Measure and integration theory

Lecture notes, University (of Technology) Graz
based on the lecture by Wolfgang Wöss

Lukas Prokop

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Contents

1 Course **1**

2 Sigma algebras and measures **2**

2.1 Definition 3

2.2 Simple properties of sigma-algebras 3

2.3 Simple properties of measures 9

2.4 Construction of the Lebesgue measures and similar ones 11

2.5 Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ 14

2.6 Sigma-algebra generated by maps 16

2.7 Product space 17

3 Integration of non-negative functions **17**

4 Integrable functions **23**

5 Lebesgue and Riemann integral **26**

1 Course

1. Thursday, 16:15–18:00
2. Monday, 12:15–14:00
3. Exam: oral, date negotiation per email, 3 examinees at once
4. In this document, $\subset$ denotes $\subseteq$ or $\subsetneq$

5. Literature: "Measure Theory" by Paul R. Halmos

↓ *This lecture took place on 2018/10/01.*

2 Sigma algebras and measures

A measure represents the content of a set. In R^2, it represents the area. In R^3, it represents the volume. In R^d, we can consider the content of a subspace as dimensionwise combination of intervals:

$$[a_1, b_1] \times \cdots \times [a_d, b_d]$$

To determine the "size" of this space, we can use the product of the individual interval sizes:

$$(b_1 - a_1) \cdot \cdots \cdot (b_d - a_d)$$

Consider an geometric object as in Figure 1. We can approximate the size of B by considering inner or outer axis-parallel boundary. The approximation using the infimum of the outer and supremum of the inner boundary defines the Jordan measure.

The indicator function of this area (1_B) is Riemann-integrable.

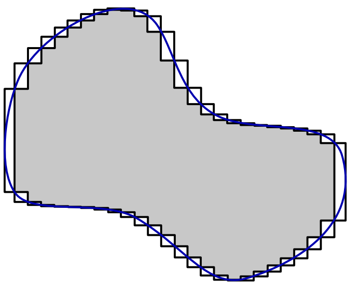

Figure 1: Jordan measurability of this area B

A one-point set is Jordan measurable with measure/content 0. However, $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable, because the indicator function is not Riemann integrable. It is desirable that the measure $\dot\bigcup_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} \text{measure}(A_n)$ (using pairwise disjoint union) holds true.

Modern measure theory was established by Lebesgue (1901):

1. Union of countable sets (σ-additivity)
2. arbitrary base set χ instead of $\mathbb{R}^d$, integration theory for $f : \chi \to \mathbb{R}$

2.1 Definition

Let (δ, ρ) be the non-empty base set. $\mathcal{A} \subset P(\chi)$. A set system of subsets of χ is called *sigma-algebra* (σ-algebra) if

1. $\chi \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^C = \chi \setminus A \in \mathcal{A}$
3. $A_n \in \mathcal{A}\ (n \in \mathbb{N}) \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Properties 1 and 2 implies that $\emptyset \in \mathcal{A}$.

A *measurable space* is given by $(\chi, \mathcal{A})$. A *measure* $\mu : \mathcal{A} \to [0, \infty]$ is defined by

1. $\mu(\emptyset) = 0$
2. If $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$), pairwise disjoint, then

$$\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

A *measure space* is given with $(\chi, \mathcal{A}, \mu)$.

Remark.

- *μ is called probability space if* $\mu(\chi) = 1$
- *μ is called finite measure if* $\mu(\chi) < \infty$
- *μ is called σ-finite if* $\chi = \bigcup_{n=1}^{\infty} A_n$ *with* $A_n \in \mathcal{A}$ *and* $\mu(A_n) < \infty$ *(e.g. real axes decomposes into intervals of length* 1*)*

Examples:

1. χ is at most countable, then mostly $\mathcal{A} = \mathbb{P}(\chi)$. Then it suffices to know, $\mu(\{x\}) \in [0, \infty)$. Then we denote $\mu(x) = \mu(\{x\})$ with $x \in \chi$.

$$\mu(A) = \sum_{x \in A} \mu(x)$$

   e.g. $\mu(x) = 1 \forall x \in \chi$ in case of a *counting measure*.

2. If χ is uncountable, e.g. $\mathbb{R}^d$, then it is not recommended to use $\mathbb{P}(\chi)$. So what about A? Consider for example $\mathbb{R}^d$. All $[a_1, b_1] \times \cdots \times [a_d, b_d]$ should be elements of $\mathcal{A}$

2.2 Simple properties of sigma-algebras

1. $\emptyset \in \mathcal{A}$
2. $A_1, \dots, A_n \in \mathcal{A} \implies \bigcup_{k=1}^{n} A_k \in \mathcal{A}$

3. $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$) or $,\dots, N \implies \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ (deMorgan) $\bigcap_n A_n = \left(\bigcup_n A_n^C\right)^C$
4. $A, B \in \mathcal{A} \implies A \setminus B \in \mathcal{A}, A \triangle B = A \setminus B \vee B \setminus A$

Definition 2.1 (Generating set)**.** *Let $\mathcal{E} \neq \emptyset$ with $\mathcal{E} \subset \mathbb{P}(\chi)$ be the* generator *(generating set) of the σ-algebra. $\sigma(\mathcal{E})$ is the smallest σ-algebra over χ which contains $\mathcal{E}$.*

$$= \bigcap \left\{ \tilde{A} : \tilde{A} \text{ is the } \sigma - \text{algebra over } \chi \text{ with } \mathcal{E} \subset \tilde{\mathcal{A}} \right\}$$

This set is non-empty because $\mathbb{P}(\chi)$ is the σ-algebra for all χ and $\mathcal{E} \subset \mathbb{P}(\chi)$

Lemma 2.2. *If $\mathcal{A}_i$ with $i \in I$ is a family of σ-algebras, then $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra over χ.*

Immediate:

1. if $\mathcal{E}_1 \subset \mathcal{E}_2$ ($\implies \mathcal{E}_1 \subset \sigma(\mathcal{E}_2)$), then $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$
2. if additionally $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$, then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$

Example:

$$\chi = \bigcup_{n\in I} E_n \neq \emptyset \qquad I = \mathbb{N} \text{ or } \{1, \dots, N\}$$

$$\mathcal{E} = \{E_n \mid n \in I\} \qquad \sigma(\mathcal{E}) = \left\{ \bigcup_{n\in J} E_n \mid J \subset I \right\}$$

1. Is a σ-algebra
2. If $\mathcal{E} \subset \tilde{\mathcal{A}}$, then $\left\{\bigcup_{n\in J} E_n \mid J \subset I\right\} \subset \tilde{\mathcal{A}}$

Definition. *(χ, d) is a metric space. Borel-σ-algebra $\sigma(\mathcal{O})$. $\mathcal{O}$ is the set of open sets in a metric space*

Example. *Consider $\mathbb{R}^d$. $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel σ-algebra.*

1. *$\mathcal{E}_1 = \{\text{open sets}\}$*
2. *$\mathcal{E}_2 = \{\text{closed sets}\}$*
3. *$\mathcal{E}_3 = \{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$ is a parallelepiped*
4. *$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$*
5. *$\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ because every open set is a countable union of open (or left half-open) parallelepipeds*

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$
$$\mathcal{E}_4 = \{(a_1, b_1] \times \cdots \times (a_d, b_d] \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$$

$$(a,b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$$

$$\mathcal{E}_5 = \{(-\infty, b_1) \times \cdots \times (-\infty, b_d] \mid b_i \in \mathbb{R}\} \text{ because } \mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$$

If $d = 1$, $(a, b] = (-\infty, b] \setminus (-\infty, a]$. Recognize that $A \setminus B = (A \cap B^C)$.
If $d = 2$,

$$(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus (-\infty, a_1] \times (-\infty, b_2) \setminus (-\infty, b_1] \times (-\infty, a_2]$$

Definition. *$(\chi, \mathcal{A})$ is a measurable space, $B \in \mathbb{A}$ is a trace σ-algebra over B. $\{A \in \mathcal{A} \mid A \subset B\}$*

Definition. *$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$ is called measurable $\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$*

Remark. *In general φ is a map from χ_1 to χ_2. $\mathcal{A}_1$ and $\mathcal{A}_2$ are mentioned to clarify that the map depends on the chosen algebra.*

Remark. *$(\chi_1, d_1) \to (\chi_1, d_2)$ on metric spaces is continuous iff $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ where $\mathcal{O}_1, \mathcal{O}_2$ are sets of open sets.*

Remark. *Measurable maps are a much stronger statement than continuity, because they cover much more sets than open ones.*

Lemma 2.3. *The composition of measurable maps is measurable.*

$$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2) \textit{ measurable}$$
$$\psi : (\chi_2, \mathcal{A}_2) \to (\chi_2, \mathcal{A}_2) \textit{ measurable}$$
$$\implies \psi \circ \varphi : (\chi_1, \mathcal{A}_1) \to (\chi_3, \mathcal{A}_3) \textit{ measurable}$$

Proof. Show that $(\psi \circ \varphi)^{-1}(\mathcal{A}_3) \in \mathcal{A}_1$ is trivial. □

Theorem 2.3.1. *Let $\mathcal{E}_2$ be a generator of $\mathcal{A}_2$. Then $\varphi : (\chi_1, \mathcal{A}_1) \to \chi_2$ is measurable in regards of $\mathcal{A}_2$ iff $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$*

Proof. $\implies$ is immediate

$\impliedby$ $\tilde{\mathcal{A}}_2 = \{A_2 \in \mathcal{A}_2 \mid \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over χ_2. $\mathcal{E}_2$ is a TM of this σ algebra.

$$\implies \mathcal{A}_2 = \sigma(\mathcal{E}_2) \subset \tilde{\mathcal{A}}_2$$

□

Example 2.4. *$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing*

$$f^{-1}(-\infty, b] = \{x \mid f(x) \le b\} = (-\infty, c) \in \mathcal{B}$$

Thus, f is measurable.

