Mathematical analysis 2 – Lecture notes

course by Wolfgang Ring

Lukas Prokop

March to July 2016

Contents

This lecture took place on 1st of March 2016 with lecturer Wolfgang Ring.

Course organization:

- Tuesday, 1 hours 30 minutes, beginning at 8:15
- Thursday, 45 minutes, beginning at 8:15
- Friday, 1 hours 30 minutes, beginning at 8:15

Literature:

- Königsberger, Analysis 1

1 Exponential function (cont.)

Let $(z_n)_{n\in\mathbb{N}}$ be a complex series with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}(1+\frac{z_n}{n})^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$. For every complex number $z \in \mathbb{C}$ this series converges on entire $\mathbb{C}$.

$$\exp(z) = \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{\infty}\frac{z^k}{k!}$$

$$\exp(z+w) = \exp(z)\cdot\exp(w)$$

$$\lim_{z\to 0}\frac{\exp(z)-1}{z} = 1$$

$$\exp(1) = e \in \mathbb{R}$$

$$z = \frac{m}{n} \in \mathbb{Q} \wedge n \neq 0 \Rightarrow \exp\left(\frac{m}{n}\right) = e^{\frac{m}{n}}$$

So we also denote

$$\exp(z) = e^z \qquad \text{for } z \in \mathbb{C}$$

It holds that

$$\exp(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

$\exp(x)$ for $x \in \mathbb{R}$

$$e^x > 0 \qquad \forall x \in \mathbb{R}$$

$$(e^x)' = e^x$$

It follows immediately that the exponential function is strictly monotonically increasing in $\mathbb{R}$.

$$(e^x)'' = (e^x)' = e^x > 0$$

It follows that the exponential function is convex. But as usual,

$$e^0 = 1$$

Let $n \in \mathbb{N}$

$$\lim_{x\to+\infty}\frac{e^x}{x^n} = \infty$$

$$\lim_{x\to-\infty} e^x \cdot x^n = 0$$

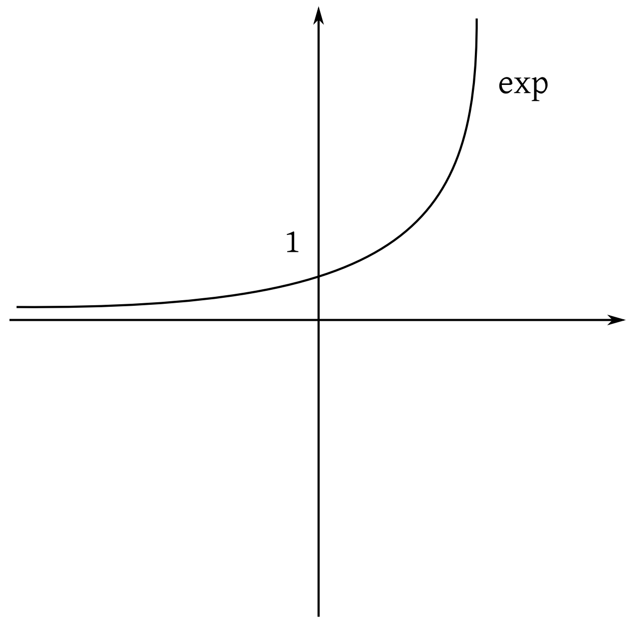


Figure 1: Graph of the exponential function

2 The natural logarithm

$$\exp : \mathbb{R} \to (0,\infty)$$

is injective, because $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$

Lemma 1. $\exp : \mathbb{R} \to (0,\infty)$ is surjective.

Proof. We need to show that the equation $e^x = y$ has some solution for every $y > 0$. We will use the Intermediate Value Theorem, we discussed in the previous course "Analysis 1".

Case 1 First of all, let $y \in [1,\infty)$. Then it holds that

$$e^0 = 1 \leq y \quad \text{and} \quad e^y = 1 + y + \underbrace{\frac{y^2}{2} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots}_{\geq 0}$$

$$\geq 1 + y > y$$

Therefore $e^0 \leq y < e^y$. Hence exp is continuous and the Intermediate Value Theorem applies:

$$\exists \xi \in [0,y]: \quad e^\xi = y$$

Case 2 Let $y \in (0,1)$. Then it holds that $w = \frac{1}{y} > 1$. The same as in Case 1 applies:

$$\exists \xi \in [0,w]: \quad e^\xi = w = \frac{1}{y}$$

$$\Rightarrow e^{-\xi} = \frac{1}{e^\xi} = y$$

So it holds that $\exp: \mathbb{R} \to (0,\infty)$ is bijective. □

Definition 1. We call the inverse function *natural logarithm*[1].

$$\exp^{-1}: (0,\infty) \to \mathbb{R}$$

$$\exp^{-1} = \ln(y) = \log(y)$$

Properties:

- It holds $\forall x \in \mathbb{R}: \ln(e^x) = x$ and $\forall y \in (0,\infty): e^{\ln(y)} = y$.
- $\ln: (0,\infty) \to \mathbb{R}$ is strictly monotonically increasing

Proof. Let $0 < y_1 < y_2$. Assume $\ln(y_1) \geq \ln(y_2) \xRightarrow{\text{monotonicity}} e^{\ln(y_1)} \geq e^{\ln(y_2)} \Rightarrow y_1 \geq y_2$. Contradiction! □

[1] In non-German literature $\ln(y)$ is almost exclusively written with the more general $\log(y)$.

2.1 Functional equations of logarithm

- For all $x, y > 0$ it holds that

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Limes:

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = 1$$

Proof.

- $$x \cdot y = e^{\ln(x \cdot y)}$$

$$e^{\ln(x)} \cdot e^{\ln(y)} = e^{\ln(x)+\ln(y)}$$

Injectivity of exp:

$$\ln(x \cdot y) = \ln(x) + \ln(y)$$

- Let $(x_n)_{n\in\mathbb{N}}$ with $x_n > 0$ be an arbitrary sequence with $\lim_{n\to\infty} x_n = 0$. Let $w_n = 1 + x_n$. Then it holds that $\lim_{n\to\infty} w_n = 1$ and $y_n = \ln(1+x_n) = \ln(w_n)$.

$$\lim_{n\to\infty} y_n = \ln(1) = 0$$

$$\lim_{n\to\infty} \frac{\ln(w_n)}{w_n - 1} = \lim_{n\to\infty} \frac{y_n}{e^{y_n} - 1} = \frac{1}{1} = 1$$

where

$$e^0 = 1 \Rightarrow \ln(1) = 0$$

□

Theorem 1 (Logarithmic growth)**.** $\forall n \in \mathbb{N}_+$ it holds that $\lim_{n\to\infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$

Proof. Let $x \in (0,\infty)$ with $x = e^{n\cdot\xi}$. That is,

$$\xi = \frac{\ln(x)}{n}$$

$$x \to \infty \Leftrightarrow \xi \to \infty$$

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[n]{x}} = \lim_{\xi\to\infty} \frac{n\cdot\xi}{\sqrt[n]{e^{n\cdot\xi}}} = \lim_{\xi\to\infty} \frac{n\cdot\xi}{e^\xi} = 0$$

because $n \cdot \xi < \xi^2$ for $\xi > n$ and $\lim_{\xi\to\infty} \frac{\xi^2}{e^\xi} = 0$. □

Theorem 2. The logarithm function is differentiable in $(0, \infty)$ and it holds that $(\ln(x))' = \frac{1}{x} \quad \forall x > 0$.

