

# Introduction to Functional Analysis

Lecture notes, University of Technology, Graz  
based on the lecture by Martin Holler

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## Contents

<b>0</b>	<b>Introduction</b>	<b>1</b>
0.1	Application examples . . . . .	2
0.2	Our first problem . . . . .	2
<b>1</b>	<b>Topological and metric spaces</b>	<b>7</b>

## 0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g.  $\mathbb{R}^n, \mathbb{C}^n \mapsto$  vector spaces  $U, V$   
matrices  $A \in \mathcal{M}^{n \times n} \mapsto$  operators  $A \in \mathcal{L}(U, V)$   
functions  $f : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto$  functionals  $f : U \rightarrow \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application

- basis of Applied Mathematics
- partial differential equations
- physical modelling
- inverse problems (operator  $A$  models some physical measurement process)
- Optimization and optimal control

A motivating example was presented with slides.

## 0.1 Application examples

Let  $K : U \rightarrow \mathbb{R}^m$  with  $U$  as vector space describe a physical model. For example,  $K$  is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or  $Ku = y(1)$  where  $y : [0, 1] \rightarrow \mathbb{R}^m$  solves  $y'(t) = y(t) + u(t)$  and  $y(0) = 0$ .

Another example is the class of so-called *inverse problems*. Given  $d = ku$ , find  $u$ . Typically inversion of  $K$  is ill-constrained. Solution is typically non-unique.

Approach: Solve  $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$  where  $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$  and  $\|\cdot\|_u$  is a norm on  $U$ . Or alternatively, let  $U = C^1([0, 1]^2)$  and solve  $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$ .

Other examples are JPEG compression and upsampling of images.

## 0.2 Our first problem

Let  $U := C^1([0, 1]^2)$  be a normed space,  $K : U \rightarrow \mathbb{R}^m$  linear. Solve  $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$ . The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let  $U : \mathbb{R}^n$  and  $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$  is a discrete gradient. In 1D, we have  $u = (u_i)_i \in \mathbb{R}^m$  and  $u_i = u(x_i) \Rightarrow u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$ . Consider  $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$  as problem.

Does there exist a solution to this problem assuming  $\lambda > 0$ ,  $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear and  $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$  linear.

*Proof. Case 1 (trivial model)* Let  $m = n$ .  $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \quad (1)$$

Take  $(u_n)_{n \in \mathbb{N}}$  in  $\mathbb{R}^n$  such that  $\lim_{n \rightarrow \infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$ . It holds that  $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$ . Without loss of generality, we can assume that  $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \forall n \in \mathbb{N}$

$$\Rightarrow \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$  is bounded. So the Bolzano-Weierstrass theorem applies and  $(u_n)_{n \in \mathbb{N}}$  admits a convergent subsequence  $(u_{n_i})_{i \in \mathbb{N}}$ . Take  $u \in \mathbb{R}^n$ .  $u_{n_i} \rightarrow u$  as  $i \rightarrow \infty$ .

Now: Show that  $u$  solves Problem (1).  $\nabla$  is continuous.  $\|\cdot\|_2$  is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \rightarrow \infty} \|\nabla u_{n_i}\|_2 + \lambda \|u_{n_i} - d\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that  $\hat{u}$  is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of  $\nabla, \|\cdot\|_2$

**Case 2 ( $K$  arbitrary)** 1.  $K$  arbitrary does not provide boundedness anymore. Define  $X := \text{kernel}(\nabla) \cap \text{kernel}(k)$  and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^n x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \quad \text{i.e. } \forall u \in \mathbb{R}^n : \exists ! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that  $X^\perp$  is called *orthogonal complement*.

**Claim 0.1.** *If  $\hat{u}$  solves  $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ . Then  $\hat{u}$  solves Problem (1).*

*Proof.* Let  $\hat{u}$  be a solution on  $X^\perp$ . Take  $u \in \mathbb{R}^n$  arbitrary. We write  $u = u_1 + u_2 \in X \times X^\perp$ . Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \|\nabla(u_1 + u_2)\|_2 + \lambda \|k(u_1 + u_2) - d\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus  $\hat{u}$  solves our problem (1). □

Take again  $(u_n)_{n \in \mathbb{N}}$  be such that  $u_n \in X^\perp \forall n$  and

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|ku - d\|_2$$

Write  $u_1 = u_n^1 + u_n^2 \in \ker(\nabla) + \ker(\nabla)^\perp$ .  $\nabla : \ker(\nabla)^\perp \rightarrow \text{image}(\nabla)$  is bijective. Since  $\nabla v = 0$  for  $v \in \ker(\nabla)^\perp \implies v \in \ker(\nabla) \implies \|v\| = (v, v) = 0$ . Thus,  $\nabla^{-1} : \text{image}(\nabla) \rightarrow \ker(\nabla)^\perp$  exists and is continuous.

