

Analysis 2

Lecture notes, University (of Technology) Graz
based on the lecture by Wolfgang Ring

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Contents

1	Mathematical Redux and topological fundamentals	2
1.1	Metric	2
1.2	Cauchy-Schwarz inequality	3
1.3	Euclidean norm	3
1.4	Metric space	4
1.5	Open sets, convergence and accumulation points	4
1.6	Norm	6
1.7	Contact point	7
1.8	Topology	11
2	Variations of continuity notions	13
3	Compactness	16
3.1	Normed vector spaces	21
4	Integral calculus	25
4.1	Regulated functions	29
5	Integration of regulated functions	37
5.1	Integration methods	52
5.1.1	Partial integration	52
5.1.2	Integration by substitution	55

5.2	Improper integrals	59
5.3	Young's inequality	63
5.4	Hölder's inequality	67
6	Elaboration on differential calculus	69

This lecture took place on 2018/03/06.

Mathematical Redux and topological fundamentals

Metric

Definition 1.1. Let $X \neq \emptyset$ be a set. We define a map $d : X \times X \rightarrow [0, \infty)$. d should behave like a geometrical distance. We require $\forall x, y, z \in X$:

- $d(x, y) = d(y, x)$ [called symmetry]
- $d(x, y) = 0 \iff x = y$ [called positive definiteness]
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$ [called triangle inequality]

Then d is called metric or distance function on X . (X, d) is called metric space.

Example 1.1.

- $X \subseteq \mathbb{C}$, $d(x, y) = |x - y|$. It satisfies $|x - z| \leq |x - y| + |y - z|$
- $X \subseteq \mathbb{R}^n$, $\|x - y\| = \langle x - y, x - y \rangle^{\frac{1}{2}}$

Claim.

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^n x_i^2}$$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

It holds that $\|x + y\| \leq \|x\| + \|y\|$ [triangle inequality].

Proof.

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y \rangle \\
&= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\
&= \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 && \text{[see Cauchy-Schwarz inequality]} \\
&= (\|x\| + \|y\|)^2 \\
\|x - y\|^2 &= \langle x - y, x - y \rangle \\
&= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2 \\
\|x + y\|^2 + \|x - y\|^2 &= 2(\|x\|^2 + \|y\|^2)
\end{aligned}$$

□

Cauchy-Schwarz inequality

Theorem 1.1 (Cauchy-Schwarz inequality).

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Proof.

$$0 \leq \langle x - \lambda y, x - \lambda y \rangle = \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|y\|^2 \quad \forall \lambda \in \mathbb{R}$$

Let $\lambda = \frac{\langle x, y \rangle}{\|y\|^2}$. Then,

$$\begin{aligned}
0 &\leq \|x\|^2 - 2 \frac{|\langle x, y \rangle|^2}{\|y\|^2} + \frac{|\langle x, y \rangle|^2}{\|y\|^4} \cdot \|y\|^2 \\
&\implies 0 \leq \|x\|^2 - \frac{|\langle x, y \rangle|^2}{\|y\|^2} \\
&\implies |\langle x, y \rangle|^2 \leq \|x\|^2 \cdot \|y\|^2
\end{aligned}$$

□

Euclidean norm

Definition 1.2. $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ is called Euclidean norm (length) of vector $x \in \mathbb{R}^n$.

$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$ It holds that

1. $\|\lambda x\| = |\lambda| \|x\| \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}$
2. $\|x\| = 0 \iff x = 0 \text{ in } \mathbb{R}^n$
3. $\|x + y\| \leq \|x\| + \|y\|$

In general: Let V be a vector space over \mathbb{R} . A map $\|\cdot\|$, which assigns every vector x a non-negative real number satisfying the properties above, is called **norm** on V . Then $(V, \|\cdot\|)$ is called a **normed vector space**.

Let $X \subseteq \mathbb{R}^n$ (V is a normed vector space), then $d(x, y) = \|x - y\|$ is a metric on X .

$$\|y - x\| = \|(-1)(x - y)\| = |-1| \cdot \|x - y\| = \|x - y\|$$

$$d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

$$d(x, z) = \|z - x\| = \|z - y + y - x\| \leq \|z - y\| + \|y - x\| = d(z, y) + d(y, x)$$

Metric space

Example 1.2 (metric space). Metric space, distance is not a norm. Consider an area in \mathbb{R}^3 .

$d(x, y)$ is the shortest path, connecting x and y in X . See Figure 1

Example 1.3 (French railway). All connections between two cities pass through Paris except one city is Paris.

Example 1.4. $X = \mathbb{R}^2$. Let $p \in \mathbb{R}^2$ be fixed.

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y, p \text{ are on one line} \\ |x - p| + |p - y| & \text{if } x, y, p \text{ are not on one line} \end{cases}$$

Open sets, convergence and accumulation points

Now we put some terminology into the context of a metric space. (X, d) is a metric space.

Definition 1.3. Let $x \in X, r \geq 0$.

$$K_r(x) = \{z \in X \mid d(x, z) < r\}$$

Is an open sphere with radius r and center x .

Definition 1.4.

$$\overline{K_r(x)} = \{z \in X \mid d(x, z) \leq r\}$$

Closed sphere with center x and radius r .

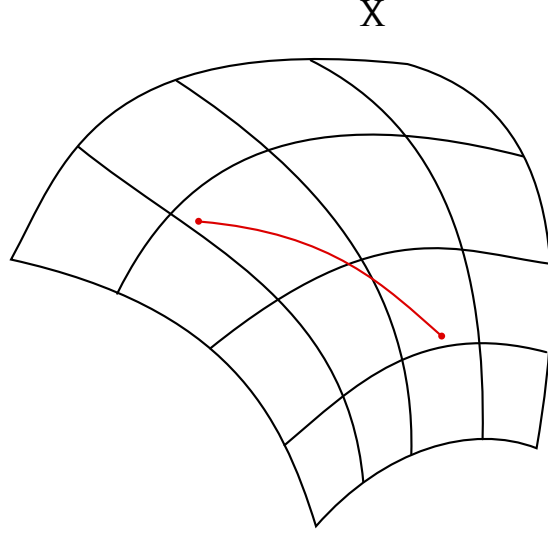


Figure 1: Example in \mathbb{R}^3 . The red line illustrates the shortest path

Definition 1.5 (Sequences in X). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X (hence, $x_n \in X \forall n \in \mathbb{N}$)

1. $(x_n)_{n \in \mathbb{N}}$ is called convergent and limit $x \in X$ if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies d(x_n, x) < \varepsilon$$

Denoted as $\lim_{n \rightarrow \infty} x_n = x$.

2. $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m \geq N \implies d(x_n, x_m) < \varepsilon$$

Every convergent sequence is also a Cauchy sequence.

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be convergent with limit x . Let $\varepsilon > 0$ be arbitrary. Because $(x_n)_{n \in \mathbb{N}}$ is convergent, there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(x_n, x) < \frac{\varepsilon}{2}$. Now let $n, m \geq N$. Then it holds that

$$d(x_n, x_m) \leq \underbrace{d(x_n, x)}_{< \frac{\varepsilon}{2}} + \underbrace{d(x, x_m)}_{< \frac{\varepsilon}{2}} < \varepsilon$$

□

Definition 1.6. (X, d) is called complete metric space if every Cauchy sequence in X is also convergent (has a limit).

\mathbb{R} is complete. \mathbb{R}^n is also complete. $\mathbb{Q} \subseteq \mathbb{R}$ is incomplete.

Definition 1.7. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of X is called "accumulation point" (dt. Häufungspunkt) of the sequence. $\forall \varepsilon > 0$, it holds that $K_\varepsilon(x)$ contains infinitely many sequence elements.

This lecture took place on 2018/03/08.

TODO

$$\begin{aligned} d(x, y) = 0 &\iff x = y \\ \forall x, y \in X : d(x, y) &= d(y, x) \\ d(x, z) &\leq d(x, y) + d(y, z) \forall x, y, z \in X \end{aligned}$$

Norm

Let V be a vector space. $\|\cdot\|$ is called norm on V .

$$\begin{aligned} \|x\| = 0 &\iff x = 0 \\ \forall \lambda \in \mathbb{R}, \mathbb{C} : \forall x \in V : \|\lambda x\| &= |\lambda| \|x\| \\ \forall x, y, z \in V : \|x + y\| &\leq \|x\| + \|y\| \end{aligned}$$

Let $X \subseteq V$ be a subset of normed vector space V . Then X is a metric space with $d(x, y) = \|x - y\|$.

For $V = \mathbb{R}^n$. Then

$$\|x\| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$$

is a norm on \mathbb{R}^n . $\|x\|_2$ is called Euclidean norm on \mathbb{R}^n .

Other norms in \mathbb{R}^n :

$$\|x\|_\infty = \max \{ |x_i| \mid i = 1, \dots, n \}$$

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

for $1 \leq p < \infty$.

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

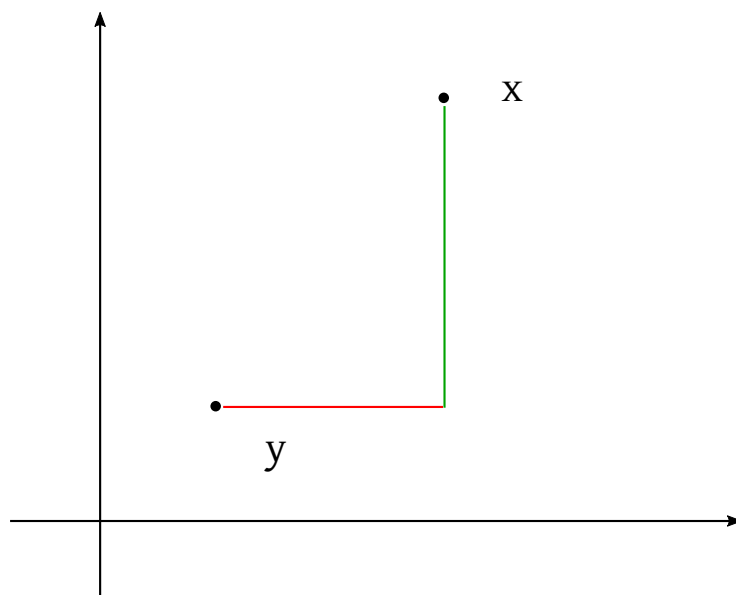


Figure 2: Visualizing $\|x\|_1$

e.g. $\|x\|_1$ in \mathbb{R}^2

$$\|x - y\| = |x_1 - y_1| + |x_2 - y_2|$$

is the so-called *Manhattan metric*.

The concepts “subsequence”, “final element of a sequence”, “reordering of a sequence” correspond one-by-one to metric spaces.

Definition 1.8 (Accumulation point). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . $x \in X$ is called accumulation point of sequence X if $\forall \varepsilon > 0$ the sphere $K_\varepsilon(x)$ contains infinitely many elements.

Lemma 1.1. $x \in X$ is accumulation point of sequence $(x_n)_{n \in \mathbb{N}}$ if and only iff there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $x = \lim_{k \rightarrow \infty} x_{n_k}$.

Proof. See Analysis 1 course

□

Contact point

Let $B \subseteq X$, X is a metric space. Then B with d is a metric space itself.

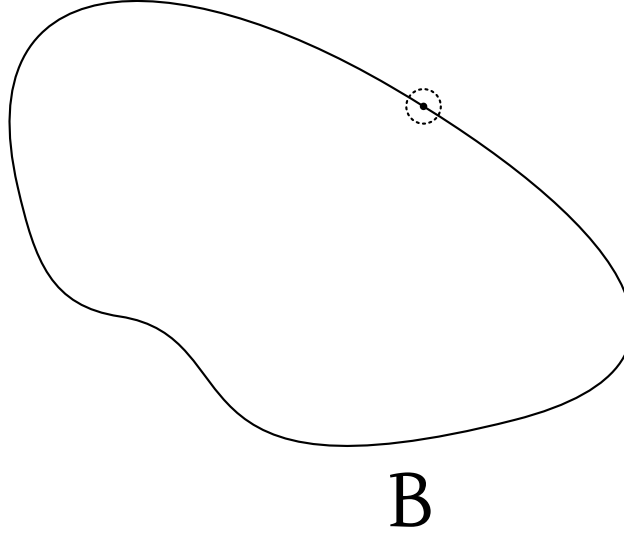


Figure 3: Contact points in set B

Definition 1.9. Let $B \subseteq X$ and $x \in X$. We say, x is a contact point of B if $\forall \varepsilon > 0 : K_\varepsilon(x) \cap B \neq \emptyset$.

[$y \in X$ is not a contact point of $B \iff \exists \varepsilon > 0 : K_\varepsilon(y) \cap B = \emptyset$]

See Figure 3.

We let $\bar{B} = \{x \in X \mid x \text{ is contact point of } B\}$.

\bar{B} is called closed hull of B .

B is called closed if $B = \bar{B}$, hence, every contact point is also element of B .

Remark 1.1. Because $\forall x \in B$ holds $K_r(x) \cap B \supseteq \{x\} \forall r > 0$ is x always contact point of B . Also $B \subseteq \bar{B}$ (always)

Lemma 1.2. x is contact point of $B \iff \exists (x_n)_{n \in \mathbb{N}}$ with $x_n \in B$ and $\lim_{n \rightarrow \infty} x_n = x$.

Proof. Let x be a contact point of B .

Direction \Rightarrow : Because $K_{\frac{1}{n}}(x) \cap B \neq \emptyset$, choose $x_n \in K_{\frac{1}{n}}(x) \cap B$. The sequence $(x_n)_{n \in \mathbb{N}}$ has property $d(x_n, x) < \frac{1}{n}$. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ sch that $N > \frac{1}{\varepsilon}$ (consider the Archimedean axiom). Then for $n \geq N$, $d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, hence $\lim_{n \rightarrow \infty} x_n = x$.

Direction \Leftarrow : Let $x = \lim_{n \rightarrow \infty} x_n$ and $x_n \in B$. Let $\varepsilon > 0$ be arbitrary and $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon \forall n \geq N$. Then $d(x_N, x) < \varepsilon$, hence

$$x_N \in \underbrace{K_\varepsilon(x) \cap B}_{\neq \emptyset}$$

So x is contact point of B . □

Lemma 1.3. *It holds that $\forall B \subseteq X : \bar{B} = \overline{\bar{B}}$, hence \bar{B} itself is closed.*

Proof. Show that $x \in \bar{B}$. Let $x \in \bar{B}$.

$$\iff \forall \varepsilon > 0 : K_\varepsilon(x) \cap \bar{B} \neq \emptyset$$

Therefore let $\varepsilon > 0$ be arbitrary and $x \in \bar{B}$.

Show that $K_\varepsilon(x) \cap B \neq \emptyset$.

Because $x \in \bar{B} : \exists y \in \bar{B} : y \in K_{\frac{\varepsilon}{2}}(x)$. Because $y \in \bar{B} : \exists z \in B : z \in K_{\frac{\varepsilon}{2}}(y)$. Hence,

$$d(z, x) \leq \underbrace{d(z, y)}_{< \frac{\varepsilon}{2}} + \underbrace{d(y, x)}_{< \frac{\varepsilon}{2}} < \varepsilon$$

so $z \in K_\varepsilon(x) \cap B$. So x is contact point of $B \implies x \in \bar{B}$. □

Lemma 1.4. *Let X be a metric space.*

- $A_i \subseteq X$ be closed $\forall i \in I$. Then $A = \bigcap_{i \in I} A_i = \{x \in X \mid x \in A_i \forall i \in I\}$ is closed itself.
- $A_1, \dots, A_n \subseteq X$ are closed. Then $\bigcup_{k=1}^n A_k$ is closed in X .
- φ is closed, X is closed.

Proof. See Analysis 1 course. □

Definition 1.10. Let $x \in X$ is called accumulation point of set $B \subseteq X$ if $\forall \varepsilon > 0 : (K_\varepsilon(x) \setminus \{x\}) \cap B \neq \emptyset$.

Remark 1.2. Accumulation points only exist in the context of sets. Accumulation values only exist in the context of sequences.

For example $(+1, -1, +1, -1, +1, \dots)$ has accumulation values $+1$ and -1 .

Lemma 1.5. Let $x \in X$ is accumulation point on $B \iff$ every sphere $K_\varepsilon(x)$ contains infinitely many points of B .

Proof. Direction \Leftarrow is trivial.

Direction \Rightarrow : Choose $x_1 \in (K_1(x) \setminus \{x\}) \cap B$, hence $x_1 \neq x$, $x_1 \in B$ and $d(x_1, x) < 1$. Let $r_1 = 1$.

Inductive: choose $r_n = \min(\frac{1}{n}, d(x_{n-1}, x))$ and $x_n \in (K_{r_n}(x) \setminus \{x\}) \cap B$. Then $d(x_n, x) > 0$ (because $x_n \neq x$) where $d(x_n, x) < r_n < \frac{1}{n}$.

$$0 < d(x_n, x) < \frac{1}{n}$$

Furthermore, $d(x_n, x) < r_n \leq d(x_{n-1}, x)$. So $x_n \neq x_{n-1}$.

Inductive: $x_n \neq x_{n-1} \neq x_{n-2} \neq \dots \neq x_1$. Now consider arbitrary $\varepsilon > 0$ and N large enough such that $\frac{1}{N} < \varepsilon$.

Then it holds that $\forall n \geq N : 0 < d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. So $K_\varepsilon(x) \cap B$ contains infinitely many points $x_N, x_{N+1}, x_{N+2}, \dots$ \square

Definition 1.11. Let $U \subseteq X$ and $x \in U$. We say x is an inner point of U if $\exists r > 0 : K_r(x) \subseteq U$. We let $\mathring{U} = \{x \in U \mid x \text{ is inner point of } U\}$ and call it interior of U (offenen Kern von U or das Innere von U). $O \subseteq X$ is called open (open set), if every point $x \in O$ is also an inner point of O . Hence $\mathring{O} = O$.

Example 1.5. Let $K_r(x)$ with $r > 0$ be an open sphere in X . Then $K_r(x)$ is an open set in X .

Why? Let $y \in K_r(x)$. Show that y is an inner point of the sphere. $d(y, x) = s < r$. Define $r' = r - s > 0$. Claim: $K_{r'}(y) \subseteq K_r(x)$.

TODO drawing

TODO

So it holds that $z \in K_{r'}(y)$ and therefore $K_{r'}(y) \subseteq K_r(x)$.

Lemma 1.6. Let $U \subseteq X$ be arbitrary. Then $\mathring{U} \subseteq X$ be an open set in X .

Proof. Let $x \in \mathring{U}$, hence x is an inner point of U . Show that x is an inner point of \mathring{U} , also $\exists r > 0 : K_r(x) \subseteq \mathring{U}$.

Because $x \in \mathring{U}$, $r > 0$ exists: $K_r(x) \subseteq U$. Claim: Every point $y \in K_r(x)$ is also an inner point of U . Obvious (previous example), because $r' > 0$ exists such that $K_{r'}(y) \subseteq K_r(x) \subseteq U$ so $y \in \mathring{U}$ and $K_r(x) \subseteq \mathring{U}$. \square

Theorem 1.2. Let X be a metric space.

$$A \subseteq X \text{ is closed in } X \iff O = X \setminus A = A^C \text{ is open}$$

Proof. Direction \Leftarrow . Let A be closed and $O = A^C$. We choose $x \in O$ and show that x is in the interior of O .

Assume the opposite.

$$\forall \varepsilon > 0 : \neg (K_\varepsilon(x) \subseteq O) \\ \iff K_\varepsilon(x) \cap O^c \neq \emptyset$$

where $O^c = A$. So x is contact point of A . Because A is closed, it holds that $x \in A$. This contradicts with $x \in O = A^c$.

Direction \Rightarrow . TODO $K_r(x) \cap \underbrace{A}_{=O^c} = \emptyset$. Hence x is not a contact point of A .

So every contact point of A is also an element of A and A is closed. \square

Theorem 1.3. *Let X be a metric space. Then it holds that*

- If $O_i \subseteq X$ is open in $X \forall i \in I$. Then also $O = \bigcup_{i \in I} O_i$ is open in X .
- If O_1, O_2, \dots, O_n is open in X , then $\bigcap_{k=1}^n O_k$ is open in X .
- X is open, \emptyset is open.

Proof. By Lemma 1.4, Theorem 1.2 and De Morgan's Laws:

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

\square

Topology

Definition 1.12. *Given a set X . If a subset $T \subseteq \mathcal{P}(X)$ is defined such that the elements $O \in T$ (hence $O \subseteq X$) satisfy the conditions of Theorem 1.3, then T is called topology on X . (X, T) is called topological space.*

The sets $O \in T$ are called open sets in terms of T . The complements $A = O^c$ for $O \in T$ are called closed sets.

Definition 1.13. *Let $x \in U \subseteq X$. We claim that U is a neighborhood of x , if $r > 0$ exists such that $x \in K_r(X) \subseteq U$*

See Figure 4

Remark 1.3. $O \subseteq X$ is open iff O is neighborhood of every point $x \in O$.

Definition 1.14. *Let X and Y be metric spaces and $x_0 \in X$. Let $f : X \rightarrow Y$ be given. We say f is continuous in x_0 if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in X : d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

Here, d_X is a metric on X and d_Y is a metric on Y .