Remark. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

Example 2.5.

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{+\infty, -\infty\} \mid B \in \mathcal{B}_{\mathbb{R}}\}$$

$f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ measurable in $(\chi, \mathcal{A})$ and $f = \max\{f_1, \dots, f_n\}$

$$f : \mathbb{R} \to \overline{\mathbb{R}} \qquad x \mapsto \max\{f_1(x), \dots, f_n(x)\}$$

$$\begin{aligned} f^{-1}([-\infty, b]) &= \{x \mid f(x) \le b\} \\ &= \{x \mid f_k(x) \le b, k = 1, \dots, n\} \\ &= \bigcap_{k=1}^{n} \{x \mid f_k(x) \le b\} \in \mathcal{B} \end{aligned}$$

Analogously for the minimum. Therefore f is measurable.

Example 2.6. *The same applies to countably many functions. Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ be measurable with $n \in \mathbb{N}$. Then $f : \sup\{f_n \mid n \in \mathbb{N}\}$ is measurable.*

$$\begin{aligned} f^{-1}([-\infty, b]) = \{x \mid \sup\{f_n(x)\} \le b\} &= \{x \mid f_n(x) \le b \forall n\} \\ = \bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in B} \in \mathcal{B} \end{aligned}$$

Analogously for the infimum.

Example 2.7.

$$\limsup_{n\to\infty} \sup f_n = \inf_n \underbrace{\sup_{k \ge n} f_k}_{\text{with } n\to\infty \text{ monotonically decreasing}} \quad \textit{is measurable}$$

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k \ge n} f_k \textit{ is measurable}$$

if all f_k are measurable. Especially if $f_n \to f$ pointwise and all f_n are measurable, then f is measurable.

Theorem 2.7.1 (Result from the previous example)**.**

$$f_n : (\chi, \mathcal{A}) \to \overline{\mathbb{R}} \textit{ measurable}, n \in \mathbb{N}$$

$$\implies \inf f_n, \sup f_n, \lim_{n\to\infty} \inf f_n, \lim_{n\to\infty} \sup f_n$$

are all measurable.

↓ *This lecture took place on 2018/10/04.*

1. Basic set χ $[\delta, \rho, \dots]$
2. σ-algebra $\mathcal{A} \subset p(\chi)$

(a) $\chi \in \mathcal{A}$

(b) $A \in \mathcal{A} \implies \mathcal{A}^C \in \mathcal{A}$

(c) $A_n \in \mathcal{A}\ (n \in \mathbb{N}) \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

$(\chi, \mathcal{A})$ is a measurable space

3. measure $\mu : \mathcal{A} \to [0, \infty]$

   (a) $\mu(\emptyset) = 0$

   (b) $A_n \in \mathcal{A}\ (n \in \mathbb{N}), A_n \cap A_m \neq 0 \forall n \neq m$

$$\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \qquad \sigma - \text{additivity}$$

   $(\chi, \mathcal{A}, \mu)$ is a measure space

4. $\mathcal{E} \subset p(\chi)$

$$\sigma(\mathcal{E}) = \bigcap \left\{ \tilde{A} : \tilde{A}\sigma - \text{algebra}, \mathcal{E} \subset \tilde{A} \right\}$$

   is the so-called $\mathcal{E}$-generated σ-algebra.

Recognize that $\mathcal{E}_1 \subset \mathcal{E}_2 \implies \sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$. If additionally, $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \implies \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

If X is a metric space, we commonly (sometimes implicitly) use the Borel-Sigma algebra as measure space.

Example: Let $\mathbb{R}^d$. Then $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel-sigma algebra.

Let $\mathcal{E}_1$ be the set of open sets. Let $\mathcal{E}_2$ be the set of closed sets. Let $\mathcal{E}_3 = \{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$. $\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ because every open set is a countable union of open (or left half-open) parallelepipeds (why countable?).

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$

$$\mathcal{E}_4 = \{(a_1, b_1] \times (a_2, b_2] \times \cdots \times (a_d, b_d]\}$$

$$(a, b) = \bigcup_{n=0}^{\infty} (a, b - \frac{1}{n})$$

$$\mathcal{E}_5 = \{(-\infty, b_1) \times (-\sigma, b_d) : b_1, \ldots, b_d \in \mathbb{R}\}$$

because $\mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$.

DeMorgan: $A \setminus B = A \cap B^C$

Let $d = 1$, $(a, b] = (-\infty, b] \setminus (-\infty, a]$.
Let $d = 2$, $(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus (-\infty, a_1] \times (-\infty, b_2] \setminus (-\infty, b_2] \times (-\infty, a_1]$.

Definition 2.8. *Let $(\chi, \mathcal{A})$ be a measure space. $B \in \mathcal{A}$. trace σ-algebra over B is defined as $\{A \in \mathcal{A} : A \subset B\}$.*

Remark (Revision on continuity). *Let $\varphi : (\chi_1, d_1) \to (\chi_2, d_2)$ be a map between metric spaces. Let φ be continuous.*

On the one hand, we know the ε-δ definition, but we also consider $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ (set of open sets)

Definition 2.9 (Measurable maps). *Let $\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$*[1]

$$\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$$

Lemma 2.10. *The composition of measurable maps is measurable.*

$$\varphi : (\chi_1, \mathcal{A}_1) \to (\chi_2, \mathcal{A}_2)$$

$$\Psi : (\chi_2, \mathcal{A}_2) \to (\chi_3, \mathcal{A}_3)$$

with φ and Ψ measurable.

$$\implies \Psi \circ \varphi : (\chi_1, \mathcal{A}_1) \to (\chi_3, \mathcal{A}_3)$$

is measurable. (trivial to prove)

Theorem 2.10.1. *Let $\mathcal{E}_2$ be the generator of some algebra $\mathcal{A}_2$. Then $\varphi : (\chi_1, \mathcal{A}_1) \to \chi_2$ in regards of $\mathcal{A}_2$ is measurable if and only if $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$.*

Proof. $\implies$ trivial

$\impliedby$ $\tilde{A}_2 := \{A_2 \in \mathcal{A}_2 : \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over χ_2 (why? left as an exercise). $\mathcal{E}_2$ is a subset of this σ-algebra. $\implies \mathcal{A}_2 = \sigma(\mathcal{E}_2) \in \tilde{A}_2 \subset \mathcal{A}_2$

□

Example. *$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. $f^{-1}(-\infty, b] = \{x : f(x) \leq b\}$ is in the Borel-sigma algebra $\mathcal{B}$. So f is measurable.*

Definition. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$
$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{\pm\infty\} : B \in \mathcal{B}_{\mathbb{R}}\}$

Example 2.11. *Let $f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ measurable. $f = \max\{f_1, \dots, f_n\}$.*

$$f^{-1}([-\infty, b]) = \{x : f(x) \leq b\} = \{x : f_k(x) \leq b, k = 1, \dots, n\} = \bigcap_{k=1}^{n} \underbrace{\{x : f_k(x) \leq b\}}_{f_k^{-1}[-\infty,b]} \in \mathcal{B}$$

Equivalently, $\min\{f_1, \dots, f_n\}$ *is measurable. Equivalently, $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable ($n \in \mathbb{N}$). $\implies f = \sup\{f_n : n \in \mathbb{N}\}$ is measurable.*

$$f^{-1}(\infty, b] = \{x : \sup f_n(x) \leq b\} = \{x : f_n(x) \leq b \forall n\}$$
$$f^{-1}[-\infty, b) = \{x : \sup f_n(x) < b\} \subset \{x : f_n(x) < b \forall n\}$$

$$\bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in \mathcal{B}} \in \mathcal{B}$$

[1]Actually, $\varphi : \chi_1 \to \chi_2$, but we don't want to forget about the associated σ-algebras

Let f_n be measurable functions.

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{k\geq n} f_k$$

The supremum of measurable functions is measurable (see Lemma 2.10). The infimum as well. So the result is measurable.

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k\geq n} f_k$$

Equivalently, the result is measurable.

Especially, if $f_n \to f$ pointwise, and all f_n are measurable, then also limit f is measurable.

How to determine measurability? Show that pre-images of generators are in the σ-algebra.

Theorem 2.11.1. *Let $f : (\chi, \mathcal{A}) \to \overline{\mathbb{R}}$ be measurable ($n \in \mathbb{N}$)*

$$\implies \inf f_n, \sup f_n, \liminf f_n, \limsup f_n$$

are also measurable.

