

# Linear Algebra 2 – Lecture Notes

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This lecture took place on 29th of Feb 2016 (Prof. Franz Lehner).

Exam: written and orally

Tutorial session:

- Every Monday, 18:30-20:00, SR 11.34

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## 3 Konversatorium:

- Every Monday, 10:00–10:45, SR 11.33

## 3 Topics, wie already discussed:

- Vector spaces
- Linear maps and their equivalence with matrices
- We introduced equivalence of matrices ( $PAQ = B$ )
- We defined the following techniques:
  - Rank
  - Linear equation system
  - Inverse matrices
  - Basis transformation

## In this semester, we will discuss:

- $PAP^{-1}$ , which is related to eigenvalues and diagonalization, hence  $\bigvee_P^2 PAP^{-1} = D$ .

## 1 Linear maps (cont.)

### 1.1 Addition to chapter 5.2.4

$\text{Hom}(V, W)$  in special case  $W = \mathbb{K}$ . We define,

$$V^* := \text{Hom}(V, \mathbb{K})$$

also denoted  $V'$  is called *dual space* of vector space  $V$ . The elements  $v^* \in V^*$  are called *linear forms* or *linear functionals*.

We denote,

$$v^*(v) =: \langle v^*, v \rangle$$

### 1.2 Example

$$V = \mathbb{K}^n$$

$v^* : V \rightarrow \mathbb{K}$  is uniquely defined with values  $v^*(e_i) =: a_i$ .

$$\langle v^*, v \rangle = \left\langle v^*, \sum_{i=1}^n v_i e_i \right\rangle = \sum_{i=1}^n v_i \langle v^*, e_i \rangle$$

$$v^* \left( \sum_{i=1}^n v_i e_i \right) = \sum_{i=1}^n v_i v^*(e_i) = \sum_{i=1}^n a_i v_i$$

### 1.3 More general

We know,  $\dim \text{Hom}(V, W) = \dim V \cdot \dim W$ .

**Theorem 1.** Let  $V$  be a vector space over  $\mathbb{K}$ .

- $\dim V =: n < \infty \Rightarrow \dim V^* = n$   
More precisely: Let  $(b_1, \dots, b_n)$  be a basis of  $V$ . Then

$$b_k^* : b_i \mapsto \delta_{ik} = \begin{cases} 1 & i = k \\ 0 & \text{else} \end{cases}$$

is a basis of  $V^*$  and is called dual basis.

- For  $v^* \in V^*$  it holds that  $v^* = \sum_{k=1}^n \langle v^*, b_k \rangle \cdot b_k^*$ .
- If  $\dim V = \infty$ ,  $(b_i)_{i \in I}$  is a basis, then it holds that  $(b_k^*)_{k \in I}$  with

$$\langle b_k^*, b_i \rangle = \delta_{ik}$$

is not a basis of  $V^*$ .

*Proof.* • Special case of 5.18

$(b_k^*)$  is linear independent, hence in  $\sum_{i=1}^n \lambda_i b_i^* = 0$  all  $\lambda_i = 0$ .

$$0 = \left\langle \sum_{i=1}^n \lambda_i b_i^*, b_k \right\rangle = \sum_{i=1}^n \lambda_i \underbrace{\langle b_i^*, b_k \rangle}_{\delta_{ik}} = \lambda_k \forall k$$

- Let  $v \in V$  with  $v = \sum_{i=1}^n v_i b_i$ . We need to show

$$\begin{aligned} \langle v^*, v \rangle &\stackrel{!}{=} \left\langle \sum_{k=1}^n \langle v^*, b_k \rangle b_k^*, v \right\rangle \\ &= \sum_{k=1}^n \langle v^*, b_k \rangle \langle b_k^*, v \rangle \\ &= \sum_{k=1}^n \langle v^*, b_k \rangle \left\langle b_k^*, \sum_{i=1}^n v_i b_i \right\rangle \\ &= \sum_{k=1}^n \sum_{i=1}^n \langle v^*, b_k \rangle \underbrace{\langle b_k^*, b_i \rangle}_{\delta_{ki}} \cdot v_i \\ &= \sum_{k=1}^n \langle v^*, b_k \rangle \langle v^*, b_k \rangle \cdot v_k \\ &= \left\langle v^*, \sum_{k=1}^n v_k b_k \right\rangle \\ &= \langle v^*, v \rangle \end{aligned}$$

- (To be done in the practicals) Consider the functional

$$\langle v^*, b_i \rangle = 1 \Rightarrow v^* \notin L((v_i^*)_{i \in I})$$

□

### 1.4 Remark and a definition for bilinearity

The mapping  $V^* \times V \rightarrow \mathbb{K}$  is linear in  $v$  (with fixed  $v^*$ ) with  $(v^*, v) \mapsto \langle v^*, v \rangle$  is linear in  $v^*$  (with fixed  $v$ ). Such a mapping is called *bilinear*.

A mapping  $F : V_1 \times \dots \times V_n \rightarrow W$  is called *multilinear* ( $n$ -linear) if it is linear in every component. Formally:

$$\begin{aligned} & F(v_1, \dots, v_{k-1}, \lambda v'_k + \mu v''_k, v_{k+1}, \dots, v_n) \\ &= \lambda F(v_1, \dots, v_{k-1}, v'_k, v_{k+1}, \dots, v_n) + \mu F(v_1, \dots, v_{k-1}, v''_k, v_{k+1}, \dots, v_n) \end{aligned}$$

### 1.5 Example

$V = \mathbb{K}[x]$  polynomials

Basis:  $\{x^k \mid k \in \mathbb{N}_0\}$  and  $\dim V = \aleph_0$

Every  $v^* \in V^*$  is uniquely defined by  $a_k := \langle v^*, x^k \rangle$

$$(a_k)_{k \in \mathbb{N}_0}$$

$V^* \cong \mathbb{K}[[t]]$  are the formal power series

$$= \left\{ \sum_{k=0}^{\infty} a_k t^k \mid a_k \in \mathbb{K} \right\}$$

$$\lambda \sum_{k=0}^{\infty} a_k t^k + \mu \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (\lambda a_k + \mu b_k) t^k$$

(Compare with Taylor series  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ )

$$\left\langle \sum_{k=0}^{\infty} a_k t^k, \sum_{k=0}^n b_k x^k \right\rangle = \sum_{k=0}^n a_k b_k \text{ is well-defined}$$

$$\rightarrow \mathbb{K}[x]^* \cong \mathbb{K}[[t]]$$

### 1.6 Example

$C[0, 1]$  continuous functions

Example:

**Example 1.**

$$x \in [0, 1] \quad \delta_x : C[0, 1] \rightarrow \mathbb{R}$$

$$f \mapsto f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$\langle \delta_x, f \rangle = f(x)$$

$$I(f) = \int_0^1 f(x) dx \text{ is linear}$$

$$\langle I_g, f \rangle = \int_0^1 f(x)g(x) dx$$

$g \in C[0, 1]$  is fixed

$$\Rightarrow I_g \in C[0, 1]$$

$$\langle I_g, \lambda f_1 + \mu f_2 \rangle' = \int_0^1 (\lambda f_1(x) + \mu f_2(x))g(x) dx$$

$$= \lambda \int_0^1 f_1(x)g(x) dx + \mu \int_0^1 f_2(x)g(x) dx$$

This also works with non-continuous  $g$  (it suffices to have  $g$  integrable). (Compare with measure theory and Riesz' theorem)

Does there exist some  $g$  such that  $f(x) = \langle \delta_x, f \rangle = \int_0^1 f(t)g(t) dt$ . (Compare with Dirac's  $\delta$  function and Schwartz/Sobder theory)

$$V^{**} = (V^*)^* \cong V \text{ if } \dim V < \infty$$

**Lemma 1.** Let  $V$  be a vector space over  $\mathbb{K}$ . It requires that  $\dim V < \infty$  and the Axiom of Choice holds.

$$\bullet v \in V \setminus \{0\} \Leftrightarrow \bigvee_{v^* \in V^*} \langle v^*, v \rangle \neq 0$$

- $\bigwedge_{v \in V} v = 0 \Leftrightarrow \bigwedge_{v^* \in V^*} \langle v^*, v \rangle = 0$

*Proof.* Addition  $v$  to a basis  $B$  of  $V$ : Define  $v^* \in V^*$  by

$$\langle v^*, b \rangle = \begin{cases} 1 & b = v \\ 0 & b \neq v \end{cases} \text{ for } b \in B$$

**Theorem 2.** Let  $V$  be a vector space over  $\mathbb{K}$ .

- The map  $\iota : V \rightarrow V^{**} := (V^*)^*$  is called *bidual space*.

$$\langle \iota(v), v^* \rangle := \langle v^*, v \rangle$$

is linear and injective.

- if  $\dim V < \infty$ , then isomorphism.

*Proof.* • Linearity

$$\iota(\lambda v + \mu w) \stackrel{!}{=} \lambda \iota(v) + \mu \iota(w)$$

must hold in every point  $v^* \in V^*$ :

$$\begin{aligned} \langle \iota(\lambda v + \mu w), v^* \rangle &= \langle v^*, \lambda v + \mu w \rangle \\ &= \lambda \langle v^*, v \rangle + \mu \langle v^*, w \rangle \\ &= \lambda \langle \iota(v), v^* \rangle + \mu \langle \iota(w), v^* \rangle \\ &= \langle \lambda \iota(v) + \mu \iota(w), v^* \rangle \end{aligned}$$

Is it injective? Let  $v \in \ker \iota$ .

$$\langle \iota(v), v^* \rangle = 0 \quad \forall v^* \in V^*$$

$$\Rightarrow \langle v^*, v \rangle = 0 \quad \forall v^* \in V^*$$

$$\xrightarrow{\text{Lemma 1}} v = 0$$

- Follows immediately, because the dimension is equal.

**Definition 1.** Let  $V, W$  be vector spaces over  $\mathbb{K}$ .  $f \in \text{Hom}(V, W)$ . We define  $f^T \in \text{Hom}(W^*, V^*)$  using  $f^T(w^*) \in V^*$  via

$$\langle f^T(w^*), v \rangle = \langle w^*, f(v) \rangle = w^*(f(v)) = w^* \circ f(v)$$

$$f^T(w^*) = w^* \circ f \text{ is linear} \Rightarrow f^T(w^*) \in V^*$$

$V$  to  $W$  (with  $f$ ) and  $W$  to  $\mathbb{K}$  (with  $w^*$ ).

□  $f^T$  is called *transposed map*.

**Example 2.** (See practicals) Let  $\dim V = n$  and  $\dim W = m$  with  $B \subseteq V$  and  $C \subseteq W$  as bases and dual bases  $B^* \subseteq V^*$  and  $C^* \subseteq W^*$

$$\Phi_{B^*}^{C^*}(f^T) = \Phi_C^B(f)^T \quad \text{transposition of matrices}$$

This lecture took place on 2nd of March 2016 (Franz Lehner).

## 2 Determinants

Leibnitz 1693 ( $3 \times 3$  matrices)

Seki Takukazu 1685 (most general version)

Gauß 1801 (“determinant”)

Cayley 1845 (on matrices)

$$n = 2$$

$$ax + by = e$$

$$cx + dy = f$$

$$\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}$$

1. Case 1:  $a \neq 0$  (multiply first row  $-\frac{a}{b}$  times second row)

$$\begin{array}{cc|c} a & b & \\ c & d & \\ \hline a & b & \\ 0 & d - \frac{bc}{a} & \end{array}$$

□

Unique solution:

$$d - \frac{bc}{a} \neq 0$$

2. Case 2:  $c \neq 0$  (multiple second row  $-\frac{a}{c}$  times first row)

$$\begin{array}{cc} a & b \\ c & d \\ 0 & b - \frac{ad}{c} \\ c & d \end{array}$$

Unique solution:

$$b - \frac{ad}{c} \neq 0$$

This gives us

$$ad - bc \neq 0$$

**Definition 2.**

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is called determinant of  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

## 2.1 Properties of determinants

- The determinant is bilinear in the columns and rows.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (v, w)$$

where  $v$  and  $w$  are column vectors of  $A$ .

$$\det(\lambda v_1 + \mu v_2, w) = \lambda \det(v_1, w) + \mu \det(v_2, w)$$

$$\det(v, \lambda w_1 + \mu w_2) = \lambda \det(v, w_1) + \mu \det(v, w_2)$$

$$\det(\lambda v_1 + \mu v_2, w) = \begin{vmatrix} \lambda a_1 + \mu a_2 & b \\ \lambda c_1 + \mu c_2 & d \end{vmatrix}$$

$$= (\lambda a_1 + \mu a_2)d - (\lambda c_1 + \mu c_2)b$$

$$= \lambda(a_1d - c_1b) + \mu(a_2d - c_2b)$$

$$= \lambda \begin{vmatrix} a_1 & b \\ c_1 & d \end{vmatrix} + \mu \begin{vmatrix} a_2 & b \\ c_2 & d \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

- $\det(v, v) = 0$ .

