Mathematical analysis 1 – Lecture notes

course by Wolfgang Ring

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Contents

1 Modern logic

This lecture took place on 1st of October 2015 with lecturer Wolfgang Ring.

- motivation for visiting university
- Kurt Gödel: Gödel's incompleteness theorem
- propositional logic (and/or/implication/equivalence operation)
  - $p \implies q$: "p implies q" ("notwendig"), "q requires p" ("hinreichend")
  - Indirect proof: $(\neg q \implies \neg p) \Leftrightarrow (p \implies q)$
  - Proof by contradiction: claim p, claim $\neg q$, show that $p \wedge \neg q$ is not possible
  - commutative law: $a \wedge b \Leftrightarrow b \wedge a$
  - associative law: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  - distributive law: $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
  - DeMorgan's law: $\neg(a \wedge b) \Leftrightarrow (\neg a) \vee (\neg b)$
- First-order logic
  - $\forall x \in \mathbb{N} : x \in \mathbb{R}$
  - $\forall x \in M : P(x)$
  - $\neg [(\forall x \in M) P(x)] \Leftrightarrow \exists x \in M : \neg P(x)$
- Peano's axioms: rationale for induction proofs

The lecture on 8th of October 2015 got cancelled spontaneously.

2 Binary operations

This lecture took place on 12th of October 2015 with lecturer Wolfgang Ring.

Literature recommendation:

- "Analysis 1 (Mathematik für das Lehramt)", Oliver Deiser

Let A and B be statements.

- Logical equivalence is given iff the truthtable of both expressions is the same.
- $\neg(\neg A) \iff A$
- $(A \vee B) \iff (B \vee A)$
- $(A \wedge B) \iff (B \wedge A)$
- $a \implies b$: implication

Boolean Laws:

$$\neg(A \implies B) \iff A \wedge \neg B \tag{1}$$
$$A \iff B \implies (A \implies B) \wedge (B \implies A) \tag{2}$$

"contraposition" or "indirect proof"

$$\neg B \implies \neg A \tag{3}$$
$$A \implies B \iff (\neg B \implies \neg A) \tag{4}$$
$$(A \iff B) \iff (\neg A \iff \neg B) \tag{5}$$
$$\neg(A \wedge B) \iff \neg A \vee \neg B \tag{6}$$
$$\neg(A \vee B) \iff \neg A \wedge \neg B \tag{7}$$
$$\neg(A \implies B) \iff (A \wedge \neg B) \tag{8}$$
$$A \wedge (B \vee C) \iff ((A \wedge B) \vee (A \wedge C)) \tag{9}$$
$$A \vee (B \wedge C) \iff ((A \vee B) \wedge (A \vee C)) \tag{10}$$
$$(A \implies B) \iff (\neg A \vee B) \tag{11}$$

"proof by contradiction"

$$((A \implies B) \wedge (A \implies \neg B)) \implies \neg A \tag{12}$$

"conclusion"

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C) \tag{13}$$

$$A \vee B \Leftrightarrow \neg(\neg A) \vee \neg(\neg B) \Leftrightarrow \neg(\neg A \wedge \neg B)$$
$$\neg(A \vee B) \Leftrightarrow \neg(\neg(\neg A) \vee (\neg B))$$

Distributive laws:

- $(A \vee B) \wedge C \Leftrightarrow (A \wedge C) \vee (A \wedge C)$
- $(A \wedge B) \vee C \Leftrightarrow (A \vee C) \wedge (B \vee C)$

2.1 Tautologies

A *tautology* is the composition of statements, which always yields the truth value true, independent of the truth value of its subexpressions.

Examples of tautologies:

"Law of excluded middle" $A \vee \neg A$

equivalences are always tautologies $A \leftrightarrow \neg(\neg A)$

implication of itself $A \to A$

Tautology with multiple statements:

implication with or and not $(A \to B) \leftrightarrow (\neg A \vee B)$

proof by contradiction $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

chain inference $[(A \to B) \wedge (B \to C)] \to (A \to C)$

This lecture took place on 14th of Oct 2015 with lecturer Wolfgang Ring.

Proof. We prove, $[(A \to B) \wedge (A \to \neg B)] \to \neg A$.

$$\begin{aligned} (A \to B) \wedge (A \to \neg B) &\iff (\neg A \vee B) \wedge (\neg A \vee \neg B) \\ &\iff \underbrace{(B \wedge \neg B)}_{\perp} \vee \neg A \\ &\iff \neg A \end{aligned}$$

Special case: $A = B$.

$$(A \to A) \wedge (A \to \neg A) \to \neg A$$

$$(A \to \neg A) \to \neg A$$

□

2.2 Negation of a tautology

- is called *contradiction*.
- has always truth value false.

Proof.
$$(A \vee B) \to C \Leftrightarrow \neg(A \vee B) \vee C \Leftrightarrow (\neg A \wedge \neg B) \vee C$$
$$(\neg A \vee C) \wedge (\neg B \vee C) \Leftrightarrow (A \to C) \wedge (B \to C)$$

$$\begin{aligned} (A \vee B) \to C &\Leftrightarrow (A \to C) \wedge (B \to C) \\ (A \wedge B) \to C &\Leftrightarrow (A \to C) \vee (B \to C) \\ A \to (B \wedge C) &\Leftrightarrow (A \to B) \wedge (A \to C) \\ A \to (B \vee C) &\Leftrightarrow (A \to B) \vee (A \to C) \end{aligned}$$

□

Example proof by contradiction: Number of prime numbers. We prove a statement by Euklid of Alexandria, 300 BC:

> The number of prime numbers is infinite.

Assume the number of prime numbers is finite. Then there exists some $N \in \mathbb{N}$ such that $\mathbb{P} = \{P_1, P_2, \dots, P_n\}$ is the set of all prime numbers.

Every integer can be represented as product of prime numbers. Therefore for every integer there exists at least one prime number that divides this number (without remainder).

Let $m = p_1 \cdot p_2 \cdot \ldots \cdot p_N + 1$. Let a be a prime number that divides m.

It holds that: Every $p_i \in \mathbb{P}$ is not a divisor of m. Because when dividing $\frac{m}{p_i}$, the remainder is always one.

So $q \in \mathbb{P}$, so there exists more than N prime numbers (at least $N+1$). This contradicts with our assumption, that only N prime numbers exist.

Therefore always one more prime number exists. So the number of prime numbers is infinite. □

2.3 Quantifiers

Quantified statements are statements, in which objects of a set occur.

Example: Let $P(x) = (x > 0)$. Its truth value cannot be determined if the set X is not defined.

Definition 1. *Let M be a set, $x \in M$ and $P(x)$ a predicate.*

The composed statement: for every $x \in M$, it holds that $P(x)$ is true, if the truth value of $P(x)$ is always true independent of the selection of $x \in M$.

Example 1. *Let $M = \mathbb{R}$ and $P(x) = (x^2 + 1 > 0)$.*

This is true for all $x \in M$. We denote: $\forall x \in M : P(x)$.

Example 2. *Let $M = \mathbb{R}$ and $P(x) = (x^2 - 1 > 0)$.*

This is not *true for all $x \in M$. We denote: $\exists x \in M : \neg P(x)$.*

Definition 2. *$\forall x \in M : P(x)$ does not hold if and only if $\exists x \in M : \neg P(x)$.*

$\forall$ is called all quantifier. *$\exists$ is called* existence quantifier.

Negation works as follows:

$$\neg (\forall x \in M : P(x)) \Leftrightarrow \exists x \in M : \neg P(x)$$

$$\neg (\exists x \in M : P(x)) \Leftrightarrow \forall x \in M : \neg P(x)$$

This lecture took place on 15th of Oct 2015 with lecturer Wolfgang Ring.

$$\forall x \in M : (P(x) \wedge Q(x)) \iff (\forall x \in X : P(x)) \wedge (\forall y \in M : Q(y)))$$

$$\forall x \in M : (P(x) \vee Q(x)) \leftrightarrow (\forall x \in M : P(x)) \wedge (\forall x \in M : Q(x))$$

Counterexample:
$$M = \mathbb{R} \quad P(x) := (x > 0)$$

A statement B is stronger than C if C

2.4 Composition of several quantifiers

1. The order of quantfiers matters.
2. For every real number x, there exists an integer $n \in \mathbb{N}$ with the property $n > x$:
$$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$$
The statement does not hold if the order is changed.
$$\exists n \in \mathbb{N} \forall x \in \mathbb{R} : n > x$$

3 Sets

We consider objects, which we call *sets*. For every set M and every element x, it holds that
$$x \in M \vee \neg(x \in M)$$

Consider the set $L = \{M : M \text{ is a set and } M \notin M\}$. Does $L \notin L$ or $L \in L$ hold?

If $L \notin L$, then L satisfies the definition and therefore $L \in L$. If $L \in L$, then elements of L satisfy the property; therefore $L \notin L$.

Set operations:

- union
- intersection
- subsets

- $\forall S : \emptyset \subseteq S$
- complete induction

Theorem 1. *(Pythagoreans, 450 BC)*

$$\forall n \in \mathbb{N}_+ : \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. **Induction base** $n = 1$

$$P(1) : 1 = \frac{1(1+1)}{2} \quad \checkmark$$

Induction step $n \to n+1$

Assume $P(n)$ is true. So $(1 + 2 + \cdots + n) = \frac{n(n+1)}{2}$.

$$[(1 + 2 + \cdots + n) + (n+1)] = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1) \cdot \frac{(n+2)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark$$

So, it simply holds that:

$$s = 1 + 2 + 3 + \cdots + n$$

$$2 \cdot s = \underbrace{n}_{\text{number of items}} \cdot \underbrace{(n+1)}_{\text{sum}} \Rightarrow s = \frac{n \cdot (n+1)}{2}$$

□

This lecture took place on 21st of October 2015 with lecturer Ring Wolfgang.

Previously: Bernoulli Inequation $(1+x)^n > 1 + nx$

- Let X be a set. $M = \{x \in X : P(x)\}$.
- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ …"enumerating set representation"
- $M = \{x \in X \mid P(x)\}$, $N = \{x \in X \mid Q(x)\}$

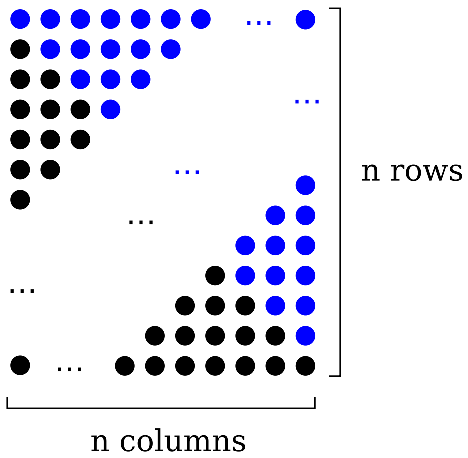

Figure 1: Illustration of the triangular number (illustrative proof)

- $M \cup N = \{x \in X \mid P(x) \vee Q(x)\}$
- Let X be a set. $A_0 \subseteq X$, $A_1 \subseteq X$, $A_2 \subseteq X$, etc
- $\forall n \in \mathbb{N} : A_n \subseteq X$
- $A_0 \cup A_1 \cup A_2 \cup \cdots = \bigcup_{n=1}^{\infty} A_n = \{x \in X \mid (x \in A_0) \vee (x \in A_1) \vee \dots\} = \{x \in X \mid \exists n \in \mathbb{N} : x \in A_n\}$
- $A_0 \cap A_1 \cap A_2 \cap \cdots = \bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall n \in \mathbb{N} : x \in A_n\}$

Definition 3. *Let A and B sets. The* cartesian product of A and B *is given as:*

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

This operation is not commutative!

Definition 4. *We denote* $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Example 3.

$$A = \{a, b, c, d, e, f, g, h\}$$
$$B = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A \times B = \{(a, 1), (a, 2), (a, 3), \dots, (a, 8), (b, 1), (b, 2), \dots\}$$

Example 4.

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$$

e.g. $(1, \frac{9}{8}) \in \mathbb{R} \times \mathbb{R}$.

Definition 5. *Let $A_1, A_2, \dots, A_n$ be sets.*

$$A_n = A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \,|\, a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

instead of $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}} = A^n$.

Definition 6. *Let X be a set. Then $\mathcal{P}(X)$ is the* power set *of x.*

$$\mathcal{P}(X) = \{A \,|\, A \subseteq X\}$$

Definition 7. *Let A and B be sets. A* mapping *f from A to B (denoted $f : A \to B$) is an assignment, such that for every $x \in A$ one $y \in B$ is assigned. We denote the corresponding $y \in B$ for some $x \in A$ with $y = f(x)$. A is called domain, B is called co-domain.*

Definition 8 (Alternative definition of mappings)**.** *A mapping f is a subset of $A \times B$ which fulfills the following properties:*

- $\forall x \in A : (\exists y \in B : (x, y) \in f)$
- $\forall x \in A \wedge (y_1, y_2 \in B) : [(x_1, y_1) \in f \wedge (x_1, y_2) \in f] \implies y_1 = y_2$

Notation:

$$(x, y) \notin f \Leftrightarrow y = f(x)$$
$$\{(x, f(x)) \in \,|\, x \in A\} \Rightarrow \textit{graph from } f$$

Definition 9. *Let $f : A \to B$ be a mapping.*

- *The mapping f is called surjective, if $\forall y \in B : \exists x \in A : y = f(x)$.*
- *The mapping f is called injective, if $\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.*
- *Let $B' \subseteq B$. Then we denote $f^{-1}(B') = \{x \in A \,|\, f(x) \in B'\}$ as the* codomain *of f.*

  Attention! The codomain distinguishes itself from the inverse function f^{-1}!
- *Let $A' \subseteq A$. Then we call $f(A') = \{f(x) \,|\, x \in A\} \subseteq B$ the* image *of A' under f.*

  Special case: $A' = A$, then $f(A) \subseteq B$ is image of A under f.

