Introduction to Functional Analysis

Lecture notes, University of Technology, Graz
based on the lecture by Martin Holler

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Contents

0 Introduction

↓ *This lecture took place on 2019/03/05.*

- Function Analysis, mostly Linear Functional Analysis

- Goal: Transfer objects and results for linear algebra and analysis to infinite-dimensional function spaces
- e.g. $\mathbb{R}^n, \mathbb{C}^n \mapsto$ vector spaces U, V
  matrices $A \in \mathcal{M}^{n \times m} \mapsto$ operators $A \in \mathcal{L}(U, V)$
  functions $f : \mathbb{R}^n \to \mathbb{R} \mapsto$ functionals $f : U \to \mathbb{R}$
- Furthermore we discuss inner products, orthogonality, connectedness, eigenvalues
- Fields of application
  - basis of Applied Mathematics
  - partial differential equations
  - physical modelling
  - inverse problems (operator A models some physical measurement process)
  - Optimization and optimal control

A motivating example was presented with slides.

0.1 Application examples

Let $K : U \to \mathbb{R}^m$ with U as vector space describe a physical model. For example, K is a Fourier/Radon/X-ray transform (MR/CT/PET imaging) or $Ku = y(1)$ where $y : [0, 1] \to \mathbb{R}^m$ solves $y'(t) = y(t) + u(t)$ and $y(0) = 0$.

Another example is the class of so-called *inverse problems*. Given $d = ku$, find u. Typically inversion of K is ill-constrained. Solution is typically non-unique.

Approach: Solve $\min_{u \in U} \lambda \left\|Ku - d\right\|_2 + \left\|u\right\|_k$ where $\left\|z\right\|_2 := \sqrt{\sum_{i=1}^n z_i^2}$ and $\left\|\cdot\right\|_u$ is a norm on U. Or alternatively, let $U = C^1([0, 1]^2)$ and solve $\min_{u \in U} \lambda \left\|ku - d\right\|_2 + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 \, dx}$.

Other examples are JPEG compression and upsampling of images.

0.2 Our first problem

Let $U := C^1([0, 1]^2)$ be a normed space, $K : U \to \mathbb{R}^m$ linear. Solve $\min_{u \in U} \lambda \left\|Ku - d\right\| + \sqrt{\int_{[0,1]^2} \left|\nabla u(x)\right|^2 \, dx}$. The question is: does such a solution exist?

We have a background in finite-dimensional vector spaces. We consider a special case to apply the theories we already know.

So we consider a discrete setting. Let $U : \mathbb{R}^n$ and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ is a discrete gradient. In 1D, we have $u = (u_i)_i \in \mathbb{R}^m$ and $u_i = u(x_i) \implies u' \approx u(x_{i+1}) - u(x_i) = u_{i+1} - u_i$. Consider $\min_{u \in \mathbb{R}^n} \left\|\nabla u\right\|_2 + \lambda \left\|Ku - d\right\|_2$ as problem.

Does there exist a solution to this problem assuming $\lambda > 0$, $K : \mathbb{R}^n \to \mathbb{R}^m$ linear and $\nabla : \mathbb{R}^n \to \mathbb{R}^k$ linear.

Proof. **Case 1 (trivial model):** Let $m = n$. $K_n = u$

$$\min_{u \in \mathbb{R}^n} \|\nabla u\|_2 + \lambda \|u - d\|_2 \tag{1}$$

Take $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^n$ such that $\lim_{n\to\infty} \|\nabla u_1\|_2 + \lambda \|u_n - d\|_2 = \inf_{u\in\mathbb{R}} \|\nabla u\|_2 + \lambda \|u - d\|_2$. It holds that $C = \lambda \|d\|_2 \geq \inf_{u\in\mathbb{R}} \|\nabla u\|_2 + \lambda \|d\|_2$. Without loss of generality, we can assume that $2C \geq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 \, \forall n \in \mathbb{N}$

$$\implies \lambda \|u_1\|_2 \leq \lambda \|u_n - d\|_2 + \lambda \|d\|_2 \leq \|\nabla u_k\|_2 + \lambda \|u_n - d\|_2 - \lambda \|d\|_2 \leq 2C + \lambda \|d\|_2$$

$(\|u_n\|_2)_n$ is bounded. So the Bolzano-Weierstrass theorem applies and $(u_n)_{n\in\mathbb{N}}$ admits a convergent subsequence $(u_{n_i})_{i\in\mathbb{N}}$. Take $u \in \mathbb{R}^n$. $u_{n_i} \to u$ as $i \to \infty$.

Now: Show that u solves Problem (1). ∇ is continuous. $\|\cdot\|_2$ is continuous.

$$\inf_{u\in U} \|\nabla u\|_2 + \lambda \|u - d\|_2 = \lim_{i\to\infty} \left\|\nabla u_{n_i}\right\| + \lambda \left\|u_{n_i} - d\right\|_2 = \|\nabla \hat{u}\|_2 + \lambda \|\hat{u} - d\|_2$$

This implies that $\hat{u}$ is the solution to the problem of this first case.

Ingredients of this proof where:

- boundedness
- compactness
- continuity of ∇, $\|\cdot\|_2$

Case 2 (K arbitrary): 1. K arbitrary does not provide boundedness anymore. Define $X := \text{kernel}(\nabla) \cap \text{kernel}(k)$ and

$$X^{\perp} := \left\{ x \in \mathbb{R}^n \mid (x, y) := \sum_{i=1}^{n} x_i y_i = 0 \forall y \in X \right\}$$

Then we apply results from linear algebra:

$$\mathbb{R}^n : X \oplus X^{\perp} \qquad \text{i.e. } \forall u \in \mathbb{R}^n : \exists! u_1 \in X, u_2 \in X^{\perp} : u = u_1 + u_2$$

Recall, that $X^{\perp}$ is called *orthogonal complement.*

Claim 0.1. *If $\hat{u}$ solves $\min_{u\in X^{\perp}} \|\nabla u\|_2 + \lambda \|Ku - d\|_2$. Then $\hat{u}$ solves Problem* (1).

Proof. Let $\hat{u}$ be a solution on $X^{\perp}$. Take $u \in \mathbb{R}^n$ arbitrary. We write $u = u_1 + u_2 \in X \times X^{\perp}$. Now we have:

$$\begin{aligned} \|\nabla u\|_2 + \lambda \|ku - d\|_2 &= \left\|\nabla(u_1 + u_2)\right\|_2 + \lambda \left\|k(u_1 + u_2) - d\right\|_2 \\ &= \|\nabla u_2\|_2 + \lambda \|ku_2 - d\|_2 \\ &\geq \|\nabla \hat{u}\|_2 + \lambda \|K\hat{u} - d\|_2 \end{aligned}$$

Thus $\hat{u}$ solves our problem (1). □

Take again $(u_n)_{n\in\mathbb{N}}$ be such that $u_n \in X^\perp \forall n$ and

$$\lim_{n\to\infty} \|\nabla u_n\|_2 + \lambda \|ku_n - d\|_2 = \inf_{u\in X^\perp} \|\nabla_u\|_2 + \lambda \|ku - d\|_2$$

Write $u_1 = u_n^1 + u_n^2 \in \text{kernel}(\nabla) + \text{kernel}(\nabla)^\perp$. $\nabla : \text{kernel}(\nabla)^\perp \to \text{image}(\nabla)$ is bijective. Since $\nabla v = 0$ for $v \in \text{kernel}(\nabla)^\perp \implies v \in \text{kernel}(\nabla) \implies \|v_2\| = (v, v) = 0$. Thus, $\nabla^{-1} : \text{image}(\nabla) \to \text{kernel}(\nabla)^\perp$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^2\right\|_2 &= \left\|\nabla^{-1}\nabla u_n^2\right\|_2 = \left\|\nabla^{-1}\right\| \cdot \left\|\nabla u_n^2\right\|_2 \leq \left\|\nabla^{-1}\right\| \\
&\leq \left\|\nabla^{-1}\right\| \left(\left\|\nabla u_n^2\right\|_2 + \lambda \|Ku_n - d\|_2\right) \\
&= \left\|\nabla^{-1}\right\| \left(\underbrace{\|\nabla u_n\|_2}_{=\|\nabla u_n\|_2} + \lambda \|Ku_n - d\|_2\right) \\
&< C \text{ for some } C > 0
\end{aligned}$$

Than $\left\|u_n^2\right\|_2$ bounded.

2. Show $(u_n^1)_n$ is bounded. $K : X^\perp \cap \ker(\nabla) \to \text{image}(K)$ is bijective. Since $Kv = 0$ for $v \in X^\perp \cap \text{kernel}(\nabla) \implies v \in \text{kernel}(K)$. Hence $v \in \text{kernel}(K) \cap \text{kernel}(\nabla) = X \implies v \in X \cap X^\perp \implies v = 0$. Hence $K^{-1} : \text{image}(K) \to X^\perp \cap \text{kernel}(\nabla)$ exists and is continuous.

$$\begin{aligned}
\implies \left\|u_n^n\right\|_2 &= \left\|K^{-1}Ku_n^n\right\|_2 \leq \left\|K^{-1}\right\| \left\|Ku_n^n\right\|_2 \\
&= \frac{\|K\|}{\lambda} \left(\lambda \left\|K(u_1^n + u_2^n) - Ku_n^n\right\|_2 + \|\nabla u_n\|_2\right) \\
&\leq \frac{\|K\|}{\lambda} \left(\underbrace{\lambda \|Ku_1 - d\|_2}_{\text{bounded}} + \underbrace{\|\nabla u_n\|_2 + \lambda \left\|d - Ku_1^2\right\|}_{\text{bounded because } u_n^2 \text{ is bounded}}\right) \\
&< D \text{ for some } D > 0
\end{aligned}$$

$\implies (u_n^n)_n$bounded $\implies (u_n) = (u_n^n + u_n^n)_n$ is bounded

$\implies (u_n)_n$ admits a subsequence converging to some $\hat{u}$. As in Case 1, $\hat{u}$ is a solution to Problem (1).

In summary,

1. $\min_{u\in U} \lambda \|Ku - d\|_2 + \sqrt{\int_{[0,1]^2} |\nabla n|^2 \, dx}$ with $U = C^1([0,1]^2)$ relevant for application.
2. Discrete version: $\min_{u\in\mathbb{R}^n} \lambda \|Ku - d\| + \|\nabla u\|_2$. We have shown existence by using:
   (a) complementary subspaces $X^\perp$
   (b) boundedness and compactness
   (c) continuity

(d) Next time: How does FA help to transfer the proof of the infinite dimensional setting?

□

About the existence of infinitely many dimensions

↓ *This lecture took place on 2019/03/07.*

Define $U = C^1([0,1]^2)$. Let Y is some Banach space and $K : U \to Y$ is linear and continuous.

Consider the problem (P_∞) given by $\min_{u \in U} \|\nabla u\|_2 + \lambda \|Ku - d\|_Y$ where $d \in Y$ and $\|\nabla u\|_2 := \sqrt{\int_{[0,1]^2} |\nabla u(x)|^2}$.

Proposition 0.2. *There exists a solution of* (P_∞).

Proof. Take $(u_n)_{n\in\mathbb{N}}$ as a sequence in U such that $\lim_{n\to\infty} \|\nabla u_1\|_2 + \lambda \|Ku_n - d\|_n \to \inf_{u\in U}(\dots)$. Now we want to show that $(u_n)_{n\in\mathbb{N}}$ is bounded.

Case 1: Assume that $Ku = u$, $Y = U$ and $\|\cdot\|_Y = \|\cdot\|_2$.

$$\implies \lambda \|u_n\|_2 = \lambda \|u_n - d\|_2 + \lambda \|d\| \leq \|\nabla u_n\|_2 + \lambda \|u_n - d\|_2 + \lambda \|d\| < C \text{ for } C > 0$$

$$\implies (u_n)_{n\in\mathbb{N}} \text{ is bounded}$$

So does $(u_n)_{n\in\mathbb{N}}$ admit a convergent subsequence? No. It requires the notion of *weak convergence* and particular spaces called *reflexive spaces*.

So we change U to $U = \left\{ u : [0,1]^2 \to \mathbb{R} \mid \sqrt{\int_{[0,1]^2}} < \infty \right\}$. Define, instead of $\|\nabla u\|_2$,

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } v \in C^2 \\ \infty & \text{else} \end{cases}$$

and consider $\min_{u\in U} R(u) + \lambda \|K_{u-d}\|_2$ instead.

In this setting, $(u_n)_{n\in\mathbb{N}}$ admits a weakly convergent subsequence converging to some $\hat{u} \in U$ (denoted by $(u_{n_i})_{i\in\mathbb{N}}$).

Our next step is to use continuity to show that $\hat{u}$ is a solution.

Problem: $u \mapsto \|u - d\|_2$ is, in general, not continuous with respect to weak convergence.

But it is always true that $\|\hat{u} - d\|_2 \leq \liminf_{i\to\infty} \left\|u_{n_i} - d\right\|_L$. Yes. We consider that as first property.

Is it also true that $R(\hat{u}) \leq \liminf_{i\to\infty} R(u_{n_i})$? No. So we apply some kind of adaption. Recall that

$$\int_0^1 \partial_x u \varphi = -\int_0^1 u \partial_x \varphi \forall \varphi \in C^\infty([0,1]^2)$$

$\varphi = 0$ in $K \setminus [0,1]^2$ for some $K \Subset (0,1)^2$.

$$\implies \int_{[0,1]^2} \nabla u \varphi = -\int_{[0,1]^2} u \cdot (\partial_{x_i}\varphi_1 + \partial_{x_2}\varphi_2)$$
$$\forall \varphi : (\varphi_1, \varphi_2) = C^\infty([0,1]^2, \mathbb{R}^2) + \text{ zero on boundary}$$

We define $w : [0,1]^2 \to \mathbb{R}^2$ is called *weak derivative* of $u \in U$.

$$\iff \int_{[0,1]^2} w\varphi = -\int_{[0,1]^2} u(\partial_{x_1}\varphi_1 + \partial_{x_2}\varphi_2) \text{ holds } \forall \varphi$$

Then w is called *weak gradient* of u. We adjust:

$$R(u) = \begin{cases} \|\nabla u\|_2 & \text{if } u \text{ is weakly differentiable} \\ \infty & \text{else} \end{cases}$$

Then $R(\hat{u}) \leq \liminf_{i\to\infty} R(u_{n_i})$. We consider this as second property.

By the two properties,

$$\begin{aligned} R(\hat{u}) + \|\hat{u} - d\| &\leq \liminf_{i\to\infty} R(u_{n_i}) + \liminf_{i\to\infty} \lambda \left\|u_{n_i} - d\right\|_2 \\ &\leq \liminf_{i\to\infty} \left(R(u_{n_i}) + \lambda \left\|u_{n_i} - d\right\|_2\right) \\ &= \inf R(u) + \lambda \|u - d\|_2 \end{aligned}$$

Case 2: Works as in the finite-dimensional setting using

- $X := \text{kernel}(A) \cap \text{kernel}(\nabla) \implies U = X \oplus X^\perp$ requires so-called *Hilbert spaces*
- $\|u\|_2 \leq C \|\nabla u\|_2 \, \forall u \in \text{kernel}(\nabla)^\perp$ is called *Poincare inequality.*

$\square$

So this content so far was a motivation. Now, which topics are we going to cover in this course:

1. Topological and metric spaces
2. Normal spaces
3. Linear operator
4. The Hahn-Banach Theorem and consequences
5. Fundamental theorems for linear operators
6. Dual spaces and reflexivity
7. Contemplementary subspaces
8. Hilbert spaces

↓ *This lecture took place on 2019/03/12.*

Remark. *1. Literature: UGU, in particular: Biezis, Werner*

2. In this lecture: always $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$ if not further specified

1 Topological and metric spaces

Remark (Motivation)**.** *Some concepts in Functional Analysis (e.g. weak convergence) cannot be associated with norms but rather with topologies*

Definition 1.1 (Topology)**.** *Let X be a set and $\tau \subset \mathcal{P}(X) = \{$"set of subsets of X"$\}$. We say that τ is a* topology *on X if*

1. $X, \emptyset \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. *For any collection of sets $(U_i)_{i \in I}$ with I as some index set. We have $U_i \in \tau \forall i \in I \implies \bigcup_{i \in I} U_i \in \tau$.*

(X, τ) *is called* topological space.

A set $U \subset X$ is called open *if $U \in \tau$ and is called closed if $U^{C} \in \tau$.*

Remark. *By the third property of topologies, $\bigcap_{i \in I} V_i$ is closed for any collection $(V_i)_{i \in I}$ of closed sets.*

Definition 1.2 (Metric)**.** *Let X be a set, $d : X \times X \to \mathbb{R}$ be such that $\forall x, y, z \in X$*

1. $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric on X *and (X, d) is called* metric space.

Definition 1.3 (Norm)**.** *Let X be a vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called* norm *if $\forall x, y \in X, \lambda \in \mathbb{K}$*

1. $\|x\| \geq 0, \|x\| = 0 \iff x = 0$
2. $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$
3. $\left\|x + y\right\| \leq \|x\| + \left\|y\right\|$

Then $(X, \|\cdot\|)$ is called normed space.

Remark. *If $\dim(x) < \infty$, all norms on X are equivalent.*

Example. 1. *Let X be a set then $\tau = \{\emptyset, X\}$ is a topology.*

2. $(X, \mathcal{P}(X))$ *is a topological space.*

3. *Define* $S^{d-1} := \left\{x \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i^2 = 1\right\}$ *and* $d(x,y) := r$ *where r is the length of the shortest connection between x and y on S^{d-1}. Then d is a metric on S^{d-1}*

4. $X := \{u : [0,1] \to \mathbb{R} \mid u$ *is continuous*$\}$ *then* $\|u\|_\infty := \sup_{x\in[0,1]} \left|u(x)\right|$ *is a norm on X*

5. $l^p := \{(X_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{K}\forall u$ *and* $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ *with* $p \in [1,\infty)$ *and* $\left\|(x_i)_{i\in\mathbb{N}}\right\|_p := (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}}$. *Then* $(l^p, \|\cdot\|_p)$ *is a normed space (the proof will be done later).*

Remark.

$$L^\infty := \left\{(X_i)_{i\in\mathbb{N}} \mid \sup_{i\in\mathbb{N}} |x_i| < \infty\right\}$$
$$\left\|(X_i)_{i\in\mathbb{N}}\right\| = \sup_i |X_i|$$

Proposition 1.4. *Let X be a set.*

1. *If (X,d) is a metric space, define for $\varepsilon > 0, x \in X$. $B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}$ and $\tau = \{U \in \mathcal{P}(x) \mid \forall x \in U \exists \varepsilon > 0 : B_\varepsilon(x) \in U\}$. Then (X,τ) is a* topological space. *We say that τ is the topology induced by d and we have that $B_\varepsilon(x) \in \tau \forall \varepsilon > 0, x \in X$*

2. *If $(X, \|\cdot\|)$ is a normed space, define $d : X \times X \to \mathbb{R}$ with $(x,y) \mapsto \left\|x - y\right\|$. Then (X,d) is a metric space and d is called the metric induced by $\|\cdot\|$.*

Remark (Consequence)**.** *Every concept introduced for topological and metric spaces transfers to metric and normed spaces, respectively. The proof is left as an exercise to the reader.*

Definition 1.5. *Let (X,τ) be a topological space. $U \subset X$. $x \in X$.*

1. *U is called a neighborhood of x if $\exists V \in \tau - x \in V \subset U$: $\mathcal{U}(x)$ is defined as the set of all neighborhoods of x*

2.
   - *x is called* interior point *of U if $U \in \mathcal{U}$*
   - *x is called* adjacent point *of U if $\forall V \in \tau$ such that $x \in V : V \cap U \neq \emptyset$*
   - *x is called* cluster point *of U if it is an adjacent point of $U \setminus \{x\}$.*

   The third property is stronger.

