Analysis 1 – Practicals

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Contents

1 Exercise 1

Exercise 1. Let p, q and r be statements. Prove the distributive law using the truth table:

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

p	q	r	$(q \vee r)$	$(p \wedge (q \vee r))$	$(p \wedge q)$	$(p \wedge r)$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Therefore the truthtable of both statements is equivalent. Two boolean statements are equivalent iff their truthtable is equivalent.

2 Exercise 2

Exercise 2. Formalize the following colloquial combination of statements p, q and r in propositional calculus. Furthermore always create the negation:

- "Under the assumption, that p or q holds, we conclude that r cannot be true."
- "It's a requirement for r, that p and q hold."
- "p or q holds, but p and q exclude each other"

- "Under the assumption, that p or q holds, we conclude that r cannot be true."

  $$(p \vee q) \rightarrow \neg r$$

  Negation: $(p \vee q) \wedge r$
- "It's a requirement for r, that p and q hold."

  $$r \rightarrow (p \wedge q)$$

  Negation: $r \wedge (\neg p \vee \neg q)$
- "p or q holds, but p and q exclude each other"

  $$(p \vee q) \wedge \neg(p \wedge q)$$

  $$\Leftrightarrow (p \dot{\vee} q) \Leftrightarrow (p \oplus q)$$

  Negation: $p \leftrightarrow q$

3 Exercise 3

Exercise 3. Mister Travelmuch bought a Eurail ticket in August 1980 and has organized a large journey. When moving flats, he list his photo album, he tries to remember, which cities of Paris, Madrid and Rome he visited.

He remembers:

- If he was not in Madrid, then he was in Paris and Rome.
- If he was in Paris, he was not in Madrid and not in Rome.
- If he was not in Paris, he was also not in Rome.

Use appropriate variables for the statements and help Mister Travelmuch determining which cities (or city) he visited in 1980.

Let M, P and R be visits to Madrid, Pairs and Rome respectively. We formalize:

$$\begin{aligned} \neg M &\implies (P \wedge R) \\ P &\implies (\neg M \wedge \neg R) \\ \neg P &\implies \neg R \end{aligned}$$

As far as all three conditions need to be satisfied, we conjoint them:

$$[\neg M \to (P \wedge R)] \wedge [P \to (\neg M \wedge \neg R)] \wedge [\neg P \to \neg R]$$

We apply $(a \to b) \Leftrightarrow (\neg a \vee b)$ to all three statements:

$$[\neg(\neg M) \vee (P \wedge R)] \wedge [\neg P \vee (\neg M \wedge \neg R)] \wedge [\neg(\neg P) \vee \neg R]$$

...and $\neg(\neg A) \Leftrightarrow A$:

$$[M \vee (P \wedge R)] \wedge [\neg P \vee (\neg M \wedge \neg R)] \wedge [P \vee \neg R]$$

...and the distributive law holds:

$$[(M \vee P) \wedge (M \vee R)] \wedge [(\neg P \wedge \neg M) \vee (\neg P \wedge \neg R)] \wedge [P \vee \neg R]$$

We reorder statements:

$$[(M \vee P) \wedge (M \vee R) \wedge (P \vee \neg R)] \wedge [(\neg P \wedge \neg M) \vee (\neg P \wedge \neg R)]$$

...and again the distributive law:

$$[(M \vee P) \wedge (M \vee R) \wedge (P \vee \neg R) \wedge (\neg P \wedge \neg M)] \vee [(M \vee P) \wedge (M \vee R) \wedge (P \vee \neg R) \wedge (\neg P \wedge \neg R)]$$

$$[(M \vee P) \wedge \neg P \wedge \neg M] \vee [(M \vee P) \wedge (M \vee R) \wedge (P \vee \neg R) \wedge (\neg P \wedge \neg R)]$$

The left-hand side cannot be satisfied, but $M \wedge \neg P \wedge \neg R$ holds for the right side. So,

- In 1980, he was in Madrid.
- In 1980, he was not in Paris.
- In 1980, he was not in Rome.

4 Exercise 4

Exercise 4. Let X be a set. Formalize the following colloquial combination of statements $p(x)$, $q(x)$, $r(x)$ and $s(x, y)$ with the help of quantifiers. Also create the negation:

1. "For all elements x of the set X for which $p(x)$ holds, also $q(x)$ or $r(x)$ holds."
2. "For all x in X, there is one y in Y such that $s(x, y)$ holds."
3. "If $p(x)$ is not wrong for all x in X, then $q(y)$ is true for at least one y in Y."

1. "For all elements x of the set X for which $p(x)$ holds, also $q(x)$ or $r(x)$ holds."

$$\forall x \in X : p(x) \to q(x) \vee r(x)$$

$$\text{negation: } \exists x \in X : p(x) \wedge (\neg q(x) \wedge \neg r(x))$$

2. "For all x in X, there is one y in Y such that $s(x, y)$ holds."

$$\forall x \in X \exists y \in Y : s(x, y)$$

$$\text{negation: } \exists x \in X \forall y \in Y : \neg s(x, y)$$

3. "If $p(x)$ is not wrong for all x in X, then $q(y)$ is true for at least one y in Y."

$$(\exists x \in X : p(x)) \to (\exists y \in X : q(y))$$

$$\text{negation: } (\exists x \in X : p(x)) \wedge (\forall y \in X : \neg q(y))$$

5 Exercise 5

Exercise 5. Prove in three ways (direct, indirect, by contradiction):

$$\forall x \in \mathbb{R} : x^3 + 2x > 0 \Rightarrow x > 0$$

Consider ϕ to be given and φ to be our conclusion. Then the three ways of proving work as follows:

Direct proof $\phi \implies \varphi$

Indirect proof $\neg\varphi \implies \neg\phi$
Because $\varphi \vee \neg\phi \Leftrightarrow \neg\phi \vee \varphi \Leftrightarrow \phi \to \varphi$.

Proof by contradiction $(\neg(\phi \implies \varphi) \implies \perp) \implies (\phi \implies \varphi)$
Because $((\phi \to \varphi) \vee \perp) \to (\phi \to \varphi) \Leftrightarrow (\phi \to \varphi) \to (\phi \to \varphi)$.

Direct proof Assume,

$$x(x^2 + 2) > 0$$

This requires that

- both factors are non-zero
- and
  - both factors are negative, or
  - both factors are positive

So,

$$(x \neq 0 \wedge (x^2 + 2) \neq 0) \wedge \left[(x < 0 \wedge (x^2 + 2) < 0) \vee (x > 0 \wedge (x^2 + 2) > 0)\right]$$

As far as a square cannot be negative, $(x^2 + 2) < 0$ does not hold.

$$(x \neq 0 \wedge (x^2 + 2) \neq 0) \wedge \left[(x > 0 \wedge (x^2 + 2) > 0)\right]$$

Therefore it must hold that

$$(x \neq 0) \wedge (x^2 + 2 \neq 0) \wedge (x > 0) \wedge (x^2 + 2 > 0)$$

And so it holds that $x > 0$.

Indirect proof Assume $x \leq 0$. Then $x \cdot x^2 \leq 0$. And also $x \cdot (x^2 + 2) \leq 0$. Which is $x^3 + 2x \leq 0$.

Proof by contradiction Assume $x(x^2+2) > 0 \implies x \leq 0$.

$$\forall x \in \mathbb{R} : x \cdot \underbrace{(x^2+2)}_{\geq 2} > 0 \implies x \leq 0$$

$$\forall x \in \mathbb{R} : \underbrace{x}_{\Rightarrow \geq 0} \cdot \underbrace{(x^2+2)}_{\geq 2} > 0 \implies x \leq 0$$

$$\forall x \in \mathbb{R} : x > 0 \implies x \leq 0$$

$$\lightning$$

$$\Rightarrow \forall x \in \mathbb{R} : x \cdot (x^2+2) > 0 \implies x > 0$$

6 Exercise 6

Exercise 6. Let p, q and r be statements. Show that

- $(p \to q) \iff \neg(p \wedge \neg q)$ "proof by contradiction"
- $[p \to (q \vee r)] \iff [(p \wedge \neg q) \to r]$

6.1 Exercise 6a

$$(p \to q) \iff \neg(p \wedge \neg q)$$
$$(\neg p \vee q) \iff (\neg p \vee q)$$

6.2 Exercise 6b

$$\begin{aligned}(p \to (q \vee r)) &\iff ((p \wedge \neg q) \to r)\\ \neg p \vee (q \vee r) &\iff \neg(p \wedge \neg q) \vee r\\ (\neg p \vee q) \vee r &\iff (\neg p \vee q) \vee r\end{aligned}$$

7 Exercise 7

Exercise 7. Let A, B, C and D be sets. Prove that

- $(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$
- $(A \setminus B) \cap (C \setminus D) = (A \setminus D) \cap (C \setminus B)$
- $B \subseteq A \implies B = A \setminus (A \setminus B)$

7.1 Exercise 7a

$$(A \setminus B) \cap (A \setminus C) = A \setminus (B \cup C)$$

Let a be a variable which is true if the considered element is contained in A. $\neg a$ analogously means not contained. Same for b and c. Then:

$$\begin{aligned}(a \wedge \neg b) \wedge (a \wedge \neg c) &= a \wedge \neg(b \vee c)\\ a \wedge \neg b \wedge a \wedge \neg c &= a \wedge (\neg b \wedge \neg c)\\ a \wedge \neg b \wedge \neg c &= a \wedge \neg b \wedge \neg c\\ \top &= \top\end{aligned}$$

7.2 Exercise 7b

$$
\begin{aligned}
(A \setminus B) \cap (C \setminus D) &= (A \setminus D) \cap (C \setminus B) \\
(a \wedge \neg b) \wedge (c \wedge \neg d) &= (a \wedge \neg d) \wedge (c \wedge \neg b) \\
a \wedge \neg b \wedge c \wedge \neg d &= a \wedge \neg b \wedge c \wedge \neg d \\
\top &= \top
\end{aligned}
$$

7.3 Exercise 7c

$$B \subseteq A \Rightarrow B = A \setminus (A \setminus B)$$

$$
\begin{aligned}
\forall x \in X : (x \in B \to x \in A) &\implies [x \in B \leftrightarrow x \in A \wedge \neg(x \in A \wedge x \notin B))] \\
\forall x \in X : (x \in B \to x \in A) &\implies \left[x \in B \leftrightarrow x \in A \wedge (\underbrace{x \notin A}_{\perp} \vee x \in B)))\right] \\
\forall x \in X : (x \in B \to x \in A) &\implies [x \in B \leftrightarrow x \in A \wedge x \in B] \\
\forall x \in X : (x \in B \to x \in A) &\implies \left[(x \in B \to x \in A \wedge \underbrace{x \in B}_{\top}) \wedge \underbrace{(x \in A \wedge x \in B \to x \in B)}_{\top}\right] \\
\forall x \in X : (x \in B \to x \in A) &\implies (x \in B \to x \in A)
\end{aligned}
$$

8 Exercise 8

Exercise 8. Let X be a set with $X \neq \emptyset$ and $X \neq \{\emptyset\}$. Of which of the following sets is (a) the set X, (b) the set $\{X\}$, element of subset?

$\downarrow S$ op $\rightarrow$	$x \in S$		$X \subseteq S$	
$\{\{X\}, X\}$	✓	2nd argument	✗	impossible to build
X	✗	impossible to build	✓	$X = X$
$\emptyset \cap \{X\} = \emptyset$	✗		✗	unless $X = \emptyset$
$\{X\} \setminus \{\{X\}\} = \{X\}$	✓	1st argument	✗	impossible to build
$\{X\} \cup X$	✓	1st argument	✓	$X = X$
$\{X\} \cup \{\emptyset\}$	✓	1st argument	✗	impossible to build

9 Exercise 9

$(0, \infty)$ is the set $\mathbb{R}_{>0}$

9.1 Exercise 9a

Prove in three ways the following statement:

$$\forall x \in (0, \infty) \forall y \in (0, \infty) : x \neq y \implies \frac{x}{y} + \frac{y}{x} > 2$$

direct proof

$$\begin{aligned}
x &\neq y \\
x - y &\neq 0 \\
(x-y)^2 &\neq 0 \\
(x-y)^2 &> 0 \\
x^2 - 2xy + y^2 &> 0 && x, y \in \mathbb{R}_{>0} \Rightarrow xy > 0 \\
\frac{x^2}{xy} - \frac{2xy}{xy} + \frac{y^2}{xy} &> 0 \\
\frac{x}{y} - 2 + \frac{y}{x} &> 0 \\
\frac{x}{y} + \frac{y}{x} &> 2
\end{aligned}$$

indirect proof

$$\forall x \in (0,\infty) \forall y \in (0,\infty) : \frac{x}{y} + \frac{y}{x} \leq 2 \Rightarrow x = y$$

$$\begin{aligned}
\frac{x^2}{xy} + \frac{y^2}{xy} &\leq 2 \\
x^2 + y^2 &\leq 2xy \\
x^2 - 2xy + y^2 &\leq 0 \\
(x-y)^2 &\leq 0 \\
(x-y)^2 &= 0 \\
x - y &= 0 \\
x &= y
\end{aligned}$$

proof by contradiction

$$\forall x \in (0,\infty) \forall y \in (0,\infty) : x \neq y \Rightarrow \frac{x}{y} + \frac{y}{x} \leq 2$$

$$\begin{aligned}
x - y &\neq 0 \\
x^2 - 2xy + y^2 &\neq 0 \\
x^2 - 2xy + y^2 &> 0 \\
\frac{x^2}{xy} - 2 + \frac{y^2}{xy} &> 0 \\
\underbrace{\frac{x}{y}}_{>0} + \underbrace{\frac{y}{x}}_{>0} &> 2 \\
&\lightning
\end{aligned}$$

9.2 Exercise 9b

Let $z = \frac{x}{y}$ and illustrate the inequality geometrically.

$$\frac{x}{y} + \frac{y}{x} > 2 \Rightarrow z + z^{-1} > 2$$

10 Reminder

If $n < m$, then the *empty sum* $\sum_{k=m}^{n} a_k$ has value 0, and the *empty product* $\prod_{k=m}^{n} a_k$ has value 1.

