Linear Algebra 2 – Practicals

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Contents

1 Solution of the last lecture exam of Analysis 1

1.1 Exam: Exercise 1

Exercise 1. Determine the limes of

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

does not help us. What about this representation?

$$\frac{1}{n^2 - 1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1} = \frac{a(n-1) + b(n+1)}{(n+1)(n-1)}$$

$$a(n-1) + b(n+1) = 1$$

$$(a+b)n + (b-a) = 1$$

$$\Rightarrow a + b = 0 \land b - a = 1$$

$$\Rightarrow a = -\frac{1}{2} \quad b = \frac{1}{2}$$

Followingly,

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right)$$

Okay, how to proceed? Let's build a pre-factor:

$$\frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \dots$$

$$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m := \frac{1}{2} \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n+1}$$

Please be aware that we explicitly define S_m because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$= \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=4}^{m+2} \frac{1}{n-1}$$

$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1} \right)$$

We already know $\frac{1}{m} \xrightarrow{m \to \infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m \to \infty} 0$. Followingly also $\frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1} \right) \xrightarrow{m \to \infty} 0$.

1.2 Exam: Exercise 2

Exercise 2. A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n \in \mathbb{N}}$$

$$a_{n+1} = \frac{1}{2} (a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \qquad a_1 = \frac{3}{2} \qquad a_2 = \frac{5}{4} \qquad a_3 \frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \qquad a_1 = 4 \qquad a_2 = \frac{5}{2} \qquad a_3 \frac{7}{4}$$

Exercise 3. a) Show that $1 \overset{!}{<} a_n \overset{!}{\leq} a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over n to prove this statement:

induction base $1 < a_0 \leq a_0$ holds trivially.

induction step We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &\leq \frac{1}{2}(a_0 + a_0) && \text{[induction hypothesis and } 1 < a_0\text{]} \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &> \frac{1}{2}(1 + 1) && \text{[induction hypothesis]} \\ &= 1 \end{aligned}$$

Exercise 4. b) Prove that $a_{n+1} \overset{!}{<} a_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &< \frac{1}{2}(a_n + a_n) && \text{[we have proven: } a_n > 1\text{]} \end{aligned}$$

Exercise 5. c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$b_n := a_n - 1 \qquad \forall n \in \mathbb{N}$$
$$b_0 := a_0 - 1$$
$$b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2} b_n$$

$$b_n = \frac{1}{2^n} b_0 \to 0 \cdot b_0 = 0$$
$$\Rightarrow b_n \to 0$$
$$\Rightarrow a_n = b_n + 1 \to 1$$

Does it work to just show: $1 = \frac{1}{2}(1 + 1)$? Nope, because in points of continuity this might be true even though 1 is not its limes.

Let $a_n \to a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$a_{n+1} \to a \qquad \frac{1}{2}(a_n + 1) \to \frac{1}{2}(a + 1) \qquad a = \frac{1}{2}(a + 1)$$

1.3 Exam: Exercise 3

Exercise 6. $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an ε-δ-proof.

If we don't need an ε-δ-proof, we would argue with the Algebraic Continuity Theorem: The function f is a composition of continuous functions, hence a continuous function itself.

ε-δ-definition:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If $|x - x_0| < \delta$,

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)\right| \\
&= \left|2x^2 + 5x - 2x_0^2 - 5x_0\right| \\
&\le 2\left|x^2 - x_0^2\right| + 5|x - x_0| \\
&= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\
&= 2|x + x_0||x - x_0| + 5|x - x_0| \\
&\le 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\
&\le 2(|x_0| + \delta + |x_0|) + 5\delta
\end{aligned}$$

Our goal: we are able to claim $\overset{!}{<} \varepsilon$

$$\begin{aligned}
&= 4|x_0|\delta + 2\delta^2 + 5\delta \\
&= 2\delta^2 + (4|x_0| + 5)\delta
\end{aligned}$$

In general (here it does not apply), that x_0 might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$\begin{aligned}
2\delta^2 &< \frac{\varepsilon}{2} \\
\delta &< \frac{\sqrt{\varepsilon}}{2} \\
(4|x_0| + 5)\delta &< \varepsilon \\
\delta &< \frac{\varepsilon}{4|x_0| + 5}
\end{aligned}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{5}, \frac{\varepsilon}{4|x_0|+6}\right)$. Then the ε-δ definition shows that f is continuous.

2 Exam: Exercise 4

Exercise 7. Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$h(0) = f(0) - f\left(\frac{1}{2}\right)$$
$$h(1) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0)$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x+\frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore h is continuous, because it is a composition of continuous functions.

Case 1: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$
$$\Rightarrow f(\xi) = f(\xi + \frac{1}{2})$$

Case 2: $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

Case 3: $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

3 Exercise 1

Exercise 8. Investigate the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(x\,|\,x\,| + x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}$$

So this is differentiable, but in case of $x = 0$, it remains questionable.

We look at the definition of differentiability:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

$$f'(x) = \begin{cases} \lim_{x \to 0} \frac{0}{x} = 0 \\ \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0 \end{cases}$$

It follows that f is differentiable one time.

$$f''(x) = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

What about $x = 0$?

$$\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \begin{cases} \lim_{x \to 0} \frac{0}{x} = 0 \\ \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} 2 = 2 \end{cases}$$

Left and right limes differ. So it is not differentiable.

4 Exercise 2

Exercise 9. Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x \to 1} \frac{\ln x}{x-1}$
2. $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1-\sin x)}$
4. $\lim_{x \to 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4.1 Exercise 2.a

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

4.2 Exercise 2.b

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

A nice hint to find out whether this function is differentiable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{\sin x - x}{x\sin x} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}} \approx x \to 0$$

This exploits, that it will take one run of L'Hôpital's rule (because each expression has at least degree 2) and its limes will be 0 (because of x).

4.3 Exercise 2.c

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\ln(\cos(x))}{\ln(1-\sin(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied. We claim that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = \infty$ is fine.

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\cos(x)}{1-\sin(x)}} = \lim_{x\to\frac{\pi}{2}^-} \frac{-\sin(x)\cdot(1-\sin(x))}{\cos(x)(-\cos(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied.

$$\lim_{x\to\frac{\pi}{2}^-} \frac{-\cos(x)(1-\sin(x)) - \sin(x)\cdot(-\cos(x))}{-\sin(x)(-\cos(x)) + \cos(x)\cdot\sin(x)} = \frac{1}{2}$$

If we want to apply the previous estimate here, we should consider

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos(y) \qquad y = \frac{\pi}{2} - x$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(y)$$

This gives us a different estimate of the result:

$$\lim_{y\to 0^+} \frac{\ln(\sin(y))}{\ln(1-\cos(y))} \approx \lim_{y\to 0^+} \frac{\ln(y)}{\ln\left(\frac{y^2}{2}\right)} = \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y) - \ln(2)} \approx \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y)} = \frac{1}{2}$$

We define neighborhoods:

$$N_\delta(x_0) = \{x : |x - x_0| < \delta\}$$

$$N_R(\infty) = \{x : x > R\}$$

4.4 Exercise 2.d

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}} = \lim_{x\to 1^-} e^{\ln(x)\frac{1}{1-x}} = \exp\left(\lim_{x\to 1^-} \underbrace{\frac{\ln(x)}{1-x}}_{(-1)\cdot\text{Exercise a}}\right) = \frac{1}{e}$$

4.5 Exercise 2.e

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\left(\exp\left(\frac{\ln n}{\sqrt{n}}\right)\right) = \exp\left(\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}\right)$$

The conditions to apply L'Hôpital's rule are satisfied („$\frac{\infty}{\infty}$")

$$\exp\left(\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}\right) = \exp\left(\lim_{n\to\infty}\frac{2\sqrt{n}}{n}\right) = \exp(0) = 1$$

4.6 Exercise 2.f

$$\lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{n\to\infty}\frac{e^x\left(1-e^{-2x}\right)}{e^x\left(1+e^{-2x}\right)} = \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{1}{e^{2x}}}{\lim_{x\to\infty} 1 + \lim_{x\to\infty}\frac{1}{e^{2x}}}$$

Remark:

$$\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\cosh(x)}{\sinh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}$$

$$y = \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} = \frac{1}{\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}} = \frac{1}{y}$$

5 Exercise 3

Exercise 10. Show that the function $f:\mathbb{R}\to\mathbb{R}$ with $x\mapsto x+e^x$ is bijective. Furthermore determine $(f^{-1})'(1)$ and $\lim_{y\to\infty}(f^{-1})'(y)$.

If the function is strictly monotonically increasing, it is injective.

$$f'(x) = 1 + e^x > 0 \qquad \forall x\in\mathbb{R}$$

We show that it is strictly monotonically increasing:

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(\alpha) \qquad \text{with } \alpha\in(x_1,x_2)$$

$$f(x_2)-f(x_1) = f'(\alpha)(x_2-x_1) > 0$$

Is f surjective?

