

Linear Algebra – Lecture Notes

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This lecture took place on 5th of Oct 2015 (Prof. Franz Lehner).

Weekly schedule:

Mon	08:15–09:45	KF 06.01
Tue	08:15–09:45	TU P2
Tue	10:15	BE 01, Konversatorium
Wed	13:00–15:00	UE + Onlinekreuzesystem, Deadline 11:00
Mon, Tue, Thu	*	Tutorien

Exams:

1. VO-Prüfung (schriftlich, 3 Termine pro Semester, ohne Unterlagen)
2. 2 UE-Prüfungen (25.11, 27.01, 1 DIN A4 Blatt)

What is linear algebra?

- Arithmetics
- Geometry
- Analysis / infinitesimal computation

100 years ago, the following branch of mathematics was introduced:

- Algebra: abstract computational operations (fields, groups, rings, etc)
 - Linear algebra (branch of algebra, related to vector computations)

Mathematics is the search for statements of the structure: *If A, then B.*

1 Set theory, logic and linear equations

1.1 Axiomatic definition of a set

Georg Kantor (1869)

Unter einer Menge verstehen wir eine Zusammenfassung von *bestimmten wohlunterschiedenen* Objekten unserer Anschauung oder unseres Denkens (welche die Objekte der Menge M genannt werden) zu einem Ganzen.

We define a set as a combination of defined well-distinguishable objects of our perception and our minds (which are denoted set M) to a whole unit.

Hence for every object x one of these statements hold:

- x is part of M : $x \in M$
- x is not part of M : $x \notin M$

1.2 Notation for set theory

Approaches for notations:

- Enumeration
 - $\{1, 2, 3\}$, $\{a, b, \text{teddy bear, lecture hall HS 06.01}\}$
 - Integers (in this lecture: without zero): $\mathbb{N} = \{0, 1, 2, \dots\}$
 - $\{1, 2, 3, \dots\}$: integers, end undetermined
 - $\{1, 2, \dots, n\}$: integers from 1 to n
 - $\{x, y, \dots, z\}$: general finite set
- Description
 - $\{1, 4, 9, 16, \dots\}$
 - $\{n | n \text{ is square of an integer}\}$
 - $\{n | \text{there exists } k \in \mathbb{N} \text{ such that } n = k^2\} = \{k^2 | k \in \mathbb{N}\}$
- Defined set with shortcuts
 - \mathbb{N}

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$
- $\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{N} \right\}$
- \mathbb{R} = complex definition, see analysis
- $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$
- $\{\} = \emptyset$ as the empty set
- M. Bourbaki, “Elements of mathematics”

1.3 Examples for custom sets

“The set of all competent politicians” Not well-defined, opinion-based

“The set of all visible fix stars” Depends on definition of visibility, are tools allowed?, opinion-based

1.4 Russell’s paradoxon

Russell 1901, Zermelo 1902

$M =$ “the set of all sets” = “the set of all sets that does not contain itself”

1.5 Berrys paradoxon

M_{12} = set of all integers describable with at most 11 words

n is the smallest number not describable with at most 11 words

So n is not contained in M_{12} . But n itself is now described with 11 words. So it’s contained? Paradoxon.

1.6 Axiomatic system of Zermelo-Frauenkel

1. For all sets A, B it holds that $A = B$ iff $x \in A$ then also $x \in B$.
2. An empty set exists. Hence for all x it holds that $x \notin \emptyset$.
3. If A and B are sets, then also $\{A, B\}$.

4. If A and B are sets, then also the union of $A \cup B$ is a set.

5. An infinite set exists.

6. If A is a set, then also the power set $\mathcal{P}(A) = \{B \mid B \subseteq A\}$

1.7 Basics of logic

Aristoteles and Organon

Organon called the system “analytics”.

A *statement* is a linguistic unit which is *true* or *false*.

Examples:

- Sokrates is a human.
- 7 is a prime number.
- 5 is an even number.
- There exists only one universe.

The last example has an unknown truth value. Constructivists: “Unknown means false”. Pragmatics: “Unknown means unknown”.

Other examples for unknown truth values:

- Today is monday.
- A. Gabalier has a beautiful voice.

Epimenides

All crets are liars.

Russell:

This statement is wrong.

1.8 Gödel's incompleteness theorem

Kurt Gödel (1930)

In every formal system statements exist that are true, but not provable.

Example: "This statement is not provable."

1.9 A correction

Due to these contradictions:

A *statement* is a linguistic unit for which it makes sense to ask: is it *true* or *false*?

1.10 Formal logic

Negation $\neg A$ means the truth value of A is inverted

Conjunction $A \wedge B$ is true, if A and B is true

Attention!

- Eating and drinking forbidden (actually: "no eating or drinking")
- Solutions for $x^2 = 1$: $x_1 = 1$ and $x_2 = -1$ ("actually: $x_1 = 1$ or $x_2 = -1$ ")

Disjunction $A \vee B$ is true, if A or B is true (latin "vel")

Exclusive disjunction $A \dot{\vee} B$ is true if A or B but not both are true (latin "out")

Equivalence $A \leftrightarrow B$ is true if both share the same truth value ($\neg(A \dot{\vee} B)$)

Implication / subjuction $A \implies B$ is true if A is false or A is true and B is false. A implies B . Deutsch: "A ist hinreichend für B. B ist notwendig für A."

1.11 Definition

Two logical statements are equivalent if for every variable assignment, the same truth value is evaluated ($P(A_1, \dots, A_n) \leftrightarrow Q(A_1, \dots, A_n)$).

1.12 Logical laws by DeMorgan

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

This lecture took place on 6th of Oct 2015 (Prof. Franz Lehner).

$$|\mathbb{N}| = \aleph_0$$

1.13 Proofs

A sentence is a statement of kind:

$$A \implies B$$

A is our requirement. B is our conclusion. A proof is showing that B holds under assumption of A .

1.14 Statement

Let $n \in \mathbb{N}$ be odd, than n^2 is odd.

Proof:

A . n is even and $n \in \mathbb{N}$, hence there exists some $k \in \mathbb{N}_0$ such that $n = 2k + 1$

B . n^2 is odd, hence it holds that $l \in \mathbb{N}_0$ such that $n^2 = 2l + 1$

We know, $n = 2k + 1$

$$\Rightarrow n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1$$

with $l = 2k^2 + 2k$, statement B holds. Direct proof.

1.14.1 Contraposition law

$$A \implies B \Leftrightarrow \neg B \implies \neg A$$

A so-called “indirect proof”.

If n^2 is even, then n is even.

A. n^2 is even

B. n is even

$\neg B$. n is odd

$\neg A$. n is odd

We already have shown,

$$\neg B \implies \neg A$$

hence also $A \implies B$ is true.

1.15 Proof by contradiction

$$A \vee \neg A$$

Tertium nondatur

hence if $\neg A$ is false, then A is true.

1.15.1 $\sqrt{2}$ is irrational

$$\sqrt{2} \notin \mathbb{Q}$$

Proof:

A. Let $x \in \mathbb{R}$ such that $x^2 = 2$ and $x > 0$ and let $\sqrt{2}$ be that number

B. $\sqrt{2} \notin \mathbb{Q}$

Assume $\neg B$ hence $\sqrt{2} \in \mathbb{Q}$. We find a contradiction.

$\sqrt{2} \in \mathbb{Q}$ then there exists some $p \in \mathbb{Z}, q \in \mathbb{N}$ such that $\sqrt{2} = \frac{p}{q}$.

Wlog (without loss of generality), we assume that the fraction is irreducible. Hence $\gcd(p, q) = 1$.

Therefore $\sqrt{2}$ has the following property.

$$\begin{aligned} \sqrt{2} &= \frac{p}{q} \\ (\sqrt{2})^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \\ \Rightarrow p^2 &= 2q^2 \\ \Rightarrow p^2 &\text{ is even} \\ \Rightarrow p &\text{ is even} \end{aligned}$$

hence there exists some $k \in \mathbb{N}$ such that $p = 2k$

$$\begin{aligned} (2k)^2 &= 2q^2 \\ 4k^2 &= 2q^2 \\ 2k^2 &= q^2 \\ \Rightarrow q^2 &\text{ is even} \\ \Rightarrow q &\text{ is even} \end{aligned}$$

hence there is some $l \in \mathbb{N}$ such that $q = 2l$.

$$\sqrt{2} = \frac{2k}{2l}$$

is not reduced. This is contradictory to our original statement.

$$\begin{aligned} \gcd(p, q) &= \gcd(2k, 2l) \\ &\geq 2 \neq 1 \end{aligned}$$

$\Rightarrow \neg B$ is wrong, so B is true.

1.16 Remark about constructivism

A few mathematicians deny “tertium non datur”. For those $A \vee \neg A$ means that there is no proof for either statement.

1.16.1 a^b is irrational with $a, b \in \mathbb{R}$

Proof: We know that $\sqrt{2} \notin \mathbb{Q}$.

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}$$

case 1: $\sqrt{2}^{\sqrt{2}}$ is irrational \Rightarrow choose $a = \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}, b = \sqrt{2} \notin \mathbb{Q}, a^b \in \mathbb{Q}$

case 2: $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ choose $a = \sqrt{2} \notin \mathbb{Q}$ and $b = \sqrt{2} \notin \mathbb{Q}$ and $a^b \in \mathbb{Q}$.

With other means means that $\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}$.

1.17 Agreement

A *predicate* is an expression which depends on variable and by insertion of values, a statement is created.

$$P(n) \Leftrightarrow n \text{ is even}$$

is not a statement unless we define n .

$$P(2) \Leftrightarrow 2 \text{ is even}$$

$$P(3) \Leftrightarrow 3 \text{ is even}$$

1.18 Quantifiers

$$Q(n) \Leftrightarrow (P(n = 2k + 1) \implies P(n^2 = 2l + 1))$$

hence the statement

$$Q(1) \wedge Q(2) \wedge Q(3) \wedge Q(4) \wedge Q(5) \dots$$

Notation:

$$\bigwedge_{n \in \mathbb{N}} Q(n) \text{ or } \forall n \in \mathbb{N} : Q(n)$$

So we can briefly write:

$$\bigwedge_{n \in \mathbb{N}} Q(n)$$

meaning for all $n \in \mathbb{N}$ it holds that “ n is odd implies n^2 is odd”.

\bigwedge is called “all quantifier”.

Analogously for $P(1) \vee P(2) \vee P(3) \vee \dots$ is true if there is some n such that $P(n)$ is true.

$$\bigvee_{n \in \mathbb{N}} P(n) \Leftrightarrow \exists n : P(n)$$

Variant:

$$\dot{\bigvee}_{x \in X} P(x)$$

there exists *exactly one* x such that $P(x)$ holds.

$$\exists! x \in X : P(x)$$

1.19 Proof using quantifiers

There exists some prime number:

- $\bigwedge_{n \in \mathbb{N}} n \in \mathbb{P}$ where \mathbb{P} is the set of prime numbers.
- An integer is a prime number, if it does not have real divisor.

$$k \mid n = k \text{ divides } n \Leftrightarrow \bigvee_{l \in \mathbb{N}} k \cdot l = n$$

$$\bigwedge_{n \in \mathbb{N}} n \in \mathbb{P} \Leftrightarrow \neg \bigvee_{k \in \mathbb{N}} (k > 1) \wedge (k < n) \wedge (k \mid n)$$

1.20 Negation with quantifiers

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

$$\neg \bigwedge_{x \in X} P(x) \Leftrightarrow \bigvee_{x \in X} \neg P(x)$$

1.21 Relation between set theory and boolean algebra

$$\begin{aligned} A \cap B &= \{x \mid x \in A \wedge x \in B\} \\ A \cup B &= \{x \mid x \in A \vee x \in B\} \\ A \triangle B &= \{x \mid x \in A \dot{\vee} x \in B\} \quad \text{“symbolic difference”} \\ A \setminus B &= \{x \mid x \in A \wedge x \notin B\} \end{aligned}$$

$$\begin{aligned} A^C &= \{x \in U \mid x \notin A\} \quad \text{“complement in } U, \text{ the universe”} \\ &= U \setminus A \end{aligned}$$

$$\begin{aligned} A \subseteq B &\Leftrightarrow \bigwedge_{x \in A} x \in B \\ &\Leftrightarrow \bigwedge_x (x \in A \implies x \in B) \end{aligned}$$

$$A = B \Leftrightarrow \bigwedge_x x \in A \Leftrightarrow x \in B$$

Let A_i with $i \in I$ (where I is the index set) be sets than

$$\bigcap_{i \in I} A_i = \left\{ x \mid \bigwedge_{i \in I} x \in A_i \right\} \quad \text{intersection of all } A_i$$

$$\bigcup_{i \in I} A_i = \left\{ x \mid \bigvee_i x \in A_i \right\} \quad \text{union of all } A_i$$

$$\bigcap_{i \in I} A_i \cap \bigcap_{j \in J} A_j = \bigcap_{i \in I \cup J} A_i = \left\{ x \mid \bigwedge_{i \in I \cup J} x \in A_i \right\}$$

What happens at $I = \emptyset$?

$$\bigwedge_{x \in \emptyset} P(x) \Leftrightarrow W \quad \text{is always true}$$

This is axiomatic:

$$\bigwedge_{x \in \emptyset} P(x) \quad \text{is always true}$$

$I = \mathbb{R}$, for every $x \in \mathbb{R}$ a set A_x is given

$$\bigcap_{x \in \mathbb{R}} A_x = \left\{ y \mid \bigwedge_{x \in \mathbb{R}} y \in A_x \right\}$$

$$\bigvee_{x \in \emptyset} Q(x) \quad \text{is always false}$$

2 Power sets

Let A be a set.

$$P(A) = 2^A = \{B \mid B \subseteq A\}$$

is called a “power set” of A .

$$P(\emptyset) = \{\emptyset\}$$

$$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$

Let A, B be sets. The following set is called “cartesian product” (lat. renatus cartesianus) (by René Descartes, 17th century)

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Followingly,

$$A^2 = A \times A$$

$$A^n = \underbrace{A \times A \times \dots}_n$$

$$A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$$

$$A^n = \{(a_1, \dots, a_n) \mid a_i \in A\}$$

$$A^I = \{(a_i)_{i \in I} \mid a_i \in A\}$$

3ary tuples are called “triples”. $(a_i)_{i \in I}$ is called family of elements (where I is an index set).

3 Relations of sets

A *relation* on a set is a subset

$$R \subseteq X \times X$$

Notation: xRy means x is in relation with y . Hence $(x, y) \in R$.

Example: X is the set of austrians. The relation is marriage. Be aware that every married couple occurs twice. Once as (x, y) and once as (y, x) .

This lecture took place on 12th of Oct 2015 (Prof. Franz Lehner).

A relation of a set X is a subset $R \subseteq X \times X$. We denote xRy iff $(x, y) \in R$.

i	set	R
0	$X = \{\text{Austrian}\}$	“married”
1	$X = \{\text{Austrian}\}$	same location of birth
2	$X = \mathbb{R}$	$x \leq y$
3	X arbitrary	$x = y$
4	$X = \mathbb{N}$	$x \mid y$
5	$X = \mathbb{Z}$, defined $n \in \mathbb{N}$	$n \mid x - y$
6	$X = \{a, b, c\}$	$R = \{(a, a), (a, c), (b, b), (c, a), (c, c)\}$

i	reflexive	symmetrical	anti-sym.	transitive	konnex
0	false	true	false	false	false
1	true	true	false	true	false
2	true	false	true	true	true
3	true	true	true	true	false
4	true	false	true	true	false
5	true	true	false	true	false
6	true	true	false	true	false

Table 1: Examples for relations and their properties

A *relation* R operating on a set X is called

reflexive

if $\bigwedge_{x \in X} xRx$ (hence $(x, x) \in R$)

symmetrical

if $\bigwedge_{x \in X} y \in X (xRy \implies yRx)$

anti-symmetrical

if $\bigwedge_{x \in X} \bigwedge_{y \in X} (xRy \wedge yRx \implies x = y)$

transitive

if $\bigwedge_{x \in X} \bigwedge_{y \in X} \bigwedge_{z \in X} (xRy \wedge yRz \implies xRz)$

konnvex

if $\bigwedge_{x \in X} \bigwedge_{y \in X} (xRy \vee yRx)$

A relation satisfying reflexivity, symmetry and transitivity is called *equivalence relation*. Examples 2, 4, 6 and 7 are equivalence relations.

A relation satisfying reflexivity, anti-symmetry and transitivity is called *order relation*. Examples 3, 4 and 5 are order relations.

A relation satisfying reflexivity, anti-symmetry, transitivity and konnvexity is called *total order*. Example 2 is a total order.

Let \sim be an equivalence relation operating on set X . For $x \in X$,

$$[x] = \{y \in X \mid x \sim y\}$$

is called equivalence class of x .

Examples:

- $[x] = \{y \mid y \text{ has the same location of birth}\}$
- $[x] = \{y \mid x = y\} = \{x\}$
- $[x] = \{y \mid n \mid x - y\} = \{y \mid x - y = q \cdot n\} = \{y \mid y = x - q \cdot n\} = \{x + k \cdot n \mid k \in \mathbb{Z}\}$
- $[a] = \{a, c\}, [b] = \{b\}, [c] = \{a, c\}$

$X/\sim = \{[x] \mid x \in X\}$ is called *factor set* or *quotient set*.

Examples:

- $X/\sim = \{\{\text{Graz}\}, \{\text{Linz}\}, \{\text{Wien}\}, \dots\}$
- $X/\sim = \{\{x\} \mid x \in X\}$
- $\mathbb{Z}/\sim = \{[0], [1], [2], \dots, [n-1]\}$

$$n = 0 + 1 \cdot n \in [0]$$

$$0 = n - 1 \cdot n \in [n]$$

A *system of representatives* is a subset $S \subseteq X$ such that

$$\bigwedge_{[x] \in X/\sim} \dot{\bigvee}_{s \in S} s \in [x]$$

Examples:

- The mayor of a city.
- $S = X$
- $S = \{0, \dots, n-1\}$

Theorem 1. Let \sim be an equivalence relation operating on X . Then it holds that

$$\bigwedge_{x, y \in X} (x \sim y \iff [x] = [y])$$

Proof: Let $x, y \in X$ be arbitrary elements such that $x \sim y$. Show that $[x] \subseteq [y] \wedge [y] \subseteq [x]$. It suffices to show that $[x] \subseteq [y]$ because x, y can be arbitrary.

Show $\bigwedge_{z \in [x]} z \in [y]$. Let $z \in [x] \implies x \sim z$. Furthermore $x \sim y \xrightarrow{\text{symmetrical}} y \sim x$. Hence $y \sim x \wedge x \sim z \xrightarrow{\text{transitive}} y \sim z \implies z \in [y]$. Hence $[x] \subseteq [y]$. Hence $[x] = [y]$.

If $[x] = [y]$, then $y \in [y]$ (because its reflexive) hence $y \in [x] \implies x \sim y$.

Let X be a set. A *partition* of X is a subset $Z \subseteq \mathcal{P}(X)$. Z is the set of subsets of X such that

- $\bigcup_{A \in Z} A = X$
- $\bigwedge_{A, B \in Z} (A \neq B \implies A \cap B = \emptyset)$

$$\iff \bigwedge_{x \in X} \bigvee_{A \in Z} x \in A$$

Theorem 2. Let X be a non-empty set.

- Let \sim be an equivalence relation operating on X , then X/\sim is a partition of X .

- Let $Z \subseteq \mathcal{P}(X)$ a partition of X . There is exactly one equivalence relation \sim on X such that $X/\sim = Z$.

Proof. Let \sim be an equivalence relation on X . Then $X/\sim = \{[x] \mid x \in X\} \subseteq \mathcal{P}(X)$

- We need to show that $\bigcup_{x \in X} [x] = X$.

$$\begin{aligned} \bigwedge_{x \in X} x \sim y &\implies \bigwedge_{x \in X} x \in [x] \\ &\implies \bigwedge_{x \in X} x \in \bigcup_{y \in X} [y] \\ &\implies X \subseteq \bigcup_{y \in X} [y] \end{aligned}$$

- Furthermore we need to show that $\bigwedge_{x,y \in X} [x] \cap [y] \neq \emptyset \implies [x] = [y] \iff x \sim y$.

$$\begin{aligned} \text{Let } [x] \cap [y] \neq \emptyset &\iff \bigvee_z z \in [x] \cap [y] \\ &\iff \bigvee_z z \in [x] \wedge z \in [y] \end{aligned}$$

definition of equivalence class $\implies x \sim z \wedge y \sim z$

$$\text{symmetrical} \implies \bigvee_z x \sim z \wedge z \sim y$$

$$\xrightarrow{\text{transitive}} x \sim y$$

$$\xrightarrow{\text{Theorem 1}} [x] = [y]$$

□

This lecture took place on 13rd of Oct 2015 (Prof. Franz Lehner).

A *function* (or mapping) between two sets X and Y

$$f : X \rightarrow Y$$

$$x \mapsto f(x)$$

is a relation assigning every element $x \in X$ some $f(x) \in Y$.

X is called domain and Y is called co-domain (also range or image). $f(x)$ is called image of x under f . We can find a symbolic expression for a function or explicitly enumerate all mappings possibilities.

Examples:

$$\begin{aligned} f_1 : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\rightarrow x^2 \\ f_2 : \{0, 1\} &\rightarrow \mathbb{R} \\ 0 &\rightarrow 11 \qquad \qquad \qquad \rightarrow \pi \\ f_3 : \mathcal{P}(x) &\rightarrow \mathcal{P}(x) \\ A &\mapsto X \setminus A \end{aligned}$$

$$\begin{aligned} f_4 : X &\rightarrow X/\sim \\ x &\mapsto [x] \end{aligned}$$

$$\begin{aligned} f_5 : \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, y) &\mapsto x + y \end{aligned}$$

Remarks:

1. Domain and codomain are part of the definition of a function. A function is unambiguously defined by some graph:

$$G_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

therefore a relation between X and Y such that every $x \in X$ occurs exactly once.

$$\bigwedge_{x \in X} \bigvee_{y \in Y} (x, y) \in G_f$$

3. Two functions $f : X \rightarrow Y$, $f : U \rightarrow V$ are equivalent iff $X = U$, $Y = V$ Analogously f indicates a function and $\bigwedge_{x \in X} f(x) = g(x)$.

Hence the domain and codomain must be equivalent.

4. The function $\text{id}_X : X \rightarrow X$ is called “identity”.

5. Let $A \subseteq X$ be a subset.

$$\mathbb{1}_A = \chi_A : X \rightarrow \{0, 1\}$$

$$x \rightarrow \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

This function is called *indicator function of A* or *characteristic function of A*.

6. Every function $f : X \rightarrow \{0, 1\}$ is the indicator function of a subset of X , namely $f = \mathbb{1}_A$ where $A = \{x \in X \mid f(x) = 1\}$.

Let $A \subseteq X$ be a subset of $f : X \rightarrow Y$. Then $f|_A : A \rightarrow Y$ with $a \mapsto f(a)$ is called *restriction* of f to A .

$f|_A$ is not defined outside A .

Let $f : X \rightarrow Y$ be a function defined for $B \subseteq Y$.

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$$

Therefore we define the domain function

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

$f^{-1}(B)$ can be empty.

If $B = \{y\}$ then we write $f^{-1}(y)$ instead of $f^{-1}(\{y\})$.

$$f^{-1}(1) = f^{-1}(\{1\}) = \{+1, -1\}$$

$$f^{-1}(-1) = \emptyset$$

$$f(\{1, 2\}) = \{1, 4\}$$

$$f(\{+1, -1\}) = \{1\}$$

$$\tilde{f} : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$$

$$A \mapsto f(A) = \{f(x) \mid x \in A\}$$

Remark:

$$f^{-1}(B) = \bigcup_{b \in B} f^{-1}(b)$$

A function $f : X \rightarrow Y$ is called *injective* iff

$$\bigwedge_{x_1, x_2 \in X} (x_1 \neq x_2 \implies f(x_1) \neq f(x_2))$$

$$\iff \bigwedge_{x_1, x_2 \in X} (f(x_1) = f(x_2) \implies x_1 = x_2)$$

A function is called *surjective* iff

$$\bigwedge_{y \in Y} \bigvee_{x \in X} f(x) = y$$

A function is called *bijective* iff a function is injective and surjective.

$$\bigwedge_{y \in Y} \bigvee_{x \in X} f(x) = y$$

For a bijective function f^{-1} is called *inverse function*.

$$f^{-1} : Y \rightarrow X$$

$$y \mapsto \text{every distinct } x \text{ such that } f(x) = y$$

Be aware that $f^{-1}(y)$ sometimes means $f^{-1}(\{y\})$.

Examples:

- $f : x \mapsto 3x$ in $\mathbb{R} \rightarrow \mathbb{R}$ is injective and surjective. Therefore it is also bijective.

- $f : x \mapsto x^2$ in $\mathbb{R} \rightarrow \mathbb{R}$ is not injective and not surjective. We have a restriction:

$$\tilde{f} : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$$

With this domain, the function is bijective.

- $f : x \mapsto x^3$ in $\mathbb{R} \rightarrow \mathbb{R}$ is bijective.
- $f : A \mapsto A^C = X \setminus A$ in $\mathcal{P}(X) \rightarrow \mathcal{P}(X)$. Injective if $A \neq B$. Wlog $x \in A$, $x \notin B$

$$\Rightarrow x \notin A^C, x \in B^C \Rightarrow B^C \neq A^C$$

Surjective: Given $B \subseteq X$, find $A \subseteq X$ such that

$$f(A) = A^C = B$$

Yes, if $A = B^C$ that $A^C = (B^C)^C = B$. The inverse function is the function itself.

A function is called *involution* if its inverse function is the function itself.

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions, the function

$$g \circ f : X \rightarrow Z$$

$$x \mapsto g(f(x))$$

is called composition of f and g .

Theorem 3. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow U$ be functions.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U$$

Then

$$h \circ (g \circ f) \stackrel{?}{=} (h \circ g) \circ f$$

Proof. $h \circ (g \circ f)$ and $(h \circ g) \circ f$ bounded from X to U .

$$(h \circ (g \circ f))(x) = h(g \circ f(x)) = h(g(f(x))) = h \circ g(f(x)) = (h \circ g) \circ f(x)$$

Theorem 4. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be functions. If f and g are injective/surjective or bijective, then $g \circ f$ has the same property.

Proof. Let f, g be injective. So $g \circ f$ must also be injective.

Let $x_1, x_2 \in X$ such that $g \circ f(x_1) = g \circ f(x_2)$. We need to show $x_1 = x_2$.

$$g \circ f(x_1) = g \circ f(x_2)$$

$$\Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow y_1 = f(x_1), y_2 = f(x_2)$$

$$g(y_1) = g(y_2) \xrightarrow{g \text{ injective}} Y_1 = Y_2$$

$$\Rightarrow f(x_1) = f(x_2) \xrightarrow{f \text{ injective}} x_1 = x_2$$

□

Remarks:

1. If $f : X \rightarrow Y$ is bijective, then $f^{-1} : Y \rightarrow X$ and it holds that

$$f \circ f^{-1} = \text{id}_Y$$

$$f^{-1} \circ f = \text{id}_X$$

2. Let f, g be bijective, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Is $g \circ f$ bijective? Is g or f bijective?

4 Solutions to linear equation systems

□ A linear equation system is an equation system of structure:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n &= b_n \end{aligned}$$

with coefficients a_{ij} , $b_i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, n\}$. x_1, x_2, \dots, x_n are the unknown variables.

$ax + b$ is linear whereas $ax^2 + bx + c$ is non-linear.

A particular solution of the equation system is an n -tuple (x_1, \dots, x_n) , which satisfies the equation.

The scheme

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

is called matrix of the equation system.

The equation system is called homogeneous if all $b_i = 0$. A homogeneous system always has at least one solution; $(0, 0, \dots, 0)$.

$$ax = b \implies x = \frac{b}{a}$$

Case distinction:

Case 1 with $a \neq 0$ $x = \frac{b}{a}$ has a distinct solution

Case 2 with $a = 0, b \neq 0$ has no solution

Case 3 with $a = 0, b = 0$ every x is a solution

Example 1. Let $n = 2$ and $m = 1$.

$$a_1x + a_2y = b$$

No distinct solution.

Case distinction:

$$a_2 \neq 0$$

$$y = \frac{-a_1x + b}{a_2}$$

x is arbitrary.

$$a_2 = 0$$

$$a_1x = b$$

y is arbitrary. Case distinction:

$$a_1 \neq 0 \quad x = \frac{b}{a_1}$$

$$a_1 = 0, b = 0 \quad 0 = 0 \implies \mathbb{R} \text{ as solution}$$

$$a_1 = 0, b \neq 0 \quad \text{no solution}$$

$$n = 2, m = 2$$

$$a_{1,1}x + a_{1,2}y = b_1$$

$$a_{2,1}x + a_{2,2}y = b_2$$

Case distinction:

Case 1 intersection between two lines (exactly one solution)

Case 2 two parallel lines (no solution)

Case 3 one line (infinite solution)

4.1 Substitution

Example 2. Example for case 1.

$$x + y = 1$$

$$x - y = 2$$

We subtract the second from the first equation.

$$\begin{aligned} 0 - 2y &= 1 \\ \Rightarrow y &= -\frac{1}{2} \\ \Rightarrow x = 1 - y &= \frac{3}{2} \end{aligned}$$

Distinct solution $(\frac{3}{2}, -\frac{1}{2})$.

Example 3. Example for case 2.

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= -1 \end{aligned}$$

We subtract equation two minus the first equation taken two times.

$$0 + 0 = -3$$

No solution.

Example 4. Example for case 3.

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned}$$

We take the second equation minus two times the first equation.

$$0 + 0 = 0$$

$0 \cdot y = 0$ is a solution for every possible $y \in \mathbb{R}$. Free variable t with $y = t$.

$$x = 1 - y = 1 - t$$

Solution set:

$$\{(1 - t, t) \mid t \in \mathbb{R}\}$$

This lecture took place on 19th of Oct 2015 (Prof. Franz Lehner).

What if there are 2 unknown variables, but more equations?

Case 4 a solution, where only two lines intersect. But not all three at one time.

Case 5 Two equations are equivalent, but other equations are parallel or intersecting.

What if there are 3 unknown variables, but only one equation?

Case 6 No unique solution. Express one variable by others. Equation describes a layer.

What if there are three variables and two equations?

Case 7 Two layers intersect in one line

Case 8 Two layers are parallel

What if there are three variables and three equations?

Case 9 Intersection of three layers in one point

Or in general: point, line, layer, no solution or \mathbb{R}^3 . On a line we have one degree of freedom whereas \mathbb{R}^3 gives us three degrees of freedom.

Example

$$\begin{aligned} -x + y + 2z &= 2 \\ 3x - y + z &= 6 \\ -x + 3y + 4z &= 4 \end{aligned}$$

We use Gauss-Jordan elimination:

$$\begin{aligned} 2 + 3 \cdot 10 \cdot 2y - 7z &= 12 \\ 3 - 12y + 2z &= 2 \end{aligned}$$

The following equation system then has the same solution:

$$\begin{aligned} -x + y + 2z &= 2 \\ 2y + 7z &= 12 \\ 2y + 2z &= 2 \end{aligned}$$

We again use Gauss-Jordan elimination:

$$2 - 30 + 5z = 10$$

Therefore we derived:

$$\begin{aligned} -x + y + 2z &= 2 \\ 2y + 2z &= 2 \\ 5z &= 10 \end{aligned}$$

Then $z = 2$, $y = -1$ and $x = 1$ follows.

Different notation (to save time & space, matrix notation):

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \\ \hline 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \\ \hline & 0 & 5 & 10 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 5 & 10 \\ \hline -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ \hline -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ \hline -x & 0 & 0 & -1 \\ 0 & y & 0 & -1 \\ 0 & 0 & z & 2 \end{array} \right)$$

Distinct solution.

Another example:

$$\begin{aligned} x + y + z &= 1 \\ x - 2z + 2z &= 2 \\ 4x + y + 3z &= 5 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 \\ 4 & 1 & 5 & 5 \\ \hline 0 & -3 & 1 & 1 \\ 0 & -3 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

We encountered a tautology $0 = 0$. We have two pivot rows left:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 \\ \hline 1 & 4 & 0 & 0 \\ 0 & -3 & 1 & 1 \\ \hline x & +4y & & = 0 \\ 0 & -3y & +z & = 1 \end{array} \right)$$

y can be chosen arbitrarily. $y = t$ once y has been defined.

$$z = 1 + 3y = 1 + 3t$$

$$x = -4y = -4t$$

The solution set is given as:

$$\{(-4t, t, 1 + 3t) \mid t \in \mathbb{R}\}$$

This represents a line in \mathbb{R}^3 .

Example without solution

$$3x + 2y + z = 3$$

$$2x + y + z = 0$$

$$6x + 2y + z = 6$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \\ \hline -1 & -1 & 0 & -3 \\ -6 & -6 & 0 & -6 \\ \hline 0 & 0 & 0 & 12 \end{array} \right)$$

There is no solution to $0 = 12$. Therefore no solution is possible for the equation system.

4.2 Gauss-Jordan elimination algorithm

1. Write matrix
2. Find $a_{ij} \neq 0$ (“pivot element” which was not a pivot element before, i -th row = pivot row, j -th row = pivot column)
 - (a) mark a_{ij}
 - (b) subtract $\frac{a_{kj}}{a_{ij}}$ times i -th row from the k -th row for every $k \neq i$. In the j -th row a zero is created.
3. If no new pivot element can be found:

- (a) Delete all rows, which only have 0s on the left and right side
- (b) If there is a row which contains only 0s on the left side
 - i. If right-hand side is not 0, NO SOLUTION!
 - ii. If right-hand side is 0, apply back substitution meaning
 - iii. Iterate over all pivot elements in reversed order and create 0 in corresponding pivot column
 - iv. All columns which look like the pivot column, are assigned to free parameters
 - v. those x_j , which are assigned to pivot columns, can be represented by the right side and free parameters

Example with 4 equations

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & -2 & -3 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 0 & -2 & -2 & -6 & -8 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -1 & -2 & -3 & -4 \end{array} \right)$$

First row is pivot row. First column is pivot column. 2nd row and 2nd column have not been pivot elements yet.

$$(\ 0 \ 0 \ 2 \ 0 \mid 0 \)$$

Therefore $2x_3 = 0$.

$$(\ 0 \ 0 \ 0 \ 0 \mid 0 \)$$

We have found an equivalent system:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 2 & 0 & 0 \end{array} \right)$$

4 is a free parameter. Therefore we set $x_4 = t$. From $2x_3 = 0$, $x_3 = 0$ follows.

