable, but in case of $x = 0$, it remains questionable.

We look at the definition of differentiability:

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$$

$$f'(x) = \begin{cases} \lim_{x\to 0} \frac{0}{x} = 0 \\ \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0 \end{cases}$$

It follows that $f$ is differentiable one time.

$$f''(x) = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

What about $x = 0$?

$$\lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} \begin{cases} \lim_{x\to 0} \frac{0}{x} = 0 \\ \lim_{x\to 0^+} \frac{2x}{x} = \lim_{x\to 0^+} 2 = 2 \end{cases}$$

Left and right limes differ. So it is not differentiable.

# 4 Exercise 2

**Exercise 9.** Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x\to 1} \frac{\ln x}{x-1}$
2. $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1-\sin x)}$
4. $\lim_{x\to 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

## 4.1 Exercise 2.a

$$\lim_{x\to 1} \frac{\ln x}{x - 1}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

## 4.2 Exercise 2.b

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

A nice hint to find out whether this function is differentiable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{\sin x - x}{x \sin x} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}} \approx x \to 0$$

This exploits, that it will take one run of L'Hôpital's rule (because each expression has at least degree 2) and its limes will be 0 (because of $x$).

## 4.3 Exercise 2.c

$$\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos(x))}{\ln(1-\sin(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied. We claim that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = \infty$ is fine.

$$\Rightarrow \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\cos(x)}{1-\sin(x)}} = \lim_{x\to \frac{\pi}{2}^-} \frac{-\sin(x)\cdot(1-\sin(x))}{\cos(x)(-\cos(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied.

$$\lim_{x\to \frac{\pi}{2}^-} \frac{-\cos(x)(1-\sin(x)) - \sin(x)\cdot(-\cos(x))}{-\sin(x)(-\cos(x)) + \cos(x)\cdot\sin(x)} = \frac{1}{2}$$

If we want to apply the previous estimate here, we should consider

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos(y) \qquad y = \frac{\pi}{2} - x$$
$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(y)$$

This gives us a different estimate of the result:

$$\lim_{y\to 0^+} \frac{\ln(\sin(y))}{\ln(1-\cos(y))} \approx \lim_{y\to 0^+} \frac{\ln(y)}{\ln\left(\frac{y^2}{2}\right)} = \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y) - \ln(2)} \approx \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y)} = \frac{1}{2}$$

We define neighborhoods:

$$N_\delta(x_0) = \{x : |\, x - x_0 \,| < \delta\}$$
$$N_R(\infty) = \{x : x > R\}$$

## 4.4 Exercise 2.d

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}} = \lim_{x\to 1^-} e^{\ln(x)\frac{1}{1-x}} = \exp\left(\lim_{x\to 1^-} \underbrace{\frac{\ln(x)}{1-x}}_{(-1)\cdot\text{Exercise a}}\right) = \frac{1}{e}$$

## 4.5 Exercise 2.e

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \left(\exp\left(\frac{\ln n}{\sqrt{n}}\right)\right) = \exp\left(\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}}\right)$$

The conditions to apply L'Hôpital's rule are satisfied („$\frac{\infty}{\infty}$")

$$\exp\left(\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}\right) = \exp\left(\lim_{n\to\infty} \frac{2\sqrt{n}}{n}\right) = \exp(0) = 1$$

## 4.6 Exercise 2.f

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{n\to\infty} \frac{e^x\left(1 - e^{-2x}\right)}{e^x\left(1 + e^{-2x}\right)} = \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{e^{2x}}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{e^{2x}}}$$

Remark:

$$\lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{\cosh(x)}{\sinh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)}$$

$$y = \lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)} = \frac{1}{\lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)}} = \frac{1}{y}$$

# 5 Exercise 3

**Exercise 10.** Show that the function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto x + e^x$ is bijective. Furthermore determine $(f^{-1})'(1)$ and $\lim_{y\to\infty}(f^{-1})'(y)$.

If the function is strictly monotonically increasing, it is injective.

$$f'(x) = 1 + e^x > 0 \qquad \forall x \in \mathbb{R}$$

We show that it is strictly monotonically increasing:

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\alpha) \qquad \text{with } \alpha \in (x_1, x_2)$$

$$f(x_2) - f(x_1) = f'(\alpha)(x_2 - x_1) > 0$$

Is $f$ surjective?

For an arbitrary $y_0 \in \mathbb{R}$ it holds that $\exists x_0 \in \mathbb{R} : f(x_0) = y_0$:

$$\exists f(a), f(b) \in \mathbb{R} : f(a) \leq y_0 < f(b)$$

It holds that

$$\lim_{x\to-\infty} x + \underbrace{e^x}_{\to 0} = -\infty$$

$$\lim_{x\to+\infty} x + e^x = \infty$$

Formally:

$$\forall y_0 \exists x_0 : \forall x < x_0 : f(x) < y_0$$

From the Intermediate Value Theorem it follows that

$$\Rightarrow \exists c \in [a,b) : f(c) = y_0 \qquad c =: x_0$$

So it is surjective.

From injectivity and surjectivity it follows that it is bijective.

### 5.1 Determine $(f^{-1})'(1)$

$$f(x) = x + e^x$$
$$f'(x) = 1 + e^x$$

We apply the inverse function theorem:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$y = 1 = f(x)$$
$$x = f^{-1}(1)$$

An educated guess gives us that $x = 0$. In general determining $x$ is more difficult.

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$$

### 5.2 Determine $\lim_{y\to\infty} \left(f^{-1}\right)'(y)$

$$\lim_{y\to\infty} \left(f^{-1}\right)'(y) = \lim_{y\to\infty} \frac{1}{1+e^x}$$

As $x$ grows to infinity, also $y$ grows to infinity. From bijectivity it follows that any value can be reached with $x$ as well as $f(x)$.

$$\underbrace{f'(\underbrace{f^{-1}(\underbrace{y}_{\to\infty})}_{\to\infty})}_{\to\infty}$$

## 6 Exercise 4

**Exercise 11.** Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. Show

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2} = f'(x_0)$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h} \\
&= \lim_{h'\to 0} \frac{1}{2}\cdot\left(f'(x_0)+\frac{f(x_0)-f(x_0+h')}{-h'}\right) \\
&= \lim_{h'\to 0} \frac{1}{2}\cdot\left(f'(x_0)+\frac{f(x_0+h')-f(x_0)}{h'}\right) \\
&= \frac{1}{2}(f'(x_0)+f'(x_0)) \\
&= f'(x_0)
\end{aligned}
$$

### 6.1 Exercise 4.b

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x_0+rh)-f(x_0+sh)}{h} &= \lim_{h\to 0}\frac{f(x_0+rh)-f(x_0)}{h} + \lim_{h\to 0}\frac{f(x_0)-f(x_0+sh)}{h} \\
h_1 &= rh \qquad h_2 = sh \\
&= \lim_{h_1\to 0}\frac{f(x_0+h_1)-f(x_0)}{\frac{1}{r}\cdot h_1} + \lim_{h_2\to 0}\frac{f(x_0)-f(x_0+h_2)}{\frac{1}{s}\cdot h_2} \\
&= r\cdot f'(x_0) - s\cdot f'(x_0) \\
&= (r-s)\cdot f'(x_0)
\end{aligned}
$$

## 7 Exercise 5

**Exercise 12.** Let $D \subseteq \mathbb{R}$ be an open interval. $f : D \to \mathbb{R}$ is differentiable and $f$ is twice differentiable in $x_0 \in D$.

### 7.1 Exercise 5.a

**Exercise 13.** Show that

$$\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = f''(x_0)$$

$f$ is differentiable, therefore continuous, and $h$ goes to 0. So we have „$\frac{0}{0}$". All conditions to apply L'Hôpital's rule are satisfied.

$$\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h} \approx \frac{0}{0}$$

We can apply L'Hôpital's Rule again or just use the result of exercise 4a.

$$\overset{4a}{\Longrightarrow} f''(x_0)$$

### 7.2 Exercise 5.b

**Exercise 14.** Show that the limes from exercise 5.a can also exist, even if $f''(x_0)$ does not exist. Use the result from Exercise 1.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

We know that it is not twice differentiable. But we want to show that the limes exists.

We are only concerned with $x = 0$.

$$\lim_{h \to 0} f(x_0) = 0$$

$$\lim_{h \to 0} \frac{h^2 - h^2}{h^2} = \frac{0}{h^2} = 0$$

So if we traverse the graph from both sides at the same time $\frac{f(x_0+h) - f(x_0-h)}{h}$.

# 8 Exercise 6

**Exercise 15.** Determine the following limit for arbitrary $c \in \mathbb{R}$:

$$\lim_{n \to \infty} \frac{n}{\ln n} \left( \sqrt[n]{n^c} - 1 \right).$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left( \sqrt[n]{n^c} - 1 \right)$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left( \sqrt[n]{n^c} - 1 \right) = \lim_{n \to \infty} \frac{e^{\frac{c}{n} \cdot \ln n} - 1}{\frac{\ln n}{n}}$$

and

$$\left( e^{\frac{c}{n} \cdot \ln n} \right)' = e^{\frac{c}{n} \cdot \ln n} \cdot \left( -\frac{c}{n^2} \cdot \ln n + \frac{c}{n} \cdot \frac{1}{n} \right) = \frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln(n))$$

All conditions are satisfied to apply L'Hôpital's rule ("$\frac{0}{0}$"):

$$\lim_{n \to \infty} \frac{\frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln n)}{\frac{\frac{1}{n} \cdot n - \ln n}{n^2}}$$

$$= \lim_{n \to \infty} \frac{c \cdot e^{\frac{c}{n} \cdot \ln n} (1 - \ln(n))}{1 - \ln n} = \lim_{n \to \infty} c \cdot e^{\frac{c}{n} \cdot \ln n} = c \cdot 1$$

# 9 Exercise 7

**Exercise 16.**

- Show that $e^x \geq 1 + x$ holds for all $x \in \mathbb{R}$.
  *Hint:* On demand, use the Mean Value Theorem.

- Prove that for all $x > 0$, the following estimates hold:

  $$\ln x \leq x - 1$$

  and for all $k \in \mathbb{N}_+$ it holds that

  $$k \left( 1 - \frac{1}{\sqrt[k]{x}} \right) \leq \ln x \leq k \left( \sqrt[k]{x} - 1 \right)$$

$x \geq 0$ Choose $f(x) = e^x$ in $[0, x)$. Mean value theorem:

$$\exists x_0 : f'(x_0) = \frac{f(b) - f(a)}{b - a} \quad \text{for } a < x_0 < b$$

$$f'(x_0) = e^{x_0} \qquad e^{x_0} \geq 1 \qquad x_0 \geq 0$$

$$e^{x_0} = \frac{f'(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x} \Rightarrow \frac{e^x - 1}{x} \geq 1$$

Or alternatively: $f$ is convex and therefoer $f''(x) > 0$.

Consider $f(x) = x - 1 - \ln x$

$$f'(x) = 1 - \frac{1}{x} \qquad f''(x) = \frac{1}{x^2}$$

$$f'(x) \stackrel{!}{=} 0$$

$$1 - \frac{1}{x} = 0 \Leftrightarrow x = -1$$

$$f''(1) = 1 > 0 \Rightarrow \text{ minimum and because } f(1) = 0 \Rightarrow \forall x : x - 1 - \ln x \geq 0$$

Or alternatively:

$$y := x - 1$$

$$x = y + 1$$

Show that $\ln(y + 1) \leq y \Leftrightarrow y + 1 \leq e^y$.

$e^x$ is monotonically increasing $\Rightarrow x \leq y \Leftrightarrow e^x \leq e^y$.

And this has been proven previously.

## 9.1 Exercise 7.b

$$\ln(x) \leq k \left(\frac{[}{k}]x - 1\right)$$

$$\ln(\sqrt[k]{x}) \leq \sqrt[k]{x} - 1 \Leftrightarrow \ln(y) \leq y - 1$$

And this has been proven in Exercise a.

The second part following analogously.

# 10 Exercise 8

**Exercise 17.** Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Show: If $f$ is continuous in an environment $U$ of $a \in D$, differentiable in $U \setminus \{a\}$ and there exists $\lim_{x \to a} f'(x)$, such that $f$ in $a$ differentiable and

$$f'(a) = \lim_{x \to a} f'(x).$$

*Hint:* On demand, use the Mean Value Theorem.

Let $h_n$ be an arbitrary zero-sequence (with $h_n(x) > 0 \quad \forall x \in D$) and due to Mean Value Theorem $\exists \xi_n \in D$ with $f'(\xi_n) = \frac{f(a+hn)-f(a)}{h_n}$.

$$\lim_{n \to \infty} f'(\xi_n) = \lim_{x \to a} f'(x) = \lim_{n \to \infty} \frac{f(a + h_n) - f(a)}{h_n} = f'(a)$$

$$\lim_{n \to \infty} \frac{f(a + h_n) - f(a)}{h_n} = \lim_{n \to \infty} f'(\xi_n) = \lim_{x \to a} f'(x) = z$$

For the arbitrary zero-sequence, we really need to consider it arbitrary (otherwise we just show it for the one sequence). Consider this counterexample:

$$f(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

## 10.1 Alternative approach

Application of "Schrankensatz".

$$\exists \lim f'(x) = \alpha$$

Hence for arbitrary $\varepsilon > 0 : \exists \delta > 0 \forall x \in (a - \delta, a + \delta) \setminus \{a\} : |f'(x) - \alpha| < \varepsilon$. Hence $\alpha - \varepsilon < f'(x) < \alpha + \varepsilon$.

