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March to July 2016

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MATHEMATICAL ANALYSIS II – LECTURE NOTES

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring. Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

• Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} (1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. For every complex number $z\in\mathbb{C}$ this series converges on entire \mathbb{C} .

$$\exp(z) = \lim_{n \to \infty} \left(1 + \frac{z}{n} \right)^n = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
$$\exp(z + w) = \exp(z) \cdot \exp(w)$$
$$\lim_{z \to 0} \frac{\exp(z) - 1}{z} = 1$$
$$\exp(1) = e \in \mathbb{R}$$
$$z = \frac{m}{n} \in \mathbb{Q} \land n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z$$
 for $z \in \mathbb{C}$

It holds that

$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

 $\exp(x)$ for $x \in \mathbb{R}$

$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in \mathbb{R} .

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^{0} = 1$$

Let $n \in \mathbb{N}$

$$\lim_{x \to +\infty} \frac{e^x}{x^n} = \infty$$
$$\lim_{x \to -\infty} e^x \cdot x^n = 0$$



Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp: \mathbb{R} \to (0, \infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. exp : $\mathbb{R} \to (0, \infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every y > 0. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

Case 1 First of all, let $y \in [1, \infty)$. Then it holds that

$$e^{0} = 1 \le y$$
 and $e^{y} = 1 + y + \underbrace{\frac{y^{2}}{2} + \frac{y^{3}}{3!} + \frac{y^{4}}{4!} + \dots}_{>0}$

$$\geq 1 + y > y$$

Therefore $e^0 \le y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0, y] : \quad e^{\xi} = y$$

Case 2 Let $y \in (0,1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0, w]: \quad e^{\xi} = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^{\xi}} = y$$

So it holds that $\exp : \mathbb{R} \to (0, \infty)$ is bijective.

Definition 1. We call the inverse function natural logarithm¹.

$$\exp^{-1}:(0,\infty)\to\mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R} : \ln(e^x) = x$ and $\forall y \in (0, \infty) : e^{\ln(y)} = y$.
- $\ln:(0,\infty)\to\mathbb{R}$ is strictly monotonically increasing

Proof. Let
$$0 < y_1 < y_2$$
. Assume $\ln(y_1) \ge \ln(y_2) \xrightarrow{\text{monotonicity}} e^{\ln(y_1)} \ge e^{\ln(y_2)} \Rightarrow y_1 \ge y_2$. Contradiction!

Functional equations of logarithm 2.1

• For all x, y > 0 it holds that

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

• Limes:

$$\lim_{x \to 1} \frac{\ln(x)}{x - 1} = 1$$

Proof.

$$\begin{split} x \cdot y &= e^{\ln(x \cdot y)} \\ e^{\ln(x)} \cdot e^{\ln(y)} &= e^{\ln(x) + \ln(y)} \end{split}$$

Injectivity of exp:

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

• Let $(x_n)_{n\in\mathbb{N}}$ with $x_n>0$ be an arbitrary sequence with $\lim_{n\to\infty}x_n=0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n \to \infty} w_n = 1$ and $y_n = \ln(1 + x_n) = 1$ $\ln(w_n)$.

$$\lim_{n \to \infty} y_n = \ln(1) = 0$$

$$\lim_{n\to\infty}\frac{\ln(w_n)}{w_n-1}=\lim_{n\to\infty}\frac{y_n}{e^{y_n}-1}=\frac{1}{1}=1$$

where

$$e^0 = 1 \Rightarrow \ln(1) = 0$$

Theorem 1 (Logarithmic growth). $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0, \infty)$ with $x = e^{n \cdot \xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{\sqrt[n]{e^{n \cdot \xi}}} = \lim_{\xi \to \infty} \frac{n \cdot \xi}{e^{\xi}} = 0$$

In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$. because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi \to \infty} \frac{\xi^2}{e^{\xi}} = 0$

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

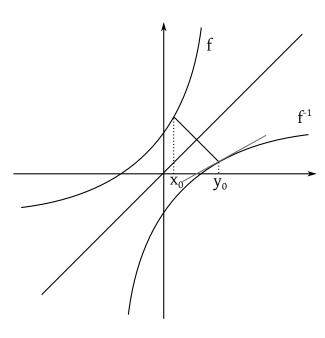


Figure 2: A geometric proof of differentiability

Proof. First approach Let x > 0, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n \to \infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n \to \infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n \to \infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \underbrace{\frac{1}{\lim_{n \to \infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}}_{(e^{\xi})' = e^{\xi}} = \frac{1}{e^{\xi}} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$

$$f'(f^{-1}(y)) \cdot (f^{-1})$$

$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let x > 0.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$
$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let x, y > 0. Then it holds that

$$\ln \frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln \frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{x \cdot \frac{h}{x}}$$
$$= \frac{1}{x} \cdot \lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x} \to 0$$

 $1 + \frac{h}{x} = w$ then it holds that $h \to 0 \Rightarrow w \to 1$.

$$\frac{h}{x} = w - 1$$

$$\lim_{h \to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{=} \lim_{h \to 0} \frac{\ln(w)}{w - 1} = 1$$

Remark 1. The exponential function can be defined from \mathbb{C} to \mathbb{C} .

$$\exp:\mathbb{C}\to\mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire \mathbb{C} (or $\mathbb{C} \setminus \{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f: I \to \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1}: f(I) \to \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

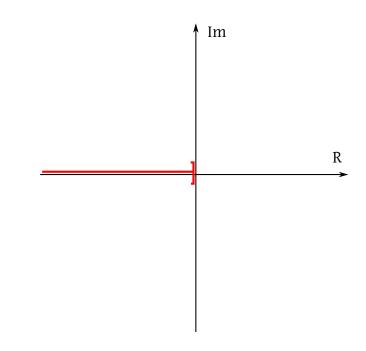


Figure 3: Continuous exponential function in $\mathbb C$

Proof. Let $y_n \to y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \Rightarrow x_n \neq x_0$.

