Linear Algebra 2 – Practicals

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summer term 2016

Contents

Exercise I did on the board: 3, 7.

1 Exercise 1

Exercise 1. Determine the matrix representation of the linear map

$$f : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$$

$$p(x) \mapsto (x-1) \cdot p(x)$$

in regards of bases $B = \{1-x, 1+x\} \subseteq \mathbb{R}_1[x]$ and $C = \left\{1, 1+x, 1+x+x^2\right\} \subseteq \mathbb{R}^2[x]$.

$$\begin{aligned} f &: \mathbb{R}_1[x] \to \mathbb{R}_2[x] \\ f &: p(x) \mapsto (x-1)p(x) \\ B &= \{1-x, 1+x\} =: \{b_1, b_2\} \\ C &= \left\{1, 1+x, 1+x+x^2\right\} =: \{c_1, c_2, c_3\} \end{aligned}$$

Find $A \in \mathbb{K}^{3\times2} =: M_C^B(f)$.

$$\forall v \in \mathbb{R}_1 : f(v) = w : \Phi_C(w) = A\Phi_B(v)$$

$$\begin{aligned} f(b_1) &= (1-x)(x-1) = -x^2 + 2x - 1 \\ f(b_2) &= (x-1)(x+1) = x^2 - 1 \end{aligned}$$

$$\Phi_C(f(b_1))$$

Coefficient comparison:

$$\begin{aligned} -x^2 + 2x - 1 &= \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2) \\ x^2 &: \lambda_3 = -1 \\ x^1 &: 2 = \lambda_2 + \lambda_3 \Rightarrow \lambda_2 = 3 \\ x^0 &: -1 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_1 = -3 \end{aligned}$$

$$\Phi_C(f(b_1)) = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\Phi_C(f(b_2)) : x^2 = 1 = \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2)$$

$$\begin{aligned} x^2 &: \lambda_3 = 1 \\ x^1 &: \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -1 \\ x^0 &: -1 = \lambda_1 + \lambda_2 + \lambda_3 \\ &-1 = \lambda_1 - 1 + 1 \\ &-1 = \lambda_1 \end{aligned}$$

$$\Phi_C(f(b_2)) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & -1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$$

2 Exercise 3

Exercise 2. Let $A_1, A_2, \dots, A_k$ be quadratic $n \times n$ matrices over the field $\mathbb{K}$. Show that the product $A_1 A_2 \dots A_k$ is invertible if and only if all A_i are invertible.

All A_i are invertible, then $\prod A_i$ is invertible.

A, B invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Generalize by induction.

If $\prod A_i$ is invertible, then all A_i are invertible.

Sidenote: We know that $\operatorname{rank}(A) = n - \dim \operatorname{kernel}(A)$.

$k = 1$ trivial

$k = 2$ $A_1 A_2$ is invertible. Let $C = (A_1 A_2)^{-1}$. Then $C A_1 A_2 = I_n$. Let $x \in \operatorname{kernel}(A_2) \Rightarrow A_2 x = 0 \Rightarrow \underbrace{C A_1}_{I_n} A_2 x =$

$C A_1 0 = 0$.

$$\operatorname{kernel}(A_2) = 0 \Rightarrow \operatorname{rank}(A_2) = n - 0 : n \Rightarrow A_2 \text{ invertible}$$

$$A_1 = \underbrace{A_1 A_2}_{\text{invertible}} \cdot \underbrace{A_2^{-1}}_{\text{invertible}}$$

$k \to k+1$ Let $A_1 \dots A_{k+1}$ is invertible $\Rightarrow$ $(A_1, \dots, A_k) A_{k+1}$ is invertible $\overset{k=2}{\Longrightarrow}$ $A_1, \dots, A_k$ is invertible, A_{k+1} invertible $\xRightarrow{\text{induction base}}$ $A_1, \dots, A_k, A_{k+1}$ is invertible.

Remark: $A, B \in \mathbb{K}^{n \times n}$. B is inverse of A

$$\Leftrightarrow AB = I = BA \Leftrightarrow AB = I \Leftrightarrow BA = I$$

3 Exercise 2

Exercise 3. Let V be a vector space and $f : V \to \mathbb{V}$ is a nilpotent linear map, hence there exists some $k \in \mathbb{N}$ such that $f^k = 0$.

3.1 Part a

Exercise 4. Show that $\mathrm{id}_V - f$ is invertible with $(\mathrm{id}_V - f)^{-1} = \mathrm{id}_V + f + f^2 + \dots + f^{k-1}$.

Show that: $(\mathrm{id}_v - f)^{-1} = \sum_{i=0}^{k-1} f^i$.

$$(\mathrm{id}_V - f) \circ \left(\sum_{i=0}^{k-1} f^i \right) = \mathrm{id}_V \circ \sum_{i=0}^{k-1} f^i - f \circ \sum_{i=0}^{k-1} f^i - \sum_{i=0}^{k-1} f^{i+1} = f^0 + \sum_{i=1}^{k-1} f^i - \sum_{i=1}^{k-1} f^i - f^k = \mathrm{id}_V - 0 = \mathrm{id}_V$$

and $\left(\sum_{i=0}^{k-1} f^i \right) \circ (\mathrm{id}_V - f)$ analogously.

3.2 Part b

Exercise 5. Use part a) to determine the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} =: A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - f_A$$

$$f_A = I_n - A = \begin{pmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_A^2 = f \cdot f = \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^3 = f^2 \cdot f = \begin{pmatrix} 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^4 = f^3 \cdot f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ f nilpotent.

$$A^{-1} = (\mathrm{id}_v - f)^{-1} = \mathrm{id}_v + f + f^2 + f^3$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot A' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4 Exercise 4

4.1 Part a

Exercise 6. Let A be an invertible $n \times n$ matrix over a field $\mathbb{K}$ and u, v are column vectors (hence $n \times 1$

matrices), such that $\sigma 1 + v^t A^{-1} u \neq 0$. Show that $(A + uv^t)$ is invertible and that

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} u v^t A^{-1}$$

4.2 Part b

Exercise 7. Apply this formula to determine the inverse of the matrix

$$A = \begin{pmatrix} 5 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

$$B = A + S$$

$$B = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

A is invertible, because it is a block matrix[1].

$$A^{-1} = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$$\sigma = 1 + \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 \neq 0$$

$$\Rightarrow B^{-1} = A^{-1} - A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 2 & -3 & 6 & -4 \\ -3 & 5 & -9 & 6 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

5 Exercise 5

Exercise 8. Show that the linear maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f : (x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2) \quad g : (x_1, x_2) \mapsto (x_1 + x_2, x_1 + x_2) \quad h : (x_1, x_2) \mapsto (x_2, x_1)$$

are linear independent, if they are considered as elements of the vector space $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ of all maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Show that

$$\lambda_1 f + \lambda_2 g + \lambda_3 h = 0 \overset{!}{=} \lambda_1 = \lambda_2 = \lambda_3 = 0$$

[1]That's why chose A and S that way

$$f : x \mapsto Ax \qquad A_f = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad A_g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Is an isomorphism, $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R}^{2\times 2}$ with $f \mapsto A_f$.

$$\lambda_1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \lambda_i = 0 \forall i \in \{1, 2, 3\}$$

6 Exercise 6

Exercise 9. Let V be a vector space with $\dim V = n < \infty$ and $U \subseteq V$ is a subspace with $\dim U = m$.