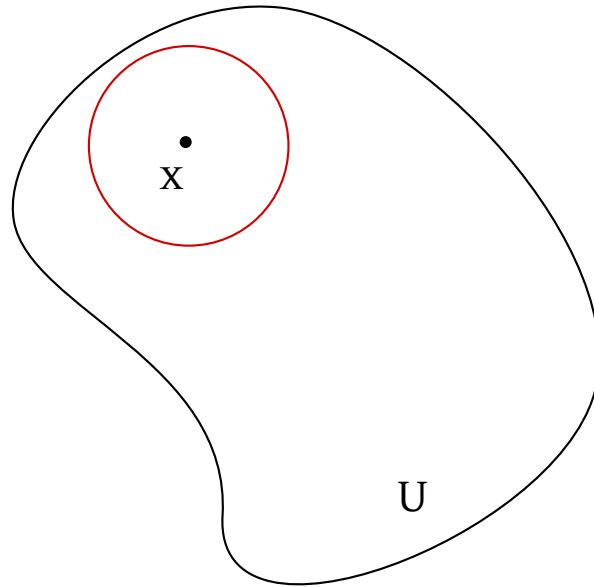


Figure 4: Neighborhood of x

This lecture took place on 2018/03/13.

TODO I missed the first twenty minutes (including Satz 3 and 4)

Proof. Direction \Rightarrow .

Let f be continuous in X and let $O \subseteq Y$ be open. Let $U = f^{-1}(O)$ and choose $x_0 \in U$. Then $f(x_0) \in O$, hence O is a neighborhood of $f(x_0)$. By Theorem 5.2 (b), it follows that $U = f^{-1}(O)$ is a neighborhood of x_0 .

Hence, U is neighborhood of every of its points, hence open in X .

Direction \Leftarrow .

Let the preimages of open sets be open and $x_0 \in X$ and $y_0 = f(x_0)$. Let V be a neighborhood of $y_0 = f(x_0)$, hence $\exists \varepsilon > 0 : K_\varepsilon(f(x_0)) \subseteq V$. Because $K_\varepsilon(f(x_0))$ is an open set, it holds that $f^{-1}(K_\varepsilon(f(x_0))) \in x_0$ is open in X .

Therefore, there exists $\delta > 0$ such that $K_\delta(x_0) \subseteq f^{-1}(K_\varepsilon(f(x_0))) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V)$ is a neighborhood of x_0 . Then by Theorem 5.2 (b), it follows that f is continuous in x_0 (chosen arbitrarily). Hence f is continuous on X . \square

Variations of continuity notions

Definition 2.1. Let $f : X \rightarrow Y$ be given. We call “ f uniformly continuous on X ” if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X \wedge d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Remark 2.1. Compare it with the definition of “continuous in X ”:

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : \forall y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

The difference is the location of the $\forall x \in X$ quantifier.

Every uniformly continuous map is continuous.

Example: $f : (0, \infty) \rightarrow (0, \infty)$ with $f(x) = \frac{1}{x}$ is continuous, but not continuously continuous.

Definition 2.2. $f : X \rightarrow Y$ is called Lipschitz continuous with Lipschitz constant $L \geq 0$ if $\forall x, y \in X : d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$.

Rudolf Lipschitz [1832–1903], University of Bonn

Theorem 2.1. Every Lipschitz continuous function is uniformly continuous.

Proof. For $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{L+1}$. Then it holds that $d_X(x, y) < \delta = \frac{\varepsilon}{L+1} \implies d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) < \frac{L}{L+1} \cdot \varepsilon < \varepsilon$. \square

- Most often $X \subseteq V$, $Y \subseteq W$. V and W are normed vector spaces and $d(x, y) = \|x - y\|$

Definition 2.3. A Lipschitz continuous map $f : X \rightarrow X$ with Lipschitz constant $L < 1$ is called contraction on X . Compare with Figure 5

Theorem 2.2 (Banach fixed-point theorem). Let $f : X \rightarrow X$ be a contraction and X be complete. Then there exists a uniquely defined $\hat{x} \in X$ such that $\hat{x} = f(\hat{x})$. \hat{x} is called fixed point on f . Furthermore it holds that $x_0 \in X$ is arbitrary and $x_n = f(x_{n-1})$ for all $n \geq 1$.

$$\lim_{n \rightarrow \infty} x_n = \hat{x}$$

TODO drawing Banach’s fixed point theorem

Proof. Let $x_0 \in X$ be arbitrary. x_n is constructed inductively by $x_n = f(x_{n-1})$ for all $n \geq 1$.

Claim. $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X .

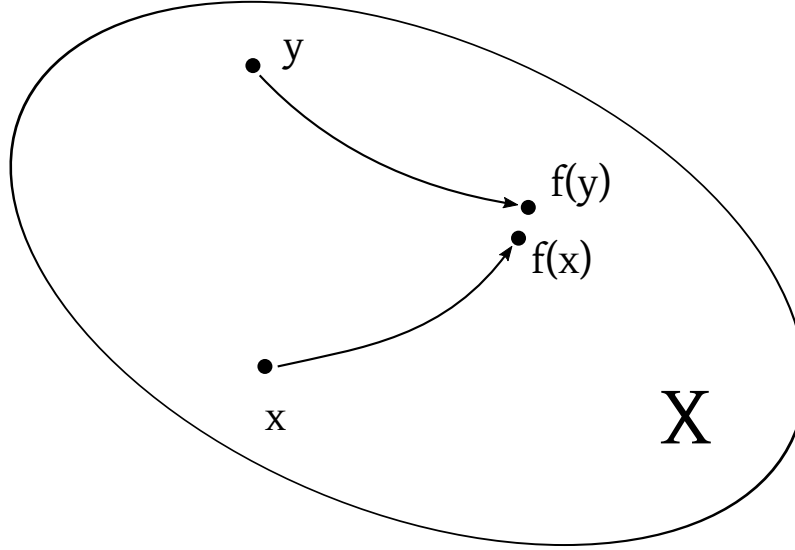


Figure 5: A contraction maps to points closer to each other

$$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k})$$

by triangle inequality

$$\begin{aligned} &= d(x_n, x_{n+1}) + d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_{n+2})) + \cdots + d(f(x_{n+k-2}), f(x_{n+k-1})) \\ &\leq d(x_n, x_{n+1}) + L(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-2}, x_{n+k-1})) \end{aligned}$$

this inequality is given by contraction

$$\begin{aligned}
&= d(x_n, x_{n+1})(1+L) + L(d(f(x_n), f(x_{n+1})) + \cdots + d(f(x_{n+k-3}), f(x_{n+k-2}))) \\
&\leq d(x_n, x_{n+1})(1+L) + L^2[d(x_n, x_{n+1}) + \cdots + d(x_{n+k-3}, x_{n+k-2})] \\
&\leq \cdots \leq d(x_n, x_{n+1})(1+L+L^2+\cdots+L^{k-1}) \\
&= d(f(x_{n-1}), f(x_n)) \left(\sum_{j=0}^{k-1} L^j \right) \leq Ld(x_{n-1}, x_n) \cdot \left(\sum_{j=0}^{k-1} L^j \right) \\
&\leq L^n d(x_0, x_1) \cdot \underbrace{\left(\sum_{j=1}^{k-1} L^j \right)}_{\leq \sum_{j=0}^{\infty} L^j = \frac{1}{1-L}} \\
&\leq \frac{L^n}{1-L} d(x_0, x_1) \\
d(x_n, x_{n+k}) &\leq \frac{L^n}{1-L} d(x_0, x_1) \forall n \in \mathbb{N} \forall k \in \mathbb{N}_0
\end{aligned}$$

with $0 \leq L < 1$.

$$\begin{aligned}
&\underbrace{\frac{L^n}{1-L} d(x_0, x_1)}_{>0} < \varepsilon \iff \\
L^n &< \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L) \quad (L > 0) \\
\iff n \underbrace{\ln L}_{<0} &< \ln \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L) \\
\iff n &> \frac{1}{\ln L} \ln \frac{\varepsilon}{d(x_0, x_1) + 1} (1-L)
\end{aligned}$$

Hence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X . X is complete, hence $\exists \hat{x} \in X$: $\hat{x} = \lim_{n \rightarrow \infty} x_n$. Because $\hat{x} = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(\hat{x})$ where the last equality is given by continuity of f . Therefore $\hat{x} = f(\hat{x})$ is a fixed point on f .

It remains to prove uniqueness:

Let $\tilde{x} = f(\tilde{x})$. Then it holds that $d(\hat{x}, \tilde{x}) = d(f(\hat{x}), f(\tilde{x})) \leq Ld(\hat{x}, \tilde{x})$ with $L < 1$. If $d(\hat{x}, \tilde{x}) > 0$, then $1 \leq L$. This is a contradiction. Hence $d(\hat{x}, \tilde{x}) = 0$ must hold, hence $\hat{x} = \tilde{x}$. \square

Remark 2.2. • The Fixed Point Theorem provides an algorithm for numeric computation of \hat{x} .

- It can reformulate problems $f(x) = 0$ (in \mathbb{R}^n) to

$$f(x) + x = g(x) = x$$

- *Attention: The conditions of the Fixed Point Theorem cannot be changed to the structure*

$$d(f(x), f(y)) < L \cdot d(x, y) \wedge L \leq 1$$

or

$$d(f(x), f(y)) \leq L \cdot d(x, y) \wedge L < 1$$

This will be discussed in the practicals.

Lemma 2.1. *Let X be a complete metric space. Let $A \subseteq X$ be closed. Then (A, d) is itself a complete, metric space.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in A ($x_n \in A$). Then $(x_n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in X . Because X is complete, there exists $\hat{x} = \lim_{n \rightarrow \infty} x_n$. Therefore \hat{x} is a contact point of A . Because A is closed, it holds that $\hat{x} \in A$.

Therefore every Cauchy sequence in A has a limit point in A , hence A is complete. \square

Compactness

Definition 3.1. *A metric space (X, d) is called compact if every sequence $(x_n)_{n \in \mathbb{N}}$ has a convergent subsequence.*

Specifically, this definition is called sequence compactness. The other definition defines compactness as closed and bounded subset of an Euclidean space. The latter definition only works for a subset of branches in mathematics. Therefore the generalization is recommended to be remembered.

Lemma 3.1. *Let X be a compact, metric space. Then X is complete.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X . By compactness, it follows that $\exists (x_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x}$. Choose $\varepsilon > 0$ arbitrary and L large enough such that $k \geq L \implies d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2}$. Furthermore choose $N \in \mathbb{N}$ large enough such that $n, m \geq N \implies d(x_n, x_m) < \frac{\varepsilon}{2}$ (satisfied, because $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence). Choose $K \geq L$ and $n_k \geq N$. Let n_k be fixed this way. Then it holds $\forall n \geq N : d(x_n, \hat{x}) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. The first summand $\frac{\varepsilon}{2}$ results from the Cauchy sequence property, the second summand $\frac{\varepsilon}{2}$ results by convergence of (x_{n_k}) . Hence $(x_n)_{n \in \mathbb{N}}$ is convergent with limit \hat{x} . \square

Definition 3.2. *A metric space X is called bounded if there exists $M \geq 0$, such that $d(x, y) \leq M \forall x, y \in X$.*

It holds for arbitrary $x \in X$ that $\forall y \in X : y \in K_M(x)$. So, $X \subseteq K_M(x)$. On the contrary, let $X \subseteq \overline{K_M(x)}$ and let $y \in X$ and $z \in X$ be arbitrary. Then it holds that $d(y, z) \leq d(y, x) + d(x, z) \leq M + M = 2M$. Hence, X is bounded.

So, X is bounded $\iff \exists x \in X \wedge M \geq 0 : X \subseteq \overline{K_M(x)}$.

Lemma 3.2. *Every compact, metric space is also bounded.*

Proof. Assume X is unbounded.

We construct a sequence of points $(x_n)_{n \in \mathbb{N}}$ with $d(x_n, x_m) \geq 1 \forall n, m \in \mathbb{N}$ with $n \neq m$.

We use the following auxiliary result: Let $B = \bigcup_{j=1}^n K_1(z_j)$ for arbitrary $n \in \mathbb{N}$ and arbitrary $z_j \in X$. Then B is bounded. This result will be part of the practicals.

We construct $(x_n)_{n \in \mathbb{N}}$ inductively. Choose arbitrary $x_0 \in X$. Assume (x_1, \dots, x_{n-1}) are already found. Then it holds that

$$\underbrace{X}_{\text{unbounded}} \not\subseteq \underbrace{\bigcup_{j=1}^{n-1} K_1(x_j)}_{\text{bounded}}$$

hence $\exists x_n \in X \setminus \bigcup_{j=1}^{n-1} K_1(x_j)$. Because $x_n \notin K_1(x_j)$ for $j = 0, \dots, n-1$ it holds that $d(x_n, x_j) \geq 1 \forall j < n$. We get $(x_n)_{n \in \mathbb{N}}$ with $d(x_n, x_m) \geq 1 \forall n \in \mathbb{N} \forall m < n$, hence $m \neq n$. Because $d(x_n, x_m) \geq 1$, i.e. $(x_n)_{n \in \mathbb{N}}$ does not contain any Cauchy sequence as subsequence, $(x_n)_{n \in \mathbb{N}}$ does not have a convergent subsequence. Therefore X is not compact. \square

This lecture took place on 2018/03/15.

Every compact metric space is bounded. Every compact metric space is complete. In $\mathbb{C}(\mathbb{R}^n)$ it holds that $A \subseteq \mathbb{C}$ is closed. Then A with metric $d(x, y) = |x - y|$ is complete as metric space.

If A is additionally bounded, then A is compact (see course Analysis 1, Bolzano-Weierstrass).

Attention! Let V be an infinite-dimensional, complete, normed vector space.

For example, $V = C([a, b], \mathbb{R}) = \{f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is continuous in } [a, b]\}$ with norm $\|f\|_\infty = \max\{|f(x)| : x \in [a, b]\}$ and metric $\|f - g\|_\infty = \max\{|f(x) - g(x)| : x \in [a, b]\}$.

$C([a, b], \mathbb{R})$ is a complete, normed vector space. It holds that $\overline{K_1(0)}$ is not compact in $C([a, b], \mathbb{R})$ (i.e. V , for every infinite-dimensional vector space).

Again: do not remember "compactness" not as closed and bounded, as this only holds in the finite-dimensional case.

In the last proof, we have shown: If a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in X$ and $d(x_n, x_m) \geq 1$ (or $\geq \varepsilon$) $\forall n \neq m \implies X$ is not compact.

Definition 3.3. X is called *totally bounded*, if for every $\varepsilon > 0$, finitely many points $X_1^\varepsilon, X_2^\varepsilon, \dots, X_{N(\varepsilon)}^\varepsilon$ such that $X \subseteq \bigcup_{i=1}^{N(\varepsilon)} K_\varepsilon(X_i^\varepsilon)$.

Hence, for every $x \in X$, there exists some X_j^ε such that $d(X, X_j^\varepsilon) < \varepsilon$.

Remark 3.1 (For the practicals). Let X be totally bounded, then there does not exist some sequence $(x_n)_{n \in \mathbb{N}}$ with $d(x_n, x_m) \geq \varepsilon \forall n \neq m$. It holds, that X is compact if and only if X is totally bounded and complete.

Theorem 3.1. Let $f : X \rightarrow Y$ be continuous. Let X be compact. Then image $f(X) \subseteq Y$ is also compact.

Be aware, that this proof is a common exam question and students often begin with the wrong order.

Proof. Let $(y_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in $f(X)$. Show that $(y_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Because $y_n \in f(X)$, there exists at least one x_n with $y_n = f(x_n)$. Then $(x_n)_{n \in \mathbb{N}}$ is a sequence in X , X is compact, hence there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x} \in X$. Because f is continuous, it holds that $\lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} y_{n_k} = f(\hat{x}) =: \hat{y}$. So $(y_n)_{n \in \mathbb{N}}$ has a convergent subsequence. Hence $f(X) \subseteq Y$ is compact. \square

Theorem 3.2 (Conclusion). Let X be compact, $f : X \rightarrow \mathbb{R}$ continuous on X . Then there exists \underline{x} and $\bar{x} \in X$, such that

$$f(\underline{x}) \leq f(x) \leq f(\bar{x}) \quad \forall x \in X$$

Hence, f has a maximum and a minimum.

Proof. $f(X) \subseteq \mathbb{R}$ is compact (Theorem 3.1), hence $f(X)$ is bounded and complete, hence closed in \mathbb{R} . There exists $\xi \in \mathbb{R}$ with $\xi = \sup f(X)$, because $f(X)$ is complete and ξ is a contact point of $f(X)$, it holds that $\xi \in f(X)$, hence $\exists \bar{x} \in X : \xi = f(\bar{x})$. Furthermore, ξ is an upper bound of $f(X) \rightarrow f(x) \leq \xi = f(\bar{x}) \forall x \in X$.

For \underline{x} , it works the same way. \square

Theorem 3.3. Let $f : X \rightarrow Y$ is continuous on X and X is compact. Then f is uniformly continuous on X .

Indirect proof. Assume X is compact, $f : X \rightarrow Y$ is continuous, but not uniformly continuous. Uniform continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Not uniformly continuous:

$$\exists \varepsilon > 0 \forall \delta_n = \frac{1}{n} (n \in \mathbb{N}) \exists x_n, y_n \in X : d_X(x_n, y_n) < \frac{1}{n} \wedge d_Y(f(x_n), f(y_n)) \geq \varepsilon$$

Now choose some (x_n) and (y_n) . We will use a specific ε later. Because X is compact, there exists a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$, hence $\lim_{k \rightarrow \infty} x_{n_k} = \hat{x}$. The sequence $(y_{n_k})_{k \in \mathbb{N}}$ has a convergent subsequence itself:

$$\lim_{l \rightarrow \infty} y_{(n_k)_l} = \hat{y}$$

Because $(x_{n_k})_{k \in \mathbb{N}}$ is convergent, the subsequence $(x_{(n_k)_l})_{l \in \mathbb{N}}$ converges towards the same limit \hat{x} .

$$\tilde{x}_l := x_{n_{k_l}} \quad \tilde{y}_l := y_{n_{k_l}}$$

because $l \leq n_{k_l}$ and

$$d_X(\tilde{x}_l, \tilde{y}_l) = d_X(x_{n_{k_l}}, y_{n_{k_l}}) \underbrace{\quad}_{\text{by assumption}} < \frac{1}{n_{k_l}} \leq \frac{1}{l}$$

Claim. For $\hat{x} = \lim_{l \rightarrow \infty} \tilde{x}_l$ and $\hat{y} = \lim_{l \rightarrow \infty} \tilde{y}_l$, it holds that $\hat{x} = \hat{y}$. Let $\varepsilon' > 0$ be arbitrary, l large enough such that

- $\frac{1}{l} < \frac{\varepsilon'}{3}$
- $d_X(\tilde{x}_l, \hat{x}) < \frac{\varepsilon'}{3}$
- $d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3}$

Therefore it holds that

$$d_X(\hat{x}, \hat{y}) \leq d_X(\hat{x}, \tilde{x}_l) + d_X(\tilde{x}_l, \tilde{y}_l) + d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3} + \frac{1}{l} + \frac{\varepsilon'}{3} < \varepsilon'$$

Therefore it holds that $d_X(\hat{x}, \hat{y}) = 0$, hence $\hat{x} = \hat{y}$. Because f is continuous and $\tilde{x}_l \rightarrow \hat{x}$ and $\tilde{y}_l \rightarrow \hat{x}$, there exists $l \in \mathbb{N}$ such that

$$d_Y(f(\tilde{x}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$

and also

$$d_Y(f(\tilde{y}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$

where ε is the epsilon from the very beginning of the proof.

$$\implies d_Y(f(\tilde{x}_l), f(\hat{x})) + d_Y(f(\tilde{y}_l), f(\hat{x})) < \varepsilon$$

This contradicts to

$$d_Y(f(\tilde{x}_l), f(\tilde{y}_l)) = d_Y(f(x_{n_{k_l}}), f(y_{n_{k_l}})) \geq \varepsilon$$

Hence, f is uniformly continuous. □

Subsets of $(\mathbb{R}^n, \|\cdot\|)$ (or $(V, \|\cdot\|)$) as metric spaces.

We consider $\Omega \subseteq V$ where V is a normed vector space. (Ω, d) is $d(x, y) = \|x - y\|$ is a metric space.

$$K_r^\Omega(x) = \{y \in \Omega \mid \|y - x\| < r\}$$

is a sphere with center x and radius r in Ω .

$$K_r^V(x) = \{y \in V \mid \|y - x\| < r\}$$

obvious: $K_r^\Omega(x) = \Omega \cap K_r^V(x)$.

TODO drawing 08

Lemma 3.3. *Let $O' \subseteq \Omega \subseteq V$.*

Then it holds that O' is open in $\Omega \iff$ there exists $O \subseteq V$ is open in V such that $O' = O \cap \Omega$.

Proof. \Rightarrow Let $O' \subseteq \Omega$ be open in Ω and $x \in O'$ be arbitrary. Then there exists $r(x) > 0 : x \in K_{r(x)}^\Omega(x) = K_{r(x)}^V(x) \cap \Omega \subseteq O'$. Then it holds that

$$O' = \bigcup_{x \in O'} \{x\} \subseteq \bigcup_{x \in O'} K_{r(x)}^\Omega(x) = \left(\bigcup_{x \in O'} (K_{r(x)}^V(x) \cap \Omega) \right) = \underbrace{\left(\bigcup_{x \in O'} K_{r(x)}^V(x) \right)}_{=O \subseteq V \text{ is open in } V} \cap \Omega \subseteq O'$$

So every \subseteq in this inclusion chain is actually an equality. So $O' = O \cap \Omega$.

