# Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

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March to July 2016

## Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

# 1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$
$$\exp(z+w) = \exp(z)\cdot\exp(w)$$
$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n} \in \mathbb{Q} \wedge n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z \qquad \text{for } z \in \mathbb{C}$$

It holds that

$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$

$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^0 = 1$$

Let $n \in \mathbb{N}$

$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$
$$\lim_{x\to-\infty} e^x\cdot x^n = 0$$



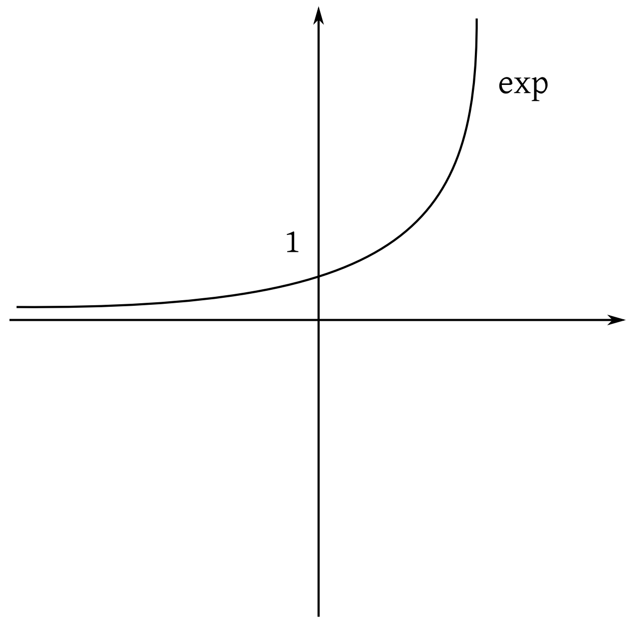


Figure 1: Graph of the exponential function

# 2 The natural logarithm

$$\exp : \mathbb{R} \to (0,\infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

**Lemma 1.** $\exp : \mathbb{R} \to (0,\infty)$ is surjective.

*Proof.* We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

**Case 1** First of all, let $y \in [1, \infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y]: \quad e^\xi = y$$

**Case 2** Let $y \in (0, 1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w]: \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp: \mathbb{R} \to (0, \infty)$ is bijective. $\square$

**Definition 1.** We call the inverse function *natural logarithm*[1].

$$\exp^{-1}: (0, \infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R}: \ln(e^x) = x$ and $\forall y \in (0, \infty): e^{\ln(y)} = y$.
- $\ln: (0, \infty) \to \mathbb{R}$ is strictly monotonically increasing

  *Proof.* Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xrightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! $\square$

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.



## 2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Limes:
$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

*Proof.*

- $$x \cdot y = e^{\ln(x \cdot y)}$$
$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x) + \ln(y)}$$
Injectivity of exp:
$$\ln(x \cdot y) = \ln(x) + \ln(y)$$
- Let $(x_n)_{n \in \mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n \to \infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = \ln(w_n)$.
$$\lim_{n \to \infty} y_n = \ln(1) = 0$$
$$\lim_{n \to \infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n \to \infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$
where
$$e^0 = 1 \Rightarrow \ln(1) = 0$$

$\square$

**Theorem 1** (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

*Proof.* Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^\xi} = 0$. $\square$

**Theorem 2.** The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

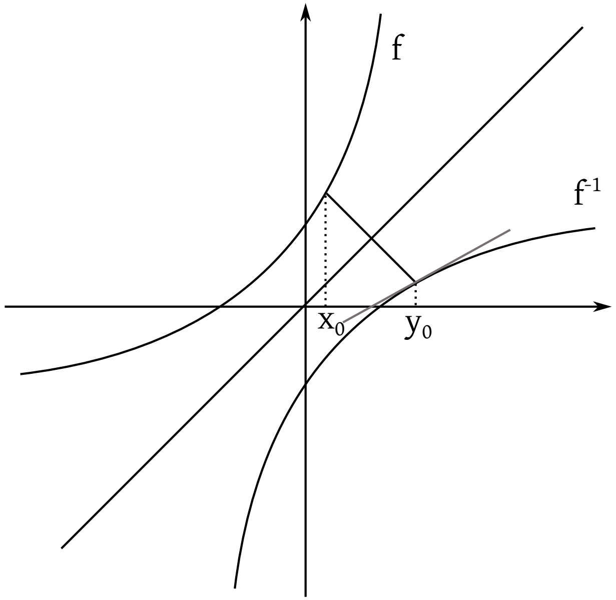


Figure 2: A geometric proof of differentiability

*Proof.* **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^{\xi})' = e^{\xi}}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

**Second approach using chain rule** Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

**Third approach** Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln\frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln\frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

## 2.2 Extension of the functional equation of logarithm

## 2.3 A different proof for the derivative of logarithm

*Proof.*

$$[\ln(x)]' = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$= \frac{1}{x}\cdot\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x}\to 0$$

$1+\frac{h}{x}=w$ then it holds that $h\to 0 \Rightarrow w\to 1$.

$$\frac{h}{x}=w-1$$

$$\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{=}\lim_{h\to 0}\frac{\ln(w)}{w-1}=1$$

□

**Remark 1.** The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp:\mathbb{C}\to\mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C}\setminus\{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C}\setminus\{x\in\mathbb{R}: x\leq 0\}$

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This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

## 2.4 Further remarks on differential calculus

**Theorem 3.** Let $f: I\to\mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where $I$ is an interval. Then $f^{-1}: f(I)\to\mathbb{R}$ is defined and the inverse function.

Let $f$ in $x_0\in I$ be differentiable and $f'(x_0)\neq 0$. Then $f^{-1}$ is in $y_0=f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0)=\frac{1}{f'(x_0)}=\frac{1}{f'(f^{-1}(y_0))}$$

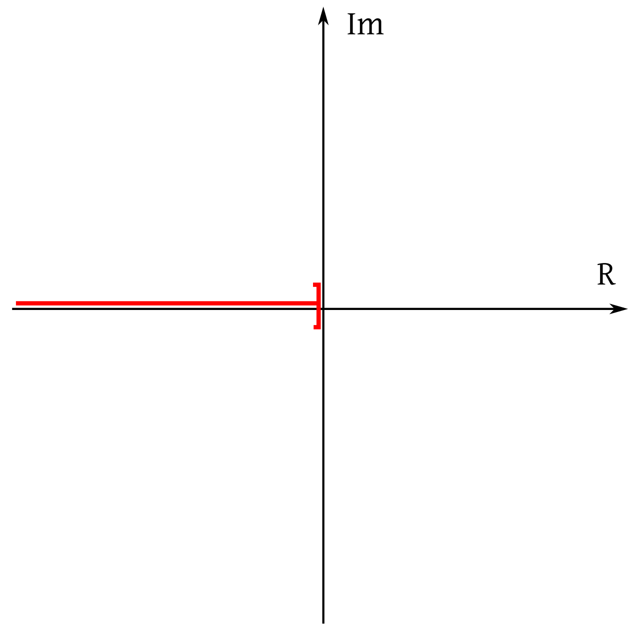


Figure 3: Continuous exponential function in $\mathbb{C}$

*Proof.* Let $y_n\to y_0$ and $y_n\in f(I)$; $y_0=f(x_0)$; $y_0\in f(I)$; $y_n=f(x_n)$. $y_n\neq y_0\Rightarrow x_n\neq x_0$.

$$\lim_{n\to\infty}\frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}$$

$$=\lim_{n\to\infty}\frac{x_n-x_0}{f(x_n)-f(x_0)}=\frac{1}{\lim_{n\to\infty}\underbrace{\frac{f(x_n)-f(x_0)}{x_n-x_0}}_{\text{ex}=f'(x_0)}}=\frac{1}{f'(x_0)}$$

□

**Lemma 2.** Let $f: I\to\mathbb{R}$ where $I$ is some interval. Then it holds that

$$f=\text{ const}\Leftrightarrow f\text{ is differentiable in } I \text{ and } f'(x)=0\forall x\in I$$

*Proof.* $\Rightarrow$ Immediate.

$\Leftarrow$ Let $f$ be differentiable and $f' \equiv 0$. Assume $f$ is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

□

**Definition 2.** Let $I$ be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of $f$ if $F$ is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

**Lemma 3.** Let $f : I \to \mathbb{R}$. Let $F_1$ and $F_2$ be two primitive functions of $f$. Then it holds that $F_1 - F_2 = \text{const}$.

*Proof.* $F_1$, $F_2$ are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

□

**Theorem 4.** Let $I$ be an interval. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions in $I$.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n\in\mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every $x$)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon\,|u - v|\,\forall u, v \in I$$

Then $f$ is differentiable in $I$ and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

*Proof.* Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right|$$
$$= \left|\frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0)\right|$$
$$= \left|\frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0)\right| + \left|f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0)\right| \forall N \in \mathbb{N}$$
$$\leq \left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right|$$
$$+ \left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0)\right| + |f_N'(x_0) - \varphi(x_0)|$$

**1st term**

$$\left|\frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0}\right| = \left|\frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0}\right|$$
$$\leq \frac{\varepsilon}{3}\frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large $N$.

**3rd term** $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large $N$.

Now let $N$ be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

**2nd term**

$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0}\right| - f_N'(x_0)$$

Differentiability of $f_N$: Therefore for $|x - x_0| < \delta$.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$f$ is differentiable in $x_0$ and $f'(x_0) = \varphi(x_0)$. □

**Theorem 5.** Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ ($n \in \mathbb{N}$) and $f_n$ is differentiable in $I$.

Assumption:

1. $f_n \to f$ converges pointwise in $I$ (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in $I$

Then $f$ is differentiable in $I$ and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

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This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

**Theorem 6** (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in $I$ and let $f_n$ be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f_n'(x) \to \varphi(x)$ for every $x$
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon\,|u - v|$

Then it holds that $f$ is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

**Theorem 7.** Let $f_n$ and $f$ be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in $I$ for $n \to \infty$
- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in $I$, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge x \in I \Rightarrow |f_n'(x) - g(x)| < \varepsilon$

Then $f$ is differentiable in $I$ and $f'(x) = g(x) \forall x \in I$.

*Proof.* We check whether the two conditions lead to the conditions of Theorem 6.

We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence

$$\forall x \in I : f_n'(x) \to g(x)$$

3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and $N$ is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$\begin{aligned} |f_n'(x) - f_m'(x)| &= |f_n'(x) - g(x) + g(x) - f_m'(x)| \\ &\leq |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.
Let $\varepsilon > 0$ be arbitrary and $N$ such that $n, m \geq N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m\to\infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$\begin{aligned} &= \lim_{m\to\infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)| \\ &= \lim_{m\to\infty} |f_m'(\xi_{m,n}) - f_n'(\xi_{m,n})| \cdot |u - v| \end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

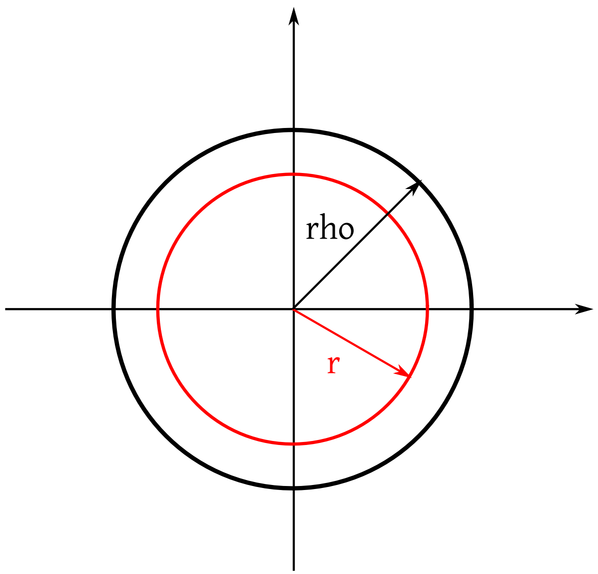


Figure 4: Convergence radius

**Remark 2** (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$



Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

□ Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of $Q$?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \underbrace{\limsup_{j\to\infty} \left( |a_{j+1}|^{\overbrace{\frac{1}{j+1}}^{\to 1}} \right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty} \left[ (j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 7 it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and $P$ is differentiable in point $x$ with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

[2] Where overline means "closed"

**Lemma 4.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then $P$ is differentiable in $x$ and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

## 2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly $g$ is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$



Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for $x$.

$$\Rightarrow 1 + x = w - wx$$
$$x(1+w) = w - 1$$
$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

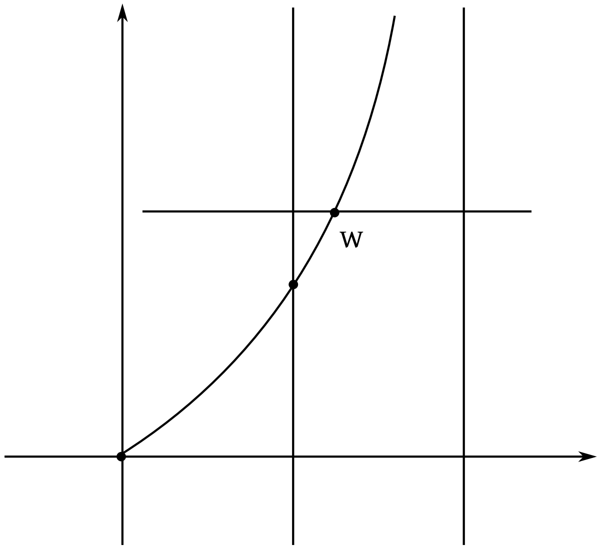


Figure 5: Plot of $\frac{1+x}{1-x}$

## 3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left( \underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[ \overline{\left(1 + \frac{it}{n}\right)} \right]^n$$



$$= \lim_{n\to\infty} \overline{\left(1+\frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty}\left(1+\frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So $e^{it}$ lies inside the complex unit circle. Compare with Figure 6.

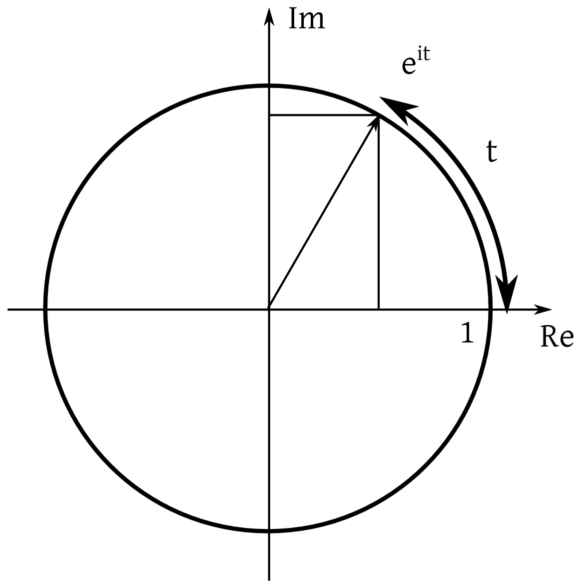


Figure 6: Unit circle in $C$ with $t$

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$ $$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$ $$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$ $$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

**Definition 3.** Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$
$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$



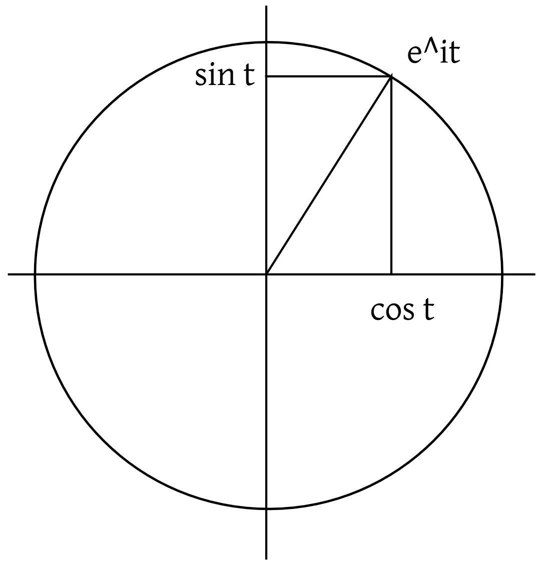


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

## 3.1 Series representation of trigonometric functions

**Lemma 5** (Addition of series of absolute convergence)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

*series sum.* Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

□



**Example 1** (Application)**.** Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0,\delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0,\delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

## 3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \dots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \dots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \dots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \dots$$

Followingly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$
$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C}$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \dots$$
$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!}$$

## 3.3 Functional equations of trigonometric functions

**Theorem 8** (Addition and substraction theorems)**.** We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i\sin(z+w)) \\ &\Rightarrow= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z \cdot \cos w) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i\left(-\cos z \sin w - \cos w \sin z\right) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i\sin(-(z+w))) \\ &\Rightarrow= \cos\left(z+w\right) - i\sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2\cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\begin{aligned} \cos(z-w) &= \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)} \\ \sin(z-w) &= -\cos z \cdot \sin(w) + \sin(z)\cos(w) \end{aligned}$$



**Corollary 1.**

$$\begin{aligned} z &= \frac{1}{2}(z+w) + \frac{1}{2}(z-w) \\ \Rightarrow \cos z &= \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2} \\ w &= \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w) \\ \cos w &= \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2} \\ \cos z - \cos w &= -2\sin\frac{z+w}{2}\sin\frac{z-w}{2} \end{aligned}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z\to 0 \\ z\neq 0}} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right) \\ &= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz} - 1}{2iz}\right) \\ &= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z\to 0}\frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz; z\to 0 \Leftrightarrow w=0 \\ \lim_{w\to 0}\frac{e^w-1}{w}=1}} \end{aligned}$$

So it holds that

$$\lim_{z\to 0}\frac{\sin z}{z} = 1$$

## 3.4 Trigonometric functions for real arguments

Subtitled "definition of $\pi$" and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \dots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{x^2}{(2k+2)(2k+1)}\right| \leq \frac{4}{3\cdot 4} = \frac{1}{3}$$

so $(c_k)_{k\geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left|\frac{s_{k+1}}{s_k}\right| = \left|\frac{x^2}{(2k+2)(2k+3)}\right| \leq \frac{4}{4\cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{ in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$



Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0}\underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1-\frac{4}{6}=\frac{1}{3}>0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = \underbrace{-2 \cdot \underbrace{\sin \underbrace{\frac{z+w}{2}}_{\in(0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in(0,2]}}_{>0}}_{<0}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that cos is a strictly monotonically decreasing functionin $[0,2)$. Hence cos has only one root because it is continuous in $(0,2]$.

**Definition 4.** The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where $\xi$ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos\frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin\frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

$$e^{i\pi} = e^{i\frac{\pi}{2}+i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi+\frac{i}{2}\pi} = e^{i\pi}\cdot e^{i\frac{\pi}{2}} = -1\cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$
$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{iz+2\pi i} + e^{-iz-2\pi i}\right)\\ &= \frac{1}{2}\left(e^{iz} + e^{-iz}\cdot\frac{1}{\underbrace{e^{2\pi i}}_{=1}}\right) = \cos z\end{aligned}$$

Therefore the cosine is periodic in $\mathbb{C}$ with period $2\pi$. Analogously, sine is periodic in $\mathbb{C}$ with period $2\pi$.

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This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

## 3.5 Periodicity and roots of trigonometric functions

$$\cos(z+2\pi) = \cos(z)$$
$$\sin(z+2\pi) = \sin(z)$$

Compare with Figure 8.

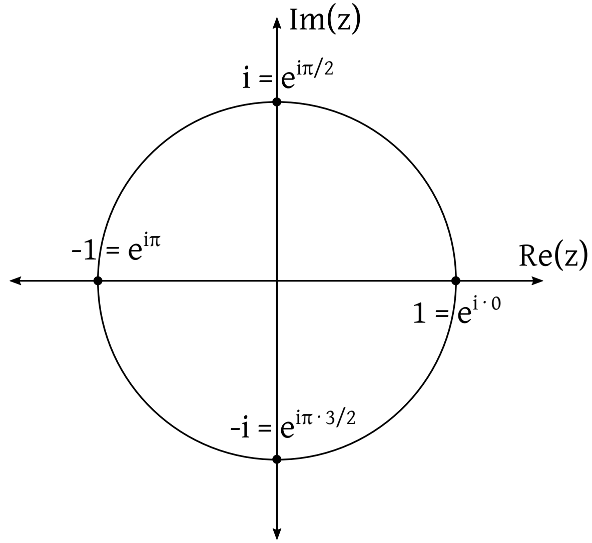
**Remark 3.** We will show: $\forall c \in (0, 2\pi)$, cos and sin are non-periodic with period $c$, hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x+c)$.

**Definition 5.** $f : \mathbb{C} \to \mathbb{C}$ (or $f : \mathbb{R} \to \mathbb{R}$) is called *periodic* with period $c \in \mathbb{C}$ ($c \in \mathbb{R}$) if $\forall z \in \mathbb{C}$ it holds that

$$f(z+c) = f(z)$$
$$(\text{or } \forall x \in \mathbb{R} : f(x+c) = f(x))$$

$c$ is called *period of $f$*.






| $t$ | 0 | $\pi/2$ | $\pi$ | $3\pi/2$ | $\pi/2$ |
|---|---|---|---|---|---|
| $e^{it}$ | 1 | i | -1 | -i | 1 |
| $\cos(t) = Re(e^{it})$ | 1 | 0 | -1 | 0 | 1 |
| $\sin(t) = Im(e^{it})$ | 0 | 1 | 0 | -1 | 0 |

Figure 8: Periodicity of cos and sin on the unit circle in the complex plane

**Remark 4.** If $f$ is periodic with period $c \in \mathbb{C}$, then $f$ is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

**Remark 5.**

$$z = u + iv$$

$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$

$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

**Remark 6.** Let $x \in \mathbb{R}$.

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \Re(e^{i(x+\frac{\pi}{2})}) \\ &= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}}) \\ &= \Re(ie^{ix}) \\ &= -\Im(e^{ix}) \\ &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \Im\left(e^{i(x+\frac{\pi}{2})}\right) \\ &= \Im(ie^{ix}) \\ &= \Re(e^{ix}) \\ &= \cos(x) \end{aligned}$$

$$\begin{aligned} \cos\left(x - \frac{\pi}{2}\right) &= \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin(x) \end{aligned}$$

$$\begin{aligned} \sin\left(x - \frac{\pi}{2}\right) &= -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\cos(x) \end{aligned}$$



Summary:

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x - \frac{\pi}{2}\right) &= \sin(x) \\ \sin\left(x - \frac{\pi}{2}\right) &= -\cos(x) \end{aligned}$$

**Remark 7** (A remark on the name "cosine")**.**

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of $x$ (Compare with Figure 9).

**Remark 8.**

$$\begin{aligned} \cos(x + \pi) &= \Re(e^{i(x+\pi)}) \\ &= \Re(-e^{ix}) \\ &= -\cos(x) \\ \sin(x + \pi) &= -\sin(x) \end{aligned}$$

**Remark 9.** Let $0 < c < 2\pi$. Assume cos is periodic with period $c$. We know that cos has exactly one root in $[0, 2]$,

$$\cos(x) = \cos(-x)$$

cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

   Therefore cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.

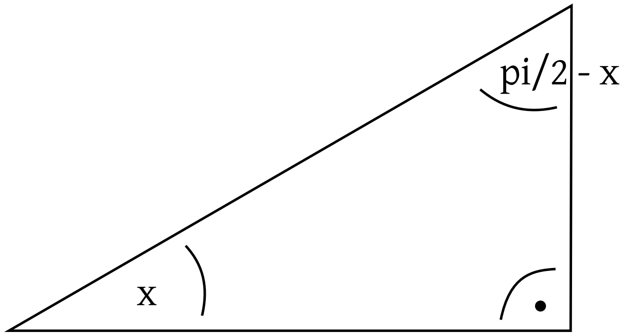



Figure 9: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$



$c$ would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of cos in } (-\frac{\pi}{2}, \frac{\pi}{2})$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period $c$. Hence $2\pi$ is indeed the smallest period of cos.

Analogously it holds for sin.

**Remark 10** (Roots of cos)**.**

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$
$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \qquad \forall l \in \mathbb{Z}$$

## 3.6 Derivatives of trigonometric functions

It holds that

$$\boxed{\lim_{z\to 0} \frac{\sin z}{z} = 1}$$

Furthermore it holds that

$$\boxed{\lim_{z\to 0} \frac{1-\cos z}{z} = 0}$$

*Proof.*

$$\begin{aligned}\frac{1-\cos z}{z} &= \frac{1}{z}\left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots\right) \\ &= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots\end{aligned}$$

is convergent in $\mathbb{C}$ and (especially) continuous in 0

$$\lim_{z\to 0}\left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots\right) = 0$$

□

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

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This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

$$\lim_{z\to 0} \frac{1-\cos z}{z} = 0$$

**Lemma 6.** The trigonometric functions sin and cos are differentiable in $\mathbb{R}$ (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x) \qquad \sin'(x) = \cos(x)$$



*Proof.*

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h) - \cos(h)}{h} &= \lim_{h\to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h} \\ &= \lim_{h\to 0} \cos x \cdot \frac{\cos(h) - 1}{h} - \lim_{h\to 0} \frac{\sin x \cdot \sin h}{h} \\ &= \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{=0} - \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{=1} \\ &= -\sin(x)\end{aligned}$$

Analogously:

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin(h)}{h} &= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} \\ &= \sin(x) \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{=0} + \cos(x) \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{=1} \\ &= \cos(x)\end{aligned}$$

□

Compare with Figure 10. We now use tools of integral calculus:

Let $I = [a,b]$ and $\gamma : I \to \mathbb{R}^n$ $(\mathbb{R}^2)$.

$$\gamma(t) = \begin{bmatrix}\gamma_1(t) \\ \vdots \\ \gamma_n(t)\end{bmatrix}$$

Assumption: $\gamma_1 : [a,b] \to \mathbb{R}^n$ is continuously differentiable.

$$\gamma'(t) = \begin{bmatrix}\gamma_1'(t) \\ \vdots \\ \gamma_n'(t)\end{bmatrix}$$

is the tangential vector in $\gamma$. Compare with Figure 11.

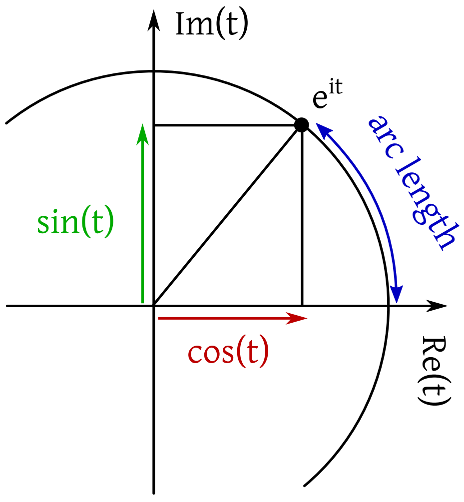


Figure 10: The arc length is related to sin and cos

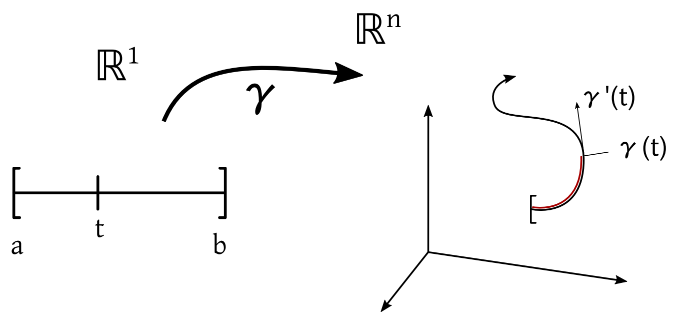


Figure 11: Mapping of the curve and tangential vector $\gamma'(t)$



Let $t \in [a, b]$. Then the arc length of $\gamma$ between $a$ and $t$ is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \, d\tau$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \to \mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

Compare with Figure 12.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \, d\tau = \int_0^t 1 \, d\tau = t$$

# 4 Integral calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses “Functional analysis” and “Measure and integration theory”. For now, we look at the basics (as discussed by Königsberger).

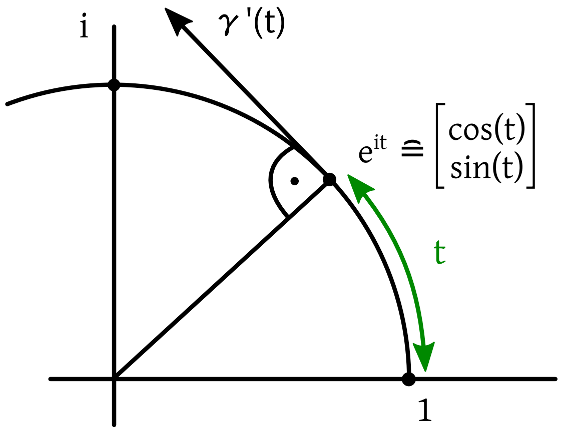


Figure 12: Derivative in $\mathbb{R}^2$

**Definition 6** (Step function)**.** Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \to \mathbb{R}$. We call $\varphi$ a *step function*, if $n \in \mathbb{N}$ and $x_0, \ldots, x_n$ exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1}, x_j)} = c_j$ is constant. The points $x_j$ define a partition of the interval $[a, b]$.

$\tau[a, b]$ defines the set of step functions of interval $[a, b]$. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 13).

**Definition 7** (Integral)**.** Let $\varphi : [a, b] \to \mathbb{R}$ be a step function and $x_0 = a < x_1 < \ldots < x_n = b$ a partition of $[a, b]$ and let $\varphi|_{(x_{j-1}, x_j)} = c_j$ for $j = 1, \ldots, n$. Then we define

$$\int_a^b \varphi \, dx = \sum_{j=1}^{n} c_j \triangle x_j$$



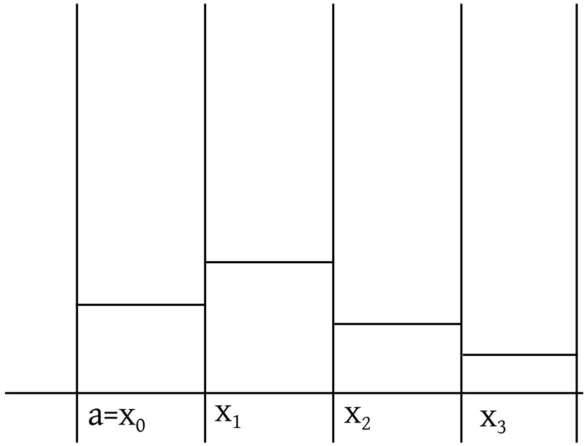


Figure 13: Partition of an area into rectangles

where $\triangle x_j = x_j - x_{j-1}$ (for $j = 1, \ldots, n$).

$$\int_a^b \varphi \, dx \text{ is called } \textit{integral} \text{ of } \varphi \text{ over } [a, b]$$

$\varphi$ is the step function in terms of the partition $\{x_0, x_1, \ldots, x_5\}$, but also a step function in terms of $\{w_0, w_1, \ldots, w_5\}$.

It remains to show that if $\varphi$ satisfies the definition of a step function in terms of partition $\{x_0, \ldots, x_n\}$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$ and $\varphi$ is a step function in terms of $\{w_0, w_1, \ldots, w_m\}$ and $\varphi|_{(w_{l-1}, w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{l=1}^{m} c'_l \triangle w_l$$

Compare with Figure 14.

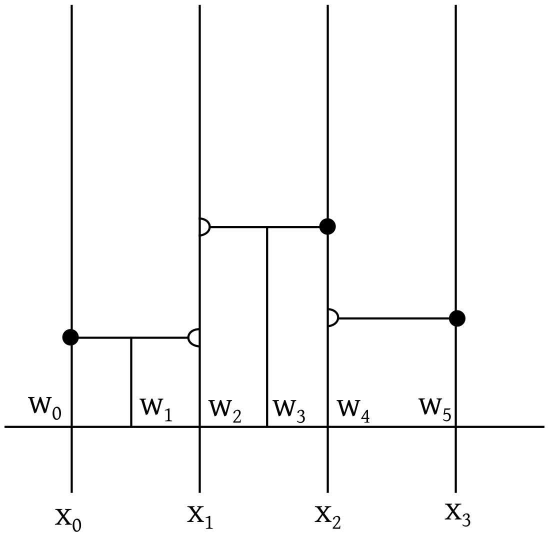


Figure 14: Step function $\varphi$

*Proof.* Let $Z = \{x_0, \dots, x_n\}$ and $Z' = \{w_0, \dots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly, $k_{j-1} < k_j$. Now let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$



So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \dots, L\}$ there exists $j \in \{1, \dots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1}, \alpha_l)} \text{ is constant}$$

Hence $\varphi$ is a step function in terms of the partition $\{\alpha_0, \dots, \alpha_L\}$.

