Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

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Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z\in\mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$
$$\exp(z+w) = \exp(z)\cdot\exp(w)$$
$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n}\in\mathbb{Q}\wedge n\neq 0 \implies \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote
$$\exp(z) = e^z \qquad \text{for } z\in\mathbb{C}$$

It holds that
$$\exp(z)\neq 0 \qquad \forall z\in\mathbb{C}$$

$\exp(x)$ for $x\in\mathbb{R}$
$$e^x > 0 \qquad \forall x\in\mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.
$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,
$$e^0 = 1$$

Let $n\in\mathbb{N}$
$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$
$$\lim_{x\to-\infty} e^x\cdot x^n = 0$$

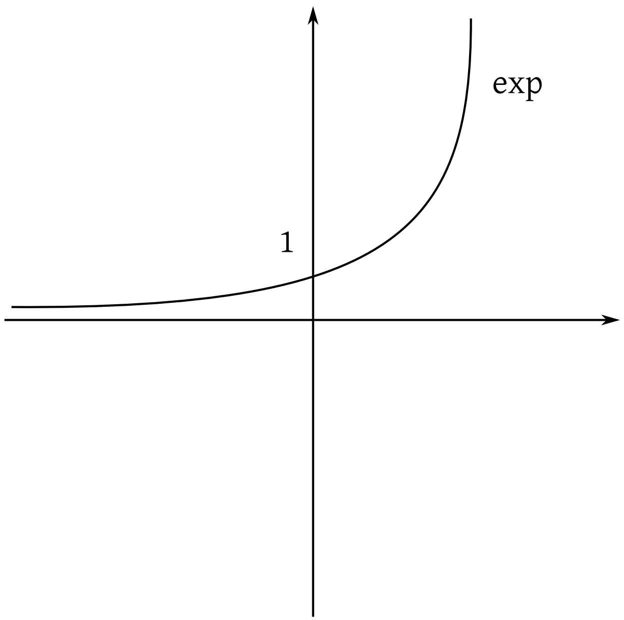


Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp : \mathbb{R}\to(0,\infty)$$

is injective, because $x_1 < x_2 \implies e^{x_1} < e^{x_2}$

Lemma 1. $\exp:\mathbb{R}\to(0,\infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y>0$. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y] : \quad e^\xi = y$$

Case 2 Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w] : \quad e^\xi = w = \frac{1}{y}$$

$$\implies e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp : \mathbb{R} \to (0, \infty)$ is bijective. $\square$

Definition 1. We call the inverse function *natural logarithm*[1].

$$\exp^{-1} : (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R} : \ln(e^x) = x$ and $\forall y \in (0, \infty) : e^{\ln(y)} = y$.
- $\ln : (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

  Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xrightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \implies y_1 \geq y_2$. Contradiction! $\square$

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Limes:
$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Proof.
- $$x \cdot y = e^{\ln(x \cdot y)}$$
$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$
Injectivity of exp:
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.
$$\lim_{n \to \infty} y_n = \ln(1) = 0$$
$$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$
where
$$e^0 = 1 \implies \ln(1) = 0$$

$\square$

Theorem 1 (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. $\square$

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

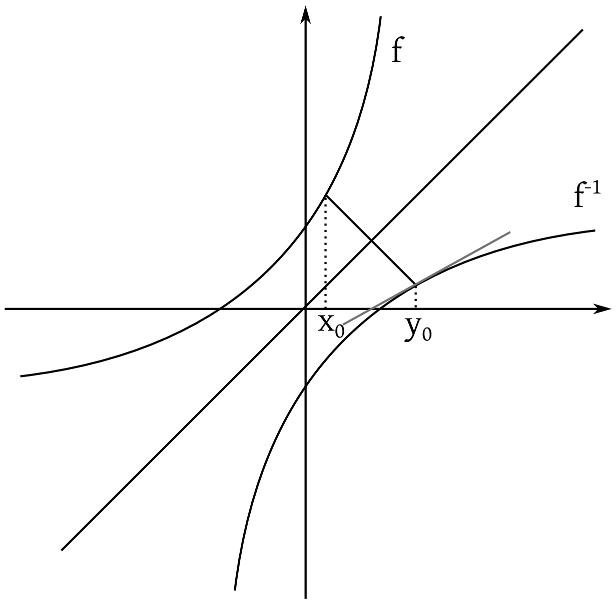


Figure 2: A geometric proof of differentiability

Proof. **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \implies \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \implies f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\implies (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\implies (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\implies \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$= \frac{1}{x}\cdot \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x} \to 0$$

$1+\frac{h}{x} = w$ then it holds that $h \to 0 \implies w \to 1$.

$$\frac{h}{x} = w-1$$

$$\lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{=} \lim_{h\to 0} \frac{\ln(w)}{w-1} = 1$$

□

Remark 1. The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C} \to \mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C}\setminus\{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C}\setminus\{x\in\mathbb{R} : x\le 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I \to \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I) \to \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

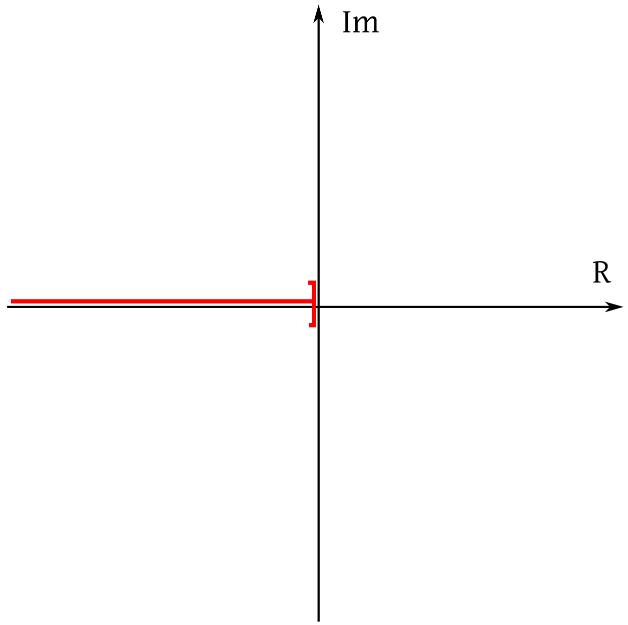


Figure 3: Continuous exponential function in $\mathbb{C}$

Proof. Let $y_n \to y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \implies x_n \neq x_0$.

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}$$

$$= \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n)-f(x_0)} = \frac{1}{\lim_{n\to\infty} \underbrace{\frac{f(x_n)-f(x_0)}{x_n-x_0}}_{\text{ex}=f'(x_0)}} = \frac{1}{f'(x_0)}$$

□

Lemma 2. Let $f : I \to \mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{ const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \forall x \in I$$

Proof. $\Rightarrow$ Immediate.

$\Leftarrow$ Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

□

Definition 2. Let I be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

□

Theorem 4. Let I be an interval. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n \in \mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every x)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon\, |u - v|\, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0) \right| + \left| f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0) \right| \forall N \in \mathbb{N}$$

$$\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right|$$

$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right| + |f_N'(x_0) - \varphi(x_0)|$$

1st term

$$\left| \frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0} \right| = \left| \frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0} \right|$$

$$\leq \frac{\varepsilon}{3} \frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0}\right| - f'_N(x_0)$$

Differentiability of f_N: Therefore for $|x - x_0| < \delta$.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$. ☐

Theorem 5. Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ $(n \in \mathbb{N})$ and f_n is differentiable in I.

Assumption:

1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f'_n \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f'_n(x) \to \varphi(x)$ for every x
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \implies |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon\,|u - v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n \to \infty$
- $\exists g : I \to \mathbb{R}$ such that $f'_n \to g$ is *uniform* in I, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \implies |f'_n(x) - g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6. We look at the conditions of Theorem 6:

2. Uniform convergences of $f'_n \to g$ implies pointwise convergence
$$\forall x \in I : f'_n(x) \to g(x)$$
3. From uniform convergence of $f'_n \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:
$$|f'_n(x) - g(x)| < \frac{\varepsilon}{2}$$
Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that
$$\begin{aligned}|f'_n(x) - f'_m(x)| &= |f'_n(x) - g(x) + g(x) - f'_m(x)| \\ &\leq |f'_n(x) - g(x)| + |g(x) - f'_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$
So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.
Let $\varepsilon > 0$ be arbitrary and N such that $n, m \geq N$ and $x \in I$:
$$|f'_n(x) - f'_m(x)| < \varepsilon$$
Consider the third condition of Theorem 6. Let $u, v \in I$
$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m\to\infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$
where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)
$$\begin{aligned}&= \lim_{m\to\infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)| \\ &= \lim_{m\to\infty} |f'_m(\xi_{m,n}) - f'_n(\xi_{m,n})| \cdot |u - v|\end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

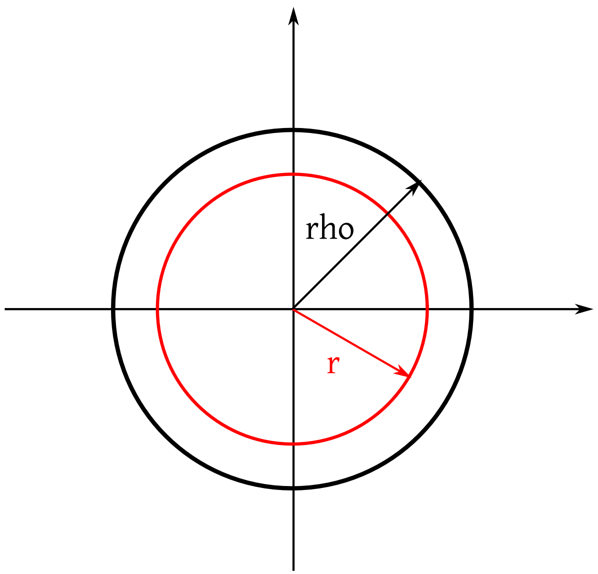


Figure 4: Convergence radius

Remark 2 (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

□

Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \limsup_{j\to\infty} \underbrace{\left(|a_{j+1}|^{\overbrace{\frac{1}{j+1}}^{\to 1}}\right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty} \left[(j+1)^{\frac{1}{j+1}}\right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 7 it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

[2] Where overline means "closed"

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1,1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\implies \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\implies \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$

Let $x \in (-1,1) \Rightarrow -x \in (-1,1)$.

$$\implies \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1,1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1,1)$$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\implies 1 + x = w - wx$$
$$x(1+w) = w - 1$$
$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

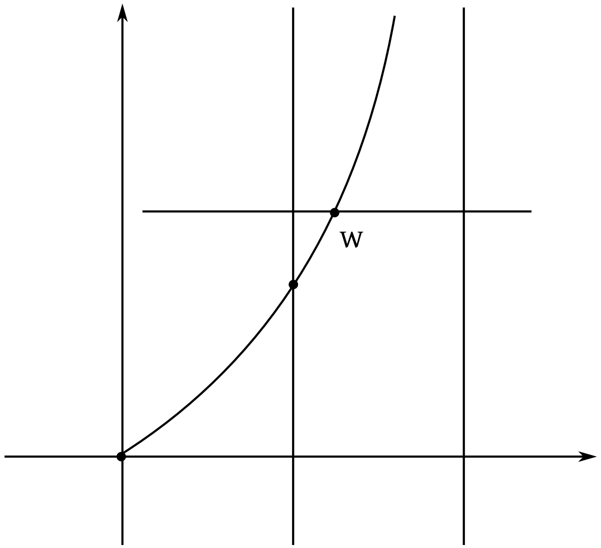


Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[\overline{\left(1 + \frac{it}{n}\right)}\right]^n$$

$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

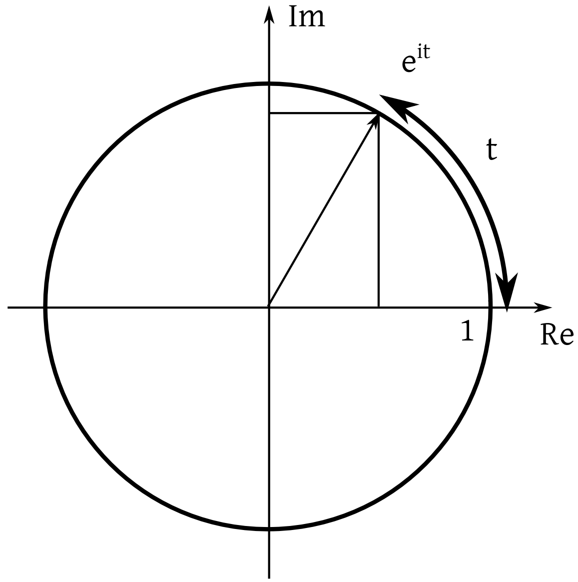


Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$
$$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$
$$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$
$$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$
$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$
$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$

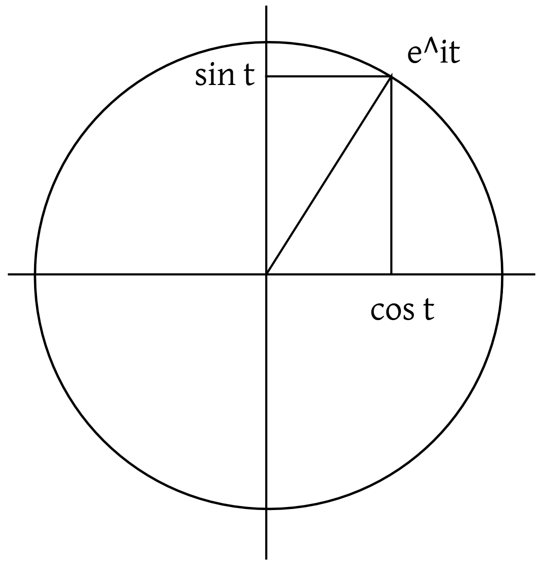


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

series sum. Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

□

Example 1 (Application)**.** Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0,\delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0,\delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \ldots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \ldots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \ldots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \ldots$$

Followingly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \ldots$$
$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C}$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \ldots$$
$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and substraction theorems)**.** We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i\sin(z+w)) \\ &\Rightarrow= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z \cdot \cos w) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i\left(-\cos z \sin w - \cos w \sin z\right) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i\sin(-(z+w))) \\ &\Rightarrow= \cos\left(z+w\right) - i\sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2\cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\begin{aligned} \cos(z-w) &= \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)} \\ \sin(z-w) &= -\cos z \cdot \sin(w) + \sin(z)\cos(w) \end{aligned}$$

Corollary 1.

$$z = \frac{1}{2}(z+w) + \frac{1}{2}(z-w)$$

$$\Rightarrow \cos z = \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2}$$

$$w = \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w)$$

$$\cos w = \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2}$$

$$\cos z - \cos w = -2\sin\frac{z+w}{2}\sin\frac{z-w}{2}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z\to 0 \\ z\neq 0}} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right) \\ &= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz} - 1}{2iz}\right) \\ &= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z\to 0} \frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz;z\to 0\Leftrightarrow w=0 \\ \lim_{w\to 0} \frac{e^w-1}{w}=1}} \end{aligned}$$

So it holds that

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled "definition of π" and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \ldots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{x^2}{(2k+2)(2k+1)}\right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k\geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left|\frac{s_{k+1}}{s_k}\right| = \left|\frac{x^2}{(2k+2)(2k+3)}\right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{ in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$

Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1-\frac{4}{6}=\frac{1}{3}>0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = \underbrace{-2 \cdot \underbrace{\sin \underbrace{\frac{z+w}{2}}_{\in(0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in(0,2]}}_{>0}}_{<0}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that cos is a strictly monotonically decreasing functionin $[0,2)$. Hence cos has only one root because it is continuous in $(0,2]$.

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos\frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin\frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i$$

$$e^{i\pi} = e^{i\frac{\pi}{2}+i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi+\frac{i}{2}\pi} = e^{i\pi}\cdot e^{i\frac{\pi}{2}} = -1\cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$
$$e^{z+2i\pi} = e^z\cdot\left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{iz+2\pi i}+e^{-iz-2\pi i}\right)\\ &= \frac{1}{2}\left(e^{iz}+e^{-iz}\cdot\frac{1}{\underbrace{e^{2\pi i}}_{=1}}\right) = \cos z\end{aligned}$$

Therefore the cosine is periodic in $\mathbb{C}$ with period 2π. Analogously, sine is periodic in $\mathbb{C}$ with period 2π.

This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

3.5 Periodicity and roots of trigonometric functions

$$\cos(z+2\pi) = \cos(z)$$
$$\sin(z+2\pi) = \sin(z)$$

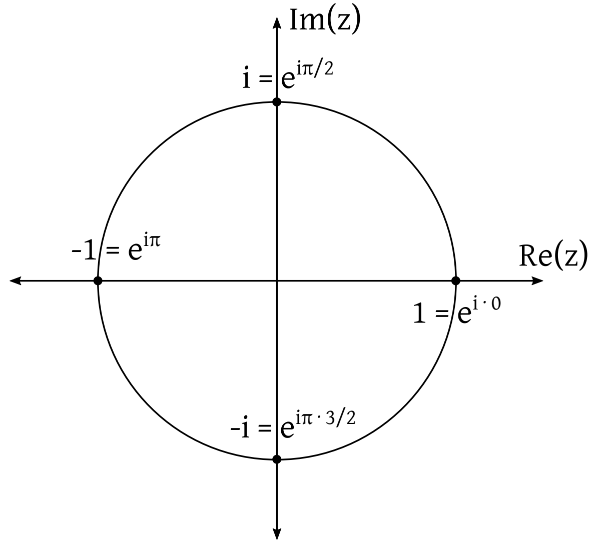
Compare with Figure 8.

Remark 3. We will show: $\forall c \in (0,2\pi)$, cos and sin are non-periodic with period c, hence $\exists x\in\mathbb{R}$ such that $\cos(x)\neq\cos(x+c)$.

Definition 5. $f:\mathbb{C}\to\mathbb{C}$ (or $f:\mathbb{R}\to\mathbb{R}$) is called *periodic* with period $c\in\mathbb{C}$ ($c\in\mathbb{R}$) if $\forall z\in\mathbb{C}$ it holds that

$$f(z+c) = f(z)$$
$$(\text{or } \forall x\in\mathbb{R}: f(x+c)=f(x))$$

c is called *period of f*.



t	0	$\pi/2$	π	$3\pi/2$	$\pi/2$
e^{it}	1	i	-1	-i	1
$\cos(t) = Re(e^{it})$	1	0	-1	0	1
$\sin(t) = Im(e^{it})$	0	1	0	-1	0

Figure 8: Periodicity of cos and sin on the unit circle in the complex plane

Remark 4. If f is periodic with period $c \in \mathbb{C}$, then f is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark 5.

$$z = u + iv$$

$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$

$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

Remark 6. Let $x \in \mathbb{R}$.

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= \Re(e^{i(x+\frac{\pi}{2})}) \\ &= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}}) \\ &= \Re(ie^{ix}) \\ &= -\Im(e^{ix}) \\ &= -\sin(x)\end{aligned}$$

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= \Im\left(e^{i(x+\frac{\pi}{2})}\right) \\ &= \Im(ie^{ix}) \\ &= \Re(e^{ix}) \\ &= \cos(x)\end{aligned}$$

$$\begin{aligned}\cos\left(x - \frac{\pi}{2}\right) &= \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin(x)\end{aligned}$$

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\cos(x)\end{aligned}$$

Summary:

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x - \frac{\pi}{2}\right) &= \sin(x) \\ \sin\left(x - \frac{\pi}{2}\right) &= -\cos(x)\end{aligned}$$

Remark 7 (A remark on the name "cosine")**.**

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of x (Compare with Figure 9).

Remark 8.

$$\begin{aligned}\cos(x + \pi) &= \Re(e^{i(x+\pi)}) \\ &= \Re(-e^{ix}) \\ &= -\cos(x) \\ \sin(x + \pi) &= -\sin(x)\end{aligned}$$

Remark 9. Let $0 < c < 2\pi$. Assume cos is periodic with period c. We know that cos has exactly one root in $[0, 2]$,

$$\cos(x) = \cos(-x)$$

cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

   Therefore cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.

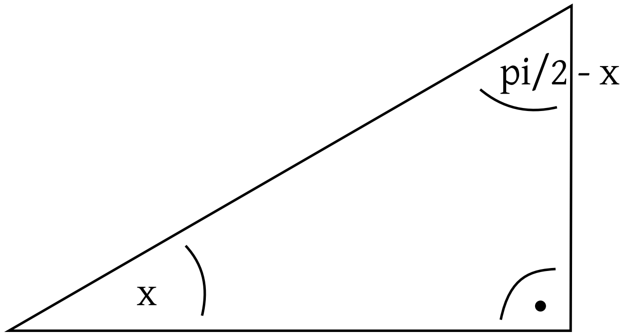


Figure 9: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$

c would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of cos in } (-\frac{\pi}{2}, \frac{\pi}{2})$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period c. Hence 2π is indeed the smallest period of cos.

Analogously it holds for sin.

Remark 10 (Roots of cos)**.**

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$
$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \qquad \forall l \in \mathbb{Z}$$

3.6 Derivatives of trigonometric functions

It holds that

$$\boxed{\lim_{z\to 0}\frac{\sin z}{z} = 1}$$

Furthermore it holds that

$$\boxed{\lim_{z\to 0}\frac{1-\cos z}{z} = 0}$$

Proof.

$$\begin{aligned}\frac{1-\cos z}{z} &= \frac{1}{z}\left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots\right)\\ &= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots\end{aligned}$$

is convergent in $\mathbb{C}$ and (especially) continuous in 0

$$\lim_{z\to 0}\left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots\right) = 0$$

□

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$

This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z\to 0}\frac{\sin z}{z} = 1$$

$$\lim_{z\to 0}\frac{1-\cos z}{z} = 0$$

Lemma 6. The trigonometric functions sin and cos are differentiable in $\mathbb{R}$ (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x) \qquad \sin'(x) = \cos(x)$$

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{\cos(x+h)-\cos(h)}{h} &= \lim_{h\to 0}\frac{\cos x\cdot\cos h - \sin x\cdot\sin h - \cos x}{h}\\ &= \lim_{h\to 0}\cos x\cdot\frac{\cos(h)-1}{h} - \lim_{h\to 0}\frac{\sin x\cdot\sin h}{h}\\ &= \cos x\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0} - \sin x\cdot\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1}\\ &= -\sin(x)\end{aligned}$$

Analogously:

$$\begin{aligned}\lim_{h\to 0}\frac{\sin(x+h)-\sin(h)}{h} &= \lim_{h\to 0}\frac{\sin x\cdot\cos h + \sin h\cdot\cos x - \sin x}{h}\\ &= \sin(x)\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0} + \cos(x)\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{h}}_{=1}\\ &= \cos(x)\end{aligned}$$

□

Compare with Figure 10. We now use tools of integral calculus:

Let $I = [a,b]$ and $\gamma : I \to \mathbb{R}^n$ $(\mathbb{R}^2)$.

$$\gamma(t) = \begin{bmatrix}\gamma_1(t)\\ \vdots\\ \gamma_n(t)\end{bmatrix}$$

Assumption: $\gamma_1 : [a,b] \to \mathbb{R}^n$ is continuously differentiable.

$$\gamma'(t) = \begin{bmatrix}\gamma_1'(t)\\ \vdots\\ \gamma_n'(t)\end{bmatrix}$$

is the tangential vector in γ. Compare with Figure 11.

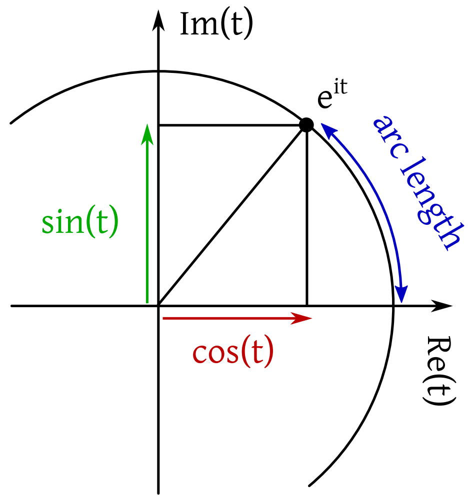


Figure 10: The arc length is related to sin and cos

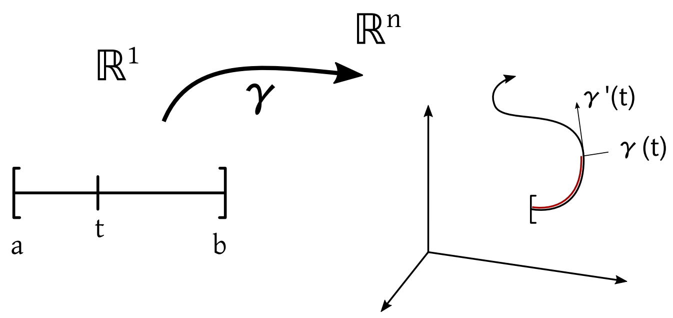


Figure 11: Mapping of the curve and tangential vector $\gamma'(t)$

Let $t \in [a, b]$. Then the arc length of γ between a and t is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \, d\tau$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \to \mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

Compare with Figure 12.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \, d\tau = \int_0^t 1 \, d\tau = t$$

4 Integral calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses “Functional analysis” and “Measure and integration theory”. For now, we look at the basics (as discussed by Königsberger).

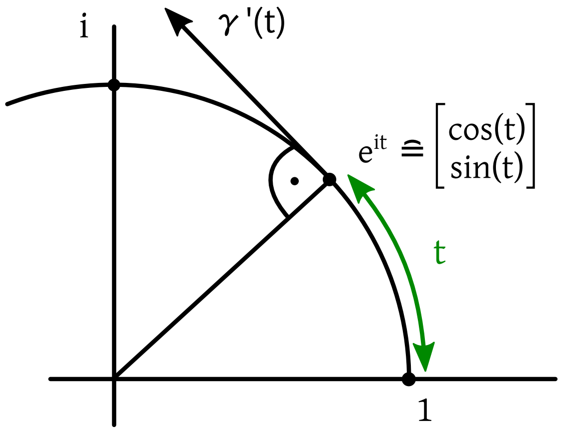


Figure 12: Derivative in $\mathbb{R}^2$

Definition 6 (Step function)**.** Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \to \mathbb{R}$. We call φ a *step function*, if $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1}, x_j)} = c_j$ is constant. The points x_j define a partition of the interval $[a, b]$.

$\tau[a, b]$ defines the set of step functions of interval $[a, b]$. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 13).

Definition 7 (Integral)**.** Let $\varphi : [a, b] \to \mathbb{R}$ be a step function and $x_0 = a < x_1 < \ldots < x_n = b$ a partition of $[a, b]$ and let $\varphi|_{(x_{j-1}, x_j)} = c_j$ for $j = 1, \ldots, n$. Then we define

$$\int_a^b \varphi \, dx = \sum_{j=1}^{n} c_j \triangle x_j$$

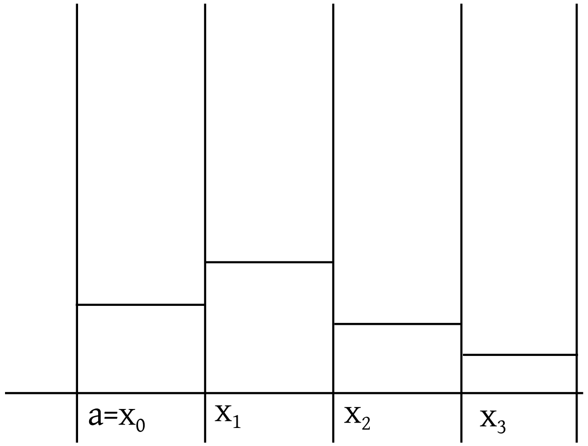


Figure 13: Partition of an area into rectangles

where $\triangle x_j = x_j - x_{j-1}$ (for $j = 1, \ldots, n$).

$$\int_a^b \varphi \, dx \text{ is called } \textit{integral} \text{ of } \varphi \text{ over } [a, b]$$

φ is the step function in terms of the partition $\{x_0, x_1, \ldots, x_5\}$, but also a step function in terms of $\{w_0, w_1, \ldots, w_5\}$.

It remains to show that if φ satisfies the definition of a step function in terms of partition $\{x_0, \ldots, x_n\}$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$ and φ is a step function in terms of $\{w_0, w_1, \ldots, w_m\}$ and $\varphi|_{(w_{l-1}, w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{l=1}^{m} c'_l \triangle w_l$$

Compare with Figure 14.

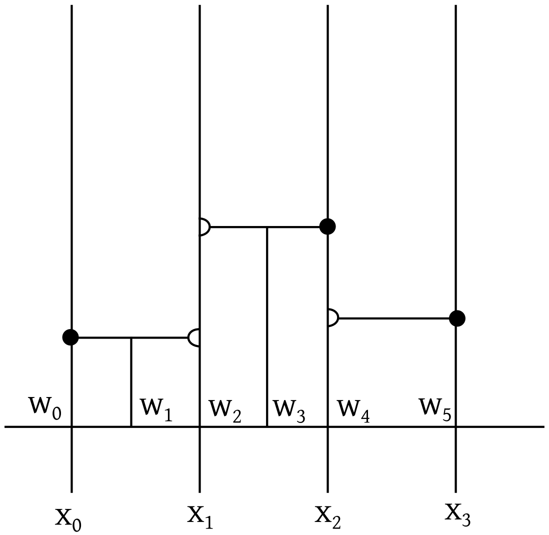


Figure 14: Step function φ

Proof. Let $Z = \{x_0, \ldots, x_n\}$ and $Z' = \{w_0, \ldots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly, $k_{j-1} < k_j$. Now let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \ldots, L\}$ there exists $j \in \{1, \ldots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1}, \alpha_l)} \text{ is constant}$$

Hence φ is a step function in terms of the partition $\{\alpha_0, \ldots, \alpha_L\}$.

Let $l \in \{0, 1, \ldots, L\}$ and j such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c''_l = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c''_l = c_j$.

$$\sum_{l=1}^{L} c''_l \cdot \triangle\alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c''_l \triangle\alpha_l$$

$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = \sum_{j=1}^{n} c_j \triangle x_j$$

Because

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1})$$

$$+(\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2}) + \ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_j - x_{j-1} = \triangle x_j$$

Analogously,

$$\sum_{l=1}^{L} c''_l \cdot \triangle\alpha_l = \sum_{k=1}^{m} c'_k \triangle w_k$$

So it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{k=1}^{m} c'_k \triangle w_k$$

□

This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

Lemma 7. Let $\varphi \in \tau[a,b]$ be a step function in terms of partition $a = x_0 < x_1 < \ldots < x_n = b$. Let $a = \alpha_0 < \alpha_1 < \ldots < \alpha_L = b$ with $Z = \{x_0, \ldots, x_n\} \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = z'$ (z' has more intervals than Z').

Then also φ is step function in terms of partition z'.

Proof. See above. □

Lemma 8. Let $\varphi_1, \varphi_2 \in \tau[a,b]$ and $\alpha, \beta \in \mathbb{C}$.

Then it holds that

- $\alpha\varphi + \beta\psi \in \tau[a,b]$ and

$$\int_a^b (\alpha\varphi + \beta\psi)\,dx = \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$

  Hence (“linearity”),

$$\int_a^b : \tau[a,b] \to \mathbb{R} \text{ is linear}$$

- $|\varphi| \in \tau[a,b]$ and it holds that

$$\left|\int_a^b \varphi\,dx\right| \leq \int_a^b |\varphi|\,dx \leq \|\varphi\|_\infty (b-a)$$

  Reminder: $\|\varphi\|_\infty = \max\{|\varphi(x)| : x \in [a,b]\}$
  This gives “boundedness”.

- Let φ and ψ be real values and it holds that

$$\forall x \in [a,b] : \varphi(x) \leq \psi(x)$$

Hence monotonicity is given. Then it holds that

$$\int_a^b \varphi\,dx \leq \int_a^b \psi\,dx$$

Proof.
- Let $\varphi|_{(x_{k-1},x_k)} = c_k$ $\psi|_{(w_{j-1},w_j)} = d_k$

$$z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = \{x_0, \ldots, x_n\} \cup \{w_0, \ldots, w_m\}$$

where α_i is sorted ascendingly. φ and ψ are step functions in terms of z'', hence

$$\varphi|_{(\alpha_{i-1},\alpha_i)} = c'_i \text{ and } \psi|_{(\alpha_{i-1},\alpha_i)} = d'_i$$

$$\Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1},\alpha_i)} = \alpha c'_i + \beta d'_i \text{ constant}$$

$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a,b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi)\,dx = \sum_{i=1}^{L} (\alpha c'_i + \beta d'_i) \cdot \triangle\alpha_i$$

$$= \alpha \sum_{i=1}^{L} c'_i \triangle\alpha_i + \beta \sum_{i=1}^{L} d'_i \triangle\alpha_i$$

$$= \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$

- Let $\varphi|_{(x_{i-1},x_i)} = c_i$ $(i = 1, \ldots, n)$. Then,

$$|\varphi|\,|_{(x_{i-1},x_i)} = |c_i|$$

$$\left|\sum_{i=1}^{n} c_i \triangle x_i\right| \leq \sum_{i=1}^{n} |c_i| \cdot \underbrace{|\triangle x_i|}_{x_i - x_{i-1} > 0} = \sum_{i=1}^{n} |c_i| \cdot \triangle x_i = \int_a^b |\varphi|\,dx$$

$$\leq \sum_{i=1}^{n} \|\varphi\|_\infty \triangle x_i = \|\varphi\|_\infty \sum_{i=1}^{n} \triangle x_i$$

$$= \|\varphi\|_\infty ((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}))$$

$$= \|\varphi\|_\infty (x_n - x_0) = \|\varphi\|_\infty (b-a)$$

- Let φ, ψ and z'' as in the linearity statement.

$$\left.\begin{aligned}\varphi|_{(\alpha_{i-1},\alpha_i)} &= c_i' \in \mathbb{R}\\ \psi|_{(\alpha_{i-1},\alpha_i)} &= d_i' \in \mathbb{R}\end{aligned}\right\}$$

$$\int_a^b \varphi\,dx = \sum_{i=1}^{L} c_i' \underbrace{\triangle\alpha_i}_{>0} \leq \sum_{i=1}^{L} d_i'\,dx$$

$$\int_a^b \varphi\,dx$$

□

Definition 8. Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\infty_A) : \mathbb{R} \to \mathbb{R}$ as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

the *characteristic function of* A. Hence $\chi_A(x)$ is 1 if and only if x is inside interval A.

Remark 11. Let $a \leq a' < b' \leq b$. Then

$$\chi_{(a',b')} \in \tau[a,b] \qquad \int_a^b \chi_{(a',b')}\,dx = 1 \cdot (b' - a')$$

Every linear combination of characteristic functions is also in $\tau[a,b]$.

On the opposite side, let $\varphi \in \tau[a,b]$ with $\varphi|_{(x_{i-1},x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \leq i \leq n$ and $0 \leq j \leq n$.

$$\Rightarrow \varphi = \sum_{i=1}^{n} c_i \chi_{(x_{i-1},x_i)} + \sum_{j=0}^{n} r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_a^b \varphi\,dx = \sum_{i=1}^{n} c_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} c_j \int_a^b \chi_{(x_{i-1},x_i)}\,dx$$

5 Regulated functions

Definition 9. Let $D \subseteq \mathbb{R}$. Let x_0 be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n\in\mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n\to\infty} z_n = x_0$. Let $f : D \to \mathbb{C}$ be given.

We state that f has left-sided limit y_0 in x_0 if

$$\forall\varepsilon > 0 \exists\delta > 0 : [x \in D \cap (-\infty, x_0) \wedge |x - x_0| < \delta]$$

$$\Rightarrow |f(x) - y_0| < \varepsilon$$

Equivalently $\forall (z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $z_n < x_0$ and $\lim_{n\to\infty} z_n = x_0$ $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} f(x_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by (x_0, ∞).

We denote: y_0 is left-sided limit of f in x_0:

$$y_0 = \lim_{x\to x_0^-} f(x)$$

and right-sided limit of f in x_0:

$$y_0 = \lim_{x\to x_0^+} f(x)$$

Definition 10. Let $a, b \in \mathbb{R}$ and $a < b$. A function $f : [a,b] \to \mathbb{C}$ is called *regulated functions* if

- $\forall x \in (a,b)$ f has a left-sided and a right-sided limes in x
- f has a right-sided limes in a
- f has a left-sided limes in b

Examples for regulated functions:

- Every continuous function in $[a,b]$ is a regulated function.

- Every step function is a regulated function.
  Why? Consider $x \in (x_{i-1}, x_i)$. Then

$$\lim_{\xi \to x^+} \varphi(\xi) = c_i = \lim_{\xi \to x^-} \varphi(\xi)$$

  Let $x = x_i$ be a partitioning point.

$$\lim_{\xi \to x_i^-} \varphi(\xi) = c_i \text{ and } \lim_{\xi \to x_i^+} \varphi(\xi) = c_{i+1}$$

  So $\tau[a,b] \subseteq R[a,b]$. Compare with Figure 15.

- Let $f : [a,b] \to \mathbb{R}$ be monotonically. Then it holds that

$$f \in R[a,b]$$

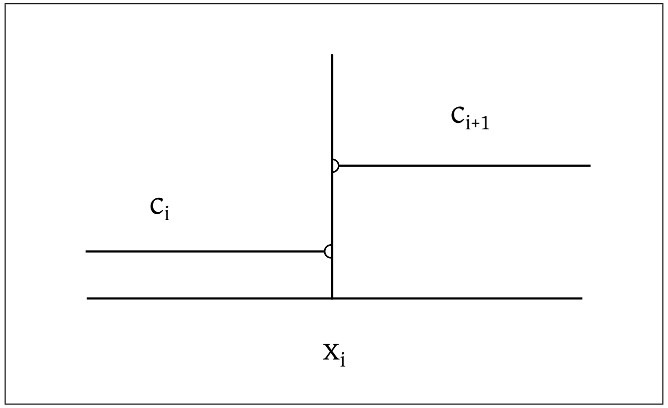


Figure 15: Step functions are also regulated functions

5.1 Approximation theorem for regulated functions

Let $f : [a,b] \to \mathbb{C}$. Then it holds that $f \in R[a,b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a,b]$ such that $\|f - \varphi\|_\infty < \varepsilon$. Hence,

$$\forall x \in [a,b] : |f(x) - \varphi(x)| < \varepsilon$$

$$\Leftrightarrow \underbrace{\sup \{|f(x) - \varphi(x)| : x \in [a,b]\}}_{\|f-\varphi\|_\infty} < \varepsilon$$

Especially $\varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a,b]$ such that

$$|\varphi_n(x) - f(x)| < \varepsilon \qquad \forall x \in [a,b]$$

hence f is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges continuously towards f.

Proof. $\Rightarrow$ Let $f \in R[a,b]$. Assume $\exists \varepsilon > 0$ fixed such that $\forall \varphi \in \tau[a,b]$

$$\exists x \in [a,b] : |\varphi(x) - f(x)| \geq \varepsilon$$

We build nested intervals such that the desired property $|\varphi(x) - f(x)| \geq \varepsilon$ holds on every subinterval $[a_n, b_n]$.

Induction:

$n = 0$ Let $a_0 = a$ and $b_0 = b$, hence the property holds in $[a_0, b_0]$.

$n \mapsto n+1$ Let $m = \frac{1}{2}(a_n + b_n)$. In $[a_n, b_n]$ the property holds.
Then the property either holds in $[a_n, m]$ or $[m, b_n]$. If the property does not hold in $[a_n, m]$:

$$\exists \varphi_1 \in \tau[a_n, m] \text{ with } |\varphi_1(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, m]$$

If the property does not hold in $[m, b_n]$:

$$\exists \varphi_2 \in \tau[m, b_n] \text{ with } |\varphi_2(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [m, b_n]$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in [a_n, m] \\ \varphi_2(x) & \text{for } x \in [m, b_n] \end{cases}$$

$$\Rightarrow \varphi \in \tau[a,b] \text{ and } |\varphi(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1}, b_{n+1}]$.
$([a_n, b_n])_{n\in\mathbb{N}}$ are nested intervals. Let $\varphi \in \bigcap_{n\in\mathbb{N}} [a_n, b_n]$.

Case $\xi \in (\mathbf{a}, \mathbf{b})$ Let ε satisfy the desired property. $f \in R[a, b]$, hence f has left-sided limit c_- in ξ and right-sided limit c_+. Hence $\exists \delta > 0$ such that

- $|x - \xi| < \delta \wedge a \leq x < \xi \Rightarrow |f(x) - c_-| < \varepsilon$
- $|x - \xi| < \delta \wedge \delta < x \leq b \Rightarrow |f(x) - c_+| < \varepsilon$

Choose δ sufficiently small such that

$$a < \xi - \delta < \xi + \delta < b$$

Let

$$\varphi(x) = \begin{cases} c_- & \text{for } x \in (\xi - \delta, \xi) \\ f(\xi) & \text{for } x = \xi \\ c_+ & \text{for } x \in (\xi, \xi + \delta) \end{cases}$$

φ is necessarily a step function in $(\xi - \delta, \xi + \delta)$ and it holds that $\forall x \in (\xi - \delta, \xi + \delta) : |\varphi(x) - f(x)| < \varepsilon$.
Let n be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi - \delta, \xi + \delta)$$

then

$$\varphi|_{[a_n, b_n]} \in \tau[a_n, b_n] \text{ and } |\varphi(x) - f(x)| < \varepsilon \quad \forall x \in [a_n, b_n]$$

This is a contradiction to our desired property.

For $\xi = a$ or $\xi = b$ only with one-sided limit.

□

This lecture took place on 17th of March 2016 with lecturer Wolfgang Ring.

We learned: All regulated functions can be approximated with step functions.

$f \in R[a, b]$ in the proof $\Leftrightarrow$ f is uniform limit of step functions. We have prove direction $\Rightarrow$.

Lemma 9 (Cauchy criterion for limits of functions)**.** Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and z_0 is a limit point of D. Then f has a limit in z_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : v, w \in D \setminus \{z_0\} \wedge |v - z_0| < \delta \wedge |w - z_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$.

If $D \subseteq \mathbb{R}$ and x_0 is limit point of $D \cap (x_0, \infty)$, then f has a *right-sided limit* in x_0 if and only if $\forall \varepsilon > 0 \exists \delta > 0 : [v, w \in D \cap (x_0, \infty) \wedge |v - x_0| < \delta \wedge |w - x_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon]$.

Analogously for left-sided limit.

Proof. This proof is done only for the first point.

$\Rightarrow$

Assume f has a limit η in z_0. Choose δ such that $v, w \in D$ with $|v - z_0| < \delta$ and $|w - z_0| < \delta$ implies that $|f(v) - \eta| < \frac{\varepsilon}{2}$ and $|f(w) - \eta| < \frac{\varepsilon}{2}$. Then $|f(v) - f(w)| \leq |f(v) - \eta| + |\eta - f(w)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Leftarrow$

Assume the Cauchy criterion holds. Show: There exists $\eta \in \mathbb{C}$ such that for every sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D \setminus \{z_0\}$ with $\lim_{n\to\infty} w_n = z_0$ it holds that $\lim_{n\to\infty} f(w_n) = \eta$.

Let $(w_n)_{n\in\mathbb{N}}$ be as above. Show: $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and δ as above. Choose $N \in \mathbb{N}$ such that $n, m \geq N$

$$\Rightarrow |w_n - z_0| < \delta \wedge |w_m - z_0| < \delta$$

The Cauchy criterion holds for $n, m \geq N$:

$$|f(w_n) - f(w_m)| < \varepsilon$$

So $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence and (because $\mathbb{C}$ is complete) is also convergent. So $\exists \eta' \in \mathbb{C} : \lim_{n\to\infty} f(w_n) = \eta'$.

It remains to show: η' is unique.

Let $(v_n)_{n\in\mathbb{N}}$ be another sequence with $\lim_{n\to\infty} v_n = z_0$ and $v_n \in D \setminus \{z_0\}$. As above: $\exists \eta'' \in \mathbb{C}$ such that $\lim_{n\to\infty} f(v_n) = \eta''$.

We construct:

$$(\xi_n)_{n\in\mathbb{N}} = (w_0, v_0, w_1, v_1, w_2, v_2, \ldots)$$

Then it holds that $\lim_{n\to\infty} \xi_n = z_0$.

We use the argument from above: $(f(\xi_n))_{n\in\mathbb{N}}$ is convergent, hence $\lim_{n\to\infty} f(\xi_n) = \eta$. Both subsequences $(f(w_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ must have the same limit, hence $\eta' = \eta = \eta''$.