\Leftarrow Let $O' = O \cap \Omega$ and $x \in O'$ be chosen arbitrarily. Because $x \in O$ and O is open in V .

$$\exists r > 0 : K_r^V(x) \subseteq O \implies \underbrace{K_r^V(x) \cap \Omega}_{=K_r^\Omega(x)} \subseteq O \cap \Omega = O'$$

So O' is open in Ω .

□

Remark 3.2. $A' \subseteq \Omega$ is closed in $\Omega \iff \exists A \subseteq V$ closed in V with $A' = A \cap \Omega$.

Remark 3.3. Let T be an arbitrary topological space with topology τ on T (a system of open sets). Furthermore let $\Omega \subseteq T$.

Then Ω itself is a topological space with $O' \subseteq \Omega$ is open $\iff \exists O \subset T$ open in T with $O' = O \cap \Omega$.

Also called “subspace topology”, “trace topology” or “relative topology”.

Attention!

$$O' \subseteq \Omega \text{ open in } \Omega \implies O' \text{ open in } V$$

does *not* hold in general.

Example 3.1.

$$\Omega = [0, 1] \cap [0, 1)$$

$K_{\frac{1}{2}}(p) \cap \Omega$ is open in Ω but not open in \mathbb{R}^2 .

Analogously,

$$A' \subseteq \Omega \text{ is closed} \implies A' \text{ closed in } V$$

does *not* hold in general.

Remark 3.4. $K \text{ is compact in } \Omega \implies K \text{ is compact in } V$

Let $(x_n)_{n \in \mathbb{N}}$ is a sequence in K . Compactness $\implies \exists (x_{n_k})_{k \in \mathbb{N}} : x_{n_k} \rightarrow \hat{x}$ for $k \rightarrow \infty$ and $K \subseteq \Omega \subseteq V$.

Then $(x_n)_{n \in \mathbb{N}}$ also has a convergent subsequence in V .

Normed vector spaces

Definition 3.4. Let V be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ are normed on V . We say, $\|\cdot\|_1$ is equivalent to norm $\|\cdot\|_2$, if $0 < m \leq M$ exist such that

$$m \|v\|_1 \leq \|v\|_2 \leq M \|v\|_1 \quad \forall v \in V$$

Remark 3.5. Equivalence of norms is an equivalence relation.

reflexivity Let $m = M = 1$. TODO

symmetry

$$\begin{aligned} m \|v\|_1 \leq \|v\|_2 &\implies \|v\|_1 \leq \frac{1}{m} \|v\|_2 \wedge \|v\|_2 \leq M \cdot \|v\|_1 \implies \frac{1}{M} \|v\|_2 \leq \|v\|_1 \\ &\implies \underbrace{\frac{1}{M}}_{m'} \|v\|_2 \leq \|v\|_1 \leq \underbrace{\frac{1}{m}}_{M'} \|v\|_2 \end{aligned}$$

hence the equivalence relations of norms are symmetrical.

transitivity Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent. Let $\|\cdot\|_2$ and $\|\cdot\|_3$ be equivalent.

$$\begin{aligned} m \cdot \|v\|_1 &\leq \|v\|_2 \leq M \|v\|_1 \quad \forall v \in V \\ m' \cdot \|v\|_2 &\leq \|v\|_3 \leq M' \|v\|_2 \quad \forall v \in V \\ \implies m \cdot m' \|v\|_1 &\leq m' \|v\|_2 \leq \|v\|_3 \leq M' \|v\|_2 \leq M \cdot M' \|v\|_1 \end{aligned}$$

This lecture took place on 2018/03/20.

Addendum:

- Let $(x_n)_{n \in \mathbb{N}}$ be in (X, d) , then it holds that

$$\underbrace{x = \lim_{n \rightarrow \infty} x_n}_{\text{in } X} \iff \underbrace{\lim_{n \rightarrow \infty} d(x_n, x) = 0}_{\text{in } \mathbb{R}}$$

$$(\iff \lim_{n \rightarrow \infty} \|x_n - x\| = 0 \text{ in normed vector spaces } V)$$

- Inversed triangle inequality: Let V be a normed vector space. Let $x, y \in V$.

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

Hence,

$$\|x\| - \|y\| \leq \|x - y\|$$

By exchanging x and y ,

$$\|y\| - \|x\| \leq \|x - y\|$$

Hence, it holds that

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|$$

- Define the map $n : V \rightarrow [0, \infty)$ on $(V, \|\cdot\|)$ with $n(x) = \|x\|$. Then n is continuous on V because

$$|n(x_1) - n(x_2)| = \left| \|x_1\| - \|x_2\| \right| \leq \|x_1 - x_2\|$$

Hence, n is Lipschitz continuous with constant 1.

Regarding the equivalence of norms:

Lemma 3.4. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on V . Then it holds that

1. $\lim_{n \rightarrow \infty} \|x_n - x\|_1 = 0 \iff \lim_{n \rightarrow \infty} \|x_n - x\|_2 = 0$, hence $(x_n)_{n \in \mathbb{N}}$ is convergent with limit x in regards of $\|\cdot\|_1 \iff (x_n)_{n \in \mathbb{N}}$ is convergent with limit x in regards of $\|\cdot\|_2$.
2. $O \subseteq V$ is open in regards of $\|\cdot\|_1 \iff O$ is open in regards of $\|\cdot\|_2$, hence $\tau_1 = \tau_2$ (topologies are equivalent).
3. $K \subseteq V$ is compact in regards of $\|\cdot\|_1 \iff K$ is compact in regards of $\|\cdot\|_2$.

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, hence $\exists m, M > 0 : m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \forall x \in V$.

1. Let $\varepsilon > 0$ and $\lim_{n \rightarrow \infty} \|x_n - x\|_1 = 0$. Choose $N \in \mathbb{N}$ such that $n \geq N \implies \|x_n - x\|_1 < \frac{\varepsilon}{M}$. For those n it holds that

$$\|x_n - x\|_2 \leq M\|x_n - x\|_1 < \frac{\varepsilon}{M} \cdot M = \varepsilon$$

Hence, $\lim_{n \rightarrow \infty} \|x_n - x\|_2 = 0$.

2. $K_r^2(x) = \{y \in V \mid \|y - x\|_2 < r\}$. For $y \in K_r^2(x)$ it holds that

$$m \|y - x\|_1 \leq \|y - x\|_2 < r$$

hence,

$$\|y - x\|_1 < \frac{r}{m} \implies y \in K_{\frac{r}{m}}^1(x)$$

hence $K_r^2(x) \subseteq K_{\frac{r}{m}}^1(x)$. Let $y \in K_{\frac{r}{m}}^1(x)$. Then it holds that,

$$\|y - x\|_2 \leq M \|y - x\|_1 < M \cdot \frac{r}{M} = r$$

hence $y \in K_r^2(x) \implies K_{\frac{r}{M}}^1(x) \subseteq K_r^2(x)$. Now let O be open in regards of $\|\cdot\|_2$, hence

$$\forall x \in O \exists r > 0 : K_r^2(x) \subseteq O \implies K_{\frac{r}{M}}^1(x) \subseteq K_r^2(x) \subseteq O$$

so O is open in regards of $\|\cdot\|_1 \implies O$ is open in regards of $\|\cdot\|_2$ analogously.

3. Let K be compact in regards of $\|\cdot\|_1$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in K . Then there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ with $\|x_{n_k} - x\|_1 \rightarrow 0$ for $k \rightarrow \infty$
by the first property $\implies \|x_{n_k} - x\|_2 \rightarrow 0$. Hence $(x_{n_k})_{k \in \mathbb{N}}$ is also a convergent subsequence in regards of $\|\cdot\|_2$.

□

Remark 3.6 (Proven in the practicals). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}^k

$$\|x\|_\infty = \max \{ |x^i| \mid i = 1, \dots, k \}$$

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix}$$

It holds that $\lim_{n \rightarrow \infty} \|x_n - x\|_\infty = 0 \iff \lim_{n \rightarrow \infty} |x_n^i - x^i| = 0$ for all $i \in \{1, \dots, k\}$.

Theorem 3.4 (Bolzano-Weierstrass theorem in \mathbb{R}^k). Let $K \subseteq \mathbb{R}^k$ be closed and bounded. Then K is compact in $(\mathbb{R}^k, \|\cdot\|_\infty)$.

Proof. Let $\|x\|_\infty \leq M \forall x \in K \iff |x^i| \leq M \forall x \in K$ and $i \in \{1, \dots, k\}$. Choose $(x_n)_{n \in \mathbb{N}}$ an arbitrary sequence in K $(x_n^i)_{n \in \mathbb{N}}$ is a bounded sequence in \mathbb{R} . Because $(x_n^1)_{n \in \mathbb{N}}$ is bounded, there exists a convergent subsequence $(x_{n_1}^1)_{l_1 \in \mathbb{N}}$

$$\lim_{l_1 \rightarrow \infty} x_{n_1}^1 = x^1$$

Consider $(x_{n_{l_1}}^2)_{l_1 \in \mathbb{N}}$, a subsequence of a bounded sequence, hence bounded itself. By the Bolzano-Weierstrass theorem in \mathbb{R} , there exists a convergent subsequence $(x_{n_{l_1 l_2}}^2)_{l_2 \in \mathbb{N}}$ with $\lim_{l_2 \rightarrow \infty} x_{n_{l_1 l_2}}^2 = x^2$. Consider $x_{n_{l_1 l_2}}^1$ as subsequence of $x_{n_{l_1}}^1$ is already convergent, hence $\lim_{l_2 \rightarrow \infty} x_{n_{l_1 l_2}}^1 = x^1$. Furthermore, up to index i , it holds that:

$$\lim_{l_k \rightarrow \infty} x_{n_{l_1 l_2 \dots l_k}} = x^i \quad \text{for } i = 1, \dots, k$$

Hence, with $\tilde{x}_{l_k} = x_{n_{l_1 l_2 \dots l_k}}$ gives a subsequence of x_n , converging by each coordinate. Thus,

$$\lim_{l_k \rightarrow \infty} \|\tilde{x}_{l_k} - x\|_\infty = 0$$

Because $\tilde{x}_{l_k} \in K$ and K be closed, it holds that $x \in K$. Hence K is compact. \square

Theorem 3.5 (Norm equivalence in \mathbb{R}^k). *In \mathbb{R}^k , all norms are equivalent.*

Proof. We show: Let $\|\cdot\|$ be an arbitrary norm on \mathbb{R}^n . Then $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. By transitivity of norm equivalence, two arbitrary norms are equivalent to each other.

1. Let (e_1, e_2, \dots, e_k) be the canonical basis in \mathbb{R}^k .

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^k \end{bmatrix} = \sum_{j=1}^k x^j e_j$$

Furthermore let $M' = \max \{\|e_j\| : j = 1, \dots, k\}$ with $\|e_j\| \neq 0$ and $M' > 0$. Then it holds that

$$\|x\| = \left\| \sum_{j=1}^k x^j e_j \right\| \leq \sum_{j=1}^k \|x^j e_j\| = \sum_{j=1}^k |x^j| \|e_j\| \leq M' \sum_{j=1}^k \underbrace{|x_j|}_{\leq \|x\|_\infty} \leq \underbrace{M' \cdot k}_M \|x\|_\infty = M \|x\|_\infty$$

2. We consider $v : \mathbb{R}^k \rightarrow [0, \infty)$. $v(x) = \|x\|$ as map on $(\mathbb{R}^k, \|\cdot\|_\infty)$.

Claim. v is continuous on $(\mathbb{R}^k, \|\cdot\|_\infty)$.

Proof. Show that,

$$|v(x) - v(y)| = \left| \|x\| - \|y\| \right| \underbrace{\leq}_{\text{inversed triangle ineq.}} \|x - y\| \underbrace{\leq}_{\text{because of (1)}} M \|x - y\|$$

Hence v is Lipschitz continuous. \square

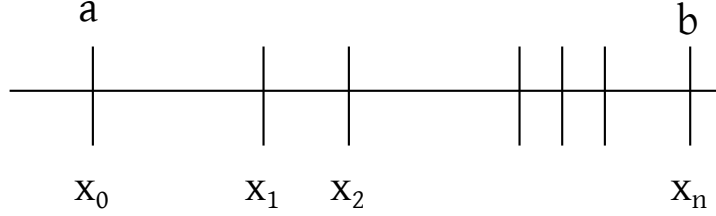


Figure 6: Illustration of a partition

We consider $S_{\infty}^{k-1} = \{x \in \mathbb{R}^k \mid \|x\|_{\infty} = 1 = \text{boundary}(K_1^{\infty}(0))$. S_{∞}^{k-1} is bounded.

Let $(x_n)_{n \in \mathbb{N}}$ is a sequence in S_{∞}^{k-1} with $x = \lim_{n \rightarrow \infty} x_n$. Because $n(x) = \|x\|_{\infty}$ is continuous, it holds that

$$\lim_{n \rightarrow \infty} \underbrace{\|x_n\|_{\infty}}_{=1} = \underbrace{\|x\|}_{=1}$$

Hence $x \in S_{\infty}^{k-1}$. Hence, S_{∞}^{k-1} is closed in $(\mathbb{R}^k, \|\cdot\|_{\infty})$. Hence S_{∞}^{k-1} is compact in $(\mathbb{R}^k, \|\cdot\|_{\infty})$, $\nu : S_{\infty}^{k-1} \rightarrow [0, \infty)$, with S_{∞}^{k-1} compact, is continuous. Has

a minimum n on S_{∞}^{k-1} . Thus there exists $\bar{x} \in S_{\infty}^{k-1} : \underbrace{m}_{>0} = \underbrace{\left\| \bar{x} \right\|}_{\neq 0} \leq$

$\|x\| \forall x \in S_{\infty}^{k-1}$. Let $x \in \mathbb{R}^k$ be arbitrary with $x \neq 0$. Then it holds that $\frac{x}{\|x\|_{\infty}} \in S_{\infty}^{k-1}$ and it holds that

$$m \leq \left\| \frac{x}{\|x\|_{\infty}} \right\| = \frac{1}{\|x\|_{\infty}} \|x\| \implies m \|x\|_{\infty} \leq \|x\|$$

Inequality also holds true for $x = 0$.

□

Integral calculus

Definition 4.1. Let $a < b$ with $a, b \in \mathbb{R}$. We consider functions of $[a, b]$. We call $(x_j)_{j=0}^n$ a partition of $[a, b]$ if $a = x_0 < x_1 < x_2 < \dots < x_n = b$. x_j decomposes $[a, b]$ in subintervals (x_{j-1}, x_j) . $\varphi : [a, b] \rightarrow \mathbb{R}$ is called step function in $[a, b]$ in regards of partition $(x_j)_{j=0}^n$ if $\varphi|_{(x_{j-1}, x_j)} = c_j$, so constant for $j = 1, \dots, n$.

φ is called step function in $[a, b]$ if there exists a partition such that φ is a subsequence.

$$\tau[a, b] = \{\varphi : [a, b] \rightarrow \mathbb{R} : \varphi \text{ is subsequence}\}$$

- Let $(\xi_i)_{i=0}^m$ be a partition of $[a, b]$ and $(x_j)_{j=0}^n$ is a partition as well. Then we call $(\xi_i)_{i=0}^m$ a refinement of $[a, b]$ and $(x_j)_{j=1}^n$ as well. Then $(\xi_i)_{i=0}^m$ is a refinement of $(x_j)_{j=0}^k$ if $\{x_0, x_1, \dots, x_n\} \subseteq \{\xi_0, \xi_1, \dots, \xi_m\}$

TODO drawing

Functions values in boundaries x_{j-1} and x_j do not have any constraints and will be relevant for an integral. A φ can be a step function in terms of many, various partitions.

Lemma 4.1. Let $\varphi \in \tau[a, b]$ be a step function in terms of partition $(x_j)_{j=0}^n$ and let $(x_i)_{i=0}^n$ be a refinement of $(x_j)_{j=0}^n$ in terms of $(x_i)_{i=0}^m$.

Proof. Refinement: For every $j \in \{0, \dots, n\}$ there exists $i_j \in \{0, \dots, m\}$ such that $X_j = \xi_{i_j}$, $i_0 = 0, i_n = m$. $i_{j-1} < i_j$.

Let $i \in \{1, \dots, m\}$. Then there exists a uniquely determined $j \in \{1, \dots, n\}$ such that $\xi_{i_{j-1}} < \xi_i \leq \xi_{i_j}$

TODO drawing

Then it holds that $(\xi_{i-1}, \xi_i) \subseteq \underbrace{(\xi_{i_{j-1}}, \xi_{i_j})}_{=(x_{j-1}, x_j)}$ and $\varphi|_{(\xi_{i-1}, \xi_i)} = c_j = \text{const.}$ So φ is a

subsequence in regards of $(\xi_i)_{i=0}^m$. □

Definition 4.2. Let $\varphi \in \tau[a, b]$ in terms of partition $(X_j)_{j=0}^n$ with $\varphi|_{(X_{j-1}, X_j)} = c_j$ and $\Delta X_j = X_j - X_{j-1} > 0$ for $g = 1, \dots, n$. Then we define ...

$$\int_a^b \varphi dx = \sum_{j=1}^n c_j \Delta x_j$$

is called integral of φ in terms of partition $(x_j)_{j=0}^n$

This lecture took place on 2018/03/22.

Step function φ . $\varphi|_{x_{j-1}, x_j} = c_j$

$$\delta x_j = x_j - x_{j-1}$$

$$\int_a^b \varphi dx = \sum_{j=1}^n c_j \cdot \delta x_j$$

Lemma 4.2. Let $(x_i)_{i=0}^n$ be a partition of $[a, b]$ and $(\xi_i)_{i=0}^m$ be a refinement of $(x_j)_{j=0}^n$. Furthermore let φ be a subsequence with respect to $(x_j)_{j=0}^n$ (so also with respect to $(\xi_j)_{i=0}^m$). Then the integrals of φ with respect to $(x_j)_{j=0}^n$ and $(\xi_i)_{i=0}^m$ are equal.

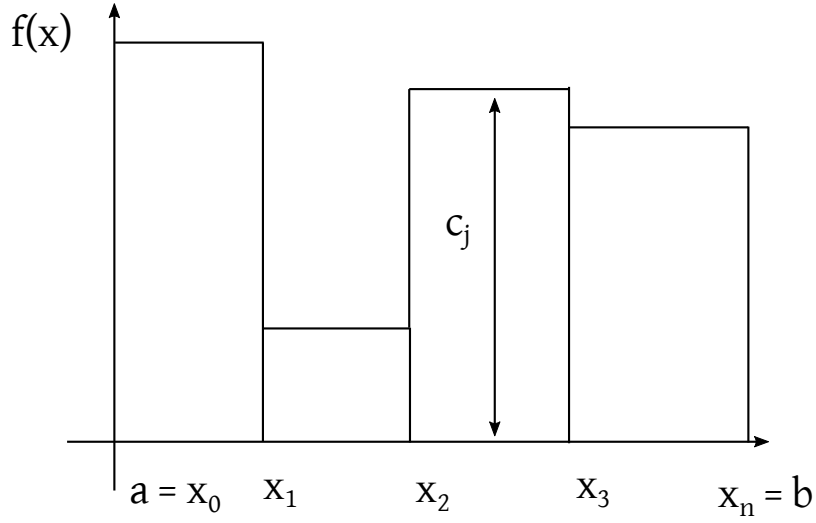


Figure 7: Integral of a step function as sum of areas of rectangles

Proof. There exist indices i_j for $j = 0, n$ such that $x_j = \xi_{i_j}$.

$$i_0 = 0 \quad i_n = m \quad i_{j-1} < i_j$$

$$\delta x_j = x_j - x_{j-1} = \xi_{i_j} - \xi_{i_{j-1}} = \xi_{i_j} - \xi_{i_{j-1}} = \underbrace{\sum_{i=i_{j-1}+1}^{i_j} (\xi_i - \xi_{i-1})}_{\text{telescoping sum}} = \sum_{i=i_{j-1}+1}^{i_j} \delta \xi_i$$

$$\varphi|_{x_{j-1}, x_j} = c_j \implies \varphi|_{(\xi_{i-1}, \xi_i)} = c_j \text{ for } i = i_{j-1} + 1, \dots, i_j$$

$$\tilde{c}_i = \varphi|_{(\xi_{i-1}, \xi_i)}$$

$$\underbrace{\sum_{i=1}^m \tilde{c}_i \delta \xi_i}_{\text{integral of } \varphi \text{ w.r.t } (\xi_i)_{i=0}^m} = \sum_{j=1}^n \sum_{i=i_{j-1}+1}^{i_j} \tilde{c}_i \delta \xi_i = \sum_{j=1}^n c_j \underbrace{\sum_{i=i_{j-1}+1}^{i_j} \delta \xi_i}_{=x_j} = \sum_{j=1}^n c_j \delta x_j$$

This is the integral of φ with respect to $(x_j)_{j=0}^n$. □

Lemma 4.3. Let φ be a step function with respect to $(x_j)_{j=0}^n$ and $(w_i)_{i=0}^L$. Then the integrals of φ with respect to $(x_j)_{j=0}^n$ and with respect to $(w_l)_{l=0}^L$ equal.