- $$\forall x \in (a, a + \delta) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$
- $$\forall x \in (a - \delta, a) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall x \in (a - \delta, a + \delta) \setminus \{a\} : \left| \frac{f(x) - f(a)}{x - a} - \alpha \right| \leq \varepsilon$$
$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha$$

## 10.2 Second alternative approach

$$\lim_{f(a+h)-f(a)} h$$

If I know $f$ is continuous, then $f(a + h) \to f(a)$. So,

$$1\frac{0}{0}\rfloor$$

$$\lim_{h \to 0} \frac{f'(a + h) - 0}{1} = \lim_{h \to 0} f'(a + h) = \lim_{x \to a} f'(x)$$

# 11 Exercise 9

**Exercise 18.** Let $f : [a, b] \to \mathbb{R}$, $a < b$, differentiable with $f(a) > 0$, $f'(a) > 0$ and $f(b) = 0$. Prove that there exists $\xi \in (a, b) : f'(\xi) = 0$.

First, we want to show that $f'(a) > 0 \Rightarrow \exists \delta > 0 \forall x \in (a, a + \delta) : f(x) > f(a)$.

$$\exists \delta > 0 \forall x \in (a, a + \delta) : \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0$$
$$\Rightarrow f(x) - f(a) > \frac{f'(a)}{2}(x - a) > 0$$

Indeed, $f(x)$ satisfies this property.

Secondly, we want to show that,

$$\exists \eta \in (a + \delta, b) : f(a) = f(\eta)$$
$$\exists \xi \in [a, \eta] \forall x_1 \in [a, \eta] : f(\xi) \geq f(x_1)$$
$$\exists \xi \in (a, \eta) : \frac{f(\eta) - f(a)}{\eta - a} = f'(\eta) = 0$$

There might be more than this one $\xi$, so the $\xi$ between the second and third line might be different. Anyways, we found a $\xi$ with the desired property.

# 12 Exercise 10

**Exercise 19**. Determine the pointwise limit of the following function sequences $f_n : [0, \infty) \to \mathbb{R}$ and determine its uniform convergence:

- $f_n(x) = \sqrt[n]{x}$
- $f_n(x) = \frac{1}{1+nx}$
- $f_n(x) = \frac{x}{1+nx}$

## 12.1 Exercise 10.a

If $x \neq 0$, $\lim_{n\to\infty} \sqrt[n]{x} = 1$.
If $x = 0$, $\lim_{n\to\infty} \sqrt[n]{x} = \lim_{n\to\infty} 0^{\frac{1}{n}} = 0$.

In terms of uniform convergence:

$$\left| \sqrt[n]{x} - 1 \right| < \varepsilon$$

$$\lim_{x\to\infty} \sqrt[n]{x} = \infty$$

Example:

$$\begin{aligned} \left| \sqrt[n]{x} - 1 \right| &< \varepsilon \\ \sqrt[n]{x} - 1 &< \varepsilon \\ \sqrt[n]{x} &< \varepsilon + 1 \\ \sqrt[n]{100} &< \varepsilon + 1 \end{aligned}$$

## 12.2 Exercise 10.b

$$f_n(x) = \frac{1}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{1}{1+nx} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{1}{1+n\cdot 0} = 1$$

Assume it it continuously convergent. Show that:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists x \in [0, \infty) : n \geq N \wedge |f_n(x) - f(x)| \geq \varepsilon$$

Does not hold for $\frac{9}{n} \geq x$.

## 12.3 Exercise 10.c

$$f_n(x) = \frac{x}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{x}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{x}+n} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{0}{1+n\cdot 0} = 0$$

$$\left|\frac{x}{1+nx}-0\right|<\varepsilon$$

$$\left|\frac{x}{1+nx}\right|<\left|\frac{x}{nx}\right|=\left|\frac{1}{n}\right|$$

Convergence is given. Uniform convergence is not given.

*Advice:* The simplest approach to show convergence is to show:

$$|f_n(x)-f(x)|\leq a_n\to 0$$

where $a_n$ is independent from $x$.

# 13 Exercise 11

**Exercise 20.** Determine $\cos\alpha$, $\sin\alpha$ and $\tan\alpha$ for $\alpha\in\{\frac{\pi}{5},\frac{2\pi}{5}\}$.

Hint: Show that $u:=\cos\frac{2\pi}{5}$ and $v:=\cos\frac{\pi}{5}$ satisfy the equations $u=2v^2-1$ and $-2u^2+1=v$. Determine $u,v$ this way.

$$\begin{aligned}u=\cos\left(\frac{2\pi}{5}\right)&=\cos\left(\frac{\pi}{5}+\frac{\pi}{5}\right)\\&=\cos^2\left(\frac{\pi}{5}\right)-\sin^2\left(\frac{\pi}{5}\right)\\&=2\cos^2\left(\frac{\pi}{5}\right)-1\\&=2v^2-1\end{aligned}$$

To show: $v+2u^2-1=0$, $\cos\left(\frac{\pi}{5}\right)+2\cos^2\left(\frac{\pi}{5}\right)-1=0$.

$$\begin{aligned}\cos\left(\frac{\pi}{5}\right)+2\cos\frac{2\pi}{5}-1&=\cos\frac{\pi}{5}+\cos\frac{4\pi}{5}\\&=\cos\frac{\pi}{5}+\cos\left(\pi-\frac{1}{5}\pi\right)\\&=\cos\frac{\pi}{5}+\cos\pi\cdot\cos\left(\frac{\pi}{5}\right)+\sin\pi\cdot\sin\frac{\pi}{5}-\cos\frac{\pi}{5}\cdot\cos\frac{\pi}{5}\\&=0\end{aligned}$$

For $u+v>0$:

$$2v^2-1=u$$
$$-2u^2+1=v$$
$$2v^2-2u^2=u+v$$
$$2(v+u)(v-u)=u+v$$
$$2(v-u)=1\Leftrightarrow v-u=\frac{1}{2}$$

$$v-2v^2+\frac{1}{2}=0$$
$$v^2-\frac{1}{2}v-\frac{1}{4}=0$$
$$v_{1,2}=\frac{1}{4}\pm\sqrt{\frac{1}{16}+\frac{4}{16}}=\frac{1\pm\sqrt{5}}{4}$$

$$0 < \cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{4}$$

$$u = \cos\frac{2\pi}{5} = v - \frac{1}{2} = \frac{-1+\sqrt{5}}{4}$$

$$\cos\left(\frac{2\pi}{5}\right) = \cos^2\frac{\pi}{5} - \sin^2\frac{\pi}{5}$$

$$\Leftrightarrow \frac{-1+\sqrt{5}}{4} = \left(\frac{\sqrt{5}+1}{4}\right)^2 - \sin^2\left(\frac{\pi}{5}\right)$$

$$\Leftrightarrow \sin^2\left(\frac{\pi}{5}\right) = \frac{5+2\sqrt{5}+1}{16} - \frac{-4+4\sqrt{5}}{16}$$

$$= \frac{5+2\sqrt{5}+1+4-4\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16} = \frac{5-\sqrt{5}}{8}$$

$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}} \approx 0.59$$

$$\sin\frac{2\pi}{5} = \sin\left(\frac{\pi}{5}+\frac{\pi}{5}\right) = \sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\cdot\sin\frac{\pi}{5} = 2\sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5}$$

$$= 2\frac{1+\sqrt{5}}{4}\sqrt{\frac{5-\sqrt{5}}{8}} = \frac{1+\sqrt{5}}{2}\cdot\frac{5-\sqrt{5}}{8} = \sqrt{\frac{5+\sqrt{5}}{8}} \approx 0.95$$

$$\tan\frac{\pi}{5} = \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = \frac{\sqrt{\frac{5-\sqrt{5}}{8}}}{\frac{\sqrt{5}+1}{4}} = \frac{\sqrt{2(5-\sqrt{5})}}{1+\sqrt{5}} = \sqrt{5-2\sqrt{5}} \approx 0.73$$

$$\tan\left(\frac{2\pi}{5}\right) = \frac{\sin\frac{2\pi}{5}}{\cos\frac{2\pi}{5}} = \frac{4}{-1+\sqrt{5}}\cdot\frac{1+\sqrt{5}}{2}\cdot\sqrt{\frac{5-\sqrt{5}}{8}} = \sqrt{5+2\sqrt{5}} \approx 3.05$$

# 14 Exercise 12

**Exercise 21.** To which order do you have to consider values in the series expansion of cosine, to approximate $\cos 1$ with an error smaller $10^{-7}$? Furthermore show that $\cos 1$ is irrational.

**Hint**: To show irrationality of $\cos 1$, assume, $p, q \in \mathbb{N}_+$ with $\cos 1 = \frac{p}{q}$. Replace that in the estimated error of

$$\cos 1 - \sum_{k=0}^{q} \frac{(-1)^k}{(2k)!},$$

multiply with $(2q)!$ and derive a contradiction.

## 14.1 Exercise 12.a

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \cdot (-1)^k$$

Consider,

$$S_{2m} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k$$

$$S_{2k+1} = \sum_{k=0}^{2k+1} \frac{1}{(2k)!} \cdot (-1)^k$$

So $S_{2k+1}$ has a negative, last expression. $S_{2m}$ has a positive last expression.

$$S_{2k+1} < \cos 1 < S_{2m}$$

$$S_{2m} - S_{2m+1} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k - \sum_{k=0}^{2m+1} \frac{1}{(2k)!}(-1)^k$$

$$\triangle \cos(1) = -\frac{1}{(2(2m+1))!} \cdot (-1)^{2m+1} = \frac{1}{(2 \cdot (2m+1))!} \overset{!}{<} 10^{-7}$$

$$N! > 10^7 \Rightarrow N > 11$$
$$2 \cdot (2m+1) > 11$$
$$2m + 1 > \frac{11}{2} = 5.5$$

$\Rightarrow$ 10-th order because every odd expression is cancelled out.

Consider paper: "The irrationality of $e$ and Others".

## 14.2 Exercise 12.b

$$\cos(1) \notin \mathbb{Q}$$

Assume $\exists p \in \mathbb{Z}$, $q \in \mathbb{N}$:

$$\cos(1) = \frac{p}{q}$$

$$\left| \cos(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \right|$$
$$= \left| \frac{p}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k}{(2k)!} \right| < \frac{1}{(2q)!}$$
$$= \left| \frac{p(2q)!}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k \cdot (2q)!}{(2k)!} \right| < 1$$
$$|\, x - y \,| < 1 \Rightarrow 0 \qquad \text{because } x \in \mathbb{Z}, y \in \mathbb{Z}$$

Leibniz criterion requires that the limes is not achieved in the sequence, because the functions need to be strictly monotonical.

# 15 Exercise 13

**Exercise 22.** Let $f : [\frac{\pi}{2}, \frac{3\pi}{2}] \to [-1, 1]$, $x \mapsto \sin x$. Show that $f$ is bijective and compute (using the formula for the derivative of the inverse function $(f^{-1})'(y)$ at all possible points $y \in [-1, 1]$). Also give an explicit representation for $f^{-1}$

$$\ldots = -\frac{1}{\sqrt{1 - y^2}}$$

It is important to recognize the negative sign.

# 16 Exercise 14

**Exercise 23.** Let $w, z \in \mathbb{R}$ with $w, z, w+z \notin \left\{\frac{\pi}{2} + k\pi \,\middle|\, k \in \mathbb{Z}\right\}$. Prove the addition theorem of the tangens function:

$$\tan(w+z) = \frac{\tan(w) + \tan(z)}{1 - \tan(w)\tan(z)}.$$

Let $x, y \in \mathbb{R}$ with $xy < 1$. Show that $\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and use it to prove the addition theorem for the arcustangens function:

$$\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}.$$

1. Show that $\tan(w+z) = \frac{\tan(w)+\tan(z)}{1-\tan(w)\tan(z)}$.

$$\tan(w+z) = \frac{\sin(w+z)}{\cos(w+z)} = \frac{\cos(w)\cdot\sin(z) + \sin(w)\cos(z)}{\cos(w)\cos(z) - \sin(w)\sin(z)}$$
$$\frac{\frac{\cos(w)\sin(w)}{\cos(w)\cos(z)} + \frac{\sin(w)\cdot\cos(z)}{\cos(w)\cdot\cos(z)}}{1 - \frac{\sin(w)\sin(z)}{\cos(w)\cos(z)}}$$
$$= \frac{\tan(z) + \tan(w)}{1 - \tan(w)\tan(z)}$$

2. $$\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

   $x, y \in \mathbb{R}, xy < 1$.