3. *Notational conventions:*

$$\mathring{U} := \{x \in U \mid x \text{ is an interior point of } U\}$$
$$\overline{U} := \{x \in U \mid x \text{ is an adjacent point of } U\}$$
$$\partial U := \overline{U} \setminus \mathring{U}$$

Proposition 1.6. *Let (X,τ) be a topological space, $U \in X$. Then*

1. *U is open $\iff \mathring{U} = U$*

2. *U is closed $\iff \overline{U} = U$*

3. *$\mathring{U} = \bigcup_{\substack{V\in\tau \\ V\subset U}} V$ and $\mathring{U}$ is open ["$\mathring{U}$ is the largest open set in U"]*

4. *$\overline{U} = \bigcap_{\substack{V \text{closed} \\ U\subset V}} V$ and $\overline{U}$ is closed ["$\overline{U}$ is the smallest closed set containing U"]*

Proof. 3. $\subset$ Let $x \in \mathring{U} \implies \exists \hat{V} \in \tau$ s.t. $x \in \hat{V} \subset U \implies x \in \bigcup_{\substack{V\in\tau \\ V\subset U}} V$

$\supset$ Let $x \in \bigcup_{\substack{V\in\tau \\ V\in U}} V \implies x \in \hat{V}$ for some $\hat{V} \in \tau, \hat{V} \in U \implies x \in \mathring{U}$

$\mathring{U}$ is open because it is the union of open sets.

1. $\implies$ $\mathring{U} \subset U$ by definition. U is open, so $U \subset \bigcup_{\substack{V\subset\tau \\ V\subset U}} V \overset{(3)}{=} \mathring{U}$

2. $\implies$ $V \subset \overline{U}$ by definition. Take $x_0 \in \overline{U}$. If $x \notin U \implies x \in U^C \in \tau$ and $U \cap U^C = \emptyset$. This contradicts to $x \in \overline{U}$.

   $\impliedby$ Take $x \in U^C = \overline{U}^C$.

   $\overset{(4)}{\implies}$ $\exists V \in \tau : x \in V \wedge V \cap \overline{U} = \emptyset$
   $\implies$ $V \cap U = \emptyset \implies V \subset U^C$
   $\implies$ U^C open $\implies$ U closed

4. We prove the fourth property without the second.

   $\subset$ Take $x \in \overline{U}$. Take closed V such that $U \subset V$ if $x \notin V \implies x \in V^C$ which is open and $V^C \cap U = \emptyset$. This contradicts to $x \in \overline{U}$.

   $\supset$ Take $x \in \bigcap_{\substack{V \text{ closed} \\ U\subset V}}$. Suppose $x \notin \overline{U}$.

   $\implies$ $\exists Z$ open such that $x \in Z$ and $Z \cap U = \emptyset$
   $\implies$ $U \subset Z^C$, Z^C closed, $x \notin Z^C$. This contradicts to $x \in \bigcap_{\substack{V \text{ closed} \\ U\subset V}} V$

   $\overline{U}$ closed follows since the intersection of closed sets is closed.

□

Definition 1.7 (Limit). *Let (X, τ) be a topological space, $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Henceforth, we write $(x_n)_n$ for $(x_n)_{n\in\mathbb{N}}$ and $\hat{x} \in X$. We say $x_n \to \hat{x}$ in τ as $n \to \infty$ ("x_n converges to x", "x is limit of x_n") if*

$$\forall U \in \tau \text{ such that } \hat{x} \in U \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in U$$

Definition 1.8 (Proposition and definition). *Let (X, τ) be a topological space. We say that (X, τ) is T_2 (or Hausdorff) if*

$$\forall x, y \in X \text{ with } x \neq y \ \exists U, V \in \tau : x \in U, v \in V \text{ and } U \cap V = \emptyset$$

- *In a T_2-sphere, the limit of any sequence is unique.*
- *If τ is induced by a metric, then (X, τ) is T_2.*

Proof. 1. Take $(x_n)_n$ to be a sequence and assume x_n converges to $\hat{x}$ and $\hat{y}$ with $\hat{x} \neq \hat{y}$. By T_2, $\exists U, V \in \tau : \hat{x} \in U, \hat{y} \in V : U \cap V = \emptyset$. By convergenc, $\exists n_x, n_y$ such that $\forall n \geq n_x : x_n \in U$ and $\forall n \geq n_y : x_n \in V$.

$$\forall n \geq \max\left\{n_x, n_y\right\} : x_i \in U \cap V$$

This gives a contradiction.

2. Take $x, y \in X : x \neq y$. Define $\varepsilon := d(x, y)$ and consider $B_{\frac{\varepsilon}{2}}(x)$ and $B_{\frac{\varepsilon}{2}}(y)$ which are open in the induced topology τ. Also $x \in B_{\frac{\varepsilon}{2}}(x)$ and $y \in B_{\frac{\varepsilon}{2}}(y)$. Assume that $z \in B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}}(y)$.

$$\varepsilon = d(x, y) \leq d(x, z) + d(z, y) > \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This gives a contradiction.

□

Definition 1.9. *Let (X, τ) be a topological space, $U \subset V \subset X$. We say that U is* dense *in V, if $V \subset \overline{U}$. We say that X is* separable *if there exists a countable, dense subset.*

Definition 1.10. *Let $(X, \tau_X), (Y, \tau_Y)$ be topological spaces and $f : X \to Y$ a function. We say f is* continuous *at $x \in X$ if $\forall V \in \mathcal{U}(f(x)) \exists U \in \mathcal{U}(x) : f(U) \subset V$. f is called* continuous *if it is continuous at any $x \in X$.*

Proposition 1.11. *With $(X, \tau_X), (Y, \tau_Y)$ and f as above, f is continuous $\iff f^{-1}(V) \in \tau_X \forall V \in \tau_Y$*

Proof. Left as an exercise to the reader. □

Definition 1.12. *Let (X, τ) be a T_2 topological space, $M \subset X$ called* compact *if for any family $(U_i)_{i \in I}$ with $U_i \in \tau$ s.t. $M \subset \bigcup_{i \in I} U_i$ ("$(U_i)_{i \in I}$ is an open covering of M"), there exists $U_{i_1}, \ldots, U_{i_n}$ such that $M \subset \bigcup_{k=1}^{n} U_{i_k}$ ("there exists a finite subcover").*

Remark. *Compactness can also be defined without T_2, this is also referred to as* quasi-compact.

Remark (Exercise)**.** *Reconsider the previous results for metric and normed spaces.*

↓ *This lecture took place on 2019/03/14.*

Definition 1.13. *Let (X, d) be a metric space, $V \subset X$ and $(x_n)_n$ a sequence in X. Then we say,*

1. *V is* bounded *if $\exists x \in X, r > 0$ such that $U \in B_r(x)$*
2. *$(x_n)_n$ is a* Cauchy sequence *if $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 : d(x_n, x_m) < \varepsilon$*

3. *X is* complete *if any Cauchy sequence in X admits a limit point*

4. *X is a* Banach space *if it is a normed space and complete*

Proposition 1.14. *Let (X, d) be a metric space. $(x_n)_n$ be a sequence in X. Then*

1. *$x_n \to x$ in the induced topology* $\iff \forall \varepsilon > 0 \exists n_0 \geq 0 \forall n \geq n_0 : d(x_n, x) < \varepsilon$

2. *If $x_n \to x$, then $(x_n)_n$ is bounded as subset of X and $(x_n)_n$ is Cauchy.*

3. *If $U \subset X$ is closed and X is complete. Then (U, d) is a complete metric space.*

Proof. 1. We prove both directions:

$\implies$ True, since $B_\varepsilon(x)$ is open $\forall \varepsilon 0$

$\impliedby$ Let $x \in V$ with V open. Show that $\exists n_0 \geq 0 \forall n \geq n_0 : x_n \in V$
V open, then $\exists \varepsilon > 0 : B_\varepsilon(x) \subset V$
$\implies \exists n_0 \forall n \geq n_0 : x_n \in B_\varepsilon(x) \subset V$

2. Using the first property, we get $\exists n_0 \forall n \geq n_0 : d(x_n, x) < 1$. Let $r := \max_{i=1,\dots,n_0} d(x, x_i) + 1$. Then

$$\forall n \in \mathbb{N} : d(x, x_n) < \begin{cases} 1 & \text{if } n \geq n_0 \\ r & \text{if } n < n_0 \end{cases} \leq r$$

$$\implies y_n \in B_r(x) \forall n \in \mathbb{N}$$

3. Take $(y_n)_n$ to be a Cauchy sequence in U, then $(y_n)_n$ is a Cauchy sequence in $X \implies \exists x \in X : y_n \to x$ as $n \to x$ if $x \notin U \implies x \in U^C \implies \exists n_0 \in N$ such that $y_{n_0} \in U^C$ due to U^C open. This is a contradiction to $(y_n)_n$ in U

□

Proposition 1.15. *Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \to Y$. The following are equivalent (TFAE):*

- *f is continuous (with respect to the induced topology)*
- *$\forall (X_n)_n$ such that $x_n \to x \implies f(x_n) \to f(x)$*

Proof. Firstly, we prove that the first statement implies the second statement.

Take $(x_n)_n$ converging to x. Take $V \in \tau_y$ such that $f(x) \in V \implies V \in \mathcal{U}(f(x))$

$\implies \exists U \in \mathcal{U} : f(U) \subset V \implies \exists \hat{U} \in \tau_X$ such that $x \in \hat{U} \subset U$

$\implies \exists n_0 \geq 0 \forall n \geq n_0 : x_n \in \hat{U} \implies \forall n > n_0 : f(x_n) \in V \implies f(x_0) \to f(x)$

Remark. 1. $\implies$ 2. *holds true in any topological space*

2. $\implies$ 1. *Not.*

Secondly, we prove that the second statement implies the first statement.

Suppose f is not continuous, find $x_n \to x$ such that $f(x_n) \to f(x)$ is wrong. If f is not contiuous, then $\exists x \in X : \exists V \in \mathcal{U}(f(x))$ such that $f(u) \not\subset V \forall U \in \mathcal{U}(x)$

$$\implies \exists \hat{V} \in \tau_Y \text{ such that } f(u) \not\subset \hat{V} \forall U \in U(x), f(x) \in \hat{V}$$

$$\implies \forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) : f(x_n) \notin \hat{V}$$

$\implies$ $(x_n)_n$ converges to x but $f(x_n) \notin \hat{V} \implies f(x_n) \not\to f(x)$. This gives a contradiction. □

Definition 1.16. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$.*

f is uniformly continuous *iff*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proposition 1.17. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$, $f : M \to Y$. If M is dense in X, Y is complete and f is uniformly continuous.*

$$\implies \exists! \hat{f} : X \to Y \textit{ such that } \hat{f} \textit{ continuous and } \hat{f}|_M = f$$

Proof. Take $x \in X$. By the practicals (and since $\overline{M} = X$), $\exists (x_n)_n$ such that $x_n \to x$ and $x_n \in M$.

We show: $(f(x_n))_n$ is Cauchy. Take $\varepsilon > 0 \implies \exists \delta > 0$ such that

$$\forall x_1, x_2 \in X : d_X(x_1, x_2) < \delta \implies d_Y(f(y_1), f(y_2)) < \varepsilon$$

Now, $(x_n)_n$ is Cauchy (why?) $\implies \exists n_0 \forall n, m \geq n_0 : d_X(x_n, x_m) < \delta$

$$\implies d_Y(f(y_n), f(x_n)) < \varepsilon \implies (f(x_n))_n \text{ is Cauchy implies convergence}$$

Now we observe: $\forall \hat{x} \in X$, there exists $(\hat{x}_n)_n$ in $M, \hat{y} \in Y$ such that $f(\hat{x}_n) \to \hat{y}$.

Now: for any $\varepsilon > 0 \exists \delta > 0 : d_Y(x_n, \hat{x}_n) < \delta \implies d_Y(f(x_n), f(\hat{x}_n)) < \varepsilon$ with $x \in X, (x_n)_n$ is a sequence in M such that $x_n \to x, f(x_n) \to y$. Now if $d(x, \hat{x}) < \delta \implies \exists n_0 \forall n \geq n_0$:

$$d(x_n, \hat{x}_n) < \delta \implies d(f(x_n), f(\hat{x}_n)) < \varepsilon \forall n \geq n_0$$

$$\implies d_Y(\hat{y}, y) < d_Y(\hat{y}, f(\hat{x}_n)) + d_Y(f(\hat{x}_n), f(x_n)) + d(f(x_n), f(x)) < 3\varepsilon$$

1. If $x = \hat{x} \implies y = \hat{y} \implies \hat{f}(x) := y$ is well-defined.
2. $\hat{f}$ is uniformly continuous.

□

↓ *This lecture took place on 2019/03/19.*

Proposition 1.18. *Let (X, d) be a metric space, $M \subset X$.*

1. *M is compact, so $\forall (X_i)_{i\in I}$ with X_i a closed set $\forall i$ such that $(\bigcap_{i\in I} X_i) \cap M = \emptyset$.*

$$\implies \exists X_{i_1}, \ldots, X_{i_n} \text{ such that } \left(\bigcap_{i=1}^{n} X_{ij}\right) \cap M = \emptyset$$

2. *M is compact, so M is closed and bounded.*

Proof.

1. We note that $\forall (X_i)_{i\in I}$ is a family of closed sets. $(X_i^C)_{i\in I}$ is a family of open sets and $\bigcap_{i\in I} X_i \cap M = \emptyset \iff M \subset \bigcup_{i\in I} X_i^C$
2. Is a special case of the next proposition.

□

Proposition 1.19. *Let (X, d) be a metric space, $M \subset X$. TFAE:*

1. *M is compact.*
2. *Every infinite subset of M admits a cluster point.*
3. *Every sequence of M admits a convergent subsequence.*
4. *M is complete and totally bounded, where totally bounded is defined as*

$$\forall \varepsilon > 0 : \exists (x_1, \ldots, x_n) \text{ in } M : M \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

Remark.

1. *totally bounded $\implies$ bounded (proof is left as an exercise)*
2. *Assume $\dim(x) < \infty$. Compact $\iff$ complete and bounded (see course Analysis I)*
3. *$\dim(x) < \infty \iff \overline{B_1}(0)$ is compact*

where the last two items imply that X is a normed space.

Proof. $1 \to 2$ If M is finite, (2) always holds true. So assume that M is infinite. Now assume that (2) does not hold. Then there is $C \subset M$ infinite which does not admit a cluster point. $[\forall x \in C \exists \varepsilon_x > 0 : B_{\varepsilon_x}(x)$ contains at most one element of $C]$. If not, $\exists x in C$ such that $\forall n \in \mathbb{N} \exists x_n \in B_{\frac{1}{n}}(x) \cap C$ such that $(X_n)_n$ is a sequence of distinct points and $x_n \to x$. This implies that x is a cluster point of C. This gives a contradiction.

Now $M \subset \bigcup_{x\in M} B_{\varepsilon_x}(x)$. If M is compact, then

$\implies \exists x_1, \ldots, x_n : M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$

$\implies C \subset M \subset \bigcup_{i=1}^{n} B_{\varepsilon_{x_i}}(x_i)$

$\implies$ C is finite

This is a contradiction.

$2 \to 3$ Let $(x_n)_n$ be a sequence in M.

Case 1: $\{x_n \mid n \in \mathbb{N}\}$ is finite $\implies$ $(x_n)_n$ admits a convergent sequence.

Case 2: $\{x_n \mid n \in \mathbb{N}\}$ is infinite. By the second property, there is a cluster point of $\{x_n \mid x \in \mathbb{N}\}$. Thus $(x_n)_n$ is a convergent subsequence to some $x \in M$.

$3 \to 4$ Suppose that M is not totally bounded. $\exists \varepsilon > 0 \forall x_1, \dots, x_n \in M \exists y \in M : y \notin \bigcup_{i=1}^n B_\varepsilon(x_i)$. Construct a sequence $(x_n)_n$ in M as follows: Given $x_1, \dots, x_n$, choose $x_1 \in M$ arbitrary and $x_{i+1} \in M \setminus \bigcup_{j=1}^i B_\varepsilon(x_j)$ arbitrary. Then $(x_i)_i$ is a sequence in M and $d(x_i, x_j) > \frac{\varepsilon}{2}$ for $i \neq j$. Hence, $(x_i)_i$ cannot admit a convenient subsequence. $G \implies M$ totally bounded.

Completeness can be shown the following way: Let $(x_n)_n$ be Cauchy in M, then there exists a subsequence $(x_{n_i})_i$ and $x \in M$ such that $x_{n_i} \to x$ as $i \to \infty$. Since $(x_n)_n$ is Cauchy, $x_n \to x$ as $n \to \infty$ [left as an exercise]. Thus M is complete.

$4 \to 1$ Let $(U_i)_{i \in I}$ be an open covering of M and assume that $(U_i)_{i \in I}$ does *not* admit a finite subsequence. For $n \in \mathbb{N}$ let $E_n \subset M$ be a finite set such that $M \subset \bigcup_{a \in E_n} B_{\frac{1}{2^n}(a)}$. Define $\Omega := \left\{\tilde{M} \subset M \mid \tilde{M} \text{ is not covered by finitely many } U_i\right\}$. We recursively define a sequence $(a_n)_n$ in M such that

$$\forall n \in \mathbb{N} : a_n \in E_n, M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega, B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n-1}) \neq \emptyset$$

Goal: Show $(a_n)_n \to a$ and then $B_{\frac{1}{2^{n_0}}}(a_{n_0}) \subset U_{i_0}$.

Step 1 $(a_n)_n$ is well defined.

$n = 1$ Since $M \in \Omega$ and $M \subset \bigcup_{a \in E_1} B_{\frac{1}{2}}(a)$, we can pick $a_1 \in E_1$ such that $M \cap B_{\frac{1}{2}}(a_1) \in \Omega$.

$n \to n+1$ Let $a_n \in E_n$ such that $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$ be given. Let $\tilde{E}_{n+1} = \left\{a \in E_{n+1} \mid B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a) \neq \emptyset\right\}$.

Since $M \cap B_{\frac{1}{2^n}}(a_n) \subset \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{2^{n+1}}}(a)$. [Take $x \in M \cap B_{\frac{1}{2^n}}(a_n) \implies x \in B_{\frac{1}{2^{n+1}}}(\hat{a})$, but if $B_{\frac{1}{2^{n-1}}}(\hat{a}) \cap B_{\frac{1}{2^n}}(a_n) = \emptyset$

$$\implies \hat{a} \in \tilde{E}_{n+1} \implies x \in \bigcup_{a \in \tilde{E}_{n+1}} B_{\frac{1}{a^{n+1}}}(a)$$

Hence there exists a_{n+1} such that $M \cap B_{\frac{1}{2^{n+1}}}(a_{n+1}) \in \Omega$ and $B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n-1}}}(a_{n+1}) \neq \emptyset$. Thus $(a_n)_n$ is well-defined.

Step 2 Show that $(a_n)_n$ converges. Take $n \in \mathbb{N}$ and $z \in B_{\frac{1}{2^n}}(a_n) \cap B_{\frac{1}{2^{n+1}}}(a_{n+1})$.

$$\implies d(a_n, a_{n+1}) < d(a_n, z) + d(z, a_{n+1}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{3}{2^{n+1}}$$

$$\forall k \geq n : d(a_k, a_n) \leq \sum_{i=n}^{k-1} d(a_{i+1}, a_i) < \sum_{i=n}^{k-1} \frac{3}{2^{i+1}} = \frac{3}{2^{n+1}} \sum_{i=0}^{k-n-1} \frac{1}{2^i} \leq \frac{3}{2^n}$$

thus, $(a_n)_n$ is Cauchy. M is complete, so $\exists a \in M : a_n \xrightarrow{n\to\infty} a$

$$\implies \exists U_{i_0} : a \in U_{i_0} and \exists i > 0 : B_r(a) \subset U_{i_0}$$

Hence, for n sufficiently large such that $d(a, a_n) < \frac{r}{2}$ and $\frac{1}{2^n} < \frac{r}{2}$. We take $x \in B_{\frac{1}{2^n}}(a_n)$ and estimate

$$d(x,a) \leq d(x, a_n) + d(a_n, a) < \frac{r}{2} + \frac{r}{2} = r$$

$$\implies B_{\frac{1}{2^n}}(a_n) \subset U_{i_0}$$

is a contradiction to $M \cap B_{\frac{1}{2^n}}(a_n) \in \Omega$.