Let $l \in \{0, 1, \dots, L\}$ and $j$ such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c''_l = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c''_l = c_j$.

$$\sum_{l=1}^{L} c''_l \cdot \triangle\alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c''_l \triangle\alpha_l$$

$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = \sum_{j=1}^{n} c_j \triangle x_j$$

Because

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1})$$

$$+(\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2}) + \ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_j - x_{j-1} = \triangle x_j$$

Analogously,

$$\sum_{l=1}^{L} c''_l \cdot \triangle\alpha_l = \sum_{k=1}^{m} c'_k \triangle w_k$$

So it holds that
$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{k=1}^{m} c'_k \triangle w_k$$
□

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This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

**Lemma 7.** Let $\varphi \in \tau[a,b]$ be a step function in terms of partition $a = x_0 < x_1 < \ldots < x_n = b$. Let $a = \alpha_0 < \alpha_1 < \ldots < \alpha_L = b$ with $Z = \{x_0, \ldots, x_n\} \subseteq \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = z'$ ($z'$ has more intervals than $Z'$).

Then also $\varphi$ is step function in terms of partition $z'$.

*Proof.* See above. □

**Lemma 8.** Let $\varphi_1, \varphi_2 \in \tau[a,b]$ and $\alpha, \beta \in \mathbb{C}$.

Then it holds that

- $\alpha\varphi + \beta\psi \in \tau[a,b]$ and
$$\int_a^b (\alpha\varphi + \beta\psi)\,dx = \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$
Hence (“linearity”),
$$\int_a^b : \tau[a,b] \to \mathbb{R} \text{ is linear}$$
- $|\varphi| \in \tau[a,b]$ and it holds that
$$\left|\int_a^b \varphi\,dx\right| \le \int_a^b |\varphi|\,dx \le \|\varphi\|_\infty (b-a)$$
Reminder: $\|\varphi\|_\infty = \max\{|\varphi(x)| : x \in [a,b]\}$
This gives “boundedness”.
- Let $\varphi$ and $\psi$ be real values and it holds that
$$\forall x \in [a,b] : \varphi(x) \le \psi(x)$$



Hence monotonicity is given. Then it holds that
$$\int_a^b \varphi\,dx \le \int_a^b \psi\,dx$$

*Proof.*
- Let $\varphi|_{(x_{k-1},x_k)} = c_k$ $\psi|_{(w_{j-1},w_j)} = d_k$
$$z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\} = \{x_0, \ldots, x_n\} \cup \{w_0, \ldots, w_m\}$$
where $\alpha_i$ is sorted ascendingly. $\varphi$ and $\psi$ are step functions in terms of $z''$, hence
$$\varphi|_{(\alpha_{i-1},\alpha_i)} = c'_i \text{ and } \psi|_{(\alpha_{i-1},\alpha_i)} = d'_i$$
$$\Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1},\alpha_i)} = \alpha c'_i + \beta d'_i \text{ constant}$$
$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a,b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi)\,dx = \sum_{i=1}^{L} (\alpha c'_i + \beta d'_i) \cdot \triangle\alpha_i$$
$$= \alpha \sum_{i=1}^{L} c'_i \triangle\alpha_i + \beta \sum_{i=1}^{L} d'_i \triangle\alpha_i$$
$$= \alpha \int_a^b \varphi\,dx + \beta \int_a^b \psi\,dx$$
- Let $\varphi|_{(x_{i-1},x_i)} = c_i$ $(i = 1, \ldots, n)$. Then,
$$|\varphi|\,|_{(x_{i-1},x_i)} = |c_i|$$
$$\left|\sum_{i=1}^{n} c_i \triangle x_i\right| \le \sum_{i=1}^{n} |c_i| \cdot \underbrace{|\triangle x_i|}_{x_i - x_{i-1} > 0} = \sum_{i=1}^{n} |c_i| \cdot \triangle x_i = \int_a^b |\varphi|\,dx$$
$$\le \sum_{i=1}^{n} \|\varphi\|_\infty \triangle x_i = \|\varphi\|_\infty \sum_{i=1}^{n} \triangle x_i$$
$$= \|\varphi\|_\infty ((x_1 - x_0) + (x_2 - x_1) + \ldots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1}))$$
$$= \|\varphi\|_\infty (x_n - x_0) = \|\varphi\|_\infty (b-a)$$

- Let $\varphi$, $\psi$ and $z''$ as in the linearity statement.

$$\left.\begin{aligned}\varphi|_{(\alpha_{i-1},\alpha_i)} &= c_i' \in \mathbb{R}\\ \psi|_{(\alpha_{i-1},\alpha_i)} &= d_i' \in \mathbb{R}\end{aligned}\right\}$$

$$\int_a^b \varphi\,dx = \sum_{i=1}^L c_i' \underbrace{\triangle\alpha_i}_{>0} \le \sum_{i=1}^L d_i'\,dx$$

$$\int_a^b \varphi\,dx$$

□

**Definition 8.** Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\infty_A) : \mathbb{R} \to \mathbb{R}$ as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

the *characteristic function of* $A$. Hence $\chi_A(x)$ is 1 if and only if $x$ is inside interval $A$.

**Remark 11.** Let $a \le a' < b' \le b$. Then

$$\chi_{(a',b')} \in \tau[a,b] \qquad \int_a^b \chi_{(a',b')}\,dx = 1 \cdot (b' - a')$$

Every linear combination of characteristic functions is also in $\tau[a,b]$.

On the opposite side, let $\varphi \in \tau[a,b]$ with $\varphi|_{(x_{i-1},x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \le i \le n$ and $0 \le j \le n$.

$$\Rightarrow \varphi = \sum_{i=1}^n c_i \chi_{(x_{i-1},x_i)} + \sum_{j=0}^n r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_a^b \varphi\,dx = \sum_{i=1}^n c_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^n c_j \int_a^b \chi_{(x_{i-1},x_i)}\,dx$$



# 5 Regulated functions

**Definition 9.** Let $D \subseteq \mathbb{R}$. Let $x_0$ be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n\in\mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n\to\infty} z_n = x_0$. Let $f : D \to \mathbb{C}$ be given.

We state that $f$ has left-sided limit $y_0$ in $x_0$ if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \cap (-\infty, x_0) \land |x - x_0| < \delta]$$
$$\Rightarrow |f(x) - y_0| < \varepsilon$$

Equivalently $\forall (z_n)_{n\in\mathbb{N}}$ with $z_n \in D$ and $z_n < x_0$ and $\lim_{n\to\infty} z_n = x_0$ $\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} f(x_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by $(x_0, \infty)$.

We denote: $y_0$ is left-sided limit of $f$ in $x_0$:

$$y_0 = \lim_{x\to x_0^-} f(x)$$

and right-sided limit of $f$ in $x_0$:

$$y_0 = \lim_{x\to x_0^+} f(x)$$

**Definition 10.** Let $a, b \in \mathbb{R}$ and $a < b$. A function $f : [a,b] \to \mathbb{C}$ is called *regulated functions* if

- $\forall x \in (a,b)$ $f$ has a left-sided and a right-sided limes in $x$
- $f$ has a right-sided limes in $a$
- $f$ has a left-sided limes in $b$

Examples for regulated functions:

- Every continuous function in $[a,b]$ is a regulated function.

- Every step function is a regulated function.
  Why? Consider $x \in (x_{i-1}, x_i)$. Then

$$\lim_{\xi \to x^+} \varphi(\xi) = c_i = \lim_{\xi \to x^-} \varphi(\xi)$$

  Let $x = x_i$ be a partitioning point.

$$\lim_{\xi \to x_i^-} \varphi(\xi) = c_i \text{ and } \lim_{\xi \to x_i^+} \varphi(\xi) = c_{i+1}$$

  So $\tau[a,b] \subseteq R[a,b]$. Compare with Figure 15.

- Let $f : [a,b] \to \mathbb{R}$ be monotonically. Then it holds that

$$f \in R[a,b]$$

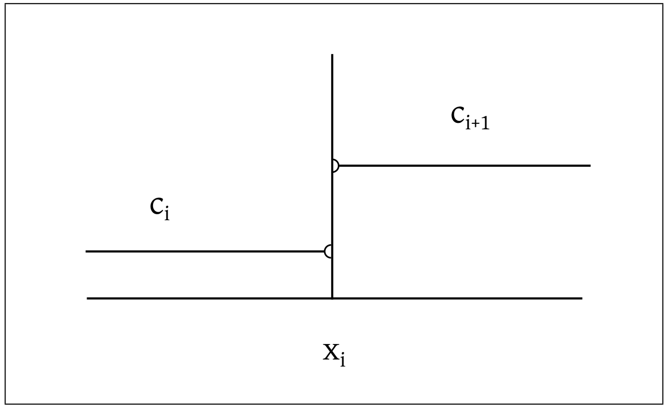


Figure 15: Step functions are also regulated functions

## 5.1 Approximation theorem for regulated functions

Let $f : [a,b] \to \mathbb{C}$. Then it holds that $f \in R[a,b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a,b]$ such that $\|f - \varphi\|_\infty < \varepsilon$. Hence,

$$\forall x \in [a,b] : |f(x) - \varphi(x)| < \varepsilon$$



$$\Leftrightarrow \underbrace{\sup\{|f(x) - \varphi(x)| : x \in [a,b]\}}_{\|f-\varphi\|_\infty} < \varepsilon$$

Especially $\varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a,b]$ such that

$$|\varphi_n(x) - f(x)| < \varepsilon \qquad \forall x \in [a,b]$$

hence $f$ is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges continuously towards $f$.

*Proof.* $\Rightarrow$ Let $f \in R[a,b]$. Assume $\exists \varepsilon > 0$ fixed such that $\forall \varphi \in \tau[a,b]$

$$\exists x \in [a,b] : |\varphi(x) - f(x)| \geq \varepsilon$$

We build nested intervals such that the desired property $|\varphi(x) - f(x)| \geq \varepsilon$ holds on every subinterval $[a_n, b_n]$.

Induction:

$n = 0$ Let $a_0 = a$ and $b_0 = b$, hence the property holds in $[a_0, b_0]$.

$n \mapsto n+1$ Let $m = \frac{1}{2}(a_n + b_n)$. In $[a_n, b_n]$ the property holds.
Then the property either holds in $[a_n, m]$ or $[m, b_n]$. If the property does not hold in $[a_n, m]$:

$$\exists \varphi_1 \in \tau[a_n, m] \text{ with } |\varphi_1(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, m]$$

If the property does not hold in $[m, b_n]$:

$$\exists \varphi_2 \in \tau[m, b_n] \text{ with } |\varphi_2(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [m, b_n]$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in [a_n, m] \\ \varphi_2(x) & \text{for } x \in [m, b_n] \end{cases}$$

$$\Rightarrow \varphi \in \tau[a,b] \text{ and } |\varphi(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1}, b_{n+1}]$.
$([a_n, b_n])_{n\in\mathbb{N}}$ are nested intervals. Let $\varphi \in \bigcap_{n\in\mathbb{N}} [a_n, b_n]$.

**Case $\xi \in (\mathbf{a}, \mathbf{b})$** Let $\varepsilon$ satisfy the desired property. $f \in R[a,b]$, hence $f$ has left-sided limit $c_-$ in $\xi$ and right-sided limit $c_+$. Hence $\exists \delta > 0$ such that

- $|x - \xi| < \delta \wedge a \leq x < \xi \Rightarrow |f(x) - c_-| < \varepsilon$
- $|x - \xi| < \delta \wedge \delta < x \leq b \Rightarrow |f(x) - c_+| < \varepsilon$

Choose $\delta$ sufficiently small such that

$$a < \xi - \delta < \xi + \delta < b$$

Let

$$\varphi(x) = \begin{cases} c_- & \text{for } x \in (\xi - \delta, \xi) \\ f(\xi) & \text{for } x = \xi \\ c_+ & \text{for } x \in (\xi, \xi + \delta) \end{cases}$$

$\varphi$ is necessarily a step function in $(\xi - \delta, \xi + \delta)$ and it holds that $\forall x \in (\xi - \delta, \xi + \delta) : |\varphi(x) - f(x)| < \varepsilon$.
Let $n$ be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi - \delta, \xi + \delta)$$

then

$$\varphi|_{[a_n,b_n]} \in \tau[a_n, b_n] \text{ and } |\varphi(x) - f(x)| < \varepsilon \quad \forall x \in [a_n, b_n]$$

This is a contradiction to our desired property.

For $\xi = a$ or $\xi = b$ only with one-sided limit.

□

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This lecture took place on 17th of March 2016 with lecturer Wolfgang Ring.

We learned: All regulated functions can be approximated with step functions.

$f \in R[a,b]$ in the proof $\Leftrightarrow$ $f$ is uniform limit of step functions. We have prove direction $\Rightarrow$.

**Lemma 9** (Cauchy criterion for limits of functions)**.** Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ and $z_0$ is a limit point of $D$. Then $f$ has a limit in $z_0$ if and only if $\forall \varepsilon > 0 \exists \delta > 0 : v, w \in D \setminus \{z_0\} \wedge |v - z_0| < \delta \wedge |w - z_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$.

If $D \subseteq \mathbb{R}$ and $x_0$ is limit point of $D \cap (x_0, \infty)$, then $f$ has a *right-sided limit* in $x_0$ if and only if $\forall \varepsilon > 0 \exists \delta > 0 : [v, w \in D \cap (x_0, \infty) \wedge |v - x_0| < \delta \wedge |w - x_0| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon]$.

Analogously for left-sided limit.

*Proof.* This proof is done only for the first point.

$\Rightarrow$

Assume $f$ has a limit $\eta$ in $z_0$. Choose $\delta$ such that $v, w \in D$ with $|v - z_0| < \delta$ and $|w - z_0| < \delta$ implies that $|f(v) - \eta| < \frac{\varepsilon}{2}$ and $|f(w) - \eta| < \frac{\varepsilon}{2}$. Then $|f(v) - f(w)| \leq |f(v) - \eta| + |\eta - f(w)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Leftarrow$

Assume the Cauchy criterion holds. Show: There exists $\eta \in \mathbb{C}$ such that for every sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in D \setminus \{z_0\}$ with $\lim_{n\to\infty} w_n = z_0$ it holds that $\lim_{n\to\infty} f(w_n) = \eta$.

Let $(w_n)_{n\in\mathbb{N}}$ be as above. Show: $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and $\delta$ as above. Choose $N \in \mathbb{N}$ such that $n, m \geq N$

$$\Rightarrow |w_n - z_0| < \delta \wedge |w_m - z_0| < \delta$$

The Cauchy criterion holds for $n, m \geq N$:

$$|f(w_n) - f(w_m)| < \varepsilon$$

So $(f(w_n))_{n\in\mathbb{N}}$ is a Cauchy sequence and (because $\mathbb{C}$ is complete) is also convergent. So $\exists \eta' \in \mathbb{C} : \lim_{n\to\infty} f(w_n) = \eta'$.

It remains to show: $\eta'$ is unique.

Let $(v_n)_{n\in\mathbb{N}}$ be another sequence with $\lim_{n\to\infty} v_n = z_0$ and $v_n \in D \setminus \{z_0\}$. As above: $\exists \eta'' \in \mathbb{C}$ such that $\lim_{n\to\infty} f(v_n) = \eta''$.

We construct:

$$(\xi_n)_{n\in\mathbb{N}} = (w_0, v_0, w_1, v_1, w_2, v_2, \ldots)$$

Then it holds that $\lim_{n\to\infty} \xi_n = z_0$.

We use the argument from above: $(f(\xi_n))_{n\in\mathbb{N}}$ is convergent, hence $\lim_{n\to\infty} f(\xi_n) = \eta$. Both subsequences $(f(w_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ must have the same limit, hence $\eta' = \eta = \eta''$.

□

*Proof of approximation theorem.* ⇐
Let $f = \lim_{n\to\infty} \varphi_n$ be uniform on $[a,b]$. Let $\varphi_n \in \tau[a,b]$ and let $x_0 \in [a,b)$. Show: $f$ has a right-sided limit in $x_0$. Let $\varepsilon > 0$ arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that

$$|f(x) - \varphi_N(x)| < \frac{\varepsilon}{2} \forall x \in [a,b]$$

$\varphi_N$ is a step function (hence interval-wise constant). Choose $\delta > 0$ such that $\varphi_N|_{(x_0,x_0+\delta)} = c$ constant. Let $v, w \in (x_0, x_0+\delta)$. Then it holds that

$$|f(v) - f(w)| \leq |f(v) - c| + |c - f(w)|$$
$$= |f(v) - \varphi_N(v)| + |f(w) - \varphi_N(w)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

The Cauchy criterion implies that $f$ has a right-sided limit in $x_0$. □

**Corollary 2.** $f \in R[a,b]$ if and only if $f(x) = \sum_{j=0}^{\infty} \psi_j(x)$ with $\psi_j \in \tau[a,b]$ and the series converges uniformly in $[a,b]$.

*Proof.* ⇐
Let $\varphi_n = \sum_{j=0}^{n} \psi_j \in \tau[a,b]$ and $\varphi_n \to f$ continuously in $[a,b]$. From the approximation theorem it follows that $f \in R[a,b]$.

⇒
Let $f \in R[a,b]$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to f$ uniform in $[a,b]$. Let $\psi_0 = \varphi_0$.

$$\psi_j := \varphi_j - \varphi_{j-1} \text{ for } j \geq 1$$

Then it holds that

$$\sum_{j=0}^{n} \psi_j = \varphi_0 + (\varphi_1 - \varphi_0) + (\varphi_2 - \varphi_1) + \ldots + (\varphi_{n-1} - \varphi_{n-2}) + (\varphi_n - \varphi_{n-1}) = \varphi_n$$

and $(\varphi_n)_{n\in\mathbb{N}}$ converges uniform if and only if the series is uniformly convergent. □

**Lemma 10** (Sidenote)**.** Let $(f_n)_{n\in\mathbb{N}}$ with $f_n : D \to \mathbb{C}$ a sequence of functions in $D$, let $z_0 \in D$ and $\forall n \in \mathbb{N}$ $f_n$ is continuous in $z_0$. Furthermore let $f : D \to \mathbb{C}$ and $f_n \to f$ is uniform in $D$. Then $f$ is continuous in $z_0$.

*Proof.* Let $\varepsilon > 0$ arbitrary. Choose $N$ sufficiently large such that $|f(z) - f_w(z)| < \frac{\varepsilon}{3}$ $\forall z \in D$ (uniform convergence). Because $f_N$ is continuous in $z_0$, $\exists \delta > 0$ such that $z \in D$ and $|z - z_0| < \delta$ then $|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3}$.

Then for $|z - z_0| < \delta$ (with $z \in D$)

$$\underbrace{|f(z) - f(z_0)|}_{<\frac{\varepsilon}{3}} \leq \underbrace{|f(z) - f_N(z)| + |f_N(z) - f_N(z_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_N(z_0) - f(z_0)|}_{<\frac{\varepsilon}{3}}$$

□

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This lecture took place on 18th of March 2016 with lecturer Wolfgang Ring.

**Theorem 9.** Let $f$ be a regulated function in $[a,b]$. Then $f$ is in at most countable infinite points of $[a,b]$ non-continuous.

*Proof.*

$$f = \sum_{k=0}^{\infty} \psi_k$$

where $\psi_k$ is a sequence of step functions and and the series is uniformly convergent. $\psi_k \in \tau[a,b]$.

Let $\{x_0^k, \ldots, x_{n(k)}^k\}$ be the partition points of $\psi_k$. Then $\psi_k$ is continuous in $[a,b] \setminus Z_k$. Let $Z = \bigcup_{k=0}^{\infty} Z_k$ be countable. Let $x \in [a,b] \setminus Z$ and $\varphi_n = \sum_{k=0}^{n} \psi_k$. Then it holds that $\varphi_n \to f$ is uniform in $[a,b]$ and $\varphi_n$ is continuous in $x$, because $x \notin Z$.

From Lemma 10 it follows that $f$ is continuous in $x$. □

## 5.2 Norms and vector spaces

**Definition 11** (Normed vector spaces)**.** Let $V$ be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). A map $n : V \mapsto [0,\infty)$ is called *norm* in $V$, if

**Definite form** $n(V) = 0 \Leftrightarrow V = 0$ ($V$ is null vector)

**Positive homogeneity** $\forall \lambda \in \mathbb{C}\ (\mathbb{R})\ \forall v \in V : n(\lambda v) = |\lambda| \cdot n(v)$

**Triangle inequality** $\forall v, w \in V : n(v+w) \leq n(v) + n(w)$

Common notation: $\|v\|$ for $n(v)$ ("norm of $v$")
A vector space satisfying the norm properties is called *normed vector space.*

**Example 2.**
- $|x|$ is a norm in $\mathbb{R}$.
- $|z|$ is a norm in $\mathbb{C}$.

$\|\vec{x}\|$ is norm in $\mathbb{R}^n$.

Let $D \subseteq \mathbb{C}$.

$$B(D) = \{f : D \to \mathbb{C} : f \text{ limited to } D\}$$

$B(D)$ is a vector space. For $f \in B(D)$ we define:

$$\|f\|_\infty = \sup\{|f(z)| : z \in D\}$$

"supremum norm" of $\infty$-norm of $f$ in $D$.

It holds that $\|\cdot\|_\infty$ is a norm in $B(D)$.

$$\begin{aligned}\|f\|_\infty = 0 &\iff \sup\{\underbrace{|f(z)|}_{\geq 0} : z \in D\} = 0\\ &\iff |f(z)| = 0 \quad \forall z \in D\\ &\implies f = 0 \text{ in } B(D)\end{aligned}$$

Homogeneity:

$$\begin{aligned}|\lambda \cdot f|_\infty &= \sup\{|\lambda f(z)| : z \in D\}\\ &= \sup\{|\lambda|\,|f(z)| : z \in D\}\\ &= |\lambda| \cdot \sup\{|f(z)| : z \in D\}\\ &= |\lambda| \cdot \|f\|_\infty\end{aligned}$$



Triangle inequality: Let $f, g \in B(D)$.

$$\begin{aligned}\|f+g\|_\infty &= \sup\{|f(z)+g(z)| : z \in D\}\\ &\leq \sup\left\{\underbrace{|f(z)|}_{\leq\|f\|_\infty} + \underbrace{|g(z)|}_{\leq\|g\|_\infty} : z \in D\right\}\\ &\leq \sup\{\|f\|_\infty + \|g\|_\infty : z \in D\}\\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

**Remark 12.** Let $V \subseteq B(D)$ be an arbitrary subvectorspace of $B(D)$. So $\|\cdot\|_\infty$ is also a norm in $V$.

Important example:

$$V = \mathcal{C}_b(D) = \{f : D \to \mathbb{C} : f \text{ is continuous and bounded in } D\}$$

Special case: $D = K$ compact in $\mathbb{C}$. Then every continuous function is also bounded.

$$\begin{aligned}\mathcal{C}(K) &= \{f : K \to \mathbb{C} : f \text{ is continuous}\}\\ &\subseteq B(K) \qquad \text{(sub vector space)}\end{aligned}$$

Another special case: $D = [a, b] \subseteq \mathbb{C}$

$$\tau[a, b] \subseteq B([a, b]) \text{ and}$$

$$R[a, b] \subseteq B([a, b])$$

**Remark 13** (Further properties of the norm)**.** The inverse triangle inequality holds:

$$\forall v, w \in V : |\|v\| - \|w\|| \leq \|v - w\|$$

*Proof.*

$$v = (v - w) + w$$

From triangle inequality it follows that

$$\|v\| \leq \|v - w\| + \|w\|$$

$$w = (w - v) + w$$
$$\|w\| \leq \|w - v\| + \|w\|$$

$$= \|(-1) \cdot (v - w)\| + \|v\|$$
$$= |(-1)| \cdot \|v - w\| + \|v\|$$
$$= \|v - w\| + \|v\|$$

$$\text{requirement 1} \Rightarrow \|v\| - \|w\| \leq \|v - w\|$$
$$\text{requirement 2} \Rightarrow \|w\| - \|v\| \leq \|v - w\|$$
$$\text{requirement 1 and 2} \Rightarrow |\|v\| - \|w\|| \leq \|v - w\|$$

□

**Definition 12.** Let $V$ be a normed vector space, $(v_n)_{n\in\mathbb{N}}$ be a sequence of elements in $V$ and $v \in V$. We define $(v_n)_{n\in\mathbb{N}}$ is convergent with limit $V$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq \mathbb{N} \Rightarrow \|V_n - V\| \leq \varepsilon]$$

**Remark 14** (Metric on $V$)**.**

$$d(v, w) = \|v - w\|$$

defines a metric on $V$. Properties of a metric:

1. $d(v, w) \geq 0$
2. $d(v, w) = 0 \Leftrightarrow v = w$
3. $\|v - w\| = 0 \Leftrightarrow v - w = 0 \Leftrightarrow v = w$

Triangle inequality of metrics: Let $v, w, u \in V$.

$$d(v, u) = \|v - u\| = \|v - w + w - u\|$$

$$\leq \|v - w\| + \|w - u\| = d(v, w) + d(w, u)$$

Works only if $d(v, w) = d(w, v)$ and can be simply proven:

$$d(v, w) = \|v - w\| = \|w - v\| = d(w, v)$$



**Remark 15.** $(V_n)_{n\in\mathbb{N}}$ is called *Cauchy sequence in* $V$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|v_n - v_m\| < \varepsilon]$$

$V$ is called *complete normed vector space* if every Cauchy sequence in $V$ is also a convergent sequence in $V$.

A complete normed vector space is called *Banach space.*

## 5.3 Integration of regulated functions

**Theorem 10.** Let $f \in \mathcal{B}[a, b]$ and $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n \in \tau[a, b]$ and $\varphi_n \to_{n\to\infty} f$ uniform in $[a, b]$ ($\Leftrightarrow \|\varphi_n - f\| \to 0$ for $n \to \infty$).

Then we define

$$\int_a^b f\, dx = \lim_{n\to\infty} \int_a^b \varphi_n\, dx$$

for the integral of $f$ in $[a, b]$. The right-sided limit exists for every sequence $(\varphi)_{n\in\mathbb{N}}$ with the property above and is independent of the choice of the sequence $(\varphi_n)_{n\in\mathbb{N}}$.

*Proof.* Let $(\varphi_n)_{n\in\mathbb{N}}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \underbrace{[n \geq N \Rightarrow |\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]]}_{\sup\{|\varphi_n(x) - f(x)| : x \in [a,b] \leq \varepsilon\}}$$

$$\Rightarrow \|\varphi_n - f\|_\infty \leq \varepsilon$$

So $\varphi_n$ converges towards $f$ in terms of $\|\cdot\|_\infty$ in $\mathcal{B}[a, b]$.

Let $N$ be sufficiently large such that

$$\forall n \geq N : \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(b - a)}$$

Then it holds for $i_n = \inf_a^b \varphi_n \, dx$ and $n, m \geq N$,

$$\begin{aligned}|i_n - i_m| &= \left| \int_a^b \varphi_n \, dx - \int_a^b \varphi_m \, dx \right| \\ &= \left| \int_a^b (\varphi_n - \varphi_m) \, dx \right| \\ &\leq \|\varphi_n - \varphi_m\|_\infty (b-a) \\ &= \|\varphi_n - f + f - \varphi_m\|_\infty (b-a) \\ &\leq (\|\varphi_n - f\|_\infty + \|f - \varphi_m\|_\infty)(b-a) \\ &< \left( \frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} \right)(b-a) \\ &= \varepsilon\end{aligned}$$

So $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ and therefore convergent. So there exists

$$\lim_{n\to\infty} \int_a^b \varphi_n \, dx$$

Let $i = \lim_{n\to\infty} \int_a^b \varphi_n \, dx$. Let $(\psi_n)_{n\in\mathbb{N}}$ be another sequence of step functions with $\psi_n \to_{n\to\infty} f$ is uniform in $[a,b]$. Analogously as above:

$$j_n = \int_a^b \psi_n \, dx$$

$(j_n)_{n\in\mathbb{N}}$ is convergent and has limes $j$.

Show that $i = j$. We again use a zip-like construction:

$$F = (\varphi_0, \psi_0, \varphi_1, \psi_1, \varphi_2, \ldots)$$

$F$ is a sequence of step functions, which converge towards $f$ uniformly. Let $l$ be the limit of integrals of this sequence of step functions. Then it holds that (subsequences have the same limit)

$$i = l = j$$

□



**Theorem 11** (Elementary properties of the integral)**.** Let $f, g \in \mathcal{B}[a,b]$ and $\alpha, \beta \in \mathbb{C}$. Then it holds that

**linearity**

$$\int_a^b (\alpha f + \beta g) \, dx = \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx$$

**boundedness**

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx \leq \|f\|_\infty (b-a)$$

**monotonicity** Let $f, g \in \mathcal{B}[a,b]$ with values in $\mathbb{R}$ and it holds that

$$f(x) \leq g(x) \qquad \forall x \in [a,b]$$

Then it holds that

$$\int_a^b f \, dx \leq \int_a^b g \, dx$$

*Proof.*
- Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be sequences of step functions with $\varphi_n \to f$ and $\psi_n \to g$ uniform in $[a,b]$. Then it holds that

$$\alpha\varphi_n + \beta\psi_n \to_{n\to\infty} \alpha f + \beta g$$

(proof left as exercise to the reader)
uniform in $[a,b]$. So it holds that

$$\begin{aligned}\int_a^b (\alpha f + \beta g) \, dx &= \lim_{n\to\infty} \int_a^b (\alpha\varphi_n + \beta\psi_n) \, dx \\ &= \alpha \lim_{n\to\infty} \int_a^b \varphi_n \, dx + \beta \lim_{n\to\infty} \int_a^b \varphi_n \, dx \\ &= \alpha \int_a^b f \, dx + \beta \int_a^b g \, dx\end{aligned}$$

- Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of step functions with $\varphi_n \to_{n\to\infty} f$ continuous in $[a,b]$. Then also $(|\varphi_n|)_{n\in\mathbb{N}}$ is a sequence of step functions and it holds that

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a,b]$$

*Proof.* Let $N$ be sufficiently large such that $\forall n \geq N \forall x \in [a,b]$ :

$$|\varphi_n(x) - f(x)| < \varepsilon \Rightarrow ||\varphi_n(x)| - |f(x)|| \leq |\varphi_n(x) - f(x)| < \varepsilon$$

$$|\varphi_n| \to_{n\to\infty} |f| \text{ uniform in } [a,b]$$

So it holds that

$$\left|\int_a^b f\,dx\right| = \left|\lim_{n\to\infty}\int_a^b \varphi_n\,dx\right| = \lim_{n\to\infty}\left|\int_a^b \varphi_n\,dx\right|$$

$$\leq \lim_{n\to\infty}\int_a^b |\varphi_n|\,dx = \int_a^b |f|\,dx$$

Because $|f - \varphi_n|_\infty \to_{n\to\infty} 0$ it follows that

$$|\|f\|_\infty - \|\varphi_n\|_\infty| \leq \|f - \varphi_n\|_\infty \to 0$$

hence $\|f\|_\infty = \lim_{n\to\infty} \|\varphi_n\|_\infty$. □

Hence,

$$\begin{aligned}\int_a^b |f|\,dx &= \lim_{n\to\infty}\int_a^b |\varphi_n|\,dx \\ &\leq \lim_{n\to\infty} \|\varphi_n\|_\infty (b-a) \\ &= \|f\|_\infty (b-a)\end{aligned}$$

**Remark 16.** We have proven that $\|\cdot\| : V \to [0,\infty)$ is a continuous map, hence $v_n \to v \Rightarrow \|v_n\| \to \|v\|$.

□

This lecture took place on 12th of April 2016 with lecturer Wolfgang Ring.

**Definition 13.** Let $f : [a,b] \to \mathbb{R}$ be given. Let $x_0 \in [a,b)$. We claim that $f$ has a right-sided derivative $f'_+(x_0)$ in $x_0$ if the function

$$\varphi(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{for } x \neq x_0 \\ 0 & \text{for } x = x_0 \end{cases}$$

has a right-sided limit in $x_0$. Then $f$ is denoted with $f'_+(x_0)$.

$$f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$$

Analogously for the left-sided derivative: Let $x_0 \in (a,b]$. $f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ if the limit exists.

**Theorem 12** (Intermediate value theorem of calculus)**.** Let $f : [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and $p : [a,b] \to \mathbb{R}$ is a regulated function with $p(x) \geq 0 \quad \forall x \in [a,b]$.