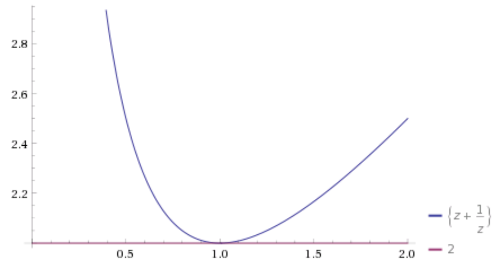

Figure 1: Plot for $z + z^{-1} > 2$ (Exercise 9b)

11 Exercise 10

Exercise 9.

- Provide a concise definition of "n is an even number" and "n is an odd number" using the existence quantifier.
- Prove $\forall n \in \mathbb{Z} : n$ is even $\Leftrightarrow n^2$ is even
  Hint: Prove $\Leftarrow$ using an indirect proof.

11.1 Exercise 10a

$$n \text{ is even} \Rightarrow \exists a \in \mathbb{Z} : n = 2a$$
$$n \text{ is odd} \Rightarrow \exists a \in \mathbb{Z} : n = 2a + 1$$

11.2 Exercise 10b

$$\forall n \in \mathbb{Z} \exists a_1 \in \mathbb{Z} \exists a_2 \in \mathbb{Z} : n = 2a_1 \Leftrightarrow n^2 = 2a_2$$

Direction $\Rightarrow$

$$n = 2a_1 \Rightarrow n^2 = 4a_1^2$$

Let $a_1 = \sqrt{\frac{a_2}{2}}$.

$$n^2 = 4\left(\sqrt{\frac{a_2}{2}}\right)^2 \Rightarrow n^2 = 2a_2$$

Such an a_2 always exists. Proof finished.

Direction $\Leftarrow$

$$n^2 \neq 2a_2 \Rightarrow n \neq 2a_1$$

Taking the square root preserves the parity of the value[1].

$$n = \sqrt{2a_2 + 1}$$

So $\sqrt{2a_2 + 1}$ gives an odd number. But this structure cannot match $2a_1$, which represents an even number. This shows a contradiction and $n \neq 2a_1$ holds.

[1]Because an even number times an even number yields an even number. An odd number times an odd number yields an odd number.

12 Exercise 11

Exercise 10. For the following statement give

1. an indirect proof
2. a proof using Exercise 6b

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R} : xy \notin \mathbb{Q} \Rightarrow x \notin \mathbb{Q} \vee y \notin \mathbb{Q}$$

12.1 Exercise 11.1

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R} : x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow xy \in \mathbb{Q}$$

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R} \exists p_0, p_1 \in \mathbb{R} \exists q_0, q_1 \in \mathbb{R} \setminus \{0\} :$$

$$\left(x = \frac{p_0}{q_0}\right) \wedge \left(y = \frac{p_1}{q_1}\right) \Rightarrow \left(\exists p_2 \in \mathbb{R} \exists q_2 \in \mathbb{R} \setminus \{0\} : xy = \frac{p_2}{q_2}\right)$$

$$xy = \frac{\overbrace{p_0 p_1}^{\in \mathbb{R}}}{\underbrace{q_0 q_1}_{\in \mathbb{R} \setminus \{0\}}} \Rightarrow xy = \frac{p_2}{q_2} \text{ for } p_2 = p_0 \cdot p_1 \text{ and } q_2 = q_0 \cdot q_1$$

12.2 Exercise 11.2

$$(p \Rightarrow (q \vee r)) \Leftrightarrow ((p \wedge \neg q) \Rightarrow r)$$

$$(xy \notin \mathbb{Q} \Rightarrow (x \notin \mathbb{Q} \vee y \notin \mathbb{Q})) \Leftrightarrow ((xy \notin \mathbb{Q} \wedge x \in \mathbb{Q}) \Rightarrow y \notin \mathbb{Q})$$

$$\forall x, y \in \mathbb{R} : \begin{pmatrix} \left(\nexists p_2 \in \mathbb{R} \nexists q_2 \in \mathbb{R} \setminus \{0\} : xy = \frac{p_2}{q_2}\right) \wedge \\ \left(\exists p_0 \in \mathbb{R} \exists q_0 \in \mathbb{R} \setminus \{0\} : x = \frac{p_0}{q_0}\right) \end{pmatrix} \Rightarrow \left(\nexists p_1 \in \mathbb{R} \nexists q_1 \in \mathbb{R} \setminus \{0\} : y = \frac{p_1}{q_1}\right)$$

Recognize $p_2 = p_0 \cdot p_1$ and $q_2 = q_0 \cdot q_1$.

Therefore the conjunction yields the $y \notin \mathbb{Q}$ because $x \in \mathbb{Q}$.

$$\forall x \in \mathbb{R} \forall y \in \mathbb{R} : \left(\nexists p_1 \in \mathbb{R} \exists q_1 \in \mathbb{R} \setminus \{0\} : y = \frac{p_1}{q_1}\right) \Rightarrow \left(\nexists p_1 \in \mathbb{R} \exists q_1 \in \mathbb{R} \setminus \{0\} : y = \frac{p_1}{q_1}\right)$$

This statement is true. The proof is complete.

13 Exercise 18

Exercise 11. Let $n \in \mathbb{N}_+$. Show that

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k}\right) = \frac{1}{n}.$$

Induction base $n = 1$

$$\prod_{k=2}^{1} \cdots = 1 = \frac{1}{1} \qquad \checkmark$$

Induction step $n \to n+1$

$$\prod_{k=2}^{n+1}\left(1-\frac{1}{k}\right) = \frac{1}{n+1}$$
$$\prod_{k=2}^{n}\left(1-\frac{1}{k}\right)\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$
$$\frac{1}{n}\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$$
$$\frac{1}{n}\cdot\frac{n+1-1}{n+1} = \frac{1}{n+1}$$
$$\frac{n}{n} = 1 \qquad \checkmark$$

Actually, can be rewritten as

$$\prod_{k=2}^{n}\left(\frac{k-1}{k}\right)$$
$$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{n-1}{n}$$
$$= \frac{1}{n}$$

So this is the multiplication equivalent of telescoping sums.

14 Exercise 19

Exercise 12. X and Y are non-empty sets and $f : X \to Y$ is a mapping. Furthermore let $A \subseteq X$ and $B \subseteq Y$.

1. Prove that $A \subseteq f^{-1}(f(A))$ and $B \supseteq f(f^{-1}(B))$
2. Show (by providing counterexamples) that in the inclusions of (1) no equivalence is given.

14.1 Exercise 19.1

Show that,

$$a \in A \Rightarrow a \in f^{-1}(f(A))$$

So we take a and map it to the codomain:

$$f(a) \in f(A)$$

We denote the result as y:

$$y := f(a)$$

Because

$$f^{-1}(x) = \{x \in A \mid f(x) \in B\}$$

we know that a originates from:

$$a \in f^{-1}(f(A))$$

It is very important here to distinguish between *domain/codomain* and *function/inverse function.* Because an inverse function implies that the corresponding function is injective. Assuming this fact, the exercise is immediate. But we are talking about domains and co-domains here.

As second exercise we need to show that,

$$B \supseteq f\left(f^{-1}(B)\right)$$

We need the definition that,

$$f^{-1}(B) = \{x' \in X \mid f(x') \in B\}$$

$$y' \in f(f^{-1}(B))$$

Does $y' \in B$ hold? Yes, because ...

$$\begin{aligned} y' \in f\left(f^{-1}(B)\right) &\Rightarrow \exists x' \in f^{-1}(B) \\ &\Rightarrow y' \in B \end{aligned}$$

14.2 Exercise 19.2

Show that,

$$\exists f : A \subsetneq f^{-1}(f(A))$$

We use a surjective, but not injective function.

$$\begin{aligned} f : \{1,2\} &\to \{a\} \\ 1 &\mapsto a \\ 2 &\mapsto a \end{aligned}$$

$$\begin{aligned} A &= \{1\} \\ f(A) &= \{a\} \\ f^{-1}(f(A)) &= \{1,2\} \end{aligned}$$

Show that,

$$\exists f : A \subsetneq f(f^{-1}(A))$$

We use an injective, but not surjective function.

$$\begin{aligned} f : \{1\} &\to \{a,b\} \\ 1 &\mapsto a \end{aligned}$$

$$\begin{aligned} B &= \{b\} \\ f^{-1}(B) &= \emptyset \\ f(f^{-1}(B)) &= \emptyset \end{aligned}$$

15 Exercise 20

Exercise 13. Prove the following variant of Bernoulli's inequality: For $x \in \mathbb{R}$ with $0 < x < 1$ and $n \in \mathbb{N}_+$ it holds that

$$(1-x)^n < \frac{1}{1+nx}.$$

$$
\begin{aligned}
(1+x)^n &\geq 1+nx \\
\frac{(1+x)^n}{1+nx} &\geq \frac{1+nx}{1+nx} \\
\frac{(1+x)^n}{1+nx} &\geq 1 \\
\frac{(1-x)^n(1+x)^n}{(1-x)^n(1+nx)} &\geq 1 \\
\frac{(1-x)^n(1+x)^n}{(1+nx)} &\geq (1-x)^n \\
\frac{((1-x)(1+x))^n}{(1+nx)} &\geq (1-x)^n \\
\frac{\left(1-x^2\right)^n}{(1+nx)} &\geq (1-x)^n \\
\frac{\overbrace{\left(1-x^2\right)^n}^{\text{interval } (0,1)}}{(1+nx)} &\geq (1-x)^n \\
\frac{1}{(1+nx)} &> (1-x)^n
\end{aligned}
$$

16 Exercise 21

Exercise 14. X and Y are nonempty sets and $f : X \to Y$ is a mapping.

a) Show that the following holds: For all $A, B \subseteq X$

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

b) Show that the following statements are equivalent:

1. f is injective.
2. For all $A, B \subseteq X$ it holds that $f(A \cap B) \supseteq f(A) \cap f(B)$
3. For all $A, B \subseteq X$ it holds that $f(A \cap B) = f(A) \cap f(B)$

16.1 Exercise 21a

Let $C = A \cap B$. Case distinction:

$A = B = C$

$$
\begin{aligned}
f(A \cap B) &= \{f(x) \mid x \in A\} \\
f(A) \cap f(B) &= f(A) \\
&= \{f(x) \mid x \in A\}
\end{aligned}
$$

$C = A \dot{\vee} C = B$ wlog. $C = A$.

$$
\begin{aligned}
f(A \cap B) &= f(A) \\
&= \{f(x) \mid x \in A\} \\
f(A) \cap f(B) &= \{f(x) \mid x \in A\} \cap \{f(x) \mid x \in B\} \\
&= \{f(x) \mid x \in A \wedge x \notin (B \setminus A)\} \\
&= \{f(x) \mid x \in A\}
\end{aligned}
$$

$C = \emptyset$

$$\begin{aligned} f(A \cap B) &= f(\emptyset) \\ &= \emptyset \\ f(A) \cap f(B) &= \{f(x) \mid x \in A\} \cap \{f(x) \mid x \in B\} \end{aligned}$$

So,

$$C \neq \emptyset \Rightarrow f(A \cap B) = f(A) \cap f(B)$$

But if $C = 0$, we get zero values on the left-hand side and zero to $|A| + |B|$ values on the right-hand side. So,

$$C = \emptyset \Rightarrow f(A \cap B) \subseteq f(A) \cap f(B)$$

16.2 Exercise 21b

We prove 3 with 1:

Let $C = A \cap B$. f is injective, meaning

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \cap f(x_2)$$

Case distinction:

$A = B = C$

$$\begin{aligned} f(A \cap B) &= \{f(x) \mid x \in A\} \\ f(A) \cap f(B) &= f(A) \\ &= \{f(x) \mid x \in A\} \end{aligned}$$

$C = A \dot{\vee} C = B$

wlog. $C = A$ meaning $A \subsetneq B$

$$\begin{aligned} f(A \cap B) &= f(A) \\ &= \{f(x) \mid x \in A\} \\ f(A) \cap f(B) &= \{f(x) \mid x \in A\} \cap \{f(x) \mid x \in B\} \\ &= \{f(x) \mid x \in A \wedge x \notin (B \setminus A)\} \\ &= \{f(x) \mid x \in A\} \end{aligned}$$

$C = \emptyset$

$$\begin{aligned} f(A \cap B) &= f(\emptyset) \\ &= \emptyset \\ f(A) \cap f(B) &= \emptyset \end{aligned}$$

Every element in A is distinct from values in B. Therefore $\forall x_1 \in A, x_2 \in B : f(x_1) \neq f(x_2)$ because of injectivity. The intersection of all $f(x_i)$ is therefore empty.

