Measure and integration theory

Lecture notes, University of Graz
based on the lecture by Wolfgang Ring

Lukas Prokop

January 17, 2018

Contents

Elementary concepts and Riemann (Cauchy) integration

This lecture took place on 2017/10/04.

Lecturer: Wolfgang Ring

Literature:

1. Knapp, “Basic Real Analysis”
2. W. Rudin, “Real & Complex Analysis”
3. Bressoud, “A Radical Approach to Lebesgue Integral”

Problem 1.1 (Containment problem)**.** *Given a geometric size (triangle, octaeder, sphere, $M \subseteq \mathbb{R}^n$). Find the corresponding volume.*

We desire certain properties:

- $A \subseteq \mathbb{R}^n$
- Let $\mu(A)$ be the volume of A. $\mu(A)$ satisfies $\mu(A) \geq 0$.
- Let $A \cap B \neq \emptyset$. $\mu(A \cup B) = \mu(A) + \mu(B)$ (“additivity” property, σ-additivity)

Theorem 1.1. *The monotonicity property follows immediately:*

$$A \subseteq A' \implies \mu(A) \leq \mu(A')$$

Proof.

$$A' = A \mathbin{\dot\cup} (A' \setminus A)$$

$$\mu(A') = \mu(A) + \underbrace{\mu(A' \setminus A)}_{\geq 0}$$

□

We desire the following property:

$$A_n \subseteq A_{n+1} \land A = \bigcup_{n=1}^{\infty} A_n$$

$$\implies \lim_{n\to\infty} \mu(A_n) = \mu(A)$$

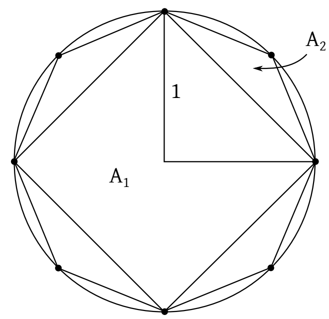

Figure 1: Given a circle of radius 1. A_1 is the rectangle. A_2 is an octaeder inside the circle. Let's assume we know the volume of these objects. Can we assign a volume to the circle? This illustrates the containment volume problem.

Limes considerations

We consider countable, infinite processes and use *sigma*-additivity.

$$(A_n)_{n\in\mathbb{N}} \qquad A_n \cap A_m = \emptyset \text{ for } n \neq m$$

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

Now we discuss another desirable property. Let $l = [a, b]$.

$$l = \bigcup_{a\leq x\leq b} \{x\}$$

$$\mu([a, b]) = b - a = \sum_{a\leq x\leq b} \mu(\{x\}) = 0$$

Informally speaking, "points should not have any content".

1. How do we define a (or *the*) volume? (a structure of Henry Lebesgue)
2. Which sets are assigned some volume?

Banach-Tarski paradox

$$K : \mathbb{R}^n \mapsto \mathbb{R}^n \text{ with } K(x) = Ox + v$$

where $O \in \mathcal{O}(n)$ and O is an orthogonal matrix. K is a congruence map.

$$\mu(A) = \mu(K(A)) \quad \forall A$$

We parameterize K with $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $v = 1$.

$$A = K(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 1) \subseteq \mathbb{R}^3$$

$$K = K(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 1) \cup K\left(\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, 1\right)$$

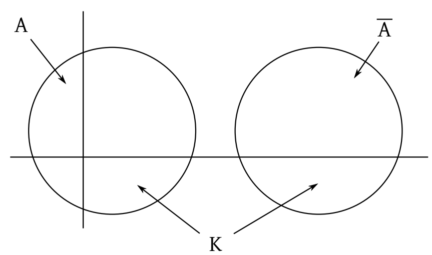

Figure 2: Banach Tarski paradox

The following sets exist:

- $A_1, A_2, \ldots, A_n \subseteq A$
- $B_1, B_2, \ldots, B_n \subseteq K$

$$A_i \cap A_j = \emptyset \text{ for } i \neq j : \bigcup_{j=1}^{k} A_j = A$$
$$B_i \cap B_j = \emptyset \text{ for } i \neq j : \bigcup_{j=1}^{k} B_j = K$$

B_j and A_j are congruent for $j = 1, \ldots, k$. A_j and B_j cannot have volumes!

It is not possible to assign an additive volme to every set. Our goal is to create the *largest* class of sets that do have volumes.

Volume is a measure.

Cauchy integral

Why are we not (entirely) confident with the Cauchy integral?

- Cauchy integration is defined on limited intervals:

$$\int_a^b f(t)\,dt \qquad a \leq b \qquad a, b \in \mathbb{R}$$

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{M\to\infty} \int_1^M \frac{1}{x^2}\,dx \qquad \textit{improper integral, boundary process} \\ &= \lim_{M\to\infty} \lim_{n\to 0} \int_1^M t_n^M(x)\,dx\end{aligned}$$

  It is desirable to compute $\int_{-\infty}^{\infty} f(x)\,dx$ directly.

- Limit theorems: Cauchy: $f_n \to f$ is uniform on $[a, b]$ (f_n converge towards f uniformly in interval $[a, b]$). Let f_n, f be regulated functions. Then,

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

  Example 1.1.

$$f_n(x) = nxe^{-nx^2}$$

  Let $[0, 1]$ *be an integration interval.* $f_n(0) = 0 \to 0$. *Let* $0 < x \leq 1$. *Then it holds that* $\lim_{n\to\infty} nxe^{-nx^2} = 0$.

$$f_n(x) \to 0 \qquad \forall x \in (0, 1]$$

  Hence, $f_n \to 0$ *is pointwise on* $[0, 1]$.

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \stackrel{?}{=} \int_0^1 \underbrace{f(x)}_{=0}\,dx = 0$$

$$\int_0^1 nx \cdot e^{-nx^2}\,dx = -\frac{1}{2} \cdot e^{-nx^2}\Big|_0^1 = \underbrace{-\frac{1}{2}e^{-n}} \to 0 + \frac{1}{2} \to \frac{1}{2}$$

$$\left(-\frac{1}{2}e^{-nx^2}\right)$$

  Example 1.2.

$$g_n(x) = \frac{n^2 x}{1 + n^3 x^2} \textit{ on } [0, 1]$$

$$\forall x \in [0, 1] : \lim_{n\to\infty} g_n(x) = 0 \checkmark \textit{ non-uniform}$$

$$x_n = \frac{1}{n} \qquad g_n(x_n) = \frac{n^2 \cdot \frac{1}{n}}{1 + n^3 \cdot \frac{1}{n^2}} = \frac{n}{1+n} = \frac{1}{1+\frac{1}{n}} \geq \frac{1}{2} \textit{ for } n \geq 1$$

$$\underbrace{\frac{1}{2n}\int_0^1 \frac{2n^3 x}{1+n}\,dx}_{\int_0^1 g_n(x)\,dx} = \frac{1}{2n}\ln\left(1+n^3x^2\right)\Big|_0^1 = \frac{1}{2n}\ln(1+n^3) \to 0$$

$$\lim_{n\to\infty}\int_0^1 g_n(x)\,dx = \int_0^1 g(x)\,dx$$

- Fundamental theorem of Calculus (dt. Hauptsatz der Differential- und Integralrechnung):

$$f : [0,1] \to \mathbb{R} \qquad \frac{d}{dx}\left[\int_0^x f(\xi)\,d\xi\right] = f(x)$$

$$f : [a,b] \to \mathbb{R} \text{ is a regulated function}$$

$$\forall x \in (a,b) \text{ exist } \lim_{\xi\to x^+} f(\xi) \text{ and } \lim_{\xi\to x^-} f(\xi)$$

Fundamental theorem:

$$\left(\int_a^x f(\xi)\,d\xi\right)_+' = \lim_{\xi\to x^+} f(\xi)$$

$$\left(\int_a^x f(\xi)\,d\xi\right)_-' = \lim_{\xi\to x^-} f(\xi)$$

Example 1.3.

$$g(x) = \begin{cases} 0 & x \in (0,1] \setminus \mathbb{Q} \cup \{0\} \\ \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q}, \gcd(p,q) = 1 \end{cases}$$

$$g(\frac{1}{\pi}) = 0; g\left(\frac{17}{24}\right) = \frac{1}{24}$$

It holds: g is a regulated function.

$$\forall x \in [0,1] \textit{ exist one-sided limits}$$

$$\lim_{\xi\to x^+} g(x) = 0 \qquad \lim_{\xi\to x^-} g(x) = 0$$

If $x \in (0,1] \setminus \mathbb{Q} \qquad \xi_n \to x \qquad \xi_n \in (0,1] \setminus \mathbb{Q} \implies g(\xi_n) = 0.$

$$g(x) = 0 : \xi_n = \frac{p_n}{q_n} \implies g_n \to \infty \implies g(\xi_n) = \frac{1}{q_n} \to 0$$

$$x \in \mathbb{Q}, x = \frac{p}{q}, \xi_n \in (0,1] \setminus \mathbb{Q} \implies g(\xi_n) = 0 \to 0$$

$$\xi_n = \frac{p_n}{q_n} = \frac{p}{q} \implies g_n \to \infty \textit{ and } g(\xi_n) = \frac{1}{q_n} \to 0$$

$$\left|\frac{p_n}{q_n} - \frac{p}{q}\right| < \varepsilon \implies 1 \leq \left|p_n q - q_n p\right| < \varepsilon q_n q \implies q_n > \frac{1}{\varepsilon \cdot q} \implies \infty \textit{ for } \varepsilon \to 0$$

Abstract measure theory

This lecture took place on 2017/10/06.

We want to:

- define abstract structures constructing the integral
- later: specific construction on $\mathbb{R}^n$ (Lebesgue measure and integral)

Topology on X

Definition 2.1. *Let $X \neq \emptyset$ be an arbitrary set. $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a* topology on *X if*

1. $\emptyset \in \mathcal{T}$; $X \in \mathcal{T}$
2. $O_i \in \mathcal{T}$ *for* $i \in I \implies \bigcup_{i \in I} O_i \in \mathcal{T}$
3. $O_1, O_2 \in \mathcal{T} \implies O_1 \cap O_2 \in \mathcal{T}$

$\mathcal{P}$ denotes the power set. Properties 2 and 3 hold for 2 or an arbitrary set of elements.
$O \in \mathcal{T}$ is called open set in *X (in terms of chosen topology $\mathcal{T}$). $\mathcal{T} = \{\emptyset, X\}$ is the so-called* indiscrete space on *X (or "trivial topology on X"). $\mathcal{T} = \mathcal{P}(X)$ is a (discrete) topology on X.*

> If you have a discrete conversation, you are disconnected from the society. Just like the points are distant from $P(X)$. Hence, discrete topologies are few elements in privacy. Indiscrete topologies include everybody (the society).

We want to reach the definition of open sets in metric spaces (but we are not there yet). In metric space $\mathbb{R}^n$, open sets are defined as:

$$O \subset \mathbb{R}^n \Leftrightarrow \forall x \in O \exists r > 0 : B(x, r) \subseteq O$$

This holds in every metric space. Compare with Figure 3.

Set algebra

Definition 2.2. *$\mathcal{A} \subseteq \mathcal{P}(X)$ is called* (set) algebra on *$X \neq 0$ if*

1. $O \in \mathcal{A}$, $x \in \mathcal{A}$
2. $\forall A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$

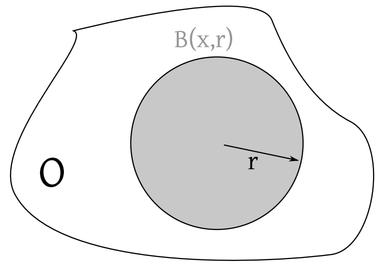

Figure 3: Topology

3. $\forall A \in \mathcal{A} \implies \underbrace{(X \setminus \mathcal{A})}_{:=A^C} \in \mathcal{A}$

$\mathcal{T}$ is closed in terms of union and finite intersection. $\mathcal{A}$ is closed in terms of union and complement.

Corollary. *Let $A, B \in \mathcal{A}$. Then,*

$$A \cap B = \underbrace{(\underbrace{\underbrace{A^C}_{\in\mathcal{A}} \cup \underbrace{B^C}_{\in\mathcal{A}}}_{\in\mathcal{A}})}_{\in\mathcal{A}}$$

Hence, $\mathcal{A}$ is closed under finite intersection.

Corollary.

$$\begin{aligned} A \triangle B &= (A \cup B) \setminus (A \cap B) \\ &= \underbrace{(A \cap \underbrace{B^C}_{\in\mathcal{A}})}_{\in\mathcal{A}} \cup \underbrace{(B \cap \underbrace{A^C}_{\in\mathcal{A}})}_{\in\mathcal{A}} \in \mathcal{A} \\ A \setminus B &= A \cap B^C \in \mathcal{A} \end{aligned}$$

$(\mathcal{A}, \cup, \cap)$ is a boolean algebra.

σ-algebra

Definition 2.3. *$\mathcal{A}$ is called σ-*algebra on X *if we take the definition of a set algebra on X (see page 7) and replace the second criterion with*

$$\forall (A_n)_{n\in\mathbb{N}} \text{ with } A_n \in \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

$\mathcal{A}$ is closed under finite union. σ-algebra is a fundamental concept to define a measure.

$\mathcal{A}$ is closed under intersection of σ-intersection (i.e. infinite intersection).

$$A_n \in \mathcal{A} \implies \bigcap_{n=1}^{\infty} A_n = \left(\overbrace{\bigcup_{n=1}^{\infty} \underbrace{A_n^C}_{\in\mathcal{A}}}^{\in\mathcal{A}} \right)^C \in \mathcal{A}$$

Set ring

Definition 2.4. *$R \subseteq \mathcal{P}(X)$ is* (set) ring *if it satisfies,*

1. $A, B \in R \implies A \cap B \in R$
2. $A, B \in R \implies A \setminus B = A \cap B^C \in R$

σ-ring

Definition 2.5. *$R \subseteq P(X)$ is called σ-*ring *if it satisfies,*

1. $\forall (A_n)_{n\in\mathbb{N}} : A_n \in R \implies \bigcup_{n=1}^{\infty} A_n \in R$
2. $A, B \in R \implies A \setminus B = A \cap B^C \in R$

Every algebra is also a ring. Every σ-algebra is also a σ-ring.

Abstract cuboid

Definition 2.6. *Let $X = \mathbb{R}^n$, $\alpha_i, \beta_i \in \mathbb{R}$ $(i = 1, \ldots, n)$. We let $[\alpha_i, \beta_i) = \{x \in \mathbb{R} : \alpha_i \le x \wedge x < \beta_i\}$. $[\alpha_i, \beta_i) = \emptyset$ if $\alpha_i \ge \beta_i$. We call Q* abstract cuboid, *if it satisfies,*

$$\begin{aligned} Q &= [\alpha_1, \beta_1) \times [\alpha_2, \beta_2) \times \ldots \times [\alpha_n, \beta_n) \\ &= \times_{i=1}^{n} [\alpha_i, \beta_i) \subseteq \mathbb{R}^n \end{aligned}$$

$Q = \emptyset$ if $\alpha_i \geq \beta_i$ for some $i \in \{1, \ldots, n\}$. Recall that, by definition, $A \times B = \emptyset$ for $A = \emptyset \vee B = \emptyset$.

$$W := \left\{ Q \subseteq \mathbb{R}^n : Q = \times_{i=1}^n [\alpha_i, \beta_i) \right\} \subseteq \mathcal{P}(\mathbb{R}^n)$$

Compare with Figure 4.

$$R_W = \left\{ V = \bigcup_{j=1}^{m} Q_j \;\middle|\; m \in \mathbb{N} \wedge Q_j \in W \text{ for } j = 1, \ldots, m \right\}$$

R_W is the set of unions of half-open abstract cuboids.

Lemma 2.1. *If $V_1, \ldots, V_n \in R_W$, then $V = \bigcup_{j=1}^k V_j \in R_W$*

Proof.

$$V_j = \bigcup_{l=1}^{m_j} Q_l^j \in R_W \implies \bigcup_{j=1}^{k} V_j = \bigcup_{j=1}^{k} \bigcup_{l=1}^{m_j} Q_l^j \in R_W$$

□

Lemma 2.2.

$$R, Q \in W \implies R \cap Q \in W$$

In words: Intersections of cuboids of W are cuboids again.

Proof.

$$Q = \times_{i=1}^n [\alpha_i, \beta_i) \qquad R = \times_{i=1}^n [\gamma_i, \delta_i)$$

Without loss of generality[1]:

$$\alpha_i < \beta_i \wedge \gamma_i < \delta_i$$

Otherwise $Q = \emptyset$ where $R = \emptyset$, then $Q \cap R = \emptyset \in W$.
Let $\hat{\alpha}_i = \max\{\alpha_i, \gamma_i\} \wedge \hat{\beta}_i = \min\{\beta_i, \delta_i\}$. Let $x \in (Q \cap R)$.

$$\begin{aligned} &\Leftrightarrow \forall i : x_i \in [\alpha_i, \beta_i) \cap [\gamma_i, \delta_i) \\ &\Leftrightarrow \forall i : \alpha_i \leq x_i < \beta_i \text{ and } \gamma_i \leq x_i < \delta_i \end{aligned}$$

Let $x = (x_1, \ldots, x_n)^t$.

$$\begin{aligned} &\Leftrightarrow \forall i : x_i \geq \hat{\alpha}_i \text{ and } \forall i : x_i < \hat{\beta}_i \\ &\Leftrightarrow x_i \in [\hat{\alpha}_i, \hat{\beta}_i) \\ &\Leftrightarrow x \in \times_{i=0}^n [\hat{\alpha}_i, \hat{\beta}_i) \in W \end{aligned}$$

□

[1] This (wlog) simplification is not really required for the proof.

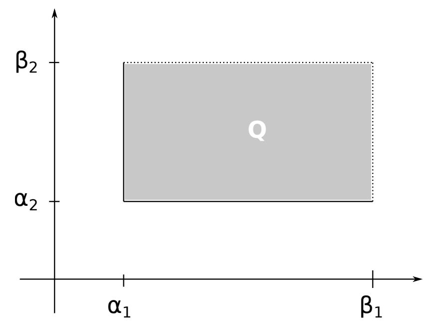

Figure 4: Abstract cuboid

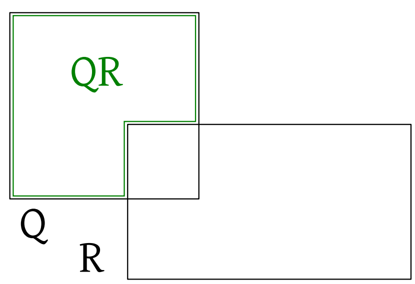

Figure 5: Illustration that the subtraction of cuboid R from Q gives another structure QR describable by two cuboids

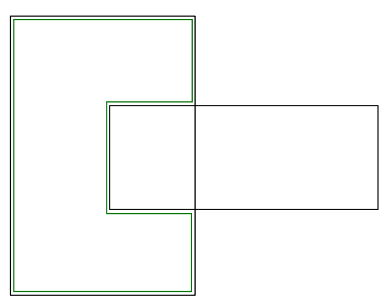

Figure 6: Also this subtraction result is describable as union of 3 cuboids

Lemma 2.3. *Let* $V, W \in R_W \implies V \cap W \in R_W$.

Proof. Let $U := \bigcup_{\nu=1}^{K} Q_\nu$ and $V := \bigcup_{\mu=1}^{L} R_\mu$. Let $Q_\nu, R_\mu \in W$.

$$\text{(by distribute law) } U \cap V = \left(\bigcup_{\nu=1}^{K} Q_\nu\right) \cap \left(\bigcup_{\mu=1}^{L} R_\mu\right)$$
$$= \bigcup_{\nu=1}^{K} \bigcup_{\mu=1}^{L} \underbrace{\left(Q_\nu \cap R_\mu\right)}_{\in W} \in R_W$$

□

Lemma 2.4. *Let $V_1, \ldots, V_L \in R_W$. Then $\bigcap_{j=1}^{L} V_j \in R_W$.*

Proof. By complete induction. Induction base $n = 2$ was just proven. Induction step $n \to n + 1$:

$$\underbrace{\left(\bigcap_{j=1}^{n} V_j\right)}_{\in R_W} \cap \underbrace{(V_{n+1})}_{\in R_W}$$

By the induction base,

$$\left(\bigcap_{j=1}^{n+1} V_j\right) \in R_W$$

□

Lemma 2.5. *Let $Q, R \in W$. Then $Q \setminus R \in R_W$. Recall that $W \in R_W$.*

Proof. Let $Q = \times_{i=1}^{n} [\alpha_i, \beta_i)$ and $R = \times_{i=1}^{n} [\gamma_i, \delta_i)$. Without loss of generality[2]:

$$\delta_i > \gamma_i \qquad \forall i = 1, \ldots, n$$

Otherwise $R = \emptyset$

$$\implies Q \setminus R = Q \in W$$

Let $x = (x_1, \ldots, x_n)^T \in Q \setminus R$.

$$\Leftrightarrow (\forall i : \alpha_i \leq x_i < \beta_i) \wedge (\exists l \in (1, \ldots, n) : (x_l < \gamma_l \vee x_l \geq \delta_l))$$

remember, that one dimension l suffices, even though multiple dimensions might be in the intervals of Q

$$\Leftrightarrow x \in \bigcup_{l=1}^{n} \left(\times_{i=1}^{n} [\alpha_i, \beta_i) \cap ((\mathbb{R} \times \ldots \times (-\infty, \gamma_l) \times \ldots \times \mathbb{R}) \cup (\mathbb{R} \times \ldots \times [\delta_l, \infty) \times \ldots \times \mathbb{R}))\right)$$

[2] This proof is always done with the loss of generality condition.

where $(-\infty, \gamma_l)$ and $[\delta_l, \infty)$ occur on the l-th index. Let $\hat{\beta}_l = \min\{\gamma_l, \beta_l\}$ and $\hat{\alpha}_l = \max\{\delta_l, \alpha_l\}$.

$$x \in \bigcup_{l=1}^{n} \left(\underbrace{[\alpha_i, \beta_i) \times \ldots \times [\alpha_l, \hat{\beta}_l) \times \ldots \times [\alpha_n, \beta_n)}_{\text{cuboid}} \cup \underbrace{[\alpha_i, \beta_i) \times \ldots \times [\hat{\alpha}_l, \beta_l) \times \ldots \times [\alpha_n, \beta_n)}_{\text{cuboid}} \right)$$

□

Lemma 2.6. *The set R_W is a ring of sets.*

Proof. By Lemma 1, it is a finite union. Let,

$$V = \bigcup_{\nu=1}^{k} Q_\nu \qquad W = \bigcup_{\mu=1}^{L} R_\mu \in R_W \qquad \text{(infinite unions)}$$

$$\implies U \setminus W = \left(\bigcup_{\nu=1}^{k} Q_\nu \setminus \bigcup_{\mu=1}^{L} R_\mu \right) = \underbrace{\bigcap_{\mu=1}^{L} \underbrace{\bigcup_{\nu=1}^{K} \underbrace{(Q_\nu \setminus R_\mu)}_{\in R_W \text{ (Lemma 5)}}}_{\in R_W \text{ (Lemma 1)}}}_{\in R_W \text{ (Lemma 4)}}$$

□

Definition 2.7. *Let $R \subseteq \mathcal{P}(x)$ is a ring on $X \neq \emptyset$. We define $\mu : R \to [-\infty, \infty] = \mathbb{R} \cup [+\infty, -\infty]$ a* set function on $\mathbb{R}$ *if $\mu(\varphi) = 0$ (which represents the volume). We use the following arithmetics:*

- $\forall x \in [-\infty, +\infty] : x + \infty = \infty$
- $\forall x \in [-\infty, +\infty] : x + (-\infty) = -\infty$
- $\forall x \in [-\infty, +\infty] \setminus \{0\} : x \cdot \infty = \text{signum}(x) \cdot \infty$
- $x_n \xrightarrow{\text{converges}} \infty \Leftrightarrow \forall m \in \mathbb{R} \exists N \in \mathbb{N} : n \geq N \implies x_n > m$

Hence every monotonic increasing sequence (x_n) has a limit in $(-\infty, +\infty]$. And every monotonic decreasing sequence (x_n) has a limit in $[-\infty, +\infty)$.

- *μ is called* non-negative *if* $\mu(A) \geq 0 \forall A \in \mathbb{R}$
- *μ is called* additive *if* $\forall A, B \in \mathbb{R} : A \cap B = \emptyset : \mu(A \cup B) = \mu(A) + \mu(B)$

Definition 2.8. *Let R be a σ-ring, μ be a non-negative set function on $\mathbb{R}$. μ is called* additive *if $\forall (A_n)_{n \in \mathbb{N}} : A_n \in R \wedge A_n \cap A_m = \emptyset$ for $n \neq m$ it holds that $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ (extension to countable infinite unions).*

Definition 2.9. *Let R be a σ algebra. A non-negative, σ-additive set function $\mu : \mathcal{A} \to [0, \infty]$ is called* measure on $\mathcal{A}$.

Definition 2.10. $(X, \mathcal{A}, \mu)$ *is called* measure space.

Definition 2.11. *Let $\mu(X) = 1$ with $x \in \mathcal{A}$, then μ is called* probability measure *(or "probability") and X is called* event system.

Definition 2.12. *Let μ be non-negative, additive. Let $A, B \in R$, $A \subseteq B$. Then so-called* monotonicity *holds, defined by,*

$$\mu(A) \leq \mu(B)$$

Proof.

$$\mu(B) = \mu(\underbrace{B \cap A}_{=A}) \cup (B \setminus A) = \mu(A) + \underbrace{\mu(B \setminus A)}_{\geq 0} \geq \mu(A)$$

$\square$

This lecture took place on 2017/10/13.

Definition 2.13. *A non-standard notation.*

Let $V = \bigcup_{j=1}^{k} Q_j \in R_W$ and $Q_j \in W$.

$$Q_j = \times_{i=1}^{n} [\alpha_i^j, \beta_i^j), \qquad \alpha_i^j < \beta_i^j \forall i, j$$

Let $i \in \{1, \dots, n\}$. We let $J_i = \left\{\alpha_i^1, \beta_i^1, \alpha_i^2, \beta_i^2, \dots, \alpha_i^k, \beta_i^k\right\}$. But this is unordered. Sort J_i in ascending order

$$J_i = \left\{\xi_i^0, \xi_i^1, \dots, \xi_i^{L_i}\right\}$$

with $\xi_i^{l-1} < x_i^l$ for $l = 1, \dots, l^i$. Duplicate entries can be skipped.

$\forall j \in \{1, \dots, k\}$, α_i^j and β_i^j occurs among ξ_i^l. This means that $\exists r_i^j, s_i^j \in \left\{0, \dots, L^i\right\}$: $\alpha_i^j = \xi_1^{n_i^j}$ and $\beta_i^j = \xi_i^{s_i^j}$ because $\alpha_i^j < \beta_i^j \implies r_i^j < s_i^j$.

$$G_v = \left\{ (\xi_i^{l_i})_{i=1}^n \in \mathbb{R}^n \,\middle|\, 0 \leq l_i \leq L_i \right\}$$

is called the *partition grid of V*. Let

$$\xi = (\xi_i^{l_i})_{i=1}^n$$

be a point of the partition grid $l_1 \geq 1 \forall i \in \{1, \dots, n\}$ (not zero, because otherwise there is no space for the cuboid left).