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ac = 0$$

- 

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det(e_1, e_2) = 1$$

**Theorem 3.** The properties 1–3 of determinants (see above) characterize the determinant.

Let  $\varphi : \mathbb{K}^2 \times \mathbb{K}^2 \rightarrow \mathbb{K}$

- bilinear
- $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$
- $\varphi(e_1, e_2) = 1$ . Then it holds that  $\varphi = \det$ .

*Proof.* To show:  $\varphi(v, w) = \det(v, w) \forall v, w \in \mathbb{K}^2$

$$v = \underbrace{ae_1 + ce_2}_{\begin{pmatrix} a \\ c \end{pmatrix}} \quad w = \underbrace{be_1 + de_2}_{\begin{pmatrix} b \\ d \end{pmatrix}}$$

$$\varphi(v, w) = \varphi(ae_1 + ce_2, be_1 + de_2)$$

$$= a\varphi(e_1, be_1 + de_2) + c \cdot \varphi(e_2, be_1 + de_2)$$

$$= ad \underbrace{\varphi(e_1, e_2)}_{=1} + \underbrace{ab\varphi(e_1, e_1)}_{=0} + cb\varphi(e_2, e_1) + cd \underbrace{\varphi(e_2, e_2)}_{=0}$$

□

**Lemma 2.** From (i) bilinearity and (ii)  $\bigwedge_{v \in \mathbb{K}^2} \varphi(v, v) = 0$  it follows that

$$\bigwedge_{v, w \in \mathbb{K}^2} \varphi(v, w) = -\varphi(w, v)$$

$$\begin{aligned} 0 &\stackrel{(ii)}{=} \varphi(v+w, v+w) \stackrel{(i)}{=} \varphi(v, v) + \varphi(v, w) + \varphi(w, v) + \varphi(w, w) \\ &\stackrel{(ii)}{=} \varphi(v, w) + \varphi(w, v) \end{aligned}$$

## 2.2 Geometric interpretation of the determinant

Consider an area with  $w$  defining its breath and  $v$  its depth (hence the area spanning vectors). Let  $e_1$  and  $e_2$  be the spanning vectors of a rectangle corresponding to the parallelogram.  $\det(v, w)$  is the surface of the spanned parallelogram. The sign defines the orientation of the pair  $(v, w)$ .

$$\det(e_1, e_2) = 1 \quad \det(e_2, e_1) = -1$$

There are surfaces where the surface is infinite if you follow a vector in some direction:

- Möbius strip
- Klein's bottle (named after Felix Klein)

$$A = |v| \cdot h$$

Consider Figure 1.  $h$  is the length of the projection of  $w$  to  $v^\perp$ .

$$\begin{aligned} v = \begin{pmatrix} a \\ b \end{pmatrix} &\rightarrow \vec{n} = \begin{pmatrix} -b \\ a \end{pmatrix} \\ \left\langle \begin{pmatrix} c \\ d \end{pmatrix}, \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle &= ad - bc \end{aligned}$$

*Second proof.*  $A(v, w)$  satisfies properties (i)–(iii).

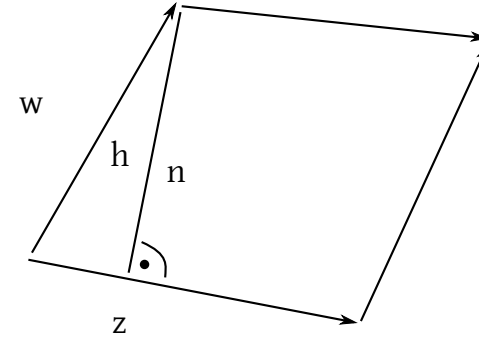


Figure 1: Parallelogram

- Property (iii) follows immediately (the area of unit vectors in two dimensions is 1).
- Property (ii) follows immediately (the area of two vectors in the same direction is 0).

Property (i) defines the linearity in  $v$

1. If  $v, w$  are linear dependent, then  $A(v, w) = 0$  (one is a multiple of the other)
2.  $n \in \mathbb{N}$  with  $A(nv, w) = nA(v, w)$

3. For  $\tilde{v} = n \cdot v$ :

$$A(\tilde{v}, w) = n \cdot A\left(\frac{\tilde{v}}{n}, w\right)$$

$$\Rightarrow A\left(\frac{\tilde{v}}{n}, w\right) = \frac{1}{n} A(\tilde{v}, w)$$

$$A(nv, w) = nA(v, w)$$

$$A\left(\frac{1}{n}v, w\right) = \frac{1}{n}A(v, w)$$

$$A\left(\frac{m}{n}v, w\right) = \frac{m}{n}A(v, w)$$

$$A(-v, w) = -A(v, w)$$

From continuity it follows that  $A(\lambda v, w) = \lambda A(v, w)$  for  $\lambda \in \mathbb{R}$ . Analogously  $A(v, \lambda w) = \lambda A(v, w)$ .

4. The sum is given with

$$A(v + w, w) = A(v, w)$$

Compare with Figure 2, where  $\text{area}(2) + \text{area}(3) = \text{area}(2) + \text{area}(1)$ .

$$\begin{aligned} A(\lambda v + \mu w, w) &= A\left(\lambda v + \mu w, \frac{1}{\mu} \mu w\right) \\ &= \frac{1}{\mu} A(\lambda v + \mu w, \mu w) \\ &= \frac{1}{\mu} A(\lambda v, \mu w) \\ &= A(\lambda v, w) \end{aligned}$$

General case:  $v, w$  are linear independent and therefore basis of  $\mathbb{R}^2$ . Besides that,  $v_1$  and  $v_2$  are arbitrary.

$$v_1 = \lambda_1 v + \mu_1 w$$

$$v_2 = \lambda_2 v + \mu_2 w$$

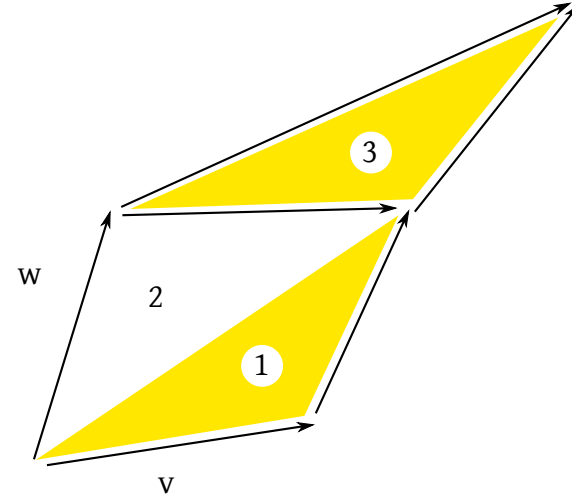


Figure 2: Translation of area 1 to area 3.

$$\begin{aligned} A(v_1 + v_2, w) &= A(\lambda_1 v + \mu_1 w + \lambda_2 v + \mu_2 w, w) \\ &= A((\lambda_1 + \lambda_2)v + (\mu_1 + \mu_2)w, w) \\ &= A((\lambda_1 + \lambda_2)v, w) \\ &= (\lambda_1 + \lambda_2)A(v, w) \\ &= A(\lambda_1 v, w) + A(\lambda_2 v, w) \end{aligned}$$

$$A(\lambda_1 v + \mu_1 w, w) + A(\lambda_2 v + \mu_2 w, w) = A(v_1, w) + A(v_2, w)$$

Additivity follows.

□

**Definition 3.** Let  $\dim V = n$ . A determinant form is a map

$$\Delta : V^n \rightarrow \mathbb{K}$$

with properties:

1.

$$\bigwedge_{\lambda} \bigwedge_k \bigwedge_{a_1, \dots, a_n \in V} \Delta(a_1, \dots, a_{k-1}, \lambda a_k, a_{k+1}, \dots, a_n) = \lambda \Delta(a_1, \dots, a_k, \dots, a_n)$$

2.

$$\begin{aligned} \bigwedge_k \bigwedge_{\substack{a_1, \dots, a_n \\ a'_k, a''_k}} \Delta(a_1, \dots, a_{k-1}, a'_k + a''_k, a_{k+1}, \dots, a_n) \\ := \Delta(a_1, \dots, a_{k-1}, a'_k + a''_k, a_{k+1}, \dots, a_n) \end{aligned}$$

3.

$$\Delta(a_1, \dots, a_n) = 0$$

if  $\bigvee_{k \neq l} a_k = e_l$  if  $\Delta \neq 0$ , i.e.  $\Delta$  is non-trivial.

Multilinearity is defined by the first two properties. Multilinearity means linearity in  $a_k$  if  $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n$  gets fixed.

**Theorem 4.**

$$\dim V = n$$

$$\Delta : V^n \rightarrow \mathbb{K} \text{ is determinant form}$$

Then,

4.

$$\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{i \neq j} \Delta(a_1, \dots, a_{i-1}, a_i + \lambda a_j, a_{i+1}, \dots, a_n) = \Delta(a_1, \dots, a_i, \dots, a_n)$$

“Addition of  $\lambda a_j$  to  $a_i$  does not change  $\Delta$ ”

5.

$$\begin{aligned} \bigwedge_{i > j} \Delta(a_1, \dots, a_{j-1}, a_i, a_{j+1}, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n) \\ = -\Delta(a_1, \dots, a_j, \dots, a_i, \dots, a_n) \end{aligned}$$

“Exchanging  $a_i$  with  $a_j$  inverts the sign”

*Proof.* 4.

$$\Delta(a_1, \dots, a_i + \lambda a_j, \dots, a_n)$$

Without loss of generality:  $i < j$ . From properties 1 and 2 it follows that:

$$= \Delta(a_1, \dots, a_i, a_j, a_n) + \lambda \Delta(a_1, \dots, a_j, a_j, \dots, a_k)$$

Oh,  $a_j$  occurs twice! Once at index  $i$  and once at index  $j$ .

$$= 0$$

due to property 3.

5.

$$\begin{aligned} 0 &\stackrel{\text{property 3}}{=} \Delta(a_1, \dots, a_{i-1}, a_i + a_j, \dots, a_{j-1}, a_i + a_j, \dots, a_n) \\ &= \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_i, \dots, a_{j-1}, \mathbf{a}_i, \dots, a_n) = \mathbf{0} \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_i, \dots, a_{j-1}, \mathbf{a}_j, \dots, a_n) \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_j, \dots, a_{j-1}, \mathbf{a}_i, \dots, a_n) \\ &+ \Delta(a_1, \dots, a_{i-1}, \mathbf{a}_j, \dots, a_{j-1}, \mathbf{a}_j, \dots, a_n) = \mathbf{0} \\ &\Rightarrow \delta \end{aligned}$$

□

**Definition 4.** A permutation of order  $n$  is a bijective mapping  $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ .

$\sigma_n =$  set of all permutations

**Remark 1.** Notation: We write the elements in the first row and their images in the second row.

**Definition 5.**  $\sigma_n$  constitutes (in terms of composition) a group with neutral element  $id$ , the so-called symmetric group.



In the previous course (Theorem 1.40) we have proven: Compositions of bijective functions are bijective. 1.

**Remark 2.** For  $n \geq 3$ ,  $\sigma_n$  is non-commutative

**Theorem 5.**

$$|\sigma_n| = n!$$

**Remark 3.** These are “a lot”!

**Example 3.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

**Definition 6.** A transposition is a permutation of the structure

$$\tau = \tau_{ij} : \begin{array}{l} i \mapsto j \\ j \mapsto i \text{ if } k \notin \{i, j\} \\ k \mapsto k \end{array}$$

Then  $\tau_{ij}^{-1} = \tau_{ij}$ , hence  $\tau_{ij}^2 = \text{id}$ .

