

Introduction to Functional Analysis

Lecture notes, University of Technology, Graz
based on the lecture by Martin Holler

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0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces

- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces U, V
matrices $A \in \mathcal{M}^{n \times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto$ functionals $f : U \rightarrow \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
 - basis of Applied Mathematics
 - partial differential equations
 - physical modelling
 - inverse problems (operator A models some physical measurement process)
 - Optimization and optimal control

A motivating example was presented with slides.

0.1 Application examples

Let $K : U \rightarrow \mathbb{R}^m$ with U as vector space describe a physical model. For example, K is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0, 1] \rightarrow \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find u . Typically inversion of K is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$ where $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$ and $\|\cdot\|_u$ is a norm on U . Or alternatively, let $U = C^1([0, 1]^2)$ and solve $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$.

Other examples are JPEG compression and upsampling of images.

0.2 Our first problem

Let $U := C^1([0, 1]^2)$ be a normed space, $K : U \rightarrow \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \implies u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ linear.

Proof. Case 1 (trivial model): Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \quad (1)$$

Take $(u_n)_{n \in \mathbb{N}}$ in \mathbb{R}^n such that $\lim_{n \rightarrow \infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \forall n \in \mathbb{N}$

$$\Rightarrow \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i \in \mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \rightarrow u$ as $i \rightarrow \infty$.

Now: Show that u solves Problem (1). ∇ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \rightarrow \infty} \|\nabla u_{n_i}\|_2 + \lambda \|u_{n_i} - d\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that \hat{u} is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of $\nabla, \|\cdot\|_2$

Case 2 (K arbitrary): 1. K arbitrary does not provide boundedness anymore. Define $X := \text{kernel}(\nabla) \cap \text{kernel}(k)$ and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^n x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \quad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that X^\perp is called *orthogonal complement*.

Claim 0.1. *If \hat{u} solves $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. Then \hat{u} solves Problem (1).*

Proof. Let \hat{u} be a solution on X^\perp . Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^\perp$. Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \|\nabla(u_1 + u_2)\|_2 + \lambda \|k(u_1 + u_2) - d\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus \hat{u} solves our problem (1). □

Take again $(u_n)_{n \in \mathbb{N}}$ be such that $u_n \in X^\perp \nabla n$ and

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_2 = \inf_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \ker(\nabla) + \ker(\nabla)^\perp$. $\nabla : \ker(\nabla)^\perp \rightarrow \text{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \ker(\nabla)^\perp \implies v \in \ker(\nabla) \implies \|v\| = (v, v) = 0$. Thus, $\nabla^{-1} : \text{image}(\nabla) \rightarrow \ker(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^2\|_2 &= \|\nabla^{-1} \nabla u_n^2\|_2 = \|\nabla^{-1}\| \cdot \|\nabla u_n^2\|_2 \leq \|\nabla^{-1}\| \\ &\leq \|\nabla^{-1}\| (\|\nabla u_n^2\|_2 + \lambda \|Ku_n - d\|_2) \\ &= \|\nabla^{-1}\| \left(\underbrace{\|\nabla u_n\|_2}_{= \|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\ &< C \text{ for some } C > 0 \end{aligned}$$

Then $\|u_n^2\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \rightarrow \text{image}(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap \ker(\nabla) \implies v \in \ker(K)$. Hence $v \in \ker(K) \cap \ker(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$. Hence $K^{-1} : \text{image}(K) \rightarrow X^\perp \cap \ker(\nabla)$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^1\|_2 &= \|K^{-1} Ku_n^1\|_2 \leq \|K^{-1}\| \|Ku_n^1\|_2 \\ &= \frac{\|K\|}{\lambda} (\lambda \|K(u_n^1 + u_n^2) - Ku_n^1\|_2 + \|\nabla u_n\|_2) \\ &\leq \frac{\|K\|}{\lambda} \left(\underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \|d - Ku_1^2\|_2}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\ &< D \text{ for some } D > 0 \end{aligned}$$

$$\implies (u_n^1)_n \text{ bounded} \implies (u_n) = (u_n^1 + u_n^2)_n \text{ is bounded}$$

$\implies (u_n)_n$ admits a subsequence converging to some \hat{u} . As in Case 1, \hat{u} is a solution to Problem (1).

In summary,

1. $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 dx}$ with $U = C^1([0,1]^2)$ relevant for application.
2. Discrete version: $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:
 - (a) complementary subspaces X^\perp
 - (b) boundedness and compactness
 - (c) continuity

- (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ This lecture took place on 2019/03/07.

Define $U = C^1([0, 1]^2)$. Let Y is some Banach space and $K : U \rightarrow Y$ is linear and continuous.

Consider the problem (P_∞) given by $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2}$.

Proposition 0.2. *There exists a solution of (P_∞) .*

Proof. Take $(u_n)_{n \in \mathbb{N}}$ as a sequence in U such that $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_n \rightarrow \inf_{u \in U} (\dots)$. Now we want to show that $(u_n)_{n \in \mathbb{N}}$ is bounded.

Case 1: Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\Rightarrow \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\Rightarrow (u_n)_{n \in \mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n \in \mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change U to $U = \left\{ u : [0, 1]^2 \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]^2} |\nabla u|^2} < \infty \right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u \in U} R(u) + \lambda \|Ku - d\|_2$ instead.

In this setting, $(u_n)_{n \in \mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i \in \mathbb{N}}$).

Our next step is to use continuity to show that \hat{u} is a solution.

Problem: $u \mapsto \|u - d\|_2$ is, in general, not continuous with respect to weak convergence.

But it is always true that $\|\hat{u} - d\|_2 \leq \liminf_{i \rightarrow \infty} \|u_{n_i} - d\|_2$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = - \int_0^1 u \partial_x \varphi \quad \forall \varphi \in C^\infty([0, 1]^2)$$

$\varphi = 0$ in $K \setminus [0, 1]^2$ for some $K \in (0, 1)^2$.

$$\begin{aligned} \implies \int_{[0,1]^2} \nabla u \varphi &= - \int_{[0,1]^2} u \cdot (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \\ \forall \varphi : (\varphi_1, \varphi_2) &= C^\infty([0, 1]^2, \mathbb{R}^2) + \text{zero on boundary} \end{aligned}$$

We define $w : [0, 1]^2 \rightarrow \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w \varphi = - \int_{[0,1]^2} u (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \text{ holds } \forall \varphi$$

Then w is called *weak gradient* of u . We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i \rightarrow \infty} R(u_{n_i}) + \liminf_{i \rightarrow \infty} \lambda \|u_{n_i} - d\|_2 \\ &\leq \liminf_{i \rightarrow \infty} (R(u_{n_i}) + \lambda \|u_{n_i} - d\|_2) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

Case 2: Works as in the finite-dimensional setting using

- $X := \text{kernel}(A) \cap \text{kernel}(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \forall u \in \text{kernel}(\nabla)^\perp$ is called *Poincare inequality*.

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normed spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Complementary subspaces
8. Hilbert spaces

↓ This lecture took place on 2019/03/12.

Remark. 1. Literature: UGU, in particular: Biezis, Werner
2. In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified

1 Topological and metric spaces

Remark (Motivation). Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies

Definition 1.1 (Topology). Let X be a set and $\tau \subset \mathcal{P}(X) = \{\text{"set of subsets of } X\}$. We say that τ is a topology on X if

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. For any collection of sets $(U_i)_{i \in I}$ with I as some index set. We have $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$.

(X, τ) is called topological space.

A set $U \subset X$ is called open if $U \in \tau$ and is called closed if $U^c \in \tau$.

Remark. By the third property of topologies, $\bigcap_{i \in I} V_i$ is closed for any collection $(V_i)_{i \in I}$ of closed sets.

Definition 1.2 (Metric). Let X be a set, $d : X \times X \rightarrow \mathbb{R}$ be such that $\forall x, y, z \in X$

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric on X and (X, d) is called metric space.

Definition 1.3 (Norm). Let X be a vector space. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is called norm if $\forall x, y \in X, \lambda \in \mathbb{K}$

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Then $(X, \|\cdot\|)$ is called normed space.

Remark. If $\dim(x) < \infty$, all norms on X are equivalent.

Example. 1. Let X be a set then $\tau = \{\emptyset, X\}$ is a topology.

2. $(X, \mathcal{P}(X))$ is a topological space.

3. Define $S^{d-1} := \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\}$ and $d(x, y) := r$ where r is the length of the shortest connection between x and y on S^{d-1} . Then d is a metric on S^{d-1}

4. $X := \{u : [0, 1] \rightarrow \mathbb{R} \mid u \text{ is continuous}\}$ then $\|u\|_\infty := \sup_{x \in [0, 1]} |u(x)|$ is a norm on X

5. $l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u \text{ and } \sum_{i=1}^\infty |x_i|^p < \infty\}$ with $p \in [1, \infty)$ and $\|(x_i)_{i \in \mathbb{N}}\|_p := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$. Then $(l^p, \|\cdot\|_p)$ is a normed space (the proof will be done later).

Remark.

$$L^\infty := \left\{ (X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|(X_i)_{i \in \mathbb{N}}\| = \sup_i |X_i|$$

Proposition 1.4. Let X be a set.

1. If (X, d) is a metric space, define for $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ and $\tau = \{U \in \mathcal{P}(X) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\}$. Then (X, τ) is a topological space. We say that τ is the topology induced by d and we have that $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$
2. If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \rightarrow \mathbb{R}$ with $(x, y) \mapsto \|x - y\|$. Then (X, d) is a metric space and d is called the metric induced by $\|\cdot\|$.

Remark (Consequence). Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.

Definition 1.5. Let (X, τ) be a topological space. $U \subset X$. $x \in X$.

1. U is called a neighborhood of x if $\exists V \in \tau - x \in X \subset U : \mathcal{U}(x)$ is defined as the set of all neighborhoods of x
2.
 - x is called interior point of U if $U \in \mathcal{U}$
 - x is called adjacent point of U if $\forall V \in \tau$ such that $x \in V : V \cap U \neq \emptyset$
 - x is called cluster point of U if it is an adjacent point of $U \setminus \{x\}$.

The third property is stronger.

3. Notational conventions:

$$\overset{\circ}{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \overset{\circ}{U}$$

Proposition 1.6. Let (X, τ) be a topological space, $U \in X$. Then

1. U is open $\iff \mathring{U} = U$
2. U is closed $\iff \overline{U} = U$
3. $\mathring{U} = \bigcup_{\substack{V \in \tau \\ V \subset U}} V$ and \mathring{U} is open [" \mathring{U} is the largest open set in U "]
4. $\overline{U} = \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$ and \overline{U} is closed [" \overline{U} is the smallest closed set containing U "]

Proof. 3. \subset Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{V \subset U} V$

\supset Let $x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \subset U \implies x \in \mathring{U}$

\mathring{U} is open because it is the union of open sets.

1. $\implies \mathring{U} \subset U$ by definition. U is open, so $U \subset \bigcup_{V \subset U} V \stackrel{(3)}{=} \mathring{U}$
2. $\implies V \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \implies x \in U^C \in \tau$ and $U \cap U^C = \emptyset$. This contradicts to $x \in \overline{U}$.
 \Leftarrow Take $x \in U^C = \overline{U}^C$.
 $\stackrel{(4)}{\implies} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$
 $\implies V \cap U = \emptyset \implies V \subset U^C$
 $\implies U^C$ open $\implies U$ closed
4. We prove the fourth property without the second.
 \subset Take $x \in \overline{U}$. Take closed V such that $U \subset V$ if $x \notin V \implies x \in V^C$ which is open and $V^C \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.
 \supset Take $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$. Suppose $x \notin \overline{U}$.
 $\implies \exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$
 $\implies U \subset Z^C, Z^C$ closed, $x \notin Z^C$. This contradicts to $x \in \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$
 \overline{U} closed follows since the intersection of closed sets is closed.

□

Definition 1.7 (Limit). Let (X, τ) be a topological space, $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Henceforth, we write $(x_n)_n$ for $(x_n)_{n \in \mathbb{N}}$ and $\hat{x} \in X$. We say $x_n \rightarrow \hat{x}$ in τ as $n \rightarrow \infty$ (" x_n converges to x ", " x is limit of x_n ") if

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

Definition 1.8 (Proposition and definition). Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or Hausdorff) if

$$\forall x, y \in X \text{ with } x \neq y \exists U, V \in \tau : x \in U, y \in V \text{ and } U \cap V = \emptyset$$

- In a T_2 -sphere, the limit of any sequence is unique.
- If τ is induced by a metric, then (X, τ) is T_2 .