17 Exercise 22

Exercise 15. Let $n \in \mathbb{N}$. Use the following idea to derive an equation for the sum of powers of three.

$$\sum_{k=1}^{n} \left(k^4 - (k-1)^4\right)$$

This sum can be written in two different ways:

- As telescoping sum (the initial and trailing value will be left)

- (First resolve the parentheses.) As combination of sums of the third, second, first and zero-th power. With that (and known equations for sums of smaller powers) we can compute $\sum_{k=1}^{n} k^3$.

We look at the telescoping sum:

$$\begin{aligned}\sum_{k=1}^{n}(k^4-(k-1)^4) &= \left(1^4-(1-1)^4\right)+\left(2^4-(2-1)^4\right)+\left(3^4-(3-1)^4\right)\\ &\quad+\cdots+\left((n-1)^4-((n-1)-1)^4\right)+\left(n^4-(n-1)^4\right)\\ &= -0^4+n^4\\ &= n^4\end{aligned}$$

Then we use the combination of sums of lower powers.

$$\begin{aligned}\sum_{k=1}^{n}(k^4-(k-1)^4) &= \sum_{k=1}^{n}(k^4-(k^4-4k^3+6k^2-4k+1))\\ &= \sum_{k=1}^{n}(k^4-k^4+4k^3-6k^2+4k-1)\\ &= \sum_{k=1}^{n}(4k^3-6k^2+4k-1)\\ &= \sum_{k=1}^{n}4k^3-\sum_{k=1}^{n}6k^2+\sum_{k=1}^{n}4k-\sum_{k=1}^{n}1\\ &= \sum_{k=1}^{n}4k^3-6\frac{2n^3+3n^2+n}{6}+4\frac{n(n+1)}{2}-n\\ &= \sum_{k=1}^{n}4k^3-2n^3-n^2\end{aligned}$$

Therefore,

$$n^4 = \sum_{k=1}^{n}4k^3-2n^3-n^2$$

$$\sum_{k=1}^{n}4k^3 = n^4+2n^3+n^2$$

$$\sum_{k=1}^{n}k^3 = \frac{n^4+2n^3+n^2}{4}$$

18 Exercise 23

Exercise 16. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

and if $n \geq 1$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

Binomial theorem with $x = 1, y = 1$:

$$\sum_{k=0}^{n} \binom{n}{k} 1^n 1^{n-k} = (1+1)^n$$
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

If $n \geq 1$,

$$\begin{aligned}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} &= \sum_{k=0}^{n} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n-1}{k} + \sum_{k=0}^{n} (-1)^k \binom{n-1}{k-1} \\
&= (-1)^n \binom{n-1}{n} + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} + \sum_{k=0}^{n} (-1)^k \binom{n-1}{k-1} \\
&= \underbrace{(-1)^n \binom{n-1}{n}}_{0} + \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} + \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} + \underbrace{(-1)^0 \binom{n-1}{-1}}_{0} \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} - (-1) \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} - \sum_{k=0}^{n-1} (-1)^{k+2} \binom{n-1}{k} \\
&= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} - \sum_{k=0}^{n-1} (-1)^{k} \binom{n-1}{k} \\
&= 0
\end{aligned}$$

19 Exercise 24

Exercise 17. Let $k, n \in \mathbb{N}_+$ with $k \leq n$. Determine the number of vectors of length k with pairwise different entries from $M_n = \{1, 2, \dots, n\}$.

This question is covered by the field of combinatorics.

$$(a_0, a_1, a_2, \dots) \neq (a_0, a_2, a_1, \dots)$$

The order of elements is relevant. Therefore a variation, not combination, is given. The number of combinations without repetitions would be given by the binomial coefficient $\binom{n}{k}$ (the number of ways to choose k of n elements disregarding their order). For variations the formula n^k holds to select k elements among n arbitrarily often (hence with repetition).

We model the given situation as

- "variation without repetition"
- i.e. "k-permutations of n"
- i.e. the k-th falling factorial power $n^{\underline{k}}$ of n

The formula is given by,

$$P_k^n = \frac{n!}{(n-k)!}$$

We can estimate it in the following way: Consider a combination without repetition represented by the formula $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

So because we have a variation, not combination, the order of elements is relevant. Therefore given some combination, there are $k!$ possible arrangements. Given the vector (and also combination) $(1,2,3)$ there are $k!$ possible arrangements (variations) $\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$. Indeed it holds that

$$\frac{3!}{(3-3)!} = \frac{6}{1} = 6$$

This argument explains why $k!$ in the denominator is omitted for variations w/o repetitions.

combinations	variations					
(123)	(123)	(132)	(213)	(231)	(312)	(321)
(124)	(124)	(142)	(214)	(241)	(412)	(421)
(134)	(134)	(143)	(314)	(341)	(413)	(431)
(234)	(234)	(243)	(324)	(342)	(423)	(432)

Table 1: Combinations and variations for $n = 4$ of $k = 3$

20 Exercise 25

Exercise 18. Let K be a field and $a, b, c \in K$. Show (using the field axioms):

(a) $-(-a) = a$.

(b) $(-a)(-b) = ab$.

(c) $a + b = a + c \Rightarrow b = c$.

(d) From $a \neq 0$ and $ab = ac$ follows $b = c$.

(e) Is $a \neq 0$, then there is exactly one $x \in K$ with $ax + b = c$.

The field axioms are defined as follows:

A1 $\forall a, b \in K : a + b = b + a$

A2 $\forall a, b, c \in K : (a + b) + c = a + (b + c)$

A3 $\exists 0 \in K \forall a \in K : a + 0 = a$

A4 $\forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$

M1 $\forall a, b \in K : a \cdot b = b \cdot a$

M2 $\forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M3 $\exists 1 \in K : a \cdot 1 = a \forall a \in K$ (neutral element)

M4 $\forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$

D $\forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$

20.1 Exercise 25.a

$$\begin{aligned}
A4 &\Rightarrow \forall a \in K \exists -a : a + (-a) = 0 \\
\text{equivalence} &\Rightarrow a + (-a) - (-a) = 0 - (-a) \\
A1 &\Rightarrow a + (-a) - (-a) = -(-a) + 0 \\
A3 &\Rightarrow a + (-a) - (-a) = -(-a) \\
A4 &\Rightarrow a + 0 = -(-a) \\
A3 &\Rightarrow a = -(-a)
\end{aligned}$$

20.2 Exercise 25.b

We have proven in the lecture: **M5**: $-a = (-1) \cdot a$

First, we show **M7**

$$\begin{aligned}
&= a \cdot (-a) \\
M5 &\Rightarrow a \cdot (-1) \cdot a \\
M1 &\Rightarrow (-1) \cdot a \cdot a \\
&\Rightarrow -(a \cdot a)
\end{aligned}$$

Secondly, we show (actually we have already shown that in the lecture) **M6**

$$\begin{aligned}
D &\Rightarrow \forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c \\
&\quad [\text{we choose} \quad a := a, \quad b := a, \quad c := (-a)] \\
&\Rightarrow a \cdot (a + (-a)) = a \cdot a + a \cdot (-a) \\
A3 &\Rightarrow a \cdot 0 = a \cdot a + a \cdot (-a) \\
\text{previous theorem} &\Rightarrow a \cdot 0 = a \cdot a + (-(a \cdot a)) \\
A4 &\Rightarrow a \cdot 0 = 0
\end{aligned}$$

Finally, we show

$$\begin{aligned}
\text{previous theorem} &\Rightarrow (-a)\cdot 0 = 0\\
A4 &\Rightarrow (-a)\cdot(b+(-b)) = 0\\
D &\Rightarrow (-a)\cdot b + (-a)(-b) = 0\\
\text{equivalence} &\Rightarrow ab + (-a)b + (-a)(-b) = ab + 0\\
M1 &\Rightarrow ab + (-a)b + (-a)(-b) = 0 + ab\\
A3 &\Rightarrow ab + (-a)b + (-a)(-b) = ab\\
M6 &\Rightarrow ab - ab + (-a)(-b) = ab\\
A4 &\Rightarrow 0 + (-a)(-b) = ab\\
A3 &\Rightarrow (-a)(-b) = ab
\end{aligned}$$

20.3 Exercise 25.c

$$\begin{aligned}
& a + b = a + c\\
\text{equivalence} &\Rightarrow a + b + (-a) = a + c + (-a)\\
A1 &\Rightarrow (a + (-a)) + b = (a + (-a)) + c\\
A4 &\Rightarrow 0 + b = 0 + c\\
A3 &\Rightarrow b = c
\end{aligned}$$

20.4 Exercise 25.d

$$\begin{aligned}
& a \neq 0 \wedge ab = ac\\
\text{equivalence} &\Rightarrow aba^{-1} = aca^{-1}\\
M1 &\Rightarrow aa^{-1}b = aa^{-1}c\\
M4 &\Rightarrow 1b = 1c\\
M3 &\Rightarrow b = c
\end{aligned}$$

20.5 Exercise 25.e

Proof by contradiction. Assume $x_1, x_2 \in K$ with $x_1 \neq x_2$ then $\exists r \in K$:

$$ax_1 = r \qquad ax_2 = r$$

$$ax_1 = ax_2$$

$$\begin{aligned}
&\Rightarrow ax_1 = ax_2\\
\text{equivalence} &\Rightarrow a^{-1}ax_1 = a^{-1}ax_2\\
M4 &\Rightarrow 1x_1 = 1x_2\\
M3 &\Rightarrow x_1 = x_2
\end{aligned}$$

This is a contradiction to our assumption $x_1 \neq x_2$. Therefore x is distinct.

21 Exercise 26

Exercise 19. Let $n \in \mathbb{N}_+$. Prove that

$$\sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=1}^{n} \binom{2n}{2k-1} = 2^{2n-1}.$$

21.1 Exercise 26.1: $\sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=1}^{n} \binom{2n}{2k-1}$ - approach 1

Proof.

$$\begin{aligned}
\sum_{k=0}^{n} \binom{2n}{2k} &= \sum_{k=1}^{n-1} \binom{2n}{2k} + 1 + 1 \\
&= \sum_{k=1}^{n-1} \left[\binom{2n-1}{2k} + \binom{2n-1}{2k-1} \right] + 1 + 1 \\
&= \sum_{k=1}^{n-1} \binom{2n-1}{2k} + \sum_{k=1}^{n-1} \binom{2n-1}{2k-1} + 1 + 1 \\
&= \sum_{k=2}^{n} \binom{2n-1}{2(k-1)} + \sum_{k=1}^{n-1} \binom{2n-1}{2k-1} + 1 + 1 \\
&= \sum_{k=1}^{n} \binom{2n-1}{2k-2} + \sum_{k=1}^{n-1} \binom{2n-1}{2k-1} + 1 \\
&= \sum_{k=1}^{n-1} \binom{2n-1}{2k-2} + \sum_{k=1}^{n-1} \binom{2n-1}{2k-1} + \binom{2n-1}{2n-2} + 1 \\
&= \sum_{k=1}^{n-1} \left[\binom{2n-1}{2k-2} + \binom{2n-1}{2k-1} \right] + \binom{2n-1}{2n-2} + 1 \\
&= \sum_{k=1}^{n-1} \binom{2n}{2k-1} + \left[1 + \binom{2n-1}{2n-2} \right] \\
&= \sum_{k=1}^{n-1} \binom{2n}{2k-1} + \left[\binom{2n-1}{2n-1} + \binom{2n-1}{2n-2} \right] \\
&= \sum_{k=1}^{n-1} \binom{2n}{2k-1} + \binom{2n}{2n-1} \\
&= \sum_{k=1}^{n} \binom{2n}{2k-1}
\end{aligned}$$

□

21.2 Exercise 26.1: $\sum_{k=0}^{n} \binom{2n}{2k} = \sum_{k=1}^{n} \binom{2n}{2k-1}$ - approach 2

Proof. Idea: Consider $(1-1)^{2n}$ and split even/odd ks.

$$\begin{aligned}
(1-1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k 1^{2n-k} && \text{[binomial theorem]} \\
0 &= \sum_{k=0}^{n} \binom{2n}{2k} (-1)^{2k} + \sum_{k=1}^{n} \binom{2n}{2k-1} (-1)^{2k-1} \\
&= \sum_{k=0}^{n} \binom{2n}{2k} + \sum_{k=1}^{n} \binom{2n}{2k-1} (-1) && [(-1)^{\text{even}} \text{ is } 1,\ (-1)^{\text{odd}} \text{ is } -1] \\
&= \sum_{k=0}^{n} \binom{2n}{2k} - \sum_{k=1}^{n} \binom{2n}{2k-1} \\
\sum_{k=1}^{n} \binom{2n}{2k-1} &= \sum_{k=0}^{n} \binom{2n}{2k}
\end{aligned}$$

□

21.3 Exercise 26.2: $\sum_{k=0}^{n} \binom{2n}{2k}$

Proof. Idea: Consider $(1+1)^{2n}$ and odd+even provides the factor 2 we need to divide with.

$$\begin{aligned}
(1+1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} 1^k 1^{2n-k} && \text{[binomial theorem]} \\
2^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} \\
&= \sum_{k=0}^{n} \binom{2n}{2k} + \sum_{k=1}^{n} \binom{2n}{2k-1} && \text{[split even and odd]} \\
&= \sum_{k=0}^{n} \binom{2n}{2k} + \sum_{k=0}^{n} \binom{2n}{2k} && \text{[from previous result]} \\
&= 2 \sum_{k=0}^{n} \binom{2n}{2k} \\
\frac{2^{2n}}{2} &= \sum_{k=0}^{n} \binom{2n}{2k} \\
2^{2n-1} &= \sum_{k=0}^{n} \binom{2n}{2k}
\end{aligned}$$

□

22 Exercise 27

> **Exercise 20.** Let $x \in \mathbb{R} \setminus \{0\}$. Show: Let $x + \frac{1}{x} \in \mathbb{Z}$, then $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \in \mathbb{N}$ (Remark: Consider $(x + \frac{1}{x})^n$.)