  Let $f : A \to B$ be a mapping. We define $f : A \to f(A) \subseteq B$ with $\tilde{f}(x) = f(x)$ for all $x \in A$. The mapping $\tilde{f}$ is surjective $\forall y \in f(A)$ there exists one $x \in A$ such that $y = f(x)$.
- *A mapping is called* bijective *iff the mapping is surjective and injective.*

3.1 Integers

Definition 10 (Bernoulli's inequality)**.** *Let $x \in \mathbb{R}$ with $x > -1$ and $x \neq 0$. Let $n \in \mathbb{N}$ with $n > 1$. Then it holds that*

$$(1 + x)^n > 1 + nx$$

Proof. Proof by complete induction.

induction base $n = 2$

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x \quad \checkmark$$

because $x^2 > 0$ for $x \neq 0$.

induction step $n \to n + 1$ Assume $(1 + x)^2 > 1 + n$, then $x > -1$ and $x \neq 0$.

$$(1 + x)^{n+1} = (1 + x)^n \cdot \underbrace{(1 + x)}_{>0} > (1 + nx) \cdot (1 + x)$$

$$= (1 + nx + x + nx^2) = (1 + (n + 1) \cdot x + \underbrace{nx^2}_{>0}) > 1 + (n + 1) \cdot x$$

Back to sets and functions:

- injective, surjective, bijective function
- composition of functions: Let $f : X \to Y$ and $g : Y \to Z$. $g \circ f : X \to Z$ is defined as $g(f(x))$ (“g after f”).
- Let f and g be mappings. If f and g are injective, $f \circ g$ is injective. If f and g are surjective, $f \circ g$ is surjective. If f and g are bijective, $f \circ g$ is bijective.
- Identity function, $f \circ \mathrm{id} = \mathrm{id} \circ f = f$
- properties of an inverse function, $f \circ f^{-1} : X \to X$, $f^{-1} \circ f : X \to X$

This lecture took place on 21st of Oct 2015 with lecturer Wolfgang Ring.

4 Summation notation

$$\sum_{k=h}^{l} a_k$$

Iteration over all values from l to h (inclusive) and evaluation of the enclosed expression with k as iteration value.

□ Laws:

$$\sum_{k=l}^{h} a_k = \sum_{i=l}^{h} a_i \tag{14}$$

$$\sum_{k=l}^{h} (a_k + b_k) = \left(\sum_{k=l}^{h} a_k\right) + \left(\sum_{k=l}^{h} b_k\right) \tag{15}$$

$$\sum_{k=0}^{h} a_k = a_0 + \sum_{k=1}^{h} a_k \qquad \text{“Extraction of the initial value”} \tag{16}$$

$$\sum_{k=0}^{h} a_k = a_h + \sum_{k=0}^{h-1} a_k \qquad \text{“Extraction of the final value”} \tag{17}$$

$$\sum_{k=u+n}^{h+n} a_k = \sum_{k=u}^{h} a_{k+n} \qquad \text{“index shifting”} \tag{18}$$

$$\sum_{k=l}^{h} \lambda \cdot a_k = \lambda \cdot \sum_{k=l}^{h} a_k \qquad \text{“extraction of a constant } \lambda\text{”} \tag{19}$$

$$\sum_{k=0}^{n} n = \frac{n(n+1)}{2} \qquad \text{“triangular sum”} \tag{20}$$

We consider $S_n = \{(a_1, a_2, \dots, a_n) : a_i \in M_n \forall i = 1, \dots, n \text{ with } a_i \neq a_j\} \subseteq M_n \times M_n \times \dots \times M_n$. S_n is the set of all arrangements of the numbers $1, \dots, n$.

Example: $\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

Theorem 2. *It holds that* $|\,S_n\,| = n!$ *for all* $n \in \mathbb{N}$

Proof. Proof by induction over n.

Induction base $n = 1$: $M_1 = \{1\}, S_1 = \{(1)\} \Rightarrow |\,S_1\,| = 1 = 1!$ ✓

Induction step $n \to n+1$:

$$S_{n+1} = \{(a_1, a_2, \dots, a_n) : a_i \in M_{n+1} \forall i \in M_{n+1}, a_i \neq a_j \text{ for } i \neq j\}$$

For $l \in M_{n+1}$:

$$W_l = \{(a_1, \dots, a_{n+1}) \in S_{n+1} : a_l = n+1\}$$

It holds that $W_l \cap W_j = \emptyset$ for $l \neq j$ and $S_{n+1} = W_i \cup W_l \cup W_j \cup \ldots \cup W_{n+1}$. Then it holds that $|\, S_{n+1} \,| = |\, W_1 \,| + |\, W_2 \,| + \ldots + |\, W_{n+1} \,| = \sum_{l=1}^{n+1} |\, W_l \,|$

Theorem 3. *Claim: For every $l \in M_{n+1}$ it holds that $|\, W_l \,| = |\, S_n \,| = n!$.*

Proof. We build a bijective map $\phi_l : W_l \to S_n$.

$$W_l = \{(a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}\}$$

$$: a_i \in M_n, \forall i \neq l, a_i \neq a_j \forall i \neq j$$

$$\phi\left((a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1})\right)$$

$$= (a_1, a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) \in S_n$$

S_n is surjective. Let $(b_1, \ldots, b_n) \in S_n$, then it holds that $(b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n) \in W_l$

$$\phi_l((b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n)) = (b_1, \ldots, b_n)$$

S_n is injective.

$$\phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$= \phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$\Rightarrow (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) = (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1})$$

ϕ is bijective. □

Therefore $|\, W_l \,| = |\, S_n \,| = n!$. Therefore $|\, S_{n+1} \,| = \sum_{l=1}^{n+1} |\, S_n \,| = \sum_{l=1}^{n+1} n! = (n+1)n! = (n+1)!$

□

Remark 1. *Let $f : M_n \to M_n$. f is represented as*

$$(1, 2, 3, 4, \ldots, n-1, n) \to (f(1), f(2), f(3), f(4), \ldots, f(n-1), f(n))$$

Therefore $(f(1), f(2), \ldots, f(n)) \in S_n$. Analogously every $(a_1, \ldots, a_n) \in S_n$ defined by $f(k) = a_k$ for $k = 1, \ldots, n$ is a bijective mapping $f : M_n \to M_n$. Therefore we set $S_n = \{f : M_n \to M_n : f \text{ is bijective}\}$. S_n is called symmetric group of n elements.

Definition 11. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ with $k \leq n$. We define*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{“binomial coefficient } n \text{ choose } k\text{”}$$

It holds that

$$\binom{n}{k} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{(1 \cdot 2 \cdot \ldots \cdot k)(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))}$$
$$= \frac{n(n-1) \cdot \ldots \cdot (k+1)}{(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))}$$

Factorial laws:

$$\binom{1}{0} = \frac{n!}{0!(n-0)!} = 1 \qquad \forall n \in \mathbb{N}$$
$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = 1$$
$$\binom{n}{n-k} = \frac{n!}{(n-k!)(n-n+k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \qquad \text{“symmetrical”}$$

A recursive definition is given by

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad n \geq 1, 1 \leq k \leq n-1$$

Proof.

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n+1}{k} &= \frac{(n-1)!}{(n-1)!(n-1-(k-1))!}\\ &= \frac{(n-1)!}{k!(n-1-k)!}\\ &= \frac{(n-1)!}{(k-1)!(n-k)!}+\left(\frac{(n-1)!}{k!(n-1-k)!}\right)\\ &= \frac{k\cdot(n-1)!+(n-k)(n-1)!}{k!(n-k)!}\\ &= \frac{n(n-1)!}{k!(n-1)!}=\frac{n!}{k!(n-k)!}\\ &= \binom{n}{k}\end{aligned}$$

$\square$

4.1 Arrangement in Pascal's triangle

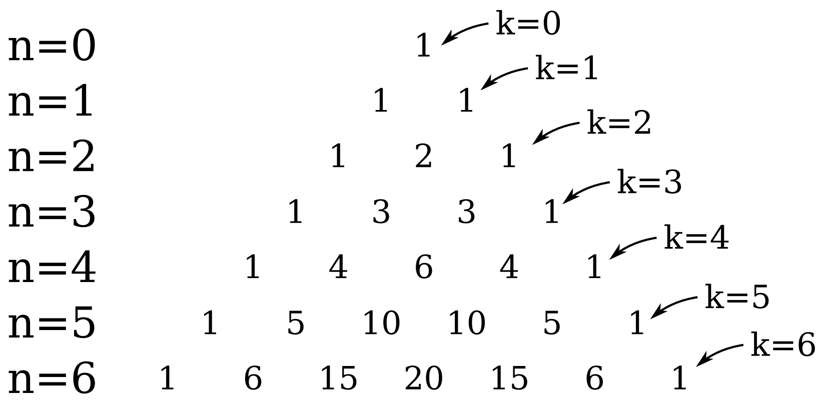

Figure 2: Pascal's triangle

Theorem 4. *Let $T_n^k = \{A \subseteq M_n : |A| = k\}$. Then it holds that $|T_n^k| = \binom{n}{k}$.*

Example: $T_3^2 = \{\{1,2\},\{1,3\},\{2,3\}\}$.

$|T_3^2| = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$

Proof. Let n be fixed. Induction for k.

Induction base $k = 0$

$$T_n^0 = \{\emptyset\}$$

$$|T_n^0| = 1 = \binom{n}{0}$$

Induction step $k \to k+1$

$$\begin{aligned}T_n^k &= \underbrace{\{\{a_1,\dots,a_k\} : a_1 \in M_n, (i=1,\dots,k), a_i \neq a_j \text{ for } i \neq j\}}_{A_1}\\ &\cup \underbrace{\{\{a_1,\dots,a_{k-1}\} \cup [n] \in M_{n-1}\}}_{A_2}\end{aligned}$$

$$|T_N^k| = |A_1| + |A_2|$$

$\square$

This lecture took place on 28th of October 2015 with lecturer Ring Wolfgang.

Let A, B be sets and define

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

Then the domain of $A \setminus B$ is "A without B".

Theorem 5.

$$T_n^x = \{x \subseteq M_x : |X| = x\}$$

Let $k \in \mathbb{N}$ and $0 \leq k \leq 1$.

$$|T_n^x| = \binom{1}{k}$$

There are exactly $\binom{n}{k}$ k-ary subsets of M_n.

Proof.

$$M_0 = \emptyset \qquad T_0^0 = \{\emptyset\} \qquad |\,T_n^0\,| = 1 = \binom{0}{n}$$

Proof by complete induction over n of the following statement:

$$\forall n \in \mathbb{N} : \forall k \in \mathbb{N} \text{ with } 0 \leq k \leq n : |\,T_n^k\,| = \binom{n}{k}$$

Induction base $n = 0$ is fine. For $n = 1$ there are two cases: $k = 0$ or $k = 1$.

$$M_1 = \{1\}$$

$$T_1^0 = \{\emptyset\} \qquad |\,T_1^0\,| = 1 = \binom{1}{0}$$

$$T_1^1 = \{\{1\}\} \qquad |\,T_1^1\,| = 1 = \binom{1}{1}$$

Is also fine.

Induction step The hypothesis is our assumption:

$$\forall 0 \leq k \leq 1 : |\,T_n^k\,| = \binom{n}{k}$$

Consider M_{n+1}. Special case $k = 0$:

$$T_{n+1}^0 = \{\emptyset\} \qquad \left|\, T_{n+1}^0 = 1 = \binom{n+1}{0} \,\right|$$

Special case $k = n + 1$:

$$T_n = \{M_{n+1}\} \qquad |\,T_N n + 1^{n+1}\,| = 1 = \binom{n+1}{n+1}$$

Let $1 \leq k \leq n$.

$$T_{n+1}^x TODO$$

Union is disjoint $\Rightarrow |\,T_{n+1}^k\,| = |\,R_{n+1}^k\,| + |\,S_{n+1}^k\,|$

$$R_{n+1}^k = \{A \subseteq M_n : |\,A\,| = k\} = T_n^k$$

$$|\,R_{n+1}^k\,| = |\,T_n^k\,| = \binom{n}{k}$$

by induction hypothesis.

$$S_{n+1}^k = \{A \subseteq M_{n+1} : A = A' \cup \{n+1\} : A' \subseteq M_n : |\,A'\,| = k - 1\}$$

We prove $|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,|$.

$$f : S_{n+1}^k \to T_n^{k-1}$$

$$f(A) = f(A' \cup \{n+1\}) = A'$$

f is bijective. f is surjective: Let $A' \in T_n^k$ define $A = A' \cup \{n+1\} \in S_{n+1}^n$ and $f(A) = A'$. f is injective: Let $f(A) = f(B)$ and $A = A' \cup \{n+1\} \in S_{n+1}^k$.
$B = B' \cup \{n+1\} \in S_{n+1}^k$. $A', B' \in T_n^{k-1}$.

$$f(A) = f(B) \Rightarrow A' = B' \Rightarrow A' \cup \{n+1\} = B' \cup \{n+1\} \Rightarrow A = B$$

$$|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,| \overset{\text{ind. hypo.}}{=} \binom{n}{k-1}$$

Therefore $|\,T_{n+1}^k\,| = \binom{n}{n} = \binom{n}{k-1} = \binom{n+1}{k}$. The last equation follows from the recursive definition of binomial coefficients.

□

Theorem 6. ***(Binomial theorem).***
Let $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$). Then it holds that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof. 1. Proof by induction over n.

Induction step $n = 0$: $(a+b)^0 = 1$

$$\sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^0 = 1$$

Induction step $n \to n+1$

$$(a+b)^{n+1} = (a+b)^n \cdot (a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^n b^{n-k}\right)(a+b)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \underbrace{\sum_{n=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\substack{\text{index shift} \\ h+1=j, h=0 \\ \Rightarrow j=1, h=j-1, h=n-1 \\ \Rightarrow j=n}} + \underbrace{\binom{n}{k} a^{n+1} \cdot b^0}_{a^{n-1}} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1}$$

$$\sum_{j=1}^{n} \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n+1}{n+1} a^{n+1} + \binom{n+1}{0} b^{n+1}$$

Renaming j to k:

$$= \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\binom{n+1}{k} \text{ by recursive definition}} a^k b^{n+1-k} + \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a^0 b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

Therefore the binomial theorem holds for $n+1$.

□

This lecture took place on 29th of October 2015 with lecturer Ring Wolfgang.

$$\forall a, b \in \mathbb{R}, n \in \mathbb{N} : (a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k}$$

Induction base $n = 0, n = 1$ follows immediately

Induction step

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)(a+b)\dots(a+b)}_{n \text{ times}}$$

When multiplying the products $a^n b^{n-k}$ are created $(0 \le k \le n)$. $a^n b^{n-k}$ are created iff a is the factor resulting from k parenthesis groups and b originates from the remaining $(n-k)$ groups. There are exactly $\binom{n}{k}$ possibilities to select from n groups. $a^k b^{n-k}$ occurs $\binom{n}{k}$ times. Therefore

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

This is a rather informal proof, but suffices at this point.

5 Arithmetics of numbers

We consider two fundamental arithmetic operators and determine fundamental properties.

Definition 12. *Let K be a set where two arithmetic operators are defined: Therefore $\forall a, b \in K$ let $a + b \in K$ and $a \cdot b \in K$.*

We require the following properties:

A1 $\forall a, b \in K : a + b = b + a$

A2 $\forall a, b, c \in K : (a+b)+c = a+(b+c)$

A3 $\exists 0 \in K \forall a \in K : a + 0 = a$

A4 $\forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$

Then $(K, +)$ is a commutative group ("abelian group"). In general we denote $\tilde{a}$ as $-a$. We define $a - b = a + (-b)$ ("subtraction").