□

Proof of approximation theorem. ⇐

Let $f = \lim_{n\to\infty} \varphi_n$ be uniform on $[a,b]$. Let $\varphi_n \in \tau[a,b]$ and let $x_0 \in [a,b)$. Show: f has a right-sided limit in x_0. Let $\varepsilon > 0$ arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that

$$|f(x) - \varphi_N(x)| < \frac{\varepsilon}{2} \forall x \in [a,b]$$

φ_N is a step function (hence interval-wise constant). Choose $\delta > 0$ such that $\varphi_N|_{(x_0,x_0+\delta)} = c$ constant. Let $v, w \in (x_0, x_0+\delta)$. Then it holds that

$$|f(v) - f(w)| \leq |f(v) - c| + |c - f(w)|$$
$$= |f(v) - \varphi_N(v)| + |f(w) - \varphi_N(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The Cauchy criterion implies that f has a right-sided limit in x_0.

□

Corollary 2. $f \in R[a,b]$ if and only if $f(x) = \sum_{j=0}^{\infty} \psi_j(x)$ with $\psi_j \in \tau[a,b]$ and the series converges uniformly in $[a,b]$.

Proof. ⇐

Let $\varphi_n = \sum_{j=0}^{n} \psi_j \in \tau[a,b]$ and $\varphi_n \to f$ continuously in $[a,b]$. From the approximation theorem it follows that $f \in R[a,b]$.

⇒

Let $f \in R[a,b]$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to f$ uniform in $[a,b]$. Let $\psi_0 = \varphi_0$.

$$\psi_j := \varphi_j - \varphi_{j-1} \text{ for } j \geq 1$$

Then it holds that

$$\sum_{j=0}^{n} \psi_j = \varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \ldots + (\varphi_{n-1} - \varphi_{n-2}) + (\varphi_n - \varphi_{n-1}) = \varphi_n$$

and $(\varphi_n)_{n\in\mathbb{N}}$ converges uniform if and only if the series is uniformly convergent.

□

Lemma 10 (Sidenote)**.** Let $(f_n)_{n\in\mathbb{N}}$ with $f_n : D \to \mathbb{C}$ a sequence of functions in D, let $z_0 \in D$ and $\forall n \in \mathbb{N}$ f_n is continuous in z_0. Furthermore let $f : D \to \mathbb{C}$ and $f_n \to f$ is uniform in D. Then f is continuous in z_0.

Proof. Let $\varepsilon > 0$ arbitrary. Choose N sufficiently large such that $|f(z) - f_w(z)| < \frac{\varepsilon}{3}$ $\forall z \in D$ (uniform convergence). Because f_N is continuous in z_0, $\exists \delta > 0$ such that $z \in D$ and $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$.

Then for $|z - z_0| < \delta$ (with $z \in D$)

$$\underbrace{|f(z) - f(z_0)|}_{<\frac{\varepsilon}{3}} \leq \underbrace{|f(z) - f_N(z)| + |f_N(z) - f_N(z_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{<\frac{\varepsilon}{3}}$$

□

This lecture took place on 18th of March 2016 with lecturer Wolfgang Ring.

Theorem 9. Let f be a regulated function in $[a,b]$. Then f is in at most countable infinite points of $[a,b]$ non-continuous.

Proof.

$$f = \sum_{k=0}^{\infty} \psi_k$$

where ψ_k is a sequence of step functions and and the series is uniformly convergent. $\psi_k \in \tau[a,b]$.

Let $\{x_0^k, \ldots, x_{n(k)}^k\}$ be the partition points of ψ_k. Then ψ_k is continuous in $[a,b] \setminus Z_k$. Let $Z = \bigcup_{k=0}^{\infty} Z_k$ be countable. Let $x \in [a,b] \setminus Z$ and $\varphi_n = \sum_{k=0}^{n} \psi_k$. Then it holds that $\varphi_n \to f$ is uniform in $[a,b]$ and φ_n is continuous in x, because $x \notin Z$.

From Lemma 10 it follows that f is continuous in x. □

5.2 Norms and vector spaces

Definition 11 (Normed vector spaces)**.** Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). A map $n : V \mapsto [0,\infty)$ is called *norm* in V, if

Definite form $n(V) = 0 \Leftrightarrow V = 0$ (V is null vector)

Positive homogeneity $\forall \lambda \in \mathbb{C}\ (\mathbb{R})\ \forall v \in V : n(\lambda v) = |\lambda| \cdot n(v)$

Triangle inequality $\forall v, w \in V : n(v + w) \leq n(v) + n(w)$

Common notation: $\|v\|$ for $n(v)$ ("norm of v")
A vector space satisfying the norm properties is called *normed vector space.*

Example 2.
- $|x|$ is a norm in $\mathbb{R}$.
- $|z|$ is a norm in $\mathbb{C}$.

$\|\vec{x}\|$ is norm in $\mathbb{R}^n$.

Let $D \subseteq \mathbb{C}$.

$$B(D) = \{f : D \to \mathbb{C} : f \text{ limited to } D\}$$

$B(D)$ is a vector space. For $f \in B(D)$ we define:

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\}$$

"supremum norm" of ∞-norm of f in D.

It holds that $\|\cdot\|_\infty$ is a norm in $B(D)$.

$$\begin{aligned} \|f\|_\infty = 0 &\iff \sup\{\underbrace{|f(z)|}_{\geq 0} : z \in D\} = 0 \\ &\iff |f(z)| = 0 \quad \forall z \in D \\ &\implies f = 0 \text{ in } B(D) \end{aligned}$$

Homogeneity:

$$\begin{aligned} |\lambda \cdot f|_\infty &= \sup\{|\lambda f(z)| : z \in D\} \\ &= \sup\{|\lambda|\,|f(z)| : z \in D\} \\ &= |\lambda| \cdot \sup\{|f(z)| : z \in D\} \\ &= |\lambda| \cdot \|f\|_\infty \end{aligned}$$

Triangle inequality: Let $f, g \in B(D)$.

$$\begin{aligned} \|f + g\|_\infty &= \sup\{|f(z) + g(z)| : z \in D\} \\ &\leq \sup\left\{\underbrace{|f(z)|}_{\leq\|f\|_\infty} + \underbrace{|g(z)|}_{\leq\|g\|_\infty} : z \in D\right\} \\ &\leq \sup\{\|f\|_\infty + \|g\|_\infty : z \in D\} \\ &= \|f\|_\infty + \|g\|_\infty \end{aligned}$$

Remark 12. Let $V \subseteq B(D)$ be an arbitrary subvectorspace of $B(D)$. So $\|\cdot\|_\infty$ is also a norm in V.

Important example:

$$V = \mathcal{C}_b(D) = \{f : D \to \mathbb{C} : f \text{ is continuous and bounded in } D\}$$

Special case: $D = K$ compact in $\mathbb{C}$. Then every continuous function is also bounded.

$$\begin{aligned} \mathcal{C}(K) &= \{f : K \to \mathbb{C} : f \text{ is continuous}\} \\ &\subseteq B(K) \qquad \text{(sub vector space)} \end{aligned}$$

Another special case: $D = [a, b] \subseteq \mathbb{C}$

$$\tau[a, b] \subseteq B([a, b]) \text{ and}$$

$$R[a, b] \subseteq B([a, b])$$

Remark 13 (Further properties of the norm)**.** The inverse triangle inequality holds:

$$\forall v, w \in V : |\|v\| - \|w\|| \leq \|v - w\|$$

Proof.

$$v = (v - w) + w$$

From triangle inequality it follows that

$$\|v\| \leq \|v - w\| + \|w\|$$

$$w = (w - v) + w$$
$$\|w\| \leq \|w - v\| + \|w\|$$

$$= \|(-1) \cdot (v - w)\| + \|v\|$$
$$= |(-1)| \cdot \|v - w\| + \|v\|$$
$$= \|v - w\| + \|v\|$$

$$\text{requirement 1} \Rightarrow \|v\| - \|w\| \leq \|v - w\|$$
$$\text{requirement 2} \Rightarrow \|w\| - \|v\| \leq \|v - w\|$$
$$\text{requirement 1 and 2} \Rightarrow |\|v\| - \|w\|| \leq \|v - w\|$$

□

Definition 12. Let V be a normed vector space, $(v_n)_{n\in\mathbb{N}}$ be a sequence of elements in V and $v \in V$. We define $(v_n)_{n\in\mathbb{N}}$ is convergent with limit V if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq \mathbb{N} \Rightarrow \|V_n - V\| \leq \varepsilon]$$

Remark 14 (Metric on V)**.**

$$d(v, w) = \|v - w\|$$

defines a metric on V. Properties of a metric:

1. $d(v, w) \geq 0$
2. $d(v, w) = 0 \Leftrightarrow v = w$
3. $\|v - w\| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$

Triangle inequality of metrics: Let $v, w, u \in V$.

$$d(v, u) = \|v - u\| = \|v - w + w - u\|$$
$$\leq \|v - w\| + \|w - u\| = d(v, w) + d(w, u)$$

Works only if $d(v, w) = d(w, v)$ and can be simply proven:

$$d(v, w) = \|v - w\| = \|w - v\| = d(w, v)$$

Remark 15. $(V_n)_{n\in\mathbb{N}}$ is called *Cauchy sequence in V* if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|v_n - v_m\| < \varepsilon]$$

V is called *complete normed vector space* if every Cauchy sequence in V is also a convergent sequence in V.

A complete normed vector space is called *Banach space.*

5.3 Integration of regulated functions

Theorem 10. Let $f \in \mathcal{B}[a, b]$ and $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n \in \tau[a, b]$ and $\varphi_n \to_{n\to\infty} f$ uniform in $[a, b]$ ($\Leftrightarrow \|\varphi_n - f\| \to 0$ for $n \to \infty$).

Then we define

$$\int_a^b f\, dx = \lim_{n\to\infty} \int_a^b \varphi_n\, dx$$

for the integral of f in $[a, b]$. The right-sided limit exists for every sequence $(\varphi)_{n\in\mathbb{N}}$ with the property above and is independent of the choice of the sequence $(\varphi_n)_{n\in\mathbb{N}}$.

Proof. Let $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [\underbrace{n \geq N \Rightarrow |\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]}_{\sup\{|\varphi_n(x) - f(x)| : x \in [a,b] \leq \varepsilon\}}]$$

$$\Rightarrow \|\varphi_n - f\|_\infty \leq \varepsilon$$

So φ_n converges towards f in terms of $\|\cdot\|_\infty$ in $\mathcal{B}[a, b]$.

Let N be sufficiently large such that

$$\forall n \geq N : \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(b - a)}$$

Then it holds for $i_n = \inf_a^b \varphi_n \, dx$ and $n, m \geq N$,

$$\begin{aligned}
|i_n - i_m| &= \left| \int_a^b \varphi_n \, dx - \int_a^b \varphi_m \, dx \right| \\
&= \left| \int_a^b (\varphi_n - \varphi_m) \, dx \right| \\
&\leq \|\varphi_n - \varphi_m\|_\infty (b-a) \\
&= \|\varphi_n - f + f - \varphi_m\|_\infty (b-a) \\
&\leq (\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty)(b-a) \\
&< \left(\frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} \right)(b-a) \\
&= \varepsilon
\end{aligned}$$

So $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and therefore convergent. So there exists

$$\lim_{n\to\infty} \int_a^b \varphi_n \, dx$$

Let $i = \lim_{n\to\infty} \int_a^b \varphi_n \, dx$. Let $(\psi_n)_{n\in\mathbb{N}}$ be another sequence of step functions with $\psi_n \to_{n\to\infty} f$ is uniform in $[a, b]$. Analogously as above:

$$j_n = \int_a^b \psi_n \, dx$$

$(j_n)_{n\in\mathbb{N}}$ is convergent and has limes j.

Show that $i = j$. We again use a zip-like construction:

$$F = (\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \ldots)$$

F is a sequence of step functions, which converge towards f uniformly. Let l be the limit of integrals of this sequence of step functions. Then it holds that (subsequences have the same limit)

$$i = l = j$$

□

Theorem 11 (Elementary properties of the integral)**.** Let $f, g \in \mathcal{B}[a, b]$ and $\alpha, \beta \in \mathbb{C}$. Then it holds that

linearity

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx$$

boundedness

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx \leq \|f\|_\infty (b-a)$$

monotonicity Let $f, g \in \mathcal{B}[a, b]$ with values in $\mathbb{R}$ and it holds that

$$f(x) \leq g(x) \qquad \forall x \in [a, b]$$

Then it holds that

$$\int_a^b f \, dx \leq \int_a^b g \, dx$$

Proof.

- Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences of step functions with $\varphi_n \to f$ and $\psi_n \to g$ uniform in $[a, b]$. Then it holds that

$$\alpha\varphi_n + \beta\psi_n \to_{n\to\infty} \alpha f + \beta g$$

(proof left as exercise to the reader)
uniform in $[a, b]$. So it holds that

$$\begin{aligned}
\int_a^b (\alpha f + \beta g) \, dx &= \lim_{n\to\infty} \int_a^b (\alpha\varphi_n + \beta\psi_n) \, dx \\
&= \alpha \lim_{n\to\infty} \int_a^b \varphi_n \, dx + \beta \lim_{n\to\infty} \int_a^b \varphi_n \, dx \\
&= \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx
\end{aligned}$$

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to_{n\to\infty} f$ continuous in $[a, b]$. Then also $(|\varphi_n|)_{n\in\mathbb{N}}$ is a sequence of step functions and it holds that

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a, b]$$

Proof. Let N be sufficiently large such that $\forall n \geq N \forall x \in [a,b]$:

$$|\varphi_n(x) - f(x)| < \varepsilon \Rightarrow ||\varphi_n(x)| - |f(x)|| \leq |\varphi_n(x) - f(x)| < \varepsilon$$

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a,b]$$

So it holds that

$$\left|\int_a^b f\,dx\right| = \left|\lim_{n\to\infty}\int_a^b \varphi_n\,dx\right| = \lim_{n\to\infty}\left|\int_a^b \varphi_n\,dx\right|$$

$$\leq \lim_{n\to\infty}\int_a^b |\varphi_n|\,dx = \int_a^b |f|\,dx$$

Because $|f - \varphi_n|_\infty \to_{n\to\infty} 0$ it follows that

$$|\|f\|_\infty - \|\varphi_n\|_\infty| \leq \|f - \varphi_n\|_\infty \to 0$$

hence $\|f\|_\infty = \lim_{n\to\infty}\|\varphi_n\|_\infty$. □

Hence,

$$\begin{aligned}\int_a^b |f|\,dx &= \lim_{n\to\infty}\int_a^b |\varphi_n|\,dx \\ &\leq \lim_{n\to\infty}\|\varphi_n\|_\infty (b-a) \\ &= \|f\|_\infty (b-a)\end{aligned}$$

Remark 16. We have proven that $\|\cdot\| : V \to [0,\infty)$ is a continuous map, hence $v_n \to v \Rightarrow \|v_n\| \to \|v\|$.

□

This lecture took place on 12th of April 2016 with lecturer Wolfgang Ring.

Definition 13. Let $f : [a,b] \to \mathbb{R}$ be given. Let $x_0 \in [a,b)$. We claim that f has a right-sided derivative $f'_+(x_0)$ in x_0 if the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

has a right-sided limit in x_0. Then f is denoted with $f'_+(x_0)$.

$$f'_+(x_0) = \lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}$$

Analogously for the left-sided derivative: Let $x_0 \in (a,b]$. $f'_-(x_0) = \lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}$ if the limit exists.

Theorem 12 (Intermediate value theorem of calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and $p : [a,b] \to \mathbb{R}$ is a regulated function with $p(x) \geq 0 \quad \forall x \in [a,b]$.

Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)\cdot p(x)\,dx = f(\xi)\cdot\int_a^b p(x)\,dx$$

Proof. Let $M = \max\{f(x) : x \in [a,b]\}$ and $m = \min\{f(x) : x \in [a,b]\}$

$$mp(x) \leq f(x)\underbrace{p(x)}_{\geq 0} \leq Mp(x) \qquad \forall x \in [a,b]$$

Due to monotonicity of the integral it holds that

$$m\int_a^b p(x)\,dx \leq \int_a^b f(x)p(x)\,dx \leq M\int_a^b p(x)\,dx$$

hence $\exists \eta \in [m,M]$ such that $\eta \cdot \int_a^b p(x)\,dx = \int_a^b f(x)p(x)\,dx$. From the Intermediate Value Theorem it follows that $\exists \xi \in [a,b] : \eta = f(\xi)$.

$$\Rightarrow f(\xi) : \int_a^b p(x)\,dx = \int_a^b f(x)p(x)\,dx$$

□

Remark 17. Consider $p \equiv 1$.

$$\exists \xi \in [a,b] : \int_a^b f(x)\cdot 1\,dx = f(\xi)\cdot\int_a^b 1\,dx = f(\xi)\cdot(b-a)$$

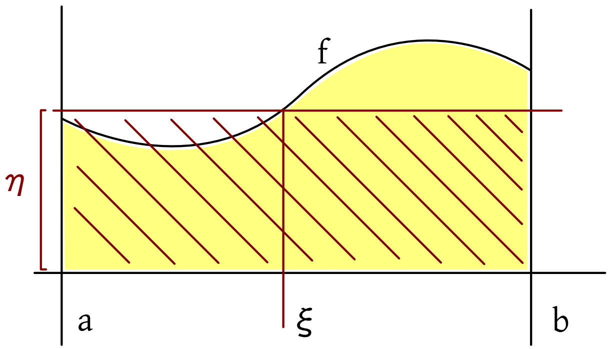


Figure 16: Intermediate value theorem where the area below the curve is given as $I = \int_a^b f(x)\,dx = \eta \cdot (b-a) = f(\xi)(b-a)$.

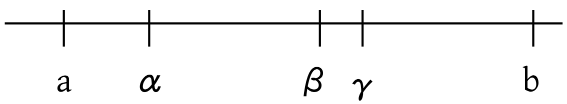


Figure 17: Relation of $a \le \alpha < \beta < \gamma \le b$

Lemma 11. Let $I = [a,b]$ and $f \in R[a,b]$ and $a \le \alpha < \beta < \gamma \le b$ (compare with Figure 17). Then $f|_{[\alpha,\gamma]} \in R[\alpha,\gamma]$.

Furthermore it holds that

$$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(x)\,dx + \int_\beta^\gamma f(x)\,dx$$

Proof. Let φ be a step function in $[\alpha,\gamma]$. Then $\varphi|_{[\alpha,\beta]} \in \tau[\alpha,\beta]$ and $\varphi|_{[\beta,\gamma]} \in \tau[\beta,\gamma]$. Furthermore it holds (proof not given here)

$$\int_\alpha^\gamma \varphi\,dx = \int_\alpha^\beta \varphi\,dx = \int_\beta^\gamma \varphi\,dx$$

For $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of subsequences with $\varphi_n \to f$ continuous in $[\alpha,\gamma]$.

$$\Rightarrow \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]} \text{ uniform in } [\alpha,\beta]$$

analogously for $[\beta,\gamma]$.

$$\int_\alpha^\gamma f dx = \lim_{n\to\infty} \int_\alpha^\gamma \varphi_n\,dx = \lim_{n\to\infty} \left[\int_\alpha^\beta \varphi_n\,dx + \int_\beta^\gamma \varphi_n\,dx\right]$$

$$= \underbrace{\lim_{n\to\infty} \int_\alpha^\beta \varphi_n\,dx}_{=\int_\alpha^\beta f\,dx} + \underbrace{\lim_{n\to\infty} \int_\beta^\gamma \varphi_n\,dx}_{=\int_\beta^\gamma f\,dx}$$

□

Remark 18. Notation ($\alpha,\beta \in [a,b]$):

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx$$

So it follows that

$$\int_\alpha^\alpha f(x)\,dx = -\int_\alpha^\alpha f(x)\,dx = 0$$

With this notation it holds that $\forall \alpha,\beta,\gamma \in I$:

$$\int_\alpha^\gamma f\,dx = \int_\alpha^\beta f\,dx + \int_\beta^\gamma f(x)\,dx$$

independent of the relation of α,β,γ towards each other. For $\alpha < \beta < \gamma$ everything is fine.

Let's also look at $\beta < \gamma < \alpha$ as an exercise.

Then it holds that

$$\int_\beta^\alpha f\,dx = \int_\beta^\gamma f\,dx + \int_\gamma^\alpha f\,dx$$

$$-\int_\alpha^\beta f\,dx = \int_\beta^\gamma f\,dx - \int_\alpha^\gamma f\,dx$$

$$\Rightarrow \int_\alpha^\gamma f\,dx = \int_\alpha^\beta f\,dx + \int_\beta^\gamma f\,dx$$

Case $\alpha = \beta$ or $\beta = \gamma$ is trivial.

Theorem 13 (Fundamental theorem of Calculus)**.** Originally formulated by Isaac Barrow (1630–1677). Followingly popularized by Newton (1642–1727) and Leibniz (1646–1716).

Let $f : I \to \mathbb{R}$ be a regulated function. I is an interval and $a \in I$ is fixed. For $x \in I$ we define

$$F(x) = \int_a^x f(\xi)\,d\xi$$

Then it holds that (two variants/characterizations)

1. F is right-sided derivable and also left-sided derivable for every $x_0 \in I$ and it holds that

$$F'_+(x) = f_+(x_0) = \lim_{x\to x_0^+} f(x) \qquad \text{and}$$

$$F'_-(x) = f_-(x_0) = \lim_{x\to x_0^-} f(x)$$

   Especially if f is continuous in x_0, then F is differentiable in x_0 with derivative $F'(x_0) = f(x_0)$.

   We call a function with the properties of F above a *primitive function* of the regulated function f.

2. Let $\Phi : I \to \mathbb{R}$ be an arbitrary primitive function of f and let $a, b \in I$. Then it holds that

$$\int_a^b f(\xi)\,d\xi = \Phi(b) - \Phi(a)$$

The first characterization claims that (informally speaking) the derivative for the upper limit of the integral of f gives f.

Let $f = \Phi'$ (Φ is our primitive function of f). The second characterization claims that the integral of a derivative of Φ gives Φ.

$$\int_a^b \Phi'\,dx = \Phi(b) - \Phi(a)$$

Proof. 1. Let $x_1, x_2 \in I$ and wlog $x_1 \leq x_2$.

$$\begin{aligned}|F(x_1) - F(x_2)| &= \left|\int_a^{x_1} f(\xi)\,d\xi - \int_a^{x_2} f(\xi)\,d\xi\right| \\ &= \left|\int_a^{x_1} f(\xi)\,d\xi + \int_{x_2}^a f(\xi)\,d\xi\right| \\ &= \left|\int_{x_2}^{x_1} f(\xi)\,d\xi\right| = \left|\int_{x_1}^{x_2} f(\xi)\,d\xi\right| \\ &\leq \int_{x_1}^{x_2} |f(\xi)|\,d\xi \leq |x_2 - x_1| \cdot \|f\|_\infty\end{aligned}$$

hence F is Lipschitz continuous in I. So F is continuous in I.

One-sided limits:

Let $\varepsilon > 0$ arbitrary and $x_0 \in I$ and δ such that $\forall x \in (x_0, x_0 + \delta)$ it holds that:

$$|f(x) - f_+(x_0)| < \varepsilon$$

$$\begin{aligned}&\left|\frac{F(x) - F(x_0)}{x - x_0} - f_+(x_0)\right| \\ &= \frac{1}{|x - x_0|}\left|\int_a^x f(\xi)\,d\xi - \int_a^{x_0} f(\xi)\,d\xi - f_+(x_0)\cdot(x - x_0)\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\,d\xi - f_+(x_0)\int_{x_0}^x 1 d\xi\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\,d\xi - \int_{x_0}^x f_+(x_0)\,d\xi\right| \\ &= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi) - f_+(x_0)\,d\xi\right| \\ &\leq \frac{1}{|x - x_0|}\int_{x_0}^x |f(\xi) - f_+(x_0)|\,d\xi\end{aligned}$$

$\xi \in (x_0, x) \subseteq (x_0, x_0 + \delta)$

$$< \frac{1}{|x - x_0|} \cdot \varepsilon \underbrace{\int_{x_0}^{x} 1 \, d\xi}_{|x-x_0|}$$
$$= \varepsilon$$
$$\Rightarrow F'_+(x_0) = f_+(x_0)$$

Analogously $F'_-(x_0) = f_-(x_0)$.

□

This lecture took place on 14th of April 2016 with lecturer Wolfgang Ring.

Theorem 14 (Addition: Lipschitz continuity of differentiable functions)**.** Let $I = [a, b]$, $f : I \to \mathbb{R}$ and f is continuous in I. Let $A \subseteq I$. Let A be countable and f is differentiable in $I \setminus A$ and $\exists L > 0 : |f'(x)| \leq L \quad \forall x \in I \setminus A$.

Then it holds that $\forall x_1, x_2 \in I$:

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$$

Proof. Without loss of generality, $x_1 < x_2$. Let $\varepsilon > 0$, define $F_\varepsilon : I \to \mathbb{R}$

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L + \varepsilon)(x - x_1)$$

Show $F_\varepsilon(x_2) \leq 0$.

Assume there is some $\varepsilon' > 0$ with $F_{\varepsilon'}(x_2) > 0$. It holds that

- $F_{\varepsilon'}(A) \subseteq \mathbb{R}$ is countable
- $0 = F_{\varepsilon'}(x_1) < F_{\varepsilon'}(x_2)$. Because $F_{\varepsilon'}$ is continuous (by Intermediate Value Theorem, $[0, F_{\varepsilon'}(x_2)] \subseteq F_{\varepsilon'}([x_1, x_2])$) and $[0, F_{\varepsilon'}(x_2)]$ contains overcountably many points, $F_{\varepsilon'}(A)$ is countable.

$$\Rightarrow \exists \gamma : 0 < \gamma < F_{\varepsilon'}(x_2)$$

and

$$\gamma \in F_{\varepsilon'}([x_1, x_2] \setminus A)$$

Let $\underbrace{F_{\varepsilon'}^{-1}(\{y\})}_{B} \cap [x_1, x_2] = \{x \in [x_1, x_2] \mid F_{\varepsilon'}(x) = y\}$.

B is bounded. Let $c = \sup B$. Let $(\xi_n)_{n\in\mathbb{N}}$, $\xi_n \in B$ with $\lim_{n\to\infty} \xi_n = c$. Then it holds that $c \in [x_1, x_2]$ and $F_{\varepsilon'}(\xi_n) = y \xRightarrow{\text{continuity of } F_{\varepsilon'}} \underbrace{\lim_{n\to\infty} F_{\varepsilon'}(\xi_n)}_{y} = F_{\varepsilon'}(c)$.

Therefore $c = \max B = \max\{x \in [x_1, x_2] : F_{\varepsilon'}(x) = y\}$. Because $F_{\varepsilon'}(x_2) > y$ and $F_{\varepsilon'}(x_1) = 0 < \gamma$, it holds that $x_1 < c < x_2$.

Consider $x \in (c, x_2]$ and let $\varphi(x) := \frac{F_{\varepsilon'}(x) - F_{\varepsilon'}(c)}{x-c}$. Furthermore $F_{\varepsilon'}(x) > \gamma = F_{\varepsilon'}(c)$ for $x \in (c, x_2]$. Because if we define $F_{\varepsilon'}(x) < \gamma$, then (due to Intermediate Value Theorem) $\exists \xi \in (x, x_2)$ with $F_{\varepsilon'}(\xi) = \gamma$, so $\exists \xi \in B$ which would be a contradiction to $c = \max B$.

$$\begin{aligned}
\varphi(x) &= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - x_1 - c + x_1)}{x - c} \\
&= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - c)}{x - c} \\
&\overset{\text{inv. triangle ineq.}}{\leq} \frac{|f(x) - f(c)|}{x - c} - (L + \varepsilon')
\end{aligned}$$

Now as far as $c \notin A$ holds, f is differentiable in c and it holds that $|f'(c)| \leq L$, hence there exists an interval (c, d), $d < x_2$ and $d > c$, such that

$$\frac{|f(x) - f(c)|}{x - c} < L + \varepsilon'$$

Because $F_{\varepsilon'}(x) > \gamma$,

$$\Rightarrow \varphi(x) > 0 \qquad \forall x \in (c, x_2]$$
$$\Rightarrow 0 < \varphi(x) \leq |f(x) - f(c)|\, x - c - (L + \varepsilon')$$
$$\Rightarrow \left|\frac{f(x) - f(c)}{x - c}\right| > L + \varepsilon'$$

This is a contradiction to the assumption that $F_{\varepsilon'}(x_2) > 0$. So $F_\varepsilon(x_2) \leq 0 \quad \forall \varepsilon > 0$

$$\Rightarrow F_0(x_2) \leq 0 \Rightarrow |f(x_2) - f(x_1)| \leq L - |x_2 - x_1|$$

□

Remark 19. Let f be differentiable in $[a,b]$ and $|f'(x)| < L \quad \forall x \in [a,b]$. Let $x_1, x_2 \in [a,b]$

$$|f(x_L) - f(x_1)| = |f'(\xi) \cdot (x_2 - x_1)| \leq L\,|x_2 - x_1|$$

by Mean Value Theorem of differential calculus.

Corollary 3. Let $f, g : I \to \mathbb{R}$. I as above and f, g are differentiable in $I \setminus A$, A countable and it holds that $f'(x) = g'(x) \quad \forall x \in I \setminus A$. There exists a constant k such that

$$f(x) = g(x) + k \qquad \forall x \in I$$

Proof. We use the previous Theorem for

$$h(x) = f(x) - g(x)$$

Then it holds that $|h'(x)| = 0 = L \quad \forall x \in I \setminus A$.

$$\Rightarrow |h(x_1) - h(x_2)| \leq 0 \cdot |x_1 - x_2| \quad \forall x_1, x_2 \in I$$

$$\Rightarrow h(x_1) = h(x_2) \quad \forall x_1, x_2 \in I$$

$$\begin{aligned} f(x_1) - g(x_1) = f(x_2) - g(x_2) \quad \forall x_1, x_2 \in I \\ = k \ldots \text{ constant} \end{aligned}$$

$\forall x_1 \in I$ it holds that $f(x_1) = g(x_1) + k$. □

This lecture took place on 15th of April 2016 with lecturer Wolfgang Ring.

cont, 2nd part. We need to show: Let f be a regulated function and Φ is a primitive function of f with the following properties

$$\Phi'(x) = f(x) \quad \forall x \in I \text{ where f is continuous}$$

$$\Phi'_+(x) = \lim_{\xi \to x_+} f(x)$$

$$\Phi'_-(x) = \lim_{\xi \to x_-} f(x) \quad \forall x \in I$$

Then it holds that

$$\int_\alpha^\beta f(x)\,dx = \Phi(\beta) - \Phi(\alpha)$$

Proof. For $\Phi(x) = \int_\alpha^x f(\xi)\,d\xi = F(x)$ (where F is also a primitive function) it holds that

$$\int_\alpha^\beta f(\xi)\,d\xi = F(\beta) - \underbrace{F(\alpha)}_{=0}$$

Because Φ and F are both primitive functions of f, Φ' and F' correspond in all continuous points, hence everywhere, but one countable set.

By the uniqueness theorem, it holds that

$$\Phi(x) = F(x) + c$$

$$F(x) = \Phi(x) - c$$

$$\int_a^b f(\xi)\,d\xi = F(b) - F(a) = \Phi(b) - c - \Phi(a) + c = \Phi(b) - \Phi(a)$$

□

□

Remark 20 (Notational remark)**.** Let f be a regulated function. Then we denote

$$\int f(x)\,dx = \begin{cases} \text{the set of all primitive function of } f \\ \text{an arbitrary primitive function of } f \end{cases}$$

$\int f(x)\,dx$ is called *indefinite integral.*

Remark 21.

$$\int x^n\,dx = \frac{1}{n+1} x^{n+1} \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

If you consider all primitive functions of the indefinite integral, you consider a constant $c \in \mathbb{R}$.

$$\int x^n\,dx = \frac{1}{n+1} x^{n+1} + c \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

Let $x > 0$: $(\ln x)' = \frac{1}{x}$.
Let $x < 0$: $(\ln -x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases} = \ln|x| \qquad \text{for } x \neq 0$$

$$\int \cos x\, dx = \sin x$$

$$\int \sin x\, dx = -\cos x$$

$$\int e^{cx}\, dx = \frac{1}{c}\cdot e^{cx} \quad (c \neq 0)$$

Lemma 12. Let f_1 and f_2 be regulated functions in $I = [a, b]$ and there exists some countable set A such that

$$f_1(x) = f_2(x) \qquad \forall x \in I \setminus A$$

Then it holds that

$$\int f_1(x)\, dx = \int f_2(x)\, dx \text{ and } \int_a^b f_1(x)\, dx = \int_a^b f_2(x)\, dx \qquad \forall a, b \in I$$

Proof. Let F_1 be a primitive function on f_1, F_2 be a primitive function of f_2. Then it holds that $F_1' = F_2'$ in $I \setminus A$. Due to identity theorem:

$$\Rightarrow F_1 = F_2 + c \Rightarrow \int f_1\, dx = \int f_2\, dx$$

□

Remark 22. Example of a function, which is differentiable everywhere. Its derivative is not a regulated function.

Let $I = [-1, 1]$ and

$$f(x) = \begin{cases} x^2 \cdot \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$ it holds that

$$f'(x) = 2x \cos\sin\frac{1}{x} - \frac{x^2}{x^2}\cdot\cos\frac{1}{x}$$

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$f'(0) = \lim_{h\to 0}\frac{1}{h}\left[h^2\cdot\sin\frac{1}{h} - 0\right] = \lim_{h\to 0}\underbrace{h}_{\to 0}\cdot\underbrace{\sin\frac{1}{h}}_{\in[-1,1]} = 0$$

$$f'(x) = \begin{cases} 0 & \text{for } x = 0 \\ \underbrace{2x\sin\frac{1}{x} - \cos\frac{1}{x}}_{\text{has no one-sided limit in } x=0} & \text{for } x \neq 0 \end{cases}$$

$$f'_+(0) \neq \lim_{x\to 0^+} f'(x)$$

5.4 Integration techniques

Theorem 15 (Integration by parts (dt. "partielle Integration")). Let $u, v : I \to \mathbb{R}$ be both primitive functions of regulated functions. Then also $u\cdot v$ is a primitive function of a regulated function and it holds that

$$\int u'v\, dx = u\cdot v - \int u\cdot v'\, dx$$

and

$$\int_a^b u'v\, dx = \underbrace{u(b)\cdot v(b) - u(a)\cdot v(a)}_{=:u\cdot v|_a^b} - \int_a^b u\cdot v'\, dx$$

Proof. u is continuous and therefore a regulated function.
v is continuous and therefore a regulated function.

u' and v' are regulated function by assumption.

$$\Rightarrow (u'\cdot v + u\cdot v') \in \mathcal{R}(I)$$

$u\cdot v$ is differentiable in every point in which u and v is differentiable. Let u be differentiable in $I \setminus A$, v is differentiable in $I \setminus B$.

$$\Rightarrow u\cdot v \text{ is differentiable in } I \setminus \underbrace{(A \cup B)}_{\text{countable}}$$

In $I \setminus (A \cup B)$ it holds that

$$(u\cdot v)'(x) = u'(x)\cdot v(x) + u(x)v'(x)$$

Hence the function $u \cdot v$ is primitive function of the regulated function $(u'v+uv')$.

$$\Rightarrow \int (u'v+uv')\,dx = u \cdot v$$

$$\Rightarrow \int_a^b (u'v+uv')\,dx = u(b)v(b) - u(a)v(a)$$

□

Example 3. Let $a \neq -1$ and $x > 0$.

$$\int x^a \ln x\,dx = \begin{vmatrix} u' = x^a & u = \frac{1}{1+a} \cdot x^{a+1} \\ v = \ln x & v' = \frac{1}{x} \end{vmatrix}$$

$$\overset{\text{int. by parts}}{=} \frac{1}{1+a} x^{1+a} \cdot \ln x - \frac{1}{1+a} \int x^a\,dx$$

$$= \frac{1}{1+a} x^{1+a} \ln x - \frac{1}{(1+a)^2} x^{1+a} = \frac{1}{1+a} x^{1+a} \left[\ln x - \frac{1}{1+a}\right]$$

Example 4.

$$\int \cos^k(x)\,dx \text{ for } k = 2,3,4,\dots$$

$$\begin{vmatrix} u' = \cos x & \Rightarrow u = \sin x \\ v = \cos^{k-1}(x) & v' = -(k-1) \cdot \cos^{k-2}(x) \cdot \sin(x) \end{vmatrix}$$

$$\int \cos^k(x)\,dx = \cos^{k-1}(x) \cdot \sin(x) + \int (k-1) \cdot \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{1-\cos^2(x)}\,dx$$

$$= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \cdot \int \cos^{k-2}(x)\,dx - (k-1) \cdot \int \cos^k(x)\,dx$$

Recognize that we have $\int \cos^k(x)\,dx$ twice in the equation (LHS and RHS, RHS with a sign).

$$k \cdot \int \cos^k(x)\,dx = \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x)\,dx$$

$$\int \cos^k(x)\,dx = \frac{1}{k} \cos^{k-1}(x) \sin(x) + \frac{k-1}{k} \int \cos^{k-2}(x)\,dx$$

Recursion formula.

Analogously,

$$\int \sin^k(x)\,dx = -\frac{1}{k} \sin^{k-1}(x) \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x)\,dx$$

Let $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x)\,dx$. Then it holds that

$$c_{2n} = \frac{(2n-1)}{2n} \cdot \frac{(2(n-1)-1)}{2(n-1)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \left(\prod_{k=1}^n \frac{2k-1}{2k}\right) \cdot \frac{\pi}{2}$$

$$c_{2n+1} = \left(\prod_{k=1}^n \frac{2k}{2k+1}\right)$$

Proof by complete induction:

Case $n = 0$

$$\int_0^{\frac{\pi}{2}} \cos^{2\cdot 0} x\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^{2\cdot 0+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos x\,dx = \sin(x)\big|_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \cos^{2(n+1)}\,dx = \frac{1}{2(n+1)} \cdot \cos^{2(n+1)-1}(x) \cdot \sin(x)\Big|_0^{\frac{\pi}{2}}$$
$$+ \frac{2(n+1)-1}{2(n+1)} \cdot \int_0^{\frac{\pi}{2}} \cos^{2n}(x)\,dx$$

$$= \frac{2n+1}{2n+2} \cdot \left(\underbrace{\prod_{k=1}^n \frac{2k-1}{2k}}_{\text{induction hypothesis}}\right) \cdot \frac{\pi}{2} = \left(\prod_{k=1}^{n+1} \frac{2k-1}{2k}\right) \cdot \frac{\pi}{2}$$

Theorem 16 (Wallis product)**.** (John Wallis, 1616–1703)

$$\frac{\pi}{2} = \lim_{n \to \infty} w_n \quad \text{with} \quad w_n = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

Proof.

$$\frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} \cdot \frac{\prod_{k=1}^{n} \frac{2k}{2k+1}}{\prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{\pi}{2}} = \prod_{k=1}^{n} \frac{(2k)^2}{(2k+1)(2k-1)} = w_n$$

It remains to show: $\lim_{n\to\infty} \frac{c_{2n+1}}{c_{2n}} = 1$.

In $\left[0, \frac{\pi}{2}\right]$ it holds that $0 \leq \cos x \leq 1$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^{2n}(x)\,dx \geq \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x)\,dx \geq \int_0^{\frac{\pi}{2}} \cos^{2n+2}(x)\,dx$$

$$c_{2n} \geq c_{2n+1} \geq c_{2n+2}$$

$$1 \geq \frac{c_{2n+1}}{c_{2n}} \geq \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1} \frac{2k-1}{2k}}{\prod_{k=1}^{n} \frac{2k-1}{2k}} = \underbrace{\frac{2n+1}{2n+2}}_{\to 1 \text{ for } n\to\infty}$$

$\Rightarrow \frac{c_{2n+1}}{c_{2n}}$ converges and limit is 1.

$$\lim_{n\to\infty} \frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} = \lim_{n\to\infty} w_n$$

□

Theorem 17 (Substitution law)**.** Let $f : I \to \mathbb{R}$ be a regulated function with primitive function F. Furthermore $t : [\alpha, \beta] \to I$ is continuously differentiable. Then $F \circ t$ is a primitive function for function $(f \circ t) \cdot t'$ and it holds that

$$\int_\alpha^\beta f(t(x)) \cdot t'(x)\,dx = \int_{t(\alpha)}^{t(\beta)} f(t)\,dt$$

Proof. The right-side integral is given (according to the Fundamental Theorem) by

$$F(t(\beta)) - F(t(\alpha))$$

The left-side integral, because of

$$F(t(x))' = F'(t(x)) \cdot t(x)$$

Hence $F = t$ is primitive function of the left-side integral. So it holds that

$$\int_a^b f(t(x)) \cdot t'(x)\,dx = F \circ t(b) - F \circ t(a) = F(t(b)) - F(t(a))$$

□

Example 5.

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_0^1 2x\sqrt{1+x^2}\,dx$$

$$\left| \begin{matrix} t(x) = 1+x^2 & t'(x) = 2x \\ f(y) = \sqrt{y} & \end{matrix} \right|$$

$$= \frac{1}{2}\int_1^2 \sqrt{x}\,dx = \frac{1}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^2 = \frac{2}{3}^{\frac{3}{2}} - \frac{1}{3}^{\frac{3}{2}} = \frac{1}{3}(\sqrt{8}-1)$$

$$\int_0^1 x \cdot \sqrt{1+x^2}\,dx = \left| \begin{matrix} \overbrace{y = x^2+1}^{\text{transform variables}} & \\ \frac{dy}{dx} = 2x & \underbrace{dy = 2x\,dx}_{\text{transformation of differences}} \\ & x\,dx = \frac{1}{2}\,dy \end{matrix} \right|$$

Transformation of limits:

$$x = 0 \Leftrightarrow y = 1 \qquad x = 1 \Leftrightarrow y = 2$$

$$= \frac{1}{2}\int_1^2 \sqrt{y}\,dy = \frac{1}{2}\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^2 = \frac{(x^2+1)^{\frac{3}{2}}}{3}\Bigg|_0^1$$

Hence it is also necessary to transform the limits.

Example 6 (Integration by parts)**.**

$$\int \ln x\,dx = \left| \begin{matrix} v' = 1 & v = x \\ u = \ln x & u' = \frac{1}{x} \end{matrix} \right| = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x$$

Theorem 18. Ivan M. Niven (published in 1947, 1915–1999)

It holds: π^2 is an irrational number. So π is irrational.

Proof by contradiction. Let $\pi^2 = \frac{a}{b} \in \mathbb{Q}$.