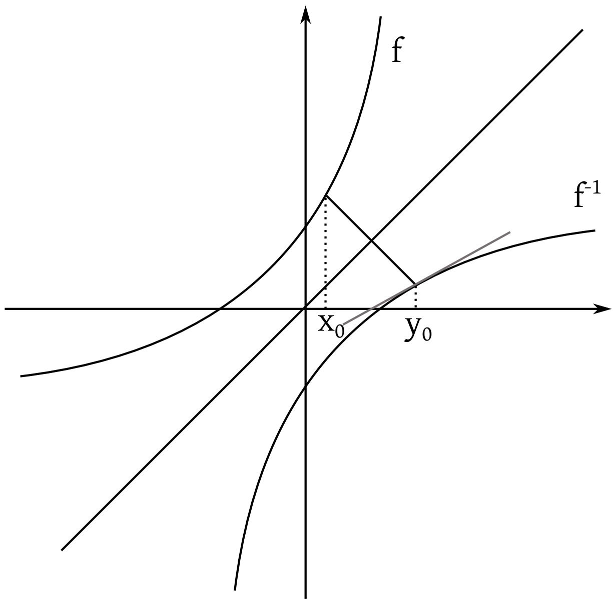


Figure 2: A geometric proof of differentiability

Proof. **First approach** Let $x > 0$, $x_n \to x$ with $x_n \neq x$, $x_n > 0$. Let $\xi_n = \ln(x_n)$ and $\xi = \ln(x) \Rightarrow \xi_n \neq \xi$.

$$e^{\xi_n} = x_n \qquad e^{\xi} = x \qquad \xi_n \to \xi$$

Then it holds that

$$\lim_{n\to\infty} \frac{\ln(x_n) - \ln(x)}{x_n - x} = \lim_{n\to\infty} \frac{\xi_n - \xi}{e^{\xi_n} - e^{\xi}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}} = \frac{1}{\underbrace{\lim_{n\to\infty} \frac{e^{\xi_n} - e^{\xi}}{\xi_n - \xi}}_{(e^\xi)' = e^\xi}} = \frac{1}{e^\xi} = \frac{1}{x}$$

Second approach using chain rule Compare with Figure 2.

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$f(f^{-1}(y)) = y \Rightarrow f(f^{-1})f(f^{-1}(y)) = y = f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$$

$$\Rightarrow (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \text{ for } f(x) = \exp(x)$$

$$\Rightarrow (\ln)'(y) = \frac{1}{\exp(\ln(y))} = \frac{1}{y}$$

$$f(f^{-1}(y)) = y$$
$$f'(f^{-1}(y)) \cdot (f^{-1})$$
$$= (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

again for $f(x) = \exp(x)$.

Third approach Let $x > 0$.

$$0 = \ln(1) = \ln\left(x \cdot \frac{1}{x}\right) = \ln(x) + \ln\left(\frac{1}{x}\right)$$

$$\Rightarrow \ln\left(\frac{1}{x}\right) = -\ln(x)$$

Let $x, y > 0$. Then it holds that

$$\ln\frac{x}{y} = \ln(x) - \ln(y)$$

because $\ln\frac{x}{y} = \ln(x \cdot \frac{1}{y}) = \ln(x) - \ln(y)$.

□

2.2 Extension of the functional equation of logarithm

2.3 A different proof for the derivative of logarithm

Proof.

$$[\ln(x)]' = \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{x\cdot\frac{h}{x}}$$

$$= \frac{1}{x}\cdot\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \text{ where } \frac{h}{x}\to 0$$

$1+\frac{h}{x} = w$ then it holds that $h \to 0 \Rightarrow w \to 1$.

$$\frac{h}{x} = w - 1$$

$$\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{=}\lim_{h\to 0}\frac{\ln(w)}{w-1} = 1$$

□

Remark 1. The exponential function can be defined from $\mathbb{C}$ to $\mathbb{C}$.

$$\exp : \mathbb{C} \to \mathbb{C}$$

It is not possible to define the logarithm *continuously* in entire $\mathbb{C}$ (or $\mathbb{C}\setminus\{0\}$). We can only define a continuous inverse function of exp in $\mathbb{C}\setminus\{x\in\mathbb{R} : x \le 0\}$

This lecture took place on 3rd of March 2016 with lecturer Wolfgang Ring.

2.4 Further remarks on differential calculus

Theorem 3. Let $f : I \to \mathbb{R}$ be strictly monotonically increasing (or s. m. decreasing) where I is an interval. Then $f^{-1} : f(I) \to \mathbb{R}$ is defined and the inverse function.

Let f in $x_0 \in I$ be differentiable and $f'(x_0) \neq 0$. Then f^{-1} is in $y_0 = f(x_0)$ differentiable and it holds that

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

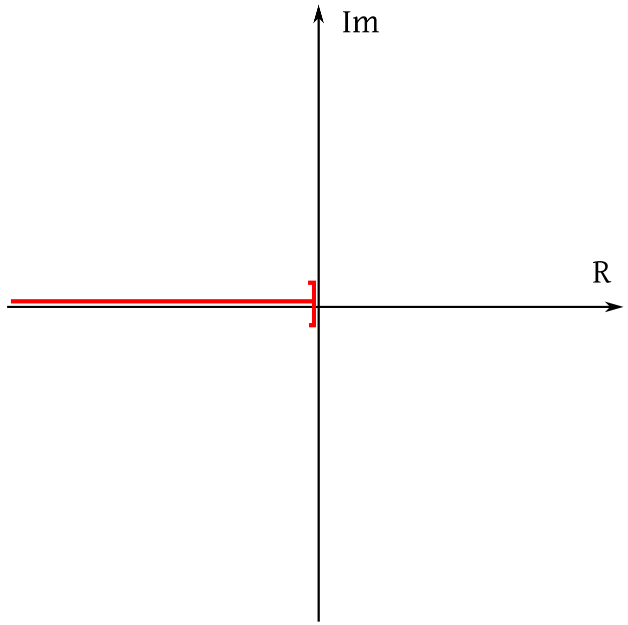


Figure 3: Continuous exponential function in $\mathbb{C}$

Proof. Let $y_n \to y_0$ and $y_n \in f(I)$; $y_0 = f(x_0)$; $y_0 \in f(I)$; $y_n = f(x_n)$. $y_n \neq y_0 \Rightarrow x_n \neq x_0$.

$$\lim_{n\to\infty}\frac{f^{-1}(y_n)-f^{-1}(y_0)}{y_n-y_0}$$

$$= \lim_{n\to\infty}\frac{x_n-x_0}{f(x_n)-f(x_0)} = \frac{1}{\lim_{n\to\infty}\underbrace{\frac{f(x_n)-f(x_0)}{x_n-x_0}}_{\text{ex}=f'(x_0)}} = \frac{1}{f'(x_0)}$$

□

Lemma 2. Let $f : I \to \mathbb{R}$ where I is some interval. Then it holds that

$$f = \text{ const} \Leftrightarrow f \text{ is differentiable in } I \text{ and } f'(x) = 0 \forall x \in I$$

Proof. $\Rightarrow$ Immediate.

$\Leftarrow$ Let f be differentiable and $f' \equiv 0$. Assume f is not constant. Then there exist $x_1, x_2 \in I$, $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$. Without loss of generality, $x_1 < x_2$. The Intermediate Value Theorem states that

$$\exists \xi \in (x_1, x_2) \subseteq I : f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$$

This is a contradiction to the assumption that $f' \equiv 0$.

□

Definition 2. Let I be an interval, $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called *primitive* or *antiderivative* of f if F is differentiable and

$$\forall x \in I : F'(x) = f(x)$$

Lemma 3. Let $f : I \to \mathbb{R}$. Let F_1 and F_2 be two primitive functions of f. Then it holds that $F_1 - F_2 = \text{const}$.

Proof. F_1, F_2 are differentiable.

$$(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

$$\xRightarrow{\text{Lemma 2}} F_1 - F_2 = \text{ const}$$

□

Theorem 4. Let I be an interval. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions in I.

$$f_n : I \to \mathbb{R} \text{ differentiable}$$

Furthermore let $f : I \to \mathbb{R}$. It holds that,

1. $\forall x \in I$ let $f(x) = \lim_{n\to\infty} f_n(x)$ ($f_n \to f$ pointwise)
2. for every $x \in I$ let $(f_n'(x))_{n\in\mathbb{N}}$ be convergent (hence $\varphi(x) = \lim_{n\to\infty} f_n'(x)$ exists for every x)
3. $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| \leq \varepsilon\, |u - v|\, \forall u, v \in I$$

Then f is differentiable in I and it holds that $f'(x) = \varphi(x) = \lim_{n\to\infty} f_n'(x)$.

$$f'(x) = [\lim_{n\to\infty} f]'(x)$$

Proof. Let $x_0 \in I$ and $x \in I$. Let $\varepsilon > 0$ arbitrary.

$$\begin{aligned}
&\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| \\
&= \left|\frac{f(x) - f(x_0)}{x - x_0} - \lim_{n\to\infty} f_N'(x_0)\right| \\
&= \left|\frac{f(x) - f(x_0)}{x - x_0} - f_N'(x_0)\right| + \left|f_N'(x_0) - \lim_{n\to\infty} f_n'(x_0)\right| \forall N \in \mathbb{N} \\
&\leq \left|\frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0}\right| \\
&\quad + \left|\frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0)\right| + |f_N'(x_0) - \varphi(x_0)|
\end{aligned}$$

1st term

$$\left|\frac{(f(x) - f_N(x)) - (f(x_0) - f_N(x_0))}{x - x_0}\right| = \left|\frac{(f - f_N)(x) - (f - f_N)(x_0)}{x - x_0}\right|$$

$$\leq \frac{\varepsilon}{3}\frac{|x - x_0|}{|x - x_0|} \overset{\text{condition 3}}{=} \frac{\varepsilon}{3}$$

for sufficiently large N.