Then there exists $\xi \in [a,b]$ such that

$$\int_a^b f(x)\cdot p(x)\,dx = f(\xi)\cdot\int_a^b p(x)\,dx$$

*Proof.* Let $M = \max\{f(x) : x \in [a,b]\}$ and $m = \min\{f(x) : x \in [a,b]\}$

$$mp(x) \leq f(x)\underbrace{p(x)}_{\geq 0} \leq Mp(x) \qquad \forall x \in [a,b]$$

Due to monotonicity of the integral it holds that

$$m\int_a^b p(x)\,dx \leq \int_a^b f(x)p(x)\,dx \leq M\int_a^b p(x)\,dx$$

hence $\exists \eta \in [m,M]$ such that $\eta \cdot \int_a^b p(x)\,dx = \int_a^b f(x)p(x)\,dx$. From the Intermediate Value Theorem it follows that $\exists \xi \in [a,b] : \eta = f(\xi)$.

$$\Rightarrow f(\xi) : \int_a^b p(x)\,dx = \int_a^b f(x)p(x)\,dx$$

□

**Remark 17.** Consider $p \equiv 1$.

$$\exists \xi \in [a,b] : \int_a^b f(x)\cdot 1\,dx = f(\xi)\cdot\int_a^b 1\,dx = f(\xi)\cdot(b-a)$$

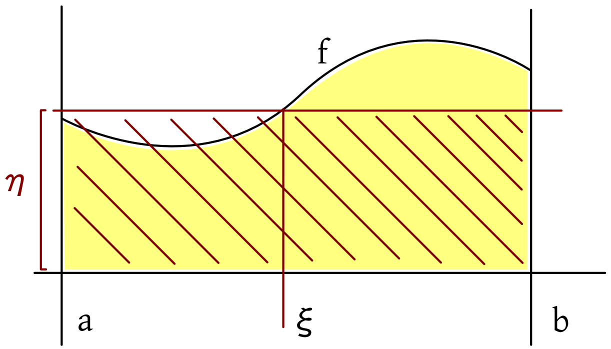


Figure 16: Intermediate value theorem where the area below the curve is given as $I = \int_a^b f(x)\,dx = \eta \cdot (b-a) = f(\xi)(b-a)$.

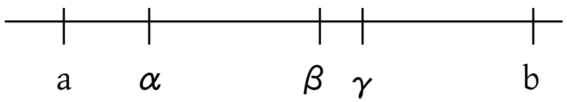


Figure 17: Relation of $a \leq \alpha < \beta < \gamma \leq b$

**Lemma 11.** Let $I = [a,b]$ and $f \in R[a,b]$ and $a \leq \alpha < \beta < \gamma \leq b$ (compare with Figure 17). Then $f|_{[\alpha,\gamma]} \in R[\alpha,\gamma]$.

Furthermore it holds that

$$\int_\alpha^\beta f(x)\,dx = \int_\alpha^\beta f(x)\,dx + \int_\beta^\gamma f(x)\,dx$$

*Proof.* Let $\varphi$ be a step function in $[\alpha,\gamma]$. Then $\varphi|_{[\alpha,\beta]} \in \tau[\alpha,\beta]$ and $\varphi|_{[\beta,\gamma]} \in$



$\tau[\beta,\gamma]$. Furthermore it holds (proof not given here)

$$\int_\alpha^\gamma \varphi\,dx = \int_\alpha^\beta \varphi\,dx = \int_\beta^\gamma \varphi\,dx$$

For $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of subsequences with $\varphi_n \to f$ continuous in $[\alpha,\gamma]$.

$$\Rightarrow \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]} \text{ uniform in } [\alpha,\beta]$$

analogously for $[\beta,\gamma]$.

$$\int_\alpha^\gamma f dx = \lim_{n\to\infty} \int_\alpha^\gamma \varphi_n\,dx = \lim_{n\to\infty} \left[\int_\alpha^\beta \varphi_n\,dx + \int_\beta^\gamma \varphi_n\,dx\right]$$

$$= \underbrace{\lim_{n\to\infty} \int_\alpha^\beta \varphi_n\,dx}_{=\int_\alpha^\beta f\,dx} + \underbrace{\lim_{n\to\infty} \int_\beta^\gamma \varphi_n\,dx}_{=\int_\beta^\gamma f\,dx}$$

□

**Remark 18.** Notation ($\alpha, \beta \in [a,b]$):

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx$$

So it follows that

$$\int_\alpha^\alpha f(x)\,dx = -\int_\alpha^\alpha f(x)\,dx = 0$$

With this notation it holds that $\forall \alpha, \beta, \gamma \in I$:

$$\int_\alpha^\gamma f\,dx = \int_\alpha^\beta f\,dx + \int_\beta^\gamma f(x)\,dx$$

independent of the relation of $\alpha, \beta, \gamma$ towards each other. For $\alpha < \beta < \gamma$ everything is fine.

Let's also look at $\beta < \gamma < \alpha$ as an exercise.

Then it holds that

$$\int_\beta^\alpha f\,dx = \int_\beta^\gamma f\,dx + \int_\gamma^\alpha f\,dx$$

$$-\int_\alpha^\beta f\,dx = \int_\beta^\gamma f\,dx - \int_\alpha^\gamma f\,dx$$

$$\Rightarrow \int_\alpha^\gamma f\,dx = \int_\alpha^\beta f\,dx + \int_\beta^\gamma f\,dx$$

Case $\alpha = \beta$ or $\beta = \gamma$ is trivial.

**Theorem 13** (Fundamental theorem of Calculus)**.** Originally formulated by Isaac Barrow (1630–1677). Followingly popularized by Newton (1642–1727) and Leibniz (1646–1716).

Let $f : I \to \mathbb{R}$ be a regulated function. $I$ is an interval and $a \in I$ is fixed. For $x \in I$ we define

$$F(x) = \int_a^x f(\xi)\,d\xi$$

Then it holds that (two variants/characterizations)

1. $F$ is right-sided derivable and also left-sided derivable for every $x_0 \in I$ and it holds that

$$F'_+(x) = f_+(x_0) = \lim_{x \to x_0^+} f(x) \qquad \text{and}$$

$$F'_-(x) = f_-(x_0) = \lim_{x \to x_0^-} f(x)$$

   Especially if $f$ is continuous in $x_0$, then $F$ is differentiable in $x_0$ with derivative $F'(x_0) = f(x_0)$.

   We call a function with the properties of $F$ above a *primitive function* of the regulated function $f$.

2. Let $\Phi : I \to \mathbb{R}$ be an arbitrary primitive function of $f$ and let $a, b \in I$. Then it holds that

$$\int_a^b f(\xi)\,d\xi = \Phi(b) - \Phi(a)$$

The first characterization claims that (informally speaking) the derivative for the upper limit of the integral of $f$ gives $f$.



Let $f = \Phi'$ ($\Phi$ is our primitive function of $f$). The second characterization claims that the integral of a derivative of $\Phi$ gives $\Phi$.

$$\int_a^b \Phi'\,dx = \Phi(b) - \Phi(a)$$

*Proof.* 1. Let $x_1, x_2 \in I$ and wlog $x_1 \leq x_2$.

$$\begin{aligned}
|F(x_1) - F(x_2)| &= \left|\int_a^{x_1} f(\xi)\,d\xi - \int_a^{x_2} f(\xi)\,d\xi\right| \\
&= \left|\int_a^{x_1} f(\xi)\,d\xi + \int_{x_2}^{a} f(\xi)\,d\xi\right| \\
&= \left|\int_{x_2}^{x_1} f(\xi)\,d\xi\right| = \left|\int_{x_1}^{x_2} f(\xi)\,d\xi\right| \\
&\leq \int_{x_1}^{x_2} |f(\xi)|\,d\xi \leq |x_2 - x_1| \cdot \|f\|_\infty
\end{aligned}$$

hence $F$ is Lipschitz continuous in $I$. So $F$ is continuous in $I$.

One-sided limits:

Let $\varepsilon > 0$ arbitrary and $x_0 \in I$ and $\delta$ such that $\forall x \in (x_0, x_0 + \delta)$ it holds that:

$$|f(x) - f_+(x_0)| < \varepsilon$$

$$\begin{aligned}
&\left|\frac{F(x) - F(x_0)}{x - x_0} - f_+(x_0)\right| \\
&= \frac{1}{|x - x_0|}\left|\int_a^x f(\xi)\,d\xi - \int_a^{x_0} f(\xi)\,d\xi - f_+(x_0) \cdot (x - x_0)\right| \\
&= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\,d\xi - f_+(x_0)\int_{x_0}^x 1 d\xi\right| \\
&= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi)\,d\xi - \int_{x_0}^x f_+(x_0)\,d\xi\right| \\
&= \frac{1}{|x - x_0|}\left|\int_{x_0}^x f(\xi) - f_+(x_0)\,d\xi\right| \\
&\leq \frac{1}{|x - x_0|}\int_{x_0}^x |f(\xi) - f_+(x_0)|\,d\xi
\end{aligned}$$

$\xi \in (x_0, x) \subseteq (x_0, x_0 + \delta)$

$$< \frac{1}{|x - x_0|} \cdot \varepsilon \underbrace{\int_{x_0}^{x} 1 \, d\xi}_{|x-x_0|}$$
$$= \varepsilon$$
$$\Rightarrow F'_+(x_0) = f_+(x_0)$$

Analogously $F'_-(x_0) = f_-(x_0)$.

□

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This lecture took place on 14th of April 2016 with lecturer Wolfgang Ring.

**Theorem 14** (Addition: Lipschitz continuity of differentiable functions)**.** Let $I = [a, b]$, $f : I \to \mathbb{R}$ and $f$ is continuous in $I$. Let $A \subseteq I$. Let $A$ be countable and $f$ is differentiable in $I \setminus A$ and $\exists L > 0 : |f'(x)| \leq L \quad \forall x \in I \setminus A$.

Then it holds that $\forall x_1, x_2 \in I$:

$$|f(x_1) - f(x_2)| \leq L \, |x_1 - x_2|$$

*Proof.* Without loss of generality, $x_1 < x_2$. Let $\varepsilon > 0$, define $F_\varepsilon : I \to \mathbb{R}$

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L + \varepsilon)(x - x_1)$$

Show $F_\varepsilon(x_2) \leq 0$.

Assume there is some $\varepsilon' > 0$ with $F_{\varepsilon'}(x_2) > 0$. It holds that

- $F_{\varepsilon'}(A) \subseteq \mathbb{R}$ is countable
- $0 = F_{\varepsilon'}(x_1) < F_{\varepsilon'}(x_2)$. Because $F_{\varepsilon'}$ is continuous (by Intermediate Value Theorem, $[0, F_{\varepsilon'}(x_2)] \subseteq F_{\varepsilon'}([x_1, x_2])$) and $[0, F_{\varepsilon'}(x_2)]$ contains overcountably many points, $F_{\varepsilon'}(A)$ is countable.

$$\Rightarrow \exists \gamma : 0 < \gamma < F_{\varepsilon'}(x_2)$$

and

$$\gamma \in F_{\varepsilon'}([x_1, x_2] \setminus A)$$



Let $\underbrace{F_{\varepsilon'}^{-1}(\{y\})}_{B} \cap [x_1, x_2] = \{x \in [x_1, x_2] \,|\, F_{\varepsilon'}(x) = y\}$.

$B$ is bounded. Let $c = \sup B$. Let $(\xi_n)_{n \in \mathbb{N}}$, $\xi_n \in B$ with $\lim_{n\to\infty} \xi_n = c$. Then it holds that $c \in [x_1, x_2]$ and $F_{\varepsilon'}(\xi_n) = y \xRightarrow{\text{continuity of } F_{\varepsilon'}} \underbrace{\lim_{n\to\infty} F_{\varepsilon'}(\xi_n)}_{y} = F_{\varepsilon'}(c)$.

Therefore $c = \max B = \max \{x \in [x_1, x_2] : F_{\varepsilon'}(x) = y\}$. Because $F_{\varepsilon'}(x_2) > y$ and $F_{\varepsilon'}(x_1) = 0 < \gamma$, it holds that $x_1 < c < x_2$.

Consider $x \in (c, x_2]$ and let $\varphi(x) := \frac{F_{\varepsilon'}(x) - F_{\varepsilon'}(c)}{x - c}$. Furthermore $F_{\varepsilon'}(x) > \gamma = F_{\varepsilon'}(c)$ for $x \in (c, x_2]$. Because if we define $F_{\varepsilon'}(x) < \gamma$, then (due to Intermediate Value Theorem) $\exists \xi \in (x, x_2)$ with $F_{\varepsilon'}(\xi) = \gamma$, so $\exists \xi \in B$ which would be a contradiction to $c = \max B$.

$$\begin{aligned}
\varphi(x) &= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - x_1 - c + x_1)}{x - c} \\
&= \frac{|f(x) - f(x_1)| - |f(c) - f(x_1)| - (L + \varepsilon')(x - c)}{x - c} \\
&\overset{\text{inv. triangle ineq.}}{\leq} \frac{|f(x) - f(c)|}{x - c} - (L + \varepsilon')
\end{aligned}$$

Now as far as $c \notin A$ holds, $f$ is differentiable in $c$ and it holds that $|f'(c)| \leq L$, hence there exists an interval $(c, d)$, $d < x_2$ and $d > c$, such that

$$\frac{|f(x) - f(c)|}{x - c} < L + \varepsilon'$$

Because $F_{\varepsilon'}(x) > \gamma$,

$$\Rightarrow \varphi(x) > 0 \qquad \forall x \in (c, x_2]$$
$$\Rightarrow 0 < \varphi(x) \leq |f(x) - f(c)| \, x - c - (L + \varepsilon')$$
$$\Rightarrow \left| \frac{f(x) - f(c)}{x - c} \right| > L + \varepsilon'$$

This is a contradiction to the assumption that $F_{\varepsilon'}(x_2) > 0$. So $F_\varepsilon(x_2) \leq 0 \quad \forall \varepsilon > 0$

$$\Rightarrow F_0(x_2) \leq 0 \Rightarrow |f(x_2) - f(x_1)| \leq L - |x_2 - x_1|$$

□

**Remark 19.** Let $f$ be differentiable in $[a,b]$ and $|f'(x)| < L \quad \forall x \in [a,b]$. Let $x_1, x_2 \in [a,b]$

$$|f(x_L) - f(x_1)| = |f'(\xi) \cdot (x_2 - x_1)| \leq L\,|x_2 - x_1|$$

by Mean Value Theorem of differential calculus.

**Corollary 3.** Let $f, g : I \to \mathbb{R}$. $I$ as above and $f, g$ are differentiable in $I \setminus A$, $A$ countable and it holds that $f'(x) = g'(x) \quad \forall x \in I \setminus A$. There exists a constant $k$ such that

$$f(x) = g(x) + k \qquad \forall x \in I$$

*Proof.* We use the previous Theorem for

$$h(x) = f(x) - g(x)$$

Then it holds that $|h'(x)| = 0 = L \quad \forall x \in I \setminus A$.

$$\Rightarrow |h(x_1) - h(x_2)| \leq 0 \cdot |x_1 - x_2| \quad \forall x_1, x_2 \in I$$

$$\Rightarrow h(x_1) = h(x_2) \quad \forall x_1, x_2 \in I$$

$$\begin{aligned} f(x_1) - g(x_1) &= f(x_2) - g(x_2) \quad \forall x_1, x_2 \in I \\ &= k \ldots \text{ constant} \end{aligned}$$

$\forall x_1 \in I$ it holds that $f(x_1) = g(x_1) + k$. □

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This lecture took place on 15th of April 2016 with lecturer Wolfgang Ring.

*cont, 2nd part.* We need to show: Let $f$ be a regulated function and $\Phi$ is a primitive function of $f$ with the following properties

$$\Phi'(x) = f(x) \quad \forall x \in I \text{ where f is continuous}$$

$$\Phi'_+(x) = \lim_{\xi \to x_+} f(x)$$

$$\Phi'_-(x) = \lim_{\xi \to x_-} f(x) \quad \forall x \in I$$

Then it holds that

$$\int_\alpha^\beta f(x)\,dx = \Phi(\beta) - \Phi(\alpha)$$



*Proof.* For $\Phi(x) = \int_\alpha^x f(\xi)\,d\xi = F(x)$ (where $F$ is also a primitive function) it holds that

$$\int_\alpha^\beta f(\xi)\,d\xi = F(\beta) - \underbrace{F(\alpha)}_{=0}$$

Because $\Phi$ and $F$ are both primitive functions of $f$, $\Phi'$ and $F'$ correspond in all continuous points, hence everywhere, but one countable set.

By the uniqueness theorem, it holds that

$$\Phi(x) = F(x) + c$$

$$F(x) = \Phi(x) - c$$

$$\int_a^b f(\xi)\,d\xi = F(b) - F(a) = \Phi(b) - c - \Phi(a) + c = \Phi(b) - \Phi(a)$$

□

□

**Remark 20** (Notational remark)**.** Let $f$ be a regulated function. Then we denote

$$\int f(x)\,dx = \begin{cases} \text{the set of all primitive function of } f \\ \text{an arbitrary primitive function of } f \end{cases}$$

$\int f(x)\,dx$ is called *indefinite integral.*

**Remark 21.**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

If you consider all primitive functions of the indefinite integral, you consider a constant $c \in \mathbb{R}$.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad \forall n \in \mathbb{R} \setminus \{-1\}\, \forall x > 0$$

Let $x > 0$: $(\ln x)' = \frac{1}{x}$.
Let $x < 0$: $(\ln -x)' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & \text{for } x > 0 \\ \ln(-x) & \text{for } x < 0 \end{cases} = \ln|x| \qquad \text{for } x \neq 0$$

$$\int \cos x \, dx = \sin x$$

$$\int \sin x \, dx = -\cos x$$

$$\int e^{cx} \, dx = \frac{1}{c} \cdot e^{cx} \quad (c \neq 0)$$

**Lemma 12.** Let $f_1$ and $f_2$ be regulated functions in $I = [a, b]$ and there exists some countable set $A$ such that

$$f_1(x) = f_2(x) \qquad \forall x \in I \setminus A$$

Then it holds that

$$\int f_1(x) \, dx = \int f_2(x) \, dx \text{ and } \int_a^b f_1(x) \, dx = \int_a^b f_2(x) \, dx \qquad \forall a, b \in I$$

*Proof.* Let $F_1$ be a primitive function on $f_1$, $F_2$ be a primitive function of $f_2$. Then it holds that $F_1' = F_2'$ in $I \setminus A$. Due to identity theorem:

$$\Rightarrow F_1 = F_2 + c \Rightarrow \int f_1 \, dx = \int f_2 \, dx$$

□

**Remark 22.** Example of a function, which is differentiable everywhere. Its derivative is not a regulated function.

Let $I = [-1, 1]$ and

$$f(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$ it holds that

$$f'(x) = 2x \cos \sin \frac{1}{x} - \frac{x^2}{x^2} \cdot \cos \frac{1}{x}$$

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$



$$f'(0) = \lim_{h \to 0} \frac{1}{h} \left[ h^2 \cdot \sin \frac{1}{h} - 0 \right] = \lim_{h \to 0} \underbrace{h}_{\to 0} \cdot \underbrace{\sin \frac{1}{h}}_{\in [-1,1]} = 0$$

$$f'(x) = \begin{cases} 0 & \text{for } x = 0 \\ \underbrace{2x \sin \frac{1}{x} - \cos \frac{1}{x}}_{\text{has no one-sided limit in } x=0} & \text{for } x \neq 0 \end{cases}$$

$$f'_+(0) \neq \lim_{x \to 0^+} f'(x)$$

## 5.4 Integration techniques

**Theorem 15** (Integration by parts (dt. "partielle Integration")). Let $u, v : I \to \mathbb{R}$ be both primitive functions of regulated functions. Then also $u \cdot v$ is a primitive function of a regulated function and it holds that

$$\int u'v \, dx = u \cdot v - \int u \cdot v' \, dx$$

and

$$\int_a^b u'v \, dx = \underbrace{u(b) \cdot v(b) - u(a) \cdot v(a)}_{=: u \cdot v|_a^b} - \int_a^b u \cdot v' \, dx$$

*Proof.* $u$ is continuous and therefore a regulated function.
$v$ is continuous and therefore a regulated function.

$u'$ and $v'$ are regulated function by assumption.

$$\Rightarrow (u' \cdot v + u \cdot v') \in \mathcal{R}(I)$$

$u \cdot v$ is differentiable in every point in which $u$ and $v$ is differentiable. Let $u$ be differentiable in $I \setminus A$, $v$ is differentiable in $I \setminus B$.

$$\Rightarrow u \cdot v \text{ is differentiable in } I \setminus \underbrace{(A \cup B)}_{\text{countable}}$$

In $I \setminus (A \cup B)$ it holds that

$$(u \cdot v)'(x) = u'(x) \cdot v(x) + u(x) v'(x)$$

Hence the function $u \cdot v$ is primitive function of the regulated function $(u'v+uv')$.

$$\Rightarrow \int (u'v + uv')\, dx = u \cdot v$$

$$\Rightarrow \int_a^b (u'v + uv')\, dx = u(b)v(b) - u(a)v(a)$$

$\square$

**Example 3.** Let $a \neq -1$ and $x > 0$.

$$\int x^a \ln x \, dx = \begin{vmatrix} u' = x^a & u = \frac{1}{1+a} \cdot x^{a+1} \\ v = \ln x & v' = \frac{1}{x} \end{vmatrix}$$

$$\overset{\text{int. by parts}}{=} \frac{1}{1+a} x^{1+a} \cdot \ln x - \frac{1}{1+a} \int x^a \, dx$$

$$= \frac{1}{1+a} x^{1+a} \ln x - \frac{1}{(1+a)^2} x^{1+a} = \frac{1}{1+a} x^{1+a} \left[ \ln x - \frac{1}{1+a} \right]$$

**Example 4.**

$$\int \cos^k(x)\, dx \text{ for } k = 2, 3, 4, \dots$$

$$\begin{vmatrix} u' = \cos x & \Rightarrow u = \sin x \\ v = \cos^{k-1}(x) & v' = -(k-1) \cdot \cos^{k-2}(x) \cdot \sin(x) \end{vmatrix}$$

$$\int \cos^k(x)\, dx = \cos^{k-1}(x) \cdot \sin(x) + \int (k-1) \cdot \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{1-\cos^2(x)} \, dx$$

$$= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \cdot \int \cos^{k-2}(x)\, dx - (k-1) \cdot \int \cos^k(x)\, dx$$

Recognize that we have $\int \cos^k(x)\, dx$ twice in the equation (LHS and RHS, RHS with a sign).

$$k \cdot \int \cos^k(x)\, dx = \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x)\, dx$$

$$\int \cos^k(x)\, dx = \frac{1}{k} \cos^{k-1}(x) \sin(x) + \frac{k-1}{k} \int \cos^{k-2}(x)\, dx$$



Recursion formula.

Analogously,

$$\int \sin^k(x)\, dx = -\frac{1}{k} \sin^{k-1}(x) \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x)\, dx$$

Let $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x)\, dx$. Then it holds that

$$c_{2n} = \frac{(2n-1)}{2n} \cdot \frac{(2(n-1)-1)}{2(n-1)} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \left( \prod_{k=1}^{n} \frac{2k-1}{2k} \right) \cdot \frac{\pi}{2}$$

$$c_{2n+1} = \left( \prod_{k=1}^{n} \frac{2k}{2k+1} \right)$$

Proof by complete induction:

**Case** $n = 0$

$$\int_0^{\frac{\pi}{2}} \cos^{2\cdot 0} x \, dx = \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^{2\cdot 0+1} x \, dx = \int_0^{\frac{\pi}{2}} \cos x \, dx = \sin(x)|_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \cos^{2(n+1)} \, dx = \frac{1}{2(n+1)} \cdot \cos^{2(n+1)-1}(x) \cdot \sin(x) \Big|_0^{\frac{\pi}{2}} + \frac{2(n+1)-1}{2(n+1)} \cdot \int_0^{\frac{\pi}{2}} \cos^{2n}(x)\, dx$$

$$= \frac{2n+1}{2n+2} \cdot \underbrace{\left( \prod_{k=1}^{n} \frac{2k-1}{2k} \right)}_{\text{induction hypothesis}} \cdot \frac{\pi}{2} = \left( \prod_{k=1}^{n+1} \frac{2k-1}{2k} \right) \cdot \frac{\pi}{2}$$

**Theorem 16** (Wallis product)**.** (John Wallis, 1616–1703)

$$\frac{\pi}{2} = \lim_{n \to \infty} w_n \quad \text{with} \quad w_n = \prod_{k=1}^{n} \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots$$

*Proof.*

$$\frac{\pi}{2}\cdot\frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2}\cdot\frac{\prod_{k=1}^{n}\frac{2k}{2k+1}}{\prod_{k=1}^{n}\frac{2k-1}{2k}\cdot\frac{\pi}{2}} = \prod_{k=1}^{n}\frac{(2k)^2}{(2k+1)(2k-1)} = w_n$$

It remains to show: $\lim_{n\to\infty}\frac{c_{2n+1}}{c_{2n}} = 1$.

In $\left[0,\frac{\pi}{2}\right]$ it holds that $0 \le \cos x \le 1$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^{2n}(x)\,dx \ge \int_0^{\frac{\pi}{2}} \cos^{2n+1}(x)\,dx \ge \int_0^{\frac{\pi}{2}} \cos^{2n+2}(x)\,dx$$

$$c_{2n} \ge c_{2n+1} \ge c_{2n+2}$$

$$1 \ge \frac{c_{2n+1}}{c_{2n}} \ge \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1}\frac{2k-1}{2k}}{\prod_{k=1}^{n}\frac{2k-1}{2k}} = \underbrace{\frac{2n+1}{2n+2}}_{\to 1 \text{ for } n\to\infty}$$

$\Rightarrow \frac{c_{2n+1}}{c_{2n}}$ converges and limit is 1.

$$\lim_{n\to\infty}\frac{\pi}{2}\cdot\frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} = \lim_{n\to\infty} w_n$$

□

**Theorem 17** (Substitution law)**.** Let $f : I \to \mathbb{R}$ be a regulated function with primitive function $F$. Furthermore $t : [\alpha,\beta] \to I$ is continuously differentiable. Then $F\circ t$ is a primitive function for function $(f\circ t)\cdot t'$ and it holds that

$$\int_\alpha^\beta f(t(x))\cdot t'(x)\,dx = \int_{t(\alpha)}^{t(\beta)} f(t)\,dt$$

*Proof.* The right-side integral is given (according to the Fundamental Theorem) by

$$F(t(\beta)) - F(t(\alpha))$$

The left-side integral, because of

$$F(t(x))' = F'(t(x))\cdot t(x)$$

Hence $F = t$ is primitive function of the left-side integral. So it holds that

$$\int_a^b f(t(x))\cdot t'(x)\,dx = F\circ t(b) - F\circ t(a) = F(t(b)) - F(t(a))$$

□

**Example 5.**

$$\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{2}\int_0^1 2x\sqrt{1+x^2}\,dx$$

$$\left|\begin{matrix} t(x) = 1+x^2 & t'(x) = 2x \\ f(y) = \sqrt{y} & \end{matrix}\right|$$

$$= \frac{1}{2}\int_1^2 \sqrt{x}\,dx = \frac{1}{2}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^2 = \frac{2}{3}^{\frac{3}{2}} - \frac{1}{3}^{\frac{3}{2}} = \frac{1}{3}(\sqrt{8}-1)$$

$$\int_0^1 x\cdot\sqrt{1+x^2}\,dx = \left|\begin{matrix} \overbrace{y = x^2+1}^{\text{transform variables}} & \\ \frac{dy}{dx} = 2x & \underbrace{dy = 2x\,dx}_{\text{transformation of differences}} \\ & x\,dx = \frac{1}{2}\,dy \end{matrix}\right|$$

Transformation of limits:

$$x = 0 \Leftrightarrow y = 1 \qquad x = 1 \Leftrightarrow y = 2$$

$$= \frac{1}{2}\int_1^2 \sqrt{y}\,dy = \frac{1}{2}\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^2 = \frac{(x^2+1)^{\frac{3}{2}}}{3}\Bigg|_0^1$$

Hence it is also necessary to transform the limits.

**Example 6** (Integration by parts)**.**

$$\int \ln x\,dx = \left|\begin{matrix} v' = 1 & v = x \\ u = \ln x & u' = \frac{1}{x} \end{matrix}\right| = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x$$

**Theorem 18.** Ivan M. Niven (published in 1947, 1915–1999)

It holds: $\pi^2$ is an irrational number. So $\pi$ is irrational.

*Proof by contradiction.* Let $\pi^2 = \frac{a}{b} \in \mathbb{Q}$.

Because $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ (practicals!) there exists $n \in \mathbb{N}$ such that $\pi \frac{a^n}{n!} < 1$.

$$f(x) = \frac{1}{n!} x^n (1-x)^n$$

is symmetrical along axis $x = \frac{1}{2}$

$$= \frac{1}{n!} \sum_{k=n}^{2n} c_k x^k \qquad \text{with } c_k = (-1)^{k-n} \binom{n}{k-n} = \pm \binom{n}{k-n} \in \mathbb{Z}$$

$$f^{(\mu)}(0) = 0 \text{ for } \mu = 0, 1, \dots, n-1 \in \mathbb{Z} \qquad \text{and also:}$$

$$f^{(\mu)}(1) \in \mathbb{Z} \text{ for } \mu = n, n+1, \dots, 2n$$

$$f^{(\mu)}(x) = \frac{1}{n!} \sum_{k=0}^{2n} \underbrace{k(k-1)\dots(k-\mu+1)}_{=\mu!} \cdot c_k \cdot x^{k-\mu}$$

$$\begin{aligned} f^{(\mu)}(0) &= \frac{1}{n!} \mu! \left( \pm \binom{n}{\mu-n} \right) \cdot 1 \\ &= \frac{1}{n!} \mu! \frac{n!}{(\mu-n)!(n-\mu+n)!} \\ &= \frac{\mu!}{(\mu-n)!(2n-\mu)!} \\ &= \frac{(\mu-n+1)(\mu-n+2)\dots\mu}{1 \cdot 2 \cdot 3 \dots (2n-\mu)} \\ &\in \mathbb{Z} \end{aligned}$$

Why does $\in \mathbb{Z}$ hold?

$$\frac{\mu!}{n!} \underbrace{\binom{n}{\mu-n}}_{\in \mathbb{Z}} \in \mathbb{Z} \qquad n \leq \mu \leq 2n$$

$$(n+1)(n+2)\dots\nu \in \mathbb{Z}$$



$$n \leq \mu \leq 2n$$

$f^{(\mu)}(0) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$, analogously $f^{(\mu)}(1) \in \mathbb{Z}$ for $\mu \in \{n, n+1, \dots, 2n\}$.

$$F(x) = b^n \left( \pi^{2n} f(x) - \pi^{2n-2} f''(x) + \pi^{2n-4} f^{(4)}(x) + (-1)^n f^{2n}(x) \pi^0 \right)$$

$$F(0) \in \mathbb{Z} \text{ because } f^{(\mu)}(0) \in \mathbb{Z} \text{ for } \mu = 0, 2, 4, 6, \dots, 2n$$

$$\pi^2 = \frac{a}{b} \qquad \pi^{2n-2l} = \frac{a^{k-l}}{b^{n-l}}$$

$$b^n \cdot \pi^{2n-2l} = a^{n-l} \cdot b^l \in \mathbb{Z}$$

Analogously for $F(1) \in \mathbb{Z}$.

$$\begin{aligned} &(F'(x) \cdot \sin(\pi x) - \pi F(x) \cdot \cos(\pi x))' \\ &= F''(x) \cdot \sin(\pi x) + \pi^2 \cdot F(x) \cdot \sin \pi x \\ &+ F'(x)(\cos(\pi x) - \pi \cos \pi x) \\ &= (F''(x) + \pi^2 F(x)) \cdot \sin(\pi x) \end{aligned}$$

$$\begin{aligned} F''(x) = b^n \cdot (&\pi^{2n} \cdot f''(x) + \pi^{2n-2} f^{(4)}(x) + \\ &\pi^{2n-4} f^{(6)}(x) - \dots + (-1)^n f^{(2n+2)}(x)) \\ \implies &F''(x) + \pi^2 \cdot F(x) \end{aligned}$$

$$\begin{aligned} F''(x) + \pi^2 \cdot F(x) = b^n(&\pi^{2n} f''(x) - \pi^{2n-2} f^{(4)}(x) + \pi^{2n-4} f^{(6)}(x) + \\ &\dots + (-1)^n f^{(2n+2)}(x)) \\ &+ b^n(\pi^{2n+2} f(x) - \pi^{2n} f''(x) + \pi^{2n-2} f^{(4)}(x) - \\ &\pi^{2n-4} f^{(6)}(x) + \dots + (-1)^n \pi^2 \cdot f^{(2n)}(x)) \end{aligned}$$

Almost all expressions cancel each other out. So it holds that

$$\begin{aligned}
&(F'(x)\cdot\sin(\pi x)-\pi F(x)\cos(\pi x))' \\
&=\pi^{2n+2}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&=\frac{a^{n+1}}{b^{n+1}}\cdot b^n\cdot f(x)\cdot\sin(\pi x) \\
&=\frac{a^{n+1}}{b}\cdot f(x)\cdot\sin(\pi x) \\
&=\pi^2\cdot a^n f(x)\cdot\sin(\pi x) \\
&=\pi\left(\pi a^n f(x)\sin(\pi x)\right) \\
I&=\pi\int_0^1 a^n f(x)\cdot\sin(\pi x)\,dx \\
&=\frac{1}{\pi}\cdot\left[F'(x)\cdot\sin(\pi x)-\pi\cdot F(x)\cos(\pi x)\right]\Big|_0^1 \\
&=F(1)+F(0)\in\mathbb{Z}
\end{aligned}$$

On the other hand it holds that

$$f(x)=\frac{1}{n!}\underbrace{x^n}_{\le 1}\underbrace{(1-x)}_{\le 1}{}^n$$

So $0\le f(x)\le\frac{1}{n!}$. Hence,

$$0\le a^n f(x)\cdot\sin(\pi x)\le\frac{a^n}{n!}<\frac{1}{\pi}$$

So $0<I<1\Rightarrow I\in\mathbb{Z}$. This is a contradiction to our assumption that $I\in\mathbb{Z}$. □

**Remark 23.** Hence $\pi$ is not rational. So there exists no linear affine function $g(x)=ax+b$ with $a,b\in\mathbb{Z}$ such that $\pi$ is root of $g$.