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0}$$

$$= \lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{\lim_{n \to \infty} \underbrace{\frac{f(x_n) - f(x_0)}{x_n - x_0}}_{\text{ex} = f'(x_0)}} = \frac{1}{f'(x_0)}$$

Lemma 2. Let $f: I \to \mathbb{R}$ where I is some interval. Then it holds that

 $f = \text{const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \forall x \in I$

 $Proof. \Rightarrow Immediate.$

 \Leftarrow Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

Definition 2. Let I be an interval, $f: I \to \mathbb{R}$. A function $F: I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f: I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const.}$

Proof. F_1 , F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xrightarrow{\text{Lemma 2}} F_1 - F_2 = \text{const}$$

Theorem 4. Let I be an interval. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n: I \to \mathbb{R}$$
 differentiable

Furthermore let $f: I \to \mathbb{R}$. It holds that,

- 1. $\forall x \in I \text{ let } f(x) = \lim_{n \to \infty} f_n(x) \ (f_n \to f \text{ pointwise})$
- 2. for every $x \in I$ let $(f'_n(x))_{n \in \mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n \to \infty} f'_n(x)$ exists for every x)

3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \ge N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \le \varepsilon |u - v| \, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n \to \infty} f'_n(x)$.

$$f'(x) = [\lim_{n \to \infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - \lim_{n \to \infty} f'_N(x_0) \right|$$

$$= \left| \frac{f(x) - f(x_0)}{x - x_0} - f'_N(x_0) \right| + \left| f'_N(x_0) - \lim_{n \to \infty} f'_n(x_0) \right| \forall N \in \mathbb{N}$$

$$\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right|$$

$$+ \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) \right| + \left| f'_N(x_0) - \varphi(x_0) \right|$$

1st term

$$\left| \frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0} \right| = \left| \frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0} \right|$$

$$\leq \frac{\varepsilon}{3} \frac{|x - x_0|}{|x - x_0|} \stackrel{\text{condition } 3}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f'_N(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left| \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| - f'_N(x_0)$$

Differentiability of f_N : Therefore for $|x - x_0| < \delta$.

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$.

Theorem 5. Let $f_n: I \to \mathbb{R}$ and $f: I \to \mathbb{R}$ $(n \in \mathbb{N})$ and f_n is differentiable in I.

Assumption:

- 1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)
- 2. There exists $g: I \to \mathbb{R}$ such that $f'_n \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem). Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f'_n(x) \to \varphi(x)$ for every x
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \ge N \Rightarrow |(f_n f)(u) (f_n f)(v)| < \varepsilon |u v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n: I \to \mathbb{R}$ and $f: I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n \to \infty$
- $\exists g: I \to \mathbb{R}$ such that $f'_n \to g$ is uniform in I, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N}: n \ge N \land x \in I \Rightarrow |f'_n(x) g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6. We look at the conditions of Theorem 6:

2. Uniform convergences of $f'_n \to g$ implies pointwise convergence

$$\forall x \in I : f'_n(x) \to g(x)$$

3. From uniform convergence of $f'_n \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$|f'_n(x) - f'_m(x)| = |f'_n(x) - g(x) + g(x) - f'_m(x)|$$

 $\leq |f'_n(x) - g(x)| + |g(x) - f'_m(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.

Let $\varepsilon > 0$ be arbitrary and N such that $n, m \ge N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f-f_n)(u)-(f-f_n)(v)| = \lim_{m\to\infty} |(f_m-f_n)(u)-(f_m-f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$= \lim_{m \to \infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)|$$

= $\lim_{m \to \infty} |f'_m(\xi_{m,n}) - f'_n(\xi_{m,n})| \cdot |u - v|$

For $m \geq N$:

$$\leq \varepsilon \cdot |u - v|$$

So the third condition of Theorem 6 is satisfied.

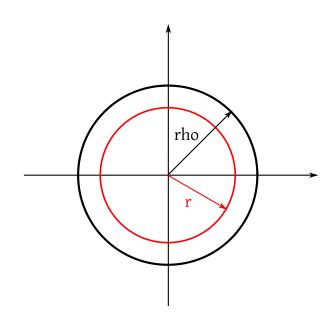


Figure 4: Convergence radius

Remark 2 (An application of Theorem 7). Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^n a_k z^k$$
 ... n-th partial sum

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ ².

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

Compare with Figure 4.

$$P'_n(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1} (j+1) x^j$$

is the n-1-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^{j}$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j \to \infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j \to \infty} |a_{j+1}|^{\frac{j+1}{j} \cdot \frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j} \cdot \frac{1}{j+1}}$$

$$= \limsup_{j \to \infty} \left(\frac{1}{|a_{j+1}|^{\frac{j+1}{j}}} \underbrace{\lim_{j \to \infty} \left[(j+1)^{\frac{1}{j+1}} \right]^{\frac{j+1}{j}}}_{1^{1}} = L \right)$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P'_n(z) = \sum_{k=1}^n k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in [-r,r].