M1 $\forall a, b \in K : a \cdot b = b \cdot a$

M2 $\forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M3 $\exists 1 \in K : a \cdot 1 = a \forall a \in K$ *(neutral element)*

M4 $\forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$

In general we denote $\hat{a}$ *as* a^{-1}.
We set $\frac{a}{b} = a \cdot b^{-1}$.

$$\frac{1}{b} = 1 \cdot b^{-1} \text{ for } b \neq 0$$

Definition 13 (Composition)**.** *Compatibility of* $+$ *and* $\cdot$*:*

D $\forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$

Under these conditions K *is called a* field*.*

Example 5. *Examples for fields:* $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

In every field it holds that

- *the inverse element of* a *is unique (*$\tilde{a}$ *is unique). Let* $-a$ *be the inverse element of* a *and* $a + b = 0 \Rightarrow b = -a$

  Proof. TODO
  $$(a + (-a)) + (b + 0) = a + b =$$
  □

- $0 \cdot a = 0$

  Proof.
  $$0 = 0 + 0$$
  follows from **D**.
  $$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$$
  $$0 \cdot a + (-0 \cdot a) = 0 \cdot a + [0 \cdot a + (-0 \cdot a)]$$
  $$0 = 0 \cdot a$$
  □

- $-a = (-1) \cdot a$

  Proof.
  $$a + (-1) \cdot a = (1 + (-1))a = 0$$
  $$a + (-1) \cdot a = 0$$
  $$-a = (-1) \cdot a$$
  □

5.1 Integers and the field of rational numbers $\mathbb{Q}$

For $\mathbb{N}$, **A1**, **A2** and **A3**. If $n \geq m$, then also $n - m \in \mathbb{N}$. $n - m = k \in \mathbb{N}$ is defined in such a way that $n = m + k$.

Corollary 1. *Extension:*

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \ldots\} = \mathbb{N}_+ \cup \{0\} \cup \{-n : n \leq \mathbb{N}_0\}$$

We define $-0 := 0$ *and* $\forall n \in \mathbb{N}_+$ *let* $n + (-n) := 0$.

Therefore for every $z \in \mathbb{Z}$ *exists some* $\tilde{z}$ *such that* $z + \tilde{z} = 0$.

- $z \in \mathbb{Z}_+ \Rightarrow \tilde{z} = -z$
- $z = 0 \Rightarrow \tilde{z} = 0$
- $z = -n$ *for* $n \in \mathbb{N}_+$
- $\tilde{z} = n$

$$\forall z \in \mathbb{Z} \exists \tilde{z} \in \mathbb{Z} : z + \tilde{z} = 0$$

In general we denote $\tilde{z} = (-z)$. *Also* $-(-z) = z$.

For $z, w \in \mathbb{Z}$*:*

$$z + w = \begin{cases} z + w & z, w \in \mathbb{N} \\ (-z) + (-w) & -z, -w \in \mathbb{N} \\ z - (-w) & z, -w \in \mathbb{N} \text{ and } z > (-w) \\ -((-w) - z) & z, -w \in \mathbb{N} \text{ and } (-w) > z \end{cases}$$

$$z \cdot w = \begin{cases} z \cdot w & z, w \in \mathbb{N} \\ (-z)(-w) & -z, -w \in \mathbb{N} \\ -((-z) \cdot w) & -z \in \mathbb{N}, w \in \mathbb{N} \end{cases}$$

*In $\mathbb{Z}$ the properties **A1**, **A2**, **A3**, **A4**, **M1**, **M2**, **M3** and **D** hold.*

Definition 14.

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

where $\frac{m}{n} = \frac{m'}{n'} \Leftrightarrow m \cdot n' = n \cdot m'$. $\mathbb{Q}$ is called the set of rational numbers.

We define

$$\frac{m}{n} + \frac{k}{l} := \frac{ml + nk}{nl}$$
$$\frac{m}{n} \cdot \frac{k}{l} = \frac{mk}{nl}$$

Show that

$$\frac{m}{n} = \frac{m'}{n'} \text{ and } \frac{k}{l} = \frac{k'}{l'}$$
$$\Rightarrow \frac{ml + nk}{nl} = \frac{m'l' + n'k'}{n'l'}$$
$$\Rightarrow (ml + nk)(n'l') = (m'l' + n'k')$$
$$\Leftrightarrow mn' \cdot ll' + nn' \cdot kl = m'n \cdot ll' + nn' \cdot k'l$$

Analogously for $\frac{m}{n} \cdot \frac{k}{l}$.

***A1**–**A4**, **M1**–**M4** and **D** hold for $\mathbb{Q}$.*

For $z \in \mathbb{Z}$ we set $z = \frac{z}{1}$. Therefore it holds that $\mathbb{Z} \subseteq \mathbb{Q}$. $0 = \frac{0}{1}$ and $\frac{m}{n} + 0 = \frac{m}{n} + \frac{0}{1} = \frac{m \cdot 1 + n \cdot 0}{n \cdot 1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$. 0 is neutral in regards of addition in $\mathbb{Q}$.

Inverse element in regards of addition:

$$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{n^2} = \frac{(m + (-m))n}{n \cdot n} = \frac{0n}{n^2} = \frac{0}{1}$$

because $0 \cdot 1 = 0 \cdot n^2$.

Concerning multiplication:

$$1 = \frac{1}{1} \qquad \frac{m}{n} \cdot \frac{1}{1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$$

1 is a neutral element in regards of multiplication in $\mathbb{Q}$.

Let $\frac{m}{n} \in \mathbb{Q} \setminus \{0\} \Rightarrow m \neq 0 \Rightarrow \frac{n}{m} \in \mathbb{Q}$ and $\frac{m}{n}\frac{n}{m} = \frac{mn}{mn} = \frac{1}{1}$. TODO: verify because $m \cdot n \cdot 1 = 1 \cdot m \cdot n$.

Corollary 2.

$$\forall \frac{m}{n} \in \mathbb{Q} : -\frac{m}{n} = \frac{-m}{n}$$
$$\forall \frac{m}{n} \in \mathbb{Q} \setminus \{0\} : \left(\frac{m}{n}\right)^{-1} = \left(\frac{n}{m}\right)$$

Therefore $\mathbb{Q}$ is a field.

This lecture took place on 30th of October 2015 with lecturer Ring Wolfgang.

Literature:

- Eblinghaus et al., "Zahlen", Springer Verlag
- E. Landau: "Grundlagen der Analysis", uses Peano axioms to build calculus

Definition 15. *Let K be a field. We assume that K is taken from two sets: $K = K_+ \cup \{0\} \cup K_-$ with $0 \notin K_+, 0 \notin K_-$. It holds that*

- *$\forall a \in K$ it holds that either $a \in K_+$ or $a = 0$ or $a \in K_+$*
  $a \in K_+ \Leftrightarrow -a \in K_-$
- *$\forall a, b \in K_+$: $a + b \in K \wedge a \cdot b \in K$*

If those properties are satisfied, such a field is called an ordered field. *Instead of $a \in K_+$ we write $a > 0$ (namely "positive numbers") and $a < 0$ for $a \in K_-$ correspondingly (namely "negative numbers").*

For arbitrary $a, b \in K$ we define

$$a > b \Leftrightarrow a - b > 0$$

It holds that $a > b \Leftrightarrow b < a$.

$$a \geq b \Leftrightarrow a > b \vee a = b$$

Lemma 1. *Let K be an ordered field. Then it holds that*

1. $a \in K_+ \wedge b \in K_- \Rightarrow a \cdot b \in K_-$
$a \in K_- \wedge b \in K_- \Rightarrow a \cdot b \in K_+$

2. $\forall a, b \in K$ *one of the following relations hold:*

$$a > b \vee a = b \vee a < b$$

Therefore $<$ *defines a total order on* K*.*

3. $\forall a, b, c \in K : [(a < b) \wedge (b < c) \implies a < c]$
Therefore $<$ *is transitive.*

4. *If* $a > b > 0$ *then* $\frac{1}{a} < \frac{1}{b}$ *If* $a > 0$ *holds, then also* $a^{-1} = \frac{1}{a} > 0$.

5. $\forall a, b, c \in K : a < b \implies a + c < b + c$

6. $\forall a, b \in K : \forall c > 0 : [a > b \implies ac > bc]$
$\forall a, b \in K : \forall c < 0 : [a > b \implies ac < bc]$

7. $\forall a \in K \setminus \{0\} : a^2 = a \cdot a > 0$

Proof. 1. We know from the practicals: $\forall a, b \in K : (-a)(-b) = ab$

$$(-a)b = -(ab)$$

Let $a \in K_+, b \in K_-$, therefore $a \in K_+$, $(-b) \in K_-$, then it holds that $ab = (-a)(-b) = -(a(-b)) \in K_-$. Let $a \in K_-$ and $b \in K_-$ therefore $(-0) \in K_+ \wedge (-b) \in K_+ \implies ab = (-a)(-b) \in K_+$.

2. Let $a, b \in K$. Then one of the following properties hold:

$$a - b > 0 \vee a - b = 0 \vee a - b < 0$$

Equivalently,

$$a > b \vee a = b \vee a < b$$

3. Let $a > b$ and $b > c$. Therefore $a - b > 0$ and $b - c > 0$.

$$\Rightarrow (a - b) + (b - c) > 0$$

$$a(-b + b) - c > 0$$

$$a - c > 0 \Leftrightarrow a > c$$

4. Let $a > 0 \Rightarrow a^{-1} \neq 0$. Assume $\frac{1}{a} = a^{-1} < 0 \Rightarrow a^{-1} \cdot a = 1 < 0$. Otherwise it holds that $1 = 1 \cdot 1 = 1^2 > 0$`

5. Let $a > b > 0$. Then it holds that

$$a^{-1}b^{-1}(b - a) = a^{-1}b^{-1}b - a^{-1}b^{-1}a = -a^{-1} \cdot b^{-1} = \frac{1}{a} \cdot \frac{1}{b} \Rightarrow a^{-1} < b^{-1}$$

6. $a < b$ therefore $a - b < 0 \Rightarrow a + c - c - b < 0 \Rightarrow (a + c) - (b + c) < 0$

$$\Leftrightarrow a + c < b + c$$

7. Let $a > b, c > 0 \Rightarrow (a - b) > 0 \Rightarrow (a - b) \cdot c > 0 \Rightarrow ac - bc > 0 \Rightarrow ac > bc$. For the second statement, it holds analogously: $a < b, c < 0 \Rightarrow (a - b) < 0 \Rightarrow (a - b) \cdot c < 0 \Rightarrow ac - bc < 0 \Rightarrow ac < bc$

8. $a > 0 \Rightarrow a \cdot a > 0$. Let $a < 0 \Rightarrow (-a) > 0$. It holds $a \cdot a = (-a)(-a) > 0$. Therefore the square of two numbers is always positive.

□

Remark 2. $\mathbb{C}$ *is not an ordered ordered field.* $\mathbb{N}$, $\mathbb{Z}$ *and* $\mathbb{Q}$ *are ordered.*

Remark 3. *Let* $q \in \mathbb{Q}$.

a) *Let* $m, n \in \mathbb{N}_+$ *such that* $q = \frac{m}{n}$ *then* $q > 0$.

b) *Let* $m, n \in \mathbb{N}_+$ *such that* $q = -\frac{m}{n}$ *then* $q < 0$.

We show that $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$. *Every* $q \in \mathbb{Q}$ *has a representation of either a) or b), but not both.* $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$.

$$q \neq 0 \Rightarrow q = \begin{cases} \frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{-n} & m, n \in \mathbb{N}_+ \end{cases}$$

$$q = \frac{n}{-m} = \frac{-n}{m}$$

because $nm = (-n)(-m)$.

$$q = \frac{-m}{-n} = \frac{m}{n}$$

because $(-m) \cdot n = m \cdot (-n)$.

Remark 4. *We want to show that* $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$. *Let* $q \in \mathbb{Q}_+ \cap \mathbb{Q}_-$.

$$q = \frac{m}{n} = -\frac{m'}{n'} \qquad m, n, m', n' \in \mathbb{N}_+$$

$$\Rightarrow n \cdot n' = (-m')n$$

$$\Rightarrow \underbrace{\underbrace{mn'}_{\in \mathbb{N}_+} + \underbrace{m'n}_{\in \mathbb{N}_+}}_{\in \mathbb{N}_+} = 0 \qquad \lightning$$

Furthermore $p \in \mathbb{Q}_+ \wedge q \in \mathbb{Q}_+$

$$\Rightarrow p + q \in \mathbb{Q}_+ \wedge pq \in \mathbb{Q}_+$$

$$\Rightarrow p = \frac{k}{l} \qquad q = \frac{m}{n} \qquad k, l, m, n \in \mathbb{N}_+$$

$$p + q = \frac{\overbrace{kn + ml}^{\in \mathbb{N}_+}}{nm} \in \mathbb{Q}_+$$

$$pq = \frac{k}{l} \cdot \frac{m}{n} = \frac{\overbrace{km}^{\in \mathbb{N}_+}}{\underbrace{ln}_{\in \mathbb{N}_+}} \in \mathbb{Q}_+$$

Definition 16. *Let* K *be an ordered field* $a \in K$. *The absolute value of* a *is defined as*

$$|a| = \begin{cases} a & \textit{if } a \in K_+ \\ 0 & \textit{if } a = 0 \\ -a & \textit{if } a \in K_- \end{cases}$$

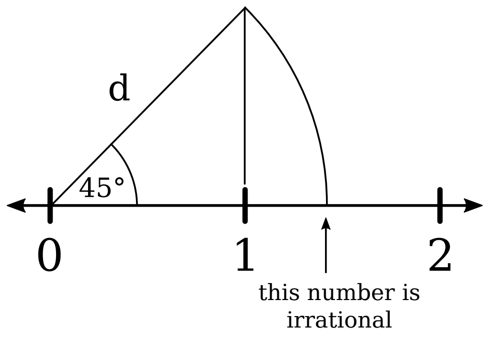

Figure 3: Illustration of an irrational number

Remark 5. *Let* K *be an ordered field. Then it holds that*

$$\mathbb{Q} \subseteq K \subseteq \mathbb{R}$$

except for isomorphism.