$$\begin{aligned} \implies \|u_n^2\|_2 &= \|\nabla^{-1} \nabla u_n^2\|_2 = \|\nabla^{-1}\| \cdot \|\nabla u_n^2\|_2 \leq \|\nabla^{-1}\| \\ &\leq \|\nabla^{-1}\| \left( \|\nabla u_n^2\|_2 + \lambda \|Ku_n - d\|_2 \right) \\ &= \|\nabla^{-1}\| \left( \underbrace{\|\nabla u_n\|_2}_{=\|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\ &< C \text{ for some } C > 0 \end{aligned}$$

Then  $\|u_n^2\|_2$  bounded.

2. Show  $(u_n^1)_n$  is bounded.  $K : X^\perp \cap \ker(\nabla) \rightarrow \text{image}(K)$  is bijective. Since  $Kv = 0$  for  $v \in X^\perp \cap \ker(\nabla) \implies v \in \ker(K)$ . Hence  $v \in \ker(K) \cap \ker(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$ . Hence  $K^{-1} : \text{image}(K) \rightarrow X^\perp \cap \ker(\nabla)$  exists and is continuous.

$$\begin{aligned} \implies \|u_n^1\|_2 &= \|K^{-1} Ku_n^1\|_2 \leq \|K^{-1}\| \|Ku_n^1\|_2 \\ &= \frac{\|K\|}{\lambda} \left( \lambda \|K(u_n^1 + u_n^2) - Ku_n^1\|_2 + \|\nabla u_n\|_2 \right) \\ &\leq \frac{\|K\|}{\lambda} \left( \underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \|d - Ku_1^2\|_2}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\ &< D \text{ for some } D > 0 \end{aligned}$$

$$\implies (u_n^1)_n \text{ bounded} \implies (u_n) = (u_n^1 + u_n^2)_n \text{ is bounded}$$

$\implies (u_n)_n$  admits a subsequence converging to some  $\hat{u}$ . As in Case 1,  $\hat{u}$  is a solution to Problem (1).

In summary,

1.  $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 dx}$  with  $U = C^1([0,1]^2)$  relevant for application.
2. Discrete version:  $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$ . We have shown existence by using:
  - (a) complementary subspaces  $X^\perp$
  - (b) boundedness and compactness
  - (c) continuity

- (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

*About the existence of infinitely many dimensions*

↓ This lecture took place on 2019/03/07.

Define  $U = C^1([0, 1]^2)$ . Let  $Y$  is some Banach space and  $K : U \rightarrow Y$  is linear and continuous.

Consider the problem  $(P_\infty)$  given by  $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$  where  $d \in Y$  and  $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2}$ .

**Proposition 0.2.** *There exists a solution of  $(P_\infty)$ .*

*Proof.* Take  $(u_n)_{n \in \mathbb{N}}$  as a sequence in  $U$  such that  $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_Y \rightarrow \inf_{u \in U} (\dots)$ . Now we want to show that  $(u_n)_{n \in \mathbb{N}}$  is bounded.

**Case 1** Assume that  $Ku = u$ ,  $Y = U$  and  $\|\cdot\|_Y = \|\cdot\|_2$ .

$$\Rightarrow \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\Rightarrow (u_n)_{n \in \mathbb{N}} \text{ is bounded}$$

So does  $(u_n)_{n \in \mathbb{N}}$  admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change  $U$  to  $U = \left\{ u : [0, 1]^2 \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]^2} |\nabla u|^2} < \infty \right\}$ . Define, instead of  $\|\nabla u\|_2$ ,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider  $\min_{u \in U} R(u) + \lambda \|Ku - d\|_2$  instead.

In this setting,  $(u_n)_{n \in \mathbb{N}}$  admits a weakly convergent subsequence converging to some  $\hat{u} \in U$  (denoted by  $(u_{n_i})_{i \in \mathbb{N}}$ ).

Our next step is to use continuity to show that  $\hat{u}$  is a solution.

Problem:  $u \mapsto \|u - d\|_2$  is, in general, not continuous with respect to weak convergence.

*But* it is always true that  $\|\hat{u} - d\|_2 \leq \liminf_{i \rightarrow \infty} \|u_{n_i} - d\|_2$ . Yes. We consider that as first property.

