Linear Algebra 2 – Practicals

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Contents

1 Solution of the last lecture exam of Analysis 1

1.1 Exam: Exercise 1

Exercise 1. Determine the limes of

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

$$\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \dots$$

does not help us. What about this representation?

$$
\begin{aligned}
\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1} &= \frac{a(n-1)+b(n+1)}{(n+1)(n-1)} \\
a(n-1)+b(n+1) &= 1 \\
(a+b)n + (b-a) &= 1 \\
\Rightarrow a+b = 0 \wedge b-a &= 1 \\
\Rightarrow a = -\frac{1}{2} \quad b &= \frac{1}{2}
\end{aligned}
$$

Followingly,

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} = \sum_{n=2}^{\infty} \left(\frac{\frac{1}{2}}{n-1} - \frac{\frac{1}{2}}{n+1} \right)$$

Okay, how to proceed? Let's build a pre-factor:

$$
\begin{aligned}
&\frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\
&= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \\
&= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}
\end{aligned}
$$

Let's describe this process of cancelling out formally as telescoping sum:

$$S_m \coloneqq \frac{1}{2} \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n+1}$$

Please be aware that we explicitly define S_m because we want to work with finite sums. Only in finite sums, we are always allowed to split up sums.

$$
\begin{aligned}
&= \frac{1}{2} \sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2} \sum_{n=4}^{m+2} \frac{1}{n-1} \\
&= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} \right) - \frac{1}{2} \left(\frac{1}{m} + \frac{1}{m+1} \right)
\end{aligned}
$$

We already know $\frac{1}{m} \xrightarrow{m \to \infty} 0$. Also $\frac{1}{m+1} \xrightarrow{m \to \infty} 0$. Followingly also $\frac{1}{2}\left(\frac{1}{m} + \frac{1}{m+1}\right) \xrightarrow{m \to \infty} 0$.

1.2 Exam: Exercise 2

Exercise 2. A recursive definition of a sequence is given:

$$a_0 \in \mathbb{R}, a_0 > 1, (a_n)_{n \in \mathbb{N}}$$

$$a_{n+1} = \frac{1}{2}(a_n + 1)$$

As an example, we look at the sequence with $a_0 = 2$:

$$a_0 = 2 \qquad a_1 = \frac{3}{2} \qquad a_2 = \frac{5}{4} \qquad a_3 \frac{9}{8}$$

Another example is $a_0 = 7$:

$$a_0 = 7 \qquad a_1 = 4 \qquad a_2 = \frac{5}{2} \qquad a_3 \frac{7}{4}$$

Exercise 3. a) Show that $1 \overset{!}{<} a_n \overset{!}{\leq} a_0 \quad \forall n \in \mathbb{N}$

Our examples suggest that this claim might hold.

We use induction over n to prove this statement:

induction base $1 < a_0 \leq a_0$ holds trivially.

induction step We are given $1 < a_n \leq a_0$ by the induction hypothesis.

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &\leq \frac{1}{2}(a_0 + a_0) && \text{[induction hypothesis and } 1 < a_0\text{]} \end{aligned}$$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &> \frac{1}{2}(1 + 1) && \text{[induction hypothesis]} \\ &= 1 \end{aligned}$$

Exercise 4. b) Prove that $a_{n+1} \overset{!}{<} a_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + 1) \\ &< \frac{1}{2}(a_n + a_n) && \text{[we have proven: } a_n > 1\text{]} \end{aligned}$$

Exercise 5. c) Does this series converge? If so, give its limit.

Yes, because it is monotonically decreasing (according to exercise b) and bounded below (according to exercise a).

$$b_n := a_n - 1 \qquad \forall n \in \mathbb{N}$$
$$b_0 := a_0 - 1$$
$$b_{n+1} = a_{n+1} - 1 = \frac{1}{2}(a_n + 1) - 1 = \frac{1}{2}(b_n + 1 + 1) - 1 = \frac{1}{2} b_n$$

$$b_n = \frac{1}{2^n} b_0 \to 0 \cdot b_0 = 0$$
$$\Rightarrow b_n \to 0$$
$$\Rightarrow a_n = b_n + 1 \to 1$$

Does it work to just show: $1 = \frac{1}{2}(1+1)$? Nope, because in points of continuity this might be true even though 1 is not its limes.