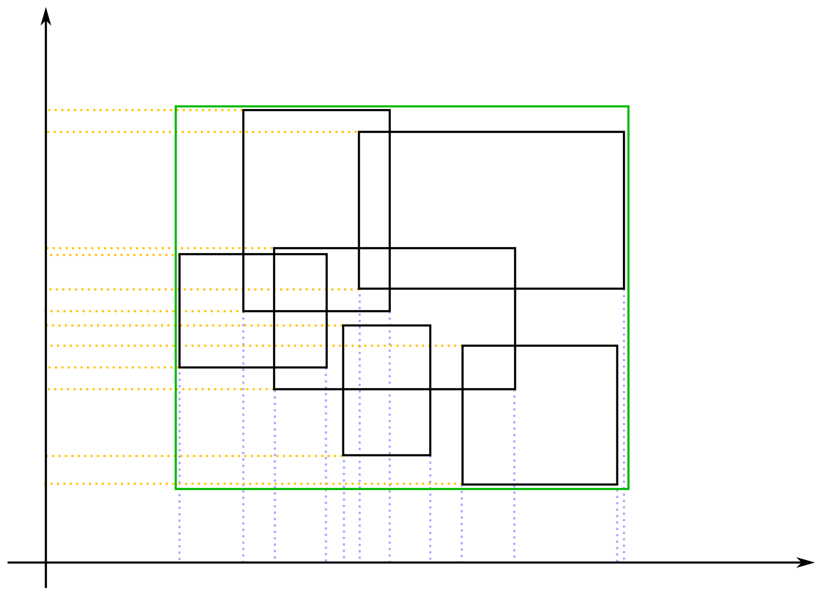

Figure 7: A *partition grid* G_V is the smallest structure containing all coordinates of its parts

$\implies$ the cuboid at the left bottom of ξ:

$$Q^{l_1 l_2 \dots l_n} = Q^{\xi} = \times_{i=1}^{n} \left[\xi_i^{l_{i-1}}, \xi^{l_i}\right) \in \mathcal{W}$$

is called. Cuboid of partition grid of V.

Lemma 2.7. *Let $\xi = (\xi_i^{l_1})_{i=1}^n$ and $\xi' = (\xi_i^{l'_i})_{i=1}^n$ in G_v with $\xi = \xi'$ (hence, at least one coordinate is different)*

- $Q^{\xi} \cap Q^{\xi'} = \emptyset$ *and* $l_i, l'_i \geq 1$
- *Let* $j \in [1, \dots, k]$. $\implies \forall \xi \in G_v : l_i \geq 1 : \left[Q_j^{\xi} \cap Q_j = \emptyset \wedge Q^{\xi} \subseteq Q_j\right]$

Proof. • Let $\xi \neq \xi'$.

$$\exists i \in \{1, \dots, n\} : l_i \neq l'_i$$

(The enumeration is different.) Assuming $x \in Q^{\xi} \wedge x \in Q^{\xi'}$

$$\implies x_i \in [\xi_i^{l_{i-1}}, \xi_i^{l_i}) \wedge x_i \in [\xi_i^{l'_{i-1}}, \xi_i^{l_i})$$

A visualization is given in Figure 8.

$$\implies [\xi_i^{l_{i-1}}, \xi_i^{l_i}) \cap [\xi_i^{l'_{i-1}}, \xi_i^{l'_i}) \neq \emptyset$$

for $l_i \neq l'_i$. This is a contradiction.

- Let $Q^\xi \cap Q_j \neq \emptyset$. Show that $Q^\xi \subseteq Q_j$. Let $x \in Q^\xi \cap Q_j$.

$$\implies \forall i : x_i \in [\xi_i^{l_{i-1}}, \xi_i^{l_i}) \wedge x_i \in [\alpha_i^j, \beta_i^j)$$

where $[\alpha_i^j, \beta_i^j) = [\xi_i^{r_i^j}, \xi_i^{s_i^j})$ with $r_i^j \leq l_{i-1} < l_i \leq s_i^j$.

$$\implies [\xi_i^{l_{i-1}}, \xi_i^{l_i}] \subseteq [\alpha_i^j, \beta_i^j]$$

$$\implies Q^\xi = \times_{i=1}^n [\xi_i^{l_i}, \xi_i^{l_i}) \subseteq \times_{i=1}^n [\alpha_i^j, \beta_i^j) = Q^j$$

□

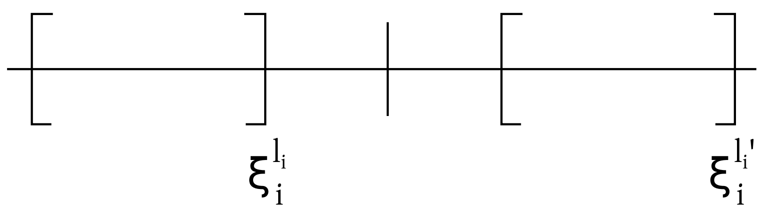

Figure 8: Lemma 7 construction, item 1

Lemma 2.8. *Let $V = \bigcup_{j=1}^k Q_j \in R_W$.*

$$\implies V = \bigcup_{\substack{\xi \in G_V, \xi_i \geq 1 \\ \text{and } Q_j \cap V \neq \emptyset}} Q^\xi$$

Hence, we wrap all cuboids of the partition grid (which are disjoint!) which have at least one point with V in common resulting precisely in V.

Proof.

$$V' = \bigcup_{\substack{\xi \in G_V, \xi_i \geq 1 \\ \text{and } Q_j \cap V \neq \emptyset}} Q^\xi$$

Show that $V' = V$.

First, we prove the relation $\subseteq$.
Let $x \in Q^{\xi} : Q^{\xi} \cap V \neq \emptyset$.

$$\implies \exists j \in [1, \dots, k] : Q_j \cap Q^{\xi} \neq \emptyset$$

By Lemma 7,

$$\implies Q^{\xi} \subseteq Q_j$$
$$\implies x \in Q_j \subseteq V$$
$$\implies x \in V$$

Second, we prove the relation $\supseteq$.
Let $x \in V$.

$$\exists j \in \{1, \dots, k\} : x \in \theta_j = [\xi_i^{r_i^j}, \xi_i^{s_i^j}) \text{ with } r_i^j < s_i^j$$
$$\exists l_i : r_i^j \leq l_i - 1 < l_i \leq s_i^j \text{ with } x_i \in [\xi_i^{l_{i-1}}, \xi_i^{l_i})$$

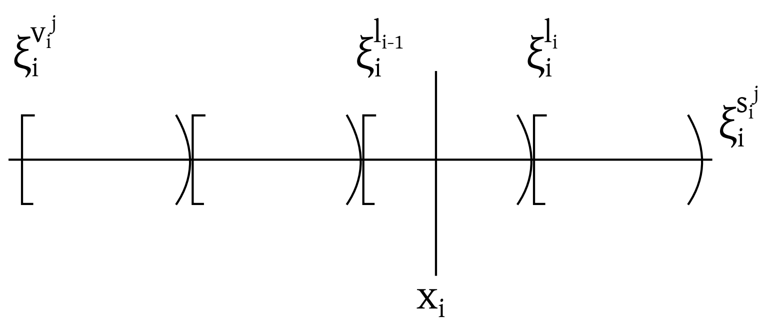

Figure 9: Lemma 8

$$\implies x \in \times_{i=1}^{n} [\xi_i^{l_{i-1}}, \xi_i^{l_i}) = Q^{\xi} \cap \{x\} \subseteq Q^{\xi} \cap Q_j$$
$$\implies Q^{\xi} \wedge V \neq \emptyset$$
$$\implies x \in V'$$

$\square$

Example 2.1.

$$Q := X_{i=1}^{n}[\alpha_i, \beta_i) \qquad \textit{cuboid} \qquad \alpha_1 \geq \beta_1 \textit{ for some } i \implies Q = \emptyset$$

$$W = \left\{ Q \subseteq \mathbb{R}^n \mid Q = X_{i=1}^{n}[\alpha_i, \beta_i) \right\}$$

$$R_W = \left\{ V = \bigcup_{j=1}^{m} Q_j \;\middle|\; m \in \mathbb{N} \wedge Q_j \in W \textit{ for } j = 1, \dots, m \right\}$$

This lecture took place on 2017/10/18.

Definition of measure

$$\mathcal{R} : \sigma\text{-ring}$$

$\mu : \mathcal{R} \mapsto [0, \infty]; \mu(\varphi) = 0, \sum_{n=1}^{\infty} \mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n) \forall A_n \in \mathcal{R}$ and $A_n \cap A_m = \varphi$ for $n \neq m$

μ is measure on $\mathcal{R}$ (usually $\mathcal{R} = \mathcal{A}$ is a σ-algebra).

$$A, B \in \mathcal{R}, A \subseteq B \implies \mu(A) \leq \mu(B) \qquad \text{monotonicity}$$

If $A \subseteq B$ and $\mu(B) < \infty$, then $\mu(A) = \mu(B) - \mu(B \setminus A)$ because of additivity: $\mu(A) + \underbrace{\mu(B \setminus A)}_{<\infty} = \underbrace{\mu(B)}_{<\infty} \implies \mu(A) = \mu(B) - \mu(B \setminus A)$.

Lemma 3.1.

1. *Let $\mathcal{R}$ be a σ-ring, let μ be a measure on $\mathcal{R}$ $(A_n)_{n \in \mathbb{N}}$, $A_n \in \mathcal{R}$. A_n is ascending (dt. "aufsteigend"), i.e., $A_n \subseteq A_{n+1}$. Then $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ and $\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n)$ and $A_n \cap A_m = \varphi$ for $n \neq m$.*

2. *Let $A_n \in \mathcal{R}$, $A_{n+1} \subseteq A_n \forall n \in \mathbb{N}$ be a descending sequence and we assume that $\exists n' \in \mathbb{N}$: $\mu(A_{n'}) < \infty$. Then $A = \bigcap_{n=1}^{\infty} A_n \in \mathbb{R}$ and $\mu(A) = \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.*

How can the intersection be zero, if the sequence is descending? Well, one example is $A_n = (n, \infty)$. The intersection is certainly zero, but each individual element has an infinite measure.

Proof. 1. We build a sequence B which represents the difference between consecutive elements of A.

$B_1 = A_1, B_n = A_n \setminus A_{n-1}$ for $n \geq 2$, $B_n \in \mathcal{R}$. $B_n \cap B_m = \emptyset$ for $n \neq m$. Suppose $n \neq m$ without loss of generality $m > n$. Let $x \in B_n \cap B_m$. Then $x \in A_m$ but $x \notin \underbrace{A_{m-1}}_{\geq n}$. $x \notin A_n$ because $A_n \subseteq A_{m-1}$. $\implies x \notin B_n \subseteq A_n$.

$A_n = \bigcup_{k=1}^n B_k$ because

$$\bigcup_{k=1}^{n} B_k = \bigcup_{k=1}^{n} (A_k \setminus A_{k-1}) = \bigcup_{k=1}^{n} A_k \setminus \underbrace{\bigcap_{k=0}^{n-1} A_k}_{=\emptyset} = \bigcup_{k=1}^{n} A_k = A_n$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} B_k = \bigcup_{n=1}^{\infty} B_n$$

By σ-additivity it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k)$$
$$= \lim_{n\to\infty} \mu\left(\overbrace{\bigcup_{k=1}^{n} B_k}^{=A_n}\right) = \lim_{n\to\infty} \mu(A_n)$$

2. $A_{n+1} \subseteq A_n$. Without loss of generality, $n' = 1$, i.e., $\mu(A_n) < \infty$. $C_k = A_1 \subseteq A_k \subseteq A \setminus A_{k+1} = C_{k+1}$. $(C_k)_{k=1}^{\infty}$ is ascending. In that sense, C_3 covers the area of A_1 and A_2 but without the area of A_3 (which contains the subsequent elements $A_4, A_5, \dots$).

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$$

Take μ on both sides

$$\overbrace{=}^{\text{due to part 1 of the proof}} \quad \mu\left(\bigcup_{n=1}^{\infty} C_n\right) = \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right)$$
$$\overbrace{=}^{\text{because } \mu(A_1)<\infty} \quad \mu(A_1) - \mu(\bigcap_{n=1}^{\infty} A_n) \qquad = -\mu(\bigcap_{n=1}^{\infty} A_n)$$

$$\lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$
$$= -\mu(\bigcap_{n=1}^{\infty} A_n)$$

$$\implies \mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$$

Appendum: If $\mathcal{R}$ is a ring and $A, B \in \mathcal{R}$.

$$\Rightarrow A \cap B = \underbrace{A \setminus \underbrace{(A \setminus B)}_{\in \mathcal{R}}}_{\in \mathcal{R}}$$

If $\mathcal{R}$ is a σ-ring, $A_n \in \mathcal{R}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.

In other words: σ-ring $\mathcal{R}$ is closed with respect to countable intersection. A ring is closed with respect to finite intersection.

$\square$

A method for generating σ-algebras

Lemma 3.2. *Suppose we have a non-empty set $X \neq \emptyset$. Let $(\mathcal{A}_i)_{i \in I}$ be a family of σ-algebra on X. Then $\mathcal{A} = \bigcap_{i \in I} \mathcal{A}_i \neq \emptyset$ is a σ-algebra on X.*

Proof. Let $x \in \mathcal{A}_i \forall i \in I$. Then $x \in \bigcap_{i \in I} \mathcal{A}_i = A$ likewise $\varphi \in \mathcal{A}$.

We need to show that $A \in \mathcal{A} \implies A^C \in \mathcal{A}$.

Let $A \in \mathcal{A}$, i. e., $\forall i \in I : A \in \mathcal{A}_i$. Because $\mathcal{A}_i$ is a σ-algebra $A^C \in \mathcal{A}_i \forall i \in I \implies A^C \in \mathcal{A} = \bigcap_{i \in I} A_i$.

We need to show that $A_n \in \mathcal{A} \forall n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Assume $\forall n \in \mathbb{N}$: $A_n \in \mathcal{A}$, i.e., $A_n \in \mathcal{A}_i \forall i \in I$. That means $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_i \forall i \in i$. $\implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. $\square$

Definition 3.1. *Let $X \neq \emptyset$. $M \subseteq \mathcal{P}(X)$ (where $\mathcal{P}$ denotes the power set). We set*

$$\mathcal{A}_m = \bigcap_{\substack{M \subseteq \mathcal{A} \subseteq \mathcal{P}(X) \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A}$$

Then A_m is a σ-algebra, A_m is not empty, because $\mathcal{A} = \mathcal{P}(X)$ is a σ-algebra which contains M.

We call $\mathcal{A}_m$ the σ-algebra generated by M. A_m is the smallest σ-algebra that contains M. This means that for every σ-algebra $\mathcal{A}$ with $M \subseteq \mathcal{A}$, we have $\mathcal{A}_m \subseteq \mathcal{A}$.

Special case: Let X be a topological space and τ is the topology on X; $\tau \subseteq \mathcal{P}(X)$. Then we call $\mathcal{B} = \mathcal{A}_\tau$ the "Borel σ-algebra on X".

Mathematician Emile Borel (1871-1956).

Example 3.1 (Examples of measures)**.** *Let $X \neq \emptyset$. We set $\mathcal{A} = \mathcal{P}(X)$. We define*

$$\mu_C(A) = \begin{cases} n \in \mathbb{N} & \text{if } A \text{ contains exactly } n \text{ elements} \\ \infty & \text{if } A \text{ has infinitely many elements} \end{cases}$$

μ_C is called the "counting measure on X" and satisfies the properties of a measure.

Proof. 1. $\mu_C(\emptyset) = 0$.

2. Proving σ-additivity is left as an exercise to the reader.

□

Example 3.2. *Let $X \neq \emptyset$ and let $x \in X$ be fixed. We define*

$$\mu_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

We call μ_x a point-measure concentrated in x.

Proof. We need to prove two statements:

1.
$$\mu_x(A) \geq 0 \qquad \mu_x(\emptyset) = 0$$

2. Let $(A_n)_{n\in\mathbb{N}}$ be pointwise disjoint.

We prove the first statement:

Case 1: $\exists n' \in \mathbb{N}$ with $x \in A_{n'}$ then $x \in \bigcup_{n=1}^{\infty} A_n$ and $\mu_x(\bigcup_{n=1}^{\infty} A_n) = 1 \forall n \neq n'$: $x \notin A_n$ because otherwise $A_n \bigcap A_{n'} \neq \emptyset$. Therefore $\sum_{n=1}^{\infty} \mu_x(A_n) = \underbrace{\mu_x(A_{n'})}_{=1} + \sum_{\substack{n=1\\n\neq n'}}^{\infty} \underbrace{\mu_x(A_n)}_{=0} =$

1

Case 2: $\forall n \in \mathbb{N} : x \notin A_n \forall n \in \mathbb{N}$

$$\implies x \notin \bigcup_{n=1}^{\infty} A_n \wedge \mu_X(\bigcup_{n=1}^{\infty} A_n) = 0$$

And also,

$$\sum_{n=1}^{\infty} \underbrace{\mu_x(A_n)}_{=0} = 0$$

Lemma 3.3 (Lemma 11)**.** *Let $\mathcal{R}$ be a σ-ring. $A_n \in \mathcal{R}$ for $n = 1, 2, 3, \ldots$ and $\mu : \mathcal{R} \mapsto [0, \infty]$ be a measure. Then*

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

This property is called "sub-additivity".

Definition 3.2. *Let $X \neq \emptyset$. $\mu^* : \mathcal{P}(X) \mapsto [0, \infty]$. Then μ^* is called "outer measure on X" if*

- $\mu^*(\varphi) = 0$

- $A \subseteq B \implies \mu^*(A) \leq \mu^*(B)$ *(monotonicity)*
- $A_n \subset X$ *for* $n = 1, 2, \ldots$. $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ *(sub-additivity)*

□

This lecture took place on 2017/10/20.

TODO

This lecture took place on 2017/10/25.

μ^* is an outer measure on X.

$$M_{\mu^*} = \{A \subseteq X : A \text{ is } \mu^* \text{ measurable}\}$$

A is measurable iff $\forall Y \subseteq X : \mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A) = \mu^*(Y \cap A) + \mu^*(Y \cap A^C)$

Theorem 3.1 (Theorem 1)**.** *Let μ^* be an outer measure on $X \neq \mu$. Let M_{μ^*} be the set of all measurable subsets of X. Then*

1. *M_{μ^*} is a σ-algebra.*
2. *$\mu^*\big|_{M_{\mu^*}}$ is a measure.*

This construction (to build a measure from an outer measure) is due to Constantin Caratheodory.

Proof. We need to prove:

1. Let $A \in M_{\mu^*} \implies A^C \in M_{\mu^*}$.
2. Let $X \in M_{\mu^*}, \varphi \in M_{\mu^*}$.
3. $(A_n)_{n\in\mathbb{N}}, A_n \in M_{\mu^*} \implies \bigcup_{n=1}^{\infty} A_n \in M_{\mu^*}$

We first need to the auxiliary statement:

$$\forall B_n \in M_{\mu^*} : B_n \cap B_m = \emptyset \quad \forall m \neq n$$

$$\mu^*\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu^*(B_n)$$

We prove the first assertion:

$$\forall Y \subseteq X \text{ and } \forall A \subseteq X$$

$$\mu^*(Y) \underbrace{\leq}_{\text{sub additivity}} \mu^*(Y \cap A) + \mu^*(Y \subseteq A)$$

because $Y = (Y \cap A) \cup (Y \setminus A)$. Let A be measurable, show that $A^C \in M_{\mu^*}$. Choose $Y \subseteq X$ arbitrary.

$$\begin{aligned}\mu^*(Y \cap A^C) + \mu^*(Y \setminus A^C) &= \mu^*(Y \cap A^C) + \mu^*(Y \cap (A^C)^C) \\ &= \mu^*(Y \cap A) + \mu^*(Y \cap A^C) \underbrace{=}_{A \in M_{\mu^*}} \mu^*(Y)\end{aligned}$$

$$\mu^*(\underbrace{Y \cap \emptyset}_{\emptyset}) + \mu^*(\underbrace{Y \setminus \emptyset}_{Y}) = \underbrace{\mu^*(\emptyset)}_{=0} + \mu^*(Y) = \mu^*(Y)$$

So $\emptyset \in M_{\mu^*}$ and $X = (\emptyset)^C \in M_{\mu^*}$. We proved the second assertion.

We prove the third assertion: Show: $A_1, A_2 \in M_{\mu^*}$ then $A_1 \cup A_2 \in M_{\mu^*}$. Let $Y \in X$ be chosen.

$$\begin{aligned}\mu^*(y) &\leq \mu^*(Y \setminus (A_1 \cup A_2)) + \mu^*(Y \cap (A_1 \cup A_2)) \\ &= \mu^*((Y \setminus A_1) \setminus A_2) + \mu^*((Y \cap A_1) \cup (Y \setminus A_1) \cap A_2) \\ &\overset{\leq}{\text{sub-additivity}}\ \mu^*((Y \setminus A_1) \setminus A_2) + \mu^*((Y \setminus A_1) \cap A_2) + \mu^*(Y \cap A_1) \\ &\underset{\substack{A_2 \in M_{\mu^*} \\ Y \setminus A_1 \text{ as testset}}}{=} \mu^*(Y \setminus A_1) + \mu^*(Y \cap A_1)\ A_1 \overset{=}{\text{is measurable}}\ \mu^*(Y)\end{aligned}$$

So, all "$\leq$" are "$=$".

$$\mu^*(Y) = \mu^*(Y \cap (A_1 \cup A_2)) + \mu^*(Y \setminus (A_1 \cup A_2)) \implies A_1 \cup A_2 \in M_{\mu^*}$$

By induction, $A_1, \ldots, A_v \in M_{\mu^*} \implies \bigcup_{n=1}^{N} A_n \in M_{\mu^*}$.

Now we want prove the auxiliary statement: Let $B_1, \ldots, B_N \in M_{\mu^*}$, $B_n \cap B_m = \emptyset$ for $n \neq m$. Then $\mu^*\left(\bigcup_{n=1}^{N} B_n\right) = \sum_{n=1}^{N} \mu^*(B_n)$. We prove this by induction. Let $N = 2$.

$$\begin{aligned}\mu^*(B_1 \cup B_2) &\underbrace{=}_{B_2 \in M_{\mu^*}} \mu^*((B_1 \cup B_2) \cap B_2) + \mu^*((B_1 \cup B_2) \setminus B_2) \\ &\underbrace{=}\ B_1 \cap B_2 = \emptyset \mu^*(B_2) + \mu^*(B_1)\end{aligned}$$

The general induction step is left as an exercise to the reader.

Let $(B_n)_{n=1}^{\infty}$ be measurable. $B_n \cap B_m = \emptyset$.

$$\begin{aligned}
\mu^*(\bigcup_{n=1}^{\infty} B_n) &\overset{\leq}{\text{sub-additivity}} \sum_{n=1}^{\infty} \mu^*(B_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \mu^*(B_n) \\
&= \lim_{N\to\infty} \mu^*(\bigcup_{n=1}^{N} B_n) \\
&\overset{\leq}{\text{monotonicity}} \lim_{N\to\infty} \mu^*(\bigcup_{n=1}^{\infty} B_n)
\end{aligned}$$

So, all "≤" are "=". So,

$$\mu^*(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu^*(B_n) = \mu^*(\bigcup_{n=1}^{\infty} B_n)$$

$B_n \cap B_m \neq \emptyset$ for $B_n \in M_{\mu^*}$. □

Let $(A_n)_{n=1}^{\infty}$ and $A_n \in M_{\mu^*}$. Check that $\bigcup_{n=1}^{\infty} A_n$ satisfies the measurability condition.

$$B_1 = A_1 \qquad B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$$

Then $B_n \cap B_m = \emptyset$ for $m \neq n$.

Check, whether $\bigcup_{n=1}^{\infty} B_n$ satisfies the measurability condition.

$$C_n = \bigcup_{n=1}^{N} B_n, \quad C_n \in M_{\mu^*}, \qquad C = \bigcup_{n=1}^{\infty} B_n$$

Let $Y \subseteq X$ be chosen arbitrarily.

Claim.

$$\mu^*(Y \cap C_N) = \sum_{n=1}^{N} \mu^*(Y \cap B_n)$$

Proof. Proof by induction: $N = 1$ follows immediately. We prove $N \to N+1$:

$$\begin{aligned}
\mu^*(Y \cap C_{N+1}) &\overset{\substack{B_{N+1} \text{ is} \\ \text{measurable}}}{=} \mu^*((Y \cap C_{N+1}) \cap B_{N+1}) + \mu^*((Y \cap C_{N+1}) \setminus B_{N+1}) \\
&= \mu^*(Y \cap B_{N+1}) + \mu^*(Y \cap C_N) \\
&\overset{\substack{\text{induction} \\ \text{hypothesis}}}{=} \mu^*(Y \cap B_{N+1}) + \sum_{n=1}^{N} \mu^*(Y \cap B_n) \\
&= \sum_{n=1}^{N+1} \mu^*(Y \cap B_n)
\end{aligned}$$

$$\sum_{n=1}^{N} \mu^*(Y \cap B_n) + \mu^*(Y \setminus C) = \mu^*(Y \cap C_N) + \mu^*(Y \setminus C) \overset{\text{monotonicity}}{\leq} \mu^*(Y \cap C_N) + \mu^*(Y \setminus C_N) = \mu^*(Y)$$

Recall that C_N are finite unions in M_{μ^*}.

$$N \to \infty \implies \sum_{n=1}^{\infty} \mu^*(Y \cap B_n) + \mu^*(Y \setminus C) \leq \mu^*(Y)$$

$$\begin{aligned}
\mu^*(Y) &\overset{\leq}{\text{sub additivity}} \mu^*(Y \cap C) + \mu^*(Y \setminus C) \\
&= \mu^*(Y \cap \bigcup_{n=1}^{\infty} B_n) + \mu^*(Y \setminus C) \\
&= \mu^*(\bigcup_{n=1}^{\infty} (Y \cap B_n)) + \mu^*(Y \setminus C) \\
&\overset{\leq}{\text{sub additivity}} \sum_{n=1}^{\infty} \mu^*(Y \cap B_n) + \mu^*(Y \setminus C) \leq \mu^*(Y)
\end{aligned}$$

Again: Every $\leq$ is an equality $=$.

$$\mu^*(Y) = \mu^*(Y \cap C) + \mu^*(Y \setminus C)$$

So, $C = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \in M_{\mu^*}$. □

$$\lambda^*(A) = \int_{\substack{(Q_j)_{j=1}^{n}, Q_j \in W \\ A \subseteq \bigcup_{j=1}^{\infty} Q_j}} \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j)$$

λ^* is an outer measure on $\mathbb{R}^n$.