Proof. 1. Take $(x_n)_n$ to be a sequence and assume x_n converges to \hat{x} and \hat{y} with $\hat{x} \neq \hat{y}$. By T_2 , $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergenc, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\{n_x, n_y\} : x_n \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x, y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology τ . Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

Definition 1.9. Let (X, τ) be a topological space, $U \subset V \subset X$. We say that U is dense in V , if $V \subset \overline{U}$. We say that X is separable if there exists a countable, dense subset.

Definition 1.10. Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \rightarrow Y$ a function. We say f is continuous at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. f is called continuous if it is continuous at any $x \in X$.

Proposition 1.11. With $(X, \tau_X), (Y, \tau_Y)$ and f as above, f is continuous $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$

Proof. Left as an exercise to the reader.

□

Definition 1.12. Let (X, τ) be a T_2 topological space, $M \subset X$ called compact if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ (“ $(U_i)_{i \in I}$ is an open covering of M ”), there exists U_{i_1}, \dots, U_{i_n} such that $M \subset \bigcup_{k=1}^n U_{i_k}$ (“there exists a finite subcover”).

Remark. Compactness can also be defined without T_2 , this is also referred to as quasi-compact.

Remark (Exercise). Reconsider the previous results for metric and normed spaces.

↓ This lecture took place on 2019/03/14.

Definition 1.13. Let (X, d) be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in X . Then we say,

1. V is bounded if $\exists x \in X, r > 0$ such that $U \in B_r(x)$
2. $(x_n)_n$ is a Cauchy sequence if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$

3. X is complete if any Cauchy sequence in X admits a limit point
4. X is a Banach space if it is a normed space and complete

Proposition 1.14. *Let (X, d) be a metric space. $(x_n)_n$ be a sequence in X . Then*

1. $x_n \rightarrow x$ in the induced topology $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$
2. If $x_n \rightarrow x$, then $(x_n)_n$ is bounded as subset of X and $(x_n)_n$ is Cauchy.
3. If $U \subset X$ is closed and X is complete. Then (U, d) is a complete metric space.

Proof. 1. We prove both directions:

\implies True, since $B_\varepsilon(x)$ is open $\forall \varepsilon > 0$

\impliedby Let $x \in V$ with V open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$
 V open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$

$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1, \dots, n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in U , then $(y_n)_n$ is a Cauchy sequence in $X \implies \exists x \in X : y_n \rightarrow x$ as $n \rightarrow \infty$ if $x \notin U \implies x \in U^c \implies \exists n_0 \in \mathbb{N}$ such that $y_{n_0} \in U^c$ due to U^c open. This is a contradiction to $(y_n)_n$ in U

□

Proposition 1.15. *Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \rightarrow Y$. The following are equivalent (TFAE):*

- f is continuous (with respect to the induced topology)
- $\forall (x_n)_n$ such that $x_n \rightarrow x \implies f(x_n) \rightarrow f(x)$

Proof. Firstly, we prove that the first statement implies the second statement.

Take $(x_n)_n$ converging to x . Take $V \in \tau_Y$ such that $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$$\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X \text{ such that } x \in \hat{U} \subset U$$

$$\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_n) \rightarrow f(x)$$

Remark. 1. \implies 2. holds true in any topological space

2. \implies 1. Not.

Secondly, we prove that the second statement implies the first statement.

Suppose f is not continuous, find $x_n \rightarrow x$ such that $f(x_n) \rightarrow f(x)$ is wrong. If f is not continuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(u) \notin V \forall U \in \mathcal{U}(x)$

$$\implies \exists \hat{V} \in \tau_Y \text{ such that } f(u) \notin \hat{V} \forall U \in \mathcal{U}(x), f(x) \in \hat{V}$$

$$\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\implies (x_n)_n$ converges to x but $f(x_n) \notin \hat{V} \implies f(x_n) \not\rightarrow f(x)$. This gives a contradiction. \square

Definition 1.16. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$. f is uniformly continuous iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proposition 1.17. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$, $f : M \rightarrow Y$. If M is dense in X , Y is complete and f is uniformly continuous.

$$\implies \exists ! \hat{f} : X \rightarrow Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

Proof. Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \rightarrow x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(x_n), f(x_m)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in M , $\hat{y} \in Y$ such that $f(\hat{x}_n) \rightarrow \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X$, $(x_n)_n$ is a sequence in M such that $x_n \rightarrow x, f(x_n) \rightarrow y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$:

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d_Y(f(x_n), y) < 3\varepsilon$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.

2. \hat{f} is uniformly continuous.

\square

↓ This lecture took place on 2019/03/19.

Proposition 1.18. Let (X, d) be a metric space, $M \subset X$.

1. M is compact, so $\forall (X_i)_{i \in I}$ with X_i a closed set $\forall i$ such that $(\bigcap_{i \in I} X_i) \cap M = \emptyset$.

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \left(\bigcap_{i=1}^n X_{i_j} \right) \cap M = \emptyset$$

2. M is compact, so M is closed and bounded.

Proof. 1. We note that $\forall (X_i)_{i \in I}$ is a family of closed sets. $(X_i^C)_{i \in I}$ is a family of open sets and $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

Proposition 1.19. Let (X, d) be a metric space, $M \subset X$. TFAE:

1. M is compact.
2. Every infinite subset of M admits a cluster point.
3. Every sequence of M admits a convergent subsequence.
4. M is complete and totally bounded, where totally bounded is defined as

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$$

Remark. 1. totally bounded \implies bounded (proof is left as an exercise)

2. Assume $\dim(x) < \infty$. Compact \iff complete and bounded (see course Analysis I)

3. $\dim(x) < \infty \iff \overline{B_1(0)}$ is compact

where the last two items imply that X is a normed space.

Proof. 1 \rightarrow 2 If M is finite, (2) always holds true. So assume that M is infinite.

Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists x \in C$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(x_n)_n$ is a sequence of distinct points and $x_n \rightarrow x$. This implies that x is a cluster point of C . This gives a contradiction.

Now $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$. If M is compact, then

$$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \subset M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$\implies C$ is finite

This is a contradiction.

2 \rightarrow 3 Let $(x_n)_n$ be a sequence in M .

Case 1: $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies (x_n)_n$ admits a convergent sequence.

Case 2: $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid n \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

3 \rightarrow 4 Suppose that M is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in M as follows: Given x_1, \dots, x_n , choose $x_{n+1} \in M \setminus \bigcup_{j=1}^n B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in M and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convergent subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in M , then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \rightarrow x$ as $i \rightarrow \infty$. Since $(x_n)_n$ is Cauchy, $x_n \rightarrow x$ as $n \rightarrow \infty$ [left as an exercise]. Thus M is complete.

4 \rightarrow 1 Let $(U_i)_{i \in I}$ be an open covering of M and assume that $(U_i)_{i \in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}}(a)$. Define $\Omega := \{M \subset M \mid \tilde{M} \text{ is not covered by finitely many } (U_i)_i\}$. We recursively define a sequence $(a_n)_n$ in M such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n}} \cap B_{\frac{1}{2^{n-1}}}(a_{n-1}) \neq \emptyset$$

Goal: Show $(a_n)_n \rightarrow a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

Step 1 $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a \in E_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \rightarrow n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let

$$\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}.$$

Since $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$. [Take $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$, but if $B_{\frac{1}{2^{n+1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$$

Hence there exists a_{n+1} such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$ and $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

Step 2 Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) \leq d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. M is complete, so $\exists a \in M : a_n \xrightarrow{n \rightarrow \infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} \text{ and } \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for n sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

Proposition 1.20. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$ compact. Let $f : X \rightarrow Y$ be continuous. Then

1. $f(M)$ is compact
2. $f|_M : M \rightarrow Y$ is uniformly continuous.

Proof. 1. Let $(U_i)_{i \in I}$ be an open covering of $f(M)$

$$\implies (f^{-1}(U_i))_{i \in I} \text{ is an open covering of } M \text{ [why!]}$$

$$\implies \exists c_1, \dots, c_n \text{ such that } M \subset \bigcup_{i=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{i=1}^n U_{i_j}$$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n, (y_n)_n$ sequences in M satisfying condition (*). M is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \rightarrow \infty} 0$$

□

↓ This lecture took place on 2019/03/21.

Proposition 1.21 (Proposition and definition). Let (X, d_X) and (Y, d_Y) be metric spaces. $g : X \rightarrow Y$ is a function. g is called Lipschitz continuous if $\exists L > 0$ such that $d_Y(g(x), g(y)) \leq L d_X(x, y) \forall x, y \in X$. Any Lipschitz continuous function is uniformly continuous.

Proof. Left as an exercise to the reader.

□

Theorem 1.22 (Arzelà-Ascoli theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Define $C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$ and $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$. Then*

1. d_C is well-defined and $(C(X, Y), d_C)$ is a complete metric space
2. A set $M \subset C(X, Y)$ is compact iff
 - (a) $\forall x \in X$ the set $M_x := \{f(x) \mid f \in M\}$ is compact
 - (b) M is equicontinuous, i.e. $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. 1. Show that: $d_C(f, g) < \infty$.

Pick $f, g \in C(X, Y)$. Because X is compact, $f(X), g(X)$ compact $\implies f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that d_C is a metric is left as an exercise.

Show that $(C(X, Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X, Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because Y is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to d_C : Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x)) \leq \lim_{m \rightarrow \infty} d(f_m, f_n) < \varepsilon$ (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X, Y) \implies f_n \rightarrow f$ with respect to d_C . Show that $f \in C(X, Y)$. Take $\varepsilon > 0$. Let n_0 such that $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x))$.

In general, we have $\forall x, y, z \in (Z, d_Z)$ with d_Z as a metric.

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Proof.

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y) \quad (2)$$

$$d(y, z) \leq d(y, x) + d(x, z) \implies d(y, z) - d(x, z) \leq d(x, y) \quad (3)$$

$$(2) \text{ and } (3) \implies |d(x, z) - d(y, z)| \leq d(x, y) \quad (4)$$

□

Consequently, $\forall z \in Z, x_n \rightarrow x$ in Z : $d(x_n, z) \rightarrow d(x, z)$ since $|d(x_n, z) - d(x, z)| \leq d(x_n, x) \rightarrow 0$.

2. We need to prove both directions.

$$\begin{aligned} \implies & \text{ (a) For } x \in X \text{ fixed, define } g_X : M \rightarrow Y \text{ with } f \mapsto f(x). \text{ Then} \\ & d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2) \\ \implies & g_X \text{ is Lipschitz continuous, in particular continuous} \\ \implies & M_X = g_X(M) \text{ compact} \\ & \text{(b) Take } \varepsilon > 0. \text{ } M \text{ is totally bounded, so } \exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i). \forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies \\ & d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}. \text{ Define } \delta := \min_i \delta_i > 0. \text{ Then } \forall x, y \in X : \\ & d(x, y) < \delta \text{ and } \forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0}) \\ \implies & d(f(x), f(y)) \leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon \end{aligned}$$

\Leftarrow We prove the other direction.

↓ This lecture took place on 2019/03/26.

B is complete since it is a closed subset of a Banach space.

Show: M is totally bounded.

Consider $\varepsilon > 0$. Show: $\exists f_1, \dots, f_n$ such that $M \subset \bigcup_{i=1}^n B_{\varepsilon}(f_i)$.

- Because M is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X , $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_{\delta}(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{j=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{i_j})$

Compare with Figure 1.

Now, for each tuple of indices $(y_{1,j_1}, \dots, y_{n,j_n})$ define $f_{y_{1,j_1}, \dots, y_{n,j_n}} \in C(x, y)$ to be such that $f_{y_{1,j_1}, \dots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an f exists. The set F of all such functions is finite. We show that $M \subset \bigcup_{q \in F} B_{\varepsilon}(q)$.

Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \dots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_{\alpha} \in F$ accordingly.