1. Show that
$$U^\perp = \{v^* \in V^* \mid U \subseteq \mathrm{kernel}(v^*)\}$$
is a subspace of V^*.
2. Determine $\dim U^\perp$.
3. Is $\{v^* \in V^* \mid U = \mathrm{kernel}\, v^*\}$ also a subspace?

$U^\perp$ is called orthogonal space or annihilation of U.

1.
$$U^\perp = \{v^* \in V^* \mid U \subseteq \mathrm{kernel}(v^*)\}$$

$v^* \in \mathrm{Hom}(V, \mathbb{K})$.
$$\mathrm{kernel}(v^*) = \{x \in V \mid v^*(x) = 0\} \supseteq U \Leftrightarrow \forall x \in U : v^*(x) = 0$$

$\mathbf{U^\perp}$ is nonempty
The constant zero-function $u : V \to \mathbb{K}$ with $x \mapsto 0 \in U^\perp$ exists. Hence $U^\perp \neq \emptyset$.

Additivity: $\bigwedge_{\mathbf{u_1}, \mathbf{u_2} \in \mathbf{U^\perp}} \mathbf{u_1} + \mathbf{u_2} \in \mathbf{U^\perp}$
Let $u_1, u_2 \in U^\perp$ be linear. Let $x \in U$.

$$(u_1 + u_2)(x) = \underbrace{u_1(x)}_{\in U^\perp} + \underbrace{u_2(x)}_{\in U^\perp} = 0 + 0 = 0$$

Multiplication: $\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{\mathbf{u} \in \mathbf{U^\perp}} \lambda \cdot \mathbf{u} \in \mathbf{U^\perp}$
Let $\lambda \in \mathbb{K}$, $u \in U^\perp$ and $x \in U$.

$$(\lambda \cdot u)(x) = \lambda \cdot \underbrace{u(x)}_{\in U^\perp} \Rightarrow \lambda \cdot 0 = 0$$

2.
$$\dim V = n \qquad \dim V^* = n \qquad \dim U = m$$

U is subspace of V, so $m \leq n$.
$$k := \dim U^\perp \leq n = \dim V^*$$

Let $(u_1, \ldots, u_m)$ be basis of U.

We apply the *basis extension theorem*: Let $(u_1, \ldots, u_m, u_{m+1}, \ldots, u_n)$ be a basis of V.

Let $(v_1^*, \ldots, v_n^*)$ the dual basis to $(v_1, \ldots, v_n)$ to V^*. Hence

$$v_1^*(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Claim: $U^\perp = L(\{v_{m+1}^*, \ldots, v_n^*\}) \Rightarrow (v_{m+1}^*, \ldots, v_n^*)$ is basis of $U^\perp \Rightarrow \dim U^\perp = n - m$.
Let $v \in V^*$ be arbitrary, $v = \lambda_1 v_1^* + \ldots + \lambda_n v_n^*$.

$$v \in U^\perp \Leftrightarrow \forall x \in U : v(x) = 0 \Leftrightarrow v|_U = 0 \overset{(u_1,\ldots,u_m) \text{ is basis of } U}{\Longleftrightarrow} v(u_i) = 0 \quad i = 1, \ldots, m$$

$$\begin{aligned} &\Leftrightarrow \forall i \in \{1, \ldots, m\}\, (\lambda_1 v_1^* + \ldots + \lambda_n v_n^*)(v_i) = 0 \\ &\Leftrightarrow \forall i \in \{1, \ldots, m\}\, v_1 v_1^*(v_i) + \ldots + \lambda_n v_n^*(v_i) = 0 \\ &\Leftrightarrow v^k \in L(v_{m+1}^*, \ldots, v_n^*) \\ &\Leftrightarrow \forall i \in \{1, \ldots, m\}\, \lambda_i = 0 \end{aligned}$$

$$\begin{aligned} \pi &: V \to V/U \\ x &\mapsto v + U \\ \pi^t &: (V/U)^* \to V^* \\ w &\to w \circ \pi \end{aligned}$$

π surjective, then π^t is injective and

$$\text{image}(\pi^t) = U^t \Rightarrow V/U^k \to U^\perp$$

3. Is $\{v^* \in V^* \mid U = \text{kernel}\, v^*\}$ also a subspace?

   Counterexample: Let $u = \{0\}$ and $V \neq \{0\}$.

   $$\text{kernel}(v^*) = \{x \in V \mid x^*(x) = 0\} = \{0\} = U$$

   If it is a subspace, then the constant null function (which is the zero element of this set) must be contained. This is a contradiction to "only $x = 0$ maps to 0".

7 Exercise 8

Exercise 10. Let $\mathbb{R}[x]$ be the vector space of real polynomials. Show that the dimension of the dual space $\mathbb{R}[x]^*$ is overcountable.

Hint: Show that linear functionals $(\delta_t)_{t \in \mathbb{R}}$ defined as $\langle \delta_t, p(x) \rangle = p(t)$ (function application) is linear independent.