   Let $x = \tan(z)$ and $y = \tan(w)$.

$$xy = \tan(z)\cdot\tan(w) = \frac{\sin(z)\cdot\sin(w)}{\cos(z)\cdot(w)} < 1$$
$$\sin(z)\cdot\sin(w) < \cos(z)\cos(w)$$
$$\Leftrightarrow 0 < \cos(z)\cdot\cos(w) - \sin(z)\cdot\sin(w)$$
$$\Leftrightarrow 0 \overset{!}{<} \cos(z+w) \Leftarrow z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \vee w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

   This proof is insufficient! A case distinction for $\cos(z)\cos(w) > 0$ is required.

3. Show that $\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}$. Let $x = \tan(z)$ and $y = \tan(w)$.

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan\left(\frac{\tan(z)+\tan(w)}{1-\tan(z)\tan(w)}\right) = \arctan(\tan(z+w)) = z + w = \arctan(x) + \arctan(y)$$

# 17 Exercise 15

**Exercise 24.** Compute the following integrals by approximating the integrands using a sequence of step functions with the given points. Let $a, b \in \mathbb{R}$ with $a < b$.

1. $\int_a^b e^x \, dx$ with points $x_k := a + k(b-a)/n$.
2. $\int_a^b x^p \, dx$ with points $x_k := aq^k$, $q := \sqrt[n]{b/a}$ and $p \in \mathbb{R} \setminus \{-1\}$.

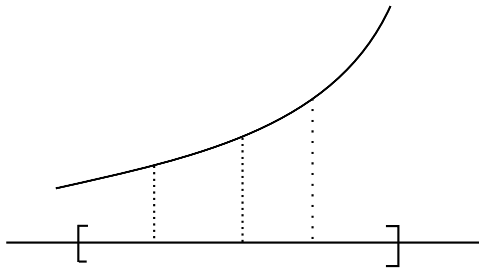

Figure 1: Illustration of 15b

### 17.1 Exercise 15.a

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} e^{a+\frac{k(b-a)}{n}} \\
&= \lim_{n\to\infty} \frac{b-a}{n} e^{a} \sum_{k=0}^{n-1} \left(e^{\frac{b-a}{n}}\right)^{k} \\
&= \lim_{n\to\infty} e^{a} \cdot \frac{b-a}{n} \frac{e^{\frac{b-a}{n}} - 1}{e^{\frac{b-a}{n}-1}} \\
&= \lim_{n\to\infty} e^{a} \left(e^{b-a} - 1\right) \cdot \underbrace{\frac{\frac{b-a}{n}}{e^{\frac{b-a}{n}} - 1}}_{\to 1} \\
&= e^{a} \cdot \frac{e^{b}}{e^{a}} - e^{a} = e^{b} - e^{a}
\end{aligned}
$$

$$
\begin{aligned}
(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in [a,b]) &: |\,\varphi_n(x) - e^{x}\,| < \varepsilon \\
e^{a+(b-a)\frac{n-1}{n}} - e^{b} = e^{a+(b-a)(1-\frac{1}{n})} - e^{b} &= e^{a+b-\frac{b}{n}-a+\frac{a}{n}} - e^{b} \\
&= e^{b-\frac{b}{n}+\frac{a}{n}-e^{b}}
\end{aligned}
$$

### 17.2 Exercise 15.b

$$
x_k \coloneqq a q^{k} \qquad q \coloneqq \left(\frac{b}{a}\right)^{\frac{1}{n}}
$$

$$
p \neq -1
$$

$$
\begin{aligned}
y_k &\coloneqq x_{k+1} - x_k \\
&= a q^{k+1} - a q^{k} \\
&= a q^{k}(q-1)
\end{aligned}
$$

$$\sum_{k=0}^{n-1} y_k x_k^p = \sum_{k=0}^{n-1} aq^k(q-1)(aq^k)^p = a^{p+1}(q-1)\sum_{k=0}^{n-1}(q^{p+1})^k$$

Is a geometric series:

$$= a^{p+1}(q-1)\frac{1-(q^{p+1})^{n-1}}{1-q^{p+1}}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} y_k x_k^p = a^{p+1}\lim_{n\to\infty}\left(\left(\frac{b}{a}\right)^{\frac{1}{n}}-1\right)\frac{1-(\frac{b}{a})^{\frac{n-1}{n}(p+1)}}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = a^{p+1}\left(1-\left(\frac{b}{a}\right)^{p+1}\right)\underbrace{\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}}}_{\imath 0/0 \jmath}$$

$$\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \imath 0/0 \jmath$$

L’Hópital’s Rule:

$$= \lim_{n\to\infty}\frac{\exp\left(\frac{1}{n}\log\left(\frac{b}{a}\right)\right)-1}{1-\exp\left(\frac{p+1}{n}\log\left(\frac{b}{a}\right)\right)} = \lim_{n\to\infty}\frac{\log\left(\frac{b}{a}\right)\cdot\frac{-1}{n^2}\exp\left(\frac{1}{n}\log\frac{b}{a}\right)}{-(p+1)\log\frac{b}{a}\cdot\frac{-1}{n^2}\exp\left(\frac{p+1}{n}\log\frac{b}{a}\right)}$$

$$= \lim_{n\to\infty}\frac{-1}{p+1}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}}{\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \frac{-1}{p+1}$$

$$\Rightarrow = (a^{p+1}-b^{p+1})\cdot\frac{-1}{p+1} = \frac{b^{p+1}-a^{p+1}}{p+1}$$

The assignment explicitly asks for a step function. This approach only verifies that

$$\int_a^b x^p\,dx$$

$$\left.\frac{x^{p+1}}{p+1}\right|_{x=a}^{x=b} = \frac{b^{p+1}}{p+1}-\frac{a^{p+1}}{p+1}$$

We only did the approximation from one side (also upper bound is needed which works analogously):

$$\sum_{k=0}^{n-1} y_k x_{k+1}^p = \dots$$

# 18 Exercise 16

**Exercise 25**. For an interval $I \subseteq \mathbb{R}$ let $f_n : I \to \mathbb{R}$ be a sequence of functions which are uniformly continuous converging towards $f : I \to \mathbb{R}$. Show that the following statements hold or provide a counterexample:

- If all $f_n$ are uniformly continuous, then $f$ is uniformly continuous.
- If all $f_n$ are Lipschitz continuous, then $f$ is Lipschitz continuous.

### 18.1 Exercise 16.a

It holds. So a proof is given in the following.

We want to show:

$$\forall \varepsilon \exists \delta : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f(x_0) + f_n(x_0) - f(x_0)| \\ &\leq \underbrace{|f(x) - f_n(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x) - f_n(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} \end{aligned}$$

We need to elaborate: For which $n$ does $\frac{\varepsilon}{3}$ hold?

$$\forall \varepsilon > 0 \exists \overset{\text{depends on } \varepsilon}{n_0} : \forall n \geq n_0 \forall x \in I : |f(x) - f_n(x)| < \frac{\varepsilon}{3}$$

$$\forall \varepsilon > 0 \forall n \exists \delta = \delta(n, \varepsilon) : \forall x, x_0 : |x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

### 18.2 Exercise 16.b

This does not hold. So we provide a counterexample.

Consider $f(x) = \sqrt{x}$. It is not differentiable at $x = 0$, but $f(0) = 0$ is defined. The function cannot be Lipschitz-continuous, because the Lipschitz constant grows as we tend towards 0. We need functions $f_n$.

Consider $f(x) = \sqrt{x + \frac{1}{n}}$. The function $f_n$ looks like $f$, but is shifted slightly to the left. As $n$ tends towards infinity, $f_n$ becomes $f$ and we get the problem at $x = 0$.

You can also consider:

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq \frac{1}{n} \\ \sqrt{\frac{1}{n}} & \text{for } x < \frac{1}{n} \end{cases}$$

## 19 Exercise 17

**Exercise 26**. Let $f : [0, 1] \to \mathbb{R}$ be a regulated function continuous in 0. Show the following relation:

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\, ds = f(0).$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\, ds = f(0) = \lim_{n\to\infty} n \cdot \left(F\left(\frac{1}{n}\right) - F(0)\right) = \lim_{n\to\infty} \frac{F\left(\frac{1}{n}\right) - F(0)}{\frac{1}{n}} = \lim_{h\to 0} \frac{F(h) - F(0)}{h} = f(0)$$

### 19.1 Other approach

Continuity at $x = 0$:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall |x| < \delta : |f(x) - f(0)| < \varepsilon$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\, dx \leq \lim_{n\to\infty} n \int_0^{\frac{1}{n}} (f(0) + \varepsilon)\, dx$$

For $\frac{1}{n} < \delta$ it holds that $f(x) < f(0) + \varepsilon$ for $x \in [0, \frac{1}{n}]$.

$$= \lim_{n\to\infty} n(f(0) + \varepsilon)\frac{1}{n} = f(0) + \varepsilon$$

holds for all $\varepsilon > 0$.

$$\Rightarrow \lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\,dx \le f(0)$$

# 20 Exercise 18

**Exercise 27.** Prove the Riemann-Lebesgue Lemma: For every regulated function $f : [a,b] \to \mathbb{R}, a < b$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx = 0.$$

*Hint:* Show the following partial results:

- For all intervals $[\alpha,\beta] \subseteq [a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_\alpha^\beta \sin(\lambda x)\,dx = 0.$$

- For all step functions $g \in \tau[a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b g(x)\sin(\lambda x)\,dx = 0.$$

## 20.1 Exercise 18.a

$$-\frac{1}{\lambda}\cos(\lambda x)\Big|_\alpha^\beta = \underbrace{\frac{1}{\lambda}}_{\to 0}\underbrace{(-\cos(\beta\lambda)+\cos(\alpha\lambda))}_{\text{bounded}}$$

## 20.2 Exercise 18.b

Because $g$ is a step function of $[a,b]$, there exists a decomposition

$$a = x_0 < x_1 < \ldots < x_n = b$$

such that $g(x)$ has a constant value $c_i$ in every subinterval $[x_{i-1}, x_i)$.

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx = \lim_{\lambda\to\infty} \sum_{i=1}^n c_i \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx$$

This can be done, because we consider a finite sum.

$$\sum_{i=1}^n c_i \underbrace{\int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx}_{\to 0 \forall \text{ subintervals } H(i)}$$

$$= \sum_{i=1}^n c_i \cdot \underbrace{\lim \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx = 0}_{\to 0}$$

## 20.3 Conclusion

Because $f(x)$ is a regulated function $\forall\varepsilon > 0$, there exists a step function $g_\varepsilon(x)$ with $|f(x) - g_\varepsilon(x)| < \varepsilon \quad \forall x \in [a,b]$.

$$\left|\int_a^b f(x)\cdot\sin(\lambda x)\,dx\right| \le \underbrace{\int_a^b \left|\underbrace{f(x)-g_\varepsilon(x)}_{<\varepsilon}\right|}_{<\varepsilon(b-a)}\cdot\left|\underbrace{\sin(\lambda x)}_{\le 1}\right|\,dx + \underbrace{\left|\int_a^b g_\varepsilon(x)\sin(\lambda x)\,dx\right|}_{\to 0 \text{ for } \lambda\to\infty}$$

$$\lim_{\lambda\to\infty}\left|\int_a^b f(x)\sin(\lambda x)\,dx\right| \le \varepsilon(b-a)$$

We can choose $\varepsilon$ arbitrary, so it must tend towards 0.

# 21 Exercise 19

**Exercise 28**. Let $I, J \subseteq \mathbb{R}$ be intervals, $f : I \to \mathbb{R}$ continuous and $g, h : J \to I$ differentiable. Furthermore it holds that $g \le h$ in $J$. Prove that

$$A : J \to \mathbb{R}, \quad x \mapsto \int_{g(x)}^{h(x)} f(\xi)\,d\xi$$

is differentiable and determine its derivative.

## 21.1 Exercise 19.a

Show differentiability.

So

$$\lim_{x\to x_0}\frac{A(x)-A(x_0)}{x-x_0}$$

exists.

$$\lim_{x\to x_0}\frac{A(x)-A(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\int_{g(x)}^{h(x)} f(\xi)\,d\xi - \int_{g(x_0)}^{h(x_0)} f(\xi)\,d\xi}{x-x_0} = \lim_{x\to x_0}\frac{F(h(x))-F(g(x))-F(h(x_0))+F(g(x_0))}{x-x_0}$$

$F(h(x))$ and $F(g(x))$ exists, because $h(x)$ is continuous, so a regulated function and regulated functions always have a primitive function.

$$\lim_{x\to x_0}\frac{F(h(x))-F(h(x_0))}{x-x_0} - \lim_{x\to x_0}\frac{F(g(x))-F(g(x_0))}{x-x_0}$$

If $h(x)$ is continuous, then $F(h(x))$ is differentiable (analogously for $g(x)$). And the composition is also differentiable.

