

Introduction to Functional Analysis

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based on the lecture by Martin Holler

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0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis
- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces U, V
matrices $A \in \mathcal{M}^{n \times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \mapsto$ functionals $f : U \rightarrow \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues

- Fields of application
 - basis of Applied Mathematics
 - partial differential equations
 - physical modelling
 - inverse problems (operator A models some physical measurement process)
 - Optimization and optimal control

A motivating example was presented with slides.

0.1 Application examples

Let $K : U \rightarrow \mathbb{R}^m$ with U as vector space describe a physical model. For example, K is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0, 1] \rightarrow \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find u . Typically inversion of K is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \|u\|_k$ where $\|z\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$ and $\|\cdot\|_u$ is a norm on U . Or alternatively, let $U = C^1([0, 1]^2)$ and solve $\min_{u \in U} \lambda \|ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$.

Other examples are JPEG compression and upsampling of images.

0.2 Our first problem

Let $U := C^1([0, 1]^2)$ be a normed space, $K : U \rightarrow \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2 dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \Rightarrow u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \rightarrow \mathbb{R}^k$ linear.

Proof. Case 1 (trivial model): Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \quad (1)$$

Take $(u_n)_{n \in \mathbb{N}}$ in \mathbb{R}^n such that $\lim_{n \rightarrow \infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u \in \mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \forall n \in \mathbb{N}$

$$\Rightarrow \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n \in \mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i \in \mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \rightarrow u$ as $i \rightarrow \infty$.

Now: Show that u solves Problem (1). ∇ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u \in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i \rightarrow \infty} \|\nabla u_{n_i}\|_2 + \lambda \|u_{n_i} - d\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that \hat{u} is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of ∇ , $\|\cdot\|_2$

Case 2 (K arbitrary): 1. K arbitrary does not provide boundedness anymore. Define $X := \text{kernel}(\nabla) \cap \text{kernel}(k)$ and

$$X^\perp := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^n x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^\perp \quad \text{i.e. } \forall u \in \mathbb{R}^n : \exists ! u_1 \in X, u_2 \in X^\perp : u = u_1 + u_2$$

Recall, that X^\perp is called *orthogonal complement*.

Claim 0.1. *If \hat{u} solves $\min_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. Then \hat{u} solves Problem (1).*

Proof. Let \hat{u} be a solution on X^\perp . Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^\perp$. Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \|\nabla(u_1 + u_2)\|_2 + \lambda \|k(u_1 + u_2) - d\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus \hat{u} solves our problem (1). □

Take again $(u_n)_{n \in \mathbb{N}}$ be such that $u_n \in X^\perp \forall n$ and

$$\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u \in X^\perp} \|\nabla u\|_2 + \lambda \|ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \text{kernel}(\nabla) + \text{kernel}(\nabla)^\perp$. $\nabla : \text{kernel}(\nabla)^\perp \rightarrow \text{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \text{kernel}(\nabla)^\perp \implies v \in \text{kernel}(\nabla) \implies \|v\|_2 = (v, v) = 0$. Thus, $\nabla^{-1} : \text{image}(\nabla) \rightarrow \text{kernel}(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^2\|_2 &= \|\nabla^{-1} \nabla u_n^2\|_2 = \|\nabla^{-1}\| \cdot \|\nabla u_n^2\|_2 \leq \|\nabla^{-1}\| \\ &\leq \|\nabla^{-1}\| (\|\nabla u_n^2\|_2 + \lambda \|Ku_n - d\|_2) \\ &= \|\nabla^{-1}\| \left(\underbrace{\|\nabla u_n\|_2}_{= \|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2 \right) \\ &< C \text{ for some } C > 0 \end{aligned}$$

Then $\|u_n^2\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \rightarrow \text{image}(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap \ker(\nabla) \implies v \in \ker(K)$. Hence $v \in \ker(K) \cap \ker(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$. Hence $K^{-1} : \text{image}(K) \rightarrow X^\perp \cap \ker(\nabla)$ exists and is continuous.

$$\begin{aligned} \implies \|u_n^n\|_2 &= \|K^{-1}Ku_n^n\|_2 \leq \|K^{-1}\| \|Ku_n^n\|_2 \\ &= \frac{\|K\|}{\lambda} \left(\lambda \|K(u_1^n + u_2^n) - Ku_n^n\|_2 + \|\nabla u_n\|_2 \right) \\ &\leq \frac{\|K\|}{\lambda} \left(\underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \|d - Ku_1^2\|}_{\text{bounded because } u_n^2 \text{ is bounded}} \right) \\ &< D \text{ for some } D > 0 \end{aligned}$$

$$\implies (u_n^n)_n \text{ bounded} \implies (u_n) = (u_n^n + u_n^n)_n \text{ is bounded}$$

$\implies (u_n)_n$ admits a subsequence converging to some \hat{u} . As in Case 1, \hat{u} is a solution to Problem (1).

In summary,

1. $\min_{u \in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla u|^2 dx}$ with $U = C^1([0,1]^2)$ relevant for application.
2. Discrete version: $\min_{u \in \mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:
 - (a) complementary subspaces X^\perp
 - (b) boundedness and compactness
 - (c) continuity
 - (d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ This lecture took place on 2019/03/07.

Define $U = C^1([0,1]^2)$. Let Y is some Banach space and $K : U \rightarrow Y$ is linear and continuous.

Consider the problem (P_∞) given by $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2}$.

Proposition 0.2. *There exists a solution of (P_∞) .*

Proof. Take $(u_n)_{n \in \mathbb{N}}$ as a sequence in U such that $\lim_{n \rightarrow \infty} \|\nabla u_n\|_2 + \lambda \|Ku_n - d\|_n \rightarrow \inf_{u \in U} (\dots)$. Now we want to show that $(u_n)_{n \in \mathbb{N}}$ is bounded.

Case 1: Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\Rightarrow \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\Rightarrow (u_n)_{n \in \mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n \in \mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change U to $U = \left\{ u : [0, 1]^2 \rightarrow \mathbb{R} \mid \sqrt{\int_{[0,1]^2} u^2} < \infty \right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u \in U} R(u) + \lambda \|K_{u-d}\|_2$ instead.

In this setting, $(u_n)_{n \in \mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i \in \mathbb{N}}$).

Our next step is to use continuity to show that \hat{u} is a solution.

Problem: $u \mapsto \|u - d\|_2$ is, in general, not continuous with respect to weak convergence.

But it is always true that $\|\hat{u} - d\|_2 \leq \liminf_{i \rightarrow \infty} \|u_{n_i} - d\|_2$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = - \int_0^1 u \partial_x \varphi \quad \forall \varphi \in C^\infty([0, 1]^2)$$

$\varphi = 0$ in $K \setminus [0, 1]^2$ for some $K \Subset (0, 1)^2$.

$$\begin{aligned} \Rightarrow \int_{[0,1]^2} \nabla u \varphi &= - \int_{[0,1]^2} u \cdot (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \\ \forall \varphi : (\varphi_1, \varphi_2) &= C^\infty([0, 1]^2, \mathbb{R}^2) + \text{zero on boundary} \end{aligned}$$

We define $w : [0, 1]^2 \rightarrow \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w \varphi = - \int_{[0,1]^2} u (\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2) \text{ holds } \forall \varphi$$

Then w is called *weak gradient* of u . We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i \rightarrow \infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i \rightarrow \infty} R(u_{n_i}) + \liminf_{i \rightarrow \infty} \lambda \|u_{n_i} - d\|_2 \\ &\leq \liminf_{i \rightarrow \infty} (R(u_{n_i}) + \lambda \|u_{n_i} - d\|_2) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

Case 2: Works as in the finite-dimensional setting using

- $X := \ker(A) \cap \ker(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \forall u \in \ker(\nabla)^\perp$ is called *Poincare inequality*.