From Theorem 6 (or 7?) it follows that P(x) is differentiable in [-r, r] and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

²Where overline means "closed"

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval [-r, r] with $0 < r < \rho(P)$.

About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k \to \infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k \to \infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(q) = 1$.

Apply the previous theorem, followingly q is differentiable in (-1,1) and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=0}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{constant}$$

Let x=0 (we determine the constant for this x=0):

$$0+0=0=$$
 constant

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$
 for $|x| < 1$

Let
$$x \in (-1, 1) \Rightarrow -x \in (-1, 1)$$
.

$$\Rightarrow \ln(1 - (-x)) = \ln(1 + x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \quad \text{for } x \in (-1,1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1 - (-1)}{(1 - x)^2} = \frac{2}{(1 - x)^2} > 0$$
 in $(-1, 1)$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\Rightarrow 1 + x = w - wx$$

$$x(1+w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

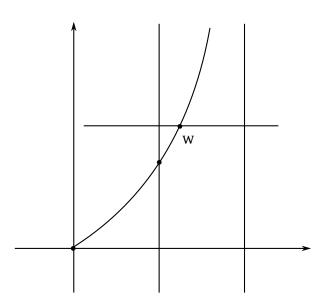


Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometic functions

We define trigonometic functions using the exponential function in \mathbb{C} . Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n \to \infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{it}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n \to \infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n \to \infty} \left[\overline{\left(1 + \frac{it}{n}\right)}\right]^n$$

$$= \lim_{n \to \infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n \to \infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$
$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$
$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

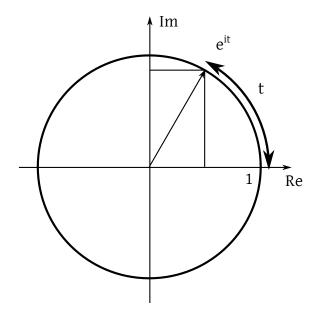


Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1.
$$e^{it} = \cos(t) + i \cdot \sin(t)$$
 (Euler's identity)

2.
$$|e^{it}|^2 = 1 = (\cos t)^2 + (\sin t)^2$$

3.

$$\Re(z) = \frac{1}{2}(z + \overline{z})$$

$$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2} \left(e^{it} + e^{-it} \right)$$

$$\Im(z) = \frac{1}{2i} [z - \overline{z}]$$

$$\sin(t) = \Im(e^{it}) = \frac{1}{2i} \left[e^{it} - e^{-it} \right]$$

4.

$$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2} \left[e^{iz} + e^{-iz} \right]$$

$$\sin(z) = \frac{1}{2i} \left[e^{iz} - e^{-iz} \right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring. Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$

$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$

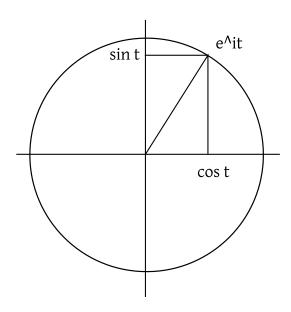


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$z \in \mathbb{C} : \cos z = \frac{1}{2} (e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

 $\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

 $\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence). Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty}a_n=a$ and $\sum_{n=0}^{\infty}b_n=s'$.

Then $\sum_{n=0}^{\infty} (a_n + b_n)$ is absolute convergent with sum s + s'.

series sum. Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n \in \mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \le \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

Example 1 (Application). Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0,\delta)$. Then also $\sum_{k=0}^{\infty} (a_k + b_k) z^k$ is convergent in $B(0,\delta)$ and it holds that $\sum_{k=0}^{\infty} (a_k + b_k) z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \dots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$
$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \dots$$
$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \dots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \dots$$

Followingly,

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots$$

$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C}$$

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \dots$$

$$= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and substraction theorems). We derive them directly: Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$= (\cos(z+w) + i\sin(z+w))$$

$$\Rightarrow = (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin zccosw)$$

Analogously,

$$e^{-(z+w)} = e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w))$$
$$= \cos z \cdot \cos w - \sin z \sin w + i (-\cos z \sin w - \cos w \sin z)$$

but also

$$= (-\cos(z+w) + i\sin(-(z+w)))$$

$$\Rightarrow = \cos(z+w) - i\sin(z+w)$$

Addition:

$$2\cos(z+w) = 2(\cos z \cdot \cos w - \sin z \sin w)$$

$$\Rightarrow \cos(z+w) = \cos z \cos w - \sin z \sin w$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\cos(z - w) = \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)}$$
$$\sin(z - w) = -\cos z \cdot \sin(w) + \sin(z)\cos(w)$$

Corollary 1.