## 21.2 Exercise 19.b

Determine its derivative.

$$(F\circ h)'(x) - (F\circ g)'(x_0) = f(h(x_0))\cdot h'(x_0) - f(g(x_0))\cdot g'(x_0)$$

# 22 Exercise 20

**Exercise 29**. Determine the following integrals for arbitrary $a, b \in \mathbb{R}, a < b$:

- $\int_a^b \frac{d}{dx}\left(x^5\cdot e^x\right)\,dx$

- $\int_a^b x^4 e^{x^5}\, dx$

### 22.1 Exercise 20.a

$$\int_a^b \frac{d}{dx}\left(x^5 e^x\right)\, dx = \int_a^b 5x^4 e^x - x^5 e^x\, dx = \int_a^b \underbrace{e^x}_{g'(x)} \underbrace{(5x^4 - x^5)}_{=f(x)}\, dx$$

$$= e^x(5x^4 - x^5)\Big|_a^b - \int_a^b e^x(20x^3 + 5x^4) = e^b b^5 - e^a a^5$$

### 22.2 Exercise 20.b

$$\int_a^b x^4 e^{x^5}\, dx$$

$$u := x^5 \Rightarrow \frac{du}{dx} = 5x^4 \qquad dx = \frac{du}{5x^4}$$

$$= \int_{a^5}^{b^5} x^4 e^u \frac{du}{5x^4} = \int_{a^5}^{b^5} e^u \frac{du}{5} = \frac{1}{5}\int_{a^5}^{b^5} e^u\, du = \frac{1}{5}\left(e^{b^5} - e^{a^5}\right)$$

Other approach for 20.b:

$$F = \frac{1}{5} e^{x^5}$$

$$F' = x^4 e^{x^5} = f$$

## 23 Exercise 21

**Exercise 30.** Consider function $f$.

$$f : [-1, 1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & x = 0, \\ \frac{1}{n+2}, & x \in \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N}_+. \end{cases}$$

Is $f$ a step function? Is $f$ a regulated function? Furthermore determine

$$\int_{-1}^1 f(x)\, dx.$$

Is not a step function, because the number of intervals is not finite.

Is it a regulated function? We can approximate $f$ using the following construction:

$$\varphi_k(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{n+2} & x \in \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N}, n \leq k \\ 0 & \text{else} \end{cases}$$

We choose a $k$ such that all elements smaller $k$ are nonzero. This approximates our function $f$.

Consider

$$\int_{-1}^1 f(x)\, dx \overset{\varphi_n \to f \text{ uniformly}}{=} \lim_{h\to\infty} \int_{-1}^1 \varphi_n(x)\, dx$$

$$\int_{-1}^1 \varphi_n(x)\, dx = \sum_{j=1}^N c_j \triangle x_j = \sum_{n=1}^k \frac{1}{n+2} \cdot \left|\frac{1}{n} - \frac{1}{n+1}\right| \cdot 2 = 2 \cdot \sum_{n=1}^k \left(\frac{1}{n(n+2)} - \frac{1}{(n+2)(n+1)}\right)$$

$$= 2 \cdot \sum_{n=1}^{k} \left( \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right) - \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \right)$$

Alternatively we can also split it up, because we estimate that a series with $\frac{1}{n^2}$ converges.

$$= 2 \sum_{n=1}^{k} \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right) - 2 \sum_{n=1}^{k} \left( \frac{1}{n+1} - \frac{1}{n+2} \right) = 1 + \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{1}{2}$$

This expression is easier to evaluate as telescoping sum.

If we take the first approach, we need to apply partial fraction decomposition.

$$\frac{1}{2} \int_k = \frac{1}{2} \int_{-1}^{1} \varphi_k(x)\,dx = \frac{3}{4} - \frac{1}{2} + \frac{1}{2} \left( \underbrace{\frac{1}{k+1}}_{\to 0} + \underbrace{\frac{1}{k+2}}_{\to 0} \right) - \frac{1}{k+1} \overset{h\to\infty}{\longrightarrow} \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

# 24 Exercise 22

**Exercise 31.** Let $I \subseteq \mathbb{R}$ be an interval. Determine with the idea from below a primitive function of

$$f : I \to \mathbb{R}, \qquad x \mapsto x^2 \sin x^3 \cos x^3.$$

- For all $x \in \mathbb{R}$ it holds that
$$\sin x \cos x = \frac{1}{2} \sin(2x).$$
- For all $x \in \mathbb{R}$ it holds that
$$\sin x \cos x = \frac{1}{2} \frac{d}{dx} \sin^2(x).$$

Explain possible differences between the results.

## 24.1 Exercise 22.a

$$\int x^2 \sin(x^3) \cos(x^3)\,dx = \int x^2 \frac{1}{2} \sin(2x^3)\,dx$$

Substitute with $u = 2x^3$ and $dx = \frac{du}{6x^2}$.

$$= \int x^2 \frac{1}{2} \sin(u) \frac{du}{6x^2} = -\frac{1}{12} \cos(u) + c = -\frac{1}{12} \cos(2x^3) + c$$

## 24.2 Exercise 22.b

$$\forall x \in \mathbb{R} : \sin(x)\cos(x) = \frac{1}{2} \frac{d}{dx} \sin^2(x)$$

$$\forall x \in \mathbb{R} : \sin(x^3)\cos(x^3) = \frac{1}{6x^2} \frac{d}{dx} \sin^2(x^3) = \frac{1}{6x^2} 2 \sin(x^3)\cos(x^3) 3x^2 = \sin(x^3)\cos(x^3)$$

$$\int x^2 \sin(x^3)\cos(x^3)\,dx = \int x^2 \frac{1}{6x^2} \frac{d}{dx} \sin^2(x^3)\,dx = \frac{1}{6} \sin^2(x^3) + c$$

## 24.3 Exercise 22.c

$$\cos(2x^3) = \cos^2(x^3) - \sin^2(x^3) = 1 - \sin^2(x^3) - \sin^2(x^3) = 1 - 2\sin^2(x^3)$$

$$\Rightarrow \frac{1}{6} \sin^2(x^3) + \tilde{c}$$

with $\tilde{c} \approx \frac{1}{12} + c$.

# 25 Exercise 23

**Exercise 32**. Determine the following integrals using integration by parts.

1. $\int e^x \sin x \, dx$
2. $\int \arcsin x \, dx$
3. $\int_0^1 x^2 \ln^3(x) \, dx$

## 25.1 Exercise 23.a

$$\int \underbrace{e^x}_{u} \underbrace{\sin(x)}_{v'} \, dx$$

with $v = -\cos x$ and $u' = e^x$.

$$= e^x(-\cos x) - \int \underbrace{e^x}_{u} \cdot \underbrace{(-\cos x)}_{v'} \, dx$$

with $u' = e^x$ and $v = -\sin x$.

$$= e^x(-\cos x) - (e^x \cdot (-\sin x)) + \int e^x(-\sin x) \, dx$$

$$= e^x (-\cos x + \sin x) - \int e^x \sin x \, dx$$

$$\Rightarrow 2 \int e^x \sin(x) \, dx = e^x (-\cos(x) + \sin(x)) + c$$

## 25.2 Exercise 23.b

$$\int \arcsin(x) \, dx = \int \underbrace{\arcsin(x)}_{v} \cdot \frac{v'}{1} \, dx$$

with $v = x$ and $u' = \frac{1}{\sqrt{1-x^2}}$.

$$= \arcsin(x) \cdot x - \int x \frac{1}{\sqrt{1-x^2}} \, dx$$

Let $t = 1 - x^2$. Hence $\frac{dt}{dx} = -2x$.

$$= \arcsin(x) \cdot x - \int \frac{x}{\sqrt{t}} \cdot \frac{1}{-2x} \, dt$$

$$= \arcsin(x) \cdot x + \frac{1}{2} \int \frac{1}{\sqrt{t}} \, dt$$

$$= \arcsin(x) \cdot x + \frac{1}{2} \cdot 2\sqrt{t} + c$$

Backsubstitution:

$$= \arcsin(x) \cdot x + \sqrt{1-x^2} + c$$

## 25.3 Exercise 23.c

$$\int_0^1 \underbrace{x^2}_{f'} \underbrace{(\ln x)^3}_{g} \, dx = \frac{1}{3} x^3 (\ln x)^3 \Big|_0^1 - \int_0^1 \frac{1}{3} x^3 (\ln x)^2 \cdot 3 \frac{1}{x} \, dx$$

$$= \frac{1}{3} \cdot 0 - \frac{1}{3} \cdot \left( \lim_{x \to 0} (x^3 \ln^3 x) - \int_0^1 \underbrace{x^2}_{f'} \underbrace{\ln^2(x)}_{g} \, dx \right)$$

In the end, we can apply L'Hôpital's Rule once we have expressions like $-\frac{1}{3}\varphi^3\frac{\ln^3\varphi}{\ln^3\varphi}$.

$$= \frac{2}{3}\left(-\int_0^1 \frac{1}{3}x^2\,dx\right) = -\frac{2}{9}\cdot\frac{1}{3}x^3\Big|_0^1 = -\frac{2}{27}$$

# 26 Exercise 24

**Exercise 33**. Determine the following integrals using appropriate substitions:

1. $\int \frac{\cos^3(x)}{1-\sin(x)}\,dx$.
2. $\int \frac{dx}{\sin^2(x)\cos^4(x)}\,dx$ using $t := \tan x$.
3. $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ using $t := \arcsin(x)$.

## 26.1 Exercise 24.a

$$\frac{\cos^3(x)}{1-\sin(x)}\,dx = \int \frac{\cos(x)\left(1-\sin^2(x)\right)}{1-\sin(x)}\,dx = \int \cos'(x)\,(1 = \sin x)\,dx$$

with $u = 1+\sin x$ and $\frac{du}{dx} = \cos x$ we get

$$= \int \cos x\cdot u\cdot\frac{du}{\cos x} = \frac{1}{2}u^2 + c = \frac{1}{2}\,(1+\sin x)^2 + c$$

Do not forget $c$ for indefinite integrals!

## 26.2 Exercise 24.b

$$\begin{aligned}
&\int \frac{1}{\sin^2(x)\cdot\cos^4(x)}\,dx \\
&= \int \frac{\sin^4(x) + 2\sin^2(x)\cos^2(x) + \cos^4(x)}{\sin^2(x)\cos^4(x)}\,dx \\
&= \int \frac{\sin^2(x)}{\cos^4(x)} + \frac{2}{\cos^2(x)} + \frac{1}{\sin^2(x)}\,dx \\
&= \int \frac{\tan^2(x)+2}{\cos^2(x)}\,dx + \int \frac{1}{\sin^2(x)}\,dx
\end{aligned}$$

Consider the left-handed expression. Consider $t = \tan(x)$ and $\frac{dt}{dx} = \frac{1}{\cos^2(x)}$.

$$\int \frac{t^2+2}{\cos^2(x)}\cdot\cos^2(x)\,dt = \int t^2 + 2\,dt = \frac{1}{3}t^3 + 2t + c = \frac{1}{3}\tan^3(x) + 2\tan(x) + c_1$$

Consider the right-handed expression.

$$\begin{aligned}
\int \frac{1}{\sin^2(x)}\,dx = \int \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}\,dx &= \int -\frac{(\cos x)'\sin x - \cos x\cdot(\sin(x))'}{\sin^2(x)}\,dx \\
&= -\int \left(\frac{\cos x}{\sin x}\right)'\,dx = -\frac{\cos x}{\sin x} + c_2
\end{aligned}$$

So for the overall expression it holds that

$$\int \frac{\tan^2(x)+2}{\cos^2(x)}\,dx + \int \frac{1}{\sin^2(x)}\,dx = \frac{1}{3}\tan^3(x) + 2\tan(x) - \frac{1}{\tan(x)} + c_3$$

### 26.3 Exercise 24.b: Alternative approach

$$\int \frac{1}{s^2c^4}\, dx$$

with $s = \sin(x)$, $c = \cos(x)$, $t = \tan(x)$ and $\frac{dt}{dx} = \frac{1}{c^2}$. It holds that

$$t^2 = \frac{s^2}{c^2} = \frac{1-c^2}{c^2} = \frac{1}{c^2} - 1$$

$$c^2 = \frac{1}{1+t^2}$$

$$t^2 = \frac{s^2}{c^2} = \frac{s^2}{1-s^2} = -1 + \frac{1}{1-s^2}$$

$$1 - s^2 = \frac{1}{1+t^2}$$

$$s^2 = 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}$$

$$\int \frac{1}{s^2c^4}\, dx = \int \frac{1}{s^2c^2}\, dt = \int \frac{1+t^2}{t^2}(1+t^2)\, dt$$

### 26.4 Exercise 24.c

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}^3}\, dx = \int_0^{\frac{1}{2}} \frac{x^2+1-1}{\sqrt{1-x^2}^3}\, dx = -\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\, dx + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}^3}\, dx$$

$$= -\arcsin(x)\big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}^3}\, dx$$

with $t = \arcsin(x)$ and $\frac{dx}{dt} = \cos(t)$ with $x = \sin(t)$. Also $\sqrt{1-x^2}^3 = \sqrt{\cos^2(t)}^3 = \cos(t)^3$.