Because $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ (practicals!) there exists $n \in \mathbb{N}$ such that $\pi \frac{a^n}{n!} < 1$.

$$f(x) = \frac{1}{n!} x^n (1-x)^n$$

is symmetrical along axis $x = \frac{1}{2}$

$$= \frac{1}{n!} \sum_{k=n}^{2n} c_k x^k \qquad \text{with } c_k = (-1)^{k-n} \binom{n}{k-n} = \pm \binom{n}{k-n} \in \mathbb{Z}$$

$$\begin{aligned}
&f^{(\mu)}(0) = 0 \text{ for } \mu = 0, 1, \dots, n-1 \in \mathbb{Z} \qquad\qquad \text{and also:} \\
&f^{(\mu)}(1) \in \mathbb{Z} \text{ for } \mu = n, n+1, \dots, 2n \\
&f^{(\mu)}(x) = \frac{1}{n!} \sum_{k=0}^{2n} \underbrace{k(k-1)\dots(k-\mu+1)}_{=\mu!} \cdot c_k \cdot x^{k-\mu} \\
&f^{(\mu)}(0) = \frac{1}{n!} \mu! \left(\pm \binom{n}{\mu-n} \right) \cdot 1 \\
&\qquad = \frac{1}{n!} \mu! \frac{n!}{(\mu-n)!(n-\mu+n)!} \\
&\qquad = \frac{\mu!}{(\mu-n)!(2n-\mu)!} \\
&\qquad = \frac{(\mu-n+1)(\mu-n+2)\dots\mu}{1\cdot 2\cdot 3\dots(2n-\mu)} \\
&\qquad \in \mathbb{Z}
\end{aligned}$$

Why does $\in \mathbb{Z}$ hold?

$$\frac{\mu!}{n!} \underbrace{\binom{n}{\mu-n}}_{\in\mathbb{Z}} \in \mathbb{Z} \qquad n \le \mu \le 2n$$

$$(n+1)(n+2)\dots\nu \in \mathbb{Z}$$

$$n \le \mu \le 2n$$

$f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$, analogously $f^{(\mu)}(1) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$.

$$F(x) = b^n \left(\pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) + (-1)^n f^{2n}(x) \pi^0 \right)$$

$$F(0) \in \mathbb{Z} \text{ because } f^{(\mu)}(0) \in \mathbb{Z} \text{ for } \mu = 0, 2, 4, 6, \dots, 2n$$

$$\pi^2 = \frac{a}{b} \qquad \pi^{2n-2l} = \frac{a^{k-l}}{b^{n-l}}$$

$$b^n \cdot \pi^{2n-2l} = a^{n-l} \cdot b^l \in \mathbb{Z}$$

Analogously for $F(1) \in \mathbb{Z}$.

$$\begin{aligned}
&(F'(x) \cdot \sin(\pi x) - \pi F(x) \cdot \cos(\pi x))' \\
&= F''(x) \cdot \sin(\pi x) + \pi^2 \cdot F(x) \cdot \sin \pi x \\
&+ F'(x)(\cos(\pi x) - \pi \cos \pi x) \\
&= (F''(x) + \pi^2 F(x)) \cdot \sin(\pi x)
\end{aligned}$$

$$\begin{aligned}
F''(x) = b^n \cdot (&\pi^{2n} \cdot f''(x) + \pi^{2n-2} f^{(4)}(x) + \\
&\pi^{2n-4} f^{(6)}(x) - \dots + (-1)^n f^{(2n+2)}(x)) \\
\implies &F''(x) + \pi^2 \cdot F(x)
\end{aligned}$$

$$\begin{aligned}
F''(x) + \pi^2 \cdot F(x) = b^n(&\pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) + \\
&\dots + (-1)^n f^{(2n+2)}(x)) \\
&+ b^n(\pi^{2n+2} f(x) - \pi^{2n} f''(x) + \pi^{2n-2} f^{(4)}(x) - \\
&\pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n \pi^2 \cdot f^{(2n)}(x))
\end{aligned}$$

Almost all expressions cancel each other out. So it holds that

$$\begin{aligned}
&(F'(x)\cdot\sin(\pi x)-\pi F(x)\cos(\pi x))' \\
&=\pi^{2n+2}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&=\frac{a^{n+1}}{b^{n+1}}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&=\frac{a^{n+1}}{b}\cdot f(x)\cdot\sin(\pi x) \\
&=\pi^2\cdot a^n f(x)\cdot\sin(\pi x) \\
&=\pi\left(\pi a^n f(x)\sin(\pi x)\right) \\
I&=\pi\int_0^1 a^n f(x)\cdot\sin(\pi x)\,dx \\
&=\frac{1}{\pi}\cdot\left[F'(x)\cdot\sin(\pi x)-\pi\cdot F(x)\cos(\pi x)\right]\Big|_0^1 \\
&=F(1)+F(0)\in\mathbb{Z}
\end{aligned}$$

On the other hand it holds that

$$f(x)=\frac{1}{n!}\underbrace{x^n}_{\leq 1}\underbrace{(1-x)}_{\leq 1}{}^n$$

So $0\leq f(x)\leq\frac{1}{n!}$. Hence,

$$0\leq a^n f(x)\cdot\sin(\pi x)\leq\frac{a^n}{n!}<\frac{1}{\pi}$$

So $0<I<1\Rightarrow I\in\mathbb{Z}$. This is a contradiction to our assumption that $I\in\mathbb{Z}$. $\square$

Remark 23. Hence π is not rational. So there exists no linear affine function $g(x)=ax+b$ with $a,b\in\mathbb{Z}$ such that π is root of g.

Remark 24. We state, $\xi\in\mathbb{R}$ is an *algebraic* number if polynomial

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_n$$

exists with $a_i\in\mathbb{Z}$ for $i=0,\ldots,n$ and $P(\xi)=0$.

Algebraic numbers are a generalization of rational numbers.

$\eta\in\mathbb{R}$ is called *transcendental*, if η is not algebraic.

Remark 25. π is transcendental.

Theorem 19 (Integration of non-compact intervals)**.**

$$\int_0^\infty e^{-x}\,dx=\lim_{c\to\infty}\int_0^c e^{-x}\,dx$$

Definition 14 (Definition of indefinite integrals)**.** Let I be an interval with boundary values a and b with $-\infty\leq a<b\leq\infty$.

Let f be a regulated function in I. Then we define

1. if $I=[a,b)$, $\int_a^b f(x)\,dx=\lim_{\beta\to b_-}\int_a^\beta f(x)\,dx$
2. if $I=(a,b]$, $\int_a^b f(x)\,dx=\lim_{\alpha\to a_+}\int_\alpha^a f(x)\,dx$
3. if $I=(a,b)$, we choose $c\in I$ and $\int_a^b f(x)\,dx=\lim_{\alpha\to a_+}\int_\alpha^c f(x)\,dx+\lim_{\beta\to b_-}\int_c^\beta f(x)\,dx$.

This lecture took place on 21st of April 2016 with lecturer Wolfgang Ring.

$$f:[a,b)\to\mathbb{R}\qquad b\in(-\infty,\infty]$$

$$\int_a^b f(x)\,dx=\lim_{\beta\to b_-}\int_a^b f(x)\,dx$$

Example 7 (Classic examples)**.** 1. Let $s>1$.

$$\begin{aligned}
\int_1^\infty\frac{1}{x^s}\,dx&=\lim\int_1^\beta x^{-s}\,dx \\
&=\frac{1}{-s+1}\cdot x^{-s+1}\Big|_1^\beta \\
&=\lim_{\beta\to\infty}\frac{1}{1-s}\cdot\frac{1}{\beta^{s-1}}-\frac{1}{1-s}
\end{aligned}$$

$s-1>0$ and $\frac{1}{1-s}\to 1$

$$=\frac{1}{s-1}\qquad\text{so indefinite integral exists}$$

2. Let $s < 1$.

$$\begin{aligned}
\int_0^1 x^{-s}\,dx &= \lim_{\alpha\to 0_+}\int_\alpha^1 x^{-s}\,dx \\
&= \lim_{\alpha\to 0_+}\frac{1}{-s+1}x^{-s+1}\Big|_\alpha^1 \\
&= \frac{1}{1-s} - \underbrace{\lim_{\alpha\to 0_+}\frac{1}{1-s}\alpha^{1-s}}_{=0} \\
&= \frac{1}{1-s}
\end{aligned}$$

Compare with Figure 19.

3.

$$\begin{aligned}
\int_0^\infty e^{-cx}\,dx &= \lim_{\beta\to\infty}\int_0^\beta e^{-cx}\,dx \\
&= \lim_{\beta\to\infty}\frac{1}{-c}\cdot e^{-cx}\Big|_0^\beta \\
&= \lim_{\beta\to\infty}\left(-\frac{1}{c}\cdot e^{-c\beta}\right)+\frac{1}{c} \\
&= \frac{1}{c}
\end{aligned}$$

4.

$$\begin{aligned}
\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{\alpha\to-\infty}\int_\alpha^0 \frac{1}{1+x^2}\,dx + \lim_{\beta\to 0}\int_0^\beta \frac{1}{1+x^2}\,dx \\
&= \arctan(0) - \underbrace{\lim_{\alpha\to-\infty}\arctan(\alpha)}_{-\frac{\pi}{2}} + \underbrace{\lim_{\beta\to\infty}\arctan(\beta)}_{\frac{\pi}{2}} - \arctan(0) \\
&= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2} \\
&= \pi
\end{aligned}$$

Compare with Figure 18.

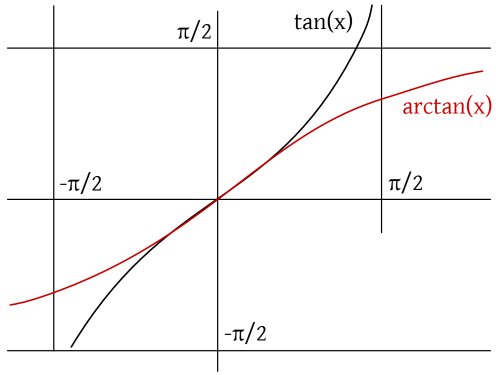


Figure 18: $\tan(x)$ and $\arctan(x)$

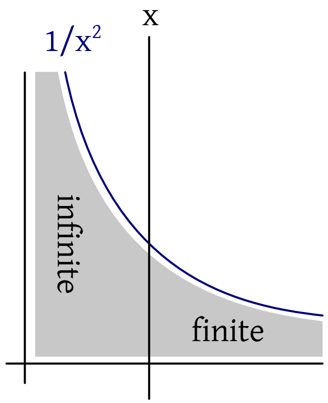


Figure 19: $\frac{1}{1-s}$

Remark 26. "Integral converges" means "an (indefinite) integral exists"

Remark 27.

$$\arctan'(x) = \frac{1}{1+x^2}$$

$$\tan'(x) = \frac{\cos x \cdot \cos x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2(x)}$$

$$\tan(x) = \frac{\sin x}{\cos x}$$

$$\begin{aligned}
\arctan'(x) &= \frac{1}{\tan'(\arctan(x))} \\
&= \left|\arctan x = s\right| \\
&= \left(\frac{1}{\cos^2(s)}\right)^{-1} \\
&= \left(\frac{\cos^2(s)+\sin^2(s)}{\cos^2(s)}\right)^{-1} \\
&= \left(1 + \left(\frac{\sin s}{\cos s}\right)^2\right)^{-1} \\
&= \left(1 + [\tan(\arctan x)]^2\right)^{-1} \\
&= (1+x^2)^{-1} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

Theorem 20 (Direct comparison test for indefinite integrals)**.** (dt. "Majorantenkriterium für uneigentliche Integrals") Let f, g be regulated functions in $[a,b]$ and $|f(x)| \leq g(x) \quad \forall x \in [a,b)$. Assume $\int_a^b g(x)\,dx$ exists. Then also $\int_a^b |f(x)|\,dx$ exists and also $\int_a^b f(x)\,dx$.

Proof.

$$G(\beta) = \int_a^\beta g(x)\,dx$$

We know that $\lim_{\beta \to b_-} G(\beta)$ exists.

Cauchy criterion: $\forall \varepsilon > 0$ there exists a left-sided environment of b such that for all u, v in this environment it holds that

$$\underbrace{|G(u) - G(v)|}_{\int_u^v g(x)\,dx} < \varepsilon$$

Because $|f| \leq g$ it holds that

$$F(\beta) = \int_a^\beta |f(x)|\,dx$$

and also that

$$\left|\int_u^v |f(x)|\,dx\right| = |F(v) - F(u)| \overset{\text{monotonicity}}{\leq} \left|\int_u^v g(x)\,dx\right| < \varepsilon$$

Hence $\lim_{\beta \to b} F(\beta)$ exists because of the Cauchy criterion. So $\int_a^b |f(x)|\,dx$ exists. Analogously for f instead of $|f|$. $\square$

Example 8.

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ exists}$$

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \text{ continuous in } 0$$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 f(x)\,dx \text{ exists because } f \text{ is continuous}$$

$$\begin{aligned}
\lim_{\beta\to\infty} \int_1^\beta \frac{\sin x}{x}\,dx &= \begin{vmatrix} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{vmatrix} \\
&= \lim_{\beta\to\infty} \frac{1}{x} \cdot (-\cos x)\Big|_1^\beta - \int_1^\beta \frac{\cos x}{x^2}\,dx \\
&= \lim_{\beta\to\infty} \left[\underbrace{-\frac{1}{\beta} \cdot \cos\beta}_{\to 0} + \cos 1 - \int_1^\beta \frac{\cos x}{x^2}\,dx\right] \\
&= \lim_{\beta\to\infty} \int_1^\beta \frac{\cos x}{x^2}\,dx
\end{aligned}$$

The last expression exists, because $\frac{1}{x^2}$ is a majorant for $\frac{\cos(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ exists.

This lecture took place on 22nd of April 2016 with lecturer Wolfgang Ring.

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \text{ does not exist}$$

$$\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|\, dx$$

$$= \frac{1}{(k+1)\pi}(\pm 1)\cdot(-\cos x)\Big|_{k\pi}^{(k+1)\pi} = \frac{1}{(k+1)\pi}(\pm 1)(\pm 2)$$

$$= \frac{2}{(k+1)\pi}$$

$$\underbrace{\int_0^{(n+1)\pi} \left|\frac{\sin x}{x}\right| dx}_{\text{unbounded}\Leftarrow} \geq \frac{2}{\pi}\cdot \underbrace{\sum_{k=0}^n \frac{1}{k+1}}_{\substack{\text{harmonic series,}\\ \text{divergent}}}$$

In terms of the Lebesgue integral, $\int_0^\infty \frac{\sin x}{x}\,dx$ does not exist.

We can define new types of integration which yield new types of function which are not representable with techniques discussed so far.

Example 9 (The Eulerian Γ-function)**.**

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \text{ for } x > 0$$

The function variable of the Γ-function is a parameter of the integrand.

The indefinite integral from above exists,

$$\lim_{\alpha\to 0_+} \int_\alpha^1 \underbrace{t^{x-1}e^{-t}}_{>0}\,dt \text{ exists}$$

of $\int_\alpha^1 t^{x-1}e^{-t}\,dt$ is bounded in terms of α.

$$\int_\alpha^1 t^{x-1} \underbrace{e^{-t}}_{<1}\,dt < \underbrace{\int_\alpha^1 t^{x-1}\,dt}_{\text{converges for } x-1>-1}$$

hence for $x > 0$.

Right-side integral boundary:

$$\int_1^\infty t^{x-1}e^{-t}\,dt \text{ converges?}$$

Example 10 (Claim)**.** There exists $c > 0$ such that

$$t^{x-1}e^{-t} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$t^{x-1}\cdot e^{-\frac{t}{2}} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$\lim_{t\to\infty}\left(t^{x-1}\cdot e^{-\frac{t}{2}}\right) = \left|\begin{matrix}\frac{t}{2} = s\\ t = 2s\end{matrix}\right|$$
$$= \lim_{s\to\infty}(2s)^x - 1e^{-s}$$
$$\leq \lim_{s\to\infty}(2s)^{\lfloor x\rfloor+1-1}\cdot e^{-s}$$

with $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$

$$= \lim_{s\to\infty}(2s)^{\lfloor x\rfloor}\cdot e^{-s}$$
$$\leq \lim_{s\to\infty} s^{\lfloor x\rfloor+1}\cdot e^{-s}$$

because $s^{n+1} > (2s)^n$ for $s > 2^n$.

Hence for $\varepsilon > 0$, $\exists t$ such that

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| < \varepsilon \text{ if } t > L$$

and

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| \leq M \text{ for } t \in \underbrace{[1, L]}_{\text{compact}}$$

$\Rightarrow$ for $t \in [1, \infty)$ it holds that

$$\left| t^{x-1} e^{-\frac{t}{2}} \right| \le \max\{M, \varepsilon\} =: c$$

$$t^{x-1} e^{-\frac{t}{2}} \le c$$

$$\int_0^\infty t^{x-1} e^{-t}\, dt \le \int_0^\infty c \cdot e^{-\frac{t}{2}}\, dt = c \cdot \left(-2 \cdot e^{-\frac{t}{2}} \right) \Big|_0^\infty = 2c$$

hence $\int_0^\infty t^{x-1} e^{-t}\, dt$ exists.

It holds that $\Gamma(1) = 1$ because,

$$\int_0^\infty e^{-t}\, dt = 1$$

Furthermore it holds that for all $x > 0$,

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{x+1-1} e^{-t}\, dt = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R t^x e^{-t}\, dt \\
&= \begin{vmatrix} u = t^x & u' = x \cdot t^{x-1} \\ v' = e^{-t} & v = -e^{-t} \end{vmatrix} \\
&= \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left[-t^x e^{-t} \Big|_{t=\varepsilon}^R + \int_\varepsilon^R x \cdot t^{x-1} \cdot e^{-t}\, dt \right]
\end{aligned}$$

$$= \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left(\underbrace{-R^x \cdot e^{-R}}_{\to 0 \text{ for } R \to \infty} + \underbrace{\varepsilon^x \cdot e^{-\varepsilon}}_{\to 0 \text{ for } \varepsilon \to 0} \right) + x \cdot \int_0^\infty t^{x-1} e^{-t}\, dt = x \cdot \Gamma(x)$$

So it holds that

$$\begin{aligned}
T(2) &= 1 \cdot T(1) = 1 \\
T(3) &= 2 \cdot T(2) = 2 \cdot 1 \\
T(4) &= 4 \cdot T(3) = 3 \cdot 2 \cdot 1 \\
T(5) &= 4 \cdot T(4) = 4 \cdot 3 \cdot 2 \cdot 1
\end{aligned}$$

By complete induction we can show that

$$\Gamma(n+1) = n! \qquad \forall n \in \mathbb{N}$$

5.5 Some important inequalities

Theorem 21 (Young's inequality)**.** Let $f : [0, \infty) \to [0, \infty)$ be continously differentiable, $f(0) = 0$; f is strictly monotonically increasing and unbounded (hence f is injective because of strong monotonicity and surjective because of unboundedness).

So there exists $f^{-1} : [0, \infty) \to [0, \infty)$.

Let $a, b \ge 0$. Then it holds that

$$a \cdot b \le \int_0^a f(x)\, dx + \int_0^b f^{-1}(y)\, dy$$

Equality holds if and only if,

$$b = f(a) \text{ i.e. } a = f^{-1}(b)$$

Compare with Figure 20

Proof.

$$\int_0^b f^{-1}(y)\, dy \overset{\text{substitution}}{=} \begin{vmatrix} y = f(x) \\ dy = f'(x)\, dx \\ y = 0 \Leftrightarrow x = f^{-1}(0) = 0 \\ y = b \Leftrightarrow x = f^{-1}(b) \end{vmatrix}$$

$$\begin{aligned}
&= \int_0^{f^{-1}(b)} \underbrace{f^{-1}(f(x))}_{x} \cdot f'(x)\, dx \\
&= \int_0^{f^{-1}(b)} x \cdot f'(x)\, dx \\
&= f(x) \cdot x \big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} 1 \cdot f(x)\, dx \\
&= f(f^{-1}(b)) \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\, dx \\
&= b \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\, dx
\end{aligned}$$

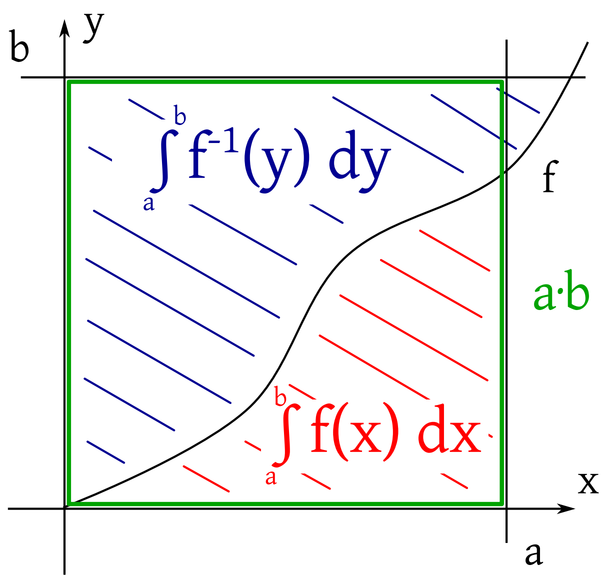


Figure 20: Young's inequality: the blue and red areas are larger than the green area

Therefore,

$$\begin{aligned} I &= \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \\ &= \int_0^a f(x)\,dx + \int_{f^{-1}(b)}^0 f(x)\,dx + b\cdot f^{-1}(b) \\ &= \int_{f^{-1}(b)}^a f(x)\,dx + b\cdot f^{-1}(b) \end{aligned}$$

Case 1: $f^{-1}(b) = a \ \ (f(a) = b)$

$$\implies I = \underbrace{\int_a^b f(x)\,dx}_{=0} + b\cdot a = ab$$

Proven.

Case 2: $b < f(a) \Leftrightarrow f^{-1}(b) < a$ f is strictly monotonically increasing, hence $f(x) > f(f^{-1}(b)) = b$ for all $x \in (f^{-1}(b), a]$.

$$\begin{aligned} \int_{f^{-1}(b)}^a f(x)\,dx &> b\cdot \int_{f^{-1}(b)}^a 1\,dx \\ &= b\cdot \left(a - f^{-1}(b)\right) \\ I > b\left(a - f^{-1}(b)\right) &+ b\cdot f^{-1}(b) = a\cdot b \end{aligned}$$

Proven.

Case 3: $b > f(a)$

$$I = \underbrace{-\int_a^{f^{-1}(b)} f(x)\,dx}_{\text{strictly mon. decreasing}} + bf^{-1}(b)$$

For $(-f(x))$ it holds that:

$$\begin{aligned} &> (-f(f^{-1}(b)) = -b) \\ &> (-b)\left(f^{-1}(b) - a\right) + b\cdot f^{-1}(b) = a\cdot b \end{aligned}$$

Proven.

$\square$

Remark 28. Young's inequality also holds if f has all the properties above but is not necessarily differentiable.

Theorem 22 (Young's inequality, special case)**.** Let $A, B \geq 0$. $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (hence p and q are "conjugate exponents"). Then it holds that

$$A\cdot B \leq \frac{A^p}{p} + \frac{B^q}{q}$$

Proof. Let $f(x) = x^{p-1}$ satisfy the requirements for Young's inequality.

$$f^{-1}(y) = y^{\frac{1}{p-1}}$$

$$\left(\frac{1}{q} = 1 - \frac{1}{q} \qquad q = \left(1 - \frac{1}{q}\right)^{-1}\right)$$

$$q - 1 = \left(1 - \frac{1}{p}\right)^{-1} - 1 = \left(\frac{p-1}{p}\right)^{-1} - 1$$

$$= \frac{p}{p-1} - 1 = \frac{p-p+1}{p-1} = \frac{1}{p-1}$$

$$f^{-1}(y) = y^{q-1}$$

Therefore

$$A \cdot B \le \int_0^A x^{p-1}\,dx + \int_0^B y^{q-1}\,dy = \left.\frac{x^p}{p}\right|_0^A + \left.\frac{y^q}{q}\right|_0^B = \frac{A^p}{p} + \frac{B^q}{q}$$

□

Remark 29. Equality holds if $A^p = B^q$. The proof is left as an exercise to the reader.

Theorem 23 (Hölder's inequality)**.** Let I be an interval, a, b are boundary values of I ($a, b \in [-\infty, \infty]$). Let p, q be conjugate exponents, hence $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let f_1 and f_2 be regulated functions in I and

$$\int_a^b |f_1(x)|^p\,dx \text{ exists and } \int_a^b |f_2(x)|^q\,dx \text{ exists}$$

Let

$$\|f_1\|_p = \left(\int_a^b |f_1(x)|^p\,dx\right)^{\frac{1}{p}}$$

and

$$\|f_2\|_q = \left(\int_a^b |f_2(x)|^q\,dx\right)^{\frac{1}{q}}$$

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \text{ exists and } \int_a^b |f_1(x) f_2(x)|\,dx \le \|f_1\|_p \cdot \|f_2\|_q$$

Proof. Let $A = \frac{|f_1(x)|}{\|f_1\|_p}$ and $B = \frac{|f_2(x)|}{\|f_2\|_q}$.

$$A \cdot b \le \frac{A^p}{p} + \frac{B^q}{q}$$

$$\overset{\text{integration}}{\Rightarrow} \int_a^b \frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q}\,dx \le \frac{1}{p}\int_a^b \frac{|f_1(x)|^p}{\|f_1\|_p^p}\,dx + \frac{1}{q}\int_a^b \frac{|f_2(x)|^q}{\|f_2\|_q^q}\,dx$$

$$\Rightarrow \frac{1}{\|f_1\|_p \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)|\,dx$$

$$\le \frac{1}{p}\frac{1}{\|f_1\|_p^p} \underbrace{\int_a^b |f_1(x)|^p\,dx}_{\|f_1\|_p^q} + \frac{1}{q}\frac{1}{\|f_2\|_q^q} \underbrace{\int_a^b |f_2(x)|^q\,dx}_{\|f_2\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$= \underbrace{\int_a^b |f_1(x) \cdot f_2(x)|\,dx}_{\text{exists}} \le \|f_1\|_p \cdot \|f_2\|_q$$

□

This lecture took place on 28th of April 2016 with lecturer Wolfgang Ring.

Example 11 (Special case $p = q = 2$)**.** Let $p = q = 2$. $\frac{1}{2} + \frac{1}{2} = 1$ holds.

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \le \left(\int_a^b |f_1(x)|^2\,dx\right)^{\frac{1}{2}} \cdot \left(\int_a^b |f_2(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \ge \left|\int_a^b f_1(x) \cdot f_2(x)\,dx\right|$$

f_1 and f_2 such that $\|f_i\|_2 < \infty$ for $i = 1, 2$, then

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) \cdot f_2(x)\, dx$$

is a scalar (= inner) product in the vector space of functions with norm:

$$\|f\| = (\langle f, f \rangle)^{\frac{1}{2}} = \|f\|_2$$

The resulting inequality is named "Cauchy-Schwarz inequality"

$$|\langle f_1, f_2 \rangle| \leq \|f_1\|_2 \cdot \|f_2\|_2$$

5.6 Elementwise integration of series

Lemma 13. Let $f_n \in R(I)$ with I as interval, f_n converges uniformly to f in I. Then also f is a regulated function and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx$$

Proof. We know f is a regulated function if and only if f can be uniformly approximated using a step function.

Let $\varepsilon > 0$ be arbitrary. Because f is the uniform limit of f_n, there exists $n \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$. Because f_N is a regulated function, there exists $\varphi \in \tau(I)$ with

$$\|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} \Rightarrow \|f - \varphi\|_\infty \leq \|f - f_N\|_\infty + \|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence f is a regulated function. Choose N such that $\forall n \geq N$:

$$\|f - f_n\|_\infty < \frac{\varepsilon}{b-a}$$

Then it holds that

$$\begin{aligned}
\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| &\leq \int_a^b |f_n(x) - f(x)|\, dx \\
&\leq \int_a^b \underbrace{\|f_n - f\|_\infty}_{< \frac{\varepsilon}{b-a}} dx \\
&< \frac{\varepsilon}{b-a} \cdot (b-a) \\
&= \varepsilon
\end{aligned}$$

□

Example 12 (Application)**.** Let $f(x) = \sum_{n=0}^{\infty} a_k x^k$ is a power series. Let ρ_f be a convergence radius of f and $0 < r < \rho_f$. Then it holds that

$$f_n(x) = \sum_{k=0}^{n} a_k x^k \text{ converges uniformly to } f \text{ in } [-r, r]$$

$$\begin{aligned}
& f_n \in R([-r, r]) \\
\Rightarrow & \int_{-r}^{r} f(x)\, dx = \lim_{n\to\infty} \int_{-r}^{r} f_n(x)\, dx
\end{aligned}$$

The integral is determined by elementwise integration

$$\int_{-r}^{r} a_k x^k\, dx = a_k \frac{x^{k+1}}{k+1}\bigg|_{-r}^{r}$$

Analogously for integration over any compact interval $[a, b] \subset (-\rho_f, \rho_f)$ i.e. for the indefinite integration. Hence,

$$\sum_{k=0}^{\infty} a_k \frac{x^{k+1}}{n+1} + c$$

is primitive function of f uniformly convergent on every interval $[-r, r] \subseteq (-\rho_f, \rho_f)$.

Example 13.

$$F : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad F(x) = \arctan(x)$$

$$F'(x) = f(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} \left(-(x^2)\right)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad \forall x \in (-1,1)$$

Elementwise integration:

$$F(x) = \arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + c$$

$$\arctan(0) = 0 = c$$

Hence,

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad \text{in } (-1,1)$$

Compare with Figure 21

6 Taylor polynomials and Taylor series

Theorem 24. Approximation of a function with polynomials or representation of a function using a power series.

$$\mathcal{C}^n((a,b)) = \{f : (a,b) \to \mathbb{R} \mid f \text{ differentiable } n \text{ times in } (a,b)\}$$

Hence $f^{(k)} : (a,b) \to \mathbb{R}$ is continuous for $k = 0, 1, \dots, n$. Choose $x_0 \in (a,b)$. Find a polynomial $T_f^a(x)$ of degree n such that

$$\left(T_f^a\right)^{(k)}(x_0) = f^{(k)}(x_0)$$

It holds that T_f^a can be determined uniquely as

$$T_f^a(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Taylor polynomial of n-th order of f in point x_0.

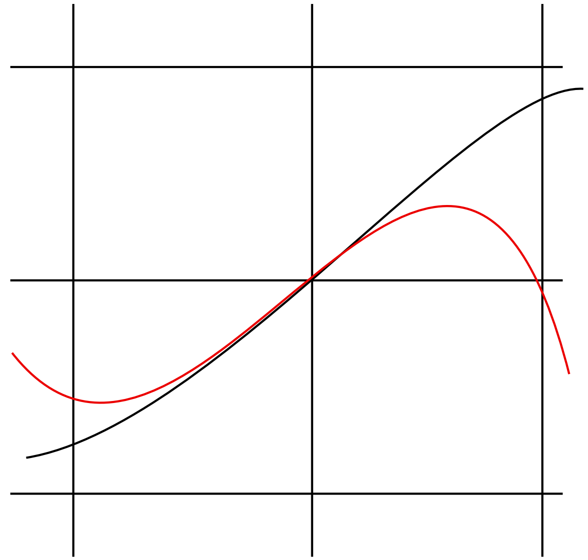


Figure 21: Approximation of $\arctan(x)$

This lecture took place on 29th of April 2016 with lecturer Kniely Michael.

Definition 15 (Additional remark to Taylor polynomials)**.** Let $P(x) := \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$. Let $k \in \{1, \dots, n\}$.

1. x_0 is called k-th root of P iff $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$.
2. It holds that x_0 is a k-th root of P iff

$$\forall j \in \{0, \dots, k-1\} : P^{(j)}(x_0) = 0 \land P^{(k)}(x_0) \neq 0$$

Complete induction over k. $\mathbf{k = 1}$

$\Rightarrow$: Let x_0 be 1st root of P.

$$P(x) = (x - x_0)Q(x) \Rightarrow P^{(0)}(x_0) = 0 \wedge P^{(1)}(x_0) = Q(x_0) \neq 0.$$

$\Leftarrow$: Let $P^{(0)}(x_0) = 0$.

$$P^{(1)}(y_0) \neq 0$$

Division with remainder $\Rightarrow$

$$P(x) = (x - x_0)Q(x) + R(x) \text{ with } \deg(R) < \deg(x - x_0) = 1$$

with R constant.

$$0 = P(y_0) = R \Rightarrow P(x) = (x - x_0)Q(x)$$

$$x \neq x_0 \Rightarrow Q(x) = \frac{P(x)}{x - x_0} = \frac{P(x) - P(x_0)}{x - x_0} \Rightarrow Q(x_0)$$

$$\overset{\text{Q continuous}}{=} \lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{P(x) - P(x_0)}{x - x_0} = P^{(1)}(x_0) \neq 0$$

$\mathbf{k \geq 2, k - 1 \to k}$ $\Rightarrow$. Let x_0 be the k-th root of P. Hence $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$. Let $\tilde{P}(x) := (x - x_0)^{k-1}Q(x)$. x_0 is $(k-1)$-th root of $\tilde{P}$.

$$\xRightarrow{\text{ind. hypo.}} \tilde{P}^{(j)}(x_0) = 0 \wedge \tilde{P}^{(k-1)}(x_0) \neq 0 \quad \forall j \in \{0, \dots, k-2\}$$

$$P(x) = (x - x_0)\tilde{P}(x) \Rightarrow P^{(j)}(x) = (x - x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x)$$

We prove the last statement using complete induction:

Proof. $j = 0$ Follows immediately.

$j \geq 0, j \to j + 1$

$$P^{(j+1)}(x) = \left(P^{(j)}\right)'(x)$$

$$= \tilde{P}^{(j)}(x) + \tilde{P}^{(j+1)}(x)(x - x_0)$$

$$+ j\tilde{P}^{(j)}(x) = (x - x_0)\tilde{P}^{(j+1)}(x) + (j + 1)P^j(x).$$

$$P^{(j)}(x_0) = j\tilde{P}^{(j-1)}(x_0)$$

$$\begin{cases} = 0 & j = 0, \dots, k - 1 \\ \neq 0 & j = k \end{cases}$$

□

We then prove the second part: $\Leftarrow$.

Let $P^{(j)}(x_0) = 0$ for $j \in \{0, \dots, k-1\}$, $P^{(k)}(x_0) \neq 0$. It holds that $P(y_0) = 0$ because of $P^{(0)}(x_0) = 0$. Like above: $P(x) = (x - x_0)\tilde{P}(x)$ and

$$P^{(j)}(x) = (x - x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x).$$

$$j \in \{1, \dots, k-1\} \implies 0 = P^{(j)}(x_0) = j \cdot \tilde{P}^{(j-1)}(x_0)$$

$$\implies \forall l \in \{0, \dots, k-2\} : \tilde{P}^{(l)}(x_0) = 0$$

$$0 \neq P^{(k)}(x_0) = k\tilde{P}^{(k-1)}(x_0) \Rightarrow \tilde{P}^{(k-1)}(x_0) \neq 0$$

induction hypothesis $\Rightarrow$

$$\tilde{P}(x) = (x - x_0)^{k-1}Q(x) \text{ with } Q(x_0) \neq 0$$

$$\implies P(x) = (x - x_0)\tilde{P}(x) = (x - x_0)^k Q(x).$$

□

Theorem 25. Let f in $\mathbb{C}^n((a, b))$ with $n \in \mathbb{N}$. Let $a, b \in [-\infty, \infty]$, $x_0 \in (a, b)$. Find a polynomial T of degree n such property

$$\forall k \in \{0, \dots, n\} : T^{(k)}(x_0) = f^{(k)}(x_0). \tag{1}$$

Claim:

$$T_f^n(x) \equiv T_f^n(x; x_0) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

where x_0 is the base point, is the only polynmial of degree n, which satisfies property 1.

T_f^n is called *Taylor polynomial* of n-th degree of f in x_0.

Proof. Let $k \in \{0, \dots, n\}$.

$$(T_g^n)^{(k)}(x) = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} j(j-1) \cdot \ldots \cdot (j - (k-1))(x - x_0)^{j-k}$$

$$\left(T_f^n\right)^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!}\underbrace{(k\cdot\ldots\cdot(k-(k-1)))}_{=k!} = f^{(k)}(x_0).$$

Let $T(x) = \sum_{j=0}^n a_j x^j$ be a polynomial, which satisfies 1. For $P := T_g^n - T$ it holds that $P^{(k)}(x_0) = 0$ for all $k \in \{0, \ldots, n\}$. And P is a polynomial of degree at most n. x_0 is at least an $(n+1)$-th root of $P \Rightarrow P \equiv 0$. ☐

Definition 16 (Deviation, error, remainder)**.**

$$R_g^{n+1}(x; x_0) \equiv R_g^{n+1}(x) := f(x) - T_g^n(x; x_0)$$

Theorem 26 (Integration form of the remainder)**.** Let $f \in C^{n+1}((a,b),\mathbb{C})$, $n \in \mathbb{N}$, $a, b \in [-\infty, \infty], x_0, x \in (a,b)$. Then it holds that

$$R_g^{n+1}(x) = \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt$$

Complete induction over n. Let $n = 0$.

$$R_g^1(x) = f(x) - T_g^0(x) = f(x) - f(x_0)$$

$$\frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt = \int_{x_0}^x f'(t)\,dt = f(x) - f(x_0).$$

Consider $n \geq 1, n-1 \to n$. From induction hypothesis we consider

$$\begin{aligned}\Rightarrow f(x) - T_g^{n-1}(x) = R_g^n(x) &= \frac{1}{(n-1)!}\int_{x_0}^x (x-1)^{n-1} f^{(n)}(t)\,dt \\ &= -\left.\frac{(x-t)^n}{n(n-1)!} f^{(n)}(t)\right|_{x_0}^x + \int_{x_0}^x \frac{(x-t)^n}{n(n-1)!} f^{(n+1)}(t)\,dt \\ &= \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt\end{aligned}$$

$$\begin{aligned}\Rightarrow R_f^{n+1}(x) = f(x) - T_g^n(x) &= f(x) - T_g^{n-1}(x) - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{n+1}(t)\,dt\end{aligned}$$

Recognize that we consider f over $\mathbb{C}$. In the next theorem we will only consider it in $\mathbb{R}$. ☐

Theorem 27 (Lagrange representation of remainder)**.** Let $f \in C^{n+1}((a,b),\mathbb{R}), n \in \mathbb{N},\ a, b \in [-\infty, \infty],\ x_0, x \in (a,b)$. Then there exists some ξ between x_0 and x such that

$$R_g^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

Proof.

$$R_f^{n+1}(x) = \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt$$

Case 1: $x \geq x_0$**:**

$$\forall t \in [x_0, x] : (x-t)^n \geq 0$$

$f \mapsto (x-1)^n$ regulated function. $t \mapsto f^{(n+1)}(t)$ continuous. Hence,

$$\exists \xi \in [x_0, x] : \int_{x_0}^x (x-1)^n f^{n+1}(t)\,dt = f^{n+1}(\xi)\int_{x_0}^x (x-t)^n\,dt$$

$$= f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{n+1}$$

$$\Rightarrow R_f^{n+1}(x) = \frac{f^{n+1}}{(n+1)!}(x-x_0)^{n+1}.$$

Case 2: $x < x_0$**:**

$$\forall t \in [x, x_0] : (t-x)^n \geq 0 \qquad \text{analogously}$$

$$\exists \xi \in [x, x_0] : \int_x^{x_0} (t-x)^n f^{(n+1)}(t)\,dt$$

$$
\begin{aligned}
&= f^{(n+1)}(\xi) \int_x^{x_0} (1-x)^n \, dt \\
&= \frac{f^{(n+1)}(\xi)}{n+1} (x_0 - x)^{n+1} \\
\Rightarrow R_g^{n+1}(x) &= \frac{(-1)^{n+1}}{n!} \int_x^{x_0} (t-x)^n f^{(n+1)}(t) \, dt \\
&= (-1)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} (x_0 - x)^{n+1} \\
&= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}
\end{aligned}
$$

□

Corollary 4 (Sufficient criterion for local extremes)**.** Let $f \in C^{n+1}((a,b),\mathbb{R}), x_0 \in (a,b)$ with $f^{(n)}(x_0) = \ldots = f^{(n)}(x_0) = 0, f^{(n+1)}(x_0) \neq 0$. Then f has the following in x_0:

- a strict local minimum, if n is odd and $f^{(n+1)}(x_0) > 0$.
- a strict local maximum, if n is odd and $f^{(n+1)}(x_0) < 0$.
- no extreme, if n is even.

Proof. **Case 1: $f^{(n+1)}(x_0) > 0$:**
$f^{(n+1)}$ is continuous $\Rightarrow$

$$\exists \varepsilon > 0 : f^{(n+1)} > 0 \text{ in } (x_0 - \varepsilon, x_0 + \varepsilon) =: I$$

by Induction hypothesis it holds that

$$\forall x \in (a,b) : f(x) = T_g^n(x) + R_g^{n+1}(x) = f(x_0) + R_f^{n+1}(x).$$

If n is even, then $n+1$ is odd, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} > 0.$$

So,

$$\forall x \in I \setminus \{x_0\} : f(x) > f(x_0)$$

If n is odd, $n+1$ is even, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

$$\begin{cases} > 0 & x > x_0 \\ < 0 & x < x_0 \end{cases}$$

$\Rightarrow f$ has no extremum in x_0.

Case 2: $f^{(n+1)}(x_0) < 0$ follows analogously like Case 1.

□

Theorem 28 (Qualitative Taylor formula)**.** Let $f \in C^n((a,b),\mathbb{C}), x, x_0 \in (a,b)$. There exists some $r \in C((a,b),\mathbb{C})$ with $r(x_0) = 0$ and

$$f(x) = T_f^n(x) + (x - x_0)^n r(x) \tag{2}$$

Proof. Equation 2 only has to be shown for $f : (a,b) \to \mathbb{R}$, because for $f : (a,b) \to \mathbb{C}$, $f = f_R + i f_I$ with $f_R, f_I : (a,b) \to \mathbb{R}$. Representations for f_R and f_I provide corresponding representations for f. Hence let $f : (a,b) \to \mathbb{R}$. Let $r : (a,b) \to \mathbb{R}$.

$$x \mapsto \frac{f(x) - T_f^n(x)}{(x - x_0)^n}, x \neq x_0 \text{ and } r(x_0) := 0$$

We only need to show:
r is continuous in x_0, hence $\lim_{x \to x_0} r(x) = r(x_0) = 0$.

$$
\begin{aligned}
x \in (a,b) \setminus \{x_0\} \Rightarrow r(x) &= \frac{1}{(x-x_0)^n} \left(f(x) - T_f^{n-1}(x) - \frac{f^{(n)}}{n!}(x - x_0)^n \right) \\
&= \frac{1}{(x-x_0)^n} \left(R_g^n(x) - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right) \\
&= \frac{1}{(x-x_0)^n} \left(\frac{f^{(n)}(\xi)}{n!}(x - x_0)^n - \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right) \\
&= \frac{1}{n!} \left(f^{(n)}(\xi) - f^{(n)}(x_0) \right)
\end{aligned}
$$

ξ is between x_0 and x. $f^{(n)}$ is continuous and $\xi \to x_0$ for $x \to x_0$

$$\Rightarrow r(x) = \frac{1}{n!}(f^{(n)}(\xi) - f^{(n)}(x_0)) \overset{x \to x_0}{\to} 0$$

$\square$

This lecture took place on 3rd of May 2016 with lecturer Wolfgang Ring.

Theorem 29. Assumption: Let $f : I \to \mathbb{R}$ be arbitrarily often continuously derivable. Hence,

$$T_f^n(x; x_0) \text{ exists for } \forall n \in \mathbb{N}$$

Therefore we can consider a power series

$$T_f(x; x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$T_f(x; x_0)$ is called Taylor series of f in x_0. Is a power sereis in $\xi = (x - x_0)$. Converges for $|\xi| = |x - x_0| < \rho(T_f)$.

If $\rho(T_f) > 0$, it holds that $T_f(x; x_0) = f(x)$?

$$\lim_{n \to \infty} T_f^n(x; x_0) = T_f(x; x_0) = f(x) \text{ for } |x - x_0| < \rho(T_f)$$

is *not* always satisfied.

Example 14.

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Compare with Figure 22.

$$f_-^{(n)}(0) = 0$$

$$f_+^{(n)}(0) = \lim_{x \to 0^+} f^{(n)}(x)$$

$$f^{(n)}(x) = R(x) \cdot e^{-\frac{1}{x}}$$

with $R(x) = \frac{P(x)}{Q(x)}$ with P and Q as polynomials. R is a rational function (i.e. division of two polynomials).

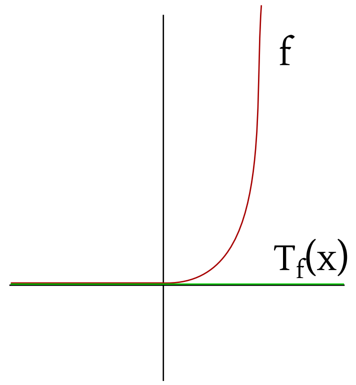


Figure 22: Plot of f

$$\lim_{x \to 0_+} R(x) \cdot e^{-\frac{1}{x}} = 0$$

Hence $f^{(n)}(0) = 0$ and therefore Taylor series $T_f(x; 0) = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0$.

Remark 30. Taylor:

$$R_f(x) = T_f(x; 0) - f(x)$$

It holds that

$$|R_f(x)| \leq c_n \cdot |x|^n \qquad \forall n \in \mathbb{N}$$

Theorem 30. Let $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series in $\xi = x - x_0$. Let $\rho(f) > 0$. We already know that f is differentiable for all $|\xi| = |x - x_0| < \rho(f)$ (differentiable by x) and f' is a power series with convergence radius $\rho(f') = \rho(f)$.

$$f'(x) = \sum_{k=1}^{\infty} a_k \cdot k x^{k-1}$$

By complete induction it follows that:

- For all $n \in \mathbb{N}$ there exists $f^{(n)}(x)$ as power series of form

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k \cdot k \cdot (k-1) \cdot (k-2) \cdot \ldots (k-n+1) \cdot x^{k-n}$$

- $f^{(n)}$ as convergent power series is a continuous function. Hence,

$$f^{(n)}(x_0) = a_n \cdot n \cdot (n-1) \cdot (n-2) \ldots (n-n+1) = a_n \cdot n!$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Backsubstitution in the power series yields

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = T_f(x; x_0)$$

Hence f has a power series representation, then the power series is the Taylor series in f.