For an arbitrary $y_0\in\mathbb{R}$ it holds that $\exists x_0\in\mathbb{R}: f(x_0)=y_0$:

$$\exists f(a), f(b)\in\mathbb{R}: f(a)\le y_0 < f(b)$$

It holds that

$$\lim_{x\to-\infty} x + \underbrace{e^x}_{\to 0} = -\infty$$

$$\lim_{x\to+\infty} x + e^x = \infty$$

Formally:

$$\forall y_0 \exists x_0 : \forall x < x_0 : f(x) < y_0$$

From the Intermediate Value Theorem it follows that

$$\Rightarrow \exists c \in [a,b) : f(c) = y_0 \qquad c =: x_0$$

So it is surjective.

From injectivity and surjectivity it follows that it is bijective.

5.1 Determine $(f^{-1})'(1)$

$$f(x) = x + e^x$$
$$f'(x) = 1 + e^x$$

We apply the inverse function theorem:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$y = 1 = f(x)$$
$$x = f^{-1}(1)$$

An educated guess gives us that $x = 0$. In general determining x is more difficult.

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$$

5.2 Determine $\lim_{y\to\infty} \left(f^{-1}\right)'(y)$

$$\lim_{y\to\infty} \left(f^{-1}\right)'(y) = \lim_{y\to\infty} \frac{1}{1+e^x}$$

As x grows to infinity, also y grows to infinity. From bijectivity it follows that any value can be reached with x as well as $f(x)$.

$$\underbrace{f'(\underbrace{f^{-1}(\underbrace{y}_{\to\infty})}_{\to\infty})}_{\to\infty}$$

6 Exercise 4

Exercise 11. Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. Show

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2} = f'(x_0)$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)+f(x_0)-f(x_0-h)}{2h} \\
&= \lim_{h'\to 0} \frac{1}{2}\cdot\left(f'(x_0)+\frac{f(x_0)-f(x_0+h')}{-h'}\right) \\
&= \lim_{h'\to 0} \frac{1}{2}\cdot\left(f'(x_0)+\frac{f(x_0+h')-f(x_0)}{h'}\right) \\
&= \frac{1}{2}(f'(x_0)+f'(x_0)) \\
&= f'(x_0)
\end{aligned}
$$

6.1 Exercise 4.b

$$
\lim_{h\to 0}\frac{f(x_0+rh)-f(x_0+sh)}{h} = \lim_{h\to 0}\frac{f(x_0+rh)-f(x_0)}{h} + \lim_{h\to 0}\frac{f(x_0)-f(x_0+sh)}{h}
$$

$$
h_1 = rh \qquad h_2 = sh
$$

$$
\begin{aligned}
&= \lim_{h_1\to 0}\frac{f(x_0+h_1)-f(x_0)}{\frac{1}{r}\cdot h_1} + \lim_{h_2\to 0}\frac{f(x_0)-f(x_0+h_2)}{\frac{1}{s}\cdot h_2} \\
&= r\cdot f'(x_0) - s\cdot f'(x_0) \\
&= (r-s)\cdot f'(x_0)
\end{aligned}
$$

7 Exercise 5

Exercise 12. Let $D \subseteq \mathbb{R}$ be an open interval. $f : D \to \mathbb{R}$ is differentiable and f is twice differentiable in $x_0 \in D$.

7.1 Exercise 5.a

Exercise 13. Show that

$$
\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} = f''(x_0)
$$

f is differentiable, therefore continuous, and h goes to 0. So we have „$\frac{0}{0}$". All conditions to apply L'Hôpital's rule are satisfied.

$$
\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h} \approx \frac{0}{0}
$$

We can apply L'Hôpital's Rule again or just use the result of exercise 4a.

$$
\stackrel{4a}{\Longrightarrow} f''(x_0)
$$

7.2 Exercise 5.b

Exercise 14. Show that the limes from exercise 5.a can also exist, even if $f''(x_0)$ does not exist. Use the result from Exercise 1.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

We know that it is not twice differentiable. But we want to show that the limes exists.

We are only concerned with $x = 0$.

$$\lim_{h \to 0} f(x_0) = 0$$

$$\lim_{h \to 0} \frac{h^2 - h^2}{h^2} = \frac{0}{h^2} = 0$$

So if we traverse the graph from both sides at the same time $\frac{f(x_0+h)-f(x_0-h)}{h}$.

8 Exercise 6

Exercise 15. Determine the following limit for arbitrary $c \in \mathbb{R}$:

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right).$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right)$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right) = \lim_{n \to \infty} \frac{e^{\frac{c}{n} \cdot \ln n} - 1}{\frac{\ln n}{n}}$$

and

$$\left(e^{\frac{c}{n} \cdot \ln n}\right)' = e^{\frac{c}{n} \cdot \ln n} \cdot \left(-\frac{c}{n^2} \cdot \ln n + \frac{c}{n} \cdot \frac{1}{n} \right) = \frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln(n))$$

All conditions are satisfied to apply L'Hôpital's rule ("$\frac{0}{0}$"):

$$\lim_{n \to \infty} \frac{\frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln n)}{\frac{\frac{1}{n} \cdot n - \ln n}{n^2}}$$

$$= \lim_{n \to \infty} \frac{c \cdot e^{\frac{c}{n} \cdot \ln n} (1 - \ln(n))}{1 - \ln n} = \lim_{n \to \infty} c \cdot e^{\frac{c}{n} \cdot \ln n} = c \cdot 1$$

9 Exercise 7

Exercise 16.
- Show that $e^x \geq 1 + x$ holds for all $x \in \mathbb{R}$.
  Hint: On demand, use the Mean Value Theorem.
- Prove that for all $x > 0$, the following estimates hold:

  $$\ln x \leq x - 1$$

  and for all $k \in \mathbb{N}_+$ it holds that

  $$k \left(1 - \frac{1}{\sqrt[k]{x}} \right) \leq \ln x \leq k \left(\sqrt[k]{x} - 1 \right)$$

$x \geq 0$ Choose $f(x) = e^x$ in $[0, x)$. Mean value theorem:

$$\exists x_0 : f'(x_0) = \frac{f(b) - f(a)}{b - a} \quad \text{for } a < x_0 < b$$

$$f'(x_0) = e^{x_0} \qquad e^{x_0} \geq 1 \qquad x_0 \geq 0$$

$$e^{x_0} = \frac{f'(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x} \Rightarrow \frac{e^x - 1}{x} \geq 1$$

Or alternatively: f is convex and therefoer $f''(x) > 0$.

Consider $f(x) = x - 1 - \ln x$

$$f'(x) = 1 - \frac{1}{x} \qquad f''(x) = \frac{1}{x^2}$$

$$f'(x) \stackrel{!}{=} 0$$

$$1 - \frac{1}{x} = 0 \Leftrightarrow x = -1$$

$$f''(1) = 1 > 0 \Rightarrow \text{ minimum and because } f(1) = 0 \Rightarrow \forall x : x - 1 - \ln x \geq 0$$

Or alternatively:

$$y := x - 1$$

$$x = y + 1$$

Show that $\ln(y + 1) \leq y \Leftrightarrow y + 1 \leq e^y$.

e^x is monotonically increasing $\Rightarrow x \leq y \Leftrightarrow e^x \leq e^y$.

And this has been proven previously.

9.1 Exercise 7.b

$$\ln(x) \leq k \left(\frac{[}{k}]x - 1\right)$$

$$\ln(\sqrt[k]{x}) \leq \sqrt[k]{x} - 1 \Leftrightarrow \ln(y) \leq y - 1$$

And this has been proven in Exercise a.

The second part following analogously.

10 Exercise 8

Exercise 17. Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Show: If f is continuous in an environment U of $a \in D$, differentiable in $U \setminus \{a\}$ and there exists $\lim_{x\to a} f'(x)$, such that f in a differentiable and

$$f'(a) = \lim_{x\to a} f'(x).$$

Hint: On demand, use the Mean Value Theorem.

Let h_n be an arbitrary zero-sequence (with $h_n(x) > 0 \quad \forall x \in D$) and due to Mean Value Theorem $\exists \xi_n \in D$ with $f'(\xi_n) = \frac{f(a+hn) - f(a)}{h_n}$.

$$\lim_{n\to\infty} f'(\xi_n) = \lim_{x\to a} f'(x) = \lim_{n\to\infty} \frac{f(a + h_n) - f(a)}{h_n} = f'(a)$$

$$\lim_{n\to\infty} \frac{f(a + h_n) - f(a)}{h_n} = \lim_{n\to\infty} f'(\xi_n) = \lim_{x\to a} f'(x) = z$$

For the arbitrary zero-sequence, we really need to consider it arbitrary (otherwise we just show it for the one sequence). Consider this counterexample:

$$f(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

10.1 Alternative approach

Application of "Schrankensatz".

$$\exists \lim f'(x) = \alpha$$

Hence for arbitrary $\varepsilon > 0 : \exists \delta > 0 \forall x \in (a-\delta, a+\delta) \setminus \{a\} : |f'(x) - \alpha| < \varepsilon$. Hence $\alpha - \varepsilon < f'(x) < \alpha + \varepsilon$.