↓ *This lecture took place on 2018/10/08.*

2.3 Simple properties of measures

A monotonically increasing sequence $(A_n)_{n\in\mathbb{N}}$ of sets is given by $A_1 \subset A_2 \subset A_3 \subset \dots$.

Theorem 2.11.2. *Let $(\chi, \mathcal{A}, \mu)$.*

1. $A_1, \dots, A_n \in \mathcal{A}, A_i \cap A_j = 0 \forall i \neq j \implies \mu\left(\dot\bigcup_{k+1}^n A_n\right) = \sum_{k=1}^n \mu(A_n)$
2. $\mu(B) = \mu(A \cap B) + \mu(A^C \cap B)$ *for* $A, B \in \mathcal{A}$
3. $A \subset B \implies \mu(A) \leq \mu(B)$ *for* $A, B \in \mathcal{A}$
4. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
5. *Let* $(A_n)_{n\in\mathbb{N}}$ *be a monotonically increasing sequence of* $\mathcal{A}$ *and* $A = \dot\bigcup_{n=1}^\infty A_n = \lim_{n\to\infty} A_n$*, then* $\mu(A) = \lim_{n\to\infty} \mu(A_n)$ *"Continuity from below"*
6. *Let* A_n *be a monotonically decreasing sequence of* $\mathcal{A}$. $A = \bigcap_{n=1}^\infty A_n = \lim A_n$.
7. A_n *arbitrary* $\implies \mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n)$

Proof of continuity from below. Consider a monotonically increasing sequence of $\mathcal{A}$. Consider $B_1 = A_1, B_k = A_k \setminus A_{k-1}$ and $k \geq 2$. Sets B_i and B_j are disjoint with $i \neq j$.

Then $B_1 \cup \dots \cup B_n = A_n$ and $\dot\bigcup_{k=1}^{\infty} B_k = A$.

$$\mu(A) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} \mu(A_n)$$

$$A'_n = A_1 \setminus A_n \qquad \nearrow A_1 \setminus A$$

$$\mu(A_1 \setminus A_n) = \mu(A'_n) \qquad \nearrow \mu(A_1 \setminus A)$$

□

What about the measure of intersected set in infinity? $A \cap B = A$ and $\mu(B) = \mu(A) + \mu(A^C \cap B)$. What happens if $\mu(A) = +\infty$ and $\mu(A^C \cap B) = -\infty$?

Remark. *How to compute algebraically with the extended real numbers?*

$$\pm\infty + a = \pm\infty \quad (a \in \mathbb{R})$$

$$+\infty \cdot a = \begin{cases} +\infty & a > 0 \\ 0 & a = 0 \\ -\infty & a < 0 \end{cases}$$

0 *for* $a = 0$ *makes sense in measure theory, but not in calculus.*

If $\mu(A_1) < \infty$, then $\mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$ and $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$.

Remark (Reminder).

$$\limsup a_n := \lim_{n\to\infty} \sup_{k\geq n} a_k$$

What about (A_n) arbitrary?

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} A_k$$

Property 7 can be shown as generalization of $\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N} \mu(A_n)$

Example (Simplest example). $\chi = \{x_n : n \in \mathbb{N}\}$. $\mathcal{A} = p(\chi)$. *Fix* $\mu(\{x_n\})$.

$$\rightsquigarrow \mu(A) = \sum_{n: x_n \in A} \mu(x_n)$$

$\mu(x_n) = 1$ *gives a counting measure.*

Let $\mathcal{E}$ be the generator of $\mathcal{A} = \sigma(\mathcal{E})$. A *stable set by intersection* is given by $E_1, E_2 \in \mathcal{E} \implies E_1 \cap E_2 \in \mathcal{E}$.

Theorem 2.11.3 (Uniqueness of measures). *Let μ, ν be measures on $\mathcal{A}$ with $\mu|_{\mathcal{E}} = \nu|_{\mathcal{E}}$. $\implies \mu = \nu$ on $\mathcal{A}$.*

$\chi \in \mathcal{E}$ and $\mu(\chi) = \nu(\chi) < \infty$ *or* $\chi = \bigcup_n E_n$ with $E_n \in \mathcal{E}$ and $\mu(E_n) = \nu(E_n) < \infty$.

Definition 2.12. *Let $\mathcal{E} \subset \mathcal{P}(\chi)$ be a semiring over χ. If*

1. $\emptyset \in \mathcal{E}$
2. $A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$
3. $A, B \in \mathcal{E}. \implies \exists C_1, \dots, C_k \in \mathcal{E}$ *pairwise disjoint* : $A \setminus B = \dot{\bigcup}_{i=1}^k C_i$.

What is the difference compared to a *ring*? Let $A, B \in \mathcal{R} \implies (A \cap B \in \mathcal{E} \wedge A \triangle B \in \mathcal{E})$.

Theorem 2.12.1 (Extension theorem by Caratheodory). $\mu : \mathcal{E}$ *(semiring)* $\to \{0, \infty\}$ *with*

1. $\mu(\emptyset) = 0$
2. $(\chi \in \mathcal{E}$ *and* $\mu(\chi) < \infty)$ *or* $\left(\chi = \bigcup_{n=1}^\infty E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty\right)$
3. *μ is σ-additive on $\mathcal{E}$, hence (A_n) is a sequence in $\mathcal{E}$, pairwise disjoint and $\bigcup_{n=1}^\infty A_n \in \mathcal{E}$*
$$\implies \mu\left(\dot{\bigcup_{n=1}^\infty} A_n\right) = \sum_n \mu(A_n)$$

Then μ has a (unique) continuation for a measure on $\mathcal{A} = \sigma(\mathcal{E})$.

2.4 Construction of the Lebesgue measures and similar ones

Let $\chi = \mathbb{R}$ or $\chi = \overline{\mathbb{R}}$.
$$\mathcal{E} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$$
is semiring.

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and right-sided continuous. Let $\mu(a, b] := F(b) - F(a)$. Properties 1 and 2 of the extension theorem are satisfied. We show finite additivity of property 3 in three steps:

1. If $(a, b] = \dot{\bigcup}_{k=1}^n (a_k, b_k]$ can be sorted. $a_1 = a, a_{k+1} = b_k$ for $k = 1, \dots, n-1$ and $b_n = b$. We get a telescoping sum such that
$$\sum_{b=1}^n (F(b_k) - F(a_k)) = F(b) - F(a)$$
2. Also $(a_1, b_1], \dots, (a_n, b_n]$. Disjoint subintervals of $(a, b]$ are
$$\implies \sum_{k=1}^n \mu(a_k, b_k] \leq \mu(a, b]$$

3. $(a,b] = \dot{\bigcup}_{n=1}^{\infty}(a_n, b_n]$

(a)
$$\dot{\bigcup_{n=1}^{N}}(a_n, b_n] \subset (a,b]$$
$$\sum_{n=1}^{N}\mu(a_n, b_n] \leq \mu(a,b] \forall N$$
$$\implies \sum_{n=1}^{\infty}\mu(a_n, b_n] \leq \mu(a,b]$$

(b) Let $\varepsilon > 0$, then $\exists a' \in (a,b] : F(a') - F(a) < \varepsilon$
$$\exists b'_n > b : F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}$$
$$[a',b] \subseteq (a,b] \subset \bigcup_n (a_n, b_n] \subset \bigcup_n (a_n, b'_n)$$
$$\implies \exists N : (a',b) \subset [a',b] \subset \bigcup_{n=1}^{N}(a_n, b'_n) \subset \bigcup_{n=1}^{N}(a_n, b'_n]$$

But these intervals in $\bigcup$ are not necessarily non-overlapping any more. But this is no problem as we can split them into disjoint sets.
$$\mu(a',b] \leq \sum_{n=1}^{N}\mu(a_n, b'_n]$$
$$\mu(a',b] = F(b) - F(a') \leq \sum_{n=1}^{N}F(b'_n) - F(a_n) \leq \sum_{n=1}^{N}\left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}\right)$$
with $F(b) - F(a') \geq F(b) - F(a) - \varepsilon$.
$$\mu(a,b] \leq \sum_{n=1}^{\infty}\mu(a_n, b_n] + 2\varepsilon \quad \forall \varepsilon > 0$$

↓ *This lecture took place on 2018/10/15.*

Theorem 2.12.2. *Let $\mathcal{E}$ be semiring over χ and $\mu : \mathcal{E} \to [0,\infty]$ on $\mathcal{E}$ be σ-additive and σ-finite. Then there exists exactly one continuaton for measure on $\sigma(\mathcal{E})$.*

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonic and right-sided continuous.
$$\mathcal{E} = \{(a,b] \mid a,b \in \mathbb{R}, a \leq b\} \qquad \mu(a,b] = F(b) - F(a)$$
Now consider the special case $F(x) = x$. This define the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

Theorem 2.12.3. *λ is the only measure on $(\mathbb{R}, \mathcal{B})$ with*

1. $\lambda(B + C) = \lambda(\{x + c \mid x \in B\}) = \lambda(B) \quad \forall B \in \mathcal{B} \forall c \in \mathbb{R}$
2. $\lambda(0, 1] = 1$