Theorem 7.

$$\forall a, b \in K : |a + b| \leq |a| + |b| \qquad \textit{“Triangle inequality”}$$

Proof. **Case 1**

$$a \cdot b > 0 \Rightarrow a \cdot b > 0 : |ab| = ab \qquad |a| \cdot |b| = ab$$

Case 2

$$a > 0, b < 0 : a \cdot b < 0 : |ab| = -ab \qquad |a| \cdot |b| = a \cdot (-b)$$

$$b < 0 \Rightarrow -b > 0 \Rightarrow b < -b \Rightarrow \underbrace{a + b}_{|a+b|} < \underbrace{a - b}_{|a|+|b|}$$

Case 3

$$a < 0, b < 0 : a \cdot b > 0 : |ab| = ab \qquad |a| = -a \qquad |b| = -b$$

$$|a| \cdot |b| = -a \cdot -b = ab$$

Case 4

$$a > 0, b < 0 : a + b < 0$$

$$|a| = a \qquad |b| = b \qquad |a + b| = -(a + b) = -a - b$$

$$a > 0 \Rightarrow -a < 0 \qquad -a - b < a - b$$

$$-(a + b) = |a + b|$$

□

This lecture took place on 4th of November 2015 with lecturer Wolfgang Ring.

5.2 Laws for absolute values

Theorem 8. *Let $y \geq 0$. Then it holds that $|x| \leq y \Leftrightarrow -y \leq x \wedge x \leq y$*

Proof. First direction $\Rightarrow$:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Case 1 Let $x \geq 0$. Then

$$|x| \leq y \Rightarrow x \leq y \Rightarrow -y \leq x$$

because $-y \leq 0 \wedge x \geq 0$ anyways.

Case 2 Let $x < 0$, therefore $|x| = -x$. Because

$$-x \leq y \Rightarrow x \geq -y$$

$x \leq y$ holds anyways because $x < 0$ and $y \geq 0$.

Second direction $\Leftarrow$:

Let $-y \leq x \leq y$.

Case 1 $x \geq 0 : |x| = x \leq y$ because of the second inequality.

Case 2 $x < 0 : |x| = -x$

$$-(-1) \Rightarrow -(-y) \geq -x \text{ or equivalently } y \geq -x = |x|$$

□

Theorem 9.

$$|x| = 0 \Leftrightarrow x = 0$$

$$\forall a \in K : |a| = |-a|$$

$$\forall \varepsilon > 0 : |x - y| \leq \varepsilon \Rightarrow x = y$$

Proof. **First direction** $\Rightarrow$ Without loss of generality: $x \geq y$.

$$x \neq y \Rightarrow \exists \varepsilon > 0 : |x - y| > \varepsilon$$

Let $x \neq y$. Because $x \geq y$ holds, so does $x > y$. Therefore $x - y > 0$. We define $\varepsilon = \frac{x-y}{2} < x - y$

$$2 = 1 + 1 > 1$$

$$2^{-1} = \frac{1}{2} < 1 = 1^{-1}$$

Therefore it holds that $\varepsilon : |x - y| = x - y > \frac{1}{2}(x - y) = \varepsilon > 0$.

Second direction $\Leftarrow$ $x = y \Rightarrow |x - y| = 0 \leq \varepsilon \forall \varepsilon > 0$

□

Theorem 10 (Inversed triangle inequality)**.** *Let $a, b \in K$. Then it holds that*

$$||a| - |b|| \leq |a - b|$$

Proof. Show that $-|a - b| \leq |a| - |b| \leq |a - b|$.

First inequality

$$|b| = |b - a + a| \leq |b - a| + |a| \Rightarrow -|a - b| \leq |a| - |b|$$

Second inequality

$$|a| = |a - b + b| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$$

□

This lecture took place on 5th of November 2015 with lecturer Wolfgang Ring.

Definition 17. ***Intervals.*** *Let $a, b \in K$.*

$$(a, b) = \{x \in K \mid (x > a) \wedge (x < b)\}$$

$$[a, b) = \{x \in K \mid (x \geq a) \wedge (x < b)\}$$

$$(a, b] = \{x \in K \mid (x > a) \wedge (x \leq b)\}$$

$$[a, b] = \{x \in K \mid (x \geq a) \wedge (x \leq b)\}$$

Theorem 11 (Laws for intervals)**.**

$$(a, b) = \emptyset \text{ if } b \leq a \tag{21}$$

$$[a, b] = \emptyset \text{ if } b < a \tag{22}$$

$$[a, a] = \{a\} \tag{23}$$

If I is an non-empty interval (hence $I \neq \emptyset$), then $|I| = b - a$ is called length of the interval. *Furthermore*

$$(a, \infty) = \{x \in K \mid x > a\} \tag{24}$$

$$[a, \infty) = \{x \in K \mid x \geq a\} \tag{25}$$

$$(-\infty, a) = \{x \in K \mid x < a\} \tag{26}$$

$$(-\infty, a] = \{x \in K \mid x \leq a\} \tag{27}$$

Theorem 12. *$\mathbb{Q}$ is arithmetically incomplete.*

Proof. We define a mapping from $\mathbb{N}_+$ to $\mathbb{N}$: Let $n \in \mathbb{N}_+$ then we know that n can be represented distinctly as product of prime numbers. Let $\mathrm{Z}(n)$ be the number of twos in the prime product representation.

Examples:

$$\mathrm{Z}(14) = \mathrm{Z}(2 \cdot 7) = 1$$

$$\mathrm{Z}(15) = \mathrm{Z}(3 \cdot 5) = 0$$

$$\mathrm{Z}(24) = \mathrm{Z}(2 \cdot 2 \cdot 2 \cdot 3) = 3$$

It holds that $\mathrm{Z}(2n) = \mathrm{Z}(n) + 1 \forall n \in \mathbb{N}_+$ and $\mathrm{Z}(n^2) = \mathrm{Z}(n) \cdot 2 \forall n \in \mathbb{N}_+$.

We claim,

$$\nexists q : q = \frac{m}{n} \text{ with } q^2 = 2$$

Proof by contradiction:

1. Assume $\left(\frac{m}{n}\right)^2 = 2$.
2. Then $\frac{m^2}{n^2} = 2$.
3. Then $m^2 = 2 \cdot n^2$.
4. With $Z(m^2) = 2 \cdot \mathrm{Z}(n)$.
5. With $\mathrm{Z}(2 \cdot n^2) = \mathrm{Z}(n^2) + 1 = 2 \cdot \mathrm{Z}(n) + 1$.
6. If $m^2 = 2n^2$, then $\mathrm{Z}(m^2)$ must be even and $\mathrm{Z}(2 \cdot n^2)$ must be odd.
7. Then equality cannot be satisfied `

□

Theorem 13. *$\mathbb{Q}$ is geometrically incomplete.*

We consider an infinite straight number line. We define $\mathbb{R}$ as ordered field with properties:

Archimedean property *$\mathbb{N} \subseteq \mathbb{R}$ with $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : x < n$*

$$\mathbb{N} \subseteq \mathbb{R} \forall n \in \mathbb{N} : -n \in \mathbb{N}$$

$$\Rightarrow \forall n \in \mathbb{N}_+ : n^{-1} \in \mathbb{R}$$

$$\Rightarrow \mathbb{Z} \subseteq \mathbb{R}$$

Therefore $\forall m \in \mathbb{N} : m \cdot \frac{1}{n} = \frac{m}{n} \in \mathbb{R} \Rightarrow \mathbb{Q} \subseteq \mathbb{R}$.

Definition 18. *Let $I_0, I_1, \dots, I_z$. $(I_n)_{n\in\mathbb{N}}$ is a sequence of closed intervals with*

1. $\forall a \in \mathbb{N} : I_{n+1} \subseteq I_n$
2. $\forall \varepsilon > 0 \exists n \in \mathbb{N} : n \geq N \Rightarrow |I_n| < \varepsilon$

Completeness axiom *Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals in $\mathbb{R}$. Then there exists some $x \in \mathbb{R} : x \in I_n : \forall n \in \mathbb{N}_+$.*

Be aware, there exists only one *$x \in \mathbb{R}$ with the property: $x \in I_n \forall n \in \mathbb{N}$.*

Assume $x \in I_n$ and $y \in I_n \forall n \in \mathbb{N}$ and $x \neq y$.

$$|\beta - \alpha| \leq b - a = |I|$$

Proof. Without loss of generality: $\alpha \leq \beta$. Then it holds that $|\beta - \alpha| = \beta - \alpha \leq \beta + (-\alpha) \leq b + (-\alpha) = b - a = |I|$.

$$a \leq \alpha \Rightarrow -a \geq -\alpha$$

Consider arbitrary small $\varepsilon > 0$ and $N \in \mathbb{N}$ sufficiently large, such that $|I_n| < \varepsilon$. Because $x, y \in I_n \Rightarrow |x - y| < \varepsilon \Rightarrow x = y$. □

Corollary 3. *From the Archimedean property it follows that,*

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

Proof. Let $x > \frac{1}{\varepsilon} \in \mathbb{R}$. Archimedean property: $\exists N \in \mathbb{N} : N > x$.

For $n \geq \mathbb{N}$ it holds that $n > x > 0 \Rightarrow \frac{1}{n} < \frac{1}{x} = \varepsilon$. □

Corollary 4. *Let $p \in \mathbb{R}, p > 1 \forall x \in \mathbb{R} : n \geq N \Rightarrow p^n > x$.*

Proof. $p > 1 + u$ with $u = p - 1$

$$p^n = (1+u)^n \underbrace{>}_{\text{Bernoulli}} 1 - nu = 1 + n(p-1)$$

Let $x \in \mathbb{R}$ arbitrary, select $N \in \mathbb{N} : \frac{x-1}{p-1} < N$.

Then it holds for $n \geq N$:

$$\underbrace{\frac{x-1}{p-1}}_{>0} \Leftrightarrow x - 1 < n \cdot (p-1) \Leftrightarrow x < 1 + n(p-1) < p^n$$

□

Theorem 14. *Let $q \in \mathbb{R}$ with $|q| < 1$. Then it holds that*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |q^n| = |q|^n < \varepsilon$$

Proof. Let $s = |q| \geq 0$. Consider $q > 0$. Then

$$q^n = 0$$
$$|q^n| = 0$$
$$|q|^n < \varepsilon \forall \varepsilon > 0 \forall n \in \mathbb{N}$$

Let $q \neq 0$, then $0 < s < 1$. Let $p = \frac{1}{s} \Rightarrow p > 1$. Choose arbitrary $\varepsilon > 0$ and $x = \frac{1}{\varepsilon}$. Because of the Completeness axiom

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow p^n > X$$

So it holds that

$$\frac{1}{p^n} = S^n < \frac{1}{x} = \varepsilon \forall n \geq N$$

$$\Rightarrow (|q|)^n = |q^n|$$

□

Theorem 15. *Let $x \in \mathbb{R}, x > 0$ and let $k \in \mathbb{N}_+$. Then there exists a distinct $y \in \mathbb{R}$ with $y \geq 0$ such that*

$$y^k = x$$

We denote $y = \sqrt[k]{x}$ and conclude there exists k-th root numbers.

Proof. Idea: Construct nested intervals.

$(I_n)_{n\in\mathbb{N}}$ such that $y \in \bigcap_{n\in\mathbb{N}} I_n$ satisfies the property that $y^k = x$.

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k$$

We define $J_0 = [a_0, b_0]$ with $a_0 = 0$ and $b_0 = 1 + x$. Then it holds that

$$a_0^k = 0^k = 0 \leq x$$

$$b_0^k = (1+x)^k = 1 + k_n + \binom{k}{2} x^2 + \dots + x^k \geq 1 + kx > 0$$

□

This lecture took place on 6th of November 2015 with lecturer Wolfgang Ring.

Theorem 16. *We prove:*

$$0 \leq y_1 < y_2 \Rightarrow y_1^k \leq y_2^k$$

Proof. A short proof by a student:

$k = 2$

$$y^{k+1} = y^k \cdot y < y_2^k x < y_2^k y_2 = y^{k+1}$$

$k \to k+1$

$$y_1^2 < y_2^2$$

□

Theorem 17. *Let $a, b \in K$ and $k \in \mathbb{N}$. Then it holds that*

$$a^k - b^k = (a-b)\left(\sum_{j=0}^{k-1} a^{k-1-j} b^j\right)$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Proof.

$$\begin{aligned}
(a-b)\left(\sum_{j=0}^{k-1} a^{k-j-1} b^j\right) &= \sum_{j=0}^{j-1} a^{k-j} b^j - \sum_{j=0}^{k-1} a^{k-j-1} b^{j+1} \\
&= a^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \underbrace{b^{k-1}}_{j=k-1} - \sum_{j=0}^{k-2} a^{k-j-1} b^{j+1} \\
&= a^k - b^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \sum_{l=1}^{k-1} a^{k-l} b^l \\
&= a^k
\end{aligned}$$

□

Theorem 18. *Let $y_2 > y_1$ then*

$$y_2^k - y_1^k = \underbrace{(y_2 - y_1)}_{>0} \underbrace{\left(\sum_{j=0}^{k-1} y_2^{k-j-1} y_1^j\right)}_{>0}$$

$$\Rightarrow y_2^k - y_1^k > 0$$

Proof.

$$\forall x \geq 0 \in \mathbb{R} : \exists y \geq 0 \in \mathbb{R} : y^k = x \text{ with } k \in \mathbb{N}_+$$

Special case $x = 0$ and $y = 0$ is the solution.

Let $x > 0$: We construct y with $y \in \bigcap_{k=0}^{\infty} I_n$ where I_n are nested intervals. Specifically I_n must have the properties:

- $I_n = [a_1, b_n]$ with $a^k \leq x, b_n^k \geq x \quad \forall n \in \mathbb{N}$
- $I_{n+1} \subseteq I_n : |I_n| = \frac{1}{2}|I_{n+1}| = \left(\frac{1}{2}\right)^n |I_0|$

$$n = 0 \qquad I_0 = [0, x-1]$$

$$a_0 = b \qquad b_0 = x + 1$$

$$a_0^k = 0 < x \qquad \checkmark$$

$$b_0^k = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots + x^k > 1 + kx > x \text{ for } k \geq 1$$

Let I_n be given: $I_n = [a_n, b_n]$. Define $m_n = \frac{1}{2}(a_n + b_n)$

Case 1

$$m_n^k \geq x \Rightarrow \text{ let } a_{n+1} = a_n, b_{n+1} = m$$

$$I_{n+1} = [a_n, m_n] \subseteq [a_n, b_n] = I_n$$

$$|I_{n+1}| = m_n - a_n = \frac{1}{2}a_n + \frac{1}{2}b_n - a_n$$

$$\frac{1}{2}(b_n - a_n) = \frac{1}{2}|I_n|$$

$$a_{n+1}^k = a^k \leq x \quad \checkmark$$

All conditions are satisfied.

Case 2 $m_n^k < x$: Let $a_{n+1} = m_1, b_{n+1} = b_n$. It holds that $a_{n+1} = m_n < x, b_{n+1} = b_n \geq x \quad \checkmark$. Furthermore it holds that $I_{n+1} \subseteq I$ and $|I_{n+1}| = \frac{1}{2}|I_n|$.