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 5 \\ 0 & -1 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & -2 & -3 \\ 0 & -1 & 0 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right)$$

$$\begin{aligned} x_4 &= t \\ x_3 &= 0 \\ -x_2 - 3x_4 &= -4 \\ x_2 &= 4 - 3x_4 = 4 - 3t \\ x_1 - 2x_4 &= -3 \\ x_1 &= -3 + 2x_4 = -3 + 2t \end{aligned}$$

Solution set: $\{(-3 + 2t, 4 - 3t, 0, t) \mid t \in \mathbb{R}\}$

5 Vector spaces

A vector is an element of \mathbb{R}^n ($\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$):

$$\left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$$

Column vectors or n-tuples in \mathbb{R}^n .

We define addition:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}$$

Multiplication for $\lambda \in \mathbb{R}$:

$$\lambda \cdot \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$$

Geometric interpretation for $n = 1, 2, 3, \dots$: For $n \leq 3$ we can think of n -tuples as points on lines, layers or within the room.

Let S be the set of all pairs of points (A, B) . Consider it as directed path from A to B . Equivalence relation on S :

$$(A, B) \sim (A', B')$$

if (A', B') comes from (A, B) using a parallel translation.

Is parallel translation an equivalence relation?

reflexivity $(A, B) \sim (A, B)$, ✓

symmetry if $(A, B) \sim (A', B')$ then also $(A', B') \sim (A, B)$, inversed parallel translation, ✓

transitivity if $(A, B) \sim (A', B')$ and $(A', B') \sim (A'', B'')$, then $(A, B) \sim (A'', B'')$, composition of parallel translations, ✓

A vector is therefore an equivalence class of directed paths.

$$\overrightarrow{PQ} = [(P, Q)]$$

The set of vectors is in bijection with the set of points. In every equivalence class there is one representative of structure $(0, A)$. $\overrightarrow{0A}$ is called position vector (dt. Ortsvektor) to A .

Addition of vectors (diagonal of a parallelogram)

Multiplication of vectors (stretching)

5.1 Properties

5.1.1 Addition

Commutativity law:

$$a + b = b + a$$

Associativity law:

$$a + (b + c) = (a + b) + c$$

Zero vector:

$$a + -a = 0$$

5.1.2 Multiplication

Associativity law:

$$\lambda \cdot (\mu \cdot a) = (\lambda \cdot \mu) \cdot a$$

Distributivity law:

$$(\lambda + \mu) \cdot a = \lambda a + \mu a$$

$$\mu \cdot (a + b) = \lambda a + \lambda b$$

5.2 Applications

5.2.1 Diagonals of a parallelogram

The diagonals of a parallelogram intersect exactly on the halfway of the whole diagonal. Hence we claim $|AS| = |SC|$ and $|BS| = |SD|$. Let M be the midpoint of \overrightarrow{AC} and N be the midpoint of \overrightarrow{BD} . Then $M = N$ must hold.

Let's assume the opposite ($M \neq N$).

$$\begin{aligned} \overrightarrow{CM} &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} \\ &= \overrightarrow{OA} - \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) \end{aligned}$$

$$\begin{aligned} \overrightarrow{ON} &= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BD} \\ &= \overrightarrow{OA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} \\ &= \overrightarrow{OA} + \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{CD}) \\ &= \overrightarrow{OA} + \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BA}) \\ &= \overrightarrow{OA} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} - \frac{1}{2}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD} \\ &= \overrightarrow{OM} \end{aligned}$$

5.2.2 Line crossing two points

The line crossing two points P_1 and P_2 is defined as

$$\begin{aligned} &\left\{ \overrightarrow{OP_1} + t \cdot \overrightarrow{P_1P_2} \mid t \in \mathbb{R} \right\} \\ &= \left\{ \overrightarrow{OP_1} + t \cdot (\overrightarrow{OP_2} - \overrightarrow{OP_1}) \mid t \in \mathbb{R} \right\} \end{aligned}$$

5.2.3 A layer can be defined by three points

A layer can be defined by three points P_1 , P_2 and P_3 .

$$\left\{ \overrightarrow{OP_1} + s \cdot \overrightarrow{P_1P_2} + t \cdot \overrightarrow{P_1P_3} \mid s, t \in \mathbb{R} \right\}$$

5.3 Algebraic structures

A set M with a mapping $\circ : M \times M \rightarrow M$ ($(x, y) \mapsto x \circ y$) is called *Magma* or *algebraic structure*.

5.3.1 Examples

Examples for M :

$$\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}, \mathbb{C}$$

Examples for mappings \circ :

$$\begin{aligned}\circ &= +, \cdot \\ x \circ y &= x + y \\ x \circ y &= x \cdot y\end{aligned}$$

1. Example $M = \mathbb{N}$ and $x \circ y = x^y$.
2. Example $M = \{\pm 1\}$ and $x \circ t = x \cdot y$.

	+1	-1
+1	+1	-1
-1	-1	+1

Table 2: composition table

3. Example $M = \mathcal{P}(X)$ and

$$A \circ B = \begin{cases} A \cap B \\ A \cup B \\ A \Delta B \end{cases}$$

4. Example $M = \{a, b, c, e\}$ and
5. Example $A = \{a, b, c, \dots\}$ where the set is the alphabet. Then $M = \{a_1, \dots, a_n \mid n \in \mathbb{N}, a_i \in A\}$ is the set of words. Then our composition is defined as

$$a_1 \dots a_m \circ b_1 \dots b_n = a_1 \dots a_m b_1 \dots b_n$$

A^* is the set of possible words. A^+ is defined as $A^* \setminus \{\varepsilon\}$ where ε is the empty word.

	a	b	c	e
a	e	c	b	a
b	c	e	a	b
c	b	a	e	c
e	a	b	c	e

Table 3: composition table

6. Example $M = X^X = \{f : X \rightarrow X\}$ of an arbitrary set. $f \circ g$ is the composition (compute f after g).

5.4 Compositions

Let (M, a) be a Magma. The composition is called

associative if

$$\bigwedge_{x, y, z \in M} (x \circ y) \circ z = x \circ (y \circ z)$$

commutative if

$$\bigwedge_{x, y \in M} x \circ y = y \circ x$$

All examples above are associative¹. The last two examples are not commutative; others are²

An element $e \in M$ is called

left-neutral if

$$\bigwedge_{x \in M} e \circ x = x$$

right-neutral if

$$\bigwedge_{x \in M} x \circ e = x$$

¹Assuming the first example uses addition. x^y is not associative.

²Assuming the first example uses addition. x^y is not commutative.

A neutral element is left- and right-neutral.

Applied to the examples:

1. 0 acts as neutral element in addition. 1 is the neutral element of multiplication.
2. 1 is the neutral element
3. $A \cap B$ (X as neutral element), $A \cup B$ (\emptyset as neutral element), $A \triangle B$ is left for the practicals
4. e as neutral element
5. ε as neutral element
6. identity function acts as neutral element, $\text{id} \circ f = f' = f \circ \text{id}$

Let (M, \circ) be a magma with a neutral element e . Let $x \in M$, then $y \in M$ is called

left-inverse if $y \circ x = e$

right-inverse if $x \circ y = e$

An *inverse* element to x is left- and right-inverse simultaneously. x is *invertible* if an inverse element exists.

Applied to examples:

1. $(\mathbb{N}_0, +)$ has no inverse element. $(\mathbb{Z}, +)$ has an inverse element to x : $-x$. Same for \mathbb{Q} and \mathbb{R} . (\mathbb{N}, \cdot) has inverse element $\{1\}$. All non-zero elements in (\mathbb{Q}, \cdot) are invertible.
2. (\mathbb{Z}, \cdot) has inverse elements $\{\pm 1\}$.
3. $A \cap B = X$: inverse elements are $\{X\}$. $A \cup B = \emptyset$: inverse elements are $\{\emptyset\}$. $A \triangle B$ is left as an exercise.
4. All elements are invertible to themselves
5. For a_1, \dots, a_m , the invertible elements are $\{\varepsilon\}$

6. The invertible elements are defined by any bijective mapping $X \rightarrow X$.

A *semigroup* is a magma with associative composition. A *monoid* is a semigroup with a neutral element. A group is a monoid where every element is invertible. An *abelian group* (or commutative group) is a semigroup, monoid or group with a commutative composition.

Niels Henrik Abel (1802–1829)

Examples:

1. $(\mathbb{N}, +)$ is a semi-group. $(\mathbb{N}_0, +)$ is a monoid. (\mathbb{N}, \cdot) is a monoid. $(\mathbb{Z}, +)$ is a group. (\mathbb{Z}, \cdot) is a monoid. $(\mathbb{Q} \setminus \{0\}, \cdot)$ is a group. $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are also groups. All of them are abelian.
2. is a group and abelian.
3. $(\mathcal{P}(X), \cap)$ and $(\mathcal{P}(X), \cup)$ are monoids. $(\mathcal{P}(X), \triangle)$ is an abelian group.
4. is an abelian group
5. (A^+, \cdot) is a semi-group (non-commutative). (A^*, \circ) is a monoid (non-commutative).

$$\mathbb{N} = A^t \text{ where } A = \{a\}$$

6. (X^X, \circ) is a non-commutative monoid

Theorem 5. A magma (G, \circ) is a group iff

G1 $\bigwedge_{x,y,z} (x \circ y) \circ z = x \circ (y \circ z)$ “associative”

G2 $\bigvee_{e \in G} \bigwedge_x e \circ x = x$ “left-neutral element”

G3 $\bigwedge_x \bigvee_y y \circ x = e$ “left-inverse element”

Neutral elements are necessarily right-neutral / right-inverse.

Proof. Show that

- i. any left-neutral element is right-neutral

ii. left-inverse elements are right-inverse

ii. Let $x, y \in G$. y is left-inverse to x : $y \circ x = e$. Show that $x \circ y = e$.

$$x \circ y = e \circ (x \circ y) = (z \circ y) \circ (x \circ y)$$

From G3 it follows that

$$\bigvee_z z \circ y = e$$

From associativity it follows that $z \circ (y \circ x) \circ y \Rightarrow z \circ (e \circ z) \Rightarrow z \circ y = e$.

i. Let $x, y \in G$ with inverse elements x^{-1} and y^{-1} . Let $z = y^{-1} \circ x^{-1}$. Then,

$$\begin{aligned} (x \circ y) \circ z &= (x \circ y) \circ (y^{-1} \circ x^{-1}) \\ &= x \circ \underbrace{y \circ y^{-1}}_e \circ x^{-1} \\ &= x \circ e \circ x^{-1} \\ &= x \circ x^{-1} \\ &= e \end{aligned}$$

So $x \circ y$ is right-invertible (analogously left-invertible)

$$\Rightarrow x \circ y \in G$$

□

Theorem 6. Let (G, \cdot) be a group.

1. The neutral element is unique
2. Inverse elements are unique (therefore every element has exactly one inverse)
3. Equivalence laws:

$$\bigwedge_{x, y, z \in G} x \circ z = y \circ z \implies x = y$$

$$\bigwedge_{x, y, z \in G} z \circ x = z \circ y \implies x = y$$

Proof. 1. Let e' be another neutral element:

$$e' \underbrace{=}_{e' \text{ is neutral}} e' \circ e \underbrace{=}_{e' \text{ is neutral}} e$$

2. Let y, y' be two inverse elements to x

$$y \circ x = e = x \circ y$$

$$y' \circ x = e = x \circ y'$$

Show that $y = y'$:

$$y = y \circ e = y \circ (x \circ y') = (y \circ x) \circ y' = e \circ y' = y'$$

3. Let $x \circ z = y \circ z$. Let w be inverse to z : $z \circ w = e$.

$$(x \circ z) \circ w = (y \circ z) \circ w$$

$$x \circ (z \circ w) = y \circ (z \circ w)$$

$$x \circ e = y \circ e$$

$$x = y$$

□

- The unique inverse element of Theorem 6 (2) of x is denoted with x^{-1} .
- Abelian groups are typically written additive. In $(G, +)$ the inverse element is denoted $-x$.

Theorem 7. Let (M, \cdot) be a monoid. Then $\{x \in M \mid x \text{ is invertible}\}$ is a group.

Proof. Let $G = \{x \in M \mid x \text{ is invertible}\}$. Show that

1. If $x, y \in G$, then also $x \circ y \in G$.
2. Associativity is inherited from M .
3. A neutral element $e \in G$ exists.

Magma	$(M, \circ), \circ : M \times M \rightarrow M$
Semigroup	+associative
Monoid	+neutral element e : $e \circ a = a = a \circ e$
Group	invertibility of all elements: $\bigwedge_x \bigvee_y x \circ y = e = y \circ x$

Table 4: Group theory cheatsheet

4. All elements are invertible in G .

Proof:

1. Let $x, y \in G$ with inverse x^{-1}, y^{-1} . Let $z = y^{-1} \circ x^{-1}$. Then it holds that

$$\begin{aligned}
 (x \circ y) \circ z &= (x \circ y) \circ (y^{-1} \circ x^{-1}) \\
 &= x \circ y \circ y^{-1} \circ x^{-1} \\
 &= x \circ e \circ x^{-1} \\
 &= x \circ x^{-1} \\
 &= e
 \end{aligned}$$

$x \circ y$ is right invertible (analogously: left invertible)

$$\Rightarrow x \circ y \in G$$

2. follows immediately

3. $e \circ e = e \Rightarrow e$ is invertible $\Rightarrow e \in G$

4. $x \in G \Rightarrow x^{-1} \in G$ because $x^{-1} \circ x = e \Rightarrow (x^{-1})^{-1} = x$

Theorem 8. Let (M, \circ) be a group.

$$\stackrel{G1}{\Rightarrow} \text{associative}$$

$$\stackrel{G2}{\Rightarrow} \bigvee_e \bigwedge_x e \circ x = x$$

$$\stackrel{G3}{\Rightarrow} \bigvee_x \bigwedge_y y \circ x = e$$

Show that

i. A left-neutral element is right-neutral

ii. Left-inverse elements are also right-inverse

Proof. ii. Let $x \in G \stackrel{G3}{\Rightarrow} \bigvee_y y \circ x = e$. Show that $x \circ y = e$.

$$\begin{aligned}
 x \circ y &\stackrel{G2}{=} e \circ (x \circ y) = (z \circ y) \circ (x \circ y) \\
 &\stackrel{G3}{\Rightarrow} \bigvee_z z \circ y = e
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{G1}{=} z \circ (y \circ x) \circ y \\
 &= z \circ (e \circ y) \\
 &= z \circ y = e
 \end{aligned}$$

i. Let $x \in G$, show that $x \circ e = x$. Let y be left-inverse to x . $e = y \circ x$.

$$\begin{aligned}
 x \circ e &= x \circ (y \circ x) \stackrel{G1}{=} (x \circ y) \circ x = e \circ x \stackrel{G2}{=} x \\
 &\Rightarrow e \text{ is also right-neutral}
 \end{aligned}$$

□

□

This lecture took place on 27th of Oct 2015 (Prof. Franz Lehner).

How do we construct groups? We select an associative (M, \circ) . $G = \{x \in M \mid x \text{ invertible}\}$ is a group.

Corollary 1.

$$(M, \circ) = (X^X, \circ) = \{f : X \rightarrow X\}$$

$$S_X = \{f : X \rightarrow X \text{ bijective}\}$$

(S_X, \circ) is a group (\circ is composition of functions) and is called symmetric group over X or permutation group (if $|X| < \infty$).

Corollary 2. Let $X = \{1, \dots, n\}$. Let $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ bijective. Then π is typically written as scheme

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

is called permutation (rearrangement).

For finite sets $f : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is bijective. $\Leftrightarrow f$ is injective. $\Leftrightarrow f$ is surjective. This does not hold for infinite sets.

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

$$f(n) = 2n$$

is injective, but not surjective

$$S_2 = S_{\{1,2\}} = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

$$= \left\{ \begin{array}{cc} 1 & \mapsto 2 \\ 1 & \mapsto 2 \end{array}, \begin{array}{cc} 1 & \mapsto 2 \\ 2 & \mapsto 1 \end{array} \right\}$$

$$S_3 = S_{\{1,2,3\}} = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \right\}$$

$$|S_n| = n!$$

S_3 is non-commutative!

$$\neg \bigwedge_{\pi, \phi \in S_3} \pi \circ \phi = \phi \circ \pi$$

Example 5.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Example 6. Symmetry group of a rectangle: The group of motions, which keeps the rectangle invariant (ie. the rectangle is mapped to itself)

- not translation
- rotation
- mirroring

Horizontal mirroring:

$$h \cong \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$$

Vertical mirroring:

$$V \cong \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$$

$$d_\pi \cong \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix}$$

Notes to create composition table:

$$v \circ h = \begin{pmatrix} A & B & C & D \\ D & C & B & A \\ C & D & A & B \end{pmatrix} = \begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix} = d_\pi$$

$$(v \circ h)^{-1} = d_\pi^{-1} = d_\pi$$

$$h^{-1} \circ v^{-1} = h \circ v$$

$$h \circ d_\pi = h \circ (h \circ v) = (h \circ h) \circ v = id \circ v = v$$

\circ	id	h	v	d_π
id	id	h	v	d_π
h	h	id	d_π	v
v	v	d_π	id	h
d_π	d_π	v	h	id

Table 5: Composition table for symmetry group of rectangles. The diagonal id represents that all elements are inverse to themselves. This table is symmetrical. Therefore this group is commutative.

Theorem 9. *Computations modulo n . The relation*

$$x \equiv y \pmod{n} \Leftrightarrow n \mid x - y$$

is an equivalence relation on \mathbb{Z} . The equivalence classes

$$[x]_n = \{x + q \circ n \mid q \in \mathbb{Z}\}$$

are called residuo modulo classes or congruence classes modulo n .

A system of representatives is

$$\{0, \dots, n-1\}$$

Factor set:

$$\mathbb{Z}_n := \mathbb{Z}/n = \mathbb{Z}/n\mathbb{Z} := \mathbb{Z}/\equiv_n$$

We define addition and multiplication

$$[x]_n + [y]_n := [x + y]_n$$

$$[x]_n \cdot [y]_n := [x \cdot y]_n$$

Are we allowed to define it like that? What about $[x]_n = [x + n]_n$? Does the definition not depend on the definition of the system of representatives?

Theorem 10. *(i) The addition on \mathbb{Z}_n is well-defined if*

$$x \equiv x' \pmod{n} \quad (\text{ie. } [x]_n = [x']_n)$$

and

$$y \equiv y' \pmod{n} \quad (\text{ie. } [y]_n = [y']_n)$$

then also $x + y \equiv x' + y' \pmod{n}$ (ie. $[x + y]_n = [x' + y']_n$).

($\mathbb{Z}_n, +$) is an abelian group with neutral element $[0]_n$ and inverse elements $-[x]_n = [-x]_n$.

(ii) The multiplication on \mathbb{Z}_n is well-defined if

$$x \equiv x' \pmod{n}$$

and

$$y \equiv y' \pmod{n}$$

then also $x \circ y \equiv x' \cdot y' \pmod{n}$ (ie. $[x \cdot y]_n = [x' \cdot y']_n$). (\mathbb{Z}_n, \cdot) is a commutative matroid with neutral element $[1]_n$. $\mathbb{Z}_n^ = \mathbb{Z}_n \setminus \{[0]_n\}$ is a group if $n \in \mathbb{P}$*

Proof. Let $x = x' \pmod{n}$ and $y = y' \pmod{n}$. Show that $x + y = x' + y'$ and $x \cdot y = x' \cdot y'$. $n \mid x - x'$ and $n \mid y - y'$. Show that

$$n \mid (x + y) - (x' + y') \text{ and } n \mid x \cdot y - x' \cdot y'$$

So for addition,

$$\bigvee_k x - x' = k \cdot n$$

$$\bigvee_l y - y' = l \cdot n$$

$$\begin{aligned} \Rightarrow (x + y) - (x' + y') &= x + y - x' - y' \\ &= x - x' + y - y' \\ &= k \cdot n + l \cdot n \\ &= (k + l) \cdot n \\ &= n \mid (x + y) - (x' + y') \end{aligned}$$

For multiplication,

$$\begin{aligned} x \cdot y &= (x' + kn) \cdot (y' + ln) \\ &= (x' \cdot y') + (k \cdot n \cdot y') + x' \cdot l \cdot n + k \cdot n \cdot l \cdot n \\ &= x' \cdot y' + n(R \cdot y' + l \cdot x' + k \cdot l \cdot n) \end{aligned}$$

$$xy - x'y' = \text{multiple of } n$$

$$\Rightarrow n \mid xy - x'y'$$

Example 7. $(\mathbb{Z}_n, +)$ is a group?

- We show G1:

$$([x]_n + [y]_n) + [z]_n \stackrel{?}{=} [x]_n + ([y]_n + [z]_n)$$

$$[x + y]_n + [z]_n \stackrel{?}{=} [x]_n + [y + z]_n$$

$$\Rightarrow [(x + y) + z]_n = [x + (y + z)]_n$$

- We show G2, by definition of $[0]_n$ as neutral element

$$[x]_n + [0]_n = [x + 0]_n = [x]_n$$

- We show G3, by definition of $[-x]_n$ as neutral element

$$[x]_n + [-x]_n = [x - x]_n = [0]_n$$

Analogously,

$$([x]_n \cdot [y]_n) \cdot [z]_n = [x]_n ([y]_n \cdot [z]_n)$$

$$[x]_n \cdot [1]_n = [x1]_n = [x]_n$$

Therefore $[1]_n$ is the neutral element for multiplication

What is the inverse for multiplication? It is immediate, that $[0]_n$ has no inverse for multiplication.

$$[0]_n \cdot [x]_n = [0]_n \neq [1]_n$$

in $\mathbb{Z}_n \setminus \{[0]_n\}$?

Case distinction:

$n \notin \mathbb{P}$

$$\Rightarrow \bigvee_{1 < n_1, n_2 < n} n = n_1 \cdot n_2$$

$$[n_1]_n \cdot [n_2]_n = [n_1 \cdot n_2]_n = [n]_n = [0]_n$$

$\Rightarrow [n_1]_n$ has not inverse element!

Assume

$$\bigvee_{[x]_n} [n_1]_n \cdot [x]_n = [1]_n$$

$$\Rightarrow [n_2] \cdot [n_1] \cdot [x]_n = [n_2]_n [1]_n$$

$$\Rightarrow [0]_n = [n_2]_n$$

This is a contradiction. No inverse can exist.

$n \in \mathbb{P}$ Beforehand, for prime numbers p it holds that

$$p \mid ab \Rightarrow p \mid a \vee p \mid b$$

Theorem 11. We claim that every $[x]_n \neq [0]_n$ has an inverse.

Proof.

$$V_X = \{[x], [2x], [3x], \dots, [(n-1)x]\} \text{ multiples of } [x]_n$$

Then $[0]_n \notin V_x$. Assume

$$\bigvee_k [k \cdot x]_n = [0]_x$$

therefore

$$\bigvee_k k \cdot x \equiv 0 \pmod n$$

$$\Rightarrow n \mid kx$$

$$\Rightarrow n \mid k \vee n \mid x$$

$$\Rightarrow n \mid x$$

$$\Rightarrow [x]_n$$

$$\Rightarrow [0]_n$$

This is a contradiction.

□

Theorem 12. *All entries of V_X are different.*

Proof. Assume

$$\begin{aligned} \bigvee_{1 \leq k, l \leq n-1} [kx]_n &= [lx]_n \\ [kx]_n - [lx]_n &= [0]_n \\ [(k-l)x] &= [0]_n \\ \Rightarrow (k-l)x &\equiv 0 \pmod n \\ \Rightarrow n &\mid (k-l)x \\ \Rightarrow n &\mid k-l \vee n \mid x \end{aligned}$$

The second condition cannot hold.

$$\Rightarrow k-l=0$$

Requirement: $[x]_n \neq [0]_n$.

$$\Rightarrow \{[x]_n, [2x]_n, \dots, [(n-1)x]_n\} \subseteq \{[1], [2], \dots, [n-1]\}$$

are all different.

$$\begin{aligned} \Rightarrow \bigvee_k [kv]_n &= [1]_n \\ \Rightarrow [k]_n &= [x]_n^{-1} \end{aligned}$$

k is constructed using the Euclidean algorithm.

Example 8.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 6: Composition table for $(\mathbb{Z}_5, +)$

In general $[x]_n$ is invertible iff $\gcd(x, n) = 1$.

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Table 7: Composition table for (\mathbb{Z}_5, \cdot) . Every row is a permutation of the first row. Every row (except 0) has a 1 element is therefore invertible.

·	1	2	3	4
1	1	2	3	4
2	2	4	0	2
3	3	0	3	0
4	4	2	0	4
4	5	4	3	2

□

Table 8: Composition table for (\mathbb{Z}_6, \cdot) . 1 and 5 have a 1-element and is therefore invertible.

+	0	1
0	0	1
1	1	0

Table 9: Composition table for $(\mathbb{Z}_2, +)$

·	+1	-1
+1	+1	-1
-1	-1	+1

Table 10: Composition table for $(\{\pm 1\}, \cdot)$

$$h : \mathbb{Z}_2 \rightarrow \{\pm 1\}$$

$$[0]_2 \rightarrow +1$$

$$[1]_2 \rightarrow -2$$

The composition table of \mathbb{Z}_2 maps to composition table of $\{\pm 1\}$.

Therefore

$$h([x] + [y]) = h([x]) \cdot h([y]) \forall [x], [y]$$

Definition 1. Let (G_1, \circ) and (G_2, \circ) be 2 groups. A map

$$h : G_1 \rightarrow G_2$$

is called group-homomorphism if it holds that $\bigwedge_{x,y \in G_1} h(x \circ_1 y) = h(x) \circ_2 h(y)$.

This lecture took place on 3rd of November 2015 (Franz Lehner).

Definition 2. Let (G_1, \circ_1) and (G_2, \circ_2) be groups. A mapping $h : G_1 \rightarrow G_2$ is called group-homomorphism if $h(a \circ_1 b) = h(a) \circ_2 h(b)$ for all $a, b \in G_1$.

Additionally

- if h is injective, the mapping is called “field embedding”.
- if h is surjective, the mapping is called “epimorphism”.
- if h is bijective, the mapping is called “isomorphism”.
- two groups are called isomorph, if there exists some isomorphism.

Example 9. $\frac{(\mathbb{Z}_2, +)}{\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}} G_1 = \mathbb{Z}_2, \circ_1 = + \quad \frac{(\{\pm 1\}, \cdot)}{\begin{array}{c|cc} & +1 & -1 \\ \hline +1 & +1 & -1 \\ -1 & -1 & +1 \end{array}} G_2 = \{+1, -1\}, \circ_2 = \cdot$

$$h : \mathbb{Z}_2 \rightarrow \{\pm 1\}$$

$$[0]_2 \mapsto +1$$

$$[1]_2 \mapsto -1$$

preserves $h([a] + [b]) = h([a]) \cdot h([b])$ are isomorphic: $(\mathbb{Z}_2, +) \cong (\{\pm 1\}, \cdot)$.

Definition 3. A homomorphism $G \rightarrow G$ is called endomorphism. An isomorphism $G \rightarrow G$ (bijective endomorphism) is called automorphism.

Example 10. 1. $(\mathbb{Z}, +)$ with fixed $n \in \mathbb{N}$.

$$h_n : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$h_n : x \mapsto n \cdot x$$

Is an endomorphism.

Show that

$$h_n(x + y) = h_n(x) + h_n(y)$$

$$n(x + y) = n \cdot x + n \cdot y$$

No epimorphism for $n \geq 2$.

2.

$$g : \mathbb{Z} \rightarrow \mathbb{Z}$$

$$x \mapsto x + 1$$

$$g(1 + 1) \stackrel{?}{=} 3$$

$$g(1) + g(1) \stackrel{?}{=} 1 + 1 + 1$$

$$4 \neq 3$$

3.

$$q_n : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}_n, +)$$

$$a \mapsto [a]_n$$

Show that

$$q_n(a + b) = q_n(a) + q_n(b)$$

$$q_n(a + b) = [a + b]_n$$

$$= [a]_n + [b]_n$$

$$= q_n(a) + q_n(b)$$

$$[0]_n = q_n(0) = q_n(n)$$

$$[1]_n = q_n(1)$$

$$\vdots$$

$$[n-1]_n = q_n(n-1)$$

Epimorphism, but no isomorphism.

4.

$$(\mathbb{R}^*, \cdot) \rightarrow (\{\pm 1\}, \cdot)$$

$$\mathbb{R}^* = \mathbb{R} \setminus \{0\}$$

$$\text{sign} : x \mapsto \text{sign}(x)$$

$$\text{sign}(x \cdot y) = \text{sign}(x) \cdot \text{sign}(y)$$

is a group homomorphism and epimorphism, but no isomorphism.

5.

$$h : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}, +)$$

$$x \mapsto -x$$

$$h(x+y) = -(x+y) = -x-y = h(x) + h(y)$$

is homomorphism.

It is surjective ($x = h(-x)$) and injective ($h(x) = h(y) \Rightarrow x = y$). Therefore it is an isomorphism.

6.

$$(\mathbb{R}^+ =]0, \infty[, \cdot) \rightarrow (\mathbb{R}, +)$$

$$x \mapsto \log(x)$$

$$\log(x \cdot y) = \log(x) + \log(y)$$

Is a group homomorphism, epimorphism and isomorphism.

Theorem 13. 1. *The composition of homomorphisms is a homomorphism.*

Let

$$q : (G_1, \circ_1) \rightarrow (G_2, \circ_2)$$

$$h : (G_2, \circ_2) \rightarrow (G_3, \circ_3)$$

be homomorphisms, then $h \circ q : (G_1, \circ_1) \rightarrow (G_3, \circ_3)$ is a homomorphism.

2. *The inverse mapping of an isomorphism is an isomorphism.*

3. *Isomorphism is an equivalence relation on the “set of all groups”. Therefore on an arbitrary set of groups the relation $G_1 \cong G_2$ is an equivalence relation.*

Proof. 1.

$$h \circ g(a \circ_1 b) = h \circ g(a) \circ_3 h \circ g(b)$$

$$(h \circ g)(a \circ_1 b) = h(g(a \circ_1 b))$$

$$\stackrel{g \text{ is homomorphous}}{=} h(g(a) \circ_2 g(b))$$

$$\stackrel{h \text{ is homomorphous}}{=} h(g(a)) \circ_3 h(g(b))$$

$$= (h \circ g)(a) \circ_3 (h \circ g)(b)$$

2. To be worked through in the practicals.

3. To be worked through in the practicals.

□

Theorem 14. *Let (G_1, \circ_1) and (G_2, \circ_2) be groups with a neutral element $e_1 \in G_1$ and $e_2 \in G_2$ and $h : G_1 \rightarrow G_2$ is a homomorphism. Then it holds that*

$$1. \ h(e_1) = e_2$$

$$2. \ h(x^{-1}) = h(x)^{-1} \forall x \in G_1$$

Proof. 1.

$$h(e_1) = h(e_1) \circ e_2$$

$$h(e_1) = h(e_1 \circ e_1)$$

$$= h(e_1) \circ h(e_1)$$

$$h(e_1) \circ e_2 = h(e_1) \circ h(e_1)$$

$$\text{Cutback law in } G_2 \Rightarrow e_2 = h(e_1)$$

2.

$$h(x^{-1}) = h(x)^{-1} \Leftrightarrow h(x) \circ h(x^{-1}) = e_2$$

$$h(x) \circ_2 h(x^{-1}) = h(x \circ_1 x^{-1}) \stackrel{\text{homomorphism}}{=} h(e_1) \stackrel{\text{bc}(1)}{=} e_2$$

Therefore $h(x^{-1}) \circ_2 h(x) = e_2$.

$\Rightarrow h(x^{-1})$ is left- and rightinverse to $h(x)$. $\Rightarrow h(x)^{-1} = h(x^{-1})$.

□

Definition 4. A subgroup of a group (G, \circ) is a non-empty subset $H \subseteq G$ such that

$$1. \bigwedge_{a,b \in H} a \circ b \in H$$

$$2. \bigwedge_{a \in H} a^{-1} \in H$$

Notation: $H \leq G$.

Example 11.

$$(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +) \quad \checkmark$$

$$(\mathbb{N}, +) \subseteq (\mathbb{Q}, +) \quad \nexists$$

$$(\mathbb{Q}, +) \subseteq (\mathbb{R}, +) \quad \checkmark$$

$$(\mathbb{Q}, +) \subseteq (\mathbb{C}, +) \quad \checkmark$$

$n \in \mathbb{N}$ is fixed:

$$n \cdot \mathbb{Z} = \{n \cdot k \mid k \in \mathbb{Z}\} \leq \mathbb{Z}$$

$$1. n \cdot k + n \cdot l = n \cdot (k + l) \in n \cdot \mathbb{Z}$$

$$2. -nk = n(-k) \in n \cdot \mathbb{Z}$$

Theorem 15.

$$S_n \leq S_{n+1}$$

$$S_n = \{f : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \text{ is bijective}\}$$

$$S_{n+1} = \{f : \{1, \dots, n+1\} \rightarrow \{1, \dots, n+1\} \text{ is bijective}\}$$

So $S_n \leq S_{n+1}$ cannot hold, right? S_n cannot be a subgroup.

Wrong, we interpreted it wrongfully: There is a subset $H \subseteq S_{n+1}$ which is a subgroup as by Theorem 4 such that $S_n \cong H$.

$$H = \{f : \{1, \dots, n+1\} \rightarrow \{1, \dots, n+1\} \mid f \text{ is bijective}\} \\ \Rightarrow H \cong S_n$$

Corollary 3.

$$\mathbb{Z} \rightarrow n \cdot \mathbb{Z} \leq \mathbb{Z}$$

$$x \mapsto n \cdot x$$

is bijective.

$$\Rightarrow \mathbb{Z} \cong n \cdot \mathbb{Z}$$

$\Rightarrow \mathbb{Z}$ is isomorphous to its own subgroup

Remark 1. 1. Let $H \leq G$ be a subgroup, then $e \in H$.

Because with $H \neq \emptyset$, let $x \in H$. From the group definition it follows that $x^{-1} \in H$ and therefore $x \circ x^{-1} \in H$ with $x \circ x^{-1} = e$.

2. (H, \circ) is a group.