Proof. Does λ satisfy these properties? Yes, λ has properties (1) and (2), because

(1) is correct $\forall (a, b] \in \mathcal{E}$

$$c \in \mathbb{R} : \{B \in \mathcal{B} \mid \lambda(B + c) = \lambda(B)\}$$

is σ-algebra and contains $\mathcal{E}$, so also $\sigma(\mathcal{E})$

(2) trivial

Is λ unique? Let λ be the measure with the two properties.

$$(0, 1] = \dot{\bigcup_{k=1}^{n}} \left(\frac{k-1}{n}, \frac{k}{n}\right]$$

$$1 = \mu(0, 1] = \sum_{k=1}^{n} \mu\left(\left(0, \frac{1}{n}\right] + \frac{k-1}{n}\right) = n\mu\left(0, \frac{1}{n}\right]$$

$$\mu\left(\frac{k-1}{n}, \frac{k}{n}\right] = \frac{1}{n} \quad \forall k \in \mathbb{Z}$$

$$\implies \mu(a, b] = b - a \qquad a, b \in \mathbb{Q}$$

$$\mu|_{\mathcal{E}_{\mathbb{Q}}} = \lambda_{\mathcal{E}_{\mathbb{Q}}} \qquad \mathcal{E}_{\mathbb{Q}} = \{(a, b] \mid a, b \in \mathbb{Q}, a \leq b\}$$

Closed under finite intersection, $\sigma(\mathcal{E}_{\mathbb{Q}}) = \mathcal{B}$:

$$(a, b) = \bigcup_{n} (a, b - \frac{1}{n}] \qquad (a, b] = \bigcap_{n} (a, b + \frac{1}{n}) \qquad \sigma\text{-finite}$$

$$\mu(-n, n] < \infty \qquad \bigcup_{n=1}^{\infty} (-n, n] = \mathbb{R} \qquad \sigma\text{-finite}$$

$$\implies \mu = \lambda \text{ (distinct extensionability)}$$

□

We apply the principle analogously to $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$.

$$\mathcal{E} = \{(a, b] = (a_1, b_1] \times \dots \times (a_n, b_n] \mid a_i \leq b_i \in \mathbb{R}\}$$

is semiring over $\mathbb{R}^d$. In $\mathbb{R}^2$, you can draw rectangles and their induced area based on their geometrical relation to each other. $F : \mathbb{R}^d \to \mathbb{R}$ complete is *monotonic* if

$$\mu(a, b] : \prod_{i=1}^{d} (F_i(b_i) - F_i(a_i)) = \sum_{x \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} \qquad F_1(x_1) F_2(x_2) \dots F_d(x_d)$$

Simplest case: $F_1, \dots, F_d : \mathbb{R} \to \mathbb{R}$ is monotonically right-sided continuous.

$$\sum_{x \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} F(x) \geq 0 \forall (a, b] \in \mathcal{E}$$

$$F(b_1, b_2) - F(a_1, b_2) - F(a_1, b_1) + F(a_1, a_2)$$

F is right-sided in every coordinate, thus $\mu(a, b] = \sum_{x \in \{a_1, b_1\} \times \dots \times \{a_d, b_d\}} (-1)^{|\{i \mid x_i = a_i\}|}$

2.5 Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$

We can extend the previous definition from $\mathbb{R}$ to $\mathbb{R}^d$. Thus λ is the only measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ with

1. $\lambda^d(B + c) = \lambda(B) \forall B \in \mathbb{B}_{\mathbb{R}^d} \forall c \in \mathbb{R}^d$
2. $\lambda((0, 1]^d) = 1$

Theorem 2.12.4. *Let $H \subset \mathbb{R}^d$ be a hyperplane. Then $\lambda_d(H) = 0$.*

Proof. Without loss of generality, $\vec{O} \in H$ is subspace with dimension $d - 1$. Why is $H \in \mathcal{B}_d$ true? The Lebesgue measure is based on open sets. The σ-algebra requires the complement, thus closed sets are also given. The measure of closed sets is zero.

$\left\{\vec{b}_1, \dots, \vec{b}_{d-1}\right\}$ is an orthonormal basis of H.

$$Q = \left\{c_1 \vec{b}_1 + \dots + c_{d-1} \vec{b}_{d-1} \mid 0 \leq c_i \leq 1\right\} \in \mathcal{B}_{\mathbb{R}^d}$$

$\vec{b}_d \perp \vec{b}_i$ $(i = 1, \dots, d - 1)$, $\left\|\vec{b}_d\right\| = 1$.

$$Q + q \cdot \vec{b}_d \qquad q \in \mathbb{Q} \cap [0, 1] \text{ pairwise disjoint}$$

$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} Q + q \vec{b}_d \subset \left\{c_1 \vec{b}_1 + \dots + c_d \vec{b}_d \mid 0 \leq c_i \leq 1\right\} \text{ compact}$$

$$\infty > \lambda_d \left(\dot{\bigcup_{q \in \mathbb{Q} \cap [0,1]}} Q + q \cdot \vec{b}_d \right) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda_d(Q)$$

$$\implies \lambda_d(Q) = 0 \qquad H \subset \bigcup_{\vec{x} \in \mathbb{Z}^d} (Q + \vec{x})$$

$$\lambda_d(H) \leq \sum_{\vec{x} \in \mathbb{Z}^d} \lambda_d(Q + \vec{x}) = 0$$

□

Theorem 2.12.5. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be linear and bijective. $\varphi(\vec{x}) = M \cdot \vec{x}$ with M as regular matrix.*

Linear implies continuous in finite dimensions. Every continuous map is measurable.

$\implies$ *φ is measurable and $\lambda_d(\varphi(B)) = \det(\varphi) \cdot \lambda_d(B)$. This holds even if φ is not bijective, because then $\det(\varphi) = 0$ and thus we have a factor zero. If φ is not bijective,* *then the matrix has lower rank. The image is a hyperplane or is contained in a hyperplane. So the measure is zero.*

Proof. $\mu_\varphi(B) := \lambda_d(\varphi(B))$ is measure on $\mathcal{B}_{\mathbb{R}^d}$ (why? left as an exercise to the reader).

$$\mu_\varphi(B + \vec{c}) = \lambda_d(\varphi(B + \vec{c})) = \lambda_d(\varphi(B) + \underbrace{\varphi(\vec{c})}_{X}) = \mu_\varphi(B)$$

$$\frac{\mu_\varphi}{\mu_\varphi((0,1]^d)} = \lambda_d$$

Show that: $\mu_\varphi((0,1]^d] = |\det\varphi|$

Case 1 $\varphi(M)$ is orthogonal $M^* = M^{-1}$.

$$\varphi(B_1(\vec{0})) = B_1(\vec{0}) \qquad 0 < \lambda_d(B_1(\vec{0})) < \infty$$

$$\frac{\lambda_d(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \frac{\mu_\varphi(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \lambda_d(B_1(\vec{0}))$$

$$\mu_\varphi = \lambda_d$$

Case 2

$$M = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_d \end{pmatrix} \qquad d_i > 0$$

$$\varphi(\vec{e}_i) = d_i \cdot e_i$$

$$\varphi((0,1]^d) = (0, d_1] \times (0, d_2] \times \ldots (0, d_d]$$

$$\varphi_\varphi((0,1]^d) = \det D$$

Generic case Let M be any matrix. We consider the singular value decomposition $M = O_1 \cdot D \cdot O_2$ with O_1, O_2 orthogonal and D is a non-negative diagonal matrix.

$$M^* M \rightsquigarrow O^* D^2 O$$

Then $\varphi = \varphi_1 \circ \psi \circ \varphi_2$. φ_1 and φ_2 are orthogonal. Let D be the representation matrix of ψ. Diagonal entries are positive because it is regular.

$$|\det\varphi| = \det(\psi)$$

Combining these results gives us the theorem.

□

↓ *This lecture took place on 2018/10/16.*

2.6 Sigma-algebra generated by maps

Definition 2.13. *$\mathcal{A}_i$ ($i \in I$) is σ-algebra over χ.*

$$\bigvee_{i\in I} \mathcal{A}_i = \sigma\left(\bigcup_{i\in I} \mathcal{A}_i\right)$$

Definition 2.14 (Image σ-algebra and Push-forward measure)**.** *Push-forward measures are called Bildmaß $(\chi, \mathcal{A})$ is a measure space. $\varphi : \chi \to \chi'$.*

$$\varphi(\mathcal{A}) = \{A' \subset \chi' \mid \varphi^{-1}(A') \in \mathcal{A}\}$$

$\varphi(\chi, \mathcal{A}) \to (\chi', \mathcal{A}')$ is measurable $\iff \varphi(\mathcal{A}) \supset \mathcal{A}'$.

$(\chi, \mathcal{A}, \mu)$ is a measure space, $\varphi : \chi \to \chi'$. μ_φ is the push-forward measure on $(\chi', \varphi(\mathcal{A}))$.

$$\mu_\varphi(A') = \mu(\varphi^{-1}(A'))$$

Definition 2.15 (Generated σ-algebra)**.**

1. *χ, $(\chi', \mathcal{A}')$ is a measurable space. $\varphi : \chi \to \chi'$*

   $$\sigma(\varphi) = \{\varphi^{-1}(A') \mid A' \in \mathcal{A}'\}$$

   Iff $\varphi : (\chi, \mathcal{A}) \to (\chi', \mathcal{A}')$ is measurable, $\sigma(\varphi) \subset \mathcal{A}$.