**Remark 24.** We state, $\xi\in\mathbb{R}$ is an *algebraic* number if polynomial

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_n$$

exists with $a_i\in\mathbb{Z}$ for $i=0,\ldots,n$ and $P(\xi)=0$.

Algebraic numbers are a generalization of rational numbers.

$\eta\in\mathbb{R}$ is called *transcendental*, if $\eta$ is not algebraic.



**Remark 25.** $\pi$ is transcendental.

**Theorem 19** (Integration of non-compact intervals)**.**

$$\int_0^\infty e^{-x}\,dx=\lim_{c\to\infty}\int_0^c e^{-x}\,dx$$

**Definition 14** (Definition of indefinite integrals)**.** Let $I$ be an interval with boundary values $a$ and $b$ with $-\infty\le a<b\le\infty$.

Let $f$ be a regulated function in $I$. Then we define

1. if $I=[a,b)$, $\int_a^b f(x)\,dx=\lim_{\beta\to b_-}\int_a^\beta f(x)\,dx$
2. if $I=(a,b]$, $\int_a^b f(x)\,dx=\lim_{\alpha\to a_+}\int_\alpha^a f(x)\,dx$
3. if $I=(a,b)$, we choose $c\in I$ and $\int_a^b f(x)\,dx=\lim_{\alpha\to a_+}\int_\alpha^c f(x)\,dx+\lim_{\beta\to b_-}\int_c^\beta f(x)\,dx$.

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This lecture took place on 21st of April 2016 with lecturer Wolfgang Ring.

$$f:[a,b)\to\mathbb{R}\qquad b\in(-\infty,\infty]$$

$$\int_a^b f(x)\,dx=\lim_{\beta\to b_-}\int_a^b f(x)\,dx$$

**Example 7** (Classic examples)**.** 1. Let $s>1$.

$$\begin{aligned}
\int_1^\infty\frac{1}{x^s}\,dx&=\lim\int_1^\beta x^{-s}\,dx \\
&=\frac{1}{-s+1}\cdot x^{-s+1}\Big|_1^\beta \\
&=\lim_{\beta\to\infty}\frac{1}{1-s}\cdot\frac{1}{\beta^{s-1}}-\frac{1}{1-s}
\end{aligned}$$

$s-1>0$ and $\frac{1}{1-s}\to 1$

$$=\frac{1}{s-1}\qquad\text{so indefinite integral exists}$$

2. Let $s < 1$.

$$\begin{aligned}\int_0^1 x^{-s}\,dx &= \lim_{\alpha\to 0_+}\int_\alpha^1 x^{-s}\,dx\\ &= \lim_{\alpha\to 0_+}\frac{1}{-s+1}x^{-s+1}\Big|_\alpha^1\\ &= \frac{1}{1-s} - \underbrace{\lim_{\alpha\to 0_+}\frac{1}{1-s}\alpha^{1-s}}_{=0}\\ &= \frac{1}{1-s}\end{aligned}$$

Compare with Figure 19.

3.

$$\begin{aligned}\int_0^\infty e^{-cx}\,dx &= \lim_{\beta\to\infty}\int_0^\beta e^{-cx}\,dx\\ &= \lim_{\beta\to\infty}\frac{1}{-c}\cdot e^{-cx}\Big|_0^\beta\\ &= \lim_{\beta\to\infty}\left(-\frac{1}{c}\cdot e^{-c\beta}\right)+\frac{1}{c}\\ &= \frac{1}{c}\end{aligned}$$

4.

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{\alpha\to-\infty}\int_\alpha^0 \frac{1}{1+x^2}\,dx + \lim_{\beta\to 0}\int_0^\beta \frac{1}{1+x^2}\,dx\\ &= \arctan(0) - \underbrace{\lim_{\alpha\to-\infty}\arctan(\alpha)}_{-\frac{\pi}{2}} + \underbrace{\lim_{\beta\to\infty}\arctan(\beta)}_{\frac{\pi}{2}} - \arctan(0)\\ &= -\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}\\ &= \pi\end{aligned}$$

Compare with Figure 18.



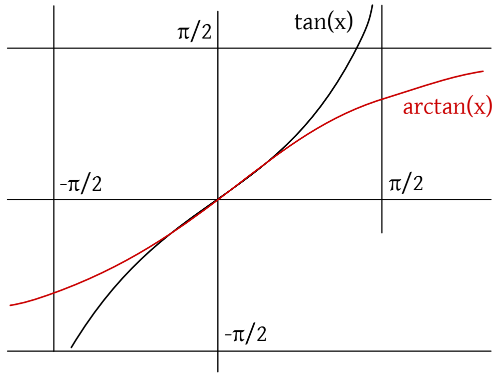


Figure 18: $\tan(x)$ and $\arctan(x)$

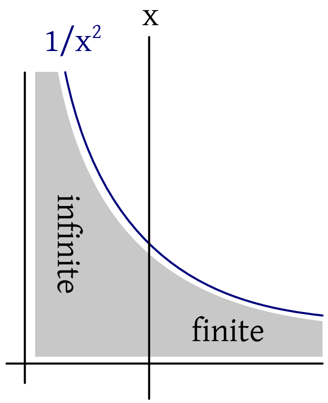


Figure 19: $\frac{1}{1-s}$

**Remark 26.** "Integral converges" means "an (indefinite) integral exists"

**Remark 27.**

$$\arctan'(x) = \frac{1}{1+x^2}$$

$$\tan'(x) = \frac{\cos x \cdot \cos x - (\sin x)(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2(x)}$$

$$\tan(x) = \frac{\sin x}{\cos x}$$

$$\begin{aligned}
\arctan'(x) &= \frac{1}{\tan'(\arctan(x))} \\
&= \left|\arctan x = s\right| \\
&= \left(\frac{1}{\cos^2(s)}\right)^{-1} \\
&= \left(\frac{\cos^2(s)+\sin^2(s)}{\cos^2(s)}\right)^{-1} \\
&= \left(1+\left(\frac{\sin s}{\cos s}\right)^2\right)^{-1} \\
&= \left(1+[\tan(\arctan x)]^2\right)^{-1} \\
&= (1+x^2)^{-1} \\
&= \frac{1}{1+x^2}
\end{aligned}$$

**Theorem 20** (Direct comparison test for indefinite integrals)**.** (dt. "Majorantenkriterium für uneigentliche Integrals") Let $f, g$ be regulated functions in $[a,b]$ and $|f(x)| \leq g(x) \quad \forall x \in [a,b)$. Assume $\int_a^b g(x)\,dx$ exists. Then also $\int_a^b |f(x)|\,dx$ exists and also $\int_a^b f(x)\,dx$.

*Proof.*

$$G(\beta) = \int_a^\beta g(x)\,dx$$

We know that $\lim_{\beta\to b_-} G(\beta)$ exists.

Cauchy criterion: $\forall \varepsilon > 0$ there exists a left-sided environment of $b$ such that for all $u, v$ in this environment it holds that

$$\underbrace{|G(u)-G(v)|}_{\int_u^v g(x)\,dx} < \varepsilon$$

Because $|f| \leq g$ it holds that

$$F(\beta) = \int_a^\beta |f(x)|\,dx$$

and also that

$$\left|\int_u^v |f(x)|\,dx\right| = |F(v)-F(u)| \overset{\text{monotonicity}}{\leq} \left|\int_u^v g(x)\,dx\right| < \varepsilon$$

Hence $\lim_{\beta\to b} F(\beta)$ exists because of the Cauchy criterion. So $\int_a^b |f(x)|\,dx$ exists. Analogously for $f$ instead of $|f|$. □

**Example 8.**

$$\int_0^\infty \frac{\sin x}{x}\,dx \text{ exists}$$

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \text{ continuous in } 0$$

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 f(x)\,dx \text{ exists because } f \text{ is continuous}$$

$$\begin{aligned}
\lim_{\beta\to\infty} \int_1^\beta \frac{\sin x}{x}\,dx &= \begin{vmatrix} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{vmatrix} \\
&= \lim_{\beta\to\infty} \frac{1}{x}\cdot(-\cos x)\Big|_1^\beta - \int_1^\beta \frac{\cos x}{x^2}\,dx \\
&= \lim_{\beta\to\infty} \left[\underbrace{-\frac{1}{\beta}\cdot\cos\beta}_{\to 0} + \cos 1 - \int_1^\beta \frac{\cos x}{x^2}\,dx\right] \\
&= \lim_{\beta\to\infty} \int_1^\beta \frac{\cos x}{x^2}\,dx
\end{aligned}$$

The last expression exists, because $\frac{1}{x^2}$ is a majorant for $\frac{\cos(x)}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ exists.

---

This lecture took place on 22nd of April 2016 with lecturer Wolfgang Ring.

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx \text{ does not exist}$$

$$\int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{1}{(k+1)\pi}\int_{k\pi}^{(k+1)\pi} \left|\sin x\right| dx$$

$$= \frac{1}{(k+1)\pi}(\pm 1)\cdot(-\cos x)\Big|_{k\pi}^{(k+1)\pi} = \frac{1}{(k+1)\pi}(\pm 1)(\pm 2)$$

$$= \frac{2}{(k+1)\pi}$$

$$\underbrace{\int_0^{(n+1)\pi}\left|\frac{\sin x}{x}\right| dx}_{\text{unbounded}\Leftarrow} \geq \frac{2}{\pi}\cdot \underbrace{\sum_{k=0}^n \frac{1}{k+1}}_{\substack{\text{harmonic series,}\\ \text{divergent}}}$$

In terms of the Lebesgue integral, $\int_0^\infty \frac{\sin x}{x}\,dx$ does not exist.

We can define new types of integration which yield new types of function which are not representable with techniques discussed so far.

**Example 9** (The Eulerian $\Gamma$-function)**.**

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \text{ for } x > 0$$

The function variable of the $\Gamma$-function is a parameter of the integrand.

The indefinite integral from above exists,

$$\lim_{\alpha\to 0_+}\int_\alpha^1 \underbrace{t^{x-1}e^{-t}}_{>0}\,dt \text{ exists}$$

of $\int_\alpha^1 t^{x-1}e^{-t}\,dt$ is bounded in terms of $\alpha$.



$$\int_\alpha^1 t^{x-1}\underbrace{e^{-t}}_{<1}\,dt < \underbrace{\int_\alpha^1 t^{x-1}\,dt}_{\text{converges for } x-1>-1}$$

hence for $x > 0$.

Right-side integral boundary:

$$\int_1^\infty t^{x-1}e^{-t}\,dt \text{ converges?}$$

**Example 10** (Claim)**.** There exists $c > 0$ such that

$$t^{x-1}e^{-t} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$t^{x-1}\cdot e^{-\frac{t}{2}} < c\cdot e^{-\frac{t}{2}} \quad \forall t \geq 1$$

$$\lim_{t\to\infty}\left(t^{x-1}\cdot e^{-\frac{t}{2}}\right) = \left|\begin{matrix}\frac{t}{2} = s\\ t = 2s\end{matrix}\right|$$

$$= \lim_{s\to\infty}(2s)^x - 1e^{-s}$$

$$\leq \lim_{s\to\infty}(2s)^{\lfloor x\rfloor+1-1}\cdot e^{-s}$$

with $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$

$$= \lim_{s\to\infty}(2s)^{\lfloor x\rfloor}\cdot e^{-s}$$

$$\leq \lim_{s\to\infty} s^{\lfloor x\rfloor+1}\cdot e^{-s}$$

because $s^{n+1} > (2s)^n$ for $s > 2^n$.

Hence for $\varepsilon > 0$, $\exists t$ such that

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| < \varepsilon \text{ if } t > L$$

and

$$\left|t^{x-1}e^{-\frac{t}{2}}\right| \leq M \text{ for } t \in \underbrace{[1, L]}_{\text{compact}}$$

$\Rightarrow$ for $t \in [1, \infty)$ it holds that

$$\left| t^{x-1} e^{-\frac{t}{2}} \right| \leq \max\{M, \varepsilon\} =: c$$

$$t^{x-1} e^{-\frac{t}{2}} \leq c$$

$$\int_0^\infty t^{x-1} e^{-t}\, dt \leq \int_0^\infty c \cdot e^{-\frac{t}{2}}\, dt = c \cdot \left( -2 \cdot e^{-\frac{t}{2}} \right) \Big|_0^\infty = 2c$$

hence $\int_0^\infty t^{x-1} e^{-t}\, dt$ exists.

It holds that $\Gamma(1) = 1$ because,

$$\int_0^\infty e^{-t}\, dt = 1$$

Furthermore it holds that for all $x > 0$,

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{x+1-1} e^{-t}\, dt = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R t^x e^{-t}\, dt \\
&= \begin{vmatrix} u = t^x & u' = x \cdot t^{x-1} \\ v' = e^{-t} & v = -e^{-t} \end{vmatrix} \\
&= \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left[ -t^x e^{-t} \Big|_{t=\varepsilon}^R + \int_\varepsilon^R x \cdot t^{x-1} \cdot e^{-t}\, dt \right]
\end{aligned}$$

$$= \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left( \underbrace{-R^x \cdot e^{-R}}_{\to 0 \text{ for } R \to \infty} + \underbrace{\varepsilon^x \cdot e^{-\varepsilon}}_{\to 0 \text{ for } \varepsilon \to 0} \right) + x \cdot \int_0^\infty t^{x-1} e^{-t}\, dt = x \cdot \Gamma(x)$$

So it holds that

$$\begin{aligned}
T(2) &= 1 \cdot T(1) = 1 \\
T(3) &= 2 \cdot T(2) = 2 \cdot 1 \\
T(4) &= 4 \cdot T(3) = 3 \cdot 2 \cdot 1 \\
T(5) &= 4 \cdot T(4) = 4 \cdot 3 \cdot 2 \cdot 1
\end{aligned}$$

By complete induction we can show that

$$\Gamma(n+1) = n! \qquad \forall n \in \mathbb{N}$$



## 5.5 Some important inequalities

**Theorem 21** (Young's inequality)**.** Let $f : [0, \infty) \to [0, \infty)$ be continously differentiable, $f(0) = 0$; $f$ is strictly monotonically increasing and unbounded (hence $f$ is injective because of strong monotonicity and surjective because of unboundedness).

So there exists $f^{-1} : [0, \infty) \to [0, \infty)$.

Let $a, b \geq 0$. Then it holds that

$$a \cdot b \leq \int_0^a f(x)\, dx + \int_0^b f^{-1}(y)\, dy$$

Equality holds if and only if,

$$b = f(a) \text{ i.e. } a = f^{-1}(b)$$

Compare with Figure 20

*Proof.*

$$\int_0^b f^{-1}(y)\, dy \overset{\text{substitution}}{=} \begin{vmatrix} y = f(x) \\ dy = f'(x)\, dx \\ y = 0 \Leftrightarrow x = f^{-1}(0) = 0 \\ y = b \Leftrightarrow x = f^{-1}(b) \end{vmatrix}$$

$$\begin{aligned}
&= \int_0^{f^{-1}(b)} \underbrace{f^{-1}(f(x))}_{x} \cdot f'(x)\, dx \\
&= \int_0^{f^{-1}(b)} x \cdot f'(x)\, dx \\
&= f(x) \cdot x \Big|_0^{f^{-1}(b)} - \int_0^{f^{-1}(b)} 1 \cdot f(x)\, dx \\
&= f(f^{-1}(b)) \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\, dx \\
&= b \cdot f^{-1}(b) - \int_0^{f^{-1}(b)} f(x)\, dx
\end{aligned}$$

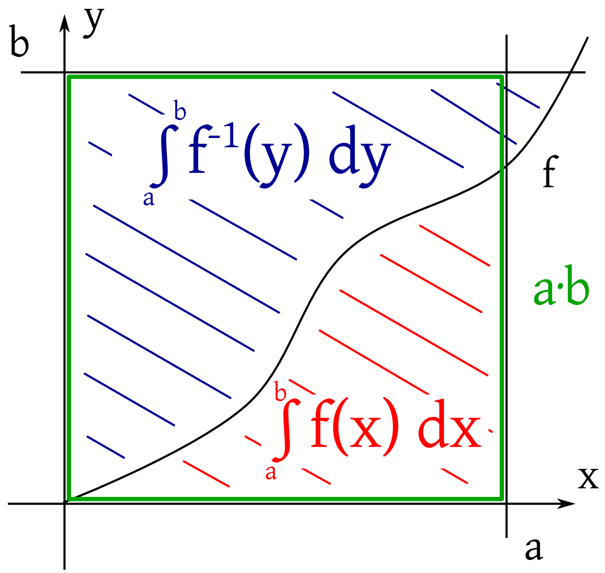


Figure 20: Young's inequality: the blue and red areas are larger than the green area

Therefore,

$$\begin{aligned} I &= \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy \\ &= \int_0^a f(x)\,dx + \int_{f^{-1}(b)}^0 f(x)\,dx + b\cdot f^{-1}(b) \\ &= \int_{f^{-1}(b)}^a f(x)\,dx + b\cdot f^{-1}(b) \end{aligned}$$



**Case 1:** $f^{-1}(b) = a \;\; (f(a) = b)$

$$\implies I = \underbrace{\int_a^b f(x)\,dx}_{=0} + b\cdot a = ab$$

Proven.

**Case 2:** $b < f(a) \Leftrightarrow f^{-1}(b) < a$ $f$ is strictly monotonically increasing, hence $f(x) > f(f^{-1}(b)) = b$ for all $x \in (f^{-1}(b), a]$.

$$\begin{aligned} \int_{f^{-1}(b)}^a f(x)\,dx &> b\cdot \int_{f^{-1}(b)}^a 1\,dx \\ &= b\cdot\left(a - f^{-1}(b)\right) \\ I &> b\left(a - f^{-1}(b)\right) + b\cdot f^{-1}(b) = a\cdot b \end{aligned}$$

Proven.

**Case 3:** $b > f(a)$

$$I = \underbrace{-\int_a^{f^{-1}(b)} f(x)\,dx}_{\text{strictly mon. decreasing}} + bf^{-1}(b)$$

For $(-f(x))$ it holds that:

$$\begin{aligned} &> (-f(f^{-1}(b)) = -b) \\ &> (-b)\left(f^{-1}(b) - a\right) + b\cdot f^{-1}(b) = a\cdot b \end{aligned}$$

Proven.

□

**Remark 28.** Young's inequality also holds if $f$ has all the properties above but is not necessarily differentiable.

**Theorem 22** (Young's inequality, special case)**.** Let $A, B \geq 0$. $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (hence $p$ and $q$ are "conjugate exponents"). Then it holds that

$$A\cdot B \leq \frac{A^p}{p} + \frac{B^q}{q}$$

*Proof.* Let $f(x) = x^{p-1}$ satisfy the requirements for Young's inequality.

$$f^{-1}(y) = y^{\frac{1}{p-1}}$$

$$\left(\frac{1}{q} = 1 - \frac{1}{q} \qquad q = \left(1 - \frac{1}{q}\right)^{-1}\right)$$

$$q - 1 = \left(1 - \frac{1}{p}\right)^{-1} - 1 = \left(\frac{p-1}{p}\right)^{-1} - 1$$

$$= \frac{p}{p-1} - 1 = \frac{p-p+1}{p-1} = \frac{1}{p-1}$$

$$f^{-1}(y) = y^{q-1}$$

Therefore

$$A \cdot B \leq \int_0^A x^{p-1}\,dx + \int_0^B y^{q-1}\,dy = \left.\frac{x^p}{p}\right|_0^A + \left.\frac{y^q}{q}\right|_0^B = \frac{A^p}{p} + \frac{B^q}{q}$$

$\square$

**Remark 29.** Equality holds if $A^p = B^q$. The proof is left as an exercise to the reader.

**Theorem 23** (Hölder's inequality)**.** Let $I$ be an interval, $a, b$ are boundary values of $I$ ($a, b \in [-\infty, \infty]$). Let $p, q$ be conjugate exponents, hence $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Let $f_1$ and $f_2$ be regulated functions in $I$ and

$$\int_a^b |f_1(x)|^p\,dx \text{ exists and } \int_a^b |f_2(x)|^q\,dx \text{ exists}$$

Let

$$\|f_1\|_p = \left(\int_a^b |f_1(x)|^p\,dx\right)^{\frac{1}{p}}$$

and

$$\|f_2\|_q = \left(\int_a^b |f_2(x)|^q\,dx\right)^{\frac{1}{q}}$$

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \text{ exists and } \int_a^b |f_1(x) f_2(x)|\,dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

*Proof.* Let $A = \frac{|f_1(x)|}{\|f_1\|_p}$ and $B = \frac{|f_2(x)|}{\|f_2\|_q}$.

$$A \cdot b \leq \frac{A^p}{p} + \frac{B^q}{q}$$

$$\overset{\text{integration}}{\Rightarrow} \int_a^b \frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q}\,dx \leq \frac{1}{p}\int_a^b \frac{|f_1(x)|^p}{\|f_1\|_p^p}\,dx + \frac{1}{q}\int_a^b \frac{|f_2(x)|^q}{\|f_2\|_q^q}\,dx$$

$$\Rightarrow \frac{1}{\|f_1\|_p \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)|\,dx$$

$$\leq \frac{1}{p}\frac{1}{\|f_1\|_p^p}\underbrace{\int_a^b |f_1(x)|^p\,dx}_{\|f_1\|_p^q} + \frac{1}{q}\frac{1}{\|f_2\|_q^q}\underbrace{\int_a^b |f_2(x)|^q\,dx}_{\|f_2\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

$$= \underbrace{\int_a^b |f_1(x) \cdot f_2(x)|\,dx}_{\text{exists}} \leq \|f_1\|_p \cdot \|f_2\|_q$$

$\square$

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This lecture took place on 28th of April 2016 with lecturer Wolfgang Ring.

**Example 11** (Special case $p = q = 2$)**.** Let $p = q = 2$. $\frac{1}{2} + \frac{1}{2} = 1$ holds.

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \leq \left(\int_a^b |f_1(x)|^2\,dx\right)^{\frac{1}{2}} \cdot \left(\int_a^b |f_2(x)|^2\,dx\right)^{\frac{1}{2}}$$

$$\int_a^b |f_1(x) \cdot f_2(x)|\,dx \geq \left|\int_a^b f_1(x) \cdot f_2(x)\,dx\right|$$

$f_1$ and $f_2$ such that $\|f_i\|_2 < \infty$ for $i = 1, 2$, then

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) \cdot f_2(x)\, dx$$

is a scalar (= inner) product in the vector space of functions with norm:

$$\|f\| = (\langle f, f \rangle)^{\frac{1}{2}} = \|f\|_2$$

The resulting inequality is named "Cauchy-Schwarz inequality"

$$|\langle f_1, f_2 \rangle| \leq \|f_1\|_2 \cdot \|f_2\|_2$$

## 5.6 Elementwise integration of series

**Lemma 13.** Let $f_n \in R(I)$ with $I$ as interval, $f_n$ converges uniformly to $f$ in $I$. Then also $f$ is a regulated function and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx$$

*Proof.* We know $f$ is a regulated function if and only if $f$ can be uniformly approximated using a step function.

Let $\varepsilon > 0$ be arbitrary. Because $f$ is the uniform limit of $f_n$, there exists $n \in \mathbb{N}$ such that $\|f - f_N\|_\infty < \frac{\varepsilon}{2}$. Because $f_N$ is a regulated function, there exists $\varphi \in \tau(I)$ with

$$\|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} \Rightarrow \|f - \varphi\|_\infty \leq \|f - f_N\|_\infty + \|f_N - \varphi\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $f$ is a regulated function. Choose $N$ such that $\forall n \geq N$:

$$\|f - f_n\|_\infty < \frac{\varepsilon}{b-a}$$



Then it holds that

$$\begin{aligned}\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| &\leq \int_a^b |f_n(x) - f(x)|\, dx \\ &\leq \int_a^b \underbrace{\|f_n - f\|_\infty}_{< \frac{\varepsilon}{b-a}}\, dx \\ &< \frac{\varepsilon}{b-a} \cdot (b-a) \\ &= \varepsilon\end{aligned}$$

□

**Example 12** (Application)**.** Let $f(x) = \sum_{n=0}^\infty a_k x^k$ is a power series. Let $\rho_f$ be a convergence radius of $f$ and $0 < r < \rho_f$. Then it holds that

$$f_n(x) = \sum_{k=0}^n a_k x^k \text{ converges uniformly to } f \text{ in } [-r, r]$$

$$\begin{aligned} f_n &\in R([-r, r]) \\ \Rightarrow \int_{-r}^r f(x)\, dx &= \lim_{n\to\infty} \int_{-r}^r f_n(x)\, dx\end{aligned}$$

The integral is determined by elementwise integration

$$\int_{-r}^r a_k x^k\, dx = a_k \frac{x^{k+1}}{k+1}\bigg|_{-r}^r$$

Analogously for integration over any compact interval $[a, b] \subset (-\rho_f, \rho_f)$ i.e. for the indefinite integration. Hence,

$$\sum_{k=0}^\infty a_k \frac{x^{k+1}}{n+1} + c$$

is primitive function of $f$ uniformly convergent on every interval $[-r, r] \subseteq (-\rho_f, \rho_f)$.

**Example 13.**

$$F : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \qquad F(x) = \arctan(x)$$

$$F'(x) = f(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} \left(-(x^2)\right)^k = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad \forall x \in (-1,1)$$

Elementwise integration:

$$F(x) = \arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + c$$

$$\arctan(0) = 0 = c$$

Hence,

$$\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad \text{in } (-1,1)$$

Compare with Figure 21

# 6 Taylor polynomials and Taylor series

**Theorem 24.** Approximation of a function with polynomials or representation of a function using a power series.

$$\mathcal{C}^n((a,b)) = \{f : (a,b) \to \mathbb{R} \mid f \text{ differentiable } n \text{ times in } (a,b)\}$$

Hence $f^{(k)} : (a,b) \to \mathbb{R}$ is continuous for $k = 0, 1, \ldots, n$. Choose $x_0 \in (a,b)$. Find a polynomial $T_f^a(x)$ of degree $n$ such that

$$\left(T_f^a\right)^{(k)}(x_0) = f^{(k)}(x_0)$$

It holds that $T_f^a$ can be determined uniquely as

$$T_f^a(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Taylor polynomial of $n$-th order of $f$ in point $x_0$.



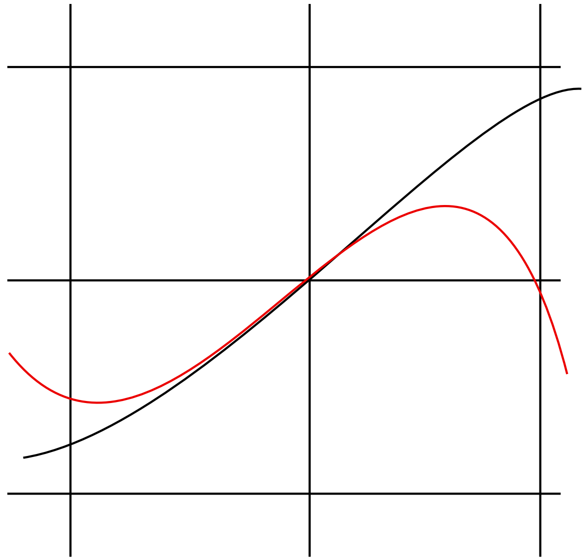

Figure 21: Approximation of $\arctan(x)$

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This lecture took place on 29th of April 2016 with lecturer Kniely Michael.

**Definition 15** (Additional remark to Taylor polynomials)**.** Let $P(x) := \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$. Let $k \in \{1, \ldots, n\}$.

1. $x_0$ is called $k$-th root of $P$ iff $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$.
2. It holds that $x_0$ is a $k$-th root of $P$ iff

$$\forall j \in \{0, \ldots, k-1\} : P^{(j)}(x_0) = 0 \land P^{(k)}(x_0) \neq 0$$

*Complete induction over $k$.* $\mathbf{k = 1}$

$\Rightarrow$: Let $x_0$ be 1st root of $P$.

$$P(x) = (x - x_0)Q(x) \Rightarrow P^{(0)}(x_0) = 0 \wedge P^{(1)}(x_0) = Q(x_0) \neq 0.$$

$\Leftarrow$: Let $P^{(0)}(x_0) = 0$.

$$P^{(1)}(y_0) \neq 0$$

Division with remainder $\Rightarrow$

$$P(x) = (x - x_0)Q(x) + R(x) \text{ with } \deg(R) < \deg(x - x_0) = 1$$

with $R$ constant.

$$0 = P(y_0) = R \Rightarrow P(x) = (x - x_0)Q(x)$$

$$x \neq x_0 \Rightarrow Q(x) = \frac{P(x)}{x - x_0} = \frac{P(x) - P(x_0)}{x - x_0} \Rightarrow Q(x_0)$$

$$\overset{\text{Q continuous}}{=} \lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{P(x) - P(x_0)}{x - x_0} = P^{(1)}(x_0) \neq 0$$

$\mathbf{k \geq 2, k - 1 \to k}$ $\Rightarrow$. Let $x_0$ be the k-th root of $P$. Hence $P(x) = (x - x_0)^k Q(x)$ with $Q(x_0) \neq 0$. Let $\tilde{P}(x) := (x - x_0)^{k-1}Q(x)$. $x_0$ is $(k-1)$-th root of $\tilde{P}$.

$$\xrightarrow{\text{ind. hypo.}} \tilde{P}^{(j)}(x_0) = 0 \wedge \tilde{P}^{(k-1)}(x_0) \neq 0 \quad \forall j \in \{0, \dots, k-2\}$$

$$P(x) = (x - x_0)\tilde{P}(x) \Rightarrow P^{(j)}(x) = (x - x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x)$$

We prove the last statement using complete induction:

*Proof.* $j = 0$ Follows immediately.

$j \geq 0, j \to j + 1$

$$P^{(j+1)}(x) = \left(P^{(j)}\right)'(x)$$

$$= \tilde{P}^{(j)}(x) + \tilde{P}^{(j+1)}(x)(x - x_0)$$

$$+ j\tilde{P}^{(j)}(x) = (x - x_0)\tilde{P}^{(j+1)}(x) + (j + 1)P^j(x).$$

$$P^{(j)}(x_0) = j\tilde{P}^{(j-1)}(x_0)$$

$$\begin{cases} = 0 & j = 0, \dots, k - 1 \\ \neq 0 & j = k \end{cases}$$

□

We then prove the second part: $\Leftarrow$.

Let $P^{(j)}(x_0) = 0$ for $j \in \{0, \dots, k - 1\}$, $P^{(k)}(x_0) \neq 0$. It holds that $P(y_0) = 0$ because of $P^{(0)}(x_0) = 0$. Like above: $P(x) = (x - x_0)\tilde{P}(x)$ and

$$P^{(j)}(x) = (x - x_0)\tilde{P}^{(j)}(x) + j\tilde{P}^{(j-1)}(x).$$

$$j \in \{1, \dots, k - 1\} \implies 0 = P^{(j)}(x_0) = j \cdot \tilde{P}^{(j-1)}(x_0)$$

$$\implies \forall l \in \{0, \dots, k - 2\} : \tilde{P}^{(l)}(x_0) = 0$$

$$0 \neq P^{(k)}(x_0) = k\tilde{P}^{(k-1)}(x_0) \Rightarrow \tilde{P}^{(k-1)}(x_0) \neq 0$$

induction hypothesis $\Rightarrow$

$$\tilde{P}(x) = (x - x_0)^{k-1}Q(x) \text{ with } Q(x_0) \neq 0$$

$$\implies P(x) = (x - x_0)\tilde{P}(x) = (x - x_0)^k Q(x).$$

□

**Theorem 25.** Let $f$ in $\mathbb{C}^n((a, b))$ with $n \in \mathbb{N}$. Let $a, b \in [-\infty, \infty]$, $x_0 \in (a, b)$. Find a polynomial $T$ of degree $n$ such property

$$\forall k \in \{0, \dots, n\} : T^{(k)}(x_0) = f^{(k)}(x_0). \tag{1}$$

Claim:

$$T_f^n(x) \equiv T_f^n(x; x_0) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

where $x_0$ is the base point, is the only polynmial of degree $n$, which satisfies property 1.