□

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normal spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contplementary subspaces
8. Hilbert spaces

↓ *This lecture took place on 2019/03/12.*

Remark. 1. *Literature: UGU, in particular: Biezis, Werner*

2. *In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified*

1 Topological and metric spaces

Remark (Motivation). *Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies*

Definition 1.1 (Topology). *Let X be a set and $\tau \subset \mathcal{P}(X) = \{\text{"set of subsets of } X\}$. We say that τ is a topology on X if*

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. For any collection of sets $(U_i)_{i \in I}$ with I as some index set. We have $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$.

(X, τ) is called topological space.

A set $U \subset X$ is called open if $U \in \tau$ and is called closed if $U^c \in \tau$.

Remark. By the third property of topologies, $\bigcap_{i \in I} V_i$ is closed for any collection $(V_i)_{i \in I}$ of closed sets.

Definition 1.2 (Metric). Let X be a set, $d : X \times X \rightarrow \mathbb{R}$ be such that $\forall x, y, z \in X$

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric on X and (X, d) is called metric space.

Definition 1.3 (Norm). Let X be a vector space. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is called norm if $\forall x, y \in X, \lambda \in \mathbb{K}$

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Then $(X, \|\cdot\|)$ is called normed space.

Remark. If $\dim(x) < \infty$, all norms on X are equivalent.

Example. 1. Let X be a set then $\tau = \{\emptyset, X\}$ is a topology.

2. $(X, \mathcal{P}(X))$ is a topological space.
3. Define $S^{d-1} := \{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i^2 = 1\}$ and $d(x, y) := r$ where r is the length of the shortest connection between x and y on S^{d-1} . Then d is a metric on S^{d-1}
4. $X := \{u : [0, 1] \rightarrow \mathbb{R} \mid u \text{ is continuous}\}$ then $\|u\|_\infty := \sup_{x \in [0, 1]} |u(x)|$ is a norm on X
5. $l^p := \{(X_i)_{i \in \mathbb{N}} \mid x_i \in \mathbb{K} \forall u \text{ and } \sum_{i=1}^\infty |x_i|^p < \infty\}$ with $p \in [1, \infty)$ and $\|(x_i)_{i \in \mathbb{N}}\|_p := (\sum_{i=1}^\infty |x_i|^p)^{\frac{1}{p}}$. Then $(l^p, \|\cdot\|_p)$ is a normed space (the proof will be done later).

Remark.

$$l^\infty := \left\{ (X_i)_{i \in \mathbb{N}} \mid \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}$$

$$\|(X_i)_{i \in \mathbb{N}}\| = \sup_i |X_i|$$

Proposition 1.4. *Let X be a set.*

1. *If (X, d) is a metric space, define for $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ and $\tau = \{U \in \mathcal{P}(X) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \subset U\}$. Then (X, τ) is a topological space. We say that τ is the topology induced by d and we have that $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$*
2. *If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \rightarrow \mathbb{R}$ with $(x, y) \mapsto \|x - y\|$. Then (X, d) is a metric space and d is called the metric induced by $\|\cdot\|$.*

Remark (Consequence). *Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.*

Definition 1.5. *Let (X, τ) be a topological space. $U \subset X$. $x \in X$.*

1. *U is called a neighborhood of x if $\exists V \in \tau - x \in X \subset U : \mathcal{U}(x)$ is defined as the set of all neighborhoods of x*
2.
 - *x is called interior point of U if $U \in \mathcal{U}(x)$*
 - *x is called adjacent point of U if $\forall V \in \tau$ such that $x \in V : V \cap U \neq \emptyset$*
 - *x is called cluster point of U if it is an adjacent point of $U \setminus \{x\}$.*

The third property is stronger.

3. *Notational conventions:*

$$\mathring{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$

$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$

$$\partial U := \overline{U} \setminus \mathring{U}$$

Proposition 1.6. *Let (X, τ) be a topological space, $U \in X$. Then*

1. *U is open $\iff \mathring{U} = U$*
2. *U is closed $\iff \overline{U} = U$*
3. *$\mathring{U} = \bigcup_{\substack{V \in \tau \\ V \subset U}} V$ and \mathring{U} is open [\mathring{U} is the largest open set in U]*
4. *$\overline{U} = \bigcap_{\substack{V \text{ closed} \\ U \subset V}} V$ and \overline{U} is closed [\overline{U} is the smallest closed set containing U]*

Proof. 3. \subset Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V$

\supset Let $x \in \bigcup_{\substack{V \in \tau \\ V \subset U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \subset U \implies x \in \mathring{U}$

\mathring{U} is open because it is the union of open sets.

1. $\implies \mathring{U} \subset U$ by definition. U is open, so $U \subset \bigcup_{\substack{V \in \tau \\ V \subset U}} V \stackrel{(3)}{=} \mathring{U}$

2. $\Rightarrow V \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \Rightarrow x \in U^C \in \tau$ and $U \cap U^C = \emptyset$. This contradicts to $x \in \overline{U}$.
- \Leftarrow Take $x \in U^C = \overline{U}^C$.
- $\xRightarrow{(4)} \exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$
- $\Rightarrow V \cap U = \emptyset \Rightarrow V \subset U^C$
- $\Rightarrow U^C$ open $\Rightarrow U$ closed
4. We prove the fourth property without the second.
- \subset Take $x \in \overline{U}$. Take closed V such that $U \subset V$ if $x \notin V \Rightarrow x \in V^C$ which is open and $V^C \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.
- \supset Take $x \in \bigcap_{U \subset V} V^{\text{closed}}$. Suppose $x \notin \overline{U}$.
- $\Rightarrow \exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$
- $\Rightarrow U \subset Z^C, Z^C$ closed, $x \notin Z^C$. This contradicts to $x \in \bigcap_{U \subset V} V^{\text{closed}}$
- \overline{U} closed follows since the intersection of closed sets is closed.

□

Definition 1.7 (Limit). Let (X, τ) be a topological space, $(X_n)_{n \in \mathbb{N}}$ be a sequence in X . Henceforth, we write $(X_n)_n$ for $(X_n)_{n \in \mathbb{N}}$ and $\hat{x} \in X$. We say $x_n \rightarrow x$ in τ as $n \rightarrow \infty$ (“ x_n converges to x ”, “ x is limit of x_n ”) if

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

Definition 1.8 (Proposition and definition). Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or Hausdorff) if

$$\forall x, y \in X \text{ with } x \neq y \exists U, V \in \tau : x \in U, y \in V \text{ and } U \cap V = \emptyset$$

- In a T_2 -space, the limit of any sequence is unique.
- If τ is induced by a metric, then (X, τ) is T_2 .

Proof. 1. Take $(x_n)_n$ to be a sequence and assume x_n converges to \hat{x} and \hat{y} with $\hat{x} \neq \hat{y}$. By T_2 , $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergenc, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\{n_x, n_y\} : x_n \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x, y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology τ . Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

Definition 1.9. Let (X, τ) be a topological space, $U \subset V \subset X$. We say that U is dense in V , if $V \subset \overline{U}$. We say that X is separable if there exists a countable, dense subset.

Definition 1.10. Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \rightarrow Y$ a function. We say f is continuous at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. f is called continuous if it is continuous at any $x \in X$.