Proof. Let $\{\xi_i | i = 1, \dots, m\} = \{x_j | j = 0, \dots, n\} \cup \{w_l | l = 0, \dots, L\}$ with $\xi_0 = a$, $\xi_m = x_n = w_L = b$ and $\xi_{i-1} < \xi_i$ for $i = 1, \dots, m$. Then $(\xi_i)_{i=0}^m$ is a refinement of $(x_j)_{j=0}^n$ as well as $(w_l)_{l=0}^L$. By Lemma 4.2, the integral of φ with respect to $(x_j)_{j=0}^n =$ integral of φ with respect to $(\xi_i)_{i=1}^m =$ integral of φ with respect to $(w_l)_{l=0}^L$. Here we discard the statement “with respect to $(x_j)_{j=0}^n$ ”. \square

Lemma 4.4. Let f, g be step functions on $[a, b]$. $f, g \in \tau[a, b]$.

- for $\alpha, \beta \in \mathbb{R}$, let $\alpha f + \beta g \in \tau[a, b]$ and

$$\int_a^b (\alpha f + \beta g) dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Hence, the integral is linear on $[a, b]$. $\tau[a, b]$ is a vector space.

- $f \leq g$ in $[a, b]$, then $\int_a^b f dx \leq \int_a^b g dx$ (monotonicity).
- $\left| \int_a^b f dx \right| \leq \int_a^b |f| dx$ ($|f(x)|$ is also a step function)

Proof. 1. Let $f, g \in \tau[a, b]$. Let $(\xi_i)_{i=0}^m$ be a partition such that $f|_{(\xi_{i-1}, \xi_i)} = c_i$ and $g|_{(\xi_{i-1}, \xi_i)} = d_i$. Then

$$\int_a^b (\alpha f + \beta g) dx = \sum_{i=1}^m (\alpha c_i + \beta d_i) \delta \xi_i = \alpha \sum_{i=1}^m c_i \delta \xi_i + \beta \sum_{i=1}^m d_i \delta \xi_i = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Furthermore,

$$(\alpha f + \beta g)|_{(\xi_{i-1}, \xi_i)} = \alpha c_i + \beta d_i = \text{const.}$$

Thus,

$$\alpha f + \beta g \in \tau[a, b]$$

2. Let $h \in \tau[a, b]$ with $h(x) \geq 0 \forall x \in [a, b]$ be a step function and $\int_a^b h dx =$

$$\sum_{i=1}^m \underbrace{h_i}_{\geq 0} \delta \xi_i \geq 0 \text{ TODO Hence, it holds that } 0 \leq \int_a^b h dx = \int_a^b (g - f) dx =$$

$$\int_a^b g dx - \int_a^b f dx.$$

3. $f \leq |f|$, hence $\int_a^b f dx \leq \int_a^b |f| dx$ and also $-f \leq |f|$, so

$$\int_a^b (-f) dx = - \int_a^b f dx \leq \int_a^b |f| dx$$

$$\implies \left| \int_a^b f dx \right| \leq \int_a^b |f| dx$$

It is left to prove: $|f| \in \tau[a, b]$ (i.e. $|f|$ is a step function)

Let $f|_{(\xi_{i-1}, \xi_i)} = c_i \implies |f|_{(\xi_{i-1}, \xi_i)} = |c_i| = \text{constant}$. Hence $|f| \in \tau[a, b]$.

□

Definition 4.3. Let $a \subseteq \mathbb{R}^k$. We call $\chi_A : \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

a characteristic function (indicator function) of set A . Often denoted as $\chi_A = \mathbb{1}$.

Remark 4.1. TODO drawings

Let $A = (a', b')$ with $a \leq a' < b' \leq b$. Then $\chi_{(a', b')} \in \tau[a, b]$. Also for $x \in [a, b]$, it holds that $\chi_{\{x\}} = \tau[a, b]$. Therefore every linear combination of characteristic functions of open subintervals (a', b') of $[a, b]$ as characteristic functions of one-point sets $\chi_{\{x\}}, x \in [a, b]$ a step function on $[a, b]$.

$$\sum_{j=1}^n \alpha_j \chi_{(a_j, b_j)} + \sum_{k=1}^m \beta_k \chi_{\{x_k\}} \in \tau[a, b]$$

On the opposite, $f \in \tau[a, b]$, hence

$$f|_{(x_{j-1}, x_j)} \underbrace{=}_{j=1, \dots, n} c_j \text{ and } f(x_j) \underbrace{=}_{j=0, \dots, n} d_j$$

$$f = \sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^n d_j \chi_{\{x_j\}} = (*)$$

for $x \in (x_{j-1}, x_j)$ it holds that $\chi_{(x_{j-1}, x_j)}(x) = 1$.

$$\chi_{(x_{l-1}, x_l)}(x) = 0 \text{ for } l \neq j$$

$$\chi_{\{x_l\}}(x) = 0 \text{ for } l = 0, \dots, n$$

i.e. $\sum_{j=1}^n c_j \chi_{(x_{j-1}, x_j)}(x) + \sum_{l=0}^n d_l \chi_{\{x_l\}}(x) = c_j \cdot 1 + 0 = c_j$ hence $(*) = c_j$ on (x_{j-1}, x_j) . Therefore $f \in \tau[a, b] \iff f$ is linear combination of characteristic functions of open intervals or one-pointed sets.

Regulated functions

Definition 4.4. Let X be a metric space $A \subseteq X$ and $x \in X$ is an accumulating point¹ of A . Let $f : A \rightarrow \mathbb{R}$. We say, f has limit $c \in \mathbb{R}$ in x ($\lim_{\xi \rightarrow x} f(\xi) = c$) if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi \neq x \text{ and } d(\xi, x) < \delta : |f(\xi) - c| < \varepsilon$$

¹An accumulation point has 3 equivalent definitions (sequence, intersection, infinitely many elements in sphere).

Remark 4.2. $x \in A$ and $c = f(x) \implies f$ is continuous in x .

We usually consider $A = [a, b] \subseteq \mathbb{R}$, $x \in [a, b]$.

It is possible, that f in x has a limit, $x \in A$ and $c = \lim_{\xi \rightarrow x} f(\xi) \neq f(x)$.

TODO drawing

Definition 4.5. Now let $A \subseteq \mathbb{R}$ and x is a accumulation point of A . Let $f : A \rightarrow \mathbb{R}$ be given. We say f has a right-sided limit c in x with $c = \lim_{\xi \rightarrow x^+} f(\xi) = c$ if $\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi > x$

$$\wedge |\xi - x| = \xi - x < \delta \implies |f(\xi) - c| < \varepsilon$$

The left-sided limit follows analogously.

$$c = \lim_{\xi \rightarrow x^-} f(\xi)$$

$$c = \lim_{\xi \rightarrow x^+} f(\xi) \quad d = \lim_{\xi \rightarrow x^-} f(\xi)$$

TODO drawing

Lemma 4.5 (Sequence criterion for limits of functions). Let $f : A \subseteq X \rightarrow \mathbb{R}$ be given. x is an accumulation point of A . Then it holds that

$$\lim_{\xi \rightarrow x} f(\xi) = c \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi_n \in A, \xi_n \neq x \text{ and } \lim_{n \rightarrow \infty} \xi_n = x \text{ it holds that } \lim_{n \rightarrow \infty} f(\xi_n) = c$$

For one-sided limits $A \subseteq \mathbb{R}$ it holds that

$$c = \lim_{\xi \rightarrow x^+} f(\xi) \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi_n \in A \quad \xi_n > x \text{ with } \lim_{n \rightarrow \infty} \xi_n = x \text{ it holds that } \lim_{n \rightarrow \infty} f(\xi_n) = c$$

Remark 4.3. Attention! We, therefore, use two different definitions of limits.

Lemma 4.6 (Cauchy criterion of limits of functions). Let $f : A \subseteq X \rightarrow \mathbb{R}$. Let x be an accumulation point of A . Let X be a metric space. Then it holds that f has a limit in x if and only if

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi, \eta \in A : \xi \neq \eta : \eta \neq x :$$

with $d(\xi, x) < \delta$ and $d(\eta, x) < \delta$ it holds that $|f(\xi) - f(\eta)| < \varepsilon$. Analogously for one-sided limits with $A \subseteq \mathbb{R}$. Additionally, we need the constraint that $\xi > x$ and $\eta > x$ for $\lim_{\xi \rightarrow x^+} f(\xi)$ or equivalently, $\xi < x$ and $\eta < x$ for $\lim_{\xi \rightarrow x^-} f(\xi)$.

TODO normalize and visualize equivalent statements for left-sided and right-sided limit (using Ring's notes)

Proof. \Leftarrow Let $c = \lim_{\xi \rightarrow x} f(\xi)$ and let $\varepsilon > 0$ be chosen arbitrarily. Then there exists $\delta > 0$ such that $d(\xi, x) < \delta$ and $\xi \neq x$

$$\implies |f(\xi) - c| < \frac{\varepsilon}{2}$$

For ξ, η : $d(\xi, x) < \delta$ and $d(\eta, x) < \delta$ with $\xi, \eta \neq x$ is therefore

$$|f(\xi) - f(\eta)| = |f(\xi) - c + c - f(\eta)| \leq |f(\xi) - c| + |f(\eta) - c| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

\Rightarrow Assume the Cauchy criterion holds. We show that

1. for every sequence $(\xi_n)_{n \in \mathbb{N}}$, $\xi_n \in A \setminus \{x\}$ with $\lim_{n \rightarrow \infty} \xi_n = x$ it holds that $(f(\xi_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} and therefore convergent in \mathbb{R} .
2. all Cauchy sequences have the *same* limit c .

We prove (1.)

Let $(\xi_n)_{n \in \mathbb{N}}$ be as above. Let $\varepsilon > 0$ be arbitrary. and N_ε large enough such that $\forall n \in N_\varepsilon$ it holds that $d(\xi_n, x) < \delta$ (δ chosen appropriately to ε according to the Cauchy criterion).

By the Cauchy criterion, $|f(\xi_n) - f(\xi_m)| < \varepsilon$ for all $m, n \geq N_\varepsilon$. Therefore $(f(\xi_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{R} . If \mathbb{R} is complete, then there exists $c = \lim_{n \rightarrow \infty} f(\xi_n)$. QED.

We prove (2.)

Let $\xi_n \rightarrow x$ as above and $\xi'_n \rightarrow x$ as above and $c = \lim_{n \rightarrow \infty} f(\xi_n)$ as well as $c' = \lim_{n \rightarrow \infty} f(\xi'_n)$. Let $\varepsilon > 0$ be arbitrary, N_ε such that $n \geq N_\varepsilon \implies |f(\xi_n) - c| < \frac{\varepsilon}{3}$ and $N'_\varepsilon \in \mathbb{N}$ such that $n \geq N'_\varepsilon \implies |f(\xi'_n) - c'| < \frac{\varepsilon}{3}$.

Furthermore choose $\delta > 0$ such that

$$d(\xi, x) < \delta \wedge d(\eta, x) < \delta \implies |f(\xi) - f(\eta)| < \frac{\varepsilon}{3}$$

(because of the Cauchy criterion). M_ε such that

$$n \geq M_\varepsilon \implies d(\xi_n, x) < \delta \wedge M'_\varepsilon : n \geq M'_\varepsilon \implies d(\xi'_n, x) < \delta$$

Let $n \geq \max\{N_\varepsilon, N'_\varepsilon, M_\varepsilon, M'_\varepsilon\}$.

This lecture took place on 2018/04/10.

Then it holds that

$$|c - c'| \leq \underbrace{|c - f(\xi_n)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(\xi_n) - f(\xi'_n)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f(\xi'_n) - c'|}_{< \frac{\varepsilon}{3}} \quad \forall \varepsilon > 0$$

Hence, $c = c'$. We have shown that $\exists c \in \mathbb{R} : \forall (\xi_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \xi_n = x$ it holds that $\lim_{n \rightarrow \infty} f(\xi_n) = c$. So $\lim_{\xi \rightarrow \infty} f(\xi) = c$ because of Lemma 4.5. QED.

□

Definition 4.6 (Regulated function). Let $a < b$, $f : [a, b] \rightarrow \mathbb{R}$. We call f a regulated function on $[a, b]$ if

1. $\forall x \in (a, b)$, f in x has a right-sided and a left-sided limit.
2. in $x = a$, f has a right-sided limit.
3. in $x = b$, f has a left-sided limit.

$$\mathcal{R}[a, b] = \{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is a regulated function} \}$$

Definition 4.7 (Equivalent definition). 1. $\forall x \in [a, b]$, f has a right-sided limit in x

2. $\forall x \in (a, b]$, f has a left-sided limit in x

Example 4.1. Let f be continuous in $[a, b]$. Let $\varphi \in \tau[a, b]$ be a regulated function. Then $\varphi \in \mathcal{R}[a, b]$.

Rationale:

Let $x_0 = a < x_1 < \dots < x_n = b$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$.

Let $x \in [a, b]$ be chosen arbitrarily.

Case 1 Let $x \in (x_{j-1}, x_j)$ for some $j \in \{1, \dots, n\}$

$$\implies \lim_{\xi \rightarrow x} \varphi(\xi) = c_j$$

Choose δ small enough such that $(x - \delta, x + \delta) \subseteq (x_{j-1}, x_j)$. $\forall \xi$ with $\xi \in (x - \delta, x + \delta)$ it holds that

$$|\varphi(\xi) - c_j| = 0$$

Case 2 Let $x = x_j$ for $j = 1, \dots, n - 1$.

$$\implies \lim_{\xi \rightarrow x_j^+} \varphi(\xi) = c_{j+1}$$

$$\lim_{\xi \rightarrow x_j^-} \varphi(\xi) = c_j$$

Compare with Figure 8.

Case 3 Let $x = x_0 = a \implies \lim_{\xi \rightarrow a^+} \varphi(\xi) = c_1$.

$$x = x_n = b \implies \lim_{\xi \rightarrow b^-} \varphi(\xi) = c_n$$

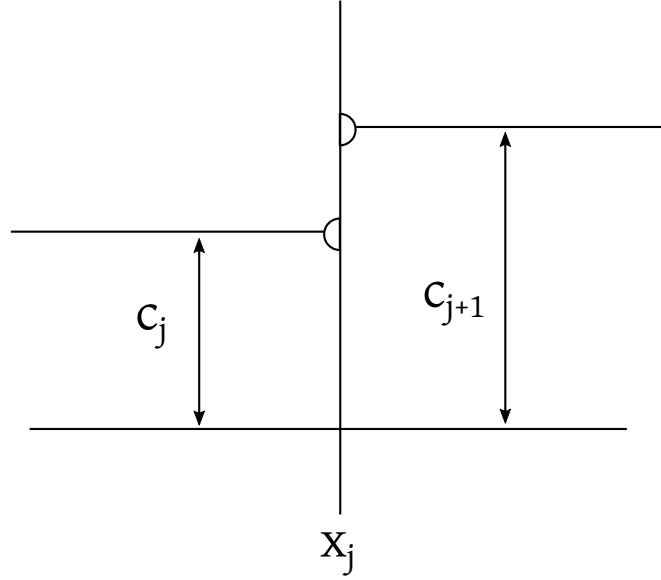


Figure 8: Regulated function

Let $f : [a, b] \rightarrow \mathbb{R}$ be monotonically increasing or monotonically decreasing. Then $f \in \mathcal{R}[a, b]$. The proof will be done in the practicals.

Definition 4.8 (Boundedness). Let $X \neq \emptyset$ be a set. $f : X \rightarrow \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say: f is bounded on X , if $f(X) \subseteq \mathbb{K}$ is a bounded set in \mathbb{K} . Hence, $\exists m \geq 0 : |f(x)| \leq m \forall x \in X$. We let,

$$\mathcal{B}(X) = \{ f : X \rightarrow \mathbb{K} \mid f \text{ is bounded} \}$$

$\mathcal{B}(X)$ has vector space structure. $f, g \in \mathcal{B}(X), \lambda \in \mathbb{K}$.

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda \cdot f)(x) = \lambda \cdot f(x)$$

$f + g \in \mathcal{B}(X)$ and $\lambda f \in \mathcal{B}(X)$. Let $|f(x)| \leq m \forall x \in X$ and $|g(x)| \leq m' \forall x \in X$. Then it holds that

$$|(f + g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq m + m'$$

Remark 4.4. It is very interesting, that X does not require any kind of algebraic structure.

We let

$$\|f\|_\infty = \sup \underbrace{\{|f(x)| : x \in X\}}_{\text{bounded in } \mathbb{R}} = \min \{m \geq 0 : |f(x)| \leq m \forall x \in X\}$$

Some work is required to show that $\|\cdot\|_\infty$ is a norm on $\mathcal{B}(X)$.

Hence, $(\mathcal{B}(X), \|\cdot\|_\infty)$ is a normed vector space. Convergence in $\mathcal{B}(X)$: It holds that $f_n \rightarrow f$ in $(\mathcal{B}(X), \|\cdot\|_\infty)$ if and only if $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies \|f_n - f\|_\infty < \varepsilon$.

$$\begin{aligned} \|f_n - f\|_\infty < \varepsilon &\iff \sup \{|f_n(x) - f(x)| : x \in X\} < \varepsilon \\ &\iff |f_n(x) - f(x)| \leq \varepsilon \forall x \in X \end{aligned}$$

Hence, $f_n \rightarrow f$ in $(\mathcal{B}(X), \|\cdot\|_\infty) \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies |f_n(x) - f(x)| \leq \varepsilon \forall x \in X$. We say “ f_n converges uniformly to f on X ”.

Theorem 4.1 (Approximation theorem for regulated function). *Let $f : [a, b] \rightarrow \mathbb{R}$. Then it holds that $f \in \mathcal{R}[a, b] \iff \forall \varepsilon > 0$ there exists some step function $\varphi \in \tau[a, b]$ such that $|\varphi(x) - f(x)| < \varepsilon \forall x \in [a, b]$ ($\|\varphi - f\|_\infty < \varepsilon$).*

Epecially $\varepsilon_n = \frac{1}{n}$ and φ_n as above. Then it holds that $\|\varphi_n - f\|_\infty < \frac{1}{n}$, hence $f = \lim_{n \rightarrow \infty} \varphi_n$ uniformly on $[a, b]$.

Proof. Direction \implies . Let $f \in \mathcal{R}[a, b]$.

Proof by contradiction. We negate our hypothesis:

$$\exists \varepsilon > 0 : \forall \varphi \in \tau[a, b] \exists x \in [a, b] : |\varphi(x) - f(x)| \geq \varepsilon \quad (1)$$

Assume (1) holds for $f \in [a, b]$. We construct nested intervals $[a_n, b_n]$ with $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$ and (1) holds on $[a_n, b_n] \forall n \in \mathbb{N}$. Hence $\forall \varphi \in \tau[a_n, b_n] \exists x \in [a_n, b_n]$ such that $|\varphi(x) - f(x)| \geq \varepsilon$. This is what we want to show.

Let $a_0 = a$ and $b_0 = b$. Then (1) holds on $[a_0, b_0]$ by assumption. $n \rightarrow n + 1$: Construction of $[a_{n+1}, b_{n+1}]$. Let $m_n = \frac{1}{2}(a_n + b_n)$. We need to prove: (1) holds either on $[a_n, m_n]$ or on $[m_n, b_n]$.

Because if the opposite of (1) holds on $[a_n, m_n]$ as well as $[m_n, b_n]$, then there exists $\varphi_n^1 \in \tau[a_n, m_n]$ with $|\varphi_n^1(x) - f(x)| < \varepsilon \forall x \in [a_n, m_n]$ and if the opposite of (1) holds on $[m_n, b_n]$:

$$\exists \varphi_n^2 \in \tau[m_n, b_n] : |\varphi_n^2(x) - f(x)| < \varepsilon \forall x \in [m_n, b_n]$$

Let

$$\varphi^n(x) = \begin{cases} \varphi_n^1(x) & \text{if } x \in [a_n, m_n] \\ \varphi_n^2(x) & \text{if } x \in (m_n, b_n] \end{cases}$$

Then φ^n is piecewise constant, hence $\varphi^n \in \tau[a_n, b_n]$ and it holds that

$$|\varphi^n(x) - f(x)| = \begin{cases} \underbrace{|\varphi_1^n(x) - f(x)|}_{< \varepsilon} & \text{for } x \in [a_n, m_n] \\ \underbrace{|\varphi_2^n(x) - f(x)|}_{< \varepsilon} & \text{for } x \in [m_n, b_n] \end{cases} < \varepsilon$$

This contradicts with (1) on $[a_n, b_n]$.

Hence: (1) holds on $[a_n, m_n]$ or on $[m_n, b_n]$.

Choose $[a_{n+1}, b_{n+1}]$ as one of the subintervals in which (1) holds. \square

Let $X \in \bigcap_{n \in \mathbb{N}} [a_n, b_n]$ (by completeness of \mathbb{R}).

1. Let $x \in (a, b)$. Let ε as above such that (1) holds on every interval $[a_n, b_n]$.

Let $c_+ = \lim_{\xi \rightarrow x^+} f(\xi)$ and $c_- = \lim_{\xi \rightarrow x^-} f(\xi)$ (possible, because $f \in \mathcal{R}[a, b]$).

Limes property: $\exists \delta > 0 : |\xi - x| < \delta$ and $\xi < x$, then $|f(\xi) - c_-| < \varepsilon$ and $|\xi - x| < \delta$ and $x < \delta$ then $|f(\xi) - c_+| < \varepsilon$.