We still need to check whether λ^* is monotone. Let $A \subseteq B \subseteq \mathbb{R}^n$. Consider a covering $(Q_j)_{j=1}^{\infty}$, $Q_j \in W$ with $B \subseteq \bigcup_{j=1}^{\infty} Q_j$. Obviously $A \subseteq \bigcup_{j=1}^{\infty} Q_j$. So $(Q_j)_{j=1}^{\infty}$ covers A.

$$\mu^*(A) = \underbrace{\inf_{\substack{Q'_j \in W \\ A \subseteq \bigcup_{j=1}^{\infty} Q'_{j'}}} TODO}_{\text{larger set}} \sum_{j=1}^{\infty} \mathrm{vol}_n(Q'_j)$$

$$\leq \underbrace{\inf_{\substack{Q_j \in W \\ B \subseteq \bigcup_{j=1}^{\infty} TODO}}}_{\text{smaller set}} \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j)$$

$$= \lambda^*(B) \text{ so } \lambda^* \text{ is monotone}$$

Definition 3.3. *Let μ^* be an outer measure on $P(X)$. We call $N \subseteq X$ with $\mu^*(N) = 0$ a* null set *(also called "zero set").*

Let (X, A, μ) be a measure space. We call $N \in A$ a null set if $\mu(N) = 0$. A measure space (X, A, μ) is called *complete* if for all null sets $N \in A$ and any $N' \subseteq N$ we have $N' \in A$ (and $\mu(N') = 0$).

Lemma 3.4 (Lemma 13)**.** *Let μ^* be an outer measure. $N \subseteq X$ with $\mu^*(N) = 0$. Then $N \in M_{\mu^*} \implies N$ is null set in (X, M_{μ^*}, μ). This means that (X, M_{μ^*}, μ) is complete.*

Proof. Let N be a μ^*-nullset, $Y \subseteq X$ be chosen.

$$\mu^*(Y) \leq \mu^*(\underbrace{Y \cap N}_{\subseteq N}) + \mu^*(\underbrace{Y \setminus N}_{\subseteq Y}) \overset{\leq}{\text{monotonicity}} \underbrace{\mu^*(N)}_{=0} + \mu^*(Y) = \mu^*(Y)$$

Again, we get = instead of $\leq$.

$N \in M_{\mu^*}$: Now let $N \subset X$ be a μ^*-nullset (also a (X, M_{μ^*}, μ)-nullset) and $N' \subseteq N$, μ^* is monotone $\implies \mu^*(N') = 0 \implies N' \in M_{\mu^*}$. □

This lecture took place on 2017/10/27.

$$\forall Q \in W \implies Q \in M_{X^*}, \lambda^*(Q) = \lambda(Q) = \mathrm{vol}_n(Q)$$

Definition 3.4. *Let $X = \mathbb{R}^n$ and λ^* is an outer Lebesgue measure on $\mathbb{R}^n$. We set $\mathcal{L} = M_{\lambda^*}$ as σ-algebra of Lebesgue measureable sets in $\mathbb{R}^n$.*

$$A \in \mathcal{L} \Leftrightarrow \forall Y \subseteq \mathbb{R}^n : \lambda^*(Y) = \lambda^*(Y \cap A) + \lambda^*(Y \setminus A)$$

$\lambda^*|_{\mathcal{L}} = \lambda$ is the Lebesgue measure on $\mathbb{R}^n$.

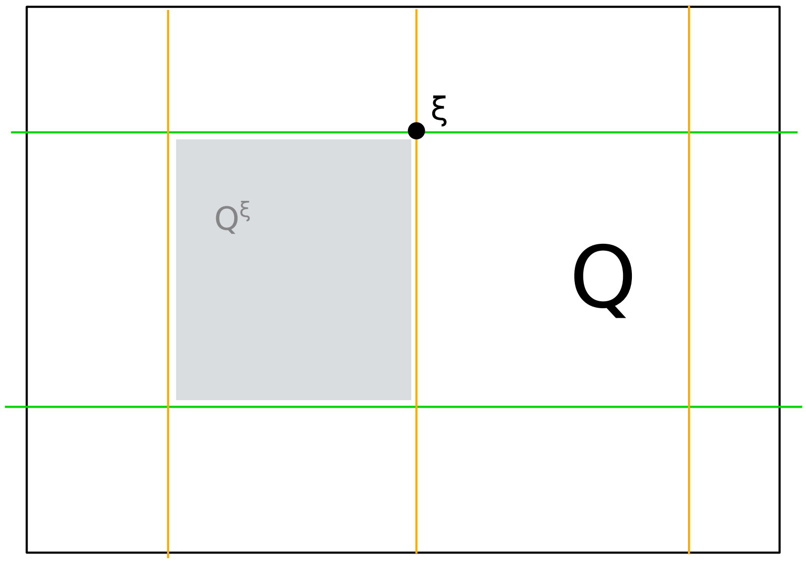

Figure 10: Construction of Lemma 14

Lemma 3.5 (Lemma 14). *Let $Q \in W$. $Q = X_{x=1}^{n}[\alpha_i, \beta_i)$ and $\alpha_i \leq \beta_i$. Let $\xi_1^0 = \alpha_1 \leq \xi_i^1 \leq \xi_i^2 \leq \ldots \leq \xi_i^{L_i} = \beta_i$ is a partition of $[\alpha_i, \beta_i)$.*

$$G = \left\{ \begin{bmatrix} \xi_1^{l_1} \\ \vdots \\ \xi_n^{l_n} \end{bmatrix} ; 0 \leq l_i \leq L_i \right\}$$

is a partition grid for Q.

$$G' = \left\{ \begin{bmatrix} \xi_1^{l_1} \\ \vdots \\ x_n^{l_n} \end{bmatrix} ; 1 \leq l_i \leq L_i \right\}$$

For $\xi = \begin{bmatrix} \xi_1^{l_1} \\ \vdots \\ \xi_n^{l_n} \end{bmatrix} \in G'$ we set $Q^{\xi} = X_{i=1}^{n}[\xi_i^{l_1 - 1}, \xi_i^{l_1})$.

Then $\mathrm{vol}_n(Q) = \sum_{\xi \in G'} \mathrm{vol}_n(Q^{\xi})$.

Proof.

$$\begin{aligned}
\sum_{\xi \in G'} \mathrm{vol}_n(Q^{\xi}) &= \sum_{l_1=1}^{L_1} \sum_{l_2=1}^{L_2} \dots \sum_{l_n=1}^{L_n} \pi_{i=1}^{n} \left(\xi_i^{l_i} - \xi_i^{l_i - 1} \right) \\
&= \left[\sum_{l_1=1}^{L_1} (\xi_1^{l_1} - \xi_1^{l_1-1}) \right] \underbrace{\left[\sum_{l_2=1}^{L_2} (\xi_2^{l_2} - \xi_2^{l_2-1}) \right]}_{\text{telescoping sum}} \dots \left[\sum_{l_n=1}^{L_n} (\xi_n^{l_n} - \xi_n^{l_n-1}) \right] \\
&= \prod_{i=1}^{n} (\xi_i^{L_i} - \xi_i^{0}) = \prod_{i=1}^{n} (\beta_i - \alpha_i) = \mathrm{vol}(Q)
\end{aligned}$$

□

Lemma 3.6 (Lemma 15). *Let* $Q \in W$, $Q = \bigcup_{j=1}^{M} Q_j$, $Q_j \in W$ *and* $Q_i \cap Q_l = \emptyset$ *for* $j = l$. *Then* $\mathrm{vol}_n(Q) = \sum_{j=1}^{M} \mathrm{vol}_n(Q_j)$.

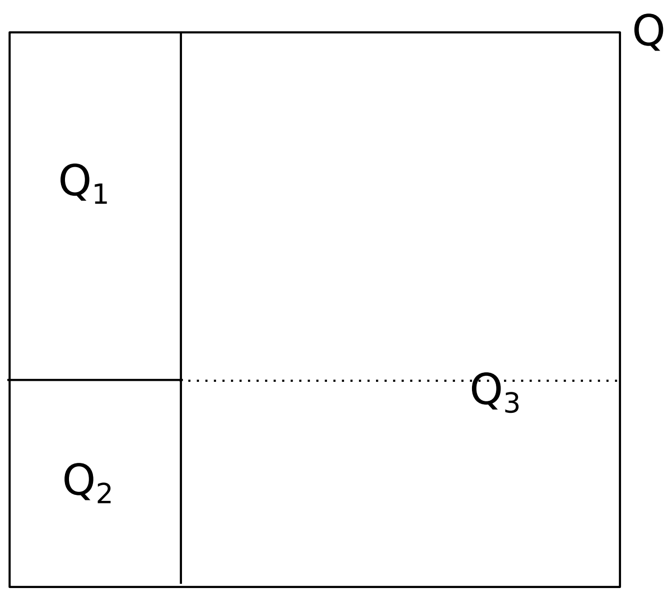

Figure 11: Construction of Lemma 15

Proof. Let G be the partitioning grid for the covering $(Q_j)_{j=1}^M$ of Q.

$$G' = \left\{ \begin{bmatrix} \xi_1^{l_1} \\ \vdots \\ \xi_n^{l_n} \end{bmatrix} ; 1 \leq l_1 \leq L_i \right\}$$

Q^ξ is above because $Q_j \subseteq Q$.

$$\alpha_1 \leq \xi_i^{l_1} \leq \beta_i \quad \forall i \in \{1, \dots, n\}, l_i \in \{0, \dots, L_i\}$$

Let $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in Q$ then $x \in Q_j$ for some $j \in \{1, \dots, M\}$. $x_i \in [\xi_1^{l_i - 1}, \xi_1^{l_i})$ for exactly one $l_i \in \{1, \dots, L_i\}$. Moreover: for the point $x = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \in Q \implies \xi_i^0 = \alpha_i$ for all i.
Also a point with $i-th$ coordinate $x_i = \beta_i - \varepsilon$ (ε sufficiently small) which lies in Q implies $\beta_i - \varepsilon < \xi_i^{L_i} \quad \forall \varepsilon > 0 \implies \beta_i \leq \xi_i^{L_i} \implies \beta_i = \xi_i^{L_i}$.
The previous lemma stated that $\mathrm{vol}_n(Q) = \sum_{\xi \in G'} \mathrm{vol}_n(Q^\xi)$. And,

$$\sum_{j=1}^M \mathrm{vol}_n(Q_j) = \sum_{j=1}^M \sum_{\substack{\xi \in G' \\ Q^\xi \cap Q_j \neq \emptyset}} \mathrm{vol}_n(Q^\xi) = \sum_{\xi \in G'} \mathrm{vol}_n(Q^\xi)$$

Hence, they are equal (because they have the same expression on the right-hand side). ☐

Lemma 3.7 (Lemma 16, a sub-additivity result)**.** *Let $Q \in W$, $Q \subseteq \bigcup_{j=1}^M Q_j; Q_j \in W$. Then we have that $\mathrm{vol}_n(Q) \leq \sum_{j=1}^m \mathrm{vol}(Q_j)$. Now we cover Q with a finite number of rectangle (possibly overlapping).*

Proof. We set $Q_0 = Q$ and construct the partitioning. Grid G for $(Q_j)_{j=0}^M$. Let $\tilde{Q} := \bigcup_{j=1}^M Q_j$.

$$Q \subseteq \bigcup_{j=0}^M Q_j = \bigcup_{j=1}^M Q_j = \bigcup_{\substack{\xi \in G' \\ Q^\xi \cap \tilde{Q} \neq \emptyset}} Q^\xi$$

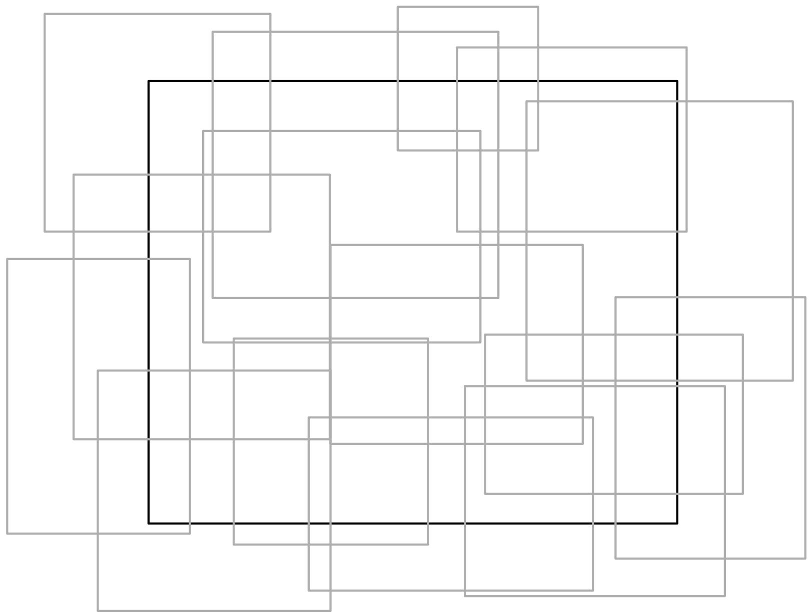

Figure 12: Lemma 16 construction

$$
\begin{aligned}
Q = \bigcup_{\substack{\xi\in G' \\ Q^{\xi}\cap Q\neq\emptyset}} Q^{\xi} \underbrace{\Longrightarrow}_{\text{Lemma 15}} \mathrm{vol}_n(Q) \\
&= \sum_{\substack{\xi\in G' \\ Q^{\xi}\cap Q\neq\emptyset}} \mathrm{vol}_n(Q^{\xi}) \\
&\leq \sum_{\substack{\xi\in G' \\ Q^{\xi}\cap \tilde{Q}\neq\emptyset}} \mathrm{vol}_n(Q^{\xi}) \\
&\leq \sum_{j=1}^{M} \sum_{\substack{\xi\in G' \\ Q^{\xi}\cap Q_j\neq\emptyset}} \mathrm{vol}_n(Q^{\xi}) \\
&\underbrace{=}_{\text{Lemma 15}} \sum_{j=1}^{M} \mathrm{vol}_n(Q_j)
\end{aligned}
$$

□

Lemma 3.8 (Lemma 17)**.** *Let* $Q \in W$*,* $Q \subseteq \bigcup_{j=1}^{\infty} Q_j, Q_j \in W$*. Then* $\mathrm{vol}_n(Q) \leq \sum_{n=1}^{\infty} \mathrm{vol}_n(Q_j)$.

Proof. Without loss of generality, $Q \neq \emptyset$ and $Q_j \neq \emptyset$.

$$\mathrm{vol}_n(Q) = \prod_{i=1}^{n}(\beta_i - \alpha_i) > 0 \qquad \mathrm{vol}_n(Q_j) > 0 \forall j \in \mathbb{N}$$

Let $\mathrm{vol}_n(Q) > \varepsilon > 0$ be arbitrary (sufficiently small). We choose $Q_\varepsilon = X_{i=1}^n[\alpha_i^\varepsilon, \beta_i^\varepsilon) \subseteq \overline{Q}_\varepsilon = X_{i=1}^n[\alpha_i^\varepsilon, \beta_i^\varepsilon] \subseteq Q$ such that

$$\mathrm{vol}_n(Q_\varepsilon) = \mathrm{vol}_n(Q) - \varepsilon$$

You will get this result if one lets,

$$\alpha_i^\varepsilon = \alpha_i + \frac{1}{2}(\alpha_i - \beta_i)\left(1 - (1 - \frac{\varepsilon}{\mathrm{vol}_n(Q)})^{\frac{1}{n}}\right)$$

Choose: $Q_j^\varepsilon \supseteq Q_j$, $Q_j^\varepsilon = X_{i=1}^n[\alpha_i^{j,\varepsilon}, \beta_i^{j,\varepsilon})$, $\alpha_i^{j,\varepsilon} < \alpha_i^j < \beta_i^j < \beta_i^{j,\varepsilon}$.

$$Q_j \subseteq \mathrm{int}(Q_j^\varepsilon) = Q_j^\varepsilon = X_{i=1}^n(\alpha_i^{j,\varepsilon}, \beta_i^{j,\varepsilon}) \text{ with } \mathrm{vol}(Q_j^\varepsilon) = \mathrm{vol}(Q_j) + \frac{\varepsilon}{2^j}$$

Then

$$Q_\varepsilon \subseteq Q \subseteq \bigcup_{j=1}^{\infty} Q_j \subseteq \bigcup_{j=1}^{\infty} Q_j^\varepsilon \underbrace{\implies}_{\text{compactness}} \exists M \subseteq N : \overline{Q}_\varepsilon \subseteq \bigcup_{j=1}^{M} Q_j$$

Q_ε is a bounded, closed set (hence, a compact set). Therefore, this result.

$$Q_3 \subseteq \overline{Q_3} \subseteq \bigcup_{j=1}^{M} Q_j^\varepsilon \subseteq \bigcup_{j=1}^{M} Q_j^\varepsilon$$

$$\underbrace{\implies}_{\text{Lemma 16}} \mathrm{vol}_n(Q_\varepsilon) \leq \sum_{n=1}^{M} \mathrm{vol}_n(Q_j^\varepsilon) = \sum_{j=1}^{M}\left(\mathrm{vol}_n(Q_j) + \frac{\varepsilon}{2^j}\right)$$

$$\leq \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j) + \varepsilon \cdot \underbrace{\sum_{j=1}^{\infty} \frac{1}{2^j}}_{=1}$$

$$\Leftrightarrow \mathrm{vol}_n(Q) \leq \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j) + 2\varepsilon \qquad \forall \varepsilon > 0$$

$$\Leftrightarrow \mathrm{vol}_n(Q) \leq \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j)$$

□

Lemma 3.9 (Lemma 18)**.** *$\forall Q \in W$ we have* $\mathrm{vol}_n(Q) = \lambda^*(Q)$.

Proof.

$$\lambda^*(Q) = \inf_{\substack{Q \subseteq \bigcup_{j=1}^{\infty} Q_j \\ Q_j \in W}} \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j)$$

Q is a covering of Q, hence $\lambda^*(Q) \leq \mathrm{vol}_n(Q)$. On the other hand, because of Lemma 17, it follows that $\mathrm{vol}_n(Q) \leq \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j)$. For any covering $(Q_j)_{j=1}^{\infty}$ of Q implies that

$$\mathrm{vol}_n(Q) \leq \inf_{Q \subseteq Q_{j=1}^{\infty} Q_j} \sum_{j=1}^{\infty} \mathrm{vol}_n(Q_j) = \lambda^*(Q)$$

So $\lambda^*(Q) = \mathrm{vol}_n(Q)$. ◻

Lemma 3.10 (Lemma 19)**.** *We have $W \subseteq \mathcal{L}$, i.e., every $Q \in W$ is measurable.*

Proof. Let $A \subseteq \mathbb{R}^n$ be given. Let $A \subseteq \bigcup_{j=1}^{\infty} Q_j$, $Q_j \in W$.

$$A \cap Q \subseteq \left(\bigcup_{j=1}^{\infty} Q_j \right) \cap Q = \bigcup_{j=1}^{\infty} \overbrace{(Q_j \cap Q)}^{\in W}$$

$$A \setminus Q \subseteq \bigcup_{j=1}^{\infty} \underbrace{(Q_j \setminus Q)}_{\in \mathcal{R}_W} \underbrace{\text{Lemma 6}} \bigcup_{j=1}^{\infty} \bigcup_{l=1}^{m_j} Q_l^j$$

So $Q_j = \underbrace{(Q_j \cap Q)}_{\in W} \cup \underbrace{(\bigcup_{l=1}^{m_j} Q_l^j)}_{\in W}$ disjoint union

$$\underbrace{\Longrightarrow}_{\text{Lemma 15}} \mathrm{vol}_n(Q_j) = \mathrm{vol}(Q \cap Q_j) + \sum_{l=1}^{m_j} \mathrm{vol}(Q_l^j)$$

$$\sum_{j=1}^{\infty} \mathrm{vol}(Q_j) = \sum_{j=1}^{\infty} \mathrm{vol}(\underbrace{Q \cap Q_j}_{\text{cover } Q \cap A}) + \sum_{j=1}^{\infty} \sum_{l=1}^{m_j} \mathrm{vol}(\underbrace{Q_l^j}_{\text{cover } A \setminus Q})$$
$$\geq \lambda^*(Q \cap A) + \lambda^*(A \setminus Q)$$

holds for every covering $(Q_j)_{j=1}^{\infty}$ of A. Taking "inf" implies that

$$\lambda^*(A) \geq \lambda^*(Q \cap A) + \lambda^*(A \setminus Q)$$

$$\lambda^*(A) \leq \lambda^*(Q \cap A) + \lambda^*(A \setminus Q)$$

$$\mathrm{vol}_n(Q_j) = \mathrm{vol}(Q \cap Q_j) + \sum_{l=1}^{m_j} \mathrm{vol}(Q_l^j)$$

Due to sub-additivity,

$$\lambda^*(A) = \lambda^*(A \cap Q) + \lambda^*(A \setminus Q)$$

so $Q \in \mathcal{L}$. □

This lecture took place on 2017/11/08.

$$\lambda(Q) = \mathrm{vol}_n(Q)$$

We let $\underbrace{Q(x,r)}_{\in W} = \times_{i=1}^n [x_i - r, x_i + r)$ for $x = (x_1, \ldots, x_n)^t \in \mathbb{R}$ and $r > 0$.

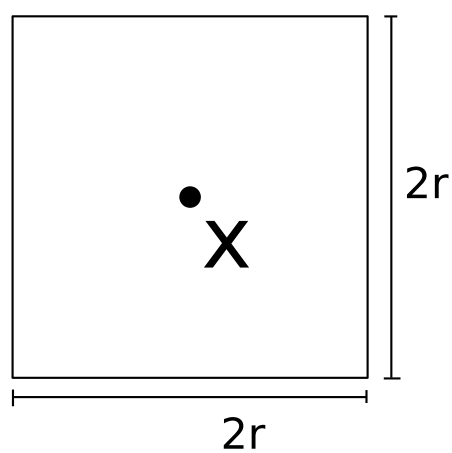

Figure 13: Q for Lemma 20

Lemma 3.11 (Lemma 20)**.** *Let $x \in \mathbb{R}^n$, $r > 0$. Then $B(x,r) \subseteq Q(x,r) \subseteq B(x, \sqrt{n}r + \varepsilon) \forall \varepsilon > 0$.*

Proof. Let $y \in B(x,r)$, i.e., $\|x - y\| < r \implies |x_i - y_i| < r \forall i \in \{1, \ldots, n\}$.

$$\left(\sum_{i=1}^n |x_i - y_i)^2 \right)^{\frac{1}{2}}$$

$$y_i \in [x_i - r, x_i + r) \implies y \in Q(x,r)$$

Let $z \in Q(x,r) \implies |x_i - z_i| \le r \implies \|x - z\| = \left(\sum_{i=1}^n |x_i - z_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^n r^2\right)^{\frac{1}{2}}$.

$$= \sqrt{n}r < \sqrt{n}r + \varepsilon \implies z \in B(x, r\sqrt{n} + \varepsilon)$$

□

Propositions occur between theorems and lemmas.

Proposition 3.1. *Any open set $O \subseteq \mathbb{R}^n$ is measurable with respect to Lebesgue measure L.*

Proof. Let $O \subseteq \mathbb{R}^n$ be open, $x \in O$ be chosen. There exists $r_x > 0$, $r_x \in \mathbb{Q}$: $B(x, r_x) \subseteq O$. $\mathbb{Q}^n \subseteq \mathbb{R}^n$ is dense in $\mathbb{R}^n$. There exists $q_x \in \mathbb{Q}^n$ such that $\|x - q_x\| < \frac{r_x}{3\sqrt{n}}$ because $x \in B(q_x, \frac{r_x}{3\sqrt{n}}) \subseteq Q(q_x, \frac{r_x}{3\sqrt{n}})$. We consider $Q(q_x, \frac{r_x}{3\sqrt{n}})$ and $Q(q_x, \frac{r_x}{3\sqrt{n}}) \subseteq B(q_x, \frac{r_x}{3} + \varepsilon)$. Let $z \in B(q_x, \frac{r_x}{3} + \varepsilon)$. Then

$$\|z - x\| \le \left\|z - q_x\right\| + \left\|q_x - x\right\| < \frac{r_x}{3} + \varepsilon + \frac{r_x}{3\underbrace{\sqrt{n}}_{>1}} \le \frac{2}{3}r_x + \varepsilon < r_x \text{ for } \varepsilon < \frac{r_x}{3}$$

$$\implies B\left(q_x, \frac{r_x}{3\sqrt{n}}\right) \subseteq B(x, r_x) \subseteq O$$

$$x \in Q\left(q_x, \frac{r_x}{3\sqrt{n}}\right) \subseteq B(x, r_x) \subseteq O$$

$$O = \bigcup_{x \in O} \{x\} \subseteq \bigcup_{x \in O} Q\left(q_x, \frac{r_x}{3\sqrt{n}}\right) \subseteq \bigcup_{x \in O} \underbrace{B(x, r_x)}_{\subseteq O} \subseteq O$$

$$O = \underbrace{\bigcup_{x \in O}}_{\text{countable union}} \underbrace{Q\left(\overbrace{q_x}^{\in \mathbb{Q}^n}, \overbrace{\frac{r_x}{3\sqrt{n}}}^{\in \mathbb{Q}}\right)}_{\in L} \in \mathbb{Q}$$

There are only countable many cubes $Q\left(q_x, \frac{r_x}{3\sqrt{n}}\right)$.

$$\implies O \in L$$

Hence the subset-equals relations are actually equalities. □

Corollary.

- *Every closed set $C \subseteq \mathbb{R}^n$ is in L, $C = \mathbb{R}^n \setminus \underbrace{\mathbb{Q}}_{\in L} \in L$.*

- *Every open half-space* $H_{n,c} = \{x \in \mathbb{R}^n : \langle x, n\rangle > c\}$ *is in* L. *With* $n \in \mathbb{R}^n, \|n\| = 1, c \in \mathbb{R}$
- *Every closed half-space* $\overline{H}_{n,c_n} = \{x \in \mathbb{R}^n : \langle x, n\rangle \geq c\}$ *is in* L
- *Every open rectangle* $\overset{o}{Q} = \times_{i=1}^n (\alpha_i, \beta_i)$ *with* $\alpha_1 \leq \beta_i$ *is in* L
- *Every closed rectangle* $\overline{Q} = \times_{i=1}^n [\alpha_i, \beta_i]$ *with* $\alpha_1 \leq \beta_i$ *is in* L
- *Let* $O_i \in \mathbb{R}^n$ *be open for* $i = 1, 2, \ldots$. *Then* $A = \bigcap_{i\in\mathbb{N}} O_i \in L$. *This is the so-called* G_δ *set.*
- *Let* $C_i \subseteq \mathbb{R}^n$ *be closed for* $i \in \mathbb{N}$. *Then* $\bigcup_{i\in\mathbb{N}} C_i \in L$. *This is the so-called* F_σ*-set.*

We know $\lambda(Q) = \mathrm{vol}_n(Q) = \prod_{i=1}^n (\beta_i - \alpha_i)$ for $Q = \times_{i=1}^n [\alpha_i, \beta_i)$ with $\alpha_i \subseteq \beta_i$.