So we need to show that,

$$x \in \mathbb{R} \setminus \{0\} : \forall n \in \mathbb{N} : x + \frac{1}{x} \in \mathbb{Z} \implies x^n + \frac{1}{x^n} \in \mathbb{Z}$$

First we need to cover some fundamentals,

- $a, b \in \mathbb{Z} \implies (a + b) \in \mathbb{Z}$
- $a, b \in \mathbb{Z} \implies (a \cdot b) \in \mathbb{Z} \implies \forall n \in \mathbb{N} : a^n \in \mathbb{Z}$

Proof. **IB**: $n = 0$

$$\begin{aligned}\forall x \in \mathbb{R} \setminus \{0\} : x + \frac{1}{x} \in \mathbb{Z} \\ &\Rightarrow x = 1 : 1 + \frac{1}{1} \in \mathbb{Z} \\ &\Rightarrow \forall x \in \mathbb{R} \setminus \{0\} : x^0 + \frac{1}{x^0} \in \mathbb{Z} \\ &\Rightarrow \forall x \in \mathbb{R} \setminus \{0\} : n = 0 : x^n + \frac{1}{x^n} \in \mathbb{Z}\end{aligned}$$

IB: $n = 1$

$$\begin{aligned}\forall x \in \mathbb{R} \setminus \{0\} : x + \frac{1}{x} \in \mathbb{Z} \\ &\Rightarrow \forall x \in \mathbb{R} \setminus \{0\} : x^1 + \frac{1}{x^1} \in \mathbb{Z} \\ &\Rightarrow \forall x \in \mathbb{R} \setminus \{0\} : n = 1 : x^n + \frac{1}{x^n} \in \mathbb{Z}\end{aligned}$$

IS: $n \to n + 1$ Okay, how does the induction step for an implication look like?

$$\begin{aligned}((a \to b) \to (a \to d)) &= \neg(\neg a \lor b) \lor (\neg a \lor d) \\ &= (a \land \neg b) \lor \neg a \lor d \\ &= ((a \lor \neg a) \land (\neg a \lor \neg b)) \lor d \\ &= (\neg a \lor \neg b) \lor d \\ &= (a \land b) \to d\end{aligned}$$

Therefore we can assume

$$\left(x + \frac{1}{x} \in \mathbb{Z}\right) \land \left(x^n + \frac{1}{x^n} \in \mathbb{Z}\right)$$

and need to prove that this follows:

$$x^{n+1} + \frac{1}{x^{n+1}} \in \mathbb{Z}$$

$$\begin{aligned}\left(x^n + \frac{1}{x^n} \in \mathbb{Z}\right) &= \left(x^n + \frac{1}{x^n}\right)\left(x + \frac{1}{x}\right) \in \mathbb{Z} \\ &= \left(x^n \cdot x + \frac{1}{x^n} \cdot x + x^n \cdot \frac{1}{x} + \frac{1}{x^n} \cdot \frac{1}{x}\right) \in \mathbb{Z} \\ &= \left(x^{n+1} + x^{-n+1} + x^{n-1} + x^{-n-1}\right) \in \mathbb{Z} \\ &= \left(x^{n+1} + x^{-n-1} + x^{n-1} + x^{-n+1}\right) \in \mathbb{Z} \\ &= \left(x^{n+1} + \frac{1}{x^{n+1}}\right) + \underbrace{\left(x^{n-1} + \frac{1}{x^{n-1}}\right)}_{\substack{\in\mathbb{Z} \text{ because of induction hypothesis} \\ \text{and we have a 2-step induction}}} \in \mathbb{Z} \\ &= \left(x^{n+1} + \frac{1}{x^{n+1}}\right) \in \mathbb{Z}\end{aligned}$$

$\square$

23 Exercise 28

Exercise 21. Let K be an ordered field and $a, b \in K_+$. Show:

$$a < b \Rightarrow a^2 < b^2$$

Especially the mapping $f : K_+ \cup \{0\} \to K_+ \cup \{0\}, a \mapsto a^2$ is injective.

We already know,

U1 $\forall a, b \in K : a < b \Leftrightarrow b > a$

U2 $\forall a \in K : a^2 = a \cdot a$

U3 $\forall c \in K_+ : a > b \Rightarrow ac > bc$

M1 $\forall a, b \in K : a \cdot b = b \cdot a$

Proof.

$$\begin{aligned}
a < b : \quad & U1 \Rightarrow b > a \\
& U1 \Rightarrow b \cdot a > a \cdot a && \text{[yes, } a \text{ originates from } K_+\text{]} \\
& U2 \Rightarrow b \cdot a > a^2 \\
b > a : \quad & U1 \Rightarrow b \cdot b > a \cdot b && \text{[yes, } b \text{ originates from } K_+\text{]} \\
& U2 \Rightarrow b^2 > a \cdot b \\
& M1 \Rightarrow b^2 > b \cdot a \\
b^2 > b \cdot a \wedge b \cdot a > a^2 : \quad & U3 \Rightarrow b^2 > a^2 \\
& U1 \Rightarrow a^2 < b^2 \\
& \Rightarrow \forall a, b \in K_+ : a < b \Rightarrow a^2 < b^2
\end{aligned}$$

□

Injectivity:

$$\forall a_1, a_2 \in K_+ \cup \{0\} : a_1 \neq a_2 \Rightarrow a_1^2 \neq a_2^2$$

Proof. First we consider $a = 0$. In this case, $a = 0$ and $a^2 = a \cdot a = 0 \cdot 0 = 0$ according to the axiom $0 \cdot a = 0$ we have proven in the lecture. So for $a = 0$, there is only one a for which the square is zero, which is 0.

We can proceed in K_+. Proof by contradiction:

$$\exists a_1, a_2 \in K_+ : a_1 \neq a_2 \Rightarrow a_1^2 = a_2^2$$

$$a_1 \neq a_2 \Leftrightarrow a_1 < a_2 \dot{\vee} a_1 > a_2$$

because a_1 and a_2 are elements of an ordered field.

Case 1: $a_1 < a_2$

$$a_1 < a_2 \Rightarrow a_1^2 < a_2^2$$

Case 2: $a_1 > a_2$

$$a_1 > a_2 \Rightarrow a_1^2 > a_2^2$$

Therefore either $a_1^2 < a_2^2$ or $a_1^2 > a_2^2$. So

$$a_1^2 \neq a_2^2$$

This contradicts and therefore $\nexists a_1, a_2 \in K_+ : a_1 \neq a_2 \Rightarrow a_1^2 = a_2^2$ or because we covered $a = 0$,

$$\nexists a_1, a_2 \in K_+ \cup \{0\} : a_1 \neq a_2 \Rightarrow a_1^2 = a_2^2$$

□

24 Exercise 29

Exercise 22. Let K be an ordered field and $a, b \in K$. Show:

$$|a+b| = |a| + |b| \Leftrightarrow ab \geq 0$$

Triangular inequality:

$$\forall a, b \in K : |a+b| \leq |a| + |b|$$

Absolute values are defined with,

$$|a| = \begin{cases} a & a \in K_+ \\ 0 & a = 0 \\ -a & a \in K_- \end{cases}$$

Proof. Case distinction:

$a = 0, b = 0$

$$\begin{aligned} |a+b| &\leq |a| + |b| \\ |a+0| &\leq |a| + |0| \\ A3 \Rightarrow |a| &\leq |a| + 0 \\ A3 \Rightarrow |a| &= |a| \end{aligned}$$

$a > 0, b = 0$

$$\begin{aligned} |a+b| &\leq |a| + |b| \\ |a+0| &\leq |a| + |0| \\ A3 \Rightarrow |a| &\leq |a| + 0 \\ A3 \Rightarrow |a| &= |a| \end{aligned}$$

$a = 0, b > 0$

$$\begin{aligned} |a+b| &\leq |a| + |b| \\ |0+b| &\leq |0| + |b| \\ A3 \Rightarrow |b| &\leq 0 + |b| \\ A3 \Rightarrow |b| &= |b| \end{aligned}$$

$a > 0, b > 0$

$$\begin{aligned} \underbrace{|a+b|}_{\in K_+} &\leq \underbrace{|a|}_{\in K_+} + \underbrace{|b|}_{\in K_+} \\ (a+b) &\leq (a) + (b) \\ A2 \Rightarrow a+b &\leq a+b \\ a+b &= a+b \end{aligned}$$

$\square$

25 Exercise 33

$$[a_n, b_n], [c_n, d_n], a_n \leq \alpha \leq b_n, c_n \leq \gamma \leq d_n$$
$$\forall \varepsilon > 0 \exists N(\varepsilon) : |a_n - b_n| < \varepsilon \forall n \geq N(\varepsilon)$$
$$\left[\frac{1}{b_n}, \frac{1}{a_n}\right] \rightarrow \frac{1}{b_n} \leq \frac{1}{b_{n+1}} \leq \frac{1}{\alpha} \leq \frac{1}{a_{n+1}} \leq \frac{1}{a_n}$$
$$\left|\frac{1}{b_n} - \frac{1}{a_n}\right| = \frac{a_n - b_n}{a_n b_n} = \frac{|a_n - b_n|}{|a_n||b_n|} \leq \frac{\varepsilon}{|a_1|\alpha} = \varepsilon'$$

Important: our approximation $a_n \geq a_1 > 0$ and $b_n \geq \alpha$ is independent of n!

$$\forall \varepsilon' > 0 \exists N(\varepsilon') : \left|\frac{1}{b_n} - \frac{1}{a_n}\right| < \varepsilon'$$

$$|a_n c_n - b_n d_n| = |a_n c_n - \alpha c_n + \alpha c_n - \alpha c_n + \alpha\gamma - b_n\gamma + b_n\gamma - b_n d_n|$$
$$\leq \underbrace{|a_n - \alpha|}_{<\varepsilon} \underbrace{|c_n|}_{\leq\gamma} + |\alpha| \underbrace{|c_n - \gamma|}_{<\varepsilon} + \underbrace{|\gamma|}_{1} \underbrace{|\alpha - b_n|}_{<\varepsilon} + \underbrace{|b_n|}_{b_1} \underbrace{|\gamma - d_n|}_{<\varepsilon} < \varepsilon \underbrace{(2\gamma + \alpha + b_1)}_{=c} = \varepsilon'$$

26 Exercise 34

Exercise 23. Let $f : X \rightarrow Y$ be a mapping. Prove that:

1. If a mapping $g : Y \rightarrow X$ with $g \circ f = \mathrm{id}_X$ exists, f is injective.
2. If a mapping $h : Y \rightarrow X$ with $f \circ h = \mathrm{id}_Y$ exists, f is surjective.

Remark. id_X is the identity function over the set X. The identity function is always defined as $f : X \rightarrow X$ with $x \mapsto x$.

26.1 Exercise 34.1

So given that $g : Y \rightarrow X$ exists with $g \circ f = \mathrm{id}_X$, let $x \in X$.

$$x \in X \Rightarrow f(x) \in Y \Rightarrow g(f(x)) = x \Leftrightarrow \mathrm{id}_X(x) = x$$

To show injectivity, we need to show for all $x_1, x_2 \in X$:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Consider two arbitrary values $x_1, x_2 \in X$.

$$f(x_1) = f(x_2)$$
$$\Rightarrow g(f(x_1)) = g(f(x_2))$$
$$\Rightarrow x_1 = x_2$$

As far as x_1 and x_2 are two arbitrary elements of X, this holds for any pair of elements of X. We have directly proven injectivity of f.

26.2 Exercise 34.2

Given that $h : Y \to X$ exists with $f \circ h = \mathrm{id}_Y$, let $y \in Y$.

$$y \in Y \Rightarrow h(y) \in X \Rightarrow f(h(y)) \in Y \Leftrightarrow \mathrm{id}_Y(y) = y$$

To show surjectivity, we need to show for all $y_1, y_2 \in Y$:

$$\forall y \in Y \exists x \in X : f(x) = y$$

Consider an arbitrary value $y \in Y$. Because of the existence of the identity function, it holds that:

$$f(h(y)) = y$$

We define $h(y)$ as an intermediate value with a different name:

$$x := h(y)$$

$$\Rightarrow \exists x \in X : f(x) = y$$

We have show that for any arbitrary value $y \in Y$. So it holds for any value of Y:

$$\Rightarrow \forall y \in Y \exists x \in X : f(x) = y$$

We have directly proven surjectivity of f.