□

Proposition 1.20. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. $M \subset X$ compact. Let $f : X \to Y$ be continuous. Then*

1. *$f(M)$ is compact*
2. *$f|_M : M \to Y$ is uniformly continuous.*

Proof. 1. Let $(U_i)_{i\in I}$ be an open covering of $f(M)$

$\implies$ $(f^{-1}(U_i))_{i\in I}$ is an open covering of M [why!]
$\implies$ $\exists c_1, \ldots, c_n$ such that $M \subset \bigcup_{j=1}^n f^{-1}(U_{i_j}) \implies f(M) \subset \bigcup_{j=1}^n U_{i_j}$

2. If $f|_M$ is not uniformly continuous, then $\exists \varepsilon > 0 \forall n \in \mathbb{N} \exists x, y \in M : d(x,y) < \frac{1}{n}$ and $d(f(x), f(y)) > \varepsilon$ (*). Now take $(x_n)_n$, $(y_n)_n$ sequences in M satisfying condition (*). M is compact, so $\exists (x_{n_i})_i$ subsequence converging to some $x \in M$.

$$d(y_{n_i}, x) < d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + d(x_{n_i}, x) \xrightarrow{i\to\infty} 0$$

□

↓ *This lecture took place on 2019/03/21.*

Proposition 1.21 (Proposition and definition)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces. $g : X \to Y$ is a function. g is called* Lipschitz continuous *if $\exists L > 0$ such that $d_Y(\varphi(x), \varphi(y)) \leq L d_X(x,y) \forall x, y \in X$. Any Lipschitz continuous function is uniformly continuous.*

Proof. Left as an exercise to the reader. □

Theorem 1.22 (Arzelà-Ascoli theorem)**.** *Let (X, d_X) and (Y, d_Y) be metric spaces and assume that X is compact. Define $C(X, Y) := \{f : X \to Y \mid f \text{ continuous}\}$ and $d_C(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$. Then*

1. *d_C is well-defined and $(C(X, Y), d_C)$ is a complete metric space*
2. *A set $M \subset C(X, Y)$ is compact iff*
   (a) *$\forall x \in X$ the set $M_X := \{f(x) \mid f \in M\}$ is compact*
   (b) *M is* equicontinuous, *i.e.* $\forall \varepsilon > 0 \exists \delta > 0$

$$\forall x, y \in X \forall f \in M : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Proof. 1. Show that: $d_C(f, g) < \infty$.

Pick $f, g \in C(X, Y)$. Because X is compact, $f(X), g(X)$ compact $\implies$ $f(X), g(X)$ bounded. Thus, $\exists x_1, x_2, D_1, D_2 : f(X) \subset B_{D_1}(x_1), g(X) \subset B_{D_2}(x_2)$. Now for $x \in X$,

$$\begin{aligned} d(f(X), g(x)) &\le d(f(x), x_1) + d(x_1, x_2) + d(x_2, g(x)) \\ &\le D_1 + d(x_1, x_2) + D_2 < \infty \\ &\implies \sup_{x\in X} d(f(x), g(x)) \end{aligned}$$

Showing that d_C is a metric is left as an exercise.

Show that $(C(X, Y), d_C)$ is a complete metric space.

Take $(f_n)_n$ be Cauchy in $C(X, Y) \implies (f_n(x))_n$ is Cauchy in $Y \forall x \in X$. Because Y is complete, $(f_n(x))_n$ is convergent and we can define $f(x) := \lim_{n\to\infty} f_n(x)$. Convergence of $(f_n)_n$ with respect to d_C: Take $\varepsilon > 0$, show

$$\exists n_0 \forall n \ge n_0 : \sup_x d(f(x), f_n(x)) < \varepsilon$$

Because it is Cauchy, $\exists n_0 \forall n, m \ge n_0 : d_C(f_n, f_m) < \varepsilon$. Consider $x \in X, n \ge n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x)) \le \lim_{m\to\infty} d(f_m, f_n) < \infty$ (the proof follows below)

$$\implies \sup_{x\in X} d(f(x), f_n(x)) < \varepsilon$$

Thus, if $f \in C(X, Y) \implies f_n \to f$ with respect to d_C. Show that $f \in C(X, Y)$. Take $\varepsilon > 0$. Let n_0 such that $\sup_{x\in X} d(f(x), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Take $\delta > 0$ such that $d(x, y) < \delta \implies d(f_{n_0}(x), f_{n_0}(y)) < \frac{\varepsilon}{3} \forall x, y$. Then $\forall x, y : d(x, y) < \delta$

$$\begin{aligned} d(f(x), f(y)) &\le d(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(y)) + d(f_{n_0}(y), f(y)) \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It remains to show: $\forall x \in X, n \ge n_0 : d(f(x), f_n(x)) = \lim_{m\to\infty} d(f_m(x), f_n(x))$.

In general, we have $\forall x, y, z \in (Z, d_Z)$ with d_Z as a metric.

$$\left| d(x, z) - d(y, z) \right| \le d(x, y)$$

Proof.

$$d(x,z) \le d(x,y) + d(y,z) \implies d(x,z) - d(y,z) \le d(x,y) \quad (2)$$
$$d(y,z) \le d(y,x) + d(x,z) \implies d(y,z) - d(x,z) \le d(x,y) \quad (3)$$
$$(2) \text{ and } (3) \implies \left|d(x,z) - d(y,z)\right| \le d(x,y) \quad (4)$$

□

Consequently, $\forall z \in Z$, $x_n \to x$ in Z: $d(x_n, z) \to d(x,z)$ since $\left|d(x_n,z) - d(x,z)\right| \le d(x_n, x) \to 0$.

2. We need to prove both directions.

$\implies$ (a) For $x \in X$ fixed, define $g_X : M \to Y$ with $f \mapsto f(x)$. Then $d_Y(g(f_1), g(f_2)) = d_Y(f_1(x), f_2(x)) \le d_C(f_1, f_2)$
$\implies$ g_X is Lipschitz continuous, in particular continuous
$\implies$ $M_X = g_X(M)$ compact

(b) Take $\varepsilon > 0$. M is totally bounded, so $\exists f_1, \dots, f_n \in M : M \subset \bigcup_{i=1}^n B_{\frac{\varepsilon}{3}}(f_i)$. $\forall i \in \{1, \dots, n\} \exists \delta_i : \forall x, y \in X : d(x,y) < \delta_i \implies d_Y(f_i(x), f_i(y)) < \frac{\varepsilon}{3}$. Define $\delta := \min_i \delta_i > 0$. Then $\forall x, y \in X : d(x,y) < \delta$ and $\forall f \in M \exists f_{i_0} : f \in B_{\frac{\varepsilon}{3}}(f_{i_0})$

$$\implies d(f(x), f(y)) \le \underbrace{d(f(x), f_{i_0}(x))}_{\le d_C(f, f_{i_0}) \le \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(x), f_{i_0}(y))}_{\le \frac{\varepsilon}{3}} + \underbrace{d(f_{i_0}(y), f(y))}_{\le d_C(f_{i_0}, f) \le \frac{\varepsilon}{3}} < \varepsilon$$

$\impliedby$ We prove the other direction.

> ↓ *This lecture took place on 2019/03/26.*

B is complete since it is a closed subset of a Banach space.
Show: M is totally bounded.
Consider $\varepsilon > 0$. Show: $\exists f_1, \dots, f_n$ such that $M \subset \bigcup_{i=1}^n B_\varepsilon(f_i)$.

- Because M is equicontinuous, $\exists \delta > 0 \forall f \in M \forall x, y \in X : d(x,y) < \delta \implies d(f(x), f(y)) < \frac{\varepsilon}{4}$.
- By compactness of X, $\exists x_1, \dots, x_n : X \subset \bigcup_{i=1}^n B_\delta(x_i)$
- $\forall i : M_{x_i}$ compact $\implies \exists (y_{i_1}, \dots, y_{i_{k_i}}) : M_{x_i} \subset \bigcup_{i=1}^{k_i} B_{\frac{\varepsilon}{4}}(y_{i_i})$

Compare with Figure 1.
Now, for each tuple of indices $(y_{1,j_1}, \dots, y_{n,j_n})$ define $f_{y_{1,j_1}, \dots, y_{n,j_n}} \in C(x,y)$ to be such that $f_{y_{1,j_1}, \dots, y_{n,j_n}}(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ if such an f exists. The set F of all such functions is finite. We show that $M \subset \bigcup_{q \in F} B_\varepsilon(q)$.
Take $f \in M$ arbitrary. Now choose $\alpha = (y_{1,j_1}, \dots, y_{n,j_n})$ such that $f(x_i) \in B_{\frac{\varepsilon}{4}}(y_{i,j_i})$ and pick $f_\alpha \in F$ accordingly.
Take $x \in X$ arbitrary and x_i such that $x \in B_\delta(x_i)$

$$\begin{aligned}\implies d(f(x), f_\alpha(x)) &\le d(f(x), f(x_i)) + d(f(x_i), f_\alpha(x_i)) + d(f_\alpha(x_i), f_\alpha(x)) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \\ \implies d_C(f, f_\alpha) &= \sup_{x \in X} d(f(y), f_\alpha(x)) < \varepsilon\end{aligned}$$

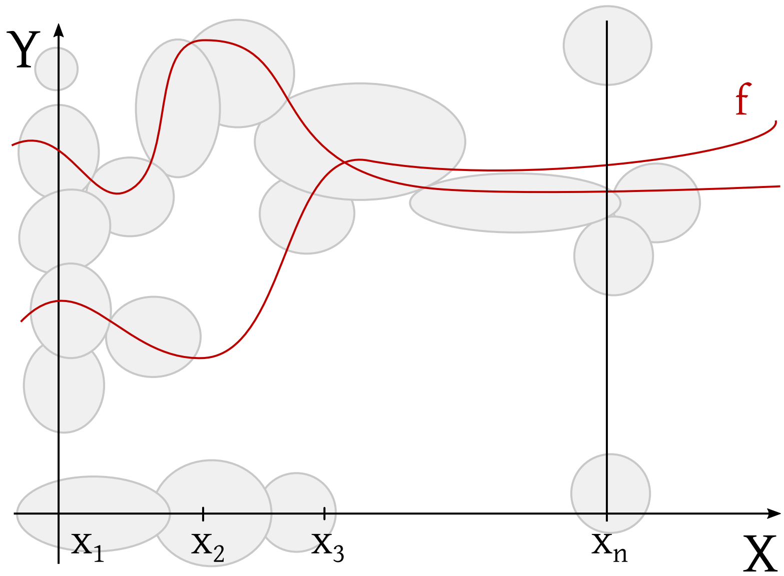

Figure 1: Covering of a function graph

$\square$

Remark. *Compare this to the fact that $B_1(0)$ in $C(X, Y)$ is* not *compact.*

To complete this chapter, we discuss an important topological assertion; the Baire category theorem.

Remark (Motivation)**.** *In general, let (X, d) be a metric space. Let A and B be open and dense, then also $A \cap B$ is dense.*

Proof. Show $\forall x \in X \forall \varepsilon : B_\varepsilon(x) \cap [A \cap B] = \emptyset$. Take $x \in Y, \varepsilon > 0 \implies \exists x_1 \in B_\varepsilon(x) \cap A$. A is dense. A is open, so $\exists \varepsilon > 0 : B_{\varepsilon_1}(x_1) \subset B(x) \cap A$. B is dense, so $B_{\varepsilon_1}(x_1) \cap X \neq \emptyset$.

$$\implies \exists z \in B_\varepsilon(x_1) \cap B$$

$$B_{\varepsilon_1}(x_1) \subset B(x) \cap A \implies z \in B_\varepsilon(x) \cap (A \cap B)$$

$\square$

More generally, $\forall A_1, \ldots, A_n$ open, dense $\implies \bigcap_{i=1}^n A_i$ is dense (this is left as an exercise). Does this also hold true for countably many A_i?

Theorem 1.23 (Baire theorem)**.** *Let (X, d) be a complete metric space. Let $(O_n)_{n\in\mathbb{N}}$ be a sequence of dense sets. Then $\bigcap O_n$ is dense.*

Proof. Let $D := \bigcap_{n\in\mathbb{N}} O_n$. Show that for $x \in X$, $\varepsilon > 0$ arbitrary we have $B_\varepsilon(x) \cap D \neq \emptyset$. We define iteratively a sequence $(x_n)_{n\in\mathbb{N}}$.

n = 1 Take x_1, ε_1 such that

$$\overline{B_{\varepsilon_1}(x_1)} \subset O_1 \cap B_\varepsilon(x) \text{ with } \varepsilon_1 < \frac{\varepsilon}{2}$$

n − 1 → n Given $x_{n-1}, \varepsilon_{n-1}$, take x_n, ε_n such that

$$\overline{B_{C_n}(x_n)} \subset O_n \cap B_{\varepsilon_{n-1}}(x_{n-1}) \quad \text{and} \quad \varepsilon_n < \frac{\varepsilon_{n-1}}{2}$$

This provides sequences $(x_n)_n, (\varepsilon_n)_n$ such that $\varepsilon_n < \frac{\varepsilon}{2^n}$ and $x_n \in B_{\varepsilon_N}(x_N) \forall n \geq N$

$$\implies (x_n)_n \text{ is Cauchy, } X \text{ complete } \implies \exists x \in X : x_n \to x$$

$$\text{since } x_n \in \overline{B_{\varepsilon_N}}(x_N) \forall n \geq N \implies x \in \overline{B_{\varepsilon_N}(x_n)} \implies x \in D \cap B_\varepsilon(x)$$

□

We consider a common, but less useful reformulation:

Definition 1.24. *Let (X, d) be a metric space, $M \subset X$. We say*

- *M is* nowhere dense*(dt. "nirgends dicht"), if* $\mathring{\overline{M}} = \emptyset$
- *M is of* first category $\iff$ *M is a countable union of nowhere dense sets*
- *M is of* second category $\iff$ *M is not of first category*

Theorem 1.25 (Baire category theorem (weaker version))**.** *Let (X, d) be a complete metric space. Then (X, d) is of second category.*

In other words (which is a useful formulation): If $X = \bigcup_{n\in\mathbb{N}} C_n \implies \exists n_0 : \mathring{\overline{C}} \neq \emptyset$*. In particular, if*

$$X = \bigcup_{n\in\mathbb{N}} C_n \textit{ with } C_n \textit{ closed } \implies \exists n_0 : \mathring{C}_{n_0} \neq \emptyset$$

Proof. Suppose that $X = \bigcup_{n\in\mathbb{N}} O_n = \bigcup_{n\in\mathbb{N}} \overline{O_n}$ with $\mathring{\overline{O}}_n = \emptyset \forall n$

$$\mathring{\overline{O}}_n = \emptyset \implies \overline{\overline{O}_n^C} = X$$

Why does this implication hold? Because consider $x \in X, \varepsilon > 0$.

$$B_\varepsilon(x) \cap \overline{O}_n^C = \emptyset \implies B_\varepsilon(x) \subset \overline{O}_n \implies \mathring{\overline{O}}_n \neq \emptyset \text{ hence } B_\varepsilon(x) \cap \overline{O}_n^C \neq \emptyset$$

Okay, then we continue by the conclusion ...

$$\implies \overline{O_n}^C \text{ is open and dense } \forall n \xrightarrow{\text{Theorem 1.23}} \bigcap_{n\in\mathbb{N}} \overline{O}_n^C \text{ is dense}$$

$$\bigcap_{n\in\mathbb{N}} \overline{O}_n^C = \left(\bigcup_{n\in\mathbb{N}} \overline{O}_n \right)^C = X^C = \emptyset$$

gives a contradiction □

Remark. 1. *This is a fundamental theorem in Functional Analysis*

2. *This can be used to show that continuous, nowhere differentiable functions exist (construction is left as an exercise, e.g. Weierstrass function)*

2 Normed space

2.1 Fundamentals

Definition 2.1. *Let X be a vector space. A function $\|\cdot\| : X \to [0, \infty)$ is called* seminorm *if*

- $x = 0 \implies \|x\| = 0$
- $\|\lambda x\| = |\lambda| \|x\| \, \forall x \in X, \lambda \in \mathbb{K}$
- $\|x + y\| \leq \|x\| + \|y\| \, \forall x, y \in X$

The first property differs between a norm and a seminorm.

The tuple $(X, \|\cdot\|)$ *is called a* semi-normed space. *We transfer the notions of convergence of sequences, Cauchy sequences and completeness verbatim to semi-normed spaces.*

Example (Not done in lecture)**.** *We found the following examples while studying:*

$$f \text{ linear}, x \mapsto |f(x)| \qquad \text{and} \qquad \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| := |y|$$

Definition 2.2 (Definition and proposition)**.** *Let $(X, \|\cdot\|)$ be a semi-normed space and $(x_n)_n$ be a sequence in X. We say that*

- $\sum_{n=1}^{\infty} x_n$ converges *to $x \in X$ and write $x = \sum_{n=1}^{\infty} x_n$ if $\lim_{m\to\infty} \sum_{n=1}^{m} x_n = x$*
- $\sum_{n=1}^{\infty} x_n$ *is* absolutely convergent *if $\sum_{n=1}^{\infty} \|x_n\|$ converges [$\iff$ $(\sum_{n=1}^{m} \|x_n\|)_m$ is bounded]*

It holds that X is complete iff any absolutely converging series converges.

Proof. $\implies$ Take $m_1 < m_2$ arbitrary, then

$$\left\| \sum_{n=1}^{m_1} x_n - \sum_{n=1}^{m_2} x_n \right\| \leq \sum_{n=m_1+1}^{m_2} \|x_1\| = \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\| \leq \left\| \sum_{n=1}^{m_1} \|x_n\| - \sum_{n=1}^{m_2} \|x_1\| \right\|$$

$$\implies \left(\sum_{n=1}^{m} x_n \right)_m \text{ is Cauchy} \implies \text{convergent}$$

$\impliedby$ Let $(x_n)_n$ be Cauchy. Show that $(x_n)_n$ converges. For $\varepsilon_k = 2^{-k}$, pick N_k such that $\|x_n - x_m\| \leq 2^{-k} \forall n, m \geq N_k$

$$\implies \exists (x_{n_k})_k \text{ a subsequence such that } \left\| x_{n_{k+1}} - x_{n_k} \right\| \leq 2^{-k}$$

Define $y_k := x_{n_{k-1}} - x_{n_k} \implies \sum_k \left\| y_{n_w} \right\| \leq \sum_k 2^{-k} < \infty$

$$\implies \exists y \in X : \sum_{k=1}^{n} y_k \to y \text{ as } n \to \infty$$

$$\sum_{k=1}^{n} y_k = x_{n_{m+1}} - x_{n_1} \implies x_{n_{m+1}} \to y - x_{n_1} \text{ as } n \to \infty$$

So $(x_n)_n$ has a convergent subsequence and $(x_n)_n$ is Cauchy, then $(x_n)_n$ is convergent.

□

Remark. *In $\mathbb{R}^n$, $\sum_n x_n$ is absolutely convergent iff every permutation converges. In general Banach spaces, only the direction $\implies$ is true.*

↓ *This lecture took place on 2019/03/28.*

Proposition 2.3 (Proposition and definition)**.** *Let X be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X. We say $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if*

$$\exists m, M > 0 \forall x \in X : m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1$$

TFAE:

1. *$\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*
2. *For any sequence $(x_n)_n$ and $x \in X, x_n \to x$ with respect to $\|\cdot\|_1 \iff x_n \to x$ with respect to $\|\cdot\|_2$*
3. *For any sequence $(x_n)_n$ we have,*

$$x_n \to 0 \text{ with respect to } \|\cdot\|_1 \iff x_n \to 0 \text{ with respect to } \|\cdot\|_2$$

Proof. (1) $\implies$ (2) $\implies$ (3) is immediate.