Is it also true that  $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$ ? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = - \int_0^1 u \partial_x \varphi \quad \forall \varphi \in C^\infty([0, 1]^2)$$

$\varphi = 0$  in  $K \setminus [0, 1]^2$  for some  $K \Subset (0, 1)^2$ .

$$\begin{aligned} \Rightarrow \int_{[0,1]^2} \nabla u \varphi &= - \int_{[0,1]^2} u \cdot (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \\ \forall \varphi : (\varphi_1, \varphi_2) &= C^\infty([0, 1]^2, \mathbb{R}^2) + \text{zero on boundary} \end{aligned}$$

We define  $w : [0, 1]^2 \rightarrow \mathbb{R}^2$  is called *weak derivative* of  $u \in U$ .

$$\Leftrightarrow \int_{[0,1]^2} w \varphi = - \int_{[0,1]^2} u (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \text{ holds } \forall \varphi$$

Then  $w$  is called *weak gradient* of  $u$ . We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then  $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$ . We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i \rightarrow \infty} R(u_{n_i}) + \liminf_{i \rightarrow \infty} \lambda \|u_{n_i} - d\|_2 \\ &\leq \liminf_{i \rightarrow \infty} (R(u_{n_i}) + \lambda \|u_{n_i} - d\|_2) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

**Case 2** Works as in the finite-dimensional setting using

- $X := \ker(A) \cap \ker(\nabla) \Rightarrow U = X \oplus X^\perp$  requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \quad \forall u \in \ker(\nabla)^\perp$  is called *Poincare inequality*.

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normal spaces

3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contemplimentary subspaces
8. Hilbert spaces

↓ This lecture took place on 2019/03/12.

**Remark.** 1. Literature: UGU, in particular: Biezis, Werner  
 2. In this lecture: always  $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$  if not further specified

## 1 Topological and metric spaces

**Remark** (Motivation). Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies

**Definition 1.1** (Topology). Let  $X$  be a set and  $\tau \subset \mathcal{P}(X) = \{\text{"set of subsets of } X\}$ . We say that  $\tau$  is a topology on  $X$  if

1.  $X, \emptyset \in \tau$
2.  $U, V \in \tau \implies U \cap V \in \tau$
3. For any collection of sets  $(U_i)_{i \in I}$  with  $I$  as some index set. We have  $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$ .

$(X, \tau)$  is called topological space.

A set  $U \subset X$  is called open if  $U \in \tau$  and is called closed if  $U^c \in \tau$ .

**Remark.** By the third property of topologies,  $\bigcap_{i \in I} V_i$  is closed for any collection  $(V_i)_{i \in I}$  of closed sets.

**Definition 1.2** (Metric). Let  $X$  be a set,  $d : X \times X \rightarrow \mathbb{R}$  be such that  $\forall x, y, z \in X$

1.  $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z)$

Then  $d$  is called a metric on  $X$  and  $(X, d)$  is called metric space.

**Definition 1.3** (Norm). Let  $X$  be a vector space. A function  $\|\cdot\| : X \rightarrow \mathbb{R}$  is called norm if  $\forall x, y \in X, \lambda \in \mathbb{K}$

1.  $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2.  $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3.  $\|x + y\| \leq \|x\| + \|y\|$

Then  $(X, \|\cdot\|)$  is called normed space.

**Remark.** If  $\dim(x) < \infty$ , all norms on  $X$  are equivalent.

**Example.** 1. Let  $X$  be a set then  $\tau = \{\emptyset, X\}$  is a topology.

2.  $(X, \mathcal{P}(X))$  is a topological space.

3. Define  $S^{d-1} := \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\}$  and  $d(x, y) := r$  where  $r$  is the length of the shortest connection between  $x$  and  $y$  on  $S^{d-1}$ . Then  $d$  is a metric on  $S^{d-1}$

4.  $X := \{u : [0, 1] \rightarrow \mathbb{R} \mid u \text{ is continuous}\}$  then  $\|u\|_\infty := \sup_{x \in [0, 1]} |u(x)|$  is a norm on  $X$

5.  $l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u \text{ and } \sum_{i=1}^\infty |x_i|^p < \infty\}$  with  $p \in [1, \infty)$  and  $\|(x_i)_{i \in \mathbb{N}}\|_p := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$ . Then  $(l^p, \|\cdot\|_p)$  is a normed space (the proof will be done later).

**Remark.**

$$l^\infty := \left\{ (X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|(X_i)_{i \in \mathbb{N}}\| = \sup_i |X_i|$$

**Proposition 1.4.** Let  $X$  be a set.

1. If  $(X, d)$  is a metric space, define for  $\varepsilon > 0, x \in X$ .  $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$  and  $\tau = \{U \in \mathcal{P}(X) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq U\}$ . Then  $(X, \tau)$  is a topological space. We say that  $\tau$  is the topology induced by  $d$  and we have that  $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$
2. If  $(X, \|\cdot\|)$  is a normed space, define  $d : X \times X \rightarrow \mathbb{R}$  with  $(x, y) \mapsto \|x - y\|$ . Then  $(X, d)$  is a metric space and  $d$  is called the metric induced by  $\|\cdot\|$ .