27 Exercise 35

This exercise was delayed until 26th of November 2015 (later then the other exercises here).

$$(a_n)_{n\in\mathbb{N}} \text{ is sequence with } \lim_{n\to\infty} a_n = a \in \mathbb{R}$$
$$(b_n)_{n\in\mathbb{N}} \text{ is sequence with } \lim_{n\to\infty} b_n = b \in \mathbb{R}$$

Furthermore $b \neq 0$.

27.1 Part 1

$$\lim_{n\to\infty} a_n = a \wedge \lim_{n\to\infty} b_n = b \neq 0$$

Let $\varepsilon > 0$ be arbitrary.

Claim: $\exists k \in \mathbb{N} \forall n \geq k : |b_n| > \frac{|b|}{2}$.

Proof: Let $\varepsilon > 0$. Consider $\varepsilon = \frac{|b|}{2}$.

For $\varepsilon = \frac{|b|}{2} > 0$:

$$\exists k \in \mathbb{N} : \forall n \geq k : |b_n - b| < \frac{|b|}{2} = \varepsilon$$

$$\begin{aligned}\forall n \geq k : |b_n| = |b_n - b + b| &\geq \left| |b| - \underbrace{|b - b_n|}_{<\frac{|b|}{2}} \right| \\ &> |b| - |b - b_n| \\ &> |b| - \frac{|b|}{2} \\ &= \frac{|b|}{2}\end{aligned}$$

Claim:

$$\text{sequence } \left(\frac{1}{b_n}\right)_{n\in\mathbb{N}} \wedge \exists \lim \left(\frac{1}{b_n}\right) = \frac{1}{b}$$

Proof: For $\frac{\varepsilon |b|^2}{2}$:

$$\exists N \in \mathbb{N} : \forall n \geq N : |b_n - b| < \frac{\varepsilon |b|^2}{2}$$

It holds that $\forall n \geq N$:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n \cdot b}\right| = \frac{|b - b_n|}{|b_n| \cdot |b|} < \frac{\varepsilon \cdot \frac{|b|^2}{2}}{\frac{|b|}{b}|b|} = \varepsilon$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = a \cdot \frac{1}{b} \cdot \frac{a}{b}$$

Or a direct proof:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - a b_n}{b_n b}\right| \\
&= \frac{|a_n b - ab + ab - ab_n|}{|b_n||b|} \\
&\leq \frac{|b||a_n - a| + |a||b_n - b|}{\frac{|b|}{2} \cdot |b|} \\
&\leq C \cdot \varepsilon
\end{aligned}$$

27.2 Part 2

28 Exercise 36

$$A = \left\{\frac{1}{2^m} + \frac{1}{n} \middle| m, n \in \mathbb{N}_+\right\}$$

Assumption: $\min a = 0$

$$0 \notin A$$

$$\frac{1}{2^N} + \frac{1}{N} < 2\varepsilon$$

$$\forall \varepsilon > 0 \exists N \in \mathbb{N}_+ : \left(m \geq N \Rightarrow \left|\frac{1}{2^m} - 0\right| < \varepsilon\right)$$

$$\frac{1}{2^N} < \varepsilon$$

$$n \geq N \Rightarrow \left|\frac{1}{n} - 0\right| < \varepsilon$$

Assume $\exists s > 0$ is our lower bound.

$$\exists m : \frac{1}{2^m} < \frac{s}{2} = \varepsilon$$

$$\varepsilon = \frac{s}{2} : \exists N : \frac{1}{N} < \frac{S}{2}$$

$$\to \underbrace{\frac{1}{sm} + \frac{1}{N}}_{\in A} < s$$

$$\Rightarrow \inf A = 0$$

Remark: When starting this exercise, always estimate whether a maximum/minimum exists. If so, you can save time to prove supremum/infimum.

$$\frac{1}{2^{m+1}} < \frac{1}{2^m} \forall m$$
$$\frac{1}{N+1} < \frac{1}{N} \forall N$$

Therefore $\max A$ is when m, n is as small as possible:

$$\frac{1}{2} + \frac{1}{1} = \frac{3}{2}$$
$$\max(A) = \frac{3}{2} = \sup(A)$$

$$B = \left\{ \frac{x}{1+x} \,\middle|\, x \in \mathbb{R}, x \geq 0 \right\}$$

$\min(B) = 0$ because $0 \leq \frac{x}{1+x} \forall x \geq 0 \wedge \frac{x}{1+x}\Big|_{x=0} = 0$.

$$\frac{x}{1+x} < 1 \Leftrightarrow x < 1 + x \Leftrightarrow 0 < 1 \forall x \geq 0$$

Is 1 an upper bound and $1 \notin B$?

Assume $\exists s < 1$:

$$\frac{x}{1+x} \leq s$$
$$x \leq s(1+x)$$
$$x(1-s) \leq 0$$
$$1 - s > 0$$
$$\Rightarrow \sup(B) = 1 \wedge \nexists \max(B)$$

29 Exercise 37

$$I = [a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$
$$a = \min{[a, b)} \Rightarrow a \text{ is } \inf([a, b))$$
$$a \in [a, b) : \forall x \in [a, b) : a \leq x \Rightarrow \min(I) = a \Rightarrow \inf(I) = a$$

b is upper bound:

$$b \notin [a, b) \text{ by definition } \forall x \in [a, b) : b > x$$

Claim: b is the smallest upper bound.
Assume: $\exists b' < b : b'$ is upper bound.

$$b' \in [a, b) \qquad \underbrace{”}_{\text{because } \mathbb{R} \text{ is complete}}$$

30 Exercise 38

Exercise 24. Let A and B two non-empty, bounded by below subseteq of $\mathbb{R}$. Prove that

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$$

Without loss of generality, $\inf A \leq \inf B$:

Let $a \in A$ and $b \in B$ arbitrary. This implies that $a \geq \inf A$ and $b \geq \inf B \geq \inf A$.

$$\Rightarrow \forall x \in (A \cup B) : x \geq \inf A$$

$$\Rightarrow \inf(A) \geq \inf(A \cup B)$$

Because extending a set A with additional elements, the infimum cannot be increased, but only decreased.

$$\Rightarrow \inf(A) \leq \inf(A \cup B)$$

$$x \in A : \inf\{\inf A, \inf B\} \leq \inf(A) \leq x$$
$$x \in B : \inf\{\inf A, \inf B\} \leq \inf(B) \leq x$$
$$\forall x \in A \cup B : \underbrace{\min\{\inf A, \inf B\}}_{\text{lower bound}} \leq x$$

$$\Rightarrow \inf(A \cup B) \leq \min\{\inf(A), \inf(B)\}$$

$$\Rightarrow \inf(A) = \inf(A \cup B)$$

31 Exercise 39

31.1 Exercise 39a

$$\sup_{y \in Y} \inf_{x \in X} f(x,y) \leq \inf_{x \in X} \sup_{y \in Y} f(x,y)$$
$$\underbrace{\inf_{x \in X} f(x,y) \leq f(x,y)}_{\sup_{y \in Y}} \leq \sup_{y \in Y} f(x,y)$$
$$\sup_{y \in Y} \inf_{x \in X} f(x,y) \leq \sup_{y \in Y} f(x,y)$$
$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} \inf_{x \in X} f(x,y) \leq \inf_{x \in X} \sup_{y \in Y} f(x,y) \quad \checkmark$$

31.2 Exercise 39b

$$f : (x,y) \mapsto 1_{\{x \geq 0, y \geq 0\} \cup \{x < 0, y < 0\}}$$
$$\sup_{y \in Y} f(x,y) = 1 \forall x$$
$$\inf_{x \in X} \sup_{y \in Y} f(x,y) = 1$$
$$\inf_{x \in X} f(x,y) = 0 \forall y \in [-1,1]$$
$$\sup_{y \in Y} \inf_{x \in X} f(x,y) = 0 < 1$$

32 Exercise 40

32.1 Exercise 40.a.1

$$\frac{5+i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{10-15i+2i-3i^2}{-6i+4+6i-9i^2} = \frac{10-13i+3}{4+9} = \frac{13-13i}{13} = 1-i$$

32.2 Exercise 40.a.2

$$\begin{aligned}
z^2 &= \frac{1+\sqrt{3}i}{2} \\
z^2 &= \pm\sqrt{\frac{1}{2}+\frac{\sqrt{3}i}{2}} \\
z^2 &= \pm\sqrt{\frac{9+6\sqrt{3}i-3}{12}} \\
z^2 &= \pm\sqrt{\frac{(3+\sqrt{3}i)^2}{12}} \\
z^2 &= \pm\frac{3+\sqrt{3}i}{\sqrt{12}} \\
z^2 &= \pm\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)
\end{aligned}$$

32.3 Exercise 40.b.1

$$M_1 = \left\{ z \in \mathbb{C} \setminus \{0\} \,\middle|\, \left|\frac{1}{z}\right| < 2 \right\}$$

$$\left|\frac{1}{z}\right| = \left|\frac{1}{a+bi}\right| = \frac{|1|}{|a+bi|} = \frac{1}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{a^2+b^2}} < 2$$

$$\Rightarrow \frac{1}{2} < \sqrt{a^2+b^2}$$

$$\Rightarrow \frac{1}{4} < a^2+b^2$$

Illustrated we draw a circle originating in $(0,0)$ with radius $\frac{1}{2}$. The solution set is the whole plane excluding everything what is part of the circle.

32.4 Exercise 40.b.2

$$M_2 = \{z \in \mathbb{C} \mid \Im((1+i)z) = 0\}$$

$$\Im(z+zi)$$

TODO

33 Exercise 41

$$A_n := (-\infty, a_n)_{n\in\mathbb{N}} \qquad A := \bigcup_{n\in\mathbb{N}} A_n$$
$$B_n := (b_n, \infty)_{n\in\mathbb{N}} \qquad B := \bigcup_{n\in\mathbb{N}} B_n$$

$$\forall n \in \mathbb{N} : x \in I_n$$

Show that $x = \sup A = \inf B$.

Because I_n are nested intervals it holds that

$$a_1 \leq \cdots \leq a_n \leq a_{n+1} \leq x$$

Because
$$\forall \varepsilon > 0 \exists N : N \geq n : 0 \leq x - a_x \leq b_n - a_n \leq \varepsilon$$
it holds that
$$x = \sup(a_n)$$

Let $y \in A$.
$$\exists n \in \mathbb{N} : y \in A_n \Rightarrow y < a_n \leq x$$
$$\Rightarrow y \in A : y < x$$

Therefore x is an upper bound. Is it the only upper bound?

Assume another upper bound x' exists.

$$x' < x = \lim) n \to \infty a_n$$
$$\Rightarrow \exists N \in \mathbb{N} : x' < a_n \qquad \forall n \geq M$$
$$\varepsilon = \frac{x - x'}{2}$$
$$\Rightarrow \exists y \in A_{n+1}$$
$$y > x'$$

This is a contradiction and therefore x is the distinct upper bound.

The proof for the infimum works analogously.

It only remains to show that $x \notin A$.

$$\forall a_n \neq x \Rightarrow \exists a_{n+k} : a_n < a_{n+k}$$

34 Exercise 42

Give the limes for the following sequences:

34.1 Exercise 42.a

$$a_n = \frac{5n+2}{3n+7}$$

$$
\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{5n+2}{3n+7} \\
&= \frac{\lim_{n\to\infty} 5n+2}{\lim_{n\to\infty} 3n+7} \\
&= \frac{\lim_{n\to\infty} 5n + \lim_{n\to\infty} 2}{\lim_{n\to\infty} 3n + \lim_{n\to\infty} 7} \\
&= \frac{n(5+\frac{2}{n})}{n(3+\frac{7}{n})} \\
&= \frac{5 + \overbrace{\frac{2}{n}}^{\to 0}}{3 + \underbrace{\frac{7}{n}}_{\to 0}} \\
&= \frac{5}{3}
\end{aligned}
$$

This works only if the denominator is non-zero. $\lim_{n\to\infty}(3+\frac{7}{n})$ turns out to be non-zero.