"In welchem Vektorraum leben wir?" (Florian Kainrath)

δ_t are linear maps.

$$\begin{aligned} \forall p \in \mathbb{R}[x] : \sum_{i=1}^n \lambda_t \delta_{t_i}(p(x)) = 0 \Rightarrow \lambda_i = 0 \forall i \in \{1, \ldots, n\} \\ \forall p \in \mathbb{R}[x] : \sum_{i=1}^n \lambda_t p(t_i) = 0 \Rightarrow \lambda_i = 0 \end{aligned}$$

Consider the polynomial $(x - t_1)(x - t_2) \ldots (x - \hat{t}_j)(x - t_{j+1}) \ldots (x - t_n) = p(x)$.

$$\Rightarrow \sum_{i=1}^n \lambda_i p_j(t_i) = 0 \Leftrightarrow \lambda_j p_j(t_j) = 0 = \lambda_j = 0$$

8 Exercise 9

Exercise 11. Let $f \in \text{Hom}(V, W)$ be a linear map between two finite-fimensional vector spaces with bases $B \subseteq V$ and $C \subseteq W$. Show that the matrix representation of the transposed map

$$f^t : W^* \to V^*$$

$$w^* \mapsto w^* \circ f$$

in regards of the dual basis C^* and B^* has the matrix representation

$$\Phi_{B^*}^{C^*}(f^t) = \Phi_C^B(f)^t$$

Show that $f \in \text{Hom}(V, W)$ and $B = (b_1, \ldots, b_m)$ is basis of V with dual basis $B^* = (b_1^*, \ldots, b_m^*)$. $C = (c_1, \ldots, c_n)$ is basis of W with dual basis $C^* = (c_1^*, \ldots, c_n^*)$.

$$\Phi_{B^*}^{C^*}(f^t) = \Phi_C^B(f)^t$$

$$A := \Phi_C^B(f)$$

$\Phi_{B^*}^{C^*}(f^t) = P = A^t \forall i \in \{1, \ldots, n\}\ j \in \{1, \ldots, m\}$ and $a_{ij} = p_{ji}$. $A \in \mathbb{K}^{n \times m}$ and $P \in \mathbb{K}^{m \times n}$.

$$(a_{ij}) = A = \Phi_C^B(f) \Leftrightarrow \forall j \in \{1, \ldots, m\}$$

$$\Phi_C(f(b_j)) = A\Phi_B(b_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \Leftrightarrow A = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \Phi_C^{-1}$$

$$f(b_j) = \sum_{i=1}^{n} a_{ij} c_i \qquad \forall j \in \{1, \ldots, m\}$$

$$(p_{ij}) = p = \Phi_{B^*}^{C^*}(f^t) \Leftrightarrow f^t(c_j^*) = \sum_{i=1}^{m} p_{ij} b_i^* \forall j \in \{1, \ldots, n\}$$

$$\Leftrightarrow f^t(c_j^*) \text{ with } j \in \{1, \ldots, n\} = \sum_{i=1}^{m} p_{ij} b_i^* \overset{w}{\Leftrightarrow} c_i \circ f = \sum_{i=1}^{m} p_{ij} b_i^* \forall j \in \{1, \ldots, n\}$$

Show that $a_{kj} = p_{ik}$ with $k \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.

$$a_{kj} = C_k^* \left(\sum_{i=1}^{n} a_{ij} c_i \right) = c_k^* \left(f(b_j) \right) = \left(f^t(c_k^*)(b_j) \right) = \left(\sum_{i=1}^{m} p_{ik} b_i^* \right) (b_i) = p_{jk}$$

9 Exercise 10

Exercise 12.

- Determine the dual basis of $(\mathbb{R}^4)^*$ to the basis.

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- Determine the matrix of the unique (why?) projection map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ with image$(\varphi) = \mathcal{L}\left\{(1,2,1,0)^t, (1,0,-1,1)^t\right\}$ and kernel$(\varphi) = \mathcal{L}\left\{(-1,-2,2,-1)^t, (2,-1,1,1)^t\right\}$.

9.1 Exercise 10.a

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -3 & 1 & 2 & 5 \\ 0 & 1 & 0 & 0 & -9 & 2 & 5 & 15 \\ 0 & 0 & 1 & 0 & -5 & 1 & 3 & 8 \\ 0 & 0 & 0 & 1 & 4 & -1 & -2 & -6 \end{pmatrix}$$

So

$$b_1^* = \begin{pmatrix} -3 \\ 1 \\ 2 \\ 5 \end{pmatrix} \quad b_2^* = \begin{pmatrix} -9 \\ 2 \\ 5 \\ 15 \end{pmatrix} \quad b_3^* = \begin{pmatrix} -5 \\ 1 \\ 3 \\ 8 \end{pmatrix} \quad b_4^* = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -6 \end{pmatrix}$$

$$B^* = \begin{pmatrix} -3 & 1 & 2 & 5 \\ -9 & 2 & 5 & 15 \\ -5 & 1 & 3 & 8 \\ 4 & -1 & -2 & -6 \end{pmatrix}$$

$$(\mathbb{R}^n)^* \cong \mathbb{R}^{1\times 4}$$
$$b_i^*(b_j) = \delta_{ij}$$

9.2 Exercise 10.b

Find a projective map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $U_1 = \varphi(\mathbb{R}^4)$. So image$(\varphi) = \mathcal{L}(U_1)$ and kernel$(\varphi) = U_2$.

$$U_1 = \mathcal{L}\left\{(1,2,1,0)^t, (1,0,-1,1)^t\right\}$$
$$U_2 = \mathcal{L}\left\{(-1,-2,2,-1)^t, (2,-1,1,1)^t\right\}$$

Why do we get a unique map?

φ is a projection map iff φ is linear and $\varphi \circ \varphi = \varphi$. Consider $b_1 \in U_1 = \varphi(\mathbb{R}^4)$ and $b_1 = \varphi(x) \quad x \in \mathbb{R}^4$. $\varphi(b_1) = \varphi(\varphi(x)) = \varphi(x) = b_1$. This isomorphism ensures that the solution is unique.

Because $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$, the linear map will be represented by a 4×4 matrix.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ -1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -12 & -6 & 6 & -9 \\ 0 & 1 & 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 7 & 4 & -3 & 5 \\ 0 & 0 & 0 & 1 & 20 & 10 & -10 & 15 \end{pmatrix}$$

$$\begin{pmatrix} -12 & 3 & 7 & 20 \\ -6 & 2 & 4 & 10 \\ 6 & -1 & -3 & -10 \\ 9 & 2 & 5 & 15 \end{pmatrix}$$

10 Exercise 11

Exercise 13. Given the permutation

$$\pi = \left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}\right)$$

- Determine π^{-1} and π^k for some $k \in \mathbb{N}$.
- Determine all inversions of π and determine sign(π).

- Decompose π in a product of transpositions.