- $$\forall x \in (a, a+\delta) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$
- $$\forall x \in (a-\delta, a) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall x \in (a-\delta, a+\delta) \setminus \{a\} : \left| \frac{f(x) - f(a)}{x - a} - \alpha \right| \leq \varepsilon$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha$$

10.2 Second alternative approach

$$\lim_{f(a+h)-f(a)} h$$

If I know f is continuous, then $f(a+h) \to f(a)$. So,

$$\text{“}\frac{0}{0}\text{”}$$

$$\lim_{h \to 0} \frac{f'(a+h) - 0}{1} = \lim_{h \to 0} f'(a+h) = \lim_{x \to a} f'(x)$$

11 Exercise 9

Exercise 18. Let $f : [a,b] \to \mathbb{R}$, $a < b$, differentiable with $f(a) > 0$, $f'(a) > 0$ and $f(b) = 0$. Prove that there exists $\xi \in (a,b) : f'(\xi) = 0$.

First, we want to show that $f'(a) > 0 \Rightarrow \exists \delta > 0 \forall x \in (a, a+\delta) : f(x) > f(a)$.

$$\exists \delta > 0 \forall x \in (a, a+\delta) : \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0$$

$$\Rightarrow f(x) - f(a) > \frac{f'(a)}{2}(x - a) > 0$$

Indeed, $f(x)$ satisfies this property.

Secondly, we want to show that,

$$\exists \eta \in (a+\delta, b) : f(a) = f(\eta)$$

$$\exists \xi \in [a,\eta] \forall x_1 \in [a,\eta] : f(\xi) \geq f(x_1)$$
$$\exists \xi \in (a,\eta) : \frac{f(\eta) - f(a)}{\eta - a} = f'(\eta) = 0$$

There might be more than this one ξ, so the ξ between the second and third line might be different. Anyways, we found a ξ with the desired property.

12 Exercise 10

Exercise 19. Determine the pointwise limit of the following function sequences $f_n : [0,\infty) \to \mathbb{R}$ and determine its uniform convergence:

- $f_n(x) = \sqrt[n]{x}$
- $f_n(x) = \frac{1}{1+nx}$
- $f_n(x) = \frac{x}{1+nx}$

12.1 Exercise 10.a

If $x \neq 0$, $\lim_{n\to\infty} \sqrt[n]{x} = 1$.
If $x = 0$, $\lim_{n\to\infty} \sqrt[n]{x} = \lim_{n\to\infty} 0^{\frac{1}{n}} = 0$.

In terms of uniform convergence:

$$\left| \sqrt[n]{x} - 1 \right| < \varepsilon$$
$$\lim_{x\to\infty} \sqrt[n]{x} = \infty$$

Example:

$$\begin{aligned} \left| \sqrt[n]{x} - 1 \right| &< \varepsilon \\ \sqrt[n]{x} - 1 &< \varepsilon \\ \sqrt[n]{x} &< \varepsilon + 1 \\ \sqrt[n]{100} &< \varepsilon + 1 \end{aligned}$$

12.2 Exercise 10.b

$$f_n(x) = \frac{1}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{1}{1+nx} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{1}{1+n\cdot 0} = 1$$

Assume it it continuously convergent. Show that:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists x \in [0,\infty) : n \geq N \wedge |f_n(x) - f(x)| \geq \varepsilon$$

Does not hold for $\frac{9}{n} \geq x$.

12.3 Exercise 10.c

$$f_n(x) = \frac{x}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{x}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{x}+n} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{0}{1+n\cdot 0} = 0$$

$$\left|\frac{x}{1+nx} - 0\right| < \varepsilon$$

$$\left|\frac{x}{1+nx}\right| < \left|\frac{x}{nx}\right| = \left|\frac{1}{n}\right|$$

Convergence is given. Uniform convergence is not given.

Advice: The simplest approach to show convergence is to show:

$$|f_n(x) - f(x)| \leq a_n \to 0$$

where a_n is independent from x.

13 Exercise 11

Exercise 20. Determine $\cos\alpha$, $\sin\alpha$ and $\tan\alpha$ for $\alpha \in \left\{\frac{\pi}{5}, \frac{2\pi}{5}\right\}$.

Hint: Show that $u := \cos\frac{2\pi}{5}$ and $v := \cos\frac{\pi}{5}$ satisfy the equations $u = 2v^2 - 1$ and $-2u^2 + 1 = v$. Determine u, v this way.

$$\begin{aligned} u = \cos\left(\frac{2\pi}{5}\right) &= \cos\left(\frac{\pi}{5} + \frac{\pi}{5}\right) \\ &= \cos^2\left(\frac{\pi}{5}\right) - \sin^2\left(\frac{\pi}{5}\right) \\ &= 2\cos^2\left(\frac{\pi}{5}\right) - 1 \\ &= 2v^2 - 1 \end{aligned}$$

To show: $v + 2u^2 - 1 = 0$, $\cos\left(\frac{\pi}{5}\right) + 2\cos^2\left(\frac{\pi}{5}\right) - 1 = 0$.

$$\begin{aligned} \cos\left(\frac{\pi}{5}\right) + 2\cos\frac{2\pi}{5} - 1 &= \cos\frac{\pi}{5} + \cos\frac{4\pi}{5} \\ &= \cos\frac{\pi}{5} + \cos\left(\pi - \frac{1}{5}\pi\right) \\ &= \cos\frac{\pi}{5} + \cos\pi\cdot\cos\left(\frac{\pi}{5}\right) + \sin\pi\cdot\sin\frac{\pi}{5} - \cos\frac{\pi}{5}\cdot\cos\frac{\pi}{5} \\ &= 0 \end{aligned}$$

For $u + v > 0$:

$$2v^2 - 1 = u$$

$$-2u^2 + 1 = v$$

$$2v^2 - 2u^2 = u + v$$
$$2(v+u)(v-u) = u + v$$
$$2(v-u) = 1 \Leftrightarrow v - u = \frac{1}{2}$$

$$v - 2v^2 + \frac{1}{2} = 0$$
$$v^2 - \frac{1}{2}v - \frac{1}{4} = 0$$
$$v_{1,2} = \frac{1}{4} \pm \sqrt{\frac{1}{16} + \frac{4}{16}} = \frac{1 \pm \sqrt{5}}{4}$$

$$0 < \cos(\frac{\pi}{5}) = \frac{1 + \sqrt{5}}{4}$$
$$u = \cos\frac{2\pi}{5} = v - \frac{1}{2} = \frac{-1 + \sqrt{5}}{4}$$

$$\cos\left(\frac{2\pi}{5}\right) = \cos^2\frac{\pi}{5} - \sin^2\frac{\pi}{5}$$
$$\Leftrightarrow \frac{-1+\sqrt{5}}{4} = \left(\frac{\sqrt{5}+1}{4}\right)^2 - \sin^2\left(\frac{\pi}{5}\right)$$
$$\Leftrightarrow \sin^2\left(\frac{\pi}{5}\right) = \frac{5 + 2\sqrt{5} + 1}{16} - \frac{-4 + 4\sqrt{5}}{16}$$
$$= \frac{5 + 2\sqrt{5} + 1 + 4 - 4\sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16} = \frac{5 - \sqrt{5}}{8}$$
$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}} \approx 0.59$$

$$\sin\frac{2\pi}{5} = \sin\left(\frac{\pi}{5} + \frac{\pi}{5}\right) = \sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5} + \cos\frac{\pi}{5}\cdot\sin\frac{\pi}{5} = 2\sin\frac{\pi}{5}\cdot\cos\frac{\pi}{5}$$
$$= 2\frac{1+\sqrt{5}}{4}\sqrt{\frac{5-\sqrt{5}}{8}} = \frac{1+\sqrt{5}}{2}\cdot\frac{5-\sqrt{5}}{8} = \sqrt{\frac{5+\sqrt{5}}{8}} \approx 0.95$$

$$\tan\frac{\pi}{5} = \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5}} = \frac{\sqrt{\frac{5-\sqrt{5}}{8}}}{\frac{\sqrt{5}+1}{4}} = \frac{\sqrt{2(5-\sqrt{5})}}{1+\sqrt{5}} = \sqrt{5 - 2\sqrt{5}} \approx 0.73$$
$$\tan\left(\frac{2\pi}{5}\right) = \frac{\sin\frac{2\pi}{5}}{\cos\frac{2\pi}{5}} = \frac{4}{-1+\sqrt{5}}\cdot\frac{1+\sqrt{5}}{2}\cdot\sqrt{\frac{5-\sqrt{5}}{8}} = \sqrt{5 + 2\sqrt{5}} \approx 3.05$$

14 Exercise 12

Exercise 21. To which order do you have to consider values in the series expansion of cosine, to approximate $\cos 1$ with an error smaller 10^{-7}? Furthermore show that $\cos 1$ is irrational.

Hint: To show irrationality of $\cos 1$, assume, $p, q \in \mathbb{N}_+$ with $\cos 1 = \frac{p}{q}$. Replace that in the estimated error of

$$\cos 1 - \sum_{k=0}^{q} \frac{(-1)^k}{(2k)!},$$

multiply with $(2q)!$ and derive a contradiction.