Take $x \in X$ arbitrary and x_i such that $x \in B_{\delta}(x_i)$

$$\begin{aligned} \implies d(f(x), f_{\alpha}(x)) & \leq d(f(x), f(x_i)) + d(f(x_i), f_{\alpha}(x_i)) + d(f_{\alpha}(x_i), f_{\alpha}(x)) \\ & < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_{\alpha}) & = \sup_{x \in X} d(f(x), f_{\alpha}(x)) < \varepsilon \end{aligned}$$

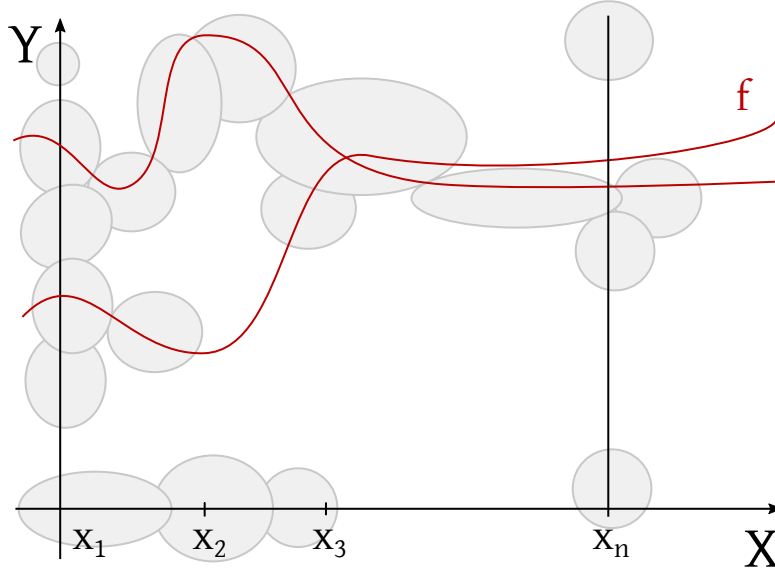


Figure 1: Covering of a function graph

□

Remark. Compare this to the fact that $B_1(0)$ in $C(X, Y)$ is not compact.

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

Remark (Motivation). In general, let (X, d) be a metric space. Let A and B be open and dense, then also $A \cap B$ is dense.

Proof. Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] \neq \emptyset$. Take $x \in X, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. A is dense. A is open, so $\exists \varepsilon_1 > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. B is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_{\varepsilon_1}(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally, $\forall A_1, \dots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many A_i ?

Theorem 1.23 (Baire theorem). Let (X, d) be a complete metric space. Let $(O_n)_{n \in \mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.

Proof. Let $D := \bigcap_{n \in \mathbb{N}} O_n$. Show that for $x \in X, \varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n \in \mathbb{N}}$.

n = 1 Take x_1, ε_1 such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

n - 1 → n Given $x_{n-1}, \varepsilon_{n-1}$, take x_n, ε_n such that

$$\overline{B_{\varepsilon_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_n}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete} \implies \exists x \in X : x_n \rightarrow x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_n}(x_N)} \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_N)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

Definition 1.24. Let (X, d) be a metric space, $M \subset X$. We say

- M is nowhere dense (dt. “nirgends dicht”), if $\overset{\circ}{M} = \emptyset$
- M is of first category $\iff M$ is a countable union of nowhere dense sets
- M is of second category $\iff M$ is not of first category

Theorem 1.25 (Baire category theorem (weaker version)). Let (X, d) be a complete metric space. Then (X, d) is of second category.

In other words (which is a useful formulation): If $X = \bigcup_{n \in \mathbb{N}} C_n \implies \exists n_0 : \overset{\circ}{C} \neq \emptyset$. In particular, if

$$X = \bigcup_{n \in \mathbb{N}} C_n \text{ with } C_n \text{ closed} \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

Proof. Suppose that $X = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O_n}$ with $\overset{\circ}{O_n} = \emptyset \forall n$

$$\overset{\circ}{O_n} = \emptyset \implies \overline{\overset{\circ}{O_n}} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overset{\circ}{O_n} = \emptyset \implies B_\varepsilon(x) \subset \overline{O_n} \implies \overset{\circ}{O_n} \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O_n}^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O_n}^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n \in \mathbb{N}} \overline{O_n}^C \text{ is dense}$$

$$\bigcap_{n \in \mathbb{N}} \overline{O_n}^C = \left(\bigcup_{n \in \mathbb{N}} \overline{O_n} \right)^C = X^C = \emptyset$$

gives a contradiction

□

Remark. 1. This is a fundamental theorem in Functional Analysis

2. This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise, e.g. Weierstrass function)

2 Normed space

2.1 Fundamentals

Definition 2.1. Let X be a vector space. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is called seminorm if

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$

The first property differs between a norm and a seminorm.

The tuple $(X, \|\cdot\|)$ is called a semi-normed space. We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.

Example (Not done in lecture). We found the following examples while studying:

$$f \text{ linear}, x \mapsto |f(x)| \quad \text{and} \quad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := |y|$$

Definition 2.2 (Definition and proposition). Let $(X, \|\cdot\|)$ be a semi-normed space and $(x_n)_n$ be a sequence in X . We say that

- $\sum_{n=1}^{\infty} x_n$ converges to $x \in X$ and write $x = \sum_{n=1}^{\infty} x_n$ if $\lim_{m \rightarrow \infty} \sum_{n=1}^m x_n = x$
- $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ converges $[\iff (\sum_{n=1}^m \|x_n\|)_m$ is bounded]

It holds that X is complete iff any absolutely converging series converges.

Proof. \implies Take $m_1 < m_2$ arbitrary, then

$$\begin{aligned} \left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| &\leq \sum_{n=m_1+1}^{m_2} \|x_n\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_1} \|x_n\| \leq \left\| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \right\| \\ &\implies \left(\sum_{n=1}^m x_n \right)_m \text{ is Cauchy} \implies \text{convergent} \end{aligned}$$

\Leftarrow Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick N_k such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$$

Define $y_k := x_{n_{k+1}} - x_{n_k} \implies \sum_k \|y_{n_w}\| \leq \sum_k 2^{-k} < \infty$

$$\implies \exists y \in X : \sum_{k=1}^n y_k \rightarrow y \text{ as } n \rightarrow \infty$$

$$\sum_{k=1}^n y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \rightarrow y - x_{n_1} \text{ as } n \rightarrow \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is convergent.

□

Remark. In \mathbb{R}^n , $\sum_n x_n$ is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction \implies is true.

↓ This lecture took place on 2019/03/28.

Proposition 2.3 (Proposition and definition). Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X . We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if

$$\exists m, M > 0 \forall x \in X : m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

TFAE:

1. $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. For any sequence $(x_n)_n$ and $x \in X$, $x_n \rightarrow x$ with respect to $\|\cdot\|_1 \iff x_n \rightarrow x$ with respect to $\|\cdot\|_2$
3. For any sequence $(x_n)_n$ we have,

$$x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_2$$

Proof. (1) \implies (2) \implies (3) is immediate.

It remains to show that:

(3) \implies (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n \|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\|y_n\|_1 = \frac{1}{n} \rightarrow 0$ hence $y_n \rightarrow 0$, but $\|y_n\|_2 > n \|y_n\|_1 = 1$.

$$\implies y_n \not\rightarrow 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

Remark. If $\dim(X) < \infty$, then any two norms on X are equivalent.

Definition 2.4 (Quotient spaces). Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation “ \sim ” on X with $x \sim y : \iff x - y \in Y$.

Then \sim defines an equivalence relation on X . We define

- $[X]_{\sim} = \{y \in X \mid x \sim y\}$ is the equivalence class of $x \in X$
- $X/Y := \{[x]_{\sim} \mid x \in X\}$ is the quotient space
- $\pi : \begin{cases} X \rightarrow X/Y \\ x \mapsto [x]_{\sim} \end{cases}$

Defining $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \quad \hat{0} := [0]$$

We get that:

1. X/Y is a vector space
2. $\|[x]\|_{X/Y} := \inf_{y \in [x]} \|y\|_X$ is a semi-norm.
3. If Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.
4. If X is complete and Y closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.

Proof. • Equivalence relation

- Vector space with “+” and “ $\lambda[x]$ ” are well-defined

This is left as an exercise to the reader.

2. – First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\|[0]\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|y\| \leq \|0\| = 0$$

- Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$.
Show that: $\|\lambda[x]\|_{X/Y} = |\lambda| \|[x]\|_{X/Y}$.
Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\|\lambda[x]\|_{X/Y} = \|[\lambda x]\|_{X/Y} = \inf_{y \in [\lambda x]} \|y\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \|y\| = \inf_{w \in [x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\|[x]\|_{X/Y}}$$

- Take $[x_1], [x_2] \in X/Y$, $\varepsilon > 0$. We note that

$$\|[x]\|_{X/Y} = \inf_{\substack{y \in X \\ w \in Y \\ w := x \cdot y}} \|y\| = \inf_{w \in Y} \|x - w\|$$

Hence we can take $y_1, y_2 \in Y$ such that $\|x_1 - y_1\| < \|[x_1]\|_{X/Y} + \varepsilon$
 $\varepsilon \in [1, 2)$.

$$\begin{aligned} \Rightarrow \|[x_1] + [x_2]\|_{X/Y} &= \|[x_1 + x_2]\|_{X/Y} \leq \|x_1 + x_2 - (y_1 + y_2)\| \\ &\leq \|x_1 - y_1\| + \|x_2 - y_2\| \leq \|[x_1]\|_{X/Y} + \|[x_2]\|_{X/Y} + 2\varepsilon \end{aligned}$$

Since ε was arbitrary, the assertion follows.

3. Suppose Y is closed if $\|[x]\|_{X/Y} = 0$, then

$$\inf_{y \in Y} \|x - y\| = 0 \implies \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n \rightarrow \infty} \|x - y_n\| = 0$$

$$Y \text{ closed} \implies x \in Y \implies [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in X/Y and suppose that $\sum_{i=1}^{\infty} \|[x_n]\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, X/Y is complete.

Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \|[x_n]\|_{X/Y} + 2^{-n}$

$$\implies \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} (\|[x_n]\|_{X/Y} + 2^{-n}) < c < \infty$$

$$X \text{ complete} \implies \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \quad \left\| [x] - \underbrace{\sum_{n=1}^m [x_n]}_{[x_n]} \right\|_{X/Y} \leq \left\| x - \underbrace{\sum_{k=0}^n \tilde{x}_k}_{\rightarrow 0} \right\|$$

□

↓ This lecture took place on 2019/04/02.

Corollary 2.5. Let X be a vector space with semi-norm $\|\cdot\|_X : X \rightarrow [0, \infty)$. Then

- $N = \{x \in X \mid \|x\|_X = 0\}$ is a subspace at X
- $\|[X]\| := \|X\|_p$ is a norm on X/N
- If X is complete, then $(X/N, \|\cdot\|)$ is a Banach space.

Proof. The proof is left as an exercise. □

Proposition 2.6. Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then

- \overline{U} is also a subspace.
- X is separable iff $\exists A \subset X$ complete such that $X = \overline{\mathcal{L}(A)}$ where $\mathcal{L}(A) = \{\sum_{i=1}^n \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathbb{K}, n \in \mathbb{N}\}$

Proof. • Left as an exercise

- \implies True since $\exists A \subset X$ countable such that $\overline{A} = X \implies \underline{X} = \overline{A} \subset \overline{\mathcal{L}(A)} \subset X$

\Leftarrow Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^n (\lambda_i + i\mu_i)x_i \mid \lambda_i, \mu_i \in \mathbb{X}, x_i \in A, n \in \mathbb{N} \right\}$$

where i is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then B is countable.

Show: $\forall x \in X \forall \varepsilon \exists x \in B : \|x - y\| < \varepsilon$.

Take $x \in X, \varepsilon > 0 \implies \exists x_0 \in \mathcal{L}(A) : \|x - x_0\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=1}^n (\lambda_i + i\mu_i)x_i$ with $\lambda_i, \mu_i \in \mathbb{R}, x_i \in A$. Choose $\lambda'_i, \mu'_i \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda'_i)^2 + (\mu_i - \mu'_i)^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^n \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let $y := \sum_{i=1}^n (\lambda'_i + i\mu'_i)x_i \in B$.

$$\begin{aligned} \implies \|x - y\| &\leq \|x - x_0\| + \|x_0 - y\| && \leq \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^n |(\lambda_i + i\varepsilon_i) - (\lambda'_i + i\mu'_i)| \|x_i\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^n \|x_i\|} = \varepsilon \end{aligned}$$

□

Proposition 2.7 (Proposition and definition). *Let $(X_i, \|\cdot\|_{x_i})$ for $i = 1, \dots, n$ be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \dots \otimes X_n = \bigotimes_{i=1}^n X_i = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

For $p \in [1, \infty]$, define

$$\|(x_1, \dots, x_n)\|_{\otimes_i X_i, p} = \begin{cases} \left(\sum_{i=1}^n \|x_i\|_{x_i}^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1, \dots, n} \|x_i\|_{x_i} & \text{if } p = \infty \end{cases}$$

Then

- $(\bigotimes_i X_i, \|\cdot\|_{\otimes_i X_i, p})$ is a normed space with respect to componentwise addition and multiplication.
- If all X_i are complete, then $\bigotimes_{i=1}^n X_i$ is complete.
- All norms $\|\cdot\|_{\otimes_i X_i, p}$ are equivalent.