$T_f^n$ is called *Taylor polynomial* of $n$-th degree of $f$ in $x_0$.

*Proof.* Let $k \in \{0, \dots, n\}$.

$$(T_g^n)^{(k)}(x) = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} j(j-1) \cdot \ldots \cdot (j - (k-1))(x - x_0)^{j-k}$$

$$\left(T_f^n\right)^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!}\underbrace{(k \cdot \ldots \cdot (k-(k-1)))}_{=k!} = f^{(k)}(x_0).$$

Let $T(x) = \sum_{j=0}^n a_j x^j$ be a polynomial, which satisfies 1. For $P := T_g^n - T$ it holds that $P^{(k)}(x_0) = 0$ for all $k \in \{0, \ldots, n\}$. And $P$ is a polynomial of degree at most $n$. $x_0$ is at least an $(n+1)$-th root of $P \Rightarrow P \equiv 0$. □

**Definition 16** (Deviation, error, remainder)**.**

$$R_g^{n+1}(x; x_0) \equiv R_g^{n+1}(x) := f(x) - T_g^n(x; x_0)$$

**Theorem 26** (Integration form of the remainder)**.** Let $f \in C^{n+1}((a,b), \mathbb{C})$, $n \in \mathbb{N}$, $a, b \in [-\infty, \infty]$, $x_0, x \in (a,b)$. Then it holds that

$$R_g^{n+1}(x) = \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt$$

*Complete induction over $n$.* Let $n = 0$.

$$R_g^1(x) = f(x) - T_g^0(x) = f(x) - f(x_0)$$

$$\frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt = \int_{x_0}^x f'(t)\,dt = f(x) - f(x_0).$$

Consider $n \geq 1, n-1 \to n$. From induction hypothesis we consider

$$\begin{aligned}
\Rightarrow f(x) - T_g^{n-1}(x) = R_g^n(x) &= \frac{1}{(n-1)!}\int_{x_0}^x (x-1)^{n-1} f^{(n)}(t)\,dt \\
&= -\left.\frac{(x-t)^n}{n(n-1)!} f^{(n)}(t)\right|_{x_0}^x + \int_{x_0}^x \frac{(x-t)^n}{n(n-1)!} f^{(n+1)}(t)\,dt \\
&= \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt
\end{aligned}$$

$$\begin{aligned}
\Rightarrow R_f^{n+1}(x) = f(x) - T_g^n(x) &= f(x) - T_g^{n-1}(x) - \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\
&= \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{n+1}(t)\,dt
\end{aligned}$$

Recognize that we consider $f$ over $\mathbb{C}$. In the next theorem we will only consider it in $\mathbb{R}$. □



**Theorem 27** (Lagrange representation of remainder)**.** Let $f \in C^{n+1}((a,b), \mathbb{R}), n \in \mathbb{N}$, $a, b \in [-\infty, \infty]$, $x_0, x \in (a,b)$. Then there exists some $\xi$ between $x_0$ and $x$ such that

$$R_g^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

*Proof.*

$$R_f^{n+1}(x) = \frac{1}{n!}\int_{x_0}^x (x-t)^n f^{(n+1)}(t)\,dt$$

**Case 1:** $x \geq x_0$**:**

$$\forall t \in [x_0, x] : (x-t)^n \geq 0$$

$f \mapsto (x-1)^n$ regulated function. $t \mapsto f^{(n+1)}(t)$ continuous. Hence,

$$\exists \xi \in [x_0, x] : \int_{x_0}^x (x-1)^n f^{n+1}(t)\,dt = f^{n+1}(\xi)\int_{x_0}^x (x-t)^n\,dt$$

$$= f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{n+1}$$

$$\Rightarrow R_f^{n+1}(x) = \frac{f^{n+1}}{(n+1)!}(x-x_0)^{n+1}.$$

**Case 2:** $x < x_0$**:**

$$\forall t \in [x, x_0] : (t-x)^n \geq 0 \qquad \text{analogously}$$

$$\exists \xi \in [x, x_0] : \int_x^{x_0} (t-x)^n f^{(n+1)}(t)\,dt$$

$$
\begin{aligned}
&= f^{(n+1)}(\xi) \int_x^{x_0} (1-x)^n \, dt \\
&= \frac{f^{(n+1)}(\xi)}{n+1} (x_0 - x)^{n+1} \\
\Rightarrow R_g^{n+1}(x) &= \frac{(-1)^{n+1}}{n!} \int_x^{x_0} (t-x)^n f^{(n+1)}(t) \, dt \\
&= (-1)^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} (x_0 - x)^{n+1} \\
&= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}
\end{aligned}
$$

□

**Corollary 4** (Sufficient criterion for local extremes)**.** Let $f \in C^{n+1}((a,b),\mathbb{R}), x_0 \in (a,b)$ with $f^{(n)}(x_0) = \ldots = f^{(n)}(x_0) = 0, f^{(n+1)}(x_0) \neq 0$. Then $f$ has the following in $x_0$:

- a strict local minimum, if $n$ is odd and $f^{(n+1)}(x_0) > 0$.
- a strict local maximum, if $n$ is odd and $f^{(n+1)}(x_0) < 0$.
- no extreme, if $n$ is even.

*Proof.* **Case 1: $f^{(n+1)}(x_0) > 0$:**
$f^{(n+1)}$ is continuous $\Rightarrow$

$$\exists \varepsilon > 0 : f^{(n+1)} > 0 \text{ in } (x_0 - \varepsilon, x_0 + \varepsilon) =: I$$

by Induction hypothesis it holds that

$$\forall x \in (a,b) : f(x) = T_g^n(x) + R_g^{n+1}(x) = f(x_0) + R_f^{n+1}(x).$$

If $n$ is even, then $n+1$ is odd, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} > 0.$$

So,

$$\forall x \in I \setminus \{x_0\} : f(x) > f(x_0)$$

If $n$ is odd, $n+1$ is even, then

$$\forall x \in I \setminus \{x_0\} : \exists \xi \in I : R_f^{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

$$\begin{cases} > 0 & x > x_0 \\ < 0 & x < x_0 \end{cases}$$

$\Rightarrow$ $f$ has no extremum in $x_0$.

**Case 2:** $f^{(n+1)}(x_0) < 0$ follows analogously like Case 1.

□

**Theorem 28** (Qualitative Taylor formula)**.** Let $f \in C^n((a,b),\mathbb{C}), x, x_0 \in (a,b)$. There exists some $r \in C((a,b),\mathbb{C})$ with $r(x_0) = 0$ and

$$f(x) = T_f^n(x) + (x - x_0)^n r(x) \tag{2}$$

*Proof.* Equation 2 only has to be shown for $f : (a,b) \to \mathbb{R}$, because for $f : (a,b) \to \mathbb{C}$, $f = f_R + i f_I$ with $f_R, f_I : (a,b) \to \mathbb{R}$. Representations for $f_R$ and $f_I$ provide corresponding representations for $f$. Hence let $f : (a,b) \to \mathbb{R}$. Let $r : (a,b) \to \mathbb{R}$.

$$x \mapsto \frac{f(x) - T_f^n(x)}{(x - x_0)^n}, x \neq x_0 \text{ and } r(x_0) := 0$$

We only need to show:
$r$ is continuous in $x_0$, hence $\lim_{x \to x_0} r(x) = r(x_0) = 0$.

$$
\begin{aligned}
x \in (a,b) \setminus \{x_0\} \Rightarrow r(x) &= \frac{1}{(x-x_0)^n} \left( f(x) - T_f^{n-1}(x) - \frac{f^{(n)}}{n!} (x - x_0)^n \right) \\
&= \frac{1}{(x-x_0)^n} \left( R_g^n(x) - \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right) \\
&= \frac{1}{(x-x_0)^n} \left( \frac{f^{(n)}(\xi)}{n!} (x - x_0)^n - \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \right) \\
&= \frac{1}{n!} \left( f^{(n)}(\xi) - f^{(n)}(x_0) \right)
\end{aligned}
$$

$\xi$ is between $x_0$ and $x$. $f^{(n)}$ is continuous and $\xi \to x_0$ for $x \to x_0$

$$\Rightarrow r(x) = \frac{1}{n!}(f^{(n)}(\xi) - f^{(n)}(x_0)) \overset{x \to x_0}{\to} 0$$

□

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This lecture took place on 3rd of May 2016 with lecturer Wolfgang Ring.

**Theorem 29.** Assumption: Let $f : I \to \mathbb{R}$ be arbitrarily often continuously derivable. Hence,

$$T_f^n(x; x_0) \text{ exists for } \forall n \in \mathbb{N}$$

Therefore we can consider a power series

$$T_f(x; x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$T_f(x; x_0)$ is called Taylor series of $f$ in $x_0$. Is a power sereis in $\xi = (x - x_0)$. Converges for $|\xi| = |x - x_0| < \rho(T_f)$.

If $\rho(T_f) > 0$, it holds that $T_f(x; x_0) = f(x)$?

$$\lim_{n\to\infty} T_f^n(x; x_0) = T_f(x; x_0) = f(x) \text{ for } |x - x_0| < \rho(T_f)$$

is *not* always satisfied.

**Example 14.**

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Compare with Figure 22.

$$f_-^{(n)}(0) = 0$$

$$f_+^{(n)}(0) = \lim_{x\to 0^+} f^{(n)}(x)$$

$$f^{(n)}(x) = R(x) \cdot e^{-\frac{1}{x}}$$

with $R(x) = \frac{P(x)}{Q(x)}$ with $P$ and $Q$ as polynomials. $R$ is a rational function (i.e. division of two polynomials).



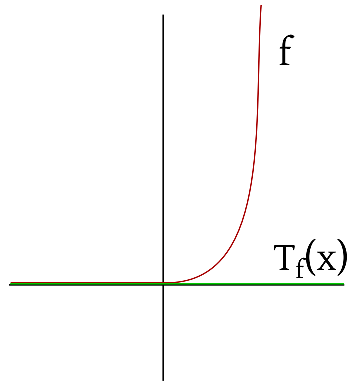


Figure 22: Plot of $f$

$$\lim_{x\to 0_+} R(x) \cdot e^{-\frac{1}{x}} = 0$$

Hence $f^{(n)}(0) = 0$ and therefore Taylor series $T_f(x; 0) = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0$.

**Remark 30.** Taylor:

$$R_f(x) = T_f(x; 0) - f(x)$$

It holds that

$$|R_f(x)| \leq c_n \cdot |x|^n \qquad \forall n \in \mathbb{N}$$

**Theorem 30.** Let $f(x) = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ be a power series in $\xi = x - x_0$. Let $\rho(f) > 0$. We already know that $f$ is differentiable for all $|\xi| = |x - x_0| < \rho(f)$ (differentiable by $x$) and $f'$ is a power series with convergence radius $\rho(f') = \rho(f)$.

$$f'(x) = \sum_{k=1}^{\infty} a_k \cdot k x^{k-1}$$

By complete induction it follows that:

- For all $n \in \mathbb{N}$ there exists $f^{(n)}(x)$ as power series of form

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k \cdot k \cdot (k-1) \cdot (k-2) \cdot \ldots (k-n+1) \cdot x^{k-n}$$

- $f^{(n)}$ as convergent power series is a continuous function. Hence,

$$f^{(n)}(x_0) = a_n \cdot n \cdot (n-1) \cdot (n-2) \ldots (n-n+1) = a_n \cdot n!$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

Backsubstitution in the power series yields

$$f(x) = \sum_{k=0}^{\infty} a_k (x-x_0)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = T_f(x; x_0)$$

Hence $f$ has a power series representation, then the power series is the Taylor series in $f$.

**Remark 31.** A function representable with a power series is called *analytical.* In the complex space, once differentiable means arbitrary often differentiable.

# 7 Curves in $\mathbb{R}^n$

**Definition 17.** A parametric curve is a map $\gamma : I \to \mathbb{R}^n$ where $I$ is an interval which has form

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

and every function $\gamma_i : I \to \mathbb{R}$ $(i = 1, \ldots, n)$ is continuous. Often we write $\gamma_i(t) = x_i(t)$. If every $\gamma_i$ is differentiable in $I$, a differentiable, parametric curve is given. $t$ is the curve parameter.

We call $\Gamma = \{\gamma(t) \,|\, t \in I\} = \gamma(I) \subseteq \mathbb{R}$ the *trace of the curve* $\gamma$.



**Example 15.**

$$\gamma : [0, 4\pi] \to \mathbb{R}^2$$
$$\gamma(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

In this example, every point on the curve is hit twice by the function.

$$\Gamma = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R} \,\middle|\, x_1^2 + x_2^2 - 1 = 0 \right\}$$

$F(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$ is called trace equation of the curve

$$\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \text{ in } I = [0, 4\pi]$$

If $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$, then

$$x_2^1 + x_2^2 - 1 = \cos^2(t) + \sin^2(t) - 1 = 1 - 1 = 0$$

On the inverse, let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ with $x_1^2 + x_2^2 = 1$. Then there exists $t \in [0, 2\pi)$ such that $x_1 = \cos t$ and $x_2 = \sin t$.

In this example it holds that $\tilde{\gamma} \neq \gamma$, but $T = \tilde{T}$.

**Example 16.** Let $\tilde{\gamma}(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ with $\tilde{\gamma} : [0, \pi] \to \mathbb{R}^2$.

$$\forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \tilde{T} : T(x_1, x_2) = x_1^2 + x_2^2 - 1 = 0$$

but

$$\tilde{T} \neq \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \,\middle|\, F(x_1, x_2) = 0 \right\}$$

**Definition 18.** Let $\gamma : I \to \mathbb{R}^n$ be a differentiable, parametric curve. We define

$$\dot{\gamma}(t) = \begin{bmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

and we call $\dot\gamma(t)$ the derivation vector of $\gamma$ in $t$. If $\gamma$ is considered as motion curve, then $\dot\gamma(t)$ is considered as speed vector of $\gamma$ in $t$.

Consider

$$\dot\gamma(t) = \lim_{h\to 0} \frac{1}{h}\left[\gamma(t+h) - \gamma(t)\right]$$

as illustrated in Figure 23.

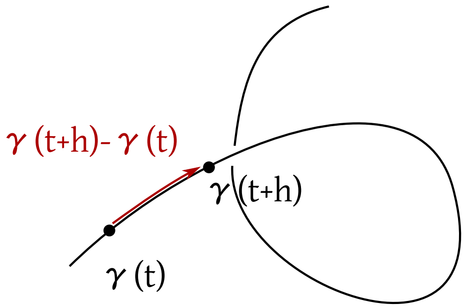


Figure 23: Curve example

If $\dot\gamma(t) \neq 0$, then $\dot\gamma$ is tangential in $\Gamma$ and we call $\dot\gamma(t)$ the *tangential vector of* $\gamma$ *in* $t$.

If $\dot\gamma(t) \neq 0$, we set

$$T_\gamma(t) = \frac{\dot\gamma(t)}{\|\dot\gamma(t)\|_2}$$

and we call $T_\gamma(t)$ the tangential unit vector of $\gamma$ in $t$.

**Example 17.**

$$\gamma : \mathbb{R} \to \mathbb{R}^2$$

$$\gamma(t) = \begin{bmatrix} t^2 - 1 \\ t^3 - 1 \end{bmatrix} \text{ differentiable}$$

$$\gamma(1) = \begin{bmatrix} 1 - 1 \\ 1 - 1 \end{bmatrix} = \vec{0}$$



$$\gamma(-1) = \begin{bmatrix} 1 - 1 \\ -1 + 1 \end{bmatrix} = \vec{0}$$

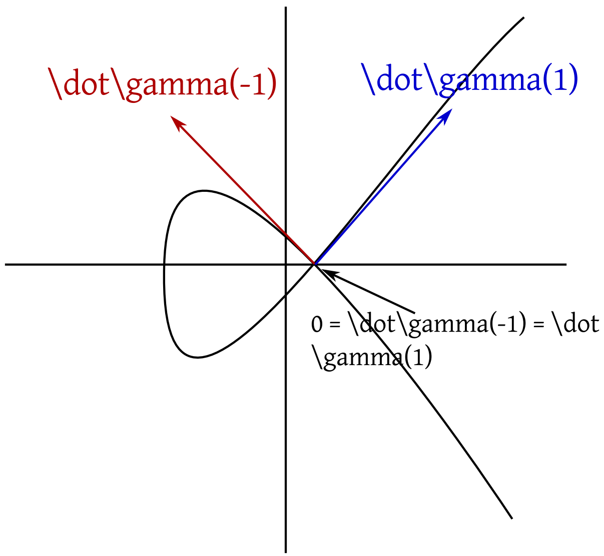


Figure 24: A double pointed curve

This curve has a double point, meaning that one point is crossed two times (compare with Figure 24).

$$\dot\gamma(t) = \begin{bmatrix} 2t \\ 3t^2 - 1 \end{bmatrix}$$

$$\dot\gamma(-1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\dot\gamma(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

**Definition 19.** Let $\gamma : I \to \mathbb{R}^n$ be a differentiable, parametric curve. $\gamma$ is called *regular* curve, if $\dot{\gamma}(t) \neq \vec{0} \quad \forall t \in I$.

**Example 18.**

$$\gamma(t) = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

is called *Neil's parabola* and non-regular.

$$\dot{\gamma}(t) = \begin{bmatrix} 2t \\ 3t^3 \end{bmatrix}$$

$$\dot{\gamma}(0) = \vec{0}$$

Has no tangent in the root.

**Example 19.**

$$\gamma(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix}$$

$a, b > 0$ and $t \in [0, 2\pi]$. We search for a trace equation of $\gamma$:

$$\frac{x_1(t)}{a} = \cos(t) \qquad \frac{x_2(t)}{b} = \sin(t)$$

We use the trace equation of the unit circle:

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 - 1 = 0$$

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

$\gamma$ has an ellipsis as trace with major axis lengths $a$ and $b$. The interpretation of curve parameter $t$ is given in Figure 26.

This lecture took place on 6th of May 2016 with lecturer Wolfgang Ring.



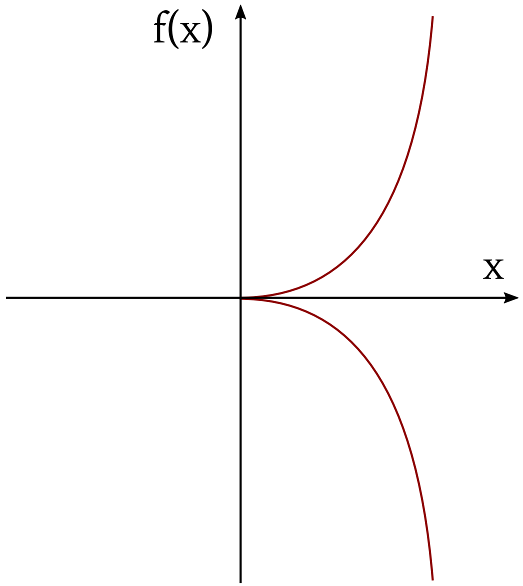


Figure 25: Neil's parabola

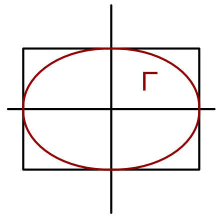

Figure 26: Curve parameter $t$

# 8 Hyperbolic functions

**Definition 20.** We define the *cosine and sinus hyperbolic functions* as follows:

$$\cosh : \mathbb{C} \to \mathbb{C} \qquad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$

$$\sinh : \mathbb{C} \to \mathbb{C} \qquad \sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$$

For real values we get Figure 27.

Properties:

$$\begin{aligned}
\cosh'(x) &= \frac{1}{2}(e^x - e^{-x}) \\
&= \sinh(x) \\
\sinh'(x) &= \frac{1}{2}(e^x + e^{-x}) \\
&= \cosh(x) \\
\cosh^2(x) - \sinh^2(x) &= \frac{1}{4}\left(e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right) \\
&- \frac{1}{4}\left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right) \\
&= \frac{1}{4} \cdot 4 \cdot 1 = 1 \\
\cosh^2(x) - \sinh(x) &= 1
\end{aligned}$$

**Example 20.** Let $y : \mathbb{R} \to \mathbb{R}^2$.

$$\gamma(t) = \begin{bmatrix} \underbrace{a\cosh(t)}_{>0} \\ b\sinh(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$\frac{(x(t))^2}{a^2} - \frac{(y(t))^2}{b^2} = \cosh^2(t) - \sinh^2(t) = 1$$



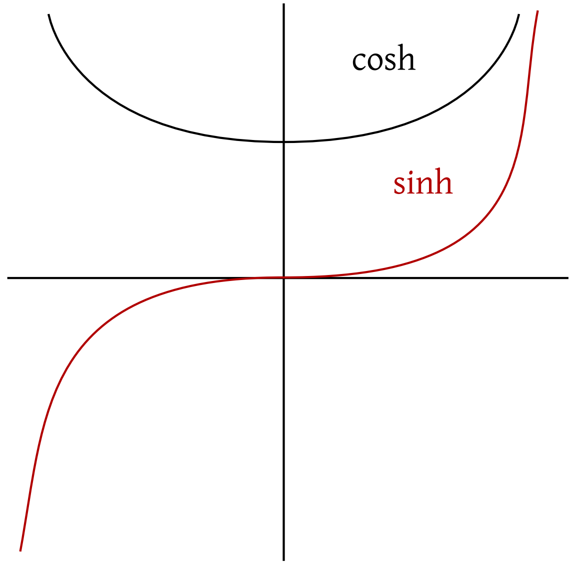


Figure 27: Plot of hyperbolic cosine and sine

hence the trace $T$ of $\gamma$ is inside the hyperbola

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\}$$

**Theorem 31.** Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ continuously differentiable. Then

$$\left.\begin{array}{c} t \mapsto \begin{bmatrix} t \\ f(t) \end{bmatrix} \\ I \to \mathbb{R}^2 \end{array}\right\}$$

a parametric, differentiable curve. The function graph is equivalent to the trace of the curve.

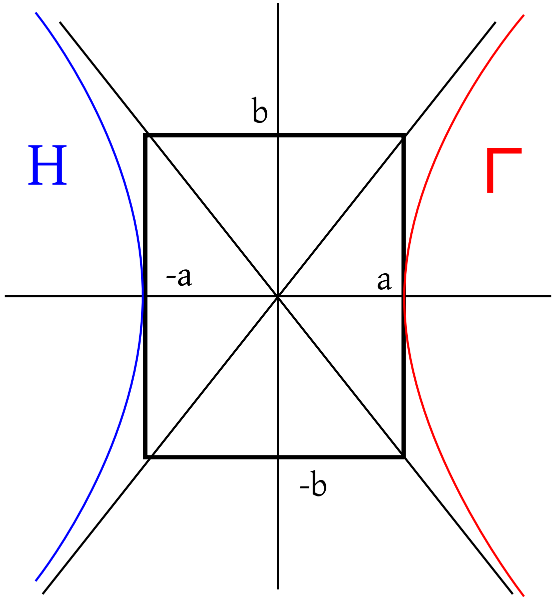


Figure 28: Hyperbola $H$

**Theorem 32** (Representation as function graph)**.** Let $\gamma : I \to \mathbb{R}^2$ a continuously differentiable curve, $I$ is an interval and

$$\gamma(t) = \begin{bmatrix}\gamma_1(t)\\ \gamma_2(t)\end{bmatrix} = \begin{bmatrix}x(t)\\ y(t)\end{bmatrix}$$

it holds that

$$\dot{\gamma}_1(t) \neq 0 \quad \forall t \in I$$

Then there exists a continuously differentiable function $f : J = \gamma_1(I) \to \mathbb{R}$ such that the graph of $f$ matches the trace of $\gamma$.

Let $x_0 = \gamma_1(t_0)$. Then it holds that

$$f'(x_0) = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}.$$



If $\gamma$ is differentiable twice in $t_0$, then $f$ is differentiable twice in $x_0$.

$$f''(x_0) = \frac{\dot{x}(t_0)\ddot{y}(t_0) - \ddot{x}(t_0)\dot{y}(t_0)}{[\dot{x}(t_0)]^3}$$

*Proof.* $\dot{y}_1$ has no root, is continuous, this means $\dot{\gamma}_1$ has a uniform sign in $I$. Hence $\gamma_1$ is strictly monotonical in $I$. So $\gamma_1 : I \to J = \gamma_1(I)$ is bijective.

Let $\gamma_1^{-1} : J \to I$ be the inverse function. Because $\dot{\gamma}_1 \neq 0$ in $I$ is $\gamma_1^{-1}$ is differentiable with

$$\left(\gamma_1^{-1}\right)'(s) = \frac{1}{\dot{\gamma}_1(\gamma_1^{-1}(s))}$$

We define

$$f(x) = \gamma_2(\gamma_1^{-1}(x))$$
$$I \to \mathbb{R}$$

Let $T_f = \{(x, f(x)) \,|\, x \in I\}$ be the graph of $f$ and $(x, f(x)) \in T_f$; $(x, f(x)) = (x, \gamma_2(\gamma_1^{-1}(x)))$. Let $\gamma_1^{-1}(x) = t \in I$ and therefore $x = \gamma_1(t)$. So it holds that

$$(x, f(x)) = (\gamma_1(t), \gamma_2(t)) \in T \qquad \ldots \text{ trace of } \gamma$$

On the opposite, we have $(\gamma_1(t), \gamma_2(t)) \in T$. Let $x = \gamma_1(t) \in J$ and $t = \gamma_1^{-1}(x)$ and $(\gamma_1(t), \gamma_2(t)) = (x, \gamma_2(\gamma_1^{-1}(x))) = (x, f(x)) \in T_f$.

$$f'(x)|_{x=x_0} = \dot{\gamma}_2(\gamma_1^{-1}(x_0)) \cdot \frac{1}{\dot{\gamma}_1^{-1}(x_0)} = \frac{\dot{y}(t_0)}{\dot{x}(t_0)}$$

Let $\gamma_1^{-1}(x_0) = t_0$.

$$\begin{aligned} f''(x_0) &= \frac{\ddot{\gamma}_2(t_0) \cdot \frac{1}{\dot{\gamma}_1(t_0)} \cdot \dot{\gamma}_1(t_0) - \dot{\gamma}_2(t_0) \cdot \ddot{\gamma}_1(t_0) \cdot \frac{1}{\dot{\gamma}_1(t_0)}}{(\dot{\gamma}_1(t_0))^2} \\ &= \frac{\ddot{\gamma}_2(t_0) \cdot \dot{\gamma}_1(t_0) - \ddot{\gamma}_1(t_0) \cdot \dot{\gamma}_2(t_0)}{(\dot{x}(t_0))^3} \\ &= \frac{\ddot{\gamma}(t_0) \cdot \dot{x}(t_0) - \dot{y}(t_0)\ddot{x}(t_0)}{(\dot{x}(t_0))^3} \end{aligned}$$

□

Figure 29 gives an example for a curve which is not representable as function graph of $x$ in $\dot{\gamma}(t_0) = 0$.

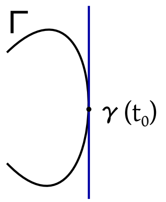


Figure 29: A parametric curve with $\dot{\gamma}(t_0) = 0$

# 9 Arc length of a parametric curve

**Theorem 33.** Let $\gamma : I = [a, b] \to \mathbb{R}^n$ be a parametric curve. Let $z = \{t_0 = a, t_1, t_2, \ldots, t_N = b\}$ with $t_{i-1} < t_i$ for $i = 1, \ldots, N$ be a partition of the interval $I$. We denote the length of the polygonal line through the partition points $\gamma(t_0)$, $\gamma(t_1)$, $\ldots$, $\gamma(t_N)$ with

$$s(z) = s_\gamma(z) = \sum_{i=1}^{N} \|\gamma(t_1) - \gamma(t_{i-1})\|$$

Let $z^*$ be more detailed than $z$. Then it holds that

$$s(z^*) \geq s(z)$$

*Insertion of partition points.* Let $z = \{t_0 < t_1 < \ldots < t_N\}$ and $z^* =$



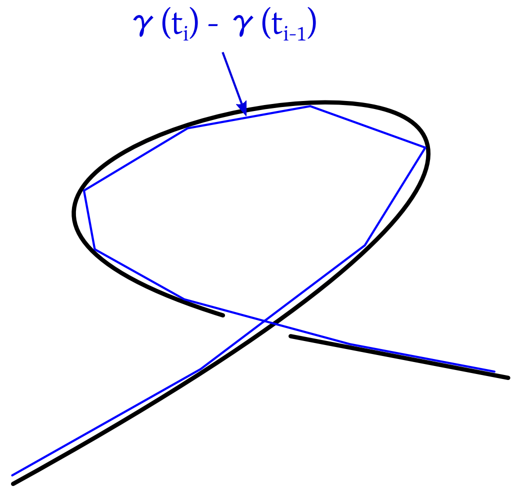


Figure 30: Approximation of the arc length

$\{t_0 < \ldots < t_{k-1} < t' < t_k < \ldots < t_N\}$.

$$\begin{aligned}
s(z) &= \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
s(z^*) &= \sum_{i=1}^{k-1} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
&\quad + \underbrace{\|\gamma(t') - \gamma(t_{k-1})\| + \|\gamma(t_k) - \gamma(t')\|}_{\geq \|\gamma(t_K) - \gamma(t') + \gamma(t') - \gamma(t_{K-1})\|} \\
&\quad + \sum_{i=K+1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\
&\geq \sum_{i=1}^{N} \|\gamma(t_1) - \gamma(t_{i-1})\| = s(z)
\end{aligned}$$

For insertion of multiple points use induction. $\square$

**Definition 21.** Let $\gamma : I = [a,b] \to \mathbb{R}^n$ be a continuous curve. $\gamma$ is call *rectifiable* if

$$s(\gamma) = \sup s(z) < \infty$$

where $z$ is a partition of $I$. In this case $s(\gamma)$ is called length of curve $\gamma$.

**Example 21.** Let $\gamma : I \to \mathbb{R}^n$ be Lipschitz continuous. Hence

$$\exists L \geq 0 : \|\gamma(s) - \gamma(t)\| \leq L(s-1)$$

for all $s, t \in I$. Then $\gamma$ is rectifiable and $s(\gamma) \leq L \cdot (b-a)$.

*Proof.* Let $z$ be a partition of $I$.

$$z = \{t_0 < t_1 < t_2 < \ldots < z_N\}$$

Then it holds that

$$\begin{aligned} s(z) &= \sum_{i=1}^{N} \|\gamma(t_i) - \gamma(t_{i-1})\| \\ &\leq \sum_{i=1}^{N} L\,|t_i - t_{i-1}| \\ &= L \sum_{i=1}^{N} (t_i - t_{i-1}) \\ &= L(t_N - t_0) = L(b-a) \end{aligned}$$

$\square$

**Theorem 34.** Let $\gamma : I \to \mathbb{R}^n$ be a continuous curve.

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$



Let every $\gamma_i : I \to \mathbb{R}$ be a primitive function of a regulated function (hence $\dot{\gamma}_i$ exists for all $t$ except for finitely many points, furthermore $\dot{\gamma}_i$ has left-sided and right-sided limits everywhere).

$$I = [a,b]$$

Then $\gamma$ is rectifiable and it holds that

$$s(\gamma) = \int_a^b \|\dot{\gamma}(t)\|\,dt$$

**Remark 32** (Some necessary preparations)**.**

$$\int_a^b \gamma(t)\,dt := \begin{bmatrix} \int_a^b \gamma_1(t)\,dt \\ \int_a^b \gamma_2(t)\,dt \\ \vdots \\ \int_a^b \gamma_n(t)\,dt \end{bmatrix}$$

**Lemma 14.** Let $\gamma : [a,b] \to \mathbb{R}^n$ be a continuous curve. Then it holds that

$$\left\| \int_a^b \gamma(t)\,dt \right\| \leq \int_a^b \|\gamma(t)\|\,dt$$

*Proof.* Let $\varphi_i$ be a step function which approximates $\gamma_i$ uniformly. Hence, we assume that $\|\varphi_i - \gamma_i\|_\infty < \varepsilon$. $z_i$ is a partition such that $\varphi_i$ is constant at every interval.