Proposition 1.11. With $(X, \tau_X), (Y, \tau_Y)$ and f as above, f is continuous $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$

Proof. Left as an exercise to the reader. \square

Definition 1.12. Let (X, τ) be a T_2 topological space, $M \subset X$ called compact if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ (" $(U_i)_{i \in I}$ is an open covering of M "), there exists U_{i_1}, \dots, U_{i_n} such that $M \subset \bigcup_{k=1}^n U_{i_k}$ ("there exists a finite subcover").

Remark. Compactness can also be defined without T_2 , this is also referred to as quasi-compact.

Remark (Exercise). Reconsider the previous results for metric and normed spaces.

↓ This lecture took place on 2019/03/14.

Definition 1.13. Let (X, d) be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in X . Then we say,

1. V is bounded if $\exists x \in X, r > 0$ such that $U \in B_r(x)$
2. $(x_n)_n$ is a Cauchy sequence if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$
3. X is complete if any Cauchy sequence in X admits a limit point
4. X is a Banach space if it is a normed space and complete

Proposition 1.14. Let (X, d) be a metric space. $(x_n)_n$ be a sequence in X . Then

1. $x_n \rightarrow x$ in the induced topology $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$
2. If $x_n \rightarrow x$, then $(x_n)_n$ is bounded as subset of X and $(x_n)_n$ is Cauchy.
3. If $U \subset X$ is closed and X is complete. Then (U, d) is a complete metric space.

Proof. 1. We prove both directions:

\implies True, since $B_\varepsilon(x)$ is open $\forall \varepsilon > 0$

\Leftarrow Let $x \in V$ with V open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$
 V open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$
 $\Rightarrow \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1, \dots, n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\Rightarrow y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in U , then $(y_n)_n$ is a Cauchy sequence in $X \Rightarrow \exists x \in X : y_n \rightarrow x$ as $n \rightarrow \infty$ if $x \notin U \Rightarrow x \in U^c \Rightarrow \exists n_0 \in \mathbb{N}$ such that $y_{n_0} \in U^c$ due to U^c open. This is a contradiction to $(y_n)_n$ in U

□

Proposition 1.15. Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \rightarrow Y$. The following are equivalent (TFAE):

- f is continuous (with respect to the induced topology)
- $\forall (x_n)_n$ such that $x_n \rightarrow x \Rightarrow f(x_n) \rightarrow f(x)$

Proof. Firstly, we prove that the first statement implies the second statement.

Take $(x_n)_n$ converging to x . Take $V \in \tau_Y$ such that $f(x) \in V \Rightarrow V \in \mathcal{U}(f(x))$

$$\Rightarrow \exists U \in \mathcal{U} : f(U) \subset V \Rightarrow \exists \hat{U} \in \tau_X \text{ such that } x \in \hat{U} \subset U$$

$$\Rightarrow \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \Rightarrow \forall n > n_0 : f(x_n) \in V \Rightarrow f(x_n) \rightarrow f(x)$$

Remark. 1. \Rightarrow 2. holds true in any topological space

2. \Rightarrow 1. Not.

Secondly, we prove that the second statement implies the first statement.

Suppose f is not continuous, find $x_n \rightarrow x$ such that $f(x_n) \rightarrow f(x)$ is wrong. If f is not continuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(x_n) \notin V \forall n \in \mathbb{N}$

$$\Rightarrow \exists \hat{V} \in \tau_Y \text{ such that } f(x_n) \notin \hat{V} \forall n \in \mathbb{N}$$

$$\Rightarrow \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\Rightarrow (x_n)_n$ converges to x but $f(x_n) \notin \hat{V} \Rightarrow f(x_n) \not\rightarrow f(x)$. This gives a contradiction. □

Definition 1.16. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \rightarrow Y$.

f is uniformly continuous iff

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

Proposition 1.17. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$, $f : M \rightarrow Y$. If M is dense in X , Y is complete and f is uniformly continuous.

$$\implies \exists \hat{f} : X \rightarrow Y \text{ such that } \hat{f} \text{ continuous and } \hat{f}|_M = f$$

Proof. Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \rightarrow x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(x_n), f(x_m)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in M , $\hat{y} \in Y$ such that $f(\hat{x}_n) \rightarrow \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X, (x_n)_n$ is a sequence in M such that $x_n \rightarrow x, f(x_n) \rightarrow y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0 :$

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d_Y(f(x_n), y) < 3\varepsilon$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.
2. \hat{f} is uniformly continuous.

□

↓ This lecture took place on 2019/03/19.

Proposition 1.18. Let (X, d) be a metric space, $M \subset X$.

1. M is compact, so $\forall (X_i)_{i \in I}$ with X_i a closed set $\forall i$ such that $\bigcap_{i \in I} X_i \cap M = \emptyset$.

$$\implies \exists X_{i_1}, \dots, X_{i_n} \text{ such that } \bigcap_{j=1}^n X_{i_j} \cap M = \emptyset$$

2. M is compact, so M is closed and bounded.

Proof. 1. We note that $\forall (X_i)_{i \in I}$ is a family of closed sets. $(X_i^C)_{i \in I}$ is a family of open sets and $\bigcap_{i \in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i \in I} X_i^C$

2. Is a special case of the next proposition.

□

Proposition 1.19. Let (X, d) be a metric space, $M \subset X$. TFAE:

1. M is compact.
2. Every infinite subset of M admits a cluster point.
3. Every sequence of M admits a convergent subsequence.
4. M is complete and totally bounded, where totally bounded is defined as

$$\forall \varepsilon > 0 : \exists (x_1, \dots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^n B_\varepsilon(x_i)$$

Remark. 1. totally bounded \implies bounded (proof is left as an exercise)

2. If $\dim(x) < \infty$, then compact \iff complete and bounded (see course Analysis I)

3. $\dim(x) < \infty \iff \overline{B_1(0)}$ is compact

where the last two items imply that X is a normed space.

Proof. 1 \rightarrow 2 If M is finite, (2) always holds true. So assume that M is infinite.

Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists x \in C$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(x_n)_n$ is a sequence of distinct points and $x_n \rightarrow x$. This implies that x is a cluster point of C . This gives a contradiction.

Now $M \subset \bigcup_{x \in M} B_{\varepsilon_x}(x)$. If M is compact, then

$$\implies \exists x_1, \dots, x_n : M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \subset M \subset \bigcup_{i=1}^n B_{\varepsilon_{x_i}}(x_i)$$

$$\implies C \text{ is finite}$$

This is a contradiction.

2 \rightarrow 3 Let $(x_n)_n$ be a sequence in M .

Case 1: $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies (x_n)_n$ admits a convergent sequence.

Case 2: $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid n \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

3 \rightarrow 4 Suppose that M is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in M as follows: Given x_1, \dots, x_n , choose $x_1 \in M$ arbitrary and $x_{i+1} \in M \setminus \bigcup_{j=1}^i B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in M and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convergent subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in M , then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \rightarrow x$ as $i \rightarrow \infty$. Since $(x_n)_n$ is Cauchy, $x_n \rightarrow x$ as $n \rightarrow \infty$ [left as an exercise]. Thus M is complete.