10.1 Exercise 11.a

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}$$
$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$$

We give a recursive definition:

$$\pi^k_{(i)} = \begin{cases} \pi^{k \bmod 4}_{(i)} & i \in \{1,2,3,5\} \\ \pi^{k \bmod 3}_{(i)} & i \in \{4,6,7\} \end{cases}$$

10.2 Exercise 11.b

Inversions are:

$$f_\pi = \{(i,j) \,|\, i < j \wedge \pi(i) > \pi(j)\}$$
$$F_\pi = \{(1,3),(2,3),(2,5),(2,7),(4,5),(4,7),(6,7)\}$$

$$\text{sign}(\pi) = (-1)^f_\pi = -1$$

10.3 Exercise 11.c

$$\pi \circ \tau_{1,3} = (1\ 5\ 2\ 6\ 3\ 7\ 4)$$
$$\pi \circ \tau_{1,3} \circ \tau_{2,3} \circ \tau_{3,5} \circ \tau_{4,7} \circ \tau_{6,7} = \text{ id}$$
$$\pi = \tau_{6,7} \circ \tau_{4,7} \circ \tau_{3,5} \circ \tau_{2,3} \circ \tau_{1,3}$$

In terms of notation, remember:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix} \circ \tau_{i,j} = \begin{pmatrix} 1 & i & j & n \\ & \pi(j) & \pi(i) & \end{pmatrix}$$

11 Exercise 12

Exercise 14. A permutation $\pi \in \mathfrak{S}_n$ is called cyclic, if there exists some $k \geq 1$ and a sequence $i_1, i_2, \dots, i_k$ such that $\pi(i_j) = i_{j+1}$ for $1 \leq j \leq k-1$, $\pi(i_k) = i_1$ and $\pi(i) = i$ for $i \notin \{i_1, i_2, \dots, i_k\}$, hence

$$i_1 \to i_2 \to \dots \to i_k \to i_1$$

and all other i are fixed. Common notation: $\pi = (i_1, i_2, \dots, i_k)$.

- Show that two cyclic permutations $\pi = (i_1, i_2, \dots, i_k)$ and $\rho = (j_1, j_2, \dots, j_l)$ commute ($\pi \circ \rho = \rho \circ \pi$) if $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_l\} = \emptyset$.
- Decompose the cycle into a product of transpositions and show that for a cyclic permutation it holds that $\text{sign}(\pi) = (-1)^{k-1}$.

11.1 Exercise 12.a

Case 1: $m \in \{i_1, i_2, \ldots, i_k\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = \pi(m)$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = \pi(m)$$

Case 2: $m \in \{j_1, j_2, \ldots, j_l\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = \rho(m)$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = \rho(m)$$

Case 3: $m \notin \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = m$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = m$$

11.2 Exercise 12.b

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_2 & i_3 \ldots & i_1 & \ldots & n \end{pmatrix}$$
$$\pi \circ \tau_{i_1, i_k} = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_1 & i_3 \ldots & i_2 & \ldots & n \end{pmatrix}$$
$$\pi \circ \tau_{i_1, i_k} \circ \tau_{i_2, i_k} = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 & i_3 & \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_1 & i_2 & i_4 & \ldots & i_3 & \ldots & n \end{pmatrix}$$
$$\tau \circ \tau_{i_1, i_k} \circ \tau_{i_2, i_k} \circ \ldots \circ \tau_{i_{k-1}, i_k} = \mathrm{id}$$
$$\pi = \tau_{i_{k-1}, i_k} \circ \ldots \circ \tau_{i_l, i_{l+1}} \circ \ldots \circ \tau_{i_1, i_k}$$

11.3 Exercise 13

Exercise 15. Let $\pi \in \mathfrak{S}_n$ be a permutation and $i \in \{1, 2, \ldots, n\}$.

- Show that the sequence i, $\pi(i)$, $\pi^2(i)$, … is periodic and the first number which occurs twice is i.
- The sequence $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$ where k is the smallest exponent such that $\pi^k(i) = i$, is called cycle of i. Show that the relation, $i \sim j :\Leftrightarrow j$ is in cycle of i, is a equivalence relation in $\{1, 2, \ldots, n\}$.
- Show that every permutation can be represented as product of commutative cycles.
- Apply this decomposition for the permutation π from exercise 11.

11.4 Exercise 13.a

- i, $\pi(i)$, …, $\pi^k(i)$ is periodic.
- the first element which occurs twice is i
- $$\left\{ \pi^k(i) \,\middle|\, k \in \{1, \ldots, n+1\} \right\}$$
  at least one elemtn must have occured twice.
- $$\pi^k(i) = \pi^l(i)$$
  wlog. $k > l$
  $$\pi^{k-l}(i) = i \qquad k - l < k$$

$$\pi^{k-l}(i) = (\pi^l)^{-1} \left(\pi^k(\tau) \right) = (\pi^e)^{-1} (\pi^e(i))$$

11.5 Exercise 13.b

reflexive

$$i \sim i \quad \Leftrightarrow \quad \exists k : \pi^k(i) = i$$

symmetrical

$$i \sim j \Rightarrow j \sim i \qquad \exists l : \pi^l(i) = j \quad \pi^k(i) = i \quad \pi^{k-l}(i) = i$$

transitive

$$i \sim j \wedge j \sim m \Rightarrow i \sim m \qquad (\exists l_1 : \pi^{l_1}(i) = j) \wedge (\exists l_2 : \pi^{l_2}(j) = m)$$
$$\Rightarrow \exists l_3 = l_1 + l_2 : \pi^{l_3}(i) = m$$

11.6 Exercise 13.c

Lengthy and therefore skipped.

11.7 Exercise 13.d

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}$$
$$\pi = (1\ 2\ 5\ 3)(4\ 6\ 7)$$

12 Exercise 14

Exercise 16. Determine the determinant of the following matrix using three different methods (Leibniz, Laplace, Gauß-Jordan).

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

Using Leibniz' definition:

$$\det(A) = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

Using Gauß' definition:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -5 & -4 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

Using Leibniz' definition:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix} = 1 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 2 + 3 \cdot 1 \cdot (-1) - 2 \cdot 1 \cdot 3 - (-1) \cdot 2 \cdot 1 - 2 \cdot 1 \cdot 2 = -1$$

13 Exercise 15

Exercise 17. The numbers 18984, 10962, 40026, 17976 and 14994 are divisible by 42. Show that the determinant of A is divisible by 42 without explicitly computing it.

$$A = \begin{pmatrix} 1 & 8 & 9 & 8 & 4 \\ 1 & 0 & 9 & 6 & 2 \\ 4 & 0 & 0 & 2 & 6 \\ 1 & 7 & 9 & 7 & 6 \\ 1 & 4 & 9 & 9 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 8 & 9 & 8 & 4 \\ 1 & 0 & 9 & 6 & 2 \\ 4 & 0 & 0 & 2 & 6 \\ 1 & 7 & 9 & 7 & 6 \\ 1 & 4 & 9 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 9 & 8 & 18984 \\ 1 & 0 & 9 & 6 & 10962 \\ 4 & 0 & 0 & 2 & 40026 \\ 1 & 7 & 9 & 7 & 17976 \\ 1 & 4 & 9 & 9 & 14994 \end{vmatrix} = 42 \cdot B$$

where B is some matrix with modified 5-th column.