3rd term $|f_N'(x_0) - \varphi(x)| < \frac{\varepsilon}{3}$ for sufficiently large N.

Now let N be fixed (with a value such that the first and third term is less than $\frac{\varepsilon}{3}$).

2nd term

$$\left|\frac{f_N(x) - f_N(x_0)}{x - x_0}\right| - f_N'(x_0)$$

Differentiability of f_N: Therefore for $|x - x_0| < \delta$.

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - \varphi(x_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

f is differentiable in x_0 and $f'(x_0) = \varphi(x_0)$. □

Theorem 5. Let $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ ($n \in \mathbb{N}$) and f_n is differentiable in I.

Assumption:

1. $f_n \to f$ converges pointwise in I (like the first statement in the previous Theorem)
2. There exists $g : I \to \mathbb{R}$ such that $f_n' \to g$ is continuous in I

Then f is differentiable in I and it holds that

$$f'(x_0) = g(x_0) \quad \forall x_0 \in I$$

This lecture took place on 4th of March 2016 with lecturer Wolfgang Ring.

Theorem 6 (Reminder of theorem)**.** Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in I and let f_n be differentiable $\forall n \in \mathbb{N}$. Furthermore,

- $f_n \to f$ pointwise
- $f_n'(x) \to \varphi(x)$ for every x
- $\forall \varepsilon > 0 \forall u, v \in I \exists N : n \geq N \Rightarrow |(f_n - f)(u) - (f_n - f)(v)| < \varepsilon\,|u - v|$

Then it holds that f is differentiable and $f'(x) = \varphi(x) \forall x \in I$.

Conclusion:

Theorem 7. Let f_n and f be differentiable as in Theorem 6: $f_n : I \to \mathbb{R}$ and $f : I \to \mathbb{R}$ and it holds that

- $f_n \to f$ pointwise in I for $n \to \infty$
- $\exists g : I \to \mathbb{R}$ such that $f_n' \to g$ is *uniform* in I, hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \land x \in I \Rightarrow |f_n'(x) - g(x)| < \varepsilon$

Then f is differentiable in I and $f'(x) = g(x) \forall x \in I$.

Proof. We check whether the two conditions lead to the conditions of Theorem 6. We look at the conditions of Theorem 6:

2. Uniform convergences of $f_n' \to g$ implies pointwise convergence

$$\forall x \in I : f_n'(x) \to g(x)$$

3. From uniform convergence of $f_n' \to g$ it follows that Let $\varepsilon > 0$ be arbitrary and N is sufficiently large enough, such that $\forall n \geq N$ and $\forall x \in I$:

$$|f_n'(x) - g(x)| < \frac{\varepsilon}{2}$$

Choose $n, m \geq N$ and $x \in I$ arbitrary. Then it holds that

$$\begin{aligned}|f_n'(x) - f_m'(x)| &= |f_n'(x) - g(x) + g(x) - f_m'(x)| \\ &\leq |f_n'(x) - g(x)| + |g(x) - f_m'(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

So $(f_n)_{n\in\mathbb{N}}$ is a uniform Cauchy sequence.
Let $\varepsilon > 0$ be arbitrary and N such that $n, m \geq N$ and $x \in I$:

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

Consider the third condition of Theorem 6. Let $u, v \in I$

$$|(f - f_n)(u) - (f - f_n)(v)| = \lim_{m\to\infty} |(f_m - f_n)(u) - (f_m - f_n)(v)|$$

where $(f_m - f_n)$ and $(f_m - f_n)$ is differentiable. Then according to the mean value theorem of differential calculus (dt. Mittelwertsatz der Differentialrechnung)

$$\begin{aligned}&= \lim_{m\to\infty} |(f_m - f_n)'(\xi_{m,n}) \cdot (u - v)| \\ &= \lim_{m\to\infty} |f_m'(\xi_{m,n}) - f_n'(\xi_{m,n})| \cdot |u - v|\end{aligned}$$

For $m \geq N$:

$$\leq \varepsilon \cdot |u-v|$$

So the third condition of Theorem 6 is satisfied.

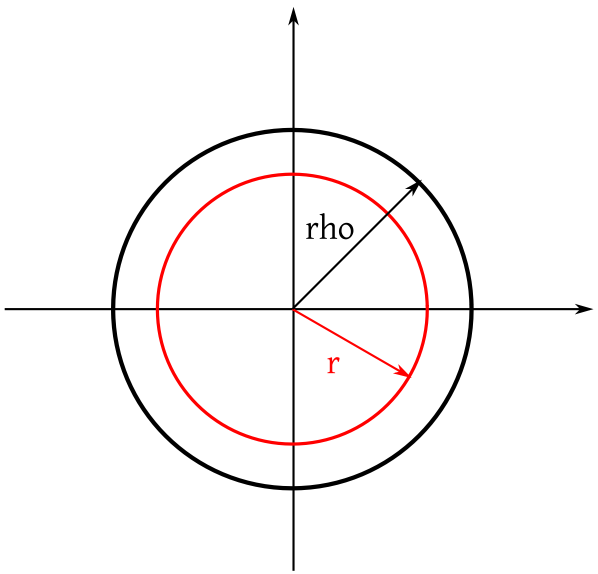


Figure 4: Convergence radius

Remark 2 (An application of Theorem 7)**.** Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P)$ with

$$\rho(P) = \frac{1}{L} \qquad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

$$P_n(z) = \sum_{k=0}^{n} a_k z^k \qquad \ldots n\text{-th partial sum}$$

Let $r < \rho(P)$. Then it holds that $P_n(z) \to P(z)$ uniform in $\overline{B(0,r)}$ [2].

$$P_n(x) \to P(x) \forall x \in [-r, r]$$

$\square$

Compare with Figure 4.

$$P_n'(x) = \sum_{k=0}^{n} a_k k \cdot x^{k-1} = \sum_{j=0}^{n-1} a_{j+1}(j+1)x^j$$

is the $n-1$-th partial sum.

$$Q(z) = \sum_{j=0}^{\infty} a_{j+1}(j+1)z^j$$

Convergence radius of Q?

$$\tilde{L} = \limsup_{j\to\infty} \sqrt[j]{a_{j+1}} \cdot \sqrt[j]{j+1} = \limsup_{j\to\infty} |a_{j+1}|^{\frac{j+1}{j}\cdot\frac{1}{j+1}} \cdot (j+1)^{\frac{j+1}{j}\cdot\frac{1}{j+1}}$$

$$= \limsup_{j\to\infty} \underbrace{\left(|a_{j+1}|^{\overbrace{\frac{1}{j+1}}^{\to 1}}\right)^{\frac{j+1}{j}}}_{L^1 = L} \cdot \underbrace{\lim_{j\to\infty}\left[(j+1)^{\frac{1}{j+1}}\right]^{\frac{j+1}{j}}}_{1^1} = L$$

In conclusion we have $\tilde{L} = L$ and $\rho(Q) = \frac{1}{L} = \rho(P)$. So $P_n'(z) = \sum_{k=1}^{n} k \cdot a_k z^{k-1}$ uniformly convergent in $\overline{B(0,r)}$ for $r < \rho$ and therefore also uniformly convergent in $[-r, r]$.