Proof. • Vector space properties: Left as an exercise

- Norm: $\|x\|_{\otimes_i X_i, n} = 0 \iff x = 0$
 $\|\lambda x\|_{\otimes_i X_i, p} = |\lambda| \|x\|_{\otimes_i X_i, p}$

- Triangle inequality: $p = 1, p = \infty$
 $p \in (1, \infty)$. Take $x, y \in \bigotimes_{i=1}^n X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i, p}$.

$$\begin{aligned}
\Rightarrow \|x + y\|_p^p &= \sum_{i=1}^n \|x_i + y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\
&\leq \sum_{i=1}^n \|x_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} + \sum_{i=1}^n \|y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\
&\leq \underbrace{\left(\sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}}}_{\text{H\"older ineq.}} \cdot \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{i=1}^n \|y_i\|_{X_i}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \|x\|_p \|x + y\|_p^{p-1} + \|y\|_p \|x + y\|_p^{p-1} \\
&= (\|x\|_p + \|y\|_p) \cdot \|x + y\|_p^{p-1}
\end{aligned}$$

$$\Rightarrow \|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ if } x + y \neq 0 \text{ (trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in \mathbb{R}^n)

□

Definition 2.8. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \rightarrow Y$ is linear such that $\|j(x)\|_Y = \|x\|_X$ (hence j is injective) then j is called *isometric embedding* from X to Y . If j is bijective, then j is called *isometric isomorphism* and we say $X = Y$ up to isomorphism.

Proposition 2.9. Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_X)$ a Banach space such that

1. \exists isometric embedding, $i : X \rightarrow \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [\hat{X} can be regarded as completion of X]
2. If $j_1 : X \rightarrow Y$ is an isometric embedding on Y , a Banach space

$$\Rightarrow \exists i_2 : \hat{X} \rightarrow Y$$

an isometric embedding such that $j_2 \circ j = j_1$ and if $\overline{j_1(X)} = Y$ then j_2 is an isometric isomorphism. Thus “the completion is essentially unique”.

Proof. 1. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is Cauchy}\}$. \hat{X} is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \quad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define $\|(x_n)_n\|_{\hat{X}} := \lim_{n \rightarrow \infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in \mathbb{R}]. Then $\|\cdot\|_{\hat{X}}$ is a semi-norm (proof is left as an exercise). Setting $N = \{(X_n)_n \mid \|(X_n)_n\|_{\hat{X}} = 0\}$. By Corollary 2.5, $\hat{X} := \hat{X} \setminus N$ with $\|[(X_n)_n]\|_{\hat{X}} = \|(X_n)_n\|_{\hat{X}}$ is a normed space. Define

$$j : X \rightarrow \hat{X} \quad x \mapsto [(x)_n]$$

then j is linear and $\|j(x)\|_{\hat{X}} = \|[(x)_n]\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x\| = \|x\|$. So j is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n := j(x_n) \in \hat{X}$.

$$\begin{aligned} \Rightarrow \|y_m - [(x_n)_n]\|_{\hat{X}} &= \|(x_m)_n - (x_n)_n\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x_m - x_n\| \\ &= \lim_{n \geq n_0} \|x_m - x_n\| < \varepsilon \end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: \hat{X} is complete.

Let $(y_n)_n$ be Cauchy in \hat{X} . Pick $X_n \in X$ such that $\|j(x_n) - y_n\|_{\hat{X}} \leq \frac{1}{n}$
($j(x) = \hat{x}$)

$$\Rightarrow \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}} \leq \|j(x_n) - y_n\|_{\hat{X}} + \|y_n - y_m\|_{\hat{X}} + \|y_m - j(x_m)\|_{\hat{X}}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \|y_n - y_m\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick n_1 such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\Rightarrow \forall n, m > \max(n_0, n_0) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\Rightarrow (x_n)_n$ is Cauchy. Let $y := (X_n)_n \in \tilde{X}$. Then

$$\|y_n - [y]\|_{\hat{X}} \leq \|y_n - j(x_n)\|_{\hat{X}} + \|j(x_n) - [y]\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n \rightarrow \infty} \|x_n - x_m\|_X \xrightarrow{n \rightarrow \infty} 0$$

2. \downarrow This lecture took place on 2019/04/04.

Let $\hat{x} \in \hat{X} \Rightarrow \exists (x_n)_n \in X$ such that $j(x_n) \rightarrow \hat{x} \Rightarrow \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}}$.

$\Rightarrow (x_n)_n$ is a Cauchy sequence.

$\Rightarrow j_1(x_n)$ is a Cauchy sequence in Y .

$\Rightarrow \exists \lim_{n \rightarrow \infty} j_1(x_n) := y$

Using this, we define $j_2 : \hat{X} \rightarrow Y$ with $\hat{x} \mapsto \lim_{n \rightarrow \infty} j_1(x_1)$ where $j(x_1) \rightarrow \hat{x}$.

Well-defined? Take $\hat{x} \in \hat{X}$ and $j(x_n) \rightarrow \hat{x}$, $j(y_n) \rightarrow \hat{x}$.

$$\Rightarrow \|j_1(x_n) - j_1(y_n)\| = \|x_n - y_n\| = \|j(x_n) - j(y_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\implies \lim_{n \rightarrow \infty} j_1(x_n) = \lim_{n \rightarrow \infty} j_1(y_n) \implies j_1 \text{ well-defined}$$

Show linearity is left as an exercise. By isometry, take $\hat{x} \in \hat{X}$,

$$\underbrace{|i_2(\hat{x})|}_{j(x_n) \rightarrow \hat{x}} = \lim_{n \rightarrow \infty} \|j_1(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|i(x_n)\| = \|\hat{x}\|$$

Show: $j_2 \circ j = j_1$. Take $x \in X \implies (x_n)$ is such that $j(x) \rightarrow j(x) \implies j_2(j(x)) = \lim_{n \rightarrow \infty} j_1(x) = j_1(x)$.

Assume that $\overline{j_1(X)} = Y$. Take $y \in Y$. Find $\hat{x} \in \hat{X}$ such that $i_2(\hat{x}) = y$. By $\overline{j_1(X)} = Y \implies \exists (x_n)_n$ in X such that $i_1(x_n) \rightarrow Y \implies (j_1(x_n))_n$ is Cauchy.

$$\implies (x_n)_n \text{ Cauchy} \implies (j(x_n))_n \text{ Cauchy}$$

$$\stackrel{\hat{X} \text{ complete}}{\implies} \exists \hat{x} \text{ such that } \lim_{n \rightarrow \infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n \rightarrow \infty} j_2(x_n) = Y$$

□

2.2 Important examples of normed spaces

Definition 2.10 (Basic notation). Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \rightarrow \mathbb{K}^M$ with $N, M \in \mathbb{N}$.

- We call Ω a domain (dt. "Gebiet") if Ω is open and connected, where connected means that $\forall x, y \in \Omega$ there is a curve in Ω connecting X and Y .
- For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$. If f is r -times continuously differentiable, we set for $\alpha \in \mathbb{N}_0^N$, $\{\alpha\} \leq r$.

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} f$$

where $\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}$ is the partial derivative of f with respect to x_i of order α_i .

Example 2.11. Let $N = 2$ and $\alpha = (1, 1)$.

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

Let $\alpha = (2, 0)$.

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- For $z \in \mathbb{K}^N$ we denote $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$.¹
- We say $E \subset \Omega$ is compact in Ω and we write $E \Subset \Omega$ if E is compact.

Remark. If $E \Subset \Omega$, then $\exists \delta > 0 : \inf \{\|x - y\| \mid x \in E, y \in \partial\Omega\} > 0$.

¹This is an abuse of notation with $|\alpha|$ for $\alpha \in \mathbb{N}_0^N$

Proof. Left as an exercise (use compactness) □

- f is compactly supported in Ω if $\text{supp}(f) \ll \Omega$.
- $\text{supp}(f) := \overline{\{x \in \Omega \mid \|f(x)\| > 0\}}$

↓ This lecture took place on 2019/04/09.

Definition 2.12 (Definition and proposition, Spaces of continuous functions).
Let $\Omega \subset \mathbb{R}^N$ be a domain. We define

$$\begin{aligned} C_b(\Omega, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ bounded}\} \text{ with } \|\varphi\|_{C_b} = \|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)| \\ C(\overline{\Omega}, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ can be continuously extended to } \overline{\Omega}\}, \|\varphi\|_C := \|\varphi\|_\infty \\ C^r(\overline{\Omega}, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid D^\alpha \varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \leq r\} \text{ and } \|\varphi\|_{C^r} = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq r}} \|D^\alpha \varphi\|_\infty \\ C_C^r(\Omega, \mathbb{K}^M) &= \{\varphi : \Omega \rightarrow \mathbb{K}^M \mid \text{supp}(\varphi) \ll \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M)\} \text{ and } \|\varphi\|_{C_C^r} = \|\varphi\|_{C^r} \\ C^\infty(\overline{\Omega}, \mathbb{K}^M) &= \bigcap_{r \in \mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M) \\ D(\Omega, \mathbb{K}^M) &= C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r \in \mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \text{ in } C^r(\overline{\Omega}, \mathbb{K}^M) \end{aligned}$$

Then for any bounded Ω , C^r, C_0^r, C_b are Banach spaces and C_C^r is a normed space.

Recall: $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

Proof. The functions $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$ are norms (proof is left as an exercise).

Show that C_b is complete: Take $(\varphi_n)_n$ in C_b to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because $|\varphi_n(x) - \varphi_m(x)| \leq \|\varphi_n - \varphi_m\|_\infty$. Hence we can define $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$.

Show: $\varphi_n \rightarrow \varphi$ in $\|\cdot\|_\infty$. Take $\varepsilon > 0$. Show that $\exists n_0 \forall n \geq n_0 : \|\varphi - \varphi_n\|_\infty < \varepsilon$.

Take n_0 such that $\forall n, m \geq n_0 : \|\varphi_n - \varphi_m\|_\infty < \varepsilon$. Take $m \geq n_0$.

$$\implies \forall x \in \Omega : |\varphi(x) - \varphi_m(x)| = \lim_{\substack{n \rightarrow \infty \\ n \geq n_0}} |\varphi_n(x) - \varphi_m(x)| < \|\varphi_n - \varphi_m\|_\infty$$

Show: φ is bounded, i.e. $\exists C > 0 : |\varphi(x)| \leq C < \|\varphi_n - \varphi_m\|_\varepsilon < \infty$. Take n such that $\|\varphi - \varphi_n\|_\infty < 1$

$$\implies \forall x \in \Omega : |\varphi(x)| > |\varphi(x) - \varphi_n(x)| + \underbrace{|\varphi_n(x)|}_{=C} \leq 1 + \|\varphi_n\|_\infty$$

Now $C^r(\overline{\Omega}, \mathbb{K}^n)$ is a subspace of $C^b(\Omega, \mathbb{K}^n)$. Also $C^r(\overline{\Omega}, \mathbb{K}^n)$ is closed, since the uniform limit of $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$ with respect to $\|\cdot\|_{C^r}$ is again in $C^r(\overline{\Omega}, \mathbb{K}^M)$ [a result from Analysis].

$\implies C^r(\overline{\Omega}, \mathbb{K}^M)$ is a Banach space

$C_C^r(\overline{\Omega}, \mathbb{K}^M)$ is closed by definition, hence Banach.