34.2 Exercise 42.b

$$
b_n = \frac{2n^2 - 4n + 5}{n^3 + 2\sqrt{n}}
$$

First, we make a remark, that $\lim_{n\to\infty} \frac{1}{n^2} = 0$. Why, because

$$
\lim_{n\to\infty} \frac{1}{n^2} = \left(\lim_{n\to\infty} \frac{1}{n}\right) \cdot \left(\lim_{n\to\infty} \frac{1}{n}\right) = 0
$$

This can be generalized for $\lim_{n\to\infty} \frac{1}{n^k} = 0$ with $k \in \mathbb{N}_+$.

$$
\begin{aligned}
\lim_{n\to\infty} b_n &= \frac{2n^2 - 4n + 5}{n^3 + 2\sqrt{n}} \\
&= \frac{n^3 \cdot \left(\frac{2}{n} - \frac{4}{n^2} + \frac{5}{n^2}\right)}{n^3 \cdot \left(1 + 2\frac{n^{0.5}}{n^3}\right)} \\
&= \frac{\frac{2}{n} - \frac{4}{n^2} + \frac{5}{n^3}}{1 + 2 \cdot \frac{1}{n^{2.5}}} \\
&= \frac{\frac{2}{n} - \frac{4}{n^2} + \frac{5}{n^3}}{\frac{n^{2.5}}{n^{2.5}}} \\
&= \frac{2n^{1.5} - 4n^{0.5} + 5n^{0.5}}{n^{2.5} + 2} \cdot \frac{\frac{1}{n^{2.5}}}{\frac{1}{n^{2.5}}} \\
&= \frac{2n^{-1} - 4n^{-2} + 5n^{-3}}{1 + 2n^{-2.5}} \\
&= \frac{0}{1}
\end{aligned}
$$

Or generally:

$$
2n^2 - 4n + 5 \le 2n^2 + 4n^2 + 5n^2 \le 11n^2
$$

$$0 \leq b_n \leq \frac{11n^2}{n^3} = \underbrace{\frac{11}{n}}_{\to 0}$$

34.3 Exercise 42.c

$$c_n = \sqrt{4n^2 + 2n + 3}$$

$$\begin{aligned} c_n &= \sqrt{4n^2 + 2n + 3} \cdot \frac{\sqrt{4n^2 + 2n + 3}}{\sqrt{4n^2 + 2n + 3} + 2n} \\ &= \ldots \\ &= \frac{2 + \frac{3}{n}}{\sqrt{4 + \frac{2}{n} + \frac{3}{n^2}} + 2} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

34.4 Exercise 42.d

$$d_n = \binom{n}{k} n^{-k} \text{ with } n \in \mathbb{N} \text{ for a fixed } k \in \mathbb{N}_+$$

$$\begin{aligned} d_n &= \binom{n}{k} n^{-k} \\ &= \frac{n!}{k!(n-k)!n^k} \\ &= \frac{n \cdot (n-1) \cdot \dots \cdot 1}{k!(n-k)!n^k} \\ &= \frac{n \cdot (n-1) \cdot \ldots \cdot 1}{k!(n-k)!n^k} \\ &= \frac{(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdot \ldots \cdot (1 - \frac{k-1}{n}) \cdot (n-k) \cdot \ldots \cdot 1}{k!(n-k)!} \\ &= \frac{(n-k)!}{k!(n-k)!} \\ &= \frac{1}{k!} \end{aligned}$$

Or better we write:

$$\begin{aligned}
\frac{n!}{k!(n-k)!} &= \frac{\prod_{i=0}^{n-1}(n-i)}{\prod_{j=k}^{n-1}(n-j)} \\
&= \frac{1}{k!}\prod_{j=0}^{k-1}(n-j)n^{-k} \\
&= \frac{1}{k!}\prod_{j=0}^{k-1}\left[(n-j)\cdot\frac{1}{n}\right] \\
&= \frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) \\
\lim_{n\to\infty}\frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) &= \frac{1}{k!}\lim_{n\to\infty}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right) && \text{[if limes exist]} \\
&= \frac{1}{k!}\prod_{j=0}^{k-1}\underbrace{\lim_{n\to\infty}\left(1-\frac{j}{n}\right)}_{=1} \\
&= \frac{1}{k!}\forall j = 0,\dots,k-1
\end{aligned}$$

35 Exercise 43

Exercise 25. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+$ with $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = q$. Prove that

$$(a_n)_{n\in\mathbb{N}}\begin{cases}\text{converges} & \text{if } q<1\\ \text{diverges} & \text{if } q>1\end{cases}$$

In case $q = 1$ no statement about the convergence of $(a_n)_{n\in\mathbb{N}}$ can be made.

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = q$$

35.1 Examples for $q = 1$

$$\begin{aligned}
a_n &= \frac{1}{n+1} & \frac{a_{n+1}}{a_n}\frac{n+1}{n+2} &\to_{n\to\infty} 1 & a_m \searrow 0 \\
a_n &= n+1 & \frac{a_{n+1}}{a_n} = \frac{n+2}{n+1} &\to_{n\to\infty} 1 & a_n \nearrow 0
\end{aligned}$$

35.2 Proof for $q < 1$

$$\exists \underbrace{\varepsilon}_{=\frac{q+1}{2}-a} > 0 : q+\varepsilon < 1$$

If n is sufficiently large:

$$\left|\frac{a_{n+1}}{a_n} - q\right| < \varepsilon \Rightarrow \frac{a_{n+1}}{a_n} \in (q-\varepsilon, q+\varepsilon)$$

$$
\begin{aligned}
0 &\le a_{n+1} \le (q+\varepsilon)a_n \\
0 &\le a_{n+2} \le (q+\varepsilon)^2 a_n \\
&\dots \\
0 &\le a_{n+k} \le (q+\varepsilon)^k a_n
\end{aligned}
$$

By induction it holds that

$$0 \le a_{n+k} \le \underbrace{(q+\varepsilon)}_{\tilde{q}<1}{}^k a_1 \to_{k\to\infty} 0$$

This follows from the squeeze theorem.

$$\forall q > 1 \exists \varepsilon > 0 : q - \varepsilon > 1$$

$$
\begin{aligned}
a_{n+1} &> (q-\varepsilon)a_n \\
a_{n+k} &> \underbrace{(q-\varepsilon)}_{\tilde{q}>1}{}^k a_n
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} \tilde{q}^k &= +\infty \\
\tilde{q} &> 1
\end{aligned}
$$

36 Exercise 44

> **Exercise 26.** Let $(a_n)_{n\in\mathbb{N}}$ be a zero sequence in $\mathbb{R}$ and $(b_n)_{n\in\mathbb{N}}$ a bounded sequence in $\mathbb{R}$. Prove that $(a_n b_n)_{n\in\mathbb{N}}$ is a zero sequence.

Because $(b_n)_{n\in\mathbb{N}}$ is bounded some d exists such that

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \ge N : |a_n - 0| < \varepsilon$$

Consider $\lim_{n\to\infty}(a_n \cdot b_n) = 0$.

We need to show that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \ge N : |a_n \cdot b_n - 0| < \varepsilon \cdot d$$

Where $\varepsilon \cdot d$ is epsilon multiplied with constant d. This is a hand-crafted value (meaning that we selected it intentionally and will turn out to solve our problem). Now we elaborate on the relation:

$$
\begin{aligned}
|a_n \cdot b_n| &< \varepsilon \cdot d \\
|a_n| \cdot |b_n| &< \varepsilon \cdot d \\
|a_n| \cdot d &< \varepsilon \cdot d \\
|a_n| &< \varepsilon
\end{aligned}
$$

Because $a_n < \varepsilon$ it holds that some constant exists for a sufficiently large N such that $|a_n \cdot b_n|$ is always smaller than some constant ε.

37 Exercise 45

Exercise 27. Let $a, b, c \in [0, \infty)$. Show that,

$$\lim_{n\to\infty} \sqrt[n]{a^n + b^n + c^n} = \max\{a, b, c\}$$

Without loss of generality, let $a = \max\{a, b, c\}$. Because a, b, c is non-negative,

$$a^n \leq a^n + b^n + c^n \leq 3a^n$$
$$\sqrt[n]{a^n} \leq \sqrt[n]{a^n + b^n + c^n} \leq \sqrt[n]{3} \cdot \sqrt[n]{a^n}$$
$$\lim_{n\to\infty} \sqrt[n]{a^n} = a$$
$$\lim_{n\to\infty} \sqrt[n]{3} \cdot \sqrt[n]{a^n} = a \lim_{n\to\infty} \sqrt[n]{3} = a \cdot 1$$

Due to the squeeze theorem, it holds that $\lim_{n\to\infty} \sqrt[n]{a^n + b^n + c^n} = a = \max\{a, b, c\}$.

38 Exercise 46

Exercise 28. Let $a_0 \in (0, 1)$ and a sequence $(a_n)_{n\in\mathbb{N}}$ is recursively defined with

$$a_{n+1} = 1 - \sqrt{1 - a_n} \text{ for } n \geq 0$$

Induction base

$$a_1 = 1 - \sqrt{1 - a_0} \Rightarrow 0 < a_1 < 1 \qquad a_1 \in (0, 1)$$

Induction step Let $a_n \in (0, 1)$.

$$0 < a_n < 1 \Rightarrow -1 < -a_n < 0 \Rightarrow 0 < \sqrt{1 - a} < 1$$
$$0 < \underbrace{1 - \sqrt{1 - a_n}}_{a_{n+1}} < 1$$

So $a_{n+1} < a_n$.

$$1 - \sqrt{1 - a_n} < a_n \Leftrightarrow (1 - a_n)^2 < 1 - a_n$$
$$\Rightarrow 1 - a_n < \sqrt{1 - a_n}$$
$$\Rightarrow x^2 < x \Leftrightarrow x \in (0, 1)$$

$$a_{n+1} - 1 - \sqrt{1 - a_n}$$
$$a = 1 - \sqrt{1 - a}$$

Therefore only 0 or 1 are possible limes for this sequence. But monotonically decreasing implies that 0 is the limes (bounded below and monotonically decreasing sequences are convergent).

39 Exercise 47

Exercise 29. For a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ we assign the sequence $(s_n)_{n\in\mathbb{N}}$, where

$$s_n = \frac{1}{n+1} \sum_{k=0}^{n} a_k \text{ for } n \geq 0$$

is the mean value of the first $n + 1$ sequence numbers.

- Show that: If $\lim_{n\to\infty} a_n = a$ with $a \in \mathbb{R}$, then $\lim_{n\to\infty} s_n = a$.

- Give an example for a divergent sequence $(a_n)_{n\in\mathbb{N}}$ for which the sequence of mean values converges anyways.

39.1 Exercise 47.a

We show that $\exists a \in \mathbb{R} : \lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n = a$.

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) = \lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}(a_n - (a_n - a_k))\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_n\right) - \lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}(a_n - a_k)\right) \\
&= \lim_{n\to\infty}\frac{1}{n+1}(n+1)a_n - \lim_{n\to\infty}\sum_{k=0}^{n}\frac{a_n - a_k}{n+1} \\
&\quad [\forall \varepsilon > 0 \exists N : n \geq N : |a_n - a| < \varepsilon] \\
&= a - \lim_{n\to\infty}\underbrace{\sum_{k=0}^{N}\frac{a_n - a_k}{n+1}}_{(N+1)\frac{C}{n+1}} + \sum_{N+1}^{n}\frac{a_n - a_k}{n+1} \\
&\overset{?}{\leq} \underbrace{\lim_{n\to\infty}\frac{(N+1)C}{n+1}}_{\to 0} + \lim_{n\to\infty}\sum_{n=N+1}^{n}\frac{a_n - a_b}{n+1} \\
&\left[\lim_{n\to\infty}\sum_{n=N+1}^{n}\frac{\varepsilon}{n+1} = \frac{n-N-1}{n+1}\varepsilon \to 1\right] \\
&= \lim_{n\to\infty}\frac{(N+1)C}{n+1} +
\end{aligned}$$

39.2 Exercise 47.a, radical variant

$$\sum_{n=N-1}^{n} a - \varepsilon \leq \cdots \leq \frac{\sum_{k=0}^{N} a_k + \sum_{k=N+1}^{n}(a+\varepsilon)}{n+1}$$

39.3 Exercise 47.b

$$\begin{aligned}
(a_n)_{n\in\mathbb{N}} &= (-1)^n \\
\Rightarrow \lim_{n\to\infty} S_n &= \lim_{n\to\infty}\frac{\sum_{k=0}^{n}(-1)^n}{n+1} = 0
\end{aligned}$$

40 Exercise 48

Exercise 30. Let $M \subseteq \mathbb{R}$ be a bounded above set and $s \in \mathbb{R}$. Prove that:

$$s = \sup(M) \Leftrightarrow \begin{cases} \forall x \in M : s \geq x & \text{and} \\ \exists (x_n)_{n\in\mathbb{N}}, x_n \in M : \lim_{n\to\infty} x_n = s \end{cases}$$

$$\exists (x_n)_{n\in\mathbb{N}}, x_n \in M : \lim_{n\to\infty} x_n = s \Leftrightarrow \varepsilon > 0 \exists N \in \mathbb{N} : |x_n - s| < \varepsilon$$

We prove the first direction $\Leftarrow$.

Let s be an upper bound of M. Let $(x_n)_{n\in\mathbb{N}}$ in M with $\lim_{n\to\infty} x_n = s$.

$$\Rightarrow \forall t < s \exists N \in \mathbb{N} : n \geq N (s - x_n) \leq |s - x_n| < \frac{\varepsilon}{2}$$

$$\Rightarrow t < x_n$$

We prove the second direction $\Rightarrow$.