Recognize that $x$ must be positive for this to hold. As it turns out, this is fine within the interval $(0, \frac{1}{2})$.

$$= -\arcsin(x)\big|_0^{\frac{1}{2}} + \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^3(t)} \cdot \cos(t)\, dt$$

$$= -\arcsin(x)\big|_0^{\frac{1}{2}} + \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^2(t)}\, dt$$

$$= -\arcsin(x)\big|_0^{\frac{1}{2}} + \tan(t)\big|_{x=0}^{x=\frac{1}{2}}$$

$$= -\arcsin(x)\big|_0^{\frac{1}{2}} + \tan(\arcsin(x))\big|_0^{\frac{1}{2}}$$

$$= -\arcsin\left(\frac{1}{2}\right) + \arcsin(0) + \tan\left(\frac{\pi}{6}\right) - \tan(0)$$

$$= -\frac{\pi}{6} + 0 + \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} - 0$$

$$= -\frac{\pi}{6} + 0 + \frac{1}{\sqrt{3}} - 0$$

## 27 Exercise 25

**Exercise 34.** Determine

- $\int \frac{\sin x}{\sin x+\cos x}\, dx$.
- $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\cos^2(x)}\, dx$.

### 27.1 Exercise 25.a

$$\int \frac{\sin x}{\sin x + \cos x} = \int \frac{\frac{1}{2}(\sin x + \cos x) - \frac{1}{2}(-\sin x + \cos x)}{\sin x + \cos x}\, dx$$

$$= \int \left(\frac{1}{2} + \frac{\sin x - \cos x}{2(\sin x + \cos x)}\right) dx$$

With $u = \sin x + \cos x$ and $dx = \frac{du}{\cos(x) - \sin x}$, we get

$$= \frac{1}{2}x + \frac{1}{2} \cdot \int -\frac{1}{u}\, du$$

$$= \frac{x}{2} - \frac{1}{2} \cdot \ln(\sin x + \cos x) + c$$

## 28 Exercise 25.b

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\cos^2(x)}\, dx$$

Consider $u = \tan x$ and $dx = du\, \cos^2 x$.

$$= \int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{3}} \sqrt{u}\, du = \frac{2}{3}(\tan(x))^{\frac{3}{2}}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{2}{3}\left(3^{\frac{3}{4}} - 1\right)$$

## 29 Exercise 27

**Exercise 35.** Investigate the following impropert integrals for convergence.

1. $\int_1^\infty \frac{x^2}{2x^4-x+1}\, dx$.
2. $\int_0^\infty x^\alpha e^{-x}\, dx$, $\alpha \in \mathbb{R}$.
3. $\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx$.
4. $\int_0^\infty \left(\frac{\pi}{2} - \arctan(x)\right) dx$.

### 29.1 Exercise 27.a

$$\int_1^\infty \frac{x^2}{2x^4 - x + 1}\, dx$$

$$\frac{x^2}{2x^4 - x + 1} = \frac{x^2}{x(2x^3 - 1) + 1} < \frac{x}{2x^3 - 1} \leq \frac{x}{x^3} = \frac{1}{x^2}$$

$$\int_1^b \frac{x^2}{2x^4 - x + 1}\, dx < \int_1^b \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_1^b = -\frac{1}{b} + 1 < 1$$

In general: Approximately $\int_1^\infty \frac{x^2}{2x^4-x+1}$ converges becomes it looks close to $\int_1^\infty \frac{x^2}{2x^4}\, dx$.

## 29.2 Exercise 27.b

$$\int_1^\infty \frac{x^\alpha}{e^x}\, dx \qquad \alpha \in \mathbb{R}$$

$$\frac{x^\alpha}{e^x} = \frac{e^{\alpha \cdot \ln x}}{e^x} = e^{\alpha \ln(x) - x}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \sum_{k=l}^{\infty} \frac{x^k}{k!} \quad l \geq \alpha$$

$$x \geq 1 : \frac{x^\alpha}{e^x} \leq \frac{x^l}{e^x} = \frac{x^l}{\sum_{k=0} \frac{x^k}{k!}} < \frac{x^l}{\frac{x^{l+2}}{(l+2)!}} = \frac{(l+2)!}{x^2}$$

$$\int_1^b \frac{x^\alpha}{e^x}\, dx < \int_1^b \frac{(l+2)!}{x^2}\, dx < (l+2)!$$

## 29.3 Exercise 27.c

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx = \int_0^1 \frac{\sqrt{x}}{(1+x)^2}\, dx + \int_1^\infty \frac{\sqrt{x}}{(1+x)^2}\, dx$$

$$x \geq 1 : \frac{\sqrt{x}}{(1+x)^2} < \frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}}$$

$$\int_1^b \frac{\sqrt{x}}{(1+x)^2}\, dx < \int_1^b x^{-\frac{3}{2}}\, dx = 2 - \frac{2}{\sqrt{b}} < 2$$

$$0 \leq x \leq 1 : \frac{\sqrt{x}}{(1+x)^2} < \frac{\sqrt{x}}{1} = \sqrt{x} < 1$$

## 29.4 Exercise 27.d

$$\int_0^\infty \left(\frac{\pi}{2} - \arctan(x)\right) dx$$

$$\int \arctan(x) \cdot 1\, dx = x \cdot \arctan(x) - \int \frac{x}{1+x^2}\, dx$$

Integration by substitution:

$$t = 1 + x^2 \qquad \frac{dt}{dx} = 2x \Rightarrow dx = \frac{dt}{2x}$$

$$= x \cdot \arctan(x) - \int \frac{*\, dx}{t \cdot 2*} = x \cdot \arctan(x) - \frac{1}{2}\ln(1+x^2) + c$$

$$\Rightarrow \int_0^b (x - \arctan(x))\, dx = \frac{\pi}{2} - x \cdot \arctan(x) + \frac{1}{2}\ln(1+x^2)\Big|_0^b$$

$$= \underbrace{b}_{>0} \underbrace{\left(\frac{\pi}{2} - \arctan(b)\right)}_{>0} + \frac{1}{2}\ln(1+b^2) > \frac{1}{2}\ln(1+b^2) > M$$

### 29.5 Remark by the tutor

$$\int_1^\infty \frac{1}{x^c}\,dx \text{ converges} \iff c > 1$$

$$\lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan(x)}{\frac{1}{x}} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = -1$$

$$\exists x_0 : \frac{\pi}{2} - \arctan(x) > \frac{1}{2} \cdot \frac{1}{x} \quad \forall x \geq x_0$$

$$\int_0^\infty \frac{\pi}{2} - \arctan(x)\,dx \geq \int_{x_0}^\infty \frac{1}{2}\frac{1}{x}\,dx$$

## 30 Exercise 28

**Exercise 36.** Find all primitive functions of $f : (-1, 1) \to \mathbb{R}$ with $x \mapsto \frac{1}{1-x^4}$ using partial fraction decomposition.

Hint: To derive the partial fraction decomposition use

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{a}{1-x} + \frac{b}{1+x} + \frac{cx+d}{1+x^2}$$

with constants $a, b, c, d \in \mathbb{R}$. Determine the values for $a, b, c, d$.

$$\int \frac{1}{1-x^4}\,dx = \int \frac{1}{4(1-x)}\,dx + \int \frac{1}{4(1+x)}\,dx + \int \frac{1}{2(1+x^2)}\,dx$$

The first resulting integrals are:

$$\frac{1}{4}\int \frac{1}{1-x}\,dx = -\frac{1}{4}\ln 1 - x + c$$

$$\frac{1}{4}\int \frac{1}{1+x}\,dx = \frac{1}{4}\ln(1+x) + c$$

$$\frac{1}{2}\int \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x) + c$$

$$\int \frac{1}{1-x^4}\,dx = -\frac{1}{4}\ln(1-x) + \frac{1}{4}\ln(1+x) + \frac{1}{2}\arctan(x) + c$$

$$\frac{1}{4}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\arctan(x) + c$$

## 31 Exercise 29

**Exercise 37.** Given a function $f : (0, \infty) \to \mathbb{R}$ with $x \mapsto \frac{1}{x+\sqrt{x}}$.

- Determine the Taylor polynomial of second degree $T_f^2(x; 1)$ of function $f$ in point $x_0 = 1$.
- Determine an upper bound for the error $\left| f(x) - T_f^2(x; 1) \right|$ in interval $[1; 2]$.

### 31.1 Exercise 29.a

$$f'(x) = -\frac{1+2\sqrt{x}}{2\cdot x^{\frac{3}{2}}(1+\sqrt{x})^2}$$
$$f''(x) = \frac{3+9\sqrt{x}+8x}{4\cdot x^{\frac{5}{2}}\cdot(1+\sqrt{x})^3}$$

$$T_f^2(x;1) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}\cdot(x-1)^2$$
$$T_f^2(x;1) = \frac{1}{2} + \frac{3}{8}(x-1) + \frac{20}{64}(x-1)^2$$

### 31.2 Exercise 29.b

$$R_f^n(x;a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-x_0)^{n+1} \qquad \xi\in[x_0,x]$$
$$f'''(x) = -\frac{3\cdot(16x^{\frac{3}{2}}+29x+20\sqrt{x}+5}{8\cdot(\sqrt{x}+1)^4\cdot x^{\frac{5}{2}}}$$
$$\left|R_f^n(x;1)\right| = \frac{(x-1)^3}{(3)!}$$

Upper bound (not the best, but works):

$$\frac{3\cdot 65x^{\frac{3}{2}}+15}{8\cdot 16\cdot x^{\frac{5}{2}}} \leq \frac{3\cdot 65}{8\cdot 16x} + \frac{15}{8\cdot 16x^{\frac{5}{2}}}$$

$f'''(x)$ is monotonically increasing and we look at the interval $[1,2]$. So we are closest to 0, if $x=1$.

## 32 Exercise 30

**Exercise 38.** Let $g:\mathbb{R}\to\mathbb{R}$, $x\mapsto\sin(2x)$.

1. For arbitrary $n\in\mathbb{N}$, determine the Taylor polynomial of $n$-th degree $T_g^n(x;0)$ of $g$ in $x_0=0$.
2. Give a Taylor polynomial $T_g^n(x;0)$ such that $\left|g(x)-T_g^n(x;0)\right|<10^{-6}$ holds for $[-\pi,\pi]$.

### 32.1 Exercise 30.a

$$g(x) = \sin(2x)$$
$$g^{(1)}(x) = \cos(2x)\cdot 2$$
$$g^{(2)}(x) = -\sin(2x)\cdot 2^2$$
$$g^{(3)}(x) = -\cos(2x)\cdot 2^3$$
$$\cdots\cdots$$

$$
\begin{aligned}
T_g^n(x;0) &= g(0) + \frac{g^{(1)}(0)(x-0)}{1!} + \dots \\
&= \underbrace{\sin(0)}_{=0} + \underbrace{\cos(0)\cdot 2x}_{=2x} + \frac{-\sin(0)\cdot 2^2x^2}{2} + \frac{-\cos(0)\cdot 2^3x^3}{3!} = -\frac{2^3x^3}{3!} + \dots \\
T_g^n(x;0) &= \sum_{k=0}^{m}(-1)^k \cdot \left(\frac{(2x)^{2k+1}}{(2k+1)!}\right) \qquad \text{s.t. } 2m+1 \leq n, 2m+3 \geq n
\end{aligned}
$$

Even easier: Consider the power series for sin:

$$
\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\sin(2x) &= (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots
\end{aligned}
$$

### 32.2 Exercise 30.b

$$
R_g^n(x;0) = \left| \frac{f^{(n+1)}(\xi)(x-0)^{n+1}}{(n+1)!} \right| < 10^{-6}
$$

with $x, \xi \in [-\pi, \pi]$. We look at the following approximation ($\sin(x)$ and $\cos(x)$ is at most 1 and the factor $2^{n+1}$ of the derivative remains). Choose $\xi$ such that $\left| f^{(n+1)}(\xi) \right| \leq 2^{n+1}$.

$$
R_g^n(x;0) \leq \left| \frac{2^{n+1}x^{n+1}}{(n+1)!} \right| < 10^{-6} \iff 2^{n+1}x^{n+1}10^6 < (n+1)! \iff 10^6 < \frac{(n+1)!}{\left| 2^{n+1}x^{n+1} \right|} < (n+1)!
$$

In the worst case, $x$ is very large. The largest value it reaches is $\pi$. Hence,

$$
\left| R_g^n \right| \leq \frac{1}{(n+1)!} \cdot 2^{n+1} \cdot \pi^{n+1} \leq 10^{-6} \qquad \forall x \in [-\pi, \pi]
$$

This holds if $n \geq 26$.

## 33 Exercise 26

**Exercise 39.** Prove the following limit criterion for improper integrals. Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ functions, which satisfy $f \geq 0$ and $g > 0$ in $[a, \infty)$. Furthermore the following limit exists:

$$
L := \lim_{x\to\infty} \frac{f(x)}{g(x)} \in [0, \infty].
$$

Then it holds that,

1. $L = 0 \Rightarrow \left[\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty\right]$.
2. $L \in (0, \infty) \Rightarrow \left[\int_a^\infty g(x)\,dx < \infty \Leftrightarrow \int_a^\infty f(x)\,dx < \infty\right]$.
3. $L = \infty \Rightarrow \left[\int_a^\infty g(x)\,dx \text{ diverges } \Rightarrow \int_a^\infty f(x)\,dx \text{ diverges}\right]$.