Additionally, choose δ sufficiently small enough such that $(x - \delta, x + \delta) \subseteq [a, b]$. Let

$$\hat{\varphi}(\xi) = \begin{cases} 0 & \text{for } \xi \in [a, b] \setminus (x - \delta, x + \delta) \\ c_- & \text{for } \xi \in (x - \delta, x) \\ c_+ & \text{for } \xi \in (x, x + \delta) \\ f(x) & \text{for } \xi = x \end{cases}$$

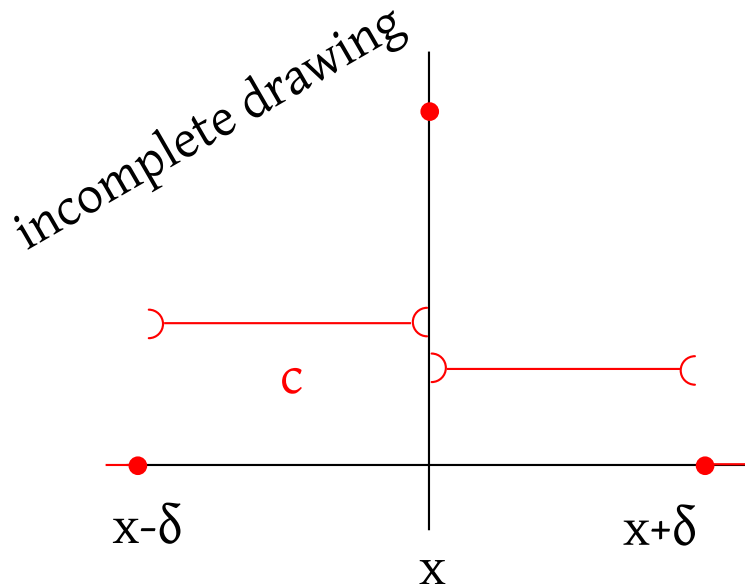
$\hat{\varphi} \in \tau[a, b]$ and it holds that

$$\forall \xi \in (x - \delta, x + \delta) : |\hat{\varphi}(\xi) - f(\xi)| = \begin{cases} \underbrace{|c_- - f(\xi)|}_{< \varepsilon} & \text{for } \xi \in (x - \delta, x) \\ \underbrace{|f(x) - f(x)|}_{=0} & \text{for } \xi = x \\ \underbrace{|c_+ - f(\xi)|}_{< \varepsilon} & \text{for } \xi \in (x, x + \delta) \end{cases} < \varepsilon$$

Now let N be sufficiently large enough such that $[a_N, b_N] \subseteq (x - \delta, x + \delta)$ (possible because $([a_n, b_n])_{n \in \mathbb{N}}$ gives nested intervals tightening on x). Then it holds on $[a_N, b_N]$ that:

$$\hat{\varphi}|_{[a_N, b_N]} \in \tau[a_N, b_N]$$

and $\forall \xi \in [a_N, b_N] \subseteq (x - \delta, x + \delta)$ it holds that $|\hat{\varphi}(\xi) - f(\xi)| < \varepsilon$. This contradicts with (1) on $[a_N, b_N]$.



We also need to cover the special cases $x = a$ and $x = b$. But this works analogously with one-sided limits.

Direction \Leftarrow : Let $f = \lim_{n \rightarrow \infty} \varphi_n$ uniform on $[a, b]$. Show that $\forall x \in [a, b]$ there exists a right-sided limit of f in x .

Let $\varepsilon > 0$ be arbitrary. $N \in \mathbb{N}$ sufficiently large such that $|f(\xi) - \varphi_N(\xi)| < \frac{\varepsilon}{2} \forall \xi \in [a, b]$. φ_N is piecewise constant. Choose $\delta > 0$ such that $\varphi_N|_{(x, x+\delta)} = c$. Now let $\xi, \eta \in (x, x + \delta)$ be chosen arbitrarily. Then it holds that

$$\begin{aligned}
 |f(\xi) - f(\eta)| &\leq \left| f(\xi) - \underbrace{c}_{=\varphi_N(\xi)} \right| + \left| \underbrace{c}_{=\varphi_N(\eta)} - f(\eta) \right| \\
 &= \left| \underbrace{f(\xi) - \varphi_N(\xi)}_{< \frac{\varepsilon}{2}} \right| + \left| \underbrace{\varphi_N(\eta) - f(\eta)}_{< \frac{\varepsilon}{2}} \right| < \varepsilon
 \end{aligned}$$

Therefore f has a right-sided limit in x by the Cauchy criterion. f has left-sided limit in every point $x \in (a, b]$ analogously.

Corollary. Every regulated function $f \in \mathcal{R}[a, b]$ is bounded. Let $\varphi \in \tau[a, b]$ with $\|f - \varphi\|_\infty < 1$. φ is bounded, hence $\exists m \in [0, \infty)$: $|\varphi(x)| \leq m \forall x \in [a, b]$. Then it holds

that $|f(x)| \leq |f(x) - \varphi(x)| + |\varphi(x)| < 1 + m \forall x \in [a, b]$, hence $f \in \mathcal{B}[a, b]$.

$$\mathcal{R}[a, b] \subseteq \mathcal{B}[a, b]$$

Corollary. Let $f \in \mathcal{R}[a, b] \iff f = \sum_{j=0}^{\infty} \psi_j$ with $\psi_j \in \tau[a, b]$ and the series converges uniformly on $[a, b]$.

Proof. Direction \Leftarrow .

Let $f = \sum_{j=0}^{\infty} \psi_j$ with uniform convergence. Let $\varphi_n = \sum_{j=0}^n \psi_j \in \tau[a, b]$ and $f = \lim_{n \rightarrow \infty} \varphi_n$ uniform on $[a, b] \implies f \in \mathcal{R}[a, b]$.
Satz 1?!

Direction \implies .

Let $f \in \mathcal{R}[a, b]$ and $f = \lim_{n \rightarrow \infty} \varphi_n$ with $\varphi_n \in \tau[a, b]$ (by Satz 1?!).

$$\begin{aligned} \psi_0 &= \varphi_0 \\ \psi_j &= \varphi_j - \varphi_{j-1} \quad \text{for } j \geq 1 \\ \sum_{j=0}^n \psi_j &= \varphi_0 + \sum_{j=1}^n (\varphi_j - \varphi_{j-1}) = \varphi_0 + \sum_{j=1}^n \varphi_j - \sum_{j=0}^{n-1} \varphi_j = \varphi_n \end{aligned}$$

converges uniformly to f . □

Integration of regulated functions

Definition 5.1 (Definition with a theorem). Let $f \in \mathcal{R}[a, b]$ and $\varphi_n \in \tau[a, b]$ with $f = \lim_{n \rightarrow \infty} \varphi_n$ is uniform on $[a, b]$. We let

$$\int_a^b f \, dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n \, dx$$

for the integral of f on $[a, b]$.

Theorem: This limit (on the right-hand side) always exists and is independent of the particular choice of the approximating sequence.

Proof. φ_n is chosen as above.

$$i_n = \int_a^b \varphi_n \, dx$$

Show: i_n is cauchy sequence in \mathbb{R} .

This lecture took place on 2018/04/12.

Let $\varepsilon > 0$ be chosen arbitrary. Choose $N \in \mathbb{N}$ such that

$$n \geq N \implies \|f - \varphi_n\|_\infty < \frac{\varepsilon}{2(b-a)}$$

For $n, m \geq N$ it holds for $x \in [a, b]$ that

$$\begin{aligned} |\varphi_n(x) - \varphi_m(x)| &\leq |\varphi_n(x) - f(x)| + |f(x) - \varphi_m(x)| \\ &\leq \| \varphi_n - f \|_\infty + \| f - \varphi_m \|_\infty < \frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{b-a} \end{aligned}$$

$|\varphi_n - \varphi_m|$ is a step function.

$$|\varphi_n - \varphi_m| \leq \frac{\varepsilon}{b-a} \cdot \underbrace{\chi_{[a,b]}}_{\in \tau[a,b]}$$

Integral for subsequence is monotonous:

$$\begin{aligned} |i_n - i_m| &= \left| \int_a^b \varphi_n dx - \int_a^b \varphi_m dx \right| = \left| \int_a^b (\varphi_n - \varphi_m) dx \right| \leq \int_a^b |\varphi_n - \varphi_m| dx \\ &\underbrace{\leq}_{\text{by monotonicity}} \int_a^b \frac{\varepsilon}{b-a} \cdot \chi_{[a,b]} dx = \frac{\varepsilon}{b-a} \underbrace{\int_a^b \chi_{[a,b]} dx}_{1 \cdot (b-a)} = \varepsilon \end{aligned}$$

So $(i_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. \mathbb{R} is complete, hence $i = \lim_{n \rightarrow \infty} i_n$ exists.

Uniqueness: (dt. mithilfe des Reissverschlussprinzips)

Let $(\varphi_n)_{n \in \mathbb{N}}, (\Phi_n)_{n \in \mathbb{N}}$ be two sequences of step functions, converging uniformly towards f .

$$\begin{aligned} i_n &= \int_a^b \varphi_n dx \quad \text{and} \quad j_n = \int_a^b \Phi_n dx \\ i &= \lim_{n \rightarrow \infty} i_n \quad \quad j = \lim_{n \rightarrow \infty} j_n \end{aligned}$$

Show that $i = j$.

Now we construct a sequence $(\mu_n)_{n \in \mathbb{N}}$ of step functions.

$$\underbrace{(\varphi_1, \Phi_1, \varphi_2, \Phi_2, \dots)}_{(\mu_n)_{n \in \mathbb{N}}}$$

μ_n is a sequence of step functions converging uniformly towards f (the proof is left as an exercise to the reader).

Because of part 1 of the proof:

$$m_n = \int_a^b \mu_n dx \text{ converges with limit } m$$

$(i_n)_{n \in \mathbb{N}}$ as well as $(j_n)_{n \in \mathbb{N}}$ are subsequences of $(m_n)_{n \in \mathbb{N}}$. Hence it holds that $i = \lim_{n \rightarrow \infty} i_n = m = \lim_{n \rightarrow \infty} j_n = j$. \square

Theorem 5.1 (Elementary properties of an integral). *Let $f, g \in \mathcal{R}[a, b]$, $\lambda, \mu \in \mathbb{R}$. Then it holds that*

Linearity

$$\lambda f + \mu g \in \mathcal{R}[a, b] \text{ and } \int_a^b (\lambda f + \mu g) dx = \lambda \int_a^b f dx + \mu \int_a^b g dx$$

Monotonicity *If $f(x) \leq g(x) \forall x \in [a, b]$ ($f \leq g$) it holds that*

$$\int_a^b f dx \leq \int_a^b g dx$$

Boundedness $|f| \in \mathcal{R}[a, b]$ and

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx$$

Proof. We prove linearity.

Let $x \in [a, b]$ and $c_+ = \lim_{\xi \rightarrow x_+} f(\xi)$ as well as $d_+ = \lim_{\xi \rightarrow x_+} g(\xi)$ ($f, g \in \mathcal{R}[a, b]$). Then it holds that

$$\lim_{\xi \rightarrow x^+} (\lambda f(\xi) + \mu g(\xi)) = \lambda \lim_{\xi \rightarrow x^+} f(\xi) + \mu \lim_{\xi \rightarrow x^+} g(\xi) = \lambda c_+ + \mu d_+$$

exists. Analogously for the left side, hence $\lambda f + \mu g \in \mathcal{R}[a, b]$.

Let $\varphi_n, \Phi_n \in \mathcal{C}[a, b]$ with $\varphi_n \rightarrow f$ and $\Phi_n \rightarrow g$ is uniform on $[a, b]$. Hence $\lambda \varphi_n + \mu \Phi_n \rightarrow \lambda f + \mu g$ is continuous on $[a, b]$.

Proof of this:

Let $\varepsilon > 0$ be arbitrary, N such that $n \geq N \implies \|\varphi_n - f\|_\infty < \frac{\varepsilon}{2(|\lambda|+1)}$ and M such that $n \geq M \implies \|\Phi_n - g\|_\infty < \frac{\varepsilon}{2(|\mu|+1)}$.

Then it holds that

$$\begin{aligned} \|\lambda \varphi_n + \mu \Phi_n - \lambda f - \mu g\|_\infty &\leq |\lambda| \|\varphi_n - f\|_\infty + |\mu| \|\Phi_n - g\|_\infty \\ &< \frac{|\lambda|}{2(|\lambda|+1)} \cdot \varepsilon + \frac{|\mu|}{2(|\mu|+1)} \cdot \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

We continue:

$$\begin{aligned}
\int_a^b (\lambda f + \mu g) dx &= \lim_{n \rightarrow \infty} \int_a^b (\lambda \varphi_n + \mu \Phi_n) dx = \lim_{n \rightarrow \infty} (\lambda \int_a^b \varphi_n dx + \mu \int_a^b \Phi_n dx) \\
&= \lambda \underbrace{\lim_{n \rightarrow \infty} \int_a^b \varphi_n dx}_{\text{exists}} + \mu \underbrace{\lim_{n \rightarrow \infty} \int_a^b \Phi_n dx}_{\text{exists}} \\
&= \lambda \int_a^b f dx + \mu \int_a^b g dx
\end{aligned}$$

We prove monotonicity.

Show: Let $h \in \mathcal{R}[a, b]$ with $h \geq 0$ in $[a, b]$. Then it holds that $\int_a^b h dx \geq 0$.

We will show that $(\tilde{\varphi}_n)_{n \in \mathbb{N}}$ exists with $\tilde{\varphi}_n \rightarrow h$ uniform on $[a, b]$ and $\tilde{\varphi}_n \geq 0$.

Therefore we prove: Let $(\varphi_n)_{n \in \mathbb{N}}$, $\varphi_n \in \tau[a, b]$ with $\varphi_n \rightarrow h$ uniform on $[a, b]$.

Define $\tilde{\varphi}_n$ such that

$$\varphi_n = \sum_{j=1}^{m_n} c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^{m_n} d_j \chi_{\{x_j\}}$$

Let

$$\tilde{\varphi}_n = \sum_{j=1}^{m_n} \underbrace{\tilde{c}_j}_{\geq 0} \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^{m_n} \underbrace{h(x_j)}_{\geq 0} \chi_{\{x_j\}}$$

and $\tilde{c}_j := \max c_j, 0 \geq 0$. So it holds that $\tilde{\varphi}_n \geq 0$.

For $x = x_l$ ($l \in \{0, \dots, m_n\}$) it holds that

$$\begin{aligned}
|\tilde{\varphi}_n(x_l) - h(x_l)| &= \left| \sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1}, x_j)}(x_l)}_{=0 \text{ bc. } x_l \notin (x_{j-1}, x_j)} + \sum_{j=0}^{m_n} h(x_j) \underbrace{\chi_{\{x_j\}}(x_l)}_{=\delta_{j,l}} - h(x_l) \right| \\
&= |h(x_l) - h(x_l)| = 0 \leq |\varphi_n(x_l) - h(x_l)|
\end{aligned}$$

For $x \in (x_{j-1}, x_j)$ it holds that

$$\begin{aligned}
|\tilde{\varphi}_n(x) - h(x)| &= \left| \sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1}, x_j)}(x)}_{\delta_{l,j}} + \sum_{j=0}^{m_n} h(x_j) \cdot \underbrace{\chi_{\{x_j\}}(x)}_{=0 \text{ bc. } x \neq x_j} - h(x) \right| \\
&= |\tilde{c}_l - h(x)| = \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ |h(x)| = h(x) & \text{if } c_l < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\leq \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ h(x) - c_l & \text{if } c_l < 0 \end{cases} \\
&= \begin{cases} |\varphi_n(x) - h(x)| & \text{if } c_l = \varphi_n(x) \geq 0 \\ |h(x) - \varphi_n(x)| & \text{if } c_l = \varphi_n(x) < 0 \end{cases} \\
&= |\varphi_n(x) - h(x)|
\end{aligned}$$

hence, $|\tilde{\varphi}_n(x) - h(x)| \leq |\varphi_n(x) - h(x)|$ for $x \in (x_{l-1}, x_l)$ as well as $x = x_l$, hence

$$\|\tilde{\varphi}_n - h\|_\infty \leq \underbrace{\|\varphi_n - h\|_\infty}_{\rightarrow 0 \text{ for } n \rightarrow \infty}$$

Hence $\|\tilde{\varphi}_n - h\|_\infty \rightarrow 0$ for $n \rightarrow \infty$, hence $\tilde{\varphi}_n$ converges uniformly to h . There exists

$$\int_a^b h \, dx = \lim_{n \rightarrow \infty} \underbrace{\int_a^b \tilde{\varphi}_n \, dx}_{\geq 0} \geq 0$$

Monotonicity: Let $f \leq g$ in $[a, b]$, hence $h = g - f \geq 0$ in $[a, b]$

$$\begin{aligned}
\Rightarrow 0 &\leq \int_a^b h \, dx = \int_a^b g \, dx - \int_a^b f \, dx \\
&\Rightarrow \int_a^b f \, dx \leq \int_a^b g \, dx
\end{aligned}$$

And finally, boundedness is left.

Consider $|f| \in \mathcal{R}[a, b]$. Proving this is left as an exercise. $f \leq |f|$ in $[a, b] \Rightarrow \int_a^b f \, dx \leq \int_a^b |f| \, dx$.

TODO

$$-f \leq |f| \text{ in } [a, b] \Rightarrow \int_a^b (-f) \, dx = - \int_a^b f \, dx \leq \int_a^b |f| \, dx \Rightarrow \left| \int_a^b f \, dx \right| \text{ TODO}$$

□

Remark 5.1. $\mathcal{R}[a, b]$ is a vector space.

1. $f, g \in \mathcal{R}[a, b] \Rightarrow \lambda f + \mu g \in \mathcal{R}[a, b]$. $\|\cdot\|_\infty$ is a norm on $\mathcal{R}[a, b]$. $(\mathcal{R}[a, b], \|\cdot\|_\infty)$ is a normed vector space. Subspace of $(\mathcal{B}[a, b], \|\cdot\|_\infty)$. We will show in the practicals that $(\mathcal{R}[a, b], \|\cdot\|_\infty)$ is complete.

Theorem 5.2 (Mean value theorem of integral calculus). *Let f be continuous on $[a, b]$ and $p \in \mathcal{R}[a, b]$ and $p \geq 0$ in $[a, b]$. Then $f \cdot p \in \mathcal{R}[a, b]$ and there exists $\xi \in [a, b]$ such that*

$$\int_a^b f \cdot p \, dx = f(\xi) \cdot \int_a^b p \, dx$$

Proof. Let $m = \min \{f(z) : z \in [a, b]\}$ (exists because f is continuous and $[a, b]$ is compact).

$$M = \max \{f(z) : z \in [a, b]\}$$

$$f([a, b]) = [m, M] \text{ (by the mean value theorem)}$$

It holds that

$$m \cdot \underbrace{p(x)}_{\geq 0} \leq f(x) \cdot p(x) \leq M \cdot p(x)$$

By monotonicity,

$$m \int_a^b p(x) \, dx \leq \int_a^b f p \, dx \leq M \int_a^b p \, dx$$

Therefore, there exists $\eta \in [m, M]$.

$$\eta \cdot \int_a^b p(x) \, dx = \int_a^b f p \, dx$$

Mean value theorem: For $\eta \in [m, M]$ there exists $\xi \in [a, b]$ such that

$$\eta = f(\xi) \text{ (f is continuous!)}$$

Hence,

$$f(\xi) \int_a^b p \, dx = \int_a^b f \cdot p \, dx$$

$f \cdot p$ is regulated function (over one-sided limits). □

Lemma 5.1. *Let $f \in \mathcal{R}[a, b]$ and $a \leq \alpha < \beta < \gamma \leq b$. Then*

$$f|_{[\alpha, \beta]} \in \mathcal{R}[\alpha, \beta], f|_{[\beta, \gamma]} \in \mathcal{R}[\beta, \gamma]$$

$$f|_{[\alpha, \gamma]} \in \mathcal{R}[\alpha, \gamma] \text{ (immediate over onesided limit)}$$

and it holds that

$$\int_{\alpha}^{\gamma} f \, dx = \int_{\alpha}^{\beta} f \, dx + \int_{\beta}^{\gamma} f \, dx$$

Compare with Figure 9.

