Measure and integration theory

Lecture notes, University (of Technology) Graz
based on the lecture by Wolfgang Wöss

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Contents

1 Course

1. Thursday, 16:15–18:00
2. Monday, 12:15–14:00
3. Exam: oral, date negotiation per email, 3 examinees at once
4. In this document, $\subset$ denotes $\subseteq$ or $\subsetneq$
5. Literature: “Measure Theory” by Paul R. Halmos

↓ *This lecture took place on 2018/10/01.*

2 Sigma algebras and measures

A measure represents the content of a set. In R^2, it represents the area. In R^3, it represents the volume. In R^d, we can consider the content of a subspace as dimensionwise combination of intervals:

$$[a_1, b_1] \times \dots \times [a_d, b_d]$$

To determine the “size” of this space, we can use the product of the individual interval sizes:

$$(b_1 - a_1) \cdot \dots \cdot (b_d - a_d)$$

Consider an geometric object as in Figure 1. We can approximate the size of B by considering inner or outer axis-parallel boundary. The approximation using the infimum of the outer and supremum of the inner boundary defines the Jordan measure.

The indicator function of this area (1_B) is Riemann-integrable.

A one-point set is Jordan measurable with measure/content 0. However, $\mathbb{Q} \cap [0, 1]$ is not Jordan measurable, because the indicator function is not Riemann integrable. It is desirable that the measure $\dot\bigcup_{n=1}^\infty A_n = \sum_{n=1}^\infty \text{measure}(A_n)$ (using pairwise disjoint union) holds true.

Modern measure theory was established by Lebesgue (1901):

1. Union of countable sets (σ-additivity)
2. arbitrary base set X instead of $\mathbb{R}^d$, integration theory for $f : X \to \mathbb{R}$

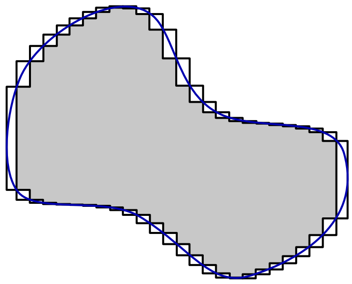

Figure 1: Jordan measurability of this area B

2.1 Definition

Let (δ, ρ) be the non-empty base set. $\mathcal{A} \subset P(X)$. A set system of subsets of X is called *sigma-algebra* (σ-algebra) if

1. $X \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^C = X \setminus A \in \mathcal{A}$
3. $A_n \in \mathcal{A}$ $(n \in \mathbb{N}) \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Properties 1 and 2 implies that $\emptyset \in \mathcal{A}$.

A *measurable space* is given by $(X, \mathcal{A})$. A *measure* $\mu : \mathcal{A} \to [0, \infty]$ is defined by

1. $\mu(\emptyset) = 0$
2. If $A_n \in \mathcal{A}$ $(n \in \mathbb{N})$, pairwise disjoint, then

$$\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

A *measure space* is given with $(X, \mathcal{A}, \mu)$.

Remark.
- *μ is called probability space if $\mu(X) = 1$*
- *μ is called finite measure if $\mu(X) < \infty$*
- *μ is called σ-finite if $X = \bigcup_{n=1}^{\infty} A_n$ with $A_n \in \mathcal{A}$ and $\mu(A_n) < \infty$ (e.g. real axes decomposes into intervals of length 1)*

Examples:

1. X is at most countable, then mostly $\mathcal{A} = \mathbb{P}(X)$. Then it suffices to know, $\mu(\{x\}) \in [0, \infty)$. Then we denote $\mu(x) = \mu(\{x\})$ with $x \in X$.

$$\mu(A) = \sum_{x \in A} \mu(x)$$

   e.g. $\mu(x) = 1 \forall x \in X$ in case of a *counting measure*.

2. If X is uncountable, e.g. $\mathbb{R}^d$, then it is not recommended to use $\mathbb{P}(X)$. So what about A? Consider for example $\mathbb{R}^d$. All $[a_1, b_1] \times \cdots \times [a_d, b_d]$ should be elements of $\mathcal{A}$

2.2 Simple properties of sigma-algebras

1. $\emptyset \in \mathcal{A}$
2. $A_1, \ldots, A_n \in \mathcal{A} \implies \bigcup_{k=1}^n A_k \in \mathcal{A}$
3. $A_n \in \mathcal{A}$ ($n \in \mathbb{N}$) or $, \ldots, N \implies \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ (deMorgan)
   $\bigcap_n A_n = \left(\bigcup_n A_n^C\right)^C$
4. $A, B \in \mathcal{A} \implies A \setminus B \in \mathcal{A}$, $A \triangle B = A \setminus B \vee B \setminus A$

Definition 2.1 (Generating set). *Let $\mathcal{E} \neq \emptyset$ with $\mathcal{E} \subset \mathbb{P}(X)$ be the* generator *(generating set) of the σ-algebra. $\sigma(\mathcal{E})$ is the smallest σ-algebra over X which contains $\mathcal{E}$.*

$$= \bigcap \left\{ \tilde{A} : \tilde{A} \text{ is the } \sigma - \text{algebra over } X \text{ with } \mathcal{E} \subset \tilde{A} \right\}$$

This set is non-empty because $\mathbb{P}(X)$ is the σ-algebra for all X and $\mathcal{E} \subset \mathbb{P}(X)$

Lemma 2.2. *If $\mathcal{A}_i$ with $i \in I$ is a family of σ-algebras, then $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra over X.*

Immediate:

1. if $\mathcal{E}_1 \subset \mathcal{E}_2$ ($\implies \mathcal{E}_1 \subset \sigma(\mathcal{E}_2)$), then $\sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$
2. if additionally $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$, then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$

Example:

$$X = \bigcup_{n\in I} E_n \neq \emptyset \qquad I = \mathbb{N} \text{ or } \{1, \ldots, N\}$$

$$\mathcal{E} = \{E_n \mid n \in I\} \qquad \sigma(\mathcal{E}) = \left\{ \bigcup_{n\in J} E_n \mid J \subset I \right\}$$

1. Is a σ-algebra
2. If $\mathcal{E} \subset \tilde{\mathcal{A}}$, then $\left\{ \bigcup_{n\in J} E_n \mid J \subset I \right\} \subset \tilde{\mathcal{A}}$

Definition. *(X, d) is a metric space. Borel-σ-algebra $\sigma(\mathcal{O})$. $\mathcal{O}$ is the set of open sets in a metric space*

Example. *Consider $\mathbb{R}^d$. $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel σ-algebra.*

1. *$\mathcal{E}_1 = \{\text{open sets}\}$*
2. *$\mathcal{E}_2 = \{\text{closed sets}\}$*
3. *$\mathcal{E}_3 = \{(a_1, b_1) \times \cdots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$ is a parallelepiped*

4. $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2) = \sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$
5. $\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ *because every open set is a countable union of open (or left half-open) parallelepipeds*

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$
$$\mathcal{E}_4 = \{(a_1, b_1] \times \cdots \times (a_d, b_d] \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$$
$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$$
$$\mathcal{E}_5 = \{(-\infty, b_1) \times \cdots \times (-\infty, b_d] \mid b_i \in \mathbb{R}\} \text{ because } \mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$$

If $d = 1$, $(a, b] = (-\infty, b] \setminus (-\infty, a]$. Recognize that $A \setminus B = (A \cap B^C)$.
If $d = 2$,

$$(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus (-\infty, a_1] \times (-\infty, b_2) \setminus (-\infty, b_1] \times (-\infty, a_2]$$

Definition. *$(X, \mathcal{A})$ is a measurable space, $B \in \mathbb{A}$ is a trace σ-algebra over B. $\{A \in \mathcal{A} \mid A \subset B\}$*

Definition. *$\varphi : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$ is called measurable $\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$*

Remark. *In general φ is a map from X_1 to X_2. $\mathcal{A}_1$ and $\mathcal{A}_2$ are mentioned to clarify that the map depends on the chosen algebra.*

Remark. *$(X_1, d_1) \to (X_1, d_2)$ on metric spaces is continuous iff $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ where $\mathcal{O}_1, \mathcal{O}_2$ are sets of open sets.*

Remark. *Measurable maps are a much stronger statement than continuity, because they cover much more sets than open ones.*

Lemma 2.3. *The composition of measurable maps is measurable.*

$$\varphi : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2) \text{ measurable}$$
$$\psi : (X_2, \mathcal{A}_2) \to (X_2, \mathcal{A}_2) \text{ measurable}$$
$$\implies \psi \circ \varphi : (X_1, \mathcal{A}_1) \to (X_3, \mathcal{A}_3) \text{ measurable}$$

Proof. Show that $(\psi \circ \varphi)^{-1}(\mathcal{A}_3) \in \mathcal{A}_1$ is trivial. □

Theorem 2.3.1. *Let $\mathcal{E}_2$ be a generator of $\mathcal{A}_2$. Then $\varphi : (X_1, \mathcal{A}_1) \to X_2$ is measurable in regards of $\mathcal{A}_2$ iff $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$*

Proof. $\implies$ is immediate

$\impliedby$ $\tilde{\mathcal{A}}_2 = \{A_2 \in \mathcal{A}_2 \mid \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over X_2. $\mathcal{E}_2$ is a TM of this σ algebra.

$$\implies \mathcal{A}_2 = \sigma(\mathcal{E}_2) \subset \tilde{\mathcal{A}}_2$$

□

Example 2.4. $f : \mathbb{R} \to \mathbb{R}$ *is monotonically increasing*

$$f^{-1}(-\infty, b] = \{x \mid f(x) \leq b\} = (-\infty, c) \in \mathcal{B}$$

Thus, f is measurable.

Remark. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

Example 2.5.

$$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{+\infty, -\infty\} \mid B \in \mathcal{B}_{\mathbb{R}}\}$$

$f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ *measurable in* $(X, \mathcal{A})$ *and* $f = \max\{f_1, \dots, f_n\}$

$$f : \mathbb{R} \to \overline{\mathbb{R}} \qquad x \mapsto \max\{f_1(x), \dots, f_n(x)\}$$

$$\begin{aligned} f^{-1}([-\infty, b]) &= \{x \mid f(x) \leq b\} \\ &= \{x \mid f_k(x) \leq b, k = 1, \dots, n\} \\ &= \bigcap_{k=1}^{n} \{x \mid f_k(x) \leq b\} \in \mathcal{B} \end{aligned}$$

Analogously for the minimum. Therefore f is measurable.

Example 2.6. *The same applies to countably many functions. Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ be measurable with $n \in \mathbb{N}$. Then $f : \sup\{f_n \mid n \in \mathbb{N}\}$ is measurable.*

$$\begin{aligned} f^{-1}([-\infty, b]) = \{x \mid \sup\{f_n(x)\} \leq b\} &= \{x \mid f_n(x) \leq b \forall n\} \\ &= \bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in B} \in \mathcal{B} \end{aligned}$$

Analogously for the infimum.

Example 2.7.

$$\limsup_{n\to\infty} \sup f_n = \inf_n \underbrace{\sup_{k\geq n} f_k}_{\text{with } n\to\infty \text{ monotonically decreasing}} \quad \textit{is measurable}$$

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k\geq n} f_k \textit{ is measurable}$$

if all f_k are measurable. Especially if $f_n \to f$ pointwise and all f_n are measurable, then f is measurable.

Theorem 2.7.1 (Result from the previous example)**.**

$$f_n : (X, \mathcal{A}) \to \overline{\mathbb{R}} \textit{ measurable}, n \in \mathbb{N}$$

$$\implies \inf f_n, \sup f_n, \lim_{n\to\infty} \inf f_n, \lim_{n\to\infty} \sup f_n$$

are all measurable.

↓ *This lecture took place on 2018/10/04.*

1. Basic set X $[\delta, \rho, ...]$
2. σ-algebra $\mathcal{A} \subset p(X)$
   (a) $X \in \mathcal{A}$
   (b) $A \in \mathcal{A} \implies \mathcal{A}^C \in \mathcal{A}$
   (c) $A_n \in \mathcal{A}$ $(n \in \mathbb{N}) \implies \bigcup_{n=1}^\infty A_n \in \mathcal{A}$

   $(X, \mathcal{A})$ is a measurable space
3. measure $\mu : \mathcal{A} \to [0, \infty]$
   (a) $\mu(\emptyset) = 0$
   (b) $A_n \in \mathcal{A}$ $(n \in \mathbb{N})$, $A_n \cap A_m \neq 0 \forall n \neq m$

   $$\implies \mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \mu(A_n) \qquad \sigma - \text{additivity}$$

   $(X, \mathcal{A}, \mu)$ is a measure space
4. $\mathcal{E} \subset p(X)$
   $$\sigma(\mathcal{E}) = \bigcap \left\{\tilde{A} : \tilde{A}\sigma - \text{algebra}, \mathcal{E} \subset \tilde{A}\right\}$$
   is the so-called $\mathcal{E}$-generated σ-algebra.

Recognize that $\mathcal{E}_1 \subset \mathcal{E}_2 \implies \sigma(\mathcal{E}_1) \subset \sigma(\mathcal{E}_2)$. If additionally, $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1) \implies \sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

If X is a metric space, we commonly (sometimes implicitly) use the Borel-Sigma algebra as measure space.

Example: Let $\mathbb{R}^d$. Then $\mathcal{B}_{\mathbb{R}^d}$ denotes the Borel-sigma algebra.

Let $\mathcal{E}_1$ be the set of open sets. Let $\mathcal{E}_2$ be the set of closed sets. Let $\mathcal{E}_3 = \{(a_1, b_1) \times \dots \times (a_d, b_d) : a_i, b_i \in \mathbb{R}, a_i < b_i\}$. $\sigma(\mathcal{E}_3) = \mathcal{B}_{\mathbb{R}^d}$ because every open set is a countable union of open (or left half-open) parallelepipeds (why countable?).

$$\mathcal{E}_3 \subset \mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$$

$$\mathcal{E}_4 = \{(a_1, b_1] \times (a_2, b_2] \times \dots \times (a_d, b_d]\}$$
$$(a, b) = \bigcup_{n=0}^\infty (a, b - \frac{1}{n})$$

$$\mathcal{E}_5 = \{(-\infty, b_1) \times (-\sigma, b_d) : b_1, \dots, b_d \in \mathbb{R}\}$$

because $\mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$.

DeMorgan: $A \setminus B = A \cap B^C$

Let $d = 1$, $(a, b] = (-\infty, b] \setminus (-\infty, a]$.
Let $d = 2$, $(a_1, b_1] \times (a_2, b_2] = (-\infty, b_1] \times (-\infty, b_2] \setminus (-\infty, a_1] \times (-\infty, b_2] \setminus (-\infty, b_2] \times (-\infty, a_1]$.

Definition 2.8. *Let $(X, \mathcal{A})$ be a measure space. $B \in \mathcal{A}$. trace σ-algebra over B is defined as $\{A \in \mathcal{A} : A \subset B\}$.*

Remark (Revision on continuity)**.** *Let $\varphi : (X_1, d_1) \to (X_2, d_2)$ be a map between metric spaces. Let φ be continuous.*

On the one hand, we know the ε-δ definition, but we also consider $\varphi^{-1}(O_2) \in \mathcal{O}_1 \forall O_2 \in \mathcal{O}_2$ (set of open sets)

Definition 2.9 (Measurable maps)**.** *Let $\varphi : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$*[1]

$$\iff \varphi^{-1}(A_2) \in \mathcal{A}_1 \forall A_2 \in \mathcal{A}_2$$

Lemma 2.10. *The composition of measurable maps is measurable.*

$$\varphi : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$$

$$\Psi : (X_2, \mathcal{A}_2) \to (X_3, \mathcal{A}_3)$$

with φ and Ψ measurable.

$$\implies \Psi \circ \varphi : (X_1, \mathcal{A}_1) \to (X_3, \mathcal{A}_3)$$

is measurable. (trivial to prove)

Theorem 2.10.1. *Let $\mathcal{E}_2$ be the generator of some algebra $\mathcal{A}_2$. Then $\varphi : (X_1, \mathcal{A}_1) \to X_2$ in regards of $\mathcal{A}_2$ is measurable if and only if $\varphi^{-1}(E_2) \in \mathcal{A}_1 \forall E_2 \in \mathcal{E}_2$.*

Proof. $\implies$ trivial

$\impliedby$ $\tilde{A}_2 := \{A_2 \in \mathcal{A}_2 : \varphi^{-1}(A_2) \in \mathcal{A}_1\}$ is a σ-algebra over X_2 (why? left as an exercise). $\mathcal{E}_2$ is a subset of this σ-algebra. $\implies$ $\mathcal{A}_2 = \sigma(\mathcal{E}_2) \in \tilde{A}_2 \subset \mathcal{A}_2$

□

Example. *$f : \mathbb{R} \to \mathbb{R}$ is monotonically increasing. $f^{-1}(-\infty, b] = \{x : f(x) \leq b\}$ is in the Borel-sigma algebra $\mathcal{B}$. So f is measurable.*

Definition. $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$
$\mathcal{B}_{\overline{\mathbb{R}}} = \{B, B \cup \{-\infty\}, B \cup \{+\infty\}, B \cup \{\pm\infty\} : B \in \mathcal{B}_{\mathbb{R}}\}$

Example 2.11. *Let $f_1, \dots, f_n : \mathbb{R} \to \overline{\mathbb{R}}$ measurable. $f = \max\{f_1, \dots, f_n\}$.*

$$f^{-1}([-\infty, b]) = \{x : f(x) \leq b\} = \{x : f_k(x) \leq b, k = 1, \dots, n\} = \bigcap_{k=1}^{n} \underbrace{\{x : f_k(x) \leq b\}}_{f_k^{-1}[-\infty, b]} \in \mathcal{B}$$

[1]Actually, $\varphi : X_1 \to X_2$, but we don't want to forget about the associated σ-algebras

Equivalently, $\min\{f_1, \dots, f_n\}$ *is measurable. Equivalently,* $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ *is measurable* *(*$n \in \mathbb{N}$*).* $\implies f = \sup\{f_n : n \in \mathbb{N}\}$ *is measurable.*

$$f^{-1}(\infty, b] = \{x : \sup f_n(x) \leq b\} = \{x : f_n(x) \leq b \forall n\}$$
$$f^{-1}[-\infty, b) = \{x : \sup f_n(x) < b\} \subset \{x : f_n(x) < b \forall n\}$$

$$\bigcap_{n=1}^{\infty} \underbrace{f_n^{-1}[-\infty, b]}_{\in \mathcal{B}} \in \mathcal{B}$$

Let f_n be measurable functions.

$$\limsup_{n\to\infty} f_n = \inf_n \sup_{k\geq n} f_k$$

The supremum of measurable functions is measurable (see Lemma 2.10). The infimum as well. So the result is measurable.

$$\liminf_{n\to\infty} f_n = \sup_n \inf_{k\geq n} f_k$$

Equivalently, the result is measurable.

Especially, if $f_n \to f$ pointwise, and all f_n are measurable, then also limit f is measurable.

How to determine measurability? Show that pre-images of generators are in the σ-algebra.

Theorem 2.11.1. *Let* $f : (X, \mathcal{A}) \to \overline{\mathbb{R}}$ *be measurable (*$n \in \mathbb{N}$*)*

$$\implies \inf f_n, \sup f_n, \liminf f_n, \limsup f_n$$

are also measurable.

↓ *This lecture took place on 2018/10/08.*

2.3 Simple properties of measures

A monotonically increasing sequence $(A_n)_{n\in\mathbb{N}}$ of sets is given by $A_1 \subset A_2 \subset A_3 \subset \dots$.