4 \rightarrow 1 Let $(U_i)_{i \in I}$ be an open covering of M and assume that $(U_i)_{i \in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}}(a)$. Define $\Omega := \{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } (U_i)_i\}$. We recursively define a sequence $(a_n)_n$ in M such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n}} \cap B_{\frac{1}{2^{n-1}}}(a_{n-1}) \neq \emptyset$$

Goal: Show $(a_n)_n \rightarrow a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

Step 1 $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a \in C_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \rightarrow n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let

$$\tilde{E}_{n+1} = \left\{ a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset \right\}.$$

Since $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$. [Take $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$, but if $B_{\frac{1}{2^{n+1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$$

Hence there exists a_{n+1} such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$ and $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

Step 2 Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) \leq d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. M is complete, so $\exists a \in M : a_n \xrightarrow{n \rightarrow \infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} \text{ and } \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for n sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x, a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

Proposition 1.20. Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$ compact. Let $f : X \rightarrow Y$ be continuous. Then

1. $f(M)$ is compact
2. $f|_M : M \rightarrow Y$ is uniformly continuous.

Proof. 1. Let $(U_i)_{i \in I}$ be an open covering of $f(M)$

$\implies (f^{-1}(U_i))_{i \in I}$ is an open covering of M [why!]

$\implies \exists c_1, \dots, c_n$ such that $M \subset \bigcup_{i=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{i=1}^n U_{i_j}$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon \in \mathbb{N} \exists x, y \in M : d(x, y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n, (y_n)_n$ sequences in M satisfying condition (*). M is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i \rightarrow \infty} 0$$

□

↓ This lecture took place on 2019/03/21.

Proposition 1.21 (Proposition and definition). *Let (X, d_X) and (Y, d_Y) be metric spaces. $g : X \rightarrow Y$ is a function. g is called Lipschitz continuous if $\exists L > 0$ such that $d_Y(\varphi(x), \varphi(y)) \leq L d_X(x, y) \forall x, y \in X$. Any Lipschitz continuous function is uniformly continuous.*

Proof. Left as an exercise to the reader. □

Theorem 1.22 (Arzelà-Ascoli theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Define $C(X, Y) := \{f : X \rightarrow Y \mid f \text{ continuous}\}$ and $d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$. Then*

1. d_C is well-defined and $(C(X, Y), d_C)$ is a complete metric space
2. A set $M \subset C(X, Y)$ is compact iff
 - (a) $\forall x \in X$ the set $M_x := \{f(x) \mid f \in M\}$ is compact
 - (b) M is equicontinuous, i.e. $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. 1. Show that: $d_C(f, g) < \infty$.

Pick $f, g \in C(X, Y)$. Because X is compact, $f(X), g(X)$ compact $\implies f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\leq d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\leq D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x \in X} d(f(x), g(x)) \end{aligned}$$

Showing that d_C is a metric is left as an exercise.

Show that $(C(X, Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X, Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because Y is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to d_C : Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \geq n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \geq n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x)) \leq \lim_{m \rightarrow \infty} d(f_m, f_n) < \varepsilon$ (the proof follows below)

$$\implies \sup_{x \in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X, Y) \implies f_n \rightarrow f$ with respect to d_C . Show that $f \in C(X, Y)$. Take $\varepsilon > 0$. Let n_0 such that $\sup_{x \in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \geq n_0 : d(f(x), f_n(x)) = \lim_{m \rightarrow \infty} d(f_m(x), f_n(x))$.

In general, we have $\forall x, y, z \in (Z, d_Z)$ with d_Z as a metric.

$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Proof.

$$d(x, z) \leq d(x, y) + d(y, z) \implies d(x, z) - d(y, z) \leq d(x, y) \quad (2)$$

$$d(y, z) \leq d(y, x) + d(x, z) \implies d(y, z) - d(x, z) \leq d(x, y) \quad (3)$$

$$(2) \text{ and } (3) \implies |d(x, z) - d(y, z)| \leq d(x, y) \quad (4)$$

□

Consequently, $\forall z \in Z, x_n \rightarrow x$ in Z : $d(x_n, z) \rightarrow d(x, z)$ since $|d(x_n, z) - d(x, z)| \leq d(x_n, x) \rightarrow 0$.

2. We need to prove both directions.

\implies (a) For $x \in X$ fixed, define $g_X : M \rightarrow Y$ with $f \mapsto f(x)$. Then

$$d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \leq d_C(f_1, f_2)$$

$\implies g_X$ is Lipschitz continuous, in particular continuous

$\implies M_X = g_X(M)$ compact

(b) Take $\varepsilon > 0$. M is totally bounded, so $\exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$. $\forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x, y) < \delta_i \implies d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$. Define $\delta := \min_i \delta_i > 0$. Then $\forall x, y \in X : d(x, y) < \delta$ and $\forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0})$

$$\begin{aligned} \implies d(f(x), f(y)) &\leq \underbrace{d(f(x), f_{i_0}(x))}_{\leq d_C(f, f_{i_0}) \leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\leq \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\leq d_C(f_{i_0}, f) \leq \frac{\varepsilon}{3}} < \varepsilon \end{aligned}$$

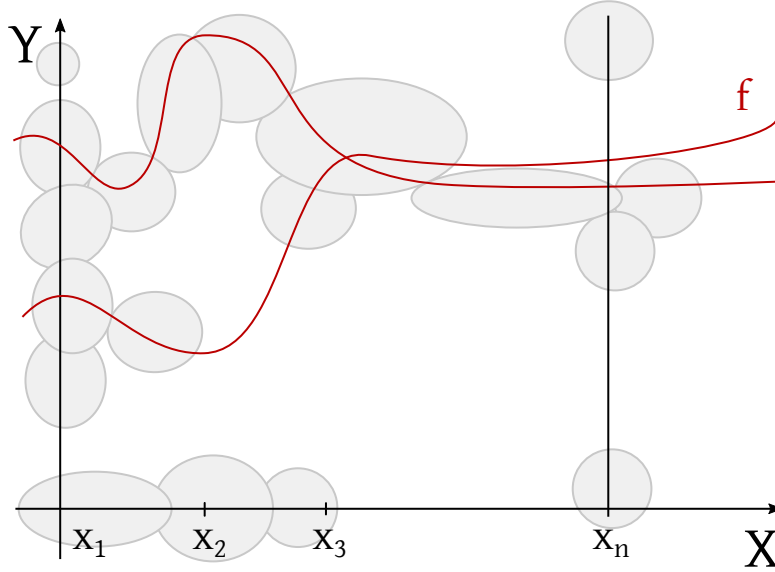


Figure 1: Covering of a function graph

\Leftarrow We prove the other direction.

\downarrow This lecture took place on 2019/03/26.

B is complete since it is a closed subset of a Banach space.

Show: M is totally bounded.

Consider $\varepsilon > 0$. Show: $\exists f_1, \dots, f_n$ such that $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$.

- Because M is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x, y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X , $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{i=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{i_i})$

Compare with Figure 1.

Now, for each tuple of indices $(y_{1,j_1}, \dots, y_{n,j_n})$ define $f_{y_{1,j_1}, \dots, y_{n,j_n}} \in C(x, y)$ to be such that $f_{y_{1,j_1}, \dots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an f exists. The set F of all such functions is finite. We show that $M \subset \bigcup_{q \in F} B_\varepsilon(q)$.

Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \dots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_\alpha \in F$ accordingly.

Take $x \in X$ arbitrary and x_i such that $x \in B_\delta(x_i)$

$$\begin{aligned}
 \implies d(f(x), f_\alpha(x)) &\leq d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\
 &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\
 \implies d_C(f, f_\alpha) &= \sup_{x \in X} d(f(x), f_\alpha(x)) < \varepsilon
 \end{aligned}$$

□

Remark. Compare this to the fact that $B_1(0)$ in $C(X, Y)$ is not compact.

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

Remark (Motivation). In general, let (X, d) be a metric space. Let A and B be open and dense, then also $A \cap B$ is dense.