It remains to show that:

(3) $\implies$ (1) Suppose no $M > 0$ exists such that $\|x\|_2 \leq M \cdot \|x\|_1 \; \forall x \in X$.

$$\implies \forall n \in \mathbb{N} \exists x_n \in X : \|x_n\|_2 > n\|x_n\|_1$$

Let $y_n := \frac{x_n}{\|x_n\|_1 n}$. Then $\left\|y_n\right\|_1 = \frac{1}{n} \to 0$ hence $y_n \to 0$, but $\left\|y_n\right\|_2 > n\left\|y_n\right\|_1 = 1$.

$$\implies y_n \not\to 0 \text{ with } \|\cdot\|_2$$

This gives a contradiction.

The second estimate is left as an exercise.

□

Remark. *If $\dim(X) < \infty$, then any two norms on X are equivalent.*

Definition 2.4 (Quotient spaces)**.** *Let $(X, \|\cdot\|)$ be a normed space and $Y \subset X$ a subspace. Define a relation "$\sim$" on X with $x \sim y :\iff x - y \in Y$.*

Then $\sim$ defines an equivalence relation on X. We define

- $[X]_\sim = \{y \in X \mid x \sim y\}$ *is the* equivalence class *of* $x \in X$
- $X/Y \coloneqq \{[x]_\sim \mid x \in X\}$ *is the* quotient space
- $\pi : \begin{cases} X \to X/y \\ x \mapsto [x]_\sim \end{cases}$

Defining $[x] + [y] \coloneqq [x + y]$

$$\lambda[x] \coloneqq [\lambda x] \qquad \hat{0} \coloneqq [0]$$

We get that:

1. *X/Y is a vector space*
2. *$\left\|[x]\right\|_{X/Y} \coloneqq \inf_{y \in [x]} \left\|y\right\|_X$ is a semi-norm.*
3. *If Y is closed, then $\|\cdot\|_{X/Y}$ is a norm.*
4. *If X is complete and Y closed, then $(X/Y, \|\cdot\|_{X/Y})$ is a Banach space.*

Proof.
- Equivalence relation
- Vector space with "+" and "$\lambda[x]$" are well-defined

This is left as an exercise to the reader.

2. – First of all, $\|\cdot\|_{X/Y} \geq 0$ is trivial.

$$\left\|[0]\right\|_{X/Y} \underbrace{=}_{\text{since } [0]=Y} \inf_{y \in Y} \|Y\| \leq \|0\| = 0$$

– Secondly, consider $\lambda \in \mathbb{K}$, $[x] \in X/Y$.
Show that: $\left\|\lambda[x]\right\|_{X/Y} = |\lambda| \left\|[x]\right\|_{X/Y}$.
Trivial, if $\lambda = 0$. Assume $\lambda \neq 0$,

$$\left\|\lambda[x]\right\|_{X/Y} = \left\|[\lambda x]\right\|_{X/Y} = \inf_{y \in [\lambda x]} \left\|y\right\| = \inf_{y \in X, \frac{y}{\lambda} \in [x]} \left\|y\right\| = \inf_{w \in [x]} \|\lambda w\| = |\lambda| \overbrace{\inf_{u \in [x]} \|u\|}^{\|[x]\|_{X/Y}}$$

– Take $[x_1], [x_2] \in X/Y, \varepsilon > 0$. We note that

$$\left\|[x]\right\|_{X/Y} = \inf_{\substack{y \in X \\ w \in Y \\ w := x \cdot y}} \left\|y\right\| = \inf_{w \in Y} \|x - w\|$$

Hence we can take $y_1, y_2 \in Y$ such that $\left\|x_1 - y_i\right\| < \left\|[x_i]\right\|_{X/Y} + \varepsilon \qquad \varepsilon \in [1, 2)$.

$$\implies \left\|[x_1] + [x_2]\right\|_{X/Y} = \left\|[x_1 + x_2]\right\|_{X/Y} \leq \left\|x_1 + x_2 - (y_1 + y_2)\right\|$$
$$\leq \left\|x_1 - y_1\right\| + \left\|x_2 - y_2\right\| \leq \left\|[x_1]\right\|_{X/Y} + \left\|[x_2]\right\|_{X/Y} + 2\varepsilon$$

Since ε was arbitrary, the assertion follows.

3. Suppose Y is closed if $\left\|[x]\right\|_{X/Y} = 0$, then

$$\inf_{y \in Y} \left\|x - y\right\| = 0 \implies \exists (y_n)_n \text{ in } Y \text{ s.t. } \lim_{n\to\infty} \left\|x - y_n\right\| = 0$$

$$Y \text{ closed } \implies x \in Y \implies [x] = [0] = \hat{0}$$

4. Take $([x_n])_n$ to be a sequence in X/Y and suppose that $\sum_{i=1}^{\infty} \left\|[x_n]\right\|_{X/Y} < \infty$. If we can show that $\exists [x] \in X/Y$ such that $\sum_{i=1}^{\infty} [x_n] = [x]$, then by Proposition 2.2, X/Y is complete.

   Choose $\forall n \in \mathbb{N} : \tilde{x}_n \in [x_n]$ such that $\|\tilde{x}_n\| \leq \left\|[x_n]\right\|_{X/Y} + 2^{-n}$

$$\implies \sum_{n=1}^{\infty} \|\tilde{x}_n\| \leq \sum_{n=1}^{\infty} \left(\left\|[x_n]\right\|_{X/Y} + 2^{-n} \right) < c < \infty$$

$$X \text{ complete } \implies \exists x \in X : \sum_{n=1}^{\infty} \tilde{x}_n = x \qquad \left\| [x] - \sum_{n=1}^{m} \underbrace{[x_n]}_{[\tilde{x}_n]} \right\|_{X/Y} \leq \left\| x - \underbrace{\sum_{k=0}^{n} \tilde{x}_k}_{\to 0} \right\|$$

□

↓ *This lecture took place on 2019/04/02.*

Corollary 2.5. *Let X be a vector space with semi-norm $\|\cdot\|_X : X \to [0, \infty)$. Then*

- $N = \{x \in X \mid \|x\|_X = 0\}$ *is a subspace at* X
- $\left\|[X]\right\| := \|X\|_p$ *is a norm on* X/N
- *If* X *is complete, then* $(X/N, \|\cdot\|)$ *is a Banach space.*

Proof. The proof is left as an exercise. □

Proposition 2.6. *Let $(X, \|\cdot\|)$ be a normed space, $U \subset X$ is a subspace. Then*

- $\overline{U}$ *is also a subspace.*
- X *is separable iff* $\exists A \subset X$ *complete such that* $X = \overline{\mathcal{L}(A)}$ *where* $L(A) = \{\sum_{i=1}^{n} \lambda_i x_i \mid x_i \in A, \lambda_i \in \mathcal{K}, n \in \mathbb{N}\}$

Proof.
- Left as an exercise
- $\implies$ True since $\exists A \subset X$ countable such that $\overline{A} = X \implies \underline{X} = \overline{A} \subset \overline{L(A)} \subset X$

$\Longleftarrow$ Let $A \subset X$ countable such that $\overline{\mathcal{L}(A)} = X$. Define

$$B = \left\{ \sum_{i=1}^{n} (\lambda_i + i\mu_i) x_i \mid \lambda_i, \mu_i \in \mathbb{X}, x \in A, n \in \mathbb{N} \right\}$$

where i is the imaginary unit if $\mathbb{K} = \mathbb{C}$ or $i = 0$ if $\mathbb{K} = \mathbb{R}$. Then B is countable.
Show: $\forall x \in X \forall \varepsilon \exists x \in B : \left\| x - y \right\| < \varepsilon$.
Take $x \in X, \varepsilon > 0 \implies \exists x_0 \in \mathcal{L}(A) : \|x - x_i\| < \frac{\varepsilon}{2}$ when $x_0 = \sum_{i=0}^{n} (\lambda_i + i\mu_i) x_i$ with $\lambda_i, \mu_i \in \mathbb{R}, x_i \in A$. Choose $\lambda', \mu_i' \in \mathbb{Q}$ such that

$$\sqrt{(\lambda_i - \lambda_i')^2 + (\mu_i - \mu_i')^2} \leq \frac{\varepsilon}{L \cdot \sum_{i=1}^{n} \|x_i\|} \forall i \in \{1, \ldots, n\}$$

Let $y := \sum_{j=1}^{n} (\lambda_i' + \mu_i') x_i \subset B$.

$$\begin{aligned}
\implies \left\| x - y \right\| \leq \|x - x_0\| + \left\| x_0 - y \right\| & & \leq \frac{\varepsilon}{2} \\
\leq \sum_{i=1}^{n} \left| (\lambda_i + i\varepsilon_i) - (\lambda_i' + i\mu_i') \right| \|x_i\| & & \\
\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|x_i\| \cdot \frac{\varepsilon}{2 \sum_{i=1}^{n} \|x_i\|} = \varepsilon & &
\end{aligned}$$

□

Proposition 2.7 (Proposition and definition)**.** *Let* $(X, \|\cdot\|_{x_i})$ *for* $i = 1, \ldots, n$ *be a normed space. Denote by*

$$X_1 \otimes X_1 \otimes \ldots \otimes X_n = \bigotimes_{i=1}^{n} X_i = X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i, i = 1, \ldots, n\}$$

For $p \in [1, \infty]$*, define*

$$\left\| (x_1, \ldots, x_n) \right\|_{\otimes_i X_i, p} = \begin{cases} \left(\sum_{i=1}^{n} \|x_i\|_{x_i}^{p} \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty] \\ \max_{i=1,\ldots,n} \|x_i\|_{x_i} & \text{if } p = \infty \end{cases}$$

Then

- $(\bigotimes_i X_i, \|\cdot\|_{\bigotimes_i X_i, p})$ *is a normed space with respect to componentwise addition and multiplication.*
- *If all* X_i *are complete, then* $\bigotimes_{i=1}^{n} X_i$ *is complete.*
- *All norms* $\|\cdot\|_{\bigotimes_i X_i, p}$ *are equivalent.*

Proof.

- Vector space properties: Left as an exercise
- Norm: $\|x\|_{\bigotimes_i X_i, n} = 0 \iff x = 0$
  $\|\lambda x\|_{\bigotimes_i X_i, p} = |\lambda| \, \|x\|_{\bigotimes_i X_i, p}$

- Triangle inequality: $p = 1$, $p = \infty$
  $p \in (1, \infty)$. Take $x, y \in \bigotimes_{i=1}^{n} X_i$ and we write $\|\cdot\|_p = \|\cdot\|_{\bigotimes_i X_i, p}$.

$$
\begin{aligned}
\implies \|x+y\|_p^p &= \sum_{i=1}^{n} \|x_i + y_i\|_{X_i} \|x_i + y_i\|_{x_i}^{p-1} \\
&\leq \sum_{i=1}^{n} \|x_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} + \sum_{i=1}^{n} \|y_i\|_{X_i} \|x_i + y_i\|_{X_i}^{p-1} \\
&\underbrace{\leq}_{\substack{\text{Hölder} \\ \text{ineq.}}} \left(\sum_{i=1}^{n} \|X_i\|_{X_i}^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} \|x_i + y_i\|_{X_i}^{(p-1)q} \right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{i=1}^{n} \|y_i\|_{x_i}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \|x_i + y_i\|^{(p-1)q} \right)^{\frac{1}{q}} \\
&= \|x\|_p \|x+y\|_p^{p-1} + \|y\|_p \|x+y\|_p^{p-1} \\
&= \left(\|x\|_p + \|y\|_p \right) \cdot \|x+y\|_p^{p-1}
\end{aligned}
$$

$$\implies \|x+y\|_p \leq \|x\|_p + \|y\|_p \quad \text{if } x + y \neq 0 \text{ (trivial otherwise)}$$

  Completeness, equivalence is trivial to show (left as an exercise) (use norm equivalence in $\mathbb{R}^n$)

□

Definition 2.8. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. If $j : X \to Y$ is linear such that $\|j(x)\|_Y = \|x\|_X$ (hence j is injective) then j is called isometric embedding from X to Y. If j is bijective, then j is called* isometric isomorphism *and we say $X = Y$ up to isomorphism.*

Proposition 2.9. *Let $(X, \|\cdot\|_X)$ be a normed space. Then $\exists(\hat{X}, \|\cdot\|_X)$ a Banach space such that*

1. *$\exists$ isometric embedding, $i : X \to \hat{X}$ such that $\overline{j(X)} = \hat{X}$ [$\hat{X}$ can be regarded as completion of X]*
2. *If $j_1 : X \to Y$ is an isometric embedding on Y, a Banach space*

$$\implies \exists i_2 : \hat{X} \to Y$$

   an isometric embedding such that $j_2 \circ j = j_1$ and if $\overline{j_1(x)} = Y$ then j_2 is an isometric isomorphism. Thus "the completion is essentially unique".

Proof. 1. Set $\hat{X} = \{(x_n)_n \mid x_n \in X \forall n, (x_n)_n \text{ is } \mathit{Cauchy}\}$. $\hat{X}$ is a vector space by

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \qquad \lambda(x_n)_n := (\lambda x_n)_n \quad \hat{0} := (0)_n$$

Define $\left\|(x_n)_n\right\|_{\tilde{X}} := \lim_{n\to\infty} \|x_n\|$ [well-defined since $(\|x_n\|)_n$ is Cauchy in $\mathbb{R}$]. Then $\|\cdot\|_{\tilde{X}}$ is a semi-norm (proof is left as an exercise). Setting $N = \left\{(X_n)_n \mid \left\|(X_n)_n\right\|_{\tilde{X}} = 0\right\}$. By Corollary 2.5, $\hat{X} := \tilde{X} \setminus N$ with $\left\|[(X_n)_n]\right\|_{\hat{X}} = \left\|(X_n)_n\right\|_{\tilde{X}}$ is a normed space. Define

$$j : X \to \hat{X} \qquad x \mapsto [(x)_n]$$

then j is linear and $\left\|j(x)\right\|_{\hat{x}} = \left\|[(x)_n]\right\|_{\hat{x}} = \lim_{n\to\infty} \|x\| = \|x\|$. So j is an isometric embedding.

Show: $\overline{j(X)} = \hat{X}$.

Take $\hat{x} = [(X_n)_n] \in \hat{X}$. Define $y_n := j(x_n) \in \hat{X}$.

$$\begin{aligned}\implies \left\|y_m - [(x_n)_n]\right\|_{\hat{X}} = \left\|(x_m)_n - (x_n)_n\right\|_{\tilde{X}} &= \lim_{n\to\infty} \|x_m - x_n\| \\ &= \lim_{n\geq n_0} \|x_m - x_n\| < \varepsilon\end{aligned}$$

Now, $\forall \varepsilon > 0 \exists n \forall n, m \geq n_0 : \|x_n - x_m\| < \varepsilon$.

Show: $\hat{X}$ is complete.

Let $(y_n)_n$ be Cauchy in $\hat{X}$. Pick $X_n \in X$ such that $\left\|j(x_n) - y_n\right\|_{\hat{X}} \leq \frac{1}{n}$ $(\overline{j(x)} = \hat{x})$

$$\implies \|x_n - x_m\|_X = \left\|j(x_n) - j(x_m)\right\|_{\hat{X}} \leq \left\|j(x_n) - y_n\right\|_{\hat{X}} + \left\|y_n - y_m\right\|_{\hat{X}} + \left\|y_n - j(x_n)\right\|_{\hat{X}}$$

Take $\varepsilon > 0$. Then $\exists n_0 \forall n, m \geq n_0 : \left\|y_n - y_m\right\|_{\hat{X}} < \frac{\varepsilon}{3}$. Pick n_1 such that $\forall n \geq n_1 : \frac{1}{n} < \frac{\varepsilon}{100}$.

$$\implies \forall n, m > \max(n_0, n_0) : \|x_n - x_m\| \leq \frac{\varepsilon}{100} + \frac{\varepsilon}{3} + \frac{\varepsilon}{100} < \varepsilon$$

$\implies (x_n)_n$ is Cauchy. Let $y := (X_n)_n \in \tilde{X}$. Then

$$\left\|y_n - [y]\right\|_{\hat{X}} \leq \left\|y_n - j(x_n)\right\|_{\hat{X}} + \left\|j(x_n) - [y]\right\|_{\hat{X}} \leq \frac{1}{n} + \lim_{n\to\infty} \|x_n - x_m\|_X \xrightarrow{n\to\infty} 0$$

2. ↓ *This lecture took place on 2019/04/04.*

Let $\hat{x} \in \hat{X} \implies \exists (x_n)_n \in X$ such that $j(x_n) \to \hat{x} \implies \|x_n - x_m\|_X = \left\|j(x_n) - j(x_m)\right\|_{\hat{X}}$.

$\implies (x_n)_n$ is a Cauchy sequence.

$\implies j_1(x_n)$ is a Cauchy sequence in Y.

$\implies \exists \lim_{n\to\infty} j_1(x_n) := y$

Using this, we define $j_2 : \hat{X} \to Y$ with $\hat{x} \mapsto \lim_{n\to\infty} j_1(x_1)$ where $j(x_1) \to \hat{x}$.

Well-defined? Take $\hat{x} \in \hat{X}$ and $j(x_n) \to \hat{x}$, $j(y_n) \to \hat{x}$.

$$\implies \left\|i_1(x_n) - j_1(y_n)\right\| = \left\|x_n - y_n\right\| = \left\|j(x_n) - j(y_n)\right\| \to 0 \text{ as } n \to \infty$$

$$\implies \lim_{n\to\infty} j_1(x_n) = \lim_{n\to\infty} j_1(y_n) \implies j_1 \text{ well-defined}$$

Show linearity is left as an exercise. By isometry, take $\hat{x} \in \hat{X}$,

$$\left|i_2(\hat{x})\right| \underbrace{=}_{j(x_n)\to\hat{x}} \lim_{n\to\infty} \left\|j_1(x_1)\right\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \left\|i(x_n)\right\| = \|\hat{x}\|$$

Show: $j_2 \circ j = j_1$. Take $x \in X \implies (x_n)$ is such that $j(x) \to j(x) \implies j_2(j(x)) = \lim_{n\to\infty} j_1(x) = j_1(x)$.

Assume that $\overline{j_1(x)} = Y$. Take $y \in Y$. Find $\hat{x} \in \hat{X}$ such that $i_2(\hat{x}) = y$. By $\overline{j_1(x)} = Y \implies \exists (x_n)_n$ in X such that $i_1(x_n) \to Y \implies (j_1(x_n))_n$ is Cauchy.

$$\implies (x_n)_n \text{ Cauchy} \implies (j(x_n))_n \text{ is Cauchy}$$

$$\xRightarrow{\hat{X}\text{ complete}} \exists \hat{x} \text{ such that } \lim_{n\to\infty} j(x_n) = \hat{x} \implies j_2(\hat{x}) = \lim_{n\to\infty} j_2(x_n) = Y$$

□

2.2 Important examples of normed spaces

Definition 2.10 (Basic notation). *Let $\Omega \subset \mathbb{R}^N$, $f : \Omega \to \mathbb{K}^M$ with $N, M \subset \mathbb{N}$.*

- *We call Ω a* domain*(dt. "Gebiet") if Ω is open and connected, where connected means that $\forall x, y \in \Omega$ there is a curve in Ω connecting X and Y.*
- *For $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}_0^N$ define $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$. If f is r-times continuously differentiable, we set for $\alpha \in \mathbb{N}_0^N$, $\{\alpha\} \leq r$.*

$$D^\infty f := \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} x_n} f$$

where $\frac{\partial^{\alpha_1}}{\partial^{\alpha_i} x_i}$ is the partial derivative of f with respect to x_i of order α_i.