Lemma 3.12. *Let* $\overset{o}{Q}$ *and* $\overline{Q}$ *be as above. Then* $\lambda(\overset{o}{Q}) = \lambda(\overline{Q}) = \mathrm{vol}_n(Q)$.

Proof. $\overset{o}{Q} \subseteq Q$.

$$\lambda^*(\overset{o}{Q}) = \inf_{\substack{Q_j \in W \\ \overset{o}{Q} \subseteq \bigcup_{j=1}^\infty Q_j}} \sum_{j=1}^\infty \mathrm{vol}_n(Q_j) \leq \mathrm{vol}_n(Q)$$

Choose $Q_\varepsilon = \times_{i=1}^n [\alpha_i + s, \beta_i) \subseteq \overset{o}{Q}$ and choose δ such that $\mathrm{vol}_n(Q_\varepsilon) = \mathrm{vol}_n(Q) - \varepsilon$ with $\mathrm{vol}_n(Q_\varepsilon) = \lambda(Q_\varepsilon)$. By monotonicity of λ it follows that $\forall \varepsilon > 0 : \lambda(Q_\varepsilon) = \mathrm{vol}_n(Q_\varepsilon) = \mathrm{vol}_n(Q_\varepsilon) - \varepsilon \leq \lambda(\overset{o}{Q})$.

$$\mathrm{vol}_n(Q) - \varepsilon \leq \lambda(\overset{o}{Q}) \leq \mathrm{vol}_n(Q)$$

$$\lambda(\overset{o}{Q}) = \mathrm{vol}_n(Q)$$

$\overline{Q}$ is similar. □

Integration

Definition 4.1. *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. We consider* $f : X \to [-\infty, \infty]$. *We endow* $[-\infty, \infty]$ *with the topology of the extended real line. We say that* f *is a measurable function if* $\forall O \subseteq [-\infty, \infty]$ *open, we have that the preimage of the open set is in* $\mathcal{A}$: $f^{-1}(O) \in \mathcal{A}$.

Topology on $[-\infty, \infty]$. *Let open intervals in* $\overline{\mathbb{R}} = [-\infty, \infty]$ *are given by the sets* $[-\infty, \alpha), (\alpha, \beta), (\beta, \infty]$ *for all* $\alpha, \beta \in \mathbb{R}$ *with* $\alpha < \beta$. $O \subset \overline{\mathbb{R}}$ *is open if* $\forall x \in O$ *there exists an open interval* I_x *such that* $x \in I_x \subseteq O$.

Easy conclusions: $O \subseteq \overline{\mathbb{R}}$ *is open iff* $O = [-\infty, \alpha) \cup O' \cup (\beta, \alpha]$ *with* $O' \subseteq \mathbb{R}$ *is open in* $\mathbb{R}$ *or* $O = [-\infty, \alpha) \cup O'$ *or* $O = O' \cup (\beta, \infty]$ *or* $O = O'$.

Lemma 4.1 (Lemma 1)**.** *$f : X \to [-\infty, \infty]$ is measurable iff $f^{-1}(C) \in \mathcal{A}$ for all $C \subseteq \overline{\mathbb{R}}$ closed.*

Proof.

$$f^{-1}(C) = f^{-1}(\underbrace{\overline{R} \setminus O}_{C=\overline{\mathbb{R}}\setminus O \text{ with } O \text{ open})} = X \setminus f^{-1}(O) \in \mathbb{A} \text{ iff } f^{-1}(O) \in \mathbb{A}$$

□

Proposition 4.1. *The following conditions are equivalent:*

- $\forall t \in \mathbb{R} : f^{-1}([-\infty, t)) \in \mathcal{A}$
- $\forall t \in \mathbb{R} : f^{-1}([-\infty, t]) \in \mathcal{A}$
- $\forall t \in \mathbb{R} : f^{-1}((t, \infty]) \in \mathcal{A}$
- $\forall t \in \mathbb{R} : f^{-1}([t, \infty]) \in \mathcal{A}$
- *f is a measurable function.*

Proof. The first condition implies the second condition.

Let the first condition be true and $t \in \mathbb{R}$ is given.

$$[-\infty, t] = \bigcap_{n=1}^{\infty} \left[-\infty, t + \frac{1}{n}\right)$$

so,

$$f^{-1}([-\infty, t]) = f^{-1}(\bigcap_{n=1}^{\infty}[-\infty, t + \frac{1}{n}))$$

$$= \underbrace{\bigcap_{n=1}^{\infty} \underbrace{f^{-1}\left([-\infty, t + \frac{1}{n})\right)}_{\in\mathcal{A} \text{ by cond. 1}}}_{\in\mathcal{A} \text{ by countable intersection}} \in \mathcal{A}$$

The second condition implies the third condition.

$$f^{-1}((t, \infty]) = f^{-1}(\overline{\mathbb{R}} \setminus [-\infty, t]) = X \setminus \overbrace{f^{-1}([-\infty, t])}^{\in\mathcal{A} \text{ by cond. 2}} \in \mathcal{A}$$

The third condition implies the fourth condition analogous to condition one implying condition two.
The fourth condition implies the first condition analogous to condition two

implying condition three.

The fifth condition implies the first condition because $[-\infty, t)$ is open in $\overline{\mathbb{R}}$ and $f^{-1}([-\infty, t]) \in \mathcal{A}$ because f is measurable.

Let conditions 1 to 4 be true. Let $\alpha < \beta$ with $\alpha, \beta \in \mathbb{R}$ then $(\alpha, \beta) = [-\infty, \beta] \cap (\alpha, \infty]$.

$$f^{-1}((\alpha, \beta)) = \underbrace{f^{-1}([-\infty, \beta))}_{\in \mathcal{A} \text{ by cond. } 1} \cap \underbrace{f^{-1}((\alpha, \infty])}_{\in \mathcal{A} \text{ by cond. } 3} \in \mathcal{A}$$

Let $O \subseteq \mathbb{R}$ be open. Then for any $x \in O$ there exists $l_x < x < r_x$ such that $x \in (l_x, r_x) \subseteq Q$ and $l_x, r_x \in \mathbb{Q}$. So we have $O = \bigcup_{x \in O} \{x\} \subseteq \bigcup_{x \in O} \underbrace{(l_x, r_x)}_{\in O} \subseteq O$.

Hence, the subset-equality operators are equalities again.

There are only countably many intervals (l_x, r_x):

$$O = \bigcup_{x \in O} \underbrace{(l_x, r_x)}$$

Thus, O is a countable union of open intervals.

$$f^{-1}(O) = f^{-1}(\bigcup_{k=1}^{\infty} (l_k, r_k)) = \bigcup_{k=1}^{\infty} \underbrace{f^{-1}((l_k, r_k))}_{\in \mathcal{A}} \in \mathcal{A}$$

For $O = [-\infty, \alpha) \cup O'$ or $O = O' \cup (\beta, \alpha]$ or $O = [-\infty, \alpha) \cup O' \cup (\beta, \infty]$. Similar! $\square$

This lecture took place on 2017/11/10.

$f : X \to [-\infty, \infty]$ is measurable $\Leftrightarrow$ every preimage of a halfline is in $\mathcal{A}$.

Remark 4.1. *$\mathcal{B} \subset \mathcal{P}(\mathbb{R}^n)$ is the smallest σ-algebra which contains all open sets, the Borel-σ-algebra. We have $\mathcal{B} \subseteq \mathcal{L}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous $\Leftrightarrow$ $f^{-1}(O)$ is open in $\mathbb{R}^n \forall O \subseteq \mathbb{R}$ open, so $f^{-1}(O) \in \mathcal{B} \subset \mathcal{L}$ so any continuous function is measureable with respect to $\mathcal{L}$.*

Definition 4.2. *Let $A \subseteq X$. We set $X_A : X \to \mathbb{R}$.*

$$X_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^C \end{cases}$$

is called the characteristic function of *A.*

Remark 4.2. *X_A is measurable with respect to $\mathcal{A} \Leftrightarrow A \in \mathcal{A}$.*

$$X_A^{-1}]([-\infty, t)) = \begin{cases} \varphi \in \mathcal{A} & t \leq 0 \\ X \setminus A & 0 < t \leq 1 \\ X \in \mathcal{A} & t \geq 1 \end{cases}$$

Let $(X, \mathcal{A}, \mu)$ be a measure space and $A \in \mathcal{A}$. Then $(A, \mathcal{A}', \mu')$ is a measure where $\mathcal{A}' = \{B \cap A : B \in \mathcal{A}\}$, $\mu'(A') = \mu(A')$ for $A' \subseteq A$. We only discuss $f : X \to [-\infty, \infty]$ but all the following results also hold for $f : A \to [-\infty, \infty]$.

Definition 4.3 (Notation)**.** *We set $f \vee g : X \to [-\infty, \infty]$ by $f \vee g(x) = \max\{f(x), g(x)\}$ the maximum of f and g. $f \wedge g$ is defined by $f \wedge g(x) = \min\{f(x), g(x)\}$.*

Lemma 4.2 (Lemma 2)**.** *Let $f, g : X \to [-\infty, \infty]$ be measurable. Then $f \vee g$ and $f \wedge g$ is measurable.*

Proof.

$$\begin{aligned} \{x \in X \mid (f \vee g)(x) < t\} &= \{x \in X : f(x) < t \text{ and } g(x) < t\} \\ &= \underbrace{\{x \in X \mid f(x) < t\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) < t\}}_{\in \mathcal{A}} \in \mathcal{A} \end{aligned}$$

□

Lemma 4.3. *Let $f, g : X \to [-\infty, \infty]$ be measurable. Then $\{x \in X \mid f(x) < g(x)\} \in \mathcal{A}$, $\{x \in X \mid f(x) \leq g(x)\} \in \mathcal{A}$ and $\{x \in X \mid f(x) = g(x)\} \in \mathcal{A}$.*

Proof.

$$f(x) < g(x) \Leftrightarrow \exists r \in \mathbb{Q} : f(x) < r < g(x)$$

$$\{x \in X \mid f(x) < g(x)\} = \{x \in X \mid \exists R \in \mathbb{Q} : f(x) < r \text{ and } g(x) > r\}$$

$$\bigcup_{r \in \mathbb{Q}} \underbrace{\left[\underbrace{\{x \in X \mid f(x) < r\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) > r\}}_{\in \mathcal{A}} \right]}_{\in \mathcal{A}} \in \mathcal{A}$$

$$\{x \in X \mid f(x) \leq g(x)\} = X \setminus \left\{ \underbrace{x \in X} g(x) < f(x)_{\in \mathcal{A}} \,\middle|\, \in \right\} \mathcal{A}$$

$$\{x \in X \mid f(x) = g(x)\} = \underbrace{\underbrace{\{x \in X \mid f(x) \leq g(x)\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) \leq f(x)\}}_{\in \mathcal{A}}}_{\in \mathcal{A}}$$

□

Proposition 4.2 (Proposition 2)**.** *Let f, g be measurable functions on X, $\alpha \in \mathbb{R}$. Then $\alpha f, f+g, f-g, f \cdot g, \frac{f}{g}$ are measurable (for the last result we assume that $g(x) \neq 0 \forall x \in X$).*

$$\mathcal{F}_{\mathcal{A}} = \left\{ f : X \to [-\infty, \infty] \,\middle|\, f \text{ is measurable} \right\} \text{ is a real vector space}$$

Proof. Consider αf.

Let $\alpha = 0$, then $\alpha f = 0$ is measurable. Let $\alpha > 0$, then $\{x \in X \mid \alpha f(x) < t\} = \left\{x \in X \mid f(x) < \frac{t}{\alpha}\right\} \in \mathcal{A}$. Let $\alpha < 0$, then $\{x \in X \mid \alpha f(x) < t\} = \left\{x \in X \mid f(x) > \frac{t}{\alpha}\right\} \in \mathcal{A}$.

Consider $f + g$.

Let $t \in \mathcal{R}$ be given.

$$f(x) + g(x) < t \Leftrightarrow \exists r \in \mathbb{Q} : f(x) < r \text{ and } g(x) < t - r$$

The direction $\Leftarrow$ follows immediately. For direction $\Rightarrow$ we show: let $f(x)+g(x) < t$, so $f(x) < \infty$, $g(x) < \infty$. Let $u = f(x)$ and $v = g(x)$. Then $u + v < t \implies u < t - v \implies \exists r \in \mathbb{Q} : \underbrace{u}_{=f(x)} < r < \underbrace{t - v}_{=t-g(x)}$.

$$\{x \in X \mid f(x) + g(x) < t\} = \{x \in X \mid \exists r \in \mathbb{Q} : f(x) < r \text{ and } g(x) < t - r\}$$

$$\bigcup_{r \in \mathbb{Q}} \underbrace{\left[\underbrace{\{x \in X \mid f(x) < r\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) < t - r\}}_{\in \mathcal{A}} \right]}_{\in \mathcal{A}}$$

Consider $f - g$.

$$f - g = f + \underbrace{(-1)g}_{\text{is measurable}}$$

is measurable.

This lecture took place on 2017/11/15.

Prove that f^2 is measurable.

$$(f^2)^{-1}([\infty, t)) = \left\{x \in X \mid f^2(x) < t\right\} = \left\{x \in X \mid -\sqrt{t} < f(x) < \sqrt{t}\right\}$$

Let $t > 0$.

$$= \left\{x \in X \mid -\sqrt{t} < f(x)\right\} \cap \left\{x \in X \mid f(x) < \sqrt{t}\right\}$$
$$= \underbrace{f^{-1}((-\sqrt{t}, \infty])}_{\in \mathcal{A}} \cap \underbrace{f^{-1}([-\infty, \sqrt{t}))}_{\in \mathcal{A}} \in \mathcal{A}$$

Prove that $f \cdot g$ is measurable.

$$\overbrace{(\underbrace{f+g}_{\text{measurable}})^2}^{\text{measurable}} - \overbrace{(\underbrace{f-g}_{\text{measurable}})^2}^{\text{measurable}} = f^2 + 2fg + g^2 - f^2 + 2fg - g^2 = 4fg$$

$$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

is measurable. Let $g(x) \neq 0$ on X.

$$\left\{x \in X : \frac{f(x)}{g(x)} < t\right\}$$

Prove that $\frac{f}{g}$ is measurable. Let $g(x) \neq 0$ on X.

$$\begin{aligned}\left\{x \in X \mid \frac{f(x)}{g(x)} < t\right\} &= \{x \in X \mid f(x) < t \cdot g(x) \text{ and } g(x) > 0\} \cup \{x \in X \mid f(x) > tg(x) \text{ and } g(x) < 0\} \\ &= \left[\underbrace{\{x \in X \mid f(x) - t \cdot g(x) < 0\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) > 0\}}_{\in \mathcal{A}}\right] \\ &\cup \left[\underbrace{\{x \in X \mid f(x) - t \cdot g(x) > 0\}}_{\in \mathcal{A}} \cap \underbrace{\{x \in X \mid g(x) < 0\}}_{\in \mathcal{A}}\right] \in \mathcal{A}\end{aligned}$$

□

Remark 4.3. *Let g be measurable on X.*

$$D_g = \{x \in X \mid g(x) \neq 0\} = \{x \in X \mid g(x) > 0\} \cup \{x \in X \mid g(x) < \infty\} \in \mathcal{A}$$

$\frac{f}{g} : D_g \to [-\infty, \infty]$ *then* $\frac{f}{g}$ *is measurable with respect to* $\left\{D_g, \mathcal{A}', \mu\big|_{D_g}\right\}$ *where* $\mathcal{A}' = \left\{A \cap D_g : A \in \mathcal{A}\right\}$ *is a σ-algebra.*

Proposition 4.3. *Let $f_n : X \to [-\infty, \infty]$ be measurable for $n \in \mathbb{N}$. Then*

1. $\overline{f}$ *is measurable with* $\overline{f}(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$
2. $\underline{f}$ *is measurable with* $\underline{f}(x) = \inf\{f_n(x) : n \in \mathbb{N}\}$
3. $\limsup_{n\to\infty} f_n$ *is measurable with* $\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\sup\left\{f_k(x) \,\middle|\, k \geq n\right\}\right]$
   $\liminf_{n\to\infty} f_n$ *is measurable with* $\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\inf\left\{f_k(x) \,\middle|\, k \geq n\right\}\right]$

4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on X a set*

$$A = \left\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\right\} \in \mathcal{A} \text{ and } f(x) = \lim_{n\to\infty} f_n(x)$$

is measurable on A.

Proof.

$$\overline{f}^{-1}([-\infty, t)) = \left\{x \in X \mid \sup\left\{f_n(x) \,\middle|\, n \in \mathbb{N}\right\} < t\right\} = \{x \in X \mid f_n(x) < t \forall n \in \mathbb{N}\}$$
$$= \bigcap_{n\in\mathbb{N}} \underbrace{\{x \in X \mid f_n(x) < t\}}_{\in\mathcal{A}} \in \mathcal{A}$$

$\underline{f}$ follows analogously.

$$\limsup_{n\to\infty} f_n(x) = \inf\left\{\underbrace{\sup\left\{f_k(x) \,\middle|\, k \geq n\right\}}_{\text{non-increasing sequence}} : n \in \mathbb{N}\right\}$$

$$\limsup_{n\to\infty} f_n = \overbrace{\inf\left\{\underbrace{\sup\left\{f_k \,\middle|\, k \geq n\right\}}_{\text{measurable by (1)}} : n \in \mathbb{N}\right\}}^{\text{measurable by (2)}}$$

$\liminf f_n$ follows analogously.

The fourth statement can be proven with the following structure: Let

$$\begin{aligned}
A &= \left\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\right\} \\
&= \{x \in X \mid (f_n(x))_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\} \\
&= \left\{x \in X : \underbrace{\forall n \in \mathbb{N}}_{\forall \varepsilon = \frac{1}{n}} : \exists N_n \in \mathbb{N} \forall m, m' \geq N_n \,\middle|\, \left|f_m(x) - f_{m'}(x)\right| < \frac{1}{n}\right\} \\
&= \bigcap_{n\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{m,m'\geq N} \underbrace{\left\{x \in X \mid -\frac{1}{n} < \underbrace{f_m(x) - f_{m'}(x)}_{\text{measurable}} < \frac{1}{n}\right\}}_{\in\mathcal{A}} \in \mathcal{A}
\end{aligned}$$

on A. $\lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ is a measurable function on A. □

Definition 4.4. *Let $f : X \to [-\infty, \infty]$ be given. We define $f_+ := f \vee 0 = \max\{f, 0\}$. Hence, f_+ is the non-negative part of f. Analogously, let $f_- := -(f \wedge 0) = -\min(f, 0) = \max(-f, 0)$ representing the non-positive part.*

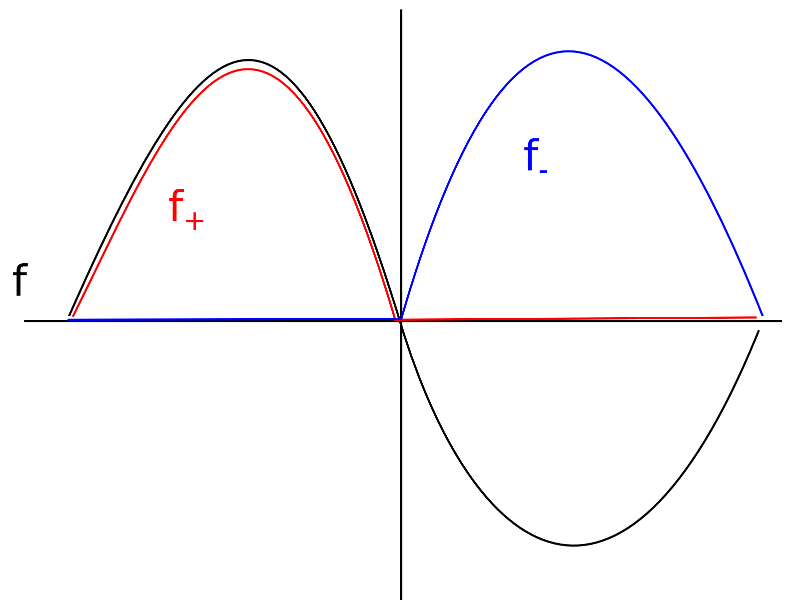

Figure 14: f, f_- and f_+

Lemma 4.4. *We have $f = f_+ - f_-$ and $\left|f\right| = f_+ + f_-$. f is measurable $\iff$ f_- and f_+ are measurable.*

Definition 4.5. *A function $S : X \to (-\infty, \infty) = \mathbb{R}$ is called a simple function iff $s(x) = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ is a finite set. We set $\mathcal{S} = \{s \mid s \text{ is simple and measurable on } X\}$.*

$$\mathcal{S}_+ = \{s : X \to [0, \infty) \mid s \text{ is simple and measurable}\}$$

For $A \subseteq X$, we set $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \in A^C \end{cases}$. We call χ_A the characteristic function of A.

Remark 4.4. *Let $S : X \to \mathbb{R}$ be simple with $S(x) = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ where $\alpha_j \neq \alpha_{j'}$ assuming $j \neq j'$. We define $A_j = s^{-1}(\{\alpha_j\})$. Then s is measurable if and only if $A_j \in \mathcal{A}$ for $j = 1, \ldots, N$ where s is measurable $\implies$ $s^{-1}(\underbrace{\{\alpha_j\}}_{closed}) \in \mathcal{A}$ if A_j is measurable $s^{-1}([-\infty, t)) = \bigcup_{\alpha_j \in [-\infty, t)} s^{-1}(\{\alpha_j\}) \in \mathcal{A}$. $s = \sum_{j=1}^{N} \alpha_j \chi_{A_j}$ because $A_j \cap A_{j'} = \emptyset$ if $j \neq j'$*

and $\bigcup_{j=1}^{N} A_j = X$ *for* $x \in A_{j'} \implies s(x) = \alpha_{j'}$ *and* $\sum_{j=1}^{N} \alpha_j \underbrace{\chi_{A_j}(x)}_{\delta_{j,j'}} = \alpha_{j'}$. *S is a linear combination of characteristic functions.*

$$s = \sum_{j=1}]^N \alpha_j \chi_{A_j}$$

Define $s' = \sum_{l=1}^{M} \beta_l \chi_x$. *TODO content missing*

Let $s = \sum_{j=1}^{N} \alpha_j \chi_{A_j}$ be simple $A_j \in \mathcal{A}$. We call the linear combination a standard representation of s if $A_j \cap A_{j'} = \emptyset$ for $j \neq j'$. A standard representation does not need to be unique.

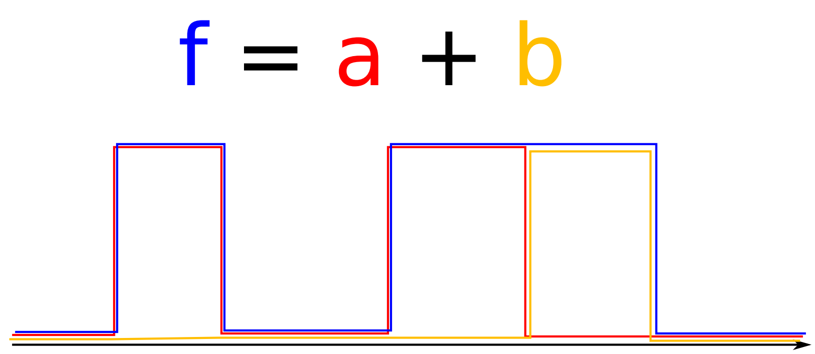

Figure 15: Sum of characteristic functions a and b

Proposition 4.4. *Let* $f : X \to [0, \infty]$ *be measurable. Then there exists a sequence* $(S_n)_{n \in \mathbb{N}}$ *of simple measurable functions* $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq s_{n+1}$ *such that* $f(x) = \lim_{n \to \infty} s_n(x) \forall x \in X$. *We say that* f *is the pointwise limit of a monotone sequence of simple functions for* $k = 0, 1, \ldots, n2^n$ *(with* $n \geq 1$*). We define* $t_k^n = \frac{k}{2^n}$, $t_0^n = 0$ *and* $t_{n2^n}^n = \frac{n2^n}{2^n} = n$

$$\begin{aligned}
t_k^n - t_{k-1}^n &= \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n} = \triangle t^n \\
t_k^n &= \frac{k}{2^n} - \frac{2k}{2^{n+1}} = t_{2k}^{n+1} < t_{2k+1}^{n+1} < t_{2k+2}^{n+1} = t_{k+1}^n \\
M_k^n &= f^{-1}([t_{k-1}^n, t_k^n)) \text{ for } k = 1, \ldots, n2^n \qquad M_k^n \in \mathcal{A}, M_\infty^n \in \mathcal{A} \\
M_\infty^n &= f^{-1}([t_{n2^n}^n, \infty]) = f^{-1}([n, \infty])
\end{aligned}$$

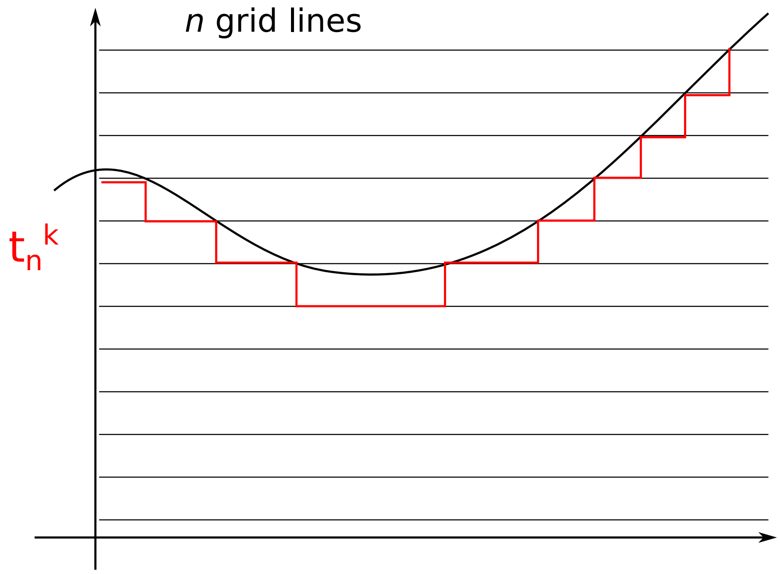

Figure 16: Construction of simple functions

because f is measurable, because $\bigcup_{k=1}^{n2^n}[t_{k-1}^n, t_k^n) \cup [n, \infty] = [0, \infty]$

$$\bigcup_{k=1}^{n2^n} M_k^n \cup M_\infty^n = X \text{ and } M_k^n \text{ are disjoint}$$

We define $f_n(x) = \begin{cases} t_{k-1}^n & x \in M_k^n \text{ for } k = 1, \dots, n2^n \\ n & x \in M_\infty^n \end{cases}$ which is simple and measurable because $M_k^n \in \mathcal{A}, M_\infty^n \in \mathcal{A}$.