Remark 31. A function representable with a power series is called *analytical.* In the complex space, once differentiable means arbitrary often differentiable.

7 Curves in $\mathbb{R}^n$

Definition 17. A parametric curve is a map $\gamma : I \to \mathbb{R}^n$ where I is an interval which has form

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

and every function $\gamma_i : I \to \mathbb{R}$ $(i = 1, \ldots, n)$ is continuous. Often we write $\gamma_i(t) = x_i(t)$. If every γ_i is differentiable in I, a differentiable, parametric curve is given. t is the curve parameter.

We call $\Gamma = \{\gamma(t) \,|\, t \in I\} = \gamma(I) \subseteq \mathbb{R}^n$ the *trace of the curve* γ.

Example 15.

$$\gamma : [0, 4\pi] \to \mathbb{R}^2$$
$$\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

In this example, every point on the curve is hit twice by the function.

$$\Gamma = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R} \,\middle|\, x_1^2 + x_2^2 - 1 = 0 \right\}$$

$F(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$ is called trace equation of the curve

$$\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \text{ in } I = [0, 4\pi]$$

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, then

$$x_2^1 + x_2^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 1 - 1 = 0$$

On the inverse, let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1$. Then there exists $t \in [0, 2\pi)$ such that $x_1 = \cos t$ and $x_2 = \sin t$.

In this example it holds that $\tilde{\gamma} \neq \gamma$, but $T = \tilde{T}$.

Example 16. Let $\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ with $\tilde{\gamma} : [0, \pi] \to \mathbb{R}^2$.

$$\forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \tilde{T} : T(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$

but

$$\tilde{T} \neq \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, F(x_1, x_2) = 0 \right\}$$

Definition 18. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable, parametric curve. We define

$$\dot{\gamma}(t) = \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

and we call $\dot\gamma(t)$ the derivation vector of γ in t. If γ is considered as motion curve, then $\dot\gamma(t)$ is considered as speed vector of γ in t.

Consider

$$\dot\gamma(t) = \lim_{h\to 0} \frac{1}{h}\left[\gamma(t+h) - \gamma(t)\right]$$

as illustrated in Figure 23.

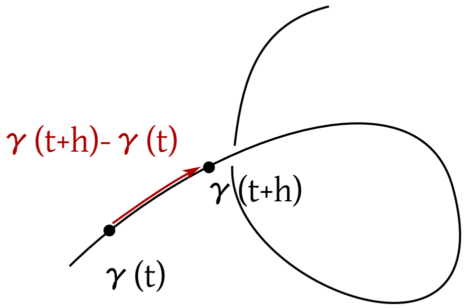


Figure 23: Curve example

If $\dot\gamma(t) \neq 0$, then $\dot\gamma$ is tangential in Γ and we call $\dot\gamma(t)$ the *tangential vector of γ in t*.

If $\dot\gamma(t) \neq 0$, we set

$$T_\gamma(t) = \frac{\dot\gamma(t)}{\|\dot\gamma(t)\|_2}$$

and we call $T_\gamma(t)$ the tangential unit vector of γ in t.

Example 17.

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

$$\gamma(t) = \begin{bmatrix} t^2 - 1 \\ t^3 - 1 \end{bmatrix} \text{ differentiable}$$

$$\gamma(1) = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \vec{0}$$

$$\gamma(-1) = \begin{bmatrix} 1-1 \\ -1+1 \end{bmatrix} = \vec{0}$$

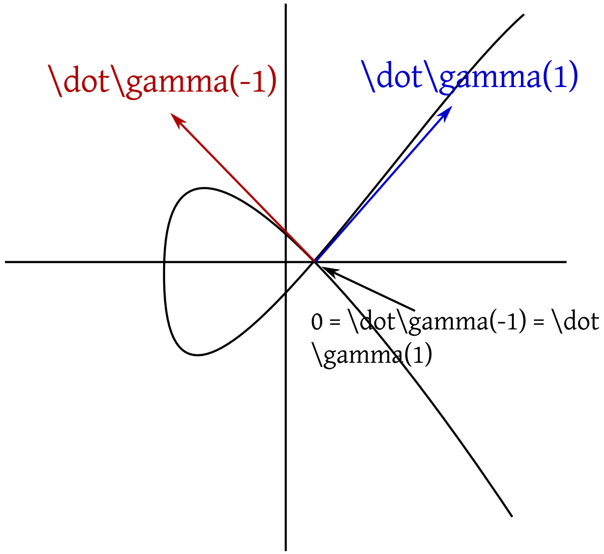


Figure 24: A double pointed curve

This curve has a double point, meaning that one point is crossed two times (compare with Figure 24).

$$\dot\gamma(t) = \begin{bmatrix} 2t \\ 3t^2 - 1 \end{bmatrix}$$

$$\dot\gamma(-1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\dot\gamma(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Definition 19. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable, parametric curve. γ is called *regular* curve, if $\dot{\gamma}(t) \neq \vec{0} \quad \forall t \in I$.

Example 18.

$$\gamma(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

is called *Neil's parabola* and non-regular.

$$\dot{\gamma}(t) = \begin{bmatrix} 2t \\ 3t^3 \end{bmatrix}$$

$$\dot{\gamma}(0) = \vec{0}$$

Has no tangent in the root.

Example 19.

$$\gamma(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}$$

$a, b > 0$ and $t \in [0, 2\pi]$. We search for a trace equation of γ:

$$\frac{x_1(t)}{a} = \cos(t) \qquad \frac{x_2(t)}{b} = \sin(t)$$

We use the trace equation of the unit circle:

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 - 1 = 0$$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

γ has an ellipsis as trace with major axis lengths a and b. The interpretation of curve parameter t is given in Figure 26.

This lecture took place on 6th of May 2016 with lecturer Wolfgang Ring.

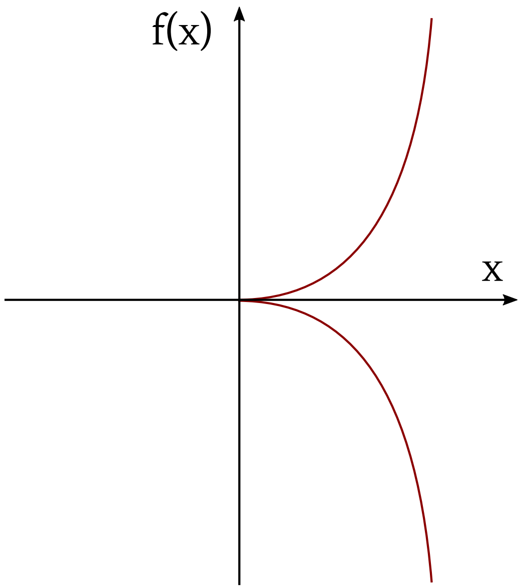


Figure 25: Neil's parabola

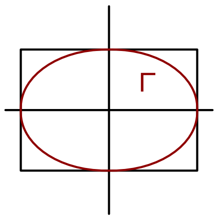


Figure 26: Curve parameter t

8 Hyperbolic functions

Definition 20. We define the *cosine and sinus hyperbolic functions* as follows:

$$\cosh : \mathbb{C} \to \mathbb{C} \qquad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$

$$\sinh : \mathbb{C} \to \mathbb{C} \qquad \sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$$

For real values we get Figure 27.

Properties:

$$\begin{aligned}
\cosh'(x) &= \frac{1}{2}(e^x - e^{-x}) \\
&= \sinh(x) \\
\sinh'(x) &= \frac{1}{2}(e^x + e^{-x}) \\
&= \cosh(x) \\
\cosh^2(x) - \sinh^2(x) &= \frac{1}{4}\left(e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right) \\
&\quad - \frac{1}{4}\left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right) \\
&= \frac{1}{4} \cdot 4 \cdot 1 = 1 \\
\cosh^2(x) - \sinh(x) &= 1
\end{aligned}$$

Example 20. Let $y : \mathbb{R} \to \mathbb{R}^2$.

$$\gamma(t) = \begin{bmatrix} \underbrace{a\cosh(t)}_{>0} \\ b\sinh(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{(x(t))^2}{a^2} - \frac{(y(t))^2}{b^2} = \cosh^2(t) - \sinh^2(t) = 1$$

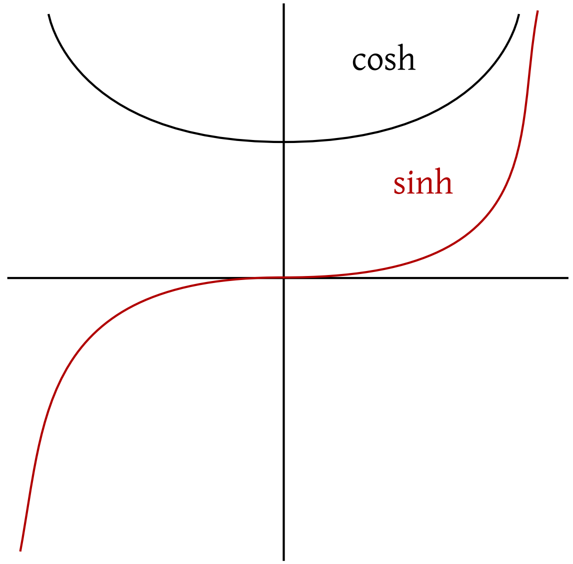


Figure 27: Plot of hyperbolic cosine and sine

hence the trace T of γ is inside the hyperbola

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$

Theorem 31. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ continuously differentiable. Then

$$\left. \begin{aligned} t &\mapsto \begin{bmatrix} t \\ f(t) \end{bmatrix} \\ I &\to \mathbb{R}^2 \end{aligned} \right\}$$

a parametric, differentiable curve. The function graph is equivalent to the trace of the curve.

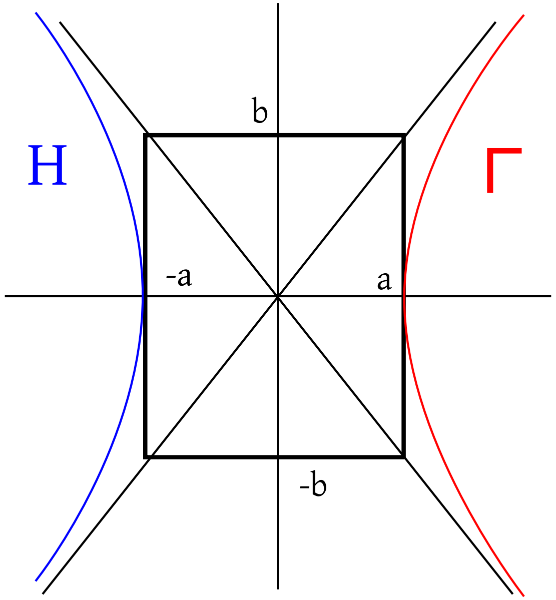


Figure 28: Hyperbola H

Theorem 32 (Representation as function graph)**.** Let $\gamma : I \to \mathbb{R}^2$ a continuously differentiable curve, I is an interval and

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

it holds that

$$\dot{\gamma}_1(t) \neq 0 \quad \forall t \in I$$

Then there exists a continuously differentiable function $f : J = \gamma_1(I) \to \mathbb{R}$ such that the graph of f matches the trace of γ.

Let $x_0 = \gamma_1(t_0)$. Then it holds that

$$f'(x_0) = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}.$$

If γ is differentiable twice in t_0, then f is differentiable twice in x_0.

$$f''(x_0) = \frac{\dot{x}(t_0)\ddot{y}(t_0) - \ddot{x}(t_0)\dot{y}(t_0)}{[\dot{x}(t_0)]^3}$$

Proof. $\dot{y}_1$ has no root, is continuous, this means $\dot{\gamma}_1$ has a uniform sign in I. Hence γ_1 is strictly monotonical in I. So $\gamma_1 : I \to J = \gamma_1(I)$ is bijective.

Let $\gamma_1^{-1} : J \to I$ be the inverse function. Because $\dot{\gamma}_1 \neq 0$ in I is γ_1^{-1} is differentiable with

$$\left(\gamma_1^{-1}\right)'(s) = \frac{1}{\dot{\gamma}_1(\gamma_1^{-1}(s))}$$

We define

$$f(x) = \gamma_2(\gamma_1^{-1}(x))$$
$$I \to \mathbb{R}$$

Let $T_f = \{(x, f(x)) \,|\, x \in I\}$ be the graph of f and $(x, f(x)) \in T_f$; $(x, f(x)) = (x, \gamma_2(\gamma_1^{-1}(x)))$. Let $\gamma_1^{-1}(x) = t \in I$ and therefore $x = \gamma_1(t)$. So it holds that

$$(x, f(x)) = (\gamma_1(t), \gamma_2(t)) \in T \qquad \dots \text{ trace of } \gamma$$

On the opposite, we have $(\gamma_1(t), \gamma_2(t)) \in T$. Let $x = \gamma_1(t) \in J$ and $t = \gamma_1^{-1}(x)$ and $(\gamma_1(t), \gamma_2(t)) = (x, \gamma_2(\gamma_1^{-1}(x))) = (x, f(x)) \in T_f$.

$$f'(x)|_{x=x_0} = \dot{\gamma}_2(\gamma_1^{-1}(x_0)) \cdot \frac{1}{\dot{\gamma}_1^{-1}(x_0)} = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}$$

Let $\gamma_1^{-1}(x_0) = t_0$.

$$\begin{aligned} f''(x_0) &= \frac{\ddot{\gamma}_2(t_0) \cdot \frac{1}{\dot{\gamma}_1(t_0)} \cdot \dot{\gamma}_1(t_0) - \dot{\gamma}_2(t_0) \cdot \ddot{\gamma}_1(t_0) \cdot \frac{1}{\dot{\gamma}_1(t_0)}}{(\dot{\gamma}_1(t_0))^2} \\ &= \frac{\ddot{\gamma}_2(t_0) \cdot \dot{\gamma}_1(t_0) - \ddot{\gamma}_1(t_0) \cdot \dot{\gamma}_2(t_0)}{(\dot{x}(t_0))^3} \\ &= \frac{\ddot{\gamma}(t_0) \cdot \dot{x}(t_0) - \dot{y}(t_0)\ddot{x}(t_0)}{(\dot{x}(t_0))^3} \end{aligned}$$

□

Figure 29 gives an example for a curve which is not representable as function graph of x in $\dot{\gamma}(t_0) = 0$.

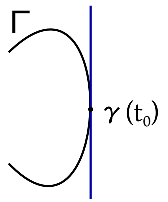


Figure 29: A parametric curve with $\dot{\gamma}(t_0) = 0$

9 Arc length of a parametric curve

Theorem 33. Let $\gamma : I = [a, b] \to \mathbb{R}^n$ be a parametric curve. Let $z = \{t_0 = a, t_1, t_2, \ldots, t_N = b\}$ with $t_{i-1} < t_i$ for $i = 1, \ldots, N$ be a partition of the interval I. We denote the length of the polygonal line through the partition points $\gamma(t_0)$, $\gamma(t_1)$, ..., $\gamma(t_N)$ with

$$s(z) = s_\gamma(z) = \sum_{i=1}^{N} \|\gamma(t_1) - \gamma(t_{i-1})\|$$

Let z^* be more detailed than z. Then it holds that

$$s(z^*) \geq s(z)$$

Insertion of partition points. Let $z = \{t_0 < t_1 < \ldots < t_N\}$ and $z^* =$

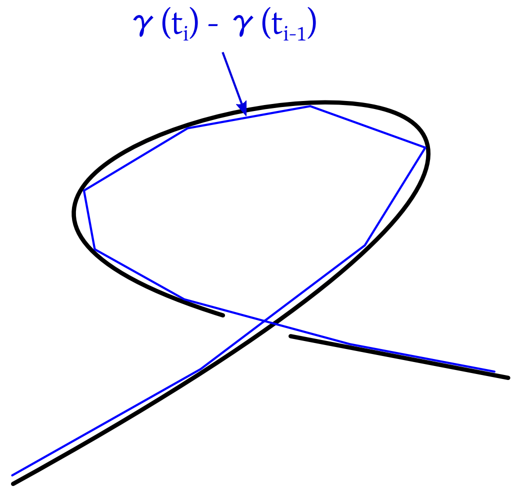


Figure 30: Approximation of the arc length

$\{t_0 < \ldots < t_{k-1} < t' < t_k < \ldots < t_N\}$.

$$\begin{aligned} s(z) &= \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ s(z^*) &= \sum_{i=1}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &\quad + \underbrace{\|\gamma(t') - \gamma(t_{k-1})\| + \|\gamma(t_k) - \gamma(t')\|}_{\geq \|\gamma(t_K) - \gamma(t') + \gamma(t') - \gamma(t_{K-1})\|} \\ &\quad + \sum_{i=K+1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &\geq \sum_{i=1}^{N} \|\gamma(t_1) - \gamma(t_{i-1})\| = s(z) \end{aligned}$$

For insertion of multiple points use induction. ☐

Definition 21. Let $\gamma : I = [a,b] \to \mathbb{R}^n$ be a continuous curve. γ is call *rectifiable* if

$$s(\gamma) = \sup s(z) < \infty$$

where z is a partition of I. In this case $s(\gamma)$ is called length of curve γ.

Example 21. Let $\gamma : I \to \mathbb{R}^n$ be Lipschitz continuous. Hence

$$\exists L \geq 0 : \|\gamma(s) - \gamma(t)\| \leq L(s-1)$$

for all $s, t \in I$. Then γ is rectifiable and $s(\gamma) \leq L \cdot (b-a)$.

Proof. Let z be a partition of I.

$$z = \{t_0 < t_1 < t_2 < \ldots < z_N\}$$

Then it holds that

$$\begin{aligned} s(z) &= \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &\leq \sum_{i=1}^{N} L\,|t_i - t_{i-1}| \\ &= L \sum_{i=1}^{N} (t_i - t_{i-1}) \\ &= L(t_N - t_0) = L(b-a) \end{aligned}$$

☐

Theorem 34. Let $\gamma : I \to \mathbb{R}^n$ be a continuous curve.

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

Let every $\gamma_i : I \to \mathbb{R}$ be a primitive function of a regulated function (hence $\dot{\gamma}_i$ exists for all t except for finitely many points, furthermore $\dot{\gamma}_i$ has left-sided and right-sided limits everywhere).

$$I = [a,b]$$

Then γ is rectifiable and it holds that

$$s(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\, dt$$

Remark 32 (Some necessary preparations)**.**

$$\int_a^b \gamma(t)\, dt := \begin{bmatrix} \int_a^b \gamma_1(t)\, dt \\ \int_a^b \gamma_2(t)\, dt \\ \vdots \\ \int_a^b \gamma_n(t)\, dt \end{bmatrix}$$

Lemma 14. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous curve. Then it holds that

$$\left\| \int_a^b \gamma(t)\, dt \right\| \leq \int_a^b \|\gamma(t)\|\, dt$$

Proof. Let φ_i be a step function which approximates γ_i uniformly. Hence, we assume that $\|\varphi_i - \gamma_i\|_\infty < \varepsilon$. z_i is a partition such that φ_i is constant at every interval.

Define $z = \bigcup_{i=1}^n z_i$ ascendingly ordered. Then every φ_i is also a step function in terms of z.

$$z = \{t_0 < t_1 < \ldots < t_N\}$$
$$\varphi_i(t) = c_k^i \quad \text{for } t \in (t_{k-1}, t_k)$$
$$\int_a^b \varphi_i(t)\, dt = \sum_{k=1}^{n} c_k^i (t_k - t_{k-1})$$

Build

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_N(t) \end{bmatrix}$$

Then it holds that

$$\left\|\int_a^b \varphi(t)\,dt\right\| = \left\|\begin{bmatrix}\sum_{k=1}^N c_k^1(t_k - t_{k-1})\\ \sum_{k=1}^N c_k^2(t_k - t_{k-1})\\ \vdots \\ \sum_{k=1}^N c_k^n(t_k - t_{k-1})\end{bmatrix}\right\| = \left\|\sum_{k=1}^N (t_k - t_{k-1})\cdot\begin{bmatrix}c_k^1\\ \vdots\\ c_k^n\end{bmatrix}\right\|$$

$$\underbrace{\leq}_{\text{triangle ineq. in } \mathbb{R}^n} \sum_{k=1}^N (t_k - t_{k-1})\left\|\begin{bmatrix}c_k^1\\ \vdots\\ c_k^n\end{bmatrix}\right\| = \int_a^b \underbrace{\|\varphi(t)\|}_{\text{step function in } \mathbb{R}}\,dt$$

$$\|\gamma(t)-\varphi(t)\| = \left(\sum_{i=1}^n |\gamma_i(t)-\varphi_i(t)|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^n \varepsilon^2\right)^{\frac{1}{2}} = \sqrt{n}\cdot\varepsilon$$

$$\Rightarrow \int_a^b \|\gamma(t)-\varphi(t)\|\,dt < \varepsilon\sqrt{n}(b-a)$$

$$\left\|\int_a^b (\gamma(t)-\varphi(t))\,dt\right\| = \left\|\begin{bmatrix}\int_a^b \gamma_1(t)-\varphi_1(t)\,dt\\ \int_a^b \gamma_2(t)-\varphi_2(t)\,dt\\ \vdots\\ \int_a^b (\gamma_n(t)-\varphi_n(t))\,dt\end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix}\left|\int_a^b (\gamma_1(t)-\varphi_1(t))\,dt\right|\\ \left|\int_a^b (\gamma_2(t)-\varphi_2(t))\,dt\right|\\ \vdots\\ \left|\int_a^b (\gamma_n(t)-\varphi_n(t))\,dt\right|\end{bmatrix}\right\| \leq \left\|\begin{bmatrix}\varepsilon(b-a)\\ \varepsilon(b-a)\\ \vdots\\ \varepsilon(b-a)\end{bmatrix}\right\| = \varepsilon(b-a)\sqrt{n}$$

Hence it holds that

$$\left\|\int_a^b \gamma(t)\,dt\right\| = \left\|\int_a^b \varphi(t)\,dt - \int_a^b (\varphi(t)-\gamma(t))\,dt\right\|$$

$$\leq \left\|\int_a^b \varphi(t)\,dt\right\| + \left\|\int_a^b (\varphi(t)-\gamma(t))\,dt\right\|$$

$$\leq \int_a^b \|\varphi(t)\|\,dt + \varepsilon(b-a)\sqrt{n}$$

$$\leq \int_a^b (\|\varphi(t)-\gamma(t)\| + \|\gamma(t)\|)\,dt + \varepsilon(b-a)\sqrt{n}$$

$$\leq \varepsilon(b-a)\sqrt{n} + \int_a^b \|\gamma(t)\|\,dt + \varepsilon(b-a)\sqrt{n}$$

Hence

$$\left\|\int_a^b \gamma(t)\,dt\right\| \leq \int_a^b \|\gamma(t)\|\,dt + 2\varepsilon(b-a)\sqrt{n} \qquad \forall \varepsilon > 0$$

Hence

$$\left\|\int_a^b \gamma(t)\,dt\right\| \leq \int_a^b \|\gamma(t)\|\,dt$$

□

This lecture took place on 10th of May 2016 with lecturer Wolfgang Ring.

Proof of the formula for the arc length. Its definition depends on the parameterization.

$$s(\gamma) = \sup_z s(z)$$

$$s(z) = \sum_{k=1}^N \|\gamma(t_k) - \gamma(t_{k-1})\|$$

We show:

1. For all decompositions of z, it holds that

$$s(z) \leq \int_a^b \|\dot\gamma(t)\|\,dt$$

$$\Rightarrow s(\gamma) \leq \int_a^b \|\dot\gamma(t)\|\,dt$$

2.

$$\forall \varepsilon > 0 \exists \text{ decomposition } z : s(\gamma) \geq s(z) \geq \int_a^b \|\dot{\gamma}(t)\| \, dt - \varepsilon$$

1. Let $z = \{t_0 < t_1 < \ldots < t_N\}$.

$$s(z) = \sum_{k=1}^{N} \|\gamma(t_k) - \gamma(t_{k-1})\|$$
$$\overset{\substack{\text{fundamental}\\ \text{theorem}}}{=} \sum_{k=1}^{N} \left\| \int_{t_{k-1}}^{t_k} \dot{\gamma}(t) \, dt \right\|$$
$$\overset{\text{Lemma}}{=} \sum_{n=1}^{n} \int_{t_{k-1}}^{t_k} \|\dot{\gamma}(t)\| \, dt$$
$$\sum_{n=1}^{N} TODO$$

2. Let $\varepsilon > 0$ be arbitrary. Find decomposition z such that $s(\gamma) \geq s(z) \geq \int_a^b \|\dot{\gamma}(t)\| \, dt - \varepsilon$. Let

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_n(t) \end{bmatrix}$$

and φ_t is a step function in $[a, b]$.

Every φ_i is constant in (t_{k-1}, t_k) for $k = 1, \ldots, N$. and we let $z = \{t_0, t_1, \ldots, t_N\}$. Let φ_i such that $\|\dot{\gamma}_i - \varphi_i\|_\infty \leq \frac{\varepsilon}{2(b-a)\sqrt{N}}$.

Then it holds that $\forall t \in [a, b]$:

$$\|\dot{\gamma}(t) - \gamma(t)\| = \left(\sum_{i=1}^{n} |\dot{\gamma}_i(t) - \varphi_i(t)| \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^{n} \|\dot{\gamma}_i - \varphi_i\|_\infty^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} \left(\frac{\varepsilon}{2(b-a)}^2 \cdot \frac{1}{n} \right) \right)^[\frac{1}{2}] = \frac{\varepsilon}{2(b-a)}$$

We let

$$\begin{aligned} \|\dot{\gamma} - \varphi\|_\infty &= \sup \{\|\dot{\gamma}(t) - \varphi(t)\|_2 : t \in [a, b]\} \\ &= \max \{\|\dot{\gamma}(t) - \varphi(t)\|_2 : t \in [a, b]\} \end{aligned}$$

It holds that

$$\|\dot{\gamma} - \varphi\|_\infty < \frac{\varepsilon}{2(b-a)}$$

$$z = \{t_0, t_1, \ldots, t_N\}$$

$$\left\| \int_{t_{k-1}}^{t_k} \dot{\gamma}(t) \, dt \right\| = \left\| \int_{t_{k-1}}^{t_k} (\dot{\gamma}(t) - \varphi(t)) \, dt + \int_{t_{k-1}}^{t_k} \varphi(t) \, dt \right\|$$

$$\geq \left\| \int_{t_{k-1}}^{t_k} \varphi(t) \, dt \right\| - \left\| \int_{t_{k-1}}^{t_k} (\dot{\gamma}(t) - \varphi(t)) \, dt \right\|$$

φ is constant and the right summand is $\leq \int_{t_{k-1}}^{t_k} \|\dot{\gamma}(t) - \varphi(t)\| \, dt$.

$$\geq \int_{t_{k-1}}^{t_k} \|\varphi(t)\| \, dt - \int_{t_{k-1}}^{t_k} \underbrace{\|\dot{\gamma}(t) - \varphi(t)\|}_{< \frac{\varepsilon}{2(b-1)}} \, dt$$

$$> \int_{t_{k-1}}^{t_k} \|\varphi(t)\| \, dt - \frac{\varepsilon}{2(b-a)} (t_k - t_{k-1})$$

$$s(z) = \sum_{k=1}^{N} \left\| \int_{t_{k-1}}^{t_k} \dot{\gamma}(t) \, dt \right\| > \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \|\varphi(t)\| - \frac{\varepsilon}{2(b-a)} (t_k - t_{k-1})$$

$$= \int_a^b \|\varphi(t)\| \, dt - \frac{\varepsilon}{2(b-a)} \underbrace{(t_N - t_0)}_{= b-a}$$

$$= \int_a^b \|\varphi(t)\| \, dt - \frac{\varepsilon}{2}$$

$$\int_a^b \|\varphi(t)\| \, dt = \int_a^b \|\varphi(t) - \dot{\gamma}(t) + \dot{\gamma}(t)\| \, dt$$

$$\geq \int_a^b \left(\|\dot\gamma(t)\| - \underbrace{\|\varphi(t) - \dot\gamma(t)\|}_{<\frac{\varepsilon}{2(b-a)}} \right) dt$$

$$= \int_a^b \|\dot\gamma(t)\| \, dt - \frac{\varepsilon}{2}$$

$$\Rightarrow s(z) > \int_a^b \|\dot\gamma(t)\| \, dt - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}$$

$$\Rightarrow s(\gamma) \geq s(z) > \int_a^b \|\dot\gamma(t)\| \, dt - \varepsilon \qquad \forall \varepsilon > 0$$

$$\Rightarrow s(\gamma) \geq \int_a^b \|\dot\gamma(t)\| \, dt$$

□

Example 22 (Circumference of a circle with radius r)**.**

$$\gamma_r(t) = r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \qquad t \in [0, 2\pi]; \dot\gamma_t(t) = r \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$s(\gamma_r) = \int_0^{2\pi} \left\| r \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \right\| dt = \int_0^{2\pi} r \, dt = 2\pi r$$

Example 23 (Ellipsis)**.**

$$\gamma(t) = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix} \qquad a, b > 0$$

$$\dot\gamma(t) = \begin{bmatrix} -a\sin(t) \\ b\cos(t) \end{bmatrix}$$

$$\|\dot\gamma(t)\| = (a^2 \underbrace{\sin^2(t)}_{1-\cos^2(t)} + b^2\cos^2(t))^{\frac{1}{2}}$$

Let $a \geq b$, $\varepsilon^2 = 1 - \frac{b^2}{a^2}$.

$$\begin{aligned} \|\dot\gamma(t)\| &= (a^2 - (a^2 - b^2)\cos^2(t))^{\frac{1}{2}} \\ &= a\left(1 - \left(1 - \frac{b^2}{a^2}\right)\cos^2(t)\right)^2 \\ &= a\left(1 - \varepsilon^2\cos^2(t)\right)^{\frac{1}{2}} \end{aligned}$$

$$s(\gamma) = a \int_0^{2\pi} \sqrt{1 - \varepsilon^2\cos^2(t)} \, dt$$

This defines a new set of functions which cannot be solved with means we discussed so far. They are called *elliptic integral.*

9.1 Change of parameters, reparameterization

Let $\sigma : I \to J$ as smooth (ie. differentiable) as required. σ is bijective and $\sigma^{-1} : J \to I$ is be part of the same differentiation class like σ. Let $\gamma : I \to \mathbb{R}^n$ be a curve. We call $\beta = \gamma \circ \sigma^{-1} : I \to \mathbb{R}^n$ a reparameterization of γ using σ. Compare with Figure 31.

σ is called parameter transformation. γ is called orientation preserving, if σ is strictly monotonically increasing.

A measure, defined by the curve (arc length, tangential vector, curvature, ...) is called *geometric*, if reparameterization can be applied without modifications.

$s(\gamma)$ is obviously a geometric measure, because

1. By definition of polygonal lines
2. Let $\beta(\tau) = \gamma \circ \sigma^{-1}(\tau)$.

$$\dot\beta(\tau) = \dot\gamma(\sigma^{-1}(\tau)) \circ (\sigma^{-1})'(\tau)$$

$$\left\|\dot\beta(\tau)\right\| = \left\|\dot\gamma(\sigma^{-1})(\tau)\right\| \cdot \left|(\sigma^{-1})'(\tau)\right|$$

Case σ is orientation preserving If and only if $\sigma' > 0 \Leftrightarrow (\sigma^{-1})' > 0$

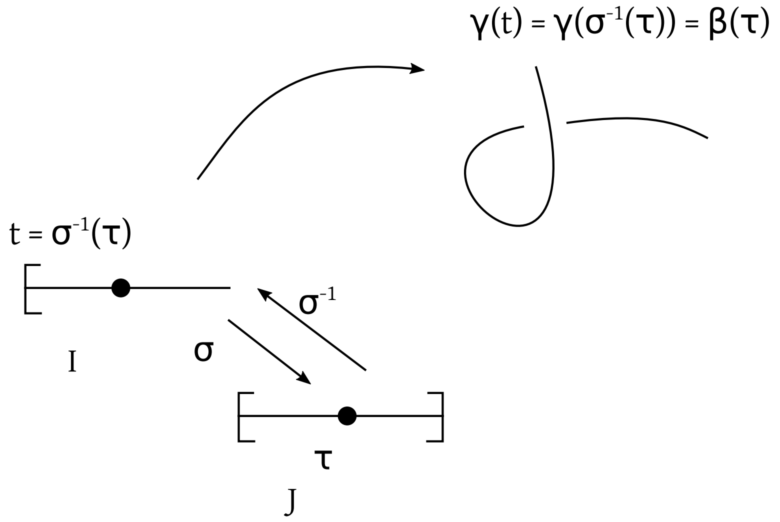


Figure 31: Reparameterization: β and γ have the same trace

Let $I = [a, b]; J = [c, d]; c = \sigma(a); d = \sigma(b)$.

$$s(\beta) = \int_c^d \left\| \dot\gamma(\underbrace{\sigma^{-1}(\tau)}_{=t}) \right\| \cdot \underbrace{(\sigma^{-1})'(\tau)\, d\tau}_{dt}$$

$$= \int_a^b \|\dot\gamma(t)\|\, dt = s(\gamma) \qquad \text{(by substitution)}$$

Case σ is orientation inversing

$$\gamma' < 0 \qquad (\gamma^{-1})' < 0$$

$$\left|(\sigma^{-1})'(\tau)\right| = -(\sigma^{-1})'(\tau) \qquad \sigma(a) = d, \sigma(b) = c$$

$$\int_c^d \left\|\dot\beta(\tau)\right\|\, d\tau = \int_c^d \left\| \dot\gamma(\underbrace{\sigma^{-1}(\tau)}_{t}) \right\| \cdot \underbrace{\left(-(\sigma^{-1})'(\tau)\right)\, d\tau}_{dt}$$

$$\begin{vmatrix} \tau = c \Leftrightarrow t = b \\ \tau = a \Leftrightarrow t = a \end{vmatrix}$$

$$= -\int_b^a \|\dot\gamma(t)\|\, dt$$

$$= \int_a^b \|\dot\gamma(t)\|\, dt = s(\gamma)$$

9.2 Reparameterization by arc length

We consider a regular curve γ, hence $\|\dot\gamma(t)\| > 0 \quad \forall t \in I$ and let $s : I \to J = S(I)$ by

$$s(t) = \int_a^t \|\dot\gamma(\tau)\|\, d\tau$$

$s(t)$ is the length of the curve γ between a and t. Let $s(a) = 0$. It holds that $\dot s(t) = \|\dot\gamma(t)\| > 0$ (by the Fundamental Theorem of Differential and Integration Theory), hence s is strictily monotonically increasing. We use s for reparameterization.

$$\beta(\xi) = \gamma \circ s^{-1}(\xi)$$

is a reparameterization of γ by the arc length.

$$\begin{aligned} \left\|\dot\beta(\xi)\right\| &= \left\|\dot\gamma(s^{-1}(\xi)) \circ (s^{-1})'(\xi)\right\| \\ &= \left\| \dot\gamma(s^{-1}(\xi)) \frac{1}{\dot s(s^{-1}(\xi))} \right\| \\ &= \left\|\dot\gamma(s^{-1})(\xi)\right\| \cdot \left| \frac{1}{\dot s(s^{-1}(\xi))} \right| \\ &= \frac{\left\|\dot\gamma(s^{-1}(\xi))\right\|}{\left\|\dot\gamma(s^{-1}(\xi))\right\|} = 1 \end{aligned}$$

Hence the tangential vector is the unit vector (in every point)

$$s_\beta(\xi) = \int_0^\xi \underbrace{\left\|\dot\beta(\eta)\right\|}_{=1} d\eta = \xi$$

So the curve parameter corresponds to the arc length. On the opposite: Let $\gamma : I \to \mathbb{R}^n$ with property $\|\dot\gamma(t)\| = 1 \quad \forall t \in I = [0, b]$. Then it holds that

$$s(t) = \int_0^t \underbrace{\|\dot\gamma(\tau)\|}_{=1} d\tau = t$$

So it holds that $s = s^{-1} = \mathrm{id}_{[0,b]}$. So γ is parameterized by the arc length.

Remark 33 (Notation)**.** We don't write $\xi = s(t)$, but $s = s(t)$.

Reparameterization by the arc length:

$$\beta(s) = \gamma(s^{-1}(s)) = \gamma(t)$$

This lecture took place on 12th of May 2016 with lecturer Wolfgang Ring.

9.3 Invariance of arc length

Let $\gamma : I \to \mathbb{R}$ be a parametric curve.

$$\sigma : I \to J \text{ orientation-preserving parameter transformation}$$

$$S_\gamma(t) = \int_a^t \|\dot\gamma(\xi)\| \, d\xi$$

$$I = [a, b] \qquad J = [c, d]$$

$$\tilde\gamma = \gamma \circ \sigma^{-1} : J \to \mathbb{R}^n \quad \text{reparameterization}$$

$$S_{\tilde\gamma}(\tau) = \int_C^\tau \left|\dot{\tilde\gamma}(\eta)\right| d\eta$$

We know that $S_{\tilde\gamma}(\tau) = S_{\tilde\gamma}(\sigma(t)) = S_\gamma(t)$.

$$S_{\tilde\gamma} \circ \sigma = S_\gamma$$

Let $S = S_\gamma(t)$ and β is a reparameterization of γ by its arc length. Hence $\beta(s) = \gamma(s_\gamma^{-1}(s))$ and $\beta = \gamma \circ S_\gamma^{-1}$.

$$\tilde\beta(s) = \tilde\gamma \circ S_{\tilde\gamma}^{-1} = \gamma \circ \sigma^{-1} \circ \sigma \circ S_\gamma^{-1} = \gamma \circ S_\gamma^{-1} = \beta(s)$$

Hence, reparametric curves γ and $\tilde\gamma$ have the same reparameterization by its arc length β.

We require orientation preservation (compare with Figure 32).

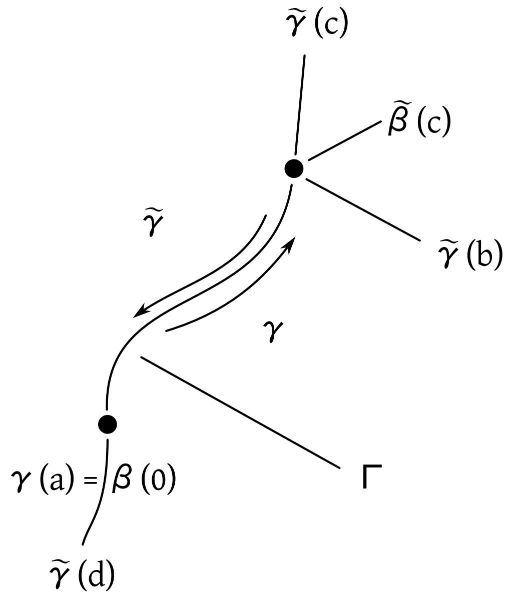


Figure 32: Invariance of arc length

Then it holds that
$$\tilde{\beta}(s) = \beta(s(\gamma) - s)$$

Consider special case $\gamma : I \to \mathbb{R}^2$.

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}; \qquad S_\gamma(t) = \int_a^t \sqrt{\dot{x}(\xi)^2 + \dot{y}(\xi)^2}\, d\xi$$

or even more special:

$$\gamma(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix} \qquad \ldots\text{function graph}$$

$$S_\gamma(t) = S_f(t) = \int_a^t \sqrt{1 + (f'(\xi))^2}\, d\xi$$

9.4 Curvature

Curvature corresponds to the rate of change of the direction of motion. This corresponds to the rate of change of

$$T(t) = \frac{\dot{\gamma}(t)}{|\gamma(t)|}$$

in regards of the arc length.

1. Let $\gamma : I \to \mathbb{R}^2$ be a parameterized regular curve. $\beta(s)$ is the reparameterization of γ by its arc length. $\beta : [0, s(\gamma)] \to \mathbb{R}^2$.

   $$\dot{\beta}(s) = T(s) \quad \text{ is a unit vector}$$

   It holds that $\langle \dot{\beta}(s), \dot{\beta}(s) \rangle = 1$, hence $\dot{\beta}_1^2(s) + \dot{\beta}_2^2(s) = 1$. β can be differentiated twice.

   So we derive $\dot{\beta}_1^2(s) + \dot{\beta}_2^2(s)$:

   $$2\dot{\beta}_1(s) \cdot \ddot{\beta}_1(s) + 2\dot{\beta}_2(s) \cdot \ddot{\beta}_2(s) = 0$$

   So it holds with

   $$\ddot{\beta}(s) = \begin{bmatrix} \ddot{\beta}_1(s) \\ \ddot{\beta}_2(s) \end{bmatrix} \qquad \langle \dot{\beta}(s), \ddot{\beta}(s) \rangle = 0$$

   $\ddot{\beta}$ is orthogonal to $\dot{\beta} = T$. We define $N = \begin{bmatrix} -\dot{\beta}_2 \\ \dot{\beta}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot T$.

Definition 22. We define a *signed* curvature κ in γ in point $\gamma(t) = \beta(s)$ by its relation
$$\frac{d^2\beta}{ds^2} = \ddot{\beta}(s) = \kappa(S) \cdot N(S)$$

κ (with this property) actually exists, because $\ddot{\beta}$ is orthogonal to T and therefore a multiple of N.

In case of reparameterization $\gamma(t) = \tilde{\gamma}(\tau)$, the arc length stays the same. Therefore the curvature in $\gamma(t)$ and $\dot{\gamma}(\tau)$ is also the same. Hence the curvature is invariant in terms of orientation-preserving reparameterization.

This lecture took place on 13th of May 2016 with lecturer Wolfgang Ring.

From now on: We denote the derivative by arc length of a function $f(s)$ using "$f'(s)$".

Again:

$$T'(s) = \kappa(S) \cdot N(S)$$
$$\langle T'(s), N(s) \rangle = \kappa(S) \cdot \underbrace{\langle N(S), N(S) \rangle}_{=1}$$
$$\kappa(S) = \langle T'(s), N(s) \rangle$$

Example 24.
$$\gamma(t) = r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

where r is the radius. It holds that $\|\dot{\gamma}(t)\| = r$.

$$S_\gamma(t) = \int_0^t r\, d\xi = r \cdot t \implies t = \frac{s}{r} \implies \beta(s) = r \begin{bmatrix} \cos \frac{s}{r} \\ \sin \frac{s}{r} \end{bmatrix}$$

$$\beta'(s) = r \cdot \frac{1}{r} \begin{bmatrix} -\sin \frac{s}{r} \\ \cos \frac{s}{r} \end{bmatrix} = \begin{bmatrix} -\sin \frac{s}{r} \\ \cos \frac{s}{r} \end{bmatrix} = T(s)$$

$$T'(s) = \frac{1}{r} \begin{bmatrix} -\cos \frac{s}{r} \\ -\sin \frac{s}{r} \end{bmatrix}$$

$$N(s) = \begin{bmatrix} -\cos(\frac{s}{r}) \\ -\sin(\frac{s}{r}) \end{bmatrix} = D \cdot T(s)$$

$$\kappa(s) = \langle T'(s), N(s) \rangle = \frac{1}{r} \cdot 1 = \frac{1}{r}$$

For an arbitrary curve γ we define the *curvature radius* in point $\gamma(t) = \beta(s)$ as $\rho(s) = \frac{1}{\kappa(s)}$ for $\kappa(s) \neq 0$. $\rho(s) = \infty$ if $\kappa(s) = 0$.

Remark 34. Hence the curvature radius of the circle line is r. $\gamma^-(t)$ goes in counter-clockwise direction along the circumference of the circle. If $\gamma^-(t) = r \cdot \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$, an analogous calculation can be made: $\kappa_{\gamma^-}(s) = -\frac{1}{r}$.