Theorem 16. Let (G_1, \circ_1) and (G_2, \circ_2) be groups.

$$h : G_1 \rightarrow G_2 \text{ is a homomorphism}$$

$$H_1 \leq G_1 \quad H_2 \leq G_2 \quad \text{are subgroups}$$

Then it holds that

$$1. h(H_1) \leq G_2$$

$$2. h^{-1}(H_2) \leq G_1$$

Proof. 1. Let $h(H_1) \leq G_2$.

$$\Rightarrow \bigwedge_{u,v \in h(H_1)} u \circ_2 v \in h(H_1)$$

$$\Rightarrow \bigwedge_{x,y \in H_1} h(x) \circ h(y) \in h(H_1)$$

$$\Rightarrow \bigwedge_{x,y \in H_1} \bigvee_{z \in H_1} h(x) \circ h(y) = h(z)$$

h is a homomorphism:

$$\Rightarrow h(x) \circ_2 h(y) = h(x \circ_1 y)$$

\Rightarrow choose $z = x \circ_1 y \in H_1$ because $H_1 \leq G_1$

2. Let $u \in h(H_1)$. We need to show that $u^{-1} \in h(H_1)$. Find $a \in H_1$ such that $u^{-1} = h(a)$. Let $b \in H_1$ with $h(b) = u$

$$\Rightarrow u^{-1} = h(b)^{-1} = h(b^{-1}) \in h(H_1)$$

then $b^{-1} \in H_1$.

□

Remark 2. Always two trivial subgroups of a group G exist, namely

$$H = G$$

$$H = \{e\}$$

One example which only has two trivial subgroups is $(\mathbb{Z}_p, +)$.

Definition 5. Let $h : G_1 \rightarrow G_2$ be a homomorphism. Then $h^{-1}(\{e_2\})$ is a subgroup of G_1 and is called kernel of a homomorphism.

$$\text{kernel}(h) = \{x \in G_1 \mid h(x) = e_2\}$$

$h(G_1) \leq G_2$ is a subgroup and is called image of h , denoted $\text{im}(h) = h(G_1)$.

Definition 6. A ring is a tuple $(R, +, \cdot)$ with $R \neq \emptyset$ and $+, \cdot$ are combinations $R \times R \rightarrow R$, such that

1. $(R, +)$ is an abelian group (“additive group”)
2. (R, \cdot) is a semigroup (“multiplicative semigroup”)
3. distributive laws hold

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Examples include: $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{R}, +, \cdot)$.

A ring is called commutative if (R, \cdot) is commutative. If (R, \cdot) is a monoid, then $(R, +, \cdot)$ is a ring with a one-element. The neutral element with respect to $+$ is called zero-element.

Inverse elements with respect to $+$ are denoted as $-x$. Inverse elements with respect to \cdot are denoted as x^{-1} .

Example 12. $(\mathbb{Z}, +, \cdot)$ is a commutative ring with a one-element. The same applies for $(\mathbb{Z}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$ and $(\mathbb{C}, +, \cdot)$.

$$\mathbb{R}[x] = \{a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N}_0, a_i \in \mathbb{R}\}$$

is the ring of polynomials with respect to addition and multiplication (as we know it in \mathbb{R}). The one element with respect to multiplication is 1 (because $a \cdot (1 \cdot x_+^0 \cdot \dots) = a$).

$$(1 + x)^{-1} = \sum_{n=0}^{\infty} (-x)^n \notin \mathbb{R}[x]$$

$$(a_0 \cdot x^0)^{-1} = \frac{1}{a_0} x^0$$

Only constant polynomials are invertible.

Theorem 17. $(\mathbb{Z}_n, +, \cdot)$ is a commutative ring with a one-element.

Proof. $(\mathbb{Z}_n, +)$ is a group. (\mathbb{Z}_n, \cdot) is a monoid. They are commutative. We have already proven that.

What remains to show is the distributive law:

$$\begin{aligned} ([a]_n + [b]_n) \cdot [c]_n &= [a + b]_n \cdot [c]_n \\ &= [(a + b) \cdot c]_n \\ &= [a \cdot c + b \cdot c]_n \\ &= [a \cdot c]_n + [b \cdot c]_n \\ &= [a]_n \cdot [c]_n + [b]_n \cdot [c]_n \end{aligned}$$

□ “Es ändert nichts an dem Ganzen, aber sie haben ein besseres Gefühl.”
(Franz Lehner)

This lecture took place on 9th of Nov 2015 (Franz Lehner).

Definition 7. Let $(R, +, \cdot)$ be a ring. An element $x \in R$ is called zero-divisor if $\bigvee_{y \in R} y \neq 0 \wedge x \cdot y = 0$. R is called zero-divisor-free if it does not contain zero-divisors.

Theorem 18. $(\mathbb{Z}_n, +, \cdot)$ is zero-divisor-free $\Leftrightarrow n \in \mathbb{P}$

Definition 8. Let $(R_1, +_1, \cdot_1)$ and $(R_2, +_2, \cdot_2)$ be rings. A mapping $h : R_1 \rightarrow R_2$ is called ring homomorphism if

$$\bigwedge_{a, b \in R} h(a +_1 b) = h(a) +_2 h(b)$$

$$\bigwedge_{a, b \in R} h(a \cdot_1 b) = h(a) \cdot_2 h(b)$$

Example 13.

$$\begin{aligned} (\mathbb{Z}, +, \cdot) &\rightarrow (\mathbb{Z}_n, +, \cdot) \\ x &\mapsto [x]_n \end{aligned}$$

Definition 9. A field is a commutative ring $(K, +, \cdot)$ with 1 in which each element $a \in K \setminus \{0\}$ has an inverse element. Therefore $(K \setminus \{0\}, \cdot)$ is an abelian group.

We denote $\frac{1}{x}$ instead of x^{-1} .

Example 14. $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Z}_p, +, \cdot)$ for $p \in \mathbb{P}$, not $(\mathbb{Z}, +, \cdot)$.

Corollary 4.

1. A field is zero-divisor-free (but not the opposite, \mathbb{Z} as example)
2. The zero-element of a non-trivial ring cannot have an inverse
3. Let $|R| \geq 2$, then

$$\underbrace{0}_{\text{zero element}} \neq \underbrace{1}_{\text{one element}}$$

Proof. One possible trivial ring is:

$$R = \{a\}$$

$$a + a := a \quad a \cdot a := a$$

3. Select $a \notin \{0\}$. Then

$$1 \cdot a = a$$

$$0 \cdot a = 0$$

$$\Rightarrow 1 \neq 0$$

1. Let $a, b \in K \setminus \{a\}$. Assume $a \cdot b = 0$.

$$\Rightarrow 0 = a^{-1} \cdot 0 \cdot b^{-1} = a^{-1} \cdot (a \cdot b) \cdot b^{-1} = (a^{-1} \cdot a) \cdot (b \cdot b^{-1}) = 1 \cdot 1 = 1$$

$$\Rightarrow 0 = 1 \quad \nexists$$

2. Let a be inverse to 0.

$$\Rightarrow a \cdot 0 = 1$$

$$\Rightarrow a = 0$$

- 4.

$$\bigwedge_{a \in R} a \cdot 0 = 0$$

$$a \cdot 0 = a \cdot (0 + 0)$$

$$a \cdot 0 = a \cdot 0 + a \cdot 0$$

$$\Rightarrow a \cdot 0 + 0 = a \cdot 0 + a \cdot 0$$

$$\Rightarrow a \cdot 0 = 0$$

□

Definition 10. (field extensions.) *The equation $x^2 - 2 = 0$ has no solution in \mathbb{Q} . We claim: $K = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field. The proof will be provided in the practicals.*

So a field K with $\mathbb{Q} \subsetneq K \subsetneq \mathbb{R}$ is a field extension for \mathbb{Q} .

Definition 11 (complex numbers). *The equation $x^2 + 1 = 0$ has no solution in \mathbb{R} because $x^2 > 0 \forall x \in \mathbb{R}$. Assume some i exists with $i^2 = -1$ (therefore $i = \sqrt{-1}$) with*

$$\begin{aligned}(a + bi) + (c + di) &= a + c + (b + d)i \\ (a + bi)(c + di) &= ac + adi + bic + bdi^2 \\ &= ac - bd + (ad + bc)i\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{a + bi} &= \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} \\ &= \frac{a - bi}{a^2 - (bi)^2} \\ &= \frac{a - bi}{a^2 + b^2}\end{aligned}$$

with $a^2 + b^2 \neq 0$ (does not hold for $a = b = 0$).

We define the complex numbers as $\mathbb{C} = \mathbb{R}^2$ with operations

$$\begin{aligned}(a, b) + (c, d) &:= (a + c, b + d) \\ (a, b) \cdot (c, d) &:= (ac - bd, ad + bc)\end{aligned}$$

We denote:

$$\begin{aligned}0 &= (0, 0) \\ 1 &= (1, 0) \\ i &= (0, 1)\end{aligned}$$

Every $z \in \mathbb{C}$ has the structure $(a, b) = a \cdot 1 + b \cdot i$.

Theorem 19. 1. $(\mathbb{C}, +, \cdot)$ is a field (proof: provided in practicals).

2. \mathbb{C} contains \mathbb{R} as subfield. Therefore

$$l : \mathbb{R} \rightarrow \mathbb{C}$$

$$x \mapsto x + 0 \cdot i = (x, 0)$$

\mathbb{R} is identified with $l(\mathbb{R})$.

Corollary 5.

$$\underbrace{\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2})}_{\mathbb{N}_0} \subseteq \underbrace{\mathbb{R} \subseteq \mathbb{C}}_{\mathbb{N}_1}$$

Also:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{3}) \subseteq \mathbb{R} \subseteq \mathbb{C}$$

Off topic: Peano curve.

Definition 12 (Fundamental Theorem of algebra). *In \mathbb{C} every polynomial $x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0$ has n solutions.*

Therefore \mathbb{C} is algebraically closed (but there exist transcendental extensions).

Definition 13 (Quaternions). \mathbb{R}^4 has a ring structure such that every element is invertible, but it is not commutative (division ring with elements called quaternions).

Definition 14. Let $z = x + iy$ be some element in \mathbb{C} . Then $\Re(z) = x$ (real part) and $\Im(z) = y$ (imaginary part) of \mathbb{Z} . $\bar{z} = x - iy$ is called complex conjugate of z . i is defined as solution of the equation $x^2 + 1 = 0$.

Geometrically, the real part is represented on the x -axis and the imaginary part is quantified on the y -axis.

- The addition of two complex numbers then geometrically corresponds to vector addition in \mathbb{R}^2 .

Complex numbers in polar coordinates are defined with

$$x + iy = r(\cos \varphi + i \cdot \sin \varphi)$$

$$\Rightarrow r = \sqrt{x^2 + y^2}$$

$$\Rightarrow \varphi = \arctan \frac{y}{x}$$

- The multiplication looks like this:

$$\begin{aligned}
 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
 &= r_1(\cos \varphi_1 + i \sin \varphi_1) \cdot r_2(\cos \varphi_2 + i \sin \varphi_2) \\
 &= r_1 r_2 (\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2 + i(\sin \varphi_1 \cos \varphi_2 + \cos \varphi_1 \sin \varphi_2)) \\
 &= r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))
 \end{aligned}$$

So geometrically this is rotation by φ with scaling by factor r .

From this the Eulerian equation follows³.

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

6 Reasoning about vector spaces and bases

Definition 15. Let $(K, +, \cdot)$ be a field. A vector space of K is a tuple (V, \oplus, \odot) if $V \neq \emptyset$.

- $V \times V \rightarrow V$
 $(\lambda, \mu) \mapsto v \oplus \mu$
- $K \times V \rightarrow V$
 $(\lambda, \mu) \rightarrow \lambda \odot v$

such that

1. (V, \oplus) is an abelian group.
2. associative law holds:

$$\bigwedge_{v \in V} \bigwedge_{\lambda \in K} \bigwedge_{\mu \in K} (\lambda \cdot \mu) \odot v = \lambda \odot (\mu \odot v)$$

3. distributive law holds:

$$\bigwedge_{\lambda \in K} \bigwedge_{v, w \in V} \lambda \odot (v \oplus w) = (\lambda \odot v) \oplus (\lambda \odot w)$$

³but can only be seen easily with the Taylor series expansion of e

$$\bigwedge_{\lambda, \mu \in K} \bigwedge_{v \in V} (\lambda + \mu) \odot v = (\lambda \odot v) \oplus (\mu \odot v)$$

4. Furthermore,

$$\bigwedge_{v \in V} 1 \odot v = v$$

Remark 3. The elements of V are called vectors. The elements of K are called scalars. Furthermore we simplify notation:

- $+$ instead of \oplus (vector addition)
- \cdot instead of \odot (vector multiplication)

Example 15. 1.

$$K^n = \left\{ \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \mid \xi \in K \right\}$$

$$\text{with } \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} + \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \begin{pmatrix} \xi_1 + \eta_1 \\ \vdots \\ \xi_n + \eta_n \end{pmatrix}$$

$$\text{and } \lambda \cdot \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} \lambda \xi_1 \\ \vdots \\ \lambda \xi_n \end{pmatrix}$$

- 2.

$$K^{m \times n} = \left\{ \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \mid a_{i,j} \in K \right\}$$

is the so-called component notation. Addition and multiplication is done component-wise.

3. Let X be an arbitrary set.

$$K^X = \{f : X \rightarrow K \mid f \text{ function}\}$$

$$(f + g)(x) := f(x) + g(x)$$

$$(\lambda f)(x) := \lambda(f(x))$$

$$\Rightarrow f + g, \lambda \cdot f \in K^X$$

Proof. (a) is a special case of (c) Specifically $X = \{1, \dots, n\}$. Every function $f : \{1, \dots, n\} \rightarrow K$ is uniquely defined by vector $\begin{pmatrix} f(1) \\ \vdots \\ f(n) \end{pmatrix}$. On the

opposite site, every vector $\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix}$ is a function $f : \{1, \dots, n\} \rightarrow K$ with

$$k \mapsto \varepsilon_k.$$

(d)

$$X = \mathbb{N} \quad K^{\mathbb{N}} = \{(\varepsilon_n)_{n \in \mathbb{N}} \mid \varepsilon_i \in \mathbb{K}\}$$

is the space of all sequences.

Definition 16. If $(K, +, \cdot)$ is a ring, the structure is called module.

Corollary 6.

$$\lambda(u + v) = \lambda u + \lambda v$$

$$(\lambda + \mu)v = \lambda v + \mu v$$

$$1 \cdot v = v$$

$$(\lambda \mu)v = \lambda(\mu v)$$

Example 16. Let $(K^n, +, \cdot)$ be a field.

$$K^X = \{f : X \rightarrow K\}$$

$$\bigwedge_{x \in X} (f + g)(x) = f(x) + g(x)$$

$$\bigwedge_{x \in X} (\lambda f)(x) = \lambda f(x)$$

Corollary 7. (e) \mathbb{R} is a vector space over \mathbb{Q} . $(\mathbb{R}, +)$ is an abelian group.

$$\cdot : \mathbb{Q} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(\lambda \in \mathbb{Q}, x \in \mathbb{R}) \mapsto \lambda \cdot x \in \mathbb{R}$$

$$\mathbb{R} = \mathbb{Q}^X$$

but \mathbb{Q} is not a vector space over \mathbb{R} .

K has a zero element denoted 0 . $(V, +)$ has a neutral element; also denoted 0 . You should infer from context which one is meant. At the beginning we denote the neutral element of $(V, +)$ with $\underline{0}$.

Theorem 20. This is a direct result following from the axioms. Let $(V, +, \cdot)$ be a vector space over K .

$$1. \bigwedge_{v \in V} 0 \cdot v = \underline{0}$$

□

$$2. \bigwedge_{\lambda \in K} \lambda \cdot \underline{0} = \underline{0}$$

$$3. \bigwedge_{v \in V} \bigwedge_{\lambda \in K} \lambda \cdot v = \underline{0} \Rightarrow \lambda = 0 \vee v = \underline{0}$$

$$4. \bigwedge_{v \in V} (-1) \cdot v = -v \text{ with } -v \text{ as neutral element in } (V, +)$$

Proof. 1. For the zero element it holds,

$$0 \cdot v = (0 + 0) \cdot v \stackrel{\text{distr. law}}{=} 0 \cdot v + 0 \cdot v$$

$$\text{but also } 0 \cdot v + \underline{0} \Rightarrow 0 \cdot v + \underline{0} = 0 \cdot v + 0 \cdot v. \underline{0} = 0 \cdot v.$$

2.

$$\lambda \cdot \underline{0} = \lambda(\underline{0} + \underline{0}) = \lambda \underline{0} + \lambda \underline{0}$$

$$\lambda \cdot \underline{0} = \lambda \cdot \underline{0} + \underline{0} \Rightarrow \underline{0} = \lambda \cdot \underline{0}$$

3.

$$\lambda v = 0 \Rightarrow \lambda = 0 \vee v = 0$$

$$A \rightarrow B \vee C \Leftrightarrow (\neg A \vee B \vee C) \Leftrightarrow \neg(A \wedge \neg B) \vee C \Leftrightarrow A \wedge \neg B \rightarrow C$$

We show: $(\lambda v = 0 \wedge \lambda \neq 0) \Rightarrow v = 0$.

Proof.

$$\begin{aligned}\lambda \cdot v = \underline{0} &\Rightarrow \lambda^{-1}(\lambda \cdot v) = \lambda^{-1} \cdot \underline{0} \\ (\lambda^{-1}\lambda) \cdot v &= \underline{0} \\ v = 1 \cdot v &= \underline{0}\end{aligned}$$

4. We need to show: $(-1) \cdot v + v = 0$

Hence, $(-1) \cdot v$ is the additive inverse to v .

$$\begin{aligned}(-1) \cdot v + v &= (-1) \cdot v + 1 \cdot v \\ &= (-1 + 1) \cdot v \\ &= 0 \cdot v \\ &\xrightarrow{\text{first law}} \underline{0}\end{aligned}$$

6.1 Subspaces, linear independence and bases

Definition 17. Let $(V, +, \cdot)$ be a vector space over K . A subset $U \subseteq V$ is called subspace of V if

U1: $U \neq \emptyset$

U2: $\bigwedge_{u,v \in U} u + v \in U$

U3: $\bigwedge_{\lambda \in K} \bigwedge_{u \in U} \lambda u \in U$

Proof.

$$\bigwedge_{u \in U} -u \in U$$

Choose $\lambda = -1$ in subspace and multiply as in Theorem 4.

Corollary 8. The trivial subspaces are $U = V$ and $U = \{0\}$.

Theorem 21. (subspace criterion.) Let $U \subseteq V$ be a subspace.

$$\Leftrightarrow U \neq \emptyset \wedge \bigwedge_{\lambda, \mu \in K} \bigwedge_{u, v \in U} \lambda u + \mu v \in U$$

Proof. Let $\lambda, \mu \in K$ and $u, v \in U$. □

$$\mathbf{U3} \Rightarrow \lambda u \in U \wedge \mu v \in U$$

$$\mathbf{U2} \Rightarrow \lambda u + \mu v \in U$$

So **U1** is immediate, **U2** follows with $\lambda = \mu = 1$ and **U3** follows with $v = 0$ and $\mu = 0$. □

Theorem 22. Let $(V, +, \cdot)$ be a vector space. $U \subseteq V$ is a subspace. Then

$$(U, +|_{U \times U}, \cdot|_{K \times U})$$

□ is a vector space.

Proof. Associativity and distributivity gets inherited. $(U, +)$ is a group.

$$-u = (-1) \cdot u \underbrace{\in}_{\mathbf{U3}} U$$

□

Example 17. 1. \mathbb{R} is a vector space over \mathbb{Q} .

$$\mathbb{Q} \subseteq \mathbb{R} \text{ is a subspace}$$

2. $V = \mathbb{R}^2$ with $U = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\} = \{(t, -t) \mid t \in \mathbb{R}\}$. Claim: U is a subspace.

Proof. **U1** $U \neq \emptyset$ because $(0, 0) \in U$.

□

$$\lambda, \mu \in \mathbb{R} \quad u, v \in U$$

Show that $\lambda u + \mu v \in U$.

Proof.

$$u = (s, -s) \text{ for some element in } \mathbb{R}$$

$$v = (t, -t) \quad t \in \mathbb{R}$$

$$\begin{aligned} \lambda u + \mu v &= \lambda(s, -s) + \mu(t, -t) \\ &= (\lambda s - \mu t, \mu t, -\mu t) \\ &= (\lambda s + \mu t, -\lambda s - \mu t) \\ &= (r, -r) \text{ with } r = \lambda s + \mu t \\ &\subseteq U \end{aligned}$$

3. $V = \mathbb{R}^2$ with $U = \{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$ is not a subspace. $U \neq \emptyset$.

$$(0, 1) \in U$$

$$(1, 0) \in U$$

$$(0, 1) + (1, 0) = (1, 1) \notin U$$

Remark 4. A subspace always contains the zero-vector:

$$U \neq \emptyset \Rightarrow \bigvee_u u \in U \xrightarrow{U3} \underline{0} = 0 \cdot u \in U$$

Remark 5. What is the usual approach to find possible subspaces?

- Is $\underline{0} \in U$? If no, no subspace exists.
- Else yes, $U \neq \emptyset$

We proceed with the subspace criterion.

6.2 Construction of subspaces

Theorem 23. Let $(V, +, \cdot)$ be vector over K . Let I be an index set. Let $(U_i)_{i \in I}$ be a family of subspaces $U_i \subseteq V$. Then $\bigcap_{i \in I} U_i$ is a subspace.

Proof. **U1**

$$\bigcap_{i \in I} U_i \neq \emptyset$$

$$\bigwedge_{i \in I} 0 \in U_i \Rightarrow 0 \in \bigcap_{i \in I} U_i = \left\{ u \mid \bigwedge_{i \in I} u \in U_i \right\}$$

$$\Rightarrow \bigcap_{i \in I} U_i \neq \emptyset$$

□

□ **UR** We need to show $\lambda, \mu \in K, a, b \in \bigcap_{i \in I} U_i$ then $\lambda a + \mu b \in \bigcap_{i \in I} U_i$.

$$\begin{aligned} \bigwedge_{i \in I} a \in U_i \wedge b \in U_i &\xrightarrow{\text{all } U_i \text{ are subspaces}} \bigwedge_{i \in I} \lambda a + \mu b \in U_i \\ &\Rightarrow \lambda a + \mu b \in \bigcap_{i \in I} U_i \end{aligned}$$

□

Remark 6. An equivalent statement for $U_1 \cup U_2$ does not hold! Unions of subspaces must not be subspaces.

- $U_1 = \{(x, 0) \mid x \in \mathbb{R}\}$
- $U_2 = \{(0, y) \mid y \in \mathbb{R}\}$

$$u = (1, 0) \in U_1 \subseteq U_1 \cup U_2$$

$$v = (0, 1) \in U_2 \subseteq U_1 \cup U_2$$

$$u + v = (1, 1) \notin U_1 \cup U_2$$

To construct a new subspace from $U_1 \cup U_2$ we need to extend it.

Definition 18. Let $(V, +, \cdot)$ be a vector space in K .

$$M \subseteq V$$

The linear hull of M is the smallest subspace of V , which contains M :

$$[M] := \bigcap \{U \subseteq V \mid U \cup R \text{ such that } M \subseteq U\}$$

This is a subspace by Theorem 23. For $M = 0$,

$$[\emptyset] = \{0\}$$

We also say $[M]$ is the subspace generated by M .

Remark 7. $[M]$ is well-defined.

At least one subspace exists which contains M :

$$U = V \Rightarrow [M] \neq \emptyset$$

Every subspace $U \subseteq V$ which contains M , contains also $[M]$ because M occurs in $M \subseteq U$ as intersection. Therefore $[M] \subseteq U$.

This construction is not constructive! We know that one smallest subspace exists, but don't know what it looks like.

There is no known method to determine whether the given vector $v \in V$ is in $[M]$ or not.

Example 18. (second most simple case.)

$$M = \{a\}$$

Case distinction:

Case 1: $a = 0$

$$[\{0\}] = \{0\}$$

Case 2: $a \neq 0$

From **U1** it follows that $[\{a\}] \neq \emptyset$ because $0, a \in [\{a\}]$.

From **U3** it follows that $\lambda, a \in [\{a\}] \forall \lambda \in K$.

$$K \cdot a := [\{a\}] = \{\lambda a \mid \lambda \in K\}$$

We look at a subfield: Let $u, v \in K \cdot a$ and $\lambda, \mu \in K$. Show that

$$\lambda u + \mu v \in K \cdot a$$

$$\bigwedge_{\alpha \in K} u = \alpha \cdot a \quad \bigwedge_{\beta \in K} v = \beta \cdot a$$

$$\lambda u + \mu v = \lambda(\alpha \cdot a) + \mu(\beta \cdot a)$$

Associativity: $(\lambda \cdot \alpha) \cdot a + (\mu \cdot \beta) \cdot a$

Distributivity: $(\lambda \cdot \alpha + \mu \cdot \beta) \cdot a \in K \cdot a$

Using these laws the subfield is actually a plane. So we look at the more general case in the next Theorem.

Theorem 24. Let $(V, +, \cdot)$ be a vector space over K with $a_1, \dots, a_n \in V$.

A linear combination of vectors a_1, \dots, a_n is a vector of structure

$$\lambda_1 \cdot a_1 + \lambda_2 \cdot a_2 + \dots + \lambda_n \cdot a_n$$

with $\lambda_i \in K$.

Let $\emptyset \neq M \subseteq V$, then a linear combination of M is a vector of structure

$$\lambda_1 \cdot a_1 + \lambda_2 \cdot a_2 + \dots + \lambda_n \cdot a_n$$

with $a_i \in M$, $\lambda_i \in K$ and $n \in \mathbb{N}$.

Construction of arbitrary finitely many vectors.

$$L(M) = \{\lambda_1 a_1 + \dots + \lambda_n a_n \mid n \in \mathbb{N}, a_i \in M, \lambda_i \in K\}$$

is the set of all linear combinations. We define $L(\emptyset) := \{0\} = [\emptyset]$.

$$L(\{a\}) \stackrel{!}{=} \{\lambda \cdot a \mid \lambda \in K\} = K \cdot a = [\{a\}]$$

Theorem 25. Let $(V, +, \cdot)$ be a vector space over K .

$$M \subseteq V \text{ as subset}$$

Then $[M] = L(M)$.

Proof. Show that,

- $[M] \subseteq L(M)$ therefore $L(M)$ is subspace which contains M .
- $L(M) \subseteq [M]$ therefore every subspace containing M , contains also $L(M)$.

We need to show $M \subseteq L(M)$. $L(M)$ is a subspace.

U1 $L(M) \neq \emptyset$ if $M = \emptyset \Rightarrow$ by definition. If $M \neq \emptyset \Rightarrow M \subseteq L(M)$.

$M \subseteq L(M)$. Let $a \in M \Rightarrow a = 1 \cdot a \in L(M)$

$$n = 1 \quad a_1 = a \quad \lambda_1 = 1$$

$M \subseteq L(M)$. $L(M)$ is a subspace.

Subfield: Let $u, v \in L(M)$ and $\lambda, \mu \in K$. Then also $\lambda u + \mu v \in L(M)$. Let $u = \lambda_1 a_1 + \dots + \lambda_m a_m$ with $\lambda_i \in K$ and $a_i \in M$. Let $v = \mu_1 b_1 + \dots + \mu_n b_n$ with $\mu_i \in K, b_i \in M$.

$$\begin{aligned} \lambda u + \mu v &= \lambda(\lambda_1 a_1 + \dots + \lambda_m a_m) + \mu(\mu_1 b_1 + \dots + \mu_n b_n) \\ &= \lambda \lambda_1 + \dots + \lambda \lambda_m a_m + \mu \mu_1 b_1 + \dots + \mu \mu_n b_n \\ &= v_1 c_1 + \dots + v_{m+n} c_n \in L(M) \end{aligned}$$

with

$$c_i = \begin{cases} a_i & i \leq m \in M \\ b_{i-m} & i \geq m+1 \end{cases}$$

$$v_i = \begin{cases} \lambda \cdot \lambda_i & i \leq i \leq n \\ \mu \mu_{i-m} & m+1 \leq i \leq m+n \end{cases}$$

This lecture took place on 16th of Nov 2015 (Franz Lehner).

6.3 Revision

$$U \subseteq V \quad U \neq \emptyset$$

(1) $U \neq \emptyset$

(UR) $a, b \in U \rightarrow \lambda a + \mu b$

Therefore every linear combination is also in U .

$M \subseteq V$ subset

$$[M] = \text{smallest vector space which contains } M := \bigcap_{U \subseteq V} U \supseteq \{0\}$$

$$L(M) = \{\lambda v_1 + \dots + \lambda_n v_n \mid n \in \mathbb{N}, \lambda \in K, v_n \in M\}$$

Theorem 26.

$$[M] = L(M)$$

$$[M] \subseteq L(M)$$

$$L(M) \subseteq [M]$$

ToDo content incomplete/incorrect

Proof. It suffices to show, that every subspace U , which contains M , contains also $L(M)$. Every U in intersection $\bigcap_{M \subseteq U} U$ contains also $L(M)$.

$$\lambda_1, \dots, \lambda_n \in K \Rightarrow L(M) \subseteq \bigcap_U U$$

Let $v_1,$

□

□

Remark 8. 1. If $M \subseteq V$ is itself a subvector space

$$\Rightarrow [M] = M$$

2. especially for arbitrary subsets $M \subseteq V$

$$[[M]] = [M]$$

3. Regarding notation: The linear combination of $M \subseteq V$ is defined as,

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$$

where $n \in \mathbb{N}$ is finite. Equivalently (but shorter) we denote,

$$\sum_{a \in M} \lambda_a \cdot a$$

If $\lambda_a = 0 \forall a \in M$, then the zero vector (trivial linear combination) is given, which is element of the linear hull of any vector space.

Example 19.

$$V = \mathbb{R}^3 \quad K = \mathbb{R}$$

$$M = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$[M] = L(M) = \left\{ \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \lambda \\ \lambda \\ \lambda + \mu \end{pmatrix} \mid \lambda, \mu \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \lambda \\ \lambda \\ \mu' \end{pmatrix} \mid \lambda, \mu' \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 = x_2 \right\}$$

Example 20.

$$V = (\mathbb{Z}_3)^3 \quad K = \mathbb{Z}_3$$

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x \in \mathbb{Z}_3 \right\}$$

$$|(\mathbb{Z}_3)^3| = 3^3 = 27$$

$$M = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\begin{aligned} L(M) &= \left\{ \lambda_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}_3 \right\} \\ &= \left\{ \begin{pmatrix} \lambda_2 + \lambda_3 \\ \lambda_1 + \lambda_2 \\ \lambda_2 + \lambda_3 \end{pmatrix} \mid \lambda_2 \in \mathbb{Z}_3 \right\} \end{aligned}$$

Let $\mu_2 = \lambda_2 + \lambda_3$ and $\mu_1 = \lambda_1 + \lambda_2$.

$$\begin{aligned} &= \left\{ \begin{pmatrix} \mu_2 \\ \mu_1 \\ \mu_2 \end{pmatrix} \mid \mu_1, \mu_2 \in \mathbb{Z}_3 \right\} \\ &= L \left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right) \end{aligned}$$

We omitted vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, because it is a linear combination of the others. Therefore we omit it.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in L \left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \right)$$

Theorem 27. Let $M \subseteq V$ subset. Let $a \in L(M)$ then $L(M) = L(M \cup \{a\})$. The linear hull does not grow, if the vector space is extended by an element of the linear hull.

Proof. We need to show:

$$a \in L(M) \Rightarrow L(M) = L(M \cup \{a\})$$

- $L(M) \subseteq L(M \cup \{a\})$ holds trivially.
- It remains to show that $L(M \cup \{a\}) \subseteq L(M)$.

In general, a linear combination w of $L(M \cup \{a\})$ is given by,

$$\bigvee_{\lambda_i \in K} \bigvee_{w_i \in M \cup \{a\}} w = \lambda_1 w_1 + \dots + \lambda_k w_k \quad i \in [1, k]$$

For $a \in L(M)$ there exist $\mu_i \in K$ and $v_i \in M$ for $i \in [1, k]$ such that,

$$a = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_k v_k$$

In the linear combination of w , a occurs as w_i for some $i \in \mathbb{N}$. Without loss of generality, $w_1 = a$.

$$\begin{aligned} w &= \lambda_1 a + \lambda_2 w_2 + \dots + \lambda_k w_k \\ &= \lambda_1 \underbrace{(\mu_1 v_1 + \dots + \mu_n v_n)}_{\text{all } \mu_i, v_i \in M} + \underbrace{\lambda_2 w_2 + \dots + \lambda_k w_k}_{\text{all } \lambda_i, w_i \in M} \\ &= (\lambda_1 \mu_1) v_1 + \dots + (\lambda_1 \mu_n) v_n + \lambda_2 w_2 + \dots + \lambda_k w_k \\ &\in L(M) \end{aligned}$$

In other words, let $a \in M$, if $a \in L(M \setminus \{a\})$ then $L(M) = L(M \setminus \{a\})$.

□

Question: Is there always a minimal generating system? Can we determine whether M is minimal?

Definition 19. Let $(V, +)$ be a vector space over K . A tuple $(v_1, \dots, v_k) \in V$ is called linear independent, iff

$$\begin{aligned} \bigwedge_{\lambda_1, \dots, \lambda_n \in K} \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n &= 0 \\ \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n &= 0 \end{aligned}$$

Example 21.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is linear independent.

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = 0 \wedge \lambda_2 = 0$$

Example 22.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

is not linear independent!

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_1 = 1 \quad \lambda_2 = 1 \quad \lambda_3 = -1$$

Theorem 28. For a family $(U_i)_{i \in I}$ with an arbitrary index set I is called linear independent iff every finite subset is linear independent.

Theorem 29. A subset $M \subseteq V$ is called linear independent if for every subfamily v_1, \dots, v_n every pairwise distinct $v_i \in M$ are linear independent. A family $(v_i)_{i \in I}$ is a mapping

$$f : I \rightarrow V$$

$$i \mapsto v_i$$

In comparison with sets elements are allowed to have duplicates. Every element has a fixed index. An n -tuple is a finite family: mapping $\{1, \dots, n\} \rightarrow V$.

Theorem 30. A rather informal statement: “The vectors v_1, \dots, v_k are linear independent” iff the tuples (v_1, \dots, v_n) are linear independent.

Definition 20. $(v_i)_{i \in \emptyset}$ is defined to be linear independent.

Corollary 9. The one-tuple (0) is linear dependent.

$$1 \cdot 0 = 0$$

with 1 as an arbitrary scalar. An n -tuple v is linear independent iff $v \neq 0$. If $v \neq 0$ and $\lambda v = 0$, then $\lambda = 0$ must hold.

Corollary 10. Let

$$(v_1, \dots, v_n) \subseteq V$$

be a tuple. If $v_k = 0$ for some k , then (v_1, \dots, v_k) is linear dependent.

$$0 \cdot v_1 + 0 \cdot v_2 + \dots + 1 \cdot v_k + 0 \cdot v_{k+1} + \dots + 0 \cdot v_n = 0$$

$$\lambda_1 = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Corollary 11. If $v_k = v_l$ for some $k \neq l$, then (v_1, \dots, v_n) is linear dependent.

$$0v_1 + \dots + 0v_{k-1} + 1 \cdot v_k + 0 \cdot v_{k+1}$$

$$\dots (-1)v_l + 0v_{l+1} + \dots + 0 \cdot v_n = 0$$

$$\lambda_1 = \begin{cases} 1 & i = k \\ -1 & i = l \\ 0 & \text{else} \end{cases}$$

Corollary 12. If $M \subseteq V$ is linear independent and $N \subseteq M$, N is also linear independent.