Therefore

$$s - \frac{1}{n} \text{ is not an upper bound of } M$$

$$\Rightarrow \exists x_n \in M : s - \frac{1}{n} < x_n$$

$$\exists x_n \in M : s - \frac{1}{n} < x_n$$

$$s - \frac{1}{n} \leq x_n < s$$

$$s \leq x_n < s \Rightarrow (x_n)_{n\in\mathbb{N}} \to s$$

41 Exercise 49

Exercise 31. Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence of non-negative real numbers with $\lim_{n\to\infty} a_n = a$ and $k \in \mathbb{N}_+$. Show that

$$\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{a}$$

Hint: $a_n - a = \sqrt[k]{a_n^k} - \sqrt[k]{a^k} = (\sqrt[k]{a_n} - \sqrt[k]{a})(\ldots)$.

$$\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{a}$$

$$\lim_{n\to\infty} a_n = a \qquad \text{i.e.} \quad \forall \varepsilon \exists N \in \mathbb{N} : (\sqrt[k]{a_n} - \sqrt[k]{a}) \left(\sum_{i=0}^{k-1} \sqrt[k]{a_n}^{k-1-j} \cdot \sqrt[k]{a} \right)$$

Case 1: $a > 0$

$$|a_n - a| < \frac{a}{2}$$

$$|a_n| > \frac{|a|}{2}$$

$\Rightarrow$ the product is always positive:

$$\underbrace{(\sqrt[k]{a_n} - \sqrt[k]{a}) \left(\sum_{i=0}^{k-1} \sqrt[k]{a_n}^{k-1-j} \cdot \sqrt[k]{a} \right)}_{b_n \geq b > 0}$$

Done.

Case 2

$$\sqrt[k]{a_n} < \varepsilon \Leftrightarrow a_n < \varepsilon^k = \tilde{\varepsilon}$$

$$\forall \tilde{\varepsilon} > 0 \exists N \in \mathbb{N} : n \geq N : |a_n - 0| < \tilde{\varepsilon}$$

$$\Rightarrow \sqrt[r]{a_n}$$

41.1 Shorter valid solution

$$b_n = \frac{a_n - a}{\left(\sqrt[k]{a_n} - \sqrt[k]{a}\right)\left(\sum_{i=0}^{k-1} \sqrt[k]{a_n}^{k-1-j} \cdot \sqrt[k]{a}\right)} = \sqrt[k]{a_n} - \sqrt[k]{a}$$

$$\sqrt[k]{a_n} - \sqrt[k]{a} = \frac{a_n - a}{b_n}$$

We already know that

$$\lim_{n\to\infty} \frac{a_n - a}{b_n} = \frac{\lim_{n\to\infty}(a_n - a)}{\lim_{n\to\infty} b_n} = \frac{0}{b} = 0$$

42 Exercise 50

Exercise 32. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ and $q \in [0,1)$ such that for all $n \geq 1$,

$$|x_{n+1} - x_n| \leq q \cdot |x_n - x_{n-1}|.$$

Prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Does the implication still hold, if only $|x_{n+1} - x_n| < |x_n - x_{n-1}|$?

This is actually the fixed point theorem by Banach. Contraction property.

42.1 Special case: q = 0 implies a Cauchy sequence

If $q = 0$, then the sequence $(x_n)_{n\in\mathbb{N}}$ is constant. A constant sequence is always a cauchy sequence.

42.2 An observation

Let $q \in [0,1)$.

$$\begin{aligned}
\forall n \geq 1 &: |x_{n+2} - x_{n+1}| \leq q\,|x_{n+1} - x_n| \\
\forall n \geq 1 &: q\,|x_{n+1} - x_n| \leq q^2\,|x_n - x_{n-1}| \\
\Rightarrow \forall n \geq 1 &: |x_{n+2} - x_{n+1}| \leq q^2\,|x_n - x_{n-1}|
\end{aligned}$$

We make a more general claim:

42.3 Complete induction over k ∈ ℕ

$$|x_{n+k+1} - x_{n+k}| \leq q^{k+1}\,|x_n - x_{n-1}| \qquad \forall n \geq N : N = 1$$

Induction base: $k = 0$

$$|x_{n+1} - x_n| \leq q^1\,|x_n - x_{n-1}| \qquad \checkmark$$

Induction step: $k \to k+1$

$$\begin{aligned}
|x_{n+k+2} - x_{n+k+1}| &\leq q^{k+2} \cdot |x_n - x_{n-1}| \\
\text{we know: } |x_{n+k+2} - x_{n+k+1}| &\leq q \cdot |x_{n+k+1} - x_{n+k}| \\
q\,|x_{n+k+1} - x_{n+k}| &\overset{!}{\leq} q^{k+2} \cdot |x_n - x_{n-1}| \\
|x_{n+k+1} - x_{n+k}| &\leq q^{k+1} \cdot |x_n - x_{n-1}| \qquad \checkmark
\end{aligned}$$

42.4 Selection of ε

Let $k \in \mathbb{N}$. We have proven that $\forall N \geq 1$, it holds that

$$|x_{N+k+1} - x_{N+k}| \leq q^{k+1} |x_N - x_{N-1}|$$

Consider $N = 1$.

$$|x_{k+2} - x_{k+1}| \leq q^{k+1} |x_1 - x_0|$$

Now consider $\varepsilon_0 = q^{k+1} |x_1 - x_0|$. Let's recognize that we can choose some $n, m \in \mathbb{N}$ without loss of generality $n \geq m$. Then it holds that,

$$|x_n - x_{n-1}| \leq \varepsilon_0 \qquad \wedge \qquad |x_m - x_{m-1}| \leq \varepsilon_0$$

Let $\varepsilon = \varepsilon_0(n - m) + 1 = q^{k+1} |x_1 - x_0| (n - m) + 1$.

TODO: $\sum_{k=0}^{m-n-1} q^k = \frac{1-q^{n-m}}{1-q}$. $\varepsilon = \frac{1}{1-q} |x_{n+1} - x_n|$

42.5 Is ε an appropriate constant?

We need to show the Cauchy criterion:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N : |a_n - a_m| < \varepsilon$$

$$\begin{aligned}
|a_n - a_m| &= |a_n - a_{n-1} + a_{n-1} - a_{n-2} + a_{n-2} - \cdots + a_{m+1} - a_m| \\
&\leq |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots + |a_{m+1} - a_m| && \text{[triangle inequality]} \\
&= \sum_{i=n}^{m+1} |a_i - a_{i-1}| \\
&\leq \sum_{i=n}^{m+1} q^{k+1} |x_1 - x_0| \\
&= (n - m) \cdot q^{k+1} |x_1 - x_0| \\
&< (n - m) \cdot q^{k+1} |x_1 - x_0| + 1 \\
&= \varepsilon
\end{aligned}$$

42.6 Does it also hold for $|x_{n+1} - x_n| < |x_n - x_{n-1}|$?

The proof stays the same (so this statement is true as well).

- Because q cannot be ≥ 1 in $q \in [0, 1)$, q cancels out with smaller-than.
- ε can be chosen differently (we can skip +1):

$$\varepsilon = q^{k+1} |x_1 - x_0| (n - m)$$

43 Exercise 51

Exercise 33. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is recursively defined by $a_0 = 0, a_1 = 1$ and

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \qquad \text{for } n \geq 2$$

Show (using the result from Exercise 50), that the sequence $(a_n)_{n\in\mathbb{N}}$ converges.

Bonus: Can you determine the limes of this sequence?

$$(a_n)_{n\in\mathbb{N}} = \left(0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{36}, \dots\right)$$

We will also rephrase the equation:

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$$
$$a_{n-1} = 2a_n - a_{n-2}$$

We revise the previous theorem:

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ and $q \in [0,1)$ such that for all $n \geq 1$,

$$|x_{n+1} - x_n| \leq q \cdot |x_n - x_{n-1}|.$$

Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
|a_{n+1} - a_n| &\leq q\,|a_n - a_{n-1}| \\
\left|\frac{a_n + a_{n-1}}{2} - \frac{a_{n-1} + a_{n-2}}{2}\right| &\leq q\left|\frac{a_{n-1} + a_{n-2}}{2} - \frac{2a_{n-1}}{2}\right| \\
\frac{1}{2}|a_n + a_{n-1} - a_{n-1} - a_{n-2}| &\leq \frac{q}{2} \cdot |a_{n-1} + a_{n-2} - 2a_{n-1}| \\
|a_n - a_{n-2}| &\leq q \cdot |a_{n-2} - a_{n-1}|
\end{aligned}$$

We have observed previously that $a_{n-1} = 2a_n - a_{n-2}$

$$\begin{aligned}
|a_n - a_{n-2}| &\leq q \cdot |2a_{n-2} - 2a_n| \\
|a_n - a_{n-2}| &\leq 2q \cdot |a_{n-2} - a_n| \\
|a_n - a_{n-2}| &\leq 2q \cdot |(-1)(a_n - a_{n-2})| \\
|a_n - a_{n-2}| &\leq 2q \cdot |a_n - a_{n-2}| \\
1 &\leq 2q
\end{aligned}$$

So this can be proven for any $q \geq \frac{1}{2}$, so we choose $q = \frac{1}{2}$.

44 Exercise 52

Exercise 34. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded above sequence in $\mathbb{R}$. Show that a number $s \in \mathbb{R}$ is the limes superior of $(a_n)_{n\in\mathbb{N}}$ if and only if for every $\varepsilon > 0$ the following conditions hold:

1. the set $\{n \in \mathbb{N} \,|\, a_n > s - \varepsilon\}$ is infinite.
2. the set $\{n \in \mathbb{N} \,|\, a_n > s + \varepsilon\}$ is finite.

We shortly point out that a limes superior must not necessarily exist, because bounded sequences in $\mathbb{C}$ have a limit point, but not sequences which are only bounded above.

44.1 Proof direction $\Leftarrow$

Given a limes superior s. So $\forall \varepsilon > 0 : |a_n - a| < \varepsilon$ for infinitely many indices.

1. If s is a limit point, it has infinitely many sequence numbers within its ε-environment.

$$\forall \varepsilon > 0 : |a_n - s| < \varepsilon \qquad \Rightarrow \qquad |\{n \in \mathbb{N} \,|\, a_n > s - \varepsilon\}| = \infty$$

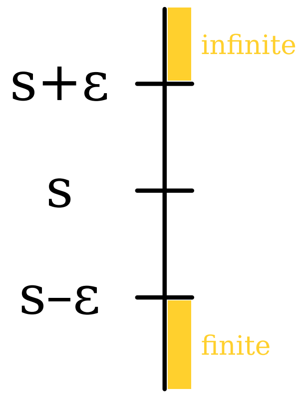

Figure 2: Illustration of the ε environment

2. So s is the supremum of all limit points. Proof by contradiction:

   Assume infinitely many elements are greater $s + \varepsilon$. Consider the upper bound of $(a_n)_{n\in\mathbb{N}}$ and $s + \varepsilon$ as lower bound. This subsequence is bounded in $\mathbb{C}$ (actually $\mathbb{R}$). Due to the Weierstrass-Bolzano theorem, a convergent subsequence must exist which necessarily provides a limit point. This limit point must be $\geq s+\varepsilon$, because of the lower bound. So we found another limit point which contradicts with our assumption that s is the supremum of all limit points.

44.2 Proof direction $\Rightarrow$

From the first property it holds that $|\{n \in \mathbb{N} \mid a_n > S - \varepsilon\}| = \infty \Rightarrow s - a_n < \varepsilon$ for infinitely many sequence numbers.

From the second property it holds that $|\{n \in \mathbb{N} \mid a_n > s + \varepsilon\}| < \infty \Rightarrow a_n > s + \varepsilon$. From that it follows that $s - a_n > \varepsilon \Rightarrow a_n - s < \varepsilon$. So $a_n - s > \varepsilon$ for infinitely many sequence numbers. So $a_n - s < \varepsilon$ for infinitely many sequence numbers.

From that it follows that $|a_n - s| < \varepsilon$ holds for infinitely many.