**Remark** (Consequence). Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.

**Definition 1.5.** Let  $(X, \tau)$  be a topological space.  $U \subset X$ .  $x \in X$ .



1.  $U$  is called a neighborhood of  $x$  if  $\exists V \in \tau - x \in X \subset U : \mathcal{U}(x)$  is defined as the set of all neighborhoods of  $x$
2.
  - $x$  is called interior point of  $U$  if  $U \in \mathcal{U}$
  - $x$  is called adjacent point of  $U$  if  $\forall V \in \tau$  such that  $x \in V : V \cap U \neq \emptyset$
  - $x$  is called cluster point of  $U$  if it is an adjacent point of  $U \setminus \{x\}$ .

The third property is stronger.

3. Notational conventions:

$$\mathring{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \mathring{U}$$

**Proposition 1.6.** Let  $(X, \tau)$  be a topological space,  $U \in X$ . Then

1.  $U$  is open  $\iff \mathring{U} = U$
2.  $U$  is closed  $\iff \overline{U} = U$
3.  $\mathring{U} = \bigcup_{V \in \tau, V \subset U} V$  and  $\mathring{U}$  is open [" $\mathring{U}$  is the largest open set in  $U$ "]
4.  $\overline{U} = \bigcap_{U \subset V, V \text{ closed}} V$  and  $\overline{U}$  is closed [" $\overline{U}$  is the smallest closed set containing  $U$ "]

*Proof.* 3.  $\subset$  Let  $x \in \mathring{U} \implies \exists \hat{V} \in \tau$  s.t.  $x \in \hat{V} \subset U \implies x \in \bigcup_{V \in \tau, V \subset U} V$

$\supset$  Let  $x \in \bigcup_{V \in \tau, V \subset U} V \implies x \in \hat{V}$  for some  $\hat{V} \in \tau, \hat{V} \subset U \implies x \in \mathring{U}$

$\mathring{U}$  is open because it is the union of open sets.

1.  $\implies \mathring{U} \subset U$  by definition.  $U$  is open, so  $U \subset \bigcup_{V \in \tau, V \subset U} V \stackrel{(3)}{=} \mathring{U}$
2.  $\implies V \subset \overline{U}$  by definition. Take  $x_0 \in \overline{U}$ . If  $x \notin U \implies x \in U^C \in \tau$  and  $U \cap U^C = \emptyset$ . This contradicts to  $x \in \overline{U}$ .  
 $\Leftarrow$  Take  $x \in U^C = \overline{U}^C$ .  
 $\stackrel{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$   
 $\implies V \cap U = \emptyset \implies V \subset U^C$   
 $\implies U^C$  open  $\implies U$  closed

4. We prove the fourth property without the second.

$\subset$  Take  $x \in \overline{U}$ . Take closed  $V$  such that  $U \subset V$  if  $x \notin V \implies x \in V^C$  which is open and  $V^C \cap U = \emptyset$ . This contradicts to  $x \in \overline{U}$ .

$\supset$  Take  $x \in \bigcap_{U \subset V} V^{\text{closed}}$ . Suppose  $x \notin \bar{U}$ .  
 $\implies \exists Z$  open such that  $x \in Z$  and  $Z \cap U = \emptyset$   
 $\implies U \subset Z^C$ ,  $Z^C$  closed,  $x \notin Z^C$ . This contradicts to  $x \in \bigcap_{U \subset V} V^{\text{closed}}$   
 $\bar{U}$  closed follows since the intersection of closed sets is closed.

□

**Definition 1.7** (Limit). Let  $(X, \tau)$  be a topological space,  $(X_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Henceforth, we write  $(X_n)_n$  for  $(X_n)_{n \in \mathbb{N}}$  and  $\hat{x} \in X$ . We say  $x_n \rightarrow x$  in  $\tau$  as  $n \rightarrow \infty$  (“ $x_n$  converges to  $x$ ”, “ $x$  is limit of  $x_n$ ”) if

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

**Definition 1.8** (Proposition and definition). Let  $(X, \tau)$  be a topological space. We say that  $(X, \tau)$  is  $T_2$  (or Hausdorff) if

$$\forall x, y \in X \text{ with } x \neq y \exists U, V \in \tau : x \in U, y \in V \text{ and } U \cap V = \emptyset$$

- In a  $T_2$ -sphere, the limit of any sequence is unique.
- If  $\tau$  is induced by a metric, then  $(X, \tau)$  is  $T_2$ .