Example 2.11. *Let $N = 2$ and $\alpha = (1, 1)$.*

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} f$$

Let $\alpha = (2, 0)$.

$$D^\infty f = \frac{\partial^{\alpha_1}}{\partial^2 x_1} f$$

- *For $z \in \mathbb{K}^N$ we denote $|z| := \sqrt{\sum_{i=1}^N |z_i|^2}$.*[1]
- *We say $E \subset \Omega$ is compact in Ω and we write $E \Subset \Omega$ if E is compact.*

Remark. *If $E \Subset \Omega$, then $\exists \delta > 0 : \inf\left\{\left\|x - y\right\| \mid x \in E, y \in \partial\Omega\right\} > 0$.*

[1] This is an abuse of notation with $|\alpha|$ for $\alpha \in \mathbb{N}_0^N$

Proof. Left as an exercise (use compactness) □

- f *is* compactly supported *in* Ω *if* $\operatorname{supp}(f) \ll \Omega$.
- $\operatorname{supp}(f) := \overline{\left\{x \in \Omega \mid \left\|f(x)\right\| > 0\right\}}$

↓ *This lecture took place on 2019/04/09.*

Definition 2.12 (Definition and proposition, Spaces of continuous functions)**.**
Let $\Omega \subset \mathbb{R}^N$ *be a domain. We define*

$$C_b(\Omega, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \varphi \textit{ bounded}\right\} \textit{ with } \left\|\varphi\right\|_{C_b} = \left\|\varphi\right\|_\infty = \sup_{x\in\Omega}\left|\varphi(x)\right|$$

$$C(\overline{\Omega}, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \varphi \textit{ can be continuously extended to } \overline{\Omega}\right\}, \left\|\varphi\right\|_C := \left\|\varphi\right\|_\infty$$

$$C^r(\overline{\Omega}, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid D^\alpha\varphi \in C(\overline{\Omega}, \mathbb{K}^M) \forall \alpha \in \mathbb{N}_0^N : |\alpha| \le r\right\} \textit{ and} \left\|\varphi\right\|_{C^r} = \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le r}} \left\|D^\infty\varphi\right\|_\infty$$

$$C_C^r(\Omega, \mathbb{K}^M) = \left\{\varphi : \Omega \to \mathbb{K}^M \mid \operatorname{supp}(\varphi) \ll \Omega, \varphi \in C^r(\overline{\Omega}, \mathbb{K}^M)\right\} \textit{ and } \left\|\varphi\right\|_{C_C^r} = \left\|\varphi\right\|_{C^r}$$

$$C^\infty(\overline{\Omega}, \mathbb{K}^M) = \bigcap_{r\in\mathbb{N}} C^r(\overline{\Omega}, \mathbb{K}^M)$$

$$D(\Omega, \mathbb{K}^M) = C_C^\infty(\Omega, \mathbb{K}^M) := \bigcap_{r\in\mathbb{N}} C_C^r(\Omega, \mathbb{K}^M), C_0^r(\Omega, \mathbb{K}^M) = \overline{C_C^r(\Omega, \mathbb{K}^M)} \textit{ in } C^r(\overline{\Omega}, \mathbb{K}^M)$$

Then for any bounded Ω, C^r, C_0^r, C_b *are Banach spaces and* C_C^r *is a normed space.*

Recall: $z \in \mathbb{K}^M \implies |z| := \sqrt{\sum_{i=1}^M |z_i|^2}$

Proof. The functions $\|\cdot\|_{C_b}, \|\cdot\|_{C^r}$ are norms (proof is left as an exercise).

Show that C_b is complete: Take $(\varphi_n)_n$ in C_b to be Cauchy.

$$\implies \forall x \in \Omega : (\varphi_n(x))_n \text{ is Cauchy in } \mathbb{K}^n$$

because $\left|\varphi_n(x) - \varphi_m(x)\right| \le \left\|\varphi_n - \varphi_m\right\|_\infty$. Hence we can define $\varphi(x) := \lim_{n\to\infty} \varphi_n(x)$.

Show: $\varphi_n \to \varphi$ in $\|\cdot\|_\infty$. Take $\varepsilon > 0$. Show that $\exists n_0 \forall n \ge n_0 : \left\|\varphi - \varphi_n\right\|_\infty < \varepsilon$. Take n_0 such that $\forall n, m \ge n_0 : \left\|\varphi_n - \varphi_m\right\|_\infty < \varepsilon$. Take $m \ge n_0$.

$$\implies \forall x \in \Omega : \left|\varphi(x) - \varphi_m(x)\right| = \lim_{\substack{n\to\infty \\ n\ge n_0}} \left|\varphi_n(x) - \varphi_m(x)\right| < \left\|\varphi_n - \varphi_m\right\|_\infty$$

Show: φ is bounded, i.e. $\exists C > 0 : \left|\varphi(x)\right| \le C < \left\|\varphi_n - \varphi_m\right\|_\varepsilon < \infty$. Take n suchthat $\left\|\varphi - \varphi_n\right\|_\infty < 1$

$$\implies \forall x \in \Omega : \left|\varphi(x)\right| > \left|\varphi(x) - \varphi_n(x)\right| + \left|\varphi_n(x)\right| \le 1 + \underbrace{\left\|\varphi_n\right\|}_{=C}$$

Now $C^r(\overline{\Omega}, \mathbb{K}^n)$ is a subspace of $C^b(\Omega, \mathbb{K}^n)$. Also $C^r(\overline{\Omega}, \mathbb{K}^n)$ is closed, since the uniform limit of $\varphi \in C^r(\overline{\Omega}, \mathbb{K}^n)$ with respect to $\|\cdot\|_{C^r}$ is again in $C^r(\overline{\Omega}, \mathbb{K}^M)$ [a result from Analysis].

$$\implies C^r(\overline{\Omega}, \mathbb{K}^M) \text{ is a Banach space}$$

$C^r_C(\overline{\Omega}, \mathbb{K}^M)$ is closed by definition, hence Banach.

$C^r_C(\Omega, \mathbb{K}^M)$ is a vector space, since $\forall \lambda \in \mathbb{K} : \varphi \in C^r_0(\Omega, \mathbb{K}^M) : \operatorname{supp}(\lambda\varphi) = \operatorname{supp}(\varphi)$ and for $\varphi, \Psi \in C^r_0(\Omega, \mathbb{K}^M) : \operatorname{supp}(\varphi + \Psi) \ll \Omega$. □

Definition 2.13 (Definition and proposition). *Let (Ω, Σ, μ) with $\Omega \subset \mathbb{R}^N$ be a measure space (i.e. Σ is a sigma algebra and μ is a measure). For $p \in [1, \infty)$, we define*

$$\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu) = \left\{ f : \Omega \to \mathbb{K}^M \mid f\ \mu-\ \textit{measurable and} \int_\Omega \left|f(x)\right|^p d\mu(x) < \infty \right\}$$

$$\left\|f\right\|_p^* = \left(\int_\Omega \left\|f(x)\right\|^p \, d\mu(x) \right)^{\frac{1}{p}}$$

$$\mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) := \left\{ f : \Omega \to \mathbb{K}^M \mid \exists N \in \Sigma : \mu(N) = 0 \wedge \sup_{x \in \Omega \setminus N} \left|f(x)\right| < \infty \right\}$$

$$\left\|f\right\|_\infty^* = \inf_{\substack{N \in \Sigma \\ \mu(N)=0}} \sup_{x \in \Omega \setminus N} \left|f(x)\right|$$

Our proposition is that these are semi-norms.

Proof. Show that $\|\cdot\|_p^*$ for $p \in [1, \infty]$ are seminorms.

They cannot be norms since $\left\|f\right\|_p^* = 0$ for

$$f(x) = \begin{cases} 1 & x \in N \\ 0 & x \notin N \end{cases}$$

$0 \neq N \in \Sigma, \mu(N) = 0.$ □

Proposition 2.14 (Hölder inequality)**.** *Let $p \in [1, \infty]$ and*

$$a = p^* = \begin{cases} \frac{p}{p-1} & \textit{if } \in (1, \infty) \\ 1 & \textit{if } p = \infty \\ \infty & \textit{if } p = 1 \end{cases}$$

$$\frac{1}{p} + \frac{1}{p^*} = 1$$

If $f \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ and $q \in \mathcal{L}^q(\Omega, \mathbb{K}^M, \mu)$ then for both

$$f \cdot g : \Omega \to \mathbb{K} \textit{ with } x \mapsto (f(x), g(x)) = \sum_{i=1}^M f_i(x) = \overline{g_i(x)}$$

$$f \otimes g : \Omega \to \mathbb{K}^M \textit{ with } x \mapsto (f_i(x), \varphi_i(x))_{i=1}^M$$

we have that $fg \in \mathcal{L}^1(\Omega, \mathbb{K}, \mu)$ *and* $f \otimes g \in L^1(\Omega, \mathbb{K}^M, \mu)$ *and* $\left\|f \otimes g\right\|_1^* \leq \left\|fg\right\|_1^* \leq \left\|f\right\|_p^* \cdot \left\|q\right\|_q^*$.

Proof. **Case** $p \in (1, \infty)$**:** Intermediate result: $\forall \sigma, \tau \geq 0, r \in (0, 1] : \sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau$ [AGM-inequality].

Proof.

Case $\sigma = 0$ or $\tau = 0$: immediate

Case $\sigma, \tau > 0$:

$$\begin{aligned} \log(\sigma^r \tau^{1-r}) &= r \log(\sigma) + (1-r)\log(\tau) \leq \log(r\sigma + (1-r)\tau) \\ &\quad \text{since } \log''(x) \leq 0 \text{ implies that log is concave} \\ \text{log is monotonic} \implies &\sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau \end{aligned}$$

$\square$

Let $A := \left(\left\|f\right\|_p^*\right)^p$ and $B := \left(\left\|\varphi\right\|_1^*\right)^q$ with $r = \frac{1}{p} \in (0, 1]$ we get

$$\forall x \in \Omega : \left(\frac{\left|f(x)\right|^p}{A}\right)^{\frac{1}{p}} \left(\frac{\varphi(x)}{B}\right)^{\frac{1}{q}} = \frac{1}{p}\frac{\left|f(x)\right|^p}{A} + \frac{1}{q}\frac{\left|q(x)\right|^q}{B}$$

$$\begin{aligned} \implies \frac{\int_\Omega \left|f(x)\right|\left|q(x)\right| \, d\mu(x)}{A^{\frac{1}{p}} B^{\frac{1}{q}}} &\leq \frac{1}{p}\frac{\int_\Omega \left|f(x)\right|^p \, d\mu(x)}{A} + \frac{1}{q}\frac{\int_\Omega \left|q(x)\right|^q \, d\mu(x)}{B} \\ \implies \int_\Omega \left|f(x)\right|\left|g(x)\right| \, d\mu(x) &\leq \left\|f\right\|_p^* \left\|g\right\|_q^* = \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

Now: $\left\|f \cdot g\right\|_x^* \leq \left\|f\right\|_p^* \cdot \left\|g\right\|_q^*$ follows since $\left|\langle x, y\rangle\right| \leq |x|\left|y\right| \forall x, y \in \mathbb{K}^M$.

Also:

$$\forall x \in \Omega : \left|f \otimes g(x)\right| = \sum_{i=1}^{M} \left|f_i(x)\right|\left|g_i(x)\right| = \begin{pmatrix} \left|f_1(x)\right| & \left|g_1(x)\right| \\ \vdots & \vdots \\ \left|f_n(x)\right| & \left|g_n(x)\right| \end{pmatrix} \leq \left|f(x)\right|\left|g(x)\right|$$

$$\implies \int_\Omega \left|f \otimes g(x)\right| \, d\mu(x) \leq \left\|f\right\|_p^* \cdot \left\|g\right\|_q^*$$

Case $p \in \{1, \infty\}$**:** Without loss of generality assume that $p = 1, q = \infty$. $\forall N \in \Sigma$ with $\mu(N) = 0$ we get

$$\begin{aligned} \int_\Omega \left|f(x)\right|\left|g(x)\right| \, d\mu(x) &= \int_{\Omega \setminus N} \left|f(x)\right|\left|q(x)\right| \, \mu(x) \\ &\leq \int_{\Omega \setminus N} \left|f(x)\right| \, d\mu(x) \cdot \sup_{x \in \Omega \setminus N} \left|q(x)\right| \quad = \int_\Omega \left|f(x)\right| \, d\mu(x) \cdot \sup_{x \in \Omega \setminus N} \left|q(x)\right| \end{aligned}$$

Taking the infimum over all such N, then

$$\int_\Omega \left|f(x)\right| \left|g(x)\right| \, d\mu(x) \leq \left\|f\right\|_1^* \cdot \left\|g\right\|_\infty^*$$

And the result follows again from $\left|\langle x, y\rangle\right| \leq |x| \cdot \left|y\right|$ and componentwise $\left|\langle x_i, y_i\rangle_i\right| \leq |x| \left|y\right| \forall x, y \in \mathbb{K}^M$

□

Proposition 2.15 (Minkowski inequality)**.** *For $p \in [1, \infty]$, $f, g \in \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$, we have that $\left\|f + g\right\|_p^* \leq \left\|f\right\|_p^* + \left\|g\right\|_p^*$ with $\left\|f\right\|_\infty := \inf_{\mu(N)\to 0} \sup_{x\in\Omega\setminus N} \left|f(x)\right|$.*

Proof. **Case $p = 1$:** trivial

Case $p \in (1, \infty)$:

$$\begin{aligned}
\left(\left\|f + g\right\|_p^*\right)^p &= \int_\Omega \left|f(x) + g(x)\right| \, d\mu(x) \\
&= \int_\Omega \left|f(x)\right| \cdot \left|f(x) + g(x)\right|^{p-1} \, d\mu(x) \\
&+ \int_\Omega \left|g(x)\right| \left|f(x) + g(x)\right|^{p-1} \, d\mu(x) \\
&\leq \left\|f\right\|_p^* \cdot \left\|\left|f + g\right|^{p-1}\right\|_q^* + \left\|g\right\|_p^* \cdot \left\|\left|f + g\right|^{p-1}\right\|_q^*
\end{aligned}$$

Recognize that $\left(\int \left|f + g\right|^p\right)^{\frac{1}{q}} = \left(\int \left|f + g\right|^{(p-1)q}\right)^{\frac{1}{q}}$ because $p = q \cdot (p - 1)$

$$\begin{aligned}
&= \left(\left\|f\right\|_p^* + \left\|q\right\|_p^*\right) \left\|\left|f + g\right|\right\|_p^* \\
\implies \quad &\left\|f + g\right\|_p^* \leq \left\|f\right\|_p^* + \left\|g\right\|_p^*
\end{aligned}$$

↓ *This lecture took place on 2019/04/11.*

Case $p = \infty$: First, note that $\forall f \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu) \exists N \in \Sigma$ such that $\mu(N) = 0$ and $\left\|f\right\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty := \sup_{x\in\Omega\setminus N} \left|f(x)\right|$.

Claim 2.16.

$$\left\|f\right\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty := \sup_{x\in\Omega\setminus N} \left|f(x)\right| = \sup_{x\in\Omega\setminus \hat{N}} \left|f(x)\right| \text{ for } \mu(\hat{N}) = 0$$

Proof. For all $n \in \mathbb{N}$, define $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $\left\|f|_{\Omega\setminus N_n}\right\|_\infty \leq \left\|f\right\|_\infty^* + \frac{1}{n}$. Thus with $N := \bigcup_{n\in\mathbb{N}^n} N_n \implies \mu(N) = 0$ and $\left\|f\right\|_\infty^* \leq \left\|f|_{\Omega\setminus N}\right\|_\infty \leq \left\|f\right\|_\infty^* + \frac{1}{n}$. $n \to \infty \implies \left\|f\right\|_\infty^* = \left\|f|_{\Omega\setminus N}\right\|_\infty$. □

For $f, g \in \mathcal{L}^\infty(\Omega, \mathbb{K}^M, \mu)$, pick N_f, N_G such that $\mu(N_C) = \mu(N_g) = 0$ and $\|f\|_\infty^* = \left\| f|_{\Omega \setminus N_\varepsilon} \right\|_\infty$ and $\|g\|_\infty^* = \left\| g|_{\Omega \setminus N_g} \right\|_\infty$.

$$\begin{aligned}
\implies \|f+g\|_\infty^* &\leq \left\|(f+g)_{\Omega \setminus (N_f \cup N_g)}\right\|_\infty \\
&\leq \left\| f|_{f \setminus (N_f \cup N_g)} \right\|_\infty + \left\| g|_{\Omega \setminus (N_f \cup N_g)} \right\|_\infty \\
&\leq \left\| f|_{\Omega \setminus N_f} \right\|_\infty + \left\| g|_{\Omega \setminus N_g} \right\|_\infty = \|f\|_\infty^* + \|g\|_\infty^*
\end{aligned}$$

□

Proposition 2.17. *Let $p \in [1, \infty]$. Then $\|\cdot\|_p^*$ is a seminorm on $\mathcal{L}^p(\Omega, \mathcal{K}^M, \mu)$ and $\mathcal{L}^n(\Omega, \mathcal{K}^M, \mu)$ is complete with the seminorm. With $M := \left\{ f \in \mathcal{L}^\infty \mid \|f\|_p^* = 0 \right\}$, we get that $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)/M$ is a Banach space with respect to $\|[f]\|_p := \|f\|_p^*$.*

Proof. Seminorm is clear by Minkowski's inequality. Give completeness of $f^p(\cdot)$, the rest follows from Corollary 2.5.

Hence, show that $\mathcal{L}^p(\Omega, \mathbb{K}^M, \mu)$ is complete.

Assume $p < \infty$. By Proposition 2.2, it suffices to show that for $f_n(t_n)_n$ in $\mathcal{L}^p(\cdot)$ such that $a := \sum_{n=1}^\infty \|f_n\|_p^* < \infty$.

$$\implies \exists f \in \mathcal{L}^p(\cdot) : f = \sum_{n=1}^\infty f_n$$

Define $\hat{q}(x) := \sum_{n=1}^\infty |f_n(x)| \in [0, \infty]$. Define $\hat{q}_n(x) := \sum_{i=1}^n |f_i(x)|$. Then q_n is measurable and by Minkowski's inequality,

$$\|q_n\|_p^* \leq \sum_{i=1}^n \|f_i\|_p^* \leq \sum_{i=1}^\infty \|f_i\|_p^* = a < \infty$$

Also $\hat{q}_n^p : x \to \hat{q}_n(x)^n$ is a sequence of positive functions and it is monotonically increasing and converging to $\hat{g}^p$.

By Beppo-Levi (from measure theory):

$$\infty_\Omega \hat{g}^p = \lim_{n \to \infty} \int_\Omega \hat{g}_n = \lim_{n \to \infty} (\|q^n\|_p^*)^p = a^p < \infty$$

$\implies \hat{g}^p < \infty$ almost everywhere (except for a μ zero-set). Define $g : \Omega \to \mathbb{R}$,

$$x \mapsto \begin{cases} \hat{g}(x) & \text{if } \hat{g}(x) < \infty \\ 0 & \text{else} \end{cases}$$

We get that $g \in \mathcal{L}^n(\Omega, \mathbb{R}, \mu)$ and $g(x) = \lim_{n \to \infty} \sum_{i=1}^n |f_i(x)|$ μ-almost everywhere. Furthermore, by completeness of $\mathbb{K}^M$, $f(x) := \sum_{i=1}^\infty f_i(x)$ exists for μ-almost everywhere. $x \in \Omega$.

Show: $f = \sum_{i=1}^{\infty} f_i$ in $\mathcal{L}^n(\cdot)$, i.e. show that $\lim_{n\to\infty} \int_\Omega \left|\sum_{i=1}^{\infty} f_i\right|_{d_N}^p = \sigma$.

$$\left\|\sum_{i=1}^{n-1} f_i - \sum_{i=1}^{\infty} f_i\right\|_p^* = \left\|\sum_{i=n}^{\infty} f_i\right\|_p^* \overset{!}{\to} 0$$

By contruction, $\left|f\right| \leq q$ almost everywhere $\implies \int_\Omega \left|f\right|^p \leq \int_\Omega g^p < \infty$. Set $h_n(x) = \left|\sum_{i=n}^{\infty} f_i(x)\right|^p$. Then $h_n(x) \to 0$ for μ-almost everywhere $x \in \Omega$ and $h_n(x) \geq 0$ and

$$0 \leq h_n(x) \leq \left(\sum_{i=n}^{\infty} \left|f_i(x)\right|\right)^p \leq q(x)^p$$

Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_\Omega h_n(x) = \int_\Omega \lim_{n\to\infty} h_n(x) = 0$$

This completes the assertion since

$$\int_\Omega h_n(x) = \int_\Omega \left|\sum_{i=n}^{\infty} f_i(x)\right|^p = \int_\Omega \left|\sum_{i=1}^{n-1} f_i(x) - f(x)\right|^p = \left(\left\|\sum_{i=1}^{n-1} f_i - f\right\|_p^*\right)^p$$

□

↓ *This lecture took place on 2019/04/30.*

Proposition (Proposition 2.15 again). *Let $p \in [1, \infty]$. Then $\|\cdot\|_{L^p}$ is a seminorm, $\mathcal{L}^p(\Omega, \mathbb{K}^n, \mu)$ is complete and $L^p(\Omega, \mathbb{K}^M, \mu) := \mathcal{L}^p(\cdot)/N$ where $N = \left\{f \mid \left\|f\right\|_{L^p} = 0\right\}$ is a Banach space.*

Proof. Assume $p \in [1, \infty]$, then the proof of the last lecture is given.