14.1 Exercise 12.a

$$\cos x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \cdot (-1)^k$$

Consider,

$$S_{2m} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k$$

$$S_{2k+1} = \sum_{k=0}^{2k+1} \frac{1}{(2k)!} \cdot (-1)^k$$

So S_{2k+1} has a negative, last expression. S_{2m} has a positive last expression.

$$S_{2k+1} < \cos 1 < S_{2m}$$

$$S_{2m} - S_{2m+1} = \sum_{k=0}^{2m} \frac{1}{(2k)!}(-1)^k - \sum_{k=0}^{2m+1} \frac{1}{(2k)!}(-1)^k$$

$$\triangle \cos(1) = -\frac{1}{(2(2m+1))!} \cdot (-1)^{2m+1} = \frac{1}{(2 \cdot (2m+1))!} \overset{!}{<} 10^{-7}$$

$$N! > 10^7 \Rightarrow N > 11$$

$$2 \cdot (2m+1) > 11$$

$$2m + 1 > \frac{11}{2} = 5.5$$

$\Rightarrow$ 10-th order because every odd expression is cancelled out.

Consider paper: “The irrationality of e and Others”.

14.2 Exercise 12.b

$$\cos(1) \notin \mathbb{Q}$$

Assume $\exists p \in \mathbb{Z}$, $q \in \mathbb{N}$:

$$\cos(1) = \frac{p}{q}$$

$$\left| \cos(1) - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \right|$$

$$= \left| \frac{p}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k}{(2k)!} \right| < \frac{1}{(2q)!}$$

$$= \left| \frac{p(2q)!}{q} - \sum_{k=0}^{q-1} \frac{(-1)^k \cdot (2q)!}{(2k)!} \right| < 1$$

$$|\, x - y \,| < 1 \Rightarrow 0 \qquad \text{because } x \in \mathbb{Z}, y \in \mathbb{Z}$$

Leibniz criterion requires that the limes is not achieved in the sequence, because the functions need to be strictly monotonical.

15 Exercise 13

Exercise 22. Let $f : [\frac{\pi}{2}, \frac{3\pi}{2}] \to [-1,1]$, $x \mapsto \sin x$. Show that f is bijective and compute (using the formula for the derivative of the inverse function $(f^{-1})'(y)$ at all possible points $y \in [-1,1]$). Also give an explicit representation for f^{-1}

$$\ldots = -\frac{1}{\sqrt{1-y^2}}$$

It is important to recognize the negative sign.

16 Exercise 14

Exercise 23. Let $w, z \in \mathbb{R}$ with $w, z, w+z \notin \left\{\frac{\pi}{2} + k\pi \,\middle|\, k \in \mathbb{Z}\right\}$. Prove the addition theorem of the tangens function:

$$\tan(w+z) = \frac{\tan(w) + \tan(z)}{1 - \tan(w)\tan(z)}.$$

Let $x, y \in \mathbb{R}$ with $xy < 1$. Show that $\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and use it to prove the addition theorem for the arcustangens function:

$$\arctan(x) + \arctan(y) = \arctan \frac{x+y}{1-xy}.$$

1. Show that $\tan(w+z) = \frac{\tan(w)+\tan(z)}{1-\tan(w)\tan(z)}$.

$$\tan(w+z) = \frac{\sin(w+z)}{\cos(w+z)} = \frac{\cos(w) \cdot \sin(z) + \sin(w)\cos(z)}{\cos(w)\cos(z) - \sin(w)\sin(z)}$$

$$\frac{\frac{\cos(w)\sin(w)}{\cos(w)\cos(z)} + \frac{\sin(w)\cdot\cos(z)}{\cos(w)\cdot\cos(z)}}{1 - \frac{\sin(w)\sin(z)}{\cos(w)\cos(z)}}$$

$$= \frac{\tan(z) + \tan(w)}{1 - \tan(w)\tan(z)}$$

2. $$\arctan(x) + \arctan(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

   $x, y \in \mathbb{R}, xy < 1$.

   Let $x = \tan(z)$ and $y = \tan(w)$.

$$xy = \tan(z) \cdot \tan(w) = \frac{\sin(z) \cdot \sin(w)}{\cos(z) \cdot (w)} < 1$$

$$\sin(z) \cdot \sin(w) < \cos(z)\cos(w)$$

$$\Leftrightarrow 0 < \cos(z) \cdot \cos(w) - \sin(z) \cdot \sin(w)$$

$$\Leftrightarrow 0 \overset{!}{<} \cos(z+w) \Leftarrow z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \vee w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

   This proof is insufficient! A case distinction for $\cos(z)\cos(w) > 0$ is required.

3. Show that $\arctan(x) + \arctan(y) = \arctan\frac{x+y}{1-xy}$. Let $x = \tan(z)$ and $y = \tan(w)$.

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan\left(\frac{\tan(z)+\tan(w)}{1-\tan(z)\tan(w)}\right) = \arctan(\tan(z+w)) = z+w = \arctan(x)+\arctan(y)$$

17 Exercise 15

Exercise 24. Compute the following integrals by approximating the integrands using a sequence of step functions with the given points. Let $a, b \in \mathbb{R}$ with $a < b$.

1. $\int_a^b e^x \, dx$ with points $x_k := a + k(b-a)/n$.
2. $\int_a^b x^p \, dx$ with points $x_k := aq^k$, $q := \sqrt[n]{b/a}$ and $p \in \mathbb{R} \setminus \{-1\}$.

17.1 Exercise 15.a

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} e^{a+\frac{k(b-a)}{n}} \\
&= \lim_{n\to\infty} \frac{b-a}{n} e^a \sum_{k=0}^{n-1} \left(e^{\frac{b-a}{n}}\right)^k \\
&= \lim_{n\to\infty} e^a \cdot \frac{b-a}{n} \frac{e^{\frac{b-a}{n}}-1}{e^{\frac{b-a}{n}}-1} \\
&= \lim_{n\to\infty} e^a \left(e^{b-a}-1\right) \cdot \underbrace{\frac{\frac{b-a}{n}}{e^{\frac{b-a}{n}}-1}}_{\to 1} \\
&= e^a \cdot \frac{e^b}{e^a} - e^a = e^b - e^a
\end{aligned}$$

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall x \in [a,b]) : |\varphi_n(x) - e^x| < \varepsilon$$

$$\begin{aligned}
e^{a+(b-a)\frac{n-1}{n}} - e^b &= e^{a+(b-a)(1-\frac{1}{n})} - e^b = e^{a+b-\frac{b}{n}-a+\frac{a}{n}} - e^b \\
&= e^{b-\frac{b}{n}+\frac{a}{n}-e^b}
\end{aligned}$$

17.2 Exercise 15.b

$$x_k := aq^k \qquad q := \left(\frac{b}{a}\right)^{\frac{1}{n}}$$

$$p \neq -1$$

$$\begin{aligned}
y_k &:= x_{k+1} - x_k \\
&= aq^{k+1} - aq^k \\
&= aq^k(q-1)
\end{aligned}$$

$$\sum_{k=0}^{n-1} y_k x_k^p = \sum_{k=0}^{n-1} aq^k(q-1)(aq^k)^p = a^{p+1}(q-1)\sum_{k=0}^{n-1}(q^{p+1})^k$$

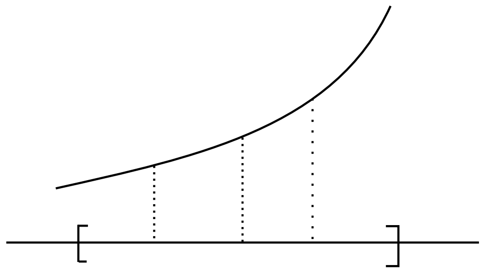

Figure 1: Illustration of 15b

Is a geometric series:

$$= a^{p+1}(q-1)\frac{1-(q^{p+1})^{n-1}}{1-q^{p+1}}$$

$$\lim_{n\to\infty}\sum_{k=0}^{n-1} y_k x_k^p = a^{p+1}\lim_{n\to\infty}\left(\left(\frac{b}{a}\right)^{\frac{1}{n}}-1\right)\frac{1-(\frac{b}{a})^{\frac{n-1}{n}(p+1)}}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = a^{p+1}\left(1-\left(\frac{b}{a}\right)^{p+1}\right)\underbrace{\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}}}_{\text{“0/0”}}$$

$$\lim_{n\to\infty}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}-1}{1-\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \text{“0/0”}$$

L'Hópital's Rule:

$$= \lim_{n\to\infty}\frac{\exp\left(\frac{1}{n}\log\left(\frac{b}{a}\right)\right)-1}{1-\exp\left(\frac{p+1}{n}\log\left(\frac{b}{a}\right)\right)} = \lim_{n\to\infty}\frac{\log\left(\frac{b}{a}\right)\cdot\frac{-1}{n^2}\exp\left(\frac{1}{n}\log\frac{b}{a}\right)}{-(p+1)\log\frac{b}{a}\cdot\frac{-1}{n^2}\exp\left(\frac{p+1}{n}\log\frac{b}{a}\right)}$$

$$= \lim_{n\to\infty}\frac{-1}{p+1}\frac{\left(\frac{b}{a}\right)^{\frac{1}{n}}}{\left(\frac{b}{a}\right)^{\frac{p+1}{n}}} = \frac{-1}{p+1}$$

$$\Rightarrow = (a^{p+1}-b^{p+1})\cdot\frac{-1}{p+1} = \frac{b^{p+1}-a^{p+1}}{p+1}$$

The assignment explicitly asks for a step function. This approach only verifies that

$$\int_a^b x^p \, dx$$

$$\left.\frac{x^{p+1}}{p+1}\right|_{x=a}^{x=b} = \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}$$

We only did the approximation from one side (also upper bound is needed which works analogously):

$$\sum_{k=0}^{n-1} y_k x_{k+1}^p = \dots$$

18 Exercise 16

Exercise 25. For an interval $I \subseteq \mathbb{R}$ let $f_n : I \to \mathbb{R}$ be a sequence of functions which are uniformly continuous converging towards $f : I \to \mathbb{R}$. Show that the following statements hold or provide a counterexample:

- If all f_n are uniformly continuous, then f is uniformly continuous.
- If all f_n are Lipschitz continuous, then f is Lipschitz continuous.