$C_C^r(\Omega, \mathbb{K}^M)$ is a vector space, since $\forall \lambda \in \mathbb{K} : \varphi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\lambda\varphi) = \text{supp}(\varphi)$ and for $\varphi, \Psi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\varphi + \Psi) \ll \Omega$. \square

Definition 2.13 (Definition and proposition). *Let (Ω, Σ, μ) with $\Omega \subset \mathbb{R}^N$ be a measure space (i.e. Σ is a sigma algebra and μ is a measure). For $p \in [1, \infty)$, we define*

$$\begin{aligned} \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) &= \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid f \mu - \text{measurable and } \int_{\Omega} |f(x)|^p d\mu(x) < \infty \right\} \\ \|f\|_p^* &= \left(\int_{\Omega} \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}} \\ \mathcal{L}^{\infty}(\Omega, \mathbb{K}^M, \mu) &:= \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} |f(x)| < \infty \right\} \\ \|f\|_{\infty}^* &= \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)| \end{aligned}$$

Our proposition is that these are semi-norms.

Proof. Show that $\|\cdot\|_p^*$ for $p \in [1, \infty]$ are seminorms.

They cannot be norms since $\|f\|_p^* = 0$ for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0$. \square

Proposition 2.14 (Hölder inequality). *Let $p \in [1, \infty]$ and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \text{if } p \in (1, \infty) \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

If $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $g \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$ then for both

$$\begin{aligned} f \cdot g : \Omega &\rightarrow \mathbb{K} \text{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)} \\ f \otimes g : \Omega &\rightarrow \mathbb{K}^M \text{ with } x \mapsto (f_i(x), g_i(x))_{i=1}^M \end{aligned}$$

we have that $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$ and $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$ and $\|f \otimes g\|_1^* \leq \|fg\|_1^* \leq \|f\|_p^* \cdot \|g\|_q^*$.

Proof. **Case $p \in (1, \infty)$:** Intermediate result: $\forall \sigma, \tau \geq 0, r \in (0, 1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$ [AGM-inequality].

Proof.

Case $\sigma = 0$ or $\tau = 0$: immediate

Case $\sigma, \tau > 0$:

$$\begin{aligned} \log(\sigma^r \tau^{1-r}) &= r \log(\sigma) + (1-r) \log(\tau) \leq \log(r\sigma + (1-r)\tau) \\ &\text{since } \log''(x) \leq 0 \text{ implies that } \log \text{ is concave} \\ \log \text{ is monotonic} &\implies \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau \end{aligned}$$

□

Let $A := \left(\|f\|_p^*\right)^p$ and $B := \left(\|g\|_q^*\right)^q$ with $r = \frac{1}{p} \in (0, 1]$ we get

$$\begin{aligned} \forall x \in \Omega : \left(\frac{|f(x)|^p}{A} \right)^{\frac{1}{p}} \left(\frac{|g(x)|^q}{B} \right)^{\frac{1}{q}} &= \frac{1}{p} \frac{|f(x)|^p}{A} + \frac{1}{q} \frac{|g(x)|^q}{B} \\ \implies \frac{\int_{\Omega} |f(x)| |g(x)| d\mu(x)}{A^{\frac{1}{p}} B^{\frac{1}{q}}} &\leq \frac{1}{p} \frac{\int_{\Omega} |f(x)|^p d\mu(x)}{A} + \frac{1}{q} \frac{\int_{\Omega} |g(x)|^q d\mu(x)}{B} \\ \implies \int_{\Omega} |f(x)| |g(x)| d\mu(x) &\leq \|f\|_p^* \|g\|_q^* = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Now: $\|f \cdot g\|_x^* \leq \|f\|_p^* \cdot \|g\|_q^*$ follows since $|\langle x, y \rangle| \leq |x| |y| \forall x, y \in \mathbb{K}^M$.

Also:

$$\begin{aligned} \forall x \in \Omega : |f \otimes g(x)| &= \sum_{i=1}^M |f_i(x)| |g_i(x)| = \begin{pmatrix} |f_1(x)| & |g_1(x)| \\ \vdots & \vdots \\ |f_n(x)| & |g_n(x)| \end{pmatrix} \leq |f(x)| |g(x)| \\ \implies \int_{\Omega} |f \otimes g(x)| d\mu(x) &\leq \|f\|_p^* \cdot \|g\|_q^* \end{aligned}$$

Case $p \in \{1, \infty\}$: Without loss of generality assume that $p = 1, q = \infty$. $\forall N \in \Sigma$ with $\mu(N) = 0$ we get

$$\begin{aligned} \int_{\Omega} |f(x)| |g(x)| d\mu(x) &= \int_{\Omega \setminus N} |f(x)| |g(x)| d\mu(x) \\ &\leq \int_{\Omega \setminus N} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| = \int_{\Omega} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |g(x)| \end{aligned}$$

Taking the infimum over all such N , then

$$\int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_1^* \cdot \|g\|_{\infty}^*$$

And the result follows again from $|\langle x, y \rangle| \leq |x| \cdot |y|$ and componentwise $|\langle x_i, y_i \rangle_i| \leq |x| |y| \forall x, y \in \mathbb{K}^M$

□

Proposition 2.15 (Minkowski inequality). *For $p \in [1, \infty]$, $f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$, we have that $\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*$ with $\|f\|_{\infty} := \inf_{\mu(N) \rightarrow 0} \sup_{x \in \Omega \setminus N} |f(x)|$.*

Proof. **Case $p = 1$:** trivial

Case $p \in (1, \infty)$:

$$\begin{aligned} (\|f + g\|_p^*)^p &= \int_{\Omega} |f(x) + g(x)|^p d\mu(x) \\ &= \int_{\Omega} |f(x)| \cdot |f(x) + g(x)|^{p-1} d\mu(x) \\ &\quad + \int_{\Omega} |g(x)| |f(x) + g(x)|^{p-1} d\mu(x) \\ &\leq \|f\|_p^* \cdot \| |f + g|^{p-1} \|_q^* + \|g\|_p^* \cdot \| |f + g|^{p-1} \|_q^* \end{aligned}$$

Recognize that $(\int |f + g|^p)^{\frac{1}{p}} = (\int |f + g|^{(p-1)q})^{\frac{1}{q}}$ because $p = q \cdot (p-1)$

$$\begin{aligned} &= (\|f\|_p^* + \|g\|_p^*) \| |f + g|^{p-1} \|_q^* \\ \implies \|f + g\|_p^* &\leq \|f\|_p^* + \|g\|_p^* \end{aligned}$$

↓ This lecture took place on 2019/04/11.

Case $p = \infty$: First, note that $\forall f \in \mathcal{L}^{\infty}(\Omega, \mathbb{K}^M, \mu) \exists N \in \Sigma$ such that $\mu(N) = 0$ and $\|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty} := \sup_{x \in \Omega \setminus N} |f(x)|$.

Claim 2.16.

$$\|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty} := \sup_{x \in \Omega \setminus N} |f(x)| = \sup_{x \in \Omega \setminus \hat{N}} |f(x)| \text{ for } \mu(\hat{N}) = 0$$

Proof. For all $n \in \mathbb{N}$, define $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $\|f|_{\Omega \setminus N_n}\|_{\infty} \leq \|f\|_{\infty}^* + \frac{1}{n}$. Thus with $N := \bigcup_{n \in \mathbb{N}} N_n \implies \mu(N) = 0$ and $\|f\|_{\infty}^* \leq \|f|_{\Omega \setminus N}\|_{\infty} \leq \|f\|_{\infty}^* + \frac{1}{n}$. $n \rightarrow \infty \implies \|f\|_{\infty}^* = \|f|_{\Omega \setminus N}\|_{\infty}$. □

For $f, g \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu)$, pick N_f, N_g such that $\mu(N_c) = \mu(N_g) = 0$ and $\|f\|_\infty^* = \|f|_{\Omega \setminus N_f}\|_\infty$ and $\|g\|_\infty^* = \|g|_{\Omega \setminus N_g}\|_\infty$.

$$\begin{aligned} \Rightarrow \|f + g\|_\infty^* &\leq \|(f + g)|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus (N_f \cup N_g)}\|_\infty + \|g|_{\Omega \setminus (N_f \cup N_g)}\|_\infty \\ &\leq \|f|_{\Omega \setminus N_f}\|_\infty + \|g|_{\Omega \setminus N_g}\|_\infty = \|f\|_\infty^* + \|g\|_\infty^* \end{aligned}$$

□

Proposition 2.17. *Let $p \in [1, \infty]$. Then $\|\cdot\|_p^*$ is a seminorm on $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$ and $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$ is complete with the seminorm. With $M := \{f \in \mathcal{L}^\infty \mid \|f\|_p^* = 0\}$, we get that $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)/M$ is a Banach space with respect to $\|[f]\|_p := \|f\|_p^*$.*

Proof. Seminorm is clear by Minkowski's inequality. Give completeness of $f^p(\cdot)$, the rest follows from Corollary 2.5.

Hence, show that $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ is complete.

Assume $p < \infty$. By Proposition 2.2, it suffices to show that for $f_n(t_n)_n$ in $\mathcal{L}^p(\cdot)$ such that $a := \sum_{n=1}^\infty \|f_n\|_p^* < \infty$.

$$\Rightarrow \exists f \in \mathcal{L}^p(\cdot) : f = \sum_{n=1}^\infty f_n$$

Define $\hat{q}(x) := \sum_{n=1}^\infty |f_n(x)| \in [0, \infty]$. Define $\hat{q}_n(x) := \sum_{i=1}^n |f_i(x)|$. Then q_n is measurable and by Minkowski's inequality,

$$\|q_n\|_p^* \leq \sum_{i=1}^n \|f_i\|_p^* \leq \sum_{i=1}^\infty \|f_i\|_p^* = a < \infty$$

Also $\hat{q}_n^p : x \rightarrow \hat{q}_n(x)^p$ is a sequence of positive functions and it is monotonically increasing and converging to \hat{g}^p .

By Beppo-Levi (from measure theory):

$$\int_\Omega \hat{g}^p = \lim_{n \rightarrow \infty} \int_\Omega \hat{q}_n^p = \lim_{n \rightarrow \infty} (\|q_n\|_p^*)^p = a^p < \infty$$

$\Rightarrow \hat{g}^p < \infty$ almost everywhere (except for a μ zero-set). Define $g : \Omega \rightarrow \mathbb{R}$,

$$x \mapsto \begin{cases} \hat{g}(x) & \text{if } \hat{g}(x) < \infty \\ 0 & \text{else} \end{cases}$$

We get that $g \in \mathcal{L}^p(\Omega, \mathbb{R}, \mu)$ and $g(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n |f_i(x)|$ μ -almost everywhere. Furthermore, by completeness of \mathbb{K}^M , $f(x) := \sum_{i=1}^\infty f_i(x)$ exists for μ -almost everywhere. $x \in \Omega$.

Show: $f = \sum_{i=1}^{\infty} f_i$ in $\mathcal{L}^n(\cdot)$, i.e. show that $\lim_{n \rightarrow \infty} \int_{\Omega} \left| \sum_{i=1}^{\infty} f_i \right|_{d_N}^p = \sigma$.

$$\left\| \sum_{i=1}^{n-1} f_i - \sum_{i=1}^{\infty} f_i \right\|_p^* = \left\| \sum_{i=n}^{\infty} f_i \right\|_p^* \xrightarrow{!} 0$$

By construction, $|f| \leq q$ almost everywhere $\implies \int_{\Omega} |f|^p \leq \int_{\Omega} q^p < \infty$. Set $h_n(x) = \left| \sum_{i=n}^{\infty} f_i(x) \right|^p$. Then $h_n(x) \rightarrow 0$ for μ -almost everywhere $x \in \Omega$ and $h_n(x) \geq 0$ and

$$0 \leq h_n(x) \leq \left(\sum_{i=n}^{\infty} |f_i(x)| \right)^p \leq q(x)^p$$

Hence, by the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\Omega} h_n(x) = \int_{\Omega} \lim_{n \rightarrow \infty} h_n(x) = 0$$

This completes the assertion since

$$\int_{\Omega} h_n(x) = \int_{\Omega} \left| \sum_{i=n}^{\infty} f_i(x) \right|^p = \int_{\Omega} \left| \sum_{i=1}^{n-1} f_i(x) - f(x) \right|^p = \left(\left\| \sum_{i=1}^{n-1} f_i - f \right\|_p^* \right)^p$$

□

↓ This lecture took place on 2019/04/30.

Proposition (Proposition 2.15 again). Let $p \in [1, \infty]$. Then $\|\cdot\|_{L^p}$ is a seminorm, $\mathcal{L}^p(\Omega, \mathbb{K}^n, \mu)$ is complete and $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\cdot)/N$ where $N = \{f \mid \|f\|_{L^p} = 0\}$ is a Banach space.