Theorem 35. Let $\gamma : I \to \mathbb{R}^2$ be a regular, twice continuously differentiable curve. Then it holds that

$$\kappa(t) = \kappa_\beta(s) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}}$$

if $\gamma(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$ (function graph) it holds that

$$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

Proof.

$$\beta(s) = \gamma(\underbrace{t(s)}_{s_\gamma^{-1}(s)})$$

$$\beta'(s) = \dot{\gamma}(t(s)) \cdot \underbrace{t'(s)}_{\frac{1}{\dot{s}_\gamma(s_\gamma^{-1}(s))}} = \dot{\gamma}(t(s)) \cdot \frac{1}{\|\dot{\gamma}(t(s))\|} = \frac{1}{\dot{s}_\gamma(t(s))} \cdot \dot{\gamma}(t(s))$$

Let $\frac{1}{\lambda} = -\frac{1}{(\dot{s}_\gamma(t(s)))^2} \cdot \ddot{s}_\gamma(t(s)) \cdot t'(s) \cdot \dot{\gamma}(t(s))$.

$$\beta''(s) = -\frac{1}{(\dot{s}_\gamma(t(s)))^2} \cdot \ddot{s}_\gamma(t(s)) \cdot t'(s) + \frac{1}{s'_\gamma(t(s))} \cdot \ddot{\gamma}(t(s)) \cdot \underbrace{\frac{1}{\dot{s}_\gamma(t(s))}}_{t'(s)}$$

$$\kappa(t) = \kappa(s(t)) = \langle N, T' \rangle = \langle D\beta'(s), \beta''(s) \rangle$$

$$= \left\langle D \cdot \frac{1}{\dot{s}_\gamma} \dot{\gamma}(t(s)), \frac{1}{\dot{s}_\gamma(t(s))^2} \cdot \ddot{\gamma}(t(s)) - \lambda \dot{\gamma}(t(s)) \right\rangle$$

where

$$\left\langle D \cdot \frac{1}{\dot{s}_\gamma} \dot{\gamma}(t(s)), \ddot{\gamma}(t(s)) - \lambda \dot{\gamma}(t(s)) \right\rangle = 0$$

$$= \underbrace{\frac{1}{(\dot{s}_\gamma(t))^3}}_{*} \left\langle \begin{bmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{bmatrix}, \begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix} \right\rangle$$

$$* = \|\dot{\gamma}(t)\| = \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}$$

$$\implies \kappa(t) = \frac{\dot{x}(t)\ddot{(}t) - \dot{\gamma}(t)\dot{x}(t)}{(\dot{x}^2(t) + \dot{\gamma}^2(t))^{\frac{3}{2}}}$$

$\square$

Definition 23. Let $\gamma : I \to \mathbb{R}^2$ be a regular, twice continuously differentiable curve. Let $\rho(t) = \frac{1}{\kappa(t)}$ be the curvature radius in $\gamma(t)$.

Then let

$$\underbrace{m(t)}_{\mathbb{R}^2} = \gamma(t) + \rho(t) \cdot N(t)$$

and denote $m(t)$ as *curvature center* of γ in $\gamma(t)$ (compare with Figure 33).

Remark 35. The error (deviation) from the osculating circle is of order $\mathcal{O}^2$.

Remark 36. The curve of curvature centers $t \to m(t)$ is called *evolute* of γ.

9.5 Situation in $\mathbb{R}^3$

Let $\gamma : I \to \mathbb{R}^3$ be parameterized by its arc length. It is twice continuously differentiable.

$$T(s) = \beta'(s) \qquad \ldots \text{tangential vector}$$

Analogously to $\mathbb{R}^2$ it holds that $\langle T'(s), T(s) \rangle = 0$ (this follows from $\langle T(s), T(s) \rangle = 1$). Hence the unit vector $N(s) := \frac{T'(s)}{\|T'(s)\|}$ is normal to $T(s)$ (only for $T'(s) \neq 0$) and we let $\kappa(s) = \|T'(s)\| \geq 0$. Also it holds that $\kappa(s)N(s) = T'(s)$ and $\langle T'(s), N(s) \rangle = \kappa(s)$.

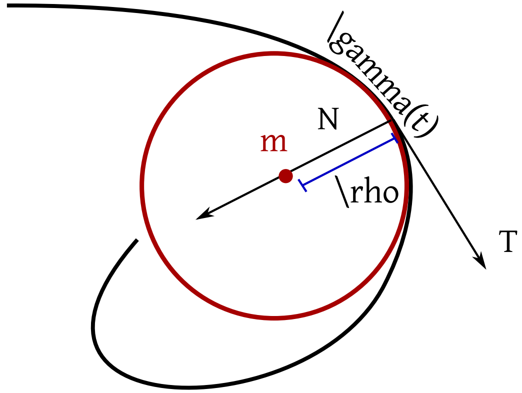


Figure 33: Curvature center m. A circle with center m and radius ρ is called curvature radius or "osculating circle" (dt. "Schmiegkreis") of γ in $\gamma(t)$.

9.6 Analysis in $\mathbb{R}^n$

We look at some important inequalities:

Theorem 36 (Jensen's inequality)**.** Let I be an interval, $f : I \to \mathbb{R}$ is convex.

$$\lambda_1, \lambda_2, \dots, \lambda_n \in [0, 1] \text{ and } \sum_{k=1}^{n} \lambda_k = 1$$

Let $x_1, \dots, x_n \in I$. Then it holds that

$$f(\lambda_1 x_1 + \dots + \lambda_n x_n) \le \lambda_1 f(x_1) + \dots + \lambda_n f(x_n)$$

Proof. By complete induction over n:

Case $n = 1$: Trivial

Case $n = 2$: convexity

Case $n \to n + 1$: left as an exercise to the reader

□

Remark 37. Reminder: f is convex if

$$\forall x_1, x_2 \in I, \lambda \in [0, 1] : f(\lambda x_1 + (1 - \lambda)x_2) \le \lambda f(x_1) + (1 - \lambda)f(x_2)$$

Remark 38. Is f strictly convex, then equality holds only for $x_1 = x_2 = \dots = x_n = x$.

Theorem 37. Inequality between the weighted geometric and the weighted arithmetic mean.

Let $x_1, \dots, x_n$ positive, real numbers. Let $\lambda_i \in [0, 1]$ for $i = 1, \dots, n$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Then it holds that

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \le \lambda_1 x_1 + \dots + \lambda_n x_n$$

Especially for $\lambda_i = \frac{1}{n} \forall i.$:

$$\underbrace{\sqrt[n]{x_1 x_2 \dots x_n}}_{\text{geometric mean}} \le \underbrace{\frac{1}{n}(x_1 + \dots + x_n)}_{\text{arithmetic mean}}$$

Equality in the equality above only holds for $x_1 = x_2 = \dots = x_n = x$.

Proof. Consider $f : (0, \infty) \to \mathbb{R}$ and $x \mapsto \ln(x)$.

$$f''(x) = -\frac{1}{x^2} < 0 \text{ hence } f \text{ is concave}$$

$-f$ is convex. Jensen's inequality for $-f$:

$$-\ln \underbrace{(\lambda_1 x_1 + \dots + \lambda_n x_n)}_{\text{convex combination}} \le -\lambda_1 \ln(x_1) - \dots - \lambda_n \ln(x_n)$$

$$\begin{aligned}\ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) &\geq \lambda_1 \ln x_1 + \ldots + \lambda_n \ln x_n \\ &= \ln x_1^{\lambda_1} + \ln x_2^{\lambda_2} + \ldots + \ln x_n^{\lambda_n} \\ &= \ln(x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n})\end{aligned}$$

Exponential function:

$$\implies \lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} \ldots x_n^{\lambda_n}$$

Equality in Jensen's inequality holds only for $x_1 = x_2 = \ldots = x_n$. □

Definition 24. Let $x \in \mathbb{R}^n$. $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We define

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \geq 1$$

$\|x\|_p$ is called *p-norm of x*. For $p = 2$, $\|x\|_2$ is called *Euclidean norm of x*.

$$\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, n\}$$

is called *maximum norm* or *∞-norm*.

This lecture took place on 19th of May 2016 with lecturer Wolfgang Ring.

Reminder:

Remark 39. Let V be a vector space. Then $\|.\| : V \to [0, \infty)$ is called *norm*, if

- $\|x\| = 0 \Leftrightarrow x = 0$ in V
- $\forall \lambda \in \mathbb{R}$ ($\mathbb{C}$) it holds that $\|\lambda x\| = |\lambda| \, \|x\| \, \forall x \in V$
- $\forall x, y \in V$ it holds that $\|x + y\| \leq \|x\| + \|y\|$

A norm defines defines a distance function (metric) in V with $d(x, y) = \|x - y\|$.

Analogously,

$$f : I \to \mathbb{R} \qquad I = [a, b]$$

$$\|p\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

Let $V \in \mathbb{R}^n$.

$$V : \{1, 2, \ldots, n\} \to \mathbb{R}$$

$$v(k) = v_k$$

$$\int v \underbrace{dz}_{\text{counting measure}} := \sum_{k=1}^n v_k$$

$$\left(\int |V|^p \, dz\right)^{\frac{1}{p}} = \left(\sum_{k=1}^n |V_k|^p\right)^{\frac{1}{p}} = \|v\|_p$$

It holds that $\|.\|_p$ for $p \geq 1$ and $\|.\|_\infty$ are norms in $\mathbb{R}^n$.

1.
$$\|.\|_\infty : \|v\|_\infty = 0 \Rightarrow \max\{|v_k| : k = 1, \ldots, n\}$$
$$\Rightarrow |v_k| = 0 \text{ for } k = 1, \ldots, n \Leftrightarrow v = \vec{0} \in \mathbb{R}^n$$

2.
$$\|\lambda v\|_\infty = \max\left\{\underbrace{|\lambda v_k|}_{|\lambda| \cdot |v_k|} : k = 1, \ldots, n\right\}$$
$$= |\lambda| \cdot \max\{|v_k| : k = 1, \ldots, n\} = |\lambda| \cdot \|v\|_\infty$$

3.
$$\begin{aligned}\|v + w\|_\infty = \max\{|v_k + w_k| : k = 1, \ldots, n\} \\ &\leq \max\{|v_k| + |w_k| : k = 1, \ldots, n\} + \max\{|w_k| : \\ &= \|v\|_\infty + \|w\|_\infty\end{aligned}$$

Hence $\|.\|_\infty$ is norm in $\mathbb{R}^n$. For $\|.\|_p$-norms it holds that $p \geq 1$.

1. Let $\|v\|_p = 0 \Rightarrow (\sum_{k=1}^n |v_k|^p)^{\frac{1}{p}} = 0$.
$$\Leftrightarrow \sum_{k=1}^n \underbrace{|v_k|^p}_{\geq 0} = 0 \Leftrightarrow |v_k|^p = 0 \text{ for } k = 1, \dots, n$$
$$\Leftrightarrow v_k = 0 \text{ for } k = 1, \dots, n \iff v = \vec{0} \text{ in } \mathbb{R}^n$$
2. Let $\lambda \in \mathbb{R}$.
$$\|\lambda \cdot v\|_p = \left(\sum_{k=1}^n |\lambda v_k|^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{k=1}^n |\lambda^p| \cdot |v_k|^p\right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{k=1}^n |v_k|^p\right)$$
$$= |\lambda| \left(\sum_{k=1}^n |v_k|^p\right)^{\frac{1}{p}} = |\lambda| \cdot \|v\|_p$$
3. Consider $p = 1$.
$$\|v + w\|_1 = \left(\sum_{k=1}^n |v_k + w_k|^1\right)^{\frac{1}{1}} \leq \left(\sum_{k=1}^n (|v_k| + |w_k|)\right) = \|v\|_1 + \|w\|_1$$
We want to consider $p > 1$. But first we need to talk about Hölder's inequality in $\mathbb{R}^n$.

Theorem 38 (Hölder's inequality)**.** Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $v, w \in \mathbb{R}^n$. Then it holds that
$$\sum_{k=1}^n |v_k \cdot w_k| \leq \|v\|_p \cdot \|w\|_q$$

Proof. $v = \vec{0}$ holds, analogously $w = \vec{0}$ holds.

Hence let $\|v\|_p \neq 0$ and $\|w\|_q \neq 0$. Inequality of geometrical and arithmetical mean for $\lambda_1 = \frac{1}{p}$, $\lambda_2 = \frac{1}{q}$ and
$$x_1 = \frac{|v_k|^p}{\|v\|_p^p} \qquad x_2 = \frac{|w_k|^q}{\|w\|_q^q}$$
$$\frac{|v_k|}{\|v\|_p} \cdot \frac{|w_k|}{\|w\|_q} \leq \frac{1}{p} \frac{|v_k|^p}{\|v\|_p^p} + \frac{1}{q} \frac{|w_k|^q}{\|w\|_q^q}$$
Sum over $k = 1, \dots, n$.
$$\frac{1}{\|v\|_p \cdot \|w\|_q} \sum_{k=1}^n |v_k \cdot w_k| \leq \frac{1}{p} \frac{\sum_{k=1}^n |v_k|^p}{\|v\|_p^p} + \frac{1}{q} \frac{\sum_{k=1}^n |w_k|^q}{\|w\|_q^q}$$
$$= \frac{1}{p} + \frac{1}{q} = 1 \implies \sum_{k=1}^n |v_k \cdot w_k| \leq \|v\|_p \cdot \|w\|_q$$
□

Theorem 39 (Minkovski inequality)**.** Let $p > 1$.
$$\forall v, w \in \mathbb{R} : \|v + w\|_p \leq \|v\|_p + \|w\|_p$$

Proof. Let $s_k = |v_k + w_k|^{\overbrace{p-1}^{>0}}$. We chose q such that $\frac{1}{p} + \frac{1}{q} = 1$.
$$\left(\frac{1}{q} = \frac{1-p}{p}; q = \frac{p}{p-1}\right)$$
$$|v_k + w_k|^p = |v_k + w_k| \cdot |s_k| \overset{\text{triangle ineq. for 1st expr}}{\leq} |v_k| \cdot |s_k| + |w_k| \cdot |s_k|$$
$$\underbrace{\sum_{k=1}^n |v_k + w_k|^p}_{=\|v+w\|_p^p} \leq \sum_{k=1}^n |v_k \cdot s_k| + \sum_{k=1}^n |w_k \cdot s_k|$$
$$\overset{\text{Hölder's ineq}}{\leq} \|v\|_p \cdot \|s\|_q + \|w\|_p \cdot \|s\|_q$$
with $s = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$.
$$|s_k|^q = |v_k + w_k|^{(p-1) \cdot q} = |v_k + w_k|^p$$

$$\|s_k\|_q = \left(\sum_{k=1}^{n} |v_k + w_k|^p\right)^{\frac{1}{q}\cdot\frac{p}{p}} = \|v+w\|_p^{\frac{p}{q}}$$

It holds that $\frac{p}{q} = p - 1$.

$$= \|v+w\|_p^{p-1}$$

Insert into $\|v\|_p \cdot \|s\|_q + \|w\|_p \cdot \|s\|_q$:

$$\|v+w\|_p \leq \|p\| \, \|v+w\|_p^{p-1} + \|w\|_p \, \|v+w\|_p^{n-1}$$

$$\Rightarrow \|v+w\|_p \leq \|v\|_p + \|w\|_p$$

□

This lecture took place on 20th of May 2016 with lecturer Wolfgang Ring.

Remark 40 (Notation)**.**

$$v \in \mathbb{R}^n \qquad v = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}$$

Coordinate index is at top.

Definition 25. Let V be a vector space and $\|.\|_1$ and $\|.\|_2$ are two norms in V. We call $\|.\|_1$ and $\|.\|_2$ equivalent if $\exists m, M > 0$ such that for all $v \in \mathbb{V}$ the relation.

$$m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1 \, \textbf{??} \tag{3}$$

Remark: Let $N = \{\|.\| \mid \|.\| \text{ is norm in } V\}$. Then Equation **??** defines an equivalence relation in N.

1. Reflexivity
2. Let $m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1$ Then it holds that $\|v\|_1 \geq \frac{1}{M}\|v\|_2$ because $\|v_2\| \leq M\,\|v\|_1$. Furthermore $\|v\|_1 \leq \frac{1}{m}\|v\|_2$ because $m\,\|v\|_1 \leq \|v\|_2$. Hence we get symmetry.
3. Let
   (a) $m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1$
   (b) and $m'\,\|v\|_2 \leq \|v\|_3$
   (c) and $\|v\|_3 \leq M'\,\|v\|_2$.

$$\Rightarrow m \cdot m'\,\|v\|_1 \overbrace{\leq}^{(1)} m'\,\|v\|_2 \overbrace{\leq}^{(2)} \|v\|_3 \overbrace{\leq}^{(3)} M'$$
$$\leq M' \cdot M\,\|v\|_1$$

Transitivity

10 Topological terms in normed vector spaces

(Convergence, continuity, open set, compactness, etc.)

Let $(V, \|.\|)$ be a normed vector space.

Definition 26 (Convergence)**.** Let $x_n \in V$ for $n \in \mathbb{N}$ and $x \in V$. We state $(x_n)_{n\in\mathbb{N}}$ converges to x in regards of $\|.\|$, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow \|x_n - x\| < \varepsilon]$$

Notation: $\lim_{n\to\infty} x_n = x$ i.e. $x_n \to x$ for $n \to \infty$ (we replace norm by vertical bars for absolute value).
Remark: We explicitly tell for which norm the expression converges. If no norm is explicitly mentioned, we mean the only norm we talk about (denoted $x_n \stackrel{\|.\|}{\to} x$).

Remark 41. Let $(\xi)_{n\in\mathbb{N}}$ and $\xi_n \in \mathbb{N}_+$ with $\xi_n = \|x_n - x\|$. Then it holds that $x_n \stackrel{\|.\|}{\to} x$ for $x \to \infty$ in $V \Leftrightarrow \xi_n \to 0$ for $n \to \infty$ in $\mathbb{N}$.

Because $\xi_n \to 0$ in $\mathbb{R}$,

$$\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow \left| \underbrace{\|x_n - x\|}_{\xi_n \text{ with } |\xi_n| = \xi} \right| < \varepsilon]$$

Definition 27 (Cauchy sequence in V)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V. We claim that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon]$$

Definition 28. $(V, \|.\|)$ is called *complete* if every Cauchy sequence in V is a convergent sequence in V, hence has a limit. A complete, normed vector space is called *Banach space.*

Stefan Banach (1892–1945)

Remark 42. Every convergent sequence is also a Cauchy sequence. Proof as in $\mathbb{R}$ (relevant for exam).

Definition 29. Let $f : D \leq V \to W$, V and W are normed vector spaces. Let $x_0 \in D$. f is called continuous in x_0, if

$$\forall\varepsilon > 0 \exists\delta > 0 : \forall x \in D : \|x - x_0\|_V < \delta \Rightarrow \|f(x) - f(x_0)\|_w < \varepsilon$$

Continuity is another concept, which depends in the norms used in V and W.

Remark 43. Let V and W be normed vector spaces, $f : D \subseteq V \to W$, $\overline{x} \in D$. Then f is continuous in $\overline{x}$

$$\Leftrightarrow [\forall (x_n)_{n\in\mathbb{N}} \text{ with } x_n \in D \wedge \lim_{n\to\infty} x_n = \overline{x} \text{ in terms of } \|.\|_v$$

$$\implies \lim_{n\to\infty} f(x_n) = f(\overline{x}) \text{ in terms of } \|.\|_W]$$

Proof. Proof as in $\mathbb{R}$ (but not relevant for exam). □

Definition 30. Let $f : D \subseteq V \to W$ and V, W are normed vector spaces. Then f is called *continuous in* D if f is continuous in every point $\overline{x} \in D$.

Definition 31. Let V be a normed vector space.

1. Let $x \in V$ and $r \geq 0$. We let $B(x,r) := \{y \in V \mid \|y - x\| < r\}$. $B(x,r)$ is a ball (in terms of $\|.\|$) with center x and radius r. $B(x,r) := \{y \in V \mid \|y - x\| \leq r\}$ is a closed ball with center x and radius r.
2. $O \subseteq V$ is called open if $\forall x \in O \exists r > 0 : B(x,r) \subseteq O$. $A \subseteq V$ is called *closed* (dt. abgeschlossen), if $V \setminus A$ is open in V.

Remark 44. $B(x,r)$ is open for $r \geq 0$. $\overline{B(x,r)}$ is closed for $r \geq 0$.

$$B(x,O) = \emptyset \qquad \overline{B(x,O)} = \{x\}$$

Proof. Left to the reader as an exercise. □

Lemma 15. Let $(V, \|.\|)$ be a normed vector space. Then it holds that,

1. φ is open in V, V is open in V
2. Let O_i with $i \in I$ be a family of open sets. Then the union
$$\bigcup_{i\in I} O_i = \{x \in V \mid \exists i \in I : x \in O_i\}$$
is open.
3. Let $O_1, O_2, \ldots, O_N$ be a set of open sets. Then $\bigcap_{k=1}^N O_k$ is open.

Proof. 1. It holds that
$$\forall x \in \varphi \exists r > 0 : B(x,r) \subseteq \varphi$$
$$\forall x \in V : \exists r > 0 : B(x,r) \subseteq V$$
For example for $r = 1$.

2. Let $O = \bigcup_{i\in I} O_i$, O_i is open and let $x \in O$. Hence $\exists i \in I : x \in O_i$, because O_i is open, there exists some $r > 0$ such that
$$B(x,r) \subseteq O_i \implies B(x,r) \subseteq O_i \subseteq \bigcup_{i\in I} O_i$$

3. Let $O = \bigcap_{k=1}^N O_k$ and $x \in O$. Hence $\forall k \in \{1, \ldots, N\} : x \in O_n$ and therefore exist $r_k > 0$ such that $B(x, r_k) \subseteq O_k$ (because every O_k is open). Choose $r = \min\left\{\underbrace{r_1}_{>0}, \underbrace{r_2}_{>0}, \ldots, \underbrace{r_N}_{>0}\right\} > 0$. Then it holds that
$$B(x,r) \subseteq B(x,r_k) \subseteq O_k \text{ for } k = 1, \ldots, N \implies B(x,r) \subseteq \bigcap_{k=1}^N O_k = O$$

□

Definition 32. A system of sets $\tau \subseteq \mathcal{P}(V)$ (where $\mathcal{P}(V)$ denotes the power set of V) which satisfy properties 1, 2 and 3 is called *topology in* V.

Lemma 16. Let $(V, \|.\|)$ be a normed vector space. Then it holds that

4. If φ is closed, then V is closed.
5. If I is an index set, $(A_i)_{i\in I}$ is a family of closed sets. Then it holds that $\bigcap_{i\in I} A_i = \{x \in V : \forall i \in I \,|\, x \in A_i\}$ is closed in V.
6. Let $A_1, A_2, \ldots, A_N$ be closed. Then the union $\bigcup_{k=1}^N A_k$ is closed in V.

Proof. Follows by generation of complement generation and DeMorgan's laws. □

Definition 33 (Limit point, Contact point, inner point)**.** Let $M \subseteq V$ where V is a normed vector space.

- $x \in M$ is called *inner point* of M if $r > 0$ exists, such that $B(x, r) \subseteq M$.
- $x \in V$ is called *contact point* M if $\forall r > 0 : B(x, r) \cap M \neq \varphi$.
- $x \in V$ is called *limit point* of M if $\forall r > 0 : (B(x, r) \setminus \{x\}) \cap M \neq 0$ $((B(x, r) \setminus \{x\})$ is also called *pointed ball* $\dot{B}(x, r))$.

Remark 45. $O \subseteq V$ is open iff $\forall x \in O : x$ is inner point of O. $A \subseteq V$ is closed iff $\forall x \in V$ where x is contact point of A, $x \in A$ holds.

Proof. Let A be closed and x is contact point of A. Assume $x \notin A$, hence $x \in V \setminus A$ is open. Because $V \setminus A$ is open, there exists $r > 0$:

$$B(x, r) \subseteq V \setminus A \implies B(x, r) \cap A = 0$$

is a contradiction to x is a contact point of A. On the opposite, let every contact point of A be element of A. Then it holds that $\forall x \notin A$ (hence $x \in V \setminus A$) that x is not a contact point of A.

$$\implies \exists r > 0 : B(x, r) \cap A = \varphi$$
$$\implies B(x, r) \subseteq V \setminus A$$

Hence $V \setminus A$ is open and therefore A is closed. □

Definition 34. Let $M \subseteq V$. Then $\mathring{M} = \{x \in M \,|\, x \text{ is inner point of } M\}$. $\mathring{M}$ is called *open kernel* of M. $\overline{M} = \{x \in V \,|\, x \in \text{ contact point}\}$ $\overline{M}$ is closed cover of M.

Lemma 17. $\forall M \subseteq V$ it holds that

- $\mathring{M}$ is open, $\mathring{M} \subseteq M$
- $\overline{M}$ is closed, $M \subseteq \overline{M}$
- M is open iff $M = \mathring{M}$
- M is closed iff $M = \overline{M}$

Proof. $\mathring{M} \subseteq M$. Let $x \in \mathring{M}$.

Show that there exists $r > 0$ such that $B(x, r) \subseteq \mathring{M}$. Choose $r > 0$ such that $B(x, r) \subseteq M$.

Show that $\forall y \in B(x, r)$ it holds that $y \in \mathring{M}$.

$$y \in B(x, r) \implies \|y - x\| < r$$

Choose $r_y = r - \|y - x\| > 0$. We show

$$B(y, r_y) \subseteq B(x, r) \subseteq M$$

hence $y \in \mathring{M}$.

Let $z \in B(y, r_y)$, hence $\|z - y\| < r_y$. It follows that

$$\|z - y\| = \|z - y + y - x\| \leq \|z - y\| + \|y - x\| < r_y + \|y - x\| = r - \|y - x\| + \|y - x\| = r$$

Therefore $\|z - x\| < r$, hence $z \in B(x, r)$. So it holds that $B(y, r_y) \subseteq B(x, r) \subseteq M$, so $y \in \mathring{M}$. □

This lecture took place on 24th of May 2016 with lecturer Wolfgang Ring.

Lemma 18. Let $M \subseteq V$. Then it holds that

1. $\mathring{M}$ is open, if $\mathring{M} \subseteq M$.
2. $\overline{M}$ is closed, if $M \subseteq \overline{M}$.

3. M is open, iff $M = \mathring{M}$.
4. M is closed, iff $M = \overline{M}$.

Proof. 1. Trivial.

2. Direction $\Rightarrow$: M is open, so every $x \in M$ is an inner point of M, so $\forall x \in M : x \in \mathring{M}$.

   Direction $\Leftarrow$: $M = \mathring{M}$ is open, so M is open.

3. Let $x \in M$. For all $r > 0$, $B(x, r) \cap M \supseteq \{x\} \neq \emptyset$, so $x \in \overline{M}$. It remains to show: $\overline{M}$ is closed. We show $V \setminus \overline{M}$ is open. Let $y \notin \overline{M}$. $\exists r > 0$ such that $B(y, r) \cap M = \emptyset$. Let $x \in B(y, r)$. So $\rho \coloneqq r - |z - y| > 0$ and

$$\forall w \in B(z, \rho) : |w - y| \leq |w - z| + |z - y| < \rho + |z - y| = r$$

$$\implies \forall w \in B(z, \rho) : w \in B(y, r) \implies B(z, \rho) \subset B(y, r).$$

$$B(z, \rho) \cap M \subseteq B(y, r) \cap M = \emptyset \text{ because } B(y, r) \cap M = \emptyset$$

$$\implies z \notin \overline{M}.$$

   Hence $B(y, r) \subset V \setminus \overline{M}$ and therefore $V \setminus \overline{M}$ is open.

4. Direction $\Leftarrow$: $M = \overline{M}$. Due to the second property, M must be closed.

   Direction $\Rightarrow$: It remains to show: $\overline{M} \subset M$. Let $x \in \overline{M}$. Assume $x \notin M$. $x \notin M \Rightarrow x \in V \setminus M$ is open.

$$\implies \exists r > 0 : B(x, r) \subset V \setminus M$$

$$\implies B(x, r) \cap M \subset V \setminus M \cap M = \emptyset$$

$$\implies x \notin \overline{M}$$

   Is a contradiction. Hence $x \in M$.

□

Lemma 19. Let $x \in V$.

1. x is a contact point of M ($x \in \overline{M}$)

$$\iff \exists (x_n)_{n \in \mathbb{N}} \subseteq M : \lim_{n \to \infty} x_n = x$$

2. x is a limit point of M

$$\iff \exists (x_n)_{n \in \mathbb{N}} \subseteq M \setminus \{x\} : \lim_{n \to \infty} x_n = x$$

Proof. We show the first property, but the second follows analogously.

Direction $\Rightarrow$: Let x be a contact point in M. We know $B\left(x, \frac{1}{n}\right) \cap M \neq \emptyset$ for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N} \exists x_n \in M : \|x_n - x\| < \frac{1}{n} \wedge \lim_{n \to \infty} x_n = x$$

Analogously for the second property:

Let x be a limit point in M. We know $B\left(x, \frac{1}{n}\right) \setminus \{x\} \cap M \neq \emptyset$ for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N} \exists x_n \in M \setminus \{x\} : \|x_n - x\| < \frac{1}{n}$$

And $\lim_{n \to \infty} x_n = x$.

Direction $\Leftarrow$: Let $x = \lim_{n \to \infty} x_n$ with $x_n \in M$. (analogously: $x_n \in M \setminus \{x\}$). Let $r > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \geq N : \|x_n - x\| < r$.

$$x_N \in B(x, r) \implies x_N \in B(x, r) \cap M$$

Analogously:

$$x_N \in B(x, r) \implies x_N \in B(x, r) \setminus \{x\} \cap M$$

Hence x is contact point (analogously: limit point) of M. □

Example 25. Consider unit circle and point $(2, 0)$ in $V \coloneqq \mathbb{R}^2$.

$$M \coloneqq \overline{B(0, 1)} \cup \{(2, 0)\}$$

$(2, 0)$ is a contact point of M, but not limit point of M.

Corollary 5. Let M be closed and $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in M. Then $x \coloneqq \lim_{n \to \infty} x_n \in M$.

Proof. From Lemma 19 it follows that $x \in \overline{M} \implies x \in M$. □

Lemma 20. Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms in V and $(x_n)_{n\in\mathbb{N}} \subset V$, $x \in V$. $m\|v\|_1 \le \|v\|_2 \le M\|v\|_1$.

$$\implies \left(\lim_{n\to\infty} \|x_n - x\|_1 = 0 \implies x_n \to x \text{ in } (V, \|.\|_1)\right)$$

$$\iff \left(\lim_{x_n - x} \|x_n - x\|_2 = 0 \implies x_n \to x \text{ in } (V, \|.\|_2)\right)$$

Proof. We only show direction $\Rightarrow$, because the other direction follows analogously.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \ge N : \|x_n - x\|_1 < \frac{\varepsilon}{M}$. Let $n \ge N$, then $\|x_n - x\|_2 \le M\|x_n - x\|_1 < \varepsilon$. So $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. □

Lemma 21. Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms in V and $O \subseteq V$. Then O is open in $(V, \|.\|_1)$ if and only if O is open in $(V, \|.\|_2)$.

Proof. Let O be open in $(V, \|.\|_1)$. Let $x \in O$.

$$\exists r > 0 : B_{\|.\|_1}(x, r) \subseteq O$$

Consider $B_{\|.\|_2}(x, r\cdot m)$. Let $y \in B_{\|.\|_2}(x, r \cdot m)$. Then

$$\|y - x\|_1 \le \frac{1}{m}\|y - x\|_2 < \frac{1}{m} r \cdot m = r$$

So $y \in B_{\|.\|_1}(x, r)$. Hence $B_{\|.\|_2}(x, r\cdot m) \subset B_{\|.\|_1}(x, r) \subseteq O$ where $r \cdot m > 0$. So O is open in $(V, \|.\|_2)$. □

Remark 46. $A \subset V$ is closed in $(V, \|.\|_1)$ if and only if A is closed in $(V, \|.\|_2)$.

Lemma 22. Let V, W be normed vector spaces. $f : D \subset V \to W$, $\overline{x} \in D$. Let $\|.\|_{1,v}$ and $\|.\|_{2,v}$ be equivalent norms with $\|.\|_{1,W}$ and $\|.\|_{2,W}$ in V and W respectively.

Then it holds that

$$f : D \subset (V, \|.\|_{1,V}) \to (W, \|.\|_{1,W}) \text{ is continuous in } \overline{x}$$
$$\iff f : D \subset (V, \|.\|_{2,V}) \to (W, \|.\|_{2,W}) \text{ is continuous in } \overline{x}$$

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset V$ with $x_n \xrightarrow[x\to\infty]{\|.\|_{2,V}} \overline{x}$, hence $\lim_{n\to\infty} \|x_n - \overline{x}\|_{2,V} = 0$. Show: $\lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{2,W} = 0$, hence $f(x_n) \xrightarrow[n\to\infty]{\|.\|_{2,W}} f(\overline{x})$. It holds that

$$\lim_{n\to\infty} \|x_n - \overline{x}\|_{1,V} = 0 \overset{\text{VS}}{\implies} \lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{1,W} = 0$$

$$\Rightarrow \lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{2,W} = 0$$

Hence, f is continuous in terms of $\|.\|_{2,V}$ and $\|.\|_{2,W}$ in $\overline{x}$ by the sequence criterion. □

Lemma 23. Consider $V := \mathbb{R}^n$. Let $1 \le i \le n$. Then the projective map

$$p_i : (\mathbb{R}^n, \|.\|_\infty) \to (\mathbb{R}, |.|)$$

$$p_i(x) = x^i \qquad \text{denotes the } i\text{-th component}$$

continuous in $\mathbb{R}^n$. Furthermore $|p_i(x)| = |x^i| \le \|x\|_\infty$.

Proof. Let $\varepsilon > 0$, $x \in \mathbb{R}^n$. Let $\delta := \varepsilon$. Let $z \in \mathbb{R}^n$ with $\|z - x\|_\infty < \delta = \varepsilon$. Then

$$|p_i(z) - p_i(x)| = |z^i - x^i| \le \max\left\{|z^i - x^i| \,\middle|\, i \in \{1, \dots, n\}\right\} = \|z - x\|_\infty < \varepsilon$$

So p_i is continuous in x. □

Lemma 24. Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{\hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$. Then it holds that

$$\left(\lim_{n\to\infty} \|x_n - x\|_\infty = 0 \implies x_n \to x \text{ in } (\mathbb{R}^{\hat{n}}, \|.\|_\infty)\right) \iff$$

$$\left(\forall i \in \{1, \dots, \hat{n}\} : \lim_{n\to\infty} |x_n^i - x^i| = 0 \implies \forall i \in \{1, \dots, \hat{n}\} : x_n^i \to x^i \text{ in } (\mathbb{R}, |\cdot|)\right)$$

Proof. Direction $\Rightarrow$: Trivial because of continuity of all p_i.

Direction $\Leftarrow$: Let $\varepsilon > 0$. Choose $N_i \in \mathbb{N}$ with $\forall n \ge N_i : |x_n^i - x^i| < \varepsilon$. Let $N := \max\{N_i \,|\, 1 \le i \le \hat{n}\}$. For $n \ge N$ it holds that

$$\|x_n - x\|_\infty = \max\left\{|x_n^i - x^i| \,\middle|\, 1 \le i \le \hat{n}\right\} < \varepsilon$$

because $n \ge N_i$ for all $1 \le i \le \hat{n}$. Hence $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$. □

Definition 35. Let $K \subseteq V$ where V is a normed vector space. Then K is called *sequence-compact* if and only if every sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, which converges and whose limit x is in K.

$M \subset V$ is called *bounded* iff $\exists R > 0 \forall x \in M : \|x\| \leq R$.

Lemma 25. Let V be a normed vector space. $K \subseteq V$. Then it holds that

$$K \text{ is compact} \implies K \text{ is closed and bounded.}$$

Proof. Let K be compact. Assume K is unbounded, then

$$\forall n \in \mathbb{N} \exists x_n \in K : \|x_n\| > n$$

$(x_n)_n$ has no bounded subsequence and especially no convergent subsequence. Hence K is not compact. This is a contradiction. So K is bounded.

It remains to show: $\overline{K} \subseteq K$. Let $x \in \overline{K}$. $\exists (x_n)_n \subseteq K : x_n \to x$. K is compact, hence

$$\exists (x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}, \overline{x} \in K : x_{n_k} \overset{k\to\infty}{\longrightarrow} \overline{x}$$

$$\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x \implies x = \overline{x} \in K$$

□

This lecture took place on 31st of May 2016 with lecturer Wolfgang Ring.

Theorem 40. Let $\mathbb{K} \subseteq \mathbb{R}^n$ $(n \in \mathbb{N})$ and we consider $\|.\|_\infty$ in $\mathbb{R}^n$. Let $\mathbb{K}$ be bounded and closed (in terms of $\|.\|_\infty$). Then $\mathbb{K}$ is compact.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathbb{K}$ $(x_n \in \mathbb{K} \forall n \in \mathbb{N})$. Construct convergent subsequences with limit in K.

$$x^j = p_j(x) \qquad (j\text{-th coordinate of } x)$$

Because $(x_n)_{n\in\mathbb{N}}$ is bounded (in terms of $\|.\|_\infty$), it holds that

$$\left|x_n^j\right| \leq \|x_n\|_\infty$$

Hence $(x_n^j)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ for $j = 1, \dots, n$. Consider $(x_n^1)_{n\in\mathbb{N}}$ bounded in $\mathbb{R}$. There exists a convergent subsequence $x_{n_{i_1}}^1 \to \xi^1$ where i_1 is the subsequence index. Consider $(x_{n_{i_1}}^2)_{i_1\in\mathbb{N}}$ which is subsequence of $(x_n^2)_{n\in\mathbb{N}}$ bounded. Hence $(x_{n_{i_1}}^2)_{i_1\in\mathbb{N}}$ is also bounded.

$$n \mapsto x_n$$

$$i_1 \mapsto n_{i_1}$$

Bolzano-Weierstrass implies that there exists a subsequence $(x_{n_{i_{1_{i_2}}}}^2)_{i_2\in\mathbb{N}}$ convergent with $\lim_{i_2\to\infty} x_{n_{i_{1_{i_2}}}}^2 = \xi_2$ for $(x_{n_{i_1}}^1)_{i_1\in\mathbb{N}}$ convergent towards ξ^2 it holds that:

$$\implies \lim_{i_2\to\infty} x_{n_{i_{1_{i_2}}}}^1 = \xi$$

We continue this construction. Assume $(x_{n_{i_{1_{i_{2_{i_k}}}}}}^j)$ convergent towards ξ^j for $j = 1, \dots, k$. Sequence $(x_{n_{i_1 \dots i_k}}^{k+1})_{i_n\in\mathbb{N}}$ bounded in $\mathbb{R}$. Bolzano-Weierstrass implies that there exists $(x_{n_{i_{1_{i_2 \dots i_{k+1}}}}}^{k+1})_{i_{k+1}\in\mathbb{N}}$ convergent to ξ^{k+1}.

Definition of subsequences do not change convergent.

$$\implies \forall j \in \{1, \dots, k\} : \lim_{i_{k+1}\to\infty} x_{n_{i_1 \dots i_{n+1}}}^j = \xi^j$$

until $k = n$ is reached. Then $(x_{n_{i_1 \dots i_n}})_{i_n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ for every coordinate sequence $(x_{n_{i_1 \dots i_n}}^j)_{i_n\in\mathbb{N}}$ converges. Hence it holds that

$$\lim_{i_n\to\infty} x_{n_{i_1 \dots i_n}} = \xi = \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^n \end{bmatrix}$$

in terms of $\|.\|_\infty$. Because $\mathbb{K}$ is bounded, it holds that $\xi \in \mathbb{K}$. □

Lemma 26. Let $\mathbb{K} \subseteq (\mathbb{R}^n, \|.\|_\infty)$ be compact, let $f : \mathbb{K} \to \mathbb{R}$ be continuous (in terms of $\|.\|_\infty$). Then f has a maximum as well as minimum in $\mathbb{K}$.

So $\exists x_{\min} \in \mathbb{K}$ such that $f(x_{\min}) \leq f(x)$ and $\exists x_{\max} \in \mathbb{K} : [\forall x \in \mathbb{K} : f(x) \leq f(x_{\max})] \forall x \in \mathbb{K}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a maximum sequence. So for $\eta = \sup\{f(x) : x \in \mathbb{K}\}$ it holds that $\eta = \lim_{n\to\infty} f(x_n)$. $(x_n)_{n\to\infty}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = \xi \in \mathbb{K}$. Because of continuity of f it holds that

$$\eta = \lim_{k\to\infty} f(x_{n_k}) = f(\xi)$$

Hence it holds that

$$f(\xi) = \eta = \sup\{f(x) : x \in \mathbb{K}\}$$

Hence $x_{\max} = \xi$ is maximum point. □

Theorem 41. In $\mathbb{R}^n$ all norms are equivalent.

Proof. Let $\|.\|$ be a norm in $\mathbb{R}^n$. We show $\|.\|$ is equivalent to $\|.\|_\infty$. We use three steps:

1. $\exists M > 0 : \|v\| \leq M\|v\|_\infty$ for every $v \in \mathbb{R}^n$. Let

$$v = \sum_{i=1}^n v^i e_i \qquad v^i \in \mathbb{R} \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 is at the i-th coordinate. Then it holds that

$$\|v\| = \left\|\sum_{i=1}^n v^i e_i\right\| \leq \sum_{i=1}^n \|v^i \cdot e_i\|$$

$$= \sum_{i=1}^n |v^i| \cdot \|e_i\| \leq \underbrace{\max\{|v^j| : j = 1, \dots, n\}}_{=\|v\|_\infty} \cdot \underbrace{\sum_{i=1}^n \|e_i\|}_{=M} = M \cdot \|v\|_\infty$$

2. We consider $f : (\mathbb{R}^n, \|.\|_\infty) \to \mathbb{R}$ with $f(v) = \|v\|$. We show f is continuous. Let $\varepsilon > 0$ be arbitrary, $v \in \mathbb{R}^n$ is arbitrary and $\delta = \frac{\varepsilon}{M}$. Then it holds that

$$\|v - w\|_\infty < \delta = \frac{\varepsilon}{M} : \|v - w\| \leq M\|v - w\|_\infty < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Hence f is continuous.

3. We consider $S_\infty^{n-1} = \{v \in \mathbb{R}^n : \|v\|_\infty = 1\}$. S_∞^{n-1} is bounded in terms of $\|.\|_\infty$. S_∞^{n-1} is closed in terms of $\|.\|_\infty$.

Let $(V_n)_{n\in\mathbb{N}}$ is a convergent (in terms of $\|.\|_\infty$) sequence in S_∞^{n-1} with limit v. Then because $\|v_n - v\|_\infty \to 0$ holds for $n \to \infty$,

$$\underbrace{|\|v_n\|_\infty - \|v\|_\infty|}_{\to 0} \leq \|v_n - v\|_\infty \to 0$$

Hence $\|v\|_\infty = \lim_{n\to\infty} \underbrace{\|v_n\|_\infty}_{1} = 1$.

So $v \in S_\infty^{n-1}$. So S_∞^{n-1} is closed. So S_∞^{n-1} is compact in terms of $\|.\|_\infty$

$$\left.\begin{matrix} f : S_\infty^{n-1} \to \mathbb{R} \\ f(v) = \|v\| \end{matrix}\right\} \text{ takes minimum } m \text{ in } S_\infty^{n-1}$$

Hence,

$$\forall v \in S_\infty^{n-1} : \|v\| \geq \underbrace{\|v_{\min}\|}_{\in S_\infty^{n-1}} = m$$

Because $\|v_{\min}\|_\infty = 1$ $(\neq 0) \Rightarrow v_{\min} \neq \vec{0}$. Hence $\|v_{\min}\| = m \neq 0$ (so > 0). Therefore $m > 0$.

Let $w \in \mathbb{R}^n \setminus \{\vec{0}\}$. Then $v = \frac{w}{\|w\|_\infty} \in S_\infty^{n-1}$ and it holds that

$$\|v\| = \left\|\frac{w}{\|w\|_\infty}\right\| = \frac{1}{\|w\|_\infty}\|w\| \geq m$$

So $\|w\| \geq m \cdot \|w\|_\infty$. For $w = 0$, $\|0\| \geq m \cdot \|0\|_\infty$ holds trivially.

Hence there exists m, $M > 0$ such that $\forall v \in \mathbb{R}^n$:

$$m\|v\|_\infty \leq \|v\| \leq M\|v\|_\infty$$

So $\|,\|$ and $\|.\|_\infty$ are equivalent.

Followingly transitivity of normequivalence shows that any two norms in $\mathbb{R}^n$ are equivalent.

This implies that all topological terms like convergence, continuity, open and closed sets, contact points, compactness, et cetera are independent of the norm choice in $\mathbb{R}^n$.

Example 26. Let $f : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$.

$$f(\xi) = \begin{bmatrix} f^1(\xi) \\ f^1(\xi) \\ \vdots \\ f^m(\xi) \end{bmatrix}$$

Let $f^k : D \to \mathbb{R}$ for $k = 1, \dots, m$. Let $\|.\|$ and $\|.\|_\infty$ be arbitrary norms in $\mathbb{R}^n$ (or equivalently $\mathbb{R}^m$). Then f is continuous in x in terms of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ if and only if f^k is continuous in x in terms of $\|.\|_{\mathbb{R}^n}$ and $|.|$ in the image set $\mathbb{R}$.