### 33.1 Exercise 26.a

We provide a counterexample:

$$
f(x) = \begin{cases} \frac{1}{x} & 0 < x < 1 \\ 0 & x = 0 \\ \frac{1}{x^2} & x > 1 \end{cases}
$$

$$g(x) = \frac{1}{x^2+1}$$
$$a = 0$$

To make this proposition work, $f$ must be continuous or even boundedness should suffice.

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \left| \frac{f(x)}{g(x)} \right| < \varepsilon$$

Both functions yield positive values:

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} < \varepsilon$$

$$\int_a^\infty f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^\infty f(x)\,dx$$

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} < \varepsilon \iff f(x) < \varepsilon \cdot g(x) \implies \int_{x_0}^\infty f(x)\,dx < \varepsilon \int^\infty g(x)\,dx$$

Because of boundedness we can provide the following estimates:

$$\int_a^\infty f(x)\,dx = \underbrace{\int_a^{x_0} f(x)\,dx}_{<\infty} + \underbrace{\int_{x_0}^\infty f(x)\,dx}_{<\infty}$$
$$\implies < \infty$$

### 33.2 Exercise 26.b

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$
$$\iff L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon$$
$$\iff (L - \varepsilon) \cdot g(x) < f(x) < (L + \varepsilon) \cdot g(x)$$
$$(L - \varepsilon) \int_{x_0}^\infty g(x)\,dx \leq \int_0^\infty f(x)\,dx \leq (L + \varepsilon) \int_{x_0}^\infty g(x)$$

### 33.3 Exercise 27.c

$$\forall n \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} > n \iff f(x) > n \cdot g(x) \implies \int_{x_0}^\infty f(x)\,dx \geq n \cdot \int_{x_0}^\infty g(x)$$

## 34 Exercise 31

**Exercise 40**. Let $I \subseteq \mathbb{R}$ be an interval, $a \in I$, $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ is differentiable $n$-times. Show: If a polynomial

$$P(x) = \sum_{k=0}^{n} a_k (x-a)^k$$

with $a_k \in \mathbb{R}$, $0 \leq k \leq n$ satisfies

$$\lim_{x \to a} \frac{f(x) - P(x)}{(x-a)^n} = 0$$

then $P(x) = T_f^n(x; a)$.

$$T_f^n(x;a) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!} = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

How do derivatives of $P$ look like?

$$P^{(1)}(x) = \sum_{k=0}^{n} k a_k (x-a)^{k-1}$$

$$P^{(2)}(x) = \sum_{k=0}^{n} a_k k(k-1)(x-a)^{k-2}$$

$$P^{(i)}(x) = \sum_{k=0}^{n} k(k-1)\cdot\ldots\cdot(k-i+1)a_k(x-a)^{k-i}$$

$$P^{(i)}(a) = a_k \cdot \underbrace{k(k-1)(k-2)\cdot\ldots\cdot(1)}_{=k!}$$

$$a_k = \frac{P^{(k)}(a)}{k!}$$

$$P(x) = \sum_{k=0}^{n} a_k(x-a)^k = \sum_{k=0}^{n} \frac{P^{(k)}(a)}{k!}(x-a)^k$$

For $k = 0$:

$$P^{(0)}(a) = P(a) = a_k(a-a)^0 = a_k$$

$$\lim_{x\to a} \frac{f(x)-P(x)}{(x-a)^n} = \lim_{n\to\infty} \frac{f'(x)-P'(x)}{n(x-a)^{n-1}} = \text{apply L'Hôpital's rule } n \text{ times} = \lim_{x\to a} \frac{f^{(n)}(x)-P^{(n)}(x)}{n!(x-a)^0} = 0$$

Because $f$ and $p$ are continuous, we have

$$f(x) - P(x) \overset{x\to a}{\to} 0$$

$$f(a) - P(a) = 0 \Leftrightarrow f(a) = P(a)$$

We need to show: $P^{(k)}(a) = f^{(k)}(a)$ for $0 \le k \le n$ and $P^{(0)}(a) = f^{(0)}(a)$.

# 35 Exercise 32

**Exercise 41.** Let $f_n : [a,b] \to \mathbb{R}$, $a < b$, be a sequence of regulated functions converging uniformly to $f : [a,b] \to \mathbb{R}$. Prove that $f$ is a regulated function and limes and integration can be exchanged as follows:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \left( = \int_a^b f(x)\,dx \right)$$

$$\forall \varepsilon > 0 \exists N = N(\varepsilon) : \forall n \ge N : \ldots$$

$$\text{Especially for } n = N(\varepsilon) : \ldots$$

$$\exists M = M(\varepsilon) : \forall m \ge M : \ldots$$

$$\text{Especially for } m = M(\varepsilon) : \ldots$$

$$\forall x : \left| f_{N(\varepsilon)}(x) - f(x) \right| < \varepsilon$$

$$\left| f_{N(\varepsilon)}(x) - f_{N(\varepsilon),M(\varepsilon)} \right| < \varepsilon$$

$$h_k \coloneqq g_{N(\frac{1}{k}),M(\frac{1}{k})}$$

$$\forall x \, | f(x) - h_k(x) | \le \frac{2}{k} \overset{k\to\infty}{\to} 0$$

# 36 Exercise 33

$$P(x) := \sum_{k=0}^{\infty} a_k x^k \qquad \rho = c > 0$$

To show: $\forall b \in (-c, c), a < b$:

$$\sum_{k=0}^{\infty} \int_a^b a_k x^k \, dx = \int_a^b \sum_{k=0}^{\infty} a_k x^k \, dx = \int_a^b P(x) \, dx$$

$$\sum_{k=0}^{\infty} \int_a^b a_k x^k \, dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_a^b a_k x^k \, dx = \lim_{n\to\infty} \int_a^b \sum_{k=0}^{n} a_k x^k \, dx \overset{\text{ex. 32}}{=} \int_a^b \lim_{n\to\infty} \sum_{k=0}^{n} a_k x^k \, dx$$

But we need to show that the requirements for the equation in exercise 32 are satisfied.

$$f(x) = \sum_{k=0}^{\infty} a_k \cdot x^k \qquad g(f) = c$$

$$f_n(x) = \sum_{k=0}^{n} a_k \cdot x^k$$

$$|f(x) - f_n(x)| = \left| \sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{n} a_k x^k \right| = \left| \sum_{k=n+1}^{\infty} a_n \cdot x^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot \left| x^k \right|$$

It holds that $\left|\frac{x}{r}\right|^k \leq \left|\frac{x}{r}\right|^{n+1}$. Then consider

$$|r < c| = \sum_{k=n+1}^{\infty} |a_k| \cdot \frac{|x|^k}{|r|^k} \cdot |r|^k \leq \left| \frac{x}{r} \right|^{n+1} \underbrace{\sum_{k=n+1}^{\infty} |a_n| \cdot |r|^k}_{=S} = \underbrace{\left| \frac{x}{r} \right|^{n+1}}_{<\varepsilon} \cdot S$$

We chose some $c$ close enough such that our desired properties are fulfilled.

## 36.1 Elaboration

Convergence radius $c$: $|x| < c$ and $|x| < r < c$. Let $a_k r^k$ be bounded and $\sum_k a_k r^k$ be convergent.

$$\sum_{k=0}^{\infty} \left| a_k x^k \right| \leq \sum_{k=0}^{\infty} \left| \frac{x}{r} \right|^k \underbrace{\left| a_k r^k \right|}_{\leq C} \leq C \sum_{k=0}^{\infty} \left| \frac{x}{r} \right|^k = \frac{c}{1 - \left|\frac{x}{r}\right|}$$

# 37 Exercise 34

> **Exercise 42**. Determine the power series representation $P(x)$ of arctan in $x_0 = 0$. Furthermore give the largest interval $(-c, c)$ with $c > 0$ in which power series $P(x)$ represents arctan.
>
> Hint: Represent the derivative of arctan as a power series and use exercise 33.

$$\arctan(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Geometrical series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \qquad \text{converges if } |x| < 1$$

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1}x^{2n+1}+c\right)'$$

$$T_n(x,0) = \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k$$

$$\begin{aligned}
\arctan^{(2)}(x) &= \sum_{n=0}^{\infty}(-1)^n 2n x^{2n-1}\\
\arctan^{(3)}(x) &= \sum_{n=0}^{\infty}(-1)^n 2n(2n-1)x^{2n-2}\\
\arctan^{(k)}(x) &= \sum_{n=0}^{\infty}(-1)^n(2n)(2n-1)\dots(2n-(k-2))x^{(2n-(k-1))}\\
\arctan^{(k)}(0) &= (-1)^{\frac{k-1}{k}}(k-1)(k-2)\dots(k-1-k+2) = (k-1)!(-1)^{\frac{k-1}{2}}
\end{aligned}$$

$$T_n(x;0) = \sum_{k=0}^{n}\frac{(-1)^l(2l)!}{(2l+1)!}x^{2l+1} = \sum_{l=0}^{n}\frac{(-1)^l x^{2l+1}}{2l+1}$$

# 38 Exercise 35

**Exercise 43.** We consider the parametric curve

$$x(t) := a\cdot\cos(t), \qquad y(t) := b\cdot\sin(t)$$

for $t\in[0,2\pi]$ and $a,b>0$.

- Interpret the image of the curve and parameter $t$ geometrically.
- Determine the tangent on the curve in point $(x(t),y(t))$ for $t\in(0,\pi/2)$. Determine also a parameter-free form of the equation.

## 38.1 Exercise 35.a

Ellipse. The minimum and maximum $x$-values are $-a$ and $a$ respectively. The minimum and maximum $y$-values are $-b$ and $b$ respectively.

## 38.2 Exercise 35.b

$$\dot\gamma(t) = \begin{bmatrix}\dot x(t)\\ \dot y(t)\end{bmatrix} = \begin{bmatrix}-a\cdot\sin(t)\\ b\cdot\cos(t)\end{bmatrix}$$

$$T(x) = \gamma(t_0)+\lambda\cdot\begin{bmatrix}-a\sin(t_0)\\ b\cos(t_0)\end{bmatrix}$$

$$\begin{aligned}
x &= a\cdot\cos(t)+\lambda\cdot(-a\cdot\sin(t_0))\\
y &= b\cdot\sin(t)+\lambda\cdot(b\cdot\cos(t_0))
\end{aligned}$$

$$\implies \lambda = -\frac{x-a\cdot\cos(t_0)}{a\cdot\sin(t_0)} = b\cdot\sin(t_0)-\frac{x-a\cdot\cos(t_0)}{a\cdot\sin(t_0)}(b\cdot\cos(t_0))$$

$$= \frac{-x\cdot b\cdot\cos(t_0)}{a\cdot\sin(t_0)}+\frac{\cos(t_0)}{\sin(t_0)}\cdot b\cdot\cos(t_0)+b\sin(t_0) = -\frac{b\cdot\cos(t_0)}{a\cdot\sin(t_0)}x+\dots$$

We can simplify handling $d$ in $y=kx+d$ if we determine $d$ using $x=0$.

## 39 Exercise 36

**Exercise 44**. Let $\gamma : [a,b] \to \mathbb{R}^2$, $a < b$, be a continuous curve. It is differentiable in $(a,b)$ and $\gamma'(t) \neq 0$ for all $t \in (a,b)$. Show the following variant of the Intermediate Value Theorem: There exists some $c \in (a,b)$ and some $\rho \in \mathbb{R}$, such that

$$\frac{\gamma(b) - \gamma(a)}{b-a} = \rho\gamma'(c)$$

So at $c$ the tangent is parallel to the chord (dt. "Sehne").

$\gamma(a) = \gamma(b)$

$$\implies \frac{\gamma(b)-\gamma(a)}{b-a} = \frac{0}{b-a} = \rho\gamma'(c)$$

$\gamma(a) \neq \gamma(b)$

$$u = \gamma(b) - \gamma(a)$$

In $\mathbb{R}^2$, we can chose a basis.

$$u = \begin{bmatrix} x(b) - x(a) \\ y(b) - y(a) \end{bmatrix}$$

$$v = \begin{bmatrix} y(a) - y(b) \\ x(b) - x(a) \end{bmatrix}$$

We chose $v$ such that $v$ is orthogonal to $u$.

$$\gamma(t) - \gamma(a) = \alpha(t) \cdot u + \beta(t) \cdot v$$

$$0 = \gamma(a) - \gamma(a) = \alpha(a) \cdot u + \beta(a) \cdot v \overset{\text{linear indep.}}{\implies} \alpha(a) = \beta(a) = 0$$

$$u = \gamma(b) - \gamma(a) = \alpha(b) + \beta(b) \cdot v \implies \alpha(b) = 1 \land \beta(b) = 0$$

Lemma: $\beta'(t)$ is differentiable.

$$\gamma(t) - \gamma(a) =: \tilde{\gamma}(t)$$

$$\tilde{\gamma}(t) = \alpha(t) \cdot u + \beta(t) \cdot v$$

$$\langle \tilde{\gamma}(t), v \rangle = \alpha(t) \cdot \underbrace{\langle u, v \rangle}_{0} + \beta(t) \underbrace{\langle v, v \rangle}_{\neq 0}$$

Let $k := \langle v, v \rangle$.

$$\tilde{\gamma}_1(t) \cdot v_1 + \tilde{\gamma}_2(t) \cdot v_2 = \beta(t) \cdot k$$

Analogously, it can be proven that $\alpha'(t)$ is differentiable.