Proof. Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] \neq \emptyset$. Take $x \in Y, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. A is dense. A is open, so $\exists \varepsilon > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. B is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_\varepsilon(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

□

More generally, $\forall A_1, \dots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many A_i ?

Theorem 1.23 (Baire theorem). Let (X, d) be a complete metric space. Let $(O_n)_{n \in \mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.

Proof. Let $D := \bigcap_{n \in \mathbb{N}} O_n$. Show that for $x \in X, \varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n \in \mathbb{N}}$.

n = 1 Take x_1, ε_1 such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

n - 1 \rightarrow n Given $x_{n-1}, \varepsilon_{n-1}$, take x_n, ε_n such that

$$\overline{B_{\varepsilon_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_n}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete} \implies \exists x \in X : x_n \rightarrow x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_n}(x_N)} \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_N)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

Definition 1.24. Let (X, d) be a metric space, $M \subset X$. We say

- M is nowhere dense (dt. “Nirgends dicht”), if $\overline{M}^\circ = \emptyset$
- M is of first category $\iff M$ is the countable union of nowhere dense sets

- M is of second category $\iff M$ is not of first category

Theorem 1.25 (Baire category theorem (weaker version)). *Let (X, d) be a complete metric space. Then (X, d) is of second category.*

In other words (which is a useful formulation): If $X = \bigcup_{n \in \mathbb{N}} C_n \implies \exists n_0 : \overset{\circ}{C} \neq \emptyset$. In particular, if

$$X = \bigcup_{n \in \mathbb{N}} C_n \text{ with } C_n \text{ closed} \implies \exists n_0 : \overset{\circ}{C}_{n_0} \neq \emptyset$$

Proof. Suppose that $X = \bigcup_{n \in \mathbb{N}} O_n = \bigcup_{n \in \mathbb{N}} \overline{O_n}$ with $\overline{O_n} = \emptyset \forall n$

$$\overline{O_n} = \emptyset \implies \overline{\overline{O_n}^c} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overline{O_n}^c = \emptyset \implies B_\varepsilon(x) \subset \overline{O_n} \implies \overline{O_n} \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O_n}^c \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O_n}^c \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n \in \mathbb{N}} \overline{O_n}^c \text{ is dense}$$

$$\bigcap_{n \in \mathbb{N}} \overline{O_n}^c = \left(\bigcup_{n \in \mathbb{N}} \overline{O_n} \right)^c = X^c = \emptyset$$

gives a contradiction □

Remark. 1. *This is a fundamental theorem in Functional Analysis*

2. *This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise)*

2 Normed space

2.1 Fundamentals

Definition 2.1. *Let X be a vector space. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is called seminorm if*

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$

The first property differs between a norm and a seminorm.

The tuple $(X, \|\cdot\|)$ is called a semi-normed space. We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.

Definition 2.2 (Definition and proposition). Let $(X, \|\cdot\|)$ be a semi-normed space and $(x_n)_n$ be a sequence in X . We say that

- $\sum_{n=1}^{\infty} x_n$ converges to $x \in X$ and write $x = \sum_{n=1}^{\infty} x_n$ if $\lim_{m \rightarrow \infty} \sum_{n=1}^m x_n = x$
- $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|$ converges [$\iff (\sum_{n=1}^m \|x_n\|)_m$ is bounded]

It holds that X is complete iff any absolutely converging series converges.

Proof. \implies Take $m_1 < m_2$ arbitrary, then

$$\begin{aligned} \left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| &\leq \sum_{n=m_1+1}^{m_2} \|x_n\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \leq \left\| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_n\| \right\| \\ &\implies \left(\sum_{n=1}^m x_n \right)_m \text{ is Cauchy} \implies \text{convergent} \end{aligned}$$

\Leftarrow Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick N_k such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}$$

$$\text{Define } y_k := x_{n_{k+1}} - x_{n_k} \implies \sum_k \|y_{n_w}\| \leq \sum_k 2^{-k} < \infty$$

$$\implies \exists y \in X : \sum_{k=1}^n y_k \rightarrow y \text{ as } n \rightarrow \infty$$

$$\sum_{k=1}^n y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \rightarrow y - x_{n_1} \text{ as } n \rightarrow \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is convergent.

□

Remark. In \mathbb{R}^n , $\sum_n x_n$ is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction \implies is true.

↓ This lecture took place on 2019/03/28.

Proposition 2.3 (Proposition and definition). Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X . We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if

$$\exists m, M > 0 \forall x \in X : m \|x\|_1 \leq \|x\|_2 \leq M \|x\|_1$$

TFAE:

1. $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
2. For any sequence $(x_n)_n$ and $x \in X$, $x_n \rightarrow x$ with respect to $\|\cdot\|_1 \iff x_n \rightarrow x$ with respect to $\|\cdot\|_2$
3. For any sequence $(x_n)_n$ we have,

$$x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \rightarrow 0 \text{ with respect to } \|\cdot\|_2$$

Proof. (1) \implies (2) \implies (3) is immediate.

It remains to show that:

(3) \implies (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n \|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\|y_n\|_1 = \frac{1}{n} \rightarrow 0$ hence $y_n \rightarrow 0$, but $\|y_n\|_2 > n \|y_n\|_1 = 1$.

$$\implies y_n \not\rightarrow 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

Remark. If $\dim(X) < \infty$, then any two norms on X are equivalent.

Definition 2.4 (Quotient spaces). Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation “ \sim ” on X with $x \sim y : \iff x - y \in Y$.

Then \sim defines an equivalence relation on X . We define

- $[X]_\sim = \{y \in X \mid x \sim y\}$...the equivalence class of $x \in X$
- $X/Y := \{[x]_\sim \mid x \in X\}$...the quotient space
- $\pi : \begin{cases} X \rightarrow X/Y \\ x \mapsto [x]_\sim \end{cases}$

Defining $[x] + [y] := [x + y]$

$$\lambda[x] := [\lambda x] \quad \hat{0} := [0]$$

We get that:

1. X/Y is a vector space
2. $\|[x]\|_{X/Y} := \inf_{y \in [x]} \|y\|_X$ is a semi-norm.
3. If Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.
4. If X is complete and Y closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.

Proof. • Equivalence relation

- Vector space with “+” and “ $\lambda[x]$ ” are well-defined

This is left as an exercise to the reader.

- First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\|[0]\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|y\| \leq \|0\| = 0$$

- Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$.
Show that: $\|\lambda[x]\|_{X/Y} = |\lambda| \|[x]\|_{X/Y}$.
Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\|\lambda[x]\|_{X/Y} = \|[\lambda x]\|_{X/Y} = \inf_{y \in [\lambda x]} \|y\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \|y\| = \inf_{w \in [x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\|[x]\|_{X/Y}}$$

- Take $[x_1], [x_2] \in X/Y$, $\varepsilon > 0$. We note that

$$\|[x]\|_{X/Y} = \inf_{\substack{y \in X \\ w \in Y \\ w := x \cdot y}} \|y\| = \inf_{w \in Y} \|x - w\|$$

Hence we can take $y_1, y_2 \in Y$ such that $\|x_1 - y_1\| < \|[x_1]\|_{X/Y} + \varepsilon$
 $\varepsilon \in [1, 2)$.

$$\begin{aligned} \Rightarrow \|[x_1] + [x_2]\|_{X/Y} &= \|[x_1 + x_2]\|_{X/Y} \leq \|x_1 + x_2 - (y_1 + y_2)\| \\ &\leq \|x_1 - y_1\| + \|x_2 - y_2\| \leq \|[x_1]\|_{X/Y} + \|[x_2]\|_{X/Y} + 2\varepsilon \end{aligned}$$

Since ε was arbitrary, the assertion follows.