I_n is set of nested intervals. Let $\varepsilon > 0$ be arbitrary. Then

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{1+x}$$

For those $n \geq N$ it holds that

$$|I_n| = \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^n (x+1) < \frac{\varepsilon}{1+x} \cdot (1+x)$$

Let $y \in I_n \forall n \in \mathbb{N}$. Further nesting of intervals:

$$(I_n)_{n\in\mathbb{N}} \text{ with } I_n = [a_n^k, b_n^k]$$

It holds that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \text{ because } I_{n+1} \subseteq I_n \Rightarrow a_n^b \leq a_{n+1}^k < b_{n+1}^k \leq b_n^k$$

Length of I_n:

$$I_n = b_n^k - a_n^k = (b_n - a_n)\sum_{j=0}^{k-1} a_n^{k-1-j} b_n^j$$

Because $I_n \leq I_0 \Rightarrow a_n < b_0 \Rightarrow b_n \leq b_0$,

$$< (b_n - b_0)\sum_{j=0}^{k-1} b_0^{k-1-j} b_0^j$$

$$= (b_n - a_n)kb_0^k = (b_n - a_n)k(1+x)^k$$

Let $\varepsilon > 0$ be arbitrary. Find some $N \in \mathbb{N}$ with $n \geq N$:

$$|I_n| = (b_n - a_n) < \frac{\varepsilon}{k(1+x)^k}$$

For those n it holds that

$$|I_n| < |I_n| \cdot k(1-x)^k < \frac{\varepsilon}{k(1+x)^k} k(1+x)^k = \varepsilon$$

Therefore $(I_n)_{n\in\mathbb{N}}$ a set of nested intervals.

$\exists z \in \mathbb{R}$ with $z \in \lfloor a_n^k, b_n^k \rfloor : \forall n \in \mathbb{N}$ and z is unique. By construction of I_n it holds that $a_n^k \leq x \leq b_n^k$

$$\Rightarrow x \in I_n \forall n \in \mathbb{N} \Rightarrow x = z \in \bigcap_{n\in\mathbb{N}} I_n.$$

On the opposite side it holds that $y \in I_n$ (hence $a_n \leq y \leq b_n \Rightarrow a_n^k \leq y^k \leq b_n^k$). So $y^k \in I_n \forall n \in \mathbb{N} \Rightarrow y^k = z = x$. So we have found some y^k which is x. But is $y \geq 0$ with $y^k = x$ unique?

Let $y_1 \neq y_2$ with $y_1^k = y_2^k = x$ and without loss of generality,

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k \quad \grave{}$$

So, y is unique.

$\square$

5.3 Supremum property in $\mathbb{R}$

Definition 19. *Let $A \subseteq \mathbb{R}$.*

- *We call A to be* bounded above *if there exists some $u \in \mathbb{R}$ such that $\forall a \in A : a \leq u$.*
- *A number u with that property is called* upper bound of A.
- *We call A to be* bounded below *if there exists some $l \in \mathbb{R}$ such that $\forall a \in A : a \geq l$.*
- *A number l with that property is called* lower bound of A.
- *A is called* bounded *if there exists a lower and upper bound of A.*

Corollary 5. *Let (a, b) be bounded. Let u be its upper bound and let $v \geq u$. Then v is also an upper bound of (a, b).*

This lecture took place on 11th of November 2015 with lecturer Wolfgang Ring.

Definition 20. *Let A be bounded above. Assume $s \in \mathbb{R}$ has the properties*

1. *s is an upper bound for A*
2. *$\forall \sigma \in \mathbb{R} : \sigma < S$: σ is not an upper bound for A.*

If those properties are satisfied, we call s supremum of A. *A supremum s is always the smallest upper bound of A. We denote $s = \sup A$.*

There exists at most one supremum for A. Let s_1 and s_2 be two suprema, then $s_1 \neq s_2$. So wlog. $\sigma_1 < \sigma_2$. This invalidates the supremum property of $s_2 \Rightarrow s_1$ is not a supremum of A `.

Analogously an infimum of A *is the greatest lower bound of A. Let A be bounded below. $t \in \mathbb{R}$ is called* infimum of A *if*

1. *$\forall a \in A : t \leq a$ (t is a lower bound of A)*
2. *$\forall x > t$ so x is no lower bound of A*

$$\Leftrightarrow \exists a \in A : a < x$$

We denote $t = \inf A$.

Definition 21. *Let $A \subseteq \mathbb{R}$. We call $u \in \mathbb{R}$* maximum of A *denoted $u = \max A$ if*

1. *$u \in A$ (is element of A)*
2. *$\forall a \in A : a \leq u$ (is an upper bound)*

$l \in \mathbb{R}$ denoted $l = \min A$ is called minimum of A if

1. *$l \in A$ (is element of A)*
2. *$\forall a \in A : l \leq a$ (l is a lower bound)*

Theorem 19. *Let $A \subseteq R$ and u be the maximum of A. Then it holds that $u = \sup A$. If $l = \min A \Rightarrow l = \inf A$.*

Proof. We need to show, that l is an upper bound of A. This follows by definition. For $x < u$ it holds that x not an upper bound.

Let $x < u$, because $u \in A$ there exists some element y in A with $y > x$. Therefore x is not an upper bound of A. $\square$

Example 6.

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}_+\right\}$$

Then it holds that $1 \in A$ and $1 \geq \frac{1}{n} \forall n \in \mathbb{N}_+$. Therefore $1 = \max A = \sup A$.

$0 = \inf A$, because 0 is a lower bound of A ($\frac{1}{n} > 0 \forall n \in \mathbb{N}_+$). Let $\varepsilon > 0$, then $\exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} \leq \varepsilon$. Therefore ε is not a lower bound of A.

So A does not have a minimum, because otherwise $l = \max A = \inf A = 0$.

Theorem 20. *Let $A \neq \emptyset$ and $A \subseteq \mathbb{R}$ be bounded above. So some $s = \sup A \in \mathbb{R}$ exists (therefore $\mathbb{R}$ has a supremum property).*

Proof. We construct nested intervals $(I_n)n \in \mathbb{N}$ such that for $s \in \bigcap_{n \in \mathbb{N}} I_n$ gilt $s = \sup A$. We construct I_{n+1} inductively using I_n

Case $n = 0$
Because $A \neq 0$, we select $a_0 \in A$. Because A is bounded above, $\exists b_0 \in \mathbb{R}$ such that b_0 is an upper bound of A. We define $I_0 = [a_0, b_0]$.

Case $n \to n+1$
Let $a_0 = b_0$, then it holds that b_0 is upper bound and $b_0 \in A$. We call that terminating condition. Therefore $b_0 = \max A = \sup A$ and the supremum was found. Instead of n we use $n+1$. Let $I_0 = [a_n, b_n]$ with $a_n \neq b_n$ and $a_n \in A$, b_n is an upper bound of A. Furthermore it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider I_{n+1} such that the same properties are satisfied. Let $m_1 = \frac{1}{2}(a_1 + b_1)$. It holds that $a_n < m_n < b_n$.

Case m_n **is an upper bound of** A Then we set $a_{n+1} = a_n \in A$ and $b_{n+1} = m_n$ is an upper bound of A.

$$|I_{n+1}| = b_{n+1} + a_{n+1} = \frac{1}{2}(b_n + a_n) - a_n$$

$$= \frac{1}{2}b_1 - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^{n+1} |I_n| \qquad \checkmark$$

Case m_n **is not an upper bound of** A Therefore $\exists x \in A$ with $x > m_n$.

Subcase $x = b_1$ So b_1 is an upper bound. Therefore $x \in A$ and x is upper bound.

$$x = \max A = \sup A$$

We found the supremum.

Subcase $m_n < x < b_n$ Let $a_{n+1} = x \in A$ and $b_{n+1} = b_n$ is an upper bound and

$$I_{n+1} = b_{n+1} - a_{n+1} - b_n - x < b_n - m_n - b_n - \frac{1}{2}(b_n + a_n) + \frac{1}{2}(b_n - a_n)$$

$$= \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

We have found supremum $s = \sup A$.

If in any case the terminating condition holds, then we have found the supremum.

The remaining case is $\forall n \in : a_n < b_n, a_n \in A, b_n$ is upper bound of A.

$$|I_n| = b_n - a_n \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider $\varepsilon > 0$ and N such that $n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{|I_n|}$. For those n it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| < \frac{\varepsilon}{|I_0|} |I_0| = \varepsilon$$

Therefore $(I_n)_{n \in \mathbb{N}}$ are nested intervals.

□

What remains for completeness: $s \in \mathbb{R}, s \in I_n : \forall n \in N$. We need to show that $s = \sup A$.

This lecture took place on 12th of November 2015 with lecturer Wolfgang Ring.

Theorem 21. *Completeness of* $\mathbb{R}$*:*

$$\exists s \in \mathbb{R} : s \in I_n \forall n \in \mathbb{N}$$

Proof *cont.* Every set with an upper bound has a supremum.

We construct $(I_n)_{n \in \mathbb{N}}$ with $I_n = [a_n, b_n]$ and $I_{n+1} \subseteq I_n$. $\forall n \in \mathbb{N} : a_n \in A$, b_n is the upper bound of A.

$$|I_{n+1}| \leq \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

Consider $I_{n+1} \subseteq I_n$ with $a_n < b_n \forall n \in \mathbb{N}$.

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

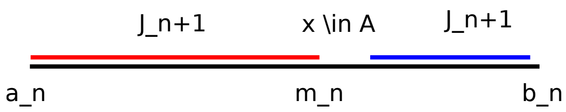

Figure 4: Relation of a_n and b_n and J_{n+1}

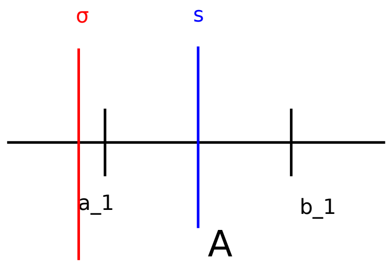

Figure 5: Illustration of s between a_n and b_n

1. Claim: s is $\sup A$.

   We need to show (by contradiction): S is upper bound of A. Assume $a \in A$ and $a > s$. Let $\varepsilon = a - s > 0$ and choose N sufficiently large such that

   $$|I_n| < \varepsilon = a - s$$

   Then it holds that

   $$b_N = \underbrace{b_n - a_n}_{\varepsilon} \not+ \underbrace{a_N}_{<s} < s + \varepsilon = a$$

   $$\Rightarrow b_N < a \in A$$

   Because b_n is an upper bound.

2. $\forall \sigma < s$ it holds that σ is not an upper bound of A. Let $\sigma < s$ and $\varepsilon = s - \sigma > 0$ and choose $n \in N$ large enough such that $b_N - a_N < \varepsilon$. Then it holds that

   $$\begin{aligned} a_N &= a_N - b_N + b_N \\ &> -\varepsilon + s \\ &= -s + \sigma + s = \sigma \qquad \checkmark \end{aligned}$$

   Therefore it holds that s is smallest upper bound of A and therefore supremum.

□

Theorem 22. *Every set with a lower bound in $\mathbb{R}$ has an infimum. Every set with an upper bound in $\mathbb{R}$ has an supremum.*

Theorem 23. *Remember that M has the same cardinality like A if $\varphi : M \to A$. φ is bijective, M is called countably infinite if M has the same cardinality like $\mathbb{N}$.*

Let $\varphi : \mathbb{N} \to M$ be bijective therefore $M = \{\varphi(1), \varphi(2), \varphi(3), \ldots\} = \{\varphi(n) \,|\, n \in \mathbb{N}\}$ and $\varphi(i) \neq \varphi(j)$ for $i \neq j$.

Notation. *$\varphi(n) = m_n$.*

$M = \{m_0, m_1, m_2, \ldots\}$ with $m_i \neq m_j$ for $i \neq j$. φ is a complete enumeration of all elements of M.

Therefore every element of M has the structure: m_n with $i \in \mathbb{N}$.

Theorem 24.
$$\mathbb{Q}^+ = \left\{\frac{m}{n}, m \in \mathbb{N}, n \in \mathbb{N}_+\right\}$$

The set $\mathbb{Q}^+$ is countably infinite.

Proof. We enumerate the elements of $\mathbb{Q}^+$.

$$\mathbb{Q}_+ = \{q_0, q_1, q_2, \ldots\}$$
$$\mathbb{Q}_- = \{-q_0, -q_1, -q_2, \ldots\}$$
$$\mathbb{Q} = \{0, q_0, -q_0, q_1, -q_1, \ldots\}$$

An enumeration exists. So $\mathbb{Q}$ is countably infinite. □

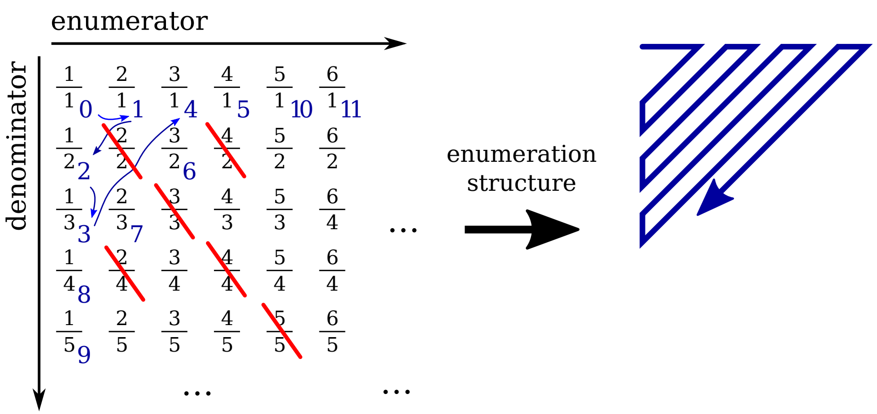

Figure 6: A complete enumeration of $\mathbb{Q}^+$ (diagonalization argument). We traverse the whole matrix diagonally. The blue numbers indicate the enumeration and red lines cross out values already enumerated. On the right-hand side the general order of the enumeration is illustrated.

Theorem 25. *There is no bijective relation $\varphi : \mathbb{N} \to \mathbb{R}$. Therefore we call $\mathbb{R}$ uncountable.*

Proof. We provide a proof by contradiction. Assume $\mathbb{R} = \{x_0, x_1, x_2, x_3, \dots\}$ is countable.

We construct nested intervals.

Case $\mathbf{n = 0}$

$$I_0 = [x_0 + 1, x_0 + 2]$$

Let $|I_0| = 1$ and $x_0 \notin I_0$.