$$z = \frac{1}{2}(z+w) + \frac{1}{2}(z-w)$$

$$\Rightarrow \cos z = \cos \frac{z+w}{2} \cos \frac{z-w}{2} - \sin \frac{z+w}{2} \sin \frac{z-w}{2}$$

$$w = \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w)$$

$$\cos w = \cos \frac{z+w}{2} \cdot \cos \frac{z-w}{2} + \sin \frac{z+w}{2} \cdot \sin \frac{z-w}{2}$$

$$\cos z - \cos w = -2\sin \frac{z+w}{2} \sin \frac{z-w}{2}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\lim_{\substack{z \to 0 \\ z \neq 0}} \frac{\sin z}{z} = \lim_{z \to 0} \frac{1}{2i} \left(\frac{e^{iz} - e^{-iz}}{z} \right)$$

$$= \lim_{z \to 0} e^{-iz} \left(\frac{e^{2iz} - 1}{2iz} \right)$$

$$= \lim_{z \to 0} e^{-iz} \cdot \lim_{z \to 0} \frac{e^{2iz} - 1}{2iz}$$

$$\lim_{w \to 0} \frac{e^{w} - 1}{w} = 1$$

So it holds that

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled "definition of π " and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \underbrace{1 - \underbrace{\frac{c_1}{x^2}}_{-2} + \underbrace{\frac{c_2}{x^4}}_{-24} - \underbrace{\frac{c_3}{x^6}}_{-720} + \underbrace{\frac{c_4}{x^8}}_{-40320} - \dots$$

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$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!}$$
 $s_k = \frac{x^{2k+1}}{(2k+1)!}$

For $x \in [0,2]$ and $k \ge 1$ it holds that

$$\left| \frac{c_{k+1}}{c_k} \right| = \left| \frac{x^2}{(2k+2)(2k+1)} \right| \le \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k>1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0, 2]$.

Similarly for $x \in (0, 2]$:

$$\left| \frac{s_{k+1}}{s_k} \right| = \left| \frac{x^2}{(2k+2)(2k+3)} \right| \le \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x$$
 in $[0, 2]$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \le w < z \le 2$,

$$0<\frac{z-w}{2}\leq \frac{z+w}{2}<\frac{z+z}{2}\leq 2$$

Let $x \in (0, 2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1 - \frac{4}{6} = \frac{1}{3} > 0} > 0$$

So it holds that sin(x) > 0 in (0, 2].

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = -2 \cdot \sin \underbrace{\frac{z+w}{2}}_{\in (0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in (0,2]}$$

 $\cos z < \cos w$ for $0 \le w < z \le 2$.

So it holds that \cos is a strictly monotonically decreasing function in [0, 2). Hence \cos has only one root because it is continuous in (0, 2].

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in (0,2].

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos \frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0, 2]$.

$$\Rightarrow \sin \frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

TODO: table missing

$$e^{i\pi} = e^{i\frac{\pi}{2} + i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi + \frac{i}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$

$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in \mathbb{C} with period $2i\pi$.

$$\cos(z + 2\pi) = \frac{1}{2} \left(e^{iz + 2\pi i} + e^{-iz - 2\pi i} \right)$$
$$= \frac{1}{2} \left(e^{iz} + e^{-iz} \cdot \underbrace{\frac{1}{e^{2\pi i}}}_{-1} \right) = \cos z$$

Therefore the cosine is periodic in \mathbb{C} with period 2π . Analogously, sine is periodic in \mathbb{C} with period 2π .

This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

$e^{i \operatorname{pi}/2}$ i $-i = e^{3\operatorname{pi}/2}$

3.5 Periodicity and roots of trigonometric functions

TODO: equations missing

$$\cos(z + 2\pi) = \cos(z)$$

$$\sin(z + 2\pi) = \sin(z)$$

Remark 3. We will show: $\forall c \in (0, 2\pi)$, cos and sin are non-periodic with period c, hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x + c)$.

Definition 5.

$$f: \mathbb{C} \to \mathbb{C}$$
 $(f: \mathbb{R} \to \mathbb{R})$

is called *periodic* with period $c \in \mathbb{C}$ $(c \in \mathbb{R})$ if $\forall z \in \mathbb{C}$ it holds that

$$f(z+c) = f(z)$$

$$(\forall x \in \mathbb{R} : f(x+c) = f(x))$$

c is called *period* of f.

Remark 4. If f is periodic with period $c \in \mathbb{C}$, then f is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark 5.

$$z = u + iv$$

$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$

$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

Remark 6. Let $x \in \mathbb{R}$.

$$\cos\left(x + \frac{\pi}{2}\right) = \Re(e^{i(x + \frac{\pi}{2})})$$

$$= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}})$$

$$= \Re(ie^{ix})$$

$$= -\Im(e^{ix})$$

$$= -\sin(x)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \Im\left(e^{i(x + \frac{\pi}{2})}\right)$$

$$= \Im(ie^{ix})$$

$$= \Re(e^{ix})$$

$$= \cos(x)$$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= \sin(x)$$

$$\sin\left(x - \frac{\pi}{2}\right) = -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right)$$
$$= -\cos(x)$$

Summary:

$$\cos\left(x + \frac{\pi}{2}\right) = -\sin(x)$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos(x)$$

$$\cos\left(x - \frac{\pi}{2}\right) = \sin(x)$$

$$\sin\left(x - \frac{\pi}{2}\right) = -\cos(x)$$

Remark 7 (A remark on the name "cosine").

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of x (Compare with Figure 8).