3. Suppose Y is closed if $\|[x]\|_{X/Y} = 0$, then

$$\inf_{y \in Y} \|x - y\| = 0 \Rightarrow \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n \rightarrow \infty} \|x - y_n\| = 0$$

$$Y \text{ closed} \Rightarrow x \in Y \Rightarrow [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in X/Y and suppose that $\sum_{i=1}^{\infty} \|[x_n]\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, X/Y is complete.

Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \|[x_n]\|_{X/Y} + 2^{-n}$

$$\Rightarrow \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} (\|[x_n]\|_{X/Y} + 2^{-n}) < c < \infty$$

$$X \text{ complete} \Rightarrow \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \quad \left\| [x] - \sum_{n=1}^m [x_n] \right\|$$

□

↓ This lecture took place on 2019/04/02.

Corollary 2.5. Let X be a vector space with semi-norm $\|\cdot\|_X : X \rightarrow [0, \infty)$. Then

- $N = \{x \in X \mid \|x\|_X = 0\}$ is a subspace at X
- $\|[X]\| := \|X\|_p$ is a norm on X/N
- If X is complete, then $(X/N, \|\cdot\|)$ is a Banach space.

Proof. The proof is left as an exercise. □

Proposition 2.6. Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then

- \overline{U} is also a subspace.
- X is separable iff $\exists A \subset X$ complete such that $X = \overline{\mathcal{L}(A)}$ where $\mathcal{L}(A) = \{\sum_{i=1}^n \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathbb{K}, n \in \mathbb{N}\}$

Proof. • Left as an exercise

- \Rightarrow True since $\exists A \subset X$ countable such that $\overline{A} = X \Rightarrow \underline{X} = \overline{A} \subset \overline{\mathcal{L}(A)} \subset X$
- \Leftarrow Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^n (\lambda_i + i\mu_i)x_i \mid \lambda_i, \mu_i \in \mathbb{K}, x_i \in A, n \in \mathbb{N} \right\}$$

where i is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then B is countable.

Show: $\forall x \in X \forall \varepsilon > 0 \exists x \in B : \|x - y\| < \varepsilon$.

Take $x \in X, \varepsilon > 0 \Rightarrow \exists x_0 \in \mathcal{L}(A) : \|x - x_0\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=1}^n (\lambda_i + i\mu_i)x_i$ with $\lambda_i, \mu_i \in \mathbb{K}, x_i \in A$. Choose $\lambda'_i, \mu'_i \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda'_i)^2 + (\mu_i - \mu'_i)^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^n \|x_i\|} \forall i \in \{1, \dots, n\}$$

Let $y := \sum_{i=1}^n (\lambda'_i + i\mu'_i)x_i \in B$.

$$\begin{aligned} \Rightarrow \|x - y\| &\leq \|x - x_0\| + \|x_0 - y\| && \leq \frac{\varepsilon}{2} \\ &\leq \sum_{i=1}^n |(\lambda_i + i\mu_i) - (\lambda'_i + i\mu'_i)| \|x_i\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^n \|x_i\|} = \varepsilon \end{aligned}$$

□

Proposition 2.7 (Proposition and definition). *Let $(X_i, \|\cdot\|_{X_i})$ for $i = 1, \dots, n$ be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \dots \otimes X_n = \bigotimes_{i=1}^n X_i = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) \mid x_i \in X_i, i = 1, \dots, n\}$$

For $p \in [1, \infty]$, define

$$\|(x_1, \dots, x_n)\|_{\bigotimes_i X_i, p} = \begin{cases} \left(\sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1, \dots, n} \|x_i\|_{X_i} & \text{if } p = \infty \end{cases}$$

Then

- $(\bigotimes_i X_i, \|\cdot\|_{\bigotimes_i X_i, p})$ is a normed space with respect to componentwise addition and multiplication.
- If all X_i are complete, then $\bigotimes_{i=1}^n X_i$ is complete.
- All norms $\|\cdot\|_{\bigotimes_i X_i, p}$ are equivalent.

Proof. • Vector space properties: Left as an exercise

- Norm: $\|x\|_{\bigotimes_i X_i, p} = 0 \iff x = 0$
 $\|\lambda x\|_{\bigotimes_i X_i, p} = |\lambda| \|x\|_{\bigotimes_i X_i, p}$
- Triangle inequality: $p = 1, p = \infty$
 $p \in (1, \infty)$. Take $x, y \in \bigotimes_{i=1}^n X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i, p}$.

$$\begin{aligned} \implies \|x + y\|_p^p &= \sum_{i=1}^n \|x_i + y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\ &\leq \sum_{i=1}^n \|x_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} + \sum_{i=1}^n \|y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\ &\leq \underbrace{\left(\sum_{i=1}^n \|x_i\|_{X_i}^p \right)^{\frac{1}{p}}}_{\text{H\"older ineq.}} \cdot \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} + \left(\sum_{i=1}^n \|y_i\|_{X_i}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\ &= \|x\|_p \|x + y\|_p^{p-1} + \|y\|_p \|x + y\|_p^{p-1} \\ &= (\|x\|_p + \|y\|_p) \cdot \|x + y\|_p^{p-1} \end{aligned}$$

$$\implies \|x + y\|_p \leq \|x\|_p + \|y\|_p \text{ if } x + y \neq 0 \text{ (trivial otherwise)}$$

Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in \mathbb{R}^n)

□

Definition 2.8. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \rightarrow Y$ is linear such that $\|j(x)\|_Y = \|x\|_X$ (hence j is injective) then j is called isometric embedding from X to Y . If j is bijective, then j is called isometric isomorphism and we say $X = Y$ up to isomorphism.

Proposition 2.9. Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_{\hat{X}})$ a Banach space such that

1. \exists isometric embedding, $i : X \rightarrow \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [\hat{X} can be regarded as completion of X]

2. If $j_1 : X \rightarrow Y$ is an isometric embedding on Y , a Banach space

$$\implies \exists i_2 : \hat{X} \rightarrow Y$$

an isometric embedding such that $j_2 \circ i_1 = j_1$ and if $\overline{j_1(X)} = Y$ then j_2 is an isometric isomorphism. Thus “the completion is essentially unique”.

Proof. 1. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is Cauchy}\}$. \hat{X} is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \quad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define $\|(x_n)_n\|_{\hat{X}} := \lim_{n \rightarrow \infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in \mathbb{R}]. Then $\|\cdot\|_{\hat{X}}$ is a semi-norm (proof is left as an exercise). Setting $N = \{(X_n)_n \mid \|(X_n)_n\|_{\hat{X}} = 0\}$. By Corollary 2.5, $\hat{X} := \hat{X} \setminus N$ with $\|[(X_n)_n]\|_{\hat{X}} = \|(X_n)_n\|_{\hat{X}}$ is a normed space. Define

$$j : X \rightarrow \hat{X} \quad x \mapsto [(x)_n]$$

then j is linear and $\|j(x)\|_{\hat{X}} = \|[x]_n\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x\| = \|x\|$. So j is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n := j(x_n) \in \hat{X}$.

$$\begin{aligned} \implies \|y_m - [(x_n)_n]\|_{\hat{X}} &= \|(x_m)_n - (x_n)_n\|_{\hat{X}} = \lim_{n \rightarrow \infty} \|x_m - x_n\| \\ &= \lim_{n \geq n_0} \|x_m - x_n\| < \varepsilon \end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: \hat{X} is complete.