$n \to n+1$ Assume $I_0 \dots I_n$ were already defined with $x_k \notin I_k$ for $0 \leq k \leq n$.

$$I_{k+1} \leq I_k \text{ for } k = 0, \dots, n-1$$

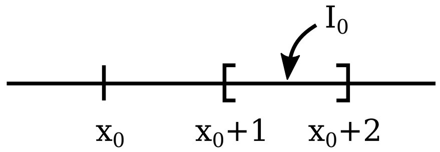

Figure 7: Construction of a nested interval and its I_0

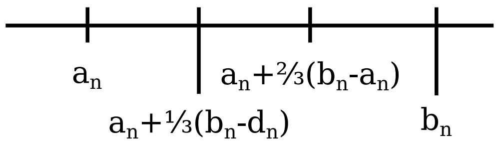

Figure 8: Construction of a nested interval and its I_n

$$|I_k| = \left(\frac{1}{3}\right)^k$$

We construct I_{n+1}. Let $I_n = [a_n, b_n]$.

$$I_n^1 = \left[a_n, \frac{2}{3}a_n + \frac{1}{3}b_n\right]$$
$$I_n^2 = \left[\frac{2}{3}a_n + \frac{1}{3}b_n, \frac{1}{3}a_n + \frac{2}{3}b_n\right]$$
$$I_n^3 = \left[\frac{1}{3}a_n + \frac{2}{3}b_n, b_n\right]$$

So x_n certainly is not contained in all three intervals I_n^1, I_n^2 and I_n^3 because

$I_n^1 \cap I_N^2 \cap I_N^3 = \emptyset$. Choose I_{n+1} as one of the three intervals I_n^l with $x_{n+1} \notin I_n^l = I_{n+1}$. $I_{n+1} < I_n$.

$$|I_{n+1}| = \frac{1}{3} I_n = \left(\frac{1}{3}\right)^{n+1}$$

For $\varepsilon > 0$ it holds that there exists some $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |I_1| = \left(\frac{1}{3}\right)^n < \varepsilon$. Therefore nested intervals I_n are given.

Let $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : X \in I_n$ (because of completeness law). Then it holds that $\forall x_n : x \neq x_n$. $x \in I_n$ and $x_n \notin I_n$. Therefore $x \in \{x_0, x_1, x_2, \ldots\} = \mathbb{R}$.

This contradicts with the assumption that $\mathbb{R}$ is countable.

□

6 Complex numbers $\mathbb{C}$

We introduce a new arithmetic unit denoted i, which extends the field $\mathbb{R}$. Elements of $\mathbb{C}$ are represented as $a + bi$ with $a, b \in \mathbb{R}$.

$$\forall a, b \in \mathbb{R} : a + bi = 0 \Leftrightarrow a = 0 \land b = 0 \tag{28}$$

$$i^2 = -1 \tag{29}$$

$$\text{associativity, commutativity etc holds} \tag{30}$$

This lecture took place on 13th of November 2015 with lecturer Wolfgang Ring.

Definition 22. *We consider an "arithmetic element" i extending $\mathbb{R}$ ("adjungiert"). Arithmetic operations are well-defined for i. Associativity and commutativity holds. It holds that*

- $a + ib = 0$ *with* $a, b \in \mathbb{R} \Leftrightarrow a = 0 \land b = 0$
- $i^2 = -1$ *i.e.* $i^2 + 1 = 0$.
- *Arithmetic operations still hold.*

By the first law,

$$a+ib = a'+ib' \Leftrightarrow (a-a')+i(b-b') = 0 \Leftrightarrow a-a' = 0 \land b-b' = 0 \text{ therefore } a = a' \land b = b'$$

By the second law, i is the solution of the quadratic equation $i^2 + 1 = 0$.

Let $z = a + ib$ a complex number. We call i the "imaginary unit".

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$

$\mathbb{C}$ is the field of complex numbers with the following properties:

- *For addition, it holds that*

$$(a + ib) + (c + id) = (a + b) + i(b + d) \subseteq \mathbb{C}$$

and

$$(a + ib) + (-a - ib) = (a - a) + i(b - b) = 0 + i \cdot 0 = 0$$

- *For multiplication, it holds that*

$$(a + ib) \cdot (c + id) = (ac + \underbrace{(i)^2}_{=-1} bd) + i(bc + ad)$$

$$(ac - bd) + i(bc + ad)$$

- *Laws $\mathbf{A_n}$ to $\mathbf{A_4}$, $\mathbf{M_1}$ to $\mathbf{M_3}$ and $\mathbf{D}$ hold.*
- *The one element exists:*

$$1 = 1 + 0 \cdot i$$

$$(a + i \cdot b)(1 + i \cdot 0) = (a + (i)^2 \cdot 0) + i(b + 0) = a + ib$$

- ***M4** holds: Let $z \in \mathbb{C}\backslash\{0\}$. Let $z = a+ib$ and $\neg(a = 0 \land b = 0) \Leftrightarrow a^2+b^2 > 0$.*

We define

$$w = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

$$z \cdot w = (a + ib)\left(\frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}\right)$$

$$= \left(\underbrace{\frac{a^2}{a^2+b^2} - \frac{b\cdot(-b)}{a^2+b^2}}_{=1} \right) + i \cdot \left(\underbrace{\frac{ba}{a^2+b^2} - \frac{a\cdot b}{a^2+b^2}}_{=0} \right)$$
$$= 1 + i\cdot 0 = 1$$

Therefore $w = z^{-1} = \frac{1}{z}$.

Therefore $\mathbb{C}$ *is a field.*

We denote

$$a = \Re(z)$$
$$b = \Im(z)$$
$$\overline{z} = a - ib$$
$$|z| = \sqrt{a^2+b^2}$$

a is called *real part of* z. b is called *imaginary part of* z. z is called complex conjugate. $|z|$ is called absolute value of z.

Theorem 26.

$$\overline{(\overline{z})} = z$$

Proof.

$$\overline{(\overline{z})} = \overline{(a-ib)} = (a-(-ib)) = a+ib = z$$

□

Theorem 27.

$$\Re(z) = \frac{1}{2}(z+\overline{z})$$

Theorem 28.

$$\frac{1}{2}(z+\overline{z}) = \frac{1}{2}(a+ib+a-ib) = \frac{1}{2}(2a) = a\checkmark$$

Theorem 29.

$$\Im(z) = \frac{1}{2i}(z-\overline{z})$$

Proof.

$$\frac{1}{2i}(a+ib-(a-ib)) = \frac{1}{2i}(2ib) = b\checkmark$$

□

Theorem 30.

$$z\in\mathbb{R} \Leftrightarrow z = \overline{z}$$

Proof.

$$z = a\in\mathbb{R} \Rightarrow \overline{z} = a = z$$

On the opposite, let $z = \overline{z}$ therefore

$$a = ib = a - ib \Rightarrow 2ib = 0 \Rightarrow b = 0$$

Therefore $z = a \in \mathbb{R}$. □

Theorem 31.

$$z \in i\mathbb{R} = \{ib : b\in\mathbb{R}\} \Leftrightarrow z = -\overline{z}$$

Proof follows analogously.

Theorem 32. *It holds that* $|z| = \sqrt{z\cdot\overline{z}}$.

Proof.

$$\sqrt{z\cdot\overline{z}} = ((a+ib)(a-ib))^{\frac{1}{2}}$$
$$= (a^2-(ib)^2)^{\frac{1}{2}} = (a^2 - i^2b^2)^{\frac{1}{2}}$$
$$= (a^2+b^2)^{\frac{1}{2}} = |z| \quad \checkmark$$

□

Theorem 33. *Let* $z, w\in\mathbb{C}$*:*

$$\overline{(zw)} = \overline{z}\cdot\overline{w}$$

Proof.

$$z = a+ib \qquad w = c+id$$
$$zw = (ac-bd) + i(bc+ad)$$
$$\overline{zw} = (ac-bd) - i(bc+ad)$$

$$\overline{zw} = a - ib \qquad \overline{w} = c - id$$

$$\overline{z} \cdot \overline{w} = (ac - (-b)(-d)) + i(-bc + a(-d)) = (ac - bd) - i(bc + ad)$$

□

Corollary 6.

$$\overline{z+w} = \overline{z} + \overline{w}$$

Theorem 34.

$$|zw| = |z| \cdot |w|$$

Proof.

$$|z \cdot w| = (zw) \cdot (\overline{z \cdot w})^{\frac{1}{2}}$$

$$= (z \cdot \overline{z} \cdot w \cdot \overline{w})^{\frac{1}{2}} = (z \cdot \overline{z})^{\frac{1}{2}} \cdot (w \cdot \overline{w})^{\frac{1}{2}} = |z| \cdot |w|$$

□

Theorem 35.

$$z = 0 \Leftrightarrow |z| = 0 \in \mathbb{R}$$

Proof.

$$z = 0 = 0 + i0 \Rightarrow |z| = \sqrt{0^2 + 0^2} = 0$$

Let $|z| = \sqrt{a^2 + b^2} = 0 \Rightarrow a^2 + b^2 = 0$.

$$\Rightarrow a = 0 \wedge b = 0$$

□

Theorem 36.

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

$$|\Im(z)| = |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| =$$

Theorem 37. *The triangle inequality holds:*

$$\forall z, w \in \mathbb{C} : |z + w| \leq |z| + |w|$$

Remark 6. *Let $0 \leq y_1 < y_2$ with $y_1, y_2 \in \mathbb{R}$. Let $k \in \mathbb{N}_+$. Then it holds that*

$$\sqrt[k]{y_1} < \sqrt[k]{y_2}$$

Proof. Indirect proof: Let $\sqrt[k]{y_1} \geq \sqrt[k]{y_2} \geq 0$.

$$\Rightarrow \left(\sqrt[k]{y_1}\right)^k \geq \left(\sqrt[k]{y_2}\right)^k$$

therefore $y_1 \geq y_2$. This is the negation of our assumption. □

Proof *of the triangle inequality.* We show that $|z + w|^2 \leq (|z| + |w|)^2$.

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = \underbrace{z\overline{z}}_{|z|^2} + w\overline{z} + z\overline{w} + \underbrace{w\overline{w}}_{|w|^2}$$

$$\begin{aligned}
&= 2\Re(w\overline{z}) \\
&= (w\overline{z} + \underbrace{\overline{(w \cdot \overline{z})}}_{\overline{w} \cdot \overline{\overline{z}} = \overline{w} \cdot z} \\
&= |z|^2 + 2\Re(w \cdot \overline{z}) + |w|^2 \\
&\leq |z|^2 + 2\,|\Re(w \cdot \overline{z})| + |w|^2 \\
&\leq |z|^2 + 2 \cdot |w \cdot \overline{z}| + |w|^2 \\
&= |z|^2 + 2 \cdot |w| \cdot |\overline{z}| + |z|^2 \\
&= |z|^2 + 2 \cdot |w| \cdot |z| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

□

Theorem 38. *In our previous proof there was a small loop hole: We need to show that*

$$|z| = |\overline{z}|$$

Proof.

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

□

6.1 Interpretation of multiplication

Multiplication with i. Let $z = a + ib$.

$$iz = i \cdot a + i^2 \cdot b = (-b) + ia$$

Multiplication with i rotates z counter-clockwise by 90° in the plane.

Let $z \in \mathbb{C}$ and $w = c + id$.

This lecture took place on 18th of November 2015 with lecturer Wolfgang Ring.

6.2 Taking roots

$$\forall a \in \mathbb{R} : a \geq 0 \forall n \in \mathbb{N}_+ : \exists x \geq 0 \in \mathbb{R} : x^n = a$$

Taking the n-th root only works for positive integers, because $\forall x \geq 0 : x^2 \geq 0$ and no solution in $\mathbb{R}$ exists for the equation $x^2 = -1$.

In $\mathbb{C}$ it holds that $\forall w \in \mathbb{C} \setminus \{0\}$. $\forall n \in \mathbb{N}$ there exist exactly n different solutions of the equation$z^n = w$.

7 Sequences of real and complex numbers

Definition 23. *Let a be a mapping $\mathbb{N} \to \mathbb{R}$ is called* sequence *of real numbers.*

$$\forall n \in \mathbb{N} : a(b) \in \mathbb{R}$$

We denote $a_n \coloneqq a(n)$. Instead of $a : \mathbb{N} \to \mathbb{C}$ we write $(a_n)_{n\in\mathbb{N}} = (a_0, a_1, \ldots)$.

Analogously for the complex numbers $\mathbb{C}$ and general sets X.

Example 7. $a_n = \sqrt[n]{2}\frac{1}{n+1}$ *with $(a_n)_{n\in\mathbb{N}}$. Or simply:*

$$\left(\sqrt[n]{2}\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$

Example 8. *Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals. Therefore $(I_n)_{n\in\mathbb{N}}$ is a sequence of elements in $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$.*

Definition 24. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. $(a_n)_{n\in\mathbb{N}}$ is called* bounded above *if $o \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \leq o$. $(a_n)_{n\in\mathbb{N}}$ is called* bounded below *if $u \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \geq u$.*

$(a_n)_{n\in\mathbb{N}}$ is called bounded, if $(a_n)_{n\in\mathbb{N}}$ is bounded above and below.

Example 9. *$(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{n}{n+1}$ is bounded below by 0 and bounded above by 1: $n \leq n + 1 \Rightarrow n\frac{1}{n+1} < \frac{n+1}{n+1} = 1$✓.*

Definition 25.
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically increasing *if $\forall n \in \mathbb{N} : a_{n+1} \geq a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically decreasing *if $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically strictly increasing *if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically strictly decreasing *if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.*

In $\mathbb{C}$, elements are not ordered, hence no complex sequences can be given. Let $(a_n)_{n\in\mathbb{N}}$ a complex sequence. We define:

- *$(a_n)_{n\in\mathbb{N}}$ is called bounded if $(|a_n|)_{n\in\mathbb{N}}$ is a bounded real sequence. Hence $\exists o \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| \leq o$.*
- *The lower bound is implicitly given by 0.*

Example 10. $a_n \coloneqq i^n$ *and* $(a_n)_{n\in\mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, i, -1, \ldots)$

$$|1| = 1 \qquad |-1| = 1 \qquad |i| = \sqrt{0^2 + 1^2} = 1 \qquad |-i| = \sqrt{0^2 + (-1)^2} = 1$$

So $(|a_n|)_{n\in\mathbb{N}} = (1, 1, 1, 1, 1, \ldots)$. It holds that

$$|z| = |-z| = |\overline{z}|$$

Definition 26. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{C}$ and let $a \in \mathbb{C}$. We state: $(a_n)_{n\in\mathbb{N}}$ has a limit (lat. limes) a if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$$

We denote

$$\lim_{n\to\infty} a_n = a$$

The distance $|a_n - a|$ becomes arbitrary small, if n is sufficiently large.