*Proof.* 1. Take  $(x_n)_n$  to be a sequence and assume  $x_n$  converges to  $\hat{x}$  and  $\hat{y}$  with  $\hat{x} \neq \hat{y}$ . By  $T_2$ ,  $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$ . By convergence,  $\exists n_x, n_y$  such that  $\forall n \geq n_x : x_n \in U$  and  $\forall n \geq n_y : x_n \in V$ .

$$\forall n \geq \max\{n_x, n_y\} : x_n \in U \cap V$$

This gives a contradiction.

2. Take  $x, y \in X : x \neq y$ . Define  $\varepsilon := d(x, y)$  and consider  $B_{\frac{\varepsilon}{2}}(x)$  and  $B_{\frac{\varepsilon}{2}}(y)$  which are open in the induced topology  $\tau$ . Also  $x \in B_{\frac{\varepsilon}{2}}(x)$  and  $y \in B_{\frac{\varepsilon}{2}}(y)$ . Assume that  $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$ .

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

**Definition 1.9.** Let  $(X, \tau)$  be a topological space,  $U \subset V \subset X$ . We say that  $U$  is dense in  $V$ , if  $V \subset \bar{U}$ . We say that  $X$  is separable if there exists a countable, dense subset.

**Definition 1.10.** Let  $(X, \tau_X), (Y, \tau_Y)$  be topological spaces and  $f : X \rightarrow Y$  a function. We say  $f$  is continuous at  $x \in X$  if  $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$ .  $f$  is called continuous if it is continuous at any  $x \in X$ .

**Proposition 1.11.** With  $(X, \tau_X), (Y, \tau_Y)$  and  $f$  as above,  $f$  is continuous  $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$

*Proof.* Left as an exercise to the reader.  $\square$

**Definition 1.12.** Let  $(X, \tau)$  be a  $T_2$  topological space,  $M \subset X$  called compact if for any family  $(U_i)_{i \in I}$  with  $U_i \in \tau$  s.t.  $M \subset \bigcup_{i \in I} U_i$  (" $(U_i)_{i \in I}$  is an open covering of  $M$ "), there exists  $U_{i_1}, \dots, U_{i_n}$  such that  $M \subset \bigcup_{k=1}^n U_{i_k}$  ("there exists a finite subcover").

**Remark.** Compactness can also be defined without  $T_2$ , this is also referred to as quasi-compact.

**Remark** (Exercise). Reconsider the previous results for metric and normed spaces.

$\downarrow$  This lecture took place on 2019/03/14.

**Definition 1.13.** Let  $(X, d)$  be a metric space,  $V \subset X$  and  $(x_n)_n$  a sequence in  $X$ . Then we say,

1.  $V$  is bounded if  $\exists x \in X, r > 0$  such that  $U \in B_r(x)$
2.  $(x_n)_n$  is a Cauchy sequence if  $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$  such that  $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$
3.  $X$  is complete if any Cauchy sequence in  $X$  admits a limit point
4.  $X$  is a Banach space if it is a normed space and complete

**Proposition 1.14.** Let  $(X, d)$  be a metric space.  $(x_n)_n$  be a sequence in  $X$ . Then

1.  $x_n \rightarrow x$  in the induced topology  $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$
2. If  $x_n \rightarrow x$ , then  $(x_n)_n$  is bounded as subset of  $X$  and  $(x_n)_n$  is Cauchy.
3. If  $U \subset X$  is closed and  $X$  is complete. Then  $(U, d)$  is a complete metric space.

*Proof.* 1. We prove both directions:

$\implies$  True, since  $B_\varepsilon(x)$  is open  $\forall \varepsilon > 0$

$\impliedby$  Let  $x \in V$  with  $V$  open. Show that  $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$

$V$  open, then  $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$

$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get  $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$ . Let  $r := \max_{i=1, \dots, n_0} d(x, x_i) + 1$ . Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take  $(y_n)_n$  to be a Cauchy sequence in  $U$ , then  $(y_n)_n$  is a Cauchy sequence in  $X \implies \exists x \in X : y_n \rightarrow x$  as  $n \rightarrow \infty$  if  $x \notin U \implies x \in U^c \implies \exists n_0 \in \mathbb{N}$  such that  $y_{n_0} \in U^c$  due to  $U^c$  open. This is a contradiction to  $(y_n)_n$  in  $U$

□

**Proposition 1.15.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.  $f : X \rightarrow Y$ . The following are equivalent (TFAE):*

- $f$  is continuous (with respect to the induced topology)
- $\forall (x_n)_n$  such that  $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$

*Proof.* Firstly, we prove that the first statement implies the second statement.