Let $(y_n)_n$ be Cauchy in \hat{X} . Pick $X_n \in X$ such that $\|j(x_n) - y_n\|_{\hat{X}} \leq \frac{1}{n}$ ($j(x) = \hat{x}$)

$$\implies \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}} \leq \|j(x_n) - y_n\|_{\hat{X}} + \|y_n - y_m\|_{\hat{X}} + \|y_m - j(x_m)\|_{\hat{X}}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \|y_n - y_m\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick n_1 such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\implies \forall n, m > \max(n_0, n_1) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\implies (x_n)_n$ is Cauchy. Let $y := (X_n)_n \in \tilde{X}$. Then

$$\|y_n - [y]\|_{\hat{X}} \leq \|y_n - j(x_n)\|_{\hat{X}} + \|j(x_n) - [y]\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n \rightarrow \infty} \|x_n - x_m\|_X \xrightarrow{n \rightarrow \infty} 0$$

2. ↓ *This lecture took place on 2019/04/04.*

Let $\hat{x} \in \hat{X} \implies \exists (x_n)_n \in X$ such that $j(x_n) \rightarrow \hat{x} \implies \|x_n - x_m\|_X = \|j(x_n) - j(x_m)\|_{\hat{X}}$.

$\implies (x_n)_n$ is a Cauchy sequence.

$\implies j_1(x_n)$ is a Cauchy sequence in Y .

$\implies \exists \lim_{n \rightarrow \infty} j_1(x_n) := y$

Using this, we define $j_2 : \hat{X} \rightarrow Y$ with $\hat{x} \mapsto \lim_{n \rightarrow \infty} j_1(x_n)$ where $j(x_n) \rightarrow \hat{x}$.

Well-defined? Take $\hat{x} \in \hat{X}$ and $j(x_n) \rightarrow \hat{x}$, $j(y_n) \rightarrow \hat{x}$.

$$\begin{aligned} \implies \|i_1(x_n) - j_1(y_n)\| &= \|x_n - y_n\| = \|j(x_n) - j(y_n)\| \rightarrow 0 \text{ as } n \rightarrow \infty \\ \implies \lim_{n \rightarrow \infty} j_1(x_n) &= \lim_{n \rightarrow \infty} j_1(y_n) \implies j_1 \text{ well-defined} \end{aligned}$$

Show linearity is left as an exercise. By isometry, take $\hat{x} \in \hat{X}$,

$$|i_2(\hat{x})| \underbrace{=}_{j(x_n) \rightarrow \hat{x}} \lim_{n \rightarrow \infty} \|j_1(x_n)\| = \lim_{n \rightarrow \infty} \|x_n\| = \lim_{n \rightarrow \infty} \|i(x_n)\| = \|\hat{x}\|$$

Show: $j_2 \circ j = j_1$. Take $x \in X \implies (x_n)$ is such that $j(x) \rightarrow j(x) \implies j_2(j(x)) = \lim_{n \rightarrow \infty} j_1(x) = j_1(x)$.

Assume that $\overline{j_1(X)} = Y$. Take $y \in Y$. Find $\hat{x} \in \hat{X}$ such that $i_2(\hat{x}) = y$. By $\overline{j_1(X)} = Y \implies \exists (x_n)_n$ in X such that $j_1(x_n) \rightarrow y \implies (j_1(x_n))_n$ is Cauchy.

$\implies (x_n)_n$ Cauchy $\implies (j(x_n))_n$ is Cauchy

$$\xrightarrow{\hat{X} \text{ complete}} \exists \hat{x} \text{ such that } \lim_{n \rightarrow \infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n \rightarrow \infty} j_2(j(x_n)) = Y$$

□

2.2 Important examples of normed spaces

Definition 2.10 (Basic notation). Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \rightarrow \mathbb{K}^M$ with $N, M \in \mathbb{N}$.

- We call Ω a domain (dt. "Gebiet") if Ω is open and connected, where connected means that $\forall x, y \in \Omega$ there is a curve in Ω connecting X and Y .
- For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ define $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_N$. If f is r -times continuously differentiable, we set for $\alpha \in \mathbb{N}_0^N$, $\{\alpha\} \leq r$.

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}} f$$

where $\frac{\partial^{\alpha_1}}{\partial_{x_1}^{\alpha_1}}$ is the partial derivative of f with respect to x_i of order α_i .

Example 2.11. Let $N = 2$ and $\alpha = (1, 1)$.

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

Let $\alpha = (2, 0)$.

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- For $z \in \mathbb{K}^N$ we denote $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$.¹
- We say $E \subset \Omega$ is compact in Ω and we write $E \Subset \Omega$ if E is compact.

Remark. If $E \Subset \Omega$, then $\exists \delta > 0 : \inf \{ \|x - y\| \mid x \in E, y \in \partial\Omega \} > 0$.

Proof. Left as an exercise (use compactness) □

- f is compactly supported in Ω if $\text{supp}(f) \Subset \Omega$.
- $\text{supp}(f) := \overline{\{x \in \Omega \mid \|f(x)\| > 0\}}$

↓ This lecture took place on 2019/04/09.

Definition 2.12 (Definition and proposition, Spaces of continuous functions).
Let $\Omega \subset \mathbb{R}^N$ be a domain. We define

$$\begin{aligned} C_b(\Omega, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ bounded} \} \text{ with } \|\varphi\|_{C_b} = \|\varphi\|_\infty = \sup_{x \in \Omega} \|\varphi(x)\| \\ C(\overline{\Omega}, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \varphi \text{ can be continuously extended to } \overline{\Omega} \}, \|\varphi\|_C := \|\varphi\|_\infty \\ C^r(\overline{\Omega}, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid D^\alpha \varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \leq r \} \text{ and } \|\varphi\|_{C^r} = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq r}} \|D^\alpha \varphi\|_\infty \end{aligned}$$

$$\begin{aligned} C_C^r(\Omega, \mathbb{K}^M) &= \{ \varphi : \Omega \rightarrow \mathbb{K}^M \mid \text{supp}(\varphi) \Subset \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M) \} \text{ and } \|\varphi\|_{C_C^r} = \|\varphi\|_{C^r} \\ C^\infty(\overline{\Omega}, \mathbb{K}^M) &= \bigcap_{r \in \mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M) \\ D(\Omega, \mathbb{K}^M) &= C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r \in \mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \text{ in } C^r(\overline{\Omega}, \mathbb{K}^M) \end{aligned}$$

Then for any bounded Ω , C^r, C_0^r, C_b are Banach spaces and C_C^r is a normed space.

Recall: $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

¹This is an abuse of notation with $|\alpha|$ for $\alpha \in \mathbb{N}_0^N$

Proof. The functions $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$ are norms (proof is left as an exercise).

Show that C_b is complete: Take $(\varphi_n)_n$ in C_b to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because $|\varphi_n(x) - \varphi_m(x)| \leq \|\varphi_n - \varphi_m\|_\infty$. Hence we can define $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$.