Remark 8.

$$\cos(x + \pi) = \Re(e^{i(x+\pi)})$$

$$= \Re(-e^{ix})$$

$$= -\cos(x)$$

$$\sin(x + \pi) = -\sin(x)$$

Remark 9. Let $0 < c < 2\pi$. Assume cos is periodic with period c. We know that cos has exactly one root in [0,2],

$$\cos(x) = \cos(-x)$$

cos has exactly two roots in [-2,2], namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \ge -\frac{\pi}{2} > -2$$

Therefore cos would have another root in [-2,2], namely $-\frac{\pi}{2}+c$. This is a contradiction.

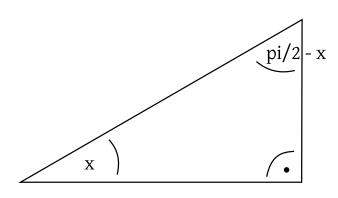


Figure 8: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$

c would be the period.

$$\Rightarrow \frac{3}{2}\pi - c$$
 is a root of cos in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period c. Hence 2π is indeed the smallest period of cos.

Analogously it holds for sin.

Remark 10 (Roots of cos).

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2} (1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2} (3 + 4k)$$

Hence for $z_l = \frac{\pi}{2} (2l+1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$

$$\sin(\pi + 2k\pi) = \sin((2k+1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \quad \forall l \in \mathbb{Z}$$

3.6 Derivatives of trigonometric functions

It holds that

MATHEMATICAL ANALYSIS II – LECTURE NOTES

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

Furthermore it holds that

$$\lim_{z \to 0} \frac{1 - \cos z}{z} = 0$$

Proof.

$$\frac{1-\cos z}{z} = \frac{1}{z} \left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots \right)$$
$$= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots$$

is convergent in \mathbb{C} and (especially) continuous in 0

$$\lim_{z \to 0} \left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots \right) = 0$$

 $\lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$

This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z \to 0} \frac{\sin z}{z} = 1$$

$$\lim_{z \to 0} \frac{1 - \cos z}{z} = 0$$

Lemma 6. The trigonometric functions \sin and \cos are differentiable in \mathbb{R} (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x)$$
 $\sin'(x) = \cos(x)$

Proof.

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos(h)}{h} = \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos(h) - 1}{h} - \lim_{h \to 0} \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

$$= -\sin(x)$$

Analogously:

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(h)}{h} = \lim_{h \to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h}$$

$$= \sin(x) \cdot \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{=0} + \cos(x) \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{n}}_{=1}$$

$$= \cos(x)$$

TODO: incomplete graphics, verify text

Figure 9. We now use tools of integral calculus:

Let I = [a, b] and $\gamma : I \to \mathbb{R}$ (\mathbb{R}^2).

$$\gamma(t) = \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

Assumption: $\gamma_1 : [a, b] \to \mathbb{R}^n$.

$$\gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

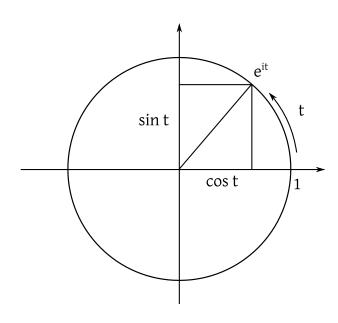


Figure 9: The arc length is related to sin and cos

TODO: graphics missing

Let $t \in [a, b]$. Then the arc length of γ between a and t is given by

$$S(t) = \int_{a}^{t} |\gamma'(\tau)| \ d\tau$$

We identify \mathbb{C} with \mathbb{R}^2 :

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma: t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma:[0,2\pi]\to\mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$
$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

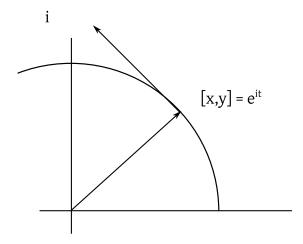


Figure 10: Derivative in \mathbb{R}^2

Compare with Figure 10.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \ d\tau = \int_0^t 1 \ d\tau = t$$

4 Integration calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses "Functional analysis" and "Measure and integration theory". For now, we look at the basis (as discussed by Königsberger).

Let [a, b] be an interval, $a, b \in \mathbb{R}$ with a < b and $\phi : [a, b] \to \mathbb{R}$. We call φ a step function, if $n \in \mathbb{N}$ and x_0, \ldots, x_n exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1},x_j)} = c_j$ is constant. The points x_j define a partition of the interval [a,b].

 $\tau[a,b]$ defines the set of step functions of interval [a,b]. The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 11).

Definition 6. Let $\varphi : [a,b] \to \mathbb{R}$ be a step function and $x_0 = a < x_1 < \ldots < x_n = b$ as partition of [a,b] and let $\varphi|_{(x_{j-1},x_j)} = c_j$ for $j = 1,\ldots,n$. Then we define

$$\int_{a}^{b} \varphi \, dx = \sum_{j=1}^{n} c_{j} \triangle x_{j}$$

where $\triangle x_j = x_j - x_{j-1}$ (for $j = 1, \dots, n$).

$$\int_{a}^{b} \varphi \, dx \text{ is called } integral \text{ of } \varphi \text{ over } [a, b]$$

 φ is the step function in terms of the partition $\{x_0, x_1, \dots, x_5\}$.