Define $z = \bigcup_{i=1}^n z_i$ ascendingly ordered. Then every $\varphi_i$ is also a step function in terms of $z$.

$$z = \{t_0 < t_1 < \ldots < t_N\}$$

$$\varphi_i(t) = c_k^i \quad \text{for } t \in (t_{k-1}, t_k)$$

$$\int_a^b \varphi_i(t)\,dt = \sum_{k=1}^{n} c_k^i (t_k - t_{k-1})$$

Build

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_N(t) \end{bmatrix}$$

Then it holds that

$$\left\|\int_a^b \varphi(t)\,dt\right\| = \left\|\begin{bmatrix}\sum_{k=1}^N c_k^1(t_k-t_{k-1})\\ \sum_{k=1}^N c_k^2(t_k-t_{k-1})\\ \vdots \\ \sum_{k=1}^N c_k^n(t_k-t_{k-1})\end{bmatrix}\right\| = \left\|\sum_{k=1}^N (t_k-t_{k-1})\cdot\begin{bmatrix}c_k^1\\ \vdots\\ c_k^n\end{bmatrix}\right\|$$

$$\underbrace{\leq}_{\text{triangle ineq. in }\mathbb{R}^n} \sum_{k=1}^N (t_k-t_{k-1})\left\|\begin{bmatrix}c_k^1\\ \vdots\\ c_k^n\end{bmatrix}\right\| = \int_a^b \underbrace{\|\varphi(t)\|}_{\text{step function in }\mathbb{R}}\,dt$$

$$\|\gamma(t)-\varphi(t)\| = \left(\sum_{i=1}^n |\gamma_i(t)-\varphi_i(t)|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{i=1}^n \varepsilon^2\right)^{\frac{1}{2}} = \sqrt{n}\cdot\varepsilon$$

$$\Rightarrow \int_a^b \|\gamma(t)-\varphi(t)\|\,dt < \varepsilon\sqrt{n}(b-a)$$

$$\left\|\int_a^b (\gamma(t)-\varphi(t))\,dt\right\| = \left\|\begin{bmatrix}\int_a^b \gamma_1(t)-\varphi_1(t)\,dt\\ \int_a^b \gamma_2(t)-\varphi_2(t)\,dt\\ \vdots\\ \int_a^b(\gamma_n(t)-\varphi_n(t))\,dt\end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix}\left|\int_a^b(\gamma_1(t)-\varphi_1(t))\,dt\right|\\ \left|\int_a^b(\gamma_2(t)-\varphi_2(t))\,dt\right|\\ \vdots\\ \left|\int_a^b(\gamma_n(t)-\varphi_n(t))\,dt\right|\end{bmatrix}\right\| \leq \left\|\begin{bmatrix}\varepsilon(b-a)\\ \varepsilon(b-a)\\ \vdots\\ \varepsilon(b-a)\end{bmatrix}\right\| = \varepsilon(b-a)\sqrt{n}$$

Hence it holds that

$$\left\|\int_a^b \gamma(t)\,dt\right\| = \left\|\int_a^b \varphi(t)\,dt - \int_a^b(\varphi(t)-\gamma(t))\,dt\right\|$$



$$\leq \left\|\int_a^b \varphi(t)\,dt\right\| + \left\|\int_a^b(\varphi(t)-\gamma(t))\,dt\right\|$$

$$\leq \int_a^b \|\varphi(t)\|\,dt + \varepsilon(b-a)\sqrt{n}$$

$$\leq \int_a^b (\|\varphi(t)-\gamma(t)\| + \|\gamma(t)\|)\,dt + \varepsilon(b-a)\sqrt{n}$$

$$\leq \varepsilon(b-a)\sqrt{n} + \int_a^b \|\gamma(t)\|\,dt + \varepsilon(b-a)\sqrt{n}$$

Hence

$$\left\|\int_a^b \gamma(t)\,dt\right\| \leq \int_a^b \|\gamma(t)\|\,dt + 2\varepsilon(b-a)\sqrt{n} \qquad \forall\varepsilon>0$$

Hence

$$\left\|\int_a^b \gamma(t)\,dt\right\| \leq \int_a^b \|\gamma(t)\|\,dt$$

□

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This lecture took place on 10th of May 2016 with lecturer Wolfgang Ring.

*Proof of the formula for the arc length.* Its definition depends on the parameterization.

$$s(\gamma) = \sup_z s(z)$$

$$s(z) = \sum_{k=1}^N \|\gamma(t_k)-\gamma(t_{k-1})\|$$

We show:

1. For all decompositions of $z$, it holds that

$$s(z) \leq \int_a^b \|\dot\gamma(t)\|\,dt$$

$$\Rightarrow s(\gamma) \leq \int_a^b \|\dot\gamma(t)\|\,dt$$

2.
$$\forall \varepsilon > 0 \exists \text{ decomposition } z : s(\gamma) \geq s(z) \geq \int_a^b \|\dot\gamma(t)\| \, dt - \varepsilon$$

1. Let $z = \{t_0 < t_1 < \ldots < t_N\}$.

$$\begin{aligned} s(z) &= \sum_{k=1}^N \|\gamma(t_k) - \gamma(t_{k-1})\| \\ &\overset{\substack{\text{fundamental}\\\text{theorem}}}{=} \sum_{k=1}^N \left\| \int_{t_{k-1}}^{t_k} \dot\gamma(t) \, dt \right\| \\ &\overset{\text{Lemma}}{=} \sum_{n=1}^n \int_{t_{k-1}}^{t_k} \|\dot\gamma(t)\| \, dt \\ & \sum_{n=1}^N TODO \end{aligned}$$

2. Let $\varepsilon > 0$ be arbitrary. Find decomposition $z$ such that $s(\gamma) \geq s(z) \geq \int_a^b \|\dot\gamma(t)\| \, dt - \varepsilon$. Let

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_n(t) \end{bmatrix}$$

and $\varphi_t$ is a step function in $[a, b]$.

Every $\varphi_i$ is constant in $(t_{k-1}, t_k)$ for $k = 1, \ldots, N$. and we let $z = \{t_0, t_1, \ldots, t_N\}$. Let $\varphi_i$ such that $\|\dot\gamma_i - \varphi_i\|_\infty \leq \frac{\varepsilon}{2(b-a)\sqrt{N}}$.

Then it holds that $\forall t \in [a, b]$:

$$\|\dot\gamma(t) - \gamma(t)\| = \left( \sum_{i=1}^n |\dot\gamma_i(t) - \varphi_i(t)| \right)^{\frac{1}{2}}$$

$$\leq \left( \sum_{i=1}^n \|\dot\gamma_i - \varphi_i\|_\infty^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^n \left( \frac{\varepsilon}{2(b-a)}^2 \cdot \frac{1}{n} \right) \right)^[ \frac{1}{2}] = \frac{\varepsilon}{2(b-a)}$$



We let

$$\begin{aligned} \|\dot\gamma - \varphi\|_\infty &= \sup \{ \|\dot\gamma(t) - \varphi(t)\|_2 : t \in [a, b] \} \\ &= \max \{ \|\dot\gamma(t) - \varphi(t)\|_2 : t \in [a, b] \} \end{aligned}$$

It holds that

$$\|\dot\gamma - \varphi\|_\infty < \frac{\varepsilon}{2(b-a)}$$

$$z = \{t_0, t_1, \ldots, t_N\}$$

$$\left\| \int_{t_{k-1}}^{t_k} \dot\gamma(t) \, dt \right\| = \left\| \int_{t_{k-1}}^{t_k} (\dot\gamma(t) - \varphi(t)) \, dt + \int_{t_{k-1}}^{t_k} \varphi(t) \, dt \right\|$$

$$\geq \left\| \int_{t_{k-1}}^{t_k} \varphi(t) \, dt \right\| - \left\| \int_{t_{k-1}}^{t_k} (\dot\gamma(t) - \varphi(t)) \, dt \right\|$$

$\varphi$ is constant and the right summand is $\leq \int_{t_{k-1}}^{t_k} \|\dot\gamma(t) - \varphi(t)\| \, dt$.

$$\geq \int_{t_{k-1}}^{t_k} \|\varphi(t)\| \, dt - \int_{t_{k-1}}^{t_k} \underbrace{\|\dot\gamma(t) - \varphi(t)\|}_{< \frac{\varepsilon}{2(b-1)}} \, dt$$

$$> \int_{t_{k-1}}^{t_k} \|\varphi(t)\| \, dt - \frac{\varepsilon}{2(b-a)} (t_k - t_{k-1})$$

$$\begin{aligned} s(z) = \sum_{k=1}^N \left\| \int_{t_{k-1}}^{t_k} \dot\gamma(t) \, dt \right\| &> \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|\varphi(t)\| - \frac{\varepsilon}{2(b-a)} (t_k - t_{k-1}) \\ &= \int_a^b \|\varphi(t)\| \, dt - \frac{\varepsilon}{2(b-a)} \underbrace{(t_N - t_0)}_{=b-a} \\ &= \int_a^b \|\varphi(t)\| \, dt - \frac{\varepsilon}{2} \end{aligned}$$

$$\int_a^b \|\varphi(t)\| \, dt = \int_a^b \|\varphi(t) - \dot\gamma(t) + \dot\gamma(t)\| \, dt$$

$$\geq \int_a^b \left( \|\dot\gamma(t)\| - \underbrace{\|\varphi(t) - \dot\gamma(t)\|}_{< \frac{\varepsilon}{2(b-a)}} \right) dt$$

$$= \int_a^b \|\dot\gamma(t)\|\, dt - \frac{\varepsilon}{2}$$

$$\Rightarrow s(z) > \int_a^b \|\dot\gamma(t)\|\, dt - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}$$

$$\Rightarrow s(\gamma) \geq s(z) > \int_a^b \|\dot\gamma(t)\|\, dt - \varepsilon \qquad \forall \varepsilon > 0$$

$$\Rightarrow s(\gamma) \geq \int_a^b \|\dot\gamma(t)\|\, dt$$

**Example 22** (Circumference of a circle with radius $r$)**.**

$$\gamma_r(t) = r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \qquad t \in [0, 2\pi]; \dot\gamma_t(t) = r \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$$

$$s(\gamma_r) = \int_0^{2\pi} \left\| r \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \right\| dt = \int_0^{2\pi} r\, dt = 2\pi r$$

**Example 23** (Ellipsis)**.**

$$\gamma(t) = \begin{bmatrix} a\cos(t) \\ b\sin(t) \end{bmatrix} \qquad a, b > 0$$

$$\dot\gamma(t) = \begin{bmatrix} -a\sin(t) \\ b\cos(t) \end{bmatrix}$$

$$\|\dot\gamma(t)\| = (a^2 \underbrace{\sin^2(t)}_{1-\cos^2(t)} + b^2\cos^2(t))^{\frac{1}{2}}$$

Let $a \geq b$, $\varepsilon^2 = 1 - \frac{b^2}{a^2}$.

$$\begin{aligned} \|\dot\gamma(t)\| &= (a^2 - (a^2 - b^2)\cos^2(t))^{\frac{1}{2}} \\ &= a\left(1 - \left(1 - \frac{b^2}{a^2}\right)\cos^2(t)\right)^2 \\ &= a\left(1 - \varepsilon^2\cos^2(t)\right)^{\frac{1}{2}} \end{aligned}$$

$$s(\gamma) = a \int_0^{2\pi} \sqrt{1 - \varepsilon^2 \cos^2(t)}\, dt$$

This defines a new set of functions which cannot be solved with means we discussed so far. They are called *elliptic integral.*

□

## 9.1 Change of parameters, reparameterization

Let $\sigma : I \to J$ as smooth (ie. differentiable) as required. $\sigma$ is bijective and $\sigma^{-1} : J \to I$ is be part of the same differentiation class like $\sigma$. Let $\gamma : I \to \mathbb{R}^n$ be a curve. We call $\beta = \gamma \circ \sigma^{-1} : I \to \mathbb{R}^n$ a reparameterization of $\gamma$ using $\sigma$. Compare with Figure 31.

$\sigma$ is called parameter transformation. $\gamma$ is called orientation preserving, if $\sigma$ is strictly monotonically decreasing.

A measure, defined by the curve (arc length, tangential vector, curvature, ...) is called *geometric*, if reparameterization can be applied without modifications.

$s(\gamma)$ is obviously a geometric measure, because

1. By definition of polygonal lines
2. Let $\beta(\tau) = \gamma \circ \sigma^{-1}(\tau)$.

$$\dot\beta(\tau) = \dot\gamma(\sigma^{-1}(\tau)) \circ (\sigma^{-1})'(\tau)$$

$$\left\|\dot\beta(\tau)\right\| = \left\|\dot\gamma(\sigma^{-1})(\tau)\right\| \cdot \left|(\sigma^{-1})'(\tau)\right|$$

**Case $\sigma$ is orientation preserving** If and only if $\sigma' > 0 \Leftrightarrow (\sigma^{-1})' > 0$

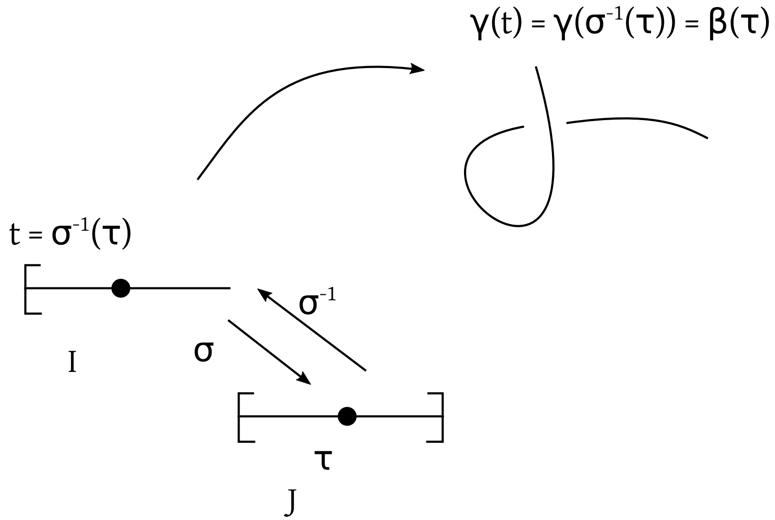


Figure 31: Reparameterization: $\beta$ and $\gamma$ have the same trace

Let $I=[a,b]; J=[c,d]; c=\sigma(a); d=\sigma(b)$.

$$s(\beta)=\int_c^d \left\|\dot\gamma(\underbrace{\sigma^{-1}(\tau)}_{=t})\right\|\cdot\underbrace{(\sigma^{-1})'(\tau)\,d\tau}_{dt}$$

$$=\int_a^b \|\dot\gamma(t)\|\;dt=s(\gamma)\qquad\text{(by substitution)}$$

**Case $\sigma$ is orientation inversing**

$$\gamma'<0\qquad(\gamma^{-1})'<0$$

$$\left|(\sigma^{-1})'(\tau)\right|=-(\sigma^{-1})'(\tau)\qquad\sigma(a)=d,\sigma(b)=c$$



$$\int_c^d \left\|\dot\beta(\tau)\right\|\;d\tau=\int_c^d\left\|\dot\gamma(\underbrace{\sigma^{-1}(\tau)}_{t})\right\|\cdot\underbrace{\left(-(\sigma^{-1})'(\tau)\right)\;d\tau}_{dt}$$

$$\left|\begin{matrix}\tau=c\Leftrightarrow t=b\\ \tau=a\Leftrightarrow t=a\end{matrix}\right|$$

$$=-\int_b^a\|\dot\gamma(t)\|\;dt$$

$$=\int_a^b\|\dot\gamma(t)\|\;dt=s(\gamma)$$

## 9.2 Reparameterization by arc length

We consider a regular curve $\gamma$, hence $\|\dot\gamma(t)\|>0\quad\forall t\in I$ and let $s:I\to J=S(I)$ by

$$s(t)=\int_a^t\|\dot\gamma(\tau)\|\;d\tau$$

$s(t)$ is the length of the curve $\gamma$ between $a$ and $t$. Let $s(a)=0$. It holds that $\dot s(t)=\|\dot\gamma(t)\|>0$ (by the Fundamental Theorem of Differential and Integration Theory), hence $s$ is strictily monotonically increasing. We use $s$ for reparameterization.

$$\beta(\xi)=\gamma\circ s^{-1}(\xi)$$

is a reparameterization of $\gamma$ by the arc length.

$$\begin{aligned}\left\|\dot\beta(\xi)\right\|&=\left\|\dot\gamma(s^{-1}(\xi))\circ(s^{-1})'(\xi)\right\|\\&=\left\|\dot\gamma(s^{-1}(\xi))\frac{1}{\dot s(s^{-1}(\xi))}\right\|\\&=\left\|\dot\gamma(s^{-1})(\xi)\right\|\cdot\left|\frac{1}{\dot s(s^{-1}(\xi))}\right|\\&=\frac{\left\|\dot\gamma(s^{-1}(\xi))\right\|}{\left\|\dot\gamma(s^{-1}(\xi))\right\|}=1\end{aligned}$$

Hence the tangential vector is the unit vector (in every point)

$$s_\beta(\xi) = \int_0^\xi \underbrace{\left\|\dot\beta(\eta)\right\|}_{=1} d\eta = \xi$$

So the curve parameter corresponds to the arc length. On the opposite: Let $\gamma : I \to \mathbb{R}^n$ with property $\|\dot\gamma(t)\| = 1 \quad \forall t \in I = [0, b]$. Then it holds that

$$s(t) = \int_0^t \underbrace{\|\dot\gamma(\tau)\|}_{=1} d\tau = t$$

So it holds that $s = s^{-1} = \mathrm{id}_{[0,b]}$. So $\gamma$ is parameterized by the arc length.

**Remark 33** (Notation)**.** We don't write $\xi = s(t)$, but $s = s(t)$.

Reparameterization by the arc length:

$$\beta(s) = \gamma(s^{-1}(s)) = \gamma(t)$$

---

This lecture took place on 12th of May 2016 with lecturer Wolfgang Ring.

## 9.3 Invariance of arc length

Let $\gamma : I \to \mathbb{R}$ be a parametric curve.

$\sigma : I \to J$ orientation-preserving parameter transformation

$$S_\gamma(t) = \int_a^t \|\dot\gamma(\xi)\|\, d\xi$$

$$I = [a, b] \qquad J = [c, d]$$

$$\tilde\gamma = \gamma \circ \sigma^{-1} : J \to \mathbb{R}^n \quad \text{reparameterization}$$

$$S_{\tilde\gamma}(\tau) = \int_C^\tau \left|\dot{\tilde\gamma}(\eta)\right| d\eta$$



We know that $S_{\tilde\gamma}(\tau) = S_{\tilde\gamma}(\sigma(t)) = S_\gamma(t)$.

$$S_{\tilde\gamma} \circ \sigma = S_\gamma$$

Let $S = S_\gamma(t)$ and $\beta$ is a reparameterization of $\gamma$ by its arc length. Hence $\beta(s) = \gamma(s_\gamma^{-1}(s))$ and $\beta = \gamma \circ S_\gamma^{-1}$.

$$\tilde\beta(s) = \tilde\gamma \circ S_{\tilde\gamma}^{-1} = \gamma \circ \sigma^{-1} \circ \sigma \circ S_\gamma^{-1} = \gamma \circ S_\gamma^{-1} = \beta(s)$$

Hence, reparametric curves $\gamma$ and $\tilde\gamma$ have the same reparameterization by its arc length $\beta$.

We require orientation preservation (compare with Figure 32).

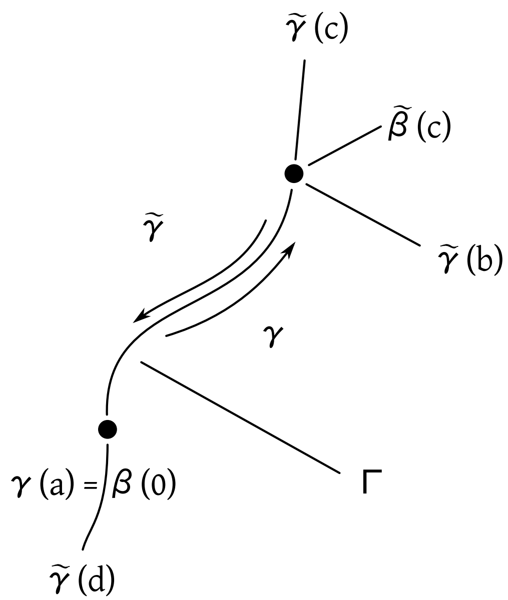


Figure 32: Invariance of arc length

Then it holds that
$$\tilde{\beta}(s) = \beta(s(\gamma) - s)$$

Consider special case $\gamma : I \to \mathbb{R}^2$.

$$\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}; \qquad S_\gamma(t) = \int_a^t \sqrt{\dot{x}(\xi)^2 + \dot{y}(\xi)^2}\, d\xi$$

or even more special:

$$\gamma(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix} \qquad \text{...function graph}$$

$$S_\gamma(t) = S_f(t) = \int_a^t \sqrt{1 + (f'(\xi))^2}\, d\xi$$

## 9.4 Curvature

Curvature corresponds to the rate of change of the direction of motion. This corresponds to the rate of change of

$$T(t) = \frac{\dot{\gamma}(t)}{|\gamma(t)|}$$

in regards of the arc length.

1. Let $\gamma : I \to \mathbb{R}^2$ be a parameterized regular curve. $\beta(s)$ is the reparameterization of $\gamma$ by its arc length. $\beta : [0, s(\gamma)] \to \mathbb{R}^2$.

   $$\dot{\beta}(s) = T(s) \quad \text{is a unit vector}$$

   It holds that $\langle \dot{\beta}(s), \dot{\beta}(s) \rangle = 1$, hence $\dot{\beta}_1^2(s) + \dot{\beta}_2^2(s) = 1$. $\beta$ can be differentiated twice.

   So we derive $\dot{\beta}_1^2(s) + \dot{\beta}_2^2(s)$:

   $$2\dot{\beta}_1(s) \cdot \ddot{\beta}_1(s) + 2\dot{\beta}_2(s) \cdot \ddot{\beta}_2(s) = 0$$

   So it holds with

   $$\ddot{\beta}(s) = \begin{bmatrix} \ddot{\beta}_1(s) \\ \ddot{\beta}_2(s) \end{bmatrix} \qquad \langle \dot{\beta}(s), \ddot{\beta}(s) \rangle = 0$$

   $\ddot{\beta}$ is orthogonal to $\dot{\beta} = T$. We define $N = \begin{bmatrix} -\dot{\beta}_2 \\ \dot{\beta}_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot T$.

**Definition 22.** We define a *signed* curvature $\kappa$ in $\gamma$ in point $\gamma(t) = \beta(s)$ by its relation
$$\frac{d^2\beta}{ds^2} = \ddot{\beta}(s) = \kappa(S) \cdot N(S)$$

$\kappa$ (with this property) actually exists, because $\ddot{\beta}$ is orthogonal to $T$ and therefore a multiple of $N$.

In case of reparameterization $\gamma(t) = \tilde{\gamma}(\tau)$, the arc length stays the same. Therefore the curvature in $\gamma(t)$ and $\dot{\gamma}(\tau)$ is also the same. Hence the curvature is invariant in terms of orientation-preserving reparameterization.

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This lecture took place on 13th of May 2016 with lecturer Wolfgang Ring.

From now on: We denote the derivative by arc length of a function $f(s)$ using "$f'(s)$".

Again:

$$T'(s) = \kappa(S) \cdot N(S)$$
$$\langle T'(s), N(s) \rangle = \kappa(S) \cdot \underbrace{\langle N(S), N(S) \rangle}_{=1}$$
$$\kappa(S) = \langle T'(s), N(s) \rangle$$

**Example 24.**
$$\gamma(t) = r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

where $r$ is the radius. It holds that $\|\dot{\gamma}(t)\| = r$.

$$S_\gamma(t) = \int_0^t r\, d\xi = r \cdot t \implies t = \frac{s}{r} \implies \beta(s) = r \begin{bmatrix} \cos \frac{s}{r} \\ \sin \frac{s}{r} \end{bmatrix}$$

$$\beta'(s) = r \cdot \frac{1}{r} \begin{bmatrix} -\sin \frac{s}{r} \\ \cos \frac{s}{r} \end{bmatrix} = \begin{bmatrix} -\sin \frac{s}{r} \\ \cos \frac{s}{r} \end{bmatrix} = T(s)$$

$$T'(s) = \frac{1}{r} \begin{bmatrix} -\cos \frac{s}{r} \\ -\sin \frac{s}{r} \end{bmatrix}$$

$$N(s) = \begin{bmatrix} -\cos(\frac{s}{r}) \\ -\sin(\frac{s}{r}) \end{bmatrix} = D \cdot T(s)$$

$$\kappa(s) = \langle T'(s), N(s) \rangle = \frac{1}{r} \cdot 1 = \frac{1}{r}$$

For an arbitrary curve $\gamma$ we define the *curvature radius* in point $\gamma(t) = \beta(s)$ as $\rho(s) = \frac{1}{\kappa(s)}$ for $\kappa(s) \neq 0$. $\rho(s) = \infty$ if $\kappa(s) = 0$.

**Remark 34.** Hence the curvature radius of the circle line is $r$. $\gamma^-(t)$ goes in counter-clockwise direction along the circumference of the circle. If $\gamma^-(t) = r \cdot \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$, an analogous calculation can be made: $\kappa_{\gamma^-}(s) = -\frac{1}{r}$.

**Theorem 35.** Let $\gamma : I \to \mathbb{R}^2$ be a regular, twice continuously differentiable curve. Then it holds that

$$\kappa(t) = \kappa_\beta(s) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{\frac{3}{2}}}$$

if $\gamma(t) = \begin{bmatrix} t \\ f(t) \end{bmatrix}$ (function graph) it holds that

$$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{\frac{3}{2}}}$$

*Proof.*

$$\beta(s) = \gamma(\underbrace{t(s)}_{s_\gamma^{-1}(s)})$$

$$\beta'(s) = \dot{\gamma}(t(s)) \cdot \underbrace{t'(s)}_{\frac{1}{\dot{s}_\gamma(s_\gamma^{-1}(s))}} = \dot{\gamma}(t(s)) \cdot \frac{1}{\|\dot{\gamma}(t(s))\|} = \frac{1}{\dot{s}_\gamma(t(s))} \cdot \dot{\gamma}(t(s))$$

Let $\frac{1}{\lambda} = -\frac{1}{(\dot{s}_\gamma(t(s)))^2} \cdot \ddot{s}_\gamma(t(s)) \cdot t'(s) \cdot \dot{\gamma}(t(s))$.

$$\beta''(s) = -\frac{1}{(\dot{s}_\gamma(t(s)))^2} \cdot \ddot{s}_\gamma(t(s)) \cdot t'(s) + \frac{1}{s'_\gamma(t(s))} \cdot \ddot{\gamma}(t(s)) \cdot \underbrace{\frac{1}{\dot{s}_\gamma(t(s))}}_{t'(s)}$$

$$\kappa(t) = \kappa(s) = \langle N, T' \rangle = \langle D\beta'(s), \beta''(s) \rangle$$



$$= \left\langle D \cdot \frac{1}{\dot{s}_\gamma} \dot{\gamma}(t(s)), \frac{1}{\dot{s}_\gamma(t(s))^2} \cdot \ddot{\gamma}(t(s)) - \lambda \dot{\gamma}(t(s)) \right\rangle$$

where

$$\left\langle D \cdot \frac{1}{\dot{s}_\gamma} \dot{\gamma}(t(s)), \ddot{\gamma}(t(s)) - \lambda \dot{\gamma}(t(s)) \right\rangle = 0$$

TODO: missing contentn

$$\Longrightarrow \kappa(t) = \frac{\dot{x}(t)\ddot{(t)} - \dot{\gamma}(t)\dot{x}(t)}{(\dot{x}^2(t) + \dot{\gamma}^2(t))^{\frac{3}{2}}}$$

□

**Definition 23.** Let $\gamma : I \to \mathbb{R}^2$ be a regular, twice continuously differentiable curve. Let $\rho(t) = \frac{1}{\kappa(t)}$ be the curvature radius in $\gamma(t)$.

Then let

$$\underbrace{m(t)}_{\mathbb{R}^2} = \gamma(t) + \rho(t) \cdot N(t)$$

and denote $m(t)$ as *curvature center* of $\gamma$ in $\gamma(t)$ (compare with Figure 33).

**Remark 35.** The error (deviation) from the osculating circle is of order $\mathcal{O}^2$.

**Remark 36.** The curve of curvature centers $t \to m(t)$ is called *evolute* of $\gamma$.

## 9.5 Situation in $\mathbb{R}^3$

Let $\gamma : I \to \mathbb{R}^3$ be parameterized by its arc length. It is twice continuously differentiable.

$$T(s) = \beta'(s) \qquad \ldots \text{tangential vector}$$

Analogously to $\mathbb{R}^2$ it holds that $\langle T'(s), T(s) \rangle = 0$ (this follows from $\langle T(s), T(s) \rangle = 1$). Hence the unit vector $N(s) := \frac{T'(s)}{\|T'(s)\|}$ is normal to $T(s)$ (only for $T'(s) \neq 0$) and we let $\kappa(s) = \|T'(s)\| \geq 0$. Also it holds that $\kappa(s)N(s) = T'(s)$ and $\langle T'(s), N(s) \rangle = \kappa(s)$.

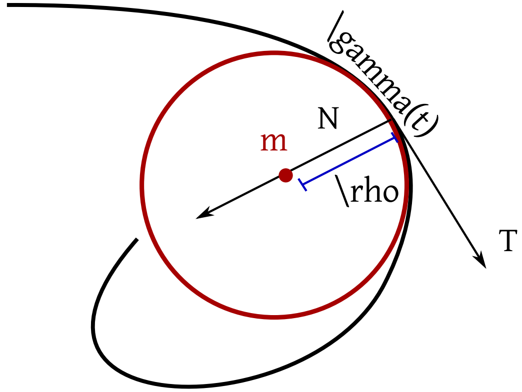


Figure 33: Curvature center $m$. A circle with center $m$ and radius $\rho$ is called curvature radius or "osculating circle" (dt. "Schmiegkreis") of $\gamma$ in $\gamma(t)$.

## 9.6 Analysis in $\mathbb{R}^n$

We look at some important inequalities:

**Theorem 36** (Jensen's inequality)**.** Let $I$ be an interval, $f : I \to \mathbb{R}$ is convex.

$$\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1] \text{ and } \sum_{k=1}^{n} \lambda_k = 1$$

Let $x_1, \ldots, x_n \in I$. Then it holds that

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n)$$

*Proof.* By complete induction over $n$:



**Case $n = 1$:** Trivial

**Case $n = 2$:** convexity

**Case $n \to n+1$:** left as an exercise to the reader

□

**Remark 37.** Reminder: $f$ is convex if

$$\forall x_1, x_2 \in I, \lambda \in [0,1] : f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

**Remark 38.** Is $f$ strictly convex, then equality holds only for $x_1 = x_2 = \ldots = x_n = x$.

**Theorem 37.** Inequality between the weighted geometric and the weighted arithmetic mean.

Let $x_1, \ldots, x_n$ positive, real numbers. Let $\lambda_i \in [0,1]$ for $i = 1, \ldots, n$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Then it holds that

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n} \leq \lambda_1 x_1 + \ldots + \lambda_n x_n$$

Especially for $\lambda_i = \frac{1}{n} \forall i.$:

$$\underbrace{\sqrt[n]{x_1 x_2 \ldots x_n}}_{\text{geometric mean}} \leq \underbrace{\frac{1}{n}(x_1 + \ldots + x_n)}_{\text{arithmetic mean}}$$

Equality in the equality above only holds for $x_1 = x_2 = \ldots = x_n = x$.

*Proof.* Consider $f : (0, \infty) \to \mathbb{R}$ and $x \mapsto \ln(x)$.

$$f''(x) = -\frac{1}{x^2} < 0 \text{ hence } f \text{ is concave}$$

$-f$ is convex. Jensen's inequality for $-f$:

$$-\ln \underbrace{(\lambda_1 x_1 + \ldots + \lambda_n x_n)}_{\text{convex combination}} \leq -\lambda_1 \ln(x_1) - \ldots - \lambda_n \ln(x_n)$$

$$\begin{aligned}\ln(\lambda_1 x_1 + \ldots + \lambda_n x_n) &\geq \lambda_1 \ln x_1 + \ldots + \lambda_n \ln x_n \\ &= \ln x_1^{\lambda_1} + \ln x_2^{\lambda_2} + \ldots + \ln x_n^{\lambda_n} \\ &= \ln(x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \ldots \cdot x_n^{\lambda_n})\end{aligned}$$

Exponential function:

$$\implies \lambda_1 x_1 + \ldots + \lambda_n x_n \geq x_1^{\lambda_1} \ldots x_n^{\lambda_n}$$

Equality in Jensen's inequality holds only for $x_1 = x_2 = \ldots = x_n$. □

**Definition 24.** Let $x \in \mathbb{R}^n$. $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We define

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \quad \text{for } p \geq 1$$

$\|x\|_p$ is called *p-norm of $x$*. For $p = 2$, $\|x\|_2$ is called *Euclidean norm of $x$*.

$$\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, n\}$$

is called *maximum norm* or *$\infty$-norm*.