Let $a_n \to a$ and $a_{n+1} = \frac{1}{2}(a_n + 1)$.

$$a_{n+1} \to a \qquad \frac{1}{2}(a_n + 1) \to \frac{1}{2}(a + 1) \qquad a = \frac{1}{2}(a+1)$$

1.3 Exam: Exercise 3

Exercise 6. $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto 2x^2 + 5x - 3$. Show continuity with an ε-δ-proof.

If we don't need an ε-δ-proof, we would argue with the Algebraic Continuity Theorem: The function f is a composition of continuous functions, hence a continuous function itself.

ε-δ-definition:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

If $|x - x_0| < \delta$,

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|2x^2 + 5x - 3 - (2x_0^2 + 5x_0 - 3)\right| \\
&= \left|2x^2 + 5x - 2x_0^2 - 5x_0\right| \\
&\leq 2\left|x^2 - x_0^2\right| + 5|x - x_0| \\
&= 2|(x + x_0)(x - x_0)| + 5|x - x_0| \\
&= 2|x + x_0||x - x_0| + 5|x - x_0| \\
&\leq 2(|x| + |x_0|)|x - x_0| + 5|x - x_0| \\
&\leq 2(|x_0| + \delta + |x_0|) + 5\delta
\end{aligned}$$

Our goal: we are able to claim $\overset{!}{<} \varepsilon$

$$\begin{aligned}
&= 4|x_0|\delta + 2\delta^2 + 5\delta \\
&= 2\delta^2 + (4|x_0| + 5)\delta
\end{aligned}$$

In general (here it does not apply), that x_0 might be zero. So division is not allowed and requires case distinctions (cumbersome!).

The following steps work only because we know $\varepsilon > 0$ and $\delta > 0$:

$$2\delta^2 < \frac{\varepsilon}{2}$$

$$\delta < \frac{\sqrt{\varepsilon}}{2}$$

$$(4|x_0| + 5)\delta < \varepsilon$$

$$\delta < \frac{\varepsilon}{4|x_0| + 5}$$

Then we can submit those results as solution:

Let $\varepsilon > 0$ and $\delta := \min\left(\frac{\sqrt{\varepsilon}}{5}, \frac{\varepsilon}{4|x_0|+6}\right)$. Then the ε-δ definition shows that f is continuous.

2 Exam: Exercise 4

Exercise 7. Let $f : [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. Show that $\exists \xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

Hint: Consider $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$.

Intuition: Let $\xi = 0$ with $f(\xi) = 0$ and $\xi = \frac{1}{2}$ with $f(\xi) = \frac{1}{16}$. Then the difference $f(0) - f(\frac{1}{2})$ is negative. At the same time $f(\frac{1}{2}) - f(1)$ is positive. So at some point between $x = 0$ and $x = 1$ the difference must be zero.

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$h(0) = f(0) - f\left(\frac{1}{2}\right)$$
$$h(1) = f\left(\frac{1}{2}\right) - f(1) = f\left(\frac{1}{2}\right) - f(0) = -h(0)$$

$f(x)$ is continuous in $[0, \frac{1}{2}]$. $f(x + \frac{1}{2})$ is continuous in $[0, \frac{1}{2}]$. Therefore h is continuous, because it is a composition of continuous functions.

Case 1: $h(0) < 0$ Then $h(\frac{1}{2}) > 0$ and $h(0) < 0 < h(\frac{1}{2})$. Due to Intermediate Value Theorem it holds that

$$\exists \xi \in [0, \frac{1}{2}] : h(\xi) = 0$$

$$\Rightarrow f(\xi) = f(\xi + \frac{1}{2})$$

Case 2: $h(0) > 0$ Then $h(\frac{1}{2}) < 0$. Remaining part analogous.