45 Exercise 53

Exercise 35. Determine the limes superior and limes inferior of the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$a_n = (1 + (-1)^n)(-1)^{\frac{n(n-1)}{2}}.$$

$$a_n = (1 + (-1)^n)(-1)^{n(n-1)/2}$$

45.1 Case 1: $n = 2k$ with $k \in \mathbb{N}$

$$\begin{aligned}
a_n &= \left(1 + (-1)^{2k}\right)(-1)^{2k(2k-1)/2} \\
&= \left(1 + \left((-1)^2\right)^k\right)(-1)^{k(2k-1)} \\
&= 2(-1)^{k(2k-1)}
\end{aligned}$$

45.2 Case 1.a: $k = 2l$ with $l \in \mathbb{N}$

$$
\begin{aligned}
&= 2(-1)^{2l(4l-1)} \\
&= 2\left((-1)^2\right)^{l(4l-1)} \\
&= 2
\end{aligned}
$$

45.3 Case 1.b: $k = 2l + 1$ with $l \in \mathbb{N}$

$$
\begin{aligned}
&= 2(-1)^{(2l+1)(4l+1)} \\
&= 2(-1)^{2l(4l+1)} \cdot (-1)^{(4l+1)} \\
&= 2(-1)^{2l(4l+1)} \cdot \left((-1)^4\right)^l \cdot (-1) \\
&= 2 \cdot 1 \cdot 1 \cdot (-1) \\
&= -2
\end{aligned}
$$

45.4 Case 2: $n = 2k + 1$ with $k \in \mathbb{N}$

$$
\begin{aligned}
a_n &= \left(1 + (-1)^{2k+1}\right)(-1)^{(2k+1)(2k)/2} \\
&= \left(1 + (-1)^{2k} \cdot (-1)\right)^{(2k+1)k} \\
&= (1 + 1 \cdot (-1))^{(2k+1)k} \\
&= 0^{(2k+1)k} \\
&= 0
\end{aligned}
$$

45.5 Putting it together

We have isolated three subsequences. Every subsequence itself is a constant sequence. Every constant sequence has a limit point. So the set of limit points is given with:

$$L = \{-2, 0, 2\}$$

$$
\begin{array}{ccccccc}
\sup(L) & = & \max(L) & = & 2 & \Rightarrow & \limsup_{n\to\infty}(a_n) = 2 \\
\inf(L) & = & \min(L) & = & -2 & \Rightarrow & \liminf_{n\to\infty}(a_n) = -2
\end{array}
$$

45.6 Limes

$$
\begin{aligned}
0 + \sum_{k=0}^{n} \frac{1}{2^{2k}} - \frac{1}{2}\sum_{k=0}^{n} \frac{1}{2^{2k}} &= \sum_{k=0}^{n} \left(\frac{1}{2^{2k}}\right)\left(1 - \frac{1}{2}\right) \\
= \sum_{k=0}^{n} \left(\frac{1}{4}\right)^k \left(\frac{1}{2}\right) &= \frac{1}{2}\frac{\left(\frac{1}{4}\right)^{n+1} - 1}{\frac{1}{4} - 1} \\
= \frac{1}{2}\frac{1}{\frac{3}{4}} &= \frac{2}{3} \\
\lim_{n\to\infty} a_n &= \frac{2}{3}
\end{aligned}
$$

46 Exercise 54

Exercise 36. Let $a \in \mathbb{R} \setminus \mathbb{C}$ be an irrational number and set $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{Z}$ or $\mathbb{N}_+$ such that it holds

$$\lim_{n\to\infty} \frac{p_n}{q_n} = a.$$

Prove that $\lim_{n\to\infty} q_n = \infty$

Hint: Prove by contradiction. Assume the sequence $(q_n)_{n\in\mathbb{N}}$ does not converge to ∞. Consider that there must exist a bounded subsequence and use the Bolzano-Weierstrass theorem.

47 Exercise 55

47.1 Exercise 55.a

$$\sum_{n=0}^{\infty} \frac{n-5}{n^2+1}$$

Let $n \geq 5$. Direct comparison test:

$$\frac{1}{2^n} \leq \left| \frac{n-5}{n^2+1} \right|$$

$$n^2 + 1 \leq 8n^2 - 10n$$

$$0 \leq n^2 - 10n - 1$$

$$n \geq 5 \pm \sqrt{26} = 11$$

So $n \geq 11$ shows that it converges.

Ratio test:

$$\lim_{n\to\infty} \left| \frac{n-4}{(n+1)^2+1} \cdot \frac{n^2+1}{n+5} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n-4)(n^2+1)}{((n+1)^2+1)(n+5)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^3 - 4n^2 + n - 4}{(n^2+2n+1)(n-5)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{n^3}{n^3} - \frac{4n^2}{n^2} + \frac{n^3}{n^3} - \frac{4}{n^3}}{\frac{n^3}{n^3} + \frac{2n^2}{n^3} - \frac{9n}{n^3} - \frac{5}{n^3}} \right|$$

Shows convergence.

But this solution is wrong.

47.1.1 Lathos' solution

$$n \geq 10$$

$$\frac{n-5}{n^2+1} \geq \frac{n - \frac{n}{2}}{n^2+n^2} = \frac{1}{2}\frac{1}{n+n} = \frac{1}{4n}$$

$$\sum_{n=1}^{\infty} \frac{1}{4n} \xRightarrow{\text{direct comparison crit.}} \sum_{n=10}^{\infty} \frac{n-5}{n^2+1} \text{ div.} \Rightarrow \sum_{n=0}^{\infty} \frac{n-5}{n^2+1} \text{ diverges}$$

47.2 Exercise 55.b

$$\sum_{n=2}^{\infty} \frac{2n+1}{n(n-1)^2}$$

Direct comparison test:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{2n+1}{n(n-1)^2} &\leq \sum_{n=2}^{\infty} \left| \frac{4n}{n(\frac{1}{4}n)^2} \right| \cdot \sum_{n=2}^{\infty} \frac{4n}{n\frac{1}{16}n^2} \\ &= \sum_{n=2}^{\infty} \left| \frac{4 \cdot 16n}{n^3} \right| \\ &= 4 \cdot 16 \sum_{n=2}^{\infty} \left| \frac{n}{n^3} \right| \leq 64 \cdot \sum_{n=2}^{\infty} \left| \frac{4 \cdot 16b}{n^3} \right|\end{aligned}$$

And this converges absolutely.

47.3 Exercise 55.c

$$\begin{aligned}&\lim_{n\to\infty} \left| \left(\frac{4}{5}\right)^{n+1} (n+1)^5 \left(\frac{4}{5}\right)^{-n} n^{-5} \right| \\ &= \lim_{n\to\infty} \left| \frac{4}{5} \left(\frac{n^5}{n^5} + 5\frac{n^4}{n^5} + 10 \cdot \frac{n^3}{n^5} + 10\frac{n^2}{n^5} + 5\frac{n}{n^5} + \frac{1}{n^5} \right) \right| \\ &= \left| \frac{4}{5} \right| < 1\end{aligned}$$

So $\sum_{n=5}^{\infty} \left(\frac{4}{5}\right)^n n^5$ converges.

47.4 Exercise 55.d

We use the Leibniz criterion.

We need to show that $|a_{n+1}| < |a_n|$.

$$\begin{aligned}\left| \frac{(-1)^{n+1}}{\sqrt{n+1}} \right| &< \left| \frac{(-1)^n}{\sqrt{n}} \right| \\ \frac{1}{\sqrt{n+1}} &< \frac{1}{\sqrt{n}} \\ \sqrt{n} &< \sqrt{n+1}\end{aligned}$$

$n < n+1$ so it is monotonically increasing.

Does it converge absolutely?

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$$

No, this does not converge. So it does not.

47.5 Exercise 55.e

TODO

48 Exercise 56

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \right)$$

$$\begin{aligned} 1 &= A(n+1)(n+2) + B(n+2)n + C(n+1)n \\ 1 &= An^2 + 3An + 2A + Bn^2 + 2Bn + Cn^2 + Cn \end{aligned}$$

For terms of degree 2 it holds that, $0 = A + B + C$. For terms of degree 1 it holds that, $0 = 3A + 2B + C$. For terms of degree 0 it holds that, $1 = 2A$.

We solve this equation system and get $A = \frac{1}{2}$, $B = -1$ and $C = \frac{1}{2}$.

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)} \right) &= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left(\left(\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} \right) - \left(\sum_{n=1}^{N} \frac{1}{n+1} - \frac{1}{n+2} \right) \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left(\left(\sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} \right) - \left(\sum_{n=2}^{N+1} \frac{1}{n} - \frac{1}{n+1} \right) \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left(\left(\sum_{n=2}^{N} \frac{1}{n} - \frac{1}{n+1} \right) + \left(\frac{1}{1} - \frac{1}{2} \right) - \left(\sum_{n=2}^{N} \frac{1}{n} - \frac{1}{n+1} \right) - \frac{1}{N+1} + \frac{1}{N+2} \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left(\left(\frac{1}{1} - \frac{1}{2} \right) - \frac{1}{N+1} + \frac{1}{N+2} \right) \\ &= \lim_{N\to\infty} \frac{1}{2} \left(\frac{1}{2} - \frac{1}{N+1} + \frac{1}{N+2} \right) \\ &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \end{aligned}$$

49 Exercise 57

50 Exercise 58

TODO: I think I only considered real numbers. Not complex numbers.

Direct comparison test: The series $\sum_{n=0}^{\infty} \left| \frac{a_n}{a_n+1} \right|$ converges if

1. $0 < \left| \frac{a_n}{a_n+1} \right| \le |\, a_n \,|$, and
2. $\sum_{n=0}^{\infty} |\, a_n \,|$ converges absolutely.

Firstly we cover a special case: Assume $\exists n \in \mathbb{N}$ such that $a_n = 0$. Followingly this means there exists some N such that all following n are 0.

$$\exists N \in \mathbb{N} : \forall n \in \mathbb{N} : n \ge N : a_n = 0$$

In this case the partial sum will not change at some sufficiently large n. So $\sum_{n=0}^{\infty} \left| \frac{a_n}{a_n+1} \right|$ converges.

In the following we can assume $\forall n \in \mathbb{N} : a_n > 0$.

1. We show that $0 < \left| \frac{a_n}{a_n+1} \right| \le |\, a_n \,|$.

Here $0 < \left| \frac{a_n}{a_n+1} \right|$ certainly holds because otherwise $a_n = 0$ for some n. This would be the special case we have already covered and we don't need to look at any more.

We show $\left| \frac{a_n}{a_n+1} \right| \le |a_n|$.

$$\frac{|a_n|}{|a_n+1|} \le |a_n|$$
$$\Rightarrow |a_n| \le |a_n||a_n+1|$$
$$\Rightarrow 1 \le |a_n+1|$$

2. This holds by precondition.

51 Exercise 59

52 Exercise 60

52.1 Exercise 60.a

wlog $|a_n| \le |b_n| \le |a_n \cdot b_n| \le \left|a_n^2\right|$.

$$\Rightarrow \left|a_n^2\right| \le |a_n b_n| \le \left|b_n^2\right|$$
$$\Rightarrow \sum_{n=0}^{\infty} |a_n b_n|$$

is absolute convergent.

52.2 Exercise 60.b

$$\sum_{n=0}^{\infty} a_n^2 + \sum_{n=0}^{\infty}(2a_n b_n) + \sum_{n=0}^{\infty} b_n^2$$

53 Exercise 61

Exercise 37. Determine the convergence radius of the following power series

53.1 Exercise 61.a: $\sum_{n=1}^{\infty} \frac{1}{n^n} z^n$

53.2 Exercise 61.b: $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} z^n$

53.3 Exercise 61.c: $\sum_{n=0}^{\infty} \frac{2^n}{n!}(z+i)^n$

53.4 Exercise 61.d: $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{z^{2n-1}}{(2n-1)!}$

Hint: Investigate convergence of this series, for fixed $z \in C$, directly with help of the quotient criterion.

54 Exercise 62

Exercise 38. We consider the function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $x \mapsto \frac{1}{x}$. Determine for appropriate subsets of $\mathbb{C} \setminus \{0\}$, representations of f as power series of structure

$$\frac{1}{x} = \sum_{n=0}^{\infty} a_n(x-1)^n \quad \text{and} \quad \frac{1}{x} = \sum_{n=0}^{\infty} b_n(x-2)^n \qquad (a_n, b_n \in \mathbb{C}).$$

Determine the convergence radius of both power series.
Hint: Geometric series.

55 Exercise 63

Exercise 39. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 2x^2 + x + 1$. Show, with the help of its ε–δ-definition of continuity, that f is continuous at -1.

56 Exercise 64

Exercise 40. Let $D \subseteq \mathbb{R}$, and $f, g : D \to \mathbb{R}$ in $x_0 \in D$ are continuous functions and it holds that $f(x_0) = g(x_0)$. Show: Let $h : D \to \mathbb{R}$ be a function, such that for all $x \in D$

$$f(x) \leq h(x) \leq g(x).$$

Prove that h is continuous in x_0.

57 Exercise 65

Exercise 41. Let $D \subseteq \mathbb{C}$, and $f, g : D \to \mathbb{C}$ are continuous in $z_0 \in D$. Show, with the use of the $\varepsilon - \delta$-definition of continuity, that also the function

$$fg : \begin{cases} D & \to \mathbb{C}, \\ z & \mapsto f(z)g(z) \end{cases}$$

is continuous in z_0.

58 Exercise 66

Exercise 42. Determine all $a \in \mathbb{R}$ such that the function is continuous in $\mathbb{R}$:

$$g : \mathbb{R} \to \mathbb{R}$$

$$g(x) = \begin{cases} x^3 + 2ax^2 + a^2 & \text{if } x \leq 1 \\ ax^2 + \frac{4a^2}{1+x^2} & \text{if } x > 1 \end{cases}$$

Let $f(x) = x^3 + 2ax^2 + a^2$ and $h(x) = ax^2 + \frac{4a^2}{1+x^2}$.

1. f is a polynomial. A polynomial is continuous.
2. g is not a polynomial. But $(ax^2)(1 + x^2) + (4a^2)$ is a polynomial. And $\frac{1}{1+x^2}$ is continuous in $\mathbb{R}$.
3. If the limes of f and g yields the same value at point x_0, then the function is continuous at this x_0.

$$\begin{aligned} \lim_{x \nearrow 1} ax^2 + \frac{4a^2}{1+x^2} &= a + \frac{4a^2}{2} \\ &= a + 2a^2 - a \end{aligned}$$

We use:

$$\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n$$

This holds only if $\lim_{n \to \infty} a_n$ and $\lim_{n \to \infty} b_n$ exist. Furthermore:

$$\lim_{x \searrow 1} ax^2 = a \cdot \lim_{x \searrow 1} x^2$$

g is continuous in

$$a = \frac{1 \pm \sqrt{5}}{2}$$

Fundamental theorem supporting this solution: f is continuous in x_0 if and only if $\lim_{x \nearrow x_0} f(k) = \lim_{x \searrow x_0} f(x)$.