From Theorem 6 (or 7?) it follows that $P(x)$ is differentiable in $[-r, r]$ and $P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$.

Let $|x| < \rho(P)$. Let $r = \frac{1}{2}(|x| + \rho(P))$, then it holds that $x \in [-r, r]$ and P is differentiable in point x with

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

[2]Where overline means "closed"

Lemma 4. Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series with convergence radius $\rho(P) > 0$. Let $x \in (-\rho(P), \rho(P))$. Then P is differentiable in x and it holds that

$$P'(x) = \sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$$

Furthermore the power series $\sum_{k=1}^{\infty} k \cdot a_k \cdot x^{k-1}$ is uniformly convergent in every interval $[-r, r]$ with $0 < r < \rho(P)$.

2.5 About logarithm functions

We consider the power series

$$g(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$$

$$\rho(g) = \frac{1}{L} \text{ with } L = \limsup_{k\to\infty} \sqrt[k]{\frac{1}{k}} = \frac{1}{\lim_{k\to\infty} \sqrt[k]{k}} = 1$$

So it holds that $\rho(g) = 1$.

Apply the previous theorem, followingly g is differentiable in $(-1, 1)$ and it holds that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{k} x^{k-1} = \sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Remark:

$$[-\ln(1-x)]' = -\frac{1}{1-x} \cdot (-1) = \frac{1}{1-x}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{x^k}{k} + \ln(1-x) = \text{ constant}$$

Let $x = 0$ (we determine the constant for this $x = 0$):

$$0 + 0 = 0 = \text{ constant}$$

$$\Rightarrow \ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad \text{for } |x| < 1$$

Let $x \in (-1, 1) \Rightarrow -x \in (-1, 1)$.

$$\Rightarrow \ln(1-(-x)) = \ln(1+x) = -\sum_{k=1}^{\infty} \frac{(-x)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Therefore: We introduce *logarithmic series*:

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1} \qquad \text{for } x \in (-1, 1)$$

$$f(x) = \frac{1+x}{1-x}$$

Compare with Figure 5.

$$f'(x) = \frac{1-(-1)}{(1-x)^2} = \frac{2}{(1-x)^2} > 0 \quad \text{ in } (-1, 1)$$

Solve $\frac{1+x}{1-x} = w$ for x.

$$\Rightarrow 1 + x = w - wx$$

$$x(1+w) = w - 1$$

$$x = \frac{w-1}{w+1}$$

$$\ln(w) = 2\sum_{l=1}^{\infty} \frac{x^{2l-1}}{2l-1}$$

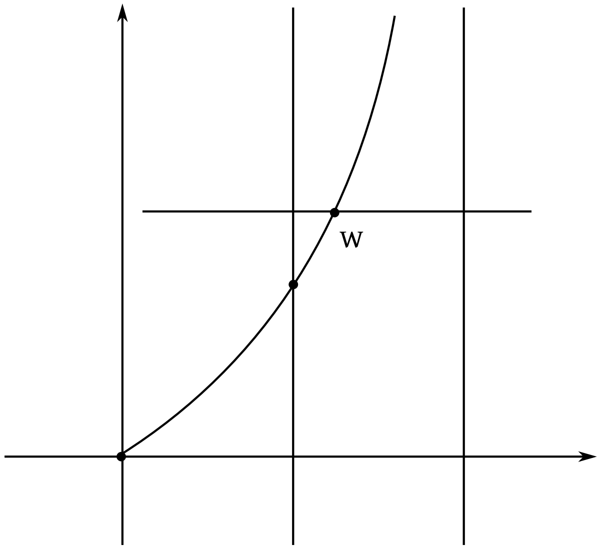


Figure 5: Plot of $\frac{1+x}{1-x}$

3 Trigonometic functions

We define trigonometic functions using the exponential function in $\mathbb{C}$.

Let $t \in \mathbb{R}$.

$$e^{it} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} = \lim_{n\to\infty} \left(\underbrace{1}_{\mathbb{R}} + \underbrace{\frac{it}{n}}_{i\mathbb{R}} \right)^n$$

$$e^{-it} = \lim_{n\to\infty} \left(1 - \frac{it}{n}\right)^n = \lim_{n\to\infty} \left[\overline{\left(1 + \frac{it}{n}\right)}\right]^n$$

$$= \lim_{n\to\infty} \overline{\left(1 + \frac{it}{n}\right)^n} = \overline{\lim_{n\to\infty} \left(1 + \frac{it}{n}\right)^n} = e^{it}$$

$$\left|e^{it}\right|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it}$$

$$e^{it-it} = e^0 = 1$$

So it holds that $\forall t \in \mathbb{R}$:

$$\left|e^{it}\right| = 1$$

So e^{it} lies inside the complex unit circle. Compare with Figure 6.

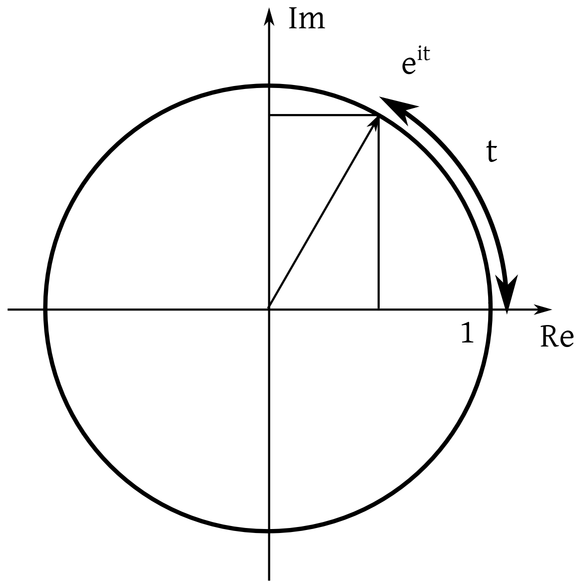


Figure 6: Unit circle in C with t

We define the cosine function $\cos : \mathbb{R} \to \mathbb{R}$ as

$$\cos(t) = \Re(e^{it})$$

and the sine function $\sin : \mathbb{R} \to \mathbb{R}$ as

$$\sin(t) = \Im(e^{it})$$

The following relations hold:

1. $e^{it} = \cos(t) + i \cdot \sin(t)$ (Euler's identity)
2. $\left|e^{it}\right|^2 = 1 = (\cos t)^2 + (\sin t)^2$
3. $$\Re(z) = \frac{1}{2}(z + \overline{z})$$ $$\Rightarrow \cos(t) = \Re(e^{it}) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$$ $$\Im(z) = \frac{1}{2i}[z - \overline{z}]$$ $$\sin(t) = \Im(e^{it}) = \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$
4. $$e^{-it} = \overline{e^{it}} = \cos t - i \cdot \sin t$$

We use property 3 to extend the domain of sine and cosine:

Definition 3. Let $z \in \mathbb{C}$. We define $\sin : \mathbb{C} \to \mathbb{C}$ and $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \frac{1}{2}\left[e^{iz} + e^{-iz}\right]$$

$$\sin(z) = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right]$$

This lecture took place on 8th of March 2016 with lecturer Wolfgang Ring.

Compare with Figure 7.

$$t \in \mathbb{R} : \cos t = \Re(e^{it}) = \frac{1}{2}(e^{it} + e^{it})$$
$$\sin t = \Im(e^{it}) = \frac{1}{2i}(e^{it} - e^{-it})$$

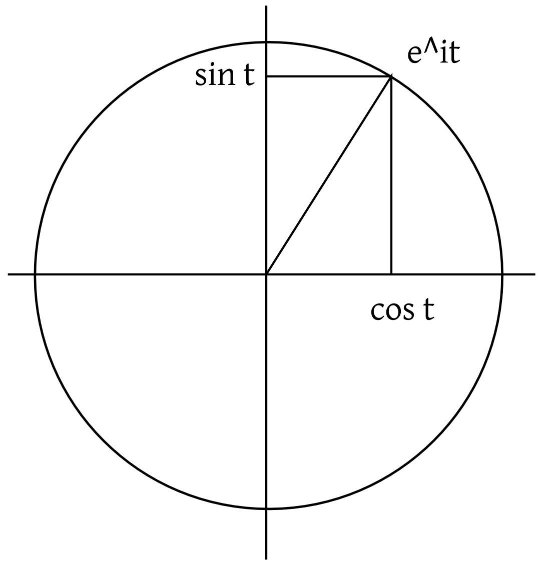


Figure 7: The trigonometric values $\sin t$ and $\cos t$ in the unit circle

$$\begin{aligned} z \in \mathbb{C} : \cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \end{aligned}$$

Properties:

$$\cos -z = \frac{1}{2}(e^{i(-z)} + e^{-i}(-z)) = \cos z$$

$\cos z$ is even

$$\sin -z = \frac{1}{2i}(e^{-iz} - e^{iz}) = -\sin z$$

$\sin z$ is odd

The cosine function in the complex space is even.