Theorem 2.11.2. *Let* $(X, \mathcal{A}, \mu)$*.*

1. $A_1, \dots, A_n \in \mathcal{A}, A_i \cap A_j = 0 \forall i \neq j \implies \mu\left(\dot\bigcup_{k+1}^{n} A_n\right) = \sum_{k=1}^{n} \mu(A_n)$
2. $\mu(B) = \mu(A \cap B) + \mu(A^C \cap B)$ *for* $A, B \in \mathcal{A}$
3. $A \subset B \implies \mu(A) \leq \mu(B)$ *for* $A, B \in \mathcal{A}$
4. $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$

5. *Let $(A_n)_{n\in\mathbb{N}}$ be a monotonically increasing sequence of $\mathcal{A}$ and $A = \dot{\bigcup}_{n=1}^{\infty} A_n = \lim_{n\to\infty} A_n$, then $\mu(A) = \lim_{n\to\infty}\mu(A_n)$ "Continuity from below"*
6. *Let A_n be a monotonically decreasing sequence of $\mathcal{A}$. $A = \bigcap_{n=1}^{\infty} A_n = \lim A_n$.*
7. *A_n arbitrary $\implies \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\mu(A_n)$*

Proof of continuity from below. Consider a monotonically increasing sequence of $\mathcal{A}$. Consider $B_1 = A_1, B_k = A_k \setminus A_{k-1}$ and $k \geq 2$. Sets B_i and B_j are disjoint with $i \neq j$. Then $B_1 \cup \dots \cup B_n = A_n$ and $\dot{\bigcup}_{k=1}^{\infty} B_k = A$.

$$\mu(A) = \sum_{k=1}^{\infty}\mu(B_k) = \lim_{n\to\infty}\sum_{k=1}^{n}\mu(B_k) = \lim_{n\to\infty}\mu\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty}\mu(A_n)$$

$$A'_n = A_1 \setminus A_n \qquad \nearrow A_1 \setminus A$$
$$\mu(A_1 \setminus A_n) = \mu(A'_n) \qquad \nearrow \mu(A_1 \setminus A)$$

□

What about the measure of intersected set in infinity? $A \cap B = A$ and $\mu(B) = \mu(A) + \mu(A^C \cap B)$. What happens if $\mu(A) = +\infty$ and $\mu(A^C \cap B) = -\infty$?

Remark. *How to compute algebraically with the extended real numbers?*

$$\pm\infty + a = \pm\infty \quad (a \in \mathbb{R})$$

$$+\infty \cdot a = \begin{cases} +\infty & a > 0 \\ 0 & a = 0 \\ -\infty & a < 0 \end{cases}$$

0 *for* $a = 0$ *makes sense in measure theory, but not in calculus.*

If $\mu(A_1) < \infty$, then $\mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$ and $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$.

Remark (Reminder).

$$\limsup a_n := \lim_{n\to\infty}\sup_{k\geq n} a_k$$

What about (A_n) arbitrary?

$$\limsup A_n = \bigcap_{n=1}^{\infty}\bigcup_{k\geq n} A_k$$
$$\liminf A_n = \bigcup_{n=1}^{\infty}\bigcap_{k\geq n} A_k$$

Property 7 can be shown as generalization of $\mu\left(\bigcup_{n=1}^{N} A_n\right) \leq \sum_{n=1}^{N}\mu(A_n)$

Example (Simplest example). $X = \{x_n : n \in \mathbb{N}\}$. $\mathcal{A} = p(X)$. *Fix* $\mu(\{x_n\})$.

$$\rightsquigarrow \mu(A) = \sum_{n:x_n \in A} \mu(x_n)$$

$\mu(x_n) = 1$ *gives a counting measure.*

Let $\mathcal{E}$ be the generator of $\mathcal{A} = \sigma(\mathcal{E})$. A *stable set by intersection* is given by $E_1, E_2 \in \mathcal{E} \implies E_1 \cap E_2 \in \mathcal{E}$.

Theorem 2.11.3 (Uniqueness of measures). *Let μ, ν be measures on $\mathcal{A}$ with $\mu|_{\mathcal{E}} = \nu|_{\mathcal{E}}$. $\implies \mu = \nu$ on $\mathcal{A}$.*

$X \in \mathcal{E}$ and $\mu(X) = \nu(X) < \infty$ *or* $X = \bigcup_n E_n$ with $E_n \in \mathcal{E}$ and $\mu(E_n) = \nu(E_n) < \infty$.

Definition 2.12. *Let $\mathcal{E} \subset \mathcal{P}(X)$ be a semiring over X. If*

1. $\emptyset \in \mathcal{E}$
2. $A, B \in \mathcal{E} \implies A \cap B \in \mathcal{E}$
3. $A, B \in \mathcal{E}. \implies \exists C_1, \dots, C_k \in \mathcal{E}$ *pairwise disjoint* : $A \setminus B = \dot{\bigcup}_{i=1}^k C_i$.

What is the difference compared to a *ring*? Let $A, B \in \mathcal{R} \implies (A \cap B \in \mathcal{E} \wedge A \triangle B \in \mathcal{E})$.

Theorem 2.12.1 (Extension theorem by Caratheodory). $\mu : \mathcal{E}$ *(semiring)* $\to \{0, \infty\}$ *with*

1. $\mu(\emptyset) = 0$
2. $(X \in \mathcal{E}$ *and* $\mu(X) < \infty)$ *or* $\left(X = \bigcup_{n=1}^\infty E_n, E_n \in \mathcal{E}, \mu(E_n) < \infty\right)$
3. μ *is σ-additive on $\mathcal{E}$, hence (A_n) is a sequence in $\mathcal{E}$, pairwise disjoint and* $\bigcup_{n=1}^\infty A_n \in \mathcal{E}$
$$\implies \mu\left(\bigcup_{n=1}^\infty A_n\right) = \sum_n \mu(A_n)$$

Then μ has a (unique) continuation for a measure on $\mathcal{A} = \sigma(\mathcal{E})$.

2.4 Construction of the Lebesgue measures and similar ones

Let $X = \mathbb{R}$ or $X = \overline{\mathbb{R}}$.

$$\mathcal{E} = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$$

is semiring.

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonically increasing and right-sided continuous. Let $\mu(a, b] := F(b) - F(a)$. Properties 1 and 2 of the extension theorem are satisfied. We show finite additivity of property 3 in three steps:

1. If $(a,b] = \dot{\bigcup}_{k=1}^{n}(a_k, b_k]$ can be sorted. $a_1 = a, a_{k+1} = b_k$ for $k = 1, \dots, n-1$ and $b_n = b$. We get a telescoping sum such that

$$\sum_{b=1}^{n}(F(b_k) - F(a_k)) = F(b) - F(a)$$

2. Also $(a_1, b_1], \dots, (a_n, b_n]$. Disjoint subintervals of $(a,b]$ are

$$\implies \sum_{k=1}^{n} \mu(a_k, b_k] \leq \mu(a,b]$$

3. $(a,b] = \dot{\bigcup}_{n=1}^{\infty}(a_n, b_n]$

   (a)

$$\dot{\bigcup_{n=1}^{N}}(a_n, b_n] \subset (a,b]$$

$$\sum_{n=1}^{N} \mu(a_n, b_n] \leq \mu(a,b] \forall N$$

$$\implies \sum_{n=1}^{\infty} \mu(a_n, b_n] \leq \mu(a,b]$$

   (b) Let $\varepsilon > 0$, then $\exists a' \in (a,b] : F(a') - F(a) < \varepsilon$

$$\exists b'_n > b : F(b'_n) - F(b_n) < \frac{\varepsilon}{2^n}$$

$$[a', b] \subseteq (a,b] \subset \bigcup_n (a_n, b_n] \subset \bigcup_n (a_n, b'_n)$$

$$\implies \exists N : (a', b) \subset [a', b] \subset \bigcup_{n=1}^{N}(a_n, b'_n) \subset \bigcup_{n=1}^{N}(a_n, b'_n]$$

   But these intervals in $\bigcup$ are not necessarily non-overlapping any more. But this is no problem as we can split them into disjoint sets.

$$\mu(a', b] \leq \sum_{n=1}^{N} \mu(a_n, b'_n]$$

$$\mu(a', b] = F(b) - F(a') \leq \sum_{n=1}^{N} F(b'_n) - F(a_n) \leq \sum_{n=1}^{N} \left(F(b_n) - F(a_n) + \frac{\varepsilon}{2^n}\right)$$

   with $F(b) - F(a') \geq F(b) - F(a) - \varepsilon$.

$$\mu(a,b] \leq \sum_{n=1}^{\infty} \mu(a_n, b_n] + 2\varepsilon \quad \forall \varepsilon > 0$$

↓ *This lecture took place on 2018/10/15.*

Theorem 2.12.2. *Let $\mathcal{E}$ be semiring over X and $\mu : \mathcal{E} \to [0, \infty]$ on $\mathcal{E}$ be σ-additive and σ-finite. Then there exists exactly one continuaton for measure on $\sigma(\mathcal{E})$.*

Let $F : \mathbb{R} \to \mathbb{R}$ be monotonic and right-sided continuous.

$$\mathcal{E} = \{(a, b] \mid a, b \in \mathbb{R}, a \leq b\} \qquad \mu(a, b] = F(b) - F(a)$$

Now consider the special case $F(x) = x$. This define the Lebesgue measure on $(\mathbb{R}, \mathcal{B})$.

Theorem 2.12.3. *λ is the only measure on $(\mathbb{R}, \mathcal{B})$ with*

1. $\lambda(B + C) = \lambda(\{x + c \mid x \in B\}) = \lambda(B) \quad \forall B \in \mathcal{B} \forall c \in \mathbb{R}$
2. $\lambda(0, 1] = 1$

Proof. Does λ satisfy these properties? Yes, λ has properties (1) and (2), because

(1) is correct $\forall (a, b] \in \mathcal{E}$

$$c \in \mathbb{R} : \{B \in \mathcal{B} \mid \lambda(B + c) = \lambda(B)\}$$

is σ-algebra and contains $\mathcal{E}$, so also $\sigma(\mathcal{E})$

(2) trivial

Is λ unique? Let λ be the measure with the two properties.

$$(0, 1] = \dot{\bigcup_{k=1}^{n}} \left(\frac{k-1}{n}, \frac{k}{n} \right]$$

$$1 = \mu(0, 1] = \sum_{k=1}^{n} \mu \left(\left(0, \frac{1}{n}\right] + \frac{k-1}{n} \right) = n\mu \left(0, \frac{1}{n}\right]$$

$$\mu \left(\frac{k-1}{n}, \frac{k}{n} \right] = \frac{1}{n} \quad \forall k \in \mathbb{Z}$$

$$\implies \mu(a, b] = b - a \qquad a, b \in \mathbb{Q}$$

$$\mu|_{\mathcal{E}_{\mathbb{Q}}} = \lambda_{\mathcal{E}_{\mathbb{Q}}} \qquad \mathcal{E}_{\mathbb{Q}} = \{(a, b] \mid a, b \in \mathbb{Q}, a \leq b\}$$

Closed under finite intersection, $\sigma(\mathcal{E}_{\mathbb{Q}}) = \mathcal{B}$:

$$(a, b) = \bigcup_{n} (a, b - \frac{1}{n}] \qquad (a, b] = \bigcap_{n} (a, b + \frac{1}{n}) \qquad \sigma\text{-finite}$$

$$\mu(-n, n] < \infty \qquad \bigcup_{n=1}^{\infty} (-n, n] = \mathbb{R} \qquad \sigma\text{-finite}$$

$$\implies \mu = \lambda \text{ (distinct extensionability)}$$

□

We apply the principle analogously to $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$.

$$\mathcal{E} = \{(a,b] = (a_1, b_1] \times \dots \times (a_n, b_n] \mid a_i \leq b_i \in \mathbb{R}\}$$

is semiring over $\mathbb{R}^d$. In $\mathbb{R}^2$, you can draw rectangles and their induced area based on their geometrical relation to each other. $F : \mathbb{R}^d \to \mathbb{R}$ complete is *monotonic* if

$$\mu(a,b] : \prod_{i=1}^{d} (F_i(b_i) - F_i(a_i)) = \sum_{x \in \{a_1,b_1\} \times \dots \times \{a_d,b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} \qquad F_1(x_1)F_2(x_2) \dots F_d(x_d)$$

Simplest case: $F_1, \dots, F_d : \mathbb{R} \to \mathbb{R}$ is monotonically right-sided continuous.

$$\sum_{x \in \{a_1,b_1\} \times \dots \times \{a_d,b_d\}} (-1)^{|\{i \mid x_i = a_i\}|} F(x) \geq 0 \forall (a,b] \in \mathcal{E}$$

$$F(b_1, b_2) - F(a_1, b_2) - F(a_1, b_1) + F(a_1, a_2)$$

F is right-sided in every coordinate, thus $\mu(a,b] = \sum_{x \in \{a_1,b_1\} \times \dots \times \{a_d,b_d\}} (-1)^{|\{i \mid x_i = a_i\}|}$

2.5 Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$

We can extend the previous definition from $\mathbb{R}$ to $\mathbb{R}^d$. Thus λ is the only measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ with

1. $\lambda^d(B + c) = \lambda(B) \forall B \in \mathbb{B}_{\mathbb{R}^d} \forall c \in \mathbb{R}^d$
2. $\lambda((0,1]^d) = 1$

Theorem 2.12.4. *Let $H \subset \mathbb{R}^d$ be a hyperplane. Then $\lambda_d(H) = 0$.*

Proof. Without loss of generality, $\vec{O} \in H$ is subspace with dimension $d-1$. Why is $H \in \mathcal{B}_d$ true? The Lebesgue measure is based on open sets. The σ-algebra requires the complement, thus closed sets are also given. The measure of closed sets is zero.

$\left\{\vec{b}_1, \dots, \vec{b}_{d-1}\right\}$ is an orthonormal basis of H.

$$Q = \left\{c_1 \vec{b}_1 + \dots + c_{d-1} \vec{b}_{d-1} \mid 0 \leq c_i \leq 1\right\} \in \mathcal{B}_{\mathbb{R}^d}$$

$\vec{b}_d \bot \vec{b}_i$ $(i = 1, \dots, d-1)$, $\left\|\vec{b}_d\right\| = 1$.

$$Q + q \cdot \vec{b}_d \qquad q \in \mathbb{Q} \cap [0,1] \text{ pairwise disjoint}$$

$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} Q + q\vec{b}_d \subset \left\{c_1 \vec{b}_1 + \dots + c_d \vec{b}_d \mid 0 \leq c_i \leq 1\right\} \text{ compact}$$

$$\infty > \lambda_d \left(\dot{\bigcup_{q \in \mathbb{Q} \cap [0,1]}} Q + q \cdot \vec{b}_d \right) = \sum_{q \in \mathbb{Q} \cap [0,1]} \lambda_d(Q)$$

$$\implies \lambda_d(Q) = 0 \qquad H \subset \bigcup_{\vec{x} \in \mathbb{Z}^d} (Q + \vec{x})$$

$$\lambda_d(H) \leq \sum_{\vec{x} \in \mathbb{Z}^d} \lambda_d(Q + \vec{x}) = 0$$

□

Theorem 2.12.5. *Let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be linear and bijective. $\varphi(\vec{x}) = M \cdot \vec{x}$ with M as regular matrix.*

Linear implies continuous in finite dimensions. Every continuous map is measurable.

$\implies$ *φ is measurable and $\lambda_d(\varphi(B)) = \det(\varphi) \cdot \lambda_d(B)$. This holds even if φ is not bijective, because then $\det(\varphi) = 0$ and thus we have a factor zero. If φ is not bijective, then the matrix has lower rank. The image is a hyperplane or is contained in a hyperplane. So the measure is zero.*

Proof. $\mu_\varphi(B) := \lambda_d(\varphi(B))$ is measure on $\mathcal{B}_{\mathbb{R}^d}$ (why? left as an exercise to the reader).

$$\mu_\varphi(B + \vec{c}) = \lambda_d(\varphi(B + \vec{c})) = \lambda_d(\varphi(B) + \underbrace{\varphi(\vec{c})}_{X}) = \mu_\varphi(B)$$

$$\frac{\mu_\varphi}{\mu_\varphi((0,1]^d)} = \lambda_d$$

Show that: $\mu_\varphi((0,1]^d] = \left|\det \varphi\right|$

Case 1 $\varphi(M)$ is orthogonal $M^* = M^{-1}$.

$$\varphi(B_1(\vec{0})) = B_1(\vec{0}) \qquad 0 < \lambda_d(B_1(\vec{0})) < \infty$$

$$\frac{\lambda_d(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \frac{\mu_\varphi(B_1(\vec{0}))}{\mu_\varphi((0,1]^d)} = \lambda_d(B_1(\vec{0}))$$

$$\mu_\varphi = \lambda_d$$

Case 2

$$M = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_d \end{pmatrix} \qquad d_i > 0$$

$$\varphi(\vec{e}_i) = d_i \cdot e_i$$

$$\varphi((0,1]^d) = (0, d_1] \times (0, d_2] \times \ldots (0, d_d]$$

$$\varphi_\varphi((0,1]^d) = \det D$$

Generic case Let M be any matrix. We consider the singular value decomposition $M = O_1 \cdot D \cdot O_2$ with O_1, O_2 orthogonal and D is a non-negative diagonal matrix.

$$M^* M \rightsquigarrow O^* D^2 O$$

Then $\varphi = \varphi_1 \circ \psi \circ \varphi_2$. φ_1 and φ_2 are orthogonal. Let D be the representation matrix of ψ. Diagonal entries are positive because it is regular.

$$|\det \varphi| = \det(\psi)$$

Combining these results gives us the theorem.

□

↓ *This lecture took place on 2018/10/16.*

2.6 Sigma-algebra generated by maps

Definition 2.13. *$\mathcal{A}_i$ ($i \in I$) is σ-algebra over X.*

$$\bigvee_{i\in I} \mathcal{A}_i = \sigma\left(\bigcup_{i\in I} \mathcal{A}_i\right)$$

Definition 2.14 (Image σ-algebra and Push-forward measure)**.** *Push-forward measures are called Bildmaß $(X, \mathcal{A})$ is a measure space. $\varphi : X \to X'$.*

$$\varphi(\mathcal{A}) = \{A' \subset X' \mid \varphi^{-1}(A') \in \mathcal{A}\}$$

$\varphi(X, \mathcal{A}) \to (X', \mathcal{A}')$ is measurable $\iff \varphi(\mathcal{A}) \supset \mathcal{A}'$.

$(X, \mathcal{A}, \mu)$ is a measure space, $\varphi : X \to X'$. μ_φ is the push-forward measure on $(X', \varphi(\mathcal{A}))$.

$$\mu_\varphi(A') = \mu(\varphi^{-1}(A'))$$

Definition 2.15 (Generated σ-algebra)**.**

1. *X, $(X', \mathcal{A}')$ is a measurable space. $\varphi : X \to X'$*

   $$\sigma(\varphi) = \{\varphi^{-1}(A') \mid A' \in \mathcal{A}'\}$$

   Iff $\varphi : (X, \mathcal{A}) \to (X', \mathcal{A}')$ is measurable, $\sigma(\varphi) \subset \mathcal{A}$.

2. *$X, (X_i, \mathcal{A}_i)$, $i \in I$ are measure spaces*

   $$\psi_i : X \to X_i \forall i$$

   The σ-algebra generated by ψ_i ($i \in I$) is the smallest σ-algebra that contains such a set. $\bigvee_{i\in I} \sigma(\psi_i)$. Is the smallest σ-algebra on X which are measurable for all ψ_i.