Proof. f is continuous in terms of $\|.\|_{\mathbb{R}^m}$ in the image set if and only if f is continuous in terms of $\|.\|_\infty$ in $\mathbb{R}^m$ if and only if $\overbrace{f(x_n) \to f(x)}^{\text{in terms of } \|.\|_\infty}$ for $n \to \infty$ if $x_n \to x$ in terms of $\|.\|_{\mathbb{R}^n}$ if and only if $\left|f^k(x_n) - f^k(x)\right| \to 0$ for $n \to \infty$ and $k = 1, \dots, m$ if $x_n \to x$ in terms of $\|.\|_{\mathbb{R}^n}$ if and only if f^k is continuous in x for $k = 1, \dots, m$. □

Lemma 27. $x_n \to x$ in $(\mathbb{R}^n, \|.\|_{\mathbb{R}^n}) \iff x_n^j \to x^j$ in $\mathbb{R}$ for $j = 1, \dots, n$.

Proof. This statement holds for $\|.\|_{\mathbb{R}^n} = \|.\|_\infty$ and therefore for any norm. □

11 Differential calculus in $\mathbb{R}^n$

Definition 36 (Landau's o-symbol)**.** With $o(x)$ we denote a function of $\vec{o} \in D \to \mathbb{R}$ if D is open and $o(x) = q(x) \cdot \|x\|$ and $\lim_{x \to \vec{0}} q(x) = 0$, $q(\vec{0}) = 0$ with $q : D \to \mathbb{R}$. Hence q is continuous in $\vec{0}$ with $q(\vec{o}) = 0$.

Definition 37. Let $D \subseteq \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^m$. Let $x_0 \in D$. We call f Fréchet-differentiable in x_0 if $A : \mathbb{R}^n \to \mathbb{R}^m$. A exists linearly such that

$$\|f(x) - f(x_0) - A(x - x_0)\| = o(x - x_0)$$

(similar to Taylor polynomial of degree 1)

□ **Remark 47.**

$$A = \lim_{x \to \infty} \frac{f(x) - f(x_0)}{x - x_0}$$

does not work, because vectors cannot be divided.

Lemma 28. The linear map A is (if it exists) by differentiability uniquely defined. In the following case we denote

$$Df(x_0) \coloneqq A$$

and $Df(x_0)$ is called *Frechet derivative* of f in x_0.

Proof. Assume A and $\tilde{A}$ both satisfy the differentiability condition. Then for arbitrary $x \in D$ it holds that

$$\left\|(A - \tilde{A})(x - x_0)\right\| = \left\|f(x) - f(x_0) - \tilde{A}(x - x_0) - (f(x) - f(x_0) - A(x - x_0))\right\|$$

$$\le \underbrace{\left\|f(x) - f(x_0) - \tilde{A}(x - x_0)\right\|}_{o(x - x_0)} + \underbrace{\|f(x) - f(x_0) - A(x - x_0)\|}_{o(x - x_0)} = o(x - x_0)$$

Hence

$$\left\|(A \cdot \tilde{A})(x - x_0)\right\| = q(x - x_0)\,\|x - x_0\|$$

with

$$\lim_{x \to x_0} q(x - x_0) = 0$$

$$\left\|(A - \tilde{A})\right\| \cdot \frac{x - x_0}{\|x - x_0\|} = q(x - x_0)$$

Let $v \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$ with $\|V\| = 1$ be arbitrary. Let $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq D$ (D is open) and choose $x = x_0 + \frac{\varepsilon}{2} V \in B(x_0, \varepsilon) \subseteq D$. Then it holds that $x - x_0 = \frac{\varepsilon}{2} \cdot v$ and $\|x - x_0\| = \frac{\varepsilon}{2} \cdot 1$. So it holds that $\frac{x - x_0}{\|x - x_0\|} = \frac{x - x_0}{\frac{\varepsilon}{2}} = v$ and $\left|(A - \tilde{A})v\right| = q(x - x_0)$. for $x \to x_0$. The left-hand side is independent of $x - x_0$ and the right side converges to 0. So $\left\|(A - \tilde{A})v\right\| = 0$. Let $w \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$.

Let $v = \frac{w}{\|w\|}$. Then it holds that $\left\|(A-\tilde{A})w\right\| = \|w\| \cdot \underbrace{\left\|(A-\tilde{A})\cdot v\right\|}_{=0}$. Hence $(A-\tilde{A})w = 0 \forall w \in \mathbb{R}^n$

$$\implies Aw = \tilde{A}w \forall w \in \mathbb{R}^n$$

□

This lecture took place on 2nd of June 2016 with lecturer Wolfgang Ring.

Remark 48. Because of norm equivalence in $\mathbb{R}^n$, differentiability and the form of the derivative $Df(x_0)$ is independent of the chosen norm.

Let D be an open set. Let D be the definition set of f. We consider only inner points as definition points of the derivative.

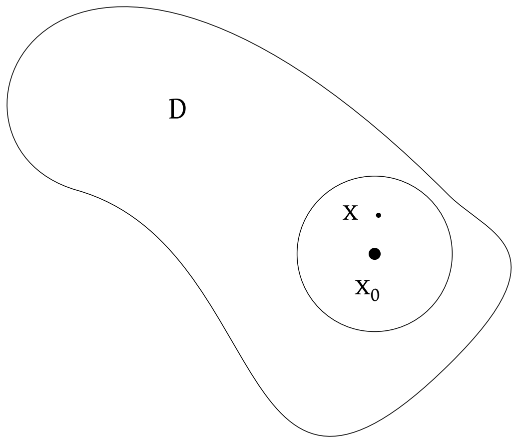


Figure 34: Open set D

Lemma 29. Let $f, g : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$ open. Let f and g be differentiable in x_0. Then for $\lambda \in \mathbb{R}$, $\lambda \cdot f$ is also differentiable in x_0. and $f+g$ is differentiable in x_0 with

$$D(\lambda f)(x_0) = \lambda Df(x_0)$$

$$D(f+g)(x_0) = Df(x_0) + Dy(x_0)$$

Hence the derivative operator D applies linearly on the function.

Proof.

$$\|\lambda f(x) - \lambda f(x_0) - \lambda Df(x_0)\cdot(x-x_0)\|_{\mathbb{K}^m} = |\lambda| \cdot \|f(x) - f(x_0) - Df(x_0)\cdot(x-x_0)\|_{\mathbb{R}^m} = \underbrace{|\lambda| \cdot}_{\lim_{x\to}}$$

$$\leq \underbrace{\|f(x) - f(x_0) - Df(x_0)(x-x_0)\|_{\mathbb{R}^m}}_{o(x-x_0)} + \underbrace{\|g(x) - g(x_0) - Dg(x_0)(x-x_0)\|_{\mathbb{R}^m}}_{o(x-x_0)} = o(x-x_0)$$

□

Remark 49. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. We identify A with the matrix representation of A in regards of canonical bases in $\mathbb{R}^m$ (or $\mathbb{R}^n$).

Definition 38 (Matrix norms)**.** Let X and Y be vector spaces. Then $\mathcal{L}(X,Y) = \operatorname{Hom}(X,Y) = \{A \mid A : X \to Y \text{ is linear}\}$ is also a vector space. A norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{m\times n}$ is called *matrix norm.*

Example 27. Let $A = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ be an $m \times n$ matrix. We let $\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n a_{i,j}^2\right)^{\frac{1}{2}}$. $\|.\|_F$ is called Forbeniusnorm in A. $\|.\|_F$ is a matrix norm.

Definition 39. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear and let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ be chosen. Then we define

$$\|A\| = \sup_{V \neq \vec{0}} \left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m}$$

and we call $\|A\|$ the *operator norm* of A in regards of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$.

Lemma 30. Let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ be chosen norms. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then it holds that

1. $h(w) = \|w\|_{\mathbb{R}^m}$ is a continuous map
$$h : (\mathbb{R}^m, \|.\|_{\mathbb{R}^m}) \to \mathbb{R}^+$$
2.
$$\|A\| = \sup_{V\neq 0} \left\| A \cdot \frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m}$$
$$\max_{\substack{x\in\mathbb{R}^n \\ \|x\|_{\mathbb{R}^n}=1}} \|A \cdot x\|_{\mathbb{R}^m}$$

Proof. 1. Let $\varepsilon > 0$ be arbitrary.
$$|h(w_1) - h(w_2)| = |\|w_1\|_{\mathbb{R}^m} - \|w_2\|_{\mathbb{R}^m}|$$
$$\stackrel{\text{inv. triangle ineq.}}{\leq} \|w_1 - w_2\|_{\mathbb{R}^m} < \varepsilon \text{ if } \|w_1 - w_2\| < \varepsilon$$

2. Let $x \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}} = \{x \in \mathbb{R}^n \mid \|x\|_{\mathbb{R}^n} = 1\}$. Then it holds that
$$\|Ax\|_{\mathbb{R}^n} = \left\| A\frac{x}{\|x\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m} \leq \sup_{\substack{V\in\mathbb{R}^n \\ V\neq 0}} \left\| A\frac{V}{\|V\|_{\mathbb{R}^n}} \right\| = \|A\|$$
So it also holds
$$\sup_{x\in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}} \|Ax\| \leq \|A\|$$
On the opposite, let $v \in \mathbb{R}^n$ with $v \neq 0$ arbitrary. Then it holds that $z = \frac{V}{\|V\|_{\mathbb{R}^n}} \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}$ and
$$\left\| A\frac{V}{\|V\|_{\mathbb{R}^n}} \right\| = \|Az\|_{\mathbb{R}^m} \leq \sup_{\|x\|_{\mathbb{R}^n}=1} \|Ax\|_{\mathbb{R}^m}$$
$$\implies \|A\| = \sup_{\substack{V\neq 0 \\ V\in\mathbb{R}^n}} \left\| A \cdot \frac{V}{\|V\|_{\mathbb{R}^n}} \right\| \leq \sup_{\|x\|_{\mathbb{R}^n}=1} \|Ax\|_{\mathbb{R}^m}$$
So it holds that
$$\sup_{\substack{V\neq 0 \\ V\in\mathbb{R}^n}} \left\| A\frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m} = \sup_{\substack{x\in\mathbb{R}^n \\ \|x\|_{\mathbb{R}^n}=1}} \|Ax\|_{\mathbb{R}^m}$$

It remains to show: right-sided sup is maximum.
We consider: $f : S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}} \to \mathbb{R}$.
$$f(x) = \|Ax\|_{\mathbb{R}^m} = h \circ A(x)$$
Linear maps $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ are continuous, h is continuous. Hence f is continuous and takes the maximum value in compact set $S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}$.

□

Remark 50. Equivalent definition: Recall: $\sup_2 \{M\}$.
$M \subseteq \mathbb{R}$ is the smallest upper bound of M.
Hence $\|A\| = \inf\left\{\tilde{m} \geq 0 : \left\| A\frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m} \leq \tilde{m} \quad \forall v \in \mathbb{R}^n \setminus \{0\}\right\}$
$$= \inf \left\{ \tilde{m} \geq 0 : \underbrace{\|Av\|_{\mathbb{R}^m} \leq \tilde{m}\,\|v\|_{\mathbb{R}^n}}_{*} \right\}$$
$*$ holds anyways for $v = 0$.
So $\|A\|$ is the smallest constant $\tilde{m}$ such that
$$\|Av\|_{\mathbb{R}^m} \leq \tilde{m}\,\|v\|_{\mathbb{R}^n}$$
holds for all $v \in \mathbb{R}^n$. Especially it holds that
$$\boxed{\|Av\|_{\mathbb{R}^n} \leq \|A\| \cdot \|v\|_{\mathbb{R}^n}}$$
(This only works for operator norms. A very important result.)

This lecture took place on 3rd of June 2016 with lecturer Wolfgang Ring.

Remark 51 (Equivalent characterization of $\|A\|$)**.**
$$\|A\| = \max_{\substack{x\in\mathbb{R}^n \\ \|x\|=1 \\ \mathbb{R}^n}} \|Ax\|_{\mathbb{R}^m} = \max_{\substack{v\in\mathbb{R}^n \\ v\neq\vec{0}}} \frac{1}{\|v\|_{\mathbb{R}^n}} \cdot \|Av\|_{\mathbb{R}^m}$$
$$= \min\{\tilde{m} : \|Av\|_{\mathbb{R}^m} \leq \tilde{m}\,\|v\|_{\mathbb{R}^n}\ \forall v \in \mathbb{R}^n\}$$

Lemma 31. Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then A is continuous in $\mathbb{R}^n$.

Proof. Consider linear $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that A is continuous if and only if component A^j is continuous as map from $\mathbb{R}^n \to \mathbb{R}$ (for $i = 1, \dots, m$).

$$A^i(x) = \sum_{j=1}^{n} a_{ij} x_j$$

Choose $\|.\|_2$ in $\mathbb{R}^n$ and $|.|$ in $\mathbb{R}$. Show: Component A^i is continuous for these norms. Let $x, y \in \mathbb{R}^n$. Then it holds that

$$\left|A^i x - A^i y\right| = \left|A^i(x-y)\right| = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$$

$$= \langle a^i, x-y \rangle \overset{\text{Cauchy-Schwarz}}{\leq} \left\|a^i\right\|_2 \cdot \|x-y\|_2$$

So A^i is Lipschitz continuous and therefore continuous. □

Corollary 6 (Conclusion of estimate $\|Ax\| \leq \|A\| \, \|x\|$)**.** The function $A : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous with Lipschitz constant $\|A\|$.

Obvious, because $\|A(x-y)\| \leq \|A\| \cdot \|x-y\|$.

Remark 52. These law only hold for operator norms. This estimate also defines *boundedness*. Bounded equals linear for operator norms.

Lemma 32. Let $\|A\|$ be an operator norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then $\|A\|$ is a norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. 1.

$$\begin{aligned}
\|A\| = 0 &\iff \max\left\{\frac{1}{\|v\|}\|Av\| : v \in \mathbb{R}^n, v \neq 0\right\} = 0 \\
&\iff \frac{1}{\|v\|}\|Av\| = 0 \forall v \in \mathbb{R}^n, v \neq \vec{0} \\
&\iff \|Av\| = 0 \forall v \neq \vec{0} \iff A = \underbrace{0}_{\text{zero matrix}}
\end{aligned}$$

2.

$$\begin{aligned}
\|\lambda A\| &= \max\{\|\lambda Ax\| : \|x\| = 1\} \\
&= |\lambda| \cdot \max \|Ax\| : \|x\| = 1 = |\lambda| \cdot \|A\|
\end{aligned}$$

3. Let $x \in \mathbb{R}^n$ with $\|x\| = 1$ arbitrary. $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then it holds that

$$\|(A+B)x\| \overset{\text{triangle ineq. in } \mathbb{R}^m}{\leq} \|Ax\| + \|Bx\| \leq \|A\| \underbrace{\|x\|}_{=1} + \|B\| \underbrace{\|x\|}_{=1}$$

$$= \|A\| + \|B\|$$

$$\implies \forall x \in \mathbb{R}^n : \|(A+B)x\| \leq \|A\| + \|B\|$$

$$\implies \underbrace{\max\{\|(A+B)x\| : \|x\| = 1\}}_{\|A+B\|} \leq \|A\| + \|B\|$$

□

Lemma 33. Let $A = \{\mathbb{R}^n, \mathbb{R}^m\}$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Choose fixed norms in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^l$. Let $\|A\|$ be the operator norm of A in regards of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$. Let $\|B\|$ be the operator norm of B in regards of $\|.\|_{\mathbb{R}^n}$ (like for $\|A\|$!) and $\|.\|_{\mathbb{R}^l}$. Then it holds that

$$\|BA\| \leq \|B\| \, \|A\|$$

Proof.

$$\|BA\| = \max\left\{\underbrace{\|BAx\|_{\mathbb{R}^l}}_{\leq \|B\| \cdot \|Ax\|} : \|x\| = 1\right\}$$

$$\leq \|B\| \cdot \underbrace{\max\{\|Ax\| : \|x\| = 1\}}_{=\|A\|} = \|B\| \cdot \|A\|$$

□

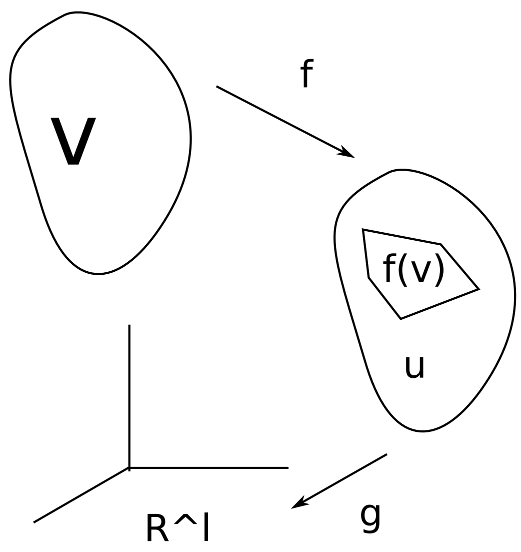


Figure 35: Chain rule in multiple dimensions

11.1 Back to differential calculus in $\mathbb{R}^n$

Theorem 42 (Chain rule in multiple dimensions)**.** Let $f : O \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Let $U \subseteq \mathbb{R}^m$ be open and $g : U \to \mathbb{R}^l$ and $f(O) \subseteq U$.

Let f in $x_0 \in O$ differentiable and y in $y_0 = f(x_0) \in U$ differentiable. Then $g \circ f$ in x_0 is differentiable and it holds that

$$D(g \circ f)(x_0) = Dg(y_0) \cdot Df(x_0) = \underbrace{Dg(f(x_0))}_{\in \mathbb{R}^{l \times m}} \cdot \underbrace{Df(x_0)}_{\in \mathbb{R}^{m \times n}}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Show that

$$\frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot Df(x_0)(x - x_0)\| = r(x - x_0) < \varepsilon$$

for sufficiently small $\|x - x_0\|$.

$$\begin{aligned}
&\frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) \mp Dg(f(x_0)) \cdot (f(x) - f(x_0)) - Dg(f(x_0)) \cdot Df(x_0)(x - x_0)\| \\
&\leq \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0))\| \\
&+ \frac{1}{\|x - x_0\|} \|Dg(f(x_0)) \cdot (f(x) - f(x_0) - Df(x_0)(x - x_0))\| \\
&\leq \frac{1}{\|x - x_0\|} \left\| \overbrace{g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0))}^{\text{small, because } g \text{ is differentiable in } f(x_0)} \right\| \\
&+ \frac{1}{\|x - x_0\|} \|Dg(f(x_0))\| \cdot \underbrace{\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\|}_{\text{small, because } f \text{ is differentiable in } x_0}
\end{aligned}$$

Remember those expressions as (#).

First, choose δ_1 such that

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \leq 1 \cdot \|x - x_0\|$$

for all $\|x - x_0\| < \delta_1$, $x \in O$. Possible, because f is differentiable in x_0.

$$\implies \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1 + \|Df(x_0)\|$$

Choose δ_g such that $\|y - y_0\| < \delta_g$ (with $f(x_0) = y_0$) such that

$$\|g(y) - g(y_0) - Dg(y_0)(y - y_0)\| \leq \frac{\varepsilon}{2} \frac{1}{\|Df(x_0)\| + 1} \|y - y_0\|$$

is possible, because g is differentiable in y_0. The inequality above also holds for $y = y_0$. Let δ_2 such that for all $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_g$. This is possible, because f is continuous in x_0 (and f differentiable in $x_0 \implies f$ is continuous in x_0).

Now let $\|x - x_0\| < \min\{S_1, S_2\}$. Then it holds that

$$\frac{1}{\|x - x_0\|} \cdot \left\| g(f(x)) - g(f(x_0)) - Dg(y_0) \left(\underbrace{f(x) - f(x_0)}_{\|f(x) - f(x_0)\| < \delta g} \right) \right\|$$

$$\leq \frac{\varepsilon}{2} \frac{1}{\|x-x_0\|} \frac{1}{\|Df(x_0)\|+1} \underbrace{\|f(x)-f(x_0)\|}_{\leq(\|Df(x_0)\|+1)\|x-x_0\|} < \frac{\varepsilon}{2}$$

The expression in underbrace holds by choice of δ_1. Now let $\delta_3 > 0$ such that

$$\|x-x_0\| < \delta_3 \implies \frac{1}{\|x-x_0\|} \|f(x)-f(x_0)-Df(x_0)(x-x_0)\|$$

$$< \frac{1}{\|Dg(f(x_0))+1\|} \frac{\varepsilon}{2}$$

is possible, because f is differentiable in x_0. So it holds that $\|x-x_0\| < \delta_3$.

$$\frac{1}{\|x-x_0\|} \|Dg(y_0)\| \cdot \|f(x)-f(x_0)-Df(x_0)(x-x_0)\|$$

$$< \|Dg(y_0)\| \cdot \frac{\varepsilon}{2} \cdot \frac{1}{\|Dg(y_0)\|+1} < \frac{\varepsilon}{2}$$

For $\|x-x_0\| < \min \delta_1, \delta_2, \delta_3$ every expression from $(\#)$ is smaller than $\frac{\varepsilon}{2}$. The desired inequality was proven. □

Lemma 34. Let $f : D \subseteq \mathbb{R}$ where D is open and $x_0 \in D$. If f is differentiable in x_0, then f is also continuous in x_0.

Proof. Let $\varepsilon > 0$ be arbitrary.

$$\|f(x)-f(x_0)\| \leq \overbrace{\|f(x)-f(x_0)-Df(x_0)(x-x_0)\|}^{r(x-x_0)\|x-x_0\|} + \|Df(x_0)(x-x_0)\|$$

$$\leq r(x-x_0)\|x-x_0\| + \|Df(x_0)\| \cdot \|x-x_0\|$$

Choose $\delta > 0$ such that

1. $\delta < \frac{\varepsilon}{2} (\|Df(x_0)\|+1)^{-1} \leq \frac{\varepsilon}{2}$
2. for $\|x-x_0\| < \delta \implies r(x-x_0) \leq 1$

Let $\|x-x_0\| < \delta$. Then it holds that

$$\|f(x)-f(x_0)\| \leq 1 \cdot \frac{\varepsilon}{2} + \|Df(x_0)\| \cdot \frac{\varepsilon}{2} \frac{1}{\|Df(x_0)\|+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence f is continous in x_0. □

Definition 40. Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ a function. We state: f is *differentiable in D*, if f is differentiable in every point $x_0 \in D$. The map

$$x \mapsto Df(x)$$

$$D \to \mathbb{R}^{m \times n}$$

is called *derivative function of f*.

We say f is *continuously differentiable in D* if the derivative function $x \mapsto Df(x)$ is continuous in terms of $\|.\|_{\mathbb{R}^n}$ in $\mathbb{R}^n$ and in terms of the operator norm in $\mathbb{R}^{m \times n}$.

Hence,

$$\forall \varepsilon > 0 \exists \delta > 0 : [\|x-x_0\| < \delta \text{ and } x \in D \implies \|Df(x)-Df(x_0)\| < \varepsilon]$$

Remark 53. f is continuously differentiable in $\overline{D}$, if every point of $\overline{D}$ is a limit point (dt. "Häufungspunkt") in D and in every point $x_0 \in \overline{D}$ the differentiability condition

$$\|f(x)-f(x_0)-Df(x_0)(x-x_0)\| = o(x-x_0)$$

holds and $x \mapsto Df(x)$ is a continuous function on $\overline{D}$.

12 Computing $Df(x_0)$

$$\lim_{v \to 0} \frac{f(x_0+v)-f(x_0)}{v} =?$$

Definition 41. Let $f : D \to \mathbb{R}^m$ be given. Let $D \subseteq \mathbb{R}^n$ be open. Let $x_0 \in D$ and $v \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$. We define

$$df(x_0; v) = \lim_{t \to 0} \frac{1}{t} (f(x_0+tv)-f(x_0))$$

if the limit exists.

In this case we call $df(x_0; v)$ the directional derivative (*Gâteaux derivative*) of f in direction v in point x_0. Compare with Figure 36.

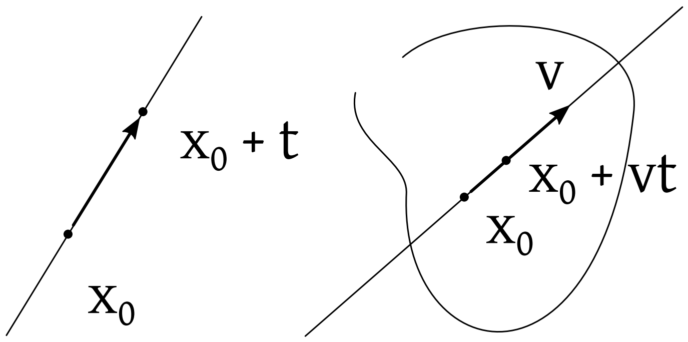


Figure 36: Directional derivative and Gateaux derivative

Remark 54. Let $l_{x_0,v}(t) = x_0 + t \cdot v$ (parameter form of a straight line).

$$l_{x_0,v} : \mathbb{R} \to \mathbb{R}^n$$

is linear affine (constant and linear). $l_{x_0,v}$ is differentiable in $\mathbb{R}$. Furthermore it holds (by chain rule) that $f(x_0 + t \cdot v) = f \circ l_{x_0,v}(t)$ defines an environment of $t = o$.

$$df(x_0; v) = D(f \circ l_{x_0,v})(0)$$
$$Dl_{x_0;v}(0) = v \in \mathbb{R}^n \qquad \text{column vector}$$

If f is Frechét differentiable in x_0, then it holds (by the chain rule)

$$df(x_0; v) = Df(l_{x_0,v}(0)) \cdot Dl_{x_0,v}(0) = Df(x_0) \cdot v$$

Lemma 35. Let f be like above and Frechét differentiable in x_0. Then it holds that

$$df(x_0; v) = Df(x_0) \cdot v$$

Now I can build the columns of Df.

This lecture took place on 7th of June 2017 with lecturer Wolfgang Ring.

12.1 Special directional derivatives

Remark 55 (Reminder)**.** We write indices for coordinates as superscript (notation originates in differential geometry). But in the following we also which to subscript, which is more convenient under some circumstances.

Definition 42. Let $v = e_k$ where 1 is at the k-th index; i.e. k-th canonical basis vector in $\mathbb{R}^n$.

$$df(x_0, e_l) = \lim_{t \to 0} \frac{1}{t} \left(f(x_0 + t \cdot e_k) - f(x_0) \right) \text{ with } \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \lim_{t \to 0} \frac{1}{t} \left(f(x_0^1, x_0^2, \ldots, x_0^k + t, x_0^{k-1}, \ldots, x_0^n) - f(x_0^1, \ldots, x_0^n) \right)$$
$$= \frac{\partial f}{\partial x^k} = f_{x^k} = f_k = \partial_k f$$

is called *k-th partial derivative of f in x_0*.

12.2 Determination of partial derivatives

In the k-th derivative, $x^1, \ldots, x^{k-1}, x^{k+1}, \ldots, x^n$ are treated like constants. Derivative rules are applied to the "variable" x^k.

An example:

$$f(x^1, x^2, x^3) = \begin{bmatrix} x^1(x^3)^2 + \sin(x^2 x^3) \\ \frac{(x^2)^2}{(x^1)^2+1} \end{bmatrix}$$

where x^2 is the derivative variable. Then:

$$\partial_2 f(x_0^1, x_0^2, x_0^3) = \begin{bmatrix} \cos(x^2 \cdot x^3) \cdot x^3 \\ \frac{2x^2}{(x^1)^2+1} \end{bmatrix}$$

Theorem 43. Let $f : D \to \mathbb{R}^m$, D is open in $x_0 \in D$ differentiable. Then it holds that

$$Df(x_0) = \left(\partial_j f^i(x_0)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,n}} \in \mathbb{R}^{m\times n}$$

$$\left(\partial_j f^i(x_0)\right)_{\substack{i=1,\dots,m\\ j=1,\dots,m}}$$

is called *Jacobi matrix* of f in x_0.

Proof. The matrix representation of the linear map $Df(x_0)$ in regards of canonical bases in $\mathbb{R}^n$ (or equivalently $\mathbb{R}^m$) contains the vector $Df(x_0)\cdot e_k = df(x_0; e_k)$ in the k-th column.

$$df(x_0, e_k) = \partial_k f = \begin{bmatrix} \partial_k f^1 \\ \partial_k f^2 \\ \vdots \\ \partial_k f^m \end{bmatrix} \implies (Df(x_0))_{i,k} = \partial_k f^i(x_0)$$

□

Example 28.

$$f : \mathbb{R}^3 \to \mathbb{R}^2 : f(\underbrace{x_1, x_2, x_3}_{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}) = \begin{bmatrix} x_1 x_3^2 + \sin(x_2 \cdot x_3) \\ \frac{x_2^2}{x_1^2+1} \end{bmatrix}$$

$$Df\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3^2 & \cos(x_2x_3)\cdot x_3 & 2x_2x_3^2 + \cos(x_2x_3)\cdot x_2 \\ -\frac{x_2^2}{(x_1^2+1)}\cdot 2x_1 & \frac{2x_2}{x_1^2+1}\cdot 2x_1 & 0 \end{bmatrix}$$

Theorem 44. Let $D \subset \mathbb{R}^n \to \mathbb{R}^m$ and D is open.

$$f(x) = \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix}$$

Then if f is differentiable in $x_0 \in D$, then $f^j : D \to \mathbb{R}$ in x_0 is differentiable for $j = 1, \dots, m$.

Proof. f is differentiable in x_0 iff

$$\forall \varepsilon > 0 \exists \delta > 0 : \|x - x_0\| < \delta \implies \|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_\infty < \varepsilon \|x - x_0\|$$

$$\iff \forall \varepsilon > 0 \exists \delta > 0 : \|x - x_0\| < \delta \implies \left| f^0(x) - f^0(x_0) - \sum_{k=1}^m \partial_k f^j(x_0)(x^k - x_0^k) \right| < \varepsilon \|x - x_0\|$$

$\iff$ f^j is Fréchet differentiable in x_0 with derivative

$$Df^j(x_0) = [\partial_1 f^1, \partial_2 f^2, \dots, \partial_n f^j](x_0)$$

Additionally,

$$Df^j(x_0) = [\partial_1 f^j(x_0), \dots, \partial_n f^j(x_0)]$$

□

Remark 56. For many considerations we can look at the special case $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

Notation: Let $f : D \subseteq \mathbb{R}^m \to \mathbb{R}$ be given and differentiable in $x_0 \in D$.

$$\begin{aligned} Df(x_0) &= [\partial_1 f(x_0), \dots, \partial_n f(x_0)] \\ &= df(x_0) \end{aligned}$$

is also called *total differential of f in x_0* (a row vector).

$$Df(x_0)\cdot v = df(x_0; v) = [\partial_1 f(x_0), \dots, \partial_n f(x_0)] \begin{bmatrix} v_1 \\ \vdots \\ v^n \end{bmatrix}$$

We let

$$\nabla f(x_0) = \begin{bmatrix} \partial_1 f(x_0) \\ \vdots \\ \partial_n f(x_0) \end{bmatrix} = (Df(x_0))^t$$

is called *gradient vector of f in x_0* (∇ of the Old Greek alphabet is called *nabla*).

With this notation it holds that the directional derivative

$$Df(x_0)\cdot v = \langle \nabla f(x_0), v \rangle$$

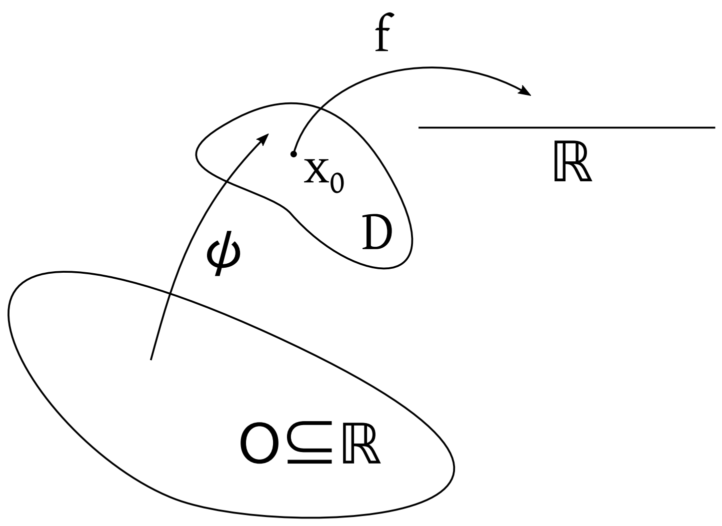


Figure 37: Coordinate transformation

The reason for different notations is coordinate transformation.

This is illustrated in Figure 37 with $f : O \to D$ is differentiable everywhere and bijective.

$$f : 0 \to D$$

is differentiable everywhere and bijective.

$$x = \varphi(y) \qquad x_0 = \varphi(y_0) \qquad \tilde{f} = f \circ \varphi : O \to \mathbb{R}$$

Coordinate transformation of f.

$$D\tilde{f}(y_0) \underbrace{=}_{\text{chain rule}} Df(x_0) \cdot \underbrace{D\varphi(y_0)}_{\text{transformation by } D\varphi}$$

$$\begin{aligned} D\tilde{f}(y_0) &= (Df(x_0) \cdot D\varphi(y_0))^t \\ &= D\varphi^t(y_0) \cdot Df(x)^t \\ &= D\varphi^t(y_0) \cdot \nabla f(x_0) \end{aligned}$$

Transformation behavior of Df and ∇f is different. Df is a covariant vector (covector) and ∇f is a contravariant vector (vector).

Question: If all partial derivatives of a function exist, is f differentiable?
Answer: In general no.

Example 29 (Counterexample)**.**

$$f(x_1, x_2) = \begin{cases} x_1 \cdot \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 0 & \text{for } (x_1, x_2) = (0,0) \end{cases}$$

$\partial_1 f$ and $\partial_2 f$ exist for $(x_1, x_2) \neq (0,0)$. Consider directional derivatives of f in point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$df(\vec{0}, v) = \lim_{t \to 0} \frac{f(t \cdot v) - 0}{t} = \lim_{t \to 0} \frac{1}{t} \left(tv_1 \frac{tv_1 \cdot tv_2}{(tv_1)^2 + (tv_2)^2} \right) = v_1 \frac{v_1 v_2}{v_1^2 + v_2^2}$$

Hence in $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ all directional derivatives in directions $v \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ exist.

$$\partial_1 f(\vec{0}) = df(\vec{0}, e_1) = 1 \cdot \frac{1 \cdot 0}{1^2 + 0^2} = 0$$

$$\partial_2 f(\vec{0}) = df(\vec{0}, e_2) = 0 \cdot \frac{0 \cdot 1}{0^2 + 1^2} = 0$$

Hence, if f would be differentiable in 0, then

$$Df(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Now consider $v = e_1 + e_2$

$$df(\vec{0}, v) = Df(0) \cdot v = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

But the determination from above of $df(\vec{0}, \vec{v})$ holds.

$$df(\vec{0}, \vec{v}) = 1 \cdot \frac{1 \cdot 1}{1^2 + 1^2} = \frac{1}{2} \neq 0$$

Recommendation: Plot this graph!

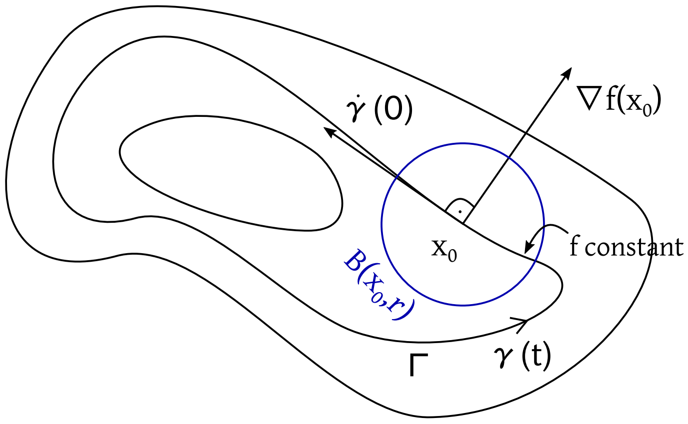


Figure 38: Directional derivative

Show that f is not differentiable.

$$f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{cases} x_1 \cdot \frac{x_1 \cdot x_2}{x_1^2 + x_2^2} & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & \text{for } \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

We already know

$$\partial_1 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) \neq \vec{0}$$

Then it holds that

$$\partial_1 f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \frac{2x_1x_2 - x_1^2 x_2 \cdot 2x_1}{(x_1^2 + x_2^2)^2}$$

$$\partial_2 f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \frac{x_1^2(x_1^2 + x_2^2) - x_1^2 \cdot x_2 \cdot 2x_2}{(x_1^2 + x_2^2)^2}$$

$$\partial_1 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = \frac{2\varepsilon^2 \cdot 2\varepsilon^2 - \varepsilon^2 \cdot \varepsilon \cdot 2\varepsilon}{(2\varepsilon^2)^2} = \frac{2\varepsilon^4}{4\varepsilon^4} = \frac{1}{2}$$

$$\partial_2 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = \frac{\varepsilon^2 \cdot 2\varepsilon^2 - \varepsilon^2 \cdot \varepsilon \cdot 2\varepsilon}{(2\varepsilon^2)^2} = 0$$

It holds that

$$\lim_{\varepsilon \to 0} \partial_2 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = 0 = \partial_2 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

but

$$\lim_{\varepsilon \to 0} \partial_1 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = \frac{1}{2} \neq \partial_1 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

Hence $\partial_1 f$ is non-continuous in $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. □

Remark 57. Let $f : D \to \mathbb{R}^2$ be differentiable. Let $\Gamma \subseteq \mathbb{R}^2$ be a *level line* (dt. Niveaulinie) of f, hence $\forall x \in \Gamma : f(x) = c = \text{const.}$.

$\exists \gamma : (a, b) \to D, x_0 = \gamma(0), 0 \in (a, b)$. Assume $\Gamma \cap B(x_0, x) = \{\gamma(t) \mid t \in (a, b)\}$, hence γ can be locally represented as parametric curve. Then it holds that $f \circ \gamma : (a, b) \to \mathbb{R}$ is constant.

$$f \circ \gamma(t) = c \quad \forall t \in (a, b)$$

Chain rule, $t = 0$:

$$Df(\underbrace{\gamma(0)}_{=x_0}) \cdot \underbrace{\dot{\gamma}(0)}_{D\gamma(0)} = 0$$

$$\langle \nabla f(x_0), \dot{\gamma}(0) \rangle = 0$$

- $\langle \nabla f(x_0), \dot{\gamma}(0) \rangle = 0$. Compare with Figure 39.
- $\nabla f(x_0)$ is normal to $\dot{\gamma}(0)$, hence normal to Γ.

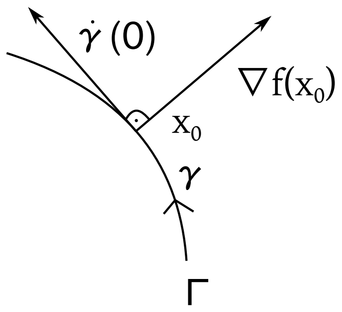


Figure 39: Tangential vector $\dot\gamma(0)$ on Γ in x_0

Theorem 45. Let $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^m$ is open. Assume $\forall x \in D$ exist all partial derivatives $\partial_j f(x)$ for $j = 1, \dots, m$ and every function $D \to \mathbb{R}$, $x \mapsto \partial_j f(x)$, is continuous. Then f is differentiable in D.

Proof. □

This lecture took place on 9th of June 2016 with lecturer Wolfgang Ring.

Now consider the counterexample above:

$$f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{cases} x_1 \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 0 & \text{for } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}$$

We know:

$$\partial_1 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \partial_2 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = 0$$

Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \vec{0}$. Then it holds that

$$\begin{aligned}
\partial_1 f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) &= \frac{\overbrace{2x_1x_2}^{(x_1^2+x_2^2)} - x_1^2 x_2 \cdot 2x_1}{(x_1^2 + x_2^2)^2} \\
\partial_2 f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) &= \frac{x^2(x_1^2 + x_2^2) - x_1^2 x_2 \cdot 2x_2}{(x_1^2 + x_2^2)^2} \\
\partial_1 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) &= \frac{2\varepsilon^2 \cdot 2\varepsilon^2 - \varepsilon^2 \varepsilon \cdot 2\varepsilon}{(2\varepsilon^2)^2} \\
&= \frac{2\varepsilon^4}{4\varepsilon^4} = \frac{1}{2} \\
\partial_2 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) &= \frac{\varepsilon^2 \cdot 2\varepsilon^2 - \varepsilon^2 \varepsilon \cdot 2\varepsilon}{(2\varepsilon^2)^2} = 0
\end{aligned}$$

It holds that

$$\lim_{\varepsilon \to 0} \partial_2 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = 0 = \partial_2 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

but

$$\lim_{\varepsilon \to 0} \partial_1 f(\begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}) = \frac{1}{2} \neq \partial_1 f(\begin{bmatrix} 0 \\ 0 \end{bmatrix})$$

Hence $\partial_1 f$ is non-continuous in $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Proof. Idea: get close to $x_0 \in D$ along lines parallel to coordinate axes. Use Mean Value Theorem. Let $x_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \\ \vdots \\ x_0^n \end{bmatrix} \in D$ and $x = \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} \in D$. We build path of x_0 to x along coordinate lines.

We let $\xi_0 = x_0$ and

$$\xi_1 \coloneqq \begin{bmatrix} x^1 \\ x_0^2 \\ \vdots \\ x_0^n \end{bmatrix} \qquad \xi_2 \coloneqq \begin{bmatrix} x^1 \\ x^2 \\ x_0^3 \\ \vdots \\ x_0^n \end{bmatrix} \qquad \dots \qquad \xi_k \coloneqq \begin{bmatrix} x^1 \\ \vdots \\ x^k \\ x_0^{k+1} \\ \vdots \\ x_0^n \end{bmatrix} \qquad \xi_n = X$$

It holds that $\xi_k + (x^{k+1} - x_0^{k+1})e_{k+1} = \xi_{k+1}$.

$$\varphi_k : (-a_k, 1 + a_k) \subseteq \mathbb{R} \to \mathbb{R} \qquad a_k > 0$$
$$\varphi_k(t) = f(\xi_k + t(x^{k+1} - x_0^{k+1}) \cdot e_{k+1})$$
$$\varphi_k(0) = f(\varphi_k) \qquad \varphi_k(1) = f(\xi_{k+1})$$

φ is continuously differentiable on $(-a_k, 1 + a_k)$ because $\varphi_k = f \circ l_k$ and $l_k(t) = \xi_k + t(x^{k+1} - x_0^{k+1})e_{k+1}$. f is (along l_k) continuously differentiable by precondition for continuity of the $(k+1)$-th partial derivative.

$$\varphi_k'(t) = \partial_{k+1} f(l_n(t)) \cdot (x^{k+1} - x_0^{k+1})$$

□

$$\lim_{h \to 0} \frac{1}{h} \left(f(l_n(t+h)) - f(l_n(t))\right)$$
$$= \lim_{h \to 0} \frac{1}{h} \left(f(\xi_k + \overbrace{(t+h)(x^{k+1} - x_0^{k+1}) \cdot e_{k+1}}^{= v_k + h(x^{k+1} - x_0^{k+1})e_{k+1}}) \right)$$
$$- f\left(\xi_n + \overbrace{t(x^{k+1} - x_0^{k+1}) \cdot e_{k+1}}^{v_k}\right)$$
$$\overset{(*)}{=} \lim_{h \to 0} \frac{1}{h}(-f(\xi_k + v_k) + f(\xi_k + h \cdot (x^{k+1} - x_0^{k+1})e_{k+1}))$$
$$\overset{\text{claim}}{=} \partial_{k+1} f(\xi_k + v_k) \cdot (x^{k+1} - x_0^{k+1})$$

This is true for $x^{k+1} = x_0^{k+1}$. For $x^{k+1} - x_0^{k+1} \neq 0$ it holds that

$$(*) = (x^{k+1} - x_0^{k+1}) \lim_{h \to 0} \frac{1}{\underbrace{h(x^{k+1} - x_0^{k+1})}_{\tilde{h}}}$$
$$\cdot (-f(\xi_k + v_k) - f(\xi_k + v_k + \overbrace{h(x^{k+1} - x_0^{k+1})}^{\tilde{h}} \cdot e_{k+1})$$
$$= (x^{k+1} - x_0^{k+1}) \cdot \underbrace{\lim_{\tilde{h} \to 0} \left(-f(\xi_k + v_k) + f(\xi + v_k + \tilde{h} e_{k+1})\right)}_{\partial_{k+1} f(\xi_k + v_k)}$$

Apply Mean Value Theorem to φ_k:

$$\exists \tau_k \in (0,1) : f(\xi_{k+1}) - f(\xi_k) = \varphi(1) - \varphi(0)$$

$$= \varphi'(\tau_k) \cdot (1 - 0) = \partial_{k+1} f(\xi_k + \tau_k(x^{k+1} - x_0^{k+1}) \cdot e_{k+1}) \cdot (x^{k+1} - x_0^{k+1})$$

Claim: f is differentiable, hence

$$Df(x_0) = \begin{bmatrix} \partial_1 f(x_0) & \dots & \partial_n f(x_0) \end{bmatrix}$$

Show that,

$$\frac{1}{\|x - x_0\|} \left\| f(x) - f(x_0) - \begin{bmatrix} \partial_1 f(x_0) & \dots & \partial_n f(x_0) \end{bmatrix} (x - x_0) \right\| \to 0 \text{ for } x \to x_0$$

$$\frac{1}{\|x-x_0\|}\left\|\underbrace{f(x)}_{\xi_n}-\underbrace{f(x_0)}_{\xi_0}-\sum_{k=1}^{n}\partial_k f(x_0)(x^k-x_0^k)\right\|$$
$$=\frac{1}{\|x-x_0\|}\|\underbrace{f(\xi_n)-f(\xi_{n-1})}_{\varphi_{k-1}(1)-\varphi_{k-1}(0)}+f(\xi_{n-1})-f(\xi_{n-2})+\ldots+f(\xi_1)$$
$$-f(\xi_1)-\sum_{k=1}^{n}\partial_k f(x_0)(x^k-x_0^k)\|$$
$$=\frac{1}{\|x-x_0\|}\|\sum_{k=1}^{n}\partial_k f(\xi_{k-1}+\tau_{k-1}(x^k-x_0^k)\cdot e_k)(x^k-x_0^k)$$
$$-\sum_{k=1}^{n}\partial_k f(x_0)(x^k-x_0^k)\|$$
$$\leq\frac{1}{\|x-x_0\|}\cdot\sum_{k=1}^{n}\left|x^k-x_0^k\right|\cdot\left\|\partial_k f(\xi_{k-1}+\tau_{k-1}(x^k-x_0^k)\cdot e_k)-\partial_k f(x_0)\right\|$$

It holds that $\left|x^k-x_0^k\right|\leq\|x-x_0\|$.

$$\leq\sum_{k=1}^{n}\underbrace{\left\|\partial_k f(\xi_{k-1}+\tau_{k-1}(x^k-x_0)e_k)-\partial_k f(x_0)\right\|}_{(**)}$$

Now let $\varepsilon>0$ be arbitrary and δ sufficiently large such that

$$\|\partial_k f(y)-\partial_k f(x_0)\|<\frac{\varepsilon}{n}$$

if $\|y-x_0\|<\delta$.