Given differentiability, we can now claim:

$$\exists \xi \in (a,b) : \frac{\beta(b) - \beta(a)}{b-a} = \beta'(\xi)$$

$$\implies \gamma'(t) = \alpha'(t) \cdot u + \beta'(t) \cdot v$$

$$\implies \gamma'(\xi) = \alpha'(\xi)(\gamma(b) - \gamma(a))$$

## 40 Exercise 37

**Exercise 45**. Let $a, b \in \mathbb{R}$ with $a < b$ and a helix

$$\gamma : [a,b] \to \mathbb{R}^3, \quad t \mapsto \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

- For all $t \in [a,b]$, determine the tangent vector of $\gamma$ in point $\gamma(t)$.

- Determine the arc length of $\gamma|_{[a,c]}$ for arbitrary $c \in [a,b]$.
- Does the variant of the Intermediate Value Theorem from Exercise 36 also hold in this case?

### 40.1 Exercise 37.a

$$\dot{\gamma}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix}$$

### 40.2 Exercise 37.b

$$s(\gamma|_{[a,c]}) = \int_a^c \| \dot{\gamma}(t) \| \, dt = \int_a^c \sqrt{\sin^2(t) + \cos^2(t) + 1^2} \, dt = \int_a^c \sqrt{2} \, dt = c\sqrt{2} - a\sqrt{2} = \sqrt{2}(c-a)$$

### 40.3 Exercise 37.c

$$\begin{pmatrix} \frac{\cos(b)-\cos(a)}{b-a} \\ \frac{\sin(b)-\sin(a)}{b-a} \\ \frac{b-a}{b-a} \end{pmatrix} = \rho \begin{pmatrix} -\sin(c) \\ \cos(c) \\ 1 \end{pmatrix}$$

Consider $a = 0, b = 2\pi$ and you will see that it cannot hold. Actually it does not hold for arbitrary $a$ and $b$.

## 41 Exercise 38

**Exercise 46.** For $a, b > 0$ and $t \in [0, 2\pi]$ define the astroid using

$$x(t) := a \cdot \cos^3(t), \qquad y(t) := b \cdot \sin^3(t)$$

- Determine a representation $y = f(x)$ of the curve $(x(t), y(t))$ with $t \in [0, \pi]$.
- Give the arc length of astroids and sketch its graph.

### 41.1 Exercise 38.a

$$y = f(x) \text{ needs to be found for } t \in [0, \pi]$$

$$x = a \cdot \cos^3(t) \Leftrightarrow t = \arccos\left(\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)$$

$$y = b \cdot \sin^3 \cdot \left(\arccos\left(\frac{c^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)\right) = b \cdot \left(\sin\left(\arccos\left(\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)\right)^3\right)$$

$$= b \cdot \left(\sqrt{1 - \frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}}\right)^3$$

$$\sin(\arccos(x)) = \sqrt{1 - x^2}$$

Simpler approach:

$$x(t) = a \cdot \cos^3(t)$$

$$\cos(t) = \left(\frac{x}{a}\right)^{\frac{1}{3}}$$

$$\sin(t) = \sqrt{1-\cos^2(t)} = \sqrt{1-\left(\frac{x}{a}\right)^{\frac{2}{3}}}$$

$$\begin{aligned}\left(\frac{x}{a}\right)^{\frac{2}{3}} &= \cos^2(t)\\ \left(\frac{y}{b}\right)^{\frac{2}{3}} &= \sin^2(t)\\ \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} &= 1\end{aligned}$$

Okay, with either approach, we come to our desired result.

### 41.2 Exercise 38.b

$$\dot\gamma(t) = \begin{pmatrix}-3a\sin(t)\cdot\cos^2(t)\\ 3b\cos(t)\sin^2(t)\end{pmatrix}$$

$$\begin{aligned}s = \int_0^{2\pi} \|\,\dot\gamma(t)\,\|\,dt &= \int_0^{2\pi}\sqrt{9a^2\sin^2(t)\cos^4(t) + 9b^2\cos^2(t)\sin^4(t)}\,dt\\ &= \int_0^{2\pi} |\,3\sin(t)\cos(t)\,|\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}\,dt\end{aligned}$$

Actually this does *not* work! Because cos and sin can be negative, we cannot take it out of the square root. This makes our expression much more complex.

We can simplify this expression. Use upper boundary value $\frac{\pi}{2}$ and take its integral/length times 4.

$$= 4\int_0^{\frac{\pi}{2}} 3\sin(t)\cos(t)\sqrt{a^2\cos^2(t)+b^2\sin^2(t)}\,dt$$

Consider $u = a^2\cos^2(t) + b^2\sin^2(t)$, then $dt = \frac{du}{2\sin(t)\cos(t)(-a^2+b^2)}$.

$$= 4\cdot\int_{t=0}^{t=\frac{\pi}{2}} \frac{3}{2(-a^2+b^2)}\sqrt{u}\,du$$

$$\begin{aligned}\frac{6}{-a^2+b^2}\int_{t=0}^{t=\frac{\pi}{2}}\sqrt{u}\,du &= \frac{6}{-a^2+b^2}\cdot\frac{2}{3}u^{\frac{3}{2}}\Big|_{t=0}^{t=\frac{\pi}{2}}\\ &= \frac{4}{b^2-a^2}\left(a^2\cos^2(t)+b^2\sin^2(t)\right)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}}\\ &= \frac{4}{b^2-a^2}\left[(b^2)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}}\right]\\ &= \frac{4}{b^2-a^2}(b^3-a^3)\end{aligned}$$

Problem if $a = b$. You need to make a case distinction before substitution. This is missing here.

Sketch: A star with center $(0,0)$. The maximum/minimum $x$-value is $-a$ and $a$ respectively. The maximum/minimum $y$-value is $-b$ and $b$ respectively.

## 42 Exercise 39

**Exercise 47.** We again consider astroids from Exercise 38.

- Determine in every point of the curve its curvature.
- Find, for the evolute of astroids, a representation of form $(\xi(t), \eta(t))$ for $t \in [0, 2\pi]$ and sketch its

curves.

$$\gamma(t) = \begin{pmatrix} a \cdot \cos^3(t) \\ b \cdot \sin^3(t) \end{pmatrix}$$

$$\dot{\gamma}(t) = \begin{pmatrix} -3a \cdot \sin(t)\cos^2(t) \\ 3b\sin^2(t)\cos(t) \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} 3b\sin^2(t)\cos(t) \\ 3a\sin(t)\cos^2(t) \end{pmatrix}$$

$$\kappa(t) = \langle \ddot{\gamma}(t), \vec{n} \rangle$$

$$\ddot{\gamma}(t) = \begin{pmatrix} 3a(2\cos(t)\sin^2(t) - \cos^3(t)) \\ 3b(2\sin(t)\cos^2(t) - \sin^3(t)) \end{pmatrix}$$

$$\kappa(t) = \left\langle \begin{pmatrix} 6a\cos(t)\sin^2(t) - 3a\cos^3(t) \\ 6b\sin(t)\cos^2(t) - 3b\sin^3(t) \end{pmatrix}, \begin{pmatrix} 3b\sin^2(t)\cos(t) \\ 3a\sin(t)\cos^2(t) \end{pmatrix} \right\rangle$$

$$18ab\sin^4(t)\cos^2(t) - 9ab\cos^4(t)\sin^2(t) + 18ab\sin^2(t)\cos^4(t) - 9ab\sin^4(t)\cos^2(t)$$
$$= 9ab\sin^2(t)\cos^2(t)\underbrace{\left(2\sin^2(t) - \cos^2(t) + 2\cos^2(t) - \sin^2(t)\right)}_{=1}$$
$$= 9ab\sin^2(t)\cos^2(t)$$

This curvature form assumes that $N$ is already normalized. This solution looks suspicious, because the usual denominator is missing (this does not necessarily mean that it is wrong).

Normalize it afterwards?! (here cubic)

$$\implies \frac{9ab\sin^2(t)\cos^2(t)}{|\vec{n}|^3}$$

Tutor's curvature equation:

$$\frac{\dot{x} \cdot \ddot{y} - \ddot{x} \cdot \dot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

$$|\vec{n}|^3 = \sqrt{9b^2\sin^4(t)\cos^2(t) + 9a^2\sin^2(t)\cos^4(t)}^{\,3}$$
$$= \left(3|\sin(t)\cos(t)|\sqrt{b^2\sin^2(t) + a^2\cos^2(t)}\right)^3$$

## 42.1 Exercise 39.b

We used norm vector

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\dot{y} \\ \dot{x} \end{pmatrix}$$

So we should have actually used

$$\vec{n} = \begin{pmatrix} -3b\sin^2(t)\cos(t) \\ -3a\sin(t)\cos^2(t) \end{pmatrix}$$

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} -y'(t) \\ x'(t) \end{pmatrix} \cdot \frac{x'(t)^2 + y'(t)^2}{x'(t)y''(t) - x''(t)y'(t)}$$
$$x'(t)^2 + y'(t)^2 = 9\cos^2(t)\sin^2(t)(a^2\cos^2(t) + b^2\sin^2(t))$$
$$x'(t)y''(t) - x''(t)y'(t) = -9ab\sin^2(t)\cos^2(t)$$

$$\xi(t) = a\cos^3(t) + 3b\sin^2(t)\cos(t) \cdot \frac{a^2\cos^2(t) + b^2\sin^2(t)}{ab}$$
$$\eta(t) = b\sin^3(t) + 3a\sin(t)\cos^2(t) \cdot \frac{a^2\cos^2(t) + b^2\sin^2(t)}{ab}$$

Sketched, this is a rectangle, but the corner have a little higher distance from the center.

# 43 Exercise 39-42

I was missing in class. Hopeful I will get access to other notes to transcribe them here.

# 44 Exercise 43

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$
$$f(x,y) = \begin{bmatrix} x^2 + 3y^2 \\ 4x - y^2 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Let $\varepsilon > 0$.

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\| < \delta$$
$$\delta := \min\left(\frac{\varepsilon}{14}, 1\right)$$
$$\left\| f(\begin{bmatrix} x \\ y \end{bmatrix}) - f(\begin{bmatrix} 2 \\ -1 \end{bmatrix}) \right\|_\infty = \left\| \begin{bmatrix} x^2 + 3y^2 - 7 \\ 4x - y^2 - 7 \end{bmatrix} \right\|_\infty$$

1st component:

$$\left| x^2 + 3y^2 - 7 \right| = \left| x^2 - 4 + 3y^2 - 3 \right| \leq |x-2||x+2| + 3|y-1||y+1|$$

$$\left| \begin{matrix} |x-2| < \delta \leq 1 \implies |x| < 3 \\ |y+1| < \delta \leq 1 \implies |y| < 2 \\ |x+2| < 5; |y-1| < 3 \end{matrix} \right|$$
$$< 5\underbrace{|x-2|}_{<\delta} + 9|y-1| < 5\delta + 9\delta = 14\delta \leq \varepsilon$$

2nd component:

$$\left| 4x - y^2 - 7 \right| = \left| 4x - 8 - y^2 + 1 \right| \leq 4\underbrace{|x-2|}_{<\delta} + |1-y|\underbrace{|1+y|}_{<\delta}$$
$$< 4\delta + 3\delta = 7\delta < \frac{\varepsilon}{2} < \varepsilon$$

# 45 Exercise 44

Show that $\| f \| := \max_{x\in[0,1]} | f(x) |$.

$$| f | := \left( \int_0^1 | f(x) |^2 \, dx \right)^{\frac{1}{2}}$$

defines a norm in $C([0,1],\mathbb{R})$.

$$\begin{aligned} \| \lambda f \| &= \max | \lambda f(x) | = \max | \lambda | | f(x) | = | \lambda | \| f \| \\ \| f + g \| &= \max | (f+g)(x) | = \max | f(x) + g(x) | \\ &\leq \max (| f(x) | + | g(x) |) \leq \max | f(x) | + \max | g(x) | \end{aligned}$$

Let $| f | = 0$, then

$$\left( \int_0^1 \underbrace{| f(x) |^2}_{\geq 0 \ \forall x \in [0,1]} \, dx \right)^{\frac{1}{2}} = 0 \iff f(x) = 0 \ \forall x \in [0,1]$$

$$\begin{aligned} | \lambda f | &= \left( \int_0^1 | \lambda f(x) |^2 \, dx \right)^{\frac{1}{2}} = \left( \int_0^1 | \lambda |^2 | f(x) |^2 \, dx \right)^{\frac{1}{2}} \\ &= \left( | \lambda |^2 \int_0^1 | f(x) |^2 \, dx \right)^{\frac{1}{2}} = | \lambda | \left( \int_0^1 | f(x) |^2 \, dx \right)^{\frac{1}{2}} \end{aligned}$$

Show: $| f + g | = | f | + | g |$.

$$\begin{aligned} | f + g |^2 &= \int_0^1 | (f+g)(x) |^2 \, dx = \int_0^1 \left| f(x)^2 + 2f(x)g(x) + g(x)^2 \right| \, dx \\ &\leq \int_0^1 | f(x) |^2 \, dx + 2 \int_0^1 | f(x)g(x) | \, dx + \int_0^1 | g(x) |^2 \, dx \\ &\leq \int_0^1 | f(x) |^2 \, dx + 2 \left( \int_0^1 | f(x) |^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^1 | g(x) |^2 \, dx \right)^{\frac{1}{2}} + \int_0^1 | g(x) |^2 \, dx \\ &= | f |^2 + 2 | f | | g | + | g |^2 = (| f | + | g |)^2 \\ &\implies | f + g | \leq | f | + | g | \end{aligned}$$

Let $f_n(x) = x^n$.

$$\begin{aligned} \lim_{n\to\infty} \| f_n \| &:= \lim_{n\to\infty} \max_{x\in[0,1]} | f(x) | = 1 \\ \lim_{n\to\infty} | f_n | &= \lim_{n\to\infty} \left( \int_0^1 | f(x) |^2 \, dx \right)^{\frac{1}{2}} \\ &= \lim_{n\to\infty} \left( \frac{x^{n+1}}{n+1} \Big|_0^1 \right)^{\frac{1}{2}} \\ \lim_{n\to\infty} \left( \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} \right)^{\frac{1}{2}} &= \lim \left( \frac{1}{n+1} \right)^{\frac{1}{2}} = 0 \end{aligned}$$

# 46 Exercise 45

**Exercise 48.** Let $I, J \subseteq \mathbb{R}$ be non-empty intervals and $\varphi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ continuous. Prove that

$$f : I \times J \to \mathbb{R}, \qquad f(x,y) := \varphi(x) + \psi(y)$$

and

$$g : I \times J \to \mathbb{R}, \qquad g(x,y) := \varphi(x)\psi(y)$$

are continuous.