Example. $\varphi : (\mathbb{R}^2, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$.

$$\varphi(x, y) = \sqrt{x^2 + y^2} \qquad \sigma(\varphi) = \{B \subset \mathbb{R}^2 \mid B \text{ rotation invariant in } 0.000001\}$$

Theorem 2.15.1. *Let $(X, \mathcal{A})$ be a measure space. Let $(X', \mathcal{A}')$ be another one. Let $(X_i, \mathcal{A}_i)$ be measure spaces with ($i \in I$). Then we can map from $(X, \mathcal{A})$ to $(X', \mathcal{A}')$ with measurable φ and we can map $(X', \mathcal{A}')$ to $(X_i, \mathcal{A}_i)$ with ψ_i such that $\mathcal{A}' = \sigma(\psi_i : i \in I)$. Then φ is measurable iff $\psi_i \circ \varphi$ is measurable $\forall i \in I$.*

Proof. $\implies$ immediate.

$\impliedby$

$$\mathcal{E}' = \bigcup \sigma(\psi_i) \text{ generates } \mathcal{A}'$$

$$A' \subset \mathcal{E}' \implies \exists i : A' \in O(\psi_i), \text{ so } A' = \psi_i^{-1}(A_i) \text{ with } A_i \in \mathcal{A}_i.$$

$$\varphi^{-1}(A) = \psi^{-1}(\psi_i^{-1}(A_i)) = (\underbrace{\psi_i \circ \varphi}_{\in\mathbb{R}})^{-1}(A_i)$$

□

2.7 Product space

Let $X_n, \mathcal{A}_n$ and $n = 1, \dots, N$ with $N < \infty$. Let $X = \prod_{n=1}^{N} X_n$ ("product $sigma$-algebra") generated by $\mathcal{E} = \left\{\prod_{n=1}^{N} A_n \cdot A_n \subset A_n \forall n\right\}$.

Consider $N = 2$. $X = X_1 \times X_2$. $\mathcal{E} = \{A_1 \times A_2 \mid A_n \in \mathcal{A}_n, n = 1, 2\}$. Product σ-algebra: $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Commonly, we use the notation $(X, \otimes \mathcal{A}_n) = \otimes(X_n, \mathcal{A}_n)$

Lemma 2.16.

$$\oplus_{n=1}^{N} \mathcal{A}_n = \sigma\left(\pi_n : n = 1, \dots, N\right)$$

where $\pi_n : X \to X_n$ *is the* n*-th projection.*

Hint: $\mathcal{E}_0 = \left\{\prod_{n=1}^{N} A_n \text{ with } A_n = X_n \forall n \text{ expect for one and this } A_n \in \mathcal{A}_n\right\}$ *also generates* $\otimes \mathcal{A}_n$.

This lemma holds obviously.

Theorem 2.16.1. $\varphi : (X, \mathcal{A}) \to \otimes_{n=1}^{N} (X_n, \mathcal{A}_n)$, *where N denotes finite or countable, is measurable* $\iff$ $\pi_n \circ \varphi : (X, \mathcal{A}) \to (X_n, \mathcal{A}_n)$ *is measurable* $\forall n$. *This is a special case of Theorem 2.15.1.*

Prospect: Product measure.

Let $(X_1, \mathcal{A}_1) \otimes (X_2, \mathcal{A}_2, \mu_2)$. How to generate this? Well,

$$= (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$$

on $\mathcal{E} : \mu_1 \otimes \mu 2(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$ (compare it to the trivial case of the area of a rectangle in $\mathbb{R}^2$) where $\mathcal{E}$ is a semiring.

3 Integration of non-negative functions

Let $(X, \mathcal{A}, \mu)$ be a measure space. Consider $f : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B})$ How about $\int_X f \, d\mu$?

First of all, $f : (X, \mathcal{A}) \to [0, \infty]$. We know construct the Lebesgue integral:

First step Consider simple functions (like step functions). f takes up only finitely many different values. $z_1, \dots, z_n (\geq 0) : [f = z_k] := \{x \in X \mid f(x) = z_k\} = f^{-1}(\{z_k\}) \in \mathcal{A}$. We restrict $z_i \geq 0$ to avoid issues like $+\infty + (-\infty)$.

$$X = \dot{\bigcup_{k=1}^{n}} [f = z_k]$$

Definition 3.1.

$$\int f \, d\mu = \sum_{k=1}^{n} z_k \mu[f = z_k]$$

Consider that $z_k \mu[f = z_k]$ *might go to infinity. We commonly denote* $\sum_{z \in \mathbb{R}} z\mu[f = z]$ *in the real-valued case to avoid indices.*

Second step Let $f : (X, \mathcal{A}) \to [0, \infty]$ be measurable.

$$\int_X f \, d\mu := \sup \left\{ \int_X s \, d\mu : s \text{ simple}, 0 \leq s \leq f \right\}$$

So the Riemann integral approximates the area with upper and lower bounds for rectangles. For the Lebesgue integral, we split the function into horizontal slices in $\mathbb{R}$. Then we consider the differences of the function values between two consecutive slices. The important point is that this does not require $\mathbb{R}$, but some X and therefore is more generic.

Third step Let $f : (X, \mathcal{A}) \to \mathbb{R}$ and $f = f^+ - f^-$. Let $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. If $\int_X f^+ \, d\mu = \int_X f^- \, d\mu = \infty$: $\int_X f \, d\mu$ is not defined. Otherwise $\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$.

Does this definition/construction of the Lebesgue integral satisfy the desired properties of linearity/monotonicity/...? In the following, we will denote "simple" functions always as s.

Definition 3.2. *Let $f : (X, \mathcal{A}) \to [0, \infty]$ be measurable. Let $A \in \mathcal{A}$.*

$$\int_A f \, d\mu := \int \mathbf{1}_A f \, d\mu$$

Lemma 3.3. *Let $s : (X, \mathcal{A}) \to [0, \infty]$ be a simple function. Then $\nu_s(A) = \int_A s \, d\mu$ is a measure on $(X, \mathcal{A})$.*

$$\nu_s(A) = \sum_{k=1}^{n} z_k \mu([s = z_k] \cap A)$$

because $\mathbf{1}_A \cdot s = \sum_{k=1}^{n} z_k \mathbf{1}_{[s=z_k]} \mathbf{1}_A + 0 \cdot \mathbf{1}_{A^C}$.

$A \mapsto \mu([s = z_k] \cap A)$ *is a measure* $\forall k$.

↓ *This lecture took place on 2018/10/22.*

Definition 3.4. *Let $(X, \mathcal{A}, \mu)$ be a measure space. $s : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is called* simple *if $s(X)$ is finite.*
$s \geq 0$.

$$\int_X s \, d\mu := \sum_z z \cdot \mu[s = z]$$

Trivial: If $s = \sum_{j=1}^{m} c_j \cdot \mathbf{1}_{A_j}$, $A_j \in \mathcal{A}$ then s is simple. A_j are not necessarily pairwise disjoint and $\int_X s \, d\mu = \sum_{j=1}^{n} c_j \mu(A_j)$.

Proof. $\vec{\varepsilon} \in \{-1, 1\}^m$ with $A^1 := A, A^{-1} := A^C$. $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$. E.g. $A_1 \cap A_2 \cap A_3^C = B_{1,1,-1}$.

$$B_{\vec{\varepsilon}} = A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \dots \cap A_m^{\varepsilon_m}$$

is pairwise disjoint. On $B_{\vec{\varepsilon}}$, s has value $\sum_{\varepsilon_j=1} c_j = b_{\vec{\varepsilon}}$

$$\implies s = \sum b_{\vec{\varepsilon}} \mathbf{1}_{B_{\vec{\varepsilon}}}$$

and $\int \leq d\mu = \sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}}) = \cdots = \sum c_j \mu(A_j)$ (where $\sum_{\vec{\varepsilon}} b_{\vec{\varepsilon}} \mu(B_{\vec{\varepsilon}})$ is the disjoint case and $\sum c_j \mu(A_j)$ is generic).

$$\sum_{\vec{\varepsilon}} \sum_{j:\varepsilon_j=1} c_j \cdot \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \sum_{\vec{\varepsilon}:\varepsilon_j=1} \mu(B_{\vec{\varepsilon}}) = \sum_j c_j \mu(A_j)$$

□

Corollary 3.5. *Let $s_1, s_2 : X \to [0, \infty]$ be simple. Then $s = \alpha \cdot s_1 + \beta \cdot s_2$ ($\alpha, \beta \geq 0$) is simple and $\int s \, d\mu = \alpha \cdot \int s_1 \, d\mu + \beta \int s_2 \, d\mu$.*

Theorem 3.5.1 (Markov inequality). *Let $z \in \mathbb{R}$. Let $f \geq 0$.*

$$z \cdot \mu[\underbrace{f \geq z}_{\{x \in X \mid f(x) \geq z\}}] \leq \int f \, d\mu$$

Proof.

$$s = z \cdot \mathbf{1}_{[f \geq z]} \leq f$$

If $x \in [f \leq z] : z \cdot 1 \leq f(x)$.
If $x \notin [f \leq z] : z \cdot 0 \leq f(x)$.

s is simple, so $z\mu[f \geq z] = \int s \, d\mu \leq \int f \, d\mu$. $s = 0 : \mathbf{1}_{[f<z]} \times z \cdot \mathbf{1}_{[f \geq z]}$. □

Definition 3.6. *A statement holds* almost everywhere *if $\forall x \in \mathcal{A} : \mu(A^C) = 0$. So A^C is a null set, i.e. of measure zero.*

Theorem 3.6.1.

$$\forall f, g : X \to [0, \infty] \text{ measurable}$$

1. $f \leq g$ *almost everywhere* $\implies \int f \, d\mu \leq \int g \, d\mu$
2. $f = g$ *almost everywhere* $\implies \int f \, d\mu = \int g \, d\mu$
3. $\int f \, d\mu = 0 \implies f = 0$ *almost everywhere*
4. $\int f \, d\mu < \infty \implies f < \infty$ *almost everywhere*

Proof.

1. Let s be simple, $0 \leq s \leq f$. $s \cdot \mathbf{1}_{[f \leq g]} \leq g$ where $s \cdot \mathbf{1}_{[f \leq g]}$ is simple. $\int s \cdot \mathbf{1}_{[f \leq g]} \, d\mu \leq \int g \, d\mu$. $\int s \cdot \mathbf{1}_{[f \leq g]} \, d\mu = \int s \, d\mu$.
   If $\forall s$ simple, $0 \leq s \leq f$, then

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu \mid 0 \leq s \leq f, s \text{ simple} \right\} \leq \int g \, d\mu$$

2. $f \leq g$ almost everywhere and $f \geq g$ almost everywhere $\implies \int f \, d\mu = \int g \, d\mu$.

3. Markov inequality with $z = \frac{1}{n}$.

$$\frac{1}{n}\mu\left[f \geq \frac{1}{n}\right] \leq \int f\,d\mu = 0 \implies \mu\left[f \geq \frac{1}{n}\right] = 0 \forall n \in \mathbb{N}$$

$x \in [f \geq \frac{1}{n}] \implies x \in [f \geq \frac{1}{n+1}]$

$$\implies \mu\left[f \geq \frac{1}{n}\right] \to \mu\left[\bigcup\left[f \geq \frac{1}{n}\right]\right] = \mu\left[f > 0\right] = 0$$

4. $z > 0$, $s = z \cdot \mathbf{1}_{[f=\infty]} \leq f$.

$$z\mu[f = \infty] = \int s\,d\mu \leq \int f\,d\mu = M < \infty$$

$$\mu[f = \infty] \leq \frac{M}{z} \qquad \forall z > 0 \implies \mu[f = \infty] = 0$$

□

Theorem 3.6.2 (Levi's theorem about monotone convergence)**.** *If $f_n : (X, \mathcal{A}) \to [0, \infty]$ is measurable and pointwise monotonically increasing ($f_1 \leq f_2 \leq \dots$) and $f = \lim_{n\to\infty} f_n$ then $\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu$*

Proof. Because of (1) in the previous theorem, $\int f_n\,d\mu$ is monotonically increasing and $\leq \int f\,d\mu$, so $\lim \int f_n\,d\mu \leq \int f\,d\mu$.

$(y)^+$ denotes the function y if $y \geq 0$ and 0 otherwise.

Show "$\geq$". Let $0 \leq s \leq f$ be simple. Let $\varepsilon > 0$. $s_{n,\varepsilon} := (s-\varepsilon)^+\mathbf{1}_{[f_n \geq f-\varepsilon]}$ is a simple function. $s - \varepsilon \leq f - \varepsilon \leq f_n$. $s_{n,\varepsilon} \leq f_n$.

$$\sum_z (z-\varepsilon)^+\mu\left[s = z, f_n > f - \varepsilon\right] = \int s_{n,\varepsilon}\,d\mu \leq \int f_n\,d\mu \leq \lim \int f_n\,d\mu$$

$$s_{n,\varepsilon} = \underbrace{\sum_{z \text{ (values of s)}} (z-\varepsilon)^+\mathbf{1}_{[s-z]}}_{(s-\varepsilon)^+}\mathbf{1}_{[f_n > f-\varepsilon]}$$

$$[f_n > f - \varepsilon] \nearrow X \qquad [s = z, f_n > f - \varepsilon] \nearrow [s - z]$$

$$\implies \sum_z (z-\varepsilon)^+\mu[s = z] \leq \lim \int f_n\,d\mu$$

$$\varepsilon \to 0 \implies \sum_z z\mu[s = z] \leq \lim \int f_n\,d\mu$$

If $z > 0$, such that $\mu[s = z] = +\infty$. $0 < \varepsilon < z$.

Let $s_{n,\varepsilon} = (s-\varepsilon)^+\mathbf{1}_{[f_n \geq M \wedge (f-\varepsilon)]}$, where $a \wedge b$ denotes the minimum of a and b. Let $M \geq \max s$. □

↓ *This lecture took place on 2018/10/29.*

Remark (Revision). *Let s be a simple function.* $s = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$.
$s = \sum_z z\mathbf{1}_{[s=z]}$ *is a finite sum*
$\int s \, d\mu = \sum_z \mu[s = z] = \sum_{i=1}^{n} c_i \mu(A_i)$
This is independent of the representation.

Let $f : (X, \mathcal{A}) \to [0, \infty]$ be measurable. Then we can approximate the integral of f using the integrals of simple functions.

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu \mid 0 \leq s \leq f, \text{ simple} \right\}$$

Remark (Properties).

1. $0 \leq f \leq f$ *almost everywhere (wrt. μ)* $\implies \int f \, d\mu \leq \int g \, d\mu$
2. $f = g$ *almost everywhere (wrt. μ)* $\implies \int f \, d\mu = \int g \, d\mu$
3. $\int f \, d\mu = 0 \iff f = 0$ *almost everywhere (wrt. μ)*
4. $\int f \, d\mu < \infty \implies f < \infty$ *almost everywhere*

It is obvious if s is simple, then $\int s \, d\mu = \max \left\{ \int t \, d\mu \mid 0 \leq t \leq s \text{ simple} \right\}$

Theorem (Monotonic convergence). *Let $f_n : (X, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$f_n \leq f_{n+1} \forall n \qquad f = \lim_{n\to\infty} f_n \qquad \implies \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

Lemma (Lemma by Fatou). *Let $f_n : (X, \mathcal{A}) \to [0, \infty]$ be measurable.*

$$\implies \int \left(\lim_{n\to\infty} \inf f_n \right) d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Proof.

$$\lim_{n\to\infty} \underbrace{\inf_{m \geq n} f_m}_{g_n} \qquad g_n \nearrow \liminf f_n$$

By the theorem of monotonic convergence,

$$\implies \int (\liminf f_n) \, d\mu = \int \lim g_n \, d\mu = \lim \int g_n \, d\mu$$

□

Lemma 3.7. *Let $f : (X, \mathcal{A}) \to [0, \infty]$ with countable $f(X)$.*

$$\implies \int f \, d\mu = \sum_{z \in f(X)} z\mu[f = z]$$

Proof.

$$f(X) = \{z_n \mid n \in \mathbb{N}\}$$

$$f_n = \sum_{k=1}^{n} z_k \mathbf{1}_{[f=z_k]} \nearrow f \implies \int f \, d\mu = \lim \int f_n \, d\mu = \lim \sum_{k=1}^{n} z_k \mu[f = z_k]$$

□

The integral should be linear. We expect this for any integral.

Theorem 3.7.1. *Let $f, g : (X, \mathcal{A}) \to [0, \infty]$ be measurable. Let $\alpha \geq 0$.*

1. $\int (\alpha f) \, d\mu = \alpha \int f \, d\mu$ *(trivial to prove)*
2. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$

Proof. 1. trivial

2. We represent f_n

$$f_n = \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} \mathbf{1}_{[\frac{k}{2^n} \leq f < \frac{k+1}{2^n})} + n \cdot \mathbf{1}_{[f \geq n]} \nearrow f$$

Compare with Figure 2. g analogously $\nearrow g$

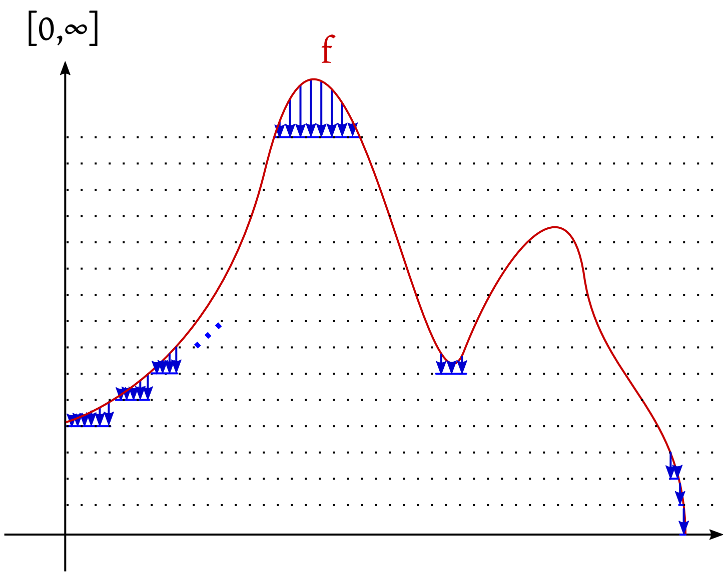

Figure 2: Illustration of the lebesgue integral

Let f_n, g_n be simple. $f_n + g_n \nearrow f + g$

$$\int (f + g) \, d\mu \overset{\text{monotonic convergence}}{=} \lim \int (f_n + g_n) \, d\mu$$

$$= \lim(\int f_n \, d\mu + \int g_n \, d\mu) \overset{\text{monotonic convergence}}{=} \int f \, d\mu + \int g \, d\mu$$

□

Unlike the Riemann integral, we use horizontal lines instead of vertical lines. Thus we partition the image, not the domain.