Corollary 13.

(v_1, \dots, v_n) is linear independent

$$\Leftrightarrow \bigvee_{\lambda_1, \dots, \lambda_n \in K} \lambda_1 v_1 + \dots + \lambda_n v_n = 0$$

$$\Rightarrow \bigvee_{k \in \{1, \dots, n\}} \bigvee_{\lambda_1, \dots, \lambda_n} v_l = \lambda_1 v_1 + \dots + \lambda_n v_n$$

Therefore one vector exists which can be represented using the other vectors.

Example 23.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are linear independent.

$$\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1 \\ \lambda_1 + \lambda_2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 24.

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is linear dependent. But

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

is linear independent.

$$\lambda_1 = 0 \quad \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0$$

Definition 21.

$$V = K^n$$

The unit vector is defined as

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 is given in row i .

(e_1, \dots, e_n) is linear independent.

$$\lambda_1 e_1 + \dots + \lambda_n e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

then for all $\lambda_i = 0$.

Theorem 31. Let $v_1, \dots, v_n \in V$. Then it holds equivalently,

1. (v_1, \dots, v_n) is linear independent.
2. $\bigwedge_{v \in L(\{v_1, \dots, v_n\})} \bigvee_{\lambda_1, \dots, \lambda_n \in K} v = \lambda_1 v_1 + \dots + \lambda_n v_n$
3. $\bigwedge_{k \in \{1, \dots, n\}} v_k \notin L(\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}) = \{v_1, \dots, v_{\hat{k}}, \dots, v_n\}$
4. $\bigwedge_{k \in \{1, \dots, n\}} L(\{v_1, \dots, v_{\hat{k}}, \dots, v_n\}) \neq L(\{v_1, v_2, \dots, v_n\})$

Proof. Circle conclusion: $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$.

$1 \rightarrow 2$ For every $v \in L(v_1, \dots, v_n)$, $\bigwedge_{\lambda_1, \dots, \lambda_n} v = \lambda_1 v_1 + \dots + \lambda_n v_n$. But is it unique? Assume $v = \mu_1 v_1 + \dots + \mu_n v_n$. Show that for all $\lambda_i = \mu_i$.

$$\Rightarrow v - v = \lambda_1 v_1 + \dots + \lambda_n v_n - (\lambda_1 v_1 + \dots + \lambda_n v_n)$$

$$0 = (\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \dots + (\lambda_n - \mu_n)v_n$$

linear independence $\Rightarrow \mu_1 - \mu = 0 \quad \lambda_n - \mu_n = 0$ Therefore for all, $\lambda_i = \mu_i$.

$2 \rightarrow 3$ Assume

$$\bigvee_k U_k \in L(\{v_1, \dots, v_{\hat{k}}, \dots, v_n\})$$

$$\Rightarrow \bigvee_{\lambda_1, \dots, \lambda_n} v_k = \lambda_1 v_1 + \dots + \lambda_{n-1} v_{n-1} + 0 + \lambda_{k+1} v_{k+1} + \dots + \lambda_n v_n$$

$$\bigvee_{\lambda_1, \dots, \lambda_n} v_k = 0v_1 + \dots + 0v_{k-1} + 1 \cdot v_k + 0v_{k+1} + \dots + 0v_n$$

So v_k has two different representations, this is a contradiction.

$3 \rightarrow 4$ ToDo content incomplete/incorrect Let $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$. Show that all $\lambda_i = 0$. Assume $\bigwedge_k v_k = 0$.

$$\Rightarrow \lambda$$

$4 \rightarrow 1$ ToDo content incomplete/incorrect

$$\Rightarrow v_k \in L(\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\})$$

$$\Rightarrow L(\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}) = L(\{v_1, \dots, v_k, \dots, v_n\})$$

This is a contradiction to (4).

□

This lecture took place on 17th of November 2015 (Franz Lehner).

$$\underbrace{[M]}_{\text{smallest subspace } \supseteq M} = \underbrace{L(M)}_{\text{set of all linear combinations}}$$

In general: $M \subseteq V$ is called linear independent, if every subfamily of p_n different element is linear independent.

$$\Leftrightarrow \bigwedge_{v \in L(\{v_1, \dots, v_n\})} \bigvee_{\lambda_1, \dots, \lambda_n} v = \lambda_1 v_1 + \dots + \lambda_n v_n$$

$$\Leftrightarrow \bigwedge_k v_k \notin L(\{v_1, \dots, v_{\hat{k}}, \dots, v_n\})$$

$$\Leftrightarrow \bigwedge_{v \in L(M)} \bigvee_{n \in \mathbb{N}} \bigvee_{v_1, \dots, v_n \in M} \bigvee_{\lambda_1, \dots, \lambda_n} v = \lambda_1 v_1 + \dots + \lambda_n v_n$$

$$L(M) = V$$

Definition 22. • A family/set $S \subseteq V$ is called generating system if $V = [S] = L(S)$. “ V is generated by S .”

• V is called finitely generated if a finite generating system exists.

- A basis of a vectorspace V is a linear independent generating system. Therefore a family $B = (b_i)_{i \in I} \subseteq V$ such that $L(B) = V$, B is linear independent.

Remark 9. • $(b_i)_{i \in I}$ is a basis of V , if

- every element is a linear combination of a finite subfamily b_{i_1}, \dots, b_{i_n} .
- every finite subfamily is linear independent.

- $(b_i)_{i \in \emptyset}$ is basis of $\{0\}$.
- if (b_1, \dots, b_n) is a basis of V then also every permutation $(b_{i_1}, \dots, b_{i_n})$ (addition is commutative).

Example 25. In K^n . Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ be the unit vector, then (e_1, e_2, \dots, e_n) is

a basis of K^n ; specifically called canonical basis (or standard basis).

Remark 10. e_i is linear independent.

$$\sum_{i=1}^n \lambda_i e_i = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$= 0 \Leftrightarrow \text{all } \lambda_i = 0$$

Every vector is reachable by a linear combination of e_i .

Example 26.

$$K[x] := V = K^{\mathbb{N}_0} = \{(a_n)_{n \geq 0} \mid a_n \in K\}$$

Is the vector space of all sequences.

$$e_i = (0, \dots, 1, 0, \dots) \quad i \in \mathbb{N}_0$$

where 1 is given on the i -th position. If $\sum \lambda_i e_i = (0, 0, \dots) \Rightarrow$ all $\lambda_i = 0$ and $(\lambda_0, \lambda_1, \dots) \Rightarrow (e_i)_{i \in \mathbb{N}_0}$ is linear independent.

Is not a basis, because 1 can never be reached.

$$(1, 1, 1, 1, \dots) \in \mathbb{R}^{\mathbb{N}_0}$$

$$\sum_{i=0}^n e_i = (1, 1, 1, \dots, 1, 0, 0, \dots) + (1, 1, 1, \dots)$$

for all $n \in \mathbb{N}$. In linear combinations only finitely many summands are allowed.

$L((e_i)_{i \in \mathbb{N}_0}) =$ vector space of all sequences $(a_n)_{n \in \mathbb{N}_0}$ with arb. many $a_n \neq 0$

is a subspace: $(a_1, \dots, a_n, 0, \dots, 0) + (b_1, \dots, b_n, 0, \dots, 0)$. Without loss of generality: $m \leq n$.

$$= (a_1 + b_1, \dots, a_m + b_m, b_{m+1}, \dots, b_n, 0, \dots, 0)$$

$(e_i)_{i \in \mathbb{Z}_0}$ is a basis of $K[x]$; the vector space of polynomials and vector space of finite sequences.

We identify the vector space of finite sequences with the vector space of formal polynomials:

$$K[x] = \{a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N}_0, a_i \in K\}$$

$$= (a_0 + a_1x + \dots + a_nx^n) + (b_0 + b_1x + \dots + b_nx^n)$$

$$= (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_m + b_m)x^m + b_{m+1}x^{m+1} + b_nx^n$$

Without loss of generality

Instead of a unit vector e_i the formal polynomial x^i occurs.

$$\Rightarrow (x^n)_{n \geq 0} \text{ is a basis of } K[x]$$

$$\deg p(x) = \max \{i \mid a_i \neq 0\} = n$$

is the degree of the polynomial.

$$p(x) = a_0 + q_1x + q_x x^2 + \dots a_n x^n$$

$$\deg 0 := -\infty$$

Every formal polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$ induces a polynomial function

$$K \rightarrow K$$

$$\xi \mapsto a_0 + a_1\xi + \dots + a_n\xi^n \in K$$

If K has infinite cardinality, then the polynomial function defines the formal polynomial uniquely.

Theorem 32. *Attention!* *This does not hold if the field is finite!*

Proof. There are $|K^K| = |K|^{|K|}$ different functions of $K \rightarrow K$. For example for $K = \mathbb{Z}_2$ there are 2^2 functions in $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2$.

$$\mathbb{Z}_2[x] = \{a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N}_0, a_n \in \mathbb{Z}_2\}$$

There are 2^{n+1} polynomials of degree n . So they cannot be unique (no bijective function can exist to map 2^2 elements to 2^{n+1} elements). \square

Does $K^{\mathbb{N}_0}$ have a basis? Does every vector space have a basis?

Theorem 33. *Every vector space has a basis.*

Proof. Case 1 V is generated finitely.

Let (v_1, \dots, v_n) be a finite generating system. If (v_1, \dots, v_n) is linear independent, we are done. Otherwise we already know that (by a previous Theorem)

$$\bigvee_{k \in \{1, \dots, n\}} v_k \in L(v_1, \dots, \hat{v}_k, v_n)$$

$$\Rightarrow L(v_1, \dots, v_n) = L(v_1, \dots, \hat{v}_k, \dots, v_n) = V$$

- is this set linear independent, then this set is a basis.
- if not, then repeat this step.

Because originally only finitely many v_i were given, this algorithm must terminate after finitely many steps. The resulting system is linear independent and a generating system. Therefore the result is a basis.

This algorithm fails for V which are not generated finitely.

Every vector space has a basis iff you believe in the axiom of choice. \square

Remark 11. *Whether every vector space has a basis depends on your faith in the Axiom of Choice (AC).*

The axiom of choice states: Let $(S_i)_{i \in I}$ be a family of non-empty sets. Then some $(x_i)_{i \in I}$ exist such that $\bigwedge_{i \in I} x_i \in S_i$.

Example 1:

$$(A)_{A \subseteq \mathbb{N}}$$

$(x_A)_{A \subseteq \mathbb{N}}$ such that $x_A = \min A$. A selection was made for every subset.

Example 2:

$$(A)_{A \subseteq \mathbb{R}}$$

$(x_A)_{A \subseteq \mathbb{R}}$ such that $x_A \in A \forall A$. Such a selection cannot be made.

Constructivists: You cannot state it explicitly, so it is not true.

General mathematicians: Well, we cannot state it, but just take one.

*A consequence of the axiom of choice is the **Hausdorff-Banach-Tarski paradox**:*

Consider a sphere in \mathbb{R}^3 . Cut the sphere in 5 parts. Then you can move the parts such that two identical copies of the original sphere are created.

The Hausdorff-Banach-Tarski paradox is equivalent to the axiom of choice.

Constructivists do not believe in the axiom of choice and therefore the Hausdorff-Banach-Tarski paradox does not hold. The majority of mathematicians assume the axiom of choice, but following they need to accept the Hausdorff-Banach-Tarski paradox.

Remark 12. *The axiom of choice is independent of the other axioms of Zermelo-Fraenkel set theory (ZF). If ZF is contradiction-free, so is $ZF + AC$.*

Theorem 34. *Let V be a vector space over K*

$$B = (b_i)_{i \in I} \subseteq V$$

Then it holds equivalently, that

1. B is a basis.

2. Every $v \in V$ can be represented uniquely as linear combination of B :

$$\bigwedge_{v \in V} \bigvee_{n \in \mathbb{N}} \bigvee_{i_1, \dots, i_n} \bigvee_{\lambda_1, \dots, \lambda_n} v = \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n}$$

3. B is a maximal linear independent family.

4. B is a minimal generating system.

Remark 13. What does minimal mean?

Minimal means no smaller generating system exists. Minimal does not mean, it is the smallest generating system.

Example:

$$\mathbb{R}^2 : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a generating system. This is also a generating system:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

is also a generating system.

Proof. We prove Theorem 34.

We use circular reasoning (dt. Zirkelschluss).

1 \rightarrow 2 Basis $\Rightarrow L(B) = V$

Let $v \in V \Rightarrow \bigvee_{\lambda_1, \dots, \lambda_n} v = \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n}$.

We need to show uniqueness of representation: Assume $v = \mu_1 b_{j_1} + \mu_2 b_{j_2} + \dots + \mu_m b_{j_m}$. We fill up the vectors such that $m = n$ and $j_k = i_k$.

Therefore

$$v = \mu_1 \cdot b_{j_1} + \dots + \mu_n b_{i_n}$$

$$\Rightarrow 0 = v - v = \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n} - (\mu_1 b_{i_1} + \dots + \mu_n b_{i_n}) = (\lambda_1 - \mu_1) b_{i_1} + \dots + (\lambda_n - \mu_n) b_{i_n}$$

$$(b_i) \text{ are linear independent} \Rightarrow \bigwedge_{k \in \{1, \dots, n\}} \lambda_k = \mu_k.$$

2 \rightarrow 1 From 2 it follows that $L(B) = V$. Show that it is linear independent.

Let $\lambda_1 + b_{i_1} + \dots + \lambda_n b_{i_n} = 0$. Condition 2 for the vector $v = 0$ implies that it is the same representation like $0b_{i_1} + \dots + 0b_{i_n} = 0$. So have two representations of the vector $v = 0$. \Rightarrow all $\lambda_k = 0$. Therefore B is linear independent and therefore a linear basis.

1 \rightarrow 3 From 1 it follows that B is linear independent. B maximal means that $\bigwedge_{v \in V \setminus B} B' = B \cup \{v\}$ is not linear independent any more.

Let $v \in V \setminus B$, but $L(B) = V$ there exists $\lambda_1, \dots, \lambda_n$ and b_{i_1}, \dots, b_{i_n} such that $v = \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n}$. Therefore $\lambda_1 b_{i_1} + \lambda_2 b_{i_2} + \dots + \lambda_n b_{i_n} - v = 0$. Then a linear combination of $B \cup \{v\}$ is the coefficient of v . $-1 \neq 0$. $\Rightarrow B' \cup \{v\}$ is not linear independent.

3 \rightarrow 4 Let B be a maximal linear independent family.

1. Show that B is generating system and minimal.

Every $v \in V$ is contained in $L(B)$. Let $v \in V$. Case distinction:

- $v \in B \Rightarrow v \in L(B)$
- $v \notin B$. From 3 it follows that $B \cup \{v\}$ is linear dependent.

$$\Rightarrow \bigvee_{\lambda_0, \lambda_1, \dots, \lambda_n} \bigvee_{b_{i_1}, \dots, b_{i_n} \in B} \lambda_0 v + \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n} = 0$$

But not all $\lambda_0, \dots, \lambda_n$ can be 0. If it would hold that $\lambda_0 = 0$, then $\lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n} = 0$.

$$\Rightarrow \lambda_i = 0 \text{ because } B \text{ is linear independent}$$

Therefore λ_0 cannot be 0.

$\lambda_i \neq 0 \Rightarrow$ division allowed.

$$\lambda_0 \cdot v = -\lambda_1 b_{i_1} - \dots - \lambda_n b_{i_n}$$

$$\Rightarrow v = -\frac{\lambda_1}{\lambda_0} b_{i_1} - \dots - \frac{\lambda_n}{\lambda_0} b_{i_n} \in L(B)$$

This holds for every $v \in V$, therefore $V = L(B)$.

- B is a minimal generating system. Assume $B' = B \setminus \{b_{i_0}\}$ is also generating system. Therefore

$$\begin{aligned} L(B \setminus \{b_{i_0}\}) &= V \\ \Rightarrow b_{i_0} &\in L(B \setminus \{b_{i_0}\}) \\ \Rightarrow \bigvee_{\lambda_1, \dots, \lambda_n} \bigvee_{i_1, \dots, i_n \neq i_0} &= \lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n} \\ \Rightarrow \lambda_n b_{i_1} + \dots + \lambda_n b_{i_n} - b_{i_0} &= 0 \end{aligned}$$

The coefficient of b_{i_0} is $\lambda_0 = -1 \neq 0$. This contradicts, because B is linear independent.

□

This lecture took place on 23rd of November 2015 (Franz Lehner).

6.4 Revision

A basis is a linear independent generating system.

$$\begin{aligned} \lambda_1 b_1 + \dots + \lambda_n b_n &= 0 \\ \Rightarrow \lambda_i &= 0 \end{aligned}$$

$v = 0$ has a unique representation as linear combination of the basis B .

Proof. We have already shown $1 \rightarrow 3 \rightarrow 4$. We prove $4 \rightarrow 1$.

Let B be a minimal generating system. Show that B is linear independent. Proof by contradiction.

Assume B is not linear independent. Then there are coefficients $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$ such that

$$\lambda_1 b_{i_1} + \dots + \lambda_n b_{i_n} = 0$$

There exists some k such that $\lambda_k \neq 0$.

$$\Rightarrow \lambda_k \cdot b_{i_k} = - \sum_{j \neq k} \lambda_j b_{i_j}$$

$$b_{i_k} = - \sum_{j \neq k} \frac{\lambda_j}{\lambda_k} b_{i_j}$$

$$\Rightarrow b_{i_k} \in L(B \setminus \{b_{i_k}\})$$

$$L(B \setminus \{b_{i_k}\}) = L(B \setminus \{b_{i_k}\}) \cup \{b_{i_k}\} = L(B) = V$$

$B \setminus \{b_{i_k}\}$ is also a generating system, but smaller. So B is not minimal. □

How can we construct/find bases?

Theorem 35 (Exchange lemma). *Let $B = (b_1, \dots, b_n)$ be basis in vector space V . Let $v \in V \setminus \{0\}$. Let*

$$v = \sum_{i=1}^n \lambda_i \cdot b_i$$

If some $\lambda_k \neq 0$ then $B' = (b_1, \dots, b_{k-1}, v, b_{k+1}, \dots, b_n)$ is also a basis of V .

Proof. We need to show that

- B' is linear independent.
- B' is generating system.

1. Let $\mu_1, \dots, \mu_k \in K$.

$$\mu_1 b_1 + \dots + \mu_{k-1} b_{k-1} + \mu_k v + \mu_{k+1} b_{k+1} + \dots + \mu_n b_n = 0$$

Show that all $\mu_i = 0$.

$$\begin{aligned} 0 &= \sum_{i \neq k} \mu_i b_i + \mu_k v \\ &= \sum_{i \neq k} \mu_i b_i + \mu_k \left(\sum_{i=1}^n \lambda_i \cdot b_i \right) \\ &= \sum_{i \neq k} \mu_i b_i + \sum_{i \neq k} \mu_k \lambda_i b_i + \mu_k \lambda_k b_k \\ &= \sum_{i \neq k} (\mu_i + \mu_k \lambda_i) b_i + \mu_k \lambda_k b_k \\ &= \text{is linear combination of } B \end{aligned}$$

$$\begin{aligned}\mu_k \cdot \lambda_k &= 0 \xrightarrow{\lambda_k \neq 0} \mu_k = 0 \\ \Rightarrow \mu_i + \mu_k \lambda_i &= 0 \Rightarrow \mu_i = 0 \text{ for all } i \neq k \\ \Rightarrow \forall \mu_i &= 0\end{aligned}$$

2. $L(B') = V$. It suffices to show that $b_k \in L(B')$.

Then it holds that

$$\begin{aligned}L(B') &= L(B' \cup \{b_k\}) \\ B' \cup \{b_k\} &= (B \setminus \{b_k\}) \cup \{b_k\} \cup \{v\} = B \cup \{v\} \\ \Rightarrow L(B \cup \{v\}) &\supseteq L(B) = V \quad \checkmark \\ v &= \sum_{i=1}^n \lambda_i b_i = \sum_{i \neq k} \lambda_i b_i + \lambda_k b_k \Rightarrow \lambda_k b_k = v - \sum_{i \neq k} \lambda_i b_i \\ \lambda_k \neq 0 &\Rightarrow b_k = \frac{1}{\lambda_k} v - \sum_{i \neq k} \frac{\lambda_i}{\lambda_k} b_i \in L(B')\end{aligned}$$

□

Theorem 36 (Steinitz exchange lemma). *Let V be a vector space over a field K . Let $B = (b_1, \dots, b_n)$ be a basis. Let $(v_1, \dots, v_r) \subseteq V$ be linear independent with $r \leq n$.*

Then it holds that the following is a basis of V :

$$\bigvee_{i_1, \dots, i_{n+1} \in \{1, \dots, n\}} (v_1, \dots, v_r, b_{i_1}, \dots, b_{i_{n-r}})$$

Followingly v_1, \dots, v_r can be exchanged as basis.

Proof. Complete induction over number of elements and using the exchange lemma.

induction base $r = 1$

1. Let (v_1) be linear independent. Then $v_1 \neq 0$. Then $B \neq \emptyset$. Then $n \geq 1$ where n is $|B|$. Because $r = 1$, $n = 1$.

2. Let $v_1 = \sum \lambda_i b_i \neq 0$. So there exists some k with $\lambda_k \neq 0$. From the exchange lemma 35 it follows that $(v_1, b_1, \dots, b_{k-1}, b_{k+1}, \dots, b_n)$ is a basis. \checkmark

induction step $r \rightarrow r + 1$

Let v_1, \dots, v_{r+1} be linear independent.

$\Rightarrow v_1, \dots, v_r$ is also linear independent

induction hypothesis $\Rightarrow \bigvee_{j_1, \dots, j_{n-r}} (v_1, \dots, v_r, b_{j_1}, \dots, b_{j_{n-r}})$ is a basis

1. It holds that $r \leq n$.

We need to show that $r + 1 \leq n$, so we need to exclude that $r = n$. In that case $r + 1 \leq n$ holds (with $r < n$).

Assume

$$r = n \Rightarrow (v_1, \dots, v_r) \text{ is a basis}$$

$\Rightarrow (v_1, \dots, v_r)$ is maximal linear independent family

$\Rightarrow (v_1, \dots, v_{r+1})$ is not linear independent

This is a contradiction to our assumption. So $r < n \Rightarrow r + 1 \leq n$.

2. By induction hypothesis V has a basis $(w_1, \dots, w_r, v_{i_1}, \dots, v_{i_{n-r}})$. The vector w_{r+1} can be written as

$$w_{r+1} = \sum_{i=1}^r \mu_i w_i + \sum_{j=1}^{n-r} \lambda_j v_{i_j}.$$

At least one k satisfies $\lambda_k \neq 0$, otherwise $w_{r+1} \in \mathcal{L}(\{w_1, \dots, w_r\})$ in contradiction to the linear independence of (w_1, \dots, w_{r+1}) . With the exchange lemma 35 we can replace v_{i_k} with w_{r+1} .

$$(w_1, \dots, w_{r+1}, v_{i_1}, \dots, v_{i_{k-1}}, v_{i_{k+1}}, \dots, v_{i_{n-r}})$$

is therefore a basis.

□

Theorem 37. *Let V be a vector space over K .*

- If V has a finite basis, then all bases are finite.
- For every two bases (b_1, \dots, b_m) and (b'_1, \dots, b'_n) it holds that $m = n$.

Proof. • Let (b_1, \dots, b_n) be a finite basis of V . Let $(v_i)_{i \in I}$ be linear independent in V .

$$\begin{aligned} \Rightarrow \bigwedge_r v_{i_1}, \dots, v_{i_r} \text{ linear independent} \\ \Rightarrow r \leq n \\ \Rightarrow |I| \leq n \end{aligned}$$

So every basis has at most n elements.

- Let (b'_1, \dots, b'_r) be another basis \Rightarrow maximal linear independent family $\Rightarrow r \leq n$. From Steinitz' exchange lemma it follows that

$$\begin{aligned} \bigvee_{j_1, \dots, j_{n-r}} (b'_1, \dots, b'_r, b_{j_1}, \dots, b_{j_{n-r}}) \text{ is a basis} \\ (b'_1, \dots, b'_r) \text{ is maximal linear independent family} \\ (b'_1, \dots, b'_r, b_{j_1}, \dots, b_{j_{n-r}}) \text{ is also linear independent} \\ \Rightarrow n - r = 0 \Rightarrow n = r \end{aligned}$$

Remark 14. V has a basis. V is finitely generated.

Proof. \Rightarrow follows immediately.

\Leftarrow use negative vectors until linear independent family remains. \square

Definition 23. Let V be a vector space over K . Assume V has a finite basis. Then the uniquely determinable number $n = \dim V$ is called dimension of the vector space. And V is called finitely dimensional.

Otherwise $\dim V = \infty$. V is called infinitely dimensional.

Example 27.

$$\dim R^3 = 3$$

$$\dim \emptyset = 0$$

$$\dim K^n = n$$

$$\dim K^m = |M|$$

$$\dim K[x] = \infty \dots \text{vector space of polynomials}$$

Remember that $K[x] = \{a_0 + a_1x + \dots + a_nx^n \mid n \in \mathbb{N} \text{ arbitrary}, a_i \in K\}$.

$$\Rightarrow (x^n)_{n \in \mathbb{N}} \text{ is basis} \Rightarrow \dim K[x] = \infty$$

Theorem 38 (Basis extension theorem). (*Steinitz' exchange lemma for finite vector spaces*)

Let V be a vector space with $\dim v = n < \infty$. Then every linear independent family (v_1, \dots, v_r) can be extended to a basis.

Proof. Let (b_1, \dots, b_n) be a basis. From Steinitz' exchange lemma it follows that $r \leq n$ and

$$\bigvee_{j_1, \dots, j_{n-r}} (v_1, \dots, v_r, b_{j_1}, \dots, b_{j_{n-r}})$$

is basis (maximal linear independent family). \square

\square **Theorem 39** (Basis selection theorem). If (v_1, \dots, v_r) is a generating system of V (with $\dim V = n$). Then $r \geq n$ and $\bigvee_{j_1, \dots, j_n} (v_{j_1}, \dots, v_{j_n})$ is a basis of V .

Proof. If (v_1, \dots, v_r) is linear independent, then it is already a basis. If it is linear dependent, then \square

$$\bigvee_k v_k \in L(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_r)$$

$$\Rightarrow L(v_1, \dots, v_r) = L(v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_r) = V$$

We iterate this step until a linear independent family remains. \square

6.5 Summary for finite vector spaces

In a finite generating vector space V

- every basis has the same number of elements ($\dim V = n$).
- every linear independent family has at most $\dim V$ elements.
- every generating system has at least $\dim V$ elements.

Theorem 40. *Let V be a vector space with $\dim V = n \in \mathbb{N}$. Let $v_1, \dots, v_n \in V$. Then the following statements are equivalent:*

1. (v_1, \dots, v_n) is basis.
2. $L(V_1, \dots, v_n) = V$
3. (v_1, \dots, v_n) is linear independent.

Proof. **1 to 2** follows immediately.

2 to 3

$$L(v_1, \dots, v_n) = V$$

From the basis extension theorem it follows that v_{i_1}, \dots, v_{i_r} is a basis.

$$\dim V = n \Rightarrow r = n \Rightarrow i = 1, \dots, n$$

So we cannot remove any elements, so (v_1, \dots, v_n) is already a basis.

3 to 1 Follows analogously with the basis extension theorem.

□

Theorem 41. *Let V be a vector space with $\dim V < \infty$ und $U \subseteq V$. Then it holds that,*

- $\dim U \leq \dim V$.
- $\dim U = \dim V \Leftrightarrow U = V$

Proof. • U is finitely dimensional.

Then every linear independent family in U is linear independent in V .
Therefore $\leq \dim V$ elements.

Let v_1, \dots, v_r be basis of U .

$$\Rightarrow r \leq \dim V \quad \checkmark$$

- Let $n := \dim U = \dim V$. Let (u_1, \dots, u_n) be basis of U .

$\Rightarrow (u_1, \dots, u_n)$ is linear independent in V

$\Rightarrow (u_1, \dots, u_n)$ is basis of V

From Theorem 40 (3) it follows that $U = L(u_1, \dots, u_n) = V$.

□

6.6 Revision

- It will turn out that vector spaces with the same dimension are isomorphic.
- The dimension of a vector is the cardinality of every basis.
- It is also the maximal cardinality of a linear independent family.
- It is also the minimal cardinality of a generating system.

How do we find a basis?

- If a generating system is given, remove elements until it is linear independent.
- Otherwise add elements as long as the system remains linear independent.

6.7 Representation of vector spaces

This lecture took place on 24th of November 2015 (Franz Lehner).

Definition 24. Let V be a vector space over K . Let $B = (b_1, \dots, b_n)$ be the basis of V . Then every $v \in V$ has a unique decomposition $v = \sum_{i=1}^n \lambda_i b_i$. The uniquely determinable coefficients λ_i are called coordinates of v with respect to B .

$$(v)_B := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called coordinates vector of v .

The mapping

$$\Phi_B : V \rightarrow K^n$$

$$v \mapsto \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called coordinate mapping.

It follows immediately that Φ_B is bijective.

Example 28.

$$V = R_3[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_i \in \mathbb{R}\}$$

$$B = (1+x, 1-x, 1+x+x^2, x^2+x^3) \text{ is basis of } V$$

To prove that B is a basis, it suffices to show that they are linear independent (because the dimension 4 reveals that 4 elements are required).

$$\lambda_1(1+x) + \lambda_2(1-x) + \lambda_3(1+x+x^2) + \lambda_4(x^2+x^3) = 0$$

$$(\lambda_1 + \lambda_2 + \lambda_3) \cdot 1 + (\lambda_1 - \lambda_2 + \lambda_3)x + (\lambda_3 + \lambda_4)x^2 + \lambda_4x^3 = 0 \text{ (zero polynomial!!)}$$

$$\text{coefficient comparison} \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 + \lambda_3 = 0$$

$$\Rightarrow \lambda_3 + \lambda_4 = 0$$

$$\Rightarrow \lambda_4 = 0$$

$$\text{coefficient comparison} \Rightarrow \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \lambda_1 - \lambda_2 = 0$$

$$\text{coefficient comparison} \Rightarrow 2\lambda_1 = 0$$

$$\Rightarrow \lambda_2 = 0$$

$\Rightarrow B$ is linear independent $\wedge |B| = \dim V \Rightarrow B$ is basis (follows from Theorem 40).

Find the coordinates of the polynomial:

$$p(x) = 3 + x - 3x^2 + x^3 \text{ with respect to } B$$

Therefore we search for $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that,

$$p(x) = \lambda_1(1+x) + \lambda_2(1-x) + \lambda_3(1+x+x^2) + \lambda_4(x^2+x^3)$$

$$= (\lambda_1 + \lambda_2 + \lambda_3) \cdot 1 + (\lambda_1 - \lambda_2 + \lambda_3) \cdot x + (\lambda_3 + \lambda_4)x^2 + \lambda_4x^3$$

Using coefficient comparison we get

$$\lambda_1 + \lambda_2 + \lambda_3 = 3$$

$$\lambda_1 - \lambda_2 + \lambda_3 = 1$$

$$\lambda_3 + \lambda_4 = -3$$

$$\lambda_4 = 1$$

$$\lambda_3 = -3 - \lambda_4 = -4$$

$$\lambda_1 + \lambda_2 = 3 - (-4) = 7$$

$$\lambda_1 - \lambda_2 = 1 - (-4) = 5$$

$$2\lambda_1 = 12 \Rightarrow \lambda_1 = 6$$

$$\lambda_2 = 7 - \lambda_1 = 1$$

So,

$$\Phi_B : \mathbb{R}_3[x] \Rightarrow \mathbb{R}^4$$

$$\Phi_B(p(x)) = \begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \end{pmatrix}$$

Theorem 42. Let B be a basis of V . $v, w \in V$ with coordinates:

$$\Phi_B(v) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \quad \Phi_B(w) = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

Then it holds that

$$\Phi_B(v+w) = \begin{pmatrix} \xi_1 + \eta_1 \\ \vdots \\ \xi_n + \eta_n \end{pmatrix} = \underbrace{\Phi_B(v) + \Phi_B(w)}_{\text{addition in } K^n}$$

$$\Phi_B(\lambda \cdot v) = \begin{pmatrix} \lambda \cdot \xi_1 \\ \vdots \\ \lambda \cdot \xi_n \end{pmatrix} = \lambda \cdot \Phi_B(v)$$

Example 29. Let V be a vector space with basis B . $v_1, \dots, v_k \in V$ are linear independent.

$$\Leftrightarrow \Phi_B(v_1) \dots \Phi_B(v_k) \text{ are linear independent in } K^n$$

7 Construction of vector spaces

Remark 15. We have already seen $U, W \subseteq \text{subspaces} \Rightarrow U \cap W$ is subspace, but not $U \cup W$.

Definition 25. V is a vector space. $U, W \subseteq V$ are subspaces. Then $[U \cup W]$ is the sum of subspaces U and W

$$=: U + W = \bigcap \{Z \mid Z \subseteq V, U \subseteq Z, W \subseteq Z\}$$

$$= L(U \cup W) = \left\{ \sum \lambda_i u_i + \mu_j w_j \mid u_i \in U, w_j \in W \right\}$$

Theorem 43.

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Proof. Let $E := \{u + w \mid u \in U, w \in W\}$. The claim is that $[U \cup W] = E$.

We want to show that E is a subspace, $U \subseteq E, W \subseteq E$.

To show that E is a subspace, we show:

(UR) Let $v \in E, v' \in E, \lambda, \mu \in K$. Show that $\lambda \cdot v + \mu v' \in E$.

$$v \in E \Rightarrow \bigvee_{u \in U} \bigvee_{w \in W} v = u + w$$

$$v' \in E \Rightarrow \bigvee_{u' \in U} \bigvee_{w' \in W} v' = u' + w'$$

$$\lambda v + \mu v' = \lambda(u + w) + \mu(u' + w')$$

$$= \underbrace{(\lambda u + \mu v')}_{\in U} + \underbrace{(\lambda w + \mu w')}_{\in W} \in E$$

$U \subseteq E$ is obvious. $u = u + 0 \in E$.

$W \subseteq E$: Every $w \in W$ is $w = 0 + w \in E$.

$[U \cup W] \supseteq E$ We need to show every subspace $Z \subseteq V$, which contains $U \cup W$, contains also E .