First case

$$f(x) = \infty \implies x \in M_\infty^n \forall n \in \mathbb{N}$$

$$f_n(x) = n \to +\infty \text{ monotone}$$

Second case

$$f(x) = t < \infty \text{ and we choose } n > t$$

$$\implies t \in [t_{k-1}^n, t_k^n) \text{ for exactly one } k \text{ and } f(x) = t, f_n(x) = t_{k-1}^n \implies x \in M_k^n$$

$$\left|f(x) - f_n(x)\right| = t - t_{k-1}^n < t_k^n - t_{k-1}^n = \frac{1}{2^n}$$

So $f_n(x) \to f(x)$ as $n \to \infty$, $f_n \leq f_{n+1}$.

$$[t_{k-1}^n, t_k^n) = [t_{2k-2}^{n+1}, t_{2k-1}^{n+1}) \cup [t_{2k-1}^{n+1}, t_{2k}^{n+1})$$

$$M_k^n = M_{2k-1}^{n+1} \cup M_{2k}^{n+1}$$

$$\text{if } x \in M_k^n, x \in M_{2k-1}^{n+1} \implies f_n(x) = t_{k-1}^n \text{ equivalent with } f_{n+1}(x) = t_{2k-2}^{n+1}.$$

This lecture took place on 2017/11/17.

$$M_X^k = f^{-1}([t_{k-1}^n, t_k^n)) \qquad t_k^n = \frac{k}{2^n} \qquad \text{where } k \in \{0, \dots, n2^n\}, t_0^n = 0, t_{n2^n}^n = n$$

$$f_1(x) = t_{k-1}^n \text{ if } x \in M_k^n, M_\infty^n = f^{-1}([n, \infty])$$

$$f_n(x) = n \text{ if } x \in M_X^n$$

$$f_n(x) \leq f_{n+1}(x) \quad \forall n \in \mathbb{N} \text{ and } x \in X, f(x) = \infty\checkmark$$

$$f(x) = t < \infty$$

a)

$$\begin{aligned}
&t < n < n+1 \text{then } \exists k \in \{0, \dots, n2^n\} : t_{k-1}^n \leq t < t_k^n \\
&\qquad \implies x \in M_k^1 \text{ and } f_n(x) = t_{k-1}^n, \quad t_{k-1}^n = t_{2k-2}^{n+1} < t_{2k-1}^{n+1} < t_{2k}^{n+1} = t_k^n \\
&\qquad \implies M_k^n = M_{2k-1}^{n+1} \cup M_{2k}^{n+1}
\end{aligned}$$

$$f_{n+1}(x) = \begin{cases} t_{2k-2}^{n+1} = t_{n-1}^n & x \in M_{2k-1}^n \implies f_{n+1}(x) = f_n(x) \\ t_{2k-1}^{n+1} & x \in M_{2k}^n \implies f_{n+1}(x) > f_n(x) \end{cases}$$

b)

$$n \leq t < n+1$$

$$x \in M_\infty^n, f_n(x) = n$$

$$f_{n+1}(x) = t_k^{n+1} \text{ with } k \geq n2^{n+1} \implies f_{n+1}(x) = k\frac{1}{2^{n+1}} \geq n\frac{2^{n+1}}{2^{n+1}} = n = f_n(x)$$

c)

$$t \geq n+1$$

$$x \in M_\infty^{n+1} \text{ and } x \in M_\infty^n$$

$$f_n(x) = n, f_{n+1}(x) = n+1$$

$$f_n(x) \leq f_{n+1}(x)\checkmark$$

$$\mathcal{M} = \left\{ f : X \to [-\infty, \infty] \,\middle|\, f \text{ is measurable with respect to } \mathcal{A} \right\}$$

$$\mathcal{M}_+ = \left\{ f \in \mathcal{M} \middle| f(x) \geq 0 \forall x \in X \right\}, \xi, \xi_+ \checkmark$$

for $s \in \xi$ we know $s = \sum_{j=1}^N \alpha_j \chi_{A_j}$ with $A_j \cap A_{j'} = \emptyset$ for $j \neq j'$ sometimes we assume that $\bigcup_{j=1}^N A_j = X$ (set $A_0 = X \setminus (\bigcup_{j=1}^N A_j)$ and $\alpha_0 = 0$). Sometimes we assume $\alpha_j \neq 0$ because we can omit a term $0.\ \chi_{A_j}$)

Definition 4.6 (integration). *Let $s \in \xi_+$, $s = \sum_{j=1}^N \alpha_j \xi_{A_j}$ ($A_j \in \mathcal{A}$ with $A_j \cap A_{j'} = \emptyset$ for $j \neq j'$). We define*

$$\int_X s\, d\mu = \sum_{j=1}^N \alpha_j \mu(A_j) = \int s\, d\mu$$

Remark 4.5. *The integral $\int_X s\, d\mu$ is independent of the chosen standard representation of s. Let*

$$s = \sum_{j=1}^N \alpha_j \chi_{A_j} = \sum_{l=1}^M \beta_l \chi_{B_i}$$

$$X = \bigcup_{j=1}^N A_j = \bigcup_{l=1}^N B_l \implies A = \bigcup_{l=1}^M (A_j \cap B_l) \text{ disjoint}, B_l = \bigcup -j = 1^N (B_l \cap A_j) \text{ disjoint}$$

$$\sum_{j=1}^N \alpha_j \mu(A_j) = \sum_{j=1}^N \alpha_j \mu \left(\bigcup_{l=1}^M (A_j \cap B_l) \right) = \sum_{j=1}^N \alpha_j \sum_{l=1}^M \underbrace{\mu(A_j \cap B_l)}_{\substack{=0 \text{ iff } A_j \cap B_l = \emptyset \\ \text{else } x \in A_j \cap B_l \implies a_j = s(x) = \beta_l}}$$

$$= \sum_{l=1}^M \beta_l \sum_{j=1}^N \mu(A_j \cap B_l) = \sum_{l=1}^M \beta_l \mu(\bigcup_{j=1}^N (A_j \cap B_l)) = \sum_{l=1}^M \beta_l \mu(B_l)$$

Proposition 4.5 (Proposition 5). *Let $(X, \mathcal{A}, \mu)$ be a measure space. $f, g \in \xi_+$. Then*

1. $\forall a \in \mathbb{R}^+ : \int_X af\, d\mu = a \int_X f\, d\mu$
2. $\int_X (f+g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$
3. $f(x) \leq g(x) \forall x \in X : (f \leq g)$ *then* $\int_X f\, d\mu \leq \int_X g\, d\mu$

Proof. 1.

$$f = \sum_{j=1}^N \alpha_j \chi_{A_j} \qquad g = \sum_{l=1}^M \beta_l \chi_{B_l} \qquad \alpha_j, \beta_l \geq 0$$

$$\int_X af\, d\mu = \sum_{j=1}^M a\alpha_j \mu(A_j) = a \sum_{j=1}^N \alpha_j \mu(A_j) = a \int_X f d\mu$$

$f+g \in \xi_+$, $f+g \geq 0$, $f+g \in \mathcal{M}_+$. $f+g$ attains only finitely many function values $(f+g)(X) \subseteq \{\alpha_0 + \beta_l, j \in \{1, \ldots, N\}, l \in \{1, \ldots, M\}\}$.

$$= \sum_{l=1}^{M} \beta_l \mu(\bigcup_{j=1}^{N}(A_j \cap B_j)) = \sum_{l=1}^{M} \beta_l \mu(B_l) \checkmark$$

$$A_j = \bigcup_{l=1}^{M}(A_j \cap B_l) \qquad B_l = \bigcup_{j=1}^{N}(B_l \cap A_j)$$

$$\begin{aligned}
(f+g) &= \sum_{j=1}^{N} \alpha_j \chi_{A_j} + \sum_{l=1}^{M} \beta_l \chi_{B_l} \\
&= \sum_{j=1}^{N} \alpha_j \sum_{l=1}^{M} \chi_{A_j \cap B_l} + \sum_{l=1}^{M} \beta_l \sum_{j=1}^{M} \chi_{B_l \cap A_j} \\
&= \sum_{j=1}^{N} \sum_{l=1}^{M} (\alpha_j + \beta_l) \chi_{\underbrace{A_j \cap B_l}_{\text{disjoint}}}
\end{aligned}$$

$$(j,l) = (j',l') \implies (A_j \cap B_l) \cap (A_{j'} \cap B_{l'}) = \emptyset$$

$$\begin{aligned}
\int_X (f+g)\, d\mu &= \sum_{j=1}^{N} \sum_{l=1}^{M} (\alpha_j + \beta_l) \mu(A_j \cap B_l) \\
&= \sum_{j=1}^{N} \alpha_j \underbrace{\sum_{l=1}^{M} \mu(A_j \cap B_l)}_{\mu(A_j)} + \sum_{l=1}^{M} \beta_l \underbrace{\sum_{j=1}^{N} \mu(B_l \cap A_j)}_{\mu(B_l)} \\
&= \int_X f\, d\mu + \int_X g\, d\mu
\end{aligned}$$

3.

$$f \leq g \implies g - f \in \xi_+ \implies \int_X (g-f)\, d\mu \geq 0$$

$$\int_X g\, d\mu = \int_X (f + (g-f))\, d\mu \overset{\text{by 2.}}{=} \int_X f\, d\mu + \underbrace{\int_X (g-f)\, d\mu}_{\geq 0} \geq \int_X f\, d\mu$$

□

This lecture took place on 2017/11/22.

Let $s \in \xi_+$.

$$\int_X s \, d\mu = \sum_{i=1}^{N} \alpha_1 \mu(A_1) \text{ where } s = \sum_{i=1}^{N} \alpha_i \chi_{A_i}$$

$s = \xi_A$ and $A \in \mathcal{A}$.

$$\int_X \chi_A \, d\mu = 1 \cdot \mu(A) = \mu(A)$$

Proposition 4.6. *Let $s_n \in \xi_+$, $S \in \xi_+$ and $\forall x \in X : s_n(x) \leq s_{n+1}(x) \leq s(x)$ and $s(x) = \lim_{n\to\infty} s_n(x)$. Then $\int_X s \, d\mu = \lim_{n\to\infty} \int_X s_n \, d\mu$ where $\int_X s \, d\mu = \int_X \lim_{n\to\infty} s_n \, d\mu$.*

Proof. By monotonicity, it follows that

$$\int_X s_n \, d\mu \leq \int_X s_{n+1} \, d\mu \leq \int_X s \, d\mu$$

$$\implies \underbrace{\lim_{n\to\infty} \int_X s_n \, d\mu}_{\exists \in [0,\infty]} \leq \int_X s \, d\mu$$

For the reverse inequality, we are going to show that $\forall \varepsilon > 0 : \lim_{n\to\infty} \int_X s_n \, d\mu \geq (1-\varepsilon) \int_X s \, d\mu$. Construct $g_n^\varepsilon \in \xi_+$ such that $g_n^\varepsilon \leq s_n$ and $\int_X g_n^\varepsilon \, d\mu \geq (1-\varepsilon) \int_X s \, d\mu$

Let $s = \sum_{j=1}^N \alpha_j \chi_{A_j} \geq 0$. Assume $\alpha_j > 0$ and $A_j \cap A_{j'} = \emptyset$ for $j \neq j'$.

$$A^\varepsilon(n,j) = \left\{ x \in A_j \,\middle|\, s_n(x) \geq (1-\varepsilon)\alpha_j \right\}$$

because for $x \in A_j$ we have $s_n(x) \to s(x) = \alpha_j \implies \exists n \in \mathbb{N} : x \in A^\varepsilon(n,j)$.

$$A_j = \bigcup_{n=1}^{\infty} A^\varepsilon(n,j)$$

because $s_{n+1} \geq s_n$ we have $x \in A^\varepsilon(n,j) \implies x \in A^\varepsilon(n+1,j)$.

$$A^\varepsilon(n,j) \subseteq A^\varepsilon(n+1,j)$$

By a previous lemma, we get $\mu(A_j) = \lim_{n\to\infty} \mu(A^\varepsilon(n,j))$. We set $g_n = \sum_{j=1}^N (1-\varepsilon)\alpha_j \chi_{A^\varepsilon}(n,j) \in \xi_X$. Then $\int_X g_n \, d\mu = (1-\varepsilon) \sum_{j=1}^N \alpha_j \mu(A^\varepsilon(n,j))$.

$$\to_{n\to\infty} (1-\varepsilon) \sum_{j=1}^{N} \alpha_j \mu(A_j) = (1-\varepsilon) \int_X s \, d\mu$$

$$\lim_{n\to\infty} \int_X g_n \, d\mu = (1-\varepsilon) \int_X s \, d\mu$$

Show that $g_n(x) \leq s_n(x)$ holds. Suppose $x \notin \bigcup_{j=1}^N A_j$ then $s(x) = 0$ and also $0 \leq s_n(x) \leq s(x) = 0 \implies s_n(x) = 0$. $x \notin \bigcup_{j=1}^N A^\varepsilon(n, j) \implies g_n(x) = 0$. In this case $g_n(x) = s_n(x) = 0$ so $g_n(x) \leq s_n(x)$ holds. Let $x \in A_j$. Then $s_n(x)$,

- if $x \in A^\varepsilon(n, j) \implies s_n(x) \geq (1 - \varepsilon)\alpha_j = g_n(x)$. QED.
- if $x \in A_j \setminus A^\varepsilon(n, j)$. Then $g_n(x) = 0 \leq s_n(x)$.

$$\lim_{n\to\infty} \int_X g_n \, d\mu \leq \int_X s_n \, d\mu$$

where $\lim_{n\to\infty} \int_X g_n \, d\mu = (1 - \varepsilon) \int_X s \, d\mu \forall \varepsilon > 0$.

□

Definition 4.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

$$f : X \to [0, \infty] \textit{ measurable}$$

$$\int_X f \, d\mu = \sup \left\{ \underbrace{\int_X s \, d\mu}_{\geq 0} \,\middle|\, s \in \xi_+ \textit{ and } s \leq f \right\} \in [0, \infty]$$

Proposition 4.7. *Let $f : X \to [0, \infty]$ be measurable. Let $s_n \in \xi_+$ with*

$$s_(x) \leq s_{n+1}(x) \leq f(x) \textit{ and } \lim_{n\to\infty} s_n(x) = f(x) \forall x \in X$$

Then

$$\lim_{n\to\infty} \int_X s_n d\mu = \int_X f \, d\mu$$

Proof. As before,

$$\int_X s_n \, d\mu \leq \int_X s_{n+1} \, d\mu \leq \lim_{n\to\infty} s_n \, d\mu \underbrace{\leq}_{\text{by def of } \int_X f \, d\mu} \int_X f \, d\mu$$

It suffices to show that, $\lim_{n\to\infty} \int_X s_n \, d\mu \geq \int_X g \, d\mu \forall g \in \xi_+$ with $g \leq f$. We set $h_n = \min(g, s_n) \in \xi_+$. Obviously, $h_n \leq s_n \forall n \in \mathbb{N}$ because $s_n \leq s_{n+1}$. Hence, $h_n \leq h_{n+1}$. Moreover, $\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\left(\underbrace{g(x)}_{\in\mathbb{R}}, h_n(x) \right) = \lim_{n\to\infty} \min(y, \psi_n)$ where $\psi_n = h_n(x)$ and $y = g(x)$.

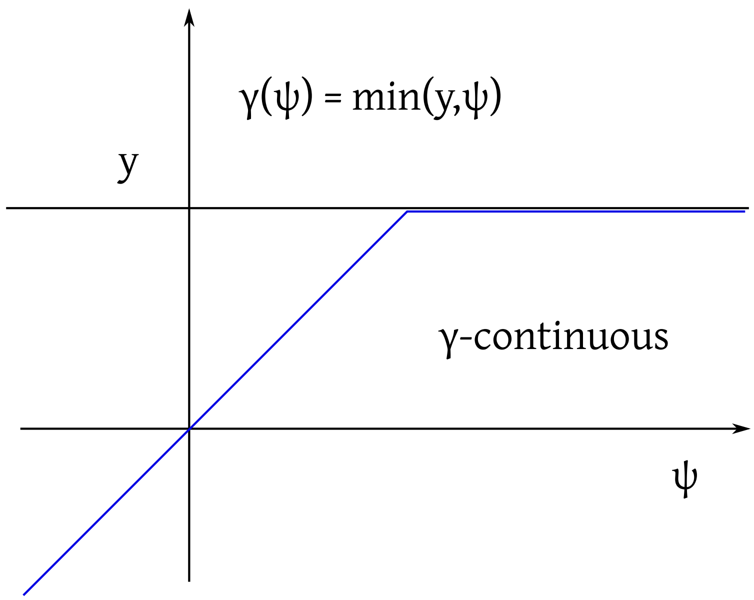

Figure 17: ψ in the proof of Proposition 7

$$\begin{aligned}\lim_{n\to\infty} \min(y, \psi_n) &= \min(g(x), \lim_{n\to\infty} s_n(x)) \\ &= \min(g(x), f(x)) = g(x)\end{aligned}$$

$$h_n(x) \to g(x) \forall x \in X$$

by proposition 6:

$$\implies \lim_{n\to\infty} \int_X h_n \, d\mu = \int_X f \, d\mu$$

Because $h_n \le s_n \implies \lim_{n\to\infty} \int_X s_n \, d\mu \ge \int_X g \, d\mu$.

$$\implies \lim_{n\to\infty} \int_X d\mu \ge \int_X f \, d\mu$$

□

Proposition 4.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $f, g : X \to [0, \infty]$ be measurable and let $\alpha \ge 0$. Then*

1. $\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu$
2. $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$

3. *If $f(x) \leq g(x) \forall x \in X$ then $\int_X f\, d\mu \leq \int_X g\, d\mu$.*

Proof. Let $(s_n)_{n\in\mathbb{N}}$, $(\sigma_n)_{n\in\mathbb{N}}$ be monotone sequences of simple functions with $\lim_{n\to\infty} s_n(x) = f(x)$ and $\lim_{n\to\infty} \sigma_n(x) = g(x)$ (Proposition 4). The last proposition 7: $\lim_{n\to\infty} s_n\, d\mu = \int_X f\, d\mu$ and $\lim_{n\to\infty} \int_X \sigma_n\, d\mu = \int_X g\, d\mu$.

$$\int_X \alpha f\, d\mu \underbrace{=}_{\text{by Prop. 7}} \lim_{n\to\infty} \int_X \alpha S_n\, d\mu = \lim_{n\to\infty} \alpha \int_X s_n\, d\mu = \alpha \int_X f\, d\mu$$

$$\int_X (f+g)\, d\mu = \lim_{n\to\infty} \int_X \underbrace{(S_n + \sigma_n)}_{\in \xi_+}\, d\mu$$

$$\lim_{n\to\infty} f\, d\mu = \sup \underbrace{\left\{ \int_X s\, \mu \,\middle|\, s \in \xi_+, s \leq f \right\}}_{\subseteq \left\{ \int_X s\, d\mu \,\middle|\, s \in \xi_+, s \leq g \right\}}$$

$$\sup \left\{ \int_X s\, d\mu \,\middle|\, s \in \xi_+, s \leq g \right\} = \int_X g\, d\mu$$

□

Definition 4.8. *Let $f \in \mathcal{M}$, $f_+ = \max(f, 0) \in \mathcal{M}_+$ and $f_- = -\min(f, 0) \in \mathcal{M}_+ \forall x \in X$. Either $f_+(x) = 0$ or $f_-(x) = 0$ holds.*

$$f = f_+ - f_- \text{ and } |f| = f_+ + f_-$$

$$f(x) = f_+(x) - f_-(x) \text{ always makes sense}$$

Assume that $\int_X f_+\, d\mu < \infty$ and $\int_X f_-\, d\mu < \infty$. Then we set $\int_X f\, d\mu = \int_X f_+\, d\mu - \int_X f_-\, d\mu$. A measurable function satisfying the previous assumption (integral f_+ and f_- are finite) is called integrable on *X*.

Remark 4.6. *If $\int_X f_+\, d\mu < \infty$ and $\int_X f_-\, d\mu < \infty$, then*

$$\int_X \underbrace{|f|}_{\geq 0}\, d\mu = \int_X (f_+ + f_-)\, d\mu = \int_X f_+\, d\mu + \int_X f_-\, d\mu$$

On the other hand, if $\int_X |f|\, d\mu < \infty$. Then $\int_X f_+\, d\mu$

$f \in \mathcal{M}$ is integrable iff $\int_X |f|\, d\mu < \infty$.

- We could define $\int_X f\, d\mu$ if only one condition $\int_X f_+\, d\mu < \infty$ or $\int_+ f_-\, d\mu < \infty$ holds.

- Let $A \subseteq X$, $A \in \mathcal{A}$. We set for $f \in \mathcal{M}$, $\int_A f \, d\mu = \int_X \underbrace{\chi_A f}_{\in \mathcal{M}} \, d\mu$ if $\chi_A f$ is integrable. We get the same integral, if we consider the measure space $(A, \mathcal{A}_A, \mu_A)$ where $\mathcal{A}_A = \{B \cap A \mid B \in \mathcal{A}\}$ is a σ-algebra and $\int_A f_A \, d\mu_A = \int_X \chi_A f \, d\mu$. For $A' \subseteq A, A' \in \mathcal{A}_A$ we set $\mu_A(A') = \mu(A')$, $f : X \to [-\infty, \infty]$ can be restricted to A, i.e. $f_A = f\big|_A$. $f_A : A \to [-\infty, \infty]$ is measurable with respect to μ_A.
- We set $\mathcal{L}^1(X, \mathcal{A}, \mu) = \mathcal{L}^1(X) = \left\{ f \in \mathcal{M} \,\middle|\, f \text{ is integrable on } X \text{, i.e., } \int_X \left| f \right| \, d\mu < \infty \right\}$

Definition 4.9. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $x \in X$. We consider $P(x)$ a statement which can be true or false. We say that $P(x)$* holds almost everywhere on X *or* holds for almost all $x \in X$ *if*

$$\mu(\{x \in X \mid \neq P(x)\}) = 0$$

Almost everywhere iff everywhere expect on a set of measure 0.

$f : X \to [0, \infty]$.

$$\int f \, d\mu = 0 \iff f(x) = 0 \; a. \; e. \; on X$$

where a.e. stands for almost everywhere*.*

This lecture took place on 2017/11/24.

Lemma 4.5. *Let $f \in \mathcal{M}_+$, $f : X \to [0, \infty]$.*

$$\implies \int_X f \, d\mu = 0 \iff \mu(\underbrace{\{x \in X \mid f(x) > 0\}}_{P}) = 0$$

Proof. Suppose $P = \{x \in X \mid f(x) > 0\}$ and $\mu(P) = 0$. Let $s \in \xi_+$ with $0 \leq s \leq f$. Then for all $x \in X \setminus P$, we have $0 \leq s(x) \leq f(x)$. Because $x \in X \setminus P$, $f(x) = 0$ holds. Hence $s(x) = 0$.

$$s = \sum_{i=1}^{N} \alpha_i \chi_{A_i} \text{ with } \alpha_i > 0 \text{ and } A_i \cap A_j = \emptyset \text{ if } i = i'$$

$$A_i \in \mathcal{A}$$

Then $s(x) > 0$ if $x \in A_i$ for one $i \in \{1, \ldots, N\}$. $x \in A_i \implies s(x) > 0 \implies f(x) > 0 \implies x \in P$. So $A_i \subseteq P$. Therefore $\mu(A) \leq \mu(P) = 0$, so $\mu(A_i) = 0$.

$$\int_X s \, d\mu = \sum_{i=1}^{N} \alpha_i \underbrace{\mu(A_i)}_{=0} = 0$$

$$\implies \int_X f\,d\mu = \sup\left\{ \int_X s\,d\mu \,\middle|\, s \in \xi_+, 0 \le s \le f \right\} = 0$$

We also need to prove the other direction: Suppose $\int_X f\,d\mu = 0$ and let $P = \{x \in X \mid f(x) > 0\}$.

$$P_n = \left\{ x \in X \mid f(x) \ge \frac{1}{n} \right\} \in \mathcal{A}$$

$$x \in P \iff \exists n \in \mathbb{N} : x \in P_n \implies P = \bigcup_{n=1}^{\infty} P_n$$

$P_n \subset P_{n+1}$. Consequently $\mu(P) = \lim_{n\to\infty} \mu(P_n)$. Assume $\mu(P) > 0$. Then $\exists n \in \mathbb{N}$. $\mu(P_n) > 0$. Let $s = \frac{1}{n} \cdot \chi_{P_n} \in \xi_+$. $x \in P_n : \frac{1}{n} = s(x) \le f(x), x \notin P_n : s(x) = 0 \le f(x)$. So $s \le f$ on X and

$$\int_X s\,d\mu = \frac{1}{n} \underbrace{\mu(P_n)}_{>0} > 0 \implies \int_x f\,d\mu > 0$$

This is a contradiction and our proof is complete. □

Remark 4.7. *Let $f : X \to [0, \infty]$ and $\int_X f\,d\mu < \infty$. Then for $S = \{x \in X \mid f(x) = \infty\}$ (where S stands for* singularity*) it holds that $\mu(S) = 0$*

Proof. because otherwise $n\chi_S$ is a simple function below f and

$$\int_X f\,d\mu \ge \int_X n\lambda_S\,d\mu = \underbrace{n \cdot \mu(\chi_S)}_{\to +\infty \text{ as } n\to\infty} \implies \int_X f\,d\mu = +\infty$$

leading to a contradiction. □

Remark 4.8. *We frequently use the following argument: Let $f \in \mathcal{M}_+$ and $E \in \mathcal{A}$ with $\mu(E) = 0$. Let*

$$\tilde{f}(x) := \begin{cases} f(x) & x \notin E \\ 0 & x \in E \end{cases}$$

The equivalent definition is given by $\tilde{f} := f \cdot \chi_{X\setminus E} \in \mathcal{M}^+$. Then $\int_X f\,d\mu = \int_X \tilde{f}\,d\mu$.

Proof. We can prove this using $g := f - \tilde{f}$.

$$g(x) = \begin{cases} 0 & x \notin E \\ f(x) \ge 0 & x \in E \end{cases}$$

$g \in \mathcal{M}_+$ and $g(x) > 0 \implies x \in E$

$$\mu(\{x \in X \mid g(x) > 0\}) = 0$$

Then by Lemma 5,

$$\int_X g\,d\mu = 0 \implies \text{ with } f = g + \tilde{f} \text{ and } \int_X f\,d\mu = \underbrace{\int_X g\,d\mu}_{=0} + \int_X \tilde{f}\,d\mu$$

$$\implies \int_X f\,d\mu = \int_X \tilde{f}\,d\mu$$

□

Lemma 4.6. *Let $f, g \in \mathcal{M}_+$ and $f = g$ almost everywhere on X. Then $\int_X f\,d\mu = \int_X g\,d\mu$.*

Proof.

$$E = \{x \in X \mid f(x) \neq g(x)\}$$

$\mu(E) = 0$. We set $\tilde{f} = f \cdot \chi_{X\setminus E}$. $\tilde{g} = g \cdot \chi_{X\setminus E}$.

$$\implies \tilde{f} = \tilde{g} \implies \int_X \tilde{f}\,d\mu = \int_X \tilde{g}\,d\mu$$

By the previous remark, it holds that

$$\int_X f\,d\mu = \int_X \tilde{f}\,d\mu = \int_X \tilde{g}\,d\mu = \int_X g\,d\mu$$

□

Let $f, g \in \mathcal{L}^1(X)$, i.e. $\int_X \left|f\right|\,d\mu < \infty$ and $\int_X \left|g\right|\,d\mu < \infty$. We define an equivalence relation on $\mathcal{L}(X)$.

$$f \sim g \iff \int_X \left|f - g\right|\,d\mu = 0 \iff \left|f - g\right| = 0 \text{ a.e. on } X \iff f = g \text{ a.e. on } X$$

It is trivial to show that $\sim$ is an equivalence relation (only transitivity is a tiny challenge). We let

$$L^1(x) := \left\{ \overline{f} \,\middle|\, f \in \mathcal{L}^1(x), \overline{f} \text{ is the equivalence class of } f \text{ with respect to } \sim \right\}$$

We will see: $L^1(x)$ is a vector space. $\left\|\overline{f}\right\|_{L^1} = \int_X \left|f\right|\,d\mu$ for some $f \in \overline{f}$. $\|\cdot\|_{L^1}$ is a norm on $L^1(X)$.