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This lecture took place on 19th of May 2016 with lecturer Wolfgang Ring.

Reminder:

**Remark 39.** Let $V$ be a vector space. Then $\|.\| : V \to [0, \infty)$ is called *norm*, if

- $\|x\| = 0 \Leftrightarrow x = 0$ in $V$
- $\forall \lambda \in \mathbb{R}$ $(\mathbb{C})$ it holds that $\|\lambda x\| = |\lambda| \, \|x\| \, \forall x \in V$
- $\forall x, y \in V$ it holds that $\|x + y\| \leq \|x\| + \|y\|$

A norm defines defines a distance function (metric) in $V$ with $d(x, y) = \|x - y\|$.

Analogously,

$$f : I \to \mathbb{R} \qquad I = [a, b]$$



$$\|p\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

Let $V \in \mathbb{R}^n$.

$$V : \{1, 2, \ldots, n\} \to \mathbb{R}$$

$$v(k) = v_k$$

$$\int v \underbrace{dz}_{\text{counting measure}} := \sum_{k=1}^n v_k$$

$$\left(\int |V|^p \, dz\right)^{\frac{1}{p}} = \left(\sum_{k=1}^n |V_k|^p\right)^{\frac{1}{p}} = \|v\|_p$$

It holds that $\|.\|_p$ for $p \geq 1$ and $\|.\|_\infty$ are norms in $\mathbb{R}^n$.

1. $$\|.\|_\infty : \|v\|_\infty = 0 \Rightarrow \max\{|v_k| : k = 1, \ldots, n\}$$
   $$\Rightarrow |v_k| = 0 \text{ for } k = 1, \ldots, n \Leftrightarrow v = \vec{0} \in \mathbb{R}^n$$

2. $$\|\lambda v\|_\infty = \max\left\{\underbrace{|\lambda v_k|}_{|\lambda| \cdot |v_k|} : k = 1, \ldots, n\right\}$$
   $$= |\lambda| \cdot \max\{|v_k| : k = 1, \ldots, n\} = |\lambda| \cdot \|v\|_\infty$$

3. $$\begin{aligned}\|v + w\|_\infty = \max\{|v_k + w_k| : k = 1, \ldots, n\} \\ &\leq \max\{|v_k| + |w_k| : k = 1, \ldots, n\} + \max\{|w_k| : \\ &= \|v\|_\infty + \|w\|_\infty\end{aligned}$$

Hence $\|.\|_\infty$ is norm in $\mathbb{R}^n$. For $\|.\|_p$-norms it holds that $p \geq 1$.

1. Let $\|v\|_p = 0 \Rightarrow (\sum_{k=1}^n |v_k|^p)^{\frac{1}{p}} = 0$.

$$\Leftrightarrow \sum_{k=1}^n \underbrace{|v_k|^p}_{\geq 0} = 0 \Leftrightarrow |v_k|^p = 0 \text{ for } k = 1, \dots, n$$

$$\Leftrightarrow v_k = 0 \text{ for } k = 1, \dots, n \iff v = \vec{0} \text{ in } \mathbb{R}^n$$

2. Let $\lambda \in \mathbb{R}$.

$$\|\lambda \cdot v\|_p = \left(\sum_{k=1}^n |\lambda v_k|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{k=1}^n |\lambda^p| \cdot |v_k|^p\right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{k=1}^n |v_k|^p\right)$$

$$= |\lambda| \left(\sum_{k=1}^n |v_k|^p\right)^{\frac{1}{p}} = |\lambda| \cdot \|v\|_p$$

3. Consider $p = 1$.

$$\|v + w\|_1 = \left(\sum_{k=1}^n |v_k + w_k|^1\right)^{\frac{1}{1}} \leq \left(\sum_{k=1}^n (|v_k| + |w_k|)\right) = \|v\|_1 + \|w\|_1$$

We want to consider $p > 1$. But first we need to talk about Hölder's inequality in $\mathbb{R}^n$.

**Theorem 38** (Hölder's inequality)**.** Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $v, w \in \mathbb{R}^n$. Then it holds that

$$\sum_{k=1}^n |v_k \cdot w_k| \leq \|v\|_p \cdot \|w\|_q$$

*Proof.* $v = \vec{0}$ holds, analogously $w = \vec{0}$ holds.

Hence let $\|v\|_p \neq 0$ and $\|w\|_q \neq 0$. Inequality of geometrical and arithmetical mean for $\lambda_1 = \frac{1}{p}$, $\lambda_2 = \frac{1}{q}$ and

$$x_1 = \frac{|v_k|^p}{\|v\|_p^p} \qquad x_2 = \frac{|w_k|^q}{\|w\|_q^q}$$



$$\frac{|v_k|}{\|v\|_p} \cdot \frac{|w_k|}{\|w\|_q} \leq \frac{1}{p}\frac{|v_k|^p}{\|v\|_p^p} + \frac{1}{q}\frac{|w_k|^q}{\|w\|_q^q}$$

Sum over $k = 1, \dots, n$.

$$\frac{1}{\|v\|_p \cdot \|w\|_q} \sum_{k=1}^n |v_k \cdot w_k| \leq \frac{1}{p}\frac{\sum_{k=1}^n |v_k|^p}{\|v\|_p^p} + \frac{1}{q}\frac{\sum_{k=1}^n |w_k|^q}{\|w\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1 \implies \sum_{k=1}^n |v_k \cdot w_k| \leq \|v\|_p \cdot \|w\|_q$$

□

**Theorem 39** (Minkovski inequality)**.** Let $p > 1$.

$$\forall v, w \in \mathbb{R} : \|v + w\|_p \leq \|v\|_p + \|w\|_p$$

*Proof.* Let $s_k = |v_k + w_k|^{\overbrace{p-1}^{>0}}$. We chose $q$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

$$\left(\frac{1}{q} = \frac{1-p}{p}; q = \frac{p}{p-1}\right)$$

$$|v_k + w_k|^p = |v_k + w_k| \cdot |s_k| \overset{\text{triangle ineq. for 1st expr}}{\leq} |v_k| \cdot |s_k| + |w_k| \cdot |s_k|$$

$$\underbrace{\sum_{k=1}^n |v_k + w_k|^p}_{=\|v+w\|_p^p} \leq \sum_{k=1}^n |v_k \cdot s_k| + \sum_{k=1}^n |w_k \cdot s_k|$$

$$\overset{\text{Hölder's ineq}}{\leq} \|v\|_p \cdot \|s\|_q + \|w\|_p \cdot \|s\|_q$$

with $s = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$.

$$|s_k|^q = |v_k + w_k|^{(p-1) \cdot q} = |v_k + w_k|^p$$

$$\|s_k\|_q = \left(\sum_{k=1}^{n} |v_k + w_k|^p\right)^{\frac{1}{q}\cdot\frac{p}{p}} = \|v+w\|_p^{\frac{p}{q}}$$

It holds that $\frac{p}{q} = p - 1$.

$$= \|v+w\|_p^{p-1}$$

Insert into $\|v\|_p \cdot \|s\|_q + \|w\|_p \cdot \|s\|_q$:

$$\|v+w\|_p \leq \|p\| \, \|v+w\|_p^{p-1} + \|w\|_p \, \|v+w\|_p^{n-1}$$

$$\Rightarrow \|v+w\|_p \leq \|v\|_p + \|w\|_p$$

□

This lecture took place on 20th of May 2016 with lecturer Wolfgang Ring.

**Remark 40** (Notation)**.**

$$v \in \mathbb{R}^n \qquad v = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}$$

Coordinate index is at top.

**Definition 25.** Let $V$ be a vector space and $\|.\|_1$ and $\|.\|_2$ are two norms in $V$. We call $\|.\|_1$ and $\|.\|_2$ equivalent if $\exists m, M > 0$ such that for all $v \in \mathbb{V}$ the relation.

$$m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1\ \textbf{??} \tag{3}$$

Remark: Let $N = \{\|.\| \mid \|.\| \text{ is norm in } V\}$. Then Equation **??** defines an equivalence relation in $N$.

1. Reflexivity
2. Let $m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1$ Then it holds that $\|v\|_1 \geq \frac{1}{M}\|v\|_2$ because $\|v_2\| \leq M\,\|v\|_1$. Furthermore $\|v\|_1 \leq \frac{1}{m}\|v\|_2$ because $m\,\|v\|_1 \leq \|v\|_2$. Hence we get symmetry.
3. Let
   (a) $m\,\|v\|_1 \leq \|v\|_2 \leq M\,\|v\|_1$
   (b) and $m'\,\|v\|_2 \leq \|v\|_3$
   (c) and $\|v\|_3 \leq M'\,\|v\|_2$.

$$\Rightarrow m \cdot m'\,\|v\|_1 \overbrace{\leq}^{(1)} m'\,\|v\|_2 \overbrace{\leq}^{(2)} \|v\|_3 \overbrace{\leq}^{(3)} M' \\ \leq M' \cdot M\,\|v\|_1$$

Transitivity



# 10 Topological terms in normed vector spaces

(Convergence, continuity, open set, compactness, etc.)

Let $(V, \|.\|)$ be a normed vector space.

**Definition 26** (Convergence)**.** Let $x_n \in V$ for $n \in \mathbb{N}$ and $x \in V$. We state $(x_n)_{n\in\mathbb{N}}$ converges to $x$ in regards of $\|.\|$, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow \|x_n - x\| < \varepsilon]$$

Notation: $\lim_{n\to\infty} x_n = x$ i.e. $x_n \to x$ for $n \to \infty$ (we replace norm by vertical bars for absolute value).
Remark: We explicitly tell for which norm the expression converges. If no norm is explicitly mentioned, we mean the only norm we talk about (denoted $x_n \overset{\|.\|}{\to} x$).

**Remark 41.** Let $(\xi)_{n\in\mathbb{N}}$ and $\xi_n \in \mathbb{N}_+$ with $\xi_n = \|x_n - x\|$. Then it holds that $x_n \overset{\|.\|}{\to} x$ for $x \to \infty$ in $V \Leftrightarrow \xi_n \to 0$ for $n \to \infty$ in $\mathbb{N}$.

Because $\xi_n \to 0$ in $\mathbb{R}$,

$$\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow \left| \underbrace{\|x_n - x\|}_{\xi_n \text{ with } |\xi_n| = \xi} \right| < \varepsilon]$$

**Definition 27** (Cauchy sequence in $V$)**.** Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $V$. We claim that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall\varepsilon > 0 \exists N \in \mathbb{N} : [n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon]$$

**Definition 28.** $(V, \|.\|)$ is called *complete* if every Cauchy sequence in $V$ is a convergent sequence in $V$, hence has a limit. A complete, normed vector space is called *Banach space.*

Stefan Banach (1892–1945)

**Remark 42.** Every convergent sequence is also a Cauchy sequence. Proof as in $\mathbb{R}$ (relevant for exam).

**Definition 29.** Let $f : D \leq V \to W$, $V$ and $W$ are normed vector spaces. Let $x_0 \in D$. $f$ is called continuous in $x_0$, if

$$\forall\varepsilon > 0 \exists\delta > 0 : \forall x \in D : \|x - x_0\|_V < \delta \Rightarrow \|f(x) - f(x_0)\|_w < \varepsilon$$

Continuity is another concept, which depends in the norms used in $V$ and $W$.

**Remark 43.** Let $V$ and $W$ be normed vector spaces, $f : D \subseteq V \to W$, $\overline{x} \in D$. Then $f$ is continuous in $\overline{x}$

$$\Leftrightarrow [\forall(x_n)_{n\in\mathbb{N}} \text{ with } x_n \in D \wedge \lim_{n\to\infty} x_n = \overline{x} \text{ in terms of } \|.\|_v$$

$$\implies \lim_{n\to\infty} f(x_n) = f(\overline{x}) \text{ in terms of } \|.\|_W]$$

*Proof.* Proof as in $\mathbb{R}$ (but not relevant for exam). ☐

**Definition 30.** Let $f : D \subseteq V \to W$ and $V, W$ are normed vector spaces. Then $f$ is called *continuous in* $D$ if $f$ is continuous in every point $\overline{x} \in D$.

**Definition 31.** Let $V$ be a normed vector space.

1. Let $x \in V$ and $r \geq 0$. We let $B(x,r) := \{y \in V \mid \|y - x\| < r\}$. $B(x,r)$ is a ball (in terms of $\|.\|$) with center $x$ and radius $r$. $B(x,r) := \{y \in V \mid \|y - x\| \leq r\}$ is a closed ball with center $x$ and radius $r$.



2. $O \subseteq V$ is called open if $\forall x \in O \exists r > 0 : B(x,r) \subseteq O$. $A \subseteq V$ is called *closed* (dt. abgeschlossen), if $V \setminus A$ is open in $V$.

**Remark 44.** $B(x,r)$ is open for $r \geq 0$. $\overline{B(x,r)}$ is closed for $r \geq 0$.

$$B(x,O) = \emptyset \qquad \overline{B(x,O)} = \{x\}$$

*Proof.* Left to the reader as an exercise. ☐

**Lemma 15.** Let $(V, \|.\|)$ be a normed vector space. Then it holds that,

1. $\varphi$ is open in $V$, $V$ is open in $V$
2. Let $O_i$ with $i \in I$ be a family of open sets. Then the union
$$\bigcup_{i\in I} O_i = \{x \in V \mid \exists i \in I : x \in O_i\}$$
is open.
3. Let $O_1, O_2, \ldots, O_N$ be a set of open sets. Then $\bigcap_{k=1}^{N} O_k$ is open.

*Proof.* 1. It holds that

$$\forall x \in \varphi \exists r > 0 : B(x,r) \subseteq \varphi$$
$$\forall x \in V : \exists r > 0 : B(x,r) \subseteq V$$

For example for $r = 1$.

2. Let $O = \bigcup_{i\in I} O_i$, $O_i$ is open and let $x \in O$. Hence $\exists i \in I : x \in O_i$, because $O_i$ is open, there exists some $r > 0$ such that
$$B(x,r) \subseteq O_i \implies B(x,r) \subseteq O_i \subseteq \bigcup_{i\in I} O_i$$
3. Let $O = \bigcap_{k=1}^{N} O_k$ and $x \in O$. Hence $\forall k \in \{1, \ldots, N\} : x \in O_n$ and therefore exist $r_k > 0$ such that $B(x, r_k) \subseteq O_k$ (because every $O_k$ is open). Choose $r = \min\left\{\underbrace{r_1}_{>0}, \underbrace{r_2}_{>0}, \ldots, \underbrace{r_N}_{>0}\right\} > 0$. Then it holds that
$$B(x,r) \subseteq B(x,r_k) \subseteq O_k \text{ for } k = 1, \ldots, N \implies B(x,r) \subseteq \bigcap_{k=1}^{N} O_k = O$$

□

**Definition 32.** A system of sets $\tau \subseteq \mathcal{P}(V)$ (where $\mathcal{P}(V)$ denotes the power set of $V$) which satisfy properties 1, 2 and 3 is called *topology in* $V$.

**Lemma 16.** Let $(V, \|.\|)$ be a normed vector space. Then it holds that

4. If $\varphi$ is closed, then $V$ is closed.
5. If $I$ is an index set, $(A_i)_{i\in I}$ is a family of closed sets. Then it holds that $\bigcap_{i\in I} A_i = \{x \in V : \forall i \in I \mid x \in A_i\}$ is closed in $V$.
6. Let $A_1, A_2, \dots, A_N$ be closed. Then the union $\bigcup_{k=1}^{N} A_k$ is closed in $V$.

*Proof.* Follows by generation of complement generation and DeMorgan's laws. □

**Definition 33** (Limit point, Contact point, inner point)**.** Let $M \subseteq V$ where $V$ is a normed vector space.

- $x \in M$ is called *inner point* of $M$ if $r > 0$ exists, such that $B(x,r) \subseteq M$.
- $x \in V$ is called *contact point* $M$ if $\forall r > 0 : B(x,r) \cap M \neq \varphi$.
- $x \in V$ is called *limit point* of $M$ if $\forall r > 0 : (B(x,r) \setminus \{x\}) \cap M \neq 0$ $((B(x,r) \setminus \{x\})$ is also called *pointed ball* $\dot{B}(x,r))$.

**Remark 45.** $O \subseteq V$ is open iff $\forall x \in O : x$ is inner point of $O$. $A \subseteq V$ is closed iff $\forall x \in V$ where $x$ is contact point of $A$, $x \in A$ holds.

*Proof.* Let $A$ be closed and $x$ is contact point of $A$. Assume $x \notin A$, hence $x \in V \setminus A$ is open. Because $V \setminus A$ is open, there exists $r > 0$:

$$B(x,r) \subseteq V \setminus A \implies B(x,r) \cap A = 0$$

is a contradiction to $x$ is a contact point of $A$. On the opposite, let every contact point of $A$ be element of $A$. Then it holds that $\forall x \notin A$ (hence $x \in V \setminus A$) that $x$ is not a contact point of $A$.

$$\implies \exists r > 0 : B(x,r) \cap A = \varphi$$
$$\implies B(x,r) \subseteq V \setminus A$$

Hence $V \setminus A$ is open and therefore $A$ is closed. □



**Definition 34.** Let $M \subseteq V$. Then $\mathring{M} = \{x \in M \mid x \text{ is inner point of } M\}$. $\mathring{M}$ is called *open kernel* of $M$. $\overline{M} = \{x \in V \mid x \in \text{ contact point}\}$ $\overline{M}$ is closed cover of $M$.

**Lemma 17.** $\forall M \subseteq V$ it holds that

- $\mathring{M}$ is open, $\mathring{M} \subseteq M$
- $\overline{M}$ is closed, $M \subseteq \overline{M}$
- $M$ is open iff $M = \mathring{M}$
- $M$ is closed iff $M = \overline{M}$

*Proof.* $\mathring{M} \subseteq M$. Let $x \in \mathring{M}$.

Show that there exists $r > 0$ such that $B(x,r) \subseteq \mathring{M}$. Choose $r > 0$ such that $B(x,r) \subseteq M$.

Show that $\forall y \in B(x,r)$ it holds that $y \in \mathring{M}$.

$$y \in B(x,r) \implies \|y-x\| < r$$

Choose $r_y = r - \|y-x\| > 0$. We show

$$B(y, r_y) \subseteq B(x,r) \subseteq M$$

hence $y \in \mathring{M}$.

Let $z \in B(y, r_y)$, hence $\|z-y\| < r_y$. It follows that

$$\|z-y\| = \|z-y+y-x\| \leq \|z-y\|+\|y-x\| < r_y+\|y-x\| = r-\|y-x\|+\|y-x\| = r$$

Therefore $\|z-x\| < r$, hence $z \in B(x,r)$. So it holds that $B(y,r_y) \subseteq B(x,r) \subseteq M$, so $y \in \mathring{M}$. □

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This lecture took place on 24th of May 2016 with lecturer Wolfgang Ring.

**Lemma 18.** Let $M \subseteq V$. Then it holds that

1. $\mathring{M}$ is open, if $\mathring{M} \subseteq M$.
2. $\overline{M}$ is closed, if $M \subseteq \overline{M}$.

3. $M$ is open, iff $M = \mathring{M}$.
4. $M$ is closed, iff $M = \overline{M}$.

*Proof.* 1. Trivial.

2. Direction $\Rightarrow$: $M$ is open, so every $x \in M$ is an inner point of $M$, so $\forall x \in M : x \in \mathring{M}$.

   Direction $\Leftarrow$: $M = \mathring{M}$ is open, so $M$ is open.

3. Let $x \in M$. For all $r > 0$, $B(x, r) \cap M \supseteq \{x\} \neq \emptyset$, so $x \in \overline{M}$. It remains to show: $\overline{M}$ is closed. We show $V \setminus \overline{M}$ is open. Let $y \notin \overline{M}$. $\exists r > 0$ such that $B(y, r) \cap M = \emptyset$. Let $x \in B(y, r)$. So $\rho := r - |z - y| > 0$ and

$$\forall w \in B(z, \rho) : |w - y| \leq |w - z| + |z - y| < \rho + |z - y| = r$$

$$\implies \forall w \in B(z, \rho) : w \in B(y, r) \implies B(z, \rho) \subset B(y, r).$$

$$B(z, \rho) \cap M \subseteq B(y, r) \cap M = \emptyset \text{ because } B(y, r) \cap M = \emptyset$$

$$\implies z \notin \overline{M}.$$

   Hence $B(y, r) \subset V \setminus \overline{M}$ and therefore $V \setminus \overline{M}$ is open.

4. Direction $\Leftarrow$: $M = \overline{M}$. Due to the second property, $M$ must be closed.

   Direction $\Rightarrow$: It remains to show: $\overline{M} \subset M$. Let $x \in \overline{M}$. Assume $x \notin M$. $x \notin M \Rightarrow x \in V \setminus M$ is open.

$$\implies \exists r > 0 : B(x, r) \subset V \setminus M$$

$$\implies B(x, r) \cap M \subset V \setminus M \cap M = \emptyset$$

$$\implies x \notin \overline{M}$$

   Is a contradiction. Hence $x \in M$.

$\square$

**Lemma 19.** Let $x \in V$.

1. $x$ is a contact point of $M$ ($x \in \overline{M}$)

$$\iff \exists (x_n)_{n \in \mathbb{N}} \subseteq M : \lim_{n \to \infty} x_n = x$$

2. $x$ is a limit point of $M$

$$\iff \exists (x_n)_{n \in \mathbb{N}} \subseteq M \setminus \{x\} : \lim_{n \to \infty} x_n = x$$

*Proof.* We show the first property, but the second follows analogously.

Direction $\Rightarrow$: Let $x$ be a contact point in $M$. We know $B\left(x, \frac{1}{n}\right) \cap M \neq \emptyset$ for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N} \exists x_n \in M : \|x_n - x\| < \frac{1}{n} \land \lim_{n \to \infty} x_n = x$$

Analogously for the second property:

Let $x$ be a limit point in $M$. We know $B\left(x, \frac{1}{n}\right) \setminus \{x\} \cap M \neq \emptyset$ for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N} \exists x_n \in M \setminus \{x\} : \|x_n - x\| < \frac{1}{n}$$

And $\lim_{n \to \infty} x_n = x$.

Direction $\Leftarrow$: Let $x = \lim_{n \to \infty} x_n$ with $x_n \in M$. (analogously: $x_n \in M \setminus \{x\}$). Let $r > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \geq N : \|x_n - x\| < r$.

$$x_N \in B(x, r) \implies x_N \in B(x, r) \cap M$$

Analogously:

$$x_N \in B(x, r) \implies x_N \in B(x, r) \setminus \{x\} \cap M$$

Hence $x$ is contact point (analogously: limit point) of $M$. $\square$

**Example 25.** Consider unit circle and point $(2, 0)$ in $V := \mathbb{R}^2$.

$$M := \overline{B(0, 1)} \cup \{(2, 0)\}$$

$(2, 0)$ is a contact point of $M$, but not limit point of $M$.

**Corollary 5.** Let $M$ be closed and $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in $M$. Then $x := \lim_{n \to \infty} x_n \in M$.

*Proof.* From Lemma 19 it follows that $x \in \overline{M} \implies x \in M$. $\square$

**Lemma 20.** Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms in $V$ and $(x_n)_{n\in\mathbb{N}} \subset V$, $x \in V$. $m\|v\|_1 \leq \|v\|_2 \leq M\|v\|_1$.

$$\implies \left(\lim_{n\to\infty} \|x_n - x\|_1 = 0 \implies x_n \to x \text{ in } (V, \|.\|_1)\right)$$

$$\iff \left(\lim_{x_n - x} \|x_n - x\|_2 = 0 \implies x_n \to x \text{ in } (V, \|.\|_2)\right)$$

*Proof.* We only show direction $\Rightarrow$, because the other direction follows analogously.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\forall n \geq N : \|x_n - x\|_1 < \frac{\varepsilon}{M}$. Let $n \geq N$, then $\|x_n - x\|_2 \leq M\|x_n - x\|_1 < \varepsilon$. So $\lim_{n\to\infty} \|x_n - x\|_2 = 0$. $\square$

**Lemma 21.** Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms in $V$ and $O \subseteq V$. Then $O$ is open in $(V, \|.\|_1)$ if and only if $O$ is open in $(V, \|.\|_2)$.

*Proof.* Let $O$ be open in $(V, \|.\|_1)$. Let $x \in O$.

$$\exists r > 0 : B_{\|.\|_1}(x, r) \subseteq O$$

Consider $B_{\|.\|_2}(x, r\cdot m)$. Let $y \in B_{\|.\|_2}(x, r\cdot m)$. Then

$$\|y - x\|_1 \leq \frac{1}{m}\|y - x\|_2 < \frac{1}{m} r \cdot m = r$$

So $y \in B_{\|.\|_1}(x, r)$. Hence $B_{\|.\|_2}(x, r\cdot m) \subset B_{\|.\|_1}(x, r) \subseteq O$ where $r \cdot m > 0$. So $O$ is open in $(V, \|.\|_2)$. $\square$

**Remark 46.** $A \subset V$ is closed in $(V, \|.\|_1)$ if and only if $A$ is closed in $(V, \|.\|_2)$.

**Lemma 22.** Let $V, W$ be normed vector spaces. $f : D \subset V \to W$, $\overline{x} \in D$. Let $\|.\|_{1,v}$ and $\|.\|_{2,v}$ be equivalent norms with $\|.\|_{1,W}$ and $\|.\|_{2,W}$ in $V$ and $W$ respectively.

Then it holds that

$$f : D \subset (V, \|.\|_{1,V}) \to (W, \|.\|_{1,W}) \text{ is continuous in } \overline{x}$$
$$\iff f : D \subset (V, \|.\|_{2,V}) \to (W, \|.\|_{2,W}) \text{ is continuous in } \overline{x}$$

*Proof.* Let $(x_n)_{n\in\mathbb{N}} \subset V$ with $x_n \xrightarrow[x\to\infty]{\|.\|_{2,V}} \overline{x}$, hence $\lim_{n\to\infty} \|x_n - \overline{x}\|_{2,V} = 0$. Show: $\lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{2,W} = 0$, hence $f(x_n) \xrightarrow[n\to\infty]{\|.\|_{2,W}} f(\overline{x})$. It holds that

$$\lim_{n\to\infty} \|x_n - \overline{x}\|_{1,V} = 0 \overset{\text{VS}}{\implies} \lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{1,W} = 0$$

$$\Rightarrow \lim_{n\to\infty} \|f(x_n) - f(\overline{x})\|_{2,W} = 0$$

Hence, $f$ is continuous in terms of $\|.\|_{2,V}$ and $\|.\|_{2,W}$ in $\overline{x}$ by the sequence criterion. $\square$

**Lemma 23.** Consider $V := \mathbb{R}^n$. Let $1 \leq i \leq n$. Then the projective map

$$p_i : (\mathbb{R}^n, \|.\|_\infty) \to (\mathbb{R}, |.|)$$

$$p_i(x) = x^i \qquad \text{denotes the } i\text{-th component}$$

continuous in $\mathbb{R}^n$. Furthermore $|p_i(x)| = |x^i| \leq \|x\|_\infty$.

*Proof.* Let $\varepsilon > 0$, $x \in \mathbb{R}^n$. Let $\delta := \varepsilon$. Let $z \in \mathbb{R}^n$ with $\|z - x\|_\infty < \delta = \varepsilon$. Then

$$|p_i(z) - p_i(x)| = |z^i - x^i| \leq \max\left\{|z^i - x^i| \,\middle|\, i \in \{1, \ldots, n\}\right\} = \|z - x\|_\infty < \varepsilon$$

So $p_i$ is continuous in $x$. $\square$

**Lemma 24.** Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{\hat{n}}$, $x \in \mathbb{R}^{\hat{n}}$. Then it holds that

$$\left(\lim_{n\to\infty} \|x_n - x\|_\infty = 0 \implies x_n \to x \text{ in } (\mathbb{R}^{\hat{n}}, \|.\|_\infty)\right) \iff$$

$$\left(\forall i \in \{1, \ldots, \hat{n}\} : \lim_{n\to\infty} |x_n^i - x^i| = 0 \implies \forall i \in \{1, \ldots, \hat{n}\} : x_n^i \to x^i \text{ in } (\mathbb{R}, |\cdot|)\right)$$

*Proof.* Direction $\Rightarrow$: Trivial because of continuity of all $p_i$.

Direction $\Leftarrow$: Let $\varepsilon > 0$. Choose $N_i \in \mathbb{N}$ with $\forall n \geq N_i : |x_n^i - x^i| < \varepsilon$. Let $N := \max\{N_i \,|\, 1 \leq i \leq \hat{n}\}$. For $n \geq N$ it holds that

$$\|x_n - x\|_\infty = \max\left\{|x_n^i - x^i| \,\middle|\, 1 \leq i \leq \hat{n}\right\} < \varepsilon$$

because $n \geq N_i$ for all $1 \leq i \leq \hat{n}$. Hence $\lim_{n\to\infty} \|x_n - x\|_\infty = 0$. $\square$

**Definition 35.** Let $K \subseteq V$ where $V$ is a normed vector space. Then $K$ is called *sequence-compact* if and only if every sequence $(x_n)_{n\in\mathbb{N}} \subseteq K$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, which converges and whose limit $x$ is in $K$.

$M \subset V$ is called *bounded* iff $\exists R > 0 \forall x \in M : \|x\| \leq R$.

**Lemma 25.** Let $V$ be a normed vector space. $K \subseteq V$. Then it holds that

$$K \text{ is compact} \implies K \text{ is closed and bounded.}$$

*Proof.* Let $K$ be compact. Assume $K$ is unbounded, then

$$\forall n \in \mathbb{N} \exists x_n \in K : \|x_n\| > n$$

$(x_n)_n$ has no bounded subsequence and especially no convergent subsequence. Hence $K$ is not compact. This is a contradiction. So $K$ is bounded.

It remains to show: $\overline{K} \subseteq K$. Let $x \in \overline{K}$. $\exists (x_n)_n \subseteq K : x_n \to x$. $K$ is compact, hence

$$\exists (x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}, \overline{x} \in K : x_{n_k} \overset{k\to\infty}{\longrightarrow} \overline{x}$$

$$\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n = x \implies x = \overline{x} \in K$$

□

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This lecture took place on 31st of May 2016 with lecturer Wolfgang Ring.

**Theorem 40.** Let $\mathbb{K} \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) and we consider $\|.\|_\infty$ in $\mathbb{R}^n$. Let $\mathbb{K}$ be bounded and closed (in terms of $\|.\|_\infty$). Then $\mathbb{K}$ is compact.

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathbb{K}$ ($x_n \in \mathbb{K} \forall n \in \mathbb{N}$). Construct convergent subsequences with limit in $K$.

$$x^j = p_j(x) \qquad (j\text{-th coordinate of } x)$$

Because $(x_n)_{n\in\mathbb{N}}$ is bounded (in terms of $\|.\|_\infty$), it holds that

$$\left|x_n^j\right| \leq \|x_n\|_\infty$$

Hence $(x_n^j)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ for $j = 1, \ldots, n$. Consider $(x_n^1)_{n\in\mathbb{N}}$ bounded in $\mathbb{R}$. There exists a convergent subsequence $x_{n_{i_1}}^1 \to \xi^1$ where $i_1$ is the subsequence index. Consider $(x_{n_{i_1}}^2)_{i_1\in\mathbb{N}}$ which is subsequence of $(x_n^2)_{n\in\mathbb{N}}$ bounded. Hence $(x_{n_{i_1}}^2)_{i_1\in\mathbb{N}}$ is also bounded.

$$n \mapsto x_n$$

$$i_1 \mapsto n_{i_1}$$

Bolzano-Weierstrass implies that there exists a subsequence $(x_{n_{i_{1_{i_2}}}}^2)_{i_2\in\mathbb{N}}$ convergent with $\lim_{i_2\to\infty} x_{n_{i_{1_{i_2}}}}^2 = \xi_2$ for $(x_{n_{i_1}}^1)_{i_1\in\mathbb{N}}$ convergent towards $\xi^2$ it holds that:

$$\implies \lim_{i_2\to\infty} x_{n_{i_{1_{i_2}}}}^1 = \xi$$

We continue this construction. Assume $(x_{n_{i_{1_{i_{2_{i_k}}}}}}^j)$ convergent towards $\xi^j$ for $j = 1, \ldots, k$. Sequence $(x_{n_{i_1 \ldots i_k}}^{k+1})_{i_n\in\mathbb{N}}$ bounded in $\mathbb{R}$. Bolzano-Weierstrass implies that there exists $(x_{n_{i_{1_{i_2 \ldots i_{k+1}}}}}^{k+1})_{i_{k+1}\in\mathbb{N}}$ convergent to $\xi^{k+1}$.