Let Z be a subspace. Let $v \in E$. Show that $v \in Z$.

$$v \in E \Rightarrow \bigvee_{u \in U} \bigvee_{w \in W} v = u + w$$

$$u \in U \subseteq Z \Rightarrow u \in Z$$

$$w \in W \subseteq Z \Rightarrow w \in Z$$

$$\Rightarrow u + w \in Z \text{ because } Z \text{ is subspace}$$

□

Example 30. Let $V = \mathbb{R}^4$.

$$U = \left\{ \begin{pmatrix} \xi \\ \eta \\ \xi \\ \eta \end{pmatrix} \mid \xi, \eta \in \mathbb{R} \right\}$$

$$W = \left\{ \begin{pmatrix} \xi \\ \xi \\ \eta \\ \eta \end{pmatrix} \mid \xi, \eta \in \mathbb{R} \right\}$$

$$U + W = ?$$

Determine the basis of $U + W$.

We guess the basis of U is $\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$. We guess the basis of W is

$$\left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

$$U = L \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right) = \left\{ \xi \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \eta \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \mid \xi, \eta \in \mathbb{R} \right\}$$

$$W = L \left(\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right) = \left\{ \xi \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \eta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \mid \xi, \eta \in \mathbb{R} \right\}$$

So... und jetzt ist das Alphabet aus! (Franz Lehner)

$$U + W = \{u + w \mid u \in U, w \in W\}$$

$$= \left\{ \xi \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \eta \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \chi \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mid \xi, \eta, \chi, w \right\}$$

$$= L \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The linear combination gives $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ is not linear independent!

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in L \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

\Rightarrow linear hull stays the same, if we remove $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$$U + W = L \left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

Linear independence:

$$\lambda \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \lambda + \gamma \\ \mu + \gamma \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \lambda = 0, \mu = 0 \Rightarrow \gamma = 0$$

$$\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right) \text{ is linear independent and basis of } U + W$$

$$\Rightarrow \dim(U + W) = 3$$

$$\dim U = 2 \quad \dim W = 2$$

Theorem 44. Let V be a vector space. $M, N \subseteq V$.

$$L(M \cup N) = L(M) + L(N)$$

We will show this in the practicals.

Example 31.

$$U \cap W = \left\{ \begin{pmatrix} \xi \\ \xi \\ \xi \\ \xi \end{pmatrix} \mid \xi \in \mathbb{R} \right\}$$

$$\dim(U \cap W) = 1$$

$$\text{Basis is } \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\dim(U + W) = 2 + 2 - 1$$

Theorem 45. Let V be a vector space. $U, W \subseteq V$ are finite-dimensional subspaces. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

Theorem 46 (Inclusion-exclusion principle). In German, it is called Siebformel.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

for $\dim(U + W + Z)$ the analogous equation is **wrong!**

Proof. Determine bases for all involved spaces.

Begin with the smallest space. Use the basis extension theorem. Let v_1, \dots, v_r be basis of $U \cap W$. The basis extension theorem for U states the $U \cap W$ is subspace of U .

$$\bigvee_{u_1, \dots, u_p} (v_1, \dots, v_r, u_1, \dots, u_p) \text{ is basis of } U$$

Analogously for W

$$\bigvee_{w_1, \dots, w_q} (v_1, \dots, v_r, w_1, \dots, w_q) \text{ is basis of } W$$

Therefore

$$U = L(\{v_1, \dots, v_r, u_1, \dots, u_p\})$$

$$W = L(v_1, \dots, v_r, w_1, \dots, w_q)$$

$$U + W = L(v_1, \dots, v_r, u_1, \dots, u_p, w_1, \dots, w_q)$$

Assume $v_1, \dots, v_r, u_1, \dots, u_p, w_1, \dots, w_q$ are linear independent.

$$\dim(U + W) = r + p + q$$

$$\dim(U) = r + p$$

$$\dim(W) = r + q$$

$$\dim(U \cap W) = r$$

\Rightarrow the equation holds.

It remains to show that B is linear independent.

Intermediate step:

$$U \cap L(w_1, \dots, w_q) = \{0\}$$

Let $v \in U \cap L(w_1, \dots, w_q) \subseteq U \cap W \Rightarrow v \in U \wedge v \in L(w_1, \dots, w_q)$.

$$\Rightarrow \bigvee_{\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_p} v = \sum_{i=1}^r \lambda_i v_i + \sum_{j=1}^p \mu_j u_j$$

$$\Rightarrow \bigvee_{\mu_1, \dots, \mu_q} v = \sum_{k=1}^q \mu_k w_k$$

$$v \in U \cap W \Rightarrow \bigvee_{\xi_1, \dots, \xi_r} v = \sum_{l=1}^r \xi_l v_l$$

Consider v in W :

$$0 = v - v = \sum_{k=1}^q \mu_k w_k - \sum_{l=1}^r \xi_l v_l$$

$(v_1, \dots, v_r, w_1, \dots, w_q)$ is basis of W

\Rightarrow linear independence

v in W is linear combination which results in 0. Therefore all coefficients are zero.

$$\Rightarrow v = 0$$

The last step remains: B is linear independent.

$$B = (v_1, \dots, v_r, u_1, \dots, u_p, w_1, \dots, w_q)$$

Let $(\lambda_i)_{i=1}^r, (\mu_j)_{j=1}^p, (\mu_k)_{k=1}^q \in K$.

$$\sum_{i=1}^r \lambda_i v_i + \sum_{j=1}^p \mu_j u_j + \sum_{k=1}^q \mu_k w_k = 0$$

Show that all λ_i , all μ_j and all μ_k are zero.

$$a := \underbrace{\sum_{i=1}^r \lambda_i v_i + \sum_{j=1}^p \mu_j u_j}_{\in U} + \underbrace{- \sum_{k=1}^q \mu_k w_k}_{\in L(w_1, \dots, w_q)}$$

$$\Rightarrow a \in U \cap L(w_1, \dots, w_q) = \{0\}$$

$$\Rightarrow a = 0 \Rightarrow \sum_{i=1}^r \lambda_i v_i + \sum_{j=1}^p \mu_j u_j = 0$$

$$\sum_{k=1}^q \mu_k w_k = 0$$

$v_1, \dots, v_r, u_1, \dots, u_p$ are bases in $U \Rightarrow$ linear independent.

From $0 \Rightarrow \sum_{i=1}^r \lambda_i v_i + \sum_{j=1}^p \mu_j u_j = 0$ it follows that $\lambda_1 = \dots = \lambda_r = 0$ and $\mu_1 = \dots = \mu_p = 0$.

$(\mu_1, \dots, \mu_r, w_1, \dots, w_q)$ is basis in W

So \Rightarrow linear independence $\Rightarrow (w_1, \dots, w_q)$ is linear independent.

From $\sum_{k=1}^q \mu_k w_k = 0$ it follows that $\mu_1, \dots, \mu_q = 0$.

So the idea of this proof was to split B into two sums. We showed that their intersection is empty. Then we showed that they result in zero individually. \square

Remark 16. In this proof we have seen that every $v \in U + W$ has a unique representation $v = a + b + c$.

$$U + W = \{u + w \mid u \in U, w \in W\}$$

$$a \in U \cap W = L(v_1, \dots, v_r)$$

$$b \in L(u_1, \dots, u_p)$$

$$c \in L(w_1, \dots, w_q)$$

The representation $v = u + w$ is not unique with $u \in U, w \in W$ (unless $U \cap W = \{0\}$).

$$v = \underbrace{(a+b)}_{\in U} + \underbrace{c}_{\in W} = \underbrace{b}_{\in U} + \underbrace{(a+c)}_{\in W}$$

Definition 26. The sum $U + W$ of two subspaces is called direct if

$$\bigwedge_{v \in U+W} \dot{\bigvee}_{u \in U} \dot{\bigvee}_{w \in W} v = u + w$$

If this holds, then we write $U \dot{+} W$ for the direct sum (or alternatively $U \oplus W$).

Theorem 47. The sum $U + W$ is direct $\Leftrightarrow U \cap W = \{0\}$.

Proof. Let $v \in U \cap W$.

$$\Rightarrow v = \underbrace{v}_{\in U} + \underbrace{0}_{\in W} = \underbrace{0}_{\in U} + \underbrace{v}_{\in W}$$

From the uniqueness of the decomposition it follows that $v = 0$.

$$u, u' \in U \quad w, w' \in W$$

We need to show that $u = u'$ and $w = w'$. Let $v \in U + W$ with the representation $v = u + w = u' + w'$.

$$0 = v - v = u + w - (u' + w') = (u - u') + (w - w')$$

$$a := \underbrace{u' - u}_{\in U} = \underbrace{w - w'}_{\in W}$$

$$\Rightarrow a \in U \cap W = \{0\}$$

$$\Rightarrow a = 0 \Rightarrow u' = u \wedge w = w'$$

Coefficient is zero, so $v = 0$. □

This lecture took place on 30th of November 2015 (Franz Lehner).

Theorem 48.

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

If $U \cap W = \{0\}$ then the dimension is directly the sum $\dim(U) + \dim(W)$. □

$$U + W = [U \cup W] = \{u + w \mid u \in U, w \in W\}$$

A sum is called direct if for all $u \in U + W$, the decomposition $u = u + w$ is unique.

Theorem 49. The sum is direct if and only if $U \cap W = \{0\}$.

Theorem 50. Vector space V , $\dim(V) < \infty$. Then $U, W \subseteq V$ are subspaces.

The following statements are equivalent:

- $V = U \dot{+} W$
- $V = U + W \wedge \dim(V) = \dim(U) + \dim(W)$
- $U \cap W = \{0\} \wedge \dim(V) = \dim(U) + \dim(W)$

Proof. **1 implies 2**

$$V = U \dot{+} W$$

$$\Rightarrow V = U + W \wedge U \cap W = \{0\} \quad \text{Theorem 47}$$

$$\xrightarrow{\text{Theorem 48}} \dim(U + W) = \dim(U) + \dim(W)$$

2 implies 3 We use theorem 48.

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\Rightarrow \dim(V) = \dim(U) + \dim(W) - \dim(U \cap W)$$

$$\dim(U + W) = \dim(V) \text{ because } U + W = V$$

$$\dim(U) + \dim(W) = \dim(V) \text{ is required}$$

$$\Rightarrow \dim(U \cap W) = 0$$

$$\Rightarrow U \cap W = \{0\}$$

3 implies 1

$$U \cap W = \{0\} \wedge \dim(U) + \dim(W) = \dim(V)$$

$$\xrightarrow{\text{Theorem refsatz-4-5b}} \dim(U + W) = \dim(U) + \dim(W) - \dim(\{0\})$$

$$\dim(U + W) = \dim(U) + \dim(W)$$

$$U + W \subseteq V \wedge \dim(U + W) = \dim(V) \Rightarrow U + W = V$$

Example 32. Consider \mathbb{R}^2 . Let U be a subspace of dimension 1 which goes through $(0,0)$. Is there some $W \subseteq \mathbb{R}^2$ such that $\mathbb{R}^2 = U \dot{+} W$. Yes, this holds for all lines $W \neq U$ (with $\dim(W) = 1$) which go through $(0,0)$. □

Theorem 51. Let V be a vector space with $\dim(V) < \infty$. Then it holds that

$$\bigwedge_{U \subseteq V} \bigvee_{\substack{\text{subspace } W \subseteq V \\ U \cap W = \{0\}}} V = U \dot{+} W$$

W is called complementary space of U .

Remark 17. 1. Complementary spaces are not uniquely defined!

2. If $\dim(V) = \infty$, then the question for existence of complementary spaces is difficult (depends on correctness of axiom of choice, covered in the complex analysis course)

Proof. Let u_1, \dots, u_n be basis of $U \subseteq V$. We use the basis extension theorem 38.

$$\Rightarrow \bigvee_{w_1, \dots, w_n \in V} (u_1, \dots, u_n, w_1, \dots, w_m) \text{ is basis of } V$$

Then $W = L(w_1, \dots, w_m)$ is a complementary space.

We need to show that $V = U \dot{+} W$. Therefore $V = U + W$ and $U \cap W = \{0\}$.

1. Let $u \in V$. Find $u \in U, w \in W$ such that $v = u + w$.

B is basis

$$\Rightarrow \bigvee_{\lambda_1, \dots, \lambda_m} \bigvee_{\mu_1, \dots, \mu_m} v = \underbrace{\lambda_1 u_1 + \dots + \lambda_r u_r}_{=u \in U} + \underbrace{\mu_1 w_1 + \dots + \mu_m w_m}_{=w \in W} = u + w \in U + W$$

2. Let $v \in U \cap W$.

$$v \in U \Rightarrow \bigvee_{\lambda_1, \dots, \lambda_r} v = \lambda_1 u_1 + \dots + \lambda_r u_r$$

$$v \in W \Rightarrow \bigvee_{\mu_1, \dots, \mu_m} v = \mu_1 w_1 + \dots + \mu_m w_m$$

$$\Rightarrow 0 = v - v = \lambda_1 u_1 + \dots + \lambda_r u_r - \mu_1 w_1 - \dots - \mu_m w_m$$

is linear combination of B , which results in 0. The basis is linear independent, therefore all $\lambda_i = 0$ and $\mu_j = 0$. Therefore $v = 0$.

Theorem 52. Let V be a vector space. Let $U_1, \dots, U_m \subseteq V$ be subspaces. Then $U_1 + \dots + U_m = [U_1 \cup \dots \cup U_m]$ is the sum of subspaces and it holds that $U_1 + \dots + U_m = \{u_1 + \dots + u_m \mid u_i \in U_i\}$.

The proof is provided in the practicals.

$$U_1 + (U_2 + U_3) = (U_1 + U_2) + U_3$$

Attention! The inclusion-exclusion principle 46 does not hold for the dimension.

Definition 27. Let $U_1, \dots, U_m \subseteq V$ be subspaces. The sum $W = U_1 + \dots + U_m$ is called direct, if

$$\bigwedge_{w \in W} \bigvee_{u_1 \in U_1} \dots \bigvee_{u_m \in U_m} w = u_1 + \dots + u_m$$

Therefore the decomposition must be unique. We denote:

$$W = U_1 \dot{+} U_2 \dot{+} \dots \dot{+} U_m$$

The resulting mapping

$$\pi_{\mathbb{R}} : W \rightarrow U_k$$

$$w \mapsto u_k$$

is called projection on U_k .

Theorem 53. The characterization $U + W$ is direct $\Leftrightarrow U \cap W = \{0\}$ cannot be generalized. It does not suffice that $U_1 \cap \dots \cap U_m = \{0\}$

Theorem 54. Let V be a vectorspace. Let $U_1, \dots, U_m \subseteq V$ be subspaces with $U_i \neq \{0\}$.

Then the sum $W = U_1 + \dots + U_m$ is direct. Therefore every family (u_1, \dots, u_m) with $u_i \in U_i \setminus \{0\}$ is linear independent.

Proof. Proof direction \Rightarrow .

Let $u_i \in U_i \setminus \{0\}$. Show that if $\sum_{i=1}^m \lambda_i u_i = 0 \Rightarrow \lambda_i = 0 \forall i$.

Followingly therefore $\lambda_i = 0 \forall i$ and then $\lambda_i \cdot u_i = 0$. From $u_i \neq 0 \forall i$ it follows that, $\lambda_i = 0$.

Assume $\sum_{i=1}^m \lambda_i u_i = 0$.

$$\sum_{i=0}^m w_i \quad w_i = \lambda_i u_i \in U_i$$

\Rightarrow decomposition of vector 0 in components from U_i .

If the sum is direct, then the decomposition must be the same.

$$0 = 0 + 0 + \dots + 0$$

Proof. Proof direction \Leftarrow .

Let $w \in W$ with $w = \sum_{i=1}^m u_i$. Show that the decomposition is unique.

Let $w = \sum_{i=1}^m w_i$ is a different decomposition. Show that all $u_i = w_i$

$$0 = w - w = \sum_{i=1}^m (u_i - w_i)$$

Let

$$w_i = \begin{cases} u_i - w_i & \text{if } u_i \neq w_i \\ z_i \in U_i \setminus \{0\} & \text{arbitrary} \end{cases} \Rightarrow w_i \neq 0$$

Correspondingly

$$\lambda_i = \begin{cases} 1 & u_i \neq w_i \\ 0 & u_i = w_i \end{cases}$$

$$\sum_{i=1}^m \lambda_i \cdot w_i = 0$$

$$= \sum_{\substack{i \\ u_i \neq w_i}} u_i - w_i + \sum_{\substack{i \\ u_i = w_i}} 0 \cdot w_i = 0$$

$$w_i \text{ is linear indep.} \Rightarrow \lambda_i = 0 \forall i \Rightarrow \bigwedge_{\substack{i \\ \lambda_i=1 \text{ does not occur}}} u_i = w_i$$

□

“Die Sache ist an sich klar. Nur wenn man sie niederschreibt, wird sie unklar.” (Franz Lehner)

Theorem 55. Let V be a vector space. $\dim(V) < \infty$.

$$U_1, \dots, U_m \subseteq V \text{ are subspaces, } U_i \neq \{0\}$$

□ Then the following statements are equivalent:

1. $W = U_1 + \dots + U_m$ is direct.
2. For every choice of basis $B_i \subseteq U_i$, $B_1 \cup \dots \cup B_m$ is basis of W .
3. $\dim(W) = \sum_{i=1}^m \dim(U_i)$

Proof 2 to 3 follows immediately.

1 to 2 Let $W = U_1 + \dots + U_m$. Let $B_i = (u_{i,1}, \dots, u_{i,\sqrt{i}})$ be basis of U_i for all i .

We need to show that $B_1 \cup \dots \cup B_m$ is basis of W . Therefore,

- (a) $L(B_1 \cup \dots \cup B_m) = W$
- (b) $B_1 \cup \dots \cup B_m$ is linear independent.

We prove those statements:

(a)

$$L(B_1 \cup \dots \cup B_m) = L(B_1) + \dots + L(B_m) = U_1 + \dots + U_m = W$$

(b) $B_1 \cup \dots \cup B_m$ is linear independent.

$$B_1 \cup \dots \cup B_m = \{b_{ij} \mid i \in \{1, \dots, m\}, j \in \{1, \dots, r_j\}\}$$

Let $\lambda_i \in K$ with $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, r_i\}$. Such that

$$\sum_{i=1}^m \sum_{j=1}^r \lambda_{ij} \mu_{ij} = 0$$

Show that all $\lambda_{ij} = 0$.

Let $w_i = \sum_{j=1}^r \lambda_{ij} u_{ij} \in U_i$.

$$\Rightarrow \sum_{i=1}^m w_i = 0$$

The sum of U_i is direct. Therefore the vector 0 has a unique decomposition. Therefore all $w_i = 0$.

$$\Rightarrow \sum_{j=1}^{r_i} \lambda_{ij} u_{ij} = 0 \forall i$$

u_{ij} is basis of U_i . So it is linear independent. So $\lambda_{ij} = 0 \forall j \in \{1, \dots, r_i\}$.

This holds for every i

$$\Rightarrow \lambda_{ij} = 0 \quad \forall i \forall j$$

3 implies 1 Let $B_i = (u_{i,1}, u_{i,2}, \dots, u_{i,r_i})$ be basis of U_i and $B = B_1 \cup \dots \cup B_m$ is basis of W .

Show that every $w \in W$ has a unique decomposition.

$$w = w_1 + \dots + w_m \text{ with } w_i \in U_i$$

Let $w = w'_1 + \dots + w'_m$ be a different decomposition.

Let $w_i = \sum_{j=1}^{r_i} \lambda_{ij} u_{ij}$ be a decomposition of w_i in regards of basis B_i .

$$\begin{aligned} w'_i &= \sum_{j=1}^{r_i} \mu_{ij} u_{ij} \\ \Rightarrow w &= \sum_{i=1}^m \left(\sum_{j=1}^{r_i} \lambda_{ij} u_{ij} \right) \\ &= \sum_{i=1}^m \left(\sum_{j=1}^{r_i} \mu_{ij}, u_{ij} \right) \end{aligned}$$

Let (u_{ij}) be basis of W . Therefore all $\lambda_{ij} = \mu_{ij}$. Therefore $w_i = w'_i$ for all i . So the decomposition is unique. □

Remark 18 (Special case).

(b_1, \dots, b_m) is basis of W

$$\Leftrightarrow w = L(b_1) + L(b_2) + \dots + L(b_m)$$

Theorem 56. Let V, W be vector spaces over K .

Given vector space X such that $X = V, W$. For example, $V = K[x]$ and $W = K^3$.

Then also

$$V \times W = \{(u, w) \mid u \in V, w \in W\}$$

with the operations

$$(v, w) + (v', w') = (v + v', w + w')$$

$$\lambda \cdot (v, w) = (\lambda v, \lambda w)$$

Given a vector space with vector 0 (which is $(0_v, 0_w)$) and an inverse element

$$-(v, w) = (-v, -w)$$

The product $V \times W$ (or denoted $V \oplus W$) is called direct product or outer sum (but not $V \otimes W$ which is the tensor product).

Theorem 57. If $\dim(V), \dim(W) < \infty$. Then $\dim(V \oplus W) = \dim(V) + \dim(W)$.

Proof. We are going to construct an appropriate basis. Let (v_1, \dots, v_m) be a basis in V . Let (w_1, \dots, w_n) be a basis in W .

Our claim is that $((u_1, 0), (u_2, 0), \dots, (u_m, 0), (0, w_1), (0, w_2), \dots, (0, w_n)) = B$ is a basis of $V \oplus W$.

Show that

1. B is linear independent.
2. $L(B) = V \oplus W$

Proof:

1. Let

$$\lambda_1, \dots, \lambda_{m+n} \in K \text{ such that } \sum_{i=1}^m \lambda_i(v_i, 0) + \sum_{j=1}^n \lambda_{m+j}(0, w_j) = (0, 0)$$

Show that all $\lambda_i = 0$.

$$\begin{aligned} &= \sum_{i=1}^m (\lambda_i v_i, 0) + \sum_{j=1}^n (0, \lambda_{m+j} w_j) \\ &= \left(\sum_{i=1}^m (\lambda_i v_i, 0) \right) + \left(0, \sum_{j=1}^n \lambda_{m+j} w_j \right) \\ &= \left(\sum_{i=1}^m \lambda_i v_i, \sum_{j=1}^n \lambda_{m+j} w_j \right) \stackrel{?}{=} (0_v, 0_w) \\ &\Rightarrow \sum_{i=1}^m \lambda_i v_i = 0_v \wedge \sum_{j=1}^n \lambda_{m+j} w_j = 0_w \end{aligned}$$

(v_1, \dots, v_m) is linear independent.

$$\Rightarrow \lambda_1 = \dots = \lambda_m = 0 \quad \Rightarrow \lambda_{m+1} = \dots = \lambda_{m+n} = 0$$

2. Let $(v, w) \in V \oplus W$.

$$\begin{aligned} \rightsquigarrow \bigvee_{\lambda_1, \dots, \lambda_m} v &= \sum_{i=1}^m \lambda_i v_i \\ \bigvee_{\mu_1, \dots, \mu_n} w &= \sum_{j=1}^n \mu_j w_j \end{aligned}$$

$$\begin{aligned} (v, w) &= \left(\sum_{i=1}^m \lambda_i v_i, \sum_{j=1}^n \mu_j w_j \right) \\ &= \left(\sum_{i=1}^m \lambda_i v_i, 0 \right) + \left(0, \sum_{j=1}^n \mu_j w_j \right) \\ &= \left(\sum_{i=1}^m \lambda_i (v_i, 0) + \sum_{j=1}^n \mu_j (0, w_j) \right) \in L(B) \end{aligned}$$

Every $(v, w) \in V \oplus W$ is in $L(B)$. $V \oplus W \subseteq L(B)$.

□

Remark 19. Let V_1 and V_2 be vector spaces.

$$V = V_1 \oplus V_2$$

Then we can identify V_1 with the subspace

$$U_1 = \{(v_1, 0) \mid v_1 \in V_1\} \subseteq V_1 \oplus V_2$$

analogously

$$V_2 \cong U_2 = \{(0, v_2) \mid v_2 \in V_2\} \subseteq V_1 \oplus V_2$$

and it holds that

$$V_1 \oplus V_2 = U_1 + U_2$$

Theorem 58. Let I be an index set. For every $i \in I$, let V_i be a vector space over K .

Direct product:

$$\prod_{i \in I} V_i = \times_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i \forall i\}$$

Direct outer sum:

$$\oplus_{i \in I} V_i = \{(v_i)_{i \in I} \mid v_i \in V_i \text{ and only finitely many } v_i \neq 0\}$$

They are vector spaces in regards of operations:

$$(v_i)_{i \in I} + (w_i)_{i \in I} = (v_i + w_i)_{i \in I} \quad \lambda \cdot (v_i)_{i \in I} = (\lambda \cdot v_i)_{i \in I}$$

$$\oplus_{i \in I} V_i \subsetneq \prod_{i \in I} V_i \text{ if } I \text{ is infinite}$$

Example 33.

$$\mathbb{R}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{R}$$

$$\begin{aligned} \oplus_{n \in \mathbb{N}} \mathbb{R} &= \left\{ (x_n)_{n \in \mathbb{N}} \mid \bigvee_{n \in \mathbb{N}} \bigwedge_{n \geq n_0} x_n = 0 \right\} \\ &= \{(x_0, x_1, \dots, x_n, 0, \dots) \mid n \in \mathbb{N}, x_i \in \mathbb{R}\} \\ &\cong \mathbb{R}[x] \end{aligned}$$

In between there are many other spaces (complex analysis discusses that).

For example, $c_0 = \{(x_n) \mid \lim_{n \rightarrow \infty} x_n = 0\}$.

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \lambda a_n = \lambda \lim_{n \rightarrow \infty} a_n$$

Because this holds, we have two operations for a vector space. This is actually a vector space (over the set of convergent sequences).

$$\mathbb{R}^{\mathbb{N}} := \oplus_{n \in \mathbb{N}} \mathbb{R} \subsetneq c_0 \subsetneq \mathbb{R}^{\mathbb{N}}$$

with

$$c = \{(x_n) \mid \lim x_n \text{ exists}\} = c_0 \oplus L((1, 1, 1, \dots)).$$

$$l^\infty = \left\{ (x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R} \wedge \sup_n (|x_n|) < \infty \right\}$$

$$\mathbb{R}^{(\mathbb{N})} \subsetneq c_0 \subsetneq c \subsetneq l^\infty \subsetneq \mathbb{R}^{\mathbb{N}}$$

Every convergent sequence (x_n) is uniquely representable as $(y_n) + \lambda(1, 1, 1, \dots)$ with $(y_n) \in c_0$.

Remark 20.

$$(\mathbb{Z}_n, +)$$

Is a factor set $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Factorization in regards of relation:

$$x \equiv_1 y \Leftrightarrow nx \mid -y \Leftrightarrow x - y \in n\mathbb{Z}$$

Let $(G, +)$ be an abelian group. $H \subseteq G$ as subgroup. So this is a equivalence relation:

$$x \equiv_H y \Leftrightarrow x - y \in H$$

Theorem 59 (Applied to vector spaces). Let V be a vector space over K . $U \subseteq V$ is a subspace.

1. The relation

$$v \sim_u w \Leftrightarrow v - w \in U$$

is an equivalence relation in V .

2. The equivalence class of a vector $v \in V$ is

$$[v]_u = \{w \mid w - v \in U\} = \{v + u \mid u \in U\} = v + U$$

is called linear manifold or affine space.

(Consider a vector v and a line U . $v + U$ is the set of all lines parallel to U and going through v .)

3.

$$\bigwedge_{v,v',w,w' \in V} v \sim_U v' \wedge w \sim_U w' \Rightarrow v + w \sim_U v' + w'$$

$$\bigwedge_{\lambda \in K} \bigwedge_{v,v' \in V} v \sim_U v' \Rightarrow \lambda v \sim_U \lambda v'$$

We therefore define

TODO content incomplete/incorrect

 Proof. 1. **reflexive** $v \sim_U v \Leftrightarrow v - v \in U$
symmetrical $v \sim_U w \Leftrightarrow v - w \in U \Rightarrow w - v \in U \Rightarrow w \sim_U v$
transitive $v \sim_U w \wedge w \sim_U z \Rightarrow v - w \in U, w - z \in U$ and $v - z = (v - w) + (w - z) \in U$.

2. Follows immediately.

3.

$$v - v' \in U, w - w' \in U \Rightarrow v - v' + w - w' \in U$$

$$(v + w') - (v' + w')$$

 Here we can see, that this will not work in non-commutative groups⁴.

$$4. v - v' \in U \Rightarrow \lambda v - \lambda v' = \lambda(v - v') \in U$$

Theorem 60. The set of equivalence classes V/U :

$$V/U := (V/\sim_U, +, \cdot)$$

with the operations

$$[v]_U + [w]_U := [v + w]_U$$

$$[\Rightarrow v + U + w + U = (v + w) + U]$$

⁴We need at least the requirement of a normal divisor.

$$xHx^{-1} = H \quad \forall x \in G$$

$$\lambda \cdot [v]_U := [\lambda v]_U$$

$$[\Rightarrow \lambda \cdot (v + U) = \lambda v + U]$$

is a vector space with neutral element

$$[0]_U = U$$

and inverse element

$$-[v]_U = [-v]_U = -v + U$$

and is called factor space or quotient space.

Proof. The operations of Theorem 59 are well-defined. The distributive laws:

$$\begin{aligned} \lambda \cdot ([v]_U + [w]_U) &\stackrel{!}{=} \lambda[v]_U + \lambda[w]_U \\ &= \lambda \cdot [v + w]_U \\ &= [\lambda(v + w)]_U \\ &= [\lambda v + \lambda w]_U \\ &= [\lambda v]_U + [\lambda w]_U \\ &= \lambda[v]_U + \lambda[w]_U \end{aligned}$$

□

Example 34.

$$V = \mathbb{R}^3$$

$$U = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} = L \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + U = \left\{ \begin{pmatrix} v_1 + x \\ v_2 + y \\ v_3 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} x' \\ y' \\ v_3 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

 V/U is the plane parallel to the x - y -plane.

$$\left(\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} + U \right) + \left(\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} + U \right) = \left(\begin{pmatrix} 0 \\ 0 \\ z_1 + z_2 \end{pmatrix} + U \right)$$

$$V/U \cong \mathbb{R}$$

Theorem 61. Let $\dim(V) < \infty$.

$U \subseteq V$ is a subspace

Then $\dim(V/U) = \dim(V) - \dim(U)$.

Proof. Let (u_1, \dots, u_r) be a basis of U . The basis extension theorem allows us to extend this set with (w_1, \dots, w_m) such that $(u_1, \dots, u_r, w_1, \dots, w_m)$ is basis of V .

Claim: $\tilde{B} = (w_1 + U, w_2 + U, \dots, w_m + U)$ is basis of V/U .

These are exactly the equivalence classes of elements with basis of V , which are not mapped to $0 + U$ ($[0]_U$).

We need to prove that this is a basis:

1. Linear independence of \tilde{B}

2. $L(\tilde{B}) = V/U$

So,

1. Let $\lambda_1, \dots, \lambda_m \in K$ such that $\lambda_1(w_1 + U) + \dots + \lambda_m(w_m + U) = [0]_U$.

$$\lambda_1 w_1 + \dots + \lambda_m w_m + U = U$$

$$\Rightarrow \lambda_1 w_1 + \dots + \lambda_m w_m \in U$$

We know: $U \cap L(w_1, \dots, w_m) = \{0\}$. So,

$$\lambda_1 w_1 + \dots + \lambda_m w_m \cap L(w_1, \dots, w_m) = \{0\}$$

because the basis of U is linear independent of $L(w_1, \dots, w_m)$.

$$\Rightarrow \lambda_1 w_1 + \dots + \lambda_m w_m = 0$$

$\Rightarrow \lambda_i = 0$ because (w_1, \dots, w_m) is linear independent (part of a basis)

2. $L(\tilde{B}) \subseteq V/U$ is obvious.

Let $v + U \in V/U$

$$\Rightarrow v = \sum_{i=1}^r \lambda_i u_i + \sum_{i=1}^m \lambda_{r+i} w_i$$

Decomposition in regards of basis B of V .

$$v + U = \underbrace{\sum_{i=1}^r \lambda_i u_i}_{\in U} + \sum_{i=1}^m \lambda_{r+i} w_i + U$$

$$= \sum_{i=1}^m \lambda_{r+i} w_i + U$$

$$= \sum_{i=1}^m \lambda_{r+i} (w_i + U) \in L(\tilde{B})$$

□

7.1 Conclusion

What did we do in this section?

- $U + W$ (sums)
- $U \dot{+} W$ (direct sums)
- $V \oplus W, V \times W$ (outer sums)
- $\prod_{i \in I} V_i, \oplus_{i \in I} V_i$
- V/U

8 Linear mappings

Definition 28. Let V, W be vector spaces over K . A mapping $f : V \rightarrow W$ is called vector space homomorphism or linear if

$$\bigwedge_{v, w \in V} f(v + w) = f(v) + f(w) \quad \text{“additivity”}$$

$$\bigwedge_{\lambda \in K} \bigwedge_v f(\lambda v) = \lambda f(v) \quad \text{“multiplicity”}$$

We denote:

$$\text{Hom}(V, W) = \{f : V \rightarrow W \mid f \text{ is linear}\}$$

Theorem 62. $f : V \rightarrow W$ is linear

$$\begin{aligned} \Leftrightarrow \bigwedge_{\lambda, \mu \in K} \bigwedge_{v, w \in V} f(\lambda v + \mu w) &= \lambda f(v) + \mu f(w) \\ \Leftrightarrow \bigwedge_{\lambda \in K} \bigwedge_{v, w \in V} f(\lambda v + w) &= \lambda f(v) + f(w) \end{aligned}$$

Example 35.

$$V = \mathbb{R} = W$$

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be linear. $x \mapsto k \cdot x$ with $k \in \mathbb{R}$ fixed.

As in high school: $f(x) = kx + d$.

Example 36.

$$\begin{aligned} id : V &\rightarrow V \\ x &\mapsto x \end{aligned}$$

Example 37. V with base (b_1, b_2, \dots, b_n) .

$$\bigwedge_{v \in V} \bigvee_{\lambda_1, \dots, \lambda_n} v = \lambda_1 b_1 + \dots + \lambda_n b_n$$

$$\Phi_B : V \rightarrow K^n \text{ is linear}$$

$$v \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

To be discussed in the practicals.

This lecture took place on 7th of December 2015 (Franz Lehner).

Homomorphisms and vector spaces:

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$$

$$f : V \rightarrow W$$

Example 38.

$$id : V \rightarrow V$$

$$v \mapsto v$$

Let V be a vector space. Let $B = (v_1, \dots, v_n)$ be our basis.

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n$$

$$\Phi_B : V \rightarrow K^n$$

$$v \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

In the practicals it is shown to be linear.