**Theorem 6.**  $\sigma_n$  is generated by transpositions. With other words, every permutation  $\pi$  can be represented as composition of transpositions

$$\pi = \tau_1 \circ \dots \circ \tau_k$$

*Proof.*

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

If  $\pi = \text{id}$ ,

$$\pi = \pi \quad \tau := \text{id}$$

If  $\pi \neq \text{id}$ ,

$$k_1 = \min \{k \mid k \neq \pi(k)\}$$

$$\tau_1 = \tau_{k_1 \pi(k_1)}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & \dots & k-1 & k_1 & \dots \\ 1 & \dots & k-1 & k_1 & \dots \end{pmatrix}$$

Example: Consider  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$ .

$$k_1 = 2$$

$$\tau_1 = \tau_{23}$$

$$\pi_1 = \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 5 & 4 & 7 & 6 & 3 \end{pmatrix}$$

2.

$$k_2 = \min \{k \mid k \neq \pi_1(k)\} > k_1$$

$$\tau_2 = \tau_{k_2, \pi(k_2)}$$

And so on and so forth.  $k_j > k_{j-1}$  ends after  $\leq n$  steps.

$$\tau_k \circ \tau_{k-1} \circ \dots \circ \tau_1 \circ \pi = \text{id}$$

$$\Rightarrow \pi = \tau_1 \circ \tau_2 \circ \dots \circ \tau_k$$

Regarding the example:

$$k_2 = 3$$

$$\tau_2 = \tau_{35}$$

$$\pi_2 = \tau_2 \circ \pi_1 = \tau_2 \circ \tau_1 \circ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 7 & 6 & 5 \end{pmatrix}$$

$$k_3 = 5 \quad \tau_3 = \tau_{57}$$

$$\Rightarrow \pi = \tau_{23} \circ \tau_{35} \circ \tau_{57}$$

□

**Definition 7.** An inversion of  $\pi$  is a pair  $(i, j)$  such that  $i < j$  with  $\pi(i) > \pi(j)$ . Let  $F_\pi$  be the set of inversions of  $\pi$ .

$$f_\pi := |F_\pi|$$

$$\text{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

**Example 4.**

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 4 & 7 & 6 & 2 \end{pmatrix}$$

$$F_\pi = \{(2, 7), (3, 4), (3, 7), (4, 7), (5, 6), (5, 7), (6, 7)\}$$

$$f_\pi = 7 \quad \text{sign}(\pi) = -1$$

This lecture took place on 7th of March 2016 (Franz Lehner).

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Recall: Determinant form:

1.  $\Delta(a_1, \dots, \lambda a_k, \dots, a_n) = \lambda \Delta(a_1, \dots, a_n)$
2.  $\Delta(a_1, \dots, a'_k + a''_k, \dots, a_n) = \Delta(a_1, \dots, a'_k, \dots, a_n) + \Delta(a_1, \dots, a''_k, \dots, a_n)$
3.  $\Delta(a_1, \dots, a_k, \dots, a_l, \dots, a_n) = 0$  if  $a_k = a_l$

Conclusions:

4.  $\Delta(a_1, \dots, a_k + \lambda a_l, \dots, a_n) = \Delta(a_1, \dots, a_n)$  if  $k \neq l$
5.  $\Delta(a_1, \dots, a_k, \dots, a_l, \dots, a_n) = -\Delta(a_1, \dots, a_l, \dots, a_k, \dots, a_n)$

$$\Delta(a_{\pi(1)}, \dots, a_{\pi(n)}) = (-1)^k \Delta(a_1, \dots, a_n)$$

Decompose  $\pi = \tau_1 \circ \dots \circ \tau_k \circ \tau_{12} \circ \tau_{12}$ . This decomposition is not distinct ( $k$  is distinct mod 2)

$$\pi \in \sigma_n \quad \text{permutation}$$

$$F_\pi = \{(i, j) \mid i < j, \pi(i) > \pi(j), \text{ inversions} \}$$

$$f_\pi = |F_\pi|$$

$$\text{sign}(\pi) := (-1)^{f_\pi} =: (-1)^\pi$$

**Theorem 7.** •  $\bigwedge_{\pi \in \sigma_n} \text{sign}(\pi) = \prod_{1 \leq i < j \leq n} \frac{\pi(j) - \pi(i)}{j - i}$

• For transposition  $\tau$  it holds that  $\text{sign}(\tau) = -1$

*Proof.* • Every pair  $\{i, j\}$  occurs in the enumerator exactly once.

$$\frac{\prod_{i < j} \pi(j) - \pi(i)}{\prod_{i < j} (j - i)}$$

Denominator:  $j > i$ , positive. Enumerator: positive if  $\pi(j) > \pi(i)$ , negative if  $\pi(i) > \pi(j)$ .

•

$$\tau = \begin{pmatrix} 1 & \dots & k & \dots & l & \dots & n \\ 1 & \dots & l & \dots & k & \dots & n \end{pmatrix}$$

$$F_\tau(\underbrace{((k, k+1), (k, k+2), \dots, (k, l-1), (k, l))}_{\text{inversions with } k, l-k \text{ times}}, \underbrace{((k+1, l), \dots, (l-1, l))}_{l-k-1 \text{ times}})$$

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 8 & 5 & 6 & 7 & 4 & 9 & 10 \end{pmatrix}$$

Yields 7 inversions (8 needs to be repositioned with 3 transpositions, 4 needs to be repositions with 4 transpositions).

□

$$\text{sign}(\pi) = \prod_{i < j} \frac{\pi(j) - \pi(i)}{j - i} \quad \binom{n}{2} \text{ factors}$$

$$\text{sign}(\tau) = -1$$

**Theorem 8.** 1.  $\text{sign}(id) = 1$

2.  $\text{sign}(\pi \circ \sigma) = \text{sign}(\pi) \cdot \text{sign}(\sigma)$ , hence

$$\text{sign } \sigma_n \rightarrow (\{+1, -1\}, \cdot)$$

is a group homomorphism. (In general: A group homomorphism  $h : G \rightarrow (\mathcal{T}, \cdot)$  is called character)

3.  $\text{sign}(\pi^{-1}) = \text{sign}(\pi)$

**Remark 4.**

$$\mathcal{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

Torus with multiplication is a group.

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2| = 1$$

*Proof.* 1. trivial

2.

$$\begin{aligned} \text{sign}(\pi \cdot \sigma) &= \prod_{i < j} \frac{\pi \circ \sigma(j) - \pi \circ \sigma(i)}{j - i} \\ &= \underbrace{\prod_{i < j} \frac{\pi(\sigma(j)) - \pi(\sigma(i))}{\sigma(j) - \sigma(i)}}_{=\text{sign}(\pi)} \cdot \underbrace{\prod_{i < j} \frac{\sigma(j) - \sigma(i)}{j - i}}_{\text{sign}(\sigma)} \end{aligned}$$

3. Group homomorphism!

**Corollary 1.** • If  $\pi = \tau_1 \circ \tau_2 \circ \dots \circ \tau_k$ , product of transpositions

$$\Rightarrow \text{sign}(\pi) = (-1)^k$$

•  $\mathfrak{a}_n := \ker(\text{sign}) = \{\pi \in \sigma_n \mid \text{sign}(\pi) = 1\}$

“even permutations”, “alternating group”

$$|\mathfrak{a}_n| = \frac{n!}{2}$$

**Corollary 2.**

$$\Delta : V^k \rightarrow \mathbb{K} \text{ determinant form}$$

then it holds that

$$\bigwedge_{\pi \in \sigma_n} \bigwedge_{a_1, \dots, a_n \in V} \Delta(a_{\pi(1)}, \dots, a_{\pi(n)}) = \text{sign}(\pi) \cdot \Delta(a_1, \dots, a_n)$$

*Proof.* • If  $\pi = \tau_{kl}$  transposition  $\xrightarrow{\text{Theorem 4}} \Delta(a_{\tau(1)}, \dots, a_{\tau(n)}) = -\Delta(a_1, \dots, a_n) = \text{sign}(\tau_{kl}) \cdot \Delta(a_1, \dots, a_n)$

• If  $\pi = \tau_1 \circ \dots \circ \tau_k = \tau_1 \circ \tilde{\pi}, \tilde{\pi} = \tau_2 \circ \dots \circ \tau_k$

$$\begin{aligned} &\Delta(a_{\tau_1 \circ \tilde{\pi}(1)}, \dots, a_{\tau_1 \circ \tilde{\pi}(n)}) \\ &= -\Delta(a_{\tilde{\pi}(1)}, \dots, a_{\tilde{\pi}(n)}) \\ &= (-1)^2 \cdot \Delta(a_{\tilde{\pi}(1)}, a_{\tilde{\pi}(n)}) \\ &\rightarrow (-1)^k \cdot \Delta(a_1, \dots, a_n) \end{aligned}$$

□

**Theorem 9** (Leibnitz’ definition of  $\det(A)$ ). Let  $B = (b_1, \dots, b_n)$  be the basis of  $V$ .  $a_1, \dots, a_n \in V$  with coordinates

$$\Phi_B(a_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$$

□

$$A := [a_{ij}]_{i,j=1,\dots,n} = [\Phi_B(a_1), \Phi_B(a_2), \dots, \Phi_B(a_n)]$$

Then it holds that for every determinant form  $\Delta : V^k \rightarrow \mathbb{K}$ :

$$\Delta(a_1, \dots, a_n) = \det(A) \cdot \Delta(b_1, \dots, b_n)$$

where

$$\det(A) := \sum_{\pi \in \sigma_n} \text{sign}_{\mathbb{K}} \pi a_{\pi(1),1} a_{\pi(2),2} \dots a_{\pi(n),n}$$

is the determinant of  $A$

**Example 5.** Example ( $n = 2$ ):

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$\text{sign} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 1$$

$$\text{sign} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = -1$$

*Proof.*

$$a_j = \sum_{i=1}^n a_{ij} b_i$$

$$\begin{aligned} \Delta(a_1, \dots, a_n) &= \Delta \left( \sum_{i=1}^n a_{i,1} b_i, \sum_{i=2}^n a_{i,2} b_i, \dots, \sum_{i=n}^n a_{i,n} b_i \right) \\ &= \sum_{i_1=1}^n a_{i_1,1} \sum_{i_2=1}^n a_{i_2,2} \dots \sum_{i_n=1}^n a_{i_n,n} \underbrace{\Delta(b_{i_1}, b_{i_2}, \dots, b_{i_n})}_{=0 \text{ if some } i_k = i_l} \end{aligned}$$

So summands with equal indices disappear. It holds that  $\sum_{i_1, \dots, i_n}$  such that  $i_1, \dots, i_n$  are different. Hence every value from  $\{1, \dots, n\}$  occurs exactly once. This is the set of all permutations  $\pi$  ( $i_j = \pi(j)$ )

$$= \sum_{\pi \in \sigma_n} a_{\pi(1),1} a_{\pi(2),2} \dots a_{\pi(n),n} \underbrace{\Delta(b_{\pi(1)}, \dots, b_{\pi(n)})}_{\text{sign}(\pi) \cdot \Delta(b_1, \dots, b_n)}$$

□

**Corollary 3.** A determinant form is uniquely defined on a basis  $(b_1, \dots, b_n)$  by the value  $\Delta(b_1, \dots, b_n)$ . Especially  $\Delta$  is nontrivial,

$$\Leftrightarrow \Delta(b_1, \dots, b_n) \neq 0 \text{ on some basis.}$$

$$\Leftrightarrow \Delta(b_1, \dots, b_n) \neq 0 \text{ in every basis } b_1, \dots, b_n.$$

Let  $\Delta(b'_1, \dots, b'_n) = 0$  for some other basis, represent  $b_1, \dots, b_n$  in basis  $b'_1, \dots, b'_n$

$$b_j = \sum a_{ij} b'_i \Rightarrow \Delta(b_1, \dots, b_n) = \det(A) \cdot \Delta(b'_1, \dots, b'_n) = 0$$

$$\Delta(a_1, \dots, a_n) = \det(A) \cdot \Delta(b_1, \dots, b_n)$$

**Theorem 10.** Let  $B = (b_1, \dots, b_n)$  be a basis of  $V$  over  $\mathbb{K}$ .  $c \in \mathbb{K}$ . For  $a_1, \dots, a_n \in V$ , let  $A = [\Phi_B(a_1), \dots, \Phi_B(a_n)]$ . Then

$$\Delta(a_1, \dots, a_n) = c \cdot \det(A)$$

defines a determinant form, specifically the unique determinant form with value

$$\Delta(b_1, \dots, b_n) = c$$

*Proof.* The 3 properties of a determinant form:

1.

$$\begin{aligned} \Delta(a_1, \dots, \lambda a_k, \dots, a_n) &= c \cdot \det[\Phi_B(a_1), \dots, \lambda \cdot \Phi_B(a_k), \dots, \Phi_B(a_n)] \\ &= c \cdot \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot a_{\pi(1),1} a_{\pi(2),2} \dots \lambda a_{\pi(k),k} \dots a_{\pi(n),n} \\ &= \lambda \cdot c \cdot \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot a_{\pi(1),1} a_{\pi(2),2} \dots a_{\pi(n),n} \\ &= \lambda \cdot \Delta(a_1, \dots, a_n) \end{aligned}$$

2.

$$\begin{aligned} \Delta(a_1, \dots, a'_k + a''_k, \dots, a_n) &= c \cdot \det[\Phi_B(a_1), \dots, \Phi_B(a'_k) + \Phi_B(a''_k), \dots, \Phi_B(a_n)] \\ &= c \cdot \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot a_{\pi(1),1} \cdot a_{\pi(2),2} \dots \left( a'_{\pi(k),k} + a''_{\pi(k),k} \right) \dots a_{\pi(n),n} \\ &= c \cdot \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot a_{\pi(1),1} \dots a'_{\pi(k),k} \dots a_{\pi(n),n} \\ &\quad + c \cdot \sum_{\pi \in \sigma_n} \text{sign}(\pi) a_{\pi(1),1} \dots a''_{\pi(k),k} \dots a_{\pi(n),n} \\ &= \Delta(a_1, \dots, a'_k, \dots, a_n) + \Delta(a_1, \dots, a''_k, \dots, a_n) \end{aligned}$$

3. Let  $a_k = a_l$  for  $k < l$ . Show that  $\Delta(a_1, \dots, a_n) = 0$

$\tau_{kl}$  = transposition exchanging  $k$  and  $l$

$$\sigma_n = \mathfrak{a}_n \dot{\cup} (\mathfrak{a}_n \cdot \tau_{kl})$$

Claim:  $\{\pi \mid \text{sign } \pi = -1\} = \{\pi \circ \tau_{kl} \mid \text{sign } \pi = +1\}$

$$\supseteq \text{ If } \text{sign } \pi = +1 \Rightarrow \text{sign}(\pi \circ \tau_{kl}) = \underbrace{\text{sign } \pi}_{+1} \cdot \underbrace{\text{sign } \tau_{kl}}_{-1} = -1$$

$$\subseteq \text{ If } \text{sign } \pi = -1 \Rightarrow \text{sign}(\pi \circ \tau_{kl}) = +1 \Rightarrow \pi = \underbrace{(\pi \circ \tau_{kl}) \circ \tau_{kl}}_{\in \mathfrak{a}_n} \in \mathfrak{a}_n \cdot \tau_{kl}$$

$$\begin{aligned} \Delta(a_1, \dots, a_n) &= c \cdot \sum_{\pi \in \sigma_n = \mathfrak{a}_n \cup \mathfrak{a}_n \cdot \tau_{kl}} \text{sign}(\pi) a_{\pi(1),1} \dots a_{\pi(n),n} \\ &= c \cdot \underbrace{\sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \dots a_{\pi(n),n}}_{\text{even}} \\ &\quad - \underbrace{\sum_{\pi \in \mathfrak{a}_n} a_{\pi \circ \tau_{kl}(1),1} \dots a_{\pi \circ \tau_{kl}(k),k} \dots a_{\pi \circ \tau_{kl}(l),l} \dots a_{\pi \circ \tau_{kl}(n),n}}_{\text{odd}} \\ &= c \cdot \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \dots a_{\pi(n),n} \\ &\quad - \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \dots \underbrace{a_{\pi(l),k}}_{a_{\pi(l),l}} \dots \underbrace{a_{\pi(k),l}}_{a_{\pi(k),k} \text{ because } a_k = a_l} \dots a_{\pi(n),n} \end{aligned}$$

What we did:

(a)  $a_{\pi(l),k} = a_{\pi(l),l}$  and  $a_{\pi(k),l} = a_{\pi(k),k}$  because  $a_k = a_l$

(b) exchange factors

$$\begin{aligned} &= c \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \dots a_{\pi(k),k} \dots a_{\pi(l),l} \dots a_{\pi(n),n} \\ &\quad - c \sum_{\pi \in \mathfrak{a}_n} a_{\pi(1),1} \dots a_{\pi(k),k} \dots a_{\pi(l),l} \dots a_{\pi(n),n} \\ &= 0 \end{aligned}$$

Value for  $(b_1, \dots, b_n)$

$$a_{ij} = \delta_{ij} \Rightarrow A = I$$

$$\det(I) = \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot \delta_{\pi(1),1} \dots \delta_{\pi(n),n} = +1$$

for all  $\pi(j) = j$  otherwise 0.

$\Rightarrow \pi = \text{id}$  is the only summand

$$\Delta(b_1, \dots, b_n) = \det(I) \cdot c = c$$

□

**Remark 5.** “ $\mathfrak{a}_n$  is the subgroup of index 2” denoted  $[\sigma_n : \mathfrak{a}_n] = 2$

You might be familiar with:

$$\begin{aligned} \mathbb{Z}_n &= \mathbb{Z} / n\mathbb{Z} \\ [\mathbb{Z} : n\mathbb{Z}] &= n \end{aligned}$$

**Theorem 11 (Summary).** • The set of determinant forms  $\Delta : V^n \rightarrow \mathbb{K}$  constructs a one-dimensional vector space,  $\Lambda^n V$

• There exists a non-trivial determinant form with  $\Delta(b_1, \dots, b_n) = 1$

This lecture took place on 9th of March 2016 (Franz Lehner).

Revision:

$$\Delta : V^n \rightarrow \mathbb{K}$$

$$\Delta(a_1, \dots, a_n) = \det A \cdot \Delta(b_1, \dots, b_n)$$

$$\phi_B(a_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$\det A = \sum_{\pi \in \sigma_n} \text{sign } \pi \cdot a_{\pi(1),1} \dots a_{\pi(n),n}$$

$$\Delta(v_1, \dots, v_n) \neq 0 \Leftrightarrow v_1, \dots, v_n \text{ linear independent } (\Leftrightarrow \text{basis})$$

**Theorem 12.**

$$\det(A \cdot B) = \det(A) \cdot \det(B)$$

**Lemma 3.** Let  $V, W$  be vector spaces over  $\mathbb{K}$  with  $\dim V = \dim W = n$ .

$$\Delta : W^n \rightarrow \mathbb{K}$$

$$f : V \rightarrow W$$

$$\Rightarrow f^n : V^n \rightarrow W^n \xrightarrow{\Delta} \mathbb{K}$$

$$(v_1, \dots, v_n) \mapsto (f(v_1), \dots, f(v_n))$$

Then  $\Delta^f : V^n \rightarrow \mathbb{K}$

$$\Delta^f(v_1, \dots, v_n) = \Delta(f(v_1), \dots, f(v_n))$$

is a determinant form in  $V$ .

*Proof.* 1.

$$\begin{aligned} \Delta f(v_1, \dots, \lambda v_k, \dots, v_n) &= \Delta(f(v_1), \dots, f(\lambda v_k), \dots, f(v_n)) \\ &= \lambda \Delta(f(v_1), \dots, f(v_n)) \\ &= \lambda \cdot \Delta^f(v_1, \dots, v_n) \end{aligned}$$

2.

$$\begin{aligned} &= \Delta^f(v_1, \dots, v'_k + v''_k, \dots, v_n) \\ &= \Delta(f(v_1), \dots, f(v'_k + v''_k), \dots, f(v_n)) \\ &= \Delta(f(v_1), \dots, f(v'_k) + f(v''_k), \dots, f(v_n)) \\ &= \Delta(f(v_1), \dots, f(v'_k), \dots, f(v_n)) + \Delta(f(v_1), \dots, f(v''_k), \dots, f(v_n)) \\ &= \Delta^f(v_1, \dots, v'_k, \dots, v_n) + \Delta^f(v_1, \dots, v''_k, \dots, v_n) \end{aligned}$$

3.

$$\begin{aligned} \Delta^f(v_1, \dots, v_k, \dots, v_l, \dots, v_n) \quad v_k = v_l &\Rightarrow f(v_k) = f(v_l) \\ &= \Delta(f(v_1), \dots, f(v_k), \dots, f(v_l), \dots, f(v_n)) \\ &= 0 \end{aligned}$$

**Corollary 4** (Conclusions for  $V = W$ ).

$$\Delta : V^n \rightarrow \mathbb{K}$$

non-trivial determinant form

$$f : V \rightarrow V$$

$\Rightarrow \Delta^f$  is a determinant form

$$\dim \bigwedge^n V = 1 \Rightarrow \bigvee_{c_f \in \mathbb{K}} \Delta^k = c_f \cdot \Delta$$

$c_f =: \det f$  is called determinant of  $f$

**Corollary 5.** Let  $V, \Delta$  and  $f$  be like above.

1. For every basis  $B = (b_1, \dots, b_n)$  it holds that

$$\Delta^f(b_1, \dots, b_n) = \Delta(f(b_1), \dots, f(b_n)) = \det(f) \cdot \Delta(b_1, \dots, b_n)$$

$$\det(f) = \frac{\Delta(f(b_1), \dots, f(b_n))}{\Delta(b_1, \dots, b_n)}$$

2. with  $a_j = f(b_j)$  it holds that

$$\det \Phi_B^B(f) = \det(f)$$

$$A = \Phi_B^B(f)$$

$a_{ij}$  =  $i$ -th coordinate of  $f(b_j)$  and  $s_j(A) = \Phi_B(f(b_j))$ .

**Theorem 13.** Let  $f : V \rightarrow V$  be an isomorphism  $\Leftrightarrow \det(f) \neq 0$ .

*Proof.* Let  $f$  be an isomorphism.

$$\begin{aligned} &\Leftrightarrow (f(b_1), \dots, f(b_n)) \text{ is basis} \\ &\Leftrightarrow \Delta(f(b_1), \dots, f(b_n)) \neq 0 \\ &\Leftrightarrow \det(f) \cdot \Delta(b_1, \dots, b_n) \\ &\Leftrightarrow \det(f) \neq 0 \end{aligned}$$

□

□

**Theorem 14.** Let  $f, g : V \rightarrow V$  be linear.

$$\Rightarrow \det(f \circ g) = \det(f) \cdot \det(g)$$

**Remark 6.** We show:  $f \circ g$  is isomorphism  $\Leftrightarrow f$  and  $g$  are isomorphisms.

If  $f, g$  are invertible, then  $f \circ g$  are invertible.

1.

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}$$

2. Attention! This is wrong, if  $\dim = \infty$ ! For example:  $\delta : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$  over  $\mathbb{K}^\infty$  is injective, but not surjective!

$$S_L : (x_1, x_2, \dots) = (x_2, x_3, \dots)$$

is not injective, but surjective.

$$S_L \circ S_R = I$$

$$S_R \circ S_L - I = P_1$$

If  $f \circ g$  is bijective, then  $g$  is injective and  $f$  surjective.

$$\xLeftrightarrow{\dim < \infty} g \text{ bijective, } f \text{ bijective}$$

*Proof.* Case distinction:

$$\det(f \circ g) = 0$$

$$\xLeftrightarrow{\text{Theorem 13}} f \circ g \text{ is not bijective}$$

$$\Leftrightarrow f \text{ is not bijective or } g \text{ not bijective}$$

$$\Leftrightarrow \det(f) = 0 \vee \det(g) = 0$$

$$\Leftrightarrow \det(f) \cdot \det(g) = 0$$

$$\det(f \circ g) \neq 0$$

$$\Leftrightarrow f \circ g \text{ is bijective}$$

$$\Rightarrow g \text{ bijective}$$

$$\Rightarrow \Delta^g \text{ non-trivial}$$

Let  $(b_1, \dots, b_n)$  be a basis of  $V$ , then  $\Delta$  is non-trivial determinant.

$$\begin{aligned} \det(f \circ g) &= \frac{\Delta(f \circ g(b_1), \dots, f \circ g(b_n))}{\Delta(b_1, \dots, b_n)} \\ &= \frac{\Delta(f(g(b_1)), \dots, f(g(b_n)))}{\Delta(g(b_1), \dots, g(b_n))} \cdot \frac{\Delta(g(b_1), \dots, g(b_n))}{\Delta(b_1, \dots, b_n)} \\ &= \frac{\Delta(f(b'_1), \dots, f(b'_n))}{\Delta(b'_1, \dots, b'_n)} \cdot \frac{\Delta(g(b_1), \dots, g(b_n))}{\Delta(b_1, \dots, b_n)} \\ &= \det(f) \cdot \det(g) \end{aligned}$$

$b'_i = g(b_i)$  are also a basis, because  $g$  is bijective.