Assume $p = \infty$. Let $(f_n)_n$ be Cauchy in $\mathcal{L}^\infty$. Remember: $\left\|f\right\|_{L^\infty} := \inf_{\mu(N)=0} \sup_{x\in\Omega\setminus N} \left|f(x)\right|$. Pick $N_{n,m}$ such that $\mu(N_{n,m}) = 0$ and $\left|f_n - f_m\right|_\infty = \left\|(f_n - f_m)\big|_{\Omega\setminus N_{n,m}}\right\|_\infty$. Set $N = \bigcup_{n,m} N_{n,m} \implies \mu(N) = 0$.

Then $\tilde{f}$ is the uniform limit of $f_n \cdot \mathbf{1}_{\Omega\setminus N}$. Hence $\tilde{f}$ is measurable. Also $\left\|\hat{f}\right\|_{L^\infty} := \inf_{\mu(M)=0} \sup_{X\subset\Omega\setminus M} \left|f(x)\right| \leq \left\|f\right\|_\infty \implies \hat{f} \in L^\infty(\Omega, \mathbb{K}^n, \mu)$. Also $\left\|f_n - \hat{f}\right\|_{L^\infty} = \left\|(f_n - f)\big|_{\Omega\setminus N}\right\|_\infty = \left\|f_n\big|_{\Omega\setminus N} - f\right\|_\infty \to 0$ as $n \to \infty$. □

Now $(f_n\big|_N)_n$ is Cauchy with respect to $\|\cdot\|_\infty$. Since $\forall n, m$:

$$\begin{aligned}\left\|f_n\big|_{N^C} - f_M\big|_{N^C}\right\|_\infty &= \left\|(f_n - f_M)\big|_{N^C}\right\|_\infty \\ &\leq \left\|(f_n - f_m)_{N_{m,n}^C}\right\|_\infty \\ &= \left\|f_n - f_m\right\|_{L^\infty}\end{aligned}$$

As in the proof of C_b being a Banach space:

$$\implies \exists f : \Omega \setminus N \to \mathbb{K}^M : \left\|f\right\|_\infty < \infty \text{ and } f_n\big|_{N^C} \to f \text{ w.r.t. } \|\cdot\|_\infty$$

Remark (Important special cases)**. Case 1** *$\mu = \mathcal{L}^N$ is the Lebesgue measure on $\Omega \subset \mathbb{R}^N$ (a domain). In this case we write $L^p(\Omega, \mathbb{K}^M) := L^p(\Omega, \mathbb{K}^M, \lambda^M)$ and $L^p(\Omega) := L^p(\Omega, \mathbb{K})$. Here the space $L^p(\Omega, \mathbb{K})$ is considered as functions which are defined almost everywhere.*

Case 2 *Set $\Omega = \mathbb{N}, \sigma = \mathbb{P}(\mathbb{N}), \mu_c(A) = |A|$.*

Then

- *$f : \Omega \to \mathbb{K}^m$ is identified with a sequence $(x_n)_n$ with $x_n \in \mathbb{K}^M$.*
- *$\int_\Omega f(x)\, d\mu(x) \sim \sum_{i\in\mathbb{N}} x_i \in \mathbb{K}^M$*
- *$\mu_c(A) = 0 \iff A = \emptyset$ and the equivalence class construction becomes obsolete.*

And we denote,

$$l^p(\mathbb{N}, \mathbb{K}^M) = \mathcal{L}^p(\mathbb{N}, \mathbb{K}^M, \mu_c) \qquad l^p := l^p(\mathbb{N}) = l^p(\mathbb{N}, \mathbb{K})$$

2.2.1 Basic properties of Lebesgue spaces

Proposition 2.18. *The space $l^p(\mathbb{N}, \mathbb{K}^M)$ is separable for $p \in [1, \infty]$ and* not *separable for $p = \infty$.*

Proof. $p < \infty$ Define $l_{i,j} \in l^p(\mathbb{N}, \mathbb{K}^M)$ as

$$(l_{ij})_k := \begin{cases} 0 & \text{if } i \neq k \\ \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix}^T & \text{if } i = k \end{cases}$$

Then $A := \left\{e_{ij} \mid i \in \mathbb{N}, j \in \{1, \dots, M\}\right\}$ is countable.

It suffices to show that $\overline{\operatorname{span}(A)} = l^p(\mathbb{N}, \mathbb{K}^M)$.

This is true since $\forall x \in l^p(\mathbb{N}, \mathbb{K}^M) : \forall \varepsilon > 0 \exists n_0 : \sum_{n_0=1}^\infty |x_i|^p < \varepsilon$ and hence $\left\|x - \sum_{i=1}^{n_0} \sum_{j=1}^M x_{ij} e_{ij}\right\|^p = \left(\sum_{i=n_0+1} |x_i|^n\right)^{\frac{1}{n}} < \varepsilon^{\frac{1}{p}}$

$p = \infty$ It suffices to show that $L^\infty(\mathbb{N})$ is not separable (why?). For $M \subset \mathbb{N}$ define $\mathbf{1}_M \in L^\infty$. Then $\triangle := \{\mathbf{1}_M \mid M \subset \mathbb{N}\}$ is uncountable.

For $A \subset L^\infty$ countable and $x \in A$ set $M_X = \left\{y \in L^\infty \mid \left\|x - y\right\|_\infty < \frac{1}{3}\right\} = B_{\frac{1}{3}}(x)$. Then each M_X contains at most one element of $\triangle$ since if $\mathbf{1}_M \neq \mathbf{1}_{M'}$ are such that $\mathbf{1}_M, \mathbf{1}_{M'} \in M_X$.

$$\implies 1 = \|\mathbf{1}_M - \mathbf{1}_{M'}\|_\infty \leq \|\mathbf{1}_M - x\| + \|\mathbf{1}_{X'} - x\| < \frac{2}{3}$$

This gives a contradiction.

$\triangle$ is uncountable, $\{M_X \mid x \in A\}$ is countable.

$$\implies \exists \hat{M} \in \mathbb{N} : \mathbf{1}_{\hat{M}} \notin M_x \forall x \in A$$

$$\implies \left\|\mathbf{1}_{\hat{M}} - x\right\|_\infty \geq \frac{1}{3} \forall x \in A$$

Hence, A is not dense. Since A was arbitrary countable. Thus L^∞ is not separable.

$\square$

2.2.2 Separability of L^p requires a density result

Proposition 2.19. *Let $f \in L^p(\mathbb{R}^N, \mathbb{K}^M)$. Let $p < \infty$. Then $\exists (f_n)_n \in \ldots$ $C_C(\mathbb{R}^N, \mathbb{R}^M)$ such that $\left\|f_n - f\right\|_{L^n} \to 0$ as $n \to \infty$.*

Proof. **Step 1** Reduction to step functions with $E \in \Sigma$.

$$\xi_E(x) := \begin{cases} 1 & x \in E \\ 0 & \text{else} \end{cases}$$

Take $f \in L^p(\ldots)$. For $\varepsilon > 0$, define

$$E_C = \lceil x : \varepsilon \leq \left|f\right| \leq \frac{1}{\varepsilon} \rceil$$

Then $E_\varepsilon \in \Sigma$ and $\int_{\mathbb{R}^N} \left|f\right|^p \geq \varepsilon^p \left|E_\varepsilon\right|$ where $\left|E_\varepsilon\right| := L^N(E_\varepsilon)$.

$$\left|E_\varepsilon\right| < \infty \text{ and } \int_{\mathbb{R}^N} \left|\mathbf{1}_{E_\varepsilon} f\right| \leq \frac{1}{\varepsilon} \cdot \left|E_\varepsilon\right| < \infty$$

$$\implies \mathbf{1}_{E_\varepsilon} f \text{ is integrable } \implies \exists (q_{n,\varepsilon})_n \text{ step functions}$$

such that $\int_{\mathbb{R}^N} \left|\mathbf{1}_{E_\varepsilon} f - q_{n,\varepsilon}\right| \to 0$ as $n \to \infty$. Define

$$f_{n,\varepsilon}(x) := \begin{cases} q_{N,\varepsilon}(x) & \text{if } x \in E_\varepsilon, \left|q_{n,\varepsilon}(x)\right| \leq \frac{2}{\varepsilon} \\ \frac{2}{\varepsilon} \frac{q_{n,\varepsilon}(x)}{\left|q_{n,\varepsilon}(x)\right|} & \text{if } x \in E_\varepsilon, \left|q_{n,\varepsilon}(x)\right| > \frac{2}{\varepsilon} \\ 0 & \text{else} \end{cases}$$

Hence $(f_{n,\varepsilon})_n$ is a sequence of step functions. For $x \in E_\varepsilon$ such that $\left|q_{n,\varepsilon}(x)\right| > \frac{2}{\varepsilon}$.

$$\implies \left|f_{n,\varepsilon}(x) - f(x)\right| \leq \frac{2}{\varepsilon} + \frac{1}{\varepsilon} = \frac{3}{\varepsilon} \leq 3 \underbrace{\left(\left|q_{n,\varepsilon}(x)\right| - \left|f(x)\right|\right)}_{\geq \frac{1}{\varepsilon}} \leq 3 \left|q_{n,\varepsilon}(x) - f(x)\right|$$

$$\int_{\mathbb{R}^n} \left|f_{n,\varepsilon}(x) - X_{E_\varepsilon}(x) f(x)\right| dx \leq 3 \int_{\mathbb{R}^N} \left|g_{n,\varepsilon}(x) - \mathbf{1}_{E_\varepsilon}(x) f(x)\right| dx \to 0 \text{ as } n \to \infty$$

$$\int_{\mathbb{R}^N} \left|f - f_{n,\varepsilon}\right|^p \leq \int_{\mathbb{R}^N \setminus E_\varepsilon} \left|f\right|^p + \left(\frac{3}{\varepsilon}\right)^{p-1} \underbrace{\int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right|}_{(*)} =: (X)$$

$$(*) = \int_{E_\varepsilon} \left|R - f_{n,\varepsilon}\right|^p = \int_{\mathbb{R}^N} \left|f \cdot \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right|^p = \int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right| \left(\underbrace{\left|f \mathbf{1}_{E_\varepsilon}\right|}_{\leq \frac{1}{3}} + \underbrace{\left|f_{n,\varepsilon}\right|^{p-1}}_{\leq \frac{2}{3}} \right)$$

Now given $\delta > 0$, we first fix $\varepsilon > 0$ such that $\int_{\mathbb{R}^N \setminus E_\varepsilon} \left|f\right|^p < \frac{\delta}{2}$. Then we find n_0 such that $\left(\frac{3}{\varepsilon}\right)^{n-1} \int_{\mathbb{R}^N} \left|f \mathbf{1}_{E_\varepsilon} - f_{n,\varepsilon}\right| < \frac{\delta}{2}$. This is possible since $\mathbb{R}^N = \bigcup_{\varepsilon > 0} E_\varepsilon$ and $\int_{\mathbb{R}^N} \left|f\right|^n < \infty$.

$$\implies (X) < \delta$$

Now suppose $\forall \varepsilon > 0 \forall E \in \Sigma : \exists \varphi \in C_C(\mathbb{R}^N, \mathbb{K}^M)$ such that $\left\|\mathbf{1}_E - \varphi\right\| < \varepsilon$. We need to show that this is true. Then for $f \in L^p(\mathbb{R}^N, \mathbb{K}), \varepsilon > 0$, we pick

$$g = \sum_{i=1}^n \underbrace{c_i}_{\in \mathbb{K}^M} \cdot \underbrace{\mathbf{1}_{E_i}}_{\in \Sigma}$$

such that $\left\|f - g\right\|_p < \frac{\varepsilon}{2}$ (possible by what we just showed). For $i \in \mathbb{N}$, pick $\varphi_i \in C_C(\mathbb{R}^N, \mathbb{R})$ such that $\left\|\mathbf{1}_{E_i} - \varphi_i\right\|_p \leq \frac{2^{-i}\varepsilon}{|C_i| 2}$

$$\begin{aligned}
\implies \left\| f - \underbrace{\sum_{i=1}^n c_i \cdot \varphi_i}_{\in C_C(\mathbb{R}^n, \mathbb{K}^n)} \right\|_p &\leq \frac{\varepsilon}{2} + \sum_{i=1}^n \left\|c_i \mathbf{1}_{E_i} - c_i \varphi_i\right\|_p \\
&\leq \frac{\varepsilon}{2} + \sum_{i=1}^n |c_i| \cdot \left\|\mathbf{1}_{E_i} - \varphi_i\right\|_p \\
&\leq \frac{\varepsilon}{2} + \sum_{i=1}^n 2^{-i} \cdot \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

□

↓ *This lecture took place on 2019/05/02.*

Proof. **Step 1** It is sufficient to approximate $f = \mathbf{1}_E$ for $E \in \Sigma$

Step 2 Reduce statement to $f = \mathbf{1}_Q$ where $Q = \bigtimes_{i=1}^N [a_i, b_i)$ with $a_i, b_i \in \mathbb{R}$. Take $f = \mathbf{1}_E$. Since Σ is generated by sets of the form $\bigtimes_{i=1}^N [a_i, b_i)$ $\forall \varepsilon > 0$ there exists $(Q_i)_{i=1}^n$, $(\lambda_i)_{i=1}^n$ such that $\left\|f - \sum_{i=1}^n \lambda_i \mathbf{1}_{Q_i}\right\|_1 < \varepsilon$ [Alt, A1 10, axiom L5].

Define $h_n(x) = \max(0, \min(1, q_n(x)))$ where $q_n := \sum_{i=1}^{n} \lambda_i \mathbf{1}_{Q_i}$, also h_n is of the form of q_n and

$$\left|f(x) - h_n(x)\right| \leq 1 \implies \left|f(x) - h_n(x)\right|^p \leq \left|f(x) - h_n(x)\right|^1 \leq \left|f(x) - q_n(x)\right|$$
$$\implies \left\|f - h_n\right\|_p \to 0 \text{ as } n \to \infty$$

As in step 1, this reduces the assertion to $f = \mathbf{1}_Q$ with $Q = \times_{i=1}^{N} [a_i, b_i)$. For such $f = \mathbf{1}_Q$, define

$$g_i(s) := \begin{cases} \frac{b_i - a_i}{2} + \left|s - \frac{b_i + a_i}{2}\right| & \text{if } s \in [a_i, b_i) \\ 0 & \text{else} \end{cases}$$

for $i \in \{1, \ldots, N\}$ and $\tilde{g}_{i,\varepsilon}(x) = \prod_{i=1}^{N} g_{i,\varepsilon}(x_i)$, we obtain that $\left\|\mathbf{1}_Q - \hat{g}_\varepsilon\right\|_p \to 0$ as $\varepsilon \to 0$.

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\mathbf{1}_Q - \hat{g}_\varepsilon\right|^p &= \int_{a_1}^{b_1} \cdots \int_{a_N}^{b_N} \prod_{i=1}^{N} \left|\mathbf{1}_{[a_i, b_i)}(x) - \tilde{g}_{i,\varepsilon}(x)\right|^p dx \\
&= \prod_{i=1}^{N} \int_{a_i}^{b_i} \left|\mathbf{1}_{[a_i, b_i)}(s) - \tilde{g}_{\varepsilon,i}(s)\right|^p ds \\
&\leq \prod_{i=1}^{N} \left|I_{i,\varepsilon}\right| \text{ where } \left|I_{i,\varepsilon}\right| \to 0 \text{ as } \varepsilon \to 0
\end{aligned}$$

□

Remark. 1. *If $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain, defining*

$$\tilde{f}(x) := \begin{cases} f(x) & x \in \Omega \\ 0 & \textit{else} \end{cases}$$

we get that $\tilde{f} \in L^p(\mathbb{R}^N, \mathbb{K}^M)$ and using Proposition 2.19 for $\tilde{f}$ we can approximate f by functions in $C(\overline{\Omega}, \mathbb{K}^M) \cap C_C(\mathbb{R}^N, \mathbb{K}^M)$.

2. *Using "Mollification" Proposition 2.19 implies density of $\mathcal{D}(\Omega, \mathbb{K}^M)$ in $L^p(\Omega, \mathbb{K}^M)$ for $\Omega \subseteq \mathbb{R}^N$ a domain.*

Proposition 2.20. *Let $\Omega \subset \mathbb{R}^N$ measurable. Then $L^p(\Omega, \mathbb{K}^M)$ is separable for $1 \leq p < \infty$ and not separable for $p = \infty$.*

Proof. **Case** $p = \infty$ Similar to l^∞, will be done in the Exercises.

Case $1 \leq p < \infty$ We show the result for $L^p(\mathbb{R}^N, \mathbb{K})$, the general case is a direct consequence. Denote $\mathcal{R} := \left\{Q \subseteq \mathbb{R}^N \mid Q = \prod_{i=1}^{N} [a_i, b_i) \text{ with } a_n, b_n \in \mathbb{Q}\right\}$. Then $\mathcal{R}$ is countable and it suffices to show that $E := \mathcal{L}(\left\{\mathbf{1}_Q \mid Q \in \mathcal{R}\right\})$ is dense. Take $f \in L^p(\mathbb{R}^N, \mathbb{K}), \varepsilon > 0$. Then $\exists \varphi \in C_C(\mathbb{R}^N, \mathbb{K})$ such that $\left\|f - \varphi\right\|_p \leq \frac{\varepsilon}{2}$. Now we need to find $h \in E$ such that $\left\|\varphi - h\right\|_p \leq \frac{\varepsilon}{2}$. Let

$M \subseteq \mathbb{R}^N$ be closed, bounded hypercube such that $\operatorname{supp}(\varphi) \subset M$. φ is uniformly continuous on M.

$$\implies \forall \delta > 0 \exists \rho > 0 \forall x, y \in M : \left|x - y\right| < \delta \implies \left|\varphi(x) - \varphi(y)\right| < \delta$$

Now we take $(Q_i)_{i=1}^K$ a disjoint covering of M with $Q_i \in \mathcal{R}$, such that $\left|x - y\right| < \delta \forall x, y \in Q_i$. Now define $\lambda_i = \varphi(z)$ for some $z \in Q_i$, $i = 1, \dots, K$. Define $h(x) := \sum_{i=1}^K \lambda_i \mathbf{1}_{Q_i}$.

$$\implies \forall x \in \mathbb{R}^M : \left|\varphi(x) - h(x)\right| \le \left|\varphi(x) - \lambda_i\right| \le \delta$$

$$\implies \left\|\varphi - h\right\|_p = \left(\int_{\mathbb{R}^N} \left|\varphi(x) - h(x)\right|^p\right)^{\frac{1}{p}} \le \delta \cdot |M|^{\frac{1}{p}}$$

Choose $\delta := \frac{\varepsilon}{2 \cdot |M|^{\frac{1}{p}}}$, then the result follows.

□

↓ *This lecture took place on 2019/05/09.*

Proposition 2.21. *Let $p \in [1, \infty]$, $(f_n)_n$, $f \in L^p(\Omega, \mathbb{K}^M)$ with $\Omega \subset \mathbb{R}^N$ a domain such that $f_n \to f$ in L^p.*

Then there exists a subsequence $(f_{n_k})_k$ such that

1. *$f_{n_k}(x) \to f(x)$ for almost every $x \in \Omega$*
2. *$\exists h \in L^p(\Omega)$ such that $(f_{n_k}(x)) \le \left|h(x)\right|$ for almost every $x \in \Omega$*

Proof. **Case** $p = \infty$ Is left as an exercise to the reader.