Remark 21. Special case: Let $V = K^n$. Let $B = (e_1, \dots, e_n)$ be your basis.

$$\Phi_i : \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \lambda_i$$

$$\Phi_i : (a + b) = \Phi_i \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = a_i + b_i = \Phi_i(a) + \Phi_i(b)$$

Remark 22. Also:

$$\Phi_i : V \rightarrow K$$

$$v \mapsto \lambda_i$$

Example 39.

$$V = K^X = \{f : X \rightarrow K\}$$

$$(f + g)(x) = f(x) + g(x)$$

$$(\lambda \cdot f)(x) = \lambda \cdot f(x)$$

Pointwise operations.

Let $x \in X$.

$$\Rightarrow \Phi_x : V \rightarrow K$$

$$f \mapsto f(x)$$

$$\Phi_x(\lambda f + \mu g) = (\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x) = \lambda \Phi_x(f) + \mu \Phi_x(g)$$

Example 40.

$$\begin{aligned}\mathbb{R}[x] &\rightarrow \mathbb{R}[x] \\ x^n &\mapsto n \cdot x^{n-1} \\ \sum_{k=0}^n a_k x^k &\mapsto \sum_{k=1}^n k \cdot a_k x^{k-1}\end{aligned}$$

The derivation of $p(x) \rightarrow p'(x)$ is additive:

$$\begin{aligned}(p+q)(x) &= p'(x) + q'(x) \\ (\lambda p)'(x) &= \lambda \cdot p'(x)\end{aligned}$$

Example 41.

$$\begin{aligned}\int_a^b : \mathbb{R}[x] &\rightarrow \mathbb{R} \\ p(x) &\mapsto \int_a^b p(x) dx \text{ is linear.}\end{aligned}$$

Example 42.

$$\begin{aligned}V &= \mathbb{R}^2 \\ T_{x_0} : x &\mapsto x + x_0 \\ x_0 = T_{x_0}(0) &= T_{x_0}(0 + 0) = T_{x_0}(0) + T_{x_1}(0) = 2x_0 \quad \nexists\end{aligned}$$

Translation in \mathbb{R}^2 is non-linear. It is only affine linear (translation together with rotation).

Example 43. Rotation itself in \mathbb{R}^2 is linear.

$$U_q : v = \text{rotated vector } q \text{ is linear}$$

Example 44.

$$A : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 \\ x_2 \end{pmatrix} \text{ is linear}$$

Dilation is linear.

Example 45.

$$A(\lambda x + y) = \begin{pmatrix} 2(\lambda x_1 + y_1) \\ \lambda x_2 + y_2 \end{pmatrix} = \lambda \begin{pmatrix} 2x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 \\ y_2 \end{pmatrix} = \lambda A(x) + A(y) \text{ is linear}$$

Example 46.

$$\begin{aligned}C &= \{(x_n)_{n \in \mathbb{N}} \mid x_n \in \mathbb{R}, x_n \text{ is convergent}\} \\ \lim_{n \rightarrow \infty} (x_n + y_n) &= \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} y_n \\ \lim_{n \rightarrow \infty} (\lambda x_n) &= \lambda \cdot \lim_{n \rightarrow \infty} x_n \\ \Rightarrow \text{the mapping } \lim_{n \rightarrow \infty} c &\rightarrow \mathbb{R} \\ (x_n)_{n \in \mathbb{N}} &\mapsto \lim_{n \rightarrow \infty} x_n\end{aligned}$$

is linear.

Example 47.

$$\begin{aligned}V = l^1 &= \left\{ (\lambda_n) \mid \sum_{n=1}^{\infty} |\lambda_n| < \infty \right\} \\ \sum_{n=1}^{\infty} |x_n + y_n| &\leq \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| < \infty \\ \sum_{n=1}^{\infty} (x_n + y_n) &= \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n \\ \sum_{n=1}^{\infty} \lambda x_n - \lambda \cdot \sum_{n=1}^{\infty} x_n \\ \Rightarrow \sum_{n=1}^{\infty} : l^1 &\rightarrow \mathbb{R} \text{ is linear} \\ (x_n)_{n \in \mathbb{N}} &\mapsto \sum_{n=1}^{\infty} x_n\end{aligned}$$

Example 48.

$$\begin{aligned}V = U \dot{+} W &\text{ is the direct sum} \\ \bigwedge_v \dot{\bigvee}_{u \in U} \dot{\bigvee}_{w \in W} v &= u + w \text{ is unambiguous} \\ \pi_U : V &\rightarrow U \quad \text{“projections on } U\text{”} \\ v &\mapsto u \\ \pi_W : V &\rightarrow W \quad \text{“projections on } W\text{”} \\ v &\mapsto w\end{aligned}$$

Theorem 63. Let V and W be vector spaces.

$f : V \rightarrow W$ is linear

1. $f(0_v) = 0_w$
2. $\bigwedge_{v \in V} f(-v) = -f(v)$
3. It holds that,

$$\bigwedge_n \bigwedge_{\lambda_1, \dots, \lambda_n \in K} \bigwedge_{v_1, \dots, v_n \in V} f(\lambda_1 v_1 + \dots + \lambda_n v_n) = \lambda_1 f(v_1) + \lambda_2 f(v_2) + \dots + \lambda_n f(v_n)$$

Proof. We prove the first statement:

$$f(0_v) = f(0_v + 0_v) = f(0_v) + f(0_v)$$

$$0_w = f(0_v)$$

We prove the second statement.

$$f(-v) = f((-1) \cdot v) = (-1) \cdot f(v) = -f(v)$$

□

Definition 29. Let V and W be vector spaces. Let $f : V \rightarrow W$. Homomorphism is an

- epimorphism if $f : V \rightarrow W$ and f is surjective.
- monomorphism if $f : V \rightarrow W$ and f is injective.
- isomorphism if $f : V \rightarrow W$ and f is bijective.

Let $V = W$, then

- endomorphism if $f : V \rightarrow V$.
- automorphism if $f : V \rightarrow V$.

We also denote

$\text{Hom}(V, W) =$ homomorphism from V to W

$$\text{End}(V) = \text{Hom}(V, V)$$

$$\text{Aut}(V) = \{f : V \rightarrow V \text{ automorphism}\}$$

Definition 30. V and W are isomorphic $V \cong W$ if there exists isomorphism $f : V \rightarrow W$.

V is called embeddable in W if there exists some monomorphism $f : V \rightarrow W$. f is called embedding.

Theorem 64. Let U, V and W be vector spaces over K .

$f : U \rightarrow V$ $g : V \rightarrow W$ is linear

1. $\Rightarrow g \circ f : U \rightarrow W$ is linear.
2. \Rightarrow if $f : U \rightarrow V$ is isomorphism, then also $f^{-1} : V \rightarrow U$ is linear.

Proof. We prove the first statement.

$$g \circ f(\lambda \cdot v + \mu w) \stackrel{!}{=} \lambda \cdot g \circ f(v) + \mu g \circ f(w)$$

$$\begin{aligned} g \circ f(\lambda \cdot v + \mu w) &= g(f(\lambda v + \mu w)) = g(\lambda f(v) + \mu f(w)) \\ &= \lambda \cdot g(f(v)) + \mu \cdot g(f(w)) = \lambda(g \circ f)(v) + \mu(g \circ f)(w) \end{aligned}$$

We prove the second statement.

$$f^{-1}(\lambda v + \mu w) = \underbrace{f^{-1}(\lambda f(f^{-1}(v))) + \mu \cdot f(f^{-1}(w))}_{f(\lambda \cdot f^{-1}(v) + \mu f^{-1}(w))}$$

$$f^{-1}(f(\lambda \cdot f^{-1}(v) + \mu f^{-1}(w))) = \lambda f^{-1}(v) + \mu f^{-1}(w)$$

□

Theorem 65. $\text{Hom}(V, W)$ with the operations $(f + g)(v) = f(v) + g(v)$ and $(\lambda f)(v) = \lambda \cdot f(v)$ is a vector space with 0-vector $0 : V \rightarrow W$ and $v \mapsto 0$.

Proof. We need to prove that $\text{Hom}(V, W)$ is a subspace of W^V . Therefore **Example 49.**
 $f, g \in \text{Hom}(V, W)$ is therefore

$$f + g \text{ and } \lambda \cdot f$$

Show that,

$$(\lambda \cdot f + \mu \cdot g)(\alpha v + \beta w) \stackrel{!}{=} \lambda \cdot (\lambda f + \mu g)(v) + \beta(\lambda f + \mu g)(w)$$

$$\begin{aligned} (\lambda f + \mu g)(\alpha v + \beta w) &= \lambda f(\alpha v + \beta w) + \mu g(\alpha v + \beta w) \\ f, g \text{ are linear} &= \lambda(\alpha f(v) + \beta f(w)) + \mu(\alpha g(v) + \beta g(w)) \\ &= \alpha(\lambda f(v) + \mu g(v)) + \beta(\lambda f(w) + \mu g(w)) \\ &= \alpha(\lambda f + \mu g)(v) + \beta(\lambda f + \mu g)(w) \end{aligned}$$

$\Rightarrow (\text{Hom}(V, W), +, \cdot)$ is a vector space over K . \square

Theorem 66. Let $V = W$, then $(\text{End}(V), +, \circ)$ where \circ denotes composition is a ring.

- Proof.* 1. $(\text{End}(V), +)$ is an abelian group \checkmark
 2. $(\text{End}(V), \circ)$ is a semi-group (sub-semigroup of (V^V, \circ))
 3. Distributive law is shown in the practicals.

Definition 31. An algebra over a field K is a structure

$$\begin{aligned} (A, +, \cdot, *) \\ + : A \times A \rightarrow A \\ \cdot : K \times A \rightarrow A \\ * : A \times A \rightarrow A \end{aligned}$$

such that $(A, +, \cdot)$ is a vector space and $(A, +, *)$ is a ring.

Associativity holds,

$$\lambda(a * b) = (\lambda \cdot a) * b = a * (\lambda b)$$

$$\begin{aligned} A &= \mathbb{R}[x] \\ (p + q)(x) &= p(x) + q(x) \\ \lambda \cdot p(x) & \\ (p * q)(x) &= p(x) \cdot q(x) \end{aligned}$$

also satisfies associativity.

Theorem 67. $(\text{End}(V), +, \cdot, \circ)$ is a non-commutative algebra.

Proof. It only remains to show associativity. This is left for the practicals. \square

8.1 Linear mappings and subspaces

Theorem 68. Let V and W be vector spaces over K .

$f : V \rightarrow W$ is linear

1. if $V' \subseteq V$ is a subspace, then $f(V') \subseteq W$ is a subspace.
2. if $W' \subseteq W$ is a subspace, then $f^{-1}(W') \subseteq V$ is a subspace.

Proof. 1. Let $w_1, w_2 \in f(V)$ then also $\lambda_1 w_1 + \lambda_2 w_2 \in f(V')$. Let $w_1, w_2 \in f(V')$.

$$\begin{aligned} \Rightarrow \bigvee_{v_1 \in V'} \bigvee_{v_2 \in V'} f(v_1) = w_1 \wedge f(v_2) = w_2 \\ \lambda_1 w_1 + \lambda_2 w_2 = \lambda_1 f(v_1) + \lambda_2 f(v_2) \\ f \text{ is linear} \Rightarrow f(\underbrace{\lambda_1 v_1 + \lambda_2 v_2}_{\in V'}) \in f(V') \end{aligned}$$

2. Show that $v_1, v_2 \in f^{-1}(W')$ then also $\lambda_1 v_1 + \lambda_2 v_2 \in f^{-1}(W')$. Show that if $f(v_1), f(v_2) \in W'$ then $f(\lambda_1 v_1 + \lambda_2 v_2) \in W'$.

$$f(\lambda_1 v_1 + \lambda_2 v_2) = \underbrace{\lambda_1 f(v_1)}_{\in W'} + \underbrace{\lambda_2 f(v_2)}_{\in W'} \in W' \text{ because its a subspace}$$

\square

Theorem 69. Let V and W be vector spaces over K .

$f : V \rightarrow W$ is linear

$$(v_i)_{i \in I} \subseteq V$$

$$1. f(L((v_i)_{i \in I})) = L((f(v_i))_{i \in I})$$

$$M \subseteq V$$

$$f(L(M)) = L(f(M))$$

$$2. (f(v_i))_{i \in I} \text{ linear independent} \Rightarrow (v_i)_{i \in I} \text{ linear independent}$$

The inverse of the second statement does not hold (think about the zero-element).

Proof. 1.

$$\begin{aligned} w \in f(L((v_i)_{i \in I})) &\Leftrightarrow \bigvee_{v \in L((v_i)_{i \in I})} w = f(v) \\ &\Leftrightarrow \bigvee_m \bigvee_{i, \dots, i_n} \bigvee_{\lambda_1, \dots, \lambda_n} w = f(\lambda_1 v_{i,1} + \dots + \lambda_n v_{i,n}) \\ &\Leftrightarrow \bigvee_m \bigvee_{i, \dots, i_n} \bigvee_{\lambda_1, \dots, \lambda_n} w = \lambda_1 f(v_{i,1}) + \dots + \lambda_n f(v_{i,n}) \\ &\Leftrightarrow w \in L((f(v_i))_{i \in I}) \end{aligned}$$

$$2. \text{ Let } \lambda_1 v_{i,1} + \dots + \lambda_n v_{i,n} = 0 \stackrel{!}{\Rightarrow} \text{ all } \lambda_i = 0.$$

$$f(\lambda_1 v_{i,1} + \dots + \lambda_n v_{i,n}) = 0_w$$

$$f \text{ linear} \Rightarrow \lambda_1 f(v_{i,1}) + \dots + \lambda_n f(v_{i,n}) = 0$$

$$f(v_i) \text{ linear independent} \Rightarrow \text{ all } \lambda_i = 0$$

Theorem 70. Let V, W be vector spaces. Let $f : V \rightarrow W$ be linear.

$$1. f \text{ is surjective and } L(M) = V, \text{ then } L(f(M)) = W.$$

$$2. f \text{ is injective and } M \subseteq V \text{ is linear independent, then } f(M) \text{ is linear independent in } W.$$

$$3. f \text{ is bijective and } B \text{ is basis then } B \text{ is basis of } W.$$

This lecture took place on 14th of December 2015 (Franz Lehner).

Proof. 1. If f is surjective and $L(M) = V$, then $L(f(M)) = W$. If f is surjective, then the image of the generating system is also a generating system.

$$L(f(M)) \stackrel{\text{Theorem 69}}{=} f(L(M)) = f(V) \stackrel{\text{surj.}}{=} W$$

$$2. \text{ Let } f(v_i) \in f(M). \text{ Let } \sum \lambda_i f(v_i) = 0. \text{ Then } f(\sum \lambda_i v_i) = 0_W = f(0_V).$$

$$f \text{ inj.} \Rightarrow \sum \lambda_i v_i = 0_v$$

$$M \text{ is linear indep.} \Rightarrow \text{ all } \lambda_i = 0$$

$$3. \text{ If } f \text{ is bijective and } B \subseteq V \text{ is basis, then } f(B) \text{ is basis.}$$

□

Theorem 71. Let $f : V \rightarrow W$ be linear.

- If f is injective, then $\dim V \leq \dim W$.
- If f is surjective, then $\dim V \geq \dim W$.
- If f is bijective, then $\dim V = \dim W$.

Proof. Let $(b_i)_{i \in I}$ be a basis of V .

□

1. If $\dim W = \infty$, we are done. $\dim W < \infty$, then from Theorem 70 it follows that, $(f(b_i))_{i \in I}$ is linear in W . $\dim W$ is given by maximal size of a linear independent family in W .

$$\Rightarrow \dim W \geq [I] = \dim V$$

2. If $\dim V = \infty$, we are done. If $\dim V < \infty \Rightarrow |I| < \infty$. From Theorem 70 (1) it follows that $(f(b_i))_{i \in I}$ generates W . $\dim W$ is given by maximal size of a linear independent family in W .

$$\Rightarrow \dim W \leq |I| = \dim V$$

3. Follows from the previous two items or directly from Theorem 70 (2).

□

Corollary 14. *If V and W are isomorphic (ie. if an isomorphism $f : V \rightarrow W$ exists), then $\dim V = \dim W$. Therefore the dimension of a vector space is an invariant.*

We show the inverse: If $\dim V = \dim W$, then isomorphism is given.

Theorem 72. *Abstract definition: “In the category of vector spaces, all objects are free.”*

Given two vector spaces V and W . Let $(b_i)_{i \in I} \subseteq V$ be basis of V . $(w_i)_{i \in I} \subseteq W$ is arbitrary.

Then there exists a distinct linear mapping $f : V \rightarrow W$, such that $f(b_i) = w_i$ for all i .

Corollary 15. *Two linear mappings $f, g : V \rightarrow W$ are equal (ie. $\bigwedge_{v \in V} f(v) = g(v)$).*

$$\Leftrightarrow f|_B = g|_B \text{ for a basis of } V$$

Proof. A linear mapping with $f(b_i) = w_i$ and linear combination $v = \sum \lambda_i b_i$ must give

$$f(v) = f\left(\sum \lambda_i b_i\right) = \sum \lambda_i f(b_i) = \sum \lambda_i w_i$$

We therefore define

$$f(v) = \sum_{j=1}^n \lambda_j w_{ij}$$

If $v = \sum_{j=1}^n \lambda_j b_{ij}$ (decomposition in regards of basis).

This define a function $f : V \rightarrow W$. So for every decomposition in regards of the basis, this decomposition is distinct. Therefore f is well-defined.

We now need to show: f is linear.

$$v = \sum_{i=1}^n \alpha_j b_{ij} \quad v = \sum_{j=1}^n \beta_j b_{ij}$$

Without loss of generality in both vectors we have the same basis vectors b_{ij} (in other case we extend them using zero coefficients).

$$\begin{aligned} f(\lambda u + \mu v) &= f\left(\lambda \sum_{j=1}^n \alpha_j b_{ij} + \mu \sum_{j=1}^n \beta_j b_{ij}\right) \\ &= f\left(\sum_{j=1}^n (\lambda \alpha_j + \mu \beta_j) b_{ij}\right) \\ &= \sum_{j=1}^n (\lambda \alpha_j + \mu \beta_j) w_{ij} \\ &= \lambda \sum_{j=1}^n \alpha_j w_{ij} + \mu \sum_{j=1}^n \beta_j w_{ij} \\ &= \lambda f(u) + \mu f(v) \end{aligned}$$

Therefore it is linear. But is it distinct?

Let $g : V \rightarrow W$ be linear with $g(b_i) = w_i$ for all i . We need to show that $g = f$. Therefore $g(v) = f(v)$ (for all $v \in V$). Let $v \in V \Rightarrow v = \sum_{j=1}^n \lambda_j b_{ij}$ be a decomposition in regards of the basis. Therefore $g(v) = g\left(\sum_{j=1}^n \lambda_j b_{ij}\right) = \sum_{j=1}^n \lambda_j g(b_{ij}) = \sum_{j=1}^n \lambda_j w_{ij} = f(v)$. □

Theorem 73. *Let V and W be finite-dimensional vector spaces. Then $V \cong W \Leftrightarrow \dim V = \dim W$.*

$$(\delta_x)_{x \in \mathbb{R}} \subseteq \mathbb{R}^{\mathbb{R}}$$

is linear independent, where

$$\delta_x(t) = \begin{cases} 1 & \text{if } t = x \\ 0 & \text{else} \end{cases}$$

Proof. Proof \Rightarrow Let $f : V \rightarrow W$ be an isomorphism. Then from Theorem 71 (3) it follows that $\dim V = \dim W$.

Proof \Leftarrow Let (v_1, \dots, v_n) be a basis of V and (w_1, \dots, w_n) be basis of W . Let $f : V \rightarrow W$ be a linear mapping from Theorem 72 for which $f(v_i) = w_i$ for all $1 \leq i \leq n$.

We need to show that f is bijective; injective and surjective.

Injectivity: Let $v, v' \in V$ with $f(v) = f(v')$. We need to show that $v = v'$.

$$\begin{aligned} 0 &= f(v) - f(v') = f\left(\sum_{i=1}^n \lambda_i v_i\right) - f\left(\sum_{i=1}^n \mu_i v_i\right) \\ &= \sum_{i=1}^n \lambda_i f(v_i) - \sum_{i=1}^n \mu_i f(v_i) \\ &= \sum_{i=1}^n \lambda_i w_i - \sum_{i=1}^n \mu_i w_i \\ &= \sum_{i=1}^n (\lambda_i - \mu_i) w_i = 0 \quad \Rightarrow \lambda_i - \mu_i = 0 \quad \forall i \\ &\Rightarrow \text{all } \lambda_i = \mu_i \Rightarrow v = v' \end{aligned}$$

Surjectivity: Let $w \in W$. We need to show that

$$\bigvee_{v \in V} f(v) = w$$

(w_1, \dots, w_i) generates W . Therefore,

$$\bigvee_{\lambda_1, \dots, \lambda_n} w = \sum_{i=1}^n \lambda \cdot w_i$$

Then

$$\begin{aligned} f(v) &= w \text{ for } v = \sum_{i=1}^n \lambda_i v_i \in V \\ f\left(\sum_{i=1}^n \lambda_i v_i\right) &= \sum_{i=1}^n \lambda_i f(v_i) = \sum_{i=1}^n \lambda_i w_i = w \end{aligned}$$

We have shown that if $f : b_i \rightarrow w_i$ is extended to a linear mapping $f : V \rightarrow W$, then it holds that

1. if (w_1, \dots, w_n) is linear independent, then f is injective.
2. if $L(w_1, \dots, w_n) = W$, then f is surjective.

□

Corollary 16.

$$\dim V = n \Leftrightarrow V \cong K^n$$

Isomorphism: Let (b_1, \dots, b_n) be a basis of V . Then,

$$f : V \rightarrow K^n$$

$$b_i \mapsto e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with the 1 in the i -th row,

$$f\left(\sum_{i=1}^n \lambda_i b_i\right) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is an isomorphism.

Corollary 17.

$$\text{Hom}(V, W) \supsetneq \{0\} \text{ if } V, W \neq \{0\}$$

$\text{Hom}(V, W)$ is vector space (and ring, hence algebra).

$$(\lambda f + \mu g)(v) = \lambda f(v) + \mu g(v)$$

It follows that $\dim \text{Hom}(V, W) = \dim V \cdot \dim W$.

Theorem 74.

$$\dim \operatorname{Hom}(V, W) = \dim V \cdot \dim W$$

Proof. Every $f : V \rightarrow W$ is uniquely defined by the values of the basis of V . Let (v_1, \dots, v_m) be a basis of V . Let (w_1, \dots, w_n) be a basis of W .

Claim: The mapping $f_{ij} : V \rightarrow W$ such that

$$f_{ij}(v_k) = \begin{cases} w_j & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

Is distinct according to Theorem 72. This is a basis of $\operatorname{Hom}(V, W)$. So we need to shown linear independence and that it is a generating system.

Let B such that,

$$B = (f_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \subseteq \operatorname{Hom}(V, W)$$

1.

$$L(B) = \operatorname{Hom}(V, W)$$

Let $f \in \operatorname{Hom}(V, W)$ be searched $\lambda_{ij} \in K$ such that $f = \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} f_{ij}$.

$$\bigwedge_h \bigvee_{\lambda_1, \dots, \lambda_n \in K} f(v_k) = \sum_{i=1}^n \lambda_{\alpha_j} w_j$$

Decomposition of $f(v_k)$ in regards of the basis (w_1, \dots, w_n) .

Claim:

$$f = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} f_{ij} = g$$

To show that $f = g$ (hence $f(v) = g(v)$), it suffices to show that $f(v_k) = g(v_k)$ for all k (Theorem 72).

$$\begin{aligned} g(v_k) &= \left(\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} f_{ij} \right) (v_k) \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} f_{ij}(v_k) \end{aligned}$$

$$f_{ij}(v_k) = \begin{cases} w_j & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

$$\Rightarrow \sum_{j=1}^n \alpha_{kj} w_j = f(v_k).$$

$$\Rightarrow g|_{\{v_1, \dots, v_m\}} = f|_{\{v_1, \dots, v_m\}}$$

$$\xrightarrow{\text{Theorem 72}} g = f$$

And finally we need to show linear independence.

Let $\lambda_{ij} \in K$ such that $\sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} f_{ij} = 0$. Therefore $\sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} f_{ij}(v) = 0$ for all $v \in V$. Show that for all $\lambda_{ij} = 0$.

$$\begin{aligned} 0 &= \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij} f_{ij}(v_k) \\ &= \sum_{j=1}^n \lambda_{kj} w_j = 0 \Rightarrow \bigwedge_j \lambda_{kj} = 0 \end{aligned}$$

where

$$f_{ij}(v_k) = \begin{cases} w_j & i = k \\ 0 & i \neq k \end{cases}$$

so (w_j) are linear independent and this holds for all k . So,

$$\bigwedge_k \bigwedge_j \lambda_{kj} = 0$$

□

This lecture took place on 15th of December 2015 (Franz Lehner).

8.2 Revision

A factor set satisfies:

$$\begin{aligned} V/U \quad U \subseteq V \text{ is a subspace} \\ = \{v + U \mid v \in V\} = \{[v] \mid v \in V\} \\ v \sim_U v' \Leftrightarrow v - v' \in U \Leftrightarrow v \in v' + U \\ \dim(V/U) = \dim U = \dim V \end{aligned}$$

Constructing a basis for V/U :

$$\begin{aligned} u_1, \dots, u_m \text{ is basis of } U \rightarrow \text{extend to basis of } V \\ u_1, \dots, u_m, w_1, \dots, w_{n-m} \text{ is basis of } V \\ w_1 + U, \dots, w_{n-m} + U \text{ is basis of } V/U \end{aligned}$$

Images and preimages of subspaces are subspaces.

Definition 32. Let $f : V \rightarrow W$ be linear. The subspace

$$\ker(f) := f^{-1}(\{0\}) = \{v \mid f(v) = 0\} \subseteq V$$

is called kernel of the linear mapping f . The image of the linear mapping f is defined as

$$\operatorname{im}(f) = f(V)$$

Example 50.

$$f : K^n \rightarrow K^n$$

Consider some fixed m .

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\operatorname{im}(f) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \\ \vdots \\ 0 \end{pmatrix} \mid X \in K \right\} \cong K^m$$

$$\ker(f) = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_{n+1} \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in K \right\} \cong K^{n-m}$$

In this example:

$$\ker(f) \dot{+} \operatorname{im}(f) = K^n$$

$$\dim \ker(f) + \dim \operatorname{im}(f) = \dim V$$

Theorem 75. Let $f : V \rightarrow W$ be linear.

- f is surjective $\Leftrightarrow \operatorname{im}(f) = W$
- f is injective $\Leftrightarrow \ker(f) = \{0_V\}$

Proof. • Follows immediately.

- \Rightarrow : Let $v \in \ker(f) \Rightarrow f(v) = 0_W = f(0_V)$ and f is injective $\Rightarrow v = 0_V$.
- \Leftarrow : Let $v, v' \in V$ with $f(v) = f(v')$.

$$0 = f(v) - f(v') = f(v - v')$$

$$\Rightarrow v - v' \in \ker(f) = \{0\}$$

$$\Rightarrow v = v'$$

□

Theorem 76 (homomorphism theorem). *Let $g : V \rightarrow V/U$ be linear. $v \mapsto v + U$. Then it holds that:*

$$\tilde{f} : V/\ker f \rightarrow \text{im}(f) \text{ is linear}$$

$$v + \ker(f) \mapsto f(v)$$

This gives an isomorphism.

Proof. We need to show,

1. Is it well-defined?
2. Is it linear?
3. Is it bijective?

1. So it must hold that $\tilde{f}(v + \ker(f))$ does not depend on the selection of the representative.

So we need to show: If $v \sim_{\ker(f)} v'$ ($v - v' \in \ker(f)$) then $f(v) = f(v')$.

$$\begin{aligned} v - v' \in \ker(f) &\Rightarrow f(v - v') = 0 \\ &\Rightarrow f(v) - f(v') = 0 \\ &\Rightarrow f(v) = f(v') \end{aligned}$$

Definition of $\tilde{f}(v + \ker(f))$ is consistent.

2.

$$\begin{aligned} &\bigwedge_{v, v' \in V} \bigwedge_{\lambda, \mu \in K} \tilde{f}(\lambda(v + \ker(f)) + \mu(v' + \ker(f))) \\ &= \tilde{f}((\lambda v + \mu v') + \ker f) \\ &= f(\lambda v + \mu v') \\ f \text{ is linear} &\Rightarrow = \lambda f(v) + \mu f(v') \\ &= \lambda \tilde{f}(v + \ker(f)) + \mu \tilde{f}(v' + \ker(f)) \end{aligned}$$

3. \tilde{f} is surjective? Let $w \in \text{im}(f)$, choose $v \in V$ with $w = f(v) = \tilde{f}(v + \ker(f))$. Therefore $w \in \text{im}(\tilde{f})$.

\tilde{f} is injective? We need to show that $\ker(\tilde{f}) = \{0 + \ker(f)\}$.

Let $\tilde{f}(v + \ker(f)) = 0$. So $v \in \ker(f) \Rightarrow v + \ker(f) = \ker(f) = 0 + \ker(f)$.

□

Corollary 18. *Let $f : V \rightarrow W$ be linear. So $\dim V < \infty$. Then $\dim \ker(f) + \dim \text{im}(f) = \dim V$.*

Proof.

$$\dim(V/\ker(f)) \stackrel{\text{Theorem 61}}{=} \dim V - \dim \ker(f)$$

$$\tilde{f} : V/\ker(f) \rightarrow \text{im}(f) \text{ is isomorphism}$$

$$\Rightarrow \dim(V/\ker(f)) = \dim(\text{im}(f))$$

□

Alternative, more comprehensible proof.

$$\ker(f) \subseteq V \text{ is subspace}$$

From Theorem 51 it follows that subspace $U \subseteq V$ exists such that $\ker(f) \dot{+} U = V$.

$$\dim U = \dim V - \dim \ker(f)$$

Claim. $f|_U : U \rightarrow \text{im}(f)$ is bijective.

Claim. $f|_U$ is surjective.

Let $w \in \text{im}(f)$

$$\Rightarrow \bigvee_{v \in V} f(v) = w$$

$$V = \ker(f) \dot{+} U \Rightarrow \bigvee_{u \in U} \bigvee_{v_0 \in \ker(f)} v = v_0 + u$$

$$w = f(v) = f(v_0) + f(u) \Rightarrow w \in f(U)$$

f_U : is bijective. We need to show that $\ker(f|_U) = \{0\}$.

$$\ker(f|_U) = \ker(f) \cap U = \{0\}$$

Is $\{0\}$, because $V = \ker(f) \dot{+} U$ is a direct sum.

Remark 23. Also the mapping

$$U \rightarrow V/\ker(f)$$

$$u \mapsto u + \ker(f)$$

is an isomorphism.

The proof will be provided in the practicals.

Theorem 77.

$$\dim V = \dim W < \infty$$

$$f : V \rightarrow W \text{ is linear}$$

then it holds equivalently,

1. f is a monomorphism
2. f is epimorphism
3. f is isomorphism

Proof. 1. $\Leftrightarrow f$ is injective $\Leftrightarrow \ker f = \{0\}$

$$\Leftrightarrow \dim \ker(f) = 0$$

$$\xLeftrightarrow{\text{Corollary 18}} \dim \operatorname{im}(f) = \dim V = \dim W$$

$$\operatorname{im}(f) \subseteq W \text{ subspace}$$

$$\text{and } \dim \operatorname{im}(f) = \dim W.$$

$$\Leftrightarrow \operatorname{im}(f) = W$$

$$\Leftrightarrow f \text{ is surjective}$$

$$\dim V = n \Leftrightarrow V \cong K^n$$

$$\text{basis } f_{i,j}, v_k \rightarrow \begin{cases} w_j & \text{if } k = i \\ 0 & \text{else} \end{cases}$$

Every $f : V \rightarrow W$ has the structure

$$f = \sum \alpha_{ij} f_{ij}$$

□

9 Matrix computations

We have already dealt with matrices when discussing linear mappings and linear equation systems.

Definition 33. An $m \times n$ matrix over K is a number scheme:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

with m rows and n columns.

a_{ij} is the number of the i -th row and j -th column.

$$M_{m,n}(K) = K^{m \times n}$$

is the set of all $m \times n$ matrices. If $m = n$:

$$M_n(K) = K^{n \times n}$$

is called a quadratic matrix.

$z_i = (a_{i1}, a_{i2}, \dots, a_{in})$ is the i -th row vector. $s_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$ is the j -th column vector.

The sequence $a_{11}, a_{22}, \dots, a_{kk}$ with $k = \min m, n$ is called main diagonal of A . If all entries are contained outside the main diagonal, A is called diagonal matrix.

$$A = \text{diag}(a_{11}, a_{22}, \dots, a_{kk}) = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{kk} \end{bmatrix}$$

$I_n = \text{diag}(1, \dots, 1)$ is called unit matrix.

$$= [\delta_{ij}]_{i,j \in 1, \dots, n}$$

Kronecker- δ :

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

If all entries outside the main diagonal are 0, then A is called a triangular matrix. If all entries below the main diagonal are 0, then A is called an lower triangular matrix. If all entries above the main diagonal are 0, then A is called an upper triangular matrix.

Matrix units are defined as

$$(E_{kl}^{(n)})_{ij} = \delta_{ki} \cdot \delta_{lj} = \begin{cases} 1 & \text{if } k = i \wedge l = j \\ 0 & \text{else} \end{cases}$$

Examples:

$$E_{11} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$E_{12} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The transposed matrix of $A \in K^{m \times n}$ is denoted $A^t \in K^{n \times m}$ with entries:

$$(A^t)_{ij} = a_{ji}$$

So we mirror along the main diagonal.

$$(A^t)^t = A$$

Remark 24. A column vector can be identified with a $1 \times n$ matrix. A row vector can be identified with a $n \times 1$ matrix.

Theorem 78. $(K^{m \times n}, +, \cdot)$ with

$$[a_{ij}]_{i=1, \dots, m; j=1, \dots, n} + [b_{ij}]_{i=1, \dots, m; j=1, \dots, n} = [a_{ij} + b_{ij}]_{i=1, \dots, m; j=1, \dots, n}$$

$$\lambda[a_{ij}]_{i=1, \dots, m; j=1, \dots, n} = [\lambda a_{ij}]_{i=1, \dots, m; j=1, \dots, n}$$

Is a vector space of dimension $m \cdot n$ with basis $(E_{ij})_{i=1, \dots, m; j=1, \dots, n}$.

Remark 25.

$$K^{m \times n} \rightarrow K^{n \times m}$$

$$A \mapsto A^t$$

is a vector space isomorphism.