Proof. Assume $p \in [1, \infty]$, then the proof of the last lecture is given.

Assume $p = \infty$. Let $(f_n)_n$ be Cauchy in \mathcal{L}^{∞} . Remember: $\|f\|_{L^{\infty}} := \inf_{\mu(N)=0} \sup_{x \in \Omega \setminus N} |f(x)|$.

Pick $N_{n,m}$ such that $\mu(N_{n,m}) = 0$ and $|f_n - f_m|_{\infty} = \left\| (f_n - f_m)|_{\Omega \setminus N_{n,m}} \right\|_{\infty}$. Set $N = \bigcup_{n,m} N_{n,m} \implies \mu(N) = 0$.

Then \tilde{f} is the uniform limit of $f_n \cdot \mathbf{1}_{\Omega \setminus N}$. Hence \tilde{f} is measurable. Also $\|\tilde{f}\|_{L^{\infty}} := \inf_{\mu(M)=0} \sup_{x \in \Omega \setminus M} |f(x)| \leq \|f\|_{\infty} \implies \tilde{f} \in L^{\infty}(\Omega, \mathbb{K}^n, \mu)$. Also $\|f_n - \tilde{f}\|_{L^{\infty}} = \left\| (f_n - f)|_{\Omega \setminus N} \right\|_{\infty} = \|f_n|_{\Omega \setminus N} - \tilde{f}\|_{\infty} \rightarrow 0$ as $n \rightarrow \infty$. □

Now $(f_n|_N)_n$ is Cauchy with respect to $\|\cdot\|_{\infty}$. Since $\forall n, m$:

$$\begin{aligned} \|f_n|_N - f_m|_N\|_{\infty} &= \|(f_n - f_m)|_N\|_{\infty} \\ &\leq \|(f_n - f_m)|_{N_{m,n}^c}\|_{\infty} \\ &= \|f_n - f_m\|_{L^{\infty}} \end{aligned}$$

As in the proof of C_b being a Banach space:

$$\implies \exists f : \Omega \setminus N \rightarrow \mathbb{K}^M : \|f\|_\infty < \infty \text{ and } f_n|_{N^c} \rightarrow f \text{ w.r.t. } \|\cdot\|_\infty$$

Remark (Important special cases). **Case 1** $\mu = \mathcal{L}^N$ is the Lebesgue measure on $\Omega \subset \mathbb{R}^N$ (a domain). In this case we write $L^p(\Omega, \mathbb{K}^M) := L^p(\Omega, \mathbb{K}^M, \lambda^M)$ and $L^p(\Omega) := L^p(\Omega, \mathbb{K})$. Here the space $L^p(\Omega, \mathbb{K})$ is considered as functions which are defined almost everywhere.

Case 2 Set $\Omega = \mathbb{N}, \sigma = \mathbb{P}(\mathbb{N}), \mu_c(A) = |A|$.

Then

- $f : \Omega \rightarrow \mathbb{K}^M$ is identified with a sequence $(x_n)_n$ with $x_n \in \mathbb{K}^M$.
- $\int_\Omega f(x) d\mu(x) \sim \sum_{i \in \mathbb{N}} x_i \in \mathbb{K}^M$
- $\mu_c(A) = 0 \iff A = \emptyset$ and the equivalence class construction becomes obsolete.

And we denote,

$$\ell^p(\mathbb{N}, \mathbb{K}^M) = \mathcal{L}^p(\mathbb{N}, \mathbb{K}^M, \mu_c) \quad \ell^p := \ell^p(\mathbb{N}) = \ell^p(\mathbb{N}, \mathbb{K})$$

2.2.1 Basic properties of Lebesgue spaces

Proposition 2.18. The space $\ell^p(\mathbb{N}, \mathbb{K}^M)$ is separable for $p \in [1, \infty]$ and not separable for $p = \infty$.

Proof. $p < \infty$ Define $l_{i,j} \in \ell^p(\mathbb{N}, \mathbb{K}^M)$ as

$$(l_{ij})_k := \begin{cases} 0 & \text{if } i \neq k \\ \left(0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0\right)^T & \text{if } i = k \end{cases}$$

Then $A := \{e_{ij} \mid i \in \mathbb{N}, j \in \{1, \dots, M\}\}$ is countable.

It suffices to show that $\overline{\text{span}(A)} = \ell^p(\mathbb{N}, \mathbb{K}^M)$.

This is true since $\forall x \in \ell^p(\mathbb{N}, \mathbb{K}^M) : \forall \varepsilon > 0 \exists n_0 : \sum_{i=n_0+1}^\infty |x_i|^p < \varepsilon$ and hence

$$\left\|x - \sum_{i=1}^{n_0} \sum_{j=1}^M x_{ij} e_{ij}\right\|^p = \left(\sum_{i=n_0+1}^\infty |x_i|^p\right)^{\frac{1}{p}} < \varepsilon^{\frac{1}{p}}$$

$p = \infty$ It suffices to show that $L^\infty(\mathbb{N})$ is not separable (why?). For $M \subset \mathbb{N}$ define $\mathbf{1}_M \in L^\infty$. Then $\Delta := \{\mathbf{1}_M \mid M \subset \mathbb{N}\}$ is uncountable.

For $A \subset L^\infty$ countable and $x \in A$ set $M_x = \left\{y \in L^\infty \mid \|x - y\|_\infty < \frac{1}{3}\right\} = B_{\frac{1}{3}}(x)$. Then each M_x contains at most one element of Δ since if $\mathbf{1}_M \neq \mathbf{1}_{M'}$ are such that $\mathbf{1}_M, \mathbf{1}_{M'} \in M_x$.

$$\implies 1 = \|\mathbf{1}_M - \mathbf{1}_{M'}\|_\infty \leq \|\mathbf{1}_M - x\| + \|\mathbf{1}_{M'} - x\| < \frac{2}{3}$$

This gives a contradiction.

Δ is uncountable, $\{M_x \mid x \in A\}$ is countable.

$$\implies \exists \hat{M} \in \mathbb{N} : \mathbf{1}_{\hat{M}} \notin M_x \forall x \in A$$

$$\implies \|\mathbf{1}_{\hat{M}} - x\|_\infty \geq \frac{1}{3} \forall x \in A$$

Hence, A is not dense. Since A was arbitrary countable. Thus L^∞ is not separable.

□

2.2.2 Separability of L^p requires a density result

Proposition 2.19. *Let $f \in L^p(\mathbb{R}^N, \mathbb{K}^M)$. Let $p < \infty$. Then $\exists (f_n)_n \in \dots C_c(\mathbb{R}^N, \mathbb{R}^M)$ such that $\|f_n - f\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Step 1 Reduction to step functions with $E \in \Sigma$.

$$\xi_E(x) := \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$$

Take $f \in L^p(\dots)$. For $\varepsilon > 0$, define

$$E_\varepsilon = \{x : \varepsilon \leq |f| \leq \frac{1}{\varepsilon}\}$$

Then $E_\varepsilon \in \Sigma$ and $\int_{\mathbb{R}^N} |f|^p \geq \varepsilon^p |E_\varepsilon|$ where $|E_\varepsilon| := L^N(E_\varepsilon)$.

$$|E_\varepsilon| < \infty \text{ and } \int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f| \leq \frac{1}{\varepsilon} \cdot |E_\varepsilon| < \infty$$

$$\implies \mathbf{1}_{E_\varepsilon} f \text{ is integrable} \implies \exists (q_{n,\varepsilon})_n \text{ step functions}$$

such that $\int_{\mathbb{R}^N} |\mathbf{1}_{E_\varepsilon} f - q_{n,\varepsilon}| \rightarrow 0$ as $n \rightarrow \infty$. Define

$$f_{n,\varepsilon}(x) := \begin{cases} q_{n,\varepsilon}(x) & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| \leq \frac{2}{\varepsilon} \\ \frac{2}{\varepsilon} \frac{q_{n,\varepsilon}(x)}{|q_{n,\varepsilon}(x)|} & \text{if } x \in E_\varepsilon, |q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon} \\ 0 & \text{else} \end{cases}$$

Hence $(f_{n,\varepsilon})_n$ is a sequence of step functions. For $x \in E_\varepsilon$ such that $|q_{n,\varepsilon}(x)| > \frac{2}{\varepsilon}$.

$$\implies |f_{n,\varepsilon}(x) - f(x)| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} = \frac{3}{\varepsilon} \leq 3 \underbrace{(|q_{n,\varepsilon}(x)| - |f(x)|)}_{\geq \frac{1}{\varepsilon}} \leq 3 |q_{n,\varepsilon}(x) - f(x)|$$

$$\int_{\mathbb{R}^N} |f_{n,\varepsilon}(x) - X_{E_\varepsilon}(x) f(x)| dx \leq 3 \int_{\mathbb{R}^N} |g_{n,\varepsilon}(x) - \mathbf{1}_{E_\varepsilon}(x) f(x)| dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\int_{\mathbb{R}^N} |f - f_{n,\varepsilon}|^p \leq \int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p + \underbrace{\left(\frac{3}{\varepsilon}\right)^{p-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|}_{(*)} =: (X)$$

$$(*) = \int_{E_\varepsilon} |f - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \cdot \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}|^p = \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| \left(\underbrace{|f \mathbf{1}_{E_\varepsilon}|}_{\leq \frac{1}{3}} + \underbrace{|f_{n,\varepsilon}|^{p-1}}_{\leq \frac{2}{3}} \right)$$

Now given $\delta > 0$, we first fix $\varepsilon > 0$ such that $\int_{\mathbb{R}^N \setminus E_\varepsilon} |f|^p < \frac{\delta}{2}$. Then we find n_0 such that $\left(\frac{3}{\varepsilon}\right)^{n-1} \int_{\mathbb{R}^N} |f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}| < \frac{\delta}{2}$. This is possible since $\mathbb{R}^N = \bigcup_{\varepsilon>0} E_\varepsilon$ and $\int_{\mathbb{R}^N} |f|^n < \infty$.

$$\implies (X) < \delta$$

Now suppose $\forall \varepsilon > 0 \forall E \in \Sigma : \exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K}^M)$ such that $\|\mathbf{1}_E - \varphi\| < \varepsilon$. We need to show that this is true. Then for $f \in L^p(\mathbb{R}^N, \mathbb{K})$, $\varepsilon > 0$, we pick

$$g = \sum_{i=1}^n \underbrace{c_i}_{\in \mathbb{K}^M} \cdot \underbrace{\mathbf{1}_{E_i}}_{\in \Sigma}$$

such that $\|f - g\|_p < \frac{\varepsilon}{2}$ (possible by what we just showed). For $i \in \mathbb{N}$, pick $\varphi_i \in C_c(\mathbb{R}^N, \mathbb{R})$ such that $\|\mathbf{1}_{E_i} - \varphi_i\|_p \leq \frac{2^{-i}\varepsilon}{|c_i|^2}$

$$\begin{aligned} \implies \left\| f - \underbrace{\sum_{i=1}^n c_i \cdot \varphi_i}_{\in C_c(\mathbb{R}^n, \mathbb{K}^n)} \right\|_p &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|c_i \mathbf{1}_{E_i} - c_i \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n |c_i| \cdot \|\mathbf{1}_{E_i} - \varphi_i\|_p \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n 2^{-i} \cdot \frac{\varepsilon}{2} \leq \varepsilon \end{aligned}$$

□

↓ This lecture took place on 2019/05/02.

Proof. Step 1 It is sufficient to approximate $f = \mathbf{1}_E$ for $E \in \Sigma$

Step 2 Reduce statement to $f = \mathbf{1}_Q$ where $Q = \times_{i=1}^N [a_i, b_i]$ with $a_i, b_i \in \mathbb{R}$. Take $f = \mathbf{1}_E$. Since Σ is generated by sets of the form $\times_{i=1}^N [a_i, b_i] \forall \varepsilon > 0$ there exists $(Q_i)_{i=1}^n, (\lambda_i)_{i=1}^n$ such that $\|f - \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}\|_1 < \varepsilon$ [Alt, A1 10, axiom L5].