4 Integrable functions

Definition 4.1. *Let $f : (X, d) \to \overline{\mathbb{R}}$ is measurable. If not $\int f^+ \, d\mu = \int f^- \, d\mu = +\infty$, integral exists:*

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

$$f^+ = \max\{f, 0\} \qquad f^- = \max\{-f, 0\} \qquad f = f^+ - f^- \qquad |f| = f^+ + f^-$$

f is called integrable, if $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$

$$\iff \int f \, d\mu \text{ exists and is finite}$$

Remark 4.2 (Properties)**.**

1. *f is integrable $\iff$ $|f|$ is integrable and $|\int f \, d\mu| \leq \int |f| \, d\mu$*
2. *f, g are integrable with $f \leq g$ almost everywhere wrt. μ $\implies$ $\int f \, d\mu \leq \int g \, d\mu$*
3. *f is integrable, $\alpha \in \mathbb{R}$ $\implies$ $\alpha \cdot f$ is integrable and $\int (\alpha \cdot f) \, d\mu = \alpha \cdot \int f \, d\mu$*
4. *f, g are integrable $\implies$ $f + g$ is integrable and $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$*

Proof. 1. f is integrable

$$:\iff \int f^\pm \, d\mu < \infty \iff \underbrace{\int f^+ \, d\mu + \int f^- \, d\mu < \infty}_{\int |f| \, d\mu < \infty}$$

$$\begin{aligned}\left|\int f \, d\mu\right| &= \left|\int f^+ \, d\mu - \int f^- \, d\mu\right| \\ &\leq \int f^+ \, d\mu + \int f^- \, d\mu \\ &= \int |f| \, d\mu\end{aligned}$$

2. $f^+ - f^- \overset{\substack{\text{almost}\\\text{everywhere}}}{\leq} g^+ - g^- \implies f^+ + g^- \overset{\text{a.e.}}{\leq} f^- + g^+$

$$\int f^+ \, d\mu + \int g^- \, d\mu = \int (f^+ + g^-) \, d\mu \leq \int (f^- + g^+) \, d\mu = \int f^- \, d\mu + \int g^+ \, d\mu$$

$$\int f^+ \, d\mu - \int f^- \, d\mu \leq \int g^+ \, d\mu - \int g^- \, d\mu$$

It is important to recognize that all integrals are finite.

3. For $\alpha = 0$, the statement is true. Consider $\alpha > 0$.

$$(\alpha f)^\pm = \alpha \cdot f^\pm \qquad \int \alpha f \, d\mu = \int \alpha \cdot f^+ \, d\mu - \int \alpha \cdot f^- \, d\mu = \alpha \int f^+ \, d\mu - \alpha \int f^- \, d\mu$$

Now consider $\alpha < 0$, or more simply $\alpha = -1$ (any negative number is the product of a positive number and -1):

$$(-f)^+ = f^- (-f)^- = f^+ \qquad \dots$$

4. $(f+g)^+ - (f+g)^- = f + g = f^+ + g^+ - (f^- + g^-)$

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

$$\int (f+g)^+ \, d\mu + \int f^- \, d\mu + \int g^- \, d\mu = \int (f+g)^- \, d\mu + \int f^+ \, d\mu + \int g^+ \, d\mu$$

$$\int (f+g)^+ \, d\mu - \int (f+g)^- \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu + \int g^+ \, d\mu - \int g^- \, d\mu$$

□

Riemann integral only works for $\mathbb{R}^n$. The Lebesgue integral works for any measure space.

Example 4.3. *We consider the Riemann integral:*

$$\pi = \int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx \overset{\text{Riemann}}{=} \lim_{c,d \to \infty} \int_{-c}^{d} \frac{\sin x}{x} \, dx \text{ exists}$$

If you consider $\frac{\sin x}{x}$ for one π, we have a positive and negative area. By Leibniz criterion, we have an alternating series and its limit is zero.

We consider the Lebesgue integral:

$$\int_{\mathbb{R}} \left| \frac{\sin x}{x} \right| \, dx = +\infty$$

$\frac{\sin x}{x}$ is not Lebesgue-integrable. Because in case of the Lebesgue integral, we don't consider an alternating series, but need to consider $|f|$, which is non-negative and the series does not converge.

Theorem 4.3.1 (Dominated convergence theorem by Lebesgue)**.** *Let $f_n : (X, \mathcal{A}) \to \overline{\mathbb{R}}$ be a sequence of measurable functions. $f_n \to f$ pointwise [almost everywhere wrt. μ]. There exists $g : (X, \mathcal{A}) \to [0, \infty]$ integrable [$\int g \, d\mu < \infty$].*

$$|f_n| \leq g \text{ almost everywhere wrt. } \mu \forall n \implies \int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$$

Proof. Without loss of generality, almost everywhere implies everywhere.

$$|f| = \lim |f_n| \leq g \qquad \text{all of them are integrable}$$
$$g_n = 2g - |f_n - f| \geq 0 \qquad g_n \to 2g$$
$$\liminf \int g_n \, d\mu \geq \int (\liminf g_n) \; d\mu \overset{g_n \to 2g}{=} \int (\lim g_n) \; d\mu = 2 \int g \, d\mu$$
$$\int g \, d\mu - \limsup \int |f_n - f| \; d\mu = \liminf \int g_n \; d\mu = 2 \int g \, d\mu$$
$$\limsup \left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \limsup \int |f_n - f| \; d\mu = 0$$

Again:

$$\int g_n = \left(\int 2g - \int |f_n - f| \right)$$
$$\implies \limsup \int g_n = \limsup \left(\int 2g - \int |f_n - f| \right)$$
$$= \int 2g + \limsup \left(- \int |f_n - f| \right) = \int 2g - \liminf \left(\int |f_n - f| \right)$$

□

↓ *This lecture took place on 2018/11/05.*

Remark (Dominated convergence theorem by Lebesgue)**.** $f_n, f, g : (X, \mathcal{A}, \mu) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}$.

$$\begin{cases} f_n \to f & \mu \textit{ almost everywhere} \\ \|f_n\| \leq g & \mu \textit{ almost everywhere} \end{cases} g \geq 0, \int_X g \, d\mu < \infty$$
$$\implies \int X f \, dx = \lim_{n \to \infty} \int_X f_n \, d\mu$$

Example 4.4. $([0,1], \mathcal{B}_{[0,1]}, \lambda)$ *with* $f_n(x) \to 0$ *and* $\int_{[0,1]} f_n \, d\lambda = 1 \not\Longrightarrow \int_{[0,1]} 0 \, d\mu$. *Compare with Figure 3.*

The theorem of convergence is a generalization of the following theorem (based on Analysis 1 and Analysis 2 courses):

Example 4.5 (Monotonic convergence)**.**

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad a_n \geq 0$$

Convergence raidus: $R < \infty$.

$$x_k \nearrow R \implies f_k(x) \nearrow f(R)$$

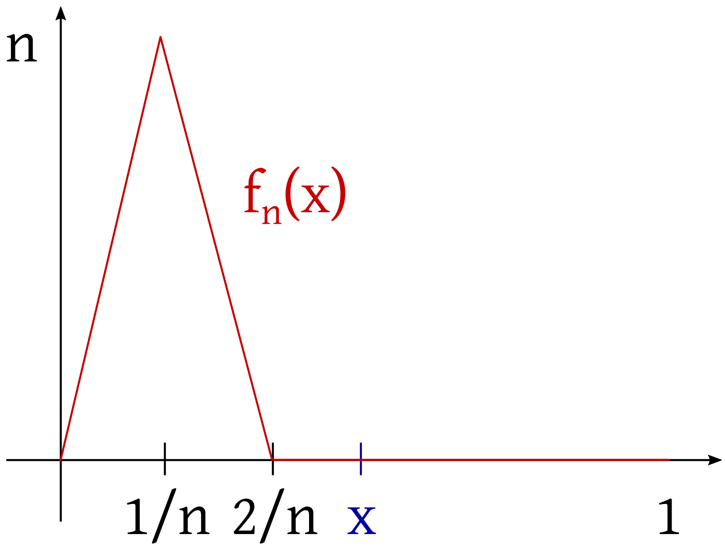

Figure 3: $f_n(x)$

$$X = \mathbb{N}_0, \mathcal{A} = \mathcal{P}(\mathbb{N}_0), \mu$$

"counting measure"

$$f_k(n) = a_n x_k^n$$

$$f : \mathbb{N}_0 \to [0, \infty]$$

$$\int_{\mathbb{N}_0} f \, d\mu = \sum_{n=0}^{\infty} f(n)\mu(n)$$

for $k \to \infty$: $f_k(n) \nearrow f(n) = a_n R^n$. By monontonic convergence, $\int f_k \, d\mu \nearrow \int f \, d\mu$.

$$\sum_{n=0}^{\infty} a_n x_k^n \nearrow \sum_{n=0}^{\infty} a_n R^n$$

5 Lebesgue and Riemann integral

λ is defined on $(\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$. The Lebesgue measure allows null sets. Lebesgue measure is also defined on completion of sigma algebras. Lebesgue measure $\mathcal{L}$ is defined on σ-algebras of Lebesgue sets.

Remark (One characterization of the Axiom of Choice)**.** *A non-empty product of non-empty sets is non-empty.*

Remark. $\mathcal{L} \setminus \mathcal{B} \neq \emptyset$ *[Axiom of Choice].*

Remark (Number representation)**.** *Basis* $q \in \{2, 3, \dots\}$.

$$x = \sum_{n=1}^{\infty} \frac{x_n}{q^n} \qquad x_n \in \{0, 1, \dots, q-1\}$$

This represents a number $0.x_1 x_2 x_3 x_4 \dots$.

Because $0.7\overline{9} = 0.8$*, there is some ambiguity between the numbers and their representation (non-bijective, two sums represent the same* x*).*

Remark (Cantor set)**.** *Consider* $[0, 1]$*. Split the interval into 3 thirds. We remove the middle third as open set. We consider the remaining two intervals and again extract the middle third. We iteratively continue this process to infinity. The remaining set is called Cantor set and is uncountable.*

The Cantor set $\mathcal{C}$ *is the set of numbers in* $[0, 1]$ *with some number representation, with respect to basis* 3*, which does not contain some* 1 *and has a unique number representation. Unique number representation because*

$$\frac{2}{3} = 0.2 = 0.1\overline{2}$$

$$\frac{1}{3} = 0.1 = 0.0\overline{2} \qquad \frac{1}{9} = 0.01 = 0.00\overline{2} \qquad \frac{2}{9} = 0.02 = 0.01\overline{2}$$

A linear combination of Borel-measurable functions is Borel-measurable.

Remark. *Riemann integral (U are lower sums, O are upper sums):*

$$f : [a, b] \to \mathbb{R} \text{ bounded}$$

$$Z = \{a = x_0 < x_1 < \dots < x_k = b\} \qquad \|Z\| = \max_{j=1,k} (x_j - x_{j-1})$$

$$m_j = \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad M_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$

$$U(Z, f) = \sum_{j=1}^{k} m_j (x_j - x_{j-1})$$

$$g_Z = \sum_{j=1}^{k} m_j \mathbf{1}_{(x_{j-1}, x_j]} \text{ are both } \mathcal{B}\text{-measurable}$$

$$U(Z, f) = \sum_{j=1}^{l} M_j (x_j - x_{j-1})$$

$$h_Z = \sum_{j=1}^{k} M_j \mathbf{1}_{(x_{j-1}, x_j]}$$

$$U(Z, f) = \int_{[a,b]} g_Z \, d\lambda \qquad O(Z, f) = \int_{[a,b]} h_Z \, d\lambda$$

Theorem 5.0.1. *f as above (if necessary, not Borel-measurable)*

$C = \{x \in [a,b] : f \text{ continuous in } x\} \quad D = \{x \in [a,b] \mid f \text{ in } x \text{ is non-continuous}\}$

1. *Then $C, D \in \mathcal{B}_{[a,b]}$, $f \cdot \mathbf{1}_C$ is Borel-measurable*
2. *f is Riemann integrable $\iff$ $\lambda(D) = 0$ and*

$$\int_a^b f(x)\,dx = \int_{[a,b]} f \cdot \mathbf{1}_C \,d\lambda$$

Proof.

$$Z_n = \left\{a = x_0^{(n)} < x_1^{(n)} < \dots < x_{k(n)}^{(n)} = b\right\}$$

A sequence of decompositions such that

1. Z_{n+1} is refinement of Z_n
2. $\|Z_n\| \to 0$

except for point a (so the intervals are left-sided half-open) (you can also close the first interval of the left side)

$$\inf f = m \leq g_{Z_n} \nearrow g \leq f \leq h \swarrow h_{Z_n} \leq M = \sup f$$

where g and h are Borel-measurable. By the dominated convergence theorem by Lebesgue,

$$U(Z_n, f) = \int_{[a,b]} g_{Z_n}\,d\lambda \nearrow \int_{[a,b]} g\,d\lambda$$

$$O(Z_n, f) = \int h_{Z_n}\,d\lambda \searrow \int_{[a,b]} h\,d\lambda$$

$R = \left\{x_j^{(n)} \mid n \in \mathbb{N}, j = 1, \dots, k(n)\right\}$ is countable, $\lambda(R) = 0$. □

Claim 5.1. *For $x \in [a,b] \setminus R$, f is continuous at x $\iff$ $g(x) = h(x)$.*

Proof. Let $I_n(x)$ be the interval of Z_n with $x \in I_n(x)$. Recognize that $I_{n+1}(x) \subset I_n(x)$ and $\lambda(I_n(x)) \searrow 0$.

$$\begin{aligned} f \text{ cont. in } x &\iff \forall \varepsilon > 0 \exists k : f(x) - \varepsilon < f_{I_k(x)} < f(x) + \varepsilon \\ &\iff \forall \varepsilon > 0 \forall n \geq k : f(x) - \varepsilon < f_{I_k(x)} < f(x) + \varepsilon \\ &\iff \forall \varepsilon > 0 \exists k \forall n \geq k : f(x) - \varepsilon \leq g_{Z_1}\big|_{I_n(x)} \leq h_{Z_n}\big|_{I_n(x)} \leq f(x) + \varepsilon \\ &\Rightarrow g(x) = h(x) \qquad [= f(x)] \end{aligned}$$

"$\Rightarrow$" is "$\iff$" assuming $x \notin R$, so $[g < h] \subset D \subset [g < h] \cup R$ where $[g < h]$ is the Borel set.

$D \setminus [g < h]$ is at most countable (because R is countable) $\implies$ D Borel set, C Borel set

$$\lambda(D) = \lambda[g < h]$$

□

$$f \text{ is R-integrable} \iff \int_{[a,b]} g \, d\lambda = \int_{[a,b]} h \, d\lambda, h \geq g$$
$$\iff \lambda[g < h] = 0 \iff \lambda(D) = 0$$

in this case (because $g \leq f \leq h$ except in a).

$$g \cdot \mathbf{1}_C = f \cdot \mathbf{1}_C$$

where g and $\mathbf{1}_C$ is Borel-measurable and thus $f \cdot \mathbf{1}_C$ is Borel-measurable.

$$\int_{[a,b]} f \cdot \mathbf{1}_C \, d\lambda = \int_{[a,b]} g \cdot \mathbf{1}_C \, d\lambda = \int_{[a,b]} g \, d\lambda = \int_a^b f(x) \, dx$$

$$g \cdot \mathbf{1}_C = g \qquad \text{almost everywhere wrt. } \lambda$$

Example 5.2. 1. *$\mathbf{1}_{\mathbb{Q}}$ is nowhere continuous.*

$$\int_0^1 \mathbf{1}_{\mathbb{Q}}(x) \, dx \textit{ does not exist}$$

$$\mathbf{1}_{\mathbb{Q}} = 0 \qquad \textit{almost everywhere wrt. } \lambda \implies \int_{[0,1]} \mathbf{1}_{\mathbb{Q}} \, d\lambda = 0$$

2.

$$\int_a^b \frac{\sin x}{x} \, dx = \int_{[a,b]} \frac{\sin x}{x} \, dx$$
$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx = \pi \qquad \nexists \int_{\mathbb{R}} \frac{\sin x}{x} \, d\lambda(x)$$

Theorem 5.2.1 (Substitution theorem). *Let $\varphi : (X, \mathbb{A}, \mu) \to (X', \mathcal{A}')$ is measurable.*

$$\mu_\varphi(A') = \mu(\varphi^{-1}(A')) \qquad A' \in \mathcal{A}' \qquad \textit{pushforward measure}$$

$$f : (X', \mathcal{A}') \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}}) \textit{ measurable}$$

Then,

$$\int_X f \circ \varphi \, d\mu \textit{ exists} \iff \int_{X'} f \, d\mu_\varphi \textit{ exists}$$

and in this case, they are the same.

↓ *This lecture took place on 2018/11/12.*

Proof. 1a. $f = \mathbf{1}_A$, $A' \in \mathcal{A}' \implies f \circ \varphi = \mathbf{1}_{\varphi^{-1}A'}$

$$\int_X f \circ \varphi \, d\mu = \mu(\varphi^{-1}A') = \mu_\varphi(A') = \int_{X'} f \, d\mu_\varphi$$

1b. $f = \sum_{k=1}^n c_k \cdot \mathbf{1}_{A'_k}$

$$(c_k \geq 0) \implies f \circ \varphi = \sum_{k=1}^n c_k \cdot \mathbf{1}_{\varphi^{-1}A'_k}$$

$$A'_k = [f = c_k] \implies \text{statement is correct}$$

2. $f \geq 0$ then $0 \leq f_n \nearrow f$ and f_n is simple. Also $f_n \circ \varphi \nearrow f \circ \varphi$, $f_n \circ \varphi$ is simple.

$$\int_X f \circ \varphi \, d\mu = \lim \int_X f_n \circ \varphi \, d\mu = \lim \int_{X'} f_n \, d\mu_\varphi = \int_{X'} f \, d\mu_\varphi$$

3.

$$f = f^+ - f^- \qquad f \circ \varphi = f^+ \circ \varphi - f^- \circ \varphi$$
$$(f \circ \varphi)^\pm = f^\pm \circ \varphi$$

The remaining parts are obvious.

□

Example 5.3.

$$\varphi : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B}) \qquad f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B}) \text{ } \lambda\text{-}integrable$$
$$\lambda_\varphi(A) = \lambda(\varphi^{-1}(A))$$
$$\int_\mathbb{R} (A) = \lambda(\varphi^{-1}(A))$$
$$\int_\mathbb{R} f(\varphi(x)) \underbrace{d\lambda(x)}_{\lambda(dx)} = \int_\mathbb{R} f(t) \, d\lambda_\varphi(t)$$

by substitution $\varphi(x) = t$.

Remark. φ *is strictly monotonic.* $\varphi' \neq 0$, $\varphi(\mathbb{R})$.

$$\psi = \varphi^{-1} : \varphi(\mathbb{R}) \to \mathbb{R} \qquad \psi' \neq 0$$

$x = \psi(t)$ *Riemann:* $dx = \psi'(t) \, dt$

$$\int_{-\infty}^{\infty} f(\varphi(x)) \, dx = \int_c^d f(t) \bigg|_{\psi'(t)}$$

$$\lim \frac{\varphi(x+h) - \varphi(\lambda)}{x}$$

TODO

6 Construction of measures

6.1 Dynkin systems, Uniqueness theorem

Definition 6.1. *$\mathcal{D} \subset p(X)$ is called* Dynkin system *or π-system, if*

1. $X \in \mathcal{D}$
2. $A \in \mathcal{D} \implies A^C \in \mathcal{D}$
3. $A_n \in \mathcal{D}$ *($n \in \mathbb{N}$) is pairwise disjoint then* $\dot\bigcup_n A_n \in \mathcal{D}$

$\mathcal{D}$ is a non-empty family of subsets of X that is closed under finite intersections.