Definition 34. Let $A = [a_{ij}]_{i=1, \dots, m; j=1, \dots, n} \in K^{m \times n}$.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in K^n$$

is a column vector. Then

$$Ax = A \cdot x = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{pmatrix} \in K^m$$

This is called the product of the matrix A with the vector x .

So instead of a linear equation system with all entries listed explicitly, we will only write Ax in this section.

Example 51.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \\ 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 32 \end{pmatrix}$$

Remark 26.

$$e_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A \cdot e_k = s_k = k\text{-th column vector}$$

Theorem 79. 1. Let $A \in K^{m \times n}$. Then the mapping

$$f_A : K^n \rightarrow K^m$$

$$x \mapsto Ax$$

is linear.

2. For every $f \in \text{Hom}(K^n, K^m)$ there exists a distinct matrix $A \in K^{m \times n}$ such that $f = f_A$.

Namely the k -th column of $A = f(e_k) = A \cdot e_k$.

3.

$$K^{m \times n} \rightarrow \text{Hom}(K^n, K^m)$$

$A \mapsto f_A$ is an isomorphism.

This lecture took place on 11th of Jan 2016 (Franz Lehner).

9.1 Revision

We look at homomorphisms between vector spaces:

$$f : V \rightarrow W$$

$$+/\cdot : \text{Hom}(V, W)$$

$$f(v + w) = f(v) + f(w)$$

$$f(\lambda w) = \lambda \cdot f(w)$$

Images and preimages of subspaces are subspaces. Especially,

$$\ker f = f^{-1}(\{0\})$$

$$\text{im } f = f(V)$$

$$\dim \ker(f) + \dim \text{im}(f) = \dim V$$

Every vector space has basis. Let $B \subseteq V$ be a basis

$$\bigwedge_{f:B \rightarrow W} \bigvee_{\tilde{f}:V \rightarrow W} \tilde{f} \text{ linear} \wedge \tilde{f}|_B = f$$

Followingly, if two mappings $f, g \in \text{Hom}(V, W)$ are equivalent if and only if $f|_B = g|_B$.

If $\dim V < \infty$, $V \cong W \Leftrightarrow \dim V = \dim W$.

9.2 Matrix

$$A = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix} \in K^{m \times n}$$

$$f_A : K^n \rightarrow K^m$$

$$f_A : x \mapsto A \cdot x = \begin{pmatrix} \sum_{j=1}^n a_{1,j}x_j \\ \sum_{j=1}^n a_{2,j}x_j \\ \vdots \\ \sum_{j=1}^n a_{m,j}x_j \end{pmatrix}$$

Remark 27. $A \cdot e_k = s_k(A)$ ⁵

Theorem 80. 1. The mapping $f_A : K^n \rightarrow K^m$ is linear.

⁵where s_k refers to the k -th column?

2. For every $f \in \text{Hom}(K^n, K^m)$ there is one unique matrix $A \in K^{m \times n}$, such that $f = f_A$. Therefore $f(x) = A \cdot x$ for all $x \in K^n$.

3. The mapping $K^{m \times n} \rightarrow \text{Hom}(K^n, K^m)$ with $A \mapsto f_A$ is a homomorphism.

Remark 28. So linear mappings and matrices are semantically equivalent.

Proof. We prove the three theorems.

1. Basic calculations.
2. Because of Remark 27 it must have a matrix with a column $s_k(A) = f(e_k)$, which satisfies $f = f_A$. Therefore it holds that $f(e_k) = f_A(e_k)$ and followingly, $f = f_A$ on the canonical basis from which $f = f_A$ on K^n follows.
Basis of $\text{Hom}(K^n, K^m)$? f_{ij} follows from Theorem 74:

$$f_{ij} : K^n \rightarrow K^m$$

$$e_k \mapsto \begin{cases} e_j & k = i \\ 0 & k \neq i \end{cases}$$

which is equivalent to

$$s_k(H_{ij}) = \begin{cases} e_j & \text{if } k = i \\ 0 & \text{else} \end{cases}$$

$$H_{ij} = j \begin{bmatrix} \cdot & \cdot & \cdot \\ \vdots & 1 & \vdots \\ \cdot & \cdot & \cdot \end{bmatrix} = E_{ji}$$

Basis of $K^{n \times m}$.

We elaborate:

$$(f_{ij})_{\substack{i \in \{1, \dots, n\} \\ j \in \{1, \dots, m\}}}$$

is basis of (K^n, K^m) .

$$f_{ij} = f_{E_{ji}} \text{ where } E_{ji} = \text{elementary matrix}$$

$$f = \sum \alpha_{ij} f_{ij} \in \text{Hom}(K^n, K^m)$$

$$(E_{ji})_{\substack{i=1, \dots, n \\ j=1, \dots, m}} \text{ build basis in } K^{m \times n}$$

$$\Rightarrow f = f_{\sum \alpha_{ij} E_{ji}} = f_A$$

$$A = \sum \alpha_{ij} E_{ji}$$

The mapping

$$K^{m \times n} \rightarrow \text{Hom}(K^n, K^m)$$

$$A \mapsto f_A$$

is linear and build a basis (E_{ij}) maps to the basis (f_{ij}) . Therefore it holds that

$$K^{m \times n} \cong \text{Hom}(K^n, K^m)$$

□

Example 52.

$$f = \text{id} : K^n \rightarrow K^n$$

$$f(e_k) = e_k \rightarrow a = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_n$$

$$f_{\lambda A + \mu B} = \lambda \cdot f_A + \mu \cdot f_B$$

Composition:

$$f_A \cdot f_B = f_C$$

$$K^p \rightarrow K^m \rightarrow K^n$$

Definition 35. Let $A \in K^{n \times m}$ and $B \in K^{m \times p}$. Then the matrix $C := A \cdot B \in K^{n \times p}$ with $C_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$ for $i \in \{1, \dots, n\}, j \in \{1, \dots, p\}$ is the product of A and B

$$A \cdot x = \begin{pmatrix} \sum_{k=1}^m a_{1k} \cdot x_k \\ \vdots \\ \sum_{k=1}^m a_{nk} \cdot x_k \end{pmatrix}$$

where $x \in K^m$. Therefore $s_j(C) = A \cdot s_j(B)$ is column of C ; A times the j -th column of B .

Example 53.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}$$

Use the schema,

$$\begin{array}{ccc|cc} & & & 1 & 4 \\ & & & 2 & 5 \\ & & & 3 & 6 \\ \hline 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{array}$$

Remark 29. $A \cdot B \neq B \cdot A$.

$$A \cdot B = \begin{array}{ccc|cc} & & & 0 & 0 \\ & & & 1 & 0 \\ \hline 0 & 1 & & 1 & 0 \\ 0 & 0 & & 0 & 0 \end{array}$$

$$B \cdot A = \begin{array}{ccc|cc} & & & 0 & 1 \\ & & & 0 & 0 \\ \hline 0 & 0 & & 0 & 0 \\ 1 & 0 & & 0 & 1 \end{array}$$

Example 54.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$A^2 = A$ shows an idempotent property (infinitely many solutions).

Theorem 81.

$$f_A \circ f_B = f_{A \cdot B}$$

Proof. It suffices to check the basis.

$$\begin{aligned} f_A \cdot f_B(e_k) &\stackrel{\text{Remark 27}}{=} A \cdot f_B(e_k) \\ &= A \cdot s_k(B) \stackrel{\text{def. of } A \cdot B}{=} s_k(A \cdot B) = f_{A \cdot B}(e_k) \end{aligned}$$

Alternative, more educational, proof: direct

Corollary 19. The matrix product is associative:

$$A \in K^{n \times m} \quad B \in K^{m \times p} \quad C \in K^{p \times q}$$

$$\underbrace{(A \cdot B) \cdot C}_{n \times q} = A \cdot \underbrace{(B \cdot C)}_{n \times q}$$

Proof.

$$\begin{aligned} f_{A \cdot (B \cdot C)} &= f_A \circ f_{B \cdot C} \\ &= f_A \circ (f_B \circ f_C) \\ &= (f_A \circ f_B) \circ f_C \\ &= f_{A \cdot B} \circ f_C \\ &= f_{(A \cdot B) \cdot C} \end{aligned}$$

□

Theorem 82. 1.

$$\bigwedge_{A \in K^{n \times m}} \bigwedge_{B, C \in K^{m \times p}} A(B + C) = A \cdot B + A \cdot C$$

2.

$$\bigwedge_{A, B \in K^{n \times m}} \bigwedge_{C \in K^{m \times p}} (A + B) \cdot C = A \cdot C + B \cdot C$$

3.

$$\bigwedge_{\lambda \in K} \bigwedge_{A \in K^{n \times m}} \bigwedge_{B \in K^{m \times p}} \lambda(A \cdot B) - (\lambda A) \cdot B = A \cdot (\lambda B)$$

4.

$$\bigwedge_{A \in K^{n \times m}} \bigwedge_{B \in K^{m \times p}} (A \cdot B)^T = B^T \cdot A^T$$

5.

$$\bigwedge_{A \in K^{n \times m}} I_n \cdot A = A = A \cdot I_m$$

□

Proof. 1. Immediate.

2. Immediate.

3. Immediate.

4.

$$\begin{aligned}
 ((A \cdot B)^T)_{ij} &= (A \cdot B)_{ji} \\
 &= \sum_{k=1}^m a_{jk} b_{ki} \\
 &= \sum_{k=1}^m b_{ki} a_{jk} \\
 &= \sum_{k=1}^m (B^T)_{ik} (A^T)_{kj} \\
 &= (B^T \cdot A^T)_{ij}
 \end{aligned}$$

$$\Rightarrow \text{for all } i, j : (A \cdot B)^T = B^T \cdot A^T$$

Corollary 20. $(K^{n \times n}, +, \cdot_{\text{scalar product}}, \cdot_{\text{matrix product}})$ is a K -algebra⁶ isomorphic to $\text{End}(K^n)$.

Definition 36. A matrix $A \in K^{n \times n}$ is called regular if it is invertible hence if

$$\bigvee_{B \in K^{n \times n}} A \cdot B = B \cdot A = I$$

A matrix which is not regular, is called singular.

Theorem 83. A matrix $A \in K^{n \times n}$ has at most one inverse. If it exists, the inverse of A is denoted A^{-1} .

⁶Scalar product is given with $K \times K^{n \times n} \rightarrow K^{n \times n}$

Proof. Let B and B' be two inverse matrices.

$$B = B \cdot I = B \cdot (A \cdot B') = (B \cdot A) \cdot B' = I \cdot B' = B'$$

□

Remark 30. For finite-dimensional matrices it suffices to find either a left-inverse or a right-inverse matrix. For infinite-dimensional matrices this does not work any more.

Theorem 84. 1. I_n is regular. $I_n \cdot I_n = I_n$

2. $A, B \in K^{n \times n}$ is regular $\Rightarrow A \cdot B$ is regular.

$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$

3. $A \in K^{n \times n}$ is regular, then A^{-1} is also regular.

$$(A^{-1})^{-1} = A$$

4. $A \in K^{n \times n}$ is regular, then A^T is regular with

$$(A^T)^{-1} = (A^{-1})^T$$

□

5. A is regular if and only if $f_A : K^n \rightarrow K^n$ is automorphism,

$$(f_A)^{-1} = f_{A^{-1}}$$

Proof. 2.

$$(A \cdot B) \cdot (B^{-1} \cdot A^{-1}) = A \cdot (B \cdot B^{-1}) \cdot A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$$

Also it holds that

$$(B^{-1} \cdot A^{-1}) \cdot (A \cdot B) = I$$

3. $A^{-1} \cdot A = I$. $A \cdot A^{-1} = I$. A^{-1} has A as inverse.

4. $A^T \cdot (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I$

5. $f_A \circ f_{A^{-1}} = f_{A \cdot A^{-1}} = f_I = \text{id}$. So $f_A \circ f_B = \text{id} \Leftrightarrow A \cdot B = I$

□

Example 55. 1. $(\lambda \cdot I)^{-1} = \frac{1}{\lambda} I$

$$(\lambda \cdot A)^{-1} = \frac{1}{\lambda} \cdot A^{-1} \quad (\lambda \neq 0)$$

2.

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_n \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

because

$$\begin{bmatrix} a_1 b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n b_n \end{bmatrix} = \begin{bmatrix} b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_n \end{bmatrix} \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix}$$

$$\text{If } \begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix} \text{ is regular} \Leftrightarrow \text{all } a_i \neq 0$$

$$\begin{bmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{bmatrix} = \begin{bmatrix} \frac{1}{a_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{1}{a_n} \end{bmatrix}$$

3. Let $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ be bijective (hence it is a permutation).

$$f : \underbrace{\{e_1, \dots, e_n\}}_{\text{canonical basis}} \rightarrow \{e_1, \dots, e_n\}$$

$$f(e_i) = e_{\sigma(i)}$$

Let $\tilde{f} : K^n \rightarrow K^n$ be a linear extension. It is also bijective. The corresponding matrix P is regular.

$$s_k(P) = f(e_k) = e_{\sigma(k)}$$

$$\sigma = (123)$$

We use the cyclic notation here. So we map 1 to 2, 2 to 3 and 3 to 1.

$$P_{\sigma} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

P_{σ} is the permutation matrix. P_{σ} is regular.

$$(P_{\sigma})^{-1} = P_{\sigma^{-1}}$$

This lecture took place on 12th of January 2016 (Franz Lehner).

Example 56. 3.

$$K^{m \times n} \rightarrow \text{Hom}(K^n, K^m)$$

$$A \mapsto f_A$$

$f_A(x) = Ax$ is linear.

$$\begin{aligned} f_{\lambda A + \mu B}(x) &= (\lambda A + \mu B) \cdot x \\ &= \lambda A \cdot x + \mu Bx \\ &= \lambda f_A(x) + \mu f_B(x) \end{aligned}$$

$$\leadsto f_{\lambda A + \mu B} = \lambda f_A + \mu f_B$$

$$f_{E_{ij}} = f_{ji}$$

E_j is a basis of $K^{m \times n}$. Therefore homomorphism. f_{ij} is basis of $\text{Hom}(K^n, K^m)$.

4. Rotation in \mathbb{R}^2 .

$$H_{\alpha} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

$$H_{\alpha} H_{\beta} = H_{\alpha + \beta}$$

$$\begin{aligned} & \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \cdot \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta - \sin \alpha \sin \beta & -\cos \alpha \sin \beta - \sin \alpha \cos \beta \\ \sin \alpha \cos \beta + \cos \alpha \sin \beta & -\sin \alpha \sin \beta + \cos \alpha \cos \beta \end{pmatrix} \\ &= \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{pmatrix} \end{aligned}$$

Corollary 21 (was covered already yesterday). If A, B is regular, then $A \cdot B$ is regular. I is regular. If A is regular, then A^{-1} regular.

Definition 37.

$$\text{GL}(n, K) = \{A \in K^{n \times n} \mid A \text{ regular}\}$$

build a group in regards of matrix multiplication. We will show later that this is a superset of $\text{SL}(n, K)$. GL stands for general linear group.

Remark 31. A is regular if and only if f_A is automorphism in K^n . So you could apply the basis exchange theorem.

Definition 38. 1. Two matrices $A, B \in K^{m \times n}$ are called equivalent if

$$\bigvee_{P \in \text{GL}(m, K)} \bigvee_{Q \in \text{GL}(n, K)} A = P \cdot B \cdot Q$$

2. Two matrices $A, B \in K^{n \times m}$ are called similar if

$$\bigvee_{P \in \text{GL}(n, K)} A = P \cdot B \cdot P^{-1}$$

In the following, we will show that

1. Equivalence is equivalence relation on $K^{m \times n}$
2. Similarity is equivalence relation on $K^{n \times n}$

Definition 39. $A \in K^{m \times n}$.

1. The linear hull of row vectors

$$L(z_1(A), \dots, z_m(A))$$

is called row space of A . Its dimension is called row rank of A : $\text{zrg}(A)$.

2. The linear hull of column vectors

$$L(s_1(A), s_2(A), \dots, s_n(A))$$

is called column space of A . Its dimension is called column rank of A : $\text{srg}(A)$.

Remark 32. 1. Because of Remark 27 and Theorem 5.12 the column vectors of A build the image space of f_A . Therefore,

$$\text{srg}(A) = \dim \text{im}(f_A)$$

$$2. \text{zrg}(A) = \text{srg}(A^T)$$

Theorem 85. For all $A \in K^{m \times n}$, it holds that $\text{zrg}(A) = \text{srg}(A)$ and is called rank of A :

$$\text{rk}(A) = \dim \text{im}(f_A)$$

(in English f_A is called range of f_A)

Proof. It suffices to show that

$$\text{srg}(A) \leq \text{zrg}(A)$$

$$\dim \text{zrg}(A) = \text{srg}(A^T) \leq \text{zrg}(A^T) = \text{srg}(A)$$

Let $r = \text{zrg}(A)$. We need to find a generating system of column vectors with $\leq r$ elements. From the basis selection theorem it follows that $z_{i_1}(A) \dots z_{i_r}(A)$ are basis of row space. All other rows are linear combinations of these vectors:

$$\begin{aligned} \bigvee_{\substack{\beta_{ij} \in K \\ 1 \leq i \leq m \\ 1 \leq j \leq r}} z_1 &= \beta_{i_1} z_{i_1} + \dots + \beta_{i_r} z_{i_r} \\ z_2 &= \beta_{i_2} z_{i_2} + \dots + \beta_{i_r} z_{i_r} \\ &\vdots \\ z_m &= \beta_{i_m} z_{i_m} + \dots + \beta_{i_r} z_{i_r} \end{aligned}$$

We denote coordinatewise $(z_i)_j = a_{ij}$.

$$\begin{aligned} a_{1j} &= (z_1)_j = (\beta_{i_1} z_{i_1} + \dots + \beta_{i_r} z_{i_r})_j \\ &= \beta_{i_1} a_{ij} + \dots + \beta_{i_r} a_{ij} \\ a_{2j} &= \beta_{i_1} a_{2j} + \dots + \beta_{i_r} a_{ij} \end{aligned}$$

$$a_{mj} = \beta_{mi} a_{ij} + \beta_{m2} a_{2j} + \dots + \beta_{mr} a_{ij}$$

j-th column:

$$s_j(A) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = \begin{pmatrix} \beta_{1i} \\ \beta_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} a_{i1,j} + \begin{pmatrix} \beta_{12} \\ \beta_{22} \\ \vdots \\ a_{m2} \end{pmatrix} a_{i2,j} + \dots + \begin{pmatrix} \beta_{1r} \\ \beta_{2r} \\ \vdots \\ a_{mr} \end{pmatrix} \cdot a_{ir,j}$$

$$\in L\left(\begin{pmatrix} \beta_m \\ \vdots \\ \beta_{m,1} \end{pmatrix}, \begin{pmatrix} \beta_{12} \\ \vdots \\ \beta_{m,2} \end{pmatrix}, \dots, \begin{pmatrix} \beta_{ir} \\ \vdots \\ \beta_{m,r} \end{pmatrix}\right)$$

All column vectors are contained $L(b_1, \dots, b_r)$, where b_1 to b_r are the vectors we wrote used for $s_j(A)$ above.

$$\Rightarrow \text{column space} \subseteq L(b_1, \dots, b_r)$$

$$\Rightarrow \text{srg}(A) = \dim \text{column space} \leq r = \text{zrg}(A)$$

Our next goal is to determine its rank.

Approach: Gaussian elimination.

We recognize that

$$\text{rank} \left(\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \right) = n \text{ where } n \text{ denotes the number of column vectors}$$

$$\text{rank} \left(\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \right) = m \text{ where } m \text{ denotes the number of non-zero columns}$$

□

Definition 40. Elementary row (column) transformations are defined as

1. Addition of a row (column) to another row (column)

2. Multiplication of a row (column) with $\lambda \in K$, $\lambda \neq 0$

Remark 33. These operations are reversible⁷.

Theorem 86. With a sequence of these elementary row (column) transformations of type 1 and 2, the following operations are possible:

3. Exchange of two rows (columns)

4. Addition of a row (column) λ times another one

$$(4.) : [S_i, S_j] \xrightarrow{2} [\lambda s_i, s_j]$$

$$\xrightarrow{1} [\lambda s_i, s_j + \lambda s_i]$$

$$\xrightarrow{2} [s_i, s_j + \lambda s_i]$$

$$(3.) : [s_i, s_j] \xrightarrow{1} [s_i, s_i + s_j]$$

$$\xrightarrow{2} [-s_i, s_i + s_j]$$

$$\xrightarrow{2} [s_j, s_i + s_j]$$

$$\xrightarrow{2} [-s_j, s_i + s_j]$$

$$\xrightarrow{2} [-s_j, s_i]$$

$$\xrightarrow{2} [s_j, s_i]$$

Theorem 87. Every matrix $A \in K^{m \times n}$ can be written as sequence of elementary row and column transformations with structure

$$I_{m \times n}^{(r)} = \begin{bmatrix} 1 & \dots & \dots & 0 \\ \dots & \ddots & \dots & 0 \\ \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & 0 \end{bmatrix}$$

where r denotes the number of non-zero columns.

⁷Multiplication with -1 , etc.

Example 57.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow{4} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof. 1. $A = 0$, nothing to do

2. At least one $a_{ij} \neq 0$ exists, then apply a recursive algorithm:

- (a) exchange first with i -th row and first with j -th column.
- (b) multiply first row with $\frac{1}{a_{ij}}$.

$$\begin{bmatrix} 1 & a'_{12} & a'_{13} & \dots & a'_{1n} \\ a'_{21} & \dots & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a'_{m1} & \dots & \dots & \dots & 1 \end{bmatrix}$$

- (c) Subtract a_{1j} times the first column from j -th column for all $j \geq 2$.
- (d) Subtract for all $2 \leq i \leq m$, a_{i1} times the first row from the i -th row.

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{m,2} & \dots & b_{mn} \end{bmatrix}$$

- (e) Repeat steps with row $(1, 0, \dots, 0)$ and column $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ excluded until no one is left.

Remark 34. Applying only row transformations, we can achieve an upper triangular matrix. Applying only column transformations, we can achieve a lower triangular matrix

Theorem 88. Let $A \in K^{m \times n}$. The following matrices are invertible and implement row and column transformations.

$$T \cdot A$$

$$1. T = I + E_{ij}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + z_2 \\ z_2 \end{pmatrix}$$

Addition of j -th row to i -th row.

$$2. I + E_{ii} \cdot (\lambda - 1)$$

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \\ & & & \ddots \\ & & & & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ \lambda z_i \\ \vdots \\ z_m \end{pmatrix}$$

Multiplies the i -th row with λ

$$3. T_{(i,j)} = \text{permutation matrix which exchanges } i \text{ and } j.$$

Exchanges i -th and j -th row.

$$4. T = I + \lambda \cdot E_{ij}$$

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 + \lambda \cdot z_2 \\ z_2 \end{pmatrix}$$

Add the λ times j -th row to the i -th row.

$$A \cdot T$$

$$1. (I + E_{ij})^{-1} = I - E_{ij}$$

$$\begin{pmatrix} s_1 & s_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s_1 & s_1 + s_2 \end{pmatrix}$$

Add the i -th column to the j -th column.

□

$$2. (I + E_{ii}(\lambda - i)) = I + \left(\frac{1}{\lambda} - 1\right) E_{ii}$$

Multiplies the i -th column with λ .

$$3. T_{(i,j)}^{-1} = T_{(i,j)}$$

Exchange the i -th and j -th column.

$$4. (I + \lambda E_{ij})^{-1} = I - \lambda E_{ij}$$

Adds the λ times j -th column to the i -th column.

$$(I + \lambda E_{ij})(I - \lambda E_{ij}) = I - \lambda E_{ij} + \lambda E_{ij} = I$$

$$E_{ij} \cdot E_{kl} = \begin{cases} 0 & j \neq k \\ E_{i,l} & j = k \end{cases}$$

Corollary 22. Every matrix $A \in K^{m \times n}$ is equivalent to the matrix of the structure $I_{m \times n}^{(r)}$.

Proof. Apply the corresponding row and column transformations. Every row (column) transformation corresponds to multiplication with an invertible matrix L_i (R_j) from left (right).

$$L_k \dots L_2 L_1 A R_1 R_2 \dots R_l = I_{m \times n}^{(r)}$$

$$\rightarrow R = L_k \dots L_2 L_1 \text{ is invertible}$$

$$\rightarrow Q = R_1 R_2 \dots R_l \text{ is invertible}$$

$$\Rightarrow P \cdot A \cdot Q = I_{m \times n}^{(r)} \text{ are equivalent}$$

□

Theorem 89. Let $A \in K^{n \times m}$, $B \in K^{m \times p}$. Then

$$\text{rank}(A \cdot B) \leq \min(\text{rank}(A), \text{rank}(B))$$

A is invertible.

$$\text{rank}(A \cdot B) \leq \text{rank}(B)$$

$$B = A^{-1} \cdot A \cdot B$$

$$\Rightarrow \text{rank}(B) \leq \text{rank}(A \cdot B)$$

Or more to the point: Column and row transformations do not change the rank.

This lecture took place on 18th of January 2016 (Franz Lehner).

$\text{srg}(A) = \text{zrg}(A) = \text{rk}(A)$ is the dimension of the column (row) space.

Elementary row and column operations (multiplication with regular matrix from left or (right)):

- exchange
- addition of a multiple
- permutation matrix

Permutation matrix:

$$I + \lambda E_{ij} = \begin{bmatrix} 1 & \dots & \dots \\ & \ddots & \\ \lambda & \ddots & \ddots \\ & \ddots & \ddots \\ & \ddots & \ddots & 1 \end{bmatrix}$$

- Add the i -th row to the j -th row
- Add the j -th column to the i -row

$$A \rightsquigarrow I_{mn}^* = \begin{matrix} & 1 & & 0 \\ & & \ddots & 0 \\ & & & 1 & 0 \\ \underbrace{0}_{r} & 0 & 0 & 0 \end{matrix}$$

with $\text{rk}(A) = r$.

Theorem 90.

$$A \in K^{n \times m}, B \in K^{m \times p} \rightarrow A \cdot B \in K^{n \times p}$$

$$\text{rank}(A \cdot B) \leq \min(\text{rank}(A), \text{rank}(B))$$

Proof.

$$\begin{aligned} \text{im}(AB) &\subseteq \text{im}(A) \\ (f(g(X))) &\subseteq f(Y) \\ K^p &\rightarrow K^m \rightarrow K^n \\ \Rightarrow \dim \text{im}(AB) &\leq \dim \text{im}(A) \\ \Rightarrow \text{rk}(AB) &\leq \text{rk}(A) \\ \text{rk}(A \cdot B) &= \text{rk}((A \cdot B)^T) = \text{rk}(B^T \cdot A^T) \leq \text{rk}(B^T) = \text{rk}(B) \end{aligned}$$

□

Theorem 91. *Equivalent matrices have the same rank. If $B = PAQ$ with $B \in K^{m \times n}$ and $A \in K^{n \times m}$, then $\text{rk}(A) = \text{rk}(B)$.*

$$P \in \text{GL}(m, K) \quad Q \in \text{GL}(n, K)$$

Proof. Let $A' = P \cdot A \Rightarrow \text{rk}(A') \leq \text{rk}(A)$.
 P invertible $\Rightarrow A = P^{-1} \cdot A' \Rightarrow \text{rk}(A) \leq \text{rk}(A')$.

So $\text{rk}(A) = \text{rk}(A')$.

$$A' = A' \cdot Q \Rightarrow \text{rk}(A') = \text{rk}(A)$$

□

Corollary 23. *Elementary row and column operations do not change the rank of the matrix.*

Remark 35. *Especially at the end of operations from Theorem 87, always the same number of ones is left (i.e. $r = \text{rank}(A)$); Independent of the order of the steps.*

Corollary 24. *Two matrices $A, B \in K^{m \times n}$ are equivalent iff $\text{rk}(A) = \text{rk}(B)$.*

(There are $mm(m, n) + 1$ equivalence classes)

Proof. Consider Theorem 91.

$$\begin{aligned} \text{rk}(A) &= a \leq \text{rk}(B) \\ \Rightarrow \bigvee_{P, Q \text{ invertible}} P \cdot A \cdot Q &= I_{m,n}^{(r)} \Rightarrow A \sim I_{m,n}^{(r)} \sim B \\ \bigvee_{P', Q' \text{ invertible}} P' \cdot B \cdot Q' &= I_{m,n}^{(r)} \\ A \in K^{m \times n} &\Rightarrow \text{rk}(A) \in \{0, \dots, \min(m, n)\} \end{aligned}$$

□

Theorem 92. *$A \in K^{m \times n}$ is regular if and only if $\text{rk}(A) = n$.*

Proof.

$$\begin{aligned} \text{rk}(A) &= \text{rk}(A^{-1} \cdot A) \\ [A \sim A^{-1} \cdot A &= I_n] \\ [P \cdot A \cdot I] \\ &= \text{rk}(I_n) = n \end{aligned}$$

$$\text{rk}(A) = n = \text{rk}(I_n) \xrightarrow{\text{Corollary 24}} A \sim I_n \Rightarrow \bigvee_{P, Q \in \text{GL}(n, K)} A = P \cdot I_n \cdot Q = P \cdot Q \in \text{GL}(n, K)$$

$\Rightarrow A$ is regular

Regular A is equivalent to I_n by row operations from left and column operations from right. □

Corollary 25. *Every regular matrix can be written as product of elementary transformation matrices.*

$$\begin{aligned} L_i \cdot \dots \cdot L_1 \cdot A \cdot R_1 \cdot \dots \cdot R_l &= I_n \\ A &= L_1^{-1} \cdot L_2^{-1} \cdot \dots \cdot L_r^{-1} \cdot I_n \cdot R_l^{-1} \cdot \dots \cdot R_1^{-1} \end{aligned}$$

Corollary 26. *Every regular matrix can be transformed into the unit matrix (only!) by elementary row operation.*

Proof. The matrices L and R have the same structure, particularly permutation matrices $I + \lambda E_{ij}$.

$$A = L_1^{-1} \cdot L_2^{-1} \dots L_n^{-1} \cdot R_l^{-1} \dots R_1^{-1} \cdot I_n$$

$$L_k \dots L_2 L_1 A = R_l^{-1} \dots R_1^{-1} \cdot I_n$$

$$R_1 \cdot R_2 \cdot \underbrace{R_l L_n L_{n-1} \dots L_2 L_1}_{\text{only row operations}} \cdot A = I_n$$

$$\text{and } A^{-1} = R_1 R_2 \dots R_l L_k \dots L_1$$

$$\begin{array}{l} \text{Algorithm:} \\ L_1 \cdot A \\ L_2 \cdot L_1 \cdot A \\ \vdots \\ R_1 \dots L_1 A = I_n \end{array} \left| \begin{array}{l} I_n \\ L_1 \\ L_2 L_1 \\ \vdots \\ A^{-1} \end{array} \right.$$

Example 58. *We are only allowed to use row operations!*

$$\begin{array}{ccc|ccc} 0 & -1 & 1 & 1 & & \\ 2 & 0 & 1 & & 1 & \\ 3 & 1 & 1 & & & 1 \\ \hline 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & -\frac{3}{2} & 1 \end{array}$$

The operations we applied are given with L_1 , $L_2 = L_1$, $L_3 = L_2 \cdot L_1$.

$$L_2 = \begin{array}{ccc} \frac{1}{2} & & \\ & 1 & \\ & & 1 \end{array} \quad L_3 = \begin{array}{ccc} 1 & & \\ & 1 & \\ & -3 & 1 \end{array} \quad L_1 = \begin{array}{ccc} 1 & & \\ & -1 & \\ & & 1 \end{array}$$

$$A \cdot A^{-1} = I$$

$$\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & -\frac{3}{2} & 1 \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 2 \end{array}$$

We continue:

$$L_5 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$$

□

Hence the inverse matrix is given with

$$\begin{pmatrix} -1 & 2 & -1 \\ 1 & -3 & 2 \\ 2 & -3 & 2 \end{pmatrix} = A^{-1}$$

Theorem 93. *Let $A \in K^{n \times m}$ and $B \in K^{m \times p}$ ($\det(AB) \leq \det(A)$). Then it holds that*

$$\bullet \operatorname{im}(AB) \leq \operatorname{im}(A)$$

$$L(s_1(AB), \dots, s_p(AB)) \subseteq L(s_1(A), \dots, s_m(A))$$

If B is regular, then it holds that $\operatorname{im}(AB) = \operatorname{im}(A)$.

$$\bullet \text{ Analogous for rows:}$$

$$\text{rows of } (AB) \subseteq \text{rows of } (B)$$

Proof.

$$\text{im}(f_A \cdot f_B) \subseteq \text{im}(f_A)$$

All columns of $A \cdot B$ are in column space of A .

$$s_j(A \cdot B)_i = (A \cdot B)_{ij} = \sum_{k=1}^m a_{ik} b_{kj}$$

$$\Rightarrow s_j(AB) = \sum_{k=1}^m s_k(A) b_{kj} \in L(s_1(A), \dots, s_m(A))$$

If B is regular:

$$\text{im}(A) = \text{im}(A \cdot B \cdot B^{-1}) \subseteq \text{im}(A \cdot B)$$

$$\text{im}(A' \cdot B') \subseteq \text{im}(A')$$

□

Corollary 27. *Elementary column transformations do not change the column space. Elementary row transformations do not change the row space.*

Theorem 94 (Method for determiner of a basis of a column space of a matrix). *Use column transformations to achieve a lower triangular matrix. This lower triangular matrix is also the basis of the column space of the original matrix (because the matrix does not semantically change after column transformations).*

Example 59. *Determine the basis of*

$$L \left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\} \right)$$

We compute,

$$\text{im} \left(\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & -1 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & -2 & 0 & 0 & 2 \end{pmatrix} \right) = \text{im} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{pmatrix} \right)$$

$$\begin{aligned} &= \text{im} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 \end{pmatrix} \right) = \text{im} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 2 \end{pmatrix} \right) \\ &= \text{im} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{pmatrix} \right) \end{aligned}$$

The lower left triangular matrix of the most-right matrix is the basis of U .

With

$$R_1^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Applying only row transformations is the same as applying as only column transformations to the transposed matrix.

Theorem 95 (Linear equation systems).

$$Ax = b \quad A \in K^{m \times n}, b \in K^m$$

If $b = 0$, then the system is called homogeneous. Otherwise inhomogeneous.

Remark 36. *If A is invertible, then $x = A^{-1}b$ is the distinct solution*

- holds for every b
- the solution is distinct.