Now choose $\|x-x_0\|_\infty<\frac{\delta}{M}$ where M will be defined later and consider $\xi_{k-1}+\tau_{k-1}(x^k-x_0^k)\cdot e_k-x_0$,

$$\begin{bmatrix}x^1\\x^2\\ \vdots\\x_0^k\\x_0^{k+1}\end{bmatrix}+\tau_{n-1}\begin{bmatrix}0\\ \vdots\\x^k-x_0^k\end{bmatrix}-\begin{bmatrix}x_0^1\\ \vdots\\x_0^n\end{bmatrix}$$

where $x^k-x_0^k$ can be found on the k-th index.

$$=\begin{bmatrix}x^1-x_0^1\\x^2-x_0^2\\ \vdots\\ \tau_{k-1}(x^k-x_0^k)\\0\\ \vdots\\0\end{bmatrix}$$

$$\left\|\xi_{k-1}+\tau_{k-1}(x^k-x_0^k)e_k-x_0\right\|\leq\sum_{j=1}^{k-1}\left|x^j-x_0^j\right|\cdot\|e_j\|+\tau_{k-1}\left|x^k-x_0^k\right|\cdot\|e_k\|$$

$$\leq\underbrace{\max_{k=1,\ldots,n}\left|x^j-x_0^j\right|}_{\underbrace{\|x-x_0\|_\infty}_{\delta/M}}\cdot\underbrace{\max_{l=1,\ldots,n}\|e_k\|}_{M>0}<\delta$$

$$\Rightarrow(**)=\left\|\partial_k f(\xi_{k-1}+\tau_{k-1}(x^k-x_0^k)e_k)\partial_k f(x_0)\right\|<\frac{\varepsilon}{n}$$

If $\|x-x_0\|_\infty<\frac{\delta}{M}$, it holds that

$$\frac{1}{\|x-x_0\|}\|f(x)-f(x_0)-[\partial_1 f(x_0),\ldots,\partial_n f(x_0)](x-x_0)\|<\varepsilon$$

12.3 Generalization

Let $f:D\subseteq\mathbb{R}^n\to\mathbb{R}^m$, $D\subseteq\mathbb{R}^n$ open. Assume all partial derivatives $\partial_j f^i(x)$ exist and be continuous in x in D with $j=1,\ldots,n$ and $j=1,\ldots,m$. Then f is differentiable in D and it holds that

$$Df(x)=\left(\partial_i f^j(x)\right)_{\substack{i=1,\ldots,m\\j=1,\ldots,n}}$$

12.4 Maxima and Minima

Let $D \subseteq \mathbb{R}^n$ and $f : D \to \mathbb{R}$. We call $x_0 \in D$ a *relative maximum* (*relative minimum*) or *local maximum* (*local minimum*) of f if $\varepsilon > 0$ exists such that $\forall x \in B(x_0, \varepsilon) \cap D$ such that $f(x) \leq f(x_0)$ ($f(x) \geq f(x_0)$). We call the maximum (minimum) *strict*, if for $x \neq x_0$, the inequality holds strictly.

12.5 Necessary optimality criterion

Theorem 46 (Necessary optimality criterion)**.** Let $D \subseteq \mathbb{R}^n$ open, let $x_0 \in D$ be a local maximum or a local minimum. Let f be differentiable in x_0. Then for all $v \in \mathbb{R}^n \setminus \{\vec{0}\}$:

$$df(x_0; v) = Df(x_0) \cdot v = 0$$

Especially $Df(x_0) \cdot e_k = \partial_k f(x_0) = 0$ for $k = 1, \dots, n$. Hence,

$$Df(x_0) = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{or equivalently} \quad \nabla f(x_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Proof. Let $l_{v,x_0}(t) = x_0 + t \cdot v$. If x_0 is a local maximum, it holds that

$$f(l_{v,x_0}(t)) \leq f(l_{v,x_0}(0)) = f(x_0)$$

for sufficiently small t. So $f \circ l_{v,x_0}$ has a local maximum at $t = 0$.

$$(f \circ l_{v,x_0})'(0) = df(x_0; v) = 0$$

by necessary optimality criteria for functions in $\mathbb{R}$. □

This lecture took place on 14th of June 2016 with lecturer Wolfgang Ring.

Recall: A set X with a map $d : X \times X \to [0, \infty)$ is called *metrical space* if

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $\forall x, y : d(x, y) = d(y, x)$
3. $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$

Using metrics, we can define convergent sequences and Cauchy sequences.

$(x_n)_{n \in \mathbb{N}}$ is convergent with limit $x \in X$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies d(x_n, x) < \varepsilon]$$

Cauchy sequence:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \implies d(x_n, x_m) < \varepsilon]$$

Definition 43. A metric space (X, d) is called *complete*, if every Cauchy sequence in X is also a convergent sequence, hence has a limit.

Lemma 36. Every normed vector space $(V, \|.\|)$ induces also a metric space $d(v, w) = \|v - w\|$.

Let V be complete and $A \subseteq V$ bounded. Then A with metric $d(v, w) = \|v - w\|$ is a complete metric space.

Metric properties of $d(v, w) = \|v - w\|$. Let $A \subseteq V$ be bounded and let $(v_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in A, hence $v_0 \in A \forall n \in \mathbb{N}$. Then $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in V. Because V is complete, $\exists v \in V : v = \lim_{n \to \infty} v_n$.

This likewise means that v is a contact point of A. Because A is bounded, it holds that $v \in A$ ($A = \overline{A}$). Hence $(v_n)_{n \in \mathbb{N}}$ has a limit point in A. □

Theorem 47 (Banach's fixed point theorem)**.** Let X be a complete metrical space. $F : X \to X$ with property $\exists \sigma \in [0, 1)$ such that $d(F(x), F(y)) \leq \sigma \cdot d(x, y)$ (a derivative with this property is called *contraction*).

Then a uniquely defined point $\overline{x} \in X$ with $F(\overline{x}) = \overline{x}$. $\overline{x}$ is called *fixed point* of F.

Proof. Let $x_0 \in X$ be arbitrary. We define

$$x_n = F(x_{n-1}) = \underbrace{F(F(\dots(F(x_0))))}_{\text{nested } n \text{ times}} = F^n(x_0) = \underbrace{(F \circ F \circ F \dots \circ F)}_{\text{times}}(x_0)$$

for $n \in \mathbb{N}$ with $n \geq 1$.

We show: $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. It holds that

$$d(\underbrace{x_n}_{=F(x_n)}, \dots, \underbrace{x_n}_{F(x_{n-1})} \leq \sigma \cdot \underbrace{d(x_n, x_{n-1})}_{F(x_{n-1})}$$

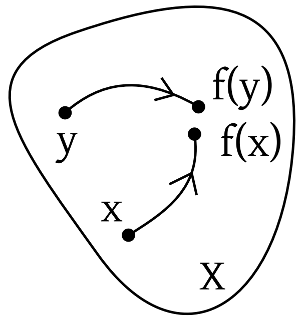


Figure 40: Illustration for Banach's fixed point theorem

$$\leq \vartheta^2 d(x_{n-1}, x_{n-2}) \leq \ldots \leq \vartheta^n d(x_1, x_0)$$

Then for every $m, n \in \mathbb{N}$ with $m > n$.

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, d_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leq \vartheta^m d(x_1, x_0) + \vartheta^{m-1} \cdot d(x_1, x_0) + \ldots + \vartheta^{n+1} d(x_1, x_0) \\ &= \vartheta^n \frac{1 - \vartheta^{m-n+1}}{1 - \vartheta} d(x_1, x_0) \\ &< \vartheta^n \frac{1}{1 - \vartheta} d(x_1, x_0) \\ &< \varepsilon \text{ if } \vartheta^n < \frac{\varepsilon(1 - \vartheta)}{d(x_1, x_0) + 1} \end{aligned}$$

Hence $\vartheta^n \to 0$ because $\vartheta < 1$ for sufficiently large n.

So $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\overline{x}$ be the limit of this Cauchy sequence (completeness of X).

F is Lipschitz continuous, hence continuous in X. So it holds that $\lim_{n\to\infty} F(x_n) = F(\overline{x})$.

$$\lim_{n\to\infty} x_{n+1} = \overline{x}$$

Followingly $\overline{x} = F(\overline{x}$, so $\overline{x}$ is a fixed point of F.

It remains to show uniqueness:

Let $\overline{y} = F(\overline{y})$. Assume $\overline{y} \neq \overline{x}$, hence $d(\overline{x}, \overline{y}) > 0$. Hence it holds that

$$\begin{aligned} d(\overline{x}, \overline{y}) &= d(F(\overline{x}), F(\overline{y})) \\ &\overset{\text{contraction}}{\leq} \vartheta d(\overline{x}, \overline{y}) \\ &\implies 1 \leq \sigma \end{aligned}$$

This a contradiction. □

Theorem 48 (Local Inverse Theorem, Theorem of the inverse map)**.** Let $O \subseteq \mathbb{R}^n$ be open and let $f : O \to \mathbb{R}^m$ be continuously differentiable in O. Let $x_0 \in O$ such that $\underbrace{Df(x_0)}_{\mathbb{R}^{n\times n}}$ is regular (invertible).

Then there exists an environment U of x_0 such that $f : U \to f(U) \subseteq \mathbb{R}^m$ is bijective.

Even more it holds that $f : U \to f(U)$ is a *Diffeomorphism*. Hence f is bijective and $f^{-1} : f(U) = V \to U$ is continuously differentiable. Furthermore it holds that

$$D(f^{-1})(\underbrace{y_0}_{f(x_0)}) = [Df(x_0)]^{-1}$$

Proof. Consider $\tilde{f}(x) = f(x + x_0) - f(x_0)$.

$$\tilde{f} : O - x_0 = \{z - x_0 : z \in O\} \to \mathbb{R}^n$$

$$\tilde{f}(0) = 0$$

$$D\tilde{f}(0) = Df(x_0)$$

It follows immediately that $\tilde{f}$ is a Diffeomorphism iff f is a Diffeomorphism.

Let $F : \tilde{O} \to \mathbb{R}^n$ with $F(x) = \underbrace{[Df(x_0)]^{-1}}_{\text{exists by hypothesis}} \cdot \tilde{f}(x)$. Then it holds that $F(0) = 0$.

$$DF(0) = [Df(x_0)]^{-1} \cdot \underbrace{D\tilde{f}(0)}_{Df(x_0)} = I$$

So it holds that, F is continuously differentiable in $\tilde{O}$ with $DF(0) = I$. We show: F is a Diffeomorphism.

We let $y \in \mathbb{R}^n$ such that,

$$\varphi_y(x) = y + x - F(x)$$

Then it holds that $\varphi_y(x) = x \iff x = y + x - F(x) \iff y = F(x)$. Hence x is the solution of $F(x) = y \iff x$ is fixed point of φ_y.

F is continuously differentiable. Because F is continuously differentiable, it holds that φ_y is continuously differentiable:

$$D\varphi_y(X) = I - DF(x)$$

Because of continuous differentiability, $\underbrace{DF(x)}_{\mathbb{R}^{m\times n}}$ depends continuously on x.

So, $\forall \varepsilon > 0 \exists \delta > 0$:

$$\|x - 0\| < \delta \implies \underbrace{\left\| DF(x) - \underbrace{DF(0)}_{=I} \right\|}_{\text{operator norm}} < \varepsilon$$

Choose $r > 0$ such that

$$\|x\| < 2r \implies \|DF(x) - I\| < \frac{1}{2}$$

and we choose r sufficiently small such that $B(O, 2r) \subseteq \tilde{O}$. This works only because $\tilde{O}$ is open.

Let $x_1, x_2 \in B(O, 2r)$.

$$l_{x_1,x_2} : (-\varepsilon, 1 + \varepsilon) \to B(O, 2r)$$

$$l_{x_1,x_2}(t) = (1 - t)x_1 + tx_2$$

$$l_{x_1,x_2}(0) = x_1 \qquad l_{x_1,x_2}(1) = x_2$$

Consider

$$\|\varphi_y(x_2) - \varphi_y(x_1)\| = \|\varphi_y \circ l_{x_1,x_2}(1) - \varphi_y \circ l_{x_1,x_2}(0)\|$$

$$= \left\| \int_0^1 \frac{d}{dt}[\varphi_y \circ l_{x_1,x_2}(t)]\, dt \right\|$$

$$= \left\| \int_0^1 D\varphi_y(l_{x_1,x_2}(t)) \cdot (x_2 - x_1)\, dt \right\|$$

$$\leq \int_0^1 \|D\varphi_y(l_{x_1,x_2}(t)) \cdot (x_2 - x_1)\|\, dt$$

$$\leq \int_0^1 \|D\varphi_y(l_{x_1,x_2}(t))\| \, \|x_2 - x_1\|\, dt$$

$$= \int_0^1 \overbrace{\left\| I - DF(\underbrace{l_{x_1,x_2}(t)}_{\in B(O,2r)}) \right\|}^{\leq \frac{1}{2}} dt \cdot \|x_2 - x_1\|$$

$$\leq \frac{1}{2}\|x_1 - x_2\|$$

Hence φ_y is a contraction with $\sigma = \frac{1}{2}$. It remains to show that $\varphi_y : B(O, r) \to B(O, r)$ (φ_y maps to itself) for sufficiently small y.

Let $y \in B(O, r)$ and $x \in B(O, 2r)$. Then it holds that

$$\|\varphi_y(x)\| \leq \underbrace{\|\varphi_y(x) - \varphi_y(0)\|}_{\leq \frac{1}{2}\|x-0\| \text{ by contraction}} + \underbrace{\|\varphi_y(0)\|}_{=\|y\|}$$

$$\leq \frac{1}{2} \cdot 2r + r = 2r$$

$$\varphi_y(x) \in B(O, 2r) \subseteq B(O, 2r)$$

So for $y \in B(O, r)$ it holds tat

$$\varphi_y : \overline{B(O, 2r)} \to \overline{B(y, 2r)}$$

Then by Banach's fixed point theorem, there exists some unique $x \in \overline{B(O, 2r)}$ with $\varphi_y(x) = x$, hence $F(x) = y$.

We show: $DF(x)$ is invertible for all $x \in U = F^{-1}(B(O, r)) \cap B(O, 2n)$.

We know:

$$\|v - DF(x)\cdot v\| = \|(I - DF(x))\cdot v\|$$
$$\leq \|I - DF(x)\| \cdot \|v\| \leq \frac{1}{2}\|v\| \qquad \forall x \in U$$

Let $v \in \operatorname{kern}(I - DF(x))$. Then it holds that

$$\|v\| = \|v - DF(x)\cdot v + DF(x)\cdot v\|$$
$$\leq \underbrace{\|v - DF(x)\cdot v\|}_{\leq\frac{1}{2}\|v\|} + \|DF(x)\cdot v\|_0$$
$$\implies \frac{1}{2}\|v\| \leq 0 \implies v = 0$$

For $x \in U$ it holds that $\operatorname{kern} DF(x) = \{0\} \implies DF(x)$ is invertible. ☐

This lecture took place on 16th of June 2016 with lecturer Wolfgang Ring.

cont. because (by $\|\varphi_y(x)\| \leq 2r$, the conclusion using Banach's fixed point theorem) for the fixed point x in φ_y, it holds that

$$\|x\| = \|\varphi_y(x)\| < 2r$$

Hence for every $y \in B(O,r)$ there exists a unique $x \in B(O,2r)$ such that $F(x) = y$.

Hence in $U = B(O,2r) \cap F^{-1}(B(O,r))$ the map f is injective. So $F : U \to F(U) = B(O,r)$ bijective. Furthermore $F^{-1}(B(O,r))$ is open[3]. Hence U is open. Therefore f (bijective backtransformation) bijective in some environment of x_0.

We use a lemma, which is not proven here:

Lemma 37. Let $g : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable in D. Let g be bijective and $Dg(x)$ invertible for all $x \in D$. Then $g^{-1} : g(D) \to D$ is continuously differentiable in $g(D)$ and it holds that

$$Dg^{-1}(\underbrace{y}_{g(x)}) = [Dg(x)]^{-1}$$

[3] Preimages of open sets in continuous maps are also open. Will be shown later.

The lemma is proven in Königsberger volume 2, 3rd edition, page 105.

This implies that F^{-1} is differentiable in $B(O,1)$ with given Jacobimatrix. Therefore also f^{-1} is differentiable. ☐

12.6 A theorem about implicit functions

Theorem 49. Let $U \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$ and let $f : U \to \mathbb{R}^m$ continuously differentiable.

Notation: $(x,y) \in U$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. For some $(x_0,y_0) \in U$ let it hold that $f(x_0,y_0) = \vec{0}$.

Furthermore let

$$Df(x_0,y_0) = \begin{bmatrix} \partial x_1 f^1 & \partial x_2 f^1 & \dots & \partial x_n f^1 & \partial y_1 f^1 & \dots & \partial y_m f^1 \\ \partial x_1 f^2 & \partial x_2 f^2 & \dots & \partial x_n f^2 & \partial y_1 f^2 & \dots & \partial y_m f^2 \\ \vdots & & \ddots & & & \ddots & \vdots \\ \partial x_1 f^m & \partial x_2 f^m & \dots & \partial x_n f^m & \partial y_1 f^m & \dots & \partial y_m f^m \end{bmatrix}$$

The left half side of the matrix is $D_x f(x_0,y_0) \in \mathbb{R}^{m\times n}$ whereas the right half of the matrix is $D_y f(x_0,y_0) \in \mathbb{R}^{m\times m}$.

Assumption:$D_y f(x_0,y_0)$ is invertible.

Then there exists some environment U' of x_0 in $\mathbb{R}^n$ and a continuously differentiable function $g : U' \to \mathbb{R}^m$ with $g(x_0) = y_0$ and $f(x,g(x)) = \vec{0} \in \mathbb{R}^m$.

$y = g(x)$ solves the implicit equation $f(x,y) = 0$.

Proof. Let $F(x,y) = \begin{bmatrix} x \\ f(x,y) \end{bmatrix} \in \mathbb{R}^{n+m}$. $F : U \to \mathbb{R}^{n+m}$.

$$DF(x,y) = \begin{bmatrix} I & 0 \\ D_x f(x,y) & D_y f(x,y) \end{bmatrix}$$

$$DF(x_0,y_0) = \begin{bmatrix} I & 0 \\ D_x f(x_0,y_0) & D_y f(x_0,y_0) \end{bmatrix}$$

$$\det(DF(x_0,y_0)) = \det I \cdot \underbrace{\det D_y f(x_0,y_0)}_{\neq 0} \neq 0$$

So $DF(x_0, y_0)$ is invertible. From the local invertibility theorem it follows that

$$\implies V \subset \mathbb{R}^{n+m} : V \text{ is open, environment of } (x_0, y_0)$$

$$F : V \to F(V)$$

is a Diffeomorphism. Let $r, r' > 0$ such that

$$B(x_0, r) \times B(y_0, r') \subseteq V$$

$U \times U' := B(x_0, r) \times B(y_0, r')$ is an open ball in $\mathbb{R}^{n+m}$ with norm $\|(x, y)\|_{\mathbb{R}^{n+m}} = \max \{\|x\|, \|y\|\}$ in $\mathbb{R}^{n+m}$. V is open in regards of this norm.

For all $(\xi, \eta) \in F(U \times U')$ there exists a unique $(x, y) \in U \times U'$ with $(\xi, \eta)^t = F(x, y)$ (local invertibility).

$$\iff \xi = x \wedge \eta = f(x, y)$$

x and y are uniquely defined by (ξ, η). Now let

$$\begin{bmatrix} x \\ y \end{bmatrix} = F^{-1}(\xi, \eta) = \begin{bmatrix} \xi \\ G(\xi, \eta) \end{bmatrix}$$

We let $g(x) = G(x, \vec{0})$. Then it holds that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = F(F^{-1}(x, \vec{0})) = F(x, \underbrace{g(x)}_{=G(x,0)}) = \begin{bmatrix} x \\ f(x, g(x)) \end{bmatrix}$$

So $\vec{0} = f(x, g(x))$.

Continuous differentiability of $g(x) = G(x, 0)$ follows from continuous differentiability of G (over F^{-1}) by the local invertibility theorem. □

Example 30. Consider $f(x, y) = x^2y - x^3y^2 - 2$.

$$\begin{aligned} f(-1, 1) &= 1 - (-1) - 2 = 0 \\ \partial_x f(x, y) &= 2xy - 3x^2y^2 \\ \partial_y f(x, y) &= x^2 - 2x^3y \\ \partial_x f(-1, 1) &= -5 \neq 0 \\ \partial_y f(-1, 1) &= 3 \neq 0 \end{aligned}$$

Hence, equation

$$x^2y - x^3y^2 - 2 = 0$$

in environment $(-1, 1)$ can uniquely solve the problem with x as well as y.

13 Higher partial derivatives

Consider $f : D \to \mathbb{R}$ (we consider $f : D \to \mathbb{R}^m$ component-wise). Let $D \subseteq \mathbb{R}^n$ be open. Let f in D be continuously differentiable, hence $\partial x_j f(x)$ is a continuous function in D. Assume $\partial x_j f$ is continuously differentiable in D. Let $\partial x_i x_j f := \partial x_i(\partial x_j f)$. Analogously for higher derivatives: If $\partial x_{i_1} x_{i_2} x_{i_3} \dots x_{i_n} f = \partial x_{i_1}(\partial x_{i_2}(\dots(\partial x_{i_n} f)\dots))$ if $\partial x_{i_2} x_{i_3} \dots x_{i_n} f$ is continuously differentiable.

Example 31.

$$f(x, y) = x^2y - x^3y^2 - 2$$

$$\begin{aligned} \partial x f &= 2xy - 3x^2y^2 \\ \partial_y f &= x^2 - 2x^3y \\ \partial_{xx} f &= 2y - 6xy^2 \\ \partial_{yx} f &= 2x - 6x^2y \\ \partial_{xy} f &= 2x - 6x^2y \\ \partial_{yy} f &= 0 - 2x^3 \end{aligned}$$

Apparently the 3 derivatives are equal, so the order of x and y does not seem to be relevant.

Especially for 2 derivatives.

$$D^2 f(x) = \begin{bmatrix} \partial x_1 x_2 f & \partial x_1 x_2 f & \dots & \partial x_1 x_n f \\ \partial x_2 x_1 f & \partial x_2 x_2 f & \dots & \partial x_2 x_n f \\ \vdots & \ddots & \ddots & \vdots \\ \partial x_n x_1 f & \dots & \dots & \partial x_1 x_n f \end{bmatrix} (x)$$

$D^2 f(x)$ is called *Hesse matrix* of f. Under general conditions (see Theorem by Schwarz) $D^2 f(x)$ is symmetrical. Hence eigenvalues of $D^2 f(x)$ are real and there exists some orthonormal basis of eigenvectors.

Theorem 50 (Sufficient optimality criterion)**.** Let $f : D \to \mathbb{R}$ such that the partial derivatives up to second degree in D exist and are continuous. Let

$x_0 \in D$ such that $Df(x_0) = 0$ (i.e. $\nabla f(x_0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$). If $D^2 f(x_0)$ is negative definite (i.e. all eigenvalues are negative)

$$\langle v, D^2 f(x_0) v \rangle < 0 \qquad \forall v \neq 0$$

then f in x_0 has a local maximum. If $D^2 f(x_0)$ is positive definite, then f in x_0 has a local minimum. If $D^2 f(x_0)$ is indefinite, then f in x_0 has *no* local extremum. We say: f in x_0 has a *saddle point*.

Proof. Will be given later. □

Existence of local or global maxima/minima is commonly shown by compactness.

Definition 44. Let $f : D \to \mathbb{R}$. D is open and all partial derivatives of f up to degree k exist and are continuous in D. Then f is called continuously differentiable k times in D and we let

$$\mathcal{C}^k(D) = \{f : D \to \mathbb{R} : f \text{ is continuously differentiable } k \text{ times}\}$$

$\mathcal{C}^k(D)$ is a real vector space.

Hermann Amandus Schwarz (1843–1921)

Definition 45. Let $D \subseteq \mathbb{R}^n$ be open, $f \in \mathcal{C}^2(D)$. Then it holds that $\partial x_i x_j f = \partial x_j x_i f$ in D.

Schwarz' Theorem, Symmetry of second derivatives. Let $x_0 \in D$ be arbitrary. Let $r > 0$ such that $B(x_0, r) \subseteq D$ and let $h, k \in \mathbb{R}$ be sufficiently small such that $x_0 + he_i \in B(x_0, \frac{r}{2}) \iff \|h \cdot e_i\| \leq \frac{r}{2}$ and $x_0 + ke_j \in B(x, \frac{r}{2}) \iff \|ke_j\| < \frac{r}{2}$. Then it holds that $x_0 + he_i + ke_j \in B(x_0, r) \iff \|he_i + ke_j\| < r$ follows immediately because $\|he_i + ke_j\| \leq \|he_i\| + \|ke_j\| < \frac{r}{2} + \frac{r}{2}$.

This lecture took place on 17th of June 2016 with lecturer Wolfgang Ring.

Let $F(h,k) = f(x_0 + he_i + ke_j) - f(x_0 + he_i) - f(x_0 + ke_j) + f(x_0)$. Let x_0 and k be fixed.

$$\varphi(\lambda) = f(x_0 + \lambda e_i + ke_j) - f(x_0 + \lambda e_i)$$

$$\varphi : I \to \mathbb{R} \qquad I = (-1, 1)$$

Compare with Figure 41.

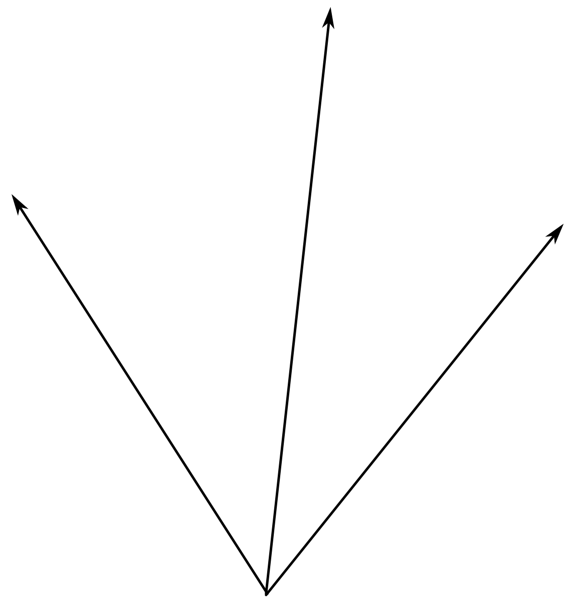


Figure 41: φ in the proof of Schwarz' Theorem

$$\begin{aligned} \varphi(0) &= f(x_0 + ke_j) - f(x_0) \\ \varphi(h) &= f(x_0 + h \cdot e_i + k \cdot e_j) - f(x_0 + h \cdot e_i) \\ \varphi(h) - \varphi(0) &= F(h, k) \end{aligned}$$

$$\varphi : (-r, r) \to \mathbb{R}$$

is continuously differentiable. The Intermediate Value Theorem gives us:

$$\exists\lambda \text{ with } |\lambda| < |h| : F(h,k) = \varphi(h) - \varphi(0) = h' \cdot \varphi'(\lambda)$$

$$= Df(x_0 + \lambda \cdot e_i + k \cdot e_j) \cdot e_i - Df(x_0 + \lambda e_i) \cdot e_i$$

$$= h \cdot [\partial x_i f(x_0 + \lambda \cdot e_i + k \cdot e_j) - \partial x_i f(x_0 + \lambda \cdot e_i)]$$

Consider $\psi : (-r, r) \to \mathbb{R}$.

$$\psi(\mu) = \partial x_i f(x_0 + \lambda e_i + \mu e_j)$$

ψ is continuously differentiable, because $\partial x_i f$ is continuously differentiable. Hence

$$F(h,k) = h \cdot [\psi(k) - \psi(0)] \underbrace{=}_{\text{IVT},|\mu|<|\lambda|} h \cdot k \cdot \varphi'(\mu) = h \cdot k \partial x_i x_j f(x_0 + \lambda e_i + \mu e_j).$$

Analogously: $\tilde{\varphi}(\mu) = f(x_0 + h \cdot e_i + \mu \cdot e_j) - f(x_0 + \mu \cdot e_j)$.

Then it holds again that

$$F(h,k) = \tilde{\varphi}(k) - \tilde{\varphi}(0) \overbrace{=}^{\text{IVT},|\tilde{\mu}|<|k|} k \cdot \tilde{\varphi}'(\mu)$$

$$= k \cdot [\partial x_j f(x_0 + h e_i + \tilde{\mu} e_j) - \partial x_j f(x_0 + \tilde{\mu} \cdot e_j)]$$

$$\underbrace{=}_{\text{IVT},\left|\tilde{\lambda}\right|<|h|} k \cdot h \cdot \left[\partial x_i x_j f(x_0 + \tilde{\lambda} e_i + \tilde{\mu} e_j)\right]$$

$$\implies h \cdot k \partial x_j \partial x_i f(x_0 + \lambda e_i + \mu e_j)$$

$$= k \cdot h \partial x_i \partial x_j f(x_0 + \tilde{\lambda} e_i + \tilde{\mu} e_j)$$

$$\left.\begin{matrix} |\lambda| \left|\tilde{\lambda}\right| < |h| \\ |\mu| \, |\tilde{\mu}| < |k| \end{matrix}\right\} \text{let } h,k \text{ go to zero (2nd derivatives are non-continuous}$$

$$\implies \partial x_j \partial x_i f(x_0) = \partial x_i \partial x_j f(x_0)$$

□

Example 32 (Counterexamples)**.**

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 0 & \text{else} \end{cases}$$

has second derivatives at every point. In $(0,0)$ it holds that $\partial x_1 \partial x_2 f(0,0) \neq \partial x_2 \partial x_1 f(0,0)$.

Theorem 51 (Generalization of Schwarz' Theorem)**.** Let $D \subseteq \mathbb{R}^n$ be open. $f \in \mathcal{C}^k(D)$. Then consider two variables you derive equal times with. Consider the corresponding resulting derivatives. They correspond (order of derivatives muss be smaller-equal k).

For example:

$$\partial x_1 \partial x_3 \partial x_3 \partial x_1 = \partial x_1 \partial x_1 \partial x_3 \partial x_3$$

Proof. Can be done by complete induction. □

Remark 58 (Multiindex notation)**.** Let $I = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n, \alpha_i \geq 0$. I is called *multiindex*. The number $k = |I| = \sum_{i=1}^n \alpha_i$ is called *order of I*.

Notation: $\partial_{x_l}^\alpha = \underbrace{\partial_{x_l} \partial_{x_l} \dots \partial_{x_l}}_{\alpha \text{ times}}$ for $\alpha \in \mathbb{N}$.

$$\partial_{x_k}^0 f = f \quad \text{and} \quad \partial_I f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_n}^{\alpha_n}$$

Further notational remarks:

$$I! := (\alpha_1!)(\alpha_2!) \dots (\alpha_n!) = \prod_{i=1}^n (\alpha_i!)$$

Let $x = (x_1, x_2, \dots, x_n)^t$. Then we let $x^I = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} = \prod_{i=1}^n x_i^{\alpha_i}$.

Theorem 52 (Taylor's theorem)**.** Let $f \in \mathcal{C}^{k+1}(D), D \subseteq \mathbb{R}^n$ be open. Let $x_0 \in D$ and v sufficiently small such that the path between x_0 and $x_0 + v$ is entirely in D.

Then some $w \in (0,1)$ exists such that

$$f(x_0 + v) = \underbrace{f(x_0) + \sum_{j=1}^{k} \sum_{i_1, i_2, \dots, i_j = 1}^{n} \frac{1}{j!} \partial i_1, i_2, \dots, i_j f(x_0) \cdot v_{i_1} \cdot v_{i_2} \cdot \dots \cdot v_{i_m}}_{\text{Taylor polynomial of } f \text{ in } x_0}$$

$$+\underbrace{\frac{1}{(j+1)!}\sum_{i_1,i_2,\dots,i_{k+1}=1}^{n}\partial_{i_1,\dots,i_{k+1}}f(x_0+vw)\cdot v_{i_1}\cdot v_{i_2}\dots v_{i_{k+1}}}_{R_{k+1}(f,v)\text{ as remaining term}}$$

Alternatively, we can write:

$$f(x_0+v)=f(x_0)+\sum_{j=1}^{k}\sum_{|I|=j}\frac{1}{I!}\partial_I f(x_0)\cdot v^I+\sum_{|I|=k+1}\frac{1}{I!}\partial_I f(x_0+vw)\cdot v^I$$

Proof. Let $B(x_0,r)\subseteq D$ and let $|v|<r$. $\varphi:(-r,r)\to\mathbb{R}$.

$$\varphi(t)=f(x_0+t\cdot v)=f\circ l_{x_0,v}(t)$$

$$l_{x_0,v}(t)=x_0+t\cdot v$$

Because k is $(k+1)$ times continuously differentiable and the linear function $l_{x_0,v}$ can be differentiated arbitrary many times, so is φ $k+1$ times differentiable.

$$\varphi'(t)=df(x_0+t\cdot v)\cdot v=\sum_{i_1=1}^{n}\underbrace{\partial i_1 f(x_0+tv)}_{\text{cont. diff. by } t}\cdot v_{i_1}$$

$$\begin{aligned}\varphi''(t)&=\sum_{i_1=1}^{n}\left(\sum_{i_2=1}^{n}\partial_{i_2}\partial_{i_1}f(x_0+t\cdot v)\cdot v_{i_2}\right)\cdot v_{i_1}\\&=\sum_{i_1,i_2=1}^{n}\partial_{i_1,i_2}f(x_0+t\cdot v)\cdot v_{i_1}v_{i_2}\end{aligned}$$

$$\varphi^{(j)}(t)=\sum_{i_1,i_2,\dots,i_j=0}^{n}\partial_{i_1,i_2,\dots,i_j}f(x_0+tv)\cdot v_{i_1}v_{i_2}\dots v_{i_j}$$

Remark 59 (Taylor's theorem in the scalar case for φ)**.**

$$\varphi(1)=\varphi(0)+\sum_{j=1}^{k}\frac{1}{j!}\varphi^{(j)}(0)+\frac{1}{(j+1)!}\varphi^{(j+1)}(v)\cdot 1^{j+1}$$

with $v\in(0,1)$.

Remark 60 (Lagrange form of remaining term)**.** For φ and $\varphi^{(j)}$ we insert,

$$f(x_0+v)=f(x_0)+\sum_{j=1}^{k}\frac{1}{j!}\sum_{i_1,i_2,\dots,i_j=1}^{n}\partial_{i_1,i_2,\dots,i_j}f(x_0)\cdot v_{i_1}\cdot v_{i_2}\cdot\ldots\cdot v_{i_j}$$

$$+\sum_{i_1,\dots,i_{k+1}=1}^{n}\partial i_1\dots i_{k+1}f(x_0+vw)\cdot v_{i_1}v_{i_2}\dots v_{i_{k+1}}$$

Taylor formula:

$$\partial_{i_1i_2\dots i_j}f=\partial_{i'_1i'_2\dots i'_j}f$$

if $(i_1,i_2,\dots,i_j)\in\mathbb{N}^j$ differs from $(i'_1,i'_2,\dots,i'_j)$ only by its order. Let $I=(\alpha_1,\alpha_2,\dots,\alpha_n)\in\mathbb{N}^n$ be a multiindex I corresponding to derivative index

$$(\underbrace{1,1,\dots,1}_{\alpha_1\text{ times}},\underbrace{2,2,\dots,2}_{\alpha_2\text{ times}},\dots,\underbrace{n,\dots,n}_{\alpha_n\text{ times}})$$

$$|I|=j=\sum_{i=1}^{n}\alpha_i\qquad\tilde{I}\in\mathbb{N}^j$$

How many permutations of $\tilde{I}$ are there in $\mathbb{N}$?

Possibilities to distribute α_1 ones over j positions: $\binom{j}{\alpha_1}$. For every position of ones, $j-\alpha_1$ positions for twos remain. There exists $\binom{j-\alpha_1}{\alpha_2}$ possibilities to distribute the threes. $\binom{j-\alpha_1-\alpha_1}{\alpha_3}$, etc.

Possibilities to distribute α_1 ones, α_2 twos, …, α_n n-es among j positions:

$$A_I:=\binom{j}{\alpha_1}\binom{j-\alpha_1}{\alpha_2}\binom{j-\alpha_1-\alpha_2}{\alpha_3}\dots\binom{j-\alpha_1-\alpha_2-\ldots-\alpha_{n-1}}{\alpha_n}$$

□ is the number partial derivatives equivalent to ∂_I.

$$A_I=\frac{j!}{\alpha_1!(j-\alpha_1)!}\frac{(j-\alpha_1)!}{\alpha_2!(j-\alpha_1-\alpha_2)!}\frac{(j-\alpha_1-\alpha_2)!}{\alpha_3!(j-\alpha_1-\alpha_2-\alpha_3)!}$$

$$\dots\frac{(j-\alpha_1-\ldots-\alpha_{n-2})!}{\alpha_{n-1}!(j-\alpha_1-\ldots-\alpha_{n-1})!}\frac{(j-\alpha_1-\ldots-\alpha_{n-1})!}{\alpha_n!\underbrace{(j-\alpha_1-\ldots-\alpha_n)!}_{=0!=1}}$$

Many terms cancel out. What remains is,

$$= \frac{j!}{\prod_{i=1}^{n}(\alpha_i)!} = \frac{j!}{I!}$$

If $(i_1, \dots, i_j)$ and $(i'_1, \dots, i'_j)$ only distinguish by their order, then it holds that

$$v_{i_1} v_{i_2} \dots v_{i_j} = v_{i'_1} \dots v_{i'_j} = v^I$$

Insert all these expressions in the Taylor formula and sum up equivalent expressions.

$$f(x_0+v) = f(x_0) + \sum_{j=1}^{k} \frac{1}{j!} \sum_{|I|=j} \frac{j!}{I!} \partial_I f(x_0) \cdot v^I + \frac{1}{(j+1)!} \sum_{|I|=j+1} \frac{(j+1)!}{I!} \partial_I f(x_0+vw) \cdot v^K$$

Example 33. Let $f : \mathbb{R}^2 \to \mathbb{R}$, f arbitrary often continuously differentiable.

$$f(x_0+v) = f(x_0) + \partial_1 f(x_0) \cdot v_1 + \partial_2 f(x_0) v_2 \frac{1}{2!0!} \partial_{(2,0)} f(x_0) \cdot v_1^2 + \frac{1}{1!1!} \partial_{(1,1)} f(x_0) v_1 v_2$$

The multiindex for $(1,1)$ is $(2,0)$.

$$+\frac{1}{0!2!}\partial_{(0,2)} f(x_0) v_2^2 + \frac{1}{3!0!}\partial_{(3,0)} f(x_0) v_1^3 + \frac{1}{2!1!}\partial_{(2,1)} f(x_0) v_1^2 v_2$$

$$+\frac{1}{1!2!}\partial_{(1,2)} f(x_0) v_1 v_2^2 + \frac{1}{0!3!}\partial_{(0,3)} f(x_0) v_2^3 + \dots$$

$$= f(x_0) + \partial_1 f(x_0) v_1 + \partial_2 f(x_0) v_2 + \frac{1}{2}\partial_{11} f(x_0) v_1^2$$

$$+\partial_{12} f(x_0) v_1 v_2 + \frac{1}{2}\partial_{22} f(x_0) v_2^2 + \frac{1}{6}\partial_{111} f(x_0) v_1^3 + \frac{1}{2}\partial_{112} f(x_0) v_1^2 v_2$$

$$+\frac{1}{2}\partial_{122} f(x_0) v_1 v_2^2 + \frac{1}{6}\partial_{222} f(x_0) v_2^3 + \dots$$

This lecture took place on 23rd of June 2016 with lecturer Wolfgang Ring.

Formula of 2nd degree: $|I| = 2, j = 2$

$$\sum_{i_1,i_2=1}^{n} \underbrace{\partial i_1 i_2 f(x_0)}_{[D_2 f(x_0)]_{i_1,i_2}} \cdot v_{i_1} v_{i_2} = \frac{1}{2} v^t \cdot D_2 f(x_0) \cdot v$$

where

$$D_2 f(x_0) = \begin{bmatrix} \sum_{i_2=1}^{n} \partial_{1,i_2} f(x_0) \cdot v_{i_2} \\ \sum_{i_2=1}^{n} \partial_{2,i_2} f(x_0) \cdot v_{i_2} \\ \sum_{i_2=1}^{n} \partial_{n,i_2} f(x_0) \cdot v_{i_2} \end{bmatrix}$$

$$v^t = (v_1, v_2, \dots, v_n)$$

Term of 2nd order in Taylor series:

$$\frac{1}{2} v^t D_2 f(x_0) \cdot v$$

Theorem 53 (Taylor's Theorem, qualitative consideration)**.** Let $f \in C^K(D), x_0 \in D, D \subseteq \mathbb{R}^n$ open. Let $r > 0$ such that $B(x_0, r) \subseteq D$ and let $|v| < r$. Then it holds that

$$f(x_0 + v) = \sum_{j=0}^{k} \sum_{|I|=j} \frac{1}{I!} \partial_I f(x_0) \cdot v^I + \underbrace{o(\|v\|^k)}_{R_k(x_0;v)}$$

where $o(\|v\|^k) = g(v) \cdot \|v\|^k$ with $\lim_{v \to 0} g(v) = 0$.

Proof. For this proof we use the Euclidean norm (then some estimates can be without constants). Because of equivalence of all norms, the proof follows for all norms.