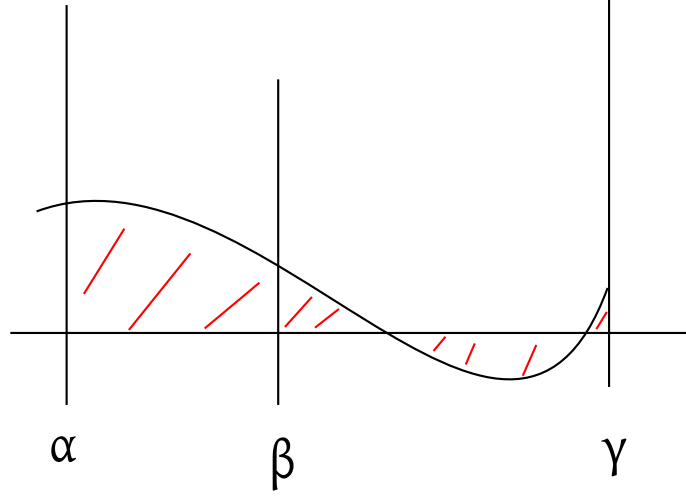


Figure 9: Positive and negative area covered by the integral

Proof. Show that this statement holds for $\varphi \in \tau[a, b]$. Without loss of generality, $\alpha = a, \gamma = b$.

$$\gamma = \sum_{j=1}^m c_j \chi_{(x_{j-1}, x_j)} + \sum_{j=0}^m \underbrace{0}_{\text{it does not matter for the integral}} \cdot \chi_{x_j}$$

Case 1 $\beta = x_l$ for some $l \in \{1, \dots, m-1\}$

$$\int_{\alpha}^{\gamma} \varphi dx = \sum_{j=1}^m c_j (x_j - x_{j-1})$$

$$\int_{\alpha}^{\beta} \varphi dx = \int_{\alpha}^{x_l} \varphi dx = \sum_{j=1}^l c_j (x_j - x_{j-1})$$

$$\int_{\beta}^{\gamma} \varphi dx = \int_{x_l}^{\gamma} \varphi dx = \sum_{j=l+1}^m c_j (x_j - x_{j-1})$$

And now,

$$\sum_{j=l+1}^m c_j (x_j - x_{j-1}) + \sum_{j=1}^l c_j (x_j - x_{j-1}) = \sum_{j=1}^m c_j (x_j - x_{j-1})$$

Case 2 $\beta \in (x_{l-1}, x_l)$ for some $l \in \{1, \dots, m\}$.

$$\begin{aligned}
\int_{\beta}^{\gamma} \varphi \, dx &= c_l(x_l - \beta) + \sum_{j=l+1}^m c_j(x_j - x_{j-1}) \\
\int_{\alpha}^{\beta} \varphi \, dx + \int_{\beta}^{\gamma} \varphi \, dx &= \sum_{j=1}^{l-1} c_j(x_j - x_{j-1}) \\
&\quad + c_l(\beta - x_{l-1}) + c_l(x_l - \beta) + \sum_{j=l+1}^m c_j(x_j - x_{j-1}) \\
&= \sum_{j=1}^m c_j(x_j - x_{j-1}) = \int_{\alpha}^{\gamma} \varphi \, dx
\end{aligned}$$

TODO verify previous lines Let $\varphi_n \in \tau[\alpha, \beta]$ with $\varphi_n \rightarrow f$ uniform on $[\alpha, \beta] \implies \varphi_n|_{[\alpha, \beta]} \rightarrow f|_{[\alpha, \beta]}$ uniform on $[\alpha, \beta]$ and also $\varphi_n|_{[\beta, \gamma]} \rightarrow f|_{[\beta, \gamma]}$ uniform on $[\beta, \gamma]$.

$$\begin{aligned}
\int_{\alpha}^{\gamma} f \, dx &= \lim_{n \rightarrow \infty} \int_{\alpha}^{\gamma} \varphi_n \, dx = \lim_{n \rightarrow \infty} \left(\int_{\alpha}^{\beta} \varphi_n \, dx + \int_{\beta}^{\gamma} \varphi_n \, dx \right) \\
&= \underbrace{\lim_{n \rightarrow \infty} \int_{\alpha}^{\beta} \varphi_n \, dx}_{\text{exists because } \varphi_n|_{[\alpha, \beta]} \rightarrow f|_{[\alpha, \beta]} \text{ uniform}} + \lim_{n \rightarrow \infty} \int_{\beta}^{\gamma} \varphi_n \, dx \\
&= \int_{\alpha}^{\beta} f \, dx + \int_{\beta}^{\gamma} f \, dx
\end{aligned}$$

□

Remark 5.2 (Notation). Let $\alpha < \beta$, $\alpha, \beta \in [a, b]$ and $f \in \mathcal{R}[a, b]$. We let

$$\int_{\beta}^{\alpha} f \, dx := - \int_{\alpha}^{\beta} f \, dx$$

By this convention, it holds that

$$\int_{\alpha}^{\alpha} f \, dx = - \int_{\alpha}^{\alpha} f \, dx \implies \int_{\alpha}^{\alpha} f \, dx = 0$$

Lemma 5.2. Let $f \in \mathcal{R}[a, b]$ and $\alpha, \beta, \gamma \in [a, b]$ (without particular order). Then it holds that

$$\int_{\alpha}^{\gamma} f \, dx = \int_{\alpha}^{\beta} f \, dx + \int_{\beta}^{\gamma} f \, dx$$

Proof. Special case: 2 points are equal

$$\begin{aligned}\alpha = \gamma &\implies \int_a^\alpha f dx = 0 \\ \int_\alpha^\beta f dx + \int_\beta^\alpha f dx &= \int_\alpha^\beta f dx - \int_\alpha^\beta f dx = 0 \\ \beta = \gamma \quad \beta = \alpha\end{aligned}$$

Case: $\alpha < \beta < \gamma$ follows immediately

And just as a representative other case: $\alpha < \gamma < \beta$

$$\begin{aligned}\int_\alpha^\beta f dx &\stackrel{\text{by Lemma 2.1}}{=} \int_\alpha^\gamma f dx + \underbrace{\int_\gamma^\beta f dx}_{-\int_\beta^\gamma f dx} \\ \int_\alpha^\beta f dx + \int_\beta^\gamma f dx &= \int_\alpha^\gamma f dx\end{aligned}$$

□

This lecture took place on 2018/04/17.

Lemma 5.3. Let $f \in \mathcal{R}[a, b]$. Then there exists an at most countable set $A \subseteq [a, b]$ such that f is continuous in every point $x \in [a, b] \setminus A$.

Proof. Let $f \in \mathcal{R}[a, b]$ and $(\varphi_n)_{n \in \mathbb{N}}$ with $\varphi_n \in \tau[a, b]$ and $\varphi \rightarrow f$ converging uniformly on $[a, b]$.

$$\begin{aligned}\varphi_n &= \sum_{j=1}^{m_n} c_j^n \chi_{(X_{j-1}^n, X_j^n)} + \sum_{j=0}^{m_n} d_j^n \chi_{\{x_j^n\}} \\ x_0^n &= a < x_1^n < \dots < x_{m_n}^n = b\end{aligned}$$

are separating points for φ_n

$$A = \{X_j^n : n \in \mathbb{N}, j \in \{0, \dots, m_n\}\}$$

A is a countable union of finite sets $A_n = \{x_0^n, x_{m_n}^n\}$. A is countable (as unions of finite sets are).

Now we show: f is continuous in every point $x \in [a, b] : x \notin A$. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that $\|\varphi_N - f\|_\infty < \frac{\varepsilon}{2}$. Because $x \in A$, there exists $j \in \{1, \dots, m_N\}$ such that $x \in (x_{j-1}^N, x_j^N)$ is open. Choose $\delta > 0$

such that $(x - \delta, x + \delta) \subset (x_{j-1}^N, x_j^N)$, hence $\forall \xi \in (x - \delta, x + \delta)$ it holds that $\varphi_N(\xi) = c_j^N$. Now consider $\xi \in (x - \delta, x + \delta)$, hence $|\xi - x| < \delta$. Then it holds that

$$\begin{aligned} |f(\xi) - f(x)| &= \left| f(\xi) - \underbrace{\varphi_N(x)}_{c_j^N = \varphi_N(\xi)} + \varphi_N(x) - f(x) \right| \\ &\leq \underbrace{|f(\xi) - \varphi_N(\xi)|}_{\leq \|f - \varphi_N\|_\infty} + \underbrace{|\varphi_N(x) - f(x)|}_{\leq \|\varphi_N - f\|_\infty} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence f is continuous in x . \square

Remark 5.3 (Notation). Let $f \in \mathcal{R}[a, b]$. For $x \in [a, b)$, there exists $f_+(x) := \lim_{\xi \rightarrow x_+} f(\xi)$. For $x \in (a, b]$, there exists $f_-(x) := \lim_{\xi \rightarrow x_-} f(\xi)$. Because of Lemma 5.3, it holds that $f_+(x) = f_-(x) = f(x)$ for all $x \in [a, b] \setminus A$ and A is at most countable.

Definition 5.2 (One-sided derivatives). Let $g : [a, b] \rightarrow \mathbb{R}$ and $x \in [a, b]$. We say g has the right-sided derivative $g'_+(x)$ if

$$\lim_{\xi \rightarrow x_+} \frac{g(\xi) - g(x)}{\xi - x} =: g'_+(x)$$

exists. Analogously we define the left-sided derivative

$$g'_-(x) = \lim_{\xi \rightarrow x_-} \frac{g(\xi) - g(x)}{\xi - x}$$

for $x \in (a, b]$. Compare with Figure 10.

Remark 5.4. If g in x has a one-sided derivative, then it holds that

$$\lim_{\xi \rightarrow x_\pm} (g(\xi) - g(x)) = 0$$

Hence g is continuous in x .

Remark 5.5. $g : [a, b] \rightarrow \mathbb{R}$ is differentiable in point $x \in (a, b)$ with derivative $g'(x)$ \iff g has a left- and right-sided derivative in x and it holds that $g'_-(x) = g'_+(x)$ ($= g'(x)$).

Theorem 5.3 (Fundamental theorem of differential/integration calculus, variation 1). Isaac Barrow (1630–1677), Isaac Newton (1642–1726), Gottfried Wilhelm von Leibniz (1646–1716).

Let $f \in \mathcal{R}[a, b]$, $\alpha \in [a, b]$ and we define

$$F(x) = \int_\alpha^x f \, d\xi$$

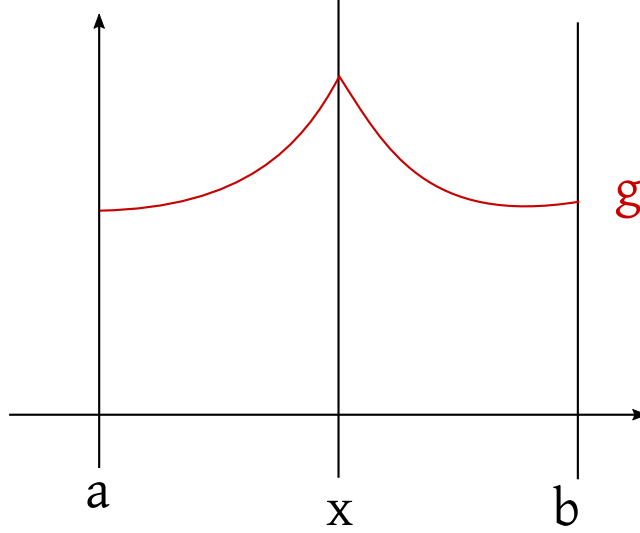


Figure 10: In this example, the left- and right-sided derivatives are not equal.
 $f'_+(x) \neq f'_-(x)$

Then F is right-sided differentiable in every point $x \in [a, b]$ and in every $x \in (a, b]$ left-sided differentiable. Furthermore it holds that

$$F'_+(x) = f_+(x) \forall x \in [a, b] \quad (2)$$

$$F'_-(x) = f_-(x) \forall x \in (a, b] \quad (3)$$

Remark 5.6.

$$\frac{d}{dx} \left(\int_a^x f \, d\xi \right) = f(x)$$

for all x such that f is continuous in x . For those x , $F'(x)$ is differentiable in x with $F'(x) = f(x)$.

Definition 5.3. Let $f \in \mathcal{R}[a, b]$ and $\varphi : [a, b] \rightarrow \mathbb{R}$ such that φ is one-sided differentiable on $[a, b]$. If $\Phi'_+(x) = f_+(x) \forall x \in [a, b]$ and $\Phi'_-(x) = f_-(x) \forall x \in (a, b]$ then we call Φ an antiderivative of regulated function f .

Proof of the Theorem 5.3. Let $x_1, x_2 \in [a, b]$ be arbitrary. Let F be defined as above. Then it holds that

$$|F(x_2) - F(x_1)| = \left| \int_a^{x_2} f \, d\xi - \int_a^{x_1} f \, d\xi \right|$$

$$\begin{aligned}
&= \left| \int_{\alpha}^{x_2} f d\xi + \int_{x_1}^{\alpha} f d\xi \right| = \left| \int_{x_1}^{x_2} f d\xi \right| \\
&\leq \int_{x_1}^{x_2} |f| d\xi \leq \int_{x_1}^{x_2} \underbrace{\|f\|_{\infty}}_{\text{const independent of } \xi} d\xi = \|f\|_{\infty} \cdot |x_2 - x_1|
\end{aligned}$$

Hence F is Lipschitz continuous with Lipschitz constant $\|f\|_{\infty}$. So F is continuous in $[a, b]$.

One-sided derivatives: Let $x \in [a, b)$ and $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\forall \xi \in [x, x + \delta)$ it holds that $|f(\xi) - f_+(x)| < \varepsilon$. For $\xi \in (x, x + \delta)$ it holds that

$$\begin{aligned}
\left| \frac{F(\xi) - F(x)}{\xi - x} - f_+(x) \right| &= \frac{1}{|\xi - x|} \left| \underbrace{\int_x^{\xi} f dy}_{F(\xi) - F(x)} - \underbrace{f_+(x)(\xi - x)}_{\int_x^{\xi} \underbrace{f_+(x)}_{\text{const.}} dy} \right| = \frac{1}{|\xi - x|} \left| \int_x^{\xi} (f - f_+(x)) dy \right| \leq \frac{1}{|\xi - x|} \int_x^{\xi} \underbrace{|f(y) - f_+(x)|}_{< \varepsilon} dy \\
&\quad y \in (x, \xi) \subseteq (x, x + \delta) \\
&< \frac{1}{\xi - x} \varepsilon \cdot \underbrace{\int_x^{\xi} 1 dy}_{|\xi - x|} = \varepsilon
\end{aligned}$$

Hence, $F'_+(x) = f_+(x)$. Analogously, $F'_-(x) = f_-(x)$ for $x \in (a, b]$. \square

Theorem 5.4 (Fundamental theorem of differential/integration calculus, variation 2). *Let $f \in \mathcal{R}[a, b]$ and ϕ is an arbitrary antiderivative of f according to Definition 5.3. For $\alpha, \beta \in [a, b]$ arbitrary, it holds that*

$$\int_{\alpha}^{\beta} f dx = \phi(\beta) - \phi(\alpha)$$

Remark 5.7. *Let f be continuous and ϕ be an antiderivative of f . Hence, $\Phi'(x) = f(x) \forall x \in [a, b]$. Then it holds that*

$$\int_{\alpha}^{\beta} \Phi' dx = \Phi(\beta) - \Phi(\alpha)$$

"Integral of a derivative of Φ gives $\Phi(\beta) - \Phi(\alpha)$ ".

Lemma 5.4. *Let $A \subseteq [a, b]$ countable. $f : [a, b] \rightarrow \mathbb{R}$ is continuous and f is differentiable in every point $x \in [a, b] \setminus A$. Furthermore let $|f'(x)| \leq L$ ($L \geq 0$) for all $x \in [a, b] \setminus A$. Then f is Lipschitz continuous on $[a, b]$ with constant L , hence*

$$|f(x_2) - f(x_1)| \leq L|x_2 - x_1| \forall x_1, x_2 \in [a, b]$$

Remark 5.8. Some people call it differentiable almost everywhere, but this expression collides with a different definition pronounced the same way from measure theory.

Proof. Let $x_1, x_2 \in [a, b]$, wlog. $x_1 < x_2$. Let $\varepsilon > 0$ be arbitrary. We define

$$F_\varepsilon(x) = |f(x) - f(x_1)| - (L + \varepsilon)(x - x_1)$$

for $x \in [x_1, b]$.

Let $\varepsilon > 0$ be arbitrary. We prove: $F_\varepsilon(x) \leq 0 \forall x \in [x_1, b]$. In particular: $F_\varepsilon(x_2) \leq 0$. Hence,

$$|f(x_2) - f(x_1)| \leq (L + \varepsilon) \underbrace{(x_2 - x_1)}_{|x_2 - x_1|}$$

We prove by contradiction: Assume there exists $\varepsilon > 0$ and $x_\varepsilon > x_1$ such that $F_\varepsilon(x_\varepsilon) > 0$.

We recognize: Let $A' = [x_1, b] \cap A$ be countable.

1. hence $F_\varepsilon(A') \subseteq \mathbb{R}$ is countable
2. $F_\varepsilon(x_1) = 0, F_\varepsilon(x_\varepsilon) > 0 \implies x_\varepsilon > x_1$
3. F_ε is continuous on $[x_1, b]$. It holds that $0 \in F_\varepsilon([x_1, x_\varepsilon])$ and because $0 = F_\varepsilon(x_1)$ and $\varepsilon \in F_\varepsilon([x_1, x_\varepsilon])$ because $\varepsilon = F_\varepsilon(x_\varepsilon)$.

By the Intermediate Value Theorem, it follows that $[0, \varepsilon] \subseteq \text{range of } F_\varepsilon$ on $[x_1, x_\varepsilon]$. By the Intermediate Value Theorem, it follows that $\underbrace{[0, \eta]}_{\text{uncountable}} \subseteq F_\varepsilon([x_1, x_\varepsilon])$.

$F_\varepsilon(A')$ is countable, hence there exists $\gamma \in (0, \eta]$ such that $\gamma = F_\varepsilon(y)$ and $\gamma \notin A'$ ($\gamma > 0$)². Hence, $y \notin A'$. So f in y is differentiable. Let $B := F_\varepsilon^{-1}(\{\gamma\}) \cap ([x_1, x_\varepsilon] \setminus A')$. Then $B \neq \emptyset$.

$B \subseteq [x_1, x_\varepsilon]$ is therefore bounded, $B \neq \emptyset$. Hence, B has a supremum. Let $x = \sup B$. Choose $(y_n)_{n \in \mathbb{N}}$ with $y_n \in B$ and $y_n \rightarrow x$ for $n \rightarrow \infty$. Because F_ε is continuous, it holds that

$$\lim_{n \rightarrow \infty} \underbrace{F_\varepsilon(y_n)}_{\gamma} = F_\varepsilon(x)$$

hence $F_\varepsilon(x) = \gamma$. This implies $x \notin A$.

Furthermore it holds for $w \in (x, x_\varepsilon]$ that $F_\varepsilon(w) > \gamma$. Because assume the opposite ($F_\varepsilon(w) \leq \gamma$ for $w > x$). Furthermore it holds that $F_\varepsilon(x_\varepsilon) = \varepsilon \geq \gamma$. Because of the Intermediate Value Theorem, $\exists y \geq w$ with $F_\varepsilon(y) = \gamma$. This contradicts with the supremum property of x .

²remember this as reference (*)

Now let $y \in (x, x_\varepsilon]$.

$$\begin{aligned}
\varphi(y) &= \frac{F_\varepsilon(y) - F_\varepsilon(x)}{y - x} \\
&\stackrel{\substack{\text{definition of} \\ F_\varepsilon}}{=} \frac{|f(y) - f(x_1)| - |f(x) - f(x_1)|}{y - x} - \frac{(L + \varepsilon)(y - x_1 - x + x_1)}{y - x} \\
&\stackrel{\substack{\leq \\ \text{inversed triangle ineq.}}}{\leq} \frac{f(y) - f(x)}{y - x} - (L + \varepsilon)
\end{aligned}$$

Because $F_\varepsilon(y) > \gamma = F_\varepsilon(x)$ it holds that $\varphi(y) > 0$ for $y > x$. So,

$$\frac{|f(y) - f(x)|}{y - x} \geq L + \varepsilon$$

$$|f'(x)| = \lim_{y \rightarrow x_+} \left| \frac{f(y) - f(x)}{y - x} \right| \geq L + \varepsilon$$

This contradicts with the boundedness of the derivative by L and f is in $x \notin A$ differentiable.

So, equations 2 do not hold. Therefore $\forall x_1, x_2$ with $x_1 < x_2$ in $[a, b]$ and $\forall \varepsilon > 0$,

$$\begin{aligned}
|f(x_2) - f(x_1)| &\leq (L + \varepsilon)|x_2 - x_1| \\
\implies |f(x_2) - f(x_1)| &\leq L|x_2 - x_1|
\end{aligned}$$

□

Corollary (Corollary to Lemma 5.4). *Let $f, g : [a, b] \rightarrow \mathbb{R}$ differentiable for all points $x \in [a, b] \setminus A$ and A is countable. Furthermore let $f'(x) = g'(x) \forall x \notin A$. Then there exists $K \in \mathbb{R}$ such that $f(x) = g(x) + K \forall x \in [a, b]$.*

Proof. Let $h = f - g$. Then it holds that

$$h'(x) = f'(x) - g'(x) = 0 \forall x \in [a, b] \setminus A$$

By Lemma 5.4 with $L = 0$, it follows that

$$\begin{aligned}
|h(x_1) - h(x_2)| &\leq 0 \cdot |x_1 - x_2| = 0 \\
\implies h(x_1) &= h(x_2) \forall x_1, x_2 \in [a, b]
\end{aligned}$$

Hence, $h(x) = K \in \mathbb{R}$.

$$\implies f(x) = g(x) + h(x) = g(x) + K$$

□

This lecture took place on 2018/04/19.

By reference (*), $\gamma \in [0, \eta]$ (uncountable) and $\gamma \notin f(A)$ (countable).

$$\implies \forall u \in [x_1, b) \text{ with } F_\varepsilon(u) = \gamma$$

it holds that $u \notin A$, hence f is differentiable in u .

Proof of Theorem 5.4. Let $f \in \mathcal{R}[a, b]$, ϕ is an antiderivative of f , hence $\phi'_+ = f_+$, $\phi'_- = f_-$. Let $\alpha \in [a, b]$ be arbitrary. By the Theorem variant 1, $F(x) = \int_\alpha^x f d\xi$ is also an antiderivative of f . By Lemma ??, $\exists K \in \mathbb{R} : F(x) = \int_\alpha^x f d\xi = \phi(x) + K$. Determine K : Let $x = \alpha \implies F(\alpha) = \int_\alpha^\alpha f dx = 0 = \phi(\alpha) - K$ hence $K = \phi(\alpha)$. Hence,

$$\int_\alpha^x f d\xi = \phi(x) - \phi(\alpha)$$

Let $x = \beta$. □

Remark 5.9 (Remark for the previous corollary). F, ϕ are differentiable on all points x for which f is continuous (all of them except for countable many). For those x , it holds that $F'(x) = \phi'(x) = f(x)$.