We discuss this norm briefly:

$$\left\|\overline{f}\right\|_{L^1} = 0 \iff \int_X \left|f\right|\,d\mu = 0 \iff f = 0 \text{ a.e. on } X \iff \overline{f} = \overline{0}$$

Triangle inequality:

$$\left\|\overline{f + g}\right\|_{L^1} = \int_X \left|f + g\right|\,d\mu \leq \int_X (\left|f\right| + \left|g\right|)\,d\mu = \left\|\overline{f}\right\|_{L^1} + \left\|\overline{g}\right\|_{L^1}$$

The relation $\leq$ holds because of monotonicity.

This lecture took place on 2017/11/29.

$$f : X \to [-\infty, \infty] \qquad f = f_+ - f_- \qquad \left|f\right| = f_+ + f_-$$

$$\int_X \left|f\right| \, d\mu < \infty \iff \int_X f_+ \, d\mu < \infty \qquad \int_X f_- \, d\mu < \infty \iff f \text{ integrable}$$

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu$$

Lemma 4.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space $f_1, f_2, g_1, g_2 \in \mathcal{M}_+$ and $f_1 - f_2 = g_1 - g_2$ on X and f_1, f_2, g_1, g_2 are supposed to be integrable. Then $\int_X f_1 \, d\mu - \int_X f_2 \, d\mu = \int_X g_1 \, d\mu - \int_X g_2 \, d\mu$.*

Proof.

$$\begin{aligned} f_1 - f_2 = g_1 - g_2 &\implies \underbrace{f_1 + g_2}_{\in \mathcal{M}_+} = \underbrace{g_1 + f_2}_{\in \mathcal{M}_+} \\ &\implies \int_X (f_1 + g_2) \, d\mu = \int_X f_1 \, d\mu + \int_X g_2 \, d\mu \\ &\implies \int_X (g_1 + f_2) \, d\mu = \int_X g_1 d\mu + \int_X f_2 \, d\mu \end{aligned}$$

all integrals are finite

$$\implies \int_X f_1 \, d\mu - \int_X f_2 \, d\mu = \int_X g_1 \, d\mu - \int_X g_2 \, d\mu$$

□

Proposition 4.9. *Let f, g be integrable on X, $f : X \to [-\infty, \infty]$, $g : X \to [-\infty, \infty]$. Let $\alpha \in \mathbb{R}$. Then*

1. *αf and $f + g$ are integrable functions.*
2. *$\int_X \alpha f \, d\mu = \alpha \int_X f \, d\mu$*
3. *$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$*
4. *$f \le g$ on X then $\int_X f \, d\mu \le \int_X g \, d\mu$*
5. *$\left|\int_X f \, d\mu\right| \le \int_X \left|f\right| \, d\mu$*

Proof. 1.

$$\int_X |\alpha f| \, d\mu = \int_X \underbrace{|\alpha| |f|}_{\in \mathcal{M}_+} \, d\mu$$
$$= |\alpha| \underbrace{\int_X |f| \, d\mu}_{<\infty} < \infty$$

so αf is integrable.

$$\int_X |f+g| \, d\mu \leq \int_X \left(|f| + |g|\right) d\mu = \int_X |f| \, d\mu + \int_X |g| \, d\mu > \infty$$

The inequality $\leq$ holds because of monotonicity for functions in $\mathcal{M}_+$.

2. $\alpha = 0 \implies \alpha f = 0$.

$$\int_X \alpha f \, d\mu = 0 \qquad \alpha \int_X f \, d\mu = 0$$

Consider $\alpha > 0$.

$$(\alpha f)_+ = \begin{cases} \alpha f(x) & \alpha f(x) \geq 0 \\ 0 & \alpha f(x) < 0 \end{cases}$$
$$= \alpha \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases} = \alpha f_+$$

Analogously: $(\alpha f)_- = \alpha \cdot f_-$.

$$\int_X (\alpha f) \, d\mu = \int_X (\alpha f)_+ \, d\mu - \int_X (\alpha f)_- \, d\mu$$
$$= \alpha \int_X f_+ \, d\mu - \alpha \int_X f_- \, d\mu$$
$$= \alpha \left(\int_X f \, d\mu - \int_X f_- \, d\mu \right)$$
$$= \alpha \int_X f \, d\mu$$

Consider $\alpha < 0$.

$$(\alpha f)_+(x) = \begin{cases} \alpha f(x) & \alpha f(x) \geq 0 \\ 0 & \text{else} \end{cases}$$
$$\alpha \begin{cases} f(x) & f(x) \leq 0 \\ 0 & \text{else} \end{cases} = \alpha(-f_-)$$

$$(\alpha f)_-(x) = \begin{cases} -\alpha f(x) & \alpha f(x) \leq 0 \\ 0 & \text{else} \end{cases}$$

$$-\alpha \begin{cases} f(x) & f(x) \geq 0 \\ 0 & \text{else} \end{cases} = (-\alpha) f_+$$

$$\begin{aligned} \int_X \alpha f \, d\mu &= \int_X (\alpha f)_+ \, d\mu - \int_X (\alpha f)_- \, d\mu \\ &= \int_X \underbrace{(-\alpha)(-\alpha) f_-}_{\in \mathcal{M}_+} \, d\mu - \int_X \underbrace{(-\alpha) f_+}_{\in \mathcal{M}_+} \, d\mu \\ &= -\alpha \int_X f_- \, d\mu + \alpha \int_X f_+ \, d\mu \\ &= \alpha \int_X f \, d\mu \end{aligned}$$

3.

$$\begin{aligned} f + g &= (f+g)_+ - (f+g)_- \\ f + g &= f_+ - f_- + g_+ - g_- = (f_+ + g_+) - (f_- + g_-) \end{aligned}$$

Now we apply Lemma 6:

$$\begin{aligned} \int_X (f+g)_+ \, d\mu - \int_X (f+g)_- \, d\mu &= \int_X (f_+ + g_+) \, d\mu - \int_X (f_- + g_-) \, d\mu \\ &= \int_X f_+ \, d\mu + \int_X g_+ \, d\mu - \int_X f_- \, d\mu - \int_X g_- \, d\mu \\ &= \left(\int_X f_+ \, d\mu - \int_X f_- \, d\mu \right) + \left(\int_X g_+ \, d\mu - \int_X g_- \, d\mu \right) \\ &= \int_X f \, d\mu + \int_X g \, d\mu \checkmark \end{aligned}$$

4.

$$f \leq g \implies g - f \geq 0 \implies g - f = (g-f)_+ \wedge (g-f)_- = 0$$

$$\int_x (g - f) \, d\mu = \int_X (g-f)_+ \, d\mu \geq 0$$

$$\int_x (g - f) \, d\mu = \int_X (g + (-f)) \, d\mu = \int_X g \, d\mu + \int_X (-f) \, d\mu = \int_X g \, d\mu - \int_X f \, d\mu \checkmark$$

5.

$$f \leq |f| \overset{(4)}{\implies} \int_X f \, d\mu \leq \int_X |f| \, d\mu$$

$$-f \le |f| \overset{(4)}{\implies} \int_X (-f)\, d\mu = -\int_X f\, d\mu \le \int_X |f|\, d\mu$$
$$\implies \left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu$$

□

Convergence theorems

Theorem 4.1 (Monotone convergence theorem)**.** *By Beppo Levi (1875–1961)*
Let $f_n, f \in \mathcal{M}_+$ and $f_n(x) \le f_{n+1}(x) \le f(x)$ (monotonicity) for all $n \in \mathbb{N}$ and for almost every $x \in X$. Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere on X. Then $\lim_{n\to\infty} f_n\, d\mu = \int_X f\, d\mu = \int_X \lim_{n\to\infty} f_n\, d\mu$.

Proof. First, we replace almost everywhere by $\forall x \in X$. $\left(\int_X f_n\, d\mu\right)_{n\in\mathbb{N}}$ is a monotone sequence in $[0,\infty]$ $\lim_{n\to\infty}\int_X f_n\, d\mu$ exists in $[0,\infty]$ and by monotonicity of the integral
$$f_n \le f \implies \lim_{n\to\infty}\int_X f_n\, d\mu \le \int_X f\, d\mu$$
We know by Proposition 4: $\exists (g_n^k)_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$ such that $g_n^k \in \xi_+$, $g_n^k \le g_n^{k+1} \le f_n$ and $f_n(x) = \lim_{k\to\infty} g_n^k(x) \forall x \in X$.
$$h_n := \max\left\{g_1^n, g_2^n, \ldots, g_n^n\right\} \in \xi_+$$
$$h_{n+1} \ge \underbrace{g_i^{n+1}}_{\text{for } i\in\{1,\ldots,n+1\}} \overset{\substack{\text{by monotonicity}\\ \text{of } (g_i^k)}}{\ge} g_i^n \qquad \text{for } i = 1,\ldots,n$$
So $h_{n+1} \ge \max\left\{g_1^n, g_2^n, \ldots, g_n^n\right\} = h_n$ where $(h_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of simple functions.
$$g_i^n \le f_i \overset{\substack{\text{by monotonicity}\\ \text{of } (f_n)}}{\le} f_n \le f \text{ for } i = 1,\ldots,n$$
$$\overset{\max}{\implies} h_n \le f_n \le f \text{ on } X$$
Claim. $\lim_{n\to\infty} h_n(x) = f(x) \forall x \in X$.

Proof. Case 1 is assuming $f(x) < \infty$.

Let $\varepsilon > 0$ be given. Choose $N_1 \in \mathbb{N}$ sufficiently large such that
$$\underbrace{\left|f_n(x) - f(x)\right| < \frac{\varepsilon}{2}}_{\text{pointwise convergence}} \text{ for all } n \ge N_1$$

Choose $N_2 \in \mathbb{N}$ such that $f_{N_1}(x) - g_{N_1}^{N_2}(x) < \frac{\varepsilon}{2}$. It is less than $\frac{\varepsilon}{2}$, because $g_{N_1}^k$ approximates f_{N_1}. Let $n \geq \max(N_1, N_2)$.

$$\underbrace{f(x) - h_n(x)}_{\geq 0} = \underbrace{f(x) - f_{N_1}(x)}_{<\frac{\varepsilon}{2}} + f_{N_1}(x) - h_n(x) \leq \frac{\varepsilon}{2} + f_{N_1}(x) - g_{N_1}^n(x)$$

Then by monotonicity of $g_{N_1}^k$:

$$\leq \frac{\varepsilon}{2} + f_{N_1}(x) - g_{N_1}^{N_2}(x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So $f(x) = \lim_{n\to\infty} h_n(x)$

Case 2 is assuming $f(x) = \infty$.

Let $c > 0$ be given. $f_n(x) \to f(x)$. Choose $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies f_n(x) > 2c$. $g_{N_1}^k \to f_{N_1}$. Choose $N_2 \in \mathbb{N}$ such that $g_{N_1}^{N_2}(x) > c$ for $n \geq \max(N_1, N_2) \cdot h_n(x) \geq g_n^{N_2}(x) \geq g_{N_1}^{N_2}(x) > c$.

$$\lim_{n\to\infty} h_n(x) = \infty = f(x)$$

$\square$

Proposition 7 implies that $\int_X f\, d\mu = \lim_{n\to\infty} \int_X h_n\, d\mu$.

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X h_n\, d\mu \underbrace{\leq}_{h_n \leq f_n} \lim_{n\to\infty} \int_X f_n \in d\mu \leq \int_X f\, d\mu \checkmark$$

Because LHS and RHS are the same, all inequalities must be equalities.

Regarding the almost everywhere situation: Let $X' = \{x \in X \mid f_n(x) \leq f_{n+1}(x) \text{ and } f(x) = \lim_{n\to\infty} f_n(x)\}$.

$$E = X \setminus X' \qquad \mu(E) = 0 \qquad \tilde{f_n} = f_n \cdot \chi_{X'} \qquad \tilde{f} = f \cdot \chi_{X'}$$

Then $\int_X f_n\, d\mu = \int_X \tilde{f_n}\, d\mu$ because $\tilde{f_n} = f_n$ almost everywhere.

$$\int_X \tilde{f}\, d\mu = \int f\, d\mu$$

$\tilde{f_n}$ and $\tilde{f}$ satisfy the condition of the theorem.

$$\forall x \in X \implies \int_X f\, d\mu = \int_X \tilde{f}\, d\mu = \lim_{n\to\infty} \int_X \tilde{f_n}\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$$

$\square$

Corollary (Beppo-Levi Theorem). *Let $f_n \in \mathcal{M}_+$ and consider $g = \sum_{n=1}^{\infty} f_n$. $\sigma_N = \sum_{n=1}^{N} f_n$, $g = \sup\{\sigma_N \mid N \in \mathbb{N}\}$ so $g \in \mathcal{M}_+$. Then $\int_X g\, d\mu = \sum_{n=1}^{\infty} f_n\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu = \lim_{N\to\infty} \sum_{n=1}^{N} \int_X f_n\, d\mu$*

Proof. Apply monotone convergence theorem to sequence σ_N. □

An important conclusion:

Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in \mathbb{M}_+$. We define $\nu : \mathcal{A} \to [0, \infty]$ as $\nu(A) = \int_A f \, d\mu = \int_X f \cdot \chi_A \, d\mu$. Then ν is a measure on $\mathcal{A}$.

If f is the constant function 1, then we get $\nu = \mu$.

$$\nu(\emptyset) = 0 \checkmark$$

We only need to prove σ-additivity of ν: Let $(A_n)_{n \in \mathbb{N}}$.

$$A_n \in \mathcal{A}, A_n \cap A_m = \emptyset \text{ if } n \neq n' \text{ show that } \nu(A) = \sum_{n=1}^{\infty} \nu(A_n)$$

$$\chi_A = \begin{cases} 1 & x \in A_n \text{ for exactly one } n \\ 0 & \text{else} \end{cases}$$

$$\sum_{n=1}^{\infty} \chi_{A_n} = \begin{cases} 1 & x \in A_n \text{ for one } n \in \mathbb{N} \\ 0 & x \notin \bigcup_{n=1}^{\infty} A_n \end{cases}$$

So $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$.

$$\sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \int_X f \cdot \chi_{A_n} \, d\mu \overset{\text{Beppo Levi}}{=} \int_X f \sum_{n=1}^{\infty} \chi_{A_n} \, d\mu = \int_X f \chi_A \, d\mu = \nu(A) \checkmark$$

This lecture took place on 2017/12/06.

TODO I was missing the first half an hour

Theorem 4.2 (Fatou's Lemma)**.** *Pierre Fatou (1878-1929)*

Let $f_n \in \mathcal{M}_+$ and we set $f = \liminf_{n\to\infty} f_n \in \mathcal{M}_+$ (is known) $(:= \lim_{n\to\infty}(\inf\{f_k \mid k \geq n\}))$.

$$\implies \int_x f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$

Proof.

$$g_n = \inf\{f_k \mid k \geq n\}$$

g_n is non-decreasing, measurable. Monotone convergence implies that $f = \lim_{n\to\infty} g_n$ limit is pointwise, monotone.

$$\int_X f \, d\mu = \lim_{n\to\infty} \int_X g_n \, d\mu = \liminf_{n\to\infty} \int_X g_n \, d\mu \underbrace{\leq}_{g_n \leq f_n} \liminf_{n\to\infty} \int_X f_n \, d\mu$$

□

Theorem 4.3 (Dominated convergence theorem, Lebesgue)**.** *Let* $f_n \in \mathcal{M}$ *and* $f \in \mathcal{M}$ *with* $f(x) = \lim_{n\to\infty} f_n(x)$ *almost everywhere on* X*. Suppose that* $\exists g \in \mathcal{M}_+$ *with* $0 \le \int_x f\,d\mu < \infty$ $(g \in \mathcal{L}^1(x))$ *and* $\left|f_n(x)\right| \le g(x)$ *almost everywhere on* X $\forall n \in \mathbb{N}$*.* g *is the so-called* integratable majorant*.*

Then $f_n \in \mathcal{L}^1(X) \forall n \in \mathbb{X}$ *and* $\lim_{n\to\infty} \int_X \left|f_n - f\right| d\mu = 0$*.* $f \in \mathcal{L}^1(X)$ *and* $\int_x f\,d\mu = \lim_{n\to\infty} \int_X f_n\,d\mu$*.*

Proof.

$$\left|f_n\right| \le g \qquad \left|f\right| = \lim_{n\to\infty} \left|f_n\right| \le g \text{ almost everywhere on } X$$

i.e. $\int_X \left|f_n\right| d\mu \le \int_X g\,d\mu < \infty$ and $\int_x \left|f\right| d\mu \le \int_X g\,d\mu < \infty$.

$$\left|f_n\right| \le g \wedge \left|f\right| \le g \implies \left|f_n - f\right| \le \left|f_n\right| + \left|f\right| \le 2g$$

$$\begin{aligned}
\underbrace{2g - \left|f_n - f\right|}_{\in\mathcal{M}_+} \ge 0 \text{ Fatou } \implies & \int_X \liminf_{n\to\infty}(2g - \underbrace{\left|f_n - f\right|}_{\to 0})\,d\mu \\
& \le \liminf_{n\to\infty} \int_X (2g - \left|f_n - f\right|)\,d\mu \\
& = \liminf_{n\to\infty}(\int_X 2g\,d\mu - \underbrace{\int_X (f_n - f)\,d\mu)}_{\ge 0} \\
& = \int_X 2g\,d\mu - \limsup_{n\to\infty} \int_X \left|f_n - f\right| d\mu \\
0 \le & -\limsup_{n\to\infty} \int_X \left|f_n - f\right| d\mu \iff \limsup_{n\to\infty} \int_X \left|f_n - f\right| d\mu \le 0 \\
& \le \liminf_{n\to\infty} \underbrace{\int_X \left|f_n - f\right| d\mu}_{\ge 0} \le \limsup_{n\to\infty} \int_X \left|f_n - f\right| d\mu \le 0 \\
& \implies \lim_{n\to\infty} \int_X \left|f_n - f\right| d\mu = 0 \\
0 \le & \liminf_{n\to\infty} \left(\left| \underbrace{\int_X f_n\,d\mu - \int_X f\,d\mu}_{\left|\int_X (f_n - f)\,d\mu\right|} \right| \right) \\
& \le \limsup_{n\to\infty} \left(\left| \int_X (f_n - f)\,d\mu \right| \right) \\
& \le \lim_{n\to\infty} \int_X \left|f_n - f\right| d\mu = 0
\end{aligned}$$

$$(a_n)_{n\in\mathbb{N}} \qquad a_n \leq 0 \qquad \liminf_{n\to\infty}(a_n) = -\limsup_{n\to\infty}(\underbrace{-a_n}_{\geq 0})$$

□

TODO

Definition 4.10 (L-spaces)**.** *Let $(X, \mathcal{A}, \mu)$ be a given measure space.*

$$\mathcal{L}^1(X) = \left\{ f \in \mathcal{M} \middle| \int_X |f| \, d\mu < \infty \right\}$$

$$L^1(X) = \left\{ \overline{f} \,\middle|\, f \in L^1(x) \right\}$$

$$\overline{f} \sim f \iff \overline{f} = f \text{ almost everywhere on } X$$

Usually we write $f \in L^1(X)$ instead of $\overline{f} \in L^1(X)$.

$$\|f\|_{L^1(X)} = \int_X |f| \, d\mu \text{ is a norm on } L^1(X)$$

$L^1(X)$ is a vector space.

Definition 4.11. *Let $-\infty \leq a < b \leq \infty$. A function $\varphi : (a, b) \to \mathbb{R}$ is called* convex *iff $\forall x, y \in (a, b)$ we have $\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y) \forall \lambda \in [0, 1]$.*

$$\iff \forall r, s \in [0, 1] \land r + s = 1 : \varphi(rx + sy) \leq r\varphi(x) + s\varphi(y)$$

Graph of φ is below the secant between x and y.

Remark 4.9 (Exercise)**.**

$$\varphi \text{ is convex} \iff \forall a < s < t < u < b$$

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

The left-hand side represents the slope of the secant between s and t. The right-hand side represents the slope of the secant between t and u.

Proposition 4.10. *Let φ be convex on (a, b). Then φ is continuous on (a, b).*

Informal proof sketch. $x \in (a, b)$, $y \to x$ $(y > x)$. Let $s < x < y < t$. X is below SY. Y is above the line SX. Y is below the line XT.

TODO drawing □

Theorem 4.4 (Jensen's inequality)**.** *Johann Jensen (1859-1925)*

Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e. $(X, \mathcal{A}, \mu)$ is a measure space with $\mu(X) = 1$). Let $f \in \mathcal{L}^1(X)$ and $a < f(x) < b$ for all $x \in X$. $a = -\infty$, $b = \infty$ is possible. Moreover, let $\varphi : (a, b) \to \mathbb{R}$ be convex. Then

$$\varphi\left(\int_X f \, d\mu\right) \leq \int_X \varphi \circ f \, d\mu$$

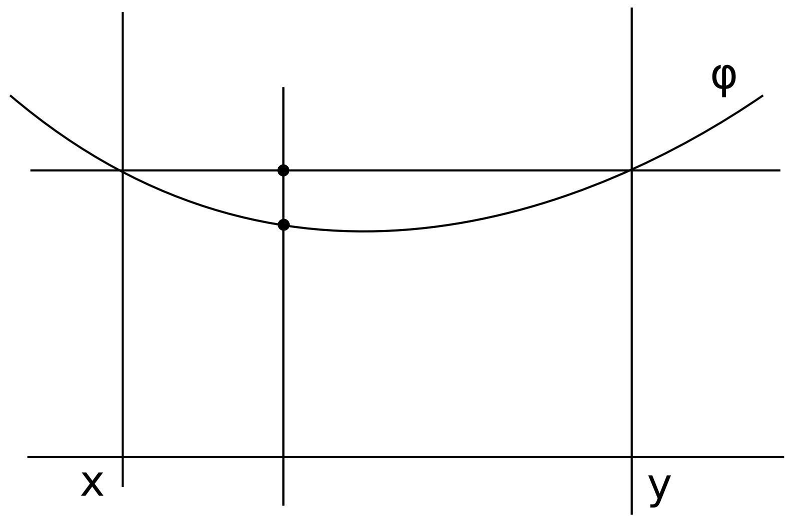

Figure 18: L-Space definition

Proof. Let $t = \int_X f \, d\mu$

$$a = a \int_X 1 \, d\mu = \int_X a \, d\mu < \int_X f \, d\mu < b$$

$t \in (a, b)$. Choose $a < s < t$ and $t < u < b$. Then,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

$$\beta = \sup \left\{ \frac{\varphi(t) - \varphi(s)}{t - s} \;\middle|\; a < s < t \right\} < \infty$$

We get,

$$\frac{\varphi(t) - \varphi(s)}{t - s} \overbrace{\leq}^{\text{supremum}} \beta \iff \varphi(t) \leq \beta(t-s) + \varphi(s) \iff \varphi(t) + \beta(s-t) \leq \varphi(s) \forall s \in (a, t)$$

For $u > t$,

$$\frac{\varphi(u) - \varphi(t)}{u - t} \geq \beta \iff \varphi(u) \geq \varphi(t) + \beta(u - t)$$

So $\forall y \in (a, b) : \varphi(y) \geq \varphi(t) + \beta(y - t)$. $y \in (a, t)$ and $y \in (t, b)$. Because $f(x) \in (a, b) \implies \varphi(f(x)) > \varphi(t) + \beta(f(x) - t)$.

$$\varphi(f(x)) - \varphi(t) - \beta(f(x) - t) \geq 0$$

φ is continuous. $\varphi \circ f$ is measurable. Integrate:

$$\int_X \varphi \circ f \, d\mu - \int_X \varphi(t) \, d\mu - \beta \left(\int_X f \, d\mu - \int_X t \, d\mu \right) \geq 0$$

Recognize that $\int_X t \, d\mu = t \int_X 1 \, d\mu = \int_X f \, d\mu$. Furthermore,

$$\left(\int_X f \, d\mu - \int_X t \, d\mu \right) = 0$$

$$\int_X \varphi(t) \, d\mu = \varphi(t) \cdot \int_X 1 \, d\mu = \varphi(\int_X f \, d\mu)$$

Hence,

$$\implies \int_X \varphi \circ f \, d\mu \geq \varphi(\int_X f \, d\mu)$$

□

Example 4.1.

$$\varphi(x) = e^x = \exp(x)$$

$$\exp(\int_X f \, d\mu) \leq \int_X \exp(f) \, d\mu$$

$$X = \{p_1, p_2, \dots, p_n\} \qquad \mu(\{p_i\}) = \frac{1}{n} \qquad \mu$$

$$f(p_i) = x_i \qquad \varphi\left(\frac{1}{n} \sum_{i=1}^n x_i\right) \leq \frac{1}{n} \sum_{i=1}^n \varphi(x_i)$$

$$\varphi = \exp \implies \exp(\frac{1}{n} \sum_{i=1}^n x_i) \leq \frac{1}{n} \sum_{i=1}^n e^{x_i}$$

TODO something is missing here

This lecture took place on 2017/12/13.

$$\varphi(x) = e^x = \exp(x) \implies \exp\left(\int_X f \, d\mu \right) \leq \int_X \exp(f) \, d\mu$$

$X = \{p_1, p_2, \dots, p_n\}$ is finite, $\mu(\{p_i\}) = \frac{1}{n}$ for $i = 1, \dots, n$.

$$f : X \to \mathbb{R}, f(p_i) = X_i$$

$$\int_X f \, d\mu = \sum_{i=1}^n \underbrace{f(p_i)}_{=x_i} \cdot \underbrace{\mu(\{p_1\})}_{=\frac{1}{n}} = \frac{1}{n} \sum_{i=1}^n x_i$$

$$\int_x \exp \circ f \, d\mu = \sum_{i=1}^{n} e^{x_i} \cdot \frac{1}{n}$$

From Jensen's inequality, it follows,

$$\implies \quad \underbrace{\exp\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)}_{\left(\exp(\sum_{i=1}^{n} x_i)\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n} e^{x_i}\right)^{\frac{1}{n}}} \quad \le \frac{1}{n}\sum_{i=1}^{n} e^{x_i}$$

Let $y_i = e^{x_i} > 0$.

$$\left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{i=1}^{n} y_i$$

Is the inequality of arithmetic and geometric mean. In some way, Jensen's inequality can be considered as generalization of it.