18.1 Exercise 16.a

It holds. So a proof is given in the following.

We want to show:

$$\forall \varepsilon \exists \delta : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f(x_0) + f_n(x_0) - f(x_0)| \\ &\leq \underbrace{|f(x) - f_n(x)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x) - f_n(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} \end{aligned}$$

We need to elaborate: For which n does $\frac{\varepsilon}{3}$ hold?

$$\forall \varepsilon > 0 \exists \overset{\text{depends on } \varepsilon}{n_0} : \forall n \geq n_0 \forall x \in I : |f(x) - f_n(x)| < \frac{\varepsilon}{3}$$
$$\forall \varepsilon > 0 \forall n \exists \delta = \delta(n, \varepsilon) : \forall x, x_0 : |x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

18.2 Exercise 16.b

This does not hold. So we provide a counterexample.

Consider $f(x) = \sqrt{x}$. It is not differentiable at $x = 0$, but $f(0) = 0$ is defined. The function cannot be Lipschitz-continuous, because the Lipschitz constant grows as we tend towards 0. We need functions f_n.

Consider $f(x) = \sqrt{x + \frac{1}{n}}$. The function f_n looks like f, but is shifted slightly to the left. As n tends towards infinity, f_n becomes f and we get the problem at $x = 0$.

You can also consider:

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq \frac{1}{n} \\ \sqrt{\frac{1}{n}} & \text{for } x < \frac{1}{n} \end{cases}$$

19 Exercise 17

Exercise 26. Let $f : [0,1] \to \mathbb{R}$ be a regulated function continuous in 0. Show the following relation:

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\,ds = f(0).$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(s)\,ds = f(0) = \lim_{n\to\infty} n \cdot \left(F\left(\frac{1}{n}\right) - F(0)\right) = \lim_{n\to\infty} \frac{F\left(\frac{1}{n}\right) - F(0)}{\frac{1}{n}} = \lim_{h\to 0} \frac{F(h) - F(0)}{h} = f(0)$$

19.1 Other approach

Continuity at $x = 0$:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall |x| < \delta : |f(x) - f(0)| < \varepsilon$$

$$\lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\,dx \leq \lim_{n\to\infty} n \int_0^{\frac{1}{n}} (f(0) + \varepsilon)\,dx$$

For $\frac{1}{n} < \delta$ it holds that $f(x) < f(0) + \varepsilon$ for $x \in [0, \frac{1}{n}]$.

$$= \lim_{n\to\infty} n(f(0) + \varepsilon)\frac{1}{n} = f(0) + \varepsilon$$

holds for all $\varepsilon > 0$.

$$\Rightarrow \lim_{n\to\infty} n \int_0^{\frac{1}{n}} f(x)\,dx \leq f(0)$$

20 Exercise 18

Exercise 27. Prove the Riemann-Lebesgue Lemma: For every regulated function $f : [a,b] \to \mathbb{R}, a < b$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx = 0.$$

Hint: Show the following partial results:

- For all intervals $[\alpha, \beta] \subseteq [a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_\alpha^\beta \sin(\lambda x)\,dx = 0.$$

- For all step functions $g \in \tau[a,b]$ it holds that

$$\lim_{\lambda\to\infty} \int_a^b g(x)\sin(\lambda x)\,dx = 0.$$

20.1 Exercise 18.a

$$-\frac{1}{\lambda}\cos(\lambda x)\Big|_\alpha^\beta = \underbrace{\frac{1}{\lambda}}_{\to 0}\underbrace{(-\cos(\beta\lambda) + \cos(\alpha\lambda))}_{\text{bounded}}$$

20.2 Exercise 18.b

Because g is a step function of $[a,b]$, there exists a decomposition

$$a = x_0 < x_1 < \ldots < x_n = b$$

such that $g(x)$ has a constant value c_i in every subinterval $[x_{i-1}, x_i)$.

$$\lim_{\lambda\to\infty} \int_a^b f(x)\sin(\lambda x)\,dx = \lim_{\lambda\to\infty} \sum_{i=1}^n c_i \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx$$

This can be done, because we consider a finite sum.

$$\sum_{i=1}^n c_i \underbrace{\int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx}_{\to 0 \forall \text{ subintervals } H(i)}$$

$$= \sum_{i=1}^n c_i \cdot \underbrace{\lim \int_{x_{i-1}}^{x_i} \sin(\lambda x)\,dx = 0}_{\to 0}$$

20.3 Conclusion

Because $f(x)$ is a regulated function $\forall \varepsilon > 0$, there exists a step function $g_\varepsilon(x)$ with $|f(x) - g_\varepsilon(x)| < \varepsilon \quad \forall x \in [a, b]$.

$$\left| \int_a^b f(x)\cdot\sin(\lambda x)\,dx \right| \leq \underbrace{\int_a^b \left| \underbrace{f(x) - g_\varepsilon(x)}_{<\varepsilon} \right|}_{<\varepsilon(b-a)} \cdot \left| \underbrace{\sin(\lambda x)}_{\leq 1} \right| dx + \underbrace{\left| \int_a^b g_\varepsilon(x)\sin(\lambda x)\,dx \right|}_{\to 0 \text{ for } \lambda\to\infty}$$

$$\lim_{\lambda\to\infty} \left| \int_a^b f(x)\sin(\lambda x)\,dx \right| \leq \varepsilon(b-a)$$

We can choose ε arbitrary, so it must tend towards 0.

21 Exercise 19

Exercise 28. Let $I, J \subseteq \mathbb{R}$ be intervals, $f : I \to \mathbb{R}$ continuous and $g, h : J \to I$ differentiable. Furthermore it holds that $g \leq h$ in J. Prove that

$$A : J \to \mathbb{R}, \quad x \mapsto \int_{g(x)}^{h(x)} f(\xi)\,d\xi$$

is differentiable and determine its derivative.

21.1 Exercise 19.a

Show differentiability.

So

$$\lim_{x\to x_0} \frac{A(x) - A(x_0)}{x - x_0}$$

exists.

$$\lim_{x\to x_0} \frac{A(x) - A(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\int_{g(x)}^{h(x)} f(\xi)\,d\xi - \int_{g(x_0)}^{h(x_0)} f(\xi)\,d\xi}{x - x_0} = \lim_{x\to x_0} \frac{F(h(x)) - F(g(x)) - F(h(x_0)) + F(g(x_0))}{x - x_0}$$

$F(h(x))$ and $F(g(x))$ exists, because $h(x)$ is continuous, so a regulated function and regulated functions always have a primitive function.

$$\lim_{x\to x_0} \frac{F(h(x)) - F(h(x_0))}{x - x_0} - \lim_{x\to x_0} \frac{F(g(x)) - F(g(x_0))}{x - x_0}$$

If $h(x)$ is continuous, then $F(h(x))$ is differentiable (analogously for $g(x)$). And the composition is also differentiable.

21.2 Exercise 19.b

Determine its derivative.

$$(F \circ h)'(x) - (F \circ g)'(x_0) = f(h(x_0)) \cdot h'(x_0) - f(g(x_0)) \cdot g'(x_0)$$

22 Exercise 20

Exercise 29. Determine the following integrals for arbitrary $a, b \in \mathbb{R}, a < b$:

- $\int_a^b \frac{d}{dx}\left(x^5 \cdot e^x\right) dx$
- $\int_a^b x^4 e^{x^5} dx$

22.1 Exercise 20.a

$$\int_a^b \frac{d}{dx}\left(x^5 e^x\right) dx = \int_a^b 5x^4 e^x - x^5 e^x \, dx = \int_a^b \underbrace{e^x}_{g'(x)} \underbrace{(5x^4 - x^5)}_{=f(x)} dx$$

$$= e^x(5x^4 - x^5)\Big|_a^b - \int_a^b e^x(20x^3 + 5x^4) = e^b b^5 - e^a a^5$$

22.2 Exercise 20.b

$$\int_a^b x^4 e^{x^5} dx$$

$$u := x^5 \Rightarrow \frac{du}{dx} = 5x^4 \qquad dx = \frac{du}{5x^4}$$

$$= \int_{a^5}^{b^5} x^4 e^u \frac{du}{5x^4} = \int_{a^5}^{b^5} e^u \frac{du}{5} = \frac{1}{5}\int_{a^5}^{b^5} e^u \, du = \frac{1}{5}\left(e^{b^5} - e^{a^5}\right)$$

Other approach for 20.b:

$$F = \frac{1}{5} e^{x^5}$$

$$F' = x^4 e^{x^5} = f$$

23 Exercise 21

Exercise 30. Consider function f.

$$f : [-1,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & x = 0, \\ \frac{1}{n+2}, & x \in \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N}_+. \end{cases}$$

Is f a step function? Is f a regulated function? Furthermore determine

$$\int_{-1}^{1} f(x)\,dx.$$

Is not a step function, because the number of intervals is not finite.