Remark 5.10 (Notation). Let $f \in \mathcal{R}[a, b]$. Then

$$\int f dx$$

- is some particular antiderivative of f (usually some arbitrary chosen)
- the set of all antiderivatives of f

$$\int f dx = \{F : F \text{ is antiderivative of } f\}$$

If F_0 is some fixed antiderivative, then

$$\int f dx = \{F_0 + K : K \in \mathbb{R}\}$$

Then $\int f dx$ is the so-called indefinite integral of f . Notation:

$$\int x^k dx = \frac{x^{k+1}}{k+1} + c \quad (k \neq -1)$$

f	F	remark
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + c$	$\alpha \in \mathbb{R} \setminus \{-1\}$; restrict x such that x^α and $x^{\alpha+1}$ are defined
x^{-1}	$\ln x + c \ (x > 0)$	
$\left(\frac{1}{-x}\right) \cdot (-1) = x^{-1}$	$\ln -x + c \ (x < 0)$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sinh x$	$\cosh x$	
$\cosh x$	$\sinh x$	
$\frac{1}{1+x^2}$	$\arctan x$	
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$ x < 1$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$	

Table 1: Table of antiderivatives

Integration methods

In this chapter, we discuss how to determine the antiderivative of a function. Usually they are composites of basic functions. Some of these are given in Table 1.

Remark 5.11. Let $F, G : [a, b] \rightarrow \mathbb{R}$ in $x \in [a, b)$ right-sided differentiable. Then also $F \cdot G$ in x is right-sided differentiable and it holds that

$$(F \cdot G)'_+(x) = F'_+(x) \cdot G(x) + F(x) \cdot G'_+(x)$$

hence the product law holds.

Analogously, the same holds for the left-sided derivative.

Look up the proof in the course Analysis 1.

Partial integration

Definition 5.4 (Partial integration). Let f, g be given. Let F, G be its antiderivatives respectively. Then $F \cdot G$ is an antiderivative of $F \cdot g + f \cdot G$.

This is immediate, because

$$(F \cdot G)'_+ = F'_+ \cdot G + F \cdot G'_+ = f_+ \cdot G + F \cdot g_+ = f_+ G_+ + F_+ \cdot g_+$$

Hence, it holds that

$$\int_a^b (Fg + fG) dx = \underbrace{F(b) \cdot G(b) - F(a)G(a)}_{=: F \cdot G|_a^b}$$

Usually, this is rewritten as

$$\int_a^b F \cdot g \, dx = F \cdot G|_a^b - \int_a^b fG \, dx$$

If $F = u$ is continuously differentiable and $G = v$ as well, then $f = u'$ and $g = v'$ and the law has the structure

$$\int_a^b uv' \, dx = u \cdot v|_a^b - \int_a^b u'v \, dx$$

Example 5.1. Let $a \neq -1$ and $x > 0$.

$$\begin{aligned} \int \underbrace{x^a}_{v'} \cdot \underbrace{\ln x}_u \, dx &= \underbrace{\left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^a & v = \frac{x^{\alpha+1}}{\alpha+1} \end{array} \right|}_{\text{scribble notes}} \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^{\alpha+1}}{\alpha+1} \, dx \\ &= \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{\alpha+1} \int x^{\alpha} \, dx = \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{(\alpha+1)^2} x^{\alpha+1} \end{aligned}$$

Example 5.2. Let $k \in \{2, 3, 4, \dots\}$.

$$\begin{aligned} \int \cos^k(x) \, dx &= \left| \begin{array}{ll} u = \cos^{k-1}(x) & u' = (k-1) \cdot \cos^{k-2}(x) \cdot (-\sin x) \\ v' = \cos x & v = \sin x \end{array} \right| \\ &\quad \cos^{k-1}(x) \sin x + (k-1) \int \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{(1-\cos^2 x)} \, dx \\ &= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx - (k-1) \int \cos^k(x) \, dx \end{aligned}$$

Then we can use the following identity:

$$k \int \cos^k(x) \, dx = \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx$$

This gives a recursive formula:

$$\int \cos^k(x) \, dx = \frac{1}{k} \cos^{k-1}(x) \cdot \frac{k-1}{k} \sin(x) + (k-1) \int \cos^{k-2}(x) \, dx$$

Analogously,

$$\int \sin^k(x) \, dx = -\frac{1}{k} \sin^{k-1}(x) \cdot \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x) \, dx$$

Let $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x) dx$. Then the following formula holds:

$$\begin{aligned} c_{2n} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2} \\ c_{2n+1} &= \prod_{k=1}^n \frac{2k}{2k+1} \end{aligned}$$

Proof by induction. Let $n = 1$.

$$\begin{aligned} c_2 &= \int_0^{\frac{\pi}{2}} \cos^2 x dx = \frac{1}{2} \cos x \sin x \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 dx = 0 - 0 + \frac{\pi}{4} \\ &= \underbrace{\prod_{k=1}^1 \frac{2k-1}{2k}}_{\frac{1}{2}} \cdot \frac{\pi}{2} \end{aligned}$$

$$c_1 = \int_0^{\frac{\pi}{2}} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

$$\underbrace{\prod_{k=1}^0 \frac{2k}{2k+1}}_{\text{empty product}} = 1$$

We make the induction step $n \rightarrow n+1$:

$$\begin{aligned} c_{2(n+1)} &= \frac{1}{2n+2} \cdot \underbrace{\cos^{2n+1}(x)}_{=0 \text{ for } x=\frac{\pi}{2}} \cdot \underbrace{\sin(x)}_{=0 \text{ for } x=0} \Big|_0^{\frac{\pi}{2}} + \frac{2n+1}{2n+2} \int_0^{\frac{\pi}{2}} \cos^{2n}(x) dx \\ &= \frac{2n+1}{2n+2} \prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2} = \prod_{k=1}^{n+1} \frac{2k-1}{2k} \cdot \frac{\pi}{2} \end{aligned}$$

$c_{2(n+1)+1}$ analogously. □

Theorem 5.5 (Wallis product). *John Wallis (1616–1703), result from 1655*

Let $w_n = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdots$. Then it holds that $\lim_{n \rightarrow \infty} w_n = \frac{\pi}{2}$.

Proof.

$$\frac{\pi}{2} \cdot \frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2} \cdot \prod_{k=1}^n \frac{\frac{2k}{2k+1}}{\prod_{k=1}^n \frac{2k-1}{2k} \cdot \frac{\pi}{2}} = \prod_{k=1}^n \frac{(2k)^2}{(2k-1)(2k+1)} = w_n$$

It remains to show that $\lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = 1$ in $[0, \frac{\pi}{2}]$ it holds that $0 \leq \cos x \leq 1$.

$$\implies \cos^{2n+2}(x) \leq \cos^{2n+1}(x) \leq \cos^{2n}(x)$$

So, $c_{2n+2} \leq c_{2n+1} \leq c_{2n}$ for $n \geq 1$.

$$\begin{aligned} 1 &\geq \frac{c_{2n+1}}{c_{2n}} \\ \implies 1 &\geq \frac{c_{2n+1}}{c_{2n}} \geq \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1} \frac{2k-1}{2k} \frac{\pi}{2}}{\prod_{k=1}^n \frac{2k-1}{2k} \frac{\pi}{2}} \\ &= \frac{2n+2-1}{2n+2} \rightarrow 1 \text{ for } n \rightarrow \infty \end{aligned}$$

Because of the sandwich lemma for convergent sequences, the intermediate expression must also converge to 1, hence

$$\lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = 1 \quad \wedge \quad \frac{\pi}{2} \cdot \lim_{n \rightarrow \infty} \frac{c_{2n+1}}{c_{2n}} = \underbrace{\lim_{n \rightarrow \infty} w_n}_{=1}$$

□

Integration by substitution

Definition 5.5 (Integration by substitution). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Let $t : [\alpha, \beta] \rightarrow [a, b]$ be continuously differentiable. Let F be an antiderivative of f (F is therefore continuously differentiable). Then $F \circ t : [\alpha, \beta] \rightarrow \mathbb{R}$ is also continuously differentiable and the chain rule holds:*

$$(F \circ t)' = (F' \circ t) \cdot t' = (f \circ t) \cdot t'$$

Hence $F \circ t$ is an antiderivative of $(f \circ t) \cdot t'$. We apply it to integration:

$$\int_{\alpha}^{\beta} (f \circ t)(u) \cdot t'(u) du = (F \circ t)(\beta) - (F \circ t)(\alpha) = F(t(\beta)) - F(t(\alpha)) = \int_{t(\alpha)}^{t(\beta)} f(x) dx$$

Then we get the substitution integration method:

$$\int_{t(\alpha)}^{t(\beta)} f(x) dx = \int_{\alpha}^{\beta} f(t(u)) \cdot t'(u) du$$

Remark 5.12 (Mnemonic). Consider the left-hand side and right-hand side simultaneously. Let $x = t(u)$ (expressions inside parentheses). Then $dx = t'(u) \cdot du$ (expressions on the right). Let $u = \alpha \implies x = t(\alpha)$ and $u = \beta \implies x = t(\beta)$ (interval boundaries).

Example 5.3.

$$\int_0^1 2x \sqrt{1-x^2} dx$$

Usually we have some expression, we want to substitute with u .

$$1 - x^2 = u \quad x = \sqrt{1-u} = t(u)$$

$$x = 0 = t(1) \quad x = 1 = t(0)$$

$$dx = \frac{1}{2} \cdot \frac{1}{\sqrt{1-u}} \cdot (-1) du$$

$$\int_0^1 2x \sqrt{1-x^2} dx = \int_1^0 2 \cdot \sqrt{1-u} \cdot u \cdot \frac{1}{2}(-1) \frac{1}{\sqrt{1-u}} du = \int_0^1 \sqrt{u} du = \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_0^1 = \frac{2}{3}$$

$$\int_0^1 2x \sqrt{\underbrace{1-x^2}_u} dx = \left| \begin{array}{ll} u = 1 - x^2 & \\ x = 0 & \Leftrightarrow u = 1 \\ x = 1 & \Leftrightarrow u = 0 \\ 1 \cdot du & = -2x dx \end{array} \right| = - \int_1^0 \sqrt{u} du = \int_0^1 \sqrt{u} du$$

In general: we set $h(u) = g(x)$, then it holds that $h'(u) du = g'(x) dx$.

Theorem 5.6. Let $f, \tilde{f} \in \mathcal{R}[a, b]$ and $A \subseteq [a, b]$ countable. Furthermore $f(x) = \tilde{f}(x) \forall x \in [a, b] \setminus A$. Then it holds that

$$\int_a^b |f - \tilde{f}| dx = 0$$

Then it follows especially that

$$\int_a^b f dx = \int_a^b \tilde{f} dx$$

This lecture took place on 2018/04/24.

Proof. Show: $r \in \mathcal{R}[a, b], r \geq 0$. $\int_a^b r dx = 0$ and $r(x) = 0$ for $x \in [a, b] \setminus A$. Then it holds that $\int_a^b r dx = 0$. Let r be as above. First, we show: $r_+(x) = \lim_{\xi \rightarrow x+} r(\xi) = 0 \forall x \in [a, b)$ and also $r_-(x) = 0 \forall x \in (a, b]$.

Proof of that: Let $x \in [a, b)$ and $y = r_+(x)$ (exists because $r \in \mathcal{R}[a, b]$). Choose $\delta_n = \frac{1}{n}$. $(x, x + \frac{1}{n}) \cap [a, b)$ is an open interval with uncountable many points, so

there is certainly one point in A . So there exists $\xi_n \in ((x, x + \frac{1}{n}) \cap [a, b]) \setminus A$ and $|\xi_n - x| < \delta_n = \frac{1}{n}$. Hence, $\lim_{n \rightarrow \infty} \xi_n = x$ and $r(\xi_n) = 0$. Therefore, $\lim_{n \rightarrow \infty} r(\xi_n) = 0$ where $r(\xi_n) = y = r_+(x)$.

Analogously, $r_-(x) = 0$ on $(a, b]$.

Let $\varepsilon > 0$ be arbitrary. We let $A_\varepsilon = \{w \in [a, b] \mid r(w) > \varepsilon\}$. We show: A_ε is finite.

Assume A_ε would have infinitely many points. Choose a sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in A_\varepsilon$ and $w_n \neq w_m$ for $n \neq m$ (works because A_ε is infinite). $(w_n)_{n \in \mathbb{N}}$ is bounded, hence there exists a convergent subsequence $(w_{n_k})_{k \in \mathbb{N}}$ with $x = \lim_{k \rightarrow \infty} w_{n_k} \in [a, b]$ and $w_{n_k} \in [a, b]$.

Either (w_{n_k}) contains infinitely many sequence element $w_{n_k} < x$ (variant (a)) or infinitely many $w_{n_k} > x$ (variant (b)). Let variant b hold without loss of generality.

Combine all $w_{n_k} > x$ to one subsequence $(w_{n_{k_l}})_{l \in \mathbb{N}}$. This gives $\lim_{l \rightarrow \infty} w_{n_{k_l}} = x$ and $w_{n_{k_l}} > x$, thus $\lim_{l \rightarrow \infty} \underbrace{r(w_{n_{k_l}})}_{\geq \varepsilon \text{ because } w_{n_{k_l}} \in A_\varepsilon} = r_+(x) = 0$. This gives a contradiction.

A_ε must be finite.

Consider

$$A_{\frac{1}{n}} = \{w_1^n, \dots, w_{m_n}^n\}$$

finite. Let $\varphi_n = \sum_{k=1}^{m_n} r(w_k^n) \cdot \chi_{\{w_k^n\}} \in \tau[a, b]$.

For $x = w_k^n \in A_{\frac{1}{n}}$ it holds that

$$\varphi_n(w_k^n) = \sum_{k=1}^{m_n} r(w_k^n) \cdot \underbrace{\chi_{\{w_k^n\}}(w_j^n)}_{\delta_{jk}} = r(w_j^n)$$

so $|\varphi_n(x) - r(x)| = 0 \forall x \in A_{\frac{1}{n}}$. Let $x \in [a, b] \setminus A_{\frac{1}{n}}$. Then it holds $0 \leq r(x) < \frac{1}{n}$ and for $x \notin A_{\frac{1}{n}}$ it holds that $\varphi(x) = 0$. Therefore,

$$|r(x) - \varphi(x)| = r(x) < \frac{1}{n}$$

hence $\|r - \varphi_n\|_\infty < \frac{1}{n}$. This means that $\varphi_n \rightarrow r$ uniformly on $[a, b]$. Therefore

$$\lim_{n \rightarrow \infty} \underbrace{\int_a^b \varphi_n dx}_{=0} = \int_a^b r dx = 0$$

Now we want to finish the proof of our theorem: Let $r(x) = |f(x) - \tilde{f}(x)| \geq 0$ and $r(x) = 0$ for $x \notin A$. So, $\int_a^b |f - \tilde{f}| dx = 0$ (first part proven).

$$\left| \int_a^b f dx - \int_a^b \tilde{f} dx \right| = \left| \int_a^b (f - \tilde{f}) dx \right| \leq \int_a^b |f - \tilde{f}| dx = 0$$

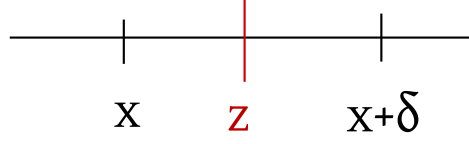


Figure 11: x and z

$$\Rightarrow \int_a^b f \, dx = \int_a^b \tilde{f} \, dx$$

Second part proven. □

Lemma 5.5. Let $f \in \mathcal{R}[a, b]$. Then it holds that $f_+ \in \mathcal{R}[a, b]$ and also $f_- \in \mathcal{R}[a, b]$.

Proof. Only for f_+ : First, we show: Let $x \in [a, b]$.

$$f_+(x) = \lim_{\xi \rightarrow x_+} f(\xi) = \lim_{\xi \rightarrow x_+} f_+(\xi)$$

(the plus is important on the right-hand side!).

Proof of this: Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\forall \xi \in (x, x + \delta)$: $|f(\xi) - f_+(x)| < \frac{\varepsilon}{2}$. Now let $z \in (x, x + \delta)$ be arbitrary chosen. For z there exists $\xi \in (z, x + \delta)$.

ξ sufficiently close enough to z such that $|f(\xi) - f_+(z)| \leq \frac{\varepsilon}{2}$ because $f_+(z)$ exists.

$$|f_+(z) - f_+(x)| \leq |f_+(z) - f(\xi)| + |f(\xi) - f_+(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

TODO some content missing here

It remains to show: f_+ has left-sided limits. Let $x \in (a, b]$ be arbitrary and $f_-(x) = \lim_{\xi \rightarrow x_-} f(\xi)$. We show: $f_-(x) = \lim_{\xi \rightarrow x_-} f_+(\xi)$ (again: the plus is important).

Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\forall z \in (x - \delta, x)$ it holds that $|f(z) - f_-(x)| < \frac{\varepsilon}{2}$.

Now let $\xi \in (x - \delta, x)$ (compare with Figure 12) and choose $x > z > \xi$ with the property that $|f(z) - f_+(\xi)| < \frac{\varepsilon}{2}$ (possible because f in ξ has a right-sided limit):

$$|f_+(\xi) - f_-(x)| \leq \underbrace{|f_+(\xi) - f(z)|}_{< \frac{\varepsilon}{2}} + \underbrace{|f(z) - f_-(x)|}_{< \frac{\varepsilon}{2}}$$

because of the choice of δ and $z \in (\xi, x) \subseteq (x - \delta, x)$.

Hence, $\lim_{\xi \rightarrow x_-} f_+(\xi) = f_-(x)$. Analogously for f_- □

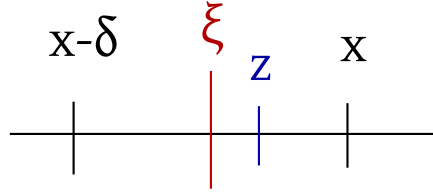


Figure 12: ξ and z

Remark 5.13.

$$\lim_{\xi \rightarrow x_+} f_+(\xi) = f_+(x)$$

$$\lim_{\xi \rightarrow x_-} f_-(\xi) = f_-(x)$$

from the proof. So f_+ is right-sided continuous and f_- is left-sided continuous.

Lemma 5.6. Let $f \in \mathcal{R}[a, b]$. Then it holds that

$$\int_a^b f \, dx = \int_a^b f_+ \, dx = \int_a^b f_- \, dx$$

Proof. For f_+ :

$$f, f_+ \in \mathcal{R}[a, b]$$

$\forall x \in [a, b]$ with f is continuous in x it holds that

$$f(x) = \lim_{\xi \rightarrow x} f(\xi) = \lim_{\xi \rightarrow x_+} f(\xi) = f_+(x)$$

f has at most countable many discontinuity points. By Satz 5.6,

$$\int_a^b |f - f_+| \, dx = 0 \quad \text{or equivalently} \quad \int_a^b f \, dx = \int_a^b f_+ \, dx$$

□

Improper integrals

Let I be an interval in \mathbb{R} with marginal points a and b with $-\infty \leq a < b \leq +\infty$. Let f be a regulated function on I . We define

$$1. \text{ If } I = [a, b), \int_a^b f \, dx = \lim_{\beta \rightarrow b_-} \int_a^\beta f \, dx$$

2. If $I = (a, b]$, $\int_a^b f dx = \lim_{\alpha \rightarrow a+} \int_{\alpha}^b f dx$
3. If $I = (a, b)$, $\int_a^b f dx = \lim_{\alpha \rightarrow a+} \int_{\alpha}^c f dx + \lim_{\beta \rightarrow b-} \int_c^{\beta} f dx$

for an arbitrarily chosen $c \in (a, b)$ under the constraint that the corresponding limits in \mathbb{R} exist.

Standard examples will follow:

Example 5.4. Let $s > 1$.

$$\begin{aligned} \int_1^{\infty} x^{-s} dx &= \lim_{\beta \rightarrow \infty} \int_1^{\beta} x^{-s} dx = \lim_{\beta \rightarrow \infty} \left(\frac{1}{-s+1} x^{-s+1} \right) \Big|_1^{\beta} \\ &= \frac{1}{1-s} \cdot \underbrace{\lim_{\beta \rightarrow \infty} \frac{1}{s-1}}_{\substack{\beta > 0 \\ =0}} - \frac{1}{1-s} \cdot 1 = \frac{1}{s-1} \end{aligned}$$

TODO drawing

Example 5.5. Let $s < 1$.

$$\begin{aligned} \int_0^1 x^{-s} dx &= \lim_{\alpha \rightarrow 0+} \int_{\alpha}^1 x^{-s} dx = \lim_{\alpha \rightarrow 0+} \frac{1}{-s+1} x^{-s+1} \Big|_{\alpha}^1 \\ &= \frac{1}{1-s} - \frac{1}{1-s} \cdot \underbrace{\lim_{\alpha \rightarrow 0} \alpha^{\overbrace{1-s}^{>0}}}_{=0} = \frac{1}{1-s} \end{aligned}$$

TODO drawing

For $s = 1$, neither $\int_0^1 \frac{1}{x} dx$ nor $\int_1^{\infty} \frac{1}{x} dx$ exists.