3.1 Series representation of trigonometric functions

Lemma 5 (Addition of series of absolute convergence)**.** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be complex sequences and the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolute convergent with series value $\sum_{n=0}^{\infty} a_n = a$ and $\sum_{n=0}^{\infty} b_n = s'$.

Then $\sum_{n=0}^{\infty}(a_n + b_n)$ is absolute convergent with sum $s + s'$.

series sum. Absolute convergence. Show that $\sum_{k=0}^{n} = |a_k + b_k| = t_n$ and $(t_n)_{n\in\mathbb{N}}$ is bounded.

Follows immediately, because

$$\sum_{k=0}^{n} |a_k k + b_k| \leq \underbrace{\sum_{k=0}^{n} |a_k|}_{\text{bounded}} + \underbrace{\sum_{k=0}^{n} |b_k|}_{\text{bounded}}$$

□

Example 1 (Application)**.** Let $P(z) := \sum_{k=0}^{\infty} a_k z^k$ and $Q(z) := \sum_{k=0}^{\infty} b_k z^k$ be power series. Both are convergent in $B(0,\delta)$. Then also $\sum_{k=0}^{\infty}(a_k + b_k)z^k$ is convergent in $B(0,\delta)$ and it holds that $\sum_{k=0}^{\infty}(a_k + b_k)z^k = P(z) + Q(z)$.

3.2 Application to trigonometric functions

$$e^{iz} = \sum_{k=0}^{\infty} \frac{(iz)^k}{k!} = \sum_{k=0}^{\infty} i^k \cdot \frac{z^k}{k!}$$

$$i^0 = 1 \qquad i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 = i^0 \qquad i^5 = i \qquad \dots$$

$$\Rightarrow = 1 + i\frac{z}{1!} - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!}$$

$$e^{-iz} = \sum_{k=0}^{\infty} \frac{(-iz)^k}{k!} = \sum_{k=0}^{\infty} (-i)^k \frac{z^k}{k!}$$

$$(-i)^0 = 1 \qquad (-i)^1 = -i \qquad (-i)^2 = -1 \qquad (-i)^3 = i \qquad (-i^4) = 1 \qquad \dots$$

$$\Rightarrow = 1 - i\frac{z}{1!} - 1\frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} - \frac{z^6}{6!} + \dots$$

$$\frac{1}{2}(e^{iz} + e^{-iz}) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \dots$$

Followingly,

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \dots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l}}{(2l)!} \text{ convergent in } \mathbb{C} \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \dots \\ &= \sum_{l=0}^{\infty} (-1)^l \frac{z^{2l+1}}{(2l+1)!} \end{aligned}$$

3.3 Functional equations of trigonometric functions

Theorem 8 (Addition and substraction theorems)**.** We derive them directly:

Let $z, w \in \mathbb{C}$.

$$e^{z+w} = e^z \cdot e^w = (\cos z + i \cdot \sin z)(\cos w + i \cdot \sin w)$$

but also

$$\begin{aligned} &= (\cos(z+w) + i\sin(z+w)) \\ &\Rightarrow= (\cos z \cdot \cos w - \sin z \cdot \sin w) + i(\cos z \cdot \sin w + \sin z c\dot{c}osw) \end{aligned}$$

Analogously,

$$\begin{aligned} e^{-(z+w)} &= e^{-z} \cdot e^{-w} = (\cos(-z) + i \cdot \sin(-z))(\cos(-w) + i \cdot \sin(-w)) \\ &= \cos z \cdot \cos w - \sin z \sin w + i\,(-\cos z \sin w - \cos w \sin z) \end{aligned}$$

but also

$$\begin{aligned} &= (-\cos(z+w) + i\sin(-(z+w))) \\ &\Rightarrow= \cos\,(z+w) - i\sin(z+w) \end{aligned}$$

Addition:

$$\begin{aligned} 2\cos(z+w) &= 2(\cos z \cdot \cos w - \sin z \sin w) \\ \Rightarrow \cos(z+w) &= \cos z \cos w - \sin z \sin w \end{aligned}$$

Subtraction:

$$\Rightarrow \sin(z+w) = \cos z \sin w + \sin z \cos w \forall z, w \in \mathbb{C}$$

Variations: $w \leftrightarrow -w$

$$\begin{aligned} \cos(z-w) &= \cos z \cdot \underbrace{\cos w}_{=\cos(-w)} + \sin z \underbrace{\sin w}_{=-\sin(-w)} \\ \sin(z-w) &= -\cos z \cdot \sin(w) + \sin(z)\cos(w) \end{aligned}$$

Corollary 1.

$$\begin{aligned} z &= \frac{1}{2}(z+w) + \frac{1}{2}(z-w) \\ \Rightarrow \cos z &= \cos\frac{z+w}{2}\cos\frac{z-w}{2} - \sin\frac{z+w}{2}\sin\frac{z-w}{2} \\ w &= \frac{1}{2}(w+z) + \frac{1}{2}(w-z) = \frac{1}{2}(z+w) - \frac{1}{2}(z-w) \\ \cos w &= \cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2} + \sin\frac{z+w}{2} \cdot \sin\frac{z-w}{2} \\ \cos z - \cos w &= -2\sin\frac{z+w}{2}\sin\frac{z-w}{2} \end{aligned}$$

Analogously,

$$\sin z - \sin w = 2\cos\frac{z+w}{2} \cdot \cos\frac{z-w}{2}$$

We consider

$$\begin{aligned} \lim_{\substack{z\to 0 \\ z\neq 0}} \frac{\sin z}{z} &= \lim_{z\to 0} \frac{1}{2i}\left(\frac{e^{iz} - e^{-iz}}{z}\right) \\ &= \lim_{z\to 0} e^{-iz}\left(\frac{e^{2iz} - 1}{2iz}\right) \\ &= \underbrace{\lim_{z\to 0} e^{-iz}}_{=e^0=1} \cdot \underbrace{\lim_{z\to 0} \frac{e^{2iz}-1}{2iz}}_{\substack{e=2iz;z\to 0\Leftrightarrow w=0 \\ \lim_{w\to 0}\frac{e^w-1}{w}=1}} \end{aligned}$$

So it holds that

$$\lim_{z\to 0}\frac{\sin z}{z} = 1$$

3.4 Trigonometric functions for real arguments

Subtitled "definition of π" and "periodicity".

Let $x \in \mathbb{R}$.

$$\cos x = \overbrace{1}^{=c_0} - \overbrace{\frac{x^2}{2}}^{=c_1} + \overbrace{\frac{x^4}{24}}^{=c_2} - \overbrace{\frac{x^6}{720}}^{=c_3} + \overbrace{\frac{x^8}{40320}}^{=c_4} - \ldots$$

$$\sin x = \underbrace{x}_{=s_0} - \underbrace{\frac{x^3}{6}}_{=s_1} + \underbrace{\frac{x^5}{120}}_{=s_2} - \underbrace{\frac{x^7}{5040}}_{=s_3} + \dots$$

$$c_n = \frac{x^{2k}}{(2k)!} \qquad s_k = \frac{x^{2k+1}}{(2k+1)!}$$

For $x \in [0,2]$ and $k \geq 1$ it holds that

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{x^2}{(2k+2)(2k+1)}\right| \leq \frac{4}{3 \cdot 4} = \frac{1}{3}$$

so $(c_k)_{k\geq 1}$ is strictly monotonically decreasing.