□

**Corollary 6.** Let  $A, B \in \mathbb{K}^{n \times n}$ .

$$1. \det(A \cdot B) = \det(A) \cdot \det(B)$$

$$2. A \text{ is regular} \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

$$3. \det(A) = 0 \Leftrightarrow \text{rank}(A) < n$$

$$4. \det(A^t) = \det(A)$$

*Proof.* 1. A first proof follows from Theorem 14.

A second proof approach is:

$$A = [s_1, \dots, s_n] \quad \text{column vectors}$$

$$A \cdot B = \left[ \sum_{i_1=1}^n s_{i_1} \cdot b_{i_1,1}, \sum_{i_2=1}^n s_{i_2} b_{i_2,2}, \dots, \sum_{i_n=1}^n s_{i_n} b_{i_n,n} \right]$$

Select determinant form such that  $\Delta(e_1, \dots, e_n) = 1$ .

$$\det(A \cdot B) = \Delta \left( \sum_{i_1=1}^n s_{i_1} b_{i_1}, \dots, \sum_{i_n=1}^n s_{i_n} b_{i_n,n} \right)$$

From multilinearity it follows that

$$\sum_{i_1=1}^n \sum_{i_2=1}^n \cdots \sum_{i_n=1}^n b_{i_1,1} b_{i_2,2} \cdots b_{i_n,n} \Delta(s_{i_1}, \dots, s_{i_n})$$

If two indices satisfy  $i_k = i_l \Rightarrow \Delta = 0$ .

$$\begin{aligned} &\Rightarrow \sum_{\text{different indices}} = \sum_{\text{permutations}} \\ &= \sum_{\pi \in \sigma_n} \underbrace{b_{\pi(1),1} b_{\pi(2),2} \cdots b_{\pi(n),n}}_{\det(B)} \underbrace{\Delta(s_{\pi(1)}, \dots, s_{\pi(n)})}_{\text{sign}(\pi) \Delta(s_1, \dots, s_n)} \\ &= \det A \cdot \det B \end{aligned}$$

Be aware that  $\det(B)$  also includes  $\text{sign}(\pi)$  from the right-hand side.

2.

$$\begin{aligned} A \cdot A^{-1} = I &\Leftrightarrow \det(A \cdot A^{-1}) = \det I = 1 \\ \det(A \cdot A^{-1}) &\stackrel{!}{=} \det(A) \cdot \det(A^{-1}) \end{aligned}$$

3.  $\det(A) = 0$  and  $\det(A) = \det(f_A)$ .

$$\Leftrightarrow f_A \text{ is not bijective} \Leftrightarrow \text{rank}(A) < n$$

4.

$$\begin{aligned} \det(A^T) &= \sum_{\pi \in \sigma_n} \text{sign}(\pi) a_{\pi(1),1}^T \cdots a_{\pi(n),n}^T \\ &= \sum_{\pi \in \sigma_n} \text{sign}(\pi) a_{1,\pi(1)} \cdots a_{n,\pi(n)} \\ &= \sum_{\pi \in \sigma_n} \text{sign} \pi a_{\pi^{-1}(1),1} \cdots a_{\pi^{-1}(n),1} \\ &= \sum_{\rho} \text{sign} \rho^{-1} a \end{aligned} \quad \rho = \pi^{-1}$$

For fixed  $\pi$ :

$$\begin{aligned} \prod_{j=1}^n a_{j,\pi(j)} &= \prod_{k=1}^n a_{\pi^{-1}(k),k} \\ \pi(j) = 1 &\Leftrightarrow j = \pi'(1) \\ \pi(j) = k &\Leftrightarrow j = \pi'(k) \end{aligned}$$

$$\begin{aligned} &\sum_{\pi} \text{sign} \pi a_{\pi^{-1}(1),1} \cdots a_{\pi^{-1}(n),n} \\ &= \sum \text{sign}(\rho^{-1}) a_{\rho(1),1} \cdots a_{\rho(n),n} = \sum_{\rho} \text{sign}(\rho) a_{\rho(1),1} \cdots a_{\rho(n),n} = \det A \end{aligned}$$

Remark:

$\sigma_n \rightarrow \sigma_n$  is bijective

$$\pi \mapsto \pi^{-1}$$

$\text{sign}(\rho) = (-1)^k$  where  $\rho = \tau_1, \dots, \tau_k$

$$\Rightarrow \rho^{-1} = \tau_k \circ \dots \circ \tau_1$$

$$\text{sign} \rho^{-1} = (-1)^k$$

□

**Remark 7** (Determination of determinants).  $\dim \leq 3$

For  $n = 2$ :

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For  $n = 3$ :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{\pi \in \sigma_3} \text{sign}(\pi) a_{\pi(1),1} a_{\pi(2),2} a_{\pi(3),3}$$



General linear group:

$$\begin{aligned} \text{GL}(n, \mathbb{K}) &= \text{group of invertible matrices} \\ &= \{A \in \mathbb{K}^{n \times n} \mid \det(A) \neq 0\} \\ \text{SL}(n, \mathbb{K}) &= \text{special linear group} \\ &= \{A \in \mathbb{K}^{n \times n} \mid \det(A) = 1\} \end{aligned}$$

$\sigma_3$  is a coxeter group.

$$\sigma_3 = \langle \tau_{12}, \tau_{23} \rangle$$

Is created by two transpositions.

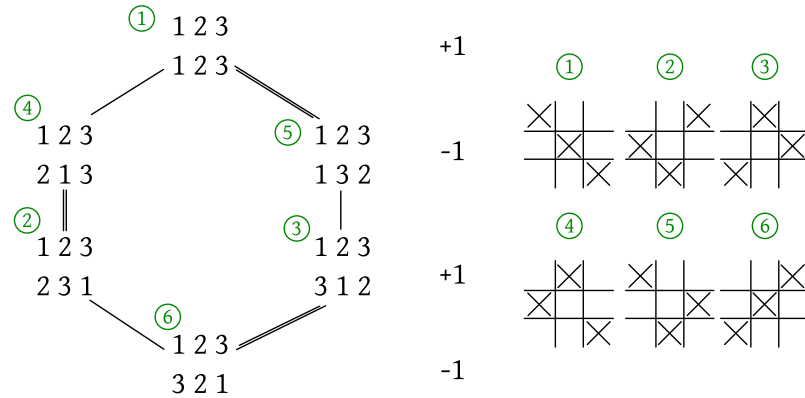


Figure 3: Sign of a permutation

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{21}a_{12}a_{33} - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13}$$

corresponding to (1) + (2) + (3) + (4) + (5) + (6) in Figure 3.

**Remark 8** (Rule of Sarrus). Compare with Figure 4.

You write the first two columns next to right side of the matrix. You add up all 3 diagonals (the product of their values) from top left diagonally to the right bottom and subtract all 3 diagonals from left bottom to the top right.

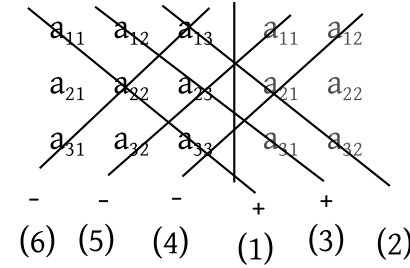


Figure 4: Rule of Sarrus visualized

The rule of Sarrus does not hold for  $n = 4$ !

**Example 6.**

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{pmatrix} &= 1 \cdot 5 \cdot 42 + 2 \cdot 14 \cdot 5 + 5 \cdot 2 \cdot 14 - 5 \cdot 5 \cdot 5 - 14 \cdot 14 \cdot 1 - 2 \cdot 2 \cdot 42 \\ &= 14(1 \cdot 5 \cdot 3 + 2 \cdot 5 + 5 \cdot 2 - 14 - 2 \cdot 2 \cdot 3) - 125 = 14 \cdot 9 - 125 = 1 \end{aligned}$$

It turns out, if we use Catalan numbers, we always end up with determinant 1.

**Lemma 4.** Let  $A$  be an upper triangular matrix, hence  $a_{ij} = 0 \forall i > j$ . Then it holds that  $\det A = a_{11}a_{22} \dots a_{nn}$ .

*Proof.*

$$\det A = \sum_{\pi \in \sigma_n} \text{sign } \pi a_{\pi(1),1} \dots a_{\pi(n),n}$$

it must hold that

$$\pi(j) \leq j \quad \forall j$$

$$\Rightarrow \pi(1) = 1, \pi(2) = 2, \dots, \pi(n) = n$$

The only permutation which contributes something is the identity. And  $\text{sign id} = 1$ , hence

$$= 1 \cdot a_{11}a_{22} \dots a_{nn}$$

**Lemma 5** (Elementary row and column transformations).

$$A = [a_{ij}] \in \mathbb{K}^{n \times n}$$

1.

$$s_i = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} \text{ column vectors}$$

$$\Rightarrow \det[as_1, \dots, s_i + \lambda s_j, \dots, s_n] = \det(A) \quad i \neq j$$

2. Let  $z_i = [a_{i1}, \dots, a_{in}]$  rows of  $A$ .

$$\det \begin{bmatrix} z_1 \\ \vdots \\ z_i + \lambda z_j \\ \vdots \\ z_n \end{bmatrix} = \det A \quad \text{for } i \neq j$$

*Proof.* 1. compare with determinant form

$$2. \det A = \det A^T$$

**Example 7.**

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 4 & 17 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

This lecture took place on 14th of March 2016 (Franz Lehner).

**Lemma 6.** Recall: The following operations do not change the determinant:

- $\Delta(s_1, \dots, s_i + \lambda s_j, \dots, s_n) = \Delta(s_1, \dots, s_n)$   
Addition of a multiple of a column (or row) to another
- Gauss-Jordan operations (elementary row/column transformations)

**Example 8.**

$$\begin{vmatrix} 1 & 0 & 3 & -2 \\ 2 & 6 & 4 & 1 \\ 3 & 3 & -1 & -1 \\ -1 & 2 & 4 & 1 \end{vmatrix} \rightsquigarrow \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 3 & -10 & 5 \\ 0 & 2 & 7 & -1 \end{vmatrix} \rightsquigarrow \frac{1}{3} \frac{1}{2} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 6 & -20 & 10 \\ 0 & 6 & 21 & -3 \end{vmatrix}$$

We multiplied the third row times 2 and the fourth row times 3. Be aware that this way we avoided fractions in the matrix.

$$\rightsquigarrow \frac{1}{6} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 0 & -18 & 5 \\ 0 & 0 & 23 & -8 \end{vmatrix} \cdot \frac{23}{18} = \frac{1}{6} \begin{vmatrix} 1 & 0 & 3 & -2 \\ 0 & 6 & -2 & 5 \\ 0 & 0 & -8 & 5 \\ 0 & 0 & 0 & -8 + 5 \frac{23}{18} \end{vmatrix}$$

Even though we have a fraction  $\frac{1}{6}$  at the front, our result will remain to be integral (i.e. without decimal points).