Define $h_n(x) = \max(0, \min(1, q_n(x)))$ where $q_n := \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}$, also h_n is of the form of q_n and

$$\begin{aligned} |f(x) - h_n(x)| \leq 1 &\implies |f(x) - h_n(x)|^p \leq |f(x) - h_n(x)| \leq |f(x) - q_n(x)| \\ &\implies \|f - h_n\|_p \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

As in step 1, this reduces the assertion to $f = \mathbf{1}_Q$ with $Q = \times_{i=1}^N [a_i, b_i]$. For such $f = \mathbf{1}_Q$, define

$$g_i(s) := \begin{cases} \frac{b_i - a_i}{2} + \left|s - \frac{b_i + a_i}{2}\right| & \text{if } s \in [a_i, b_i] \\ 0 & \text{else} \end{cases}$$

for $i \in \{1, \dots, N\}$ and $\tilde{g}_{i,\varepsilon}(x) = \prod_{i=1}^N g_{i,\varepsilon}(x_i)$, we obtain that $\|\mathbf{1}_Q - \hat{g}_\varepsilon\|_p \rightarrow 0$ as $\varepsilon \rightarrow 0$.

$$\begin{aligned} \int_{\mathbb{R}^N} |\mathbf{1}_Q - \hat{g}_\varepsilon|^p &= \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{i=1}^N |\mathbf{1}_{[a_i, b_i]}(x) - \tilde{g}_{i,\varepsilon}(x)|^p dx \\ &= \prod_{i=1}^N \int_{a_i}^{b_i} |\mathbf{1}_{[a_i, b_i]}(s) - \tilde{g}_{i,\varepsilon}(s)|^p ds \\ &\leq \prod_{i=1}^N |I_{i,\varepsilon}| \text{ where } |I_{i,\varepsilon}| \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \end{aligned}$$

□

Remark. 1. If $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain, defining

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & \text{else} \end{cases}$$

we get that $\tilde{f} \in L^p(\mathbb{R}^N, \mathbb{K}^M)$ and using Proposition 2.19 for \tilde{f} we can approximate f by functions in $C(\bar{\Omega}, \mathbb{K}^M) \cap C_c(\mathbb{R}^N, \mathbb{K}^M)$.

2. Using “Mollification” Proposition 2.19 implies density of $\mathcal{D}(\Omega, \mathbb{K}^M)$ in $L^p(\Omega, \mathbb{K}^M)$ for $\Omega \subseteq \mathbb{R}^N$ a domain.

Proposition 2.20. Let $\Omega \subset \mathbb{R}^N$ measurable. Then $L^p(\Omega, \mathbb{K}^M)$ is separable for $1 \leq p < \infty$ and not separable for $p = \infty$.

Proof. **Case** $p = \infty$ Similar to l^∞ , will be done in the Exercises.

Case $1 \leq p < \infty$ We show the result for $L^p(\mathbb{R}^N, \mathbb{K})$, the general case is a direct consequence. Denote $\mathcal{R} := \{Q \subseteq \mathbb{R}^N \mid Q = \prod_{i=1}^N [a_i, b_i] \text{ with } a_n, b_n \in \mathbb{Q}\}$. Then \mathcal{R} is countable and it suffices to show that $E := \mathcal{L}(\{\mathbf{1}_Q \mid Q \in \mathcal{R}\})$ is dense. Take $f \in L^p(\mathbb{R}^N, \mathbb{K}), \varepsilon > 0$. Then $\exists \varphi \in C_c(\mathbb{R}^N, \mathbb{K})$ such that $\|f - \varphi\|_p \leq \frac{\varepsilon}{2}$. Now we need to find $h \in E$ such that $\|\varphi - h\|_p \leq \frac{\varepsilon}{2}$. Let

$M \subseteq \mathbb{R}^N$ be closed, bounded hypercube such that $\text{supp}(\varphi) \subset M$. φ is uniformly continuous on M .

$$\implies \forall \delta > 0 \exists \rho > 0 \forall x, y \in M : |x - y| < \delta \implies |\varphi(x) - \varphi(y)| < \delta$$

Now we take $(Q_i)_{i=1}^K$ a disjoint covering of M with $Q_i \in \mathcal{R}$, such that $|x - y| < \delta \forall x, y \in Q_i$. Now define $\lambda_i = \varphi(z)$ for some $z \in Q_i$, $i = 1, \dots, K$. Define $h(x) := \sum_{i=1}^K \lambda_i \mathbf{1}_{Q_i}$.

$$\implies \forall x \in \mathbb{R}^M : |\varphi(x) - h(x)| \leq |\varphi(x) - \lambda_i| \leq \delta$$

$$\implies \|\varphi - h\|_p = \left(\int_{\mathbb{R}^N} |\varphi(x) - h(x)|^p \right)^{\frac{1}{p}} \leq \delta \cdot |M|^{\frac{1}{p}}$$

Choose $\delta := \frac{\varepsilon}{2 \cdot |M|^{\frac{1}{p}}}$, then the result follows.

□

↓ This lecture took place on 2019/05/09.

Proposition 2.21. Let $p \in [1, \infty]$, $(f_n)_n$, $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain such that $f_n \rightarrow f$ in L^p .

Then there exists a subsequence $(f_{n_k})_k$ such that

1. $f_{n_k}(x) \rightarrow f(x)$ for almost every $x \in \Omega$
2. $\exists h \in L^p(\Omega)$ such that $(f_{n_k}(x)) \leq |h(x)|$ for almost every $x \in \Omega$

Proof. **Case $p = \infty$** Is left as an exercise to the reader.

Case $p \in [1, \infty)$ Pick $(n_k)_k$ such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq \frac{1}{2^k}$. Define $g_n := \sum_{k=1}^n |f_{n_{k+1}}(x) - f_{n_k}(x)|$. Then $g_n(x)$ is increasing, $g_n(x) \geq 0 \forall n$.

$\implies g_n(x)$ is convergent for almost every $x \in \Omega$. Hence we can define $g(x) := \lim_{n \rightarrow \infty} g_n(x) \in [0, \infty]$.

Also, $\|g_n\|_p \leq \sum_{i=1}^n \|f_{n_{i+1}} - f_{n_i}\|_p \leq 1$. By Beppo-Levi,

$$\int_{\Omega} |g(x)|^n dx = \lim_{n \rightarrow \infty} \int_{\Omega} |g_n(x)|^n = \lim_{n \rightarrow \infty} \|g_n\|_p^n \leq 1 \implies g \in L^p(\Omega)$$

especially $g(x) < \infty$ for almost every $x \in \Omega$.

$$\forall l \geq k \geq 1 : |f_{n_l}(x) - f_{n_k}(x)| \leq \sum_{i=k}^{l-1} |f_{n_{i+1}}(x) - f_{n_i}(x)| \leq g_{n_l}(x) - g_{n_k}(x) \stackrel{\text{monot.}}{\leq} g(x) - g_{n_k}(x)$$

$\implies (f_{n_k}(x))_k$ is Cauchy for almost every $x \in \Omega$ such that we can define $\hat{f}(x) := \lim_{k \rightarrow \infty} f_{n_k}(x)$.

$$|\hat{f}(x) - f_{n_k}(x)| \leq g(x) \text{ for almost every } x \in \Omega$$

By the Dominated convergence theorem, $\|f_{n_k} - \tilde{f}\|_n \rightarrow 0$ for $k \rightarrow \infty$. $\implies f = \tilde{f}$ almost every and hence $f_{n_k}(x) \rightarrow f(x)$ for almost every $x \in \Omega \implies (1)$. Also

$$|f_{n_k}(x)| \leq |f_{n_k}(x) - f(x)| + |f(x)| \leq q(x) + |f(x)| =: h(x)$$

□

3 Linear Operators

Definition 3.1. Let X, Y be normed spaces and $D \subset X$ is a subspace. A linear operator with domain $\text{dom}(T) = D$ is a linear mapping $T : D \rightarrow Y$. We define: $\text{range}(T) = \text{rg}(T) := T(D)$. Graph of T , $\text{gr}(T) := \{(x, y) \mid x \in \text{dom}(T), y = Tx\} \subset X \times Y$.

We say that T is decently define, if $\overline{\text{dom}(T)} = X$.

Example 3.2. 1. $X = Y = C([0, 1], \mathbb{R})$ and $\text{dom}(T) := C^1([0, 1], \mathbb{R})$ $T : \text{dom}(T) \rightarrow Y$ with $u \mapsto u'$.

2. $X = Y = \mathbb{R}^n, T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $x \mapsto Ax$ with $A \in \mathbb{R}^{n \times n}$

3. Fixed $u \in L^p(\Omega)$ and $p \in [1, \infty)$.

$$q := \begin{cases} \frac{p}{p-1} & p = 1 \\ \infty & \text{else} \end{cases}$$

$$T : L^q(\Omega) \rightarrow \mathbb{R} \quad v \mapsto \int_{\Omega} u \cdot v$$

4. $X = L^2(\Omega), Y = \mathbb{R}, \text{dom}(T) = C(\overline{\Omega})$ with $x \in \Omega$ fixed, $T : \text{dom}(T) \rightarrow Y$ with $u \mapsto u(x_0)$

Definition 3.3. Let X, Y be normed spaces and $T : X \rightarrow Y$ a linear operator ($\text{dom}(T) = X$). We say that T is bounded $\iff \exists M > 0 \forall x \in X : \|Tx\|_Y \leq M \|x\|_X$. In this case, we define $\|T\| = \|T\|_{\mathcal{L}(X, Y)} := \inf \{M > 0 \mid \|Tx\| \leq M \|x\| \forall x\}$.

$$\mathcal{L}(X, Y) := \{T : X \rightarrow Y \mid T \text{ bounded, linear operator}\}$$

$$\mathcal{L}(X) := \mathcal{L}(X, X)$$

Proposition 3.4. Let X, Y be normed spaces, $T : X \rightarrow Y$ be linear. The following are equivalent:

1. T is continuous
2. T is continuous at 0
3. $\exists M > 0$ such that $\|Tx\| \leq M \|x\| \forall x \in X$ (T bounded)
4. T is uniformly continuous

Also:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|T(x)\| \quad \text{and} \quad \|Tx\| \leq \|T\| \|x\| \quad \forall x \in X$$

Proof. (3) \rightarrow (4) Is true since $\forall x, y \in X : \|Tx - Ty\| = \|T(x - y)\| \leq M \|x - y\|$

(4) \rightarrow (1) \rightarrow (2) trivial

(2) \rightarrow (3) Assume (3) is not true, then

$$\exists (x_n)_n \text{ in } X : \forall n \in \mathbb{N} : \|Tx_n\| > n \|x_n\|$$

Define $y_n := \frac{x_n}{\|x_n\|n} \Rightarrow \|y_n\| = \frac{1}{n} \Rightarrow y_n \rightarrow 0$ but $\|Ty_n\| = \frac{\|Tx_n\|}{\|x_n\|n} > 1$.
This gives a contradiction to continuity at 0 since $T0 = 0$.

□

Additionally,

$$M := \sup_{x \neq 0} \frac{\|Tx\| \|x\|}{\|x\|^2} = \sup_{x \neq 0} \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|\leq 1} \|Tx\|$$

But also,

$$\sup_{\|x\|\leq 1} \|Tx\| = \sup_{\lambda \in [0,1]} \sup_{\|x\|=1} \|T(\lambda x)\| = \sup_{\lambda \in [0,1]} \lambda \left(\sup_{\|x\|=1} \|Tx\| \right) = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \left\| \frac{Tx}{\|x\|} \right\|$$

We also get that

$$\begin{aligned} M_0 &\geq \frac{\|Tx\|}{\|x\|} \quad \forall x \in X, x \neq 0 \\ \Rightarrow \|Tx\| &\leq M_0 \|x\| \quad \forall x \in X : x \neq 0 \quad \text{and also for } x = 0 \Rightarrow \|T\| \leq M_0 \\ M_0(1 - \varepsilon) &\leq \frac{\|Tx_\varepsilon\|}{\|x_\varepsilon\|} \end{aligned}$$

For $\varepsilon > 0$ pick $x_\varepsilon \neq 0$ such that

$$\begin{aligned} \|Tx_\varepsilon\| &\geq M_0(1 - \varepsilon) \|x_\varepsilon\| \\ \Rightarrow \|T\| &\geq M_0(1 - \varepsilon) \end{aligned}$$

since $\varepsilon > 0$ was arbitrary $\Rightarrow \|T\| \geq M_0$.

↓ This lecture took place on 2019/05/10.