Obviously,

1. Every σ-algebra is a Dynkin system
2. $\varepsilon \subset \mathcal{P}(X)$, $D(\varepsilon)$ is generated D-system

$$\implies \varepsilon \subset \mathcal{D}(\varepsilon) \subset \sigma(\varepsilon)$$

The intersection of Dynkin systems is a Dynkin system

Lemma 6.2. *$\mathcal{D}$ as Dynkin system is a σ-algebra $\iff$ closed under intersection ($A, B \in \mathcal{D} \implies A \cap B \in \mathcal{D}$)*

Proof. $\implies$ is immediate

$\impliedby$ Let $A_n \in \mathcal{D}$ ($n \in \mathbb{N}$)

$$B_1 = A_1 \in \mathcal{D}, B_2 = A_2 \setminus (B_1 \cap A_2) = \underbrace{A_2}_{\in \mathcal{D}} \cap \underbrace{(B_1 \cap A_2)^C}_{\in \mathcal{D}} \in \mathcal{D}$$

$$B_n = A_n \setminus \underbrace{\left(\underbrace{\dot\bigcup_{i=1}^{n-1} B_i}_{\in \mathcal{D}} \cap \underbrace{A_n}_{\in \mathcal{D}} \right)}_{\in \mathcal{D}}$$

$$\bigcup_{n=1}^{\infty} A_n = \dot\bigcup_{n=1}^{\infty} B_n \in \mathcal{D}$$

□

Lemma 6.3. *Let $\varepsilon \subset \mathcal{P}(X)$ is closed under intersection $\implies \mathcal{D}(\varepsilon) = \sigma(\varepsilon)$*

Good set principle. We have to show $\mathcal{D}(\varepsilon)$ is closed under intersection. Let $D \in \mathcal{D}(\varepsilon)$.

$$\mathcal{D}_D \coloneqq \{A \subset X \mid A \cap D \subset \mathcal{D}(\varepsilon)\} \text{ is a } \mathcal{D}\text{-system?}$$

1. Trivial

2. $A \in \mathcal{D}_D$. $A^C \cap D = \cdots = \left((A \cap D) \dot{\cup} D^C\right)^C \in \mathcal{D}(\varepsilon)$

3. Trivial

Yes, it is. Especially:

$$E \in \underbrace{\varepsilon}_{\subset \mathcal{D}(\varepsilon)} \qquad A \in \varepsilon \implies A \cap E \in \underbrace{\varepsilon}_{\subset \mathcal{D}(\varepsilon)} \qquad A \in \mathcal{D}_E$$

$$\implies \varepsilon \subset \mathcal{D}_E \implies \mathcal{D}(\varepsilon) \subset \mathcal{D}_E \forall E \in \varepsilon$$

$$D \in \mathcal{D}(\varepsilon) \implies D \in \mathcal{D}_E \implies E \cap D \in \mathcal{D}(\varepsilon) \implies E \in \mathcal{D}_D \forall E \in \varepsilon$$

$$\varepsilon \subset \mathcal{D}_D, \text{ so } \mathcal{D}(\varepsilon) \subset \mathcal{D}_D \implies D' \cap D \in \mathcal{D}(\varepsilon) \forall D' \in \mathcal{D}(\varepsilon) \text{ is true } \forall D \in \mathcal{D}(\varepsilon)$$

□

Theorem 6.3.1 (Uniqueness theorem)**.** *$\mathcal{A} = \sigma(\varepsilon)$, ε is closed under intersection. μ and ν are measure on $\mathcal{A}$.*

1. $\mu|_\varepsilon = \nu|_\varepsilon$
2. $\mu(X) = \nu(X) < \infty$ *or* $\varepsilon \ni E_n \nearrow X, \mu(E_n) = \nu(E_n) < \infty \forall n$

Then $\mu = \nu$ on $\mathcal{A}$.

Proof. Let $E \in \varepsilon$ with $\mu(E) = \nu(E) < \infty$ and $S_E = \{D \in \mathcal{A} \mid \mu(D \cap E) = \nu(D \cap E)\} \subset \mathcal{A}$

- $\varepsilon \in S_E$ (because of closure under intersection)
- S_E is a Dynkin system
  1. trivial
  2. $D \in S_E$, $E = (D \cap E) \dot{\cup} (D^C \cap E)$

$$\mu(E) = \mu(D \cap E) + \mu(D^C \cap E) : \mu(D^C \cap E) = \mu(E) - \mu(D \cap E)$$
$$\nu(D^C \cap E) = \nu(E) - \nu(D \cap E)$$

$$\mu(D^C \cap E) = \nu(D^C \cap E), D^C \in S_E$$

  3. $D_n \in S_E$ pairwise disjoint

$$\mu\left(\left(\bigcup_{n=1}^{\infty} D_n\right) \cap E\right) = \sum_{n=1}^{\infty} \mu\left(D_n \cap E\right) = \sum_{n=1}^{\infty} \nu(D_n \cap E)$$

$$= \nu\left(\left(\dot{\bigcup_{n=1}^{\infty}} D_n\right) \cap E\right) \implies \dot{\bigcup_{n=1}^{\infty}} D_n \in S_E$$

$$\varepsilon \subset \mathcal{A} = \mathcal{D}(\varepsilon) \subset S_E$$

$$\mu(A \cap E) = \nu(A \cap E) \forall A \in \mathcal{A}$$

1. $E = X$: $\mu = \nu$
2. $\mu(A \cap E_n) \nearrow \mu(A \cap X)$
   $\nu(A \cap E_n) \nearrow \nu(A \cap X) \forall A \in \mathcal{A}$

$$\implies \mu = \nu$$

□

↓ *This lecture took place on 2018/11/19.*

6.2 Outer measure and extension theorem

Definition 6.4 (Outer measure). $\eta : P(X) \to [0, \infty]$ *is called* outer measure *if*

1. $\eta(\emptyset) = 0$
2. *σ-additivity*

$$\implies A \subset \bigcup_{n=1}^{\infty} A_n \implies \eta(A) \leq \sum_{n=1}^{\infty} \eta(A_n)$$

$A \subset X$ is called η-measurable if $\forall C \subset X : \eta(C \cap A) + \eta(C \cap A^C) = \eta(C)$.

Is not a measure.

Remark. *Third property: $A \subset B \implies \eta(A) \leq \eta(B)$*

Theorem 6.4.1 (Carathéodory). *$A_\eta = \{A \subset \mathcal{P}(X) \mid A$ measurable$\}$ is a σ-algebra and $\eta|_{\mathcal{A}_\eta}$ is a measure, $(X, \mathcal{A}_\eta, \eta)$ is complete.*

Proof.

$$\forall C : \eta(C \cap X) + \underbrace{\eta(C \cap \emptyset)}_{\sigma(i)} = \eta(C) \implies X \in \mathcal{A}_\eta$$

Thus, the first statement follows. For the second property, we have:

A is measurable, then A^C is measurable. By finite union and additivity:

$$A_1, A_2 \in \mathcal{A}_\eta \implies \forall C \in \mathcal{P}(X)$$

$$\begin{aligned}
\eta(C) &= \eta(C \cap A_1) + \eta(C \cap A_1^C) \\
&= \eta(C \cap A_1) + \eta\left((C \cap A_1^C) \cap A_2\right) + \eta\left((C \cap A_1^C) \cap A_2^C\right) \\
&= \eta\left(C \cap (A_1 \cup A_2) \cap A_1\right) + \eta(C \cap (A_1 \cup A_2) \cap A_1^C) + \eta(C \cap (A_1 \cup A_2)^C) \\
&\overset{A_1 \text{ measurable}}{=} \eta\left(C \cap (A_1 \cup A_2)\right) + \eta(C \cap (A_1 \cup A_2)^C) \\
&\implies A_1 \cup A_2 \text{ measurable}
\end{aligned}$$

If $A_1 \cap A_2 = \emptyset$

$$\eta(C \cap (A_1 \cup A_2) \cap A_1) + \mu(C \cap (A_1 \cup A_2) \cap A_1^C) = \eta(C \cap A_1) + \eta(C \cap A_2)$$

Especially, $C = X$, $\eta(A_1 \mathbin{\dot\cup} A_2) = \eta(A_1) + \eta(A_2)$.

Also $A_1, A_2 \in \mathcal{A}_\eta \implies A_1 \cap A_2 = (A_1^C \cup A_2^C)^C \in \mathcal{A}_\eta$.

σ: $A_n \in \mathcal{A}_\eta$ ($n \in \mathbb{N}$) with p_n disjoint. $\bigcup_{n=1}^N A_n \in \mathcal{A}_\eta$. Let $C \subset X$.

$$\begin{aligned}
\eta(C) &= \eta\left(C \cap \bigcup_{n=1}^N A_n\right) + \eta\left(C \cap \left(\bigcup_{n=1}^N A_n\right)^C\right) \\
&\geq \underbrace{\sum_{n=1}^N \eta(C \cap A_n)}_{\forall N} + \eta\left(C \cap \left(\bigcup_{n=1}^\infty A_n\right)^C\right) \\
&\overset{\text{by } \sigma\text{-subadd.}}{\geq} \eta\left(C \cap \dot{\bigcup_{n=1}^\infty} A_n\right) + \eta\left(C \cap \left(\bigcup_{n=1}^\infty A_n\right)^C\right) \\
&\implies \dot{\bigcup_{n=1}^\infty} A_n \text{measurable}, \eta\left(\dot{\bigcup_{n=1}^\infty} \eta(A_n)\right) + 0
\end{aligned}$$

We know: $\mathcal{A}_\eta$ is a Dynkin system and closed under intersection (thus, a σ-algebra). η is σ-additive on $\mathcal{A}_\eta$.

Completeness is obvious. □

Theorem 6.4.2 (Carathéodory's extension theorem)**.** *Let $\varepsilon \subset \mathcal{P}(X)$ be the generator of $\mathcal{A}$. Let $\mu : \varepsilon \to [0, \infty]$ be a set functions. $A \subset X$.*

$$\mu^*(A) = \inf\left\{\sum_{n=1}^\infty \mu(E_1) \mid E_1 \in \varepsilon, A \subset \bigcup_{n=1}^\infty E_1\right\}$$

1. *μ^* is an outer measure $\iff \mu^*(\emptyset) = 0$ (especially true, if $\emptyset \in \varepsilon$ and $\mu(\emptyset) = 0$)*
2. *If $\mu^*(\emptyset) = 0, \varepsilon \subset \mathcal{A}_{\mu^*} \iff \mu^*(D \cap E) + \mu^*(D \cap E^C) \leq \mu(D) \forall D, E \in \varepsilon$*
3. *If the first two properties are satisfied, then $\mu^*|_\varepsilon = \mu$ iff $\mu^*(E) \geq \mu(E) \forall E \in \varepsilon$, so $\mu(E) \leq \sum_{n=1}^\infty \mu(E_n)$ if $E_n \in \varepsilon$ and $E \subset \bigcup_{n=1}^\infty E_n$*

μ^* is σ-subadditive in every case.

$\forall \delta > 0 \exists E_n \in \varepsilon : \sum \mu(E_n) < \delta$
$E \subset E \cup \bigcup_n E_n$
$\mu^*(E) \leq \mu(E) + \sum_n \mu(E_n) \leq \mu(E) + \delta$.
Thus, $\mu^*(\emptyset) = 0 \implies \mu^*(E) \leq \mu(E) \forall E \in \varepsilon$

Proof. 1. Direction $\implies$ is immediate.
For direction $\impliedby$, we need to show σ-subadditivity.

$$A, A_n \in X, A \subset \bigcup_{n=1}^{\infty} A_n$$

$$\mu^*(A_n) = \inf\left\{\sum_{m=1}^{\infty} \mu(E_{m,n}) \mid E_{m,n} \in \varepsilon, A_n \subset \bigcup_{m=1}^{\infty} E_{m,n}\right\}$$

$\delta > 0 : \exists E_{m,n} \in \varepsilon : \mu^*(A_n) + \frac{\delta}{2^n} \geq \sum_{m=1}^{\infty} \mu(E_{m,n}) \geq \mu^*(A_n)$

$$A \subset \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{m,n}$$

$$\begin{aligned}
\mu^*(A) &\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu\left(E_{m,n}\right) \\
&\leq \sum_{n=1}^{\infty} \left(\mu^*(A_n) + \frac{\delta}{2^n}\right) \\
&= \sum_{n=1}^{\infty} \mu^*(A_n) + \delta \forall \delta > 0 \\
&\implies \mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)
\end{aligned}$$

2. Direction $\implies$ is immediate.
   For direction $\impliedby$, let $E \in \varepsilon$. Show that E is measurable with respect to μ^*.
   Let $C \subset X$ be arbitrary. $D_n \in \varepsilon$ such that $C \subset \bigcup_{n=1}^{\infty} D_n$ and $\mu^*(C) + \delta \geq \sum_{n=1}^{\infty} \mu(D_n) \geq \mu^*(C)$

$$\begin{aligned}
\mu^*(C) &\leq \mu^*(C \cap E) + \mu^*(C \cap E^C) \\
&\leq \sum_{n=1}^{\infty} (\mu^*(D_n \cap E) + \mu^*(D_n \cap E)) \\
&\overset{!}{\leq} \sum_{n=1}^{\infty} \mu(D_n) \leq \mu^*(C) + \delta \forall \delta > 0
\end{aligned}$$

   So, $\mu^*(C \cap E) + \mu^*(C \cap E^C) = \mu^*(C)$ where E is measurable.

□

Theorem 6.4.3. *Let ε be a semiring. $\mu : \varepsilon \to [0, \infty]$.*

1. $\mu(\emptyset) = 0$
2. $X \in \varepsilon$ *and* $\mu(X) < \infty$ *or* $X = \bigcup_n E_n, E_n \in \varepsilon, \mu(E_n) < \infty$
3. μ *is σ-additive on ε $\implies$ unique extension on $\sigma(\varepsilon)$*

ε is a semiring over X, if

1. $\emptyset \in \varepsilon$

2. $D, E \in \varepsilon \implies D \cap E \in \varepsilon$

3. $D \setminus E = C_1 \dot{\cup} \dots \dot{\cup} C_k, C_k \in \varepsilon$

$\mu : \varepsilon \to [0, \infty]$ is σ-additive on ε.

where $\mu(\emptyset) = 0$, $\mu\left(\dot{\bigcup}_n E_n\right) = \sum_n \mu(E_n)$ if $\dot{\bigcup}_n E_n \in \varepsilon$

$$X = \bigcup_n E_n \qquad E_n \in \varepsilon \qquad \mu(E_n) < \infty$$

Thus, the theorem is applicable, so μ has a unique extension for measure to $\mathcal{A}_{\mu^*} \supset \mathcal{A} \supset \varepsilon$. $\mathcal{A}_{\mu^*}$ complete.

Proof.

- μ also finitely additive on ε.
- $D, E \subset \varepsilon$. $E \subset D \implies D \setminus E = C_1 \dot{\cup} \dots \dot{\cup} C_k$. $D = E \dot{\cup} C_1 \dot{\cup} \dots \dot{\cup} C_k$

$$\mu(D) = \mu(E) + \sum \mu(C_j) \geq \mu(E)$$

1. Immediate.
2. $D, E \in \varepsilon$. $D \cap E \in \varepsilon . D \cap E^C = C_1 \dot{\cup} \dots \dot{\cup} C_k$ with $C_j \in \varepsilon$. $D = (D \cap E) \dot{\cup} C_1 \dot{\cup} \dots \dot{\cup} C_k$. $\mu^*(D \cap E) + \mu^*(D \cap E^C) \leq \mu(D \cap E) + \sum_{j=1}^k \mu(C_j) = \mu(D)$.
3. $E, E_n \in \varepsilon : E \subset \bigcup_{n=1}^\infty E_n$. $E = \bigcup_{n=1}^\infty \underbrace{(E_n \cap E)}_{\in \varepsilon}$ it suffices to show that $\mu(E) \leq \sum_n \mu(E_n \cap E)$. Without loss of generality, $E = \bigcup_{n=0}^\infty E_n$. $D_1 = E_1$, $D_2 = E_2 \cap E_1^C = C_{2,1} \dot{\cup} \dots \dot{\cup} C_{2,k_2}$. $D_n = E_n \cap E_{n-1}^C \cap \dots \cap E_1^C = C_{n,1} \dot{\cup} C_{n,2} \dot{\cup} \dots \dot{\cup} C_{n,k_n}$. $C_{n,j} \in \varepsilon$.

$$E = \bigcup_{n=1}^\infty E_n = \dot{\bigcup_{n=1}^\infty} D_n = \dot{\bigcup_{n=1}^\infty} \dot{\bigcup_{j=1}^{k_n}} C_{n,j}$$

$$\mu(E) = \sum_{n=1}^\infty \underbrace{\sum_{j=1}^{k_n} \mu(C_{n,j})}_{\leq \mu(E_n)} \leq \sum_n \mu(E_n)$$

□

6.3 Regular measures

Definition 6.5. *Let $(X, \mathcal{A}, \mu)$ is a measurable space. Let ε be a generator of $\mathcal{A}$. μ is called* outer regular *with respect to ε, if $\mu = \mu^*$ on d where μ^* is defined like*

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^\infty \mu(E_1) \mid E_1 \in \varepsilon, A \subset \bigcup_{n=1}^\infty E_1 \right\}$$

(dt. "μ is regulär von außen")

Theorem 6.5.1. *Let ε be a generator closed under intersection. Let $\emptyset \in \varepsilon$. Let μ be a measure on $\mathcal{A} = \sigma(\varepsilon)$, σ-finite with respect to ε.*

$$\mu^*(A) := \inf\left\{\sum_{n=1}^{\infty} \mu(E_1) \mid E_1 \in \varepsilon, A \subset \bigcup_{n=1}^{\infty} E_1\right\}$$

Then $\mu = \mu^*$ on $\mathcal{A}$ (μ is outer regular) $\iff$

$$\mu(D \cap E^C) = \mu^*(D \cap E^C) \forall D, E \in \varepsilon$$

Proof. 1. trivial

2. $\mu(E) \geq \mu^*(E)$ with $E \in \varepsilon$. By (3), we have $\mu(E) \leq \mu^*(E)$. $\mu(E) = \mu^*(E)$ on ε. $D, E \in \varepsilon : \mu^*(D \cap E) + \mu^*(D \cap E^C) = \mu(D \cap E) + \mu(D \cap E^C) = \mu(D)$. Thus $E \subseteq D$.

   Remark.
   $$D \cap E^C = D \cap (E \cap D)^C$$
   $$\mu^*(D \cap E) + \mu^*(D \cap E^C) = \mu(D \cap E) + \mu^*(\underbrace{D}_{\in\varepsilon} \cap (\underbrace{E \cap D}_{\in\varepsilon})^C)$$
   $$= \mu(D \cap E) + \mu(D \cap (E \cap D)^C) = \mu(D \cap E) + \mu(D \cap E^C) = \mu(D)$$

3. trivial

$\square$

Theorem 6.5.2. *Let X be a metric space. $\varepsilon = \{$open sets$\}$. $\mathcal{B}_X = \sigma(\varepsilon)$ is a Borel-σ-algebra. μ is a σ-finite measure on $(X, \mathcal{B}_X)$.*

$$\mu(B) = \inf\{\mu(O) \mid O \supset B \text{ open}\}$$
$$= \sup\{\mu(A) \mid A \subset B \text{ closed}\}$$

Proof. Without loss of generality: $\mu(X) < \infty$.

$$B \subset O = \bigcup_{n=1}^{\infty} E_n \text{ open}$$

$$\mu(B) \leq \mu(O) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

In combination,

$$\mu^*(B) = \inf\{\mu(O) \mid O \supset B\}$$

ε is closed under intersection. $\emptyset \in \varepsilon$. $A = E^C$ is closed. Is $E \subset D$ open? $\mu^*(D \cap E^C) = \mu(D \cap E^C)$. $A_n = \{x \mid d(x, A) < \frac{1}{n}\}$ open, $A_n \searrow A$ where $\searrow$ denotes a monotonically decreasing sequence converging from the LHS to the RHS.

$$D \cap E^C = D \cap A \swarrow \underbrace{D \cap A_n}_{\in\varepsilon,\text{ open}}$$

$$\mu(D \cap A_n) \searrow \mu(D \cap A)$$

Like above: In ε, we have $\mu = \mu^*$.

$$\underbrace{\mu^*(D \cap A_n)}_{\geq \mu^*(D \cap E^C)} = \underbrace{\mu(D \cap A_n)}_{\to \mu(D \cap E^C)}$$

Thus, $\mu^*(D \cap E^C) \leq \mu(D \cap E^C)$. In contrast, we already have μ on entire σ-algebra. So $\forall B \in \mathcal{B} : \mu(B) \leq \mu(O) \forall O \supset B$ open. $\mu(B) \leq \mu^*(B)$ and therefore $\mu^*(D \cap E^C) \leq \mu(D \cap E^C) \leq \mu^*(D \cap E^C)$. So $\mu = \mu^*$. □

6.4 Product measures and Fubini's Theorem

↓ *This lecture took place on 2018/12/03.*

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure space. Let μ and ν be σ-finite. $A \otimes B = \sigma(\varepsilon)$. $\varepsilon = \{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$. $\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$.