2. *χ, $(\chi_i, \mathcal{A}_i)$, $i \in I$ are measure spaces*

   $$\psi_i : \chi \to \chi_i \forall i$$

   The σ-algebra generated by ψ_i ($i \in I$) is the smallest σ-algebra that contains such a set. $\bigvee_{i\in I} \sigma(\psi_i)$. Is the smallest σ-algebra on χ which are measurable for all ψ_i.

Example. $\varphi : (\mathbb{R}^2, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$.

$$\varphi(x, y) = \sqrt{x^2 + y^2} \qquad \sigma(\varphi) = \{B \subset \mathbb{R}^2 \mid B \textit{ rotation invariant in } 0.000001\}$$

Theorem 2.15.1. *Let $(\chi, \mathcal{A})$ be a measure space. Let $(\chi', \mathcal{A}')$ be another one. Let $(\chi_i, \mathcal{A}_i)$ be measure spaces with ($i \in I$). Then we can map from $(\chi, \mathcal{A})$ to $(\chi', \mathcal{A}')$ with measurable φ and we can map $(\chi', \mathcal{A}')$ to $(\chi_i, \mathcal{A}_i)$ with ψ_i such that $\mathcal{A}' = \sigma(\psi_i : i \in I)$. Then φ is measurable iff $\psi_i \circ \varphi$ is measurable $\forall i \in I$.*

Proof. $\implies$ immediate.

$\impliedby$

$$\mathcal{E}' = \bigcup \sigma(\psi_i) \text{ generates } \mathcal{A}'$$

$$A' \subset \mathcal{E}' \implies \exists i : A' \in O(\psi_i), \text{ so } A' = \psi_i^{-1}(A_i) \text{ with } A_i \in \mathcal{A}_i.$$

$$\varphi^{-1}(A) = \psi^{-1}(\psi_i^{-1}(A_i)) = (\underbrace{\psi_i \circ \varphi}_{\in\mathbb{R}})^{-1}(A_i)$$

$\square$

2.7 Product space

Let $\chi_n, \mathcal{A}_n$ and $n = 1, \dots, N$ with $N < \infty$. Let $\chi = \prod_{n=1}^{N} \chi_n$ ("product $sigma$-algebra") generated by $\mathcal{E} = \left\{\prod_{n=1}^{N} A_n \cdot A_n \subset A_n \forall n\right\}$.

Consider $N = 2$. $\chi = \chi_1 \times \chi_2$. $\mathcal{E} = \{A_1 \times A_2 \mid A_n \in \mathcal{A}_n, n = 1, 2\}$. Product σ-algebra: $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Commonly, we use the notation $(\chi, \otimes \mathcal{A}_n) = \otimes(\chi_n, \mathcal{A}_n)$

Lemma 2.16.

$$\oplus_{n=1}^{N} \mathcal{A}_n = \sigma\left(\pi_n : n = 1, \dots, N\right)$$

where $\pi_n : \chi \to \chi_n$ is the n-th projection.

Hint: $\mathcal{E}_0 = \left\{\prod_{n=1}^{N} A_n \text{ with } A_n = \chi_n \forall n \text{ expect for one and this } A_n \in \mathcal{A}_n\right\}$ also generates $\otimes \mathcal{A}_n$.

This lemma holds obviously.

Theorem 2.16.1. *$\varphi : (\chi, \mathcal{A}) \to \otimes_{n=1}^{N}(\chi_n, \mathcal{A}_n)$, where N denotes finite or countable, is measurable $\iff$ $\pi_n \circ \varphi : (\chi, \mathcal{A}) \to (\chi_n, \mathcal{A}_n)$ is measurable $\forall n$. This is a special case of Theorem 2.15.1.*

Prospect: Product measure.

Let $(\chi_1, \mathcal{A}_1) \otimes (\chi_2, \mathcal{A}_2, \mu_2)$. How to generate this? Well,

$$= (\chi_1 \times \chi_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$$

on $\mathcal{E} : \mu_1 \otimes \mu 2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ (compare it to the trivial case of the area of a rectangle in $\mathbb{R}^2$) where $\mathcal{E}$ is a semiring.

3 Integration of non-negative functions

Let $(\chi, \mathcal{A}, \mu)$ be a measure space. Consider $f : (\chi, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B})$ How about $\int_\chi f \, d\mu$?

First of all, $f : (\chi, \mathcal{A}) \to [0, \infty]$. We know construct the Lebesgue integral:

First step Consider simple functions (like step functions). f takes up only finitely many different values. $z_1, \dots, z_n (\geq 0) : [f = z_k] \coloneqq \{x \in \chi \mid f(x) = z_k\} = f^{-1}(\{z_k\}) \in \mathcal{A}$. We restrict $z_i \geq 0$ to avoid issues like $+\infty + (-\infty)$.

$$\chi = \dot{\bigcup_{k=1}^{n}} [f = z_k]$$

Definition 3.1.

$$\int f \, d\mu = \sum_{k=1}^{n} z_k \mu[f = z_k]$$

Consider that $z_k \mu[f = z_k]$ might go to infinity. We commonly denote $\sum_{z \in \mathbb{R}} z\mu[f = z]$ in the real-valued case to avoid indices.

Second step Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.

$$\int_\chi f \, d\mu := \sup \left\{ \int_\chi s \, d\mu : s \text{ simple}, 0 \leq s \leq f \right\}$$

So the Riemann integral approximates the area with upper and lower bounds for rectangles. For the Lebesgue integral, we split the function into horizontal slices in $\mathbb{R}$. Then we consider the differences of the function values between two consecutive slices. The important point is that this does not require $\mathbb{R}$, but some χ and therefore is more generic.

Third step Let $f : (\chi, \mathcal{A}) \to \mathbb{R}$ and $f = f^+ - f^-$. Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. If $\int_\chi f^+ \, d\mu = \int_\chi f^- \, d\mu = \infty$: $\int_\chi f \, d\mu$ is not defined. Otherwise $\int_\chi f \, d\mu = \int_\chi f^+ \, d\mu - \int_\chi f^- \, d\mu$.

Does this definition/construction of the Lebesgue integral satisfy the desired properties of linearity/monotonicity/…? In the following, we will denote "simple" functions always as s.

Definition 3.2. *Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Let $A \in \mathcal{A}$.*

$$\int_A f \, d\mu := \int \mathbf{1}_A f \, d\mu$$

Lemma 3.3. *Let $s : (\chi, \mathcal{A}) \to [0, \infty]$ be a simple function. Then $\nu_s(A) = \int_A s \, d\mu$ is a measure on $(\chi, \mathcal{A})$.*

$$\nu_s(A) = \sum_{k=1}^n z_k \mu([s = z_k] \cap A)$$

because $\mathbf{1}_A \cdot s = \sum_{k=1}^n z_k \mathbf{1}_{[s=z_k]} \mathbf{1}_A + 0 \cdot \mathbf{1}_{A^C}$.

$A \mapsto \mu([s = z_k] \cap A)$ *is a measure* $\forall k$.

↓ *This lecture took place on 2018/10/22.*

Definition 3.4. *Let $(\chi, \mathcal{A}, \mu)$ be a measure space. $s : (\chi, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is called* simple *if $s(\chi)$ is finite.*
$s \geq 0$.

$$\int_\chi s \, d\mu := \sum_z z \cdot \mu[s = z]$$

Trivial: If $s = \sum_{j=1}^m c_j \cdot \mathbf{1}_{A_j}, A_j \in \mathcal{A}$ then s is simple. A_j are not necessarily pairwise disjoint and $\int_\chi s \, d\mu = \sum_{j=1}^n c_j \mu(A_j)$.