Show: $\varphi_n \rightarrow \varphi$ in $\|\cdot\|_\infty$. Take $\varepsilon > 0$. Show that $\exists n_0 \forall n \geq n_0 : \|\varphi - \varphi_n\|_\infty < \varepsilon$.
Take n_0 such that $\forall n, m \geq n_0 : \|\varphi_n - \varphi_m\|_\infty < \varepsilon$. Take $m \geq n_0$.

$$\implies \forall x \in \Omega : |\varphi(x) - \varphi_m(x)| = \lim_{\substack{n \rightarrow \infty \\ n \geq n_0}} |\varphi_n(x) - \varphi_m(x)| < \|\varphi_n - \varphi_m\|_\infty$$

Show: φ is bounded, i.e. $\exists C > 0 : |\varphi(x)| \leq C < \|\varphi_n - \varphi_m\|_\varepsilon < \infty$. Take n such that $\|\varphi - \varphi_n\|_\infty < 1$

$$\implies \forall x \in \Omega : |\varphi(x)| > |\varphi(x) - \varphi_n(x)| + |\varphi_n(x)| \leq 1 + \underbrace{\|\varphi_n\|}_{=C}$$

Now $C^r(\overline{\Omega}, \mathbb{K}^n)$ is a subspace of $C^b(\Omega, \mathbb{K}^n)$. Also $C^r(\overline{\Omega}, \mathbb{K}^n)$ is closed, since the uniform limit of $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$ with respect to $\|\cdot\|_{C^r}$ is again in $C^r(\overline{\Omega}, \mathbb{K}^M)$ [a result from Analysis].

$$\implies C^r(\overline{\Omega}, \mathbb{K}^M) \text{ is a Banach space}$$

$C_c^r(\overline{\Omega}, \mathbb{K}^M)$ is closed by definition, hence Banach.

$C_c^r(\Omega, \mathbb{K}^M)$ is a vector space, since $\forall \lambda \in \mathbb{K} : \varphi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\lambda\varphi) = \text{supp}(\varphi)$ and for $\varphi, \Psi \in C_0^r(\Omega, \mathbb{K}^M) : \text{supp}(\varphi + \Psi) \ll \Omega$. \square

Definition 2.13 (Definition and proposition). *Let (Ω, Σ, μ) with $\Omega \subset \mathbb{R}^N$ be a measure space (i.e. Σ is a sigma algebra and μ is a measure). For $p \in [1, \infty)$, we define*

$$\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) = \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid f \mu - \text{measurable and } \int_\Omega |f(x)|^p d\mu(x) < \infty \right\}$$

$$\|f\|_p^* = \left(\int_\Omega \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}}$$

$$\mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) := \left\{ f : \Omega \rightarrow \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} |f(x)| < \infty \right\}$$

$$\|f\|_\infty^* = \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)|$$

Our proposition is that these are semi-norms.

Proof. Show that $\|\cdot\|_p^*$ for $p \in [1, \infty]$ are seminorms.

They cannot be norms since $\|f\|_p^* = 0$ for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0$. □

Proposition 2.14 (Hölder inequality). *Let $p \in [1, \infty]$ and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \text{if } p \in (1, \infty) \\ 1 & \text{if } p = \infty \\ \infty & \text{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

If $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $g \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$ then for both

$$f \cdot g : \Omega \rightarrow \mathbb{K} \text{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)}$$

$$f \otimes g : \Omega \rightarrow \mathbb{K}^M \text{ with } x \mapsto (f_i(x), \varphi_i(x))_{i=1}^M$$

we have that $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$ and $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$ and $\|f \otimes g\|_1^* \leq \|fg\|_1^* \leq \|f\|_p^* \cdot \|g\|_q^*$.

Proof. **Case $p \in (1, \infty)$:** Intermediate result: $\forall \sigma, \tau \geq 0, r \in (0, 1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$ [AGM-inequality].

Proof.

Case $\sigma = 0$ or $\tau = 0$: immediate

Case $\sigma, \tau > 0$:

$$\log(\sigma^r \tau^{1-r}) = r \log(\sigma) + (1-r) \log(\tau) \leq \log(r\sigma + (1-r)\tau)$$

since $\log''(x) \leq 0$ implies that \log is concave

$$\log \text{ is monotonic} \implies \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$$

□

Let $A := \left(\|f\|_p^*\right)^p$ and $B := \left(\|g\|_q^*\right)^q$ with $r = \frac{1}{p} \in (0, 1]$ we get

$$\forall x \in \Omega : \left(\frac{|f(x)|^p}{A} \right)^{\frac{1}{p}} \left(\frac{|g(x)|^q}{B} \right)^{\frac{1}{q}} = \frac{1}{p} \frac{|f(x)|^p}{A} + \frac{1}{q} \frac{|g(x)|^q}{B}$$

$$\implies \frac{\int_{\Omega} |f(x)| |g(x)| d\mu(x)}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \leq \frac{1}{p} \frac{\int_{\Omega} |f(x)|^p d\mu(x)}{A} + \frac{1}{q} \frac{\int_{\Omega} |g(x)|^q d\mu(x)}{B}$$

$$\implies \int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_p^* \|g\|_q^* = \frac{1}{p} + \frac{1}{q} = 1$$

Now: $\|f \cdot g\|_x^* \leq \|f\|_p^* \cdot \|g\|_q^*$ follows since $|\langle x, y \rangle| \leq |x| |y| \forall x, y \in \mathbb{K}^M$.

Also:

$$\begin{aligned} \forall x \in \Omega : |f \otimes g(x)| &= \sum_{i=1}^M |f_i(x)| |g_i(x)| = \begin{pmatrix} |f_1(x)| & |g_1(x)| \\ \vdots & \vdots \\ |f_n(x)| & |g_n(x)| \end{pmatrix} \leq |f(x)| |g(x)| \\ \implies \int_{\Omega} |f \otimes g(x)| d\mu(x) &\leq \|f\|_p^* \cdot \|g\|_q^* \end{aligned}$$

Case $p \in \{1, \infty\}$: Without loss of generality assume that $p = 1, q = \infty$. $\forall N \in \Sigma$ with $\mu(N) = 0$ we get

$$\begin{aligned} \int_{\Omega} |f(x)| |g(x)| d\mu(x) &= \int_{\Omega \setminus N} |f(x)| |q(x)| \mu(x) \\ &\leq \int_{\Omega \setminus N} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |q(x)| = \int_{\Omega} |f(x)| d\mu(x) \cdot \sup_{x \in \Omega \setminus N} |q(x)| \end{aligned}$$

Taking the infimum over all such N , then

$$\int_{\Omega} |f(x)| |g(x)| d\mu(x) \leq \|f\|_1^* \cdot \|g\|_{\infty}^*$$

And the result follows again from $|\langle x, y \rangle| \leq |x| \cdot |y|$ and componentwise $|\langle x_i, y_i \rangle_i| \leq |x| |y| \forall x, y \in \mathbb{K}^M$

□

Proposition 2.15 (Minkowski inequality). *For $p \in [1, \infty]$, $f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$, we have that $\|f + g\|_p^* \leq \|f\|_p^* + \|g\|_p^*$.*

Proof. **Case $p = 1$:** trivial

Case $p \in (1, \infty)$:

$$\begin{aligned} (\|f + g\|_p^*)^p &= \int_{\Omega} |f(x) + g(x)|^p d\mu(x) \\ &= \int_{\Omega} |f(x)| \cdot |f(x) + g(x)|^{p-1} d\mu(x) \\ &\quad + \int_{\Omega} |g(x)| |f(x) + g(x)|^{p-1} d\mu(x) \\ &\leq \|f\|_p^* \cdot \| |f + g|^{p-1} \|_q^* + \|g\|_p^* \cdot \| |f + g|^{p-1} \|_q^* \end{aligned}$$

Recognize that $\left(\int |f+g|^p\right)^{\frac{1}{q}} = \left(\int |f+g|^{(p-1)q}\right)^{\frac{1}{q}}$ because $p = q \cdot (p-1)$

$$\begin{aligned}
 &= \left(\|f\|_p^* + \|g\|_p^*\right) \|f+g\|_p^* \\
 \Rightarrow \|f+g\|_p^* &\leq \|f\|_p^* + \|g\|_p^*
 \end{aligned}$$

□

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