Take  $(x_n)_n$  converging to  $x$ . Take  $V \in \tau_Y$  such that  $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$$\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X \text{ such that } x \in \hat{U} \subset U$$

$$\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_n) \rightarrow f(x)$$

**Remark.** 1.  $\implies$  2. holds true in any topological space

2.  $\implies$  1. Not.

Secondly, we prove that the second statement implies the first statement.

Suppose  $f$  is not continuous, find  $x_n \rightarrow x$  such that  $f(x_n) \rightarrow f(x)$  is wrong. If  $f$  is not continuous, then  $\exists x \in X : \exists V \in \mathcal{U}(f(x))$  such that  $f(x_n) \notin V \forall n \in \mathbb{N}$

$$\implies \exists \hat{V} \in \tau_Y \text{ such that } f(x_n) \notin \hat{V} \forall n \in \mathbb{N}, f(x) \in \hat{V}$$

$$\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\implies (x_n)_n$  converges to  $x$  but  $f(x_n) \notin \hat{V} \implies f(x_n) \not\rightarrow f(x)$ . This gives a contradiction. □

**Definition 1.16.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Let  $f : X \rightarrow Y$ .*

*$f$  is uniformly continuous iff*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

**Proposition 1.17.** Let  $(X, d_X), (Y, d_Y)$  be metric spaces.  $M \subset X$ ,  $f : M \rightarrow Y$ . If  $M$  is dense in  $X$ ,  $Y$  is complete and  $f$  is uniformly continuous.

$$\implies \exists! \hat{f} : X \rightarrow Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

*Proof.* Take  $x \in X$ . By the practicals (and since  $\overline{M} = X$ ),  $\exists (x_n)_n$  such that  $x_n \rightarrow x$  and  $x_n \in M$ .

We show:  $(f(x_n))_n$  is Cauchy. Take  $\varepsilon > 0 \implies \exists \delta > 0$  such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

Now,  $(x_n)_n$  is Cauchy (why?)  $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(x_n), f(x_m)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe:  $\forall \hat{x} \in X$ , there exists  $(\hat{x}_n)_n$  in  $M$ ,  $\hat{y} \in Y$  such that  $f(\hat{x}_n) \rightarrow \hat{y}$ .

Now: for any  $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$  with  $x \in X, (x_n)_n$  is a sequence in  $M$  such that  $x_n \rightarrow x, f(x_n) \rightarrow y$ . Now if  $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$ :

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d_Y(f(x_n), y) < 3\varepsilon$$

1. If  $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$  is well-defined.

2.  $\hat{f}$  is uniformly continuous.

□

↓ This lecture took place on 2019/03/19.

**Proposition 1.18.** Let  $(X, d)$  be a metric space,  $M \subset X$ .

1.  $M$  is compact, so  $\forall (X_i)_{i \in I}$  with  $X_i$  a closed set  $\forall i$  such that  $\bigcap_{i \in I} X_i \cap M = \emptyset$ .

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \bigcap_{j=1}^n X_{i_j} \cap M = \emptyset$$

2.  $M$  is compact, so  $M$  is closed and bounded.

*Proof.* 1. We note that  $\{X_i\}_{i \in I}$  is a family of closed sets.  $(X_i^C)_{i \in I}$  is a family of open sets and  $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

**Proposition 1.19.** *Let  $(X, d)$  be a metric space,  $M \subset X$ . TFAE:*

1.  $M$  is compact.
2. Every infinite subset of  $M$  admits a cluster point.
3. Every sequence of  $M$  admits a convergent subsequence.
4.  $M$  is complete and totally bounded, where totally bounded is defined as

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$$

**Remark.** 1. totally bounded  $\implies$  bounded (proof is left as an exercise)

2. If  $\dim(x) < \infty$ , then compact  $\iff$  complete and bounded (see course Analysis I)

3.  $\dim(x) < \infty \iff \overline{B_1(0)}$  is compact

where the last two items imply that  $X$  is a normed space.

*Proof.* 1  $\rightarrow$  2 If  $M$  is finite, (2) always holds true. So assume that  $M$  is infinite.

Now assume that (2) does not hold. Then there is  $C \subset M$  infinite which does not admit a cluster point.  $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$  contains at most one element of  $C]$ . If not,  $\exists x \in C$  such that  $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$  such that  $(x_n)_n$  is a sequence of distinct points and  $x_n \rightarrow x$ . This implies that  $x$  is a cluster point of  $C$ . This gives a contradiction.