Is it a regulated function? We can approximate f using the following construction:

$$\varphi_k(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{n+2} & x \in \left[-\frac{1}{n}, -\frac{1}{n+1}\right) \cup \left(\frac{1}{n+1}, \frac{1}{n}\right], n \in \mathbb{N}, n \le k \\ 0 & \text{else} \end{cases}$$

We choose a k such that all elements smaller k are nonzero. This approximates our function f.

Consider

$$\int_{-1}^{1} f(x)\,dx \overset{\varphi_n \to f \text{ uniformly}}{=} \lim_{h\to\infty} \int_{-1}^{1} \varphi_n(x)\,dx$$

$$\int_{-1}^{1} \varphi_n(x)\,dx = \sum_{j=1}^{N} c_j \triangle x_j = \sum_{n=1}^{k} \frac{1}{n+2} \cdot \left| \frac{1}{n} - \frac{1}{n+1} \right| \cdot 2 = 2 \cdot \sum_{n=1}^{k} \left(\frac{1}{n(n+2)} - \frac{1}{(n+2)(n+1)} \right)$$

$$= 2 \cdot \sum_{n=1}^{k} \left(\frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) - \left(\frac{1}{n+1} - \frac{1}{n+2} \right) \right)$$

Alternatively we can also split it up, because we estimate that a series with $\frac{1}{n^2}$ converges.

$$= 2 \sum_{n=1}^{k} \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right) - 2 \sum_{n=1}^{k} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = 1 + \frac{1}{2} - 2 \cdot \frac{1}{2} = \frac{1}{2}$$

This expression is easier to evaluate as telescoping sum.

If we take the first approach, we need to apply partial fraction decomposition.

$$\frac{1}{2} \int_k = \frac{1}{2} \int_{-1}^{1} \varphi_k(x)\,dx = \frac{3}{4} - \frac{1}{2} + \frac{1}{2} \left(\underbrace{\frac{1}{k+1}}_{\to 0} + \underbrace{\frac{1}{k+2}}_{\to 0} \right) - \frac{1}{k+1} \overset{h\to\infty}{\longrightarrow} \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

24 Exercise 22

Exercise 31. Let $I \subseteq \mathbb{R}$ be an interval. Determine with the idea from below a primitive function of

$$f : I \to \mathbb{R}, \qquad x \mapsto x^2 \sin x^3 \cos x^3.$$

- For all $x \in \mathbb{R}$ it holds that
$$\sin x \cos x = \frac{1}{2} \sin(2x).$$
- For all $x \in \mathbb{R}$ it holds that
$$\sin x \cos x = \frac{1}{2} \frac{d}{dx} \sin^2(x).$$

Explain possible differences between the results.

24.1 Exercise 22.a

$$\int x^2 \sin(x^3)\cos(x^3)\,dx = \int x^2 \frac{1}{2}\sin(2x^3)\,dx$$

Substitute with $u = 2x^3$ and $dx = \frac{du}{6x^2}$.

$$= \int x^2 \frac{1}{2}\sin(u)\frac{du}{6x^2} = -\frac{1}{12}\cos(u) + c = -\frac{1}{12}\cos(2x^3) + c$$

24.2 Exercise 22.b

$$\forall x \in \mathbb{R} : \sin(x)\cos(x) = \frac{1}{2}\frac{d}{dx}\sin^2(x)$$

$$\forall x \in \mathbb{R} : \sin(x^3)\cos(x^3) = \frac{1}{6x^2}\frac{d}{dx}\sin^2(x^3) = \frac{1}{6x^2}2\sin(x^3)\cos(x^3)3x^2 = \sin(x^3)\cos(x^3)$$

$$\int x^2 \sin(x^3)\cos(x^3)\,dx = \int x^2 \frac{1}{6x^2}\frac{d}{dx}\sin^2(x^3)\,dx = \frac{1}{6}\sin^2(x^3) + c$$

24.3 Exercise 22.c

$$\cos(2x^3) = \cos^2(x^3) - \sin^2(x^3) = 1 - \sin^2(x^3) - \sin^2(x^3) = 1 - 2\sin^2(x^3)$$

$$\Rightarrow \frac{1}{6}\sin^2(x^3) + \tilde{c}$$

with $\tilde{c} \approx \frac{1}{12} + c$.

25 Exercise 23

Exercise 32. Determine the following integrals using integration by parts.

1. $\int e^x \sin x\,dx$
2. $\int \arcsin x\,dx$
3. $\int_0^1 x^2 \ln^3(x)\,dx$

25.1 Exercise 23.a

$$\int \underbrace{e^x}_{u} \underbrace{\sin(x)}_{v'}\,dx$$

with $v = -\cos x$ and $u' = e^x$.

$$= e^x(-\cos x) - \int \underbrace{e^x}_{u} \cdot \underbrace{(-\cos x)}_{v'}\,dx$$

with $u' = e^x$ and $v = -\sin x$.

$$= e^x(-\cos x) - (e^x \cdot (-\sin x)) + \int e^x(-\sin x)\,dx$$

$$= e^x\,(-\cos x + \sin x) - \int e^x \sin x\,dx$$

$$\Rightarrow 2\int e^x \sin(x)\,dx = e^x\,(-\cos(x) + \sin(x)) + c$$

25.2 Exercise 23.b

$$\int \arcsin(x)\, dx = \int \underbrace{\arcsin(x)}_{v} \cdot \frac{v'}{1}\, dx$$

with $v = x$ and $u' = \frac{1}{\sqrt{1-x^2}}$.

$$= \arcsin(x) \cdot x - \int x \frac{1}{\sqrt{1-x^2}}\, dx$$

Let $t = 1 - x^2$. Hence $\frac{dt}{dx} = -2x$.

$$= \arcsin(x) \cdot x - \int \frac{x}{\sqrt{t}} \cdot \frac{1}{-2x}\, dt$$

$$= \arcsin(x) \cdot x + \frac{1}{2} \int \frac{1}{\sqrt{t}}\, dt$$

$$= \arcsin(x) \cdot x + \frac{1}{2} \cdot 2\sqrt{t} + c$$

Backsubstitution:

$$= \arcsin(x) \cdot x + \sqrt{1-x^2} + c$$

25.3 Exercise 23.c

$$\int_0^1 \underbrace{x^2}_{f'} \underbrace{(\ln x)^3}_{g}\, dx = \frac{1}{3} x^3 (\ln x)^3 \Big|_0^1 - \int_0^1 \frac{1}{3} x^3 (\ln x)^2 \cdot 3 \frac{1}{x}\, dx$$

$$= \frac{1}{3} \cdot 0 - \frac{1}{3} \cdot \left(\lim_{x \to 0} (x^3 \ln^3 x) - \int_0^1 \underbrace{x^2}_{f'} \underbrace{\ln^2(x)}_{g}\, dx \right)$$

In the end, we can apply L'Hôpital's Rule once we have expressions like $-\frac{1}{3}\varphi^3 \frac{\ln^3 \varphi}{\ln^3 \varphi}$.

$$= \frac{2}{3} \left(- \int_0^1 \frac{1}{3} x^2\, dx \right) = -\frac{2}{9} \cdot \frac{1}{3} x^3 \Big|_0^1 = -\frac{2}{27}$$

26 Exercise 24

Exercise 33. Determine the following integrals using appropriate substitions:

1. $\int \frac{\cos^3(x)}{1-\sin(x)}\, dx$.
2. $\int \frac{dx}{\sin^2(x)\cos^4(x)}\, dx$ using $t \coloneqq \tan x$.
3. $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx$ using $t \coloneqq \arcsin(x)$.

26.1 Exercise 24.a

$$\frac{\cos^3(x)}{1-\sin(x)}\, dx = \int \frac{\cos(x)\left(1-\sin^2(x)\right)}{1-\sin(x)}\, dx = \int \cos'(x)\,(1 = \sin x)\, dx$$

with $u = 1 + \sin x$ and $\frac{du}{dx} = \cos x$ we get

$$= \int \cos x \cdot u \cdot \frac{du}{\cos x} = \frac{1}{2} u^2 + c = \frac{1}{2}(1 + \sin x)^2 + c$$

Do not forget c for indefinite integrals!