A sequence, which has a limit, is called convergent. *A sequence, which does not have a limit, is called* divergent.

Remark 7. *Sometimes we consider mappings $a : \mathbb{N}_+ \to \mathbb{C}$, which we also call sequences:*

$$a \leftrightarrow (a_1, a_2, \dots)$$

Example 11.

$$a_n = \frac{1}{n}$$

We know:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \to \frac{1}{n} < \varepsilon$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Let $q \in \mathbb{C}$, $|q| < 1$.

We know $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \to |q^n - 0| < \varepsilon$.

$$\lim_{n\to\infty} q^n = 0$$

This lecture took place on 19th of November 2015 with lecturer Wolfgang Ring.

Remark 8. *Consider $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$ as a circle with radius ε. So if n is sufficiently large, all new sequence numbers are located inside the circle.*

Lemma 2. *A sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$ can have at most one limit.*

Proof. Assume a and b are limes of $(a_n)_{n\in\mathbb{N}}$. Then we prove:

$$\forall \varepsilon > 0 : |a - b| < \varepsilon$$

$$\Rightarrow a = b$$

Let $\varepsilon > 0$ arbitrary: Because $a = \lim_{n\to\infty} a_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\right]$$

Because $b = \lim_{n\to\infty} b_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}\right]$$

Let $N = \max(N_1, N_2)$, hence $N \geq N_1 \wedge N \geq N \geq N_2$.

$$\Rightarrow |a_N - a| < \frac{\varepsilon}{2} \wedge |a_N - b| < \frac{\varepsilon}{2}$$

$$|a - b| = |a \underbrace{-a_N + a_N}_{0} - b| \leq \underbrace{|a - a_N|}_{<\frac{\varepsilon}{2}} + \underbrace{|a_N - b|}_{<\frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem 39 (Well-known convergent sequences.)**.**

1. *Let $s = \frac{p}{q} \in \mathbb{Q}_+$ and $n \in \mathbb{N}_+$. Consider $\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$.*

$$n^s = n^{\frac{p}{q}} := \sqrt[q]{n^q}$$

It holds that

$$\lim_{n\to\infty} \frac{1}{n^s} = 0$$

2. *Let $q \in \mathbb{C}, |q| < 1$. Then it holds that*

$$\lim_{n\to\infty} q^n = 0$$

3. *Let $a \in \mathbb{R}, a > 0, n \in \mathbb{N}_+$. Then it holds that*

$$\lim_{n\to\infty} \sqrt[n]{a} = 0$$

4. *It holds that $(n \in \mathbb{N}_+)$*

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

5. *Let $z \in \mathbb{C} : |z| > 1$. Let $k \in \mathbb{N}$. Then it holds that*

$$\lim_{n\to\infty} \frac{n^k}{z^n} = 0$$

Remark 9 (Remark to sequence 5)**.** *$|z^n|$ grows faster then n^k.*

Proof *of sequence 1.* Let $0 \leq x_n < x_2$.

$$\Rightarrow 0 \leq x_1^p < x_2^p \Rightarrow \sqrt[q]{x_1^p} < \sqrt[q]{x_2^p}$$

Therefore $f(x) = x^s$ is strongly monotonic rising for $x \in (0, \infty)$. Let $\varepsilon > 0$ arbitrary and $N > \frac{1}{\varepsilon^{\frac{1}{s}}} = \varepsilon^{\frac{1}{s}} = \varepsilon^{-\frac{q}{p}}$. Then it holds that $n \geq N$:

$$\left|\frac{1}{n^s} - 0\right| = \frac{1}{n^s} \leq \frac{1}{N^s}$$

$$\frac{1}{n^s} < \frac{1}{N^s} \implies n^s \geq N^s$$

$$\frac{1}{n^s} \leq \frac{1}{N^s} < \frac{1}{\left(\frac{1}{\varepsilon^{\frac{1}{s}}}\right)^s} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

□

Proof *of sequence 2.* Already done. □

Proof *of sequence 3.* **case** $a > 1$ Let $a > 1$. Consider $\varepsilon > 0$. Show that $|\sqrt[n]{a} - 1| < \varepsilon$ for sufficiently large n.

$$x_n = \sqrt[n]{a} - 1 = \left|\sqrt[n]{a} - 1\right|$$

$$a > 1 \implies \sqrt[n]{a} > \sqrt[n]{1} = 1 \implies \sqrt[n]{a} - 1 > 0$$

It holds that $x_n + 1 = \sqrt[n]{q}$, i.e. $(x_n + 1)^n = a$.

$$a = (\underbrace{x_1}_{>0} + 1)^n \underbrace{>}_{\text{Bernoulli}} 1 + n \cdot x_n$$

$$\Rightarrow x_n < \frac{a-1}{n}$$

$$N > \frac{a-1}{\varepsilon} \xRightarrow{\text{for } x \geq N} \left|\sqrt[n]{a} - 1\right| = x_n$$

$$< \frac{a-1}{n} \leq \frac{a-1}{N} < \frac{a-1}{\frac{a-1}{2}} = \varepsilon$$

case $a = 1$

$$\sqrt[n]{a} = \sqrt[n]{1} = 1$$

$$\left(\sqrt[n]{a}\right)_{n \in \mathbb{N}} = (1, 1, 1, 1, \dots)$$

has the limit 1.

case $0 < a < 1$ Let $0 < a < 1 \Rightarrow 0 < \sqrt[n]{q} < \sqrt[n]{1} = 1$.

$$x_n = 1 - \sqrt[n]{a} > 0$$

Show that $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow x_n < \varepsilon]$.

$$x_n = 1 - \sqrt[n]{a} = \sqrt[n]{a}\left(\frac{1}{\sqrt[n]{a}} - 1\right) = \sqrt[n]{a}\left(\sqrt[n]{\frac{1}{a}} - 1\right) < \left(\sqrt[n]{a'} - 1\right)$$

with $a' = \frac{1}{a} > 1$. From case $a > 1$ we already know

$$\exists N \in \mathbb{N} : \left[n \geq N \Rightarrow \left|\sqrt[n]{a'} - 1\right| = \sqrt[n]{a'} - 1 < \varepsilon\right]$$

$$\Rightarrow x_n < \varepsilon$$

□

Proof *of sequence 4.* This proof works similar to the proof of sequence 3.

$$x_n = \sqrt[n]{n} - 1 > 0 \text{ for } n \geq 2$$

Therefore $|x_n| = x_n$. Let $\varepsilon > 0$ be arbitrary.

$$x_n + 1 = \sqrt[n]{n} \quad \text{i.e.} \quad (x_n + 1)^n = n$$

$$n = (1 + x_n)^n = 1 + \underbrace{nx_n}_{>0} + \underbrace{\binom{n}{2} x_n^2}_{>0} + \underbrace{\binom{n}{3} x_n^3}_{>0} + \underbrace{\cdots + x_n^n}_{>0} > 1 + \binom{n}{2} x_n^2$$

All expressions we remove are positive (but we don't remove all positive expressions).

$$x_n^2 < \frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2 \cdot 1}} = \frac{2}{n}$$

$$x_n < \sqrt{\frac{2}{n}}$$

Choose $N > \frac{2}{\varepsilon^2}$. Then it holds for $n \geq N$ that

$$x_n < \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} < \sqrt{\frac{2}{\frac{2}{\varepsilon^2}}} = \varepsilon$$

Consider $\sqrt{\frac{2}{n}} < \varepsilon$ hence $\frac{2}{n} < \varepsilon^2$ hence $n > \frac{2}{\varepsilon^2}$. □

Proof *of sequence 5.*

$$|z| > 1 \text{ thus } x = |z| - 1 > 0 \text{ it holds that } |z| = 1 + x$$

We show that for $\varepsilon > 0$ arbitrary, there exists $N \in \mathbb{N}$:

$$n \geq N \implies \left|\frac{n^k}{z^n} - 0\right| = \left|\frac{n^k}{z^n}\right| = \frac{n^k}{|z|^n} < \varepsilon$$

Let $\varepsilon > 0$ be given,

- For $n > 2k$ it holds that $n - k > n - \frac{n}{2} = \frac{n}{2}$.

$$|z|^n = (1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j > \underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1}$$

$$n > 2k \geq k+1$$

$$\underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1} = \frac{\overbrace{n}^{>\frac{n}{2}}\overbrace{(n-1)}^{>\frac{n}{2}}\overbrace{(n-2)}^{>\frac{n}{2}}\ldots\overbrace{(n-k)}^{>\frac{n}{2}}}{(k+1)!} x^{k+1} > \frac{\frac{n^{k+1}}{2^{n+1}}}{(k+1)!} x^{n+1}$$

Therefore $|z|^n > \frac{n^{k+1}}{2^{k+1}(k+1)!} x^{k+1}$. So,

$$\frac{n^k}{|z|^n} < \frac{n^k \cdot 2^{k+1}(k+1)!}{n^{k+1} \cdot x^{k+1}} = \underbrace{\frac{2^{k+1}(k+1)!}{x^{n+1}}}_{\substack{= \text{constant} \wedge >0 \\ =: M}} \cdot \frac{1}{n} = M \cdot \frac{1}{n}$$

$$\frac{n^k}{|z|^n} < M \cdot \frac{1}{n} \text{ for } n > 2k$$

Consider N such that $N > \frac{M}{\varepsilon}$ and $N > 2k$. Then it holds that

$$\frac{n^k}{|z|^n} < M\frac{1}{n} \leq \frac{M}{N} < \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$$

□

Lemma 3. *Every convergent sequence is bounded (in $\mathbb{C}$).*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. This means especially e.g. $\varepsilon = 13$.

$$\exists N \in \mathbb{N} \text{ s.t. } [n \geq N \implies |a_n - a| < 13]$$

Consider $O > 0$ such that

$$O = \max\{|a_0|, |a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 13\}$$

So $O \geq |a_n|$ for $n \in \{0, \ldots, N\}$. Then for $0 \leq n < N$ it holds that $|a_n| < O$. ✓

For $n \geq N$ it holds that

$$|a_n| = |a_n - a + a| \leq \underbrace{|a_n - a|}_{<13} + |a| < \underbrace{13 + |a|}_{\leq O}$$

Therefore $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and followingly $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$. □

Theorem 40. *Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then the following laws hold:*

1. *$\lim_{n\to\infty}(a_n + b_n)$ is convergent with limes $a + b$*
2. *$\lim_{n\to\infty}(a_n \cdot b_n)$ is convergent with limes $a \cdot b$*
3. *$\lim_{n\to\infty} \frac{a_n}{b_n}$ is convergent with limes $\frac{a}{b}$ if $\forall n \in \mathbb{N} : b_n \neq 0 \wedge b \neq 0$.*

Proof. 1. Let $\varepsilon > 0$ arbitrary. Because $(a_n)_{n\in\mathbb{N}}$ is convergent,

$$\exists N_1 : \left[n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\right]$$

(b_n) is convergent hence

$$\exists N_2 : \left[n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}\right]$$

$N = \max\{N_1, N_2\}$, hence for $n \geq N$ both statements above hold. Let $n \geq N$, then the triangle inequality holds:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

2. $(a_n)_{n\in\mathbb{N}}$ is convergent and therefore also bounded. Therefore,

$$\exists m \geq 0 : \forall n \in \mathbb{N} : |a_n| \leq m$$

$(b_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_1 : n \geq N_1 :\Rightarrow |b_n - b| < \frac{\varepsilon}{2} \cdot \frac{1}{m+1}$$

$(a_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_2 \leq N : n \geq N_2 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\frac{1}{|b|+1}$$

$N = \max\{N_1, N_2\}$. For $n \geq N$ both relations above hold. Let $n \geq N$:

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$$

$$\leq |a_n(b_n - b)| + |b(a_n - a)| = |a_n|\,|b_n - b| + |b|\,|a_n - a|$$

$$\leq m\frac{\varepsilon}{2}\frac{1}{m+1} + |b|\frac{\varepsilon}{2}\frac{1}{|b|+1} < \frac{\varepsilon}{2}\cdot 1 + \frac{\varepsilon}{2}\cdot 1 = \varepsilon$$

3. Left for the practicals.

□

7.1 Laws for convergent complex sequences

Theorem 41. *Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a, $(a_n \to a)$. Then it holds that*

- $(\Re(a_n))_{n\in\mathbb{N}}$ *is convergent.*

$$\lim_{n\to\infty} (\Re(a_n)) = \Re(a)$$

- $(\Im(a_n))_{n\in\mathbb{N}}$ *is convergent.*

$$\lim_{n\to\infty} (\Im(a_n)) = \Im(a)$$

- $(|a_n|)_{n\in\mathbb{N}}$ *is a convergent real sequence.*

$$\lim_{n\to\infty} |a_n| = |a|$$

- $(\overline{a_n})_{n\in\mathbb{N}}$ *is convergent with*

$$\lim_{n\to\infty} \overline{a_n} = \overline{a}$$

On the opposite, let $(a_n)_{n\in\mathbb{N}}$ with $a_n = \alpha_n + i\beta_n$ a sequence of complex numbers. Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be convergent with limes α i.e. β. Then $(a_n)_{n\in\mathbb{N}}$ is a convergent complex sequence with limes $a = \alpha + \beta i$.

Proof. Let $\varepsilon > 0$. Consider N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$.

$$\underbrace{|a_n - a|}_{(\alpha_n - \alpha)+(\beta_n - \beta)i} = \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2}$$

TODO

Therefore $(\alpha_n) = (\Re(a_n))_{n\in\mathbb{N}}$ is convergent. $(\beta_n) = (\Im(a_n))_{n\in\mathbb{N}}$ is convergent.

Let $\varepsilon > 0$. Consider N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$.

$$||a_n| - |a|| \underbrace{\leq}_{\text{inverse triangular inequality}} |a_n - a| < \varepsilon \text{ for } n \geq N$$

Now we need to show $\alpha_n \to \alpha$ and $\beta_n \to \beta$

$$\Rightarrow a_n \to a$$

Let $\varepsilon > 0$ be arbitrary. Because $(\alpha_n)_{n\in\mathbb{N}}$ be convergent, there exists $N_1 \in \mathbb{N}$:

$$n \geq N_1 \Rightarrow |\alpha_1 - \alpha| < \frac{\varepsilon}{\sqrt{2}}$$

$(\beta_n)_{n\in\mathbb{N}}$ is convergent. So,

$$\exists N_2 \in \mathbb{N} : n \geq N_2$$

$$|\beta_n - \beta| < \frac{\varepsilon}{\sqrt{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$ both relations hold.
Let $n \geq N$:

$$|a_n - a| = |(\alpha_n - \alpha) + i(\beta_n - \beta)|$$

$$= \sqrt{(\alpha_n - \alpha)^2 + (\beta_n - \beta)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon$$

Let $a_n = \alpha_n + i\beta_n$ is convergent with limes $\alpha + i\beta$ which is a.

$$\Rightarrow \lim_{n\to\infty} \alpha_n = \alpha \wedge \lim_{n\to\infty} \beta_n = \beta$$

$$\Rightarrow \lim_{n\to\infty} (-\beta_n) = -\beta \qquad \text{“multiplication rule”}$$

$$\Rightarrow (\overline{a_n})_{n\in\mathbb{N}} = (\underbrace{\alpha_n}_{\text{convergent}} - \underbrace{i\beta_n}_{\text{convergent}})_{n\in\mathbb{N}}$$

$$\Rightarrow \lim_{n\to\infty} \overline{a_n} = \alpha - i\beta = \overline{a}$$

□

7.2 Other laws for complex sequences

Theorem 42. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$ with limes a (i.e. b) and it must hold that $\forall n \in \mathbb{N} : a_n \leq b_n$. Then also $a \leq b$.*

Proof. Consider $a - b = \varepsilon > 0$.