Now  $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$ . If  $M$  is compact, then

$$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \subset M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \text{ is finite}$$

This is a contradiction.

2  $\rightarrow$  3 Let  $(x_n)_n$  be a sequence in  $M$ .

**Case 1**  $\{x_n \mid n \in \mathbb{N}\}$  is finite  $\implies (x_n)_n$  admits a convergent sequence.

**Case 2**  $\{x_n \mid n \in \mathbb{N}\}$  is infinite. By the second property, there is a cluster point of  $\{x_n \mid n \in \mathbb{N}\}$ . Thus  $(x_n)_n$  is a convergent subsequence to some  $x \in M$ .

3  $\rightarrow$  4 Suppose that  $M$  is not totally bounded.  $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$ . Construct a sequence  $(x_n)_n$  in  $M$  as follows: Given  $x_1, \dots, x_n$ , choose  $x_1 \in M$  arbitrary and  $x_{i+1} \in M \setminus \bigcup_{j=1}^i B_\varepsilon(x_j)$  arbitrary. Then  $(x_i)_i$  is a sequence in  $M$  and  $d(x_i, x_j) > \frac{\varepsilon}{2}$  for  $i \neq j$ . Hence,  $(x_i)_i$  cannot admit a convenient subsequence.  $G \implies M$  totally bounded.

Completeness can be shown the following way: Let  $(x_n)_n$  be Cauchy in  $M$ , then there exists a subsequence  $(x_{n_i})_i$  and  $x \in M$  such that  $x_{n_i} \rightarrow x$  as  $i \rightarrow \infty$ . Since  $(x_n)_n$  is Cauchy,  $x_n \rightarrow x$  as  $n \rightarrow \infty$  [left as an exercise]. Thus  $M$  is complete.

4  $\rightarrow$  1 Let  $(U_i)_{i \in I}$  be an open covering of  $M$  and assume that  $(U_i)_{i \in I}$  does *not* admit a finite subsequence. For  $n \in \mathbb{N}$  let  $E_n \subset M$  be a finite set such that  $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}}(a)$ . Define  $\Omega := \{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } (U_i)_i\}$ . We recursively define a sequence  $(a_n)_n$  in  $M$  such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n} \cap B_{\frac{1}{2^{n-1}}}}(a_{n-1}) \neq \emptyset$$

**Goal:** Show  $(a_n)_n \rightarrow a$  and then  $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$ .

**Step 1**  $(a_n)_n$  is well defined.

$n = 1$  Since  $M \in \Omega$  and  $M \subset \bigcup_{a \in E_1} B_{\frac{1}{2}}(a)$ , we can pick  $a_1 \in E_1$  such that  $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$ .

$n \rightarrow n+1$  Let  $a_n \in E_n$  such that  $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$  be given. Let

$$\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}.$$

Since  $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$ . [Take  $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$ , but if  $B_{\frac{1}{2^{n+1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$$

Hence there exists  $a_{n+1}$  such that  $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$  and  $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \neq \emptyset$ . Thus  $(a_n)_n$  is well-defined.

**Step 2** Show that  $(a_n)_n$  converges. Take  $n \in \mathbb{N}$  and  $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$ .

$$\implies d(a_n, a_{n+1}) \leq d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus,  $(a_n)_n$  is Cauchy.  $M$  is complete, so  $\exists a \in M : a_n \xrightarrow{n \rightarrow \infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} \text{ and } \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for  $n$  sufficiently large such that  $d(a, a_n) < \frac{r}{2}$  and  $\frac{1}{2^n} < \frac{r}{2}$ . We take  $x \in B_{\frac{1}{2^n}}(a_n)$  and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to  $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ .

□

**Proposition 1.20.** *Let  $(X, d_X), (Y, d_Y)$  be metric spaces.  $M \subset X$  compact. Let  $f : X \rightarrow Y$  be continuous. Then*

1.  $f(M)$  is compact
2.  $f|_M : M \rightarrow Y$  is uniformly continuous.

*Proof.* 1. Let  $(U_i)_{i \in I}$  be an open covering of  $f(M)$

$$\implies (f^{-1}(U_i))_{i \in I} \text{ is an open covering of } M \text{ [why!]}$$

$$\implies \exists c_1, \dots, c_n \text{ such that } M \subset \bigcup_{i=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{i=1}^n U_{i_j}$$

2. If  $f|_M$  is not uniformly continuous, then  $\exists \varepsilon \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$  and  $d(f(x), f(y)) > \varepsilon$  (\*). Now take  $(x_n)_n, (y_n)_n$  sequences in  $M$  satisfying condition (\*).  $M$  is compact, so  $\exists (x_{n_i})_i$  subsequence converging to some  $x \in M$ .