Triangular matrix:

$$\frac{1}{6} \cdot 1 \cdot 6 \cdot (-18) \cdot \left( -8 + \frac{5 \cdot 23}{18} \right)$$

$$= -(-18 \cdot 8 + 5 \cdot 23) = -(-144 + 115) = 29$$

**Lemma 7.** 1.

$$\begin{array}{c|ccc} a_{11} & * & \dots & * \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & B & \end{array} = a_{11} \cdot \det B$$

2.

$$\begin{array}{cc|c} & & 0 \\ & & 0 \\ & & \vdots \\ B & & \\ \hline * & \dots & * \\ & & a_{nn} \end{array} = \det B \cdot a_{nn}$$

*Proof.*

$$\det A = \sum_{\pi \in \sigma_n} (-1)^\pi a_{\pi(1),1} \dots a_{\pi(n),n}$$

2.

$$\begin{aligned} a_{\pi(n),n} &= 0 \text{ except when } \pi(n) = n \\ &= \sum_{\pi \in \sigma_n} (-1)^\pi a_{\pi(1),1} \dots a_{\pi(n),n} \\ &= \sum_{\rho \in \sigma_{n-1}} (-1)^\rho a_{\rho(1),1} \dots a_{\rho(n-1),n-1} a_{\rho(n),n} = \det B \cdot a_{nn} \end{aligned}$$

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,l-1} & a_{1,l+1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,l-1} & a_{2,l+1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-1,1} & \dots & a_{k-1,l-1} & a_{k-1,l+1} & \dots & a_{k-1,n} \\ a_{k+1,1} & \dots & a_{k+1,l-1} & a_{k+1,l+1} & \dots & a_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,l-1} & a_{n,l+1} & \dots & a_{n,n} \end{vmatrix}$$

**Theorem 15** (Generative theorem of Laplace (dt. Entwicklungssatz von Laplace)). *Let  $A \in K^{n \times n}$ , then it holds that*

$$\det(A) = \sum_{k=1}^n a_{k,l} \cdot (-1)^{k+l} \cdot \det A_{k,l}$$

*Generation to  $l$ -th column.*

$$\det A = \sum_{l=1}^n a_{k,l} \cdot (-1)^{k+l} \cdot \det A_{k,l}$$

*Generation to  $k$ -th row.*

□

**Definition 8.** *Let  $A \in \mathbb{K}^{n \times n}$ .*

$$1 \leq k, l \leq n$$

$A_{k,l}$  (dimension  $(n-1) \times (n-1)$ ) which is generated by  $A$  if you cancel out row  $k$  and column  $l$ .

*Proof.*  $l$ -th column is

$$a_l = \sum_{k=1}^n a_{kl} e_k$$

$$\begin{aligned}
 \det(A) &= \Delta(a_1, \dots, a_n) \\
 &= \Delta(a_1, \dots, a_{l-1}, \sum_{k=1}^n a_{kl} e_k, \dots, a_n) \\
 &= \sum_{k=1}^n a_{kl} \Delta(a_1, \dots, a_{l-1}, e_k, \dots, a_n) \\
 &= \sum_{k=1}^n a_{kl} \begin{vmatrix} a_{11} & \dots & a_{1,l-1} & 0 & a_{1,l+1} & \dots & a_{1,n} \\ & & & \vdots & & & \\ & & & 0 & & & \\ & & & 1 & & & \vdots \\ & & & 0 & & & \\ & & & \vdots & & & \\ a_{n1} & \dots & a_{n,l-1} & 0 & a_{n,l+1} & \dots & a_{n,n} \end{vmatrix}
 \end{aligned}$$

where 1 is given on the  $k$ -th row and the  $l$ -th column which is  $e_k$ .

We exchange the  $l$ -th column with the  $(l-1)$ -th, then  $(l-2)$ -th and so on and so forth ... This requires  $(l-1)$  transpositions.

$$\sum_{k=1}^n a_{kl} (-1)^{l-1} \begin{vmatrix} 0 & a_{11} & \dots & a_{1,l-1} & a_{1,l-1} & \dots & a_{1,n} \\ \vdots & a_{21} & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_{n1} & \dots & a_{n,l-1} & a_{n,l-1} & \dots & a_{n,n} \end{vmatrix}$$

where 1 is given on the  $k$ -th row.

Exchange  $k$ -th and  $(k-1)$ -th row, then  $(k-2)$ -th and so on and so forth ... This requires  $k-1$  transpositions.

$$= \sum_{k=1}^n a_{kl} (-1)^{k-1+l-1} \begin{vmatrix} 1 & \\ 0 & \\ \vdots & \\ \dots & A_{k,l} \\ \vdots & \\ 0 & \end{vmatrix} = \sum_{l=1}^n a_{k,l} (-1)^{k+l} \det A_{k,l}$$

□

**Example 9.**

$$\begin{vmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 14 \\ 14 & 42 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 14 \\ 5 & 42 \end{vmatrix} + 5 \cdot \begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix}$$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

where the top right + refers to the third summand (submatrix) and the top middle − refers to the second summand (submatrix).

$$= (5 \cdot 42 - 14 \cdot 14) - 2 \cdot (2 \cdot 42 - 5 \cdot 14) + 5 \cdot (2 \cdot 14 - 5 \cdot 5) = 14 - 2 \cdot 14 + 5 \cdot 3 = 1$$

**Theorem 16.**  $A$  is invertible iff  $\det A \neq 0$ .

Let  $A \in K^{n \times n}$ ,  $\hat{A} := [\hat{a}_{kl}]_{k,l=1,\dots,n}$  is the complementary matrix or adjoint matrix.

$$\hat{a}_{kl} = (-1)^{k+l} \det A_{lk}$$

Then

$$A^{-1} = \frac{1}{\det A} \cdot \hat{A}$$

*Proof.* Show that  $B := \hat{A} \cdot A = \det A \cdot I$ .

$$b_{k,l} = \sum_{j=1}^n \hat{a}_{kj} a_{jl} = \sum_{j=1}^n (-1)^{k+j} \det A_{jk} a_{jl}$$

**Case**  $k = l$

$$b_{kk} = \sum_{j=1}^n (-1)^{k+j} a_{jk} \det A_{jk} = \det A \text{ (Laplace generation to } k\text{-th column)}$$

**Case**  $k \neq l$  Without loss of generality  $k < l$ .

$$0 = \det \begin{bmatrix} a_{11} & \dots & a_{1l} & \dots & a_{1l} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nl} & \dots & a_{nl} & \dots & a_{nn} \end{bmatrix}$$

We replace the  $k$ -th column (left column with  $a_{1l}$  in the middle) by the  $l$ -th column (right column with  $a_{1l}$  in the middle).

Laplace generation by  $k$ -th column:

$$= \sum_{j=1}^n a_{jl} \det \begin{bmatrix} a_{11} & \dots & 0 & \dots & a_{1l} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & 1 & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & 0 & \dots & a_{nl} & \dots & a_{nn} \end{bmatrix}$$

Similar to Laplace:

$$= \sum_{j=1}^n a_{jl} (-1)^{j+l} \det A_{jk} = \sum_{j=1}^n a_{jl} \hat{a}_{kj} = b_{kl}$$

**Example 10** (Cayley 1855). *Cayley considered it as partial derivations:*

$$\frac{1}{\nabla} \begin{vmatrix} \partial_a \nabla & \partial_c \nabla \\ \partial_b \nabla & \partial_d \nabla \end{vmatrix}$$

Consider  $n = 2$ :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Consider  $n = 3$ :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} \begin{vmatrix} a_{22} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

**Example 11.**

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 5 & 14 \\ 5 & 14 & 42 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 5 & 14 \\ 14 & 42 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 14 & 42 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 5 & 14 \end{vmatrix} \\ -\begin{vmatrix} 2 & 14 \\ 5 & 42 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 5 & 42 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 2 & 14 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 5 & 14 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 5 & 14 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 14 & -14 & 3 \\ -14 & 17 & -4 \\ 3 & -4 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 5 & 14 \\ 14 & 42 \end{vmatrix} = 5 \cdot 3 \cdot 14 - 14 \cdot 14 = 14$$

$$\begin{vmatrix} 2 & 5 \\ 14 & 42 \end{vmatrix} = 2 \cdot 3 \cdot 14 - 5 \cdot 14 = 14$$

**Theorem 17** (Arnold's hypothesis). *"No theorem in mathematics is named after it's original author"*

*Proof.* No proof provided here.  $\square$

$\square$  **Theorem 18** (Cramer's rule). *Originally by McLansin (1748) based on work by Leibniz (1678) and reformulated by G. Cramer (1750).*

A regular  $n \times n$  matrix with column vectors  $a_1, \dots, a_n \in \mathbb{K}^n$ .

Then the unique solution to the equation system  $Ax = b$  is given by

$$x_i := \frac{\Delta(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n)}{\Delta(a_1, \dots, a_n)} = \frac{\det(a_1, \dots, b, \dots, a_n)}{\det A}$$

Its complexity is given by  $n + 1$  determinants!

*Proof.*

$$\begin{aligned}
 b &= \sum_{j=1}^n b_j e_j \\
 x &= A^{-1}b = \frac{1}{\det A} \hat{A} \cdot b \\
 x_i &= \frac{1}{\det A} \sum_{j=1}^n \hat{a}_{ij} b_j = \frac{1}{\det A} \sum_{j=1}^n (-1)^{i+j} \det(A_j) b_j \\
 &= \frac{1}{\det A} \sum_{j=1}^n \Delta(a_1, \dots, a_{i-1}, \dots, a_{j-1}, e_j, a_{j+1}, \dots, a_n) \cdot b_j \\
 &= \frac{1}{\det A} \Delta(a_1, \dots, a_{i-1}, \underbrace{\sum_{j=1}^n b_j e_j}_{=b}, \dots, a_n)
 \end{aligned}$$

**Example 12.**

$$\begin{aligned}
 2x_1 + 2x_2 &= 7 \\
 x_1 - 3x_2 &= 0 \\
 A &= \begin{bmatrix} 2 & 2 \\ 1 & -3 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \\
 \det A &= -8 \quad x_1 = \frac{\begin{vmatrix} 7 & 2 \\ 0 & -3 \end{vmatrix}}{-8} = \frac{21}{8} \quad x_2 = \frac{\begin{vmatrix} 2 & 7 \\ 1 & 0 \end{vmatrix}}{-8} = \frac{7}{8}
 \end{aligned}$$

**Remark 9.** • in higher dimensions ( $n \geq 4$ ) Cramer's rule is disallowed.

1. too computationally intense
2. numerically unstable (small errors have large effects)

• Anyways, still useful for theoretical considerations

1. the map  $A \mapsto \det A$  is  $C^\infty$  (polynomial!) (this denotes infinite differentiability)

2. The set of invertible matrices in  $\mathbb{R}^{n \times n}$  is open, because if  $\det A \neq 0$ , then also  $\det \tilde{A} \neq 0$  as long as  $|a_{ij} - \tilde{a}_{ij}| < \delta$ .
3. The solution of the equation system  $Ax = b$ , for invertible  $A$ , depends continuously and differentiable on  $A$  and  $b$ :

$$x_i = \underbrace{\frac{1}{\det A}}_{\text{continuous as long as } \det A \neq 0} \underbrace{\hat{A}b}_{\text{polynomial}}$$

4. The map  $\text{GL}(n, \mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R})$

$$A \mapsto A^{-1}$$

is continuous.

$$A^{-1} = \frac{1}{\det A} \cdot \hat{A}$$

So  $\text{GL}(n, \mathbb{R})$  is a Lie group.

□

This lecture took place on 16th of March 2016 (Franz Lehner).

### 3 Inner products

Descartes introduced “La Géometrie” (1637).