It remains to show that if φ satisfies the definition of a step function in terms of partition $\{x_0,\ldots,x_n\}$ and $\varphi|_{(x_{j-1},x_j)}=c_j$ (TODO: text missing: "but ...") and φ is a step function in terms of $\{w_0,w_1,\ldots,w_m\}$ and $\varphi|_{(w_{l-1},w_l)}=c'_l$, then it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{l=1}^{m} c_l' \triangle w_l$$

Compare with Figure 12.

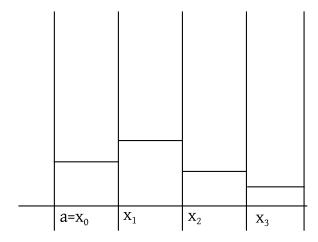


Figure 11: Partition of an area into rectangles

Proof. Let $Z = \{x_0, \ldots, x_n\}$ and $Z' = \{w_0, \ldots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \ldots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly,

$$k_{j-1} < k_j$$

Let $k_{j-1} < l \le k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l-1 \ge k_{j-1} \Rightarrow \alpha_{l-1} \ge \alpha_{k_{j-1}} = x_{j-1}$ and $l \le k_j$.

$$\Rightarrow \alpha_l \le \alpha_{k_j} = x_j$$

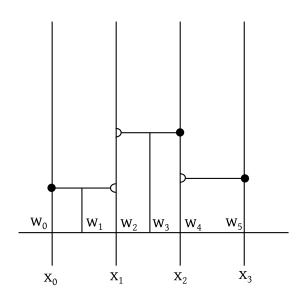


Figure 12: Step function φ

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$. $k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \ldots, L\}$ there exists $j \in \{1, \ldots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1},\alpha_l)}$$
 is constant

Hence φ is a step function in terms of the partition $\{\alpha_0, \ldots, \alpha_L\}$. Let $l \in \{0, 1, \ldots, L\}$ and j such that

$$k_{j-1} < l \le k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c_l'' = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c_l'' = c_i$.

$$\sum_{l=1}^{L} c_l'' \cdot \triangle \alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c_l'' \triangle \alpha_l$$
$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}}^{k_j} \triangle \alpha_l$$

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle \alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1}) + (\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2})$$

$$+\ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$$x_i - x_{i-1} = \triangle x_i$$

Analogously,

$$\sum_{l=1}^{L} c_l'' \cdot \triangle \alpha_l = \sum_{k=1}^{m} c_k' \triangle w_k$$

So it holds that

$$\sum_{j=1}^{n} c_j \triangle x_j = \sum_{k=1}^{m} c_k' \triangle w_k$$

This lecture took place on 15th of March 2016 with lecturer Wolfgang Ring.

Lemma 7. Let $\varphi \in \tau[a,b]$ be a step function in terms of partition $a=x_0 < x_1 < \ldots < x_n = b$. Let $a=\alpha_0 < \alpha_1 < \ldots < \alpha_L = b$ with $Z=\{x_0,\ldots,x_n\} \subseteq \{\alpha_0,\alpha_1,\ldots,\alpha_L\} = z'$ (z' has more intervals than Z').

Then also φ is step function in terms of partition z'.

Proof. see above \Box

Lemma 8. Let $\varphi_1, \varphi_2 \in \tau[a, b]$ and $\alpha, \beta \in \mathbb{C}$.

Then it holds that

• $\alpha \varphi + \beta \psi \in \tau[a, b]$ and

$$\int_{a}^{b} (\alpha \varphi + \beta \psi) \, dx = \alpha \int_{a}^{b} \varphi \, dx + \beta \int_{a}^{b} \psi \, dx$$

Hence ("linearity"),

$$\int_a^b:\tau[a,b]\to\mathbb{R} \text{ is linear }$$

• $|\varphi| \in \tau[a,b]$ and it holds that

$$\left| \int_a^b \varphi \, dx \right| \le \int_a^b |\varphi| \, dx \le \|\varphi\|_\infty \, (b-a)$$

Reminder: $\|\varphi\|_{\infty} = \max\{|\varphi(x)| : x \in [a, b]\}$ This gives "boundedness".

• Let φ and ψ be real values and it holds that

$$\forall x \in [a, b] : \varphi(x) \le \psi(x)$$

Then TODO Monotonicity

Proof.
$$\bullet$$
 Let $\varphi|_{(x_{k-1},x_k)} = c_k \ \psi|_{(w_{j-1},w_j)} = d_k$

$$z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\} = \{x_0, \dots, x_n\} \cup \{w_0, \dots, w_m\}$$

where α_i is sorted ascendingly. φ and ψ are step functions in terms of z'', hence

$$\varphi|_{(\alpha_{i-1},\alpha_i)} = c_i' \text{ and } \psi|_{(\alpha_{i-1},\alpha_i)} = d_i'$$

$$\Rightarrow (\alpha\varphi + \beta\psi)|_{(\alpha_{i-1},\alpha_i)} = \alpha c_i' + \beta d_i' \text{ constant}$$

$$\Rightarrow \alpha\varphi + \beta\psi \in \tau[a,b] \text{ and } \int_a^b (\alpha\varphi + \beta\psi) \, dx = \sum_{i=1}^L (\alpha c_i' + \beta d_i') \cdot \triangle \alpha_i$$

$$= \alpha \sum_{i=1}^{L} c'_i \triangle \alpha_i + \beta \sum_{i=1}^{L} d'_i \triangle \alpha_i$$
$$= \alpha \int_a^b \varphi \, dx + \beta \int_a^b \psi \, dx$$