Definition 6.6. *Let $C \in \mathcal{A} \otimes \mathcal{B}$. $C_x = \{y \in Y \mid (x, y) \in C\}$ with $x \in X$. $C^Y = \{x \in X \mid (x, y) \in C\}$.*

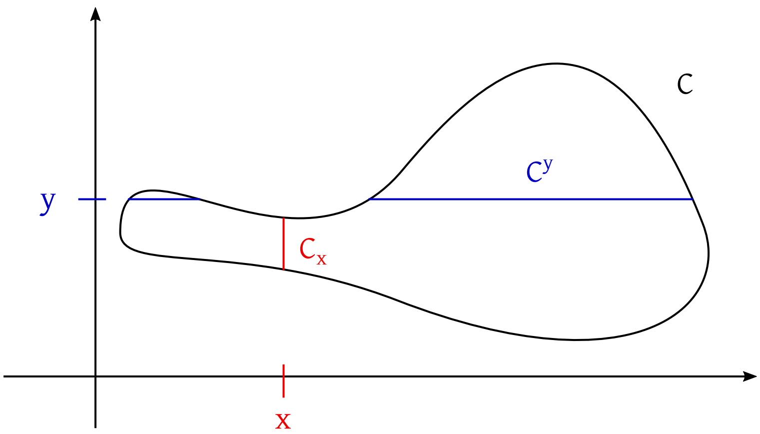

Figure 4: Setting in Definition 6.6

$$f : X \times Y \to \overline{\mathbb{R}}$$
$$f_x : Y \to \overline{\mathbb{R}} \qquad f_x(y) = f(x, y)$$
$$f^y : X \to \overline{\mathbb{R}} \qquad f^y(x) = f(x, y)$$

Theorem 6.6.1.

$$C \in \mathcal{A} \otimes \mathcal{B} \implies C_x \in \mathcal{B} \forall x \in X, C^y \in \mathcal{A} \forall y \in Y$$

$$f \textit{ measurable } \implies f_x, f^y \textit{ measurable } \forall x \in X, y \in Y$$

Proof.

$$\mathcal{D} = \{C \in \mathcal{A} \otimes \mathcal{B} \mid C_x \in \mathcal{B} \forall x \in X\}$$

$$\varepsilon \subset \mathcal{D}(\text{closed under intersection}) \qquad (A \times B)_x = \begin{cases} B & x \in A \\ \emptyset & x \notin A \end{cases}$$

$$(C_x)^C = (C^c)_x \text{ so } C \in \mathcal{D} \implies C^C \in \mathcal{D}$$

$$C_1, C_2, \dots \in \mathcal{D}, \text{ pairwise disjoint } \implies \left(\bigcup_n^+ C_n \right)_x = \bigcup_n^+ \underbrace{C_{n,x}}_{\in \mathcal{B}} \in \mathcal{B}$$

$$\mathcal{D} \text{ Dynkin } \implies \mathcal{D} = \mathcal{A} \otimes \mathcal{B}$$

$$f \geq 0 : f_n = \sum_k c_{n,k} \mathbf{1}_{c_{n,k}}$$

$$C_{n,k} \in \mathcal{A} \otimes \mathcal{B} \text{ for fixed } n$$

pairwise disjoint

$$f_n \nearrow f \qquad f_{n,x} \nearrow f_x$$

$$f_{n,x} = \sum_k c_{n,k} \underbrace{\mathbf{1}_{(C_{n,k})x}}_{\in \mathcal{B}}$$

$\mathcal{B}$-measurable $\implies f_x$ $\mathcal{B}$-measurable. ◻

Remark. *In general, $f = f^+ - f^-$, $f^\pm$ measurable, $f_x = f_x^+ - f_x^-$ (f_x^+ and f_x^- are measurable).*

Definition 6.7 (Definition and theorem)**.** *There exists exactly one measure g on $\mathcal{A} \otimes \mathcal{B}$ such that $g(A \times B) = \mu(A)\nu(B) \forall A \in \mathcal{A}, B \in \mathcal{B}$ and*

$$g(C) = \int_X \nu(C_x) \, d\mu(x) = \int_Y \nu(C^y) \, d\nu(y) \forall C \in \mathcal{A} \otimes \mathcal{B}$$

Remark. *This implies that $x \mapsto \nu(C_x)$ is measurable and $y \mapsto \nu(C^y)$ is measurable.*

Proof. The uniqueness follows from the uniqueness theorem.

What about existence? Without loss of generality, $\mu(X) < \infty$ and $\nu(Y) < \infty$.

$$\mathcal{D} = \{C \in \mathcal{A} \otimes \mathcal{B} \mid x \mapsto \nu(C_x), y \mapsto \mu(C^y) \text{ measurable } \forall x, y \text{ and G}\}$$

where G means that the following property (from before) is true:

$$g(C) = \int_X \nu(C_x) \, d\mu(x) = \int_Y \nu(C^y) \, d\nu(y) \forall C \in \mathcal{A} \otimes \mathcal{B}$$

$$\begin{aligned}\varepsilon \subset \mathcal{D} :& \nu((A \times B)_x) = \nu(B)\mathbf{1}_A(x)\\ & \mu((A \times B)^y) = \mu(A)\mathbf{1}_B(x)\end{aligned}$$

$$\int_X \nu((A \times B)_x)\, d\mu(x) = \nu(B)\mu(A)$$
$$\int_Y \mu((A \times B)^y)\, d\nu(y) = \mu(A)\nu(B)$$

$$C \in \mathcal{D} \qquad \nu\left((C^C)_x\right) = \nu(C_x^C) = \underbrace{\nu(Y)}_{\text{measurable}} - \nu(C_x)$$
$$\mu((C^C)^y) = \underbrace{\mu(X) - \mu(C^y)}_{\text{measurable}}$$

$$\begin{aligned}\int_X \nu((C^C)_x)\, d\mu(x) &= \int_X \left(\nu(y) - \nu(C_x)\right)\, d\mu(x) = \mu(X)\nu(Y) - \int_X \nu(C_x)\, d\mu(x)\\ &= \nu(Y)\mu(X) - \int_Y \mu(C^y)\, d\nu(y)\\ &= \dots\\ &\implies C^C \in \mathcal{D}]\end{aligned}$$

$C_1 \in \mathcal{D}$ is pairwise disjoint.

$$\nu\left(\left(\overset{+}{\bigcup_n} C_n\right)_x\right) = \sum_n \nu(C_{n,x}) \qquad \mu\left(\left(\overset{+}{\bigcup_n} C_n\right)^y\right) = \sum_n \mu(C_n^y)$$

where the elements of both inner parentheses are measurable as function of x and y respectively.

$$\begin{aligned}\int \nu\left(\left(\overset{+}{\bigcup_n} C_1\right)_x\right) d\mu(x) &= \sum_n \int_X \nu(C_{n,x})\, d\mu(x)\\ &\underbrace{=}_{C_n \in \mathcal{D}} \sum_n \int_Y \mu(C_n^y)\, d\nu(y)\\ &= \int \mu\left(\left(\overset{+}{\bigcup_n} C_n\right)^y\right) \nu(dy)\\ &\implies \overset{+}{\bigcup} C_n \in \mathcal{D}, \mathcal{D} \text{ Dynkin system}\\ &\implies \mathcal{D} = \mathcal{A} \times \mathcal{B}\end{aligned}$$

Furthermore $\rho(C) \coloneqq \int_X \nu(C_x)\, d\mu(x)$ is a measure with $\rho(A \times B) = \mu(A)\nu(B)$. □

Remark. *From now on:* $g \longleftrightarrow \mu \otimes \nu$

Theorem 6.7.1 (Fubini-Tonelli Theorem). *Let $f : X \times Y \to [0, \infty]$ be measurable. Then*

$$\int_{X\times Y} f \, d\mu \times \nu = \int_X \left[\int_Y f_X \, d\nu\right] d\mu(x) = \int_Y \left[\int_X f^Y \, d\mu\right] d\nu(x)$$

Equivalently, we can formulate this theorem as

$$\int_X \left[\int_Y f(x,y) \, d\nu(y)\right] d\mu(x) = \int_Y \left[\int_X f(x,y) \, d\mu(x)\right] d\nu(y)$$

Remark 6.8. *This theorem implies that $x \mapsto \int_Y f_X \, d\nu$ and $y \mapsto \int_X f^Y \, d\mu$ are measurable.*

Proof.

1. $f = \mathbf{1}_C$ with $C \in \mathcal{A} \otimes \mathcal{B}$
   corresponds to the previous theorem $\quad C_k \in \mathcal{A} \otimes \mathcal{B}$

2. $f = \sum_{k=1}^N c_k \mathbf{1}_{C_k}$ is pairwise disjoint, $c_k \geq 0$
   follows by linearity of the integral

$$\int_Y f_x \, d\nu = \sum_k c_k \nu(C_{k,x})$$

$$\begin{aligned}\int_X \left(\int_Y f_X \, d\nu\right) d\mu(x) &= \sum_k c_k \int \nu(C_{k,x}) \, d\mu(x) \\ &= \sum_k c_k \mu \otimes \nu(C_k) = \int_{X\times Y} f \, d\mu \otimes \nu = \text{ vice versa}\end{aligned}$$

3. $0 \leq f_n \nearrow f$. f_n is simple. Monotone convergence.

□

Theorem 6.8.1 (Fubini).

$$f : X \times Y \to \mathbb{R} \textit{ measurable}$$

Then the following statements are equivalent:

1. $\int_{X\otimes Y} |f| \, d\mu \otimes \nu < \infty$
2. $\int_X \left[\int_Y |f_X| \, d\nu\right] d\mu(x) < \infty$
3. $\int_Y \left[\int_X |f^Y| \, d\mu\right] d\nu(y) < \infty$

Especially, $|f|_X = |f_X|$. Fubini's theorem is obvious due to Fubini-Tonelli's theorem.

Proof. and then

$$\begin{aligned}\int f\,d\mu\otimes\nu &= \int_X \left[\int_Y \hat{f}(x,y)\,d\nu(y)\right] d\mu(x)\\ &= \int_Y \left[\int_X \hat{\hat{f}}(x,y)\,d\mu(x)\right] d\nu(y)\end{aligned}$$

□

Remark 6.9 (Reminder)**.** *f is integrable iff $\int f^+ < \infty$ and $\int f^- < \infty$ and then $\int f = \int f^+ - \int f^-$, $\int |f| = \int f^+ + \int f^-$. Especially: $|\int f| \leq \int |f|$*

Proof.

$$\int \left[\int f_x^\pm\,d\nu\right] d\mu(x) < \infty \implies \int_Y f_x^\pm\,d\nu < \infty \qquad \mu \text{ almost everywhere}$$

$$A_0 = \left\{x \mid \int_Y |f_X|\,d\nu = \infty\right\} \in \mathcal{A} \qquad \mu(A_0) = 0$$

$$B_0 = \left\{y \mid \int_X |f^Y|\,d\mu = \infty\right\} \in \mathcal{B} \qquad \nu(B_0) = 0$$

$$C_0 = (A_0 \otimes Y) \cup (X \times B_0) \qquad \mu\otimes\nu(C_0) = 0$$

$\mu\otimes\nu(C_0) = 0$

$$f = \hat{\hat{f}} = f\cdot \mathbf{1}_{X\times Y:C_0}$$

$\mu\otimes\nu$ almost everywhere.

$$\begin{aligned}\hat{f}_x &= f_x && \nu \text{ almost everywhere}\\ \hat{\hat{f}}^y &= f^y && \mu \text{ almost everywhere}\end{aligned}$$

almost for $f^\pm$. Apply Fubini-Tonelli's Theorem to $\hat{f}^\pm$ and accordingly $\hat{\hat{f}}^\pm$

Remark. *The distinction into $\hat{f}$ and $\hat{\hat{f}}$ was not necessary!*

□

↓ *This lecture took place on 2018/12/06.*

Remark. *$(\xi_1, \mathcal{A}_1, \mu_1), (X_2, \mathcal{A}_2, \mu_2), (X_3, \mathcal{A}_3, \mu_3)$ σ-finite.*
$(X_1 \times X_2 \times X_3, \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3, \mu_1 \otimes \mu_2 \otimes \mu_3)$
Also $(X_1 \times \cdots \times X_n, \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n, \mu_1 \otimes \cdots \otimes \mu_n)$

Remark. *Construction of a product space is associative.*

$$\mu_1(A_1)\mu_2(A_2)\mu_3(A_3)$$

$(X_n, \mathcal{A}_n, \mu_n)$ and $\mu_n(X_n) = 1$
$X = X_1 \times X_2 \times \dots = \prod_{n=1}^{\infty} X_n$
$\varepsilon_n = \{A_1 \times \dots \times A_n \times X_{n+1} \times X_{n+2} \times \dots \mid A_i \in \mathcal{A}_i, i = 1, \dots, n\} \subset \varepsilon_{n+1}$

$$\begin{aligned}\varepsilon = \bigcup \varepsilon_n \qquad \mathcal{A} &= \sigma(\bigcup \varepsilon_i) \\ &= \mu(A_1 \times A_2 \times \dots \times A_n \times X) \\ &= \mu_1(A_1)\mu_2(A_2) \dots \mu_n(A_n) \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \\ \sigma(\varepsilon_1^n) = A_1^n &= \{A \times X_{n+1} \times \dots \mid A \in \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n\}\end{aligned}$$

$\mu_1^n \rightsquigarrow \sigma$-additive on $\mathcal{A}_1^n \subset \mathcal{A}_1^{n+1}$.

$$\mu_1^{n+1}\big|_{A_1^n} = \mu_1^n \qquad \sigma - \text{algebra over } X_1^{\infty}$$

$$A_1^n \nearrow, \bigcup_{n=2}^{\infty} \mathcal{A}_1^n = \mathcal{F}_1^{\infty}$$

$$\emptyset \in \mathcal{F}, A \in \mathcal{F} \implies A^C \in \mathcal{F}$$
$$A_1, \dots, A_k \in \mathcal{F} \implies A_1 \cup \dots \cup A_k \in \mathcal{F}$$

"ring", $X \in \mathcal{F} \rightsquigarrow$ algebra.

$\mu = \mu_1^{\infty}$ on $\mathcal{F}$:

$$A \in \mathcal{F} \implies \exists n : A \in \mathcal{A}_1^n \implies \mu_1^{\infty}(A) := \mu_1^n(A) \text{ is well-defined}$$

$A_1, \dots, A_k \in \mathcal{F}$ is pairwise disjoint. $A_j \in \mathcal{A}_1^{n_j}$, $n \le \max_j n_j : A_j \in \mathcal{A}_1^n$

$$\begin{aligned}\implies \mu_1^{\infty}(A_1 \uplus \dots \uplus A_k) \qquad \infty \leftrightarrow n \\ = \mu(A_1) + \dots + \mu(A_k) \\ \sigma(\mathcal{F}) = \sigma(\varepsilon) = \mathcal{A}_1^{\infty}\end{aligned}$$

Show: μ_1^{∞} on $\mathcal{F}$ is σ-additive.

Lemma 6.10. *Via practicals: in order to satisfy this, it suffices that μ at $\emptyset$ is continuous from above.*

Why? $A_n \in \mathcal{F}$ is pairwise disjoint. $\biguplus A_n \in \mathcal{F}$

$$\implies \sum_n \mu(A_n) = \mu(A)$$

$$B_N = \bigcup_{n=1}^{N} A_n \nearrow \bigcup_n A_n = A$$

$$\mathcal{F} \ni A \setminus B_N \searrow \emptyset \implies \dots$$

Show that:

$$E_n \in \mathcal{F}(m \in \mathbb{N}) \quad E_n \searrow \emptyset \qquad E_{m+1} \subset E_m, \bigcap_m E_m = \emptyset \overset{?}{\Rightarrow} \mu(E_m) \to 0$$

Show: If $E_{m+1} \subset E_m$ and $\mu(E_m) \geq \varepsilon > 0 \forall n$.

$$\overset{?}{\Rightarrow} \bigcap_m E_m \neq \emptyset$$

$$n(m) = \min \{n : E_m \in \mathcal{A}_1^n\}$$

Case 1 $(n(m))_{m \in \mathbb{N}}$ bounded by N

$$\implies E_m \in \mathcal{A}_1^N \qquad \forall m : \mu(E_m) = \underbrace{\mu_1^N(E_m)}_{\sigma-\text{additive}}$$

Then $\bigcap E_m \neq \emptyset$.

Case 2 $n(m)$ is not bounded
Without loss of generality, $n(m) \nearrow \infty$ (more than $n(m)$)
Without loss of generality, $E_m \in \mathcal{A}_1^m \forall m$

Now we finished some initial constructions. Now the actual proof starts:

$$x_1 \in X_1 : (E_m)_{x_1} =: E_m(x_1) = \left\{ \underline{y} \in X_2^\infty \mid (x_1, \underline{y}) \in E_m \right\} \in \mathcal{A}_2^m \subset \mathcal{F}_2^\infty$$

$$\mu_1^m(E_m) = \int_{X_1} \underbrace{\mu_2^m(E_m(x_1))}_{\leq 1} \, d\mu(x_1)$$

$$F_m = \left\{ x_1 \in X_1 \mid \mu_2^m(E_m(x_1)) \geq \frac{\varepsilon}{2} \right\} \in \mathcal{A}$$

$$\varepsilon \leq \mu_1^\infty(E_m) = \mu_1^m(E_m) = \int_{F_m^C} + \int_{F_m} \leq \frac{\varepsilon}{2} \cdot \underbrace{\mu_1(F_m^C)}_{\leq 1} + \mu_1(F_m)$$

$$\mu_1(F_m) \geq \frac{\varepsilon}{2} \qquad E_m(x_1) \searrow \qquad F_m \searrow \qquad \implies \bigcap_m F_m \neq \emptyset$$

$$\exists \xi_1 \in \bigcap_m F_m \subset X_1 : \mu_2^m(E_m(\xi_1)) \geq \frac{\varepsilon}{2} \qquad \forall m \geq 2$$

recursive:

$$\exists \xi_2 \in X_2 : \mu_3^m(E_m(\xi_1, \xi_2)) \geq \frac{\varepsilon}{4} \forall m \geq 3$$

$$\implies \dots \exists (\xi_1, \dots, \xi_n) \in (X_1 \times \dots \times X_n) : \mu_{n+1}^m(E_m(\xi_1, \dots, \xi_n)) \geq \frac{\varepsilon}{2^n} \forall m \geq n+1$$

$$E_m(\xi_1, \dots, \xi_n) = \left\{ \underline{y} \in X_{n+1}^\infty \mid (\xi_1, \dots, \xi_n, \underline{y}) \in E_m \right\} \implies (\xi_1, \xi_2, \xi_3, \dots) \in \bigcap E_m \neq \emptyset$$

This is in some way related to compactness in Analysis (but this is an advanced technical detail)

6.5 Signed measures

Definition 6.11. *Signed measure on* $(X, \mathcal{A})$. $\nu : \mathcal{A} \to [-\infty, \infty)$ *or* $\nu : \mathcal{A} \to (-\infty, \infty]$. $\nu(\emptyset) = 0$ *and* σ*-additive:*

$$\nu\left(\dot{\bigcup_n} A_n\right) = \sum_n \nu(A_n) \text{ absolute convergent}$$

Example 6.12. μ_1, μ_2 *are measures. One of them is finite.* $\nu = \mu_1 - \mu_2$.