Proof. $\vec{\varepsilon} \in \{-1, 1\}^m$ with $A^1 := A, A^{-1} := A^C$. $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$. E.g. $A_1 \cap A_2 \cap A_3^C = B_{1,1,-1}$.

$$B_{\vec{\varepsilon}} = A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \dots \cap A_m^{\varepsilon_m}$$

is pairwise disjoint. On $B_{\vec{\varepsilon}}$, s has value $\sum_{\varepsilon_j=1} c_j = b_{\vec{\varepsilon}}$

$$\implies s = \sum b_{\vec{\varepsilon}} \mathbf{1}_{B_{\vec{\varepsilon}}}$$

and $\int \leq d\mu = \sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}}) = \cdots = \sum c_j \mu(A_j)$ (where $\sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}})$ is the disjoint case and $\sum c_j \mu(A_j)$ is generic).

$$\sum_{\vec{\varepsilon}} \sum_{j:\varepsilon_j=1} c_j \cdot \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \sum_{\vec{\varepsilon}:\varepsilon_j=1} \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \mu(A_j)$$

□

Corollary 3.5. *Let $s_1, s_2 : \chi \to [0, \infty]$ be simple. Then $s = \alpha \cdot s_1 + \beta \cdot s_2$ ($\alpha, \beta \geq 0$) is simple and $\int s \, d\mu = \alpha \cdot \int s_1 \, d\mu + \beta \int s_2 \, d\mu$.*

Theorem 3.5.1 (Markov inequality). *Let $z \in \mathbb{R}$. Let $f \geq 0$.*

$$z \cdot \mu[\underbrace{f \geq z}_{\{x \in \chi \mid f(x) \geq z\}}] \leq \int f \, d\mu$$

Proof.

$$s = z \cdot \mathbf{1}_{[f \geq z]} \leq f$$

If $x \in [f \leq z] : z \cdot 1 \leq f(x)$.
If $x \notin [f \leq z] : z \cdot 0 \leq f(x)$.

s is simple, so $z\mu[f \geq z] = \int s \, d\mu \leq \int f \, d\mu$. $s = 0 : \mathbf{1}_{[f<z]} \times z \cdot \mathbf{1}_{[f \geq z]}$. □

Definition 3.6. *A statement holds* almost everywhere *if $\forall x \in \mathcal{A} : \mu(A^C) = 0$. So A^C is a null set, i.e. of measure zero.*

Theorem 3.6.1.

$$\forall f, g : \chi \to [0, \infty] \textit{ measurable}$$

$f \leq g$ *almost everywhere* $\implies \int f \, d\mu \leq \int g \, d\mu$

2. $f = g$ *almost everywhere* $\implies \int f \, d\mu = \int g \, d\mu$
3. $\int f \, d\mu = 0 \implies f = 0$ *almost everywhere*
4. $\int f \, d\mu < \infty \implies f < \infty$ *almost everywhere*

Proof.
1. Let s be simple, $0 \leq s \leq f$. $s \cdot \mathbf{1}_{[f \leq g]} \leq g$ where $s \cdot \mathbf{1}_{[f \leq g]}$ is simple. $\int s \cdot \mathbf{1}_{[f \leq g]} \, d\mu \leq \int g \, d\mu$. $\int s \cdot \mathbf{1}_{[f \leq g]} \, d\mu = \int s \, d\mu$.
   If $\forall s$ simple, $0 \leq s \leq f$, then

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu \mid 0 \leq s \leq f, s \text{ simple} \right\} \leq \int g \, d\mu$$

2. $f \leq g$ almost everywhere and $f \geq g$ almost everywhere $\implies \int f \, d\mu = \int g \, d\mu$.

3. Markov inequality with $z = \frac{1}{n}$.

$$\frac{1}{n}\mu\left[f \geq \frac{1}{n}\right] \leq \int f\,d\mu = 0 \implies \mu\left[f \geq \frac{1}{n}\right] = 0 \forall n \in \mathbb{N}$$

$x \in [f \geq \frac{1}{n}] \implies x \in [f \geq \frac{1}{n+1}]$

$$\implies \mu\left[f \geq \frac{1}{n}\right] \to \mu\left[\bigcup\left[f \geq \frac{1}{n}\right]\right] = \mu\left[f > 0\right] = 0$$

4. $z > 0$, $s = z \cdot \mathbf{1}_{[f=\infty]} \leq f$.

$$z\mu[f = \infty] = \int s\,d\mu \leq \int f\,d\mu = M < \infty$$

$$\mu[f = \infty] \leq \frac{M}{z} \qquad \forall z > 0 \implies \mu[f = \infty] = 0$$

□

Theorem 3.6.2 (Levi's theorem about monotonic convergence). *If $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ is measurable and pointwise monotonically increasing ($f_1 \leq f_2 \leq \dots$) and $f = \lim_{n\to\infty} f_n$ then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$*

Proof. Because of (1) in the previous theorem, $\int f_n\,d\mu$ is monotonically increasing and $\leq \int f\,d\mu$, so $\lim \int f_n\,d\mu \leq \int f\,d\mu$.

$(y)^+$ denotes the function y if $y \geq 0$ and 0 otherwise.

Show "$\geq$". Let $0 \leq s \leq f$ be simple. Let $\varepsilon > 0$. $s_{n,\varepsilon} := (s-\varepsilon)^+ \mathbf{1}_{[f_n \geq f-\varepsilon]}$ is a simple function. $s - \varepsilon \leq f - \varepsilon \leq f_n$. $s_{n,\varepsilon} \leq f_n$.

$$\sum_z (z-\varepsilon)^+ \mu\left[s = z, f_n > f - \varepsilon\right] = \int s_{n,\varepsilon}\,d\mu \leq \int f_n\,d\mu \leq \lim \int f_n\,d\mu$$

$$s_{n,\varepsilon} = \underbrace{\sum_{z\text{ (values of s)}} (z-\varepsilon)^+ \mathbf{1}_{[s-z]}}_{(s-\varepsilon)^+} \mathbf{1}_{[f_n > f-\varepsilon]}$$

$$[f_n > f - \varepsilon] \nearrow \chi \qquad [s = z, f_n > f - \varepsilon] \nearrow [s - z]$$

$$\implies \sum_z (z-\varepsilon)^+ \mu[s = z] \leq \lim \int f_n\,d\mu$$

$$\varepsilon \to 0 \implies \sum_z z\mu[s = z] \leq \lim \int f_n\,d\mu$$

If $z > 0$, such that $\mu[s = z] = +\infty$. $0 < \varepsilon < z$.

Let $s_{n,\varepsilon} = (s-\varepsilon)^+ \mathbf{1}_{[f_n \geq M \wedge (f-\varepsilon)]}$, where $a \wedge b$ denotes the minimum of a and b. Let $M \geq \max s$. □

↓ *This lecture took place on 2018/10/29.*

Remark (Revision). *Let s be a simple function.* $s = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$.
$s = \sum_z z \mathbf{1}_{[s=z]}$ *is a finite sum*
$\int s \, d\mu = \sum_z \mu[s = z] = \sum_{i=1}^{n} c_i \mu(A_i)$
This is independent of the representation.

Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Then we can approximate the integral of f using the integrals of simple functions.

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu \mid 0 \leq s \leq f, \text{ simple} \right\}$$

Remark (Properties).

1. $0 \leq f \leq f$ *almost everywhere (wrt. μ)* $\implies \int f \, d\mu \leq \int g \, d\mu$
2. $f = g$ *almost everywhere (wrt. μ)* $\implies \int f \, d\mu = \int g \, d\mu$
3. $\int f \, d\mu = 0 \iff f = 0$ *almost everywhere (wrt. μ)*
4. $\int f \, d\mu < \infty \implies f < \infty$ *almost everywhere*

It is obvious if s is simple, then $\int s \, d\mu = \max \{ \int t \, d\mu \mid 0 \leq t \leq s \text{ simple} \}$

Theorem (Monotonic convergence). *Let $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$f_n \leq f_{n+1} \forall n \qquad f = \lim_{n\to\infty} f_n \qquad \implies \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

Lemma (Lemma by Fatou). *Let $f_n : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$\implies \int \left(\lim_{n\to\infty} \inf f_n \right) d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Proof.

$$\lim_{n\to\infty} \underbrace{\inf_{m \geq n} f_m}_{g_n} \qquad g_n \nearrow \liminf f_n$$

By the theorem of monotonic convergence,

$$\implies \int (\liminf f_n) \, d\mu = \int \lim g_n \, d\mu = \lim \int g_n \, d\mu$$

□

Lemma 3.7. *Let $f : (\chi, \mathcal{A}) \to [0, \infty]$ with countable $f(\chi)$.*

$$\implies \int f \, d\mu = \sum_{z \in f(\chi)} z \mu[f = z]$$

Proof.

$$f(\chi) = \{z_n \mid n \in \mathbb{N}\}$$

$$f_n = \sum_{k=1}^{n} z_k \mathbf{1}_{[f=z_k]} \nearrow f \implies \int f \, d\mu = \lim \int f_n \, d\mu = \lim \sum_{k=1}^{n} z_k \mu[f = z_k]$$

□

The integral should be linear. We expect this for any integral.

Theorem 3.7.1. *Let $f, g : (\chi, \mathcal{A}) \to [0, \infty]$ be measurable. Let $\alpha \geq 0$.*

1. $\int (\alpha f) \, d\mu = \alpha \int f \, d\mu$ *(trivial to prove)*
2. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Proof. 1. trivial

2. We represent f_n

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbf{1}_{[\frac{k}{2^n} \leq f < \frac{k+1}{2^n})} + n \cdot \mathbf{1}_{[f \geq n]} \nearrow f$$

Compare with Figure 2. g analogously $\nearrow g$

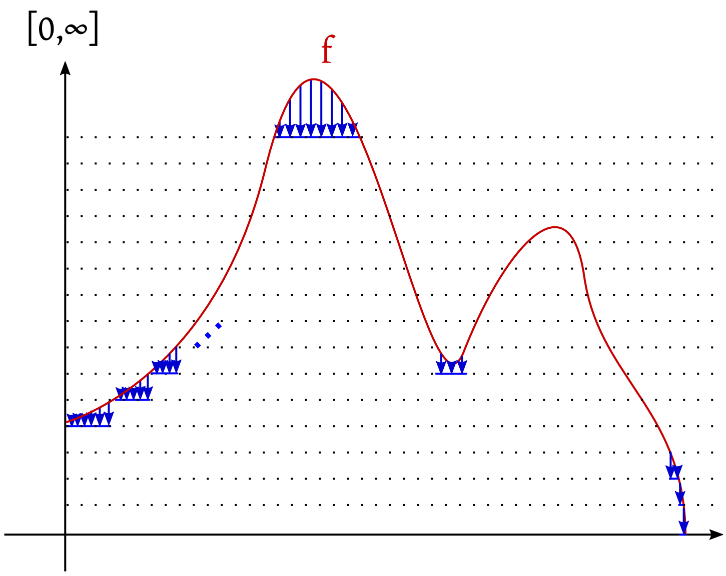

Figure 2: Illustration of the lebesgue integral

Let f_n, g_n be simple. $f_n + g_n \nearrow f + g$

$$\int (f + g) \, d\mu \overset{\text{monotonic convergence}}{=} \lim \int (f_n + g_n) \, d\mu$$

$$= \lim(\int f_n\, d\mu + \int g_n\, d\mu) \overset{\text{monotonic convergence}}{=} \int f\, d\mu + \int g\, d\mu$$

□

Unlike the Riemann integral, we use horizontal lines instead of vertical lines. Thus we partition the image, not the domain.