• Let $\varphi|_{(x_{i-1},x_i)} = c_i \ (i = 1,\ldots,n)$. Then,

$$|\varphi||_{(x_{i-1},x_i)} = |c_i|$$

$$\left| \sum_{i=1}^{n} c_{i} \triangle x_{i} \right| \leq \sum_{i=1}^{n} |c_{i}| \cdot \underbrace{|\triangle x_{i}|}_{x_{i} - x_{i-1} > 0} = \sum_{i=1}^{n} |c_{i}| \cdot \triangle x_{i} = \int_{a}^{b} |\varphi| \, dx$$

$$\leq \sum_{i=1}^{n} \|\varphi\|_{\infty} \, \triangle x_{i} = \|\varphi\|_{\infty} \sum_{i=1}^{n} \triangle x_{i}$$

$$= \|\varphi\|_{\infty} \left((x_{1} - x_{0}) + (x_{2} - x_{1}) + \dots + (x_{n-1} - x_{n-2}) + (x_{n} - x_{n-1}) \right)$$

$$= \|\varphi\|_{\infty} \left((x_{n} - x_{0}) + \|\varphi\|_{\infty} \right) \left((b - a) \right)$$

• Let φ , ψ and z'' as in the linearity statement.

$$\varphi|_{(\alpha_{i-1},\alpha_i)} = c_i' \in \mathbb{R}$$

$$\psi|_{(\alpha_{i-1},\alpha_i)} = d_i' \in \mathbb{R}$$

$$\int_a^b \varphi \, dx = \sum_{i=1}^L c_i' \, \triangle \alpha_i \le \sum_{i=1}^L d_i' \, dx$$

$$\int_a^b \varphi \, dx$$

Definition 7. Let $A \subseteq \mathbb{R}$. Then we call $\chi_A = (\infty_A) : \mathbb{R} \to \mathbb{R}$ as

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

the characteristic function of A. Hence $\chi_A(x)$ is 1 if and only if x is inside interval A.

Remark 11. Let $a \le a' < b' \le b$. Then

$$\chi_{(a',b')} \in \tau[a,b]$$
 $\int_{a}^{b} \chi_{(a',b')} dx = 1 \cdot (b'-a')$

Every linear combination of characteristic functions is also in $\tau[a, b]$.

On the opposite side, let $\varphi \in \tau[a,b]$ with $\varphi|_{(x_{i-1},x_i)} = c_i$ and $\varphi(x_i) =: r_j$ with $1 \le i \le n$ and $0 \le j \le n$.

$$\Rightarrow \varphi = \sum_{i=1}^{n} c_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^{n} r_j \chi_{\{x_j\}}$$

The step function is a linear combination of characteristic functions of open intervals and of characteristic functions of one-point sets.

$$\int_{a}^{b} \varphi \, dx = \sum_{i=1}^{n} c_{i} \cdot (x_{i} - x_{i-1}) = \sum_{i=1}^{n} c_{j} \int_{a}^{b} \chi_{(x_{i-1}, x_{i})} \, dx$$

5 Regulated functions

Definition 8. Let $D \subseteq \mathbb{R}$. Let x_0 be a limit point of $D \cap (-\infty, x_0)$ hence $\exists (z_n)_{n \in \mathbb{N}}$ with $z_n \in D \cap (-\infty, x_0)$, hence $z_n < x_0$, and $\lim_{n \to \infty} z_n = x_0$. Let $f : D \to \mathbb{C}$ be given.

We state that f has left-sided limit y_0 in x_0 if

$$\forall \varepsilon > 0 \exists \delta > 0 : [x \in D \cap (-\infty, x_0) \land |x - x_0| < \delta]$$
$$\Rightarrow |f(x) - y_0| < \varepsilon$$

Equivalently $\forall (z_n)_{n\in\mathbb{N}}$ with $z_n\in D$ and $z_n< x_0$ and $\lim_{n\to\infty}z_n=x_0 \ \forall n\in\mathbb{N}$

$$\lim_{n \to \infty} f(x_n) = y_0$$

Analogously for the right-sided limes, we replace $(-\infty, x_0)$ by (x_0, ∞) .

We denote: y_0 is left-sided limit of f in x_0 :

$$y_0 = \lim_{x \to x_0^-} f(x)$$

and right-sided limit of f in x_0 :

$$y_0 = \lim_{x \to x_0^+} f(x)$$

Definition 9. Let $a, b \in \mathbb{R}$ and a < b. A function $f : [a, b] \to \mathbb{C}$ is called regulated functions if

- $\forall x \in (a, b)$ f has a left-sided and a right-sided limes in x
- f has a right-sided limes in a
- f has a left-sided limes in b

Examples for regulated functions:

- Every continuous function in [a, b] is a regulated function.
- Every step function is a regulated function. Why? Consider $x \in (x_{i-1}, x_i)$. Then

$$\lim_{\xi \to x^+} \varphi(\xi) = c_i = \lim_{\xi \to x^-} \varphi(\xi)$$

Let $x = x_i$ be a partitioning point.

 $\lim TODO$ and $\lim TODO$

So $\tau[a,b] \subseteq R[a,b]$. Compare with Figure 13.