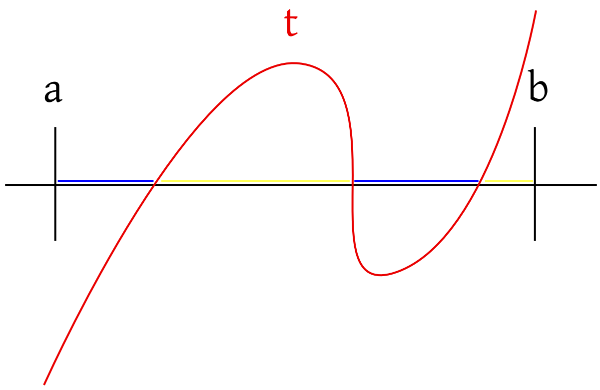

Figure 5: Signed measure of f

$$\nu(B) = \int_B f \, d\lambda = \int_{B \cap [f \geq 0]} f^+ \, d\lambda - \int_{B \cap [f < 0]} f^- \, d\lambda$$

Theorem 6.12.1 (Hahn-Jordan). *Let* ν *be signed.*

$$X = P \cup N \qquad P, N \in \mathcal{A}$$

$$\nu^+(A) = \nu(A \cap P) \qquad \nu^-(A) = -\nu(A \cap N)$$

Non-negative measures are $\nu^+ - \nu^- = \nu$.

↓ *This lecture took place on 2018/12/10.*

Lemma 6.13. *Let* $A, B \in \mathcal{A}$, $B \subset A$ *and* $|\nu(A)| < \infty \implies |\nu(B)| < \infty$.

Proof.

$\nu(A)$	$= \nu(B)$	$+\nu(A \setminus B)$
$+\infty$	$+\infty$	$+\infty$
$+\infty$	$+\infty$	$\in \mathbb{R}$
$-\infty$	$-\infty$	$-\infty$
$-\infty$	$-\infty$	$\in \mathbb{R}$

This shows a contradiction. $+\infty$ and $-\infty$ is not possible by definition of ν.

□

Lemma 6.14.
$$A_n \nearrow A \implies \nu(A_n) \to \nu(A)$$

Lemma 6.15.
$$A_n \searrow A, |\nu(A_1)| < \infty \implies \nu(A_n) \to \nu(A)$$

Proof. Immediate. □

Definition 6.16. *Let ν be a signed measure.*

- *$P \in \mathcal{A}$ is called positive if $\nu(A) \geq 0 \forall A \in \mathcal{A}, A \subset P$.*
- *$N \in \mathcal{A}$ is called negative if $\nu(A) \leq 0 \forall A \in \mathcal{A}, A \subset N$.*
- *Z is a ν-zero set if $\nu(A) = 0 \forall A \in \mathcal{A}$ with $A \subset Z$.*

Remark. *The subset of a positive set is positive.*

Lemma 6.17. *Let $\nu : A \to [-\infty, \infty)$ be a signed measure. $A \in \mathcal{A} \implies \exists P \in \mathcal{A}, P \subset A$ positive, $\nu(P) \geq \nu(A)$.*

Proof. Case distinction of relation of P and A.

1. $\nu(A) \leq 0 : P = \emptyset$. Done.
2. $\nu(A) = 0$

   Claim 6.18. $\forall \varepsilon > 0 \exists A_\varepsilon \subset A$ *with*

   (a) $\nu(A_\varepsilon) \geq \nu(A)$
   (b) $\nu(B) > -\varepsilon \forall B \subset A_\varepsilon$

   Proof. Proof by contradiction. Assume $\exists \varepsilon > 0 : \forall C \subset A$ with $\nu(C) \geq \nu(A) : \exists B = B_C \subset C$ with $\nu(B) \leq -\varepsilon$. Inductive construction:

   (a) $C_0 = A \exists B_1 \subset A : \nu(B_1) \leq -\varepsilon$.
   (b) $C_1 = A \setminus B_1 : \nu(C_1) + \underbrace{\nu(B_1)}_{\leq -\varepsilon} = \nu(A)$
   (c) $\nu(C_1) > \nu(C_0) = \nu(A) : \exists B_2 \subset C_1 : \nu(B_2) \leq -\varepsilon$. $C_2 = C_1 \setminus B_2$
   (d) $\nu(C_2) + \underbrace{\nu(B_2)}_{\leq -\varepsilon} = \nu(C_1) \geq \nu(A)$. $\nu(C_2) > \nu(A)$.

(e) $B_1, B_2, \ldots$ is pairwise disjoint. $\dot\bigcup_n B_n \subset A$.

$$0 < \nu(A) < \infty \implies \left|\nu\left(\dot\bigcup_n B_n\right)\right| < \infty$$

$$\sum_n \underbrace{\nu(B_n)}_{\leq -\varepsilon} = -\infty$$

This is a contradiction.

□

Now let $\varepsilon = \frac{1}{n}$. A_1 for $\varepsilon = 1$ as above. $A_1 \subset A$ ($0 < \nu(A) < \infty$). $A_2 \subset A_1$ with respect to A_1 for $\varepsilon = \frac{1}{2}$. $\nu(A_2) \geq \nu(A_1) \geq \nu(A)$. $A_3 \subset A_2$ with respect to A_2 for $\varepsilon = \frac{1}{3}$. $\nu(B) > -\frac{1}{2} \forall B \subset A_2$. $A_3 \subset A_2$ with respect to A_2 for $\varepsilon = \frac{1}{3}$. $A_n \subset A$, $A_{n+1} \subset A_n$. $\nu(B) > -\frac{1}{n+1} \forall B \in A_{n+1}$. $\nu(A_{n+1}) \geq \nu(A_n)$.

$P \coloneqq \bigcap_n A_n$. So $\nu(A_n) \nearrow \nu(P)$. This implies that $\nu(P) \geq \nu(A)$.

P positive. $B \subset P \implies B \subset A_n$. $\nu(B) \geq -\frac{1}{n} \forall n$. Therefore ≥ 0.

Analogously for negative sets: If $\nu : \mathcal{A} \to (-\infty, \infty]$, then $\nu \leftrightarrow -\nu$ and $N \leftrightarrow P$.

□

Theorem 6.18.1 (Hahn decomposition theorem)**.** *Let ν be a signed measure. Then $X = P \mathbin{\dot\cup} N$ where P is ν-positive and N is ν-negative. $P, N \in \mathcal{A}$. Furthermore let P, N be almost everywhere distinct. Hence $X = P' \mathbin{\dot\cup} N'$.*

$$\nu(P' \triangle P) = \nu(N' \triangle N) = 0$$

where the symmetric unions are ν-zero sets.

Proof. Without loss of generality, $\nu : A \to [-\infty, +\infty)$. $\alpha = \sup\{\nu(A) \mid A \in \mathcal{A}\} \geq 0$. $\exists$ sequence (A_n) in $\mathcal{A}$ such that $\nu(A_n) \nearrow \alpha$. By Theorem 6.17, $\exists P_n \subset A_n$ positive. $\alpha \geq \nu(P_n) \geq \nu(A_n)$. $\nu(P_n) \to \alpha$. $P = \bigcup_n P_n \supset P_n$.

$$\alpha \geq \nu(P) \geq \lim \nu(P_n) = \alpha$$

Why does the second inequality hold? Because P is positive (refer to the practicals, we might do this as an exercise). P is disjoint countable union of positive sets.

$$P_1' = P_1, \qquad P_n' = P_n \setminus \bigcup_{k=2}^{n-1} P_k$$

$$\alpha = \nu(P) = \max\{\nu(A) \mid A \in \mathcal{A}\} < \infty$$

□

Claim 6.19. *$N = P^C$ is negative.*

Remark. $\exists A \subset N : \nu(A) > 0$

Proof.
$$\nu(A \cup P) = \nu(A) + \alpha > \alpha$$
This is a contradiction. □

$$\nu^+(A) := \nu(A \cap P) \qquad \nu^-(A) := -\nu(A \cap N)$$
is a non-negative measure.
$$\nu = \nu^+ - \nu^-$$
one of them is finite.

$$|\nu| := \nu^+ + \nu^-$$
$$|\nu|(A) = \nu^+(A) + \nu^-(A) \geq |\nu(A)|$$

6.6 Total variation

Definition 6.20. *Let ν_1, ν_2 be signed measures are singular with respect to each other. $\nu_1 \perp \nu_2$, if they are concentrated on two disjoint sets:*
$$\chi = A_1 \cup A_2, A_i \in \mathcal{A} : \nu_2(B) = \nu_C(B \cap A_i)$$
So A_2 is a ν_1-zero set and A_2 is a ν_2 is a zero set.

Example 6.21 ($\nu^+ \perp \nu^-$ on $\mathbb{R}$)**.**
$$\lambda \perp \textit{ counting measure over } \mathbb{Z}$$
$$\mu(A) = |A \cap \mathbb{Z}|$$

Theorem 6.21.1 (Jordan's decomposition theorem)**.** *Every signed measure ν has a unique decomposition $\nu = \nu^+ - \nu^-$ with $\nu^+ \geq 0, \nu^+ \perp \nu^-$.*

Remark (Minimality)**.** *If $\nu = \nu_1 - \nu_2$ with non-negative measures ν_i then $\nu^+ \leq \nu_1$, $\nu^- \leq \nu_2$. Is trivial to show.*

7 Theorem of Radon-Nikodym

Related to Johann Radon: https://en.wikipedia.org/wiki/Radon_transform

Let μ be a positive measure on (X, μ). $\varphi : X \to \mathbb{R}$ measurable. $\int f \, d\mu$ exists.
$$\nu(A) = \int_A f \, d\mu = \int f \cdot \mathbf{1}_A \, d\mu$$
Signed measure $\nu^+(A) = \int f^\pm \cdot \mathbf{1}_A \, d\mu$.

$\mu(A) = 0 \implies A$ is a ν-zero set.

Definition. *ν (signed measure) is called* absolutely continuous *with respect to μ ($\nu \ll \mu$) if $\mu(A) = 0 \implies A$ is a ν-zero set$\forall A \in \mathcal{A}$.*

Theorem 7.0.1. *Let μ be σ-finite measure. Let ν be a signed measure. $\nu \ll \mu$.*

$$\implies \exists f : X \to \overline{\mathbb{R}} \text{ measurable}$$

such that $\int_X f \, d\mu$ exists and $\nu(A) = \int_A f \, d\mu \forall A \in \mathcal{A}$. f is μ-almost everywhere distinct; i.e. if $\nu(A) = \int_A \tilde{f} d\mu \forall A \in \mathcal{A}$ then $\mu\left(\left\{x \mid \tilde{f}(x) \neq f(x)\right\}\right) = 0$. f is called Radon-Nikodyn density *of ν with respect to μ, $f = \frac{d\nu}{d\mu}$.*

Then $\int g \, d\nu = \int gf \, d\mu$ for "nice" $g : X \to \mathbb{R}$.

$\downarrow$ *This lecture took place on 2019/01/10.*

Theorem 7.0.2 (Hahn)**.** *Let ν be a signed measure.*

$$X = P \cup N$$

$P \in \mathbb{A}$ positive $[\nu(A) \geq 0 \forall A \subset P, A \in \mathcal{A}]$ $\quad$ $N \in \mathbb{A}$ negative $\forall B \subseteq N, B \in \mathcal{A}$

P, N are unique (up to zero sets)

$$\nu^+(A) = \nu(A \cap P) \qquad \nu^-(A) = -\nu(A \cap N)$$

$$\nu = \nu^+ - \nu^- (\textit{Jordan decomposition}) \qquad \nu^+ \perp \nu^- \textit{ on disjoint sets}$$

and if $\nu = \nu_1 - \nu_2$ and ν_i are non-negative measures, then $\nu^+ \leq \nu_1$ and $\nu^- \leq \nu_2$.

Remark 7.1. *μ is a positive measure, $f : (X, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ such that $\int f \, d\mu$ exists. $\nu := \mu_f$, $\mu_f(A) = \int_A f \, d\mu$ signed measure and $\mu(A) = 0 \implies A$ is a μ_f-zero set.*

Definition 7.2. *$\nu \ll \mu$ (ν absolutely continuous w.r.t. μ) if every μ-zero set is also a ν-zero set.*

Theorem 7.2.1 (Radon-Nikodyn)**.** *μ (positive measure!) σ-finite, $\nu \ll \mu$ (signed) $\implies \exists f : \nu = \mu_f$, f is μ almost everywhere unique.*

$$f := \frac{d\nu}{d\mu}$$

Proof of Radon-Nikodyn theorem. Without loss of generality, $\nu \geq 0$. Why? Because $\nu^+ \ll \nu \ll \mu$. $\nu^+ = \mu_{f^+}$. $\nu^- = \nu_{f^-}$. $\nu = \mu_{f^+ - f^-}$.

1. μ, ν is finite. $\mu(X), \nu(X) < \infty$. Let

$$\mathcal{R} = \left\{g \mid X \to [0, \infty] \text{ measurable}, \int_A g \, d\mu \leq \nu(A) \forall A \in \mathcal{A}\right\}$$

$\mathcal{R} \neq \emptyset$ because $0 \in \mathcal{R}$.

$$\alpha := \sup\left(\int g \, d\mu : g \in \mathcal{R}\right)$$

$$\exists g_n \in \mathcal{R} : \int g_n \, d\mu \to \alpha$$

$$f, g \in \mathbb{R} \overset{\text{claim}}{\Longrightarrow} \max\{f, g\} \in \mathcal{R}$$

$$A_0 = [f \geq g] = \{x \in X \mid f(x) \geq g(x)\}$$

$$h(x) = \max\{f(x), g(x)\} \qquad A_0^C = [f < g]$$

$$A_0^C = [f < g]$$

$$h = f \cdot 1_{A_0} + g \cdot 1_{A_0^C}$$

Let $A \in \mathcal{A}$ be arbitrary.

$$\int_A h \, d\mu = \int_A f \cdot \mathbf{1}_{A_0} \, d\mu + \int_A g \cdot \mathbf{1}_{A_0} c \, d\mu = \int_{A \cap A_0} f \, d\mu + \int_{A \cap A_0^C} g \, d\mu$$

$$\leq \nu(A \cap A_0) + \nu(A \cap A_0^C) = \nu(A)$$

$$f_n = \max g_1, \ldots, g_n \in \mathcal{R} \qquad 0 \leq f_n \nearrow$$

$$\lim_{n \to \infty} \int f_n \, d\mu = \alpha \overset{\substack{\text{monotonic} \\ \text{convergence}}}{=} \int f \, d\mu \leq \nu(X) < \infty \qquad \text{and by } f \in \mathcal{R}$$

Side condition: $\int_A f \, d\mu = \lim \int_A f_n \, d\mu \leq \nu(A)$.
Now show: $\nu = \mu_f$. We know: $\nu \geq \mu_f$.

Let $\tau = \nu - \mu_f$ is a non-negative measure, $\tau \ll \mu$. We want: $\tau \equiv 0$ suffices $\tau(X) = 0$. $\tau(X) = \nu(X) - \alpha$. $\varepsilon > 0$.

$$\sigma_\varepsilon = \tau - \varepsilon \cdot \mu \qquad \text{signed measure}$$

$$= \underbrace{\sigma_\varepsilon^+}_{P_\varepsilon} - \underbrace{\sigma_\varepsilon^-}_{N_\varepsilon}$$

$$P_\varepsilon \cupdot N_\varepsilon = X$$

Assumption:

$$\sigma_\varepsilon^+ \neq 0 : \sigma_\varepsilon(P_\varepsilon) > 0 \implies \tau(P_\varepsilon) > 0 \implies \mu(P_\varepsilon) > 0$$

Let $f_\varepsilon = f + \varepsilon \mathbf{1}_{P_\varepsilon}$.

$$\int_A f_\varepsilon \, d\mu = \mu_f(A) + \varepsilon\mu(A \cap P_\varepsilon) < \mu_f(A) + \tau(A \cap P_\varepsilon)$$

$$\tau(A \cap P_\varepsilon) - \varepsilon_\mu(A \cap P_\varepsilon) = \sigma_\varepsilon(A \cap P_\varepsilon) = \sigma_\varepsilon^+ \geq 0$$

$$= \mu_f(A \cap N_\varepsilon) + \mu_\varepsilon(A \cap P_\varepsilon) + \nu(A \cap P_\varepsilon) - \mu_f(A \cap P_\varepsilon)$$

$$\leq \nu(A \cap N_\varepsilon) + \nu(A \cap P_\varepsilon) = \nu(A) \forall A \in \mathcal{A}$$

So $f_\varepsilon \in \mathcal{R}$. So $\int f_\varepsilon \, d\mu \leq \alpha$ and $\underbrace{\int f \, d\mu}_{\alpha} + \underbrace{\varepsilon\mu(P_\varepsilon)}_{>0} > \alpha$ with $\int f_\varepsilon \, d\mu = \int f \, d\mu$.
Contradiction! So $\sigma_\varepsilon^+ = 0$. $\sigma_\varepsilon = -\sigma_\varepsilon^-$.

$$(\tau - \varepsilon\mu)(X) \leq 0 \qquad T(X) \leq \varepsilon \cdot \mu(X) \forall \varepsilon > 0$$

Also $\tau(X) = 0$.

2. Let μ be finite. Let $\nu \geq 0$ be arbitrary. Let $\beta = \sup\{\mu(B) : B \in \mathcal{A}, \nu(B) < \infty\} \leq \mu(X) < \infty$.