Leibniz criterion:

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for $x \in (0,2]$.

Similarly for $x \in (0,2]$:

$$\left|\frac{s_{k+1}}{s_k}\right| = \left|\frac{x^2}{(2k+2)(2k+3)}\right| \leq \frac{4}{4 \cdot 5} = \frac{1}{5} < 1$$

So the Leibniz criterion tells us that

$$x - \frac{x^3}{6} < \sin x < x \quad \text{ in } [0,2]$$

So it holds that

$$\cos(0) = 1$$

$$\cos(2) < 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3}$$

Intermediate value theorem (power series is continuous):

$$\exists \xi \in (0,2) \text{ with } \cos(\xi) = 0$$

Let $0 \leq w < z \leq 2$,

$$0 < \frac{z-w}{2} \leq \frac{z+w}{2} < \frac{z+z}{2} \leq 2$$

Let $x \in (0,2]$, then it holds that

$$\sin(x) > x - \frac{x^3}{6} = \underbrace{x}_{>0} \underbrace{\left(1 - \frac{x^2}{6}\right)}_{>1-\frac{4}{6}=\frac{1}{3}>0} > 0$$

So it holds that $\sin(x) > 0$ in $(0,2]$.

Functional equation for $\cos z - \cos w$.

$$\cos z - \cos w = \underbrace{-2 \cdot \underbrace{\sin \underbrace{\frac{z+w}{2}}_{\in(0,2]} \cdot \sin \underbrace{\frac{z-w}{2}}_{\in(0,2]}}_{>0}}_{<0}$$

$\cos z < \cos w$ for $0 \leq w < z \leq 2$.

So it holds that cos is a strictly monotonically decreasing functionin $[0,2)$. Hence cos has only one root because it is continuous in $(0,2]$.

Definition 4. The number $\pi \in \mathbb{R}$ is defined as $\pi = 2\xi$, where ξ is the uniquely defined root of the cosine in $(0,2]$.

Some further important function values:

$$0 < \frac{\pi}{2} < 2 \text{ and } \cos\frac{\pi}{2} = 0$$

because $\cos^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{\pi}{2}\right) = 1$.

$$\Rightarrow \left|\sin\frac{\pi}{2}\right| = 1$$

We know that $\sin x > 0$ for $x \in (0,2]$.

$$\Rightarrow \sin\frac{\pi}{2} = 1$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

TODO: table missing

$$e^{i\pi} = e^{i\frac{\pi}{2}+i\frac{\pi}{2}} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$
$$e^{i\frac{3}{2}\pi} = e^{i\pi+\frac{i}{2}\pi} = e^{i\pi} \cdot e^{i\frac{\pi}{2}} = -1 \cdot i = -i$$

Furthermore,

$$e^{z+i\pi} = e^z \cdot \underbrace{e^{i\pi}}_{=-1} = -e^z$$
$$e^{z+2i\pi} = e^z \cdot \left(e^{i\pi}\right)^2 = e^z$$

So the exponential function is periodic in $\mathbb{C}$ with period $2i\pi$.

$$\begin{aligned}\cos(z+2\pi) &= \frac{1}{2}\left(e^{iz+2\pi i} + e^{-iz-2\pi i}\right)\\ &= \frac{1}{2}\left(e^{iz} + e^{-iz} \cdot \frac{1}{\underbrace{e^{2\pi i}}_{=1}}\right) = \cos z\end{aligned}$$

Therefore the cosine is periodic in $\mathbb{C}$ with period 2π. Analogously, sine is periodic in $\mathbb{C}$ with period 2π.

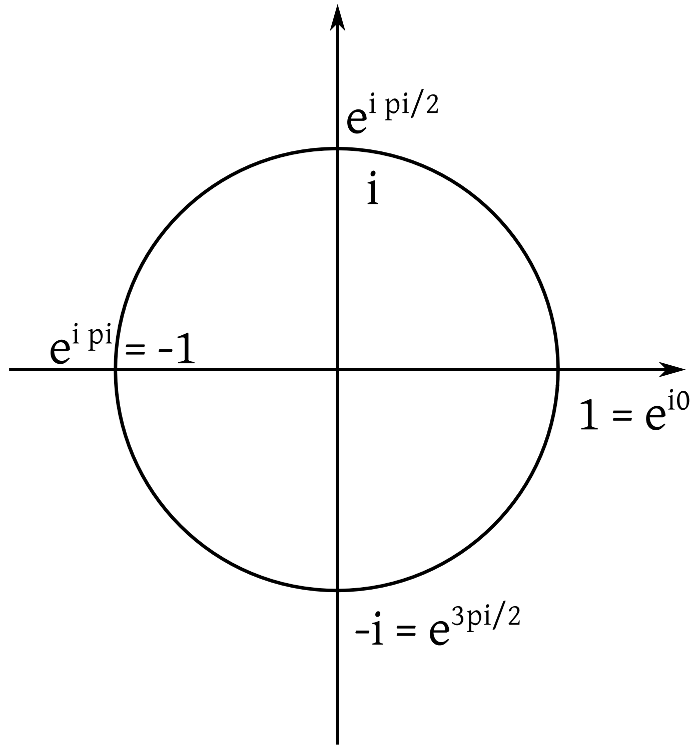
This lecture took place on 10th of March 2016 with lecturer Wolfgang Ring.

3.5 Periodicity and roots of trigonometric functions

TODO: equations missing

$$\cos(z+2\pi) = \cos(z)$$
$$\sin(z+2\pi) = \sin(z)$$

Remark 3. We will show: $\forall c \in (0, 2\pi)$, cos and sin are non-periodic with period c, hence $\exists x \in \mathbb{R}$ such that $\cos(x) \neq \cos(x+c)$.



Definition 5.

$$f : \mathbb{C} \to \mathbb{C} \qquad (f : \mathbb{R} \to \mathbb{R})$$

is called *periodic* with period $c \in \mathbb{C}$ ($c \in \mathbb{R}$) if $\forall z \in \mathbb{C}$ it holds that

$$f(z+c) = f(z)$$
$$(\forall x \in \mathbb{R} : f(x+c) = f(x))$$

c is called *period of* f.

Remark 4. If f is periodic with period $c \in \mathbb{C}$, then f is also periodic with period $k \cdot c$ for every $k \in \mathbb{Z} \setminus \{0\}$.

Remark 5.

$$z = u + iv$$
$$\Re(i \cdot z) = \Re(iu - v) = -v = -\Im(z)$$
$$\Im(i \cdot z) = \Im(iu - v) = u = \Re(z)$$

Remark 6. Let $x \in \mathbb{R}$.

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= \Re(e^{i(x+\frac{\pi}{2})}) \\ &= \Re(e^{ix} \cdot e^{i\frac{\pi}{2}}) \\ &= \Re(ie^{ix}) \\ &= -\Im(e^{ix}) \\ &= -\sin(x)\end{aligned}$$

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= \Im\left(e^{i(x+\frac{\pi}{2})}\right) \\ &= \Im(ie^{ix}) \\ &= \Re(e^{ix}) \\ &= \cos(x)\end{aligned}$$

$$\begin{aligned}\cos\left(x - \frac{\pi}{2}\right) &= \sin\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= \sin(x)\end{aligned}$$

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= -\cos\left(x - \frac{\pi}{2} + \frac{\pi}{2}\right) \\ &= -\cos(x)\end{aligned}$$

Summary:

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin\left(x + \frac{\pi}{2}\right) &= \cos(x) \\ \cos\left(x - \frac{\pi}{2}\right) &= \sin(x) \\ \sin\left(x - \frac{\pi}{2}\right) &= -\cos(x)\end{aligned}$$

Remark 7 (A remark on the name "cosine")**.**

$$\sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right) = \cos(x)$$

The sine of the complementary angle is the co-sine of x (Compare with Figure 8).