Case $p \in [1, \infty)$ Pick $(n_k)_k$ such that $\left\|f_{n_{k+1}} - f_{n_k}\right\|_p \le \frac{1}{2^k}$. Define $g_n := \sum_{k=1}^n \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$. Then $g_n(x)$ is increasing, $g_n(x) \ge 0 \forall n$.

$\implies$ $g_n(x)$ is convergent for almost every $x \in \Omega$. Hence we can define $g(x) := \lim_{n\to\infty} g_n(x) \in [0, \infty]$.

Also, $\left\|g_n\right\|_p \le \sum_{i=1}^n \left\|f_{n_{k+1}} - f_{n_k}\right\| \le 1$. By Beppo-Levi,

$$\int_\Omega \left|q(x)\right|^n \, dx = \lim_{n\to\infty} \int_\Omega \left|q_n(x)\right|^n = \lim_{n\to\infty} \left\|g_n\right\|_p^p \le 1 \implies g \in L^p(\Omega)$$

especially $g(x) < \infty$ for almost every $x \in \Omega$.

$$\forall l \ge k \ge 1 : \left|f_{n_l}(x) - f_{n_k}(x)\right| \le \sum_{i=k}^{l-1} \left|f_{n_{j+1}}(x) - g_{k-1}(x)\right| g_{l-1}(x) - g_{k-1}(x) \overset{\text{monot.}}{\le} g(x) - g_{k-1}(x)$$

$\implies$ $(f_{n_k}(x))_k$ is Cauchy for almost every $x \in \Omega$ such that we can define $\hat{f}(x) := \lim_{k\to\infty} f_{n_k}(x)$.

$$\left|\tilde{f}(x) - f_{n_k}(x)\right| \le g(x) \text{ for almost every } x \in \Omega$$

By the Dominated convergence theorem, $\left\|f_{n_k} - \tilde{f}\right\|_n \to 0$ for $k \to \infty$. $\implies$ $f = \tilde{f}$ almost every and hence $f_{n_k}(x) \to f(x)$ for almost every $x \in \Omega$ $\implies$ (1). Also

$$\left|f_{n_k}(x)\right| \leq \left|f_{n_k}(x) - f(x)\right| + \left|f(x)\right| \leq q(x) + \left|f(x)\right| =: h(x)$$

□

3 Linear Operators

Definition 3.1. *Let X, Y be normed spaces and $D \subset X$ is a subspace. A linear operator with domain* $\mathrm{dom}(T) = D$ *is a linear mapping $T : D \to Y$. We define:* $\mathrm{range}(T) = \mathrm{rg}(T) := T(D)$. *Graph of T,* $\mathrm{gr}(T) := \{(x, y) \mid x \in \mathrm{dom}(T), y = Tx\} \subset X \times Y$.

We say that T is decently define, if $\overline{\mathrm{dom}(T)} = X$.

Example 3.2. 1. *$X = Y = C([0, 1], \mathbb{R})$ and $\mathrm{dom}(T) := C^1([0, 1], \mathbb{R})$ $T : \mathrm{dom}(T) \to Y$ with $u \mapsto u'$.*

2. *$X = Y = \mathbb{R}^N, T : \mathbb{R}^n \to \mathbb{R}^n$ with $x \mapsto Ax$ with $A \in \mathbb{R}^{n\times n}$*

3. *Fixed $u \in L^p(\Omega)$ and $p \in [1, \infty)$.*

$$q := \begin{cases} \frac{p}{p-1} & p = 1 \\ \infty & \textit{else} \end{cases}$$

$$T : L^q(\Omega) \to \mathbb{R} \qquad v \mapsto \int_\Omega u \cdot v$$

4. *$X = L^2(\Omega), Y = \mathbb{R}, \mathrm{dom}(T) = C(\overline{\Omega})$ with $x \in \Omega$ fixed, $T : \mathrm{dom}(T) \to Y$ with $u \mapsto u(x_0)$*

Definition 3.3. *Let X, Y be normed spaces and $T : X \to Y$ a linear operator ($\mathrm{dom}(T) = X$). We say that T is bounded $\iff$ $\exists M > 0 \forall x \in X : \|Tx\|_y \leq M \|x\|_X$. In this case, we define $\|T\| = \|T\|_{\mathcal{L}(X,Y)} := \inf\{M > 0 \mid \|Tx\| \leq M \|x\| \forall x\}$.*

$$\mathcal{L}(X, Y) := \{T : X \to Y \mid T \textit{ bounded, linear operator}\}$$

$$\mathcal{L}(X) := \mathcal{L}(X, X)$$

Proposition 3.4. *Let X, Y be normed spaces, $T : X \to Y$ be linear. The following are equivalent:*

1. *T is continuous*
2. *T is continuous at* 0
3. *$\exists M > 0$ such that $\|Tx\| \leq M \|x\| \forall x \in X$ (T bounded)*
4. *T is uniformly continuous*

Also:

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \left\|T(x)\right\| \ \textit{and}\ \|Tx\| \leq \|T\|\,\|x\|\,\forall x \in X$$

Proof. (3) $\to$ (4) Is true since $\forall x, y \in X : \left\|Tx - Ty\right\| = \left\|T(x-y)\right\| \leq M\left\|x-y\right\|$

(4) $\to$ (1) $\to$ (2) trivial

(2) $\to$ (3) Assume (3) is not true, then

$$\exists (x_n)_n \text{ in } X : \forall n \in \mathbb{N} : \|Tx_n\| > n\,\|x_n\|$$

Define $y_n := \frac{x_n}{\|x_n\|n} \implies \left\|y_n\right\| = \frac{1}{n} \implies y_n \to 0$ but $\left\|Ty_n\right\| = \frac{\|Tx_n\|}{\|x_n\|n} > 1$. This gives a contradiction to continuity at 0 since $T0 = 0$.

□

Additionally,

$$M := \sup_{x\neq 0} \frac{\|Tx\|\,\|x\|}{=} \sup_{x\neq 0} \left\|T\left(\frac{x}{\|x\|}\right)\right\| \leq \sup_{\|x\|=1} \|Tx\| \leq \sup_{\|x\|\leq 1} \|Tx\|$$

But also,

$$\sup_{\|x\|\leq 1} \|Tx\| = \sup_{\lambda\in[0,1]} \sup_{\|x\|=1} \left\|T(\lambda x)\right\| = \sup_{\lambda\in[0,1]} \lambda \left(\sup_{\|x\|=1} \|Tx\|\right) = \sup_{\|x\|=1} \|Tx\| = \sup_{x\neq 0} \left\|\frac{Tx}{\|x\|}\right\|$$

We also get that

$$M_0 \geq \frac{\|Tx\|}{\|x\|} \forall x \in X, x \neq 0$$

$$\implies \|Tx\| \leq M_0\,\|x\|\,\forall x \in X : x \neq 0 \text{ and also for } x = 0 \implies \|T\| \leq M_0$$

$$M_0(1-\varepsilon) \leq \frac{\|Tx_\varepsilon\|}{\|x_\varepsilon\|}$$

For $\varepsilon > 0$ pick $x_\varepsilon \neq 0$ such that

$$\|Tx_\varepsilon\| \geq M_0(1-\varepsilon)\,\|x_\varepsilon\|$$

$$\implies \|T\| \geq M_0(1-\varepsilon)$$

since $\varepsilon > 0$ was arbitrary $\implies \|T\| \geq M_0$.

↓ *This lecture took place on 2019/05/10.*

Proposition 3.5. *Let X and Y be normed spaces. Then*

1. $\mathcal{L}(X, Y)$ *is a vectorspace with*

$$(T+S)(x) := T(x) + S(x) \qquad (\lambda T)(x) := \lambda T(x) \qquad 0(x) := 0$$

2. $T \mapsto \|T\|$ *is a norm on* $\mathcal{L}(X, Y)$ *(the operator norm)*

3. *If Y is complete, then $\mathcal{L}(X, Y)$ is complete. In particular, $\mathcal{L}(X, \mathbb{K})$ is complete for any X and is also called the* space of bounded linear functionals

Proof. 1. Left as an exercise to the reader

2. **(N1)** $\|0\| = \sup_{\|x\|\le 1} \|0(x)\| = 0$.
   Also $\|T\| = 0 \implies \|Tx\| \le 0 \|x\| = 0 \forall x \implies T = 0$

   (N2)
$$\|\lambda T\| = \sup_{\|x\|\le 1} \|\lambda T(x)\| = \sup_{\|x\|\le 1} \underbrace{|\lambda|}_{\ge 0} \|Tx\| = |\lambda| \cdot \|Tx\|$$

   (N3)
$$\forall x : \|(T+S)(x)\| = \|Tx + Sx\| \le \|Tx\| + \|Sx\| \le (\|T\| + \|S\|) \|x\|$$
$$\implies \|T + S\| \le \|T\| + \|S\|$$

3. Let $(T_n)_n$ be Cauchy in $\mathcal{L}(X, Y)$ and Y a Banach space. Since $\|(T_n - T_m)(x)\| \le \|T_n - T_m\| \|x\| \implies (T_n x)_n$ is Cauchy in $Y \forall x \in X \implies Tx := \lim_{n\to\infty} T_n x$ is well defined.

Furthermore, we want to show

Linearity:
$$\forall x, y \in X, \lambda \in \mathbb{K} : T(\lambda x + y) = \lim_{n\to\infty} T_n(\lambda x + y) = \lim_{n\to\infty} \lambda T_n x + \lim_{n\to\infty} T_n y = \lambda Tx + Ty$$

$\|\mathbf{T_n} - \mathbf{T}\| \to \mathbf{0}$: Take $\varepsilon > 0$, $n_0 \in \mathbb{N} : \|T_n - T_m\| \le \varepsilon \forall n, m \ge n_0$
Show: $\exists n_1 \forall n \ge n_1 : \|T_n - T\| \le 2\varepsilon$. For $x \in X : \|x\| \le 1$ fix $m_x \ge n_0$: $\|T_{m_x} x - Tx\| \le \varepsilon \implies \forall n \ge n_1 =: n_0$:
$$\begin{aligned} \|T_n x - Tx\| &\le \|T_n x - T_{m_x} x\| + \|T_{m_x} x - Tx\| \\ &\le \|T_n - T_{m_x}\| + \varepsilon \le 2\varepsilon \\ \implies \|T_n - T\| &= \sup_{\|x\|\le 1} \|T_n x - Tx\| < 2\varepsilon \end{aligned}$$
$$\implies \forall x \in X : \|Tx\| \le \|T_n x - Tx\| + \|T_n x\| \le \|T_n - T\| + \|T_n\| \forall n \text{ fixed}$$

□

Proposition 3.6. *Let X, Y be normed spaces. $D \subset X$ is a subspace such that $\overline{D} = X$, $T \in \mathcal{L}(D, Y)$.*
$$\exists! \hat{T} \in \mathcal{L}(X, Y) : \hat{T}|_D = T$$
In addition: $\|\hat{T}\| = \|T\|$.

Proof. Unique extension is clear for T is uniformly continuous.

Also:
$$\|\hat{T}\| = \sup_{\substack{x\in X \\ \|x\|\ne 0}} \frac{\|\hat{T}x\|}{\|x\|} \overset{\text{by density}}{=} \sup_{\substack{x\in D \\ \|x\|\ne 0}} \frac{\|\hat{T}x\|}{\|x\|} = \|T\|$$
To show the density equality is left as an exercise to the reader. □

Proposition 3.7. *Let X, Y, Z be normed spaces. $S \in \mathcal{L}(X,Y)$. $T \in \mathcal{L}(Y,Z)$. Then $T_0 S \in \mathcal{L}(X,Z)$ and $\|N_0 S\| \leq \|T\| \, \|S\|$.*

Proof. $T_0 S$ is linear (show as an exercise).

Take $x \in X$. $\left\|T_0 S(x)\right\| = \left\|T(Sx)\right\| = \|T\| \, \|Sx\| \leq \|T\| \, \|Sx\| \leq \|T\| \, \|S\| \, \|x\|$. $\implies$ $\|T_0 S\| \leq \|T\| \, \|S\|$ □

Remark. *If* $\dim(X) < \infty$, *$T : X \to Y$ is linear, then $T_C \mathcal{L}(X,Y)$ (left as an exercise).*

Proposition 3.8 (Neumann series)**.** *Let X be a normed space. $T \in \mathcal{L}(X)$. If $\sum_{n=0}^{\infty} T^n$ is convergent in $\mathcal{L}(X)$, then $(I - T)$ is invertible and $(I-T)^{-1} = \sum_{n=0}^{\infty} T^n$.*

Here: $T^n := T_0 \cdot T_0 \cdot T_0 \cdot \ldots$ n times

In particular, if X is Banach and $\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} =: a < 1$ *then $\sum_{i=0}^{\infty} T^n$ is convergent. Also if $\|T\| < 1$, then $a < 1$ holds true. In case of $a < 1$, then $\left\|(I - T)^{-1}\right\| \leq \frac{1}{1-a}$.*

Proof. Let $S_m := \sum_{n=0}^{m} T^n$ and $S := \lim_{m\to\infty} S_m$. Then $(I - T)S_m = I - T^{m+1} = S_m(I - T)$ (compute!).

$$\|T^m\| = \left\| \sum_{n=0}^{m} T^n - \sum_{n=0}^{m-1} T^n \right\| = \|S_m - S_{m-1}\| \to 0$$

for $m \to \infty$ since $(S_m)_n$ is Cauchy. ($RS := R_0 S$)

Now note that for fixed $R \subset \mathcal{L}(X)$ the mappings

$$S \mapsto RS \qquad S \mapsto SR$$

are continuous since $\|S_n R - SR\| \leq \|S_n S\| \, \|R\| \to 0$ for $S_n \to S$. Continuity implies that

$$I = \lim_{m\to\infty} I - T^{m+1} = \begin{cases} \lim_{m\to\infty}(I-T)S_m = (I-T)S \\ \lim_{m\to\infty} S_m(I-T) = S(I-T) \end{cases}$$

$$\implies (I-T)^{-1} = S$$

Now if $\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq a < 1 \forall \varepsilon > 0 \implies \exists n_0 \forall n \geq n_0 : \|T^n\| \leq (a+\varepsilon)^n$

$$\implies \sum_{n=0}^{\infty} \|T^n\| \leq c + \sum_{n=0}^{\infty} (a+\varepsilon)^n = \frac{1}{1-(a+\varepsilon)} + c < \infty \text{ for } c > 0$$

X is Banach, so $\sum_{n=0}^{\infty} T^n$ is convergent and

$$\left\|(I-T)^{-1}\right\| = \left\| \sum_{n=0}^{\infty} T^n \right\| \leq \frac{1}{1-(a+\varepsilon)}$$

Since ε was arbitrary, $\left\|(I-T)^{-1}\right\| \leq \frac{1}{1-a}$.

If $\|T\| < 1$, then

$$\begin{aligned}\limsup_{m\to\infty} \|T^n\|^{\frac{1}{n}} &\leq \limsup(\|T\| \cdot \|T\| \ldots \|T\|)^{\frac{1}{n}} \\ &\leq \limsup\left(\|T\|^n\right)^{\frac{1}{n}} \\ &= \limsup \|T\| \\ &= \|T\|\end{aligned}$$

□

Remark. $(I - T)^{-1}$ *is linear (left as an exercise). However:* $(I - T)^{-1} \notin \mathcal{L}(X)$ in general*!*

4 The Hahn-Banach Theorem and its consequences

Apparently the Hahn-Banach Theorem of this chapter is very central to Functional Analysis. This section deals with an extension of linear functionals and separation of sets.

First, consider $\mathbb{K} = \mathbb{R}$.

Definition 4.1. *Let* X *be a vector space.* $p : X \to \mathbb{R}$ *is called* sublinear *iff*

1. $p(\lambda x) = \lambda p(x) \forall \lambda \geq 0, x \in X$
2. $p(x + y) \leq p(x) + p(y) \forall x, y \in X$

Example 4.2. $p(x) = \|x\|$*,* p *linear and* p *is a seminorm.*

Theorem 4.3 (Hahn-Banach Theorem, real version)**.** *Let* X *be a vector space over* $\mathbb{R}$*,* $U \subset X$*, a subspace.* $p : X \to \mathbb{R}$ *be sublinear and* $l : U \to \mathbb{R}$ *is linear such that* $l(x) \leq p(x) \forall x \in U$

Then $\exists L : X \to \mathbb{R}$ *is linear such that*

$$L|_U = l \qquad L(x) \leq p(x) \forall x \in X$$

Proof. This proof consists of two steps:

1. Method to extend l from U to $U + \operatorname{span}(x_0)$, $x_0 \notin U$
2. Iterate this step and get maximal extension (Zorn)

Step 1 For $x_0 \in X \setminus U$, let $V = U + \operatorname{span}(x_0) = \{u + \lambda x_0 \mid u \in U, \lambda \in \mathbb{R}\}$. Any $v \in V$ can be written uniquely as $v = u + \lambda x_0$ for $u \in U, x \in \mathbb{R}$ (why? left as an exercise). Thus for any $r \in \mathbb{R}$, we can define $L_p : V \to \mathbb{R}$. $v = u + \lambda x_0 \mapsto l(u) + \lambda r$. L_p is linear (why? left as an exercise).

Also: $L_r(x) \leq p(x) \forall x \in V \iff l(u) + \lambda r \leq p(u + \lambda x_0) \forall \lambda, u$ (let this statement be (*)).

$\lambda = 0$ (*) holds true

$\lambda > 0$ (*)

$$\iff r \le p\left(\frac{u}{\lambda} + x_0\right) - l\left(\frac{u}{\lambda}\right) \forall u \in U$$
$$\iff r \le \inf_{u\in U} p(u + x_0) - l(u)$$

$\lambda < 0$

$$\iff -r \le p\left(\frac{u}{-\lambda} - x_0\right) - l\left(\frac{u}{-\lambda}\right) \forall u \in U \quad \iff r \ge -p(u - x_0) + l(u) \forall u \in U$$
$$\iff r \ge \sup_{u\in U} l(u) - p(u - x_0)$$

Thus, (*) holds for $r = \sup_{u\in U} l(u) - p(u - x_0)$ if $\sup_{u\in U} l(u) - p(u - x_0) \le \inf_{u\in U} p(u + x_0) - l(u) \iff l(w) - p(w - x) \le p(u + x_0) - l(u) \forall w, u \in U \iff l(w) + l(u) \le p(u + x_0) + p(w - x_0)$.

But this holds since:

$$l(w) + l(u) = l(w + u) \le p(w + u) = p(w - x_0 + x_0 + u) \le p(w - x_0) + p(u + x_0)$$

Step 2

Revision 4.4 (Zorn's Lemma). *Let $(A, \le)$ be a partially ordered set such that every chain (every subset R of $A \forall a, b \in R : a \le b \vee b \le a$) admits an upper bound (i.e. $\exists c \in A : b \le c \forall b \in R$), then A has a maximal element, i.e. $\exists z \in A$ such that $\forall a \in A : z_0 \le a \implies a = z_0$*

Let A be a set of (V, L_V) tuples where $V \subset X$ is a subspace with $U \subset V$ and $L_V : V \to \mathbb{R}$ such that $L_V \le p$ on V and $L_V|_U = l$.

For (V_1, L_{v_1}) and $(V_2, L_{v_2}) \in A$, we say that $(V_1, L_{v_1}) \le (V_2, L_{v_2})$ if $V_1 \subset V_2$ and $L_{v_2}|_{v_1} = L_{v_1}$. Now $A \ne \emptyset$ since $(U, l) \in A$. If $(V_i, L_{v_i})_{i\in I} := R$ is a chain, define $V := \bigcup_{i\in I} V_i$, $L_V(x) := L_{V_i}(x)$ if $x \in V_i$.

This is well-defined.