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \rightarrow \infty} 0$$

□

↓ This lecture took place on 2019/03/21.

**Proposition 1.21** (Proposition and definition). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces.  $g : X \rightarrow Y$  is a function.  $g$  is called Lipschitz continuous if  $\exists L > 0$  such that  $d_Y(g(x), g(y)) \leq L d_X(x, y) \forall x, y \in X$ . Any Lipschitz continuous function is uniformly continuous.*

*Proof.* Left as an exercise to the reader.

□

**Theorem 1.22** (Arzelà-Ascoli theorem). *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and assume that  $X$  is compact. Define  $C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$  and  $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$ . Then*



1.  $d_C$  is well-defined and  $(C(X, Y), d_C)$  is a complete metric space

2. A set  $M \subset C(X, Y)$  is compact iff

(a)  $\forall x \in X$  the set  $M_x := \{f(x) \mid f \in M\}$  is compact

(b)  $M$  is equicontinuous, i.e.  $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

*Proof.* 1. Show that:  $d_C(f, g) < \infty$ .

Pick  $f, g \in C(X, Y)$ . Because  $X$  is compact,  $f(X), g(X)$  compact  $\implies f(X), g(X)$  bounded. Thus,  $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$ . Now for  $x \in X$ ,

$$\begin{aligned} d(f(X), g(x)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that  $d_C$  is a metric is left as an exercise.

Show that  $(C(X, Y), d_C)$  is a complete metric space.

Take  $(f_n)_n$  be Cauchy in  $C(X, Y) \implies (f_n(x))_n$  is Cauchy in  $Y \forall x \in X$ . Because  $Y$  is complete,  $(f_n(x))_n$  is convergent and we can define  $f(x) := \lim_{n \rightarrow \infty} f_n(x)$ . Convergence of  $(f_n)_n$  with respect to  $d_C$ : Take  $\varepsilon > 0$ , show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy,  $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$ . Consider  $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x)) \leq \lim_{m \rightarrow \infty} d(f_m, f_n) < \varepsilon$  (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if  $f \in C(X, Y) \implies f_n \rightarrow f$  with respect to  $d_C$ . Show that  $f \in C(X, Y)$ . Take  $\varepsilon > 0$ . Let  $n_0$  such that  $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$ . Take  $\delta > 0$  such that  $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$ . Then  $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show:  $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x))$ .

In general, we have  $\forall x, y, z \in (Z, d_Z)$  with  $d_Z$  as a metric.

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

*Proof.*

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y) \quad (2)$$

$$d(y, z) \leq d(y, x) + d(x, z) \implies d(y, z) - d(x, z) \leq d(x, y) \quad (3)$$

$$(2) \text{ and } (3) \implies |d(x, z) - d(y, z)| \leq d(x, y) \quad (4)$$

□

Consequently,  $\forall z \in Z, x_n \rightarrow x$  in  $Z$ :  $d(x_n, z) \rightarrow d(x, z)$  since  $|d(x_n, z) - d(x, z)| \leq d(x_n, x) \rightarrow 0$ .

2. We need to prove both directions.

$\implies$  (a) For  $x \in X$  fixed, define  $g_X : M \rightarrow Y$  with  $f \mapsto f(x)$ . Then

$$d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2)$$

$\implies g_X$  is Lipschitz continuous, in particular continuous

$\implies M_X = g_X(M)$  compact

(b) Take  $\varepsilon > 0$ .  $M$  is totally bounded, so  $\exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$ .  $\forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$ . Define  $\delta := \min_i \delta_i > 0$ . Then  $\forall x, y \in X : d(x, y) < \delta$  and  $\forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0})$

$$\implies d(f(x), f(y)) \leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon$$

$\Leftarrow$

□

## Index

Adjacent point, 9  
Banach space, 11  
Bounded sequence, 11  
Cauchy sequence, 11  
Cluster point, 9  
Compactness, 11  
Complete space, 11  
Continuity, 10  
Continuous function, 10  
Convergent sequence, 10  
Dense space, 10  
Equicontinuous set, 17  
Hausdorff space, 10  
Hilbert spaces, 6  
Interior point, 9  
Inverse problems, 2  
Limit, 10  
Lipschitz continuity, 16  
Metric, 8  
Metric space, 8  
Norm, 8  
Normed space, 8  
Orthogonal complement, 3  
Quasicompactness, 11  
Reflexive space, 5  
Separable space, 10  
 $T_2$  space, 10  
Topological space, 7, 8  
Topology, 7  
Uniform continuity, 12  
Weak convergence, 5  
Weak derivative, 6  
Weak gradient, 6