Why does this work? Well, this can be proven using Leibniz' definition of the determinant.

$$\det((a_{ij})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_1 \dots$$

14 Exercise 16

Exercise 18. Compute the $n \times n$-determinants:

1.
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & 0 & 3 & 4 & \dots & n-1 & n \\ -1 & -2 & 0 & 4 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & -4 & \dots & 0 & n \\ -1 & -2 & -3 & -4 & \dots & -n+1 & 0 \end{pmatrix}$$

2.
$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & \dots & a_{n-1} & * \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & a_2 & * & \dots & * \\ a_1 & * & \dots & & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & 0 & 3 & 4 & \dots & n-1 & n \\ -1 & -2 & 0 & 4 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & -4 & \dots & 0 & n \\ -1 & -2 & -3 & -4 & \dots & -n+1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 2 & * & * & \dots & n-1 & n \\ 0 & 0 & 3 & * & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & 0 & n \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n = n!$$

$$\begin{vmatrix} 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & \dots & a_{n-1} & * \\ \vdots & & \vdots & \vdots & * \\ 0 & a_2 & * & \dots & * \\ a_1 & * & \dots & & * \end{vmatrix} = (-1)^k \begin{vmatrix} a_1 & * & \dots & * & a_n \\ 0 & a_2 & \dots & \iddots & * \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & & a_{n-1} & * \\ 0 & 0 & \dots & 0 & a_n \end{vmatrix} = \left(\prod_{k=1}^{n} a_k\right)(-1)^k$$

where $k = \frac{n}{2}$ is n is even or $k = \frac{n-1}{2}$ is odd.

15 Exercise 17

Exercise 19. Let $A \in \mathbb{K}_{m \times m}$, $B \in \mathbb{K}_{m \times n}$, $D \in \mathbb{K}_{n \times n}$ matrices. Show that,

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \cdot \det D$$

Let $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$

If A is singular, the rows are linear dependent. So $\det T = 0$. The same applies to D.

We apply row operations to A to retrieve an upper triangular matrix A_1. If we do the same operations on T, we get B_1. We apply row oeprations to D to retrieve an upper triangular matrix D_1.

$$\hat{T} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}$$

Let a be the product of diagonal elements of A_1. Let d be the product of diagonal elements of D_1.

So $a \cdot d$ is the product of diagonal elements of $\hat{T}$.

Let p be the number of swaps in A_1. Let q be the number of swaps in A_2.

$$p + q = \hat{T}$$

Then

$$\det A = (-1)^p a \qquad \det D = (-1)^q b$$
$$\det T = (-1)^{p+q} a \cdot b$$

16 Exercise 18

Exercise 20. Compute the entry $(A^{-1})_{4,3}$ of the inverse matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2 & -1 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We compute the inverse matrix A^{-1}.

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2 & -1 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 & -2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

But we can also use the Theorem from the lecture.

Use the adjoint matrix $\hat{A}$ of A where $\hat{a}_{kl} = (-1)^{k+l} \det A_{lk}$. Then $A^{-1} = \frac{1}{\det A} \cdot \hat{A}$.

$$A^{-1} = \frac{1}{\det A} \cdot \hat{A}$$
$$A^{-1}_{43} = \frac{1}{\det A} (-1)^{3+4} \det A_{3,4} = -1$$

But we can also determine it more easily. $(A^{-1})_{4,3}$ is the element in the 4th row and 3rd column. It is also the element in the 4-th row of $A^{-1}e_3$.

So

$$A_{e_4} = -e_3$$

So -1.

17 Exercise 19

Exercise 21. Let $\mathbb{K}$ be a field and $a_1, a_2, \ldots, a_n \in \mathbb{K}$. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{vmatrix} = \prod_{i<j} (a_j - a_i)$$

Proof by complete induction over n.

Induction base: $n = 0$ Empty product.

$$\begin{vmatrix} 1 \end{vmatrix} = 1$$

Is true.

Induction step: $n \to n+1$ We start from the last column and add it to the second from last row. This goes on for all columns.

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^n \\ \vdots & & & \ddots & \vdots \\ 1 & a_{n+1} & \ldots & \ldots & a_{n+1}^n \end{vmatrix} \stackrel{!}{=} \prod_{\substack{i,j=1 \\ j>i}} (a_j - a_i) \rightsquigarrow \begin{vmatrix} 1 & (a_1 - a_{n+1}) & \ldots & a_1^{n-1}(a_1 - a_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (a_n - a_{n+1}) & \ldots & a_n^{n-1}(a_n - a_{n+1}) \\ 1 & (a_{n+1} - a_{n+1}) & \ldots & a_n^{n-1}(a_{n+1} - a_{n+1}) \end{vmatrix}$$

$$= (-1)^{n+1+1}(a_1 - a_{n+1}) \cdot (a_2 - a_{n+1}) \ldots (a_n - a_{n+1}) \cdot \begin{vmatrix} 1 & a_1 & \ldots & a_1^{n-1} \\ \vdots & & \ddots & \vdots \\ 1 & a_n & \ldots & a_n^{n-1} \end{vmatrix}$$

$$\stackrel{\text{induction hypothesis}}{=} (a_{n+1} - a_1) \ldots (a_{n+1} - a_n) \cdot \prod_{\substack{i,j=1 \\ j>i}} (a_j - a_i) = \prod_{j,i=1}^{n+1} (a_j - a_i)$$

18 Exercise 20

Exercise 22. Let $A, B \in \mathbb{K}^{n \times n}$. Show that, using elementary row and column transformations, the following identity holds for block matrices.