Proposition 3.5. Let X and Y be normed spaces. Then

1. $\mathcal{L}(X, Y)$ is a vectorspace with

$$(T + S)(x) := T(x) + S(x) \quad (\lambda T)(x) := \lambda T(x) \quad 0(x) := 0$$

2. $T \mapsto \|T\|$ is a norm on $\mathcal{L}(X, Y)$ (the operator norm)

3. If Y is complete, then $\mathcal{L}(X, Y)$ is complete. In particular, $\mathcal{L}(X, \mathbb{K})$ is complete for any X and is also called the space of bounded linear functionals

Proof. 1. Left as an exercise to the reader

2. (N1) $\|0\| = \sup_{\|x\| \leq 1} \|0(x)\| = 0$.
Also $\|T\| = 0 \implies \|Tx\| \leq 0\|x\| = 0 \forall x \implies T = 0$

$$(N2) \quad \|\lambda T\| = \sup_{\|x\| \leq 1} \|\lambda T(x)\| = \sup_{\|x\| \leq 1} \underbrace{|\lambda|}_{\geq 0} \|Tx\| = |\lambda| \cdot \|T\|$$

$$(N3) \quad \forall x : \|(T + S)(x)\| = \|Tx + Sx\| \leq \|Tx\| + \|Sx\| \leq (\|T\| + \|S\|)\|x\| \\ \implies \|T + S\| \leq \|T\| + \|S\|$$

3. Let $(T_n)_n$ be Cauchy in $\mathcal{L}(X, Y)$ and Y a Banach space. Since $\|(T_n - T_m)(x)\| \leq \|T_n - T_m\| \|x\| \implies (T_n x)_n$ is Cauchy in $Y \forall x \in X \implies Tx := \lim_{n \rightarrow \infty} T_n x$ is well defined.

Furthermore, we want to show

Linearity:

$$\forall x, y \in X, \lambda \in \mathbb{K} : T(\lambda x + y) = \lim_{n \rightarrow \infty} T_n(\lambda x + y) = \lim_{n \rightarrow \infty} \lambda T_n x + \lim_{n \rightarrow \infty} T_n y = \lambda Tx + Ty$$

$\|T_n - T\| \rightarrow 0$: Take $\varepsilon > 0$, $n_0 \in \mathbb{N} : \|T_n - T_m\| \leq \varepsilon \forall n, m \geq n_0$

Show: $\exists n_1 \forall n \geq n_1 : \|T_n - T\| \leq 2\varepsilon$. For $x \in X : \|x\| \leq 1$ fix $m_x \geq n_0$:
 $\|T_{m_x} x - Tx\| \leq \varepsilon \implies \forall n \geq n_1 =: n_0 :$

$$\|T_n x - Tx\| \leq \|T_n x - T_{m_x} x\| + \|T_{m_x} x - Tx\| \\ \leq \|T_n - T_{m_x}\| + \varepsilon \leq 2\varepsilon \\ \implies \|T_n - T\| = \sup_{\|x\| \leq 1} \|T_n x - Tx\| < 2\varepsilon$$

$$\implies \forall x \in X : \|Tx\| \leq \|T_n x - Tx\| + \|T_n x\| \leq \|T_n - T\| + \|T_n\| \forall n \text{ fixed}$$

□

Proposition 3.6. Let X, Y be normed spaces. $D \subset X$ is a subspace such that $\overline{D} = X$, $T \in \mathcal{L}(D, Y)$.

$$\exists! \hat{T} \in \mathcal{L}(X, Y) : \hat{T}|_D = T$$

In addition: $\|\hat{T}\| = \|T\|$.

Proof. Unique extension is clear for T is uniformly continuous.

Also:

$$\|\hat{T}\| = \sup_{\substack{x \in X \\ \|x\| \neq 0}} \frac{\|\hat{T}x\|}{\|x\|} \stackrel{\text{by density}}{=} \sup_{\substack{x \in D \\ \|x\| \neq 0}} \frac{\|Tx\|}{\|x\|} = \|T\|$$

To show the density equality is left as an exercise to the reader.

□

Proposition 3.7. Let X, Y, Z be normed spaces. $S \in \mathcal{L}(X, Y)$. $T \in \mathcal{L}(Y, Z)$. Then $T_0 S \in \mathcal{L}(X, Z)$ and $\|T_0 S\| \leq \|T\| \|S\|$.

Proof. $T_0 S$ is linear (show as an exercise).

Take $x \in X$. $\|T_0 S(x)\| = \|T(Sx)\| = \|T\| \|Sx\| \leq \|T\| \|S\| \|x\|$. $\Rightarrow \|T_0 S\| \leq \|T\| \|S\|$ \square

Remark. If $\dim(X) < \infty$, $T : X \rightarrow Y$ is linear, then $T_C \mathcal{L}(X, Y)$ (left as an exercise).

Proposition 3.8 (Neumann series). Let X be a normed space. $T \in \mathcal{L}(X)$. If $\sum_{n=0}^{\infty} T^n$ is convergent in $\mathcal{L}(X)$, then $(I - T)$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$.

Here: $T^n := T_0 \cdot T_0 \cdot T_0 \cdot \dots$ n times

In particular, if X is Banach and $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} =: a < 1$ then $\sum_{i=0}^{\infty} T^n$ is convergent. Also if $\|T\| < 1$, then $a < 1$ holds true. In case of $a < 1$, then $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$.

Proof. Let $S_m := \sum_{n=0}^m T^n$ and $S := \lim_{m \rightarrow \infty} S_m$. Then $(I - T)S_m = I - T^{m+1} = S_m(I - T)$ (compute!).

$$\|T^m\| = \left\| \sum_{n=0}^m T^n - \sum_{n=0}^{m-1} T^n \right\| = \|S_m - S_{m-1}\| \rightarrow 0$$

for $m \rightarrow \infty$ since $(S_m)_n$ is Cauchy. ($RS := R_0 S$)

Now note that for fixed $R \in \mathcal{L}(X)$ the mappings

$$S \mapsto RS \quad S \mapsto SR$$

are continuous since $\|S_n R - S R\| \leq \|S_n S\| \|R\| \rightarrow 0$ for $S_n \rightarrow S$. Continuity implies that

$$\begin{aligned} I &= \lim_{m \rightarrow \infty} I - T^{m+1} = \begin{cases} \lim_{m \rightarrow \infty} (I - T)S_m = (I - T)S \\ \lim_{m \rightarrow \infty} S_m(I - T) = S(I - T) \end{cases} \\ &\Rightarrow (I - T)^{-1} = S \end{aligned}$$

Now if $\limsup_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \leq a < 1 \forall \varepsilon > 0 \Rightarrow \exists n_0 \forall n \geq n_0 : \|T^n\| \leq (a + \varepsilon)^n$

$$\Rightarrow \sum_{n=0}^{\infty} \|T^n\| \leq c + \sum_{n=0}^{\infty} (a + \varepsilon)^n = \frac{1}{1 - (a + \varepsilon)} + c < \infty \text{ for } c > 0$$

X is Banach, so $\sum_{n=0}^{\infty} T^n$ is convergent and

$$\|(I - T)^{-1}\| = \left\| \sum_{n=0}^{\infty} T^n \right\| \leq \frac{1}{1 - (a + \varepsilon)}$$

Since ε was arbitrary, $\|(I - T)^{-1}\| \leq \frac{1}{1-a}$.

If $\|T\| < 1$, then

$$\begin{aligned}\limsup_{m \rightarrow \infty} \|T^m\|^{\frac{1}{m}} &\leq \limsup (\|T\| \cdot \|T\| \cdots \|T\|)^{\frac{1}{n}} \\ &\leq \limsup (\|T\|^n)^{\frac{1}{n}} \\ &= \limsup \|T\| \\ &= \|T\|\end{aligned}$$

□

Remark. $(I - T)^{-1}$ is linear (left as an exercise). However: $(I - T)^{-1} \notin \mathcal{L}(X)$ in general!

4 The Hahn-Banach Theorem and its consequences

Apparently the Hahn-Banach Theorem of this chapter is very central to Functional Analysis. This section deals with an extension of linear functionals and separation of sets.

First, consider $\mathbb{K} = \mathbb{R}$.

Definition 4.1. Let X be a vector space. $p : X \rightarrow \mathbb{R}$ is called sublinear iff

1. $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0, x \in X$
2. $p(x + y) \leq p(x) + p(y) \forall x, y \in X$

Example 4.2. $p(x) = \|x\|$, p linear and p is a seminorm.

Theorem 4.3 (Hahn-Banach Theorem, real version). Let X be a vector space over \mathbb{R} , $U \subset X$, a subspace. $p : X \rightarrow \mathbb{R}$ be sublinear and $l : U \rightarrow \mathbb{R}$ is linear such that $l(x) \leq p(x) \forall x \in U$

Then $\exists L : X \rightarrow \mathbb{R}$ is linear such that

$$L|_U = l \quad L(x) \leq p(x) \forall x \in X$$

Proof. This proof consists of two steps:

1. Method to extend l from U to $U + \text{span}(x_0)$, $x_0 \notin U$
2. Iterate this step and get maximal extension (Zorn)

Step 1 For $x_0 \in X \setminus U$, let $V = U + \text{span}(x_0) = \{u + \lambda x_0 \mid u \in U, \lambda \in \mathbb{R}\}$. Any $v \in V$ can be written uniquely as $v = u + \lambda x_0$ for $u \in U, \lambda \in \mathbb{R}$ (why? left as an exercise). Thus for any $r \in \mathbb{R}$, we can define $L_r : V \rightarrow \mathbb{R}$. $v = u + \lambda x_0 \mapsto l(u) + \lambda r$. L_r is linear (why? left as an exercise).

Also: $L_r(x) \leq p(x) \forall x \in V \iff l(u) + \lambda r \leq p(u + \lambda x_0) \forall \lambda, u$ (let this statement be (*)).

$\lambda = 0$ (*) holds true

$\lambda > 0$ (*)

$$\begin{aligned} &\iff r \leq p\left(\frac{u}{\lambda} + x_0\right) - l\left(\frac{u}{\lambda}\right) \forall u \in U \\ &\iff r \leq \inf_{u \in U} p(u + x_0) - l(u) \end{aligned}$$

$\lambda < 0$

$$\begin{aligned} &\iff -r \leq p\left(\frac{u}{-\lambda} - x_0\right) - l\left(\frac{u}{-\lambda}\right) \forall u \in U \iff r \geq -p(u - x_0) + l(u) \forall u \in U \\ &\iff r \geq \sup_{u \in U} l(u) - p(u - x_0) \end{aligned}$$

Thus, (*) holds for $r = \sup_{u \in U} l(u) - p(u - x_0)$ if $\sup_{u \in U} l(u) - p(u - x_0) \leq \inf_{u \in U} p(u + x_0) - l(u) \iff l(w) - p(w - x) \leq p(u + x_0) - l(u) \forall w, u \in U \iff l(w) + l(u) \leq p(u + x_0) + p(w - x_0)$.

But this holds since:

$$l(w) + l(u) = l(w + u) \leq p(w + u) = p(w - x_0 + x_0 + u) \leq p(w - x_0) + p(u + x_0)$$

Step 2

Revision 4.4 (Zorn's Lemma). *Let (A, \leq) be a partially ordered set such that every chain (every subset R of A $\forall a, b \in R : a \leq b \vee b \leq a$) admits an upper bound (i.e. $\exists c \in A : b \leq c \forall b \in R$), then A has a maximal element, i.e. $\exists z \in A$ such that $\forall a \in A : z_0 \leq a \implies a = z_0$*

Let A be a set of (V, L_V) tuples where $V \subset X$ is a subspace with $U \subset V$ and $L_V : V \rightarrow \mathbb{R}$ such that $L_V \leq p$ on V and $L_V|_U = l$.

For (V_1, L_{v_1}) and $(V_2, L_{v_2}) \in A$, we say that $(V_1, L_{v_1}) \leq (V_2, L_{v_2})$ if $V_1 \subset V_2$ and $L_{v_2}|_{V_1} = L_{v_1}$. Now $A \neq \emptyset$ since $(U, l) \in A$. If $(V_i, L_{v_i})_{i \in I} := R$ is a chain, define $V := \bigcup_{i \in I} V_i$, $L_V(x) := L_{v_i}(x)$ if $x \in V_i$.

This is well-defined.

$\implies (V, L_V)$ is an upper bound for R .

□

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