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$$

For $N = \max\{N_1, N_2\}$:

$$b_N = b_N - b + b \leq b + |b_N - b| < b\frac{\varepsilon}{2} = b + \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$a_N = \underbrace{a_N - a}_{\geq -|a_n - a|} + a \geq a - |a_n - a| > a - \frac{\varepsilon}{2} = a - \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$b_N < \frac{1}{2}(a+b) < d_N$$

Attention:

$$a_n < b_n \not\Rightarrow a < b$$

Example: $a_n = 0$, $b_n = \frac{1}{n}$. □

7.3 Convergence criteria

Are there criteria such that if they have a specific structure, they are obivously convergent?

7.3.1 Squeeze theorem

Theorem 43. *Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ convergent real sequence with $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = A$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $M \in \mathbb{N}$ such that*

$$\forall n \geq M : A_n \leq a_n \leq B_n$$

Then it holds that $(a_n)_{n\in\mathbb{N}}$ is also convergent and $\lim a_n = A$.

Proof. Let $\varepsilon > 0$ be arbitrary. Consider N such that,

- $M \geq M$
- $n \geq N \Rightarrow |A_n - A| < \varepsilon$
- $n \geq N \Rightarrow |B_n - A| < \varepsilon$

Then it holds that for $n \geq N$:

$$\left.\begin{array}{l} A - a_n \leq A - A_n \leq |A - A_n| < \varepsilon \\ a_n - A \leq B_n - A \leq |B_n - A| < \varepsilon \end{array}\right\} = 1$$

$$\Rightarrow |a_n - A| < \varepsilon$$

$$\lim_{n\to\infty} a_N = A$$

□

Example 12. *Let $S \in \mathbb{Q}_+$. Then it holds that*

$$\lim_{n\to\infty} \left(\sqrt[n]{n^s}\right) = 1$$

We apply the squeeze theorem:

$$n^2 \geq 1 \forall n \in \mathbb{N}$$

$$\Rightarrow \sqrt[n]{n^s} \geq 1$$

Let $k \in \mathbb{N}_+$. Then it holds that

$$\lim_{n\to\infty} \sqrt[n]{n^k} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}\sqrt[n]{n}\ldots\sqrt[n]{n}}_{k\ times}$$

$$= 1 \cdot 1 \cdot 1 \cdots = 1$$

For the last two lines we actually need to read them from right to left.

Let $s = \frac{p}{q}$.

$$\Rightarrow n^s = n^{\frac{p}{q}} \leq q \cdot \left(n\frac{p}{q}\right)^q = n^p$$

$$q \geq 1 \Rightarrow \sqrt[n]{n^s} \leq \underbrace{\sqrt[n]{n^p}}_{convergent\ with\ limes\ 1} \qquad p \in \mathbb{N}$$

Then it holds that $\lim_{n\to\infty} \sqrt[n]{n^s} = 1$ with the squeezing theorem.

Remark 10. *Let $A \subseteq \mathbb{R}$ be bounded above. Then it holds that*

$$S = \sup A \Leftrightarrow s \textit{ is upper bound of } A \wedge \forall\varepsilon > 0 \exists a \in A : a > s - \varepsilon$$

Proof. Implication from left to right: Let $s = \sup A$. Then it holds that s is upper bound of A and $s - \varepsilon < s$ is not an upper bound. Therefore $\exists a \in A : a > s - \varepsilon$.

Implication from right to left: Consider that both statements on the RHS hold. So s is an upper bound. We need to show that any t is not an upper bound with $t > s$. Let $t < s, s - t = \varepsilon > 0$. Therefore $t = s - \varepsilon$. Because of the right statement $\exists a \in A : a > s - \varepsilon = t$ therefore t is not an upper bound. □

Remark 11. *Analogously:*

$$\sigma = \inf A \Leftrightarrow \sigma \textit{ is lower bound} \wedge \forall\varepsilon > 0 \exists a \in A : a < \sigma + \varepsilon$$

Theorem 44. *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded monotonic sequence. Then $(a_n)_{n\in\mathbb{N}}$ has a limes a with*

- *$a = \sup\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing.*
- *$a = \inf\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be monotonically increasing. Let $a = \sup\{a_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ be arbitrary. Because a is a supremum, there exists $a_N \in \{a_n : n \in \mathbb{N}\}$ such that $a_N > a - \varepsilon$.

$$\Rightarrow \underbrace{a - a_N}_{\geq 0} < \varepsilon$$

because a is an upper bound. Therefore

$$|a - a_N| < \varepsilon$$

Let $n \geq N$ then it holds that

$$|a - a_n| \underbrace{=}_{a \text{ is upper bound}} a - a_n \leq a - a_N$$

because $a_N \leq a_n$ is increasing:

$$a - a_N < \varepsilon$$

Therefore $\lim_{n\to\infty} a_n = a$. □

This lecture took place on 25th of November 2015 with lecturer Wolfgang Ring.

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If $(a_n)_{n\in\mathbb{N}}$ is bounded and monotonous. Then $(a_n)_{n\in\mathbb{N}} \in \mathbb{N}$ is convergent.

Example: Wallis product John Wallis (1616–1703)

$$p_n = \frac{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}{1\cdot 3\cdot 5 \ldots \ldots \cdot (2n-1)} = \prod_{k=1}^{n} \frac{2k}{2k-1}$$

Consider

$$\alpha_n = \frac{p_n}{\sqrt{n}} \qquad \beta_n = \frac{p_n}{\sqrt{n+1}}$$

We need to show that

- (α_n) is monotonously decreasing
- (β_n) is monotonously increasing

$$\forall n \in \mathbb{N} : n \geq 1 : \alpha_n > \beta_n$$

Both are convergent.

1. Show that,

$$\alpha_{n+1} < \alpha_n \Leftrightarrow \frac{\alpha_{n+1}}{\alpha_n} < 1 \Leftrightarrow \frac{(\alpha_{n+1})^2}{(\alpha_n)^2} < 1$$

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2 = \left(\frac{\frac{2\cdot 4\cdot 6\cdot\ldots\cdot(2n+2)}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)\cdot(n+1)}}{\frac{2\cdot 4\cdot\ldots\cdot 2n}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)}} \cdot \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}}\right)^2$$

$$= \frac{(2n+2)^2\cdot n}{(2n+1)^2(n+1)} = \frac{4n^3+8n^2+4n}{(4n^2+4n+1)\cdot(n+1)} = \frac{4n^3+8n^2+4n}{4n^3+8n^2+5n+1} < 1$$

2. We show,

$$\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 = \frac{(2n+2)^2\cdot(n+1)}{(2n+1)^2\cdot(n+2)} = \frac{(4n^2+8n+4)(n+1)}{(4n^2+2n+1)(n+2)}$$

$$= \frac{4n^3+12n^2+12n+4}{4n^3+12n^2+9n+2} > 1 \Rightarrow \beta_{n+1} > \beta_n \Rightarrow \beta_n \text{ is monotonically increasing}$$

Let $p = \lim_{n\to\infty} a_n$ and $p' = \lim_{n\to\infty} b_n$.

$$\beta_n = \frac{p_n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \alpha_n \cdot \sqrt{\frac{n}{n+1}}$$

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha_n \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \alpha_n \cdot \underbrace{\lim_{n\to\infty} \sqrt{\frac{n}{n+1}}}_{=1}$$

$$\Rightarrow \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} a_n \Rightarrow p = p'$$

It holds that $p = \lim_{n\to\infty} \frac{p_n}{\sqrt{n}} = \sqrt{n}$.

7.4 Bolzano-Weierstrass theorem

Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897)

Definition 27. *Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence. The complex number a is called* limit point *(german "Häufungspunkt") of $(a)_{n\in\mathbb{N}}$ if $\forall \varepsilon > 0 : |a_n - a| < \varepsilon$ for infinitely many indices $n \in \mathbb{N}$. Hence infinitely many numbers of the sequence lie within a circle with center a and radius ε.*

Remark 12. *Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limit a. Then it holds that a is the only limit point of the sequence $(a_n)_{n\in\mathbb{N}}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. Let

$$\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a|$$

Therefore $\forall n \in \{N, N+1, N+2, \dots\}$ it holds that $|a_n - a| < \varepsilon$. Assume $a' \in \mathbb{C}$ is another limit point with $a \neq a'$. Let

$$\varepsilon = \frac{|a - a'|}{2} > 0$$

Let $N \in \mathbb{N}$ such that $\forall n \geq N : |a_n - a| < \varepsilon$.

$$\Rightarrow n \in \mathbb{N} : |a' - a_n| = |a' - a + a - a_n| = |a' - a - (a_n - a)| \geq |a' - a| - |a_n - a|$$

$$= 2\varepsilon - |a_n - a| > 2\varepsilon - \varepsilon = \varepsilon$$

At most for $n \in \{1, \dots, N-1\}$ it is possible that $|a_n - a'| < \varepsilon$. $\square$

Remark 13. *$a_n = (-1)^n$ has the limit points $+1$ and -1.*

The lecture on 26th of November 2015 got cancelled.

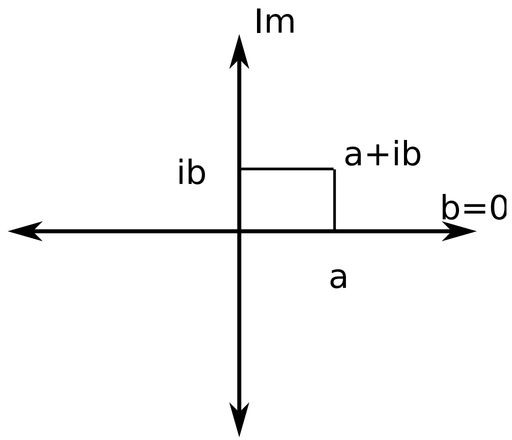

Figure 9: Illustration of complex numbers

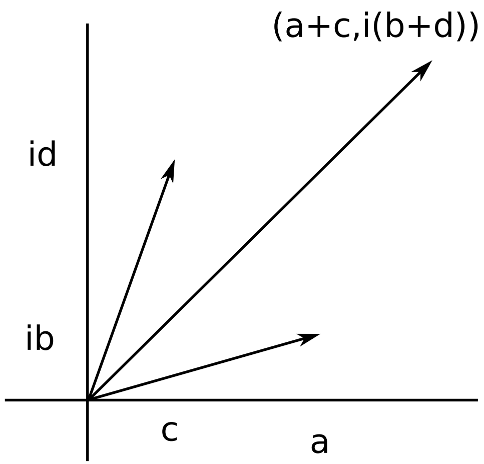

Figure 10: Illustration of complex number addition

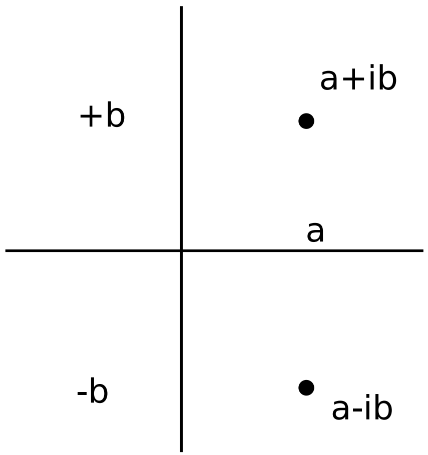

Figure 11: Illustration of the complex conjugate

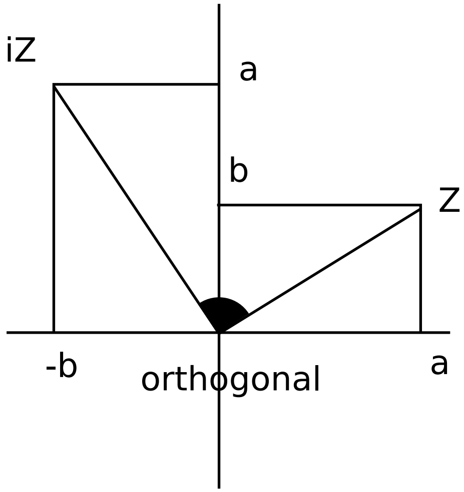

Figure 12: Multiplication corresponds to a rotation by 90°

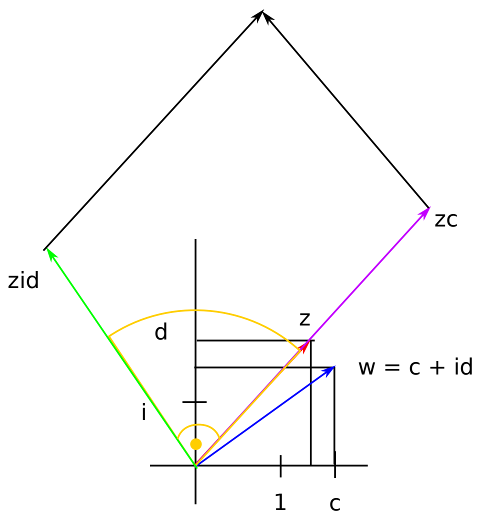

Figure 13: In regards of multiplication with w the complex number z is scaled by $|w|$ and then rotated by an angle which is given between w and the positive real axis.

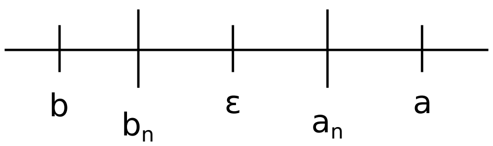

Figure 14: the sequences a_n, b_n and limes a, b and ε in relation

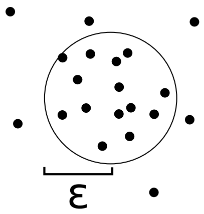

Figure 15: Illustration of a limit point in the Euclidean plane. The point is represented as circle with radius ε. Finitely many points lie outside the limit point; infinitely many inside.

German keywords

English keywords