Definition of subsequences do not change convergent.

$$\implies \forall j \in \{1, \ldots, k\} : \lim_{i_{k+1}\to\infty} x_{n_{i_1 \ldots i_{n+1}}}^j = \xi^j$$

until $k = n$ is reached. Then $(x_{n_{i_1 \ldots i_n}})_{i_n\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ for every coordinate sequence $(x_{n_{i_1 \ldots i_n}}^j)_{i_n\in\mathbb{N}}$ converges. Hence it holds that

$$\lim_{i_n\to\infty} x_{n_{i_1 \ldots i_n}} = \xi = \begin{bmatrix} \xi^1 \\ \vdots \\ \xi^n \end{bmatrix}$$

in terms of $\|.\|_\infty$. Because $\mathbb{K}$ is bounded, it holds that $\xi \in \mathbb{K}$. □

**Lemma 26.** Let $\mathbb{K} \subseteq (\mathbb{R}^n, \|.\|_\infty)$ be compact, let $f : \mathbb{K} \to \mathbb{R}$ be continuous (in terms of $\|.\|_\infty$). Then $f$ has a maximum as well as minimum in $\mathbb{K}$.

So $\exists x_{\min} \in \mathbb{K}$ such that $f(x_{\min}) \leq f(x)$ and $\exists x_{\max} \in \mathbb{K} : [\forall x \in \mathbb{K} : f(x) \leq f(x_{\max})] \forall x \in \mathbb{K}$.

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a maximum sequence. So for $\eta = \sup\{f(x) : x \in \mathbb{K}\}$ it holds that $\eta = \lim_{n\to\infty} f(x_n)$. $(x_n)_{n\to\infty}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = \xi \in \mathbb{K}$. Because of continuity of $f$ it holds that

$$\eta = \lim_{k\to\infty} f(x_{n_k}) = f(\xi)$$

Hence it holds that

$$f(\xi) = \eta = \sup\{f(x) : x \in \mathbb{K}\}$$

Hence $x_{\max} = \xi$ is maximum point. □

**Theorem 41.** In $\mathbb{R}^n$ all norms are equivalent.

*Proof.* Let $\|.\|$ be a norm in $\mathbb{R}^n$. We show $\|.\|$ is equivalent to $\|.\|_\infty$. We use three steps:

1. $\exists M > 0 : \|v\| \leq M\|v\|_\infty$ for every $v \in \mathbb{R}^n$. Let

$$v = \sum_{i=1}^n v^i e_i \qquad v^i \in \mathbb{R} \qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

where 1 is at the $i$-th coordinate. Then it holds that

$$\|v\| = \left\|\sum_{i=1}^n v^i e_i\right\| \leq \sum_{i=1}^n \|v^i \cdot e_i\|$$

$$= \sum_{i=1}^n |v^i| \cdot \|e_i\| \leq \underbrace{\max\{|v^j| : j = 1, \ldots, n\}}_{=\|v\|_\infty} \cdot \underbrace{\sum_{i=1}^n \|e_i\|}_{=M} = M \cdot \|v\|_\infty$$

2. We consider $f : (\mathbb{R}^n, \|.\|_\infty) \to \mathbb{R}$ with $f(v) = \|v\|$. We show $f$ is continuous. Let $\varepsilon > 0$ be arbitrary, $v \in \mathbb{R}^n$ is arbitrary and $\delta = \frac{\varepsilon}{M}$. Then it holds that

$$\|v - w\|_\infty < \delta = \frac{\varepsilon}{M} : \|v - w\| \leq M\|v - w\|_\infty < M \cdot \frac{\varepsilon}{M} = \varepsilon$$

Hence $f$ is continuous.

3. We consider $S_\infty^{n-1} = \{v \in \mathbb{R}^n : \|v\|_\infty = 1\}$. $S_\infty^{n-1}$ is bounded in terms of $\|.\|_\infty$. $S_\infty^{n-1}$ is closed in terms of $\|.\|_\infty$.

Let $(V_n)_{n\in\mathbb{N}}$ is a convergent (in terms of $\|.\|_\infty$) sequence in $S_\infty^{n-1}$ with limit $v$. Then because $\|v_n - v\|_\infty \to 0$ holds for $n \to \infty$,

$$\underbrace{|\|v_n\|_\infty - \|v\|_\infty|}_{\to 0} \leq \|v_n - v\|_\infty \to 0$$

Hence $\|v\|_\infty = \lim_{n\to\infty} \underbrace{\|v_n\|_\infty}_{1} = 1$.

So $v \in S_\infty^{n-1}$. So $S_\infty^{n-1}$ is closed. So $S_\infty^{n-1}$ is compact in terms of $\|.\|_\infty$

$$\left.\begin{array}{c} f : S_\infty^{n-1} \to \mathbb{R} \\ f(v) = \|v\| \end{array}\right\} \text{ takes minimum } m \text{ in } S_\infty^{n-1}$$

Hence,

$$\forall v \in S_\infty^{n-1} : \|v\| \geq \underbrace{\|v_{\min}\|}_{\in S_\infty^{n-1}} = m$$

Because $\|v_{\min}\|_\infty = 1\ (\neq 0) \Rightarrow v_{\min} \neq \vec{0}$. Hence $\|v_{\min}\| = m \neq 0$ (so $> 0$). Therefore $m > 0$.

Let $w \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$. Then $v = \frac{w}{\|w\|_\infty} \in S_\infty^{n-1}$ and it holds that

$$\|v\| = \left\|\frac{w}{\|w\|_\infty}\right\| = \frac{1}{\|w\|_\infty}\|w\| \geq m$$

So $\|w\| \geq m \cdot \|w\|_\infty$. For $w = 0$, $\|0\| \geq m \cdot \|0\|_\infty$ holds trivially.

Hence there exists $m$, $M > 0$ such that $\forall v \in \mathbb{R}^n$:

$$m\|v\|_\infty \leq \|v\| \leq M\|v\|_\infty$$

So $\|,\|$ and $\|.\|_\infty$ are equivalent.

Followingly transitivity of normequivalence shows that any two norms in $\mathbb{R}^n$ are equivalent.

This implies that all topological terms like convergence, continuity, open and closed sets, contact points, compactedness, et cetera are independent of the norm choice in $\mathbb{R}^n$.

**Example 26.** Let $f : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$.

$$f(\xi) = \begin{bmatrix} f^1(\xi) \\ f^1(\xi) \\ \vdots \\ f^m(\xi) \end{bmatrix}$$

Let $f^k : D \to \mathbb{R}$ for $k = 1, \ldots, m$. Let $\|.\|$ and $\|.\|_\infty$ be arbitrary norms in $\mathbb{R}^n$ (or equivalently $\mathbb{R}^m$). Then $f$ is continuous in $x$ in terms of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ if and only if $f^k$ is continuous in $x$ in terms of $\|.\|_{\mathbb{R}^n}$ and $|.|$ in the image set $\mathbb{R}$.

*Proof.* $f$ is continuous in terms of $\|.\|_{\mathbb{R}^m}$ in the image set if and only if $f$ is continuous in terms of $\|.\|_\infty$ in $\mathbb{R}^m$ if and only if $\overbrace{f(x_n) \to f(x)}^{\text{in terms of } \|.\|_\infty}$ for $n \to \infty$ if $x_n \to x$ in terms of $\|.\|_{\mathbb{R}^n}$ if and only if $\left|f^k(x_n) - f^k(x)\right| \to 0$ for $n \to \infty$ and $k = 1, \ldots, m$ if $x_n \to x$ in terms of $\|.\|_{\mathbb{R}^n}$ if and only if $f^k$ is continuous in $x$ for $k = 1, \ldots, m$. ☐

**Lemma 27.** $x_n \to x$ in $(\mathbb{R}^n, \|.\|_{\mathbb{R}^n}) \iff x_n^j \to x^j$ in $\mathbb{R}$ for $j = 1, \ldots, n$.

*Proof.* This statement holds for $\|.\|_{\mathbb{R}^n} = \|.\|_\infty$ and therefore for any norm. ☐

## 11 Differential calculus in $\mathbb{R}^n$

**Definition 36** (Landau's o-symbol)**.** With $o(x)$ we denote a function of $\vec{o} \in D \to \mathbb{R}$ if $D$ is open and $o(x) = q(x) \cdot \|x\|$ and $\lim_{x \to \vec{0}} q(x) = 0$, $q(\vec{0}) = 0$ with $q : D \to \mathbb{R}$. Hence $q$ is continuous in $\vec{0}$ with $q(\vec{o}) = 0$.

**Definition 37.** Let $D \subseteq \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^m$. Let $x_0 \in D$. We call $f$ Fréchet-differentiable in $x_0$ if $A : \mathbb{R}^n \to \mathbb{R}^m$. $A$ exists linearly such that

$$\|f(x) - f(x_0) - A(x - x_0)\| = o(x - x_0)$$

(similar to Taylor polynomial of degree 1)

☐ **Remark 47.**

$$A = \lim_{x \to \infty} \frac{f(x) - f(x_0)}{x - x_0}$$

does not work, because vectors cannot be divided.

**Lemma 28.** The linear map $A$ is (if it exists) by differentiability uniquely defined. In the following case we denote

$$Df(x_0) := A$$

and $Df(x_0)$ is called *Frechet derivative* of $f$ in $x_0$.

*Proof.* Assume $A$ and $\tilde{A}$ both satisfy the differentiability condition. Then for arbitrary $x \in D$ it holds that

$$\left\|(A - \tilde{A})(x - x_0)\right\| = \left\|f(x) - f(x_0) - \tilde{A}(x - x_0) - (f(x) - f(x_0) - A(x - x_0))\right\|$$

$$\leq \underbrace{\left\|f(x) - f(x_0) - \tilde{A}(x - x_0)\right\|}_{o(x - x_0)} + \underbrace{\|f(x) - f(x_0) - A(x - x_0)\|}_{o(x - x_0)} = o(x - x_0)$$

Hence

$$\left\|(A \cdot \tilde{A})(x - x_0)\right\| = q(x - x_0)\,\|x - x_0\|$$

with

$$\lim_{x \to x_0} q(x - x_0) = 0$$

$$\left\|(A - \tilde{A})\right\| \cdot \frac{x - x_0}{\|x - x_0\|} = q(x - x_0)$$

Let $v \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$ with $\|V\| = 1$ be arbitrary. Let $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq D$ ($D$ is open) and choose $x = x_0 + \frac{\varepsilon}{2} V \in B(x_0, \varepsilon) \subseteq D$. Then it holds that $x - x_0 = \frac{\varepsilon}{2} \cdot v$ and $\|x - x_0\| = \frac{\varepsilon}{2} \cdot 1$. So it holds that $\frac{x - x_0}{\|x - x_0\|} = \frac{x - x_0}{\frac{\varepsilon}{2}} = v$ and $\left|(A - \tilde{A})v\right| = q(x - x_0)$. for $x \to x_0$. The left-hand side is independent of $x - x_0$ and the right side converges to 0. So $\left\|(A - \tilde{A})v\right\| = 0$. Let $w \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$.

Let $v = \frac{w}{\|w\|}$. Then it holds that $\left\|(A - \tilde{A})w\right\| = \|w\| \cdot \underbrace{\left\|(A - \tilde{A}) \cdot v\right\|}_{=0}$. Hence $(A - \tilde{A})w = 0 \forall w \in \mathbb{R}^n$

$$\implies Aw = \tilde{A}w \forall w \in \mathbb{R}^n$$

□

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This lecture took place on 2nd of June 2016 with lecturer Wolfgang Ring.

**Remark 48.** Because of norm equivalence in $\mathbb{R}^n$, differentiability and the form of the derivative $Df(x_0)$ is independent of the chosen norm.

Let $D$ be an open set. Let $D$ be the definition set of $f$. We consider only inner points as definition points of the derivative.

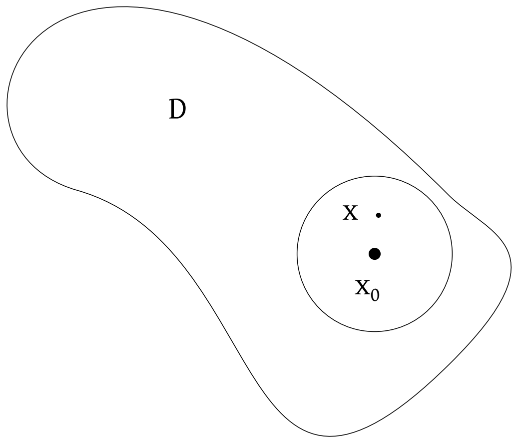


Figure 34: Open set $D$

**Lemma 29.** Let $f, g : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$ open. Let $f$ and $g$ be differentiable in $x_0$. Then for $\lambda \in \mathbb{R}$, $\lambda \cdot f$ is also differentiable in $x_0$. and $f + g$ is differentiable in $x_0$ with

$$D(\lambda f)(x_0) = \lambda Df(x_0)$$

$$D(f + g)(x_0) = Df(x_0) + Dy(x_0)$$

Hence the derivative operator $D$ applies linearly on the function.

*Proof.*

$\|\lambda f(x) - \lambda f(x_0) - \lambda Df(x_0) \cdot (x - x_0)\|_{\mathbb{K}^m} = |\lambda| \cdot \|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\|_{\mathbb{R}^m} = \underbrace{|\lambda| \cdot}_{\lim_{x \to}}$

$$\leq \underbrace{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|_{\mathbb{R}^m}}_{o(x - x_0)} + \underbrace{\|g(x) - g(x_0) - Dg(x_0)(x - x_0)\|_{\mathbb{R}^m}}_{o(x - x_0)} = o(x - x_0)$$

□

**Remark 49.** Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. We identify $A$ with the matrix representation of $A$ in regards of canonical bases in $\mathbb{R}^m$ (or $\mathbb{R}^n$).

**Definition 38** (Matrix norms)**.** Let $X$ and $Y$ be vector spaces. Then $\mathcal{L}(X, Y) = \text{Hom}(X, Y) = \{A \mid A : X \to Y \text{ is linear}\}$ is also a vector space. A norm on $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{m \times n}$ is called *matrix norm.*

**Example 27.** Let $A = [a_{ij}]_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ be an $m \times n$ matrix. We let $\|A\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n a_{i,j}^2\right)^{\frac{1}{2}}$. $\|.\|_F$ is called Forbeniusnorm in $A$. $\|.\|_F$ is a matrix norm.

**Definition 39.** Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear and let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$. Then we define

$$\|A\| = \sup_{V \neq \vec{0}} \left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m}$$

and we call $\|A\|$ the *operator norm* of $A$ in regards of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$.

**Lemma 30.** Let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ be chosen norms. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then it holds that

1. $n(w) = \|w\|_{\mathbb{R}^m}$ is a continuous map

$$n : (\mathbb{R}^m, \|.\|_{\mathbb{R}^m}) \to \mathbb{R}^+$$

2.

$$\|A\| = \sup_{V \neq 0} \left\| A \cdot \frac{V}{\|V\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m}$$

$$\max_{\substack{x \in \mathbb{R}^n \\ \|x\|_{\mathbb{R}^n} = 1}} \|A \cdot x\|_{\mathbb{R}^m}$$

*Proof.* 1. Let $\varepsilon > 0$ be arbitrary.

$$|n(w_1) - n(w_2)| = |\|w_1\|_{\mathbb{R}^m} - \|w_2\|_{\mathbb{R}^m}|$$

$$\overset{\text{inv. triangle ineq.}}{\leq} \|w_1 - w_2\|_{\mathbb{R}^m} < \varepsilon \text{ if } \|w_1 - w_2\| = \varepsilon$$

2. Let $x \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}} = \{x \in \mathbb{R}^n \,|\, \|x\|_{\mathbb{R}^n} = 1\}$. Then it holds that $\|Ax\|_{\mathbb{R}^n} = \left\| A \frac{x}{\|x\|_{\mathbb{R}^n}} \right\|_{\mathbb{R}^m}$.

$$\leq \sup_{\substack{V \in \mathbb{R}^n \\ V \neq 0}} \left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\| = \|A\|$$

So it also holds

$$\sup_{x \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}} \|Ax\| \leq \|A\|$$

On the opposite, let $v \in \mathbb{R}^n$ with $v \neq 0$ arbitrary. Then it holds that $z = \frac{V}{\|V\|_{\mathbb{R}^n}} \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}$ and

$$\left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\| = \|Az\|_{\mathbb{R}^m} \leq \sup_{\|x\|_{\mathbb{R}^n} = 1} \|Ax\|_{\mathbb{R}^m}$$

$$\implies \|A\| = \sup_{\substack{V \neq 0 \\ V \in \mathbb{R}^n}} \in S^{n-1}_{\|\cdot\|_{\mathbb{R}^n}}$$

$$\land \left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\| = \|Az\|_{\mathbb{R}^m} \leq \sup_{\|x\|_{\mathbb{R}^{n-1}}} \|Ax\|_{\mathbb{R}^m}$$



$$\implies \|A\| = \sup_{\substack{V \neq 0 \\ V \in \mathbb{R}^n}} \left\| A \frac{V}{\|V\|_{\mathbb{R}^n}} \right\|$$

So it holds that $\sup \left\| A \frac{V}{\|V\|} \right\|$.

It remains to show: right-sided sup is maximum.

We consider: $f : S^{n+1}_{\|\cdot\|_{\mathbb{R}^n}} \to \mathbb{R}$.

$$f(x) = \|Ax\|_{\mathbb{R}^m} = n \circ A(x)$$

Lineare maps $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ are continuous, $h$ is continuous. Hence $f$ is continuous and takes in compact set $S^{n+1}_{\|\cdot\|_{\mathbb{R}^n}}$ the maximum value.

□

**Remark 50.** Equivalent definition: Recall: $\sup_2 \{M\}$.

$M \subseteq \mathbb{R}$ is the smallest upper bound of $M$.

Hence $\|A\| = \inf \{\tilde{m} \geq 0\}$

TODO

So $\|A\|$ is the smallest constant $\tilde{m}$ such that

$$\|Av\|_{\mathbb{R}^m} \leq \tilde{m} \|v\|_{\mathbb{R}^n}$$

holds for all $v \in \mathbb{R}^n$. Especially it holds that

$$\boxed{\|Av\|_{\mathbb{R}^n} \leq \|A\| \cdot \|v\|_{\mathbb{R}^n}}$$

(This only works for operator norms. A very important result.)

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This lecture took place on 3rd of June 2016 with lecturer Wolfgang Ring.

**Remark 51** (Equivalent characterization of $\|A\|$)**.**

$$\|A\| = \max_{\substack{x \in \mathbb{R}^n \\ \|x\|=1 \\ \mathbb{R}^n}} \|Ax\|_{\mathbb{R}^m} = \max_{\substack{v \in \mathbb{R}^n \\ v \neq \vec{0}}} \frac{1}{\|v\|_{\mathbb{R}^n}} \cdot \|Av\|_{\mathbb{R}^m}$$

$$= \min \{\tilde{m} : \|Av\|_{\mathbb{R}^m} \leq \tilde{m} \|v\|_{\mathbb{R}^n} \, \forall v \in \mathbb{R}^n\}$$

**Lemma 31.** Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then $A$ is continuous in $\mathbb{R}^n$.

*Proof.* Consider linear $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $A$ is continuous if and only if component $A^j$ is continuous as map from $\mathbb{R}^n \to \mathbb{R}$ (for $i = 1, \dots, m$).

$$A^i(x) = \sum_{j=1}^{n} a_{ij} x_j$$

Choose $\|.\|_2$ in $\mathbb{R}^n$ and $|.|$ in $\mathbb{R}$. Show: Component $A^i$ is continuous for these norms. Let $x, y \in \mathbb{R}^n$. Then it holds that

$$\left|A^i x - A^i y\right| = \left|A^i(x-y)\right| = \sum_{j=1}^{n} a_{ij}(x_j - x_i)$$

$$= \langle a^i, x-y \rangle \overset{\text{Cauchy-Schwarz}}{\leq} \left\|a^i\right\|_2 \cdot \|x-y\|_2$$

So $A^i$ is Lipschitz continuous and therefore continuous. □

**Corollary 6** (Conclusion of estimate $\|Ax\| \leq \|A\| \, \|x\|$)**.** The function $A : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous with Lipschitz constant $\|A\|$.

Obvious, because $\|A(x-y)\| \leq \|A\| \cdot \|x-y\|$.

**Remark 52.** These law only hold for operator norms. This estimate also defines *boundedness*. Bounded equals linear for operator norms.

**Lemma 32.** Let $\|A\|$ be an operator norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then $\|A\|$ is a norm in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

*Proof.* 1.

$$\begin{aligned} \|A\| = 0 &\iff \max\left\{\frac{1}{\|v\|}\|Av\| : v \in \mathbb{R}^n, v \neq 0\right\} = 0 \\ &\iff \frac{1}{\|v\|}\|Av\| = 0 \forall v \in \mathbb{R}^n, v \neq \vec{0} \\ &\iff \|Av\| = 0 \forall v \neq \vec{0} \iff A = \underbrace{0}_{\text{zero matrix}} \end{aligned}$$

2.

$$\begin{aligned} \|\lambda A\| &= \max\left\{\|\lambda A x\| : \|x\| = 1\right\} \\ &= |\lambda| \cdot \max \|Ax\| : \|x\| = 1 = |\lambda| \cdot \|A\| \end{aligned}$$

3. Let $x \in \mathbb{R}^n$ with $\|x\| = 1$ arbitrary. $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then it holds that

$$\|(A+B)x\| \overset{\text{triangle ineq. in } \mathbb{R}^m}{\leq} \|Ax\| + \|Bx\| \leq \|A\| \underbrace{\|x\|}_{=1} + \|B\| \underbrace{\|x\|}_{=1}$$

$$= \|A\| + \|B\|$$

$$\implies \forall x \in \mathbb{R}^n : \|(A+B)x\| \leq \|A\| + \|B\|$$

$$\implies \underbrace{\max\left\{\|(A+B)x\| : \|x\| = 1\right\}}_{\|A+B\|} \leq \|A\| + \|B\|$$

□

**Lemma 33.** Let $A = \{\mathbb{R}^n, \mathbb{R}^m\}$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Choose fixed norms in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^l$. Let $\|A\|$ be the operator norm of $A$ in regards of $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$. Let $\|B\|$ be the operator norm of $B$ in regards of $\|.\|_{\mathbb{R}^n}$ (like for $\|A\|$!) and $\|.\|_{\mathbb{R}^l}$. Then it holds that

$$\|BA\| \leq \|B\| \, \|A\|$$

*Proof.*

$$\|BA\| = \max\left\{\underbrace{\|BAx\|_{\mathbb{R}^l}}_{\leq \|B\| \cdot \|Ax\|} : \|x\| = 1\right\}$$

$$\leq \|B\| \cdot \underbrace{\max\left\{\|Ax\| : \|x\| = 1\right\}}_{= \|A\|} = \|B\| \cdot \|A\|$$

□

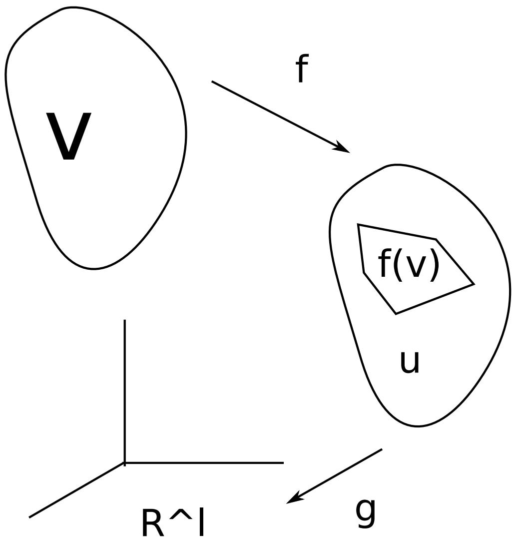


Figure 35: Chain rule in multiple dimensions

## 11.1 Back to differential calculus in $\mathbb{R}^n$

**Theorem 42** (Chain rule in multiple dimensions)**.** Let $f : O \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Let $U \subseteq \mathbb{R}^m$ be open and $g : U \to \mathbb{R}^l$ and $f(O) \subseteq U$.

Let $f$ in $x_0 \in O$ differentiable and $y$ in $y_0 = f(x_0) \in U$ differentiable. Then $g \circ f$ in $x_0$ is differentiable and it holds that

$$D(g \circ f)(x_0) = Dg(y_0) \cdot Df(x_0) = \underbrace{Dg(f(x_0))}_{\in \mathbb{R}^{l \times m}} \cdot \underbrace{Df(x_0)}_{\in \mathbb{R}^{m \times n}}$$

*Proof.* Let $\varepsilon > 0$ be arbitrary. Show that $\frac{1}{\|x-x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot Df(x_0)(x - x_0)\| = r(x - x_0) < \varepsilon$ for



sufficiently small $\|x - x_0\|$.

$$\frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot Df(x_0)(x - x_0)\|$$

$$\leq \frac{1}{\|x - x_0\|} \left\| g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0)) + \frac{1}{\|x - x_0\|} \right\| Dg(f(x_0)) \cdot \underbrace{(f(x) - f(}_{\text{small, becau}}$$

$$\leq \frac{1}{\|x - x_0\|} \|g(f(x)) - g(f(x_0)) - Dg(f(x_0))(f(x) - f(x_0))\| + \frac{1}{\|x - x_0\|} \|Dg(f(x_0))\| \cdot \underbrace{\|f(x) - f(}_{\text{small, bec}}$$

First, choose $\delta_1$ such that

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| \leq 1 \cdot \|x - x_0\|$$

for all $\|x - x_0\| \leq \delta_1$, $x \in O$. Possible, because $f$ is differentiable in $x_0$.

$$\implies \frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} < 1 + \|Df(x_0)\| + 1$$

Choose $\delta_g$ such that $\|y - y_0\| < \delta_g$ such that

$$\|g(y) - g(y_0) - Dg(y_0)(y - y_0)\| \leq \frac{\varepsilon}{2} \frac{1}{\|Df(x_0)\| + 1} \|y - y_0\|$$

is possible, because $g$ is differentiable in $y_0$. The inequality above also holds for $y = y_0$. Let $\delta_2$ such that for all $\|x - x_0\| < \delta_2 \implies \|f(x) - f(x_0)\| < \delta_g$. This is possible, because $f$ is continuous in $x_0$. (and $f$ differentiable in $x_0 \implies f$ is continuous in $x_0$).

Now let $\|x - x_0\| < \min\{S_1, S_2\}$. Then it holds that

$$\frac{1}{\|x - x_0\|} \left\| g(f(x)) - g(f(x_0)) - Dg(y_0) \left( \underbrace{f(x) - f(x_0)}_{\|f(x) - f(x_0)\| < \delta g} \right) \right\|$$

$$\leq \frac{\varepsilon}{2} \frac{1}{\|x - x_0\|} \frac{1}{\|Df(x_0)\| + 1} \underbrace{\|f(x) - f(x_0)\|}_{\leq (\|Df(x_0)\| + 1)\|x - x_0\|}$$

The expression in the brace below the line holds, because of the choice of $\delta_1$.

$$< \frac{\varepsilon}{2}$$

Now let $\delta_3 > 0$ such that

$$\|x - x_0\| < \delta_3 \implies \frac{1}{\|x - x_0\|} \|f(x) - f(x_0) - Df(x_0)(x - x_0)\|$$

$$< \frac{1}{\|Dg(f(x_0)) + 1\|} \frac{\varepsilon}{2}$$

is possible, because $f$ is differentiable in $x_0$. So it holds that $\|x - x_0\| < \delta_3$.

$$\frac{1}{\|x - x_0\|} \|Dg(y_0)\| \cdot \|f(x) - f(x_0) - Df(x_0)(x - x_0)\|$$

$$< \|Dg(y_0)\| \cdot \frac{\varepsilon}{2} \cdot \frac{1}{\|Dg(y_0)\| + 1} < \frac{\varepsilon}{2}$$

For $\|x - x_0\| < \min \delta_1, \delta_2, \delta_3$ every expression from above smaller than $\frac{\varepsilon}{2}$. The desired inequality was proven. □

**Lemma 34.** Let $f : D \subseteq \mathbb{R}.$ and $x_0 \in D$. If $f$ is differentiable in $x_0$, then $f$ is also continuous in $x_0$.

*Proof.* Let $\varepsilon > 0$ be arbitrary.

$$\|f(x) - f(x_0)\| \leq \overbrace{\|f(x) - f(x_0) - Df(x_0)(x - x_0)\|}^{r(x-x_0)\|x-x_0\| + \|Df(x_0)(x-x_0)\|}$$

$$\leq r(x - x_0)\|x - x_0\| + \|Df(x_0)\| \cdot \|x - x_0\|$$

Choose $\delta > 0$ such that

1. $\delta < \frac{\varepsilon}{2} \left(\|Df(x_0)\| + 1\right)^{-1} \leq \frac{\varepsilon}{2}$
2. for $\|x - x_0\| < \delta \implies r(x - x_0) \leq 1$

Let $\|x - x_0\| < \delta$. Then it holds that

$$\|f(x) - f(x_0)\| \leq 1 \cdot \frac{\varepsilon}{2} + \|Df(x_0)\| \cdot \frac{\varepsilon}{2} \frac{1}{\|Df(x_0)\| + 1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $f$ is continous in $x_0$. □



**Definition 40.** Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ a function. We state: $f$ is *differentiable in $D$*, if $f$ is differentiable in every point $x_0 \in D$. The map

$$x \mapsto Df(x)$$

$$D \to \mathbb{R}^{m \times n}$$

is called *derivative function of $f$*.

We say $f$ is *continuously differentiable in $D$* if the derivative function $x \mapsto Df(x)$ is continuous in terms of $\|.\|_{\mathbb{R}^n}$ in $\mathbb{R}^n$ and in terms of the operator norm in $\mathbb{R}^{m \times n}$.

Hence,

$$\forall \varepsilon > 0 \exists \delta > 0 : [\|x - x_0\| < \delta \text{ and } x \in D \implies \|Df(x) - Df(x_0)\| < \varepsilon]$$

**Remark 53.** $f$ is continuously differentiable in $\overline{D}$, if every point of $\overline{D}$ is a limit point (dt. "Häufungspunkt") in $D$ and in every point $x_0 \in \overline{D}$ the differentiability condition

$$\|f(x) - f(x_0) - Df(x_0)(x - x_0)\| = o(x - x_0)$$

holds and $x \mapsto Df(x)$ is a continuous function on $\overline{D}$.

## 12 Computing $Df(x_0)$

$$\lim_{v \to 0} \frac{f(x_0 + v) - f(x_0)}{v} =?$$

**Definition 41.** Let $f : D \to \mathbb{R}^m$ be given. Let $D \subseteq \mathbb{R}^n$ be open. Let $x_0 \in D$ and $v \in \mathbb{R}^n \setminus \left\{\vec{0}\right\}$. We define

$$df(x_0; v) = \lim_{t \to 0} \frac{1}{t} \left(f(x_0 + tv) - f(x_0)\right)$$

if the limit exists.

In this case we call $df(x_0; v)$ the directed derivative (*Gateaux derivative*) of $f$ in direction $v$ in point $x_0$.

**Remark 54.** Let $l_{x_0,v}(t) = x_0 + t \cdot v$ (parameter form of a straight line).

$$l_{x_0,v} : \mathbb{R} \to \mathbb{R}^n$$

is linear affine (constant and linear). $l_{x_0,v}$ is differentiable in $\mathbb{R}$. Furthermore it holds that $f(x_0 + t \cdot v) = f \circ l_{x_0,v}(t)$ defines an environment of $t = o$.

$$df(x_0; v) = D(f \circ l_{x_0,v})(o)$$

$$Dl_{x_0;v}(o) = v \in \mathbb{R}^n \quad \text{column vector}$$

If $f$ is Frechét differentiable in $x_0$, then it holds (by the chain rule)

$$df(x_0; v) = Df(l_{x_0,v}(o)) \cdot Dl_{x_0,v}(o) = Df(x_0) \cdot v$$

**Lemma 35.** Let $f$ like above and Frechét differentiable in $x_0$. Then it holds that

$$df(x_0; v) = Df(x_0) \cdot v$$

Now I can build the columns of $Df$.

## German keywords

## English keywords