For Euclidean norm it holds that

$$|v_i| = \sqrt{v_i^2} \leq \sqrt{\sum_{j=1}^{n} v_j^2} = \|v\|_2$$

We use the Taylor formula with remaining term from before (we look at degree up to $k-1$).

$$f(x_0 + v) = \sum_{j=1}^{k-1} \sum_{|I|=j} \frac{1}{|I|!} \cdot \partial_I f(x_0) \cdot v^I + \sum_{|I|=k} \frac{1}{I!} \cdot \partial_I f(x_0 + vw) v^I$$

$$+ \sum_{|I|=k} \frac{1}{I!} \partial_I f(x_0) v^I - \sum_{|I|=k} \frac{1}{I!} \partial_I f(x_0) \cdot v^I$$

$$= \sum_{j=0}^{k} \sum_{|I|=j} \frac{1}{I!} \partial_I f(x_0) v^I + \underbrace{\sum_{|I|=k} \frac{1}{I!} \left[\partial_I f(x_0 + vw) - \partial_I f(x_0)\right] \cdot v^I}_{R_k(x_0;v)}$$

$$|R_k(x_0, v)| \leq \sum_{|I|=k} \frac{1}{I!} \left|\partial_I f(x_0 + vw) - \partial_I f(x_0)\right| \cdot \underbrace{\left|v_1^{i_1} v_2^{i_2} \dots v_n^{i_n}\right|}_{\leq \|v\|^{i_1} \cdot \|v\|^{i_2} \dots \|v\|^{i_n}}$$

$$I = (i_1, i_2, \dots, i_n) = \|v\|^{i_1 + \dots + i_n} = \|v\|^k$$

and $D_I f(x_0 + vw) - D_I f(x_0) \to 0$ for $v \to 0$ because $f \in \mathcal{C}^k(D)$.

So $|R(x_0; v)| \leq \|v\|^k \cdot \tilde{g}(v)$ with $\tilde{g}(v) \to 0$ for $v \to 0$, $\tilde{g}(v) \geq 0$.

$$\implies g(v) = \frac{|R(x_0; v)|}{\|v\|^k} \text{ for } v \neq \vec{0}$$

$$0 \leq g(v) \leq \tilde{g}(v) \text{ and } \lim_{v \to 0} g(v) = 0$$

□

Theorem 54 (Sufficient optimality criterion)**.** Let $f : D \to \mathbb{R}$; $D \subseteq \mathbb{R}^n$ open. Let $x_0 \in D$ be a *critical point* of f, hence $df(x_0) = \vec{0}$ or equivalently $\nabla f(x_0) = \vec{0}$.

Assume

1. Let $D_2 f(x_0)$ be positive definite. Then f has a strict local minimum in x_0.
2. Let $D_2 f(x_0)$ be negative definite. Then f has a strict local maximum in x_0.
3. Let $D_2 f(x_0)$ be indefinite. Then x_0 is not an extremum. x_0 is called saddle point of f.

M is positive definite iff $v^t M v > 0 \quad \forall v \neq 0$.

Proof. 1. Choose the Euclidean norm. Let x_0 be a critical point $D_2 f(x_0)$ spd (symmetrically positive definite). Taylor:

$$f(x) = f(x_0) + \frac{1}{2}(x - x_0)^t D_2 f(x_0)(x - x_0) + o(\|x - x_0\|)$$

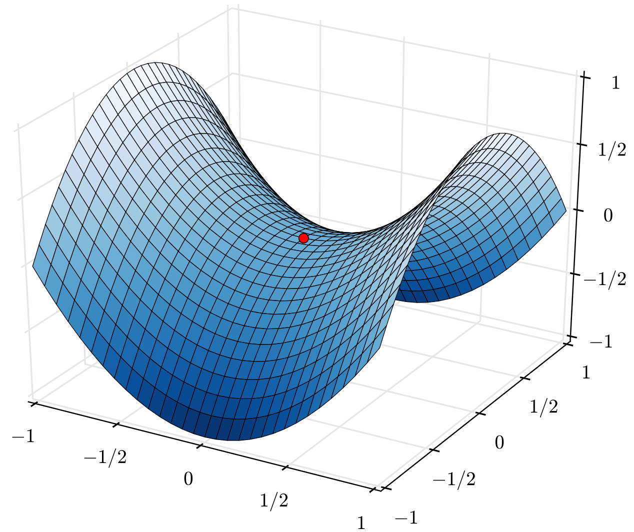


Figure 42: Saddle point in $\mathbb{R}^2$, image by Nicoguaro

$D_2 f(x_0) = \vec{0}$ because x_0 is a critical point.

$$\implies f(x) - f(x_0) = \frac{1}{2}(x - x_0)^t D_2 f(x_0)(x - x_0) \|x - x_0\|^2$$

because $\lim_{v \to 0} g(v) = 0$. Let

$$m = \min \left\{ \underbrace{\frac{1}{2} v^t D_2 f(x_0) v}_{>0 \quad \forall v \neq 0} : v \in \mathbb{R}^n, \|v\| = 1 \right\}$$

$$S^{n-1} = \{v \in \mathbb{R}^n : \|v\| = 1\} \qquad (n-1) \text{ sphere compact in } \mathbb{R}^n$$

$$v \mapsto \frac{1}{2} v^t D_2 f(x_0) v$$

is continuous. So m exists as minimum and $m > 0$. Let $\|x - x_0\|$ be sufficiently small ($< \delta$) such that $0 \leq g(x - x_0) < \frac{m}{2}$. For $\|x - x_0\| < \delta$ it holds that

$$f(x) - f(x_0) = \|x - x_0\|^2 \left(\frac{1}{2} \underbrace{\left(\frac{(x - x_0)^t}{\|x - x_0\|} \right)^t}_{=:v \in S^{n-1}} D_2 f(x_0) \frac{(x - x_0)}{\|x - x_0\|} + g(x - x_0) \right)$$

$$\geq \|x - x_0\|^2 \left(m - \frac{m}{2}\right) = \|x - x_0\|^2 \frac{m}{2} > 0$$

for $x \neq x_0$ and $\|x - x_0\| < \delta$. So $f(x) > f(x_0) \forall x \in B(x_0, \delta) \setminus \{x_0\}$. So x_0 is a strict minimum.

2. analogously

3. Let $D_2 f(x_0)$ be indefinite. Hence there exists eigenvalues $\lambda > 0$ and $\mu < 0$ with eigenvectors v, w such that

$$v^t \cdot D_2 f(x_0) \cdot v = v^t \cdot \lambda v = \lambda \|v\|^2 > 0$$

and

$$w^t \cdot D_2 f(x_0) \cdot w = w^t \cdot \mu \cdot w = \mu \cdot \|w\|^2 < 0$$

As in Taylor's formula:

$$\varphi(t) = f(x_0 + tv)$$

$$\varphi'(t) = \langle \nabla f(x_0 + tv); v \rangle$$

$$\varphi'(0) = 0 \text{ because } \nabla f(x_0) = \vec{0}$$

$$\varphi''(t) = \sum_{i,j=1}^n \partial_{ij} f(x_0 + tv) \cdot v_i \cdot v_j$$

$$= \frac{1}{2} v^t D_2 f(x_0 + tv) \cdot v \varphi''(0) \qquad = \frac{1}{2} v^t \cdot \lambda v > 0$$

Hence φ has a strict local minimum in $t = 0$ (extreme value in 1 dimensional setting).

$$\varphi(t) = f(x_0 + tw)$$

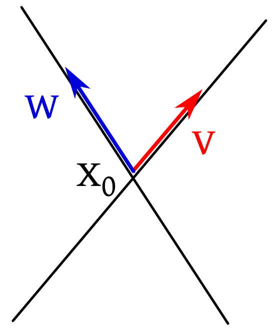


Figure 43: x_0 is maximum by approximation along w. x_0 is minimum by approximation along v.

Analogously,

$$\psi'(0) = 0 \qquad \psi''(0) = \frac{1}{2} w^t \mu w < 0$$

hence ψ has a strict local maximum in $t = 0$.

Compare with Figure 43. Hence in every arbitrary small environment of x_0 there exists a point with larger function value ($x_0 + \varepsilon v$ for sufficiently small ε) and points with smaller function values ($x_0 + \varepsilon' w$ with ε' sufficiently small). Hence x_0 is neither local maximum nor local minimum.

□

14 Additional results in topology

This lecture took place on 24th of June 2016 with lecturer Wolfgang Ring.

To discuss topology any further, we need additional results to fill some gaps in our established theory. We look at those without context.

Reminder: $x_0 \in \mathbb{R}^n$. Then $U \subseteq \mathbb{R}^n$ with $x_0 \in U$ is called *environment of* x_0, if $r > 0$ exists such that $x_0 \in B(x_0, r) \subset U$.

$O \subseteq \mathbb{R}^n$ is open if and only if O is environment of every point $x \in O$.

Lemma 38. U is environment of $x_0 \in U$ if and only if there exists some open set in $\mathbb{R}^n$ with $x_0 \in O \subseteq U$.

Proof. Let U be an environment of x_0, then $B(x_0, r) = \{x \in \mathbb{R}^n : \|x_0 - x\| < r\}$ satisfies the properties of the open set O. Hence $B(x_0, r)$ is open in $\mathbb{R}^n$. □

.

On the opposite, let $x_0 \in O \subseteq U$ with open O. Then there exists, because O is open, some $r > 0$: $B(x_0, r) < O \implies B(x_0, r) \subseteq U$ because $O \subseteq U$.

Hence "environment" is a topological term (hence can be defined with open sets only).

Let $M \subseteq \mathbb{R}^n$. We define environments and open sets *in* M (*relative* to M).

Definition 46. $O \subseteq M$ is called *open* in M if some open set $O^* \subseteq \mathbb{R}^n$ such that $O = M \cap O^*$.

$U \subseteq M$ is called environment of x_0 *in* M if some environment U^* of x_0 in $\mathbb{R}^n$ exists such that $U = M \cap U^*$.

Example 34. $M = Q \subseteq \mathbb{R}$. $O^* = (0,1)$ is open in $\mathbb{R}$. So $O = (0,1) \cap \mathbb{Q} = \{g \in \mathbb{Q} : 0 < g < 1\}$ is open in $\mathbb{Q}$.

But O is *not* an open set in $\mathbb{R}$ because in every interval $(q - \varepsilon, q + \varepsilon)$ for $q \in O$ there are also irrational numbers.

Example 35. Similarly, $M = [0,1]$. Then $\left(-\frac{1}{2}, \frac{1}{2}\right) \cap M = \left[0, \frac{1}{2}\right)$ is open in M, not open in $\mathbb{R}$.

Remark 61. Let $M \subseteq \mathbb{R}^n$ be open itself. Then $O \subseteq M$ is open in M if and only if O is open.

Proof. Let $O = M \cap O^*$ be open in M where M and O^* are open. Then O is open in $\mathbb{R}^n$ as intersection of finitely many sets.

On the opposite: Let $O \subseteq M$ be open in $\mathbb{R}^n$ itself. Then $O = M \cap O \implies O$ is open in M. □

Definition 47. Let X be a set. $\tau \subseteq \mathcal{P}(X)$. Let τ satisfy properties:

1. $\psi \in \tau, x \in \tau$.
2. Let I be an index set and for every $i \in I$ let $O_i \in \tau$. Then it holds that $\bigcup_{i \in I} O_i \in \tau$.
3. Let $O_1, \ldots, O_k \in \tau$, $k \in \mathbb{N}$. Then it holds that $\bigcap_{i=1}^k O_i \in \tau$.

Then τ is called *topology* in X. Elements O of τ are called *open sets*.

Remark: $\{O \subseteq \mathbb{R}^n : O \text{ is open}\}$ is an topology in $\mathbb{R}^n$. Let $M \subseteq \mathbb{R}^n$. Then $\tau_M := \{O \subseteq M : O \text{ is open in } M\}$ is also a topology in M. τ_M is called *relative topology* of M in regards of $\mathbb{R}^n$.

Lemma 39. Let $M \subseteq \mathbb{R}^n$, $f : M \to \mathbb{R}^m$ and $x_0 \in M$. Then f is continuous in x_0 iff for every environment V of $y_0 = f(x_0)$ in $\mathbb{R}$ it holds that $U = f^{-1}(V)$ is an environment of x_0 in M.

$f : M \to \mathbb{R}^m$ is continuous in M iff for every open set $O \subseteq \mathbb{R}^m$ it holds that $f^{-1}(O)$ is open in M.

Proof. First part:

Let $\varepsilon > 0$ be arbitrary. Assume the preimages of environments of $y_0 = f(x_0)$ are environments of x_0. Now let $V = B(y_0, \varepsilon)$ be an environment of y_0. Let

$$U = f^{-1}(V) = \{x \in M : f(x) \in V\} = \left\{ x \in M : \underbrace{\|f(x) - f(x_0)\| < \varepsilon}_{\iff f(x) \in B(y_0, \varepsilon) = V} \right\}$$

By assumption U is the environment of x_0 *in* M.

Hence $\exists \delta > 0 : B(x_0, \delta) \cap M \subset U$. So for every $x \in B(x_0, \delta) \cap M$ it holds that $x \in U = f^{-1}(V)$ or equivalently $f(x) \in V$, hence $\|f(x) - f(x_0)\| < \varepsilon$.

We rewrite the implication of the last paragraph: $\forall x \in M$ with $\|x - x_0\| < \delta$ it holds that $\|f(x) - f(x_0)\| < \varepsilon$. Hence f is continuous in x_0.

On the opposite, let f be continuous in x_0 and $V \subseteq \mathbb{R}^n$ an arbitrary environment of $y_0 = f(x_0)$. Hence $\exists \varepsilon > 0 : y_0 \in B(y_0, \varepsilon) \subseteq V$. Continuity implies $\exists \delta > 0 : \forall x \in B(x_0, \delta) \cap M$ it holds that $\|f(x) - f(x_0)\| < \varepsilon$ with $f(x_0) = y_0$. Hence for these x it holds that $f(x) \in B(y_0, \varepsilon) \subseteq V$.

Equivalently $x \in f^{-1}(V)$. So it holds that $B(x_0, \delta) \cap M \subseteq f^{-1}(V)$. Then $f^{-1}(V)$ is environment of x_0 *in* M.

Second part:

Let $f : M \to \mathbb{R}^m$ be continuous in every point of M. Let $O \subseteq \mathbb{R}^m$ be open. Then O is environment of every point y in O. Let $\tilde{O} = f^{-1}(O)$. If $\tilde{O} = \emptyset$, nothing has to be proven, because $\emptyset$ is open. Let $x \in \tilde{O}$ be arbitrary. Then it holds that $f(x) \in O$. Hence O is environment of $y = f(x)$.

By part 1, it holds that $\tilde{O}$ is environment of x. Hence $\tilde{O}$ is environment of every point $x \in \tilde{O}$, so $\tilde{O}$ is open in M.

Now it remains to prove the other direction.

On the opopsite, let the preimages of open sets be open and $x_0 \in M$ is arbitrary. Consider $y_0 = f(x_0) \in B(y_0, \varepsilon) \subseteq V$, hence V is an arbitrary environment of y_0. Then it holds that $f^{-1}(B(y_0, \varepsilon))$ is open in M. Furthermore it holds that

$$x_0 \in f^{-1}(B(y_0, \varepsilon))$$

Because $B(y_0, \varepsilon) \subseteq V$,

$$f^{-1}(B(y_0, \varepsilon)) \subseteq f^{-1}(V)$$

where $f^{-1}(B(y_0, \varepsilon))$ is a relative open set, which contains x_0, so $f^{-1}(V)$ is an environment of x_0. By part1 is f continuous in every point $x_0 \in M$. □

15 Differentiability of linear and affine maps

Let $A \in \mathbb{R}^{m \times n}$ and $f(x) = Ax + b$ ($b \in \mathbb{R}^m$). $f : \mathbb{R}^n \to \mathbb{R}^m$. Because

$$\|f(x) - f(y)\| = \|Ax - Ay\| \leq \|A\| \cdot \|x - y\|$$

Hence f is Lipschitz continuous with constant $L = \|A\|$. f is differentiable in every point $x \in \mathbb{R}^n$.

$$\|f(x + h) - f(x) - Df(x) \cdot h\|$$

We try $Df(x) = A$.

$$= \|A(x + h) + b - Ax - b - A \cdot h\|$$

$$= \left\|\vec{0}\right\| = 0 = O(\|H\|)$$

So every affine function $f(x) = Ax + b$ is differentiable in $\mathbb{R}^n$ and it holds that $Df(x) = A$.

This lecture took place on 28th of June 2016 with lecturer Wolfgang Ring.

Lemma 40. Let (X, d_X) and (Y, d_Y) be metric spaces. Then by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

with $(x_1, y_1), (x_2, y_2) \in X \times Y$ also $X \times Y$ is also a metric space.

Proof. Symmetry follows by triangle inequality. Left as an exercise to the reader. □

Definition 48. Let (K, τ) be a topological space. Let $T \subseteq \mathbb{P}(X)$ be a topological space. We call $\{U_i : i \in I\}$ with I as index set an *open cover of* K if $U_1 \in \tau$ for all $i \in I$ and $K \subseteq \bigcup_{i \in I} U_i$.

The topological space K is called compact if for every open cover $(U_i)_{i \in I}$ a finite index set $\{i_1, i_2, \ldots, i_N\} \subseteq I$ exists, such that

$$K \subseteq \bigcup_{j=1}^{N} U_{i_j}$$

$(U_{i_j})_{j=1}^N$ is called *finite partial cover* of $(U_i)_{i \in I}$.

Definition 49. A metric space K is called *sequence-compact* if every sequence $(x_n)_{n \in \mathbb{N}}$ in K has a convergent subsequence.

Theorem 55. Let (K, d) be a compact metric spac.e Then K is also sequence-compact.

In other words:

"Topologically compact implies sequence compact."

Proof. Let K be cover-compact. Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in K. Show that $(x_n)_{n \in \mathbb{N}}$ has a limit point (specifically: accumulating point).

Assume $(x_n)_{n \in \mathbb{N}}$ does not have a limit point. $\forall y \in K \exists r_y > 0 : B(y, r_y)$ contains only finitely many elements of the sequence. $B(y, r_y)$ is open. $(B(y, r_y))_{y \in K}$

is an open cover of K. Let $(B(y_i, r_{y_i}))_{i=1}^N$ a finite partial cover. The cover properties imply that $\forall n \in \mathbb{N} \exists i \in \{1, \dots, N\} : x_n \in B(y_i, r_{y_i})$. On the other hand, every $B(y_i, r_{y_i})$ contains only finitely many sequence elements, hence $\bigcup_{i=1}^N B(y_i, r_{y_i})$ contains only finitely many sequence elements.

This contradicts with statement that $\bigcup_{i=1}^N B(y_i, r_{y_i})$ contains all sequence elements. □

We also want to prove the opposite direction: Sequence compact implies (topologically) compact.

Lemma 41. Let (K, d) be a sequence-compact metric space and let $(U_i)_{i \in I}$ be an open cover of K. Then there exists some $\delta > 0$ such that $\forall x \in K \exists i \in I$ with $B(x, \delta) \subseteq U_i$.

Proof. Assume the statement does not hold. Then for every $\delta_n = \frac{1}{n}$ there exists some corresponding $x_n \in K$ such that $\forall i \in I : B(x_n, \delta_n) \not\subseteq U_i$.

Due to sequence-compactness, $(x_n)_{n \in \mathbb{N}}$ is a convergent subsequence $(x_{n_k})_{n \in \mathbb{N}}$ with $x = \lim_{k \to \infty} x_{n_k}$. Because $(U_i)_{i \in I}$ cover K, there exists $i_x \in I$ such that $x \in U_{i_x}$ where U_{i_x} is open. So it holds that: $\varepsilon > 0 : B(x, \varepsilon) \subseteq U_{i_k}$. Now let n be sufficiently large such that $\delta_{n_k} = \frac{1}{n_k} < \frac{\varepsilon}{2}$ and $d(x_{n_k}, x) < \frac{\varepsilon}{2}$.

Let $y \in B(x_{n_{k_1}}, \delta_{n_k})$.

$$d(y, x) \leq f(y, x_{n_k}) + \underbrace{d(x_{n_k}, x)}_{< \frac{\varepsilon}{2}} < \varepsilon$$

$f(y, x_{n_k})$ because $y \in B(x_{n_k}, \delta_{n_k})$ and $\delta_{n_k} < \frac{\varepsilon}{2}$. So it holds that $B(x_{n_k}, \delta_{n_k}) \subseteq B(x, \varepsilon) \subseteq U_{i_x}$. This is a contradiction to propery of $x_n - s$. □

Theorem 56. Let (K, d) be a sequence-compact topological space. Then K is complete and compact.

Proof. First we prove completeness.

Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. Let $(x_{n_k})_{n \in \mathbb{N}}$ be a convergent subsequence with limit x. Then x is also a limit of $(x_n)_{n \in \mathbb{N}}$.

Reason: Let $\varepsilon > 0$ be arbitrary. Choose N such that $n, m \geq N$. Then $d(x_n, x_m) < \frac{\varepsilon}{2}$. Choose $n_k \in \mathbb{N}$ such that $n_k \geq N$ such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$. Then for $n \geq N$ it holds that

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So it holds that $x = \lim_{n \to \infty} x_n$.

We show compactness. Sequence-compactness implies compactness.

Let $(U_i)_{i \in I}$ be an open cover of K. Let $\delta > 0$ as in the previous lemma. Hence $\forall x \in K \in i(x) \in I$ with $B(x, \delta) \subseteq U_{i(x)}$.

We show: There are finitely many points $\{y_1, y_2, \dots, y_N\}$ such that

$$K \subseteq \bigcup_{j=1}^N B(y_j, \delta) \subseteq \bigcup_{j=1}^N U_{i(y_j)}$$

Then we have built a finite partial cover.

We prove this by contradiction. Assume it does not hold. We construct $(y_i)_{i \in \mathbb{N}}$ inductively. Let $y_1 \in K$ be arbitrary. Then $K \subseteq B(y_0, \delta)$. Let $\{y_1, \dots, y_m\}$ be already constructed.

$$K \not\subseteq \bigcup_{j=1}^m B(y_j, \delta)$$

So there exists $y_{m+1} \in K \setminus \bigcup_{j=1}^m B(y_j, \delta)$, hence $y_{m+1} \notin B(y_j, \delta)$ for $j = 1, \dots, m$, hence $d(y_j, y_{m+1} > \delta$.

Consider $(y_j)_{j \in \mathbb{N}}$. For $j \neq l$ it holds that $d(y_j, y_i) > \delta$. But $(y_j)_{j \in \mathbb{N}}$ is not a Cauchy sequence. Then $(y_j)_{j \in \mathbb{N}}$ also has no convergent subsequence, because every convergent subsequence must have the Cauchy property. This contradicts with sequence-compactness.

So the proof is given and $\left(U_{i(y_j)}\right)_{j=1}^N$ is a finite partial cover of K. □

Definition 50. K has the property

$$\forall \delta > 0 \exists \{y_1, \dots, y_N\} : (N = N(\delta))$$

such that $K \subseteq \bigcup_{i=1}^N B(y_i, \delta)$. This property is called *total boundedness of* K. (K must not be compact and can be an arbitrary metric space)

Corollary 7. compact $\iff$ complete and totally bounded

15.1 Exchange theorems

Theorem 57 (Tube lemma)**.** Let X be an arbitrary metric space. Let K be a compact metric space. Let $x_0 \in X$.

We call set $\{x_0\} \times K$ the preimage (dt. Faser) over x_0 in $X \times K$. Compare with Figure 44.

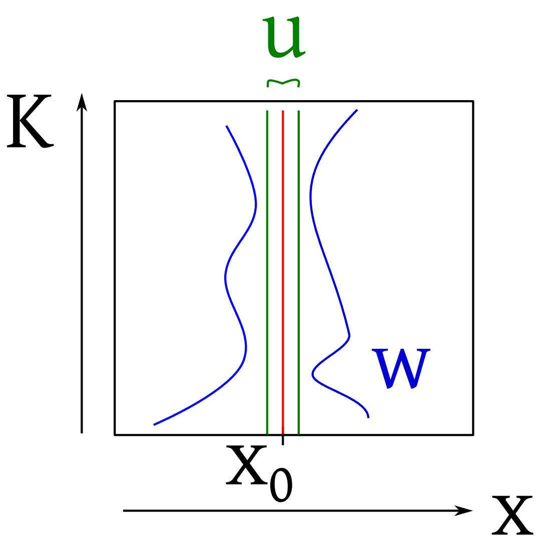


Figure 44: Faser over x_0 in $X \times K$

Let $W \subset X \times K$ be open and $\{x_0\} \times K \subseteq W$. Then there exists some environment U of x_0 in X such that

$$\{x_0\} \times K \subseteq U \times K \subseteq W.$$

A counterexample is given with $y = \frac{1}{|x|}$ and $y = -\frac{1}{|x|}$.

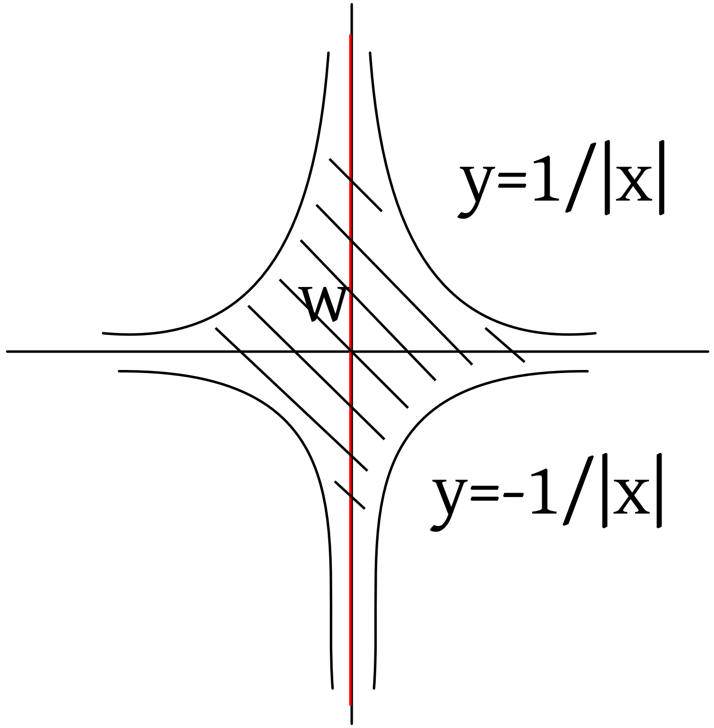


Figure 45: The counterexample

Proof. Let $(x_0, y) \in \{x_0\} \times K \subset W$ because W is open, there exists $B((x_0, y), r_y) \subseteq W$ with $r_y > 0$.

$$\begin{aligned} B((x_0, y), r_y) &= \{(x, z) : \max\{d_X(x_0, x), d_K(y, z)\} < r_y\} \\ &= \{(x, z) : d_X(x_0, x) < r_y \wedge d_K(y, z) < r_y\} \\ &= B_X(x_0, r_y) \times B_K(y, r_y) \end{aligned}$$

For $y \in K$, let $B_K(y, r_y)$ chosen as above. $(B_K(y, r_y))_{y \in K}$ cover K. Because K is compact, we choose a finite partial cover. Let $(B_k(y_i, r_{y_i}))_{i=1}^N$ be a finite partial cover. Then it holds that

$$\delta = \min\{r_{y_i} : i = 1, \dots, N\} > 0$$

$$\bigcup_{i=1}^N B_X(x_0, \delta) \times B_K(y_i, r_{y_i}) \subseteq \bigcup_{i=1}^N B_X(x_0, r_{y_i}) \times B(y_i, r_{y_i}) \subseteq W$$

where $B_k(y_i, r_{y_i})$ cover K.

$$\bigcup_{i=1}^N \{x_0\} \times B(y_i, r_{y_i}) \subseteq \bigcup_{i=1}^N B_x(x_0, \delta) \times B(y_i, r_{y_i})$$

where $\bigcup_{i=1}^N \{x_0\} \times B(y_i, r_{y_i}) = \{x_0\} \times \bigcup_{i=1}^N B(y_i, r_{y_i}) = \{x_0\} \times K$.

$$\{x_0\} \times K \subseteq B_X(x_0, \delta) \times K = B_X(x_0, \delta) \times \bigcup_{i=1}^N B(y_i, r_{y_i})$$
$$= \bigcup_{i=1}^N (B_X(x_0, \delta) \times B(y_i, r_{y_i})) \subseteq W$$

□

Corollary 8. Let K, L be compact metric spaces. Then $K \times L$ are also compact.

Proof. Let $(W_i)_{i \in I}$ be an open cover of $K \times L$.

Let $\{x\} \times L \subseteq K \times L$. Then there exists $\forall (x, y) \in F_y$ some $W_{i(y)}$ with $i(y) \in I$ such that $(x, y) \in W_{i(y)}$ where $W_{i(y)}$ is open. Hence $\exists U_{i(y)}$ and $V_{i(y)}$ as open neighborhood of x in K or equivalently of y in L such that $(x, y) \in U_{i(y)} \times V_{i(y)} \subseteq W_{i(y)}$. So for $x \in K$ it is fixed that the set system $(V_{i(y)})_{y \in L}$ is an open cover of L. Because L is compact, there exists $i(y_1), \dots, i(y_M)$ such that

$$L \subseteq \bigcup_{i=1}^M V_{i(y_j)}$$

so it holds that

$$\{x\} \times L \subseteq \{x\} \times \bigcup_{j=1}^M V_i(y_j) \subseteq \bigcup_{j=1}^M U_{i(x_j)} \times V_{i(y_j)} \subseteq \bigcup_{j=1}^M W_{i(y_j)}$$

So the preimage (dt. Faser) over x is covered by finitely many Ws. By the Tube lemma, let

$$U_x \times L \subseteq \bigcup_{j=1}^M W_{i(y_j)}$$

where $\{x\} \times L \subseteq U_x$. $(U_x)_{x \in K}$ cover K. Let $(U_{x_i})_{l=1}^N$ be a finite partial cover. Every $U_{x_l} \times L$ is covered by finitely many Ws. So

$$\underbrace{\bigcup_{i=1}^N U_{x_l}}_{K} \times L = K \times L$$

is covered by finitely many Ws. Hence $K \times L$ is compact. □

This lecture took place on 30th of June 2016 with lecturer Wolfgang Ring.

$$f : X \times [a, b] \to \mathbb{R}$$

where X is a metric space. Let f be continuous.

$$d((x, t), (y, \tau)) = \max\{d_X(x, y), |t - \tau|\}$$

is a metric on $X \times [a, b]$. f is continuous in (x_0, t_0):

$$\forall \varepsilon > 0 \exists \delta > 0 : d((x, \tau), (x_0, t_0)) < \delta \implies |f(x, \tau) - f(x_0, t_0)| < \varepsilon$$

For fixed $x \in X$, $t \mapsto f(x, t)$ is continuous in $[a, b]$. Hence,

$$F(x) = \int_a^b f(x, t)\, dt$$

exists. $F : X \to \mathbb{R}$.

Theorem 58. The function $F : X \to \mathbb{R}$ from above is continuous in X. Hence for $x = \lim_{n\to\infty} x_n$ it holds that

$$F(x) = \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} \int_a^b f(x_n, t)\, dt$$

$$F(x) = \int_a^b f(x, t)\, dt = \int_a^b \left(\lim_{n\to\infty} f(x_n, t)\right)\, dt$$

due to continuity of f.

Proof. Let $x_0 \in X$. Show continuity of F in x_0. Let $\varepsilon > 0$ be arbitrary.

$$\varphi(x,t) = f(x,t) - f(x_0,t)$$

Then φ is continuous in $X \times [a,b]$ and $\varphi = 0$ in preimage (dt. Faser) $\{x_0\} \times [a,b]$.

We consider $W = \varphi^{-1}((-\varepsilon,\varepsilon))$ open in $X \times [a,b]$ because φ is continuous and are preimages of open sets. W is an open neighborhood of preimage (dt. Faser) $\{x_0\} \times [a,b]$.

From the Tube Lemma (Theorem 57) it follows that

$$\exists \delta > 0 : \{x_0\} \times [a,b] \subset B(x_0,\delta) \times [a,b] \subseteq W = \varphi^{-1}((-\varepsilon,\varepsilon))$$

Hence for all (x,t) with $d(x,x_0) < \delta$ and $t \in [a,b]$ it holds that $\varphi(x,t) \in (-\varepsilon,\varepsilon)$, hence $|\varphi(x,t)| < \varepsilon$. Now let $d_X(x,x_0) < \delta$,

$$\begin{aligned} |F(x) - F(x_0)| &= \left| \int_a^b (f(x,t) - f(x_0,t))\, dt \right| \\ &\leq \int_a^b \underbrace{|f(x,t) - f(x_0,t)|}_{\varphi(x,t)}\, dt \\ &< \varepsilon \cdot \int_a^b 1\, dt \\ &= \underbrace{\varepsilon(b-a)}_{\text{arbitrary small}} \end{aligned}$$

So F is continuous in x_0. □

Example 36 (Application)**.** Let $X = [c,d]$ and $f : [c,d] \times [a,b] \to \mathbb{R}$ is continuous. Then $F(x) = \int_a^b f(x,t)\, dt$ is continuous in $[c,d]$. Hence, there exists

$$\int_c^d F(x)\, dx = \int_c^d \int_a^b f(x,t)\, dt\, dx = \int_{x=c}^d \left(\int_{t=a}^b f(x,t)\, dt \right) dx$$

Theorem 59 (Derivative on integral)**.** Let $U \subseteq \mathbb{R}$ be open, $f : U \times [a,b] \to \mathbb{R}$ be continuous. We let

$$F(x) = \int_a^b f(x,t)\, dt$$

f furthermore satisfies,

- For every $t \in [a,b]$ let $x \mapsto f(x,t)$ be partially differentiable in every coordinate direction.
- The partial derivatives $\partial_{x_k} f(x,t)$ are continuous functions in $U \times [a,b]$.

Then $F : U \to \mathbb{R}$ is continuously differentiable in U and it holds that

$$\partial_{x_k} F(x) = \int_a^b \partial_{x_k} f(x,t)\, dt$$

Proof. If suffices to show that every individual partial derivative exists and is continuous. Hence without loss of generality, $U \subseteq \mathbb{R}$, $f : U \times [a,b] \to \mathbb{R}$, hence $f(x,t) \in \mathbb{R}$.

We let $\psi(x,t) = \partial_x f(x,t) - \partial_x f(x_0,t)$. ψ is continuous in $U \times [a,b]$. In preimage (dt. Faser) $\{x_0\} \times [a,b]$, ψ equals zero. Continuity of ψ is given.

$$W = \psi^{-1}\left(\left(-\frac{\varepsilon}{b-a}, \frac{\varepsilon}{b-a} \right) \right)$$

is open in $U \times [a,b]$. From the Tube lemma it follows that

$$\exists \delta > 0 \forall (x,t) \in \underbrace{(x_0 - \delta, x_0 + \delta) \cap U}_{:=V} \times [a,b]$$

$$|\psi(x,t)| < \frac{\varepsilon}{b-a}$$

Now let $(x,t) \in V \times [a,b]$ with $x \neq x_0$.

$$\frac{F(x) - F(x_0)}{x - x_0} - \int_a^b \partial_x f(x_0,t)\, dt = \int_a^b \left(\frac{f(x,t) - f(x_0,t)}{x - x_0} - \partial_x f(x_0,t) \right) dt$$

where

$$\frac{f(x,t) - f(x_0,t)}{x - x_0} = \frac{\partial_x f(\xi(t),t) \cdot (x - x_0)}{(x - x_0)}$$

with $\xi(t)$ between x and x_0, hence $|\xi(x) - x_0| < |x - x_0| < \delta$ by the Intermediate Value Theorem.

$$= \int_a^b \left(\underbrace{\partial_x f(\xi(t), t) - \partial_x f(x_0, t)}_{\psi(\xi(t),t)\,dt} \right) dt$$

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - \int_a^b \partial_x f(x,t)\,dt \right|$$

$$= \underbrace{\left| \int_a^b \psi(\xi(t), t) \right|}_{< \frac{\varepsilon}{b-a} \text{ because } |\xi(t) - x_0| < \delta} dt \le \frac{\varepsilon}{b-a} = \varepsilon$$

Hence $F'(x_0) = \int_a^b \partial_x f(x,t)\,dt$. □

Example 37 (Application)**.** Let $f : [c,d] \times [a,b]$ be continuous. Then it holds that

$$\int_{x=c}^d \left[\int_{t=a}^b f(x,t)\,dt \right] dx = \int_{t=a}^b \left[\int_{x=c}^d f(x,t)\,dx \right]$$

Exchangability of iterated integration.

Proof. Define in $[c,d]$:

$$\phi_1(\xi) = \int_{x=c}^\xi \left[\int_a^b f(x,t)\,dt \right] dx$$

$$\phi_2(\xi) = \int_{t=a}^b \left[\int_c^\xi f(x,t)\,dx \right] dt$$

From the Fundamental theorem of Differential calculus it follows that:

$$\phi_1'(x) = F(x) = \int_a^b f(x,t)\,dt$$

ϕ_2 is integrand $g(\xi, t) = \int_c^\xi f(x,t)\,dx$ is continuous by ξ and differentiable with $\partial_\xi g(x,t) = f(x,t)$. By the theorem above it holds that

$$\phi_2'(x) = \int_a^b \partial_\xi g(x,t)\,dt = \int_a^b f(x,t)\,dt$$

So it holds that $\phi_1'(x) = \phi_2'(x)$. So it holds that $\phi_1(x) = \phi_2(x) + k$.

$$\phi_1(c) = \int_c^c \int_a^b f(x,t)\,dt\,dx = 0$$

$$\phi_2(c) = \int_a^b \underbrace{\int_c^c f(x,t)\,dx}_{=0}\,dt = 0$$

So it holds that $\phi_1(x) = \phi(x) \quad \forall x \in [c,d]$.

$$\phi_1(d) = \int_c^d \int_a^b f(x,t)\,dt\,dx$$

$$\phi_2(d) = \int_a^b \int_c^d f(x,t)\,dx\,dt$$

where $\phi_1(d) = \phi_2(d)$. Hence, proof completed. □

This was a variation of the theorem by Fubini.

This lecture took place on 1st of July 2016 with lecturer Wolfgang Ring.

16 Introduction to Variation Calculus

Optimization problem: The unknown we search for is a function. It is an infinite dimensional problem.

Formalization: $L : [a,b] \times \mathbb{R} \times \mathbb{R}$ open.

Find $y : [a,b] \to \mathbb{R}$ such that

$$J(y) = \int_a^b L(x, y(x), y'(x))\,dx$$

needs to be minimal (maximal).

$y \in \mathcal{C}^2([a,b])$ with property $y(a) = \alpha, y(b) = \beta$. Compare with Figure 46.

Example 38. Find $y \in \mathcal{H} = \left\{ y \in \mathcal{C}^2[a,b] : y(a) = \alpha, y(b) = \beta \right\}$ such that $F(x) \coloneqq \pi \int_a^b y(x) \sqrt{1 + (y'(x))^2}\,dx$ which represents the surface of a rotational device with $F(x) \to$ min. Compare with Figure 47.

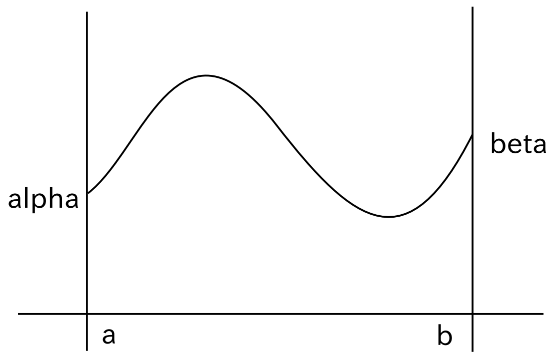


Figure 46: Setting for examples in Variation Calculus

Assume $y_{\min}(x)$ is a solution to our problem. For arbitrary h such that $y_{\min} + th \in H$(admissible set) it must holds that $F(y_{\min}+th) \geq F(y_{\min})$, hence $f(t) = F(y_{\min} + th)$ has a minimum in $t = 0$ with $f'(0) = 0$.

$$f(t) = \int_a^b L(x, y_{\min}(x) + th(x), y'_{\min}(x) + th'(x))\,dx \to \min \text{ at } t = 0$$

Necessary condition: $f'(0) = 0$, derivative under integral

$$\implies \int_a^b \left[\partial_y L(x, y_{\min}(x), y'_{\min}(x)) \cdot h(x) + \partial_p L(x, y_{\min}(x), y'_{\min}(x)) \cdot h'(x)\right] dx = 0$$

$\forall h \in \mathcal{C}^2([a,b])$ with $h(a) = h(b) = 0$.

$$\int_a^b \underbrace{\partial_p L(x, y_{\min}(x), y'_{\min}(x))}_{:=v} \cdot \underbrace{h'(x)}_{=u'}\,dx$$

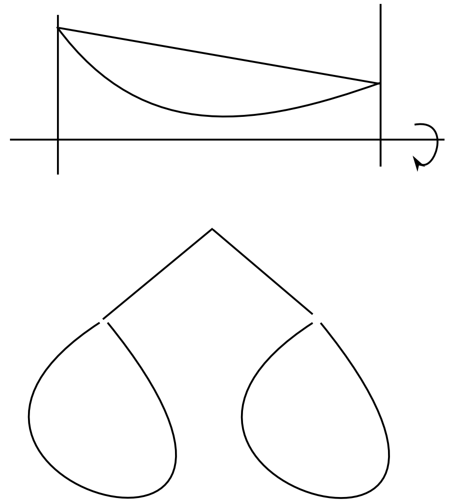


Figure 47: Rotational device in our problem setting.

We apply integration by parts:

$$= \underbrace{h(x) \cdot \partial_p L(x, y_{\min}(x), y'_{\min}(x))\Big|_a^b}_{=0} - \int_a^b \partial_{xp} L(x, y_{\min}, y'_{\min})$$

$$+\partial_{yp} L(x, y_{\min}, y'_{\min}) \cdot y'_{\min} + \partial_{pp} L(x, y_{\min}, y'_{\min}) \cdot y''_{\min}) \cdot h\,dx$$

So,

$$\int_a^b \left[\partial_y L(x,y,y') - \partial_{xp} L(x,y,y') - \partial_{yp} L(x,y,y') - \partial_{pp} L(x,y,y')\right] \cdot h\,dx = 0$$

in $y = y_{\min}$. $\forall h \in \mathcal{C}^2([a,b])$ with $h(a) = h(b) = 0$.

It holds that

$$Ly(x, y_{\min}, y'_{\min}) - \partial_{xp} L(x, y_{\min}, y'_{\min}) - \partial_{yp} L(x, y_{\min}, y'_{\min}) y'_{\min}$$

$$-\partial_{pp}L(x, y_{\min}, y'_{\min})y''_{\min} = 0$$

which is called *Euler-Lagrange Equation* (EL).

Differential equation of second order for $y_{\min}$.

$$y_{\min}(a) = \alpha \qquad y_{\min}(b) = \beta$$

L is called *Lagrange function.* $F(y_{\min} + th)$ is called *variation of* $y_{\min}$.

Special case: $L = L(y, p)$, so it does not depend on x (as for example in $L(y, p) = y\sqrt{1+p^2}$). Then it holds that

$$L_y(y, y') - L_{yp}(y, y') \cdot y' - L_{pp}(y, y') \cdot y'' = 0$$

Lemma 42. If the Lagrange function does not depend on x, then for every solution $\varphi(x)$ of Euler-Lagrange equation, the expression

$$E_\varphi(x) = L_p(\varphi(x), \varphi'(x)) \cdot \varphi'(x) - L(\varphi(x), \varphi'(x))$$

is constant in regards of x.

Proof.

$$E'_\varphi(x) = \text{(EL)} = 0$$

In our example,

$$L_p(y, p) = y \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1+p^2}} \cdot 2p = \frac{yp}{\sqrt{1+p^2}}$$

$$E_y = c \implies \frac{yy'}{\sqrt{1+y'^2}}y' - y\sqrt{1+y'^2} = c$$

At the beginning of the derivation:

$$\partial_x[\partial_p L(y, y')] - \partial_y L(y, y') = 0$$

$$\frac{d}{dx}\left[\frac{yy'}{\sqrt{1+y'^2}}\right] - \sqrt{1+y'^2} = 0$$

$$\implies \frac{y}{\sqrt{1+y'^2}} = \text{constant } c$$

$$y(x) = c\cosh\frac{1}{c}(x - x_0)$$

□

German keywords

English keywords