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$$

Use this to derive an alternative proof for the multiplicity of the determinant.

$$\det(AB) = \det(A) \cdot \det(B)$$

18.1 Exercise 20.a

$$\begin{vmatrix} 1 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & 1 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & b_{n,1} & b_{n,2} & \dots & b_{n,n} \\ -a_{11} & -a_{12} & \dots & -a_{1n} & 0 & 0 & \dots & 0 \\ -a_{21} & -a_{22} & \dots & -a_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & -a_{n,n} & 0 & 0 & \dots & 0 \end{vmatrix}$$

Add the a_{11}-multiple of the first row to the $n+1$-th row. Add the a_{21}-multiple of the first row to the $n+1$-th row. Add the a_{n1}-multiple of the first row to the $2n$-th row.

$$\begin{vmatrix} 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & B \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & \\ & 0 & & & A \cdot B \end{vmatrix}$$

18.2 Exercise 20.b

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = (-1)^n \begin{vmatrix} B & I \\ 0 & -A \end{vmatrix} = (-1)^n \cdot \det B \cdot \det -A$$

We multiply n rows by -1,

$$= (-1)^n \cdot (-1)^n \cdot \det B \cdot \det A = \det A \cdot \det B$$

19 Exercise 21

Exercise 23. Let $A, B, C, D \in \mathbb{K}_{n \times n}$ be matrices where D is invertible. Let M be a $2n \times 2n$ block matrix.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

1. Show: M is invertible iff $A - BD^{-1}C) \det D$.
2. Show: $\det M = \det(A - BD^{-1}C) \cdot \det D$.

19.1 Exercise 21.b

$$\det M = \det(A - BD^{-1}C) \cdot \det(D)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det \left[\begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} A - BD^{-1} & 0 \\ D^{-1}C & 1 \end{pmatrix} \right] = \begin{vmatrix} 1 & B \\ 0 & D \end{vmatrix} \cdot \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{vmatrix}$$

$$= \det(1) \cdot \det(D) \cdot \det(A - BD^{-1}C)$$

$$= \det(D) \cdot \det(A - BD^{-1}C)$$

19.2 Exercise 21.b

M is invertible, so $A - BD^{-1}C$ is invertible. $\det(D) \neq 0$.

$$\begin{aligned}\det M \neq 0 &\Leftrightarrow \det(A - BD^{1}C) \cdot \det(D) \neq 0 \\ &\Leftrightarrow \det(A - BD^{-1}C) \neq 0\end{aligned}$$

Corollary of this exercise:

$$\det(AD - BC) = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

20 Exercise 22

Exercise 24. Let V be an n-dimensional vector space over a field $\mathbb{K}$ and $\triangle : V^n \to \mathbb{K}$ is a non-trivial determinant form. Furthermore let $a_1, a_2, \dots, a_{n-1} \in V$ vectors. Show that

- the following element is a linear functional with $\mathcal{L}(a_1, a_2, \dots, a_{n-1}) \subseteq \operatorname{kernel} v^*$

$$\begin{aligned} v^* : V &\to \mathbb{K} \\ x &\mapsto \triangle(a_1, a_2, \dots, a_{n-1}, x) \end{aligned}$$

- $\mathcal{L}(a_1, a_2, \dots, a_{n-1}) = \operatorname{kernel} v^*$ iff $a_1, a_2, \dots, a_{n-1}$ is linear independent.
- Determine the equation (hence, a linear functional v^* such that $\operatorname{kernel} v^* = U$)

$$U = \mathcal{L}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \right)$$

20.1 Exercise 22.a

1. Firstly,

$$\begin{aligned} v^*(x_1 + x_2) = v^*(x_1) + v^*(x_2) : v^*(x + 1 + x_2) &= \triangle(a_1, a_2, \dots, a_{n-1}, x_1 + x_2) \\ &= \triangle(a_1, \dots, a_{n-1}, x_1) + \triangle(a_1, \dots, a_{n-1}, x_2) \\ &= v^*(x_1) + v^*(x_2) \end{aligned}$$

   Secondly,

$$\begin{aligned} v^*(\lambda x_1) = \lambda v^*(x_1) : v^*(\lambda x_1) &= \triangle(a_1, a_2, \dots, a_{n-1}, \lambda x_1) \\ &= \lambda \triangle(a_1, \dots, a_{n-1}, x_1) \end{aligned}$$

$\mathcal{L}(a_1, \dots, a_{n-1}) \subseteq \operatorname{kernel}(v^*)$ is by definition $\triangle(a_1, \dots, a_n) = 0$ if $i, j \in \{1, \dots, n\}$ and $i \neq j$ and a_1 and a_j are linear independent.

$$\forall i \in \{1, \dots, n-1\} : \triangle(a_1, \dots, a_{n-1}, a_i) = 0$$

20.2 Exercise 22.b

First we show $\Leftarrow$.

Let $a_1, \dots, a_{n-1}$ be linear independent.

$$\mathcal{L}(a_1, \dots, a_{n-1}) \subseteq \operatorname{kernel}(v^*)$$

Assume kernel$(v^*) \supsetneq \mathcal{L}(a_1, \ldots, a_{n-1})$. So there exists $x \in$ kernel(v^*) with $x \notin \mathcal{L}(a_1, \ldots, a_{n-1})$. So $(a_1, \ldots, a_{n-1}, x)$ are linear independent. This forms a basis of V.

$$\triangle(a_1, \ldots, a_{n-1}, x) \neq 0 \Rightarrow v^*(x) \neq 0$$

This is a contradiction to our assumption that $x \in$ kernel(v^*).

Second we show $\Rightarrow$.

Proof by contradiction. Assume $\mathcal{L}(a_1, a_2, \ldots, a_{n-1}) =$ kernel v^* and $a_1, a_2, \ldots, a_{n-1}$ linear independent.

$$\triangle(a_1, \ldots, a_{n-1}, x) = 0 \qquad \forall x \in V$$

$\Rightarrow V - \mathcal{L}(a_1, \ldots, a_{n-1})$ is a contradiction to $\dim(K) = n$.

20.3 Exercise 22.c

Use the linear functional from exercise (a).

$$\begin{aligned} v^* : \mathbb{K}^4 &\to \mathbb{K} \\ x &\mapsto \det(a_1, a_2, a_3, x) \end{aligned}$$

$$v^* = \begin{vmatrix} 1 & -1 & 3 & x_1 \\ 2 & 2 & -1 & x_2 \\ 3 & 0 & 2 & x_3 \\ 1 & 0 & 1 & x_4 \end{vmatrix} = 2x_1 + x_2 + x_3 - 7x_4$$

21 Exercise 23

Exercise 25. Let $x, y, u, v \in \mathbb{R}^3$.

1. Show that the identity $\langle x \times y, u \times v \rangle = \langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle$
2. Conclude that
$$\|u\|^2 \|v\|^2 = \|u \times v\|^2 + \langle u, v \rangle^2$$
for arbitrary vectors $u, v \in \mathbb{R}^3$.