Case 3: $h(0) = 0$ Then by definition $f(0) = f(\frac{1}{2})$, so choose $\xi = 0$.

3 Exercise 1

Exercise 8. Investigate the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto \frac{1}{2}(x\,|\,x\,|+x^2)$ in terms of multiple differentiability in all points $x_0 \in \mathbb{R}$.

$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0 \end{cases}$$

So this is differentiable, but in case of $x = 0$, it remains questionable.

We look at the definition of differentiability:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

$$f'(x) = \begin{cases} \lim_{x \to 0} \frac{0}{x} = 0 \\ \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0 \end{cases}$$

It follows that f is differentiable one time.

$$f''(x) = \begin{cases} 0 & x < 0 \\ 2x & x > 0 \end{cases}$$

What about $x = 0$?

$$\lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} \begin{cases} \lim_{x \to 0} \frac{0}{x} = 0 \\ \lim_{x \to 0^+} \frac{2x}{x} = \lim_{x \to 0^+} 2 = 2 \end{cases}$$

Left and right limes differ. So it is not differentiable.

4 Exercise 2

Exercise 9. Determine, possibly using l'Hôpital's rule, the following limits:

1. $\lim_{x \to 1} \frac{\ln x}{x-1}$
2. $\lim_{x \to 0^+} \frac{1}{x} - \frac{1}{\sin x}$
3. $\lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\cos x)}{\ln(1 - \sin x)}$
4. $\lim_{x \to 1^-} x^{\frac{1}{1-x}}$
5. $\lim_{\substack{n \to \infty \\ n \in \mathbb{N}}} n^{\frac{1}{\sqrt{n}}}$
6. $\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4.1 Exercise 2.a

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

4.2 Exercise 2.b

$$\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$$

The conditions to apply L'Hôpital's rule are satisfied.

$$\Rightarrow \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

A nice hint to find out whether this function is differentiable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\frac{\sin x - x}{x \sin x} = \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} - \dots}{x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}} \approx x \to 0$$

This exploits, that it will take one run of L'Hôpital's rule (because each expression has at least degree 2) and its limes will be 0 (because of x).

4.3 Exercise 2.c

$$\lim_{x\to \frac{\pi}{2}^-} \frac{\ln(\cos(x))}{\ln(1-\sin(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied. We claim that $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = \infty$ is fine.

$$\Rightarrow \lim_{x\to \frac{\pi}{2}^-} \frac{\frac{-\sin(x)}{\cos(x)}}{\frac{-\cos(x)}{1-\sin(x)}} = \lim_{x\to \frac{\pi}{2}^-} \frac{-\sin(x)\cdot(1-\sin(x))}{\cos(x)(-\cos(x))}$$

The conditions to apply L'Hôpital's rule are partially satisfied.

$$\lim_{x\to \frac{\pi}{2}^-} \frac{-\cos(x)(1-\sin(x)) - \sin(x)\cdot(-\cos(x))}{-\sin(x)(-\cos(x)) + \cos(x)\cdot\sin(x)} = \frac{1}{2}$$

If we want to apply the previous estimate here, we should consider

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \cos(y) \qquad y = \frac{\pi}{2} - x$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right) = \sin(y)$$

This gives us a different estimate of the result:

$$\lim_{y\to 0^+} \frac{\ln(\sin(y))}{\ln(1-\cos(y))} \approx \lim_{y\to 0^+} \frac{\ln(y)}{\ln\left(\frac{y^2}{2}\right)} = \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y) - \ln(2)} \approx \lim_{y\to 0^+} \frac{\ln(y)}{2\ln(y)} = \frac{1}{2}$$

We define neighborhoods:

$$N_\delta(x_0) = \{x : |x - x_0| < \delta\}$$

$$N_R(\infty) = \{x : x > R\}$$

4.4 Exercise 2.d

$$\lim_{x\to 1^-} x^{\frac{1}{1-x}} = \lim_{x\to 1^-} e^{\ln(x)\frac{1}{1-x}} = \exp\left(\lim_{x\to 1^-} \underbrace{\frac{\ln(x)}{1-x}}_{(-1)\cdot\text{Exercise a}}\right) = \frac{1}{e}$$