Definition 4.12. *Let $p, q > 1$ and suppose $p + q = p \cdot q \iff \frac{1}{q} + \frac{1}{p} = 1$ then p and q are called* conjugate exponents *(or* pair of conjugate exponents*).*

Special case: *$p = q = 2$ (Hilbert space).*
Special case *(or actually a definition): We also consider $\frac{1}{1} + \frac{1}{\infty} = \frac{1}{1} + 0 = 1$ so also $(1, \infty)$ is a pair of conjugate exponents.*

Theorem 4.5. *Let $p, q \in (1, \infty)$ be conjugate exponents. Let $f, g \in \mathcal{M}_+$ and let $f(x) < \infty$ and $g(x) < \infty$ almost everywhere on X. Then Hölder's inequality holds:*

$$\int_X f \cdot g \, d\mu \le \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} \left(\int_X g^q \, d\mu\right)^{\frac{1}{q}} \tag{1}$$

$$\left(\int_X (f+g)^p \, d\mu\right)^{\frac{1}{p}} \le \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p \, d\mu\right)^{\frac{1}{p}} \tag{2}$$

Equality 1 is called Hölder inequality. Equality 2 is called Minkowski inequality. Correspondingly,

$$\|fg\|_{L^1} \le \|f\|_{L^p} \cdot \|g\|_{L^q} \qquad f \in L^p(x), g \in L^q(x)$$
$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p} \qquad f, g \in L^p(x)$$

Proof. First, we prove Hölder's inequality.

Let $A = \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}}$ and $B = \left(\int_X g^q \, d\mu\right)^{\frac{1}{q}}$.

$$\begin{aligned} A = 0 &\implies \underbrace{f^p}_{\ge 0} = 0 \text{ almost everywhere on } X \\ &\implies f = 0 \text{ almost everywhere on } X \\ &\implies \int_X fg \, d\mu = 0 \text{ and right-hand side also } 0 \end{aligned}$$

$B = 0$✓. $A = \infty, B \neq 0 \implies$ right-hand side $= +\infty$✓.
$A = \infty, B = 0 \implies g = 0$ almost everywhere on $X \implies f \cdot g = 0$ almost everywhere on X.

$$\implies \int_X fg \, d\mu = 0$$

Suppose $A > 0$ and $B > 0$. Let $F = \frac{f}{A}$ and $G = \frac{g}{B}$. Then $\int_x F^p \, d\mu = \frac{1}{A^p} \int_X f^p \, d\mu = 1$.

$$\int_X G^q \, d\mu = \frac{1}{B^q} \int_X g^q \, d\mu = 1$$

Assume (without loss of generality) $G(x) < \infty, F(x) < \infty$.

Case 1 Let $x \in X$ be such that $G(x) > 0$, $F(x) > 0$. Then there exists $s, t \in \mathbb{R}$ such that $F(x) = e^{\frac{s}{p}}$ and $G(x) = e^{\frac{t}{q}}$ ($s = \log(F(x)^p)$).

$$F(x)G(x) = e^{\frac{s}{p} + \frac{t}{q}} \underbrace{\leq}_{(*)} \frac{1}{p}e^s + \frac{1}{q}e^t = \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q$$

where $(*)$ follows from the convexity of exponents with $\lambda = \frac{1}{q}, 1 - \lambda = \frac{1}{p}$.

Case 2 $F(x) = 0$ or $G(x) = 0$.

$$\implies F(x) \cdot G(x) = 0 \leq \frac{1}{p}F(x)^p + \frac{1}{g}G(x)^q$$

Integration:

$$\int_X FG \, d\mu \leq \frac{1}{p} \underbrace{\int_X F^p \, d\mu}_{=1} + \frac{1}{q} \underbrace{\int_X G^q \, d\mu}_{=1} = \frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{AB} \int_X fg \, d\mu \leq 1 \implies \int_X fg \, d\mu \leq A \cdot B ✓$$

Next, we prove the Minkowski inequality.
Let $p \in (1, \infty)$ given and $q = (1 - \frac{1}{p})^{-1} = \frac{p}{p-1}$. Then p and q are conjugate exponents.

$$(f + g)^p = (f + g)(f + g)^{p-1} = f(f + g)^{p-1} + g(f + g)^{p-1}$$

By Hölder's inequality,

$$\implies \int_X f(f+g)^{p-1} \, d\mu \leq \left(\int_X f^p d\mu\right)^{\frac{1}{p}} \cdot \left(\int_X (f + g)^{(pq-q)} \, d\mu\right)^{\frac{1}{q}} = \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} \cdot \left(\int_X (f + g)^p \, d\mu\right)^{\frac{1}{q}}$$

$$\int_X g(f+g)^{p-1}\,d\mu \leq \left(\int_X g^p\,d\mu\right)^{\frac{1}{p}} \cdot \left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{q}}$$

By the sum,

$$\implies \int_X (f+g)(f+g)^{p-1}\,d\mu \leq \left[\left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\,d\mu\right)^{\frac{1}{p}}\right]\left(\int_X (f+g)^p\,d\mu\right)^{\frac{1}{q}}$$

$$\iff \left(\int_X (f+g)^p\,d\mu\right)^{\overbrace{\scriptstyle =\frac{1}{p}}^{1-\frac{1}{q}}} \leq \left(\int_X f^p\,d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\,d\mu\right)\frac{1}{p}$$

□

TODO remark is missing

Definition 4.13. *$L^1(X)\checkmark$. let $p \in (q, \infty)$. We set*

$$L^p(X) = \left\{ f \in \mathcal{M} \,\middle|\, \int_X |f|^p\,d\mu < \infty \right\}$$

and we set,

$$L^p(X) = \left\{ \overline{f} \,\middle|\, f \in L^p(X) \right\}$$

where $\overline{f}$ is the equivalence class of f with respect to equality almost everywhere.

For $f \in \mathcal{L}^p(X)$ we set $\|f\|_{L^p} = \|f\|_p = \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}}$. For $\overline{f} \in \mathcal{L}^1(X)$ we set $\|f\|_{L^p} = \|f\|_p := \|f\|_p$ for any $f \in \overline{f}$. Notation $\|f\|_{L^p}$ instead of $\left\|\overline{f}\right\|_{L^p}$.

Remark 4.10. *$\|\cdot\|_{L^p}$ is a norm on $L^p(X)$.*

- $\left\|\overline{f}\right\|_{L^p} = 0 \iff \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}} = 0 \iff |f|^p = 0$ *almost everywhere on X* $\iff \overline{f} = \overline{0}$.
- *Triangle inequality $\iff$ Morkowski inequality.*
- $\|\lambda f\|_{L^p} = \left(\int_X |\lambda|^p |f|^p\,d\mu\right)^{\frac{1}{p}} = \left(|\lambda|^p \int_X |f|^p\,d\mu\right)^{\frac{1}{p}} = |\lambda|\,\|f\|_{L^p}$

Theorem 4.6. *Let $1 \leq p < \infty$. Then $L^p(X)$ is a complete normed space, i.e. every Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(X)$ has a limit $f \in L^p(X)$. $L^p(X)$ is called a* Banach space*.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $L^p(X)$, i.e. $\left\|f_n - f_m\right\| < \varepsilon$ if m, n are sufficiently large. We choose a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ such that $\left\|f_{n_{i+1}} - f_{n_i}\right\| < \frac{1}{2^i}$. For $k \in \mathbb{N}$, we let $g_k(x) = \sum_{i=1}^{k} \left|f_{n_{i+1}}(x) - f_{n_i}(x)\right|$ and $g(x) := \sum_{i=1}^{\infty} \left|f_{n_{i+1}} - f_{n_i}(x)\right| \in [0, \infty]$. We have $g = \lim_{k\to\infty} g_k$ pointwise on X. By Minkowski's inequality,

$$\left|g_k\right|_p \leq \sum_{i=1}^{k} \left\|f_{n_{i+1}} - f_{n_i}\right\|_p < \sum_{i=1}^{k} \frac{1}{2^i} < 1$$

By Fatou,

$$\int_X g^p \, d\mu = \int_X \lim_{k\to\infty} g_k^p \, d\mu \leq \liminf_{k\to\infty} \int_X g_k^p \, d\mu \leq \int_X \liminf_{k\to\infty} g_k^p \, d\mu \leq 1$$

$$\int_X g^p \, d\mu < \infty \implies g^p(x) < \infty \text{ almost everywhere on } X \implies g(x) < \infty \text{almost everywhere on } X$$

$$g(x) = \sum_{i=1}^{\infty} \left|f_{n_{i+1}}(x) - f_{n_i}(x)\right| < \infty \text{ almost everywhere on } X$$

Consider,

$$f(x) = f_{n_i}(x) + \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

where the sum is absolutely convergent almost everywhere on X due to $\sum_{i=1}^{\infty} \left|f_{n_{i+1}}(x) - f_{n_i}(x)\right| < \infty$ a.e. on X. We define,

$$f(x) := \begin{cases} f_{n_i}(x) + \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right) & \text{if the series converges absolutely} \\ 0 & \text{otherwise} \end{cases}$$

We have

$$f_{n_{k+1}} = f_{n_i} + \sum_{i=1}^{k} \left(f_{n_{i+1}} - f_{n_i}\right) \to f \text{ almost everywhere on } X$$

Show: $\left\|f - f_n\right\|_p \to 0$ as $n \to \infty$.

Choose $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $n, m \geq N \implies \left\|f_n - f_m\right\| < \varepsilon$. Fatou:

$$\begin{aligned} \int_X \left|f - f_n\right|^p \, d\mu &= \int_X \left|\lim_{i\to\infty} f_{n_i} - f_n\right| \, d\mu \\ &\leq \liminf_{i\to\infty} \int_X \left|f_{n_i} - f_n\right|^p \, d\mu = \liminf_{i\to\infty} \underbrace{\left\|f_{n_i} - f_n\right\|_p^p}_{<\varepsilon} \text{ for } n_i \geq N \end{aligned}$$

So $\int_X |f - f_n|^p \, d\mu \leq \varepsilon^p$ if $n \geq N$. $\implies \|f - f_n\|_{L^p} \to 0$ as $n \to \infty$.
The last remaining argument: Show that $f \in L^p(X)$.
By Minkowsky's inequality,

$$\left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} = \left(\int_X |f - f_n + f_n|^p \, d\mu\right)^{\frac{1}{p}}$$
$$\leq \left(\underbrace{\int_X |f_n - f|^p \, d\mu}_{<1 \text{ if } n \text{ is sufficiently large}}\right)^{\frac{1}{p}} + \left(\underbrace{\int_X |f_n|^p \, d\mu}_{<\infty}\right)^{\frac{1}{p}} < \infty$$

□

Corollary. *Let $f_n \to f$ in L^p where $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence. Then there exists a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ such that $f_{n_i}(x) \to f(x)$ almost everywhere on X.*

This lecture took place on 2017/12/15.

$L^p(X)$ is a complete normed vector space (Banach space) for every $1 \leq p \leq \infty$, $l(\alpha x + \beta y) = \alpha l(x) + \beta l(y)$.

Definition 4.14. *Let B be a Banach space over $\mathbb{R}$. A linear map $l : B \to \mathbb{R}$ with the property $\exists C \geq 0$ s.t. $|l(x)| \leq C \|x\| \, \forall x \in B$. l is called a bounded linear functional on B. We set $B^* = \{l \mid l$ is bounded linear functional on $B\}$. Then B^* is a vector space and $\|l\|_{B^*} = \inf\{c \geq 0 \mid |l(x)| \leq c\|x\| \, \forall x \in X\}$ is a norm on B^*. B^* is also a Banach space. We call B^* the dual space to B.*

Consider conjugate exponents $p, q \in (1, \infty)$ and fix $g \in L^q(X)$. Consider $l(f) = \int_X f \cdot g \, d\mu$ for $f \in L^p(X)$. Then l is linear on $L^p(X)$.

$$|l(f)| = \left|\int_X f \cdot g \, d\mu\right| \leq \int_X |f \cdot g| \, d\mu \underbrace{\leq}_{\text{Hölder}} \|g\|_{L^q} \, |f|_{L^p} = C \|f\|_{L^p}$$

with $C = \|g\|_{L^q}$. So $l \in (L^p(X))^*$. It holds that

- $\|l\|_{(L^p)^*} = \|g\|_{L^q}$
- $\forall l \in (L^p(X))^*$ there exists $g \in L^q(X)$ such that $l(f) = \int_X f \cdot g \, d\mu$.

We say $(L^p(X))^* = L^q(X)$.

Definition 4.15. *Let $g \in \mathcal{M}_+$. Let $\alpha \in \mathbb{R}$ such that $\mu(g^{-1}((\alpha,\beta])) = 0$. $S = \left\{\alpha \geq 0 \,|\, \mu(g^{-1}((\alpha,\infty])) = 0\right\}$. If $S \neq \emptyset$, we set $\beta = \inf S$. We say that g is essentially bounded from above if $S \neq \emptyset$ and we call β the* smallest essential upper bound for *g or* essential supremum of *g.*

$$\alpha \in S \iff g(x) \leq \alpha \text{ almost everywhere on } X$$

We have $\beta \in S$, i.e. "inf" = "min".

$$\underbrace{g^{-1}((\beta,\infty])}_{\text{nullset}} = \bigcup_{n=1}^{\infty} \underbrace{g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right)}_{\text{nullset}}$$

$\beta = \operatorname{esssup}(g)$ if $S \neq \emptyset$. $\operatorname{esssup}(g) = \infty$ if $S = \emptyset$.

Definition 4.16. *We set $\mathcal{L}^\infty(X) = \left\{f \in \mathcal{M} \,\middle|\, \operatorname{esssup}\left|f\right| < \infty\right\}$. $\mathcal{L}^\infty(X)$ is a vector space over $\mathbb{R}$ (verify!). We say that $f \in \mathcal{L}^\infty(X)$ is essentially bounded. We set $\left\|f\right\|_\infty = \operatorname{esssup}\left|f\right|$. Again we define $L^\infty = \left\{\overline{f} \,\middle|\, f \in \mathcal{L}^\infty(x)\right\}$.*

Now we what to verify the statement: $\left\|f\right\|_\infty$ is a norm on $L^\infty(X)$.

Theorem 4.7. *$L^\infty(X)$ is complete with respect to $\|\cdot\|_\infty$.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(X)$.

$$A_K = \left\{x \in X : \left|f_k(x)\right| > \left\|f_k\right\|_\infty\right\}, \mu(A_k) = 0$$

$$B_{n,m} = \left\{x \in X : \left|f_n(x) - f_m(x)\right| > \left\|f_n - f_m\right\|_\infty \Big| \,,\right\} \mu(B_{m,n}) = 0$$

$$E = \left(\bigcup_{k=1}^{\infty} A_k\right) \cup \left(\bigcup_{m,n=1}^{\infty} B_{n,m}\right) \implies \mu(E) = 0$$

Then

$$\forall x \in X \setminus E \text{ and } \forall \varepsilon > 0 \exists N \in \mathbb{N} : m,n > N \implies \left|f_n(x) - f_m(x)\right| \leq \left\|f_n - f_m\right\|_\infty < \varepsilon$$

The inequality $<$ on the right is given because (f_n) is a Cauchy sequence. i.e. $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Completeness of $\mathbb{R}$: $\exists \alpha_x \in \mathbb{R} : f_n(x) \to \alpha_x$ as $n \to \infty$. We define

$$f(x) := \begin{cases} \alpha_x & x \in E \\ 0 & x \notin E \end{cases}$$

$f_n \to f$ pointwise almost everywhere on X. Then $\forall x \in X \setminus E$ and $n \geq N$.

$$\left|f_n(x) - f(x)\right| = \lim_{m\to\infty} \left|\underbrace{f_n(x) - f_m(x)}_{<\varepsilon}\right| \leq \varepsilon$$

So, $\left\|f_n - f\right\|_\infty < \varepsilon$ if $n \geq N$. $f_n \to f$ in $L^\infty(X)$.

$$\left|f\right| \leq \left|f - f_N\right| + \left|f_N\right| \leq \varepsilon + \left|f_N\right|$$

for $x \in X \setminus E \implies \left\|f\right\|_\infty \leq \underbrace{\varepsilon + \left\|f_N\right\|}_{<\infty}$. $\square$

This lecture took place on 2018/01/10.

Does the Fundamental Theorem of Calculus hold?

$$f(x) = F'(x) \forall x \in [a, b]$$

We will discuss this now. Furthermore, usually we write

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In case of Lebesgue integral, we write instead

$$\int_{[a,b]} f\,d\lambda$$

$$\lambda(\mathbb{R}^2) \quad E = [\alpha, \beta] \times [\gamma, \delta]$$

$$f : E \to \mathbb{R}$$

$$\int_E f\,d\lambda = \int_{[\alpha,\beta]\times[\gamma,\delta]} f(x,y)\,dx\,dy = \int_{y=\gamma}^{\delta} \underbrace{\int_{x=\alpha}^{\beta} f(x,y)\,dx}_{g(y)}\,dy$$

Compare with Figure 19.

Theorem 4.8 (Transformation theorem). *Set $\mathcal{S}_b = \{s \in \mathcal{S} \mid \mu(\{x \in X \mid s(x) \neq 0\}) < \infty\}$ where $\mathcal{S}$ stands for* simple. *Then $\mathcal{S}_b$ is dense in $L^p(X)$ for all $1 \leq p \leq \infty$.*

Product measure and Fubini's Theorem

Fubini's theorem is a wonderful tool to simplify proofs.

Definition 4.17 (Dynkin class, Dynkin system[3]). *Let X be a set. A subset $D \subseteq P(X)$ is called a Dynkin class (d-system), if*

[3] In German, "Dynkin *System*" is used exclusively

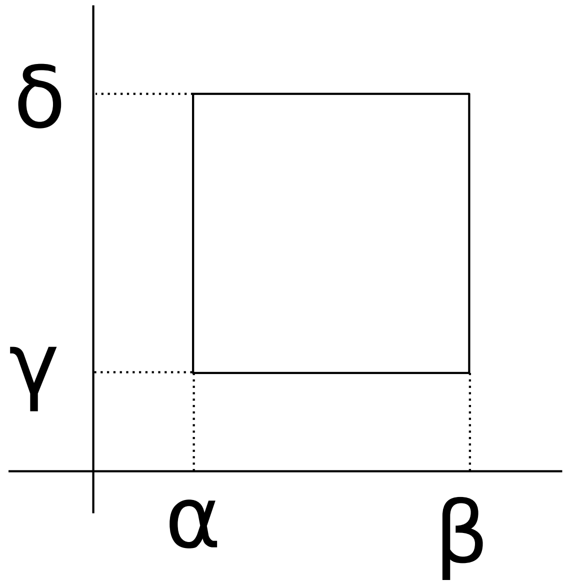

Figure 19: E

1. $X \in D$
2. $\forall A, B \in D : B \subseteq A \implies A \setminus B \in D$
3. $\forall A_n \in D : A_n \subseteq A_{n+1}, \forall n \in \mathbb{N} : \bigcup_{n=1}^{\infty} A_n \in D$

So the definition is similar to a sigma algebra, but condition 3 is different. Often it is easier to prove that a set is a Dynkin system, compared to a sigma algebra. $\mathcal{C}' \subseteq P(X)$ is called a π-system if $\forall A, B \in \mathcal{C}', A \cap B \in \mathcal{C}'$.

Let $\mathcal{C} \subseteq P(X)$. We call $D_{\mathcal{C}} := \bigcap_{\substack{\mathcal{C} \subseteq D \\ D \text{ is } d\text{-system}}} D$ a d-system (the smallest Dynkin system containing $\mathcal{C}$). $D_{\mathcal{C}}$ is a d-system generated by $\mathcal{C}$.

$$\mathcal{C} \subseteq P(X) : A_{\mathcal{C}} = \bigcap_{\substack{\mathcal{C} \subseteq A \\ A \text{ is } \sigma\text{-algebra}}} A$$

Theorem 4.9. *Let X be a set, $\mathcal{C} \subseteq P(X)$ be a π-system. Then $D_{\mathcal{C}} = A_{\mathcal{C}}$.*

Proof. Every σ-algebra is a d-system $\implies D_{\mathcal{C}} \subseteq A_{\mathcal{C}}$.
Show: $A_{\mathcal{C}} \subseteq D_{\mathcal{C}}$ by proving that $D_{\mathcal{C}}$ is itself a σ-algebra.

We start by showing $A, B \in D_{\mathcal{C}} \implies A \cap B \in D_{\mathcal{C}}$.

$$D_1 = \{A \in D_{\mathcal{C}} \mid A \cap C \in D_{\mathcal{C}} \forall C \in \mathcal{C}\}$$

if $C' \in \mathcal{C} \implies C' \cap C \in \mathcal{C} \subseteq D_{\mathcal{C}} \forall C \in \mathcal{C}$ because $\mathcal{C}$ is a π-system.

$$\implies \mathcal{C} \subseteq D_1$$

Show: D_1 is a d-class,

$$X \cap C = C \in \mathcal{C} \subseteq D_{\mathcal{C}} \forall C \in \mathcal{C}$$

$$\implies X \in D_1 \checkmark$$

First property is proven.

Let $A, B \in D_1$ with $B \subseteq A$. Then $(A \setminus B) \cap C = \underbrace{(A \cap C)}_{\in D_{\mathcal{C}}} \setminus \underbrace{(B \cap C)}_{\in D_{\mathcal{C}}} \in D_{\mathcal{C}} \qquad \forall C \in \mathcal{C}$

because $D_{\mathcal{C}}$ is a d-system.

$$\implies A \setminus B \in D_1$$

Second property is proven.

Let $A_n \in D_1, A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$.

$$(\bigcup_{n=1}^{\infty} A_n) \cap C = \bigcup_{n=1}^{\infty} \left(\overbrace{\underbrace{A_n \cap C}_{\in D_{\mathcal{C}}}}^{(A_n \cap C) \subseteq (A_{n+1} \cap C)} \right) \in D_{\mathcal{C}} \implies \bigcup_{n=1}^{\infty} A_n \in D_1$$

So D_1 is a d-system which contains $\mathcal{C} \implies D_{\mathcal{C}} \subseteq D_1$, $D_1 \subseteq D_{\mathcal{C}}$ by definition $\implies D_1 = D_{\mathcal{C}}$. For all $A \in D_{\mathcal{C}}$ and for all $C \in \mathcal{C} : A \cap C \in D_{\mathcal{C}}$.

$$D_2 = \{B \in D_{\mathcal{C}} \mid A \cap B \in D_{\mathcal{C}} \forall A \in D_{\mathcal{C}}\}$$

by first step: $\mathcal{C} \subseteq D_2 : D_2$ is a d-system. Proof as in step 1 for D_1.

$$\implies D_{\mathcal{C}} \subseteq D_2 \text{ and } D_2 \subseteq D_{\mathcal{C}} \text{ by definition } \implies D_2 = D_{\mathcal{C}}$$

$$\forall A, B \in D_{\mathcal{C}} : A \cap B \in D_{\mathcal{C}}$$

Use this, to show that $D_{\mathcal{C}}$ is a σ-algebra. Let $A \in D_{\mathcal{C}}, X \in D_{\mathcal{C}} \implies A^C = X \setminus A \in D_{\mathcal{C}}$ by the second property. Let $A_n \in D_{\mathcal{C}}$ for $n \in \mathbb{N}$. Show that $\bigcup_{k=1}^{n} A_k \in D_{\mathcal{C}} \forall n \in \mathbb{N}$.

$$\bigcup_{k=1}^{n} A_k = \left(\left(\bigcup_{k=1}^{n} A_k \right)^C \right)^C = \left(\bigcap_{k=1}^{n} \underbrace{A_k^C}_{\in D_{\mathcal{C}}} \right)^C \in D_{\mathcal{C}}$$

because D_C is closed with respect to finite intersection. So D_C is an algebra. We let $B_1 = A_1$ and $B_n = A_n \setminus \left(\bigcup_{k=1}^{n-1} A_k\right)$.

$$B_n = \bigcup_{k=1}^{n} A_n \in D_C, B_n \subseteq B_{n+1} \text{ and } \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in D_C \text{ due to property 3}$$

$$\implies \bigcup_{n=1}^{\infty} A_n \in D_C \forall A_n \in D_C$$

$\implies$ D_C is a σ-algebra and hence $D_C \subseteq A_C$. ◻

Corollary. *Let X be a set, A be a σ-algebra on X, $\mathcal{C}$ is a π-system on X. $A = A_C$. Let μ, ν be measures on $\mathcal{A}$, $\mu(X) < \infty$ and $\nu(X) < \infty$ (finite measures). Suppose $\nu(C) = \nu(C) \forall C \in \mathcal{C}$ and $\mu(X) = \nu(X)$. Then $\nu(A) = \mu(A) \forall A \in \mathcal{A}$.*

Proof. Let $D = \{A \in \mathcal{A} \mid \mu(A) = \nu(A)\}, C \in D \forall C \in \mathcal{C} \implies \mathcal{C} \subseteq D$. Show that D is a d-system. $X \in D$ because $\nu(X) = \mu(X)$. Let $A, B \in D : B \subseteq A$. Then $B \cup (A \setminus B) = A \implies \mu(B) + \mu(A \setminus B) = \mu(A)$ and $\nu(B) + \nu(A \setminus B) = \nu(A)$. In both LHS expressions, the set is finite. Hence, the RHS is finite correspondingly.

$$\implies \mu(A \setminus B) = \underbrace{\mu(A)}_{\in D} - \underbrace{\mu(B)}_{\in D} = \mu(A) - \mu(B) = \mu(A \setminus B)$$

$$\implies A \setminus B \in D$$

Let $A_n \in D, A_n \subseteq A_{n+1} \forall n \in \mathbb{N}$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) \underbrace{=}_{\text{because } A_n \in D} \lim_{n\to\infty} \nu(A_n) = \nu(\bigcup_{n=1}^{\infty} A_n)$$

D is a d-system and (remember ($D_C = A_C$ by Theorem 4.9)) $D_C \subset D \implies A = A_C \subseteq D \subseteq A$ with $A_n \in D$. $\implies D = A$, so $\forall A \in \mathcal{A} : \mu(A) = \nu(A)$. ◻

Corollary. *Let A be a σ-algebra on X, $\mathcal{C}$ be a π-system on X. $\mathcal{A} = \mathcal{A}_C$. Let μ, ν be measures on $\mathcal{A}$ wich coincide on $\mathcal{C}$ ($\nu(C) = \mu(C) \forall c \in \mathcal{C}$) and assume that $(C_n)_{n\in\mathbb{N}}$ exists with $C_n \in \mathcal{C}$, $C_n \subseteq C_{n+1}$, $X = \bigcup_{n=1}^{\infty} C_n$ and $\nu(C_n) < \infty$, $\mu(C_n) < \infty$ for all $n \in \mathbb{N}$. Then $\nu = \mu$ on $\mathcal{A}$.*

Definition 4.18. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be sets with σ-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. We consider $Z = X \times Y$. We call a set $C = A \times B \subseteq X \times Y$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ a measurable rectangle in $X \times Y$.*

$A \times B$ is the product σ-algebra of A and B. Let $C = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subseteq P(X \times Y)$. Then we set $A \times B = A_C$ is a σ-algebra in $X \times Y$ generated by the measurable rectangles.