21.1 Exercise 23.a

Case 1: $u \times v = 0$ So u and v are linear dependent, so $\exists a \in \mathbb{R} : u = av$ or $v = au$. Without loss of generality: $u = av$ ($v = au$ analogously).

$$\begin{gathered} \langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle \\ = \langle x, u \rangle \langle y, av \rangle - \langle x, au \rangle \langle y, v \rangle = a \langle x, u \rangle \langle y, u \rangle - a \langle x, u \rangle \langle y, u \rangle \\ = 0 = \langle x \times y, 0 \rangle = \langle x \times y, u \times u \rangle \end{gathered}$$

Case 2: $u \times v \neq 0$

$$\langle x \times y, u \times v \rangle \langle u \times v, u \times v \rangle = \det(x|\, y\, |uxv) \cdot \det(u|\, v\, |u \times v) = \det(x|\, y\, |u \times v)^t \cdot \det(u|\, v\, |u \times v)$$

$$= \det \begin{pmatrix} x^t \\ y^t \\ (u \times v)^t \end{pmatrix} \cdot \det(u|\, v\, |u \times v) = \det \begin{pmatrix} x^t \\ y^t \\ (u \times v)^t \end{pmatrix} (u|\, v\, |u \times v)$$

$$= \det\begin{pmatrix} xv & x^t v & x^t(u\times v) & \\ yv^t & y^t v & y & t(u\times v) \\ (u\times v)^t\cdot v & (u\times v)^t\cdot v & (u\times v)^t(u\times v) & \end{pmatrix} = \det\begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle & \langle x,u\times v\rangle \\ \langle y,u\rangle & \langle y,v\rangle & \langle y,u\times v\rangle \\ \langle u\times v,v\rangle & \langle u\times v,v\rangle & \langle u\times v,u\times v\rangle \end{pmatrix}$$

$$\langle u\times v,u\times v\rangle\cdot(\langle x,u\rangle\langle y,v\rangle-\langle x,v\rangle\langle y,u\rangle)$$

$$\Rightarrow \langle x\times y,u\times v\rangle = \langle x,u\rangle\langle y,v\rangle-\langle x,v\rangle\langle y,u\rangle$$

21.2 Exercise 23.b

$$\begin{aligned}\|u\times v\|^2 &= \langle u\times v,u\times v\rangle = \langle u,u\rangle\langle v,v\rangle-\langle u,v\rangle\langle v,u\rangle \\ &= \|u\|^2\|v\|^2-\langle u,v\rangle^2\end{aligned}$$

$$\Rightarrow \|u\times v\|^2+\langle u,v\rangle^2 = \|u\|^2\cdot\|v\|^2$$

22 Exercise 24

Exercise 26.
- Show: A (Hermitian) matrix A is positive semi-definite if and only if there exists some matrix B such that $A = B^*B$. Which property must B have such that A is positive definite?
- Let A be positive definite. Show that A^{-1} is also positive definite.
- Let A be positive semi-definite. Show that $a_{ii} \geq 0$ for all i and if for some i with diagonal value $a_{ii} = 0$ holds, then $a_{ji} = 0$ for all j.
- Does the following variation of the generalized Sylvester's criterion hold?
  "An $n\times n$ matrix A is positive semidefinite iff $\det A_r \geq 0$ for all $r = 1,2,\ldots,n$"

22.1 Exercise 24.a

Prerequisite: $(C^*)^{-1} = (C^{-1})^*$.

First direction: $\Rightarrow$.
Let A be Hermitian, show that $A \geq 0 \Leftrightarrow \exists B : A = B^*B$. So $A \geq 0$.

$$\exists C \in \mathrm{GL}(\mathbb{K}) : C^*AC = D \Leftrightarrow A = (C^*)^{-1}DC^{-1} \Leftrightarrow A = (C^{-1})^*D^2C^{-1}$$

holds because $D = D^2$.

$$\Leftrightarrow A = ((C^{-1})^*D)(DC^{-1}) = \tilde{C}^*\tilde{C}$$

Second direction: $\Leftarrow$.
$A = B^*B$. Let $x \in \mathbb{K}^n$:

$$\begin{aligned} x^tA\overline{x} &= x^tB^*B\overline{x} = (\overline{B}x)^tB\overline{x} = (\overline{B}x)^t\overline{(\overline{B}x)} \\ &= \langle\overline{B}x,\overline{B}x\rangle_2 \geq 0\end{aligned}$$

If $\mathrm{rank}(B) = n$, then $A = B^*IB \Leftrightarrow A \cong I$. If $A > 0 \Rightarrow B \in \mathrm{GL}_n(\mathbb{K})$.

$$\Leftarrow x \in \mathbb{K}^k \setminus \{0\} \Rightarrow \overline{\beta}x \neq 0 \Rightarrow \langle\overline{B}x,\overline{B}x\rangle_2 \Rightarrow x^tA\overline{x} > 0 \Rightarrow A > 0$$

22.2 Exercise 24.b

Let $A > 0$. Then $\exists C \in \mathrm{GL}_n(\mathbb{K})$:

$$A = C^*C \Leftrightarrow A^{-1} = C^{-1}(C^*)^{-1} = C^{-1}(C^{-1})^*$$

Let $B = (C^{-1})^* \Rightarrow A^{-1} = B^*B \Rightarrow A^{-1} > 0$ (because $\mathrm{rank}(B) = \max$)

22.3 Exercise 24.c

Let $A \geq 0$. Show $a_{ii} \geq 0 \quad \forall i = 1, \ldots, n$. Assume $\exists a_{ii} < 0$,

$$\forall \xi \in \mathbb{C}^n \setminus \{0\} : \xi^T A \xi \geq 0$$

Let $\xi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$$\Rightarrow \xi^T A \xi = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{ii} \\ \vdots \\ a_{in} \end{pmatrix}^T \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = a_{ii} < 0$$

Second part: A must be Hermitian.

Let $a_{ii} = 0$ and assume $a_{ij} \neq 0$.

$$\xi^T A \overline{\xi} \geq 0 \forall \xi \in \mathbb{C}^n$$

Consider

$$\xi = \begin{pmatrix} 0 \\ \vdots \\ c \\ \vdots \\ 1 \end{pmatrix} = ce_i + e_j$$

where c is in the i-th row and 1 is in the j-th row.

$$\delta^T A \overline{\delta} = \left\langle ce_i + e_j, ce_i + e_j \right\rangle_A = \langle ce_i, ce_i \rangle + \left\langle ce_i, e_j \right\rangle + \left\langle e_j, ce_i \right\rangle + \left\langle e_j, e_j \right\rangle = c\overline{c}a_{ii} + ca_{ij} + \overline{c}a_{ji} + a_{jj}$$

$$2\Re(ca_{ij})xa_{jj} \geq 0$$
$$c = -\overline{a}_{ij}$$
$$2\Re(-| a_{ij} |^2) + a_{jj}$$

As it turns out this approach should be started differently. We therefore exchange i and j.

$$= c\overline{c}a_{ii} + ca_{ij} + \overline{c}a_{ji}$$
$$= c\overline{c}a_{ii} + c\overline{a_{ji}} + \overline{c}a_{ji}$$
$$= c\overline{c}a_{ii} + 2\Re(\overline{c}a_{ji}) \geq 0$$

Goal: c cancels itself out.