Remark 8.

$$\begin{aligned}\cos(x + \pi) &= \Re(e^{i(x+\pi)}) \\ &= \Re(-e^{ix}) \\ &= -\cos(x) \\ \sin(x + \pi) &= -\sin(x)\end{aligned}$$

Remark 9. Let $0 < c < 2\pi$. Assume cos is periodic with period c. We know that cos has exactly one root in $[0, 2]$,

$$\cos(x) = \cos(-x)$$

cos has exactly two roots in $[-2, 2]$, namely $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

1. Consider $c \in (0, \pi)$. Then $\cos\left(-\frac{\pi}{2} + c\right) = \cos\left(-\frac{\pi}{2}\right) = 0$.

$$-\frac{\pi}{2} + c < -\frac{\pi}{2} + \pi = \frac{\pi}{2} < 2$$

$$-\frac{\pi}{2} + c \geq -\frac{\pi}{2} > -2$$

   Therefore cos would have another root in $[-2, 2]$, namely $-\frac{\pi}{2} + c$. This is a contradiction.

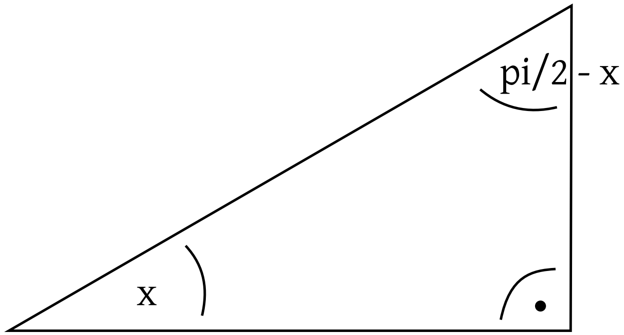


Figure 8: Complementary angle: co-sinus

2. Consider $c \in [\pi, 2\pi)$. $c = \pi$ is not a period because $\cos(0) = 1$ and $\cos(0 + \pi) = -1$. Let $\pi < c < 2\pi$. Then $\frac{3}{2}\pi - c < \frac{3}{2}\pi - \pi = \frac{\pi}{2}$ and $\frac{3}{2}\pi - c > \frac{3}{2}\pi - 2\pi = -\frac{\pi}{2}$. Hence,

$$\frac{3}{2}\pi - c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cos\left(\frac{3}{2}\pi - c\right) = \cos\left(\frac{3}{2}\pi - c + c\right) = \cos\left(\frac{3}{2}\pi\right) = 0$$

c would be the period.

$$\Rightarrow \frac{3}{2}\pi - c \text{ is a root of cos in } (-\frac{\pi}{2}, \frac{\pi}{2})$$

This is a contradiction.

Therefore it holds that

$$\forall c \in (0, 2\pi) : \exists x \in \mathbb{R} : \cos(x + c) \neq \cos(x)$$

Therefore cos is not periodic with period c. Hence 2π is indeed the smallest period of cos.

Analogously it holds for sin.

Remark 10 (Roots of cos)**.**

$$\cos\left(\frac{\pi}{2} + 2k\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$\cos\left(\frac{3}{2}\pi + 2k\pi\right) = \cos\left(\frac{3}{2}\pi\right) = 0 \qquad \forall k \in \mathbb{Z}$$

$$x_k = \frac{\pi}{2} + 2k\pi = \frac{\pi}{2}(1 + 4k)$$

$$y_k = \frac{3}{2}\pi + 2k\pi = \frac{\pi}{2}(3 + 4k)$$

Hence for $z_l = \frac{\pi}{2}(2l + 1)$ with $l \in \mathbb{Z}$ it holds that $\cos(z_l) = 0$. These are the odd multiples of $\frac{\pi}{2}$.

$$\sin(0 + 2k\pi) = \sin(0) = 0$$
$$\sin(\pi + 2k\pi) = \sin((2k + 1)\pi) = \sin(\pi) = 0$$

$$\Rightarrow (l\pi) = 0 \qquad \forall l \in \mathbb{Z}$$

3.6 Derivatives of trigonometric functions

It holds that

$$\boxed{\lim_{z\to 0} \frac{\sin z}{z} = 1}$$

Furthermore it holds that

$$\boxed{\lim_{z\to 0} \frac{1-\cos z}{z} = 0}$$

Proof.

$$\begin{aligned}\frac{1-\cos z}{z} &= \frac{1}{z}\left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \frac{z^6}{6!} - \frac{z^8}{8!} + \dots\right)\\ &= \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \frac{z^7}{8!} + \dots\end{aligned}$$

is convergent in $\mathbb{C}$ and (especially) continuous in 0

$$\lim_{z\to 0}\left(\frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \dots\right) = 0$$

□

$$\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

This lecture took place on 11th of March 2016 with lecturer Wolfgang Ring.

Recall:

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$
$$\lim_{z\to 0} \frac{1-\cos z}{z} = 0$$

Lemma 6. The trigonometric functions sin and cos are differentiable in $\mathbb{R}$ (because they can be expressed as power series with infinite convergence radius) and it holds that

$$\cos'(x) = -\sin(x) \qquad \sin'(x) = \cos(x)$$

Proof.

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h)-\cos(h)}{h} &= \lim_{h\to 0} \frac{\cos x\cdot\cos h - \sin x\cdot\sin h - \cos x}{h}\\ &= \lim_{h\to 0} \cos x\cdot\frac{\cos(h)-1}{h} - \lim_{h\to 0}\frac{\sin x\cdot\sin h}{h}\\ &= \cos x\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0} - \sin x\cdot\underbrace{\lim_{h\to 0}\frac{\sin(h)}{h}}_{=1}\\ &= -\sin(x)\end{aligned}$$

Analogously:

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h)-\sin(h)}{h} &= \lim_{h\to 0} \frac{\sin x\cdot\cos h + \sin h\cdot\cos x - \sin x}{h}\\ &= \sin(x)\cdot\underbrace{\lim_{h\to 0}\frac{\cos(h)-1}{h}}_{=0} + \cos(x)\cdot\underbrace{\lim_{h\to 0}\frac{\sin h}{n}}_{=1}\\ &= \cos(x)\end{aligned}$$

□

TODO: incomplete graphics, verify text

Figure 9. We now use tools of integral calculus:

Let $I = [a,b]$ and $\gamma : I \to \mathbb{R}$ $(\mathbb{R}^2)$.

$$\gamma(t) = \begin{bmatrix}\gamma_1(t)\\ \vdots\\ \gamma_n(t)\end{bmatrix}$$

Assumption: $\gamma_1 : [a,b] \to \mathbb{R}^n$.

$$\gamma'(t) = \begin{bmatrix}\gamma_1'(t)\\ \vdots\\ \gamma_n'(t)\end{bmatrix}$$

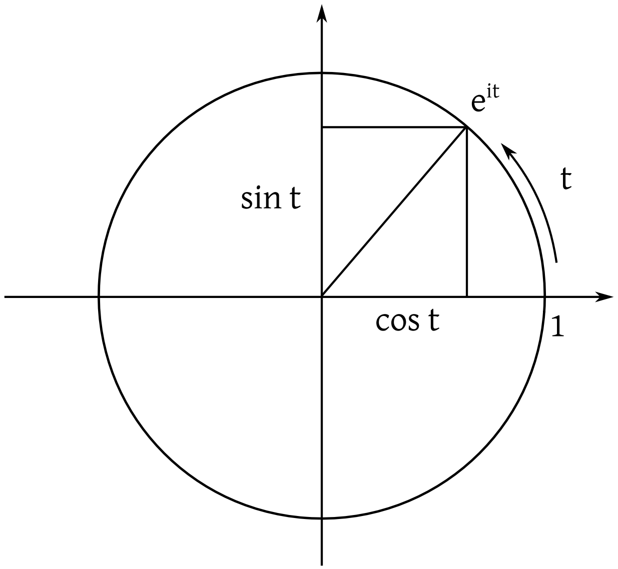


Figure 9: The arc length is related to sin and cos

TODO: graphics missing

Let $t \in [a, b]$. Then the arc length of γ between a and t is given by

$$S(t) = \int_a^t |\gamma'(\tau)| \; d\tau$$

We identify $\mathbb{C}$ with $\mathbb{R}^2$:

$$x + iy \leftrightarrow \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\gamma : t \mapsto e^{it} = \cos t + i \cdot \sin t$$

is a curve in $\mathbb{C} \cong \mathbb{R}^2$.

$$\gamma : [0, 2\pi] \to \mathbb{C}$$

$$\gamma(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$\gamma'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