4.5 Exercise 2.e

$$\lim_{n\to\infty} n^{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\left(\exp\left(\frac{\ln n}{\sqrt{n}}\right)\right) = \exp\left(\lim_{n\to\infty}\frac{\ln(n)}{\sqrt{n}}\right)$$

The conditions to apply L'Hôpital's rule are satisfied („$\frac{\infty}{\infty}$")

$$\exp\left(\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}}\right) = \exp\left(\lim_{n\to\infty}\frac{2\sqrt{n}}{n}\right) = \exp(0) = 1$$

4.6 Exercise 2.f

$$\lim_{x\to\infty}\frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{n\to\infty}\frac{e^x\left(1-e^{-2x}\right)}{e^x\left(1+e^{-2x}\right)} = \frac{\lim_{x\to\infty}1 - \lim_{x\to\infty}\frac{1}{e^{2x}}}{\lim_{x\to\infty}1 + \lim_{x\to\infty}\frac{1}{e^{2x}}}$$

Remark:

$$\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\cosh(x)}{\sinh(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}$$

$$y = \lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)} = \frac{1}{\lim_{x\to\infty}\frac{\sinh(x)}{\cosh(x)}} = \frac{1}{y}$$

5 Exercise 3

Exercise 10. Show that the function $f : \mathbb{R} \to \mathbb{R}$ with $x \mapsto x + e^x$ is bijective. Furthermore determine $(f^{-1})'(1)$ and $\lim_{y\to\infty}(f^{-1})'(y)$.

If the function is strictly monotonically increasing, it is injective.

$$f'(x) = 1 + e^x > 0 \qquad \forall x \in \mathbb{R}$$

We show that it is strictly monotonically increasing:

Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\alpha) \qquad \text{with } \alpha \in (x_1, x_2)$$

$$f(x_2) - f(x_1) = f'(\alpha)(x_2 - x_1) > 0$$

Is f surjective?

For an arbitrary $y_0 \in \mathbb{R}$ it holds that $\exists x_0 \in \mathbb{R} : f(x_0) = y_0$:

$$\exists f(a), f(b) \in \mathbb{R} : f(a) \le y_0 < f(b)$$

It holds that

$$\lim_{x\to-\infty} x + \underbrace{e^x}_{\to 0} = -\infty$$

$$\lim_{x\to+\infty} x + e^x = \infty$$

Formally:

$$\forall y_0 \exists x_0 : \forall x < x_0 : f(x) < y_0$$

From the Intermediate Value Theorem it follows that

$$\Rightarrow \exists c \in [a,b) : f(c) = y_0 \qquad c =: x_0$$

So it is surjective.

From injectivity and surjectivity it follows that it is bijective.

5.1 Determine $(f^{-1})'(1)$

$$f(x) = x + e^x$$
$$f'(x) = 1 + e^x$$

We apply the inverse function theorem:

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$

$$y = 1 = f(x)$$
$$x = f^{-1}(1)$$

An educated guess gives us that $x = 0$. In general determining x is more difficult.

$$\left(f^{-1}\right)'(1) = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}$$

5.2 Determine $\lim_{y\to\infty} \left(f^{-1}\right)'(y)$

$$\lim_{y\to\infty} \left(f^{-1}\right)'(y) = \lim_{y\to\infty} \frac{1}{1+e^x}$$

As x grows to infinity, also y grows to infinity. From bijectivity it follows that any value can be reached with x as well as $f(x)$.

$$\underbrace{f'(\underbrace{f^{-1}(\underbrace{y}_{\to\infty})}_{\to\infty})}_{\to\infty}$$

6 Exercise 4

Exercise 11. Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ be differentiable in $x_0 \in D$. Show