4 Integrable functions

Definition 4.1. *Let $f : (\chi, d) \to \overline{\mathbb{R}}$ is measurable. If not $\int f^+\, d\mu = \int f^-\, d\mu = +\infty$, integral exists:*

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu$$

$$f^+ = \max\{f, 0\} \qquad f^- = \max\{-f, 0\} \qquad f = f^+ - f^- \qquad |f| = f^+ + f^-$$

f is called integrable, if $\int f^+\, d\mu < \infty$ and $\int f^-\, d\mu < \infty$

$$\iff \int f\, d\mu \text{ exists and is finite}$$

Remark 4.2 (Properties).

1. *f is integrable $\iff$ $|f|$ is integrable and $|\int f\, d\mu| \leq \int |f|\, d\mu$*
2. *f, g are integrable with $f \leq g$ almost everywhere wrt. $\mu \implies \int f\, d\mu \leq \int g\, d\mu$*
3. *f is integrable, $\alpha \in \mathbb{R} \implies \alpha \cdot f$ is integrable and $\int (\alpha \cdot f)\, d\mu = \alpha \cdot \int f\, d\mu$*
4. *f, g are integrable $\implies$ $f + g$ is integrable and $\int (f + g)\, d\mu = \int f\, d\mu + \int g\, d\mu$*

Proof. 1. f is integrable

$$:\iff \int f^{\pm}\, d\mu < \infty \iff \underbrace{\int f^+\, d\mu + \int f^-\, d\mu < \infty}_{\int |f|\, d\mu < \infty}$$

$$\begin{aligned} \left|\int f\, d\mu\right| &= \left|\int f^+\, d\mu - \int f^-\, d\mu\right| \\ &\leq \int f^+\, d\mu + \int f^-\, d\mu \\ &= \int |f|\, d\mu \end{aligned}$$

2. $f^+ - f^- \overset{\substack{\text{almost}\\\text{everywhere}}}{\leq} g^+ - g^- \implies f^+ + g^- \overset{\text{a.e.}}{\leq} f^- + g^+$

$$\int f^+\, d\mu + \int g^-\, d\mu = \int (f^+ + g^-)\, d\mu \leq \int (f^- + g^+)\, d\mu = \int f^-\, d\mu + \int g^+\, d\mu$$

$$\int f^+ \, d\mu - \int f^- \, d\mu \leq \int g^+ \, d\mu - \int g^- \, d\mu$$

It is important to recognize that all integrals are finite.

3. For $\alpha = 0$, the statement is true. Consider $\alpha > 0$.

   $(\alpha f)^{\pm} = \alpha \cdot f^{\pm}$ $\quad \int \alpha f \, d\mu = \int \alpha \cdot f^+ \, d\mu - \int \alpha \cdot f^- \, d\mu = \alpha \int f^+ \, d\mu - \alpha \int f^- \, d\mu$

   Now consider $\alpha < 0$, or more simply $\alpha = -1$ (any negative number is the product of a positive number and -1):

   $$(-f)^+ = f^- (-f)^- = f^+ \qquad \ldots$$

4. $(f+g)^+ - (f+g)^- = f + g = f^+ + g^+ - (f^- + g^-)$

   $$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

   $$\int (f+g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int (f+g)^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu$$

   $$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu$$

□

Riemann integral only works for $\mathbb{R}^n$. The Lebesgue integral works for any measure space.

Example 4.3. *We consider the Riemann integral:*

$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \overset{\text{Riemann}}{=} \lim_{c,d \to \infty} \int_{-c}^{d} \frac{\sin x}{x} \, dx \text{ exists}$$

If you consider $\frac{\sin x}{x}$ for one π, we have a positive and negative area. By Leibniz criterion, we have an alternating series and its limit is zero.

We consider the Lebesgue integral:

$$\int_{\mathbb{R}} \left| \frac{\sin x}{x} \right| \, dx = +\infty$$

$\frac{\sin x}{x}$ is not Lebesgue-integrable. Because in case of the Lebesgue integral, we don't consider an alternating series, but need to consider $|f|$, which is non-negative and the series does not converge.

Theorem 4.3.1 (Dominated convergence theorem by Lebesgue)**.** *Let $f_n : (\chi, \mathcal{A}) \to \overline{\mathbb{R}}$ be a sequence of measurable functions. $f_n \to f$ pointwise [almost everywhere wrt. μ]. There exists $g : (\chi, \mathcal{A}) \to [0, \infty]$ integrable [$\int g \, d\mu < \infty$].*

$$|f_n| \leq g \text{ almost everywhere wrt. } \mu \forall n \implies \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proof. Without loss of generality, almost everywhere implies everywhere.

$$|f| = \lim |f_n| \leq g \qquad \text{all of them are integrable}$$
$$g_n = 2g - |f_n - f| \geq 0 \qquad g_n \to 2g$$
$$\liminf \int g_n \, d\mu \geq \int (\liminf g_n) \, d\mu \stackrel{g_n \to 2g}{=} \int (\lim g_n) \, d\mu = 2 \int g \, d\mu$$
$$\int g \, d\mu - \limsup \int |f_n - f| \, d\mu = \liminf \int g_n \, d\mu = 2 \int g \, d\mu$$
$$\limsup \left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \limsup \int |f_n - f| \, d\mu = 0$$

Again:

$$\int g_n = \left(\int 2g - \int |f_n - f| \right)$$
$$\implies \limsup \int g_n = \limsup \left(\int 2g - \int |f_n - f| \right)$$
$$= \int 2g + \limsup \left(- \int |f_n - f| \right) = \int 2g - \liminf \left(\int |f_n - f| \right)$$

□

↓ *This lecture took place on 2018/11/05.*

Remark (Dominated convergence theorem by Lebesgue). $f_n, f, g : (\chi, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\overline{\mathbb{R}}}$.

$$\begin{cases} f_n \to f & \mu \text{ almost everywhere} \\ \|f_n\| \leq g & \mu \text{ almost everywhere} \end{cases} g \geq 0, \int_\chi g \, d\mu < \infty$$
$$\implies \int \chi f \, dx = \lim_{n\to\infty} \int_\chi f_n \, d\mu$$

Example 4.4. $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ *with* $f_n(x) \to 0$ *and* $\int_{[0,1]} f_n \, d\lambda = 1 \not\Longrightarrow \int_{[0,1]} 0 \, d\mu$. *Compare with Figure 3.*

The theorem of convergence is a generalization of the following theorem (based on Analysis 1 and Analysis 2 courses):

Example 4.5 (Monotonic convergence).

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad a_n \geq 0$$

Convergence raidus: $R < \infty$.

$$x_k \nearrow R \implies f_k(x) \nearrow f(R)$$

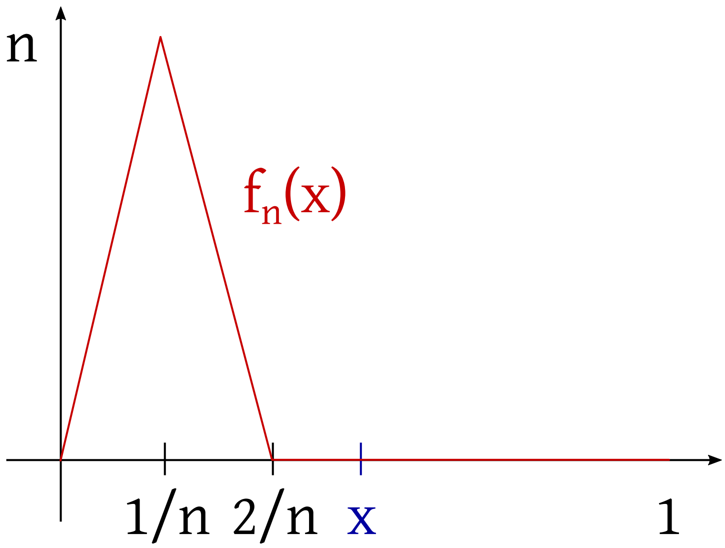

Figure 3: $f_n(x)$

$$\chi = \mathbb{N}_0, \mathcal{A} = \mathcal{P}(\mathbb{N}_0), \mu$$

"counting measure"

$$f_k(n) = a_n x_k^n$$

$$f : \mathbb{N}_0 \to [0, \infty]$$

$$\int_{\mathbb{N}_0} f \, d\mu = \sum_{n=0}^{\infty} f(n)\mu(n)$$

for $k \to \infty$: $f_k(n) \nearrow f(n) = a_n R^n$. By monontonic convergence, $\int f_k \, d\mu \nearrow \int f \, d\mu$.

$$\sum_{n=0}^{\infty} a_n x_k^n \nearrow \sum_{n=0}^{\infty} a_n R^n$$

5 Lebesgue and Riemann integral

λ is defined on $(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$. The Lebesgue measure allows null sets. Lebesgue measure is also defined on completion of sigma algebras. Lebesgue measure $\mathcal{L}$ is defined on σ-algebras of Lebesgue sets.