**Definition 9.** The length of a vector in  $\mathbb{R}^2/\mathbb{R}^3$  is:

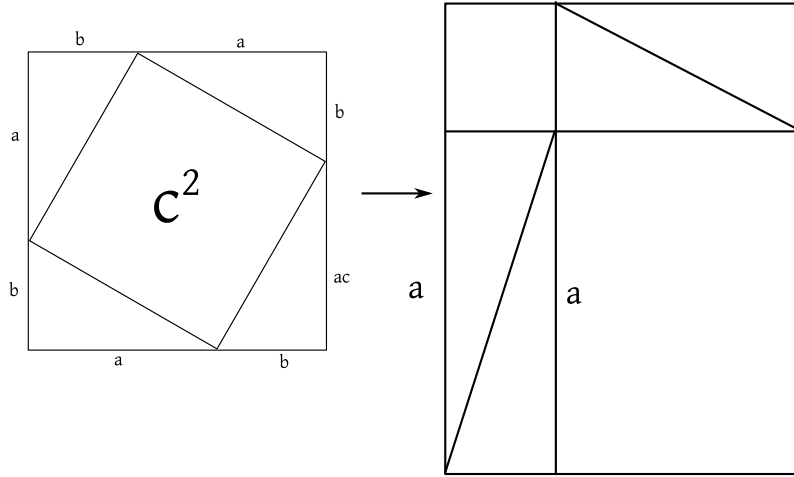
$$\left\| \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

**Definition 10** (Scalar product).

$$\cos \theta = \cos(2\pi - \theta)$$

The scalar product is defined as

$$\langle a, b \rangle = \|a\| \cdot \|b\| \cdot \cos \theta$$


 Figure 5: Pythagorean proof of  $c^2 = a^2 + b^2$ 

**Theorem 19.** *The following properties hold:*

- $\|\lambda \cdot a\| = |\lambda| \cdot \|a\|$
- $\|a + b\| \leq \|a\| + \|b\|$  (*triangle inequality*)
- $\langle a, a \rangle = \|a\|^2 \geq 0$
- $\langle a, a \rangle = 0 \Leftrightarrow a = 0$
- $\langle a, b \rangle = 0 \Leftrightarrow a = 0 \vee b = 0$

$$\langle a, b \rangle > 0 \Leftrightarrow \text{acute angle}$$

$$\langle a, b \rangle < 0 \Leftrightarrow \text{obtuse angle}$$

**Theorem 20.**

$$\langle a, b \rangle = \langle b, a \rangle \quad (1)$$

$$\langle \lambda a, b \rangle = \lambda \langle a, b \rangle \quad (2)$$

$$\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle \quad (3)$$

*So it actually describes a bilinear map.*

*Proof.* • immediate

- $\lambda > 0$  immediate
- $\lambda < 0$  Angle  $\theta$  becomes  $\pi - \theta$ .

$$\cos(\pi - \theta) = -\cos \theta$$

$$\langle \lambda a, b \rangle = |\lambda| \cdot \|a\| \cdot \|b\| \cos(\pi - \theta) = -|\lambda| \cdot \|a\| \cdot \|b\| \cdot \cos \theta = \lambda \langle a, b \rangle$$

- Let  $b = e$ ,  $\|e\| = 1$ .

$$\langle a, e \rangle = \|a\| \cdot \cos \theta$$

$$\langle a + b, c \rangle = \|c\| \left\langle a + b, \frac{c}{\|c\|} \right\rangle = \|c\| \left( \left\langle a, \frac{c}{\|c\|} \right\rangle + \left\langle b, \frac{c}{\|c\|} \right\rangle \right) = \langle a, c \rangle + \langle b, c \rangle$$

Compare with Figure 6.

□

**Theorem 21.**

$$\left\langle \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

*Proof.*

$$\begin{aligned} \langle a, b \rangle &= \langle a_1 e_1 + a_2 e_2 + a_3 e_3, b \rangle \\ &= a_1 \langle e_1, b \rangle + a_2 \langle e_2, b \rangle + a_3 \langle e_3, b \rangle \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

$$\begin{aligned} \langle e_i, b \rangle &= \langle e_i, b_1 e_1 + b_2 e_2 + b_3 e_3 \rangle \\ &= b_1 \langle e_i, e_1 \rangle + b_2 \langle e_i, e_2 \rangle + b_3 \langle e_i, e_3 \rangle \\ &= b_1 \delta_{i1} + b_2 \delta_{i2} + b_3 \delta_{i3} \\ &= b_i \end{aligned}$$

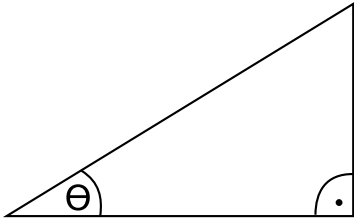


Figure 6:  $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$

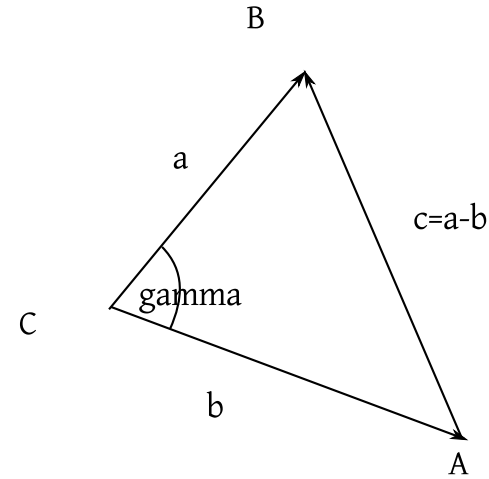


Figure 7: Law of cosines

with  $\dim \langle e_i, e_j \rangle = \delta_{ij}$ .

**Example 13** (Law of cosines).

$$a^2 + b^2 = c^2 + 2ab \cos \gamma$$

Compare with Figure 7.

$$\begin{aligned} \|c\|^2 &= \langle a - b, a - b \rangle \\ &= \langle a, a \rangle - \langle a, b \rangle - \langle b, a \rangle + \langle b, b \rangle \\ &= \|a\|^2 - 2 \cdot \|a\| \|b\| \cos \gamma + \|b\|^2 \end{aligned}$$

□

**Theorem 22.** *Theorem by Thales TODO: image*

$$\begin{aligned} \langle a - b, -a - b \rangle &= \|a - b\| \|a + b\| \cos \theta \\ \langle a - b, -a - b \rangle &= -\langle a - b, a + b \rangle \\ &= -(\langle a, a \rangle - \langle b, a \rangle + \langle a, b \rangle - \langle b, b \rangle) \\ &= -(\|a\|^2 - \|b\|^2) \\ &= 0 \\ \Rightarrow \theta &= \frac{\pi}{2} \end{aligned}$$



**Remark 10.** How do we find the normal vector?

$$\vec{n} = \begin{pmatrix} a_2 \\ -a_1 \end{pmatrix}$$

$$\left\langle \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_2 & -a_1 \end{pmatrix} \right\rangle = a_1 a_2 - a_2 a_1 = 0$$

**Definition 11** (Outer product). “Outer product”, “cross product” or “vector product”

TODO: image missing

This is only available in  $\mathbb{R}^3$ .

Let  $a, b \in \mathbb{R}^3$ , then  $a \times b$  is the vector with properties:

- $\|a \times b\| = \|a\| \cdot \|b\| \cdot \sin \theta$

This corresponds to the area of a parallelogram.

$$\|b\| \cdot \sin \theta = \text{height of a parallelogram}$$

- $a \times b \perp a, b$

$$\langle a \times b, a \rangle = 0$$

$$\langle a \times b, b \rangle = 0$$

- $(a, b, a \times b)$  are clockwise (consider a screw coming out of Figure)

$$a \times b = 0 \Leftrightarrow a = 0 \vee b = 0 \vee a, b \text{ are linear dependent}$$

**Theorem 23.** 1.  $b \times a = -a \times b$  (counter-clockwise)

2.  $(\lambda a) \times b = \lambda \cdot a \times b = a \times (\lambda b)$

3.  $(a + b) \times c = a \times c + b \times c$

So it is bilinear in  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$

Proof.

$$a \times c, b \times c, (a + b) \times c \in E$$

Let  $a', b', (a + b)'$  be the projection of  $a, b$  and  $a + b$  in the plane.

TODO: image missing

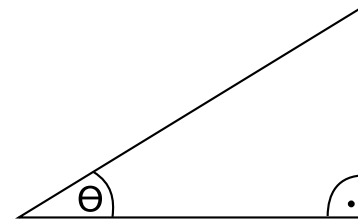


Figure 8: Theorem 23, third statement

1.

$$(a + b)' = a' + b'$$

Projection of the sum = sum of projections.

2.

$$a \times c = a' \times c$$

$$\|a' \times c\| = \|a'\| \cdot \|c\|$$

$$\begin{aligned} \|a \times c\| &= \|a\| \cdot \|c\| \cdot \sin \theta \\ &= \|a'\| \cdot \|c\| \end{aligned}$$

$$\|a'\| = \|c\| \cdot \sin \theta$$

and they have the same direction.

TODO: image missing

3.

$$(a' + b') \times c = c' \times c + b' \times c$$

From above:

TODO: image missing

$$\|a' \times c\| = \|c\| \cdot \|a'\|$$

So this operation is linear.

$$\begin{aligned} (a + b) \times c &\stackrel{2}{=} (a + b)' \times c \\ &\stackrel{1}{=} (a' + b') \times c \\ &\stackrel{3}{=} (a' \times c + b' \times c) \\ &\stackrel{2}{=} a \times c + b \times c \end{aligned}$$

□

**Corollary 7.** *The cross product is a map  $x : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$  with properties:*

• *bilinear*

• *anti-symmetric*

• *“chiral”, namely*

$$e_1 \times e_2 = e_3$$

$$e_2 \times e_3 = e_1$$

$$e_3 \times e_1 = e_2$$

**Corollary 8.**

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} \end{bmatrix}$$

$$\stackrel{\text{Laplace}}{=} \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

*Formally, matrices in a vector of values are disallowed, but as far as it boils down to addition, this is fine.*

*Proof.*

$$\begin{aligned} &(a_1 e_1 + a_2 e_2 + a_3 e_3) \times (b_1 e_1 + b_2 e_2 + b_3 e_3) \\ &= a_1 b_1 e_1 \times e_1 + a_1 b_2 e_1 \times e_2 + a_1 b_3 e_1 \times e_3 \\ &\quad + a_2 b_1 e_2 \times e_1 + a_2 b_2 e_2 \times e_2 + a_2 b_3 e_2 \times e_3 \\ &\quad + a_3 b_1 e_3 \times e_1 + a_3 b_2 e_3 \times e_2 + a_3 b_3 e_3 \times e_3 \\ &= a_1 b_2 e_3 + a_1 b_3 (-e_2) + a_2 b_1 (-e_3) + a_2 b_3 e_1 + a_3 b_1 e_2 + a_3 b_2 (-e_1) \\ &= (a_2 b_3 - a_3 b_2) e_1 + (a_3 b_1 - a_1 b_3) e_2 + (a_1 b_2 - a_2 b_1) e_3 \end{aligned}$$

□

**Theorem 24** (Scalar triple product). *The three-dimensional parallelepiped is called “Spat” in German (compare with Figure 9).*

$$\langle a \times b, c \rangle = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{volume of spanned 3-dimensional parallelepiped}$$

$\|a \times b\|$  is the area of the parallelogram.  $\langle a \times b, c \rangle = \|a \times b\| \cdot \|c\| \cdot \cos \theta$  where

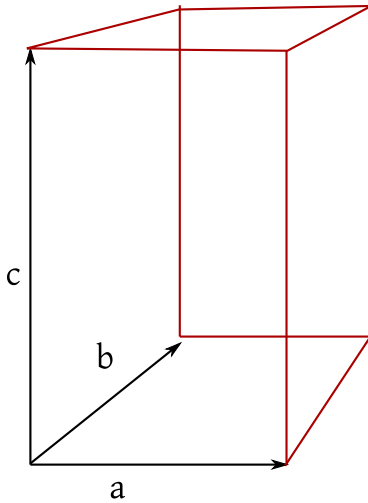


Figure 9: Three-dimensional parallelepiped

$\|c\| \cdot \cos \theta$  is the height of the 3-dimensional parallelepiped.

$$\langle a \times b, c \rangle = \left\langle \begin{pmatrix} \begin{vmatrix} a_1 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{pmatrix}, \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \right\rangle$$

$$\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \cdot c_1 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \cdot c_2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot c_3 = \text{Laplace generated by third column}$$

**Example 14.** *Given a plane in parameter representation:*

$$E = \{v_0 + \lambda a + \mu b \mid \lambda, \mu \in \mathbb{R}\}$$

Find  $\alpha_1, \alpha_2, \alpha_3$  and  $\beta$  with (“implicit representation”)

$$E = \{x \mid \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta\}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = a \times b$$

TODO: image missing

$$\beta = \langle v_0, a \times b \rangle$$

In the following chapters we always consider  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ .

**Definition 12.** *An inner product over a vector space in  $\mathbb{R}$  or  $\mathbb{C}$  is a map  $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$  with properties:*

- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \quad \forall x, y, z \in V$
- $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \forall x, y \in V \forall \lambda \in \mathbb{K}$
- $\langle y, x \rangle = \overline{\langle x, y \rangle} \quad \forall x, y \in V$

where  $\overline{\langle x, y \rangle}$  denotes the complex conjugate. Especially  $\langle x, x \rangle \in \mathbb{R} \forall x \in V$ .