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2} = f'(x_0)$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) + f(x_0) - f(x_0-h)}{2h} \\
&= \lim_{h'\to 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0) - f(x_0+h')}{-h'} \right) \\
&= \lim_{h'\to 0} \frac{1}{2} \cdot \left(f'(x_0) + \frac{f(x_0+h') - f(x_0)}{h'} \right) \\
&= \frac{1}{2}(f'(x_0) + f'(x_0)) \\
&= f'(x_0)
\end{aligned}
$$

6.1 Exercise 4.b

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(x_0+rh) - f(x_0+sh)}{h} &= \lim_{h\to 0} \frac{f(x_0+rh) - f(x_0)}{h} + \lim_{h\to 0} \frac{f(x_0) - f(x_0+sh)}{h} \\
& h_1 = rh \qquad h_2 = sh \\
&= \lim_{h_1\to 0} \frac{f(x_0+h_1) - f(x_0)}{\frac{1}{r} \cdot h_1} + \lim_{h_2\to 0} \frac{f(x_0) - f(x_0+h_2)}{\frac{1}{s} \cdot h_2} \\
&= r \cdot f'(x_0) - s \cdot f'(x_0) \\
&= (r-s) \cdot f'(x_0)
\end{aligned}
$$

7 Exercise 5

Exercise 12. Let $D \subseteq \mathbb{R}$ be an open interval. $f : D \to \mathbb{R}$ is differentiable and f is twice differentiable in $x_0 \in D$.

7.1 Exercise 5.a

Exercise 13. Show that

$$\lim_{h\to 0} \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0)$$

f is differentiable, therefore continuous, and h goes to 0. So we have „$\frac{0}{0}$". All conditions to apply L'Hôpital's rule are satisfied.

$$\lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} \approx \frac{0}{0}$$

We can apply L'Hôpital's Rule again or just use the result of exercise 4a.

$$\overset{4a}{\Longrightarrow} f''(x_0)$$

7.2 Exercise 5.b

Exercise 14. Show that the limes from exercise 5.a can also exist, even if $f''(x_0)$ does not exist. Use the result from Exercise 1.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

We know that it is not twice differentiable. But we want to show that the limes exists.

We are only concerned with $x = 0$.

$$\lim_{h \to 0} f(x_0) = 0$$

$$\lim_{h \to 0} \frac{h^2 - h^2}{h^2} = \frac{0}{h^2} = 0$$

So if we traverse the graph from both sides at the same time $\frac{f(x_0+h)-f(x_0-h)}{h}$.

8 Exercise 6

Exercise 15. Determine the following limit for arbitrary $c \in \mathbb{R}$:

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right).$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right)$$

$$\lim_{n \to \infty} \frac{n}{\ln n} \left(\sqrt[n]{n^c} - 1 \right) = \lim_{n \to \infty} \frac{e^{\frac{c}{n} \cdot \ln n} - 1}{\frac{\ln n}{n}}$$

and

$$\left(e^{\frac{c}{n} \cdot \ln n} \right)' = e^{\frac{c}{n} \cdot \ln n} \cdot \left(-\frac{c}{n^2} \cdot \ln n + \frac{c}{n} \cdot \frac{1}{n} \right) = \frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln(n))$$

All conditions are satisfied to apply L'Hôpital's rule ("$\frac{0}{0}$"):

$$\lim_{n \to \infty} \frac{\frac{c}{n^2} e^{\frac{c}{n} \cdot \ln n} \cdot (1 - \ln n)}{\frac{\frac{1}{n} \cdot n - \ln n}{n^2}}$$

$$= \lim_{n \to \infty} \frac{c \cdot e^{\frac{c}{n} \cdot \ln n} (1 - \ln(n))}{1 - \ln n} = \lim_{n \to \infty} c \cdot e^{\frac{c}{n} \cdot \ln n} = c \cdot 1$$

9 Exercise 7

Exercise 16.