Theorem 96.

$$A \in K^{m \times n}, b \in K^m$$

Then it holds equivalently,

- $Ax = b$ is solvable.
- $b \in \text{im } f_A$

- $\text{rk}(A) = A|b$ where $A|b$ is the extended matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix}$$

Proof. 1. Then $Ax = b$ is solvable.

$$\begin{aligned} \Leftrightarrow \bigvee_{x \in K^n} Ax &= b \\ \Leftrightarrow \bigvee_{x \in K^n} f_A(x) &= b \\ \Leftrightarrow b \in f_A(K^n) &= \text{im } f_A \\ \Leftrightarrow b \in L(s_1(A), \dots, s_n(A)) & \\ \Leftrightarrow L(s_1(A), \dots, s_n(A), b) &= L(s_1(A), \dots, s_n(A)) \\ \Leftrightarrow \dim L(s_1(A), \dots, s_n(A), b) &= \dim L(s_1(A), \dots, s_n(A)) \\ \Leftrightarrow \text{rk}(A|b) &= \text{rk}(A) \end{aligned} \quad (2.)$$

Theorem 97. • Let $A \in K^{m \times n}$.

The solution set of the homogeneous equation system

$$Ax = 0$$

is a subspace with $\dim(L) = n - \text{rk}(A)$.

- For every subspace $U \subseteq K^n$ with $\dim U = r$ and for all $m \geq n - r$ it holds that a matrix $A \in K^{m \times n}$ exists such that $U = \{x \mid Ax = 0\}$ (A is not distinct).
- $L = \{x \mid Ax = b\}$ is linear manifold.

$$L = x_0 + \{x \mid Ax = 0\}$$

where x_0 is a solution in $Ax = b$.

This lecture took place on 19th of January 2015 (Franz Lehner).

9.3 Summary for row and column transformations

- Represents multiplication from left or right:

$$A \mapsto PAQ$$

- Determine the rank
- Determine base of column or row space
- Determine inverse matrix
- Solution of $Ax = b$

$A = PBP^{-1}$ is a much more difficult problem involving eigenvalues and determinants.

Yesterday, we saw:

$$Ax = b \text{ solvable} \Leftrightarrow b \in f_A \Leftrightarrow \text{rk}(A|b) = \text{rk}(A)$$

□ **Theorem 98.** 1. $A \in K^{m \times n}$,

$$L = \{x \mid Ax = 0\}$$

(m equations and n unknown variables)

L is supspace with

$$\dim L = n - \text{rk}(A)$$

“Number of free parameters”

2. For every subspace $U \subseteq K^n$ with $\dim U = r$ and for all m, n with $m \geq n - r$ there exists some matrix $A \in K^{m \times n}$ (multiple solutions possible) such that $U = \ker f_A = \{x \mid Ax = 0\}$.

3. Let $A \in K^{m \times n}, b \in K^m$. Let $x_0 \in K^n$ be a solution such that $Ax_0 = b$

$$\Rightarrow L = \{x \mid Ax = b\} = x_0 + \ker A$$

\Rightarrow linear manifold.

Proof of Theorem 98. 1. $\ker A$ is subspace. Because of Corollary 18, it holds that

$$\begin{aligned}\dim \ker f_A + \dim \operatorname{im} f_A &= \dim K^n \\ \Leftrightarrow \dim L + \operatorname{rk}(A) &= n\end{aligned}$$

2. Given $U \subseteq K^n$. Let u_1, \dots, u_r be basis of U . Extend basis to basis of K^n : $u_1, \dots, u_r, \dots, u_n$. (Theorem 72 tells us that every $f : B \rightarrow W$ on basis has distinct extension to linear mapping $f : V \rightarrow W$).

$$\begin{aligned}f : K^n &\rightarrow K^m \\ f(u_i) &= 0 \quad 1 \leq i \leq r \\ f(u_{r+j}) &= v_j \quad 1 \leq j \leq n-r\end{aligned}$$

where $v_1, \dots, v_{n-r} \in K^n$ is linear independent.

$$\Rightarrow U \subseteq \ker f$$

and $U = \ker f$ because v_1, \dots, v_{n-r} is linear independent. Choose $A \in K^{m \times n}$ such that $f = f_A$

$$\Rightarrow U = \{x \mid Ax = 0\}$$

3. Let $Ax_0 = b$. Let $x \in K^n$, then it holds that

$$Ax = b \Leftrightarrow Ax = Ax_0 \Leftrightarrow A \cdot (x - x_0) = 0 \Leftrightarrow x - x_0 \in \ker A \Leftrightarrow x \in x_0 + \ker A$$

□

9.4 Remarks on Gauss-Jordan elimination

Theorem 99 (Remarks on Gauss-Jordan elimination). 1. Elementary row transformations correspond to multiplication from left with invertible matrices, namely

- Row exchange $T_{j,i}$

- Addition of vectors row to other rows

$$\begin{bmatrix} 1 & \dots & \lambda_2 \\ & \ddots & \\ & & \lambda_1 \end{bmatrix}$$

- L is regular and in $K^{m \times n}$.

$$Ax = b \Leftrightarrow LAx = Lb$$

\rightarrow Elementary row transformations do not change the solution set.

- Row transformations

$$(A \cdot Q) \cdot y = b \Leftrightarrow A \cdot (Q \cdot y) = b \Leftrightarrow y = Q^{-1}x$$

If you want to solve $Ax = b$ for b_i with $i = 1, \dots, k$.

$$AX = B$$

For example, $B = I$.

$$X = \begin{pmatrix} x_1 & \dots & x_k \\ \vdots & & \vdots \end{pmatrix}$$

$$B = \begin{pmatrix} b_1 & \dots & b_k \\ \vdots & & \vdots \end{pmatrix}$$

$$AX = I \Rightarrow X = A^{-1} \cdot I = A^{-1}$$

\rightarrow also k -th column of A^{-1} meaning the solution stays the same: $Ax = e_k$.

$$Ax = b \quad Ay \Leftrightarrow A(\lambda x + y) = \lambda b + \mu c$$

A^{-1} is a linear mapping.

What do you get in the general case? ($m \neq n$ if we transform $(A|I_m)$)

Theorem 100 (LU-decomposition). Let $A \in K^{m \times n}$. Then it holds that:

- $P \in K^{m \times m}$ is permutation matrix

- $L \in K^{m \times m}$ is regular lower-left triangular matrix
- $R \in K^{m \times n}$ is upper-right triangular matrix

such that

$$P \cdot A = L \cdot R$$

Example 60 (Application of LU decomposition).

$$A = P^{-1}LR$$

$$Ax = b \Leftrightarrow PAx = Pb$$

$$\Leftrightarrow LRx = Pb$$

$$\Leftrightarrow \begin{cases} c := Ly = Pb \\ y := Rx \end{cases}$$

$$y_1 = \frac{1}{l_{1,1}}c_1$$

$$y_2 = \frac{1}{l_{2,2}}(c_2 - l_{2,1}y_1)$$

...

y is the vector which remains after application of row transformations.

$$Ax = b \quad L^{-1}Ax = L^{-1}b \quad Rx = y$$

This is recursively solvable from the bottom to the top. In the upper-right triangular matrix R , the value closest to the bottom needs to be zero.

Theorem 101. • Let $L_1, L_2 \in K^{m \times m}$ be a lower triangular matrix

$$\Rightarrow L_1 \cdot L_2 \text{ is lower triangular matrix}$$

$$\Rightarrow \text{lower triangular matrices build a subalgebra of } K^{m \times m}$$

- If L is a triangular matrix, then L^{-1} is triangular matrix.

Proof. • Left for the reader.

- Left for the reader. Look how the matrix looks like after inverting it.

□

Theorem 102. The set

$$F_k = \left\{ \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & 1 & 0 & 0 \\ \dots & \dots & \lambda_{k+1} & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \lambda_m & 0 & 1 \end{pmatrix} \right\}$$

builds a group in regards of multiplication (“Frobenius matrices”).

Proof.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \lambda_{k+1} & 1 & 0 \\ \dots & \dots & \dots & \lambda_n & 0 & 1 \end{bmatrix}}_{k\text{-th column}} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \mu_{k+1} & 1 & 0 \\ \dots & \dots & \dots & \mu_n & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \lambda_{k+1}\mu_{k+1} & 1 & 0 \\ \dots & \dots & \dots & \mu_n\mu_m & 0 & 1 \end{bmatrix}$$

□

Alternative proof.

$$\begin{aligned} F_k &= \left\{ I + \sum_{i=k+1}^n \lambda_{k+i} E_{i,k} \mid \lambda_j \in K \right\} \\ &= \left(I + \sum_{i=k+1}^n \lambda_i E_k \right) \cdot \left(I + \sum_{j=k+1}^n \mu_j E_{j,k} \right) \\ &= I + \sum_{j=k+1}^n \mu_j I \cdot E_{j,k} + \sum_{i=k+1}^n \lambda_i E_k \cdot I \cdot \left(\sum_{i=k+1}^n \lambda_i E_{i,k} \right) \cdot \left(\sum_{j=k+1}^n \mu_j E_{j,k} \right) \\ &= I + \sum_{i=k+1}^n (\lambda_i + \mu_i) E_{i,k} + \sum_{i=k+1}^n \sum_{j=k+1}^n \lambda_i \mu_j \underbrace{E_{i,k} \cdot E_{j,k}}_{=0 \text{ because } j > k} \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & 1 & \dots & 0 & 0 \\ \dots & \dots & \ddots & \lambda_{k+1} & \dots & 0 & 0 \\ \dots & \dots & \ddots & \vdots & \dots & 0 & 0 \\ \dots & \dots & \ddots & \vdots & \dots & 1 & 0 \\ \dots & \dots & \ddots & \lambda_n & \dots & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \ddots & 1 & \dots & 0 & 0 \\ \dots & \dots & \ddots & -\lambda_{k+1} & \dots & 0 & 0 \\ \dots & \dots & \ddots & \vdots & \dots & 0 & 0 \\ \dots & \dots & \ddots & \vdots & \dots & 1 & 0 \\ \dots & \dots & \ddots & -\lambda_n & \dots & 0 & 1 \end{pmatrix} \bullet$$

□

$$\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & & \\ 2 & 2 & 5 & & 1 & \\ 4 & 6 & 8 & & & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 & 1 & 0 \\ 0 & 2 & 4 & -4 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & -4 & 0 & 1 \\ 0 & 0 & 3 & -2 & 1 & 0 \end{array}$$

$\underbrace{\hspace{10em}}_{\text{gives } R \text{ on the left}}$

Example 61 (Example for LU decomposition). •

This is not a triangular matrix!

$$\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & & \\ 2 & 3 & 5 & & 1 & \\ 4 & 6 & 8 & & & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 4 & -4 & 0 & 1 \end{array}$$

$\underbrace{\hspace{10em}}_{\text{gives } L_1 \cdot A \text{ on the left and } L_1 \text{ on the right}}$

$$\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 & -2 & 1 \end{array}$$

$\underbrace{\hspace{10em}}_{\text{gives } R \text{ on the left and } L_2 \cdot L_1 \text{ on the right}}$

$$L_2 \cdot L_1 \cdot A = R$$

$$A = L_1^{-1} \cdot L_2^{-1} \cdot R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix}}_{LU \text{ decomposition of } A}$$

$$L_1^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & & 1 \end{bmatrix} \quad L_2^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -4 & & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2 L_1^{-1} P_2^{-1}$$

$$P_2 L_1 A = R$$

$$\Rightarrow A = L_1^{-1} P_2^{-1} R$$

$$= P_2^{-1} P_2 \cdot L_1^{-1} P_2^{-1} \cdot R$$

$$L_1^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & & 1 \end{bmatrix}$$

$$P_2 L_1^{-1} P_2^{-1} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 2 & & 1 \end{bmatrix}$$

$$P_2 \cdot A = \underbrace{P_2 \cdot L_1^{-1} \cdot P_2^{-1}}_{(L_1')^{-1}} \cdot R = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ & 2 & 4 \\ & & 3 \end{bmatrix}$$

•

$$\begin{array}{ccc|ccc}
 1 & 1 & 1 & 1 & 0 & 0 \\
 2 & 2+\varepsilon & 5 & 0 & 1 & 0 \\
 4 & 6 & 8 & 0 & 0 & 1 \\
 \hline
 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & \varepsilon & 3 & -2 & 1 & 0 \\
 0 & 2 & 4 & -4 & 0 & 1 \\
 \hline
 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & \varepsilon & 3 & -2 & 1 & 0 \\
 0 & 0 & 4-\frac{6}{\varepsilon} & -4+\frac{4}{\varepsilon} & -\frac{2}{\varepsilon} & 1
 \end{array}$$

$$L_1 = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ -1 & & 1 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\frac{2}{\varepsilon} & 1 \end{bmatrix}$$

$$A = L_1^{-1} \cdot L_2^{-1} \cdot R$$

$$L_1^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & & 1 \end{bmatrix} \quad L_2^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & \frac{2}{\varepsilon} & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & \frac{2}{\varepsilon} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & \varepsilon & 3 \\ 0 & 0 & 4-\frac{6}{\varepsilon} \end{bmatrix}$$

Better row exchange:

$$\begin{array}{ccc|ccc}
 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 2 & 4 & -4 & 0 & 1 \\
 0 & \varepsilon & 3 & -2 & 1 & 0 \\
 \hline
 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 2 & 4 & -4 & 0 & 1 \\
 0 & 0 & 3-2\varepsilon & -2-2\varepsilon & 1 & -\frac{\varepsilon}{2}
 \end{array}$$

$$P_2 = \begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ & -\frac{\varepsilon}{2} & 1 \end{bmatrix}$$

$$L_2 P_2 L_1 A = R$$

$$\Rightarrow A = L_1^{-1} P_2^{-1} L_2^{-1} R$$

$$= P_2^{-1} \cdot P_2 \cdot L_1^{-1} \cdot P_2^{-1} \cdot L_2^{-1} \cdot R$$

$$P_2 \cdot A = P_2 L_1^{-1} P_2^{-1} \cdot L_2^{-1} \cdot R$$

$$\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 2 & \frac{\varepsilon}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3-2\varepsilon \end{bmatrix}$$

The error does not increase!

Remark 37. In numerics the rank is pointless, because a small error makes zero non-zero. This can change the rank of the matrix.

Remark 38. To achieve a small error, always select a greatest possible pivot element!

This lecture took place on 25th of January 2016 (Franz Lehner).

Let $P \cdot A = L \cdot R$.

$$Ax = b \Rightarrow x = R^{-1} L^{-1} b$$

Example 62.

$$P \cdot A \stackrel{!}{=} L \cdot R$$

$$A^{(0)} = A$$

Search for column $\neq 0 \rightarrow$ column number j .

Heuristic: Choose the greatest value $a_{i,j} \neq 0$ as pivot element.

1. exchange such that $a_{i,j}$ is in the first row.

$$T_{(1,i_1)} \cdot A = \begin{array}{cccc}
 & 0 & 0 & a_{i,j_1} & 0 & \dots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \dots \\
 & 0 & 0 & 0 & 0 & \dots
 \end{array}$$

2. Produce 0 underneath a_{i,j_1} .

$$A^{(i)} = F_1 \cdot T_{(1,j_1)} \cdot A^{(0)}$$

$$F_1 = \begin{pmatrix} 1 & \ddots & & \\ \lambda_2^{(1)} & \ddots & & \\ \vdots & \vdots & \ddots & \\ \lambda_m^{(i)} & \dots & \dots & 1 \end{pmatrix}$$

$$\lambda_2^{(i)} = -\frac{a_{2,j}}{a_{i,j_1}}$$

$$\lambda_i^{(i)} = \begin{cases} -\frac{a_{i,j_1}}{a_{i,j_1}} & i \neq i_1 \\ -\frac{a_{1,j_1}}{a_{i,j_1}} & i = i_1 \end{cases}$$

3. Repeat procedure for matrix B

(a) Search column $j_2 \neq 0$

(b) Exchange largest element a_{i_2,j_2} to second element of A .

$$T_{(2j_2)}A^{(i)} = \begin{pmatrix} 0 & a_{i,j} & \dots & \dots & \dots & \dots \\ 0 & 0 & a'_{i_2,j_2} & \dots & \dots & \dots \\ \vdots & & & & & \end{pmatrix}$$

$$\text{with } a'_{i_2,j_2} = a_{i_2,j_2} - \lambda_{i_2}^{(i)} \cdot a_{i,j_2}$$

$$F_2 = \begin{pmatrix} 1 & 0 & & \\ \lambda_j^{(2)} & 1 & & \\ \vdots & & \ddots & \\ \lambda_m^{(2)} & \dots & \dots & \dots \end{pmatrix}$$

$$\lambda_i^{(2)} = \left(-\frac{a_{i,j_2}}{a'_{i_2,j_2}} \right)$$

The first column is kept unmodified \rightarrow Frobenius matrix.

$$F_2 \cdot T_{(2,i_2)} \cdot F_1 \cdot T_{(1,i_1)} \cdot A = \begin{pmatrix} \infty & a_{i,j_1} & \dots & \dots & \dots & \dots & \dots & \dots \\ \vdots & 0 & 0 & 0 & a_{i_2,j_2} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \dots & \dots & \dots \end{pmatrix}$$

where at the bottom-right a $m-2 \times n-j_2$ submatrix is given.

(c) Multiplication with T_{ij} and F_j does not change the rank of the matrix. Therefore if $r = \text{rk}(A)$, then the zero matrix remains after r steps.

$$A^{(r)} = F_r T_{(r,i_r)} \dots F_1 T_{(1,i_1)} A = R$$

$$A = \underbrace{T_{(1,i_1)} F_1^{-1} \dots T_{(1,i_r)}}_{\text{not a triangular matrix!}} F_r^{-1} \cdot R$$

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & -\lambda_{n,i_n} & \\ & & \vdots & \\ & & -\lambda_m & 1 \end{pmatrix}$$

$$T_{(i,j)} \cdot F_1^{-1} \cdot T_{2,i_2} = T_{(1,i_1)} \cdot T_{(2,i_2)} \cdot \underbrace{T_{(2,i_2)}^{-1} \cdot F_1^{-1} T_{(2,i_2)}}_{F'_1}$$

Theorem 103.

$$F = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \lambda_{n+1} & & \\ & \vdots & & \\ & \lambda_m & \dots & 1 \end{pmatrix} \in \mathcal{F}_k^{m \times m}$$

$\pi \in \sigma_m$ permutation with $\sigma(i) = i \forall i \leq k$. T_π is the permutation matrix such that $T_\pi \cdot e_i = e_{\pi(i)}$

$$\Rightarrow T_\pi^{-1} F T_\pi = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda_{\pi(n+1)} & \\ & & \vdots & \\ & & \lambda_{\pi(m)} & \dots & 1 \end{pmatrix} \in \mathcal{F}_k$$

Proof.

$$F = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{m-k} \end{array} \right] \quad T_\pi = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P \end{array} \right]$$

$$\begin{aligned}
 T_\pi^{-1} &= \left[\begin{array}{c|c} T_k & 0 \\ \hline 0 & P^{-\pi} \end{array} \right] \\
 T'_\pi F T_\pi &= \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P^{-1} \end{array} \right] \cdot \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{n-k} \end{array} \right] \cdot \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P \end{array} \right] \\
 &= \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P^{-1} \end{array} \right] \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & P \end{array} \right] = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{m-k} \end{array} \right] \\
 A &= T_{(i,i_1)} F_1^1 \dots T_{(i,i_r)} F_r^{-1} R \\
 &= \dots T_{(r-1,i_{r-1})} \cdot T_{r-1}^{-1} \cdot T_{(r,i_r)} F_r^{-1} \cdot R \\
 &= \dots T_{(r-1,i_{r-1})} T_{r,i_r} \underbrace{T_{(r,i_r)}^{-1} F_{r-1}^{-1} T_{r,i_r} F_r^{-1}}_{\text{lemma } F'_{r-1} \in \mathcal{F}_r} \cdot R \\
 &= T_{(1,i_1)} F_1^1 \dots T_{(r-2,i_{r-2})} F_{r-2}^{-1} \cdot T_{(r-1,i_{r-1})} T_{r,i_r} F'_{r-1} F_r^{-1} \cdot R \\
 &= \dots T_{(r-2,i_{r-2})} T_\pi \cdot F'_{r-2} \cdot F'_{r-1} \cdot F_r^{-1} \cdot R \\
 &= P \cdot F'_1 F'_2 \dots F'_{r-2} F'_{r-1} F_r^{-1} \cdot R \\
 &\Rightarrow P^{-1} \cdot A = L \cdot R
 \end{aligned}$$

Theorem 104 (Matrix representation of linear maps).

$$f : K^n \rightarrow K^m \leftrightarrow \text{matrix } A \in K^{m \times n} \text{ such that } \underbrace{f = f_A}_{f(x) = A \cdot x}$$

$$\text{Hom}(K^n, K^m) \cong K^{m \times n}$$

Let V, W with $\dim V = n$, $\dim W = m$.

$$V \cong K^n, \quad W \cong K^m \quad \Rightarrow \text{Hom}(V, W) \cong \text{Hom}(K^n, K^m) \cong K^{m \times n}$$

How does this isomorphism look like?

Choose basis $B \subseteq V$, $B = (b_1, \dots, b_n)$ and $C \subseteq W$. Isomorphism:

$$\Phi_B : V \rightarrow K^n$$

$$v \mapsto (v)_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

$$\Phi_C : W \rightarrow K^m$$

$$w \mapsto (w)_C = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$$\Phi_B^{-1} = K^n \rightarrow V$$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \sum_{i=1}^n \lambda_i b_i$$

Let f be a linear map from V to W ($f \in \text{Hom}(V, W)$). There exists a distinct linear mapping $\tilde{f} \in \text{Hom}(K^n, K^m)$ such that $\Phi_C \circ f = \tilde{f} \circ \Phi_B$, specifically $\Phi_C \circ f \circ \Phi_B^{-1}$ the corresponding matrix (Theorem 79) in $K^{m \times n}$. b_i is computed in matrix representation in f in regards of B and C . Notation: $\Phi_C^B(f) \in K^{m \times n}$.

□

Theorem 105. $\Phi_C^B(f)$ is the matrix for which it holds that

$$\Phi_C(f(v)) = \Phi_C^B(f) \cdot \Phi_B(v)$$

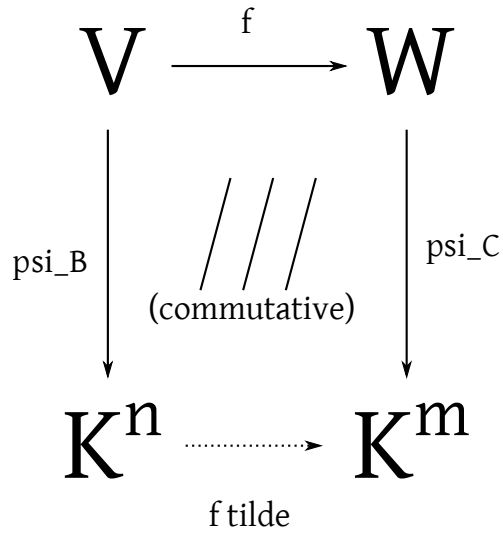
$$f(v)_C = A \cdot (v)_B$$

$$S_i(\Phi_C^B(f)) = \Phi_C(f(b_i))$$

Corollary 28.

$$\Phi_C^B(f) = \begin{bmatrix} \Phi_C(f(b_1)) & \Phi_C(f(b_2)) & \dots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Columns of $\Phi_C^B(f)$ are the coordinate vectors of the images of the base vectors in regards of basis C .


 Figure 1: Linear mapping in terms of f

Proof.

$$S_i(\Phi_C^B(f)) = \Phi_C^B(f) \cdot e_i = \Phi_C^B(f) \Phi_B(b_i) \stackrel{\text{Theorem 105 for } v=b_i}{=} \Phi_C(f(b_i))$$

$$e_i = \Phi_B(b_i)$$

□

Example 63.

$$V = \mathbb{R}^3 \text{ with basis } \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right) = B$$

$$W = \mathbb{R}^2 \text{ with basis } \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) = C$$

$$f : V \rightarrow W$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x + 3y - z \\ 2y + 3z \end{pmatrix}$$

$$\Phi_C^B(f) = ?$$

i -th column is image of b_i in basis C .

$$f(b_1) = f \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$f(b_2) = f \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$f(b_3) = f \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 1 \\ 5 & 3 & 5 \end{pmatrix} = \Phi_{std \text{ basis}}^B(f)$$

$$\Phi_C(f(b_i)) : \text{solve } \lambda_1 c_1 + \lambda_2 c_2 = f(b_i)$$

$$\begin{pmatrix} C_1 & C_2 \\ \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = f(b_i)$$

$$\begin{array}{c|ccc}
 & 2 & 3 & 0 & 1 \\
 1 & 0 & 5 & 3 & 5 \\
 \hline
 0 & 2 & -2 & -3 & -4 \\
 & 1 & -1 & -\frac{3}{2} & -2
 \end{array}$$

$$\rightsquigarrow \Phi_C^B(f) = \begin{pmatrix} 5 & 3 & 5 \\ -1 & -\frac{3}{2} & -2 \end{pmatrix}$$

Test:

$$\Phi_C^B(t) \cdot \Phi_B(b_i) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$5 \cdot c_1 - c_2 = 5 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} = t \cdot (b_1)$$

Theorem 106. $\Phi_C^B : \text{Hom}(V, W) \rightarrow K^{m \times n}$ is linear where B, C are bases of V, W . Hence,

$$\Phi_C^B(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \Phi_C^B(f) + \mu \Phi_C^B(g)$$

Proof. Will be provided in the practicals for basis elements. \square

Theorem 107. Let $B = (b_1, \dots, b_n)$ be basis of V . Let $C = (c_1, \dots, c_m)$ be basis of W . Let $D = (d_1, \dots, d_p)$ be basis of Z .

$$f : V \rightarrow W \quad g : W \rightarrow Z \quad \text{linear}$$

$$\Rightarrow \Phi_D^B(g \circ f) = \Phi_D^C(g) \cdot \Phi_C^B(f)$$

Proof.

$$((g \circ f)(v))_D \stackrel{!}{=} \Phi_D^C(g) \cdot \Phi_C^B(f) \circ (v)_B$$

$$\begin{aligned} \Phi_D((g \circ f)(v)) &= \Phi_D(g(f(v))) \\ &= \Phi_D^C(g) \cdot \Phi_C(f(v)) \\ &= \Phi_D^C(g) \cdot \Phi_C^B(f) \cdot \Phi_B(v) \end{aligned}$$

\square

This lecture took place on 26th of January 2016 (Wolfgang Wöss).

$$\begin{aligned} V &\cong K^m & W &\cong K^m \\ B &= (b_1, \dots, b_n) & C &= (c_1, \dots, c_n) \end{aligned}$$

$$\Phi_B : V \rightarrow K^n$$

$$v \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

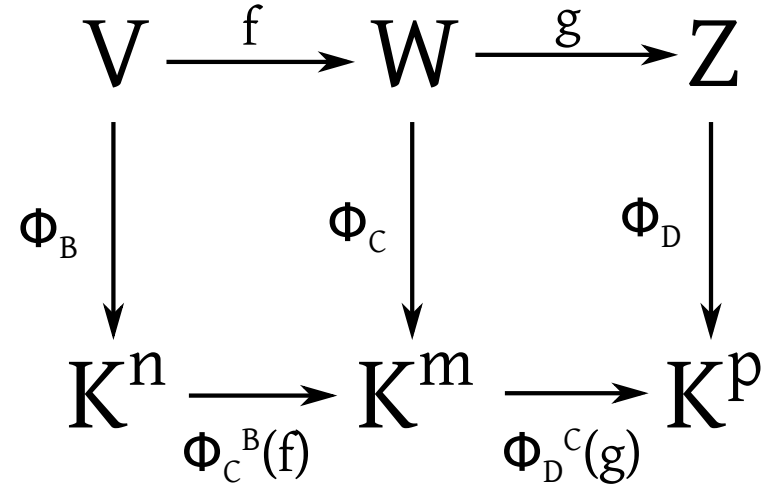


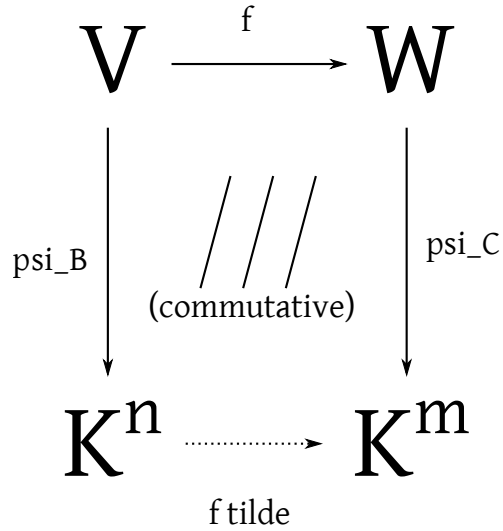
Figure 2: Mapping f and g

$$\Phi_C : W \rightarrow K^m$$

$$w \mapsto \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$$v = \sum_{j=1}^n \lambda_j b_j$$

$$w = \sum_{i=0}^m \mu_i c_i$$


 Figure 3: Linear mapping in terms of f

$$\tilde{f} = \Phi_B \circ f \circ \Phi_B^{-1} = f_A$$

$m \times n$ matrix:

$$A = \Phi_C^B(f)$$

$$\Phi_C^B(f) = \left(\underbrace{\Phi_C(f(b_1))}_{\text{1st column}}, \Phi_C(f(b_2)), \dots, \underbrace{\Phi_C(f(b_n))}_{\text{n-th column}} \right)$$

$$f \mapsto \Phi_C^B(f)$$

$$\text{Hom}(V, W) \rightarrow K^{m \times n}$$

is the vector space of $m \times n$ matrices over K .

Theorem 108. 1. Given

$$V \cong K^n \quad W \cong K^m$$

$$B = (b_1, \dots, b_n) \quad C = (c_1, \dots, c_m)$$

$$\Phi_B : V \rightarrow K^n \quad \Phi_C : W \rightarrow K^m$$

Then

$$\text{rank } \Phi_C^B(f) = \dim \underbrace{\text{im}(f)}_{f(r) \subset W}$$

2. f is an isomorphism if and only if $m = n$ and $\Phi_C^B(f)$ is regular. So $\Phi_B^C(f^{-1}) = \Phi_C^B(f)^{-1}$ holds.

Proof. 1.

$$V = L(b_1, \dots, b_n)$$

$$\text{im } V = L(f(b_1), \dots, f(b_n))$$

$$\cong \Phi_C(f(b_1), \dots, f(b_n))$$

$$= L(\underbrace{\Phi_C(f(b_1)), \dots, \Phi_C(f(b_n))}_{\text{columns of } \Phi_C^B})$$

So,

$$\dim \text{im } V = \dim L(\Phi_C(f(b_1)), \dots, \Phi_C(f(b_n))) = \text{rank}(\Phi_C^B)$$

Why is Φ_C an isomorphism? $\Phi_C : W \rightarrow K^m$ (bijective and linear).

$$U = L(f(b_1), \dots, f(b_n))$$

$$\Phi_C|_U : U \rightarrow \Phi_C(U) \subset K^m$$

2. $m = n$ is trivial.

$$f \text{ is an isomorphism} \Leftrightarrow \text{im } f = W$$

$$\Leftrightarrow \dim \text{im } f = n$$

$$\Leftrightarrow \text{rank} \left(\underbrace{\Phi_C^B(f)}_{n \times n \text{ matrix}} \right) = n \Leftrightarrow \underbrace{\Phi_C^B(f)}_{\text{regular}}$$

$$\Phi_C^B(f) \cdot \Phi_B^C(f^{-1}) \stackrel{\text{Theorem 107}}{=} \Phi_C^C(f \cdot f^{-1}) = \Phi_C^C(\text{id}_W) = I_n$$

□

Definition 41.

$$V \cong K^n$$

Bases $B = (b_1, \dots, b_n)$ and $B' = (b'_1, \dots, b'_n)$.

1.

$$\begin{aligned} \Phi_{B'}^B(\text{id}_V) &\leftrightarrow \Phi_{B'} \circ \Phi_B^{-1} \\ \Phi_{B'}^B(\text{id}_V) &= T_{B'}^B \end{aligned}$$

“basis transformation matrix”

So,

$$T_{B'}^B = \left(\underbrace{\Phi_B(b_1)}_{\text{column 1}}, \dots, \underbrace{\Phi_B(b_n)}_{\text{column n}} \right)$$

2. $T_{B'}^B$ is invertible and (follows from Theorem 108)

$$(T_{B'}^B)^{-1} = T_B^{B'}$$

3. Given

$$\begin{aligned} V &\cong K^n & W &\cong K^m \\ B &= (b_1, \dots, b_n) & C &= (c_1, \dots, c_m) \\ \Phi_B : V &\rightarrow K^n & \Phi_C : W &\rightarrow K^m \end{aligned}$$

Then we have new bases

$$B' = (b'_1, \dots, b'_n) \text{ of } V$$

$$C' = (c'_1, \dots, c'_m) \text{ of } W$$

$$\begin{aligned} \Phi_{C'}^B(f) &= \underbrace{T_{C'}^C}_{m \times m} \cdot \underbrace{\Phi_C^B(f)}_{m \times n} \cdot \underbrace{T_B^{B'}}_{n \times n} \\ &= (T_{C'}^C)^{-1} \cdot \Phi_C^B(f) \cdot T_B^{B'} \end{aligned}$$

Figure 4 follows from Theorem 107.

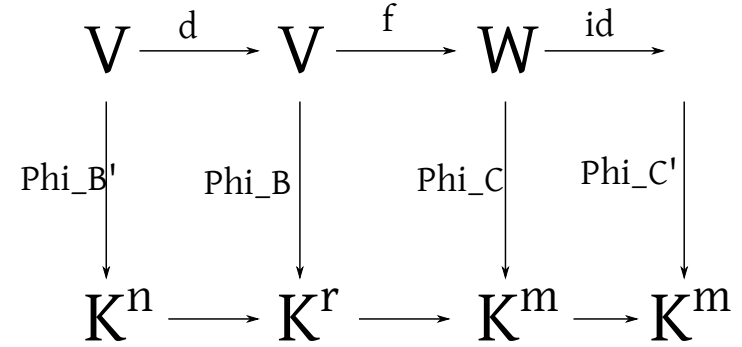


Figure 4: This structure follows from Theorem 107

Corollary 29. 1. Matrix representations $\Phi_C^B(f)$ and $\Phi_{C'}^{B'}(f)$ of a linear mapping $f : V \rightarrow W$ are pairwise equivalent.

2. Two matrix representations $\Phi_B^B(f)$ and $\Phi_{B'}^{B'}(f)$ of $f \in \text{End}(V)$ are pairwise similar

$$\Phi_B^B(f) = (T_B^B)^{-1} \Phi_{B'}^{B'}(f) T_B^B$$

3. f as previously. $K \cong V \rightarrow W \cong K^n$. $B = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_n)$.

Then bases B of V and C of W exist such that

$$\Phi_C^B(f) = I_{m \times n}^{(r)}$$

Hence we have r diagonal ones.

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