Does not work out. According to Mr. Kainrath we need to choose:

$$c \in \mathbb{C}$$
$$d \in \mathbb{R} \setminus \{0\}$$
$$\xi = ce_i + de_j$$

22.4 Exercise 24.c: fixed proof

Assume $a_{ii} = 0$. By $A = B^*B$ with $B \in \mathbb{K}^{n\times n}$.

$$
\begin{aligned}
0 = a_{ii} = e_i^t A e_i = e_i^t B^t B e_i = (\overline{B}e_i)^T \overline{(\overline{B}e_i)} \\
= \langle \overline{B}e_i, \overline{B}e_i \rangle_2 = \left\|\overline{B}e_i\right\|_2^2 \Rightarrow \overline{B}e_i = 0 \\
a_{ij} = e_i A e)j^T = e_i B^t B e_j^t = (\overline{B}e_i)^t B e_j^t = 0 \cdot B e_j^t = 0
\end{aligned}
$$

22.5 Exercise 24.d

Wrong.

Assume $\det(Ar) \geq 0 \quad \forall r = 1, \dots, n \Rightarrow A \geq 0$.

$$
A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \qquad x = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x^T A x = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1 < 0
$$

This is a contradiction.

However, direction $\Rightarrow$ holds.

22.6 Exercise 25

Exercise 27. Show that the relation $A \leq B \Leftrightarrow B - A \geq 0$ (hence $B - A$ is positive semidefinite) defines an order relation on the set of self-adjoint matrices.

22.7 Reflexivity

Show: $A \leq A$.

$$
\begin{gathered}
A - A \stackrel{?}{\geq} 0 \\
0 \geq 0
\end{gathered}
$$

The bilinear form by the zero-matrix maps every value to 0. So this holds.

22.8 Anti-symmetry

Show: $A \leq B \land B \leq A \Rightarrow A = B$.

$$
B - A \geq 0 \land A - B \geq 0
$$

From $A - B \geq 0$ it follows that $B - A \leq 0$. From that $B - A = 0$ follows.

In more detail: I know $\forall x \in \mathbb{K}^n : x^t C \overline{x} \geq 0$ and $x^t C \overline{x} \leq 0$. Then $\forall x \in \mathbb{K}^n : x^t C \overline{x} = 0$. From exercise 24 it follows that $C = 0$.

22.9 Transitivity

Show: $A \leq B \land B \leq C \Rightarrow A \leq C$.

$$
\begin{aligned}
A \le B &\Leftrightarrow B - A \ge 0 \\
B \le C &\Leftrightarrow C - B \ge 0
\end{aligned}
$$

$$\forall x \in \mathbb{K}^n : x^t(C-A)\overline{x} \ge 0$$

Let $x \in \mathbb{K}^n$:

$$x^t(C-A)\overline{x} = x^t(C-B+B-A)\overline{x} = \underbrace{x^t(C-B)\overline{x}}_{\ge 0, \in \mathbb{R}} + \underbrace{x^t(B-A)\overline{x}}_{\ge 0} \ge 0$$

23 Exercise 26

Exercise 28. Determine index and signature of the matrix A such that $C^*AC = D$:

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

D is a diagonal matrix with entries in $\{+1, -1, 0\}$ and a basis B of $\mathbb{R}^4$ such that $x^t Ay = \Phi_B(x)^t$

Is certainly not positive semidefinite. Compare with Exercise 26 (c). So at least one -1 must occur in our result.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = A_1$$

$$C_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} = A_2$$

Don't forget to also apply the column transformations as well. This algorithm is based on quadratic extension. The number of iteration depends on the transformation you choose.

$$C = C_1 \cdot C_2 \cdot \ldots \cdot C_8$$

$$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{sig} = 1$$

Determine basis of $\mathbb{R}^4$: $x^t Ay = \Phi_B(x) \cdot D \cdot \Phi_B(y)$.

$$C^t AC = D \Leftrightarrow A = (C^{-1})^t DC^{-1}$$

$$x^t Ay = x^t (C^{-1})^t DC^{-1} y = (C^{-1} x)^t DC^{-1} y \underbrace{=}_{\Phi_B(x)=B^{-1}x} \Phi_C(x)^t \cdot D \cdot \Phi_C(y)$$

We will also consider other methods in advanced courses, but keep in mind those methods require the roots of polynomials, which are not always possible to determine.

24 Exercise 27

Exercise 29. Show: The block matrix A is positive definite iff $I - B^* B$ is positive definite.

$$A = \begin{pmatrix} I & B \\ B^* & I \end{pmatrix}$$

$$I, B, B^* \in \mathbb{K}^{n \times n} \qquad \Rightarrow A \in \mathbb{K}^{2n \times 2n}$$

Consider

$$C = \begin{pmatrix} I & 0 \\ -B^* & I \end{pmatrix} \qquad \text{and} \qquad C^* = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$$

$$\Rightarrow C^* AC = \begin{pmatrix} I - BB^* & 0 \\ 0 & I \end{pmatrix} =: M$$

Hence A is congruent to M (according to lecture, reference 8.23) with $A \hat{=} M$.

Then it holds that (lecture, reference 8.29)

$$A > 0 \overset{(ii)}{\Leftrightarrow} \text{index}(A) = 2n \overset{(ii)}{\Leftrightarrow} \text{index}(M) = 2n \,\text{(iii)} \Leftrightarrow M > 0$$

Hence, it suffices to show: $M > 0 \Leftrightarrow I - BB^* > 0$.

Direction $\Rightarrow$. Let $M > 0$. Then the leading minor theorem yields (lecture reference 8.32)

$$M > 0 \Leftrightarrow \det M_r > 0 \qquad \forall r = \{1, \ldots, 2n\}$$

Especially it holds that

$$\det(M_r) = \det((I - BB^*)_2) \qquad r = 1, \ldots, n$$

$$\overset{8.32}{\Leftrightarrow} I - BB^* > 0$$

Direction $\Leftarrow$. Let $I - BB^* > 0$. Then by 8.29 (iii) it holds that

$$\text{ind}(I - BB^*) = n$$

$$\text{ind}(I) = n$$

$$\Rightarrow \text{ind}(M) = 2n$$

because the index corresponds to the number of ones.