- Show that $e^x \geq 1 + x$ holds for all $x \in \mathbb{R}$.
  Hint: On demand, use the Mean Value Theorem.
- Prove that for all $x > 0$, the following estimates hold:

$$\ln x \leq x - 1$$

and for all $k \in \mathbb{N}_+$ it holds that

$$k \left(1 - \frac{1}{\sqrt[k]{x}} \right) \leq \ln x \leq k \left(\sqrt[k]{x} - 1 \right)$$

$x \geq 0$ Choose $f(x) = e^x$ in $[0, x)$. Mean value theorem:

$$\exists x_0 : f'(x_0) = \frac{f(b) - f(a)}{b - a} \qquad \text{for } a < x_0 < b$$
$$f'(x_0) = e^{x_0} \qquad e^{x_0} \geq 1 \qquad x_0 \geq 0$$
$$e^{x_0} = \frac{f'(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x} \Rightarrow \frac{e^x - 1}{x} \geq 1$$

Or alternatively: f is convex and therefoer $f''(x) > 0$.

Consider $f(x) = x - 1 - \ln x$

$$f'(x) = 1 - \frac{1}{x} \qquad f''(x) = \frac{1}{x^2}$$
$$f'(x) \stackrel{!}{=} 0$$
$$1 - \frac{1}{x} = 0 \Leftrightarrow x = -1$$
$$f''(1) = 1 > 0 \Rightarrow \text{ minimum and because } f(1) = 0 \Rightarrow \forall x : x - 1 - \ln x \geq 0$$

Or alternatively:

$$y \coloneqq x - 1$$
$$x = y + 1$$

Show that $\ln(y + 1) \leq y \Leftrightarrow y + 1 \leq e^y$.

e^x is monotonically increasing $\Rightarrow x \leq y \Leftrightarrow e^x \leq e^y$.

And this has been proven previously.

9.1 Exercise 7.b

$$\ln(x) \leq k \left(\frac{[}{k}]x - 1\right)$$
$$\ln(\sqrt[k]{x}) \leq \sqrt[k]{x} - 1 \Leftrightarrow \ln(y) \leq y - 1$$

And this has been proven in Exercise a.

10 Exercise 8

Exercise 17. Let $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Show: If f is continuous in an environment U of $a \in D$, differentiable in $U \setminus \{a\}$ and there exists $\lim_{x \to a} f'(x)$, such that f in a differentiable and

$$f'(a) = \lim_{x \to a} f'(x).$$

Hint: On demand, use the Mean Value Theorem.

Let h_n be an arbitrary zero-sequence (with $h_n(x) > 0 \quad \forall x \in D$) and due to Mean Value Theorem $\exists \xi_n \in D$ with $f'(\xi_n) = \frac{f(a+hn)-f(a)}{h_n}$.

$$\lim_{n\to\infty} f'(\xi_n) = \lim_{x \to a} f'(x) = \lim_{n\to\infty} \frac{f(a + h_n) - f(a)}{h_n} = f'(a)$$
$$\lim_{n\to\infty} \frac{f(a + h_n) - f(a)}{h_n} = \lim_{n\to\infty} f'(\xi_n) = \lim_{x \to a} f'(x) = z$$

For the arbitrary zero-sequence, we really need to consider it arbitrary (otherwise we just show it for the one sequence). Consider this counterexample:

$$f(x) = \begin{cases} 0 & x = \frac{1}{n} \text{ for } n \in \mathbb{N} \\ 1 & \text{else} \end{cases}$$

10.1 Alternative approach

Application of “Schrankensatz”.

$$\exists \lim f'(x) = \alpha$$

Hence for arbitrary $\varepsilon > 0 : \exists \delta > 0 \forall x \in (a-\delta, a+\delta) \setminus \{a\} : |f'(x) - \alpha| < \varepsilon$. Hence $\alpha - \varepsilon < f'(x) < \alpha + \varepsilon$.