26.2 Exercise 24.b

$$
\begin{aligned}
&\int \frac{1}{\sin^2(x)\cdot\cos^4(x)}\,dx \\
&= \int \frac{\sin^4(x) + 2\sin^2(x)\cos^2(x) + \cos^4(x)}{\sin^2(x)\cos^4(x)}\,dx \\
&= \int \frac{\sin^2(x)}{\cos^4(x)} + \frac{2}{\cos^2(x)} + \frac{1}{\sin^2(x)}\,dx \\
&= \int \frac{\tan^2(x)+2}{\cos^2(x)}\,dx + \int \frac{1}{\sin^2(x)}\,dx
\end{aligned}
$$

Consider the left-handed expression. Consider $t = \tan(x)$ and $\frac{dt}{dx} = \frac{1}{\cos^2(x)}$.

$$
\int \frac{t^2+2}{\cos^2(x)}\cdot\cos^2(x)\,dt = \int t^2 + 2\,dt = \frac{1}{3}t^3 + 2t + c = \frac{1}{3}\tan^3(x) + 2\tan(x) + c_1
$$

Consider the right-handed expression.

$$
\begin{aligned}
\int \frac{1}{\sin^2(x)}\,dx = \int \frac{\sin^2(x)+\cos^2(x)}{\sin^2(x)}\,dx &= \int -\frac{(\cos x)'\sin x - \cos x\cdot(\sin(x))'}{\sin^2(x)}\,dx \\
&= -\int \left(\frac{\cos x}{\sin x}\right)'\,dx = -\frac{\cos x}{\sin x} + c_2
\end{aligned}
$$

So for the overall expression it holds that

$$
\int \frac{\tan^2(x)+2}{\cos^2(x)}\,dx + \int \frac{1}{\sin^2(x)}\,dx = \frac{1}{3}\tan^3(x) + 2\tan(x) - \frac{1}{\tan(x)} + c_3
$$

26.3 Exercise 24.b: Alternative approach

$$
\int \frac{1}{s^2c^4}\,dx
$$

with $s = \sin(x)$, $c = \cos(x)$, $t = \tan(x)$ and $\frac{dt}{dx} = \frac{1}{c^2}$. It holds that

$$
\begin{aligned}
t^2 &= \frac{s^2}{c^2} = \frac{1-c^2}{c^2} = \frac{1}{c^2} - 1 \\
c^2 &= \frac{1}{1+t^2} \\
t^2 &= \frac{s^2}{c^2} = \frac{s^2}{1-s^2} = -1 + \frac{1}{1-s^2} \\
1 - s^2 &= \frac{1}{1+t^2} \\
s^2 &= 1 - \frac{1}{1+t^2} = \frac{t^2}{1+t^2}
\end{aligned}
$$

$$
\int \frac{1}{s^2c^4}\,dx = \int \frac{1}{s^2c^2}\,dt = \int \frac{1+t^2}{t^2}(1+t^2)\,dt
$$

26.4 Exercise 24.c

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}^3}\,dx = \int_0^{\frac{1}{2}} \frac{x^2+1-1}{\sqrt{1-x^2}^3}\,dx = -\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}^3}\,dx$$

$$= -\arcsin(x)|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}^3}\,dx$$

with $t = \arcsin(x)$ and $\frac{dx}{dt} = \cos(t)$ with $x = \sin(t)$. Also $\sqrt{1-x^2}^3 = \sqrt{\cos^2(t)}^3 = \cos(t)^3$.

Recognize that x must be positive for this to hold. As it turns out, this is fine within the interval $(0, \frac{1}{2})$.

$$= -\arcsin(x)|_0^{\frac{1}{2}} + \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^3(t)} \cdot \cos(t)\,dt$$

$$= -\arcsin(x)|_0^{\frac{1}{2}} + \int_{x=0}^{x=\frac{1}{2}} \frac{1}{\cos^2(t)}\,dt$$

$$= -\arcsin(x)|_0^{\frac{1}{2}} + \tan(t)|_{x=0}^{x=\frac{1}{2}}$$

$$= -\arcsin(x)|_0^{\frac{1}{2}} + \tan(\arcsin(x))|_0^{\frac{1}{2}}$$

$$= -\arcsin\left(\frac{1}{2}\right) + \arcsin(0) + \tan\left(\frac{\pi}{6}\right) - \tan(0)$$

$$= -\frac{\pi}{6} + 0 + \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} - 0$$

$$= -\frac{\pi}{6} + 0 + \frac{1}{\sqrt{3}} - 0$$

27 Exercise 25

Exercise 34. Determine

- $\int \frac{\sin x}{\sin x + \cos x}\,dx$.
- $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\cos^2(x)}\,dx$.

27.1 Exercise 25.a

$$\int \frac{\sin x}{\sin x + \cos x} = \frac{\frac{1}{2}(\sin x + \cos x) - \frac{1}{2}(-\sin x + \cos x)}{\sin x + \cos x}\,dx$$

$$= \int \left(\frac{1}{2} + \frac{\sin x - \cos x}{2(\sin x + \cos x)}\right)\,dx$$

With $u = \sin x + \cos x$ and $dx = \frac{du}{\cos(x) - \sin x}$, we get

$$= \frac{1}{2}x + \frac{1}{2} \cdot \int -\frac{1}{u}\,du$$

$$= \frac{x}{2} - \frac{1}{2} \cdot \ln(\sin x + \cos x) + c$$

28 Exercise 25.b

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\cos^2(x)}\,dx$$

Consider $u = \tan x$ and $dx = du\,\cos^2 x$.

$$= \int_{x=\frac{\pi}{4}}^{x=\frac{\pi}{3}} \sqrt{u}\,du = \frac{2}{3}(\tan(x))^{\frac{3}{2}}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{2}{3}\left(3^{\frac{3}{4}} - 1\right)$$

29 Exercise 27

Exercise 35. Investigate the following impropert integrals for convergence.

1. $\int_1^\infty \frac{x^2}{2x^4-x+1}\,dx$.
2. $\int_0^\infty x^\alpha e^{-x}\,dx$, $\alpha \in \mathbb{R}$.
3. $\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$.
4. $\int_0^\infty \left(\frac{\pi}{2} - \arctan(x)\right)\,dx$.

29.1 Exercise 27.a

$$\int_1^\infty \frac{x^2}{2x^4-x+1}\,dx$$

$$\frac{x^2}{2x^4-x+1} = \frac{x^2}{x(2x^3-1)+1} < \frac{x}{2x^3-1} \le \frac{x}{x^3} = \frac{1}{x^2}$$

$$\int_1^b \frac{x^2}{2x^4-x+1}\,dx < \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = -\frac{1}{b} + 1 < 1$$

In general: Approximately $\int_1^\infty \frac{x^2}{2x^4-x+1}$ converges becomes it looks close to $\int_1^\infty \frac{x^2}{2x^4}\,dx$.

29.2 Exercise 27.b

$$\int_1^\infty \frac{x^\alpha}{e^x}\,dx \qquad \alpha \in \mathbb{R}$$

$$\frac{x^\alpha}{e^x} = \frac{e^{\alpha\cdot\ln x}}{e^x} = e^{\alpha\ln(x)-x}$$

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!} > \sum_{k=l}^\infty \frac{x^k}{k!} \qquad l \ge \alpha$$

$$x \ge 1: \frac{x^\alpha}{e^x} \le \frac{x^l}{e^x} = \frac{x^l}{\sum_{k=0} \frac{x^k}{k!}} < \frac{x^l}{\frac{x^{l+2}}{(l+2)!}} = \frac{(l+2)!}{x^2}$$

$$\int_1^b \frac{x^\alpha}{e^x}\,dx < \int_1^b \frac{(l+2)!}{x^2}\,dx < (l+2)!$$

29.3 Exercise 27.c

$$\int_0^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx = \int_0^1 \frac{\sqrt{x}}{(1+x)^2}\,dx + \int_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$$

$$x \geq 1 : \frac{\sqrt{x}}{(1+x)^2} < \frac{\sqrt{x}}{x^2} = x^{-\frac{3}{2}}$$

$$\int_1^b \frac{\sqrt{x}}{(1+x)^2}\,dx < \int_1^b x^{-\frac{3}{2}}\,dx = 2 - \frac{2}{\sqrt{b}} < 2$$

$$0 \leq x \leq 1 : \frac{\sqrt{x}}{(1+x)^2} < \frac{\sqrt{x}}{1} = \sqrt{x} < 1$$

29.4 Exercise 27.d

$$\int_0^\infty \left(\frac{\pi}{2} - \arctan(x)\right)\,dx$$

$$\int \arctan(x) \cdot 1\,dx = x \cdot \arctan(x) - \int \frac{x}{1+x^2}\,dx$$

Integration by substitution:

$$t = 1 + x^2 \qquad \frac{dt}{dx} = 2x \Rightarrow dx = \frac{dt}{2x}$$

$$= x \cdot \arctan(x) - \int \frac{*\,dx}{t \cdot 2*} = x \cdot \arctan(x) - \frac{1}{2}\ln(1+x^2) + c$$

$$\Rightarrow \int_0^b (x - \arctan(x))\,dx = \frac{\pi}{2} - x \cdot \arctan(x) + \frac{1}{2}\ln(1+x^2)\bigg|_0^b$$

$$= \underbrace{b}_{>0}\underbrace{\left(\frac{\pi}{2} - \arctan(b)\right)}_{>0} + \frac{1}{2}\ln(1+b^2) > \frac{1}{2}\ln(1+b^2) > M$$

29.5 Remark by the tutor

$$\int_1^\infty \frac{1}{x^c}\,dx \text{ converges} \iff c > 1$$

$$\lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan(x)}{\frac{1}{x}} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = -1$$

$$\exists x_0 : \frac{\pi}{2} - \arctan(x) > \frac{1}{2} \cdot \frac{1}{x} \qquad \forall x \geq x_0$$

$$\int_0^\infty \frac{\pi}{2} - \arctan(x)\,dx \geq \int_{x_0}^\infty \frac{1}{2}\frac{1}{x}\,dx$$

30 Exercise 28

Exercise 36. Find all primitive functions of $f : (-1,1) \to \mathbb{R}$ with $x \mapsto \frac{1}{1-x^4}$ using partial fraction decomposition.