   There exists some sequence (B_n) of A with $B_n \subset B_{n+1}$. $\nu(B_n) < \infty$. $\nu(B_n) \nearrow \beta$. Let $E = \bigcup_n B_n, F = E^C$. If $A \subset F, A \in \mathcal{A}$, we assume $\nu(A) < \infty \implies \nu(B_n \cup A) < \infty$

   $$\implies \mu(B_n \cup A) \leq \beta \implies \mu(B_n) + \mu(A) \leq \beta \qquad n \to \infty$$

   $$\mu(A) = 0 \implies \nu(A) = 0$$

   Either $\mu(A) = \nu(A) = 0$ or $\mu(A) > 0, \nu(A) = \infty$.

   $$E = \dot{\bigcup_n} E_n \qquad E_1 = B_1 \qquad E_n = B_n \setminus B_{n-1} \qquad (n \geq 2)$$

   $$\nu|_{E_n} =: \nu_n \qquad \mu|_{E_n} =: \mu_n \qquad \text{are finite measures}$$

   $$\nu_n(A) = \nu(E_n \cap A) \qquad \nu_n \ll \mu_n$$

   $$\exists \tilde{f}_n : \nu_n(A) = \int_A \tilde{f}_n \, d\mu_n = \int_A \underbrace{\tilde{f}_n \cdot \mathbf{1}_{E_n}}_{f_n} \, d\mu \forall A \in \mathcal{A}$$

   $$f = \sum_{n=1}^{\infty} f_n + \infty \cdot \mathbf{1}_F$$

   $$\nu(A) = \nu(A \cap E) + \nu(A \cap F) = \int_{A \cap E} f \, d\mu + \int_{A \cap F} \infty \, d\mu = \int_A f \, d\mu \forall A \in \mathcal{A}$$

3. Let μ be σ-finite.

   $$X = \bigcup_n^{+} D_n \mu(D_n) < \infty \text{ then use } \mu|_{D_n}, \nu|_{D_n}$$

   The following remark is left as an exception to the reader: $\mu_f = \mu_{f'} \implies f = f'$ is $\mu -$ almost everywhere

□

↓ *This lecture took place on 2019/01/14.*

7.1 Convergence rates of sequences of functions

Definition 7.3. *Let $(X, \mathcal{A}, \mu)$ is a measure space. $f_n, f : (X, \mathcal{A}) \to (\overline{R}, \mathcal{B}_{\overline{R}})$*

1. *$f_n \to f$ μ-almost-everywhere if $\mu(\{x \in X\} f_n(x) \not\to f(x)) = 0$.*

   $$\mu[f_n \not\to f]$$

2. $f_n \to f$ *in the measure if* $\lim_{n\to\infty}(\{x \mid |f_n(x) - f(x)| \geq a\}) = 0$

$$\mu[|f_n - f| \geq a] \forall a > 0$$

TODO: there was another board with notes

Lemma 7.4. *In both cases, f μ-almost-everywhere unambiguous. So, if $f_n \to f$ and $f_n \to g$ μ-almost-everywhere in the measure.*

$$\implies f = g \mu\text{-almost-everywhere} : \mu[f \neq g] = 0$$

$$|f - g| \leq |f - f_n| + |f_n - g|$$
$$[|f - g| \geq a] \subset \left[|f_n - f| \geq \frac{a}{2}\right] \cup \left[|f_n - g| \geq \frac{a}{2}\right]$$
$$\mu\left[|f - g| \geq a\right] \leq \mu\left[|f_n - f| \geq \frac{a}{2}\right] + \mu\left[|f_n - g| \geq \frac{a}{2}\right]$$

Also $\mu[|f - g| \geq a] = 0 \forall a = \frac{1}{r} > 0$ with $r \in \mathbb{N}$.

$$\implies \mu[|f - g| > 0] = 0$$

Theorem 7.4.1. *If $\mu(X) < \infty$, then $f_n - f$ μ-almost-everywhere $\iff$ $g_k = \sup_{n\geq k} |f_n - f| \to 0$ in the measure.*

Proof.

$$A = [f_n - f] = \left[\forall r \exists k \forall n \geq k : |f_n - f| \leq \frac{1}{r}\right]$$
$$= \bigcap_r A_r, A_r = \left[\exists k \forall n \geq k : |f_n - f| \leq \frac{1}{r}\right] \supset A_{r+1}$$
$$A = \bigcap_r A_r \qquad A^C = \bigcap_r A_r^C$$
$$f_n \to f \mu\text{-almost-everywhere} \iff \mu(A^C) = 0 \iff \forall r : \mu(A_r^C) = 0$$
$$A_r^C = \bigcap_k \left[\exists n \geq k : |f_n - f| > \frac{1}{r}\right] = \bigcap_k \underbrace{\left[g_k > \frac{1}{r}\right]}_{B_{k,r} \supset B_{k+1,r}}$$

with $g_k > g_{k+1}$,

$$\iff \forall r : \mu\left(\bigcap_k B_{k,r}\right) = 0 \qquad B_{k,r} \searrow A^C$$

$$\iff \forall r : \mu(B_{k,r}) \xrightarrow{k\to\infty} 0$$

where $\implies$ is given because $\mu(X) < \infty$.

$$\forall r : \mu\left[g_k < \frac{1}{r}\right] \xrightarrow{k\to\infty} 0$$

$g_k \to 0$ in the measure. ◻

Definition 7.5. *Let (f_n) be a Cauchy sequence in the measure if*

$$\lim_{m,n\to\infty} \mu\left[|f_n - f_m| > a\right] = 0 \forall a > 0, a = \frac{1}{r}, r \in \mathbb{N}$$

$$\lim_{n\to\infty} \sup_{m>n} \mu\left[(f_n - f_m) > a\right] = 0 \forall a > 0$$

Theorem 7.5.1.
1. *$\exists f : f_n \to f$ in the measure $\iff$ (f_n) in Cauchy sequence in the measure*
2. *$f_n \to f$ in the measure $\implies$ $f_{n_k} \to f$ μ-almost-everywhere for some subsequence.*

Example 7.6. *Let $X = (0,1], \mathbb{B}_{(0,1]}, \mu = \lambda$ is a Lebesgue measure.*

$$I_1 = (0,1] \qquad I_2 = (0,\frac{1}{2}] \qquad I_3 = (\frac{1}{2},1] \qquad I_4 = (0,\frac{1}{4}] \qquad \dots$$

$$I_7 = (\frac{3}{4},1] \qquad I_8 = (0,\frac{1}{8}] \qquad I_9 = (\frac{1}{8},\frac{2}{8}] \qquad I_{16} = (0,\frac{1}{16}]$$

TODO

$$f_{2^k} = \mathbf{1}_{(0,\frac{1}{2^k}]}$$

$x \in (0,1]$*: if* $\frac{1}{2^k} < x$*, then* $f_{2^k}(x) = 0$*.* $f_{2^k} \to 0$ $(\lambda - f)$ *everywhere.*

Proof. **1.** $\implies$ $f_n \to f$ in the measure. $\forall a > 0$

$$\mu\left(|f_n - f_m| \geq a\right) \leq \mu\left(|f_n - f| \geq \frac{a}{2}\right) + \mu\left(|f_n - f| \geq \frac{a}{2}\right) \xrightarrow{n,m\to\infty} 0$$

1. $\impliedby$ **, and 2.**

$$a_k > \varepsilon_k = \frac{1}{2^k} : \exists n_k \forall n, m \geq n_k$$

$$\mu\left[|f_n - f_m| \geq \frac{1}{2^k}\right] < \frac{1}{2^k}$$

wlog. $n_{k+1} > n_k \implies$

$$\sum_{k=1}^{\infty} \mu\left[\left|f_{n_{k+1}} - f_{n_k}\right| \geq \frac{1}{2^k}\right] < \sum_k \frac{1}{2^k} < \infty$$

□

Excursion

Lemma 7.7 (Borel-Cantelli Lemma)**.**

$$\sum_{k=1}^{\infty} \mu(A_k) < \infty \implies \mu\left(\limsup A_k = 0\right)$$

$$\mu\left(\bigcap_{n=1}^{\infty}\underbrace{\bigcup_{k=m}^{\infty}A_k}_{B_m\supset B_{m+1}}\right)=0$$

$$\mu(B_1)\leq\sum_k\mu(A_k)<\infty\qquad B_m\searrow\limsup A_k$$

So, $\mu(\limsup A_k)>\lim_m\mu(B_m)$

$$\mu(B_m)\leq\sum_{k=m}^{\infty}\mu(A_k)\xrightarrow{m\to\infty}0$$

$$\implies\underbrace{\mu\left(\limsup\underbrace{[\left|f_{n_{k+1}}-f_{n_k}\right|\geq\frac{1}{2^k}]}_{D_k^C}\right)}_{D^C}=0$$

$$D=\liminf D_k=\bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty}D_k$$

$$x\in D\implies\exists m:x\in D_k\forall k\geq m$$

$$\left|f_n(x)-f_{n_k}(x)\right|<\frac{1}{2^k}\forall k\geq m$$

$$\sum_{k=1}^{\infty}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right|<\infty$$

$$\sum_{k=1}^{\infty}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right)\text{ converges as well}$$

$$\sum_{k=1}^{k-1}(\dots)=f_{n_k}(x)-f_{n_1}(1)\text{ converges, }k\to\infty$$

so,

$$\exists\lim_{k\to\infty}f_{n_k}(x)=f(x),\mu(D^C)=0,f|_{D^C}=0$$

$$f_{n_k}\to f\mu\text{-almost-everywhere}$$

It remains to show that $f_n\to f$ in the measure.

$$\mu\left[|f_n-f|\geq a\right]=\mu\left[\liminf_{k\to\infty}\left|f_n-f_{n_k}\right|\geq a\right]$$

$$\overset{\text{Fatou}}{\leq}\liminf_{k\to\infty}\underbrace{\mu\left[\left|f_n-f_{n_k}\right|\geq a\right]}_{<\varepsilon,\text{ if }n,n_k\text{ sufficiently large}}$$

In the next lecture, we will discuss:

L^p convergence:

Definition 7.8.

$$\|f\|_p := \int_X \left(|f(x)|^p\right)^{\frac{1}{p}} \qquad 1 \le 1 < \infty$$

$$\begin{aligned} \|f\|_\infty &:= \text{ess sup}\,|f| = \inf\{c > 0 \mid |f| \le c \quad \mu - \textit{almost-everwhere}\} \\ &= \inf\left\{\sup_A |f(x)| \mid A \in \mathcal{A} : \mu(A^C) = 0\right\} \end{aligned}$$

$$f \sim g \iff \mu[f \neq g] = 0, \text{ so } f = g \text{ } \mu\text{-almost-everywhere}$$

↓ *This lecture took place on 2019/01/17.*

Definition 7.9. *Let $(X, \mathcal{A}, \mu)$. Consider $f, g : (X, \mathcal{A}) \to (\overline{R}, \mathcal{B}_{\overline{R}})$. $f \sim g :\iff \mu[f \neq g] = 0$. $f = g$ μ-almost-everywhere.*

$$1 \le p < \infty : L^p(X, \mu) = \left\{f : X \to \overline{\mathbb{R}} \mid \int |f|^p \, d\mu < \infty\right\} / \sim$$

Equivalence classes: μ-almost-everywhere equal functions.

$$\|f\|_\infty = \text{ess sup}\,|f| = \inf\left\{\sup_A |f| \mid \mu(A^C) = 0\right\}$$

$$L^\infty(X, \mu) = \left\{f : X \to \overline{\mathbb{R}} \text{ measurable} \mid \|f\|_\infty < \infty\right\} / \sim$$

Side remark: $\|f\|_p = 0 \iff f \equiv 0 \qquad \mu$-almost everywhere $(1 \le p \le \infty)$

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$$

Immediate: $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$.

Theorem 7.9.1 (Hölder's inequality)**.** $1 < p < \infty$. $q : \frac{1}{p} + \frac{1}{q} = 1$ *(conjugated exponent). $f, g : X \to \mathbb{R}$ measurable, f, g integrable.*

$$\left|\int f \cdot g \, d\mu\right| \le \int |f \cdot g| \, d\mu \le \|f\|_p \|g\|_q$$

Remark. *For $a, b > 0$, $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{1}{p}a + \frac{1}{q}b$. Why? $\frac{1}{p} = t \in (0,1)$. $\frac{1}{q} = 1 - t$*

$$a = e^x \qquad b = e^y$$

$$a^{\frac{1}{p}} b^{\frac{1}{q}} = e^{tx + (1-t)y}$$

e^x is convex.

$X = \{1, \dots, d\}$. $\mu(\{k\}) = 1$. $f : X \to \mathbb{R} \iff (f(1), f(2), \dots, f(d))$.

$$\|f\|_p = \left(\sum_{k=1}^{d} f(k)^p\right)^{\frac{1}{p}}$$

$$\|f\|_\infty = \max\{|f(1)|, \dots, |f(d)|\}$$

$$L^p(X, \mu) = (\mathbb{R}^d, \|\cdot\|_p)$$

Proof. If $\|f\|_p \in \{0, \infty\}$ or $\|g\|_q \in \{0, \infty\}$, then immediate.

Without loss of generality: $0 < \|f\|_p, \|g\|_q < \infty$. Let $F = \frac{|f|}{\|f\|_p}$ and $G = \frac{|g|}{|g|_q}$. $\|F\|_p = \|G\|_q = 1$.

$$FG = (F^p)^{\frac{1}{p}}(G^q)^{\frac{1}{q}} \le \frac{1}{p}F^p + \frac{1}{q}G^q$$

$$\int FG\, d\mu \le \frac{1}{p}\underbrace{\int F^p\, d\mu}_{=1} + \frac{1}{q}\underbrace{\int G^q\, d\mu}_{=1} = 1$$

$$\implies \int \frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_q}\, d\mu \le 1$$

□

Theorem 7.9.2 (Minkowski inequality)**.** $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

Proof.

$$\|f+g\|_p^p = \int |f+g|^p\, d\mu \le \int |f| \cdot |f+g|^{p-1}\, d\mu + \int |g|\, |f+g|^{p-1}\, d\mu$$

$$\overset{\text{Hölder ineq.}}{\le} \left(\|f\|_p + \|g\|_p\right)\left(\int |f+g|^{(p-1)q}\, d\mu\right)^{\frac{1}{q}}$$

□

Remark.

$$\frac{p}{q} + 1 = p \qquad (p-1)q = p \qquad p - \frac{p}{q} = 1$$

$$(\dots)^{\frac{1}{q}} = \left(\int |f+g|^p\, d\mu\right)^{\frac{1}{p}\frac{p}{q}} = \|f+g\|_p^{\frac{p}{q}}$$

Without loss of generality $\notin \{0, \infty\}$.

$$\|f+g\|_p^{p - \frac{p}{q}} \le \|f\|_p + \|g\|_p$$

Definition 7.10 (Convergence in the p-th mean)**.** $f_n \to f$ *in* L^p *(in p-th mean) if* $\|f_n - f\|_p \to 0$.

Lemma 7.11. $1 \leq p < \infty$. *If* $f_n \geq 0$,

$$\left\|\sum_{n=0}^{\infty} f_n\right\|_p \leq \sum_{n=0}^{\infty} \|f_n\|_p$$

Proof.

$$\left\|\sum_{n=0}^{N} f_n\right\|_p \leq \sum_{n=0}^{N} \|f_n\|_p \leq \sum_{n=0}^{N} \|f_n\|_p$$

$$\left\|\sum_{n=0}^{N} f_n\right\|_p^A = \int \left(\sum_{n=0}^{N} f_n\right) d\mu \nearrow_{N\to\infty} \int \left(\sum_{n=0}^{\infty} f_n\right)^p d\mu$$

□

Theorem 7.11.1. 1. $\sum_{n=0}^{\infty} \|g_n\|^p < \infty \implies \exists g \in L_p(X,\mu) : \sum_{n=0}^{N} g_n \xrightarrow{N\to\infty} g$ *μ-almost everywhere and in L^p.*

2. *$(f_n)_{n\geq 0}$ Cauchy sequence in $L^p(X,\mu)$ then $\exists f \in L^p(X,\mu)$ is a subsequence f_{n_k} such that $f_n \to f$ is μ-almost everywhere and $f_n \to f$ in $L^p(X,\mu)$. $\|f_n - f\|_p \to 0$.*

3. *$f_n \to f$ in $L^p \implies \exists f_{n_k} \to f$ μ-almost everywhere.*

Proof. $1 \leq p < \infty$

1. $\sum_{n=0}^{N} |g_n| \nearrow \varphi : X \to [0,\infty]$ measurable.

$$\|q\|_p \leq \sum_{n=0}^{\infty} \|g_n\|_p < \infty$$

$$\implies \varphi = \sum_{n=0}^{\infty} |g_n| < \infty \text{ } \mu\text{-almost everywhere}$$

so, $\sum_{n=0}^{\infty} g_n$ μ-almost everywhere (pointwise) absolute convergent; thus convergent.

$$\exists g : \sum_{n=0}^{N} g_n \to_{N\to\infty} g \, \mu - \text{almost everywhere}$$

$$\left|\sum_{n=0}^{\infty} g_n\right| \leq \sum_{n=0}^{\infty} |g_n| = \varphi$$

$$\|g\|_p \leq \|\varphi\|_p < \infty$$

$g \in L_p$. If you think this through,

$$\left\|\sum_{n=0}^{N} g_n - g\right\|_p \to 0$$

2. $k \in \mathbb{N}_0$:

$$\exists n_k : \forall m, n \geq n_k : \|f_n - f_m\|_p < \frac{1}{2^k}$$

Without loss of generality: $n_{k+1} > n_k$

$$\sum_{k=0}^{\infty} \underbrace{\left|f_{n_{k+1}} - f_{n_k}\right|}_{g_k} < 2 < \infty$$

$$\exists g \in L^p \implies \underbrace{\sum_{k=0}^{K-1} g_k}_{f_{n_k} - f_0} \to_{k\to\infty} g \ \mu\text{-almost everywhere and in } L^p$$

$$\implies f_{n_k} \to f = g + f_{n_0} \ \mu\text{-almost-everywhere and in } L^p$$

$$\|f_n - f\|_p < \underbrace{\left\|f_n - f_{n_k}\right\|_p}_{<\varepsilon} + \underbrace{\left\|f_{n_k} - f\right\|_p}_{<\varepsilon} < \varepsilon$$

For n sufficiently large.

□

Theorem 7.11.2.

$S(X, \mu) = \{s : X \to \mathbb{R}$ simple, measurable and $\mu[s \neq 0] < \infty\} \subset L^p(X, \mu)$, dense

Proof.

$$s \in S : s = \sum_{k=1}^{n} c_k \overbrace{\mathbf{1}_{A_k}}^{\in L^p}, c_k \neq 0, \mu(A_k) < \infty$$

$$\|\mathbf{1}_A\|_p = \mu(A)^{\frac{1}{p}} < \infty \implies s \in L^p$$

$$f \in L^p, f \geq 0 \qquad \infty > \|f\|_p^p \geq \int_{[f \geq \frac{1}{n}] = A_n} f^p \, d\mu \geq \frac{1}{n}\mu(A_n)$$

$$\mu(A_n) < \infty \qquad A_n \subset A_{n+1} \nearrow [f > 0] = A$$

$$\exists (t_n) \text{ simple functions: } t_n \nearrow f$$

$$s_n = t_n \cdot \mathbf{1}_{A_n} \nearrow f \cdot \mathbf{1}_{\mathbf{A}} = f \qquad s_1 \in S(X, \mu)$$

$$0 \leq f - s_n \leq f \qquad \|f - s_1\|_p^p \leq \|f\|_p^p < \infty$$

We can apply dominated convergence: $\|f - s_n\|_p \to 0$. □

Theorem 7.11.3. *$L^\infty(X,\mu)$ is complete.*

Proof. Let (f_n) be a Cauchy sequence in L^∞.

$$\forall k \in \mathbb{N} \exists N_k : \forall m,n \geq N_k : \|f_n - f_m\|_\infty < \frac{1}{k}$$

$$\implies \exists A_{n,m,k} : \mu(A^C_{n,m,k}) = 0 \text{ and } |f_n - f_m| < \frac{1}{k} \text{ on } A_{n,m,k}$$

$$A^C = \bigcup_k \bigcup_{m,n \geq N_k} A_{m,n,k} \qquad \mu(A^C) = 0$$

for $x \in A = \bigcap_k \bigcap_{m,n \geq N(k)} A_{m,n,k}$:

$$\forall k \forall m,n \geq N(k) : |f_n(k) - f_m(x)| < \frac{1}{k}$$

$$(f_n(x))_{n\in\mathbb{N}} \text{ Cauchy sequence } \implies f(x) = \lim_{n\to\infty} f_n(x)$$

$$f(x) \coloneqq 0 \qquad x \in A^C : f_n \to f \text{ } \mu\text{-almost everywhere (on } A^C)$$

Think about: $\|f_n - f\|_\infty \to 0$ ∎

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