- $$\forall x \in (a, a+\delta) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$
- $$\forall x \in (a-\delta, a) : \alpha - \varepsilon \leq \frac{f(x) - f(a)}{x - a} \leq \alpha + \varepsilon$$

$$\Rightarrow \forall x \in (a-\delta, a+\delta) \setminus \{a\} : \left| \frac{f(x) - f(a)}{x - a} - \alpha \right| \leq \varepsilon$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha$$

10.2 Second alternative approach

$$\lim_{f(a+h)-f(a)} h$$

If I know f is continuous, then $f(a+h) \to f(a)$. So,

$$\text{“}\frac{0}{0}\text{”}$$

$$\lim_{h \to 0} \frac{f'(a+h) - 0}{1} = \lim_{h \to 0} f'(a+h) = \lim_{x \to a} f'(x)$$

11 Exercise 9

Exercise 18. Let $f : [a, b] \to \mathbb{R}$, $a < b$, differentiable with $f(a) > 0$, $f'(a) > 0$ and $f(b) = 0$. Prove that there exists $\xi \in (a, b) : f'(\xi) = 0$.

First, we want to show that $f'(a) > 0 \Rightarrow \exists \delta > 0 \forall x \in (a, a+\delta) : f(x) > f(a)$.

$$\exists \delta > 0 \forall x \in (a, a+\delta) : \frac{f(x) - f(a)}{x - a} > \frac{f'(a)}{2} > 0$$

$$\Rightarrow f(x) - f(a) > \frac{f'(a)}{2}(x - a) > 0$$

Indeed, $f(x)$ satisfies this property.

Secondly, we want to show that,

$$\exists \eta \in (a+\delta, b) : f(a) = f(\eta)$$

$$\exists \xi \in [a, \eta] \forall x_1 \in [a, \eta] : f(\xi) \geq f(x_1)$$

$$\exists \xi \in (a, \eta) : \frac{f(\eta) - f(a)}{\eta - a} = f'(\eta) = 0$$

There might be more than this one ξ, so the ξ between the second and third line might be different. Anyways, we found a ξ with the desired property.

12 Exercise 10

Exercise 19. Determine the pointwise limit of the following function sequences $f_n : [0, \infty) \to \mathbb{R}$ and determine its uniform convergence:

- $f_n(x) = \sqrt[n]{x}$
- $f_n(x) = \frac{1}{1+nx}$
- $f_n(x) = \frac{x}{1+nx}$

12.1 Exercise 10.a

If $x \neq 0$, $\lim_{n\to\infty} \sqrt[n]{x} = 1$.
If $x = 0$, $\lim_{n\to\infty} \sqrt[n]{x} = \lim_{n\to\infty} 0^{\frac{1}{n}} = 0$.

In terms of uniform convergence:

$$\left| \sqrt[n]{x} - 1 \right| < \varepsilon$$

$$\lim_{x\to\infty} \sqrt[n]{x} = \infty$$

Example:

$$\begin{aligned} \left| \sqrt[n]{x} - 1 \right| &< \varepsilon \\ \sqrt[n]{x} - 1 &< \varepsilon \\ \sqrt[n]{x} &< \varepsilon + 1 \\ \sqrt[n]{100} &< \varepsilon + 1 \end{aligned}$$

12.2 Exercise 10.b

$$f_n(x) = \frac{1}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{1}{1+nx} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{1}{1+n\cdot 0} = 1$$

Assume it it continuously convergent. Show that:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists x \in [0, \infty) : n \geq N \land | f_n(x) - f(x) | \geq \varepsilon$$

Does not hold for $\frac{9}{n} \geq x$.

12.3 Exercise 10.c

$$f_n(x) = \frac{x}{1+nx}$$

If $x \neq 0$,

$$\lim_{n\to\infty} \frac{x}{1+nx} = \lim_{n\to\infty} \frac{1}{\frac{1}{x} + n} = 0$$

If $x = 0$,

$$\lim_{n\to\infty} \frac{0}{1+n\cdot 0} = 0$$

$$\left| \frac{x}{1+nx} - 0 \right| < \varepsilon$$

$$\left| \frac{x}{1+nx} \right| < \left| \frac{x}{nx} \right| = \left| \frac{1}{n} \right|$$

Convergence is given. Uniform convergence is not given.

Advice: The simplest approach to show convergence is to show:

$$| f_n(x) - f(x) | \leq a_n \to 0$$

where a_n is independent from x.