Example 5.6. For $c > 0$,

$$\int_0^{\infty} e^{-cx} dx = \lim_{\beta \rightarrow \infty} \int_0^{\beta} e^{-cx} dx = \lim_{\beta \rightarrow \infty} \left(-\frac{1}{c} \right) \cdot e^{-cx} \Big|_0^{\beta} = \frac{1}{c} \cdot \underbrace{\lim_{\beta \rightarrow \infty} e^{-c\beta}}_{=0} + \frac{1}{c} = \frac{1}{c}$$

Theorem 5.7 (Direct comparison test for improper integrals). In German, “Majorantenkriterium für uneigentliche Integrale”.

Let f, g be regulated functions on I and it holds that

$$|f(x)| \leq g(x) \forall x \in I$$

Assume $\int_a^b g \, dx$ exists as improper integral. Then also the following improper integrals exist:

$$\int_a^b |f| \, dx \text{ and } \int_a^b f \, dx$$

In German, g is called Majorante of f (there is no equivalent terminology in English).

Proof. Without loss of generality, let $I = [a, b)$. Let $G(\beta) = \int_a^\beta g \, dx$. We know that $\lim_{\beta \rightarrow b^-} G(\beta)$ exists. By Lemma 4.6 (Cauchy criterion for existence of limits): Let $\varepsilon > 0$ be arbitrary, then there exists a right-sided neighborhood U of b ($U = (b - \delta, b)$ if $b < \infty$ and $U = (M, \infty)$ if $b = \infty$) with $u, v \in U$, then it holds that $|G(v) - G(u)| < \varepsilon$.

$$|G(v) - G(u)| = \left| \int_a^v g \, dx - \int_a^u g \, dx \right| = \left| \int_u^v g \, dx \right| = \left| \int_u^v |g| \, dx \right|$$

Let $F(\beta) = \int_a^\beta |f| \, dx$. Analogously as for G , it holds that $F(v) - F(u) = \int_u^v |f| \, dx$. Let $u, v \in U$. Then it holds that

$$|F(v) - F(u)| = \left| \int_u^v |f| \, dx \right| \leq \left| \int_u^v g \, dx \right| = |G(v) - G(u)| < \varepsilon$$

hence by the Cauchy criterion for F : $\lim_{\beta \rightarrow b^-} F(\beta)$ exists, so there exists $\int_a^b |f| \, dx$ as improper integral. The same applies for the existence of $\int_a^b f \, dx$. \square

Example 5.7. The cardinal sine function is defined as

$$\text{sinc}(x) = \frac{\sin x}{x}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad \text{sinc}(0) = 1$$

So $\text{sinc}(x)$ is continuous on \mathbb{R} .

$$\int_0^\infty \frac{\sin x}{x} \, dx = \underbrace{\int_0^1 \frac{\sin x}{x} \, dx + \int_1^\infty \frac{\sin x}{x} \, dx}_{\substack{\text{continuous} \\ \text{exists}}}$$

How about $\int_1^\infty \frac{\sin(x)}{x} \, dx$?

$$\begin{aligned} \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\sin x}{x} \, dx &= \left| \begin{array}{cc} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{array} \right| = \lim_{\beta \rightarrow \infty} \left[-\frac{1}{x} \cos x \right]_1^\beta - \int_1^\beta \frac{\cos x}{x^2} \, dx \\ &= \cos(1) - \lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\cos(x)}{x^2} \, dx \end{aligned}$$

$$\left| \frac{\cos(x)}{x^2} \right| \leq \frac{1}{x^2} \text{ on } [1, \beta]$$

and $\int_1^\infty \frac{1}{x^2} dx$ exists. So $g(x) = \frac{1}{x^2}$ is a majorant of $\frac{\cos(x)}{x^2}$ and by Theorem 5.7, $\lim_{\beta \rightarrow \infty} \int_1^\beta \frac{\cos(x)}{x^2} dx$ exists.

Attention! $\int_0^\infty \left| \frac{\sin(x)}{x} \right| dx$ does not exist. Is not Lebesgue integrable.

Definition 5.6. Let $x > 0$. We call Γ Euler's Gamma function.

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt$$

Remark 5.14. The improper integral in the definition of the Γ -function exists for all $x > 0$.

This lecture took place on 2018/04/26.

TODO I missed the first 15 minutes

Proof of this:

Proof.

$$\lim_{t \rightarrow \infty} \underbrace{t^{x-1}}_{\text{polynomially in } t} \cdot \underbrace{e^{-t}}_{\text{exponentially } \rightarrow 0} = 0$$

Also there exists $L > 1$, such that $\forall x > L$ it holds that $t^{x-1} e^{-t/2} < 1$ on $[1, L]$ (which is a compact interval) continuous. So there exists $M > 0$ such that $t^{x-1} e^{-t/2} \leq M \forall t \in [1, L]$. Let $c = \max\{M, 1\}$. Therefore it holds on $[1, L]$ and also on (L, ∞) .

$$t^{x-1} e^{-t/2} \leq c$$

Multiply with $e^{-t/2} > 0$, then it holds that $t^{x-1} \cdot e^{-t} \leq c e^{-t/2} \forall t \in [1, \infty)$.

$$c \int_1^\infty e^{-t/2} dt$$

exists. By the direct comparison test, we get $\int_1^\infty t^{x-1} e^{-t} dt$ exists. \square

Lemma 5.7. For all $x > 0$ it holds that

$$\Gamma(x+1) = x \cdot \Gamma(x) \quad (\text{functional equation of the } \Gamma\text{-function})$$

Especially with $\Gamma(1) = 1$ it holds that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$.

Proof.

$$\Gamma(x+1) = \int_0^\infty t^{x+1-1} e^{-t} dt = \int_0^\infty t^x e^{-t} dt$$

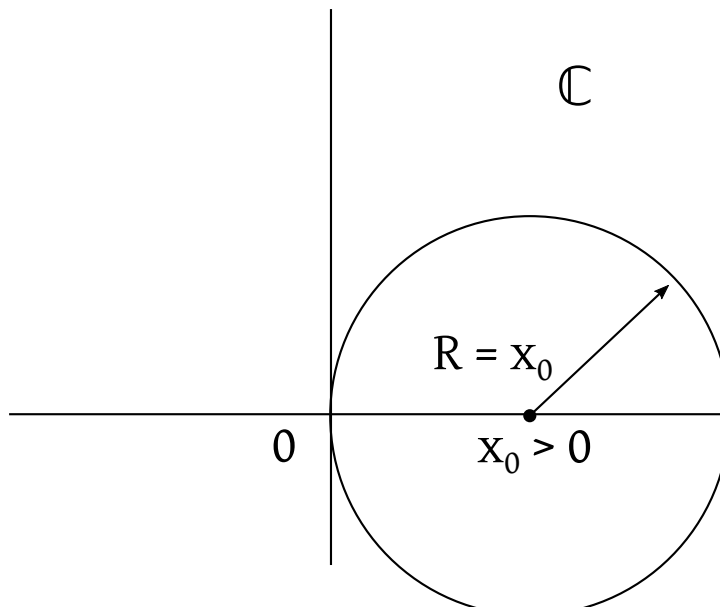


Figure 13: Γ on \mathbb{C}

$$\begin{aligned}
 &= \left| \begin{array}{ll} u = t^x & u' = x \cdot t^{x-1} \\ v' = e^{-t} & v = -e^{-t} \end{array} \right| \\
 &= \underbrace{-t^x \cdot e^{-t} \Big|_0^\infty}_{\substack{=0 \text{ on the upper bound} \\ =0 \text{ on the lower bound}}} + \int_0^\infty x \cdot t^{x-1} \cdot e^{-t} dt = x \int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)
 \end{aligned}$$

$$\Gamma(1) = \int_0^\infty \underbrace{t^{1-1}}_{=1} \cdot e^{-t} dt = -e^{-t} \Big|_0^\infty = 1$$

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1)\Gamma(n-1) = n \cdot (n-1) \cdot \dots \cdot 1 \cdot \underbrace{\Gamma(1)}_{=1} = n!$$

□

Remark 5.15. There exists a power series $\Gamma(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$. $\Gamma(z)$ is also defined for $z \in \mathbb{C}$ with $\Re z > 0$. Compare with Figure 13.

Young's inequality

Some important inequalities in integration theory follow.

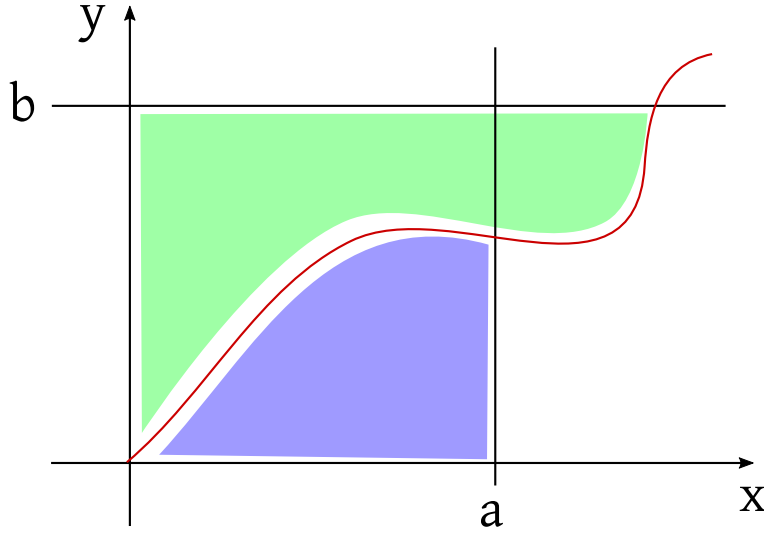


Figure 14: Young's inequality visualized. The blue area denotes $\int_0^a f(x) dx$ and $\int_0^b f^{-1}(y) dy$ is the green area.

Theorem 5.8 (Young's inequality). *Let $f : [0, \infty) \rightarrow [0, \infty)$ be continuous differentiable, strictly monotonically increasing with $f(0) = 0$ and f is unbounded. Then $f : [0, \infty) \rightarrow [0, \infty)$ bijective and $f^{-1} : [0, \infty) \rightarrow [0, \infty)$ is strictly monotonically increasing and continuous. Let $a, b \geq 0$ be given. Then it holds that*

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy$$

Equality is given if and only if, $b = f(a)$ or $a = f^{-1}(b)$. Compare with Figure 14.

Proof. Let $f : [0, \infty) \rightarrow [0, \infty)$ be as above. Let $x_1 \neq x_2$. Without loss of generality $x_1 < x_2$. Then it holds that $f(x_1) < f(x_2) \implies f$ is injective. Surjectivity: $f(0) = 0$, hence $0 \in f([0, \infty))$. Let $\eta > 0$ be arbitrary. Because f is unbounded, there exists $z \in (0, \infty)$ with $f(z) > \eta$. $f(0) = 0 < \eta < f(z)$.

By the Intermediate Value Theorem (f is continuous), there exists $\xi \in (0, z)$ with $f(\xi) = \eta$. So f is surjective.

$$f^{-1} : [0, \infty) \rightarrow [0, \infty)$$

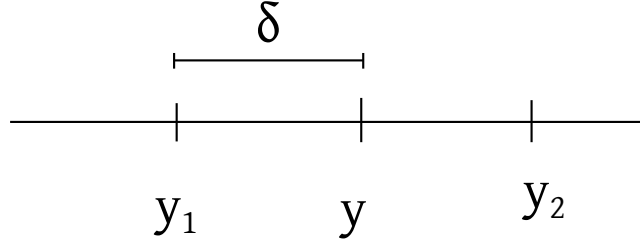


Figure 15: δ , y , y_1 and y_2

Monotonicity: Let $y_1 < y_2$. Then it holds that $x_1 = f^{-1}(y_1) < x_2 = f^{-1}(y_2)$. If this would not be true (hence, $x_2 \leq x_1$) then $y_2 = f(x_2) \leq y_1 = f(x_1)$ gives a contradiction.

Continuity of f^{-1} : Let $\varepsilon > 0$ be arbitrary. Let $y \in (0, \infty)$ be chosen arbitrarily. We show f^{-1} is continuous in y . Let $x = f^{-1}(y) > 0$ and choose $\varepsilon = \min\left\{\frac{x}{2}, \frac{\varepsilon}{2}\right\}$.

$$x_1 = x - \varepsilon > 0 \quad x_2 = x + \varepsilon > 0$$

Let $y_1 = f(x_1)$, $y_2 = f(x_2)$, $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. By monotonicity of f : $x_1 < x < x_2 \implies y_1 < y < y_2$.

Choose $\delta = \min\{y - y_1, y_2 - y\} > 0$ (compare with Figure 15). Hence $(y - \delta, y + \delta) \subseteq (y_1, y_2) \forall \eta \in (y - \delta, y + \delta)$ it holds that

$$f^{-1}(\eta) < f^{-1}(y + \delta) < f^{-1}(y_2) = x_2 = x + \varepsilon$$

$$f^{-1}(\eta) < f^{-1}(y - \delta) < f^{-1}(y_1) = x_1 = x - \varepsilon$$

So $f^{-1}(\eta) \in (x - \varepsilon, x + \varepsilon)$, or equivalently

$$|\eta - y| < \delta \implies \left| f^{-1}(\eta) - \underbrace{f^{-1}(y)}_{=x} \right| < c \leq \frac{\varepsilon}{2} < \varepsilon$$

So f^{-1} is continuous in y and f^{-1} is continuous in y_0 analogously. \square

Consider

$$\begin{aligned} \int_0^b f^{-1}(y) dy &= \left| \begin{array}{l} y \\ y=0 \\ y=b \end{array} \right| \begin{array}{l} = f(x) \\ \Rightarrow x = f^{-1}(0) = 0 \\ \Rightarrow x = f^{-1}(b) \end{array} \left| \begin{array}{l} = f'(x) dx \\ \\ \end{array} \right| = \int_0^{f^{-1}(b)} \underbrace{f^{-1}(f(x))}_{=x} \cdot f'(x) dx = \int_0^{f^{-1}(b)} x \cdot f'(x) dx \\ &= \underbrace{x \cdot f(x) \Big|_0^{f^{-1}(b)}}_{\text{integration by parts}} - 0 \int_0^{f^{-1}(b)} 1 \cdot f(x) dx \\ &= f^{-1}(b) \cdot b - \int_0^{f^{-1}(b)} f(x) dx \end{aligned}$$

So

$$\begin{aligned} I &= \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy = \int_{f^{-1}(b)}^0 f(x) dx + b \cdot f^{-1}(b) \\ &= \int_{f^{-1}(b)}^a f(x) dx + b \cdot f^{-1}(b) \end{aligned}$$

Case 1 $a = f^{-1}(b)$

$$\Rightarrow I = \underbrace{\int_a^a f(x) dx}_{=0} + b \cdot a$$

Case 2 $b < f(a)$, or equivalently $f^{-1}(b) < a$

$$\Rightarrow \int_{f^{-1}(b)}^a \underbrace{f(x)}_{f(f^{-1}(b)) \text{ for } x > f^{-1}(b)} dx > \overbrace{b}^{\text{minimal value}} \cdot \underbrace{(a - f^{-1}(b))}_{\text{length of integration interval}}$$

Therefore $I > b(a - f^{-1}(b)) + b \cdot f^{-1}(b) = ab$.

Case 3 $b > f(a)$, or equivalently $f^{-1}(b) > a$

$$\begin{aligned} \int_{f^{-1}(b)}^a f(x) dx &= \int_a^{f^{-1}(b)} \underbrace{(-f(x))}_{\substack{\text{monotonically decreasing} \\ > -f(f^{-1}(b)) \forall x \in [a, f^{-1}(b)]}} dx > -f(f^{-1}(b)) \cdot (f^{-1}(b) - a) \\ &= -b(f^{-1}(b) - a) \\ I &> -b(f^{-1}(b) - a) + b \cdot f^{-1}(b) = ab \end{aligned}$$

Remark 5.16. Young's inequality also holds without requiring differentiability of f (but the proof is more complex).

Lemma 5.8 (Special case of Young's inequality). Let $A, B \geq 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = p \cdot q$. Then p and q are called conjugate exponents. Then it holds that $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$.

Proof.

$$f(x) = x^{p-1} \text{ in Young's inequality}$$

$$y = x^{p-1} \iff x = y^{\frac{1}{p-1}}$$

$$\frac{1}{p-1} = q-1 \text{ is immediate, because}$$

$$\frac{1}{p-1} = q-1 \iff 1 = pq - p - q + 1 \iff p + q = pq$$

So $f^{-1}(y) = y^{\frac{1}{p-1}} = y^{q-1}$. By Young's inequality:

$$\begin{aligned} AB &\leq \int_0^A x^{p-1} dx + \int_0^B y^{q-1} dy \\ &= \frac{x^p}{p} \Big|_0^A + \frac{y^q}{q} \Big|_0^B = \frac{A^p}{p} + \frac{B^q}{q} \end{aligned}$$

□

Remark 5.17.

$$AB = \frac{A^p}{p} + \frac{B^q}{q}$$

Equality holds if and only if $B = A^{p-1} \iff B^q = \overbrace{A^{pq} - q}^p = A^p$.

Hölder's inequality

Theorem 5.9 (Hölder's inequality). Let I be an interval with boundary values a and b . $-\infty \leq a < b \leq +\infty$. Let p and q be conjugate exponents. Let f_1 and f_2 be regulated function on I such that

$$\int_a^b |f_1(x)|^p dx < \infty$$

$$\int_a^b |f_2(x)|^q dx < \infty$$

both exist.

We let $\|f_1\|_p := \left(\int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}}$ and $\|f_2\|_q := \left(\int_a^b |f_2(x)|^q dx \right)^{\frac{1}{q}}$. They are called L^p -norm of f_1 and L^q -norm of f_2 .

Then it holds that

$$\int_a^b |f_1(x) \cdot f_2(x)| dx < \infty$$

exists and

$$\int_a^b |f_1(x) \cdot f_2(x)| dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

Proof. Assume that $\|f_1\|_p > 0$ and $\|f_2\|_q > 0$. Let $A = \frac{|f_1(x)|}{\|f_1\|_p}$ and $B = \frac{|f_2(x)|}{\|f_2\|_q}$. By Lemma 5.8,

$$\frac{|f_1(x)|}{\|f_1\|_p} \cdot \frac{|f_2(x)|}{\|f_2\|_q} \leq \frac{1}{q} \cdot \frac{|f_1(x)|^p}{\|f_1\|_p^p} + \frac{1}{q} \cdot \frac{|f_2(x)|^q}{\|f_2\|_q^q}$$

We integrate the inequality,

$$\begin{aligned} & \frac{1}{\|f_1\|_p \cdot \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| dx \\ & \leq \frac{1}{p} \cdot \frac{1}{\|f_1\|_p^p} \cdot \underbrace{\int_a^b |f_1(x)|^p dx}_{=\|f_1\|_p^p} + \frac{1}{q} \cdot \frac{1}{\|f_2\|_q^q} \underbrace{\int_a^b |f_2(x)|^q dx}_{=\|f_2\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

$$\frac{1}{\|f_1\|_p \cdot \|f_2\|_q} \cdot \int_a^b |f_1(x) \cdot f_2(x)| dx \implies \int_a^b |f_1(x) f_2(x)| dx \leq \|f_1\|_p \cdot \|f_2\|_q$$

Special case: Let $\|f_1\|_p = 0$

$$\implies \left(\int_a^b |f_1(x)|^p dx \right)^{\frac{1}{p}} = 0 \implies \int_a^b \underbrace{|f_1(x)|^p}_{\geq 0} dx = 0$$

By Theorem 5.6, $f_1(x) = 0 \forall x \in [a, b] \setminus A$ and A is at most countable.

$$\implies f_1(x) \cdot f_2(x) = 0 \forall x \in [a, b] \setminus A$$

$$\implies \int_a^b |f_1(x) \cdot f_2(x)| dx = 0$$

$$\implies 0 = 0 \text{ in Hölder's inequality}$$

□

Remark 5.18 (Special case of Hölder's inequality). Let $p = q = 2$, $\frac{1}{2} + \frac{1}{2} = 1$.

$$\int_a^b |f_1(x) \cdot f_2(x)| \, dx \leq \|f_1\|_2 \|f_2\|_2$$

is called *Cauchy-Schwarz inequality for L^2 functions*.

$$\int_a^b f_1(x)f_2(x) \, dx = \langle f_1, f_2 \rangle_2 = \langle f_1, f_2 \rangle_{L^2}$$

is an *inner product on a proper space of functions*.

Elaboration on differential calculus

We consider a metric space X and functions $f : X \rightarrow \mathbb{C}$. We define a concept of uniform convergence of such sequences:

$$f_n : X \rightarrow \mathbb{C} \quad (n \in \mathbb{N}) \text{ and } f : X \rightarrow \mathbb{C}$$

We say, $(f_n)_{n \in \mathbb{N}}$ converges uniformly towards f if $\forall \varepsilon > 0 \forall N \in \mathbb{N}$ such that $\forall x \in X$ and $\forall n \geq N$ it holds that

$$\underbrace{|f_n(x) - f(x)|}_{\text{absolute value in } \mathbb{C}} < \varepsilon$$

$$\iff \sup \{|f_n(x) - f(x)| : x \in X\} < \varepsilon$$

Remark 6.1. Do not use $\|f\|_\infty$ for the definition of uniform convergence, because f_n and f must not be necessarily bounded. Hence,

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}$$

must not be finite.

Theorem 6.1. Let X be a metric space, $f_n : X \rightarrow \mathbb{C}$ be a sequence of continuous functions and $f : X \rightarrow \mathbb{C}$ such that $f_n \rightarrow f$ uniform on X . Then f is also continuous on X .