### 46.1 Part 1

We use the $\varepsilon$-$\delta$-criterion.

$$\forall x_0 \in I \forall \varepsilon > 0 \exists \delta_1 > 0 : \forall x \in I : |x - x_0| < \delta_1 \implies |\varphi(x) - \varphi(x_0)| < \frac{\varepsilon}{2}$$
$$\forall y_0 \in J \forall \varepsilon > 0 \exists \delta_2 > 0 : \forall y \in J : |y - y_0| < \delta_2 \implies |\psi(y) - \psi(y_0)| < \frac{\varepsilon}{2}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$.

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right\|_1 = |x - x_0| + |y - y_0| < \delta \implies |x - x_0| < \delta \wedge |y - y_0| < \delta$$

$$\begin{aligned} \left| f\begin{pmatrix} x \\ y \end{pmatrix} - f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right| &= |\varphi(x) + \psi(y) - (\varphi(x_0) - \psi(y_0))| \\ &\leq |\varphi(x) - \varphi(x_0)| + |\psi(y) - \psi(y_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

### 46.2 Part 2

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $I \times J$. $v = \lim_{n\to\infty}((v_n)_{n\in\mathbb{N}})$.

$$v'_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}_{n\in\mathbb{N}} \qquad v = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} \right\|_1 = |x_n - x| + |y_n - y| < \varepsilon \qquad \text{for } n \geq N$$
$$x = \lim_{n\to\infty}(x_n)_{n\in\mathbb{N}} \qquad y = \lim_{y\to\infty}(y_n)_{n\in\mathbb{N}}$$

$\varphi$ is continuous in $x$ and $\psi$ is continuous in $y$.

$$\lim_{n\to\infty}(\varphi(x_n)_{n\in\mathbb{N}}) = \varphi(x) \wedge \lim_{n\to\infty}(\psi(y_n)_{n\in\mathbb{N}}) = \psi(y)$$
$$\lim_{n\to\infty} g(x_n, y_n)_{n\in\mathbb{N}} = \lim_{n\to\infty}(\varphi(x_n)_{n\in\mathbb{N}} \cdot \varphi(y_n)_{n\in\mathbb{N}}) = \lim_{n\to\infty}(\varphi(x_n)_{n\in\mathbb{N}}) \cdot \lim_{n\to\infty}(\psi(y_n)_{n\in\mathbb{N}}) = \varphi(x) \cdot \psi(y)$$

Remark: Part 2 can also be proven using the $\varepsilon$-$\delta$-definition. It is a bit more cumbersome, but feasible.

## 47 Exercise 46

**Exercise 49**. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) := \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Show the $f$ satisfies the following properties:

- $f$ is continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$, but non-continuous in $(0,0)$.

- For all $h \in \mathbb{R}^2$ the directed derivative $f'(0,0;h)$ exists.

### 47.1 Exercise 46.a

We use the sequence criterion.

$$\lim_{h\to\infty} f(\vec{x}_k) = f(\vec{a})$$

$$\lim_{h\to\infty} \vec{x}_k = \vec{a}$$

We need to prove that the relationship between both equations with $\vec{a}$ holds.

$$\lim_{k\to\infty} \frac{x_k \cdot y_k^2}{x_k^2 + y_k^4} = \frac{\lim_{k\to\infty} x_k \cdot (\lim_{k\to\infty} y_k)^2}{(\lim_{k\to\infty} x_k)^2 + (\lim_{k\to\infty} y_k)^4} = \frac{x_0 y_0^2}{x_0^2 + y_0^2} = f(\vec{a})$$

$$(x_k)_{k\geq 1} = \frac{1}{k} \to_{k\to\infty} 0$$

$$(y_k)_{k\geq 1} = \frac{1}{k} \to_{k\to\infty} 0$$

$$\lim_{k\to\infty} \frac{\frac{1}{k^2}\cdot\frac{1}{k^2}}{\frac{1}{k^4}+\frac{1}{k^4}} = \lim_{k\to\infty} \frac{\frac{1}{k^4}}{\frac{2}{k^4}} = \frac{1}{2}$$

### 47.2 Exercise 46.b

$$\lim_{t\to 0} \frac{f(\vec{a} + t\cdot\vec{h}) - f(\vec{a})}{t} \in \mathbb{R} \qquad \text{with } \| h \| = 1$$

Let $h \neq (0,0)$.

$$\lim_{t\to 0} \frac{\frac{tk_x\cdot(tk_y)^2}{(tk_x)^2+(tk_y)^4)}}{t} = \lim_{t\to 0} \frac{k_x k_y^2}{k_x^2 + t^2 k_y^4} = \begin{cases} \frac{k_y^2}{k_x} & k_x \neq 0 \\ 0 & k_x = 0 \end{cases}$$

Let $h = (0,0)$.

$$\lim_{t\to 0} \frac{f(0,0) - f(0,0)}{t}$$

## 48 Exercise 47

**Exercise 50.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given with

$$f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Prove the following statements:

- $f$ is continuous in $\mathbb{R}^2$.
- For all $h \in \mathbb{R}^2$ there exists a directed derivative $f'(0,0;h)$. But it does not depend linearly on direction $h$.

## 48.1 Exercise 47.a

In general, this is a composition of continuous functions as long as the denominator is non-zero. However, this is only informally, but suffices for now. We now need to look at $(0,0)^T$ more carefully.

Show:

$$\forall \varepsilon > 0 \exists \delta > 0 : \| \vec{x} - \vec{x_0} \| < \delta \implies \| f(\vec{x}) - f(\vec{x_0}) \| < \varepsilon$$

$$\| x - x_0 \| = \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \sqrt{x^2 + y^2} < \delta \implies |x| < \delta \wedge |y| < \delta$$

Let $\delta := \varepsilon$:

$$\| f(x) - f(x_0) \| = \left| \frac{x^3}{x^2 + y^2} \right| \leq \begin{cases} \frac{|x^3|}{|x^2|} \cdot |x| < \delta = \varepsilon, & x \neq 0 \\ 0 < \varepsilon, & x = 0. \end{cases}$$

## 48.2 Exercise 47.b

$\forall h \in \mathbb{R}^2$ there exists $f'(0,0;h)$. $h := \begin{pmatrix} r \\ s \end{pmatrix}$.

$$\lim_{t \to 0} \frac{f\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} r \\ s \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{t} = \lim_{t \to 0} \frac{f\begin{pmatrix} tr \\ ts \end{pmatrix}}{t} = \lim_{t \to 0} \frac{1}{t} \cdot \frac{t^3 x^3}{t^2 r^2 + f^2 s^2} = \lim_{t \to 0} \frac{r^3}{r^2 + s^2} = \frac{r^3}{r^2 + s^2}$$

Is it linearly dependent?

$$\partial f(0,0;h_1) + \partial f(0,0;h_2) = \partial f(0,0;h_1 + h_2)$$

Let $h_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\frac{1^3}{1^2 + 0^2} = 1$$
$$\frac{0^3}{0^2 + 1^2} = 0$$

But $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1^3}{1^2 + 1^2} = \frac{1}{2}$.

# 49 Exercise 48

**Exercise 51.** Determine the derivative of $f$ using Fréchet differentiability.

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) := \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

in every point $(x_0, y_0) \in \mathbb{R}^2$.

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \left\| f\left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right) - f\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - Df \cdot h \right\|$$

$$Df\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2x_0 & -2y_0 \\ 2y_0 & 2x_0 \end{pmatrix}$$

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} (x_0 + h_1)^2 - (y_0 + h_2)^2 \\ 2(x_0 + h_1)(x_0 + h_2) \end{pmatrix} - \begin{pmatrix} x_0^2 - y_0^2 \\ 2x_0 y_0 \end{pmatrix} - \begin{pmatrix} 2x_0 h_1 - 2y_0 h_2 \\ 2y_0 h_1 + 2x_0 h_2 \end{pmatrix} \right\|$$

$$= \lim_{\|h\|\to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} 2h_1x_0 + h_1^2 - h_2^2 - 2h_2y_0 \\ 2h_1y_0 + 2h_2x_0 + 2h_1h_2 \end{pmatrix} - \begin{pmatrix} 2x_0h_1 - 2x_0h_2 \\ 2y_0h_1 + 2x_0h_2 \end{pmatrix} \right\|$$
$$= \lim_{\|h\|\to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} h_1^2 - h_2^2 \\ 2h_1h_2 \end{pmatrix} \right\| = \lim_{\|h\|\to 0} \frac{1}{\|h\|} \sqrt{h_1^4 - 2h_1^2h_2^2 + h_2^4 + 2h_1^2h_2^2}$$
$$= \lim_{\|h\|\to 0} \frac{1}{\|h\|}(h_1^2 + h_2^2) = 0$$

Or very informally, we can estimate:

$$\lim_{\|h\|\to 0} \frac{1}{\|h\|} \left\| \begin{pmatrix} h_1^2 - h_2^2 \\ 2h_1h_2 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} h_1^2 \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 \\ h_2 \end{pmatrix} \right\|^2 + 2 \left\| \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right\| \left\| \begin{pmatrix} 0 \\ h_2 \end{pmatrix} \right\| \leq \|h\|^2 + \|h\|^2 + 2\|h\|^2$$

# 50 Exercise 49

**Exercise 52.** Determine the matrix norm induced by the $\infty$-norm in $\mathbb{R}^n$

$$\|A\|_\infty := \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

for arbitrary $A \in \mathbb{R}^{n\times n}$

First, we show:

$$\|Ax\|_\infty = \max_{i=1,\dots,m} \left| \sum_{j=1}^n a_{ij} \cdot x_j \right| \leq \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}||x_j| \leq \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}| \|x\|_\infty$$
$$\|A\|_\infty \leq \sup_{x\neq 0} \frac{\max_{i=1,\dots,m} \sum_{j=1}^n |a_{ij}| \|x\|_\infty}{\|x\|_\infty} = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$$

Now we show the other side:

Let $i_0 \in \{1,\dots,n\}$.

$$\sum_{j=1}^n |a_{i_0j}| = \max_{i=1,\dots,n} \sum_{i=1}^n |a_{ij}|$$
$$\tilde{x} = \left( \frac{a_{i_01}}{|a_{i_01}|}, \dots, \frac{a_{i_0n}}{|a_{i_0n}|} \right) \qquad \|\tilde{x}\|_\infty = 1$$

Remark:

$$\begin{cases} \frac{a}{|a|} =: \mathrm{sgn}(a) & \text{for } a \neq 0 \\ 0 & \text{for } a = 0 \end{cases}$$

$$\|A\tilde{x}\|_\infty = \max_{i=1,\dots,m} \left| \sum a_{ij}\tilde{x}_2 \right| \geq \left| \sum_{j=1}^n a_{i_0j}\tilde{x}_j \right| = \left| \sum_{j=1}^n |a_{i_0j}| \right| = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$$

Tutor's approach:

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} = \left\| A\left( \frac{x}{\|x\|_\infty} \right) \right\|_\infty$$
$$\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \sup_{x\neq 0} \left\| A\left( \frac{x}{\|x\|_\infty} \right) \right\|_\infty = \sup_{\|x\|_\infty = 1} \|Ax\|_\infty$$
$$\sup_{x:|x_i|\leq 1 \forall i} \max_{j=1,\dots,n} \left| \sum_{j=1}^n a_{ij}x_j \right| = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$$