An inner product is called

**positive semidefinite** if  $\langle x, x \rangle \geq 0 \quad \forall x$

**positive definite** if  $\langle x, x \rangle > 0 \quad \forall x \neq 0$

**negative semidefinite** if  $\langle x, x \rangle \leq 0 \quad \forall x$

**negative definite** if  $\langle x, x \rangle < 0 \quad \forall x \neq 0$

**indefinite** if  $\exists x : \langle x, x \rangle > 0 \wedge \exists y : \langle y, y \rangle < 0$

**Definition 13.** *Scalar product* if  $\mathbb{K} = \mathbb{R}$   
*Hermitian product (or unitary product)* if  $\mathbb{K} = \mathbb{C}$

*Quadratic form* if  $\mathbb{K} = \mathbb{R}$   
*Hermitian form* if  $\mathbb{K} = \mathbb{C}$

**Lemma 8.** •  $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

$$\bullet \langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$$

$$\bullet \langle x, 0 \rangle = 0$$

*Linear in x and anti-linear in y!*

*Sesquilinear*

This lecture took place on 11th of April 2016 (Franz Lehner).

Scalar product.

1. Not bilinear, but sesquilinear

$$\bullet \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

$$\bullet \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$$

$$\bullet \langle x, y \rangle = \overline{\langle y, x \rangle} \text{ (hermitian)}$$

2.  $\langle, \rangle$  is called positive definite, if  $\langle x, x \rangle > 0 \quad \forall x \neq 0$   
 $\geq 0$  positive semidefinite  
 $< 0$  negative definite  
 $\leq$  negative semidefinite  
 $\neq$  indefinite

### 3.1 Examples

•  $\mathbb{R}^n$ :

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle = \sum_{i=1}^n x_i y_i = x^t y$$

•  $\mathbb{C}^n$

$$\langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i = x^t \bar{y}$$

•  $A = [a_j]_{j=1, \dots, n}$  because  $\langle x, y \rangle_A = x^t A y$  is complex.

Exercise: is symmetrical if and only if  $A = A^t$ , hence  $a_{ij} = a_{ji}$ .

Exercise: is hermitian if and only if  $a_{ij} = \bar{a}_{ji}$  ( $A$  is hermitian)

$\dim = \infty$ .

$$\mathbb{R}^\infty : \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

Development on  $l^2 = \left\{ (x_i) \mid \sum |x_i|^2 < \infty \right\}$  where  $l$  stands for Lebesgue.

$\Rightarrow$  Hilbert space.

$$V = C([a, b], \mathbb{C})$$

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

### 3.2 Norm

**Definition 14.** A norm on a vector space  $V$  over  $\mathbb{R}$  or  $\mathbb{C}$  is a mapping  $\| \cdot \| \rightarrow [0, \infty[$  with properties

$$N1. \|X\| \geq 0 \quad \forall X, \quad \|X\| = 0 \Leftrightarrow X = 0$$

$$N2. \|\lambda X\| = \|\lambda\| \cdot \|X\| \text{ (homogeneous)}$$

$$N3. \|X + Y\| \leq \|X\| + \|Y\| \text{ (triangle inequality)}$$

**Remark 11.** *A norm induces a metric.*

$$d(x, y) = \|x - y\|$$

*The induced metric satisfies*

$$d(x + z, y + z) = d(x, y)$$

**Example 15.** • In  $\mathbb{R}^n / \mathbb{C}^n$

$$\|X\|_\infty = \max(\|X_1\|, \dots, \|X_n\|)$$

*The euclidean norm is given by:*

$$\|X\|_2 = \left( \sum \|X\|^2 \right)^{\frac{1}{2}}$$

*The  $L^1$ -norm is given by (compare it with possible paths in a grid)*

$$\|X\|_1 = \sum_{i=1}^n \|X_i\|$$

• Analogously for  $V = C[a, b]$

$$\|f\|_\infty = \sup_{x \in [a, b]} \|f(x)\|$$

$$\|f\|_2 = \left( \int_a^b \|f(x)\|^2 dx \right)^{\frac{1}{2}}$$

$$\|f\|_1 = \int_a^b \|f(x)\| dx$$

**Theorem 25.** *Let  $\langle \cdot, \cdot \rangle$  be a positive definite scalar product in  $V$ . Then  $\|X\| = \sqrt{\langle X, X \rangle}$  defines a norm in  $V$ .*

*Proof.* N1.

$$\begin{aligned} \langle x, x \rangle \geq 0 &\Rightarrow \sqrt{\cdot} \text{ is well-defined in } \mathbb{R}^+ \\ \|X\| = 0 &\Leftrightarrow \langle X, X \rangle = 0 \xrightarrow{\text{positive definite}} X = 0 \end{aligned}$$

N2.

$$\|\lambda \cdot X\| = \sqrt{\langle \lambda X, \lambda X \rangle} = \sqrt{\lambda \bar{\lambda} \langle X, X \rangle} = \|\lambda\| \sqrt{\langle X, X \rangle} = \|\lambda\| \cdot \|X\|$$

$$\text{because } \langle x, y \rangle = \overline{\langle \lambda y, x \rangle} = \bar{\lambda} \overline{\langle y, x \rangle} = \bar{\lambda} \cdot \overline{\langle y, x \rangle} = \bar{\lambda} \cdot \langle x, y \rangle.$$

□

**Lemma 9** (Cauchy-Bunjakovsky-Schwarz Inequality). *Cauchy (1789–1857), Bunjakovsky (1804–1880), Schwarz (1843–1921)*

*For a positive definite scalar product, the following inequality holds:*

$$\|\langle x, y \rangle\| \leq \|x\| \cdot \|y\|$$

*Equality holds if and only if  $x, y$  are linear independent.*

**Lemma 10.** *Cauchy (in “Cours d’Analyse”, 1815)*

$$\left| \sum_{i=1}^n x_i \bar{y}_i \right| \leq \left( \sum \|x_i\|^2 \right)^{\frac{1}{2}} \left( \sum \|y_i\|^2 \right)^{\frac{1}{2}}$$

*Bunjakovsky (1859)*

$$\left| \int_a^b f(x) \overline{g(x)} dx \right| \leq \left( \int_a^b \|f(x)\|^2 dx \right)^{\frac{1}{2}} \cdot \left( \int_a^b \|g(x)\|^2 dx \right)^{\frac{1}{2}}$$

*Schwarz (1882), abstract*

*Lagrange (17???)*

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m (x_i y_i - x_j y_j)^2 &= \sum_{x_i^2 y_j^2} -2 \sum_{i,j} x_i y_j x_j y_i + \sum_{i,j} x_j^2 y_i^2 \\ &= 2 \left( \sum x_i^2 \right) \left( \sum x_i^2 \right) \left( \sum y_j^2 \right) - 2 \left( \sum_{i=1}^n x_i \cdot y_i \right)^2 \end{aligned}$$

$$\Rightarrow \left( \sum_{i=1}^n x_i^2 \right) \left( \sum_{j=1}^m y_j^2 \right) = \left( \sum_{i=1}^n x_i \cdot y_i \right)^2 + \frac{1}{2} \sum_{i,j} (x_i y_j - x_j y_i)^2 \geq \left( \sum_{i=1}^n x_i y_i \right)^2$$

$h = 3$

$$\|X\|^2\|y\|^2 = \|\langle x, y \rangle\|^2 + \|x \cdot y\|^2$$

A geometrical proof is left as an exercise to the reader.

*General proof. Case 1:*  $y = 0$  trivial,  $\langle x, y \rangle = 0$

**Case 2:**  $y \neq 0$

$$\begin{aligned} 0 &\leq \langle x - \lambda y, x - \lambda y \rangle = \langle x, x \rangle - \langle x, \lambda y \rangle - \langle \lambda y, x \rangle + \langle \lambda y, \lambda y \rangle \\ &= \langle x, x \rangle - \overline{\lambda} \langle x, y \rangle - \lambda \underbrace{\langle y, x \rangle}_{=\lambda \langle x, y \rangle} - \|\lambda\|^2 \langle y, y \rangle \end{aligned}$$

holds for all  $\lambda$ . Especially:

$$\begin{aligned} \lambda &= \frac{\langle x, y \rangle}{\langle y, y \rangle} \\ 0 &\leq \langle x, x \rangle - \frac{\overline{\langle x, y \rangle}}{\langle y, y \rangle} \cdot \langle x, y \rangle - \frac{\langle x, y \rangle}{\langle y, y \rangle} \overline{\langle x, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle^2} \langle y, y \rangle \\ &= \langle x, x \rangle - \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \\ \langle x, x \rangle \cdot \langle y, y \rangle &\geq |\langle x, y \rangle|^2 \end{aligned}$$

Equality  $\Rightarrow \|x - \lambda y\|^2 = 0 \Rightarrow x = \lambda y \Rightarrow$  linear independent. Inequality if  $x = \lambda y$ ,  $|\langle x, y \rangle| = |\langle \lambda y, y \rangle| = |\lambda| \|y\|^2 = \|x\| \cdot \|y\| = \|\lambda y\| \cdot \|y\|$ .

The triangle inequality can be proven this way:

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \|x\|^2 + 2\Re \langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

□

**Remark 12.**

$$\|X\|_p = \left( \sum_{i=1}^n \|x_i\|^2 \right)^{\frac{1}{p}}$$

with  $1 \leq p < \infty$  is the  $L^p$ -norm

$\Rightarrow$  Höldische Ungleichung

$$|\sum x_i y_i| \leq \left( \sum |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum |y_i|^q \right)^{\frac{1}{q}}$$

where  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Theorem 26.** Let  $V$  be a vector space over  $\mathbb{R}$  or  $\mathbb{C}$  with an inner product  $\langle, \rangle$ . Let  $B = \{b_1, \dots, b_n\}$  be the basis of  $V$ .

Then there exists exactly one hermitian matrix  $A \in \mathbb{K}^{n \times n}$  such that

$$\langle x, y \rangle = \Phi_B(x)^t A \overline{\Phi_B(y)}$$

then  $\langle, \rangle$  is positive definite,  $A$  is regular.

*Proof.* Let  $x = \sum_{i=1}^n \xi_i b_i$  and  $y = \sum_{i=1}^n \eta_i b_i$ .

$$\Phi_B(x) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

$$\Phi_B(y) = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\begin{aligned} \langle x, y \rangle &= \left\langle \sum_{i=1}^n \xi_i b_i, \sum_{j=1}^n \eta_j b_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \xi_i \overline{\eta_j} \underbrace{\langle b_i, b_j \rangle}_{=: a_{ij}} \\ &= \sum_{i,j} \xi_i a_{ij} \overline{\eta_j} = \xi^t A \overline{\eta} \\ a_{ji} &= \langle b_j, b_i \rangle = \overline{\langle b_i, b_j \rangle} = \overline{a_{ij}} \end{aligned}$$

$\Rightarrow A$  is regular.

It suffices to show that  $\ker A = \{0\}$ . Let  $A\xi = 0 \Rightarrow \xi^t A \xi = 0 \Rightarrow \sum \xi a_i = 0$ .  
And also  $\xi^t A \xi = \langle \sum \xi b_i, \sum \xi b_i \rangle$  for all  $\xi_i = 0$ .  $\square$

**Definition 15.** Let  $A \in \mathbb{C}^{n \times n}$ . Then the matrix

$$A^* := \overline{A^t}$$

$$(A^*)_{ij} = \overline{a_{ji}}$$

is the conjugate matrix to  $A$  (german: adjungiert).

$A$  is called self-conjugate if  $A = A^*$ , symmetrical if  $K = \mathbb{R}$  and hermitian if  $K = \mathbb{C}$ .

$A$  is called positive/negative semidefinite/definite or indefinite if the inner product

$$\langle \xi, \eta \rangle_A := \xi^t A \bar{\eta}$$

has the corresponding property, hence  $A$  is positive positive definite if  $\xi^t A \bar{\xi} > 0 \quad \forall \xi \neq 0$ .

$$\langle x, x \rangle > 0 \quad x$$

We want to determine how “positive” a given matrix is

Analogously to the rank, we consider: Every rank is equivalent to some matrix of the form

$$\exists P, Q \in GL(n, \mathbb{K}) : PAQ = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

**Definition 16.** Two matrices  $A, B \in \mathbb{C}^{m \times n}$  is called congruent if

$$\exists C \in GL(n, \mathbb{C}) : C^* A C = B$$

(strong condition for equivalence)

**Theorem 27.** Every hermetian matrix is congruent to the diagonal matrix

$$D = \text{diag}(+1, \dots, +1, -1, \dots, -1, 0, \dots, 0)$$

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