Hint: To derive the partial fraction decomposition use

$$\frac{1}{(1-x)(1+x)(1+x^2)} = \frac{a}{1-x} + \frac{b}{1+x} + \frac{cx+d}{1+x^2}$$

with constants $a, b, c, d \in \mathbb{R}$. Determine the values for a, b, c, d.

$$\int \frac{1}{1-x^4}\,dx = \int \frac{1}{4(1-x)}\,dx + \int \frac{1}{4(1+x)}\,dx + \int \frac{1}{2(1+x^2)}\,dx$$

The first resulting integrals are:

$$\frac{1}{4}\int \frac{1}{1-x}\,dx = -\frac{1}{4}\ln 1 - x + c$$
$$\frac{1}{4}\int \frac{1}{1+x}\,dx = \frac{1}{4}\ln(1+x) + c$$
$$\frac{1}{2}\int \frac{1}{1+x^2}\,dx = \frac{1}{2}\arctan(x) + c$$

$$\int \frac{1}{1-x^4}\,dx = -\frac{1}{4}\ln(1-x) + \frac{1}{4}\ln(1+x) + \frac{1}{2}\arctan(x) + c$$
$$\frac{1}{4}\ln\left(\frac{1+x}{1-x}\right) + \frac{1}{2}\arctan(x) + c$$

31 Exercise 29

Exercise 37. Given a function $f : (0,\infty) \to \mathbb{R}$ with $x \mapsto \frac{1}{x+\sqrt{x}}$.

- Determine the Taylor polynomial of second degree $T_f^2(x;1)$ of function f in point $x_0 = 1$.
- Determine an upper bound for the error $\left| f(x) - T_f^2(x;1) \right|$ in interval $[1;2]$.

31.1 Exercise 29.a

$$f'(x) = -\frac{1+2\sqrt{x}}{2\cdot x^{\frac{3}{2}}(1+\sqrt{x})^2}$$
$$f''(x) = \frac{3+9\sqrt{x}+8x}{4\cdot x^{\frac{5}{2}}\cdot(1+\sqrt{x})^3}$$

$$T_f^2(x;1) = f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}\cdot(x-1)^2$$
$$T_f^2(x;1) = \frac{1}{2} + \frac{3}{8}(x-1) + \frac{20}{64}(x-1)^2$$

31.2 Exercise 29.b

$$R_f^n(x;a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-x_0)^{n+1} \qquad \xi \in [x_0, x]$$
$$f'''(x) = -\frac{3\cdot(16x^{\frac{3}{2}} + 29x + 20\sqrt{x} + 5}{8\cdot(\sqrt{x}+1)^4\cdot x^{\frac{5}{2}}}$$
$$\left| R_f^n(x;1) \right| = \frac{(x-1)^3}{(3)!}$$

Upper bound (not the best, but works):

$$\frac{3\cdot 65x^{\frac{3}{2}} + 15}{8\cdot 16\cdot x^{\frac{5}{2}}} \le \frac{3\cdot 65}{8\cdot 16x} + \frac{15}{8\cdot 16x^{\frac{5}{2}}}$$

$f'''(x)$ is monotonically increasing and we look at the interval $[1,2]$. So we are closest to 0, if $x = 1$.

32 Exercise 30

Exercise 38. Let $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(2x)$.

1. For arbitrary $n \in \mathbb{N}$, determine the Taylor polynomial of n-th degree $T_g^n(x;0)$ of g in $x_0 = 0$.
2. Give a Taylor polynomial $T_g^n(x;0)$ such that $\left|g(x) - T_g^n(x;0)\right| < 10^{-6}$ holds for $[-\pi, \pi]$.

32.1 Exercise 30.a

$$\begin{aligned}
g(x) &= \sin(2x) \\
g^{(1)}(x) &= \cos(2x) \cdot 2 \\
g^{(2)}(x) &= -\sin(2x) \cdot 2^2 \\
g^{(3)}(x) &= -\cos(2x) \cdot 2^3 \\
&\ldots\ldots
\end{aligned}$$

$$\begin{aligned}
T_g^n(x;0) &= g(0) + \frac{g^{(1)}(0)(x-0)}{1!} + \ldots \\
&= \underbrace{\sin(0)}_{=0} + \underbrace{\cos(0) \cdot 2x}_{=2x} + \frac{-\sin(0) \cdot 2^2 x^2}{2} + \frac{-\cos(0) \cdot 2^3 x^3}{3!} = -\frac{2^3 x^3}{3!} + \ldots \\
T_g^n(x;0) &= \sum_{k=0}^{m} (-1)^k \cdot \left(\frac{(2x)^{2k+1}}{(2k+1)!}\right) \qquad \text{s.t. } 2m+1 \leq n, 2m+3 \geq n
\end{aligned}$$

Even easier: Consider the power series for sin:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots$$

32.2 Exercise 30.b

$$R_g^n(x;0) = \left|\frac{f^{(n+1)}(\xi)(x-0)^{n+1}}{(n+1)!}\right| < 10^{-6}$$

with $x, \xi \in [-\pi, \pi]$. We look at the following approximation ($\sin(x)$ and $\cos(x)$ is at most 1 and the factor 2^{n+1} of the derivative remains). Choose ξ such that $\left|f^{(n+1)}(\xi)\right| \leq 2^{n+1}$.

$$R_g^n(x;0) \leq \left|\frac{2^{n+1} x^{n+1}}{(n+1)!}\right| < 10^{-6} \iff 2^{n+1} x^{n+1} 10^6 < (n+1)! \iff 10^6 < \frac{(n+1)!}{\left|2^{n+1} x^{n+1}\right|} < (n+1)!$$

In the worst case, x is very large. The largest value it reaches is π. Hence,

$$\left|R_g^n\right| \leq \frac{1}{(n+1)!} \cdot 2^{n+1} \cdot \pi^{n+1} \leq 10^{-6} \qquad \forall x \in [-\pi, \pi]$$

This holds if $n \geq 26$.

33 Exercise 26

Exercise 39. Prove the following limit criterion for improper integrals. Let $a \in \mathbb{R}$ and $f, g : [a, \infty) \to \mathbb{R}$ functions, which satisfy $f \geq 0$ and $g > 0$ in $[a, \infty)$. Furthermore the following limit exists:

$$L := \lim_{x \to \infty} \frac{f(x)}{g(x)} \in [0, \infty].$$

Then it holds that,

1. $L = 0 \Rightarrow \left[\int_a^\infty g(x)\,dx < \infty \Rightarrow \int_a^\infty f(x)\,dx < \infty\right]$.
2. $L \in (0, \infty) \Rightarrow \left[\int_a^\infty g(x)\,dx < \infty \Leftrightarrow \int_a^\infty f(x)\,dx < \infty\right]$.
3. $L = \infty \Rightarrow \left[\int_a^\infty g(x)\,dx \text{ diverges } \Rightarrow \int_a^\infty f(x)\,dx \text{ diverges}\right]$.

33.1 Exercise 26.a

We provide a counterexample:

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x < 1 \\ 0 & x = 0 \\ \frac{1}{x^2} & x > 1 \end{cases}$$
$$g(x) = \frac{1}{x^2 + 1}$$
$$a = 0$$

To make this proposition work, f must be continuous or even boundedness should suffice.

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \left|\frac{f(x)}{g(x)}\right| < \varepsilon$$

Both functions yield positive values:

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} < \varepsilon$$

$$\int_a^\infty f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^\infty f(x)\,dx$$

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} < \varepsilon \iff f(x) < \varepsilon \cdot g(x) \implies \int_{x_0}^\infty f(x)\,dx < \varepsilon \int^\infty g(x)\,dx$$

Because of boundedness we can provide the following estimates:

$$\int_a^\infty f(x)\,dx = \underbrace{\int_a^{x_0} f(x)\,dx}_{<\infty} + \underbrace{\int_{x_0}^\infty f(x)\,dx}_{<\infty}$$
$$\implies < \infty$$

33.2 Exercise 26.b

$$\forall \varepsilon > 0 \exists x_0 \forall x \geq x_0 : \left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$$
$$\iff L - \varepsilon < \frac{f(x)}{g(x)} < L + \varepsilon$$
$$\iff (L - \varepsilon) \cdot g(x) < f(x) < (L + \varepsilon) \cdot g(x)$$
$$(L - \varepsilon) \int_{x_0}^\infty g(x)\,dx \leq \int_0^\infty f(x)\,dx \leq (L + \varepsilon) \int_{x_0}^\infty g(x)$$

33.3 Exercise 27.c

$$\forall n \exists x_0 \forall x \geq x_0 : \frac{f(x)}{g(x)} > n \iff f(x) > n \cdot g(x) \implies \int_{x_0}^{\infty} f(x)\, dx \geq n \cdot \int_{x_0}^{\infty} g(x)$$