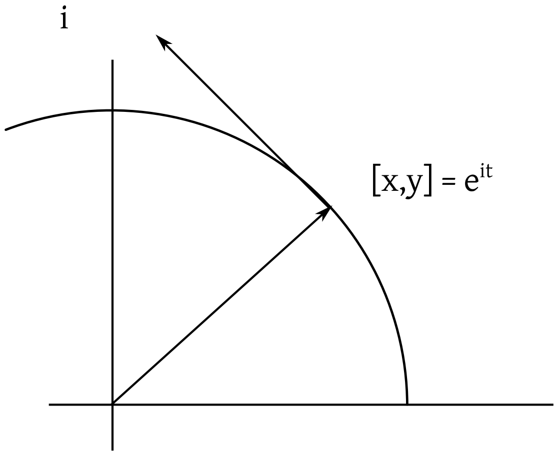


Figure 10: Derivative in $\mathbb{R}^2$

Compare with Figure 10.

$$|\gamma'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = 1$$

$$\int_0^t |\gamma'(\tau)| \; d\tau = \int_0^t 1 \, d\tau = t$$

4 Integration calculus

Integration calculus was developed to determine areas of curves regions. It was developed by Leibniz, Cauchy, Riemann and Lebeque. There are different notions of integrations and it will discussed in further details in the courses "Functional analysis" and "Measure and integration theory". For now, we look at the basis (as discussed by Königsberger).

Let $[a, b]$ be an interval, $a, b \in \mathbb{R}$ with $a < b$ and $\phi : [a, b] \to \mathbb{R}$. We call φ a *step function*, if $n \in \mathbb{N}$ and $x_0, \dots, x_n$ exist such that

$$x_0 = a < x_1 < x_2 < \ldots < x_n = b$$

and $\varphi|_{(x_{j-1}, x_j)} = c_j$ is constant. The points x_j define a partition of the interval $[a, b]$.

The function values defining the partitions do not have any constraints and are therefore irrelevant for further considerations (compare with Figure 11).

Definition 6. Let $\varphi : [a, b] \to \mathbb{R}$ a step function and $x_0 = a < x_1 < \ldots < x_n = b$ as partition of $[a, b]$ and let $\varphi|_{(x_{j-1}, x_j)} = c_j$ for $j = 1, \dots, n$. Then we define

$$\int_a^b \varphi \, dx = \sum_{j=1}^{n} c_j \delta x_j$$

where $\delta x_j = x_j - x_{j-1}$ (for $j = 1, \dots, n$).

$$\int_a^b \varphi \, dx \text{ is called } \textit{integral} \text{ of } \varphi \text{ over } [a, b]$$

φ is the step function in terms of the partition $\{x_0, x_1, \dots, x_5\}$.

It remains to show that if φ satisfies the definition of a step function in terms of partition $\{x_0, \dots, x_n\}$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$ (TODO: text missing: "but …") and φ is a step function in terms of $\{w_0, w_1, \dots, w_m\}$ and $\varphi|_{(w_{l-1}, w_l)} = c'_l$, then it holds that

$$\sum_{j=1}^{n} c_j \delta x_j = \sum_{l=1}^{m} c'_l \delta w_l$$

Compare with Figure 12.

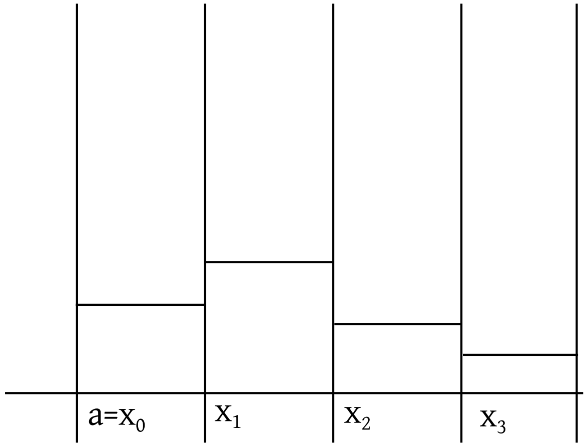


Figure 11: Partition of an area into rectangles

Proof. Let $Z = \{x_0, \dots, x_n\}$ and $Z' = \{w_0, \dots, w_m\}$. We define $Z'' = Z \cup Z'$ and $Z'' = \{\alpha_0, \alpha_1, \dots, \alpha_L\}$. Duplicates get lost in the set.

$$\alpha_0 = a < \alpha_1 < \ldots < \alpha_L = b$$

Because $Z \subseteq Z''$,

$$\forall x_j \exists k_j : x_j = \alpha_{k_j}$$

Because $x_{j-1} < x_j$, it holds that $\alpha_{k_{j-1}} < \alpha_{k_j}$. Followingly,

$$k_{j-1} < k_j$$

Let $k_{j-1} < l \leq k_j$. It holds that $(\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$, because $l > k_{j-1} = l - 1 \geq k_{j-1} \Rightarrow \alpha_{l-1} \geq \alpha_{k_{j-1}} = x_{j-1}$ and $l \leq k_j$.

$$\Rightarrow \alpha_l \leq \alpha_{k_j} = x_j$$

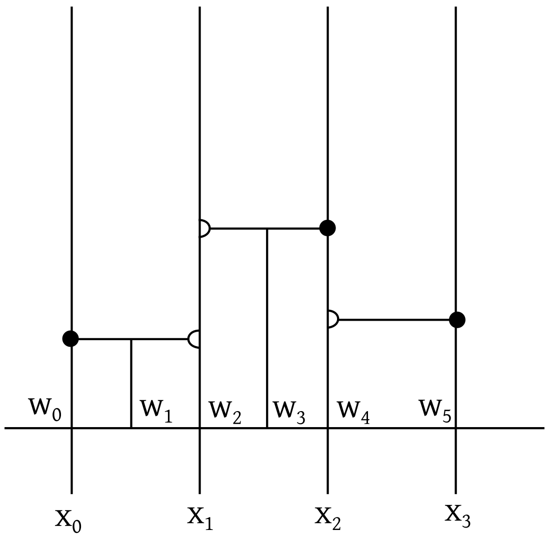


Figure 12: Step function φ

So for $x \in (\alpha_{l-1}, \alpha_l) \subseteq (x_{j-1}, x_j)$ it holds that $\varphi(x) = c_j$.

$k_0 = 0$ because $x_0 = \alpha_0 = a$ and $k_n = L$ because $x_n = \alpha_L = b$. $\forall l \in \{0, \dots, L\}$ there exists $j \in \{1, \dots, n\}$ such that $k_{j-1} \leq l \leq k_j$.

$$\Rightarrow \varphi|_{(\alpha_{l-1}, \alpha_l)} \text{ is constant}$$

Hence φ is a step function in terms of the partition $\{\alpha_0, \dots, \alpha_L\}$.

Let $l \in \{0, 1, \dots, L\}$ and j such that

$$k_{j-1} < l \leq k_j \Rightarrow (\alpha_{l-1}, \alpha_l) \subset (x_{j-1}, x_j)$$

and $c_l'' = \varphi(x)$ for $x \in (\alpha_{l-1}, \alpha_l)$, then $c_l'' = c_j$.

$$\sum_{l=1}^{L} c_l'' \cdot \delta\alpha_l = \sum_{j=1}^{n} \sum_{l=k_{j-1}+1}^{k_j} c_l'' \delta\alpha_l$$

$$= \sum_{j=1}^{n} c_j \sum_{l=k_{j-1}}^{k_j} \triangle\alpha_l$$

$$\sum_{l=k_{j-1}+1}^{k_j} \triangle\alpha_l = (\alpha_{k_{j-1}+1} - \alpha_{k_{j-1}}) + (\alpha_{k_{j-1}+2} - \alpha_{k_{j-1}+1}) + (\alpha_{k_{j-1}+3} - \alpha_{k_{j-1}+2})$$

$$+ \ldots + (\alpha_{k_j-1} - \alpha_{k_j-2}) + (\alpha_{k_j} - \alpha_{k_j-1})$$

This is a telescoping sum. What remains is:

$$= \alpha_{k_j} - \alpha_{k_{j-1}}$$

$\square$

German keywords

English keywords