$\implies (V, L_V)$ is an upper bound for R.

□

↓ *This lecture took place on 2019/05/14.*

Proof of Theorem 4.3. Let $U \subset X$, $x_0 \notin U$, $V = U + \operatorname{span}(x_0)$.

$$\implies \exists L_V : V \to \mathbb{R} : L_V|_U = l, L_V(v) = p(v) \forall v \in V$$

$$R = \{(V, L_V) \mid U \subset V, L_V|_U = l, L_V = p \text{ on } V\}$$

$$(V_1, L_{V_1}) \le (V_2, L_{V_2}) :\iff V_1 \subset V_2, L_{V_2}|_{V_1} = L_{V_1}$$

Remark. *Any chain has an upper bound.*

Let $(V_i, L_{V_i})_{i\in I}$ be a chain in R. Then we define $V = \bigcup_{i\in V} V_i$. $L_V : V \to \mathbb{R}$ with $v \mapsto L_{V_i}(v)$ if $v^* \in V_i$. Thus we showed well-definedness.

Then (V, L_V) is an upper bound of $(V_i, L_{V_i})_{i\in I}$ since $V_i \subset V$, $L_V|_{V_i} = L_{V_i} \forall i \in I$. By Zorn, there exists (V_0, L_{V_0}) a maximal element of R. It is left to show that $V_0 = X$. If not: Take some $x_0 \in X \setminus V_0$, define $\tilde{V} := V_0 + \operatorname{span}(x_0)$ and $L_{\tilde{V}}$ as an extension of L_{V_0} as in step 1.

$$\implies (V_0, L_{V_0}) \leq (\tilde{V}, L_{\tilde{V}})$$

This contradicts the maximality of (V_0, L_{V_0}). □

Remark. *If U is not dense, then the extension is unique.*

Next: Hahn-Banach Theorem for $\mathbb{K} = \mathbb{C}$.
Approach: Establish bijection between $\mathbb{R}$ vector space and $\mathbb{C}$ vector space.

Proposition 4.5. *Let X be a $\mathbb{C}$ vector space (vector space over the complex numbers).*

1. *If $l : X \to \mathbb{R}$ is $\mathbb{R}$-linear (i.e. $l(x+y) = l(x) + l(y)$ and $l(\lambda x) = \lambda l(x) \forall \lambda \in \mathbb{R}$). We set $\hat{l} : X \to \mathbb{C}$ with $x \mapsto l(x) - i \cdot l(ix)$. Then $\hat{l}$ is $\mathbb{C}$-linear and $\Re(\hat{l}) = l$.*
2. *If $h : X \to \mathbb{C}$ is $\mathbb{C}$-linear and we let $l := \Re(h)$ and $\hat{l}$ as in (1), then l is $\mathbb{R}$-linear and $\hat{l} = h$ [$l \to \hat{l}$ is surjective]*
3. *If $p : X \to \mathbb{R}$ is a seminorm and $l : X \to \mathbb{C}$ is $\mathbb{C}$-linear. Then*
$$\left|l(x)\right| \leq p(x) \forall x \iff \left|\Re(l(x))\right| \leq p(x) \forall x$$
4. *If x is normed, $l \in \mathcal{L}(X, \mathbb{C})$, then $\|L\| = \left\|\Re(x)\right\|$*

Remark. *This means that $l \mapsto [x \mapsto l(x) - il(ix)]$ is bijective and an isometry if X is normed.*

Proof. 1. By construction $\hat{l}$ is $\mathbb{R}$-linear and $\Re(\hat{l}) = l$ is obvious.
Show: $\tilde{l}(ix) = i\tilde{l}(x)$.

$$\begin{aligned}\tilde{l}(ix) &= l(ix) - il(iix) = l(ix) - il(-x)\\ &= i(l(x) - il(ix)) = i\tilde{l}(x)\end{aligned}$$

2. Define $l := \Re(h)$. Show: $\tilde{l} = h$.
Note: $\forall z \in \mathbb{C} : \Im(z) = -\Re(iz)$.

$$\begin{aligned}h(x) &= \Re(h(x)) + i \cdot \Im(h(x)) = \Re(h(x)) - i \cdot \Re(i \cdot h(x))\\ &= \Re(h(x)) - i \cdot \Re(h(ix)) = l(x) - i \cdot l(ix) = \tilde{l}(x)\end{aligned}$$

Hence $l \mapsto \tilde{l}$ is bijective.

3. Since $\left|Re(z)\right| \leq |z|$,

   $\Longrightarrow$ holds trivially

   $\Longleftarrow$ Write $l(x) = \lambda_X \left|l(x)\right|$ with $|\lambda_X| = 1$. Then $\forall x \in X : \left|l(x)\right| = \lambda_X^{-1} l(x) = l(\lambda_X^{-1} x) = \left|\Re l(\lambda_X^{-1} x)\right| \leq p(\lambda_X^{-1} x) = \left|\lambda_X^{-1}\right| p(x) = p(x)$

4. Consequence of (3) with $p(x) := \|l\| \, \|x\|$

$\square$

Theorem 4.6 (Hahn-Banach Theorem, complex version)**.** *Let X be a $\mathbb{C}$ vector space. $U \subset X$. $p : X \to \mathbb{R}$ sublinear and $l : U \to \mathbb{C}$ be linear such that $\Re l(u) < p(u) \forall u \in U$.*

$$\exists L : X \to \mathbb{C} \text{ linear such that } L|_U = l, \Re L(x) \leq p(x) \forall x$$

Proof. Applying Theorem 4.3 to $\Re l \implies \exists F : X \to \mathbb{R}$ r-linear such that $F|_U = \Re(l)$ and $F(x) \leq p(x) \forall x \in X$

Proposition 4.5 implies there exists some $L : X \to \mathbb{C}$ that io $\mathbb{C}$-linear such that $F = \Re(L)$. Now $\Re(L)|_U = F|_U = \Re(l) \implies L = l$ by Proposition 4.5 (2) and also $\Re L(x) = F(x) = p(x) \forall x \in X$. $\square$

Proposition 4.7 (Consequence)**.** *If X is a normed space, $U \subset X$ be a subspace, $u' \in \mathcal{L}(U, \mathbb{K})$, then $\exists x' \in \mathcal{L}(X, \mathbb{K})$ such that $x'|_U = u'$ with $\|x'\| = \|u'\|$.*

Proof. **Case $\mathbb{K} = \mathbb{R}$:** Let $p(x) := \|u'\| \, \|x\|$. Then p is sublinear and $u'(x) \leq \left|u'(x)\right| \leq p(x) \forall x \in U$. By Theorem 4.3, there exists $x' : X \to \mathbb{R}$ linear such that $x'|_U = u'$ and $x'(x) \leq p(x) \forall x \in X$.

$$\implies -x'(x) = x'(-x) \leq p(-x) = p(x) \implies \left|x'(x)\right| \leq p(x) = \|u'\| \, \|x\| \implies \|x'\| \leq \|u\|$$

Also:

$$\|u'\| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} \left|u'(u)\right| = \sup_{\substack{u \in U \\ \|u\| \leq 1}} \left|x'(u)\right| \leq \sup_{\substack{x \in X \\ \|x\| \leq 1}} \left\|x'(x)\right\| = \|x'\| \implies \|x'\| = \|u'\|$$

Case $\mathbb{K} = \mathbb{C}$: As before, $\exists x' : X \to \mathbb{C} : x'|_u = u'$ and $\left\|\Re x'\right\| \leq \|u'\|$. By proposition 4.5, $\|x'\| = \left\|\Re(x')\right\|$

$\square$

Remark (Next application: Separation of convex sets)**.** Motivation: *Given two (convex) sets $A, B \subset \mathbb{R}^2$. When can we find a line L separating these sets*

Compare with Figure 2.

Remark. *In $\mathbb{R}^2$, any line L can be separable as $L = \{x \in \mathbb{R}^2 : (x, n) = \alpha \mid \alpha \in \mathbb{R}, n \in \mathbb{R}^2, \|n\| = 1\}$.*

Definition. *Let X be a vector space $H \subset X$ is called a* hyperplane *if it is of the form $H = \{x \in X \mid \Re(f(x)) = \alpha\}$ with $\alpha \in \mathbb{R}$, $f : X \to \mathbb{K}$ linear.*

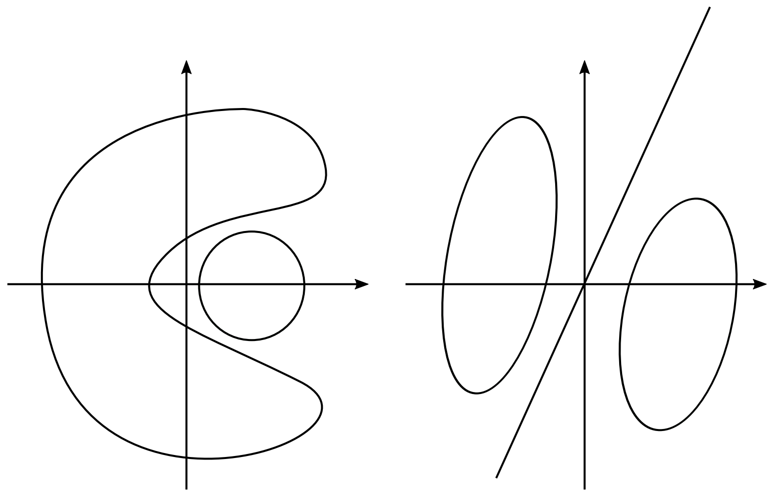

Figure 2: Inseparable convex sets (left) and separable convex sets (right)

Lemma 4.8. *Let X be a normed space, $H \subset X$ be a hyperplane of the form $H = \{x \in X \mid \Re(f(x)) = \alpha\}$ with $\alpha \in \mathbb{R}$, $f : X \to \mathbb{K}$ linear.*

Then H is closed iff $f \in \mathcal{L}(X, \mathbb{K})$.

Proof. Compare with the practicals. □

Remark (Goal)**.** *Given X as a normed vector space. $A, B \subset X$ where does some closed hyperplane H exist represented by $f \in \mathcal{L}(X, \mathbb{K})$ and α separating A and B, e.g. $\Re(f(a)) \le \alpha \le \Re(f(b)) \forall a \in A, b \in B$.*

To this aim associate a set $U \subset X$ to a sublinear functional $p : X \to \mathbb{R}$.

Definition 4.9. *Let X be a vector space. $A \subset X$. The Minkovsky functional $p_A : X \to [0, \infty]$ is defined as $p_A(x) = \inf\left\{\lambda > 0 \mid \frac{x}{\lambda} \in A\right\}$. A is called* absorbing *if $p_A(x) < \infty \forall x \in X$.*

Theorem 4.10. *Let X be a normed space. $U \subset X$ convex such that $0 \in$* interior$(U) = \mathring{U}$*. Then,*

1. *U is absorbing and $\forall \varepsilon > 0 : B_\varepsilon(0) \subseteq U \implies p_U(x) \le \frac{1}{\varepsilon} \|x\|$ [no convexity needed]*
2. *p_U is sublinear*
3. *If U is open, then $U = p_u^{-1}([0, 1))$.*

Proof. 1. Trivial

2.
   - $p_u(\lambda x) = \lambda p_u(x)$ for $\lambda > 0$. Compare with the practicals.
   - Take $x, y \in X$. Show: $p_u(x + y) \leq p_u(x) + p_u(y)$.

   Take $\varepsilon > 0$ and choose λ, μ:

$$\lambda \leq p_u(x) + \varepsilon \qquad \frac{x}{\lambda} \in U$$
$$\mu \leq p_u(y) + \varepsilon \qquad \frac{y}{\mu} \in U$$

   Since U is convex,

$$\frac{x+y}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}\left(\frac{x}{\lambda}\right) + \frac{\mu}{\lambda+\mu}\left(\frac{y}{\mu}\right) \in U$$
$$\implies p_u(x + y) \leq \lambda + \mu = p_u(x) + p_u(\mu) + 2\varepsilon$$

   ε can be arbitrary, thus the proof is complete.

3. Direction $\supset$. If $p_u(x) < 1 \implies \exists \lambda > 0 : \lambda < 1$ and $\frac{x}{\lambda} \in U$. Since $0 \in U$,

$$\implies x = \lambda\left(\frac{x}{\lambda}\right) + (1 - x)0 \in U$$
$$\implies p_u^{-1}([0, 1)) \subset U$$

   Direction $\subset$. If $p_u(x) \geq 1$, then $\frac{x}{\lambda} \notin U \forall \lambda < 1$

$$\implies x = \lim_{\substack{\lambda \to 1 \\ \lambda < 1}} \frac{x}{\lambda} \in U^C$$

□

Lemma (Fundamental lemma)**.** *Let X be a normed vector space. $V \subset X$ be convex and open. $O \notin V$.*

$$\implies \exists x' : X \to \mathbb{K} \textit{ continuous}$$

linear such that $\Re x'(x) < 0 \forall x \in V$.

Proof. Define $A \mp B = \{a + b \mid a \in A, b \in B\}$.

Case $\mathbb{K} = \mathbb{R}$: Take $x_0 \in V \setminus \{0\}$, define $y_0 := -x_0$ and $U := V - \{x_0\}$.

$$\implies U \text{ is open, convex, } 0 \in U, y_0 \notin U$$

We consider $p_u : X \to \mathbb{R}$ which is sublinear, finite and $p_u(y_0) \geq 1$. On $Y := \operatorname{span}(y_0)$ we define $y' : Y \to \mathbb{R}$ with $ty_0 \mapsto tp_u(y_0)$ and $t \in \mathbb{R}$.

$$\implies y'(y) \leq p_u(y) \forall y \in Y$$

since

$$y'(y) = y'(ty_0) = tp_u(y_0)$$

- $t \leq 0$: $\leq 0 \leq p_u(y)$
- $t > 0$: $= p_u(ty_0) = p_u(y)$

↓ *This lecture took place on 2019/05/16.*

Now by Hahn-Banach Theorem, $\exists x' : X \to \mathbb{R}$ linear such that $x'|_y = y'$ and $x'(x) \leq p_u(x) \forall x \in X$

$$\forall x \in X : \left|x'(x)\right| = \max\left\{x'(x), \underbrace{-x'(x)}_{=x'(-x)}\right\} \leq \min(p_u(x), -p_u(-x)) \leq \frac{1}{2}\|x\| \qquad \text{for } \varepsilon > 0 : B_\varepsilon(0) \subseteq U$$

$$\implies x' \in \mathcal{L}(X, \mathbb{R})$$

Also $x'(y_0) = y'(y_0) = p_u(y_0) \geq 1$.

$\implies \forall v \in V$ we can write $v = u - y_0$ with $u \in U$

$\implies x'(v) = x'(u) - x'(y_0) \leq p_u(u) - 1 < 0$

Case $\mathbb{K} = \mathbb{C}$ Lemma 4.5. Left as an exercise.

□

Theorem 4.11 (Separation 1)**.** *Let X be normed. Let $V_1, V_2 \subset X$ be convex and V_1 open. $V_1 \cap V_2 = \emptyset$*

$$\implies \exists x' \in \mathcal{L}(X, \mathbb{K}) \text{ s.t. } \Re(x'(u_1)) \leq \Re(x'(x_2)) \forall v_1 \in V_1, v_2 \in V_2$$

Proof. Define $V := V_1 - V_2$. Then V is convex (why?) and open since $V = \bigcup_{x \in V_2} V1 - \{x\}$ since $V_1 \cap V_2 = \emptyset$. Thus $0 \in V$. By Lemma 4,

$$\exists x' \in \mathcal{L}(X, \mathbb{K}) : \Re x'(v_1 - v_2) < 0 \forall v_1 \in V_1, v_2 \in V_2$$

$$\implies \Re x'(v_1) < \Re x'(v_2)$$

□

Remark. *V being open is sufficient.*

Theorem 4.12 (Separation 2)**.** *Let X be a normed spaces. $V \subset X$ is closed and convex.*

$$\hat{x} \notin V \implies \exists x' \in \mathcal{L}(X, \mathbb{K})$$

$$\Re(x'(\hat{x})) < \inf_{v \in V} \Re(x'(v))$$

i.e. $\exists \varepsilon > 0 : \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \varepsilon \leq \inf_{v \in V} \Re(x'(v))$

Proof.

$$V \text{ closed} \iff \exists \varepsilon > 0 : \underline{B_\varepsilon(\hat{x}) \cap V}_{V_1} = \emptyset$$

By Theorem 4.11, $\exists x' \in \mathcal{L}(X, \mathbb{K})$:

$$\Re(x'(\hat{x} + u)) < \Re(x'(v)) \forall v \in V, u \in X : \|u\| < \varepsilon$$

$$\Re(x'(\hat{x})) + \Re(x'(u)) < \Re(x'(v)) \forall v \in V, u \in X, \|u\| \leq \frac{\varepsilon}{2}$$

Taking the sum over u.

$$\Re(x'(\hat{x})) + \left\|\Re(x')\right\| \frac{\varepsilon}{2} \leq \Re(x'(v)) \forall v \in V$$

since

$$\left\|\Re(x')\right\| = \sup_{\|\lambda\| \leq 1} \left|\Re(x'(x))\right| \frac{\varepsilon}{2} = \sup_{\|x\| \leq \frac{\varepsilon}{2}} \left|\Re(x'(x))\right| = \sup_{\|x\| \leq \frac{\varepsilon}{2}} \Re(x'(x))$$

$$\implies \Re(x'(\hat{x})) < \Re(x'(\hat{x})) + \|x'\| \frac{\varepsilon}{2} \leq \inf_{v \in V} \Re(x'(v))$$

□

5 Fundamental theorems for operators in Banach spaces

In this chapter we are going to discuss the Baire theorem.

Theorem 5.1 (Banach-Steinhaus, uniform boundedness principle)**.** *Let X be a Banach space, Y normed. Let I be an index set. For all $i \in I$, let $T_i \in \mathcal{L}(X, Y)$. Then if $\forall x \in X : \sup_{i \in I} \|T_i x\| < \infty \implies \sup_{i \in I} \|T_i\| < \infty$*

Proof. Define $E_n := \{x \in X \mid \sup_{i \in I} \|T_i x\| \leq n\}$ since all T_i are continuous

$$\implies E_n = \bigcap_{i \in I} \left\|T_i(\cdot)\right\|^{-1}([0, n])$$

since $x \mapsto \|T_i x\|$ is continuous $\to$ closed.

$\implies E_n$ is closed as the intersection of closed sets

Also: $X = \bigcup_{n \in \mathbb{N}} E_n$

By Baire's theorem, $\exists E_n : \mathring{E}_{n_0} \neq \emptyset$.

$$\implies \exists \varepsilon > 0, y \in E_{n_0} \text{ fixed such that } \forall x \in X : \left\|x - y\right\| \leq \varepsilon \implies x \in E_{n_0}$$

Now take $x \in X : \left\|x + y\right\| \leq \varepsilon$.

$$\left\|x + y\right\| = \left\|x - (-y)\right\| = \left\|-x - y\right\| \implies -x \in E_n \implies x \in E_n$$

Also, $\forall x_1, x_2 \in X, \lambda \in [0, 1], x_1, x_2 \in E_{n_0}$.

$$\implies \lambda x_1 + (1 - \lambda) x_2 \in E_{n_0}$$

since $\forall i : \left\|T_i(\lambda x_i + (1 - \lambda) x_2)\right\| \leq \lambda \|T x_1\| + (1 - \lambda) \|T x_2\| < n_0$.

$$\forall x \in X : \|x\| \leq \varepsilon \implies x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \in E_{n_0}$$

since $x + y \in E_{n_0}$ and $x - y \in E_{n_0}$.

$$\implies \forall i \in I : \|T_i x\| \leq n_0 \forall \|x\| \leq \varepsilon$$

$$\implies \|T_i\| = \frac{1}{\varepsilon} \sup_{\|x\| \leq 1} \left\|T_i(\varepsilon x)\right\| \leq \frac{1}{\varepsilon} n_0$$

$$\implies \sup_{i \in I} \|T_i\| \leq \frac{\varepsilon}{n_0}$$

□

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