Remark 4.11. *C is a π-system: $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$.*

$$\begin{aligned}(A \times B) \cap (A' \times B')(&= \{(a, b) \in X \times Y \mid a \in A \cap A' \wedge b \in B \cap B'\} \\ &= \underbrace{(A \cap A')}_{\in\mathcal{A}} \times \underbrace{(B \cap B')}_{\in\mathcal{B}} \in C\end{aligned}$$

Definition 4.19. *A, B, X, Y as above. $E \subseteq X \times Y$. We set $E_X = \left\{y \in Y : (x, y) \in E \middle| \subseteq \right\} Y$ for some given $x \in X$.*

$$E^Y = \{x \in X \mid (x, y) \in E\} \subseteq X$$

for some given $y \in Y$. E_x, E^y are called sections of *E.*

This lecture took place on 2018/01/12.

$$E \subseteq X \times Y \qquad E_x = \left\{y \in Y \middle| (x, y) \in E\right\} \qquad E^y = \{x \in X \mid (x, y) \in E\}$$
$$\pi_x : X \times Y \to X \qquad \pi_x((x, y)) = x$$
$$\pi_y : X \times Y \to Y \qquad \pi^y((x, y)) = y$$

Suppose $f : X \times Y \to Z$. For $x \in X$ fixed, we set $f_x : Y \to Z$, $f_x(Y) = f(x, y)$. For $y \in Y$ fixed, $f^y : X \to Z$, $f^y(x) = f(x, y)$ (compare with Figure 22).

$\mathcal{A} \times \mathcal{B} \subseteq P(X \times Y)$. $\mathcal{A} \times \mathcal{B}$ is the σ-algebra generated by all measurable rectangles $\mathcal{A} \times \mathcal{B}$, $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ is a π-system.

Lemma 4.8. *Let X, Y be sets, $\mathcal{A}$ is a σ-algebra on X, $\mathcal{B}$ is a σ-algebra on Y, $\mathcal{A} \times \mathcal{B}$ as above. Then*

1. *For any $E \subset \mathcal{A} \times \mathcal{B}$, any $x \in X$ and any $y \in Y$, we have $E_x \in \mathcal{B}$, $E^y \in \mathcal{A}$.*
2. *Let $f : X \times Y \to \overline{\mathbb{R}}$ be measurable with respect to $\mathcal{A} \times \mathcal{B}$. Then for any $x \in X$ and $y \in Y$ the function f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable.*

Proof. 1. Let $\mathcal{F} = \{E \subset X \times Y \mid E_x \in \mathcal{B} \forall x \in X\}$. Show $\mathcal{A} \times \mathcal{B} \subseteq \mathcal{F}$ by $C \subseteq \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra. $\mathcal{F}$ is a σ-algebra, because

(a) $\emptyset \in \mathcal{F}$ because $\emptyset_x = \emptyset \in \mathcal{B} \forall x \in X$

(b) $X \times Y \in \mathcal{F}$ because $(X \times Y)_x = Y \in \mathcal{B} \forall x \in X$

Let $E \in \mathcal{F}$.

$$(E^C)_x = \left\{y \in Y \middle| (x, y) \notin E\right\} = \left\{y \in Y \middle| y \notin E_x\right\} = (E_x)^C \in \mathcal{B}$$

where $E_x \in \mathcal{B}$ because $E \in \mathcal{F}$. $(E_n)_{n \in \mathbb{N}}$ for $E_n \in \mathcal{F} \forall n \in \mathbb{N}$.

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_x = \left\{y \in Y \middle| \exists n \in \mathbb{N} : (x, y) \in E_n\right\} = \bigcup_{n=1}^{\infty} \left\{y \in Y \middle| (x, y) \in E_n\right\}$$

Therefore $\mathcal{F}$ is a σ-algebra (compare with Figure 23).

Show $C \subseteq \mathcal{F}$: Let $E = A \times B$. Then

$$E_x = \left\{ y \in Y \,\middle|\, (x, y) \in A \times b \right\} = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

so $A \times B \in \mathcal{F}$ and $C \subseteq \mathcal{F} \implies \mathcal{A} \times \mathcal{B} \subseteq \mathcal{F}$.

The same argument holds true for E^y.

2. Show $f_x : Y \to \overline{\mathbb{R}}$ is measurable with respect to $\mathcal{B}$. Let $\alpha \in \mathbb{R}$.

$$f_x^{-1}([\alpha, \infty]) = \left\{ y \in Y \,\middle|\, \underbrace{f_x(y)}_{=f(x,y)} \in [\alpha, \infty] \right\} = \left[f^{-1}([\alpha, \infty]) \right]_x \in \mathcal{B}$$

where $f^{-1}([\alpha, \infty]) \in \mathcal{A} \times \mathcal{B}$ because f is measurable with respect to $\mathcal{A} \times \mathcal{B}$.

The same argument holds true for f^y.

$\square$

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $(X, \mathcal{A}, \mu)$ is σ-finite if there exists $(A_n)_{n \in \mathbb{N}}, A_n \in \mathcal{A}, A_n \cap A_m = \emptyset$ for $n \neq m$, $\mu(A_n) < \infty$ and $\bigcup_{n=1}^{\infty} A_n = X$.

Set $C_n = \bigcup_{k=1}^{n} A_n$ then $\mu(C_n) = \sum_{k=1}^{n} \mu(A_k) < \infty$. $C_n \subseteq C_{n+1}, \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n = X$.

Proposition 4.11. *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be finite measure spaces. Then for any $E \in \mathcal{A} \times \mathcal{B}$ the map*

$$f_E : X \to [0, \infty] \qquad f_E(x) = \nu(E_x) \quad (x \mapsto \nu(E_x))$$

is $\mathcal{A}$-measurable. Likewise,

$$g_E : Y \to [0, \infty] \qquad g_E(y) = \mu(E^y)$$

is $\mathcal{B}$-measurable.

Proof. First case: $\nu(Y) < \infty$.

$$\mathcal{F} = \{E \subseteq X \times Y \mid x \to \nu(E_x) \text{ is } \mathcal{A}\text{-measurable}\}$$

Show: $C \subseteq \mathcal{F}$ and $\mathcal{F}$ is a d-system. Let $E = A \times B \in C$ and

$$x \mapsto \nu(E_x) = \begin{cases} \nu(B) & \text{for } x \in A \\ \nu(\emptyset) = 0 & \text{for } x \notin A \end{cases}$$

so $f_E = \nu(B)\chi_A \in \mathcal{A}$ where $\nu(B) < \infty$ because $\nu(Y) < \infty$ so $C \subseteq \mathcal{F}$.

d-system:

$$\nu(\emptyset_X) = 0 \forall x \in X \text{ and } x \mapsto 0 \text{ is measurable}$$

Let $F \subseteq E$ and $F, E \in \mathcal{F} : E \setminus F \in \mathcal{F}$. Show this membership.

$$\begin{aligned}
(E \setminus F)_X &= \left\{ y \in Y \,\middle|\, (x,y) \in E \text{ and } (x,y) \notin F \right\} \\
&= \left\{ y \in Y \,\middle|\, (x,y) \in E \right\} \setminus \left\{ y \in Y \,\middle|\, (x,y) \in F \right\} \\
&= E_x \setminus F_x \\
\nu((E \setminus F)_X) &= \nu(E_x \setminus F_x) = \underbrace{\nu(E_x)}_{<\infty} - \underbrace{\nu(F_x)}_{<\infty}
\end{aligned}$$

$x \mapsto \nu((E \setminus F)_x)$ is the difference of two measurable functions, hence it is measurable.

This lecture took place on 2018/01/17.

Revision: We have the following setting: $E \in \mathcal{A}\times\mathcal{B}$. $x \mapsto \nu(E_x)$ is $\mathcal{A}$-measurable. $X \to [0, \infty]$.

Assumption: $\nu(Y) < \infty$.

$$\mathcal{F} = \{E \subseteq X \times Y \mid x \to \nu(E_x) \text{ is measurable}\}$$

$$C = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\} \subseteq \mathcal{F}$$

$$\emptyset \in \mathcal{F}, E, F \in \mathcal{F} \text{ with } F \subseteq E \implies E \setminus F \in \mathcal{F}$$

Let $E_n \in F$, $E_n \subseteq E_{n+1}$. Show $\bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$.

$$\left(\bigcup_{n=1}^{\infty} E_n\right)_X = \left\{ y \in Y \,\middle|\, (x,y) \in E_n \text{ for one } n \in \mathbb{N} \right\} = \bigcup_{n=1}^{\infty} (E_n)_x$$

$$(E_n)_x \subseteq (E_{n+1})_x$$

$$\nu\left(\bigcup_{n+1}^{\infty} (E_n)_x\right) = \lim_{n\to\infty} \nu((E_n)_x)$$

$$x \mapsto \nu\left(\left(\bigcup_{n=1}^{\infty} E_n\right)_x\right) = \lim_{n\to\infty} \nu((E_n)_x)$$

measurable as limit of measurable function. So $\mathcal{F}$ is a d-system, $C \subseteq \mathcal{F}$, C is a π-system.

$$\xRightarrow{\text{by Theorem 4.9}} \mathcal{A} \times \mathcal{B} \subseteq \mathcal{F}$$

Case 2: Let $D_n \in \mathcal{B}$ with $\nu(D_n) < \infty$.

$D_n \cap D_m = \emptyset$ if $n \neq m$ and $Y = \bigcup_{n=1}^{\infty} D_n$ (i.e. Y is σ-finite). We define $\nu_n : \mathcal{B} \to [0, \infty)$.

$$\nu_n(B) = \nu(D_n \cap B)$$

ν_n is a measure on $\mathcal{B}$.

$$\nu_n(B) \leq \nu_n(Y) = \nu(D_n) < \infty$$

$$\forall B \in \mathcal{B} : \nu(B) = \nu\left(\underbrace{\left(\bigcup_{n=1}^{\infty} D_n\right)}_{=Y} \cap B\right) = \nu\left(\bigcup_{n=1}^{\infty} (D_n \cap B)\right) = \sum_{n=1}^{\infty} \nu\left(D_n \cap B\right) = \sum_{n=1}^{\infty} \nu_n(B)$$

$x \mapsto \nu_n(E_x)$ is measurable by part 1 for all $n \in \mathbb{N}$ and $E \in \mathcal{A} \times \mathcal{B}$. $x \mapsto \nu(E_x) = \sum_{n=1}^{\infty} \nu_n(E_x)$ is measurable as sum of measurable non-negative functions. □

Theorem 4.10. *Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be a σ-finite measure spaces. Then there exists a unique measure $\mu \times \nu$ on $\mathcal{A} \times \mathcal{B}$ such that $\forall A \times B \in C$ we have $(\mu \times \nu)(A \times B) = \mu(A) \cdot \mu(B)$. Moreover, we have*

$$(\mu \times \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E^y)\, d\nu \qquad \forall E \in \mathcal{A} \times \mathcal{B}$$

$\mu \times \nu$ is called the produce measure.

Compare with Figure 25.

Proof. $x \mapsto \nu(E_x)$ is non-negative, m-able $\forall E \in \mathcal{A} \times \mathcal{B}$. $y \mapsto \mu(E^y)$ is non-negative, m-able $\forall E \in \mathcal{A} \times \mathcal{B}$. So $\int_Y \nu(E_x)\, d\mu =: (\mu \times \nu)_1(E)$, $\int_Y \mu(E^Y)\, d\nu =: (\mu \times \nu)_2(E)$.

Check: $(\nu \times \nu)_1$ and $(\mu \times \nu)_2$ are both measures on $\mathcal{A} \times \mathcal{B}$ (exercise).

Case 1: $\mu(X) < \infty$ and $\nu(Y) < \infty \implies (\mu \times \nu)_1$ and $(\mu \times \nu)_2$ are finite. Let $E = A \times B \in C$. Then $(\mu \times \nu)_1(E) = \int_X \underbrace{\nu(E_x)}_{\nu(B)\chi_A}\, d\mu = \nu(B) \cdot \int_X \chi_A\, d\mu = \nu(B)\mu(A)$ and $(\mu \times \nu)_2(E) = \int_Y \mu(E^y)\, d\mu = \int_y \mu(A)\chi_B\, d\nu = \mu(A)\nu(B)$. So $\forall E \in C : (\mu \times \nu)_1(E) = (\mu \times \nu)_2(E)$.

$$F = \{E \in \mathcal{A} \times \mathcal{B} \mid (\mu \times \nu)_1(E) = (\mu \times \nu)_2(E)\}$$

Show that $\mathcal{F}$ is a d-systemm, $\emptyset \in \mathcal{F}$.

$$(\mu \times \nu)_1(F) = (\mu \times \nu)_2(F)$$

Let $E, F \in \mathcal{F}, F \subseteq E$.

$$\begin{aligned}(\mu \times \nu)_1(F) + (\mu \times \nu)_1(E \setminus F) &= \\ (\mu \times \nu)_1(E) = (\mu \times \nu)_2(E) &= \\ (\mu \times \nu)_2(F) + (\mu \times \nu)_2(E \setminus F)\end{aligned}$$

We get $(\mu \times \nu)_1(E \setminus F) = (\mu \times \nu)_2(E \setminus F)$. Let $E_n \in \mathcal{F}, E_n \subseteq E_{n+1}, (\mu \times \nu)_1(E_n) = \lim_{n\to\infty}(\mu\times\nu)_1(E_n) = \lim_{n\to\infty}(\mu\times\nu)_2(E_n) = (\mu\times\nu)_2(\bigcup_{n=1}^{\infty} E_n)$. So $\mathcal{F}$ is a d-system.

By Theorem 4.9, $\mathcal{A}\times\mathcal{B} \subseteq \mathcal{F} \subseteq \mathcal{A}\times\mathcal{B}$. So $\forall E \in \mathcal{A}\times\mathcal{B} : (\mu\times\nu_1)(E) = (\mu\times\nu)_2(E)$. So also any measure κ on $\mathcal{A}\times\mathcal{B}$ which coincides with $(\mu\times\nu)_1$ on C also coincides with $(\mu\times\nu)_1$ on $\mathcal{A}\times\mathcal{B} \implies$ uniqueness.

Case 2: σ-finite mesaures:

$$A_n \in \mathcal{A} \qquad X = \bigcup_{n=1}^{\infty} A_n \qquad \mu(A_n) < \infty \qquad A_n \subseteq A_{n+1}$$

$B_n \in \mathcal{B}$, $Y = \bigcup_{n=1}^{\infty} B_n$, $\nu(B_n) < \infty$, $C_n = A_n \times B_n, (\mu\times\nu)_1(C_n) = \mu(A_n)\nu(B_n) = (\mu\times\nu)_2(C_n) < \infty$. $(\mu\times\nu)_1(E) = (\mu\times\nu)_1(E \cap \bigcup_{n=1}^{\infty} C_n) = \lim_{n\to\infty}(\mu\times\nu)_1(E\cap C_n)$. $= \lim_{n\to\infty}(\mu\times\nu)_2(E\cap C_n)$ by part 1, $= (\mu\times\nu)_2(E)$. ◻

Leonida Tonelli, 1885–1946

Theorem 4.11 (Tonelli's Theorem)**.** *Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to [0, \infty]$ be $\mathcal{A}\times\mathcal{B}$ measurable. Then,*

1. *$x \mapsto \int_y f_x \, dy$ is μ-measurable. $y \mapsto \int_x f^y \, d\mu$ is $\mathcal{B}$-measurable.*
2. *$\int_X \left(\int_y f_x \, d\nu\right) d\mu = \int_Y \left(\int_X f^y \, d\mu\right) d\nu = \int_{X\times Y} f \, d(\mu\times\nu)$.*

Remark 4.12. *$f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.*

$$\int_X \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x)\, d\nu(y)$$

Proof. We know $y \mapsto f_x(y)$ is non-negative and ν-measurable so $\int_y f_x \, d\nu$ exists. Likewise: $\int_X f^y \, d\mu$ exists $\forall y \in Y$. We start with $f = \chi_E$ with $E \in \mathcal{A}\times\mathcal{B}$. Then,

$$(\chi_E)_x(y) = \begin{cases} 1 & \underbrace{(x,y)}_{\Leftrightarrow y \in E_X} \in E \\ 0 & \text{else} \end{cases}$$

and this equals $\chi_{E_x}(y)$, so $\int_Y (\chi_E)_X \, d\nu = \int_Y \chi_{E_x} \, d\nu = \nu(E_x)$. Therefore,

$$\int_X \int_Y (\chi_E)_x \, d\nu \, d\mu = \int_Y \nu(E_x) \, d\mu$$

Analogously:

$$\int_Y \int_X (\chi_E)^Y \, d\mu \, d\nu = \int_Y \mu(E^y) \, d\nu$$

but $\int_Y \nu(E_x)\, d\mu = \int_Y \mu(E^y)\, d\nu$ by Theorem 4.10.

$$\overset{\text{by def.}}{=} (\mu \times \nu)(E) = \int_{X\times Y} \chi_E d(\mu \times \nu)$$

So the two statement of this theorem hold for all characteristic functions χ_E for $E \in \mathcal{A} \times \mathcal{B}$. This implies that both statements hold for linear combinations of characteristic functions, i.e. for simple functions.

Let f be $\mathcal{A}\times\mathcal{B}$ be measurable, $f : X\times Y \to [0, \infty]$. Then $f = \lim_{n\to\infty} S_n$ pointwise $s_n \leq s_{n+1}$.

$$\int_{X\times Y} f d(\mu \times \nu) \overset{\substack{\text{by monotone}\\ \text{convergence}\\ \text{theorem}}}{=} \lim_{n\to\infty} s_n d(\mu \times \nu) = \lim_{n\to\infty} \underbrace{\int_X \int_Y \underbrace{(S_n)_x}_{\text{measurable}} \, d\nu \; d\mu}_{\text{measurable}}$$

$$\overset{\substack{\text{by monotone}\\ \text{convergence}\\ \text{theorem}}}{=} \int_X \int_Y \underbrace{\lim_{n\to\infty} (S_n)_X}_{=f_x} \, d\nu\, d\mu = \int_X \int_Y f_x \, d\nu\, d\mu$$

Then we can do a symmetric argument:

$$\int_{X\times Y} f\, d(\mu \times \nu) = \int_Y \int_X f^Y \, d\mu\, d\nu$$

□

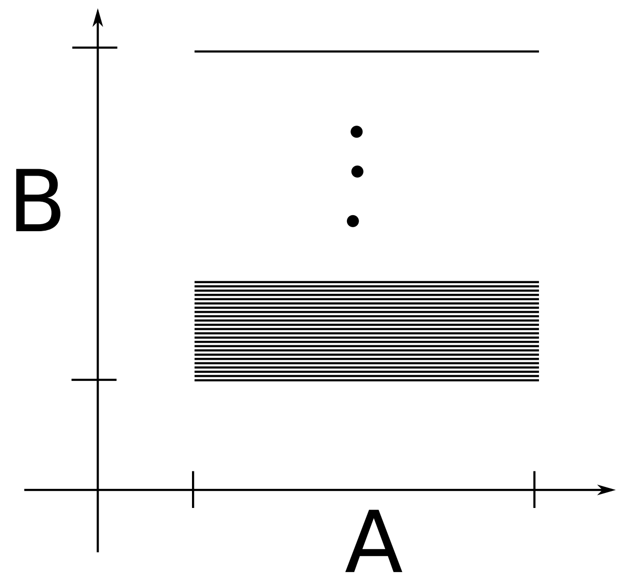

Figure 20: Area as the cartesian product of lines

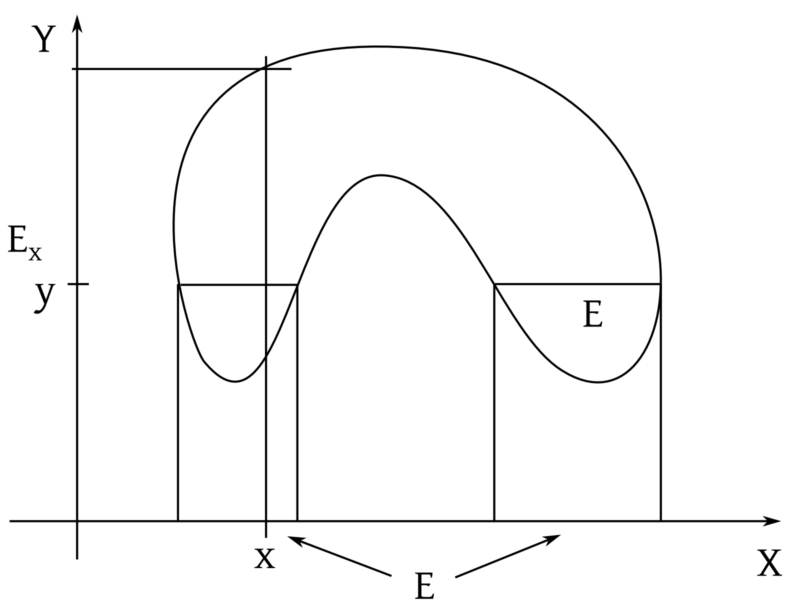

Figure 21: *E*

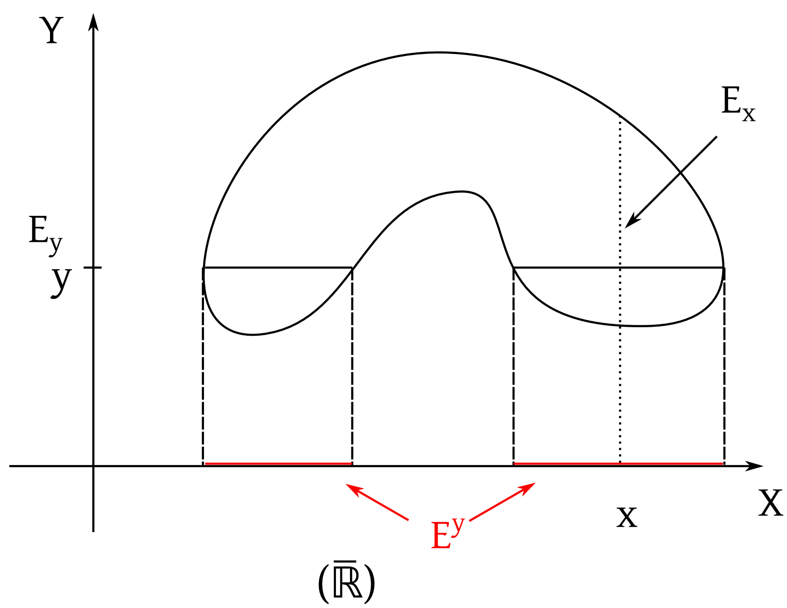

Figure 22: E_x, E^x and E^y

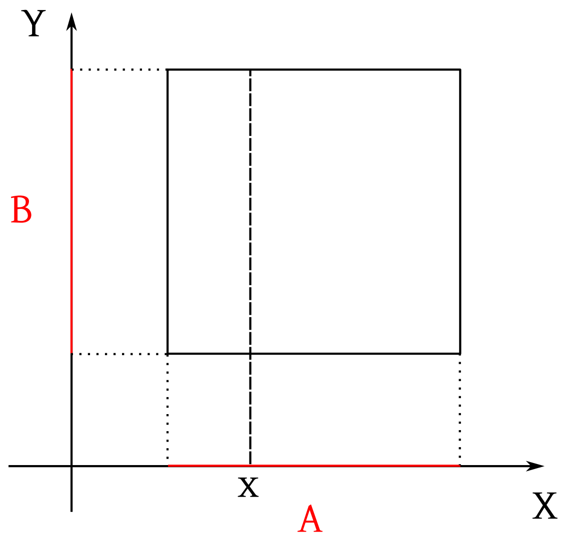

Figure 23: Rectangle $A \times B$

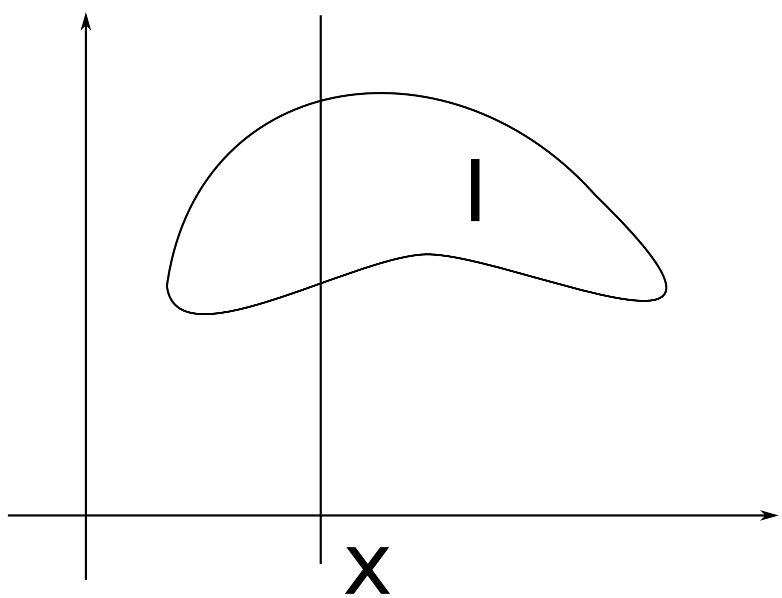

Figure 24: *I*

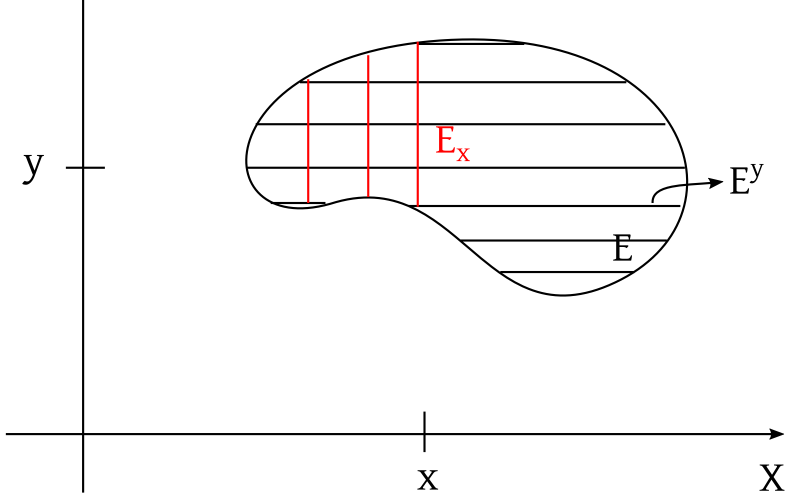

Figure 25: Setting for Theorem 4.10

Index