• Let $f:[a,b]\to\mathbb{R}$ be monotonically. Then it holds that

$$f \in R[a, b]$$

5.1 Approximation theorem for regulated functions

Let $f:[a,b] \in \mathbb{C}$. Then it holds that $f \in R[a,b] \Leftrightarrow \forall \varepsilon > 0 \exists \varphi \in \tau[a,b]$ such that $\|f - \varphi\|_{\infty} < \varepsilon$. Hence $\forall x \in [a,b] : |f(x)|$ TODO

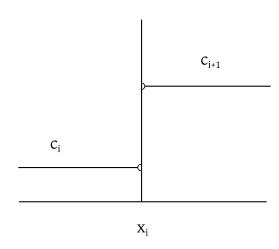


Figure 13: Step functions are also regulated functions

$$\Leftrightarrow \underbrace{\sup\{|f(x) - \varphi(x)| : x \in [a, b]\}}_{\|f - \varphi\|_{\infty}} < \varepsilon$$

 $S \varepsilon_n = \frac{1}{n} \Rightarrow \exists \varphi_n \in \tau[a, b] \text{ such that}$

$$|\varphi_n(x) - f(x)| < \varepsilon \forall x \in [a, b]$$

hence f is a continuous limit point of a sequence of step functions. Hence the function sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges continuously towards f.

Proof. \Rightarrow Let $f \in R[a,b]$. Assume $\exists \varepsilon > 0$ fixed such that $\forall \varphi \in \tau[a,b]$

$$\exists x \in [a, b] : |\varphi(x) - f(x)| \ge \varepsilon$$

We build nested intervals such that the desired property $|\varphi(x) - f(x)| \ge \varepsilon$ holds on every subinterval $[a_n, b_n]$.

Induction:

n=0 Let $a_0=a$ and $b_0=b$, hence the property holds in $[a_0,b_0]$.

$$n \mapsto n+1$$
 Let $m=\frac{1}{2}(a_n+b_n)$. In $[a_n,b_n]$ the property holds.

Then the property either holds in $[a_n, m]$ or $[m, b_n]$. If the property does not hold in $[a_n, m]$:

$$\exists \varphi_1 \in \tau[a_n, m] \text{ with } |\varphi_1(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [a_n, m]$$

If the property does not hold in $[m, b_n]$:

$$\exists \varphi_2 \in \tau[m, b_n] \text{ with } |\varphi_2(\xi) - f(\xi)| < \varepsilon \qquad \forall \xi \in [m, b_n]$$

Let

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{for } x \in [a_n, m] \\ \varphi_2(x) & \text{for } x \in [m, b_n] \end{cases}$$

$$\Rightarrow \varphi \in \tau[a, b] \text{ and } |\varphi(\xi) - f(\xi)| < \varepsilon \forall \xi \in [a_n, b_n]$$

So in at least one of the intervals the property holds. Let this interval be $[a_{n+1}, b_{n+1}]$.

 $([a_n,b_n])_{n\in\mathbb{N}}$ are nested intervals. Let $\varphi\in\bigcap_{n\in\mathbb{N}}[a_n,b_n]$.

Case $\xi \in (\mathbf{a}, \mathbf{b})$ Let ε satisfy the desired property. $f \in R[a, b]$, hence f has left-sided limit c_- in ξ and right-sided limit c_+ . Hence $\exists \delta > 0$ such that

•
$$|x - \xi| < \delta \land a \le x < \xi \Rightarrow |f(x) - c_{-}| < \varepsilon$$

•
$$|x - \xi| < \delta \land \delta < x \le b \Rightarrow |f(x) - c_+| < \varepsilon$$

Choose δ sufficiently small such that

$$a < \xi - \delta < \xi + \delta < b$$

Let

$$\varphi(x) = \begin{cases} c_{-} & \text{for } x \in (\xi - \delta, \xi) \\ f(\xi) & \text{for } x = \xi \\ c_{+} & \text{for } x \in (\xi, \xi + \delta) \end{cases}$$

MATHEMATICAL ANALYSIS II – LECTURE NOTES

 φ is necessarily a step function in $(\xi - \delta, \xi + \delta)$ and it holds that $\forall x \in (\xi - \delta, \xi + \delta) : |\varphi(x) - f(x)| < \varepsilon$. Let n be sufficiently large such that

$$[a_n, b_n] \subseteq (\xi - \delta, \xi + \delta)$$

then

$$\varphi|_{[a_n,b_n]} \in \tau[a_n,b_n] \text{ and } |\varphi(x)-f(x)| < \varepsilon \quad \forall x \in [a_n,b_n]$$

This is a contradiction to our desired property.

German keywords

Charakteristische Funktion, 45 Cosinusfunktion, 21 Einseitiger Grenzwert, 47 Integral, 41 Linearität des Integral, 43 Linksseitiger Grenzwert, 47 Logarithmische Reihe, 19 Natürlicher Logarithmus, 7 Periode, 31 Periodische Funktion, 31 Rechtsseitiger Grenzwert, 47 Regelfunktion, 47 Sinusfunktion, 21 Stammfunktion, 13 Treppenfunktion, 41

English keywords

Characteristic function, 45 Cosine function, 21

Integral, 41

Left-sided limit, 47 Linearity of integration, 43 Logarithmic series, 19

Natural logarithm, 7

One-sided limit, 47

Period, 31 Periodic function, 31 Primitive, 13

Regulated function, 47 Right-sided limit, 47

Sine function, 21 Step function, 41