Remark (One characterization of the Axiom of Choice)**.** *A non-empty product of non-empty sets is non-empty.*

Remark. $\mathcal{L} \setminus \mathcal{B} \neq \emptyset$ *[Axiom of Choice].*

Remark (Number representation). *Basis* $q \in \{2, 3, \dots\}$.

$$x = \sum_{n=1}^{\infty} \frac{x_n}{q^n} \qquad x_n \in \{0, 1, \dots, q-1\}$$

This represents a number $0.x_1 x_2 x_3 x_4 \dots$.

Because $0.7\overline{9} = 0.8$*, there is some ambiguity between the numbers and their representation (non-bijective, two sums represent the same* x*).*

Remark (Cantor set). *Consider* $[0, 1]$*. Split the interval into 3 thirds. We remove the middle third as open set. We consider the remaining two intervals and again extract the middle third. We iteratively continue this process to infinity. The remaining set is called Cantor set and is uncountable.*

The Cantor set $\mathcal{C}$ *is the set of numbers in* $[0, 1]$ *with some number representation, with respect to basis* 3*, which does not contain some* 1 *and has a unique number representation. Unique number representation because*

$$\frac{2}{3} = 0.2 = 0.1\overline{2}$$

$$\frac{1}{3} = 0.1 = 0.0\overline{2} \qquad \frac{1}{9} = 0.01 = 0.00\overline{2} \qquad \frac{2}{9} = 0.02 = 0.01\overline{2}$$

A linear combination of Borel-measurable functions is Borel-measurable.

Remark. *Riemann integral (*U *are lower sums,* O *are upper sums):*

$$f : [a, b] \to \mathbb{R} \text{ bounded}$$

$$Z = \{a = x_0 < x_1 < \dots < x_k = b\} \qquad \|Z\| = \max_{j=1,k}(x_j - x_{j-1})$$

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$

$$U(Z, f) = \sum_{j=1}^{k} m_j (x_j - x_{j-1})$$

$$g_Z = \sum_{j=1}^{k} m_j \mathbf{1}_{(x_{j-1}, x_j]} \text{ are both } \mathcal{B}\text{-measurable}$$

$$U(Z, f) = \sum_{j=1}^{l} M_j (x_j - x_{j-1})$$

$$h_Z = \sum_{j=1}^{k} M_j \mathbf{1}_{(x_{j-1}, x_j]}$$

$$U(Z, f) = \int_{[a,b]} g_Z \, d\lambda \qquad O(Z, f) = \int_{[a,b]} h_Z \, d\lambda$$

Theorem 5.0.1. *f as above (if necessary, not Borel-measurable)*

$C = \{x \in [a,b] : f \text{ continuous in } x\} \quad D = \{x \in [a,b] \mid f \text{ in } x \text{ is non-continuous}\}$

1. *Then $C, D \in \mathcal{B}_{[a,b]}$, $f \cdot \mathbf{1}_C$ is Borel-measurable*
2. *f is Riemann integrable $\iff \lambda(D) = 0$ and*

$$\int_a^b f(x)\,dx = \int_{[a,b]} f \cdot \mathbf{1}_C \,d\lambda$$

Proof.

$$Z_n = \left\{a = x_0^{(n)} < x_1^{(n)} < \dots < x_{k(n)}^{(n)} = b\right\}$$

A sequence of decompositions such that

1. Z_{n+1} is refinement of Z_n
2. $\|Z_n\| \to 0$

except for point a (so the intervals are left-sided half-open) (you can also close the first interval of the left side)

$$\inf f = m \le g_{Z_n} \nearrow g \le f \le h \swarrow h_{Z_n} \le M = \sup f$$

where g and h are Borel-measurable. By the dominated convergence theorem by Lebesgue,

$$U(Z_n, f) = \int_{[a,b]} g_{Z_n}\,d\lambda \nearrow \int_{[a,b]} g\,d\lambda$$

$$O(Z_n, f) = \int h_{Z_n}\,d\lambda \searrow \int_{[a,b]} h\,d\lambda$$

$R = \left\{x_j^{(n)} \mid n \in \mathbb{N}, j = 1, \dots, k(n)\right\}$ is countable, $\lambda(R) = 0$. □

Claim 5.1. *For $x \in [a,b] \setminus R$, f is continuous at $x \iff g(x) = h(x)$.*

Proof. Let $I_n(x)$ be the interval of Z_n with $x \in I_n(x)$. Recognize that $I_{n+1}(x) \subset I_n(x)$ and $\lambda(I_n(x)) \searrow 0$.

$$\begin{aligned}
f \text{ cont. in } x &\iff \forall \varepsilon > 0 \exists k : f(x) - \varepsilon < f_{I_k(x)} < f(x) + \varepsilon \\
&\iff \forall \varepsilon > 0 \forall n \ge k : f(x) - \varepsilon < f_{I_k(x)} < f(x) + \varepsilon \\
&\iff \forall \varepsilon > 0 \exists k \forall n \ge k : f(x) - \varepsilon \le g_{Z_1}\Big|_{I_n(x)} \le h_{Z_n}\Big|_{I_n(x)} \le f(x) + \varepsilon \\
&\Rightarrow g(x) = h(x) \qquad [= f(x)]
\end{aligned}$$

“$\Rightarrow$” is “ $\iff$ ” assuming $x \notin R$, so $[g < h] \subset D \subset [g < h] \cup R$ where $[g < h]$ is the Borel set.

$D \setminus [g < h]$ is at most countable (because R is countable) $\implies D$ Borel set, C Borel set

$$\lambda(D) = \lambda[g < h]$$

□

$$f \text{ is R-integrable} \iff \int_{[a,b]} g \, d\lambda = \int_{[a,b]} h \, d\lambda, h \geq g$$
$$\iff \lambda[g < h] = 0 \iff \lambda(D) = 0$$

in this case (because $g \leq f \leq h$ except in a).

$$g \cdot \mathbf{1}_C = f \cdot \mathbf{1}_C$$

where g and $\mathbf{1}_C$ is Borel-measurable and thus $f \cdot \mathbf{1}_C$ is Borel-measurable.

$$\int_{[a,b]} f \cdot \mathbf{1}_C \, d\lambda = \int_{[a,b]} g \cdot \mathbf{1}_C \, d\lambda = \int_{[a,b]} g \, d\lambda = \int_a^b f(x) \, dx$$

$$g \cdot \mathbf{1}_C = g \qquad \text{almost everywhere wrt. } \lambda$$

Example 5.2. 1. *$\mathbf{1}_{\mathbb{Q}}$ is nowhere continuous.*

$$\int_0^1 \mathbf{1}_{\mathbb{Q}}(x) \, dx \textit{ does not exist}$$

$$\mathbf{1}_{\mathbb{Q}} = 0 \qquad \textit{almost everywhere wrt. } \lambda \implies \int_{[0,1]} \mathbf{1}_{\mathbb{Q}} \, d\lambda = 0$$

2.
$$\int_a^b \frac{\sin x}{x} \, dx = \int_{[a,b]} \frac{\sin x}{x} \, dx$$
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi \qquad \nexists \int_{\mathbb{R}} \frac{\sin x}{x} \, d\lambda(x)$$

Theorem 5.2.1 (Substitution theorem). *Let $\varphi : (\chi, \mathbb{A}, \mu) \to (\chi', \mathcal{A}')$ is measurable.*

$$\mu_\varphi(A') = \mu(\varphi^{-1}(A')) \qquad A' \in \mathcal{A}' \qquad \textit{pushforward measure}$$

$$f : (\chi', \mathcal{A}') \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}) \textit{ measurable}$$

Then,
$$\int_\chi f \circ \varphi \, d\mu \textit{ exists} \iff \int_{\chi'} f \, d\mu_\varphi \textit{ exists}$$

and in this case, they are the same.

Index

Continuity from below, 9
Counting measure, 3

Fatou's Lemma, 21

Generating set, 4
Generator, 4

Lebesgue measure, 12

Measurable maps, 5
Measurable space, 3
Measure, 3
Measure space, 3
Monotonically increasing set sequence, 9

Push-forward measure, 16

Sigma-algebra, 3
Stable set by intersection, 10

Trace σ-algebra, 5, 7