Mathematical analysis 1 – Lecture notes

course by Wolfgang Ring

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Oct 2015 to Jan 2016

Contents

1 Propositional logic

This lecture took place on 1st of October 2015 with lecturer Wolfgang Ring.

- Discussion about motivation for visiting university
- Kurt Gödel: Gödel's incompleteness theorem
- propositional logic (and/or/implication/equivalence operation)
  - $p \implies q$: "p implies q" ("notwendig"), "q requires p" ("hinreichend")
  - Indirect proof: $(\neg q \implies \neg p) \Leftrightarrow (p \implies q)$
  - Proof by contradiction: claim p, claim $\neg q$, show that $p \wedge \neg q$ is not possible
  - commutative law: $a \wedge b \Leftrightarrow b \wedge a$
  - associative law: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
  - distributive law: $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$
  - DeMorgan's law: $\neg(a \wedge b) \Leftrightarrow (\neg a) \vee (\neg b)$
- First-order logic
  - $\forall x \in \mathbb{N} : x \in \mathbb{R}$
  - $\forall x \in M : P(x)$
  - $\neg [(\forall x \in M) P(x)] \Leftrightarrow \exists x \in M : \neg P(x)$
- Peano's axioms: rationale for induction proofs

The lecture on 8th of October 2015 got cancelled spontaneously.

2 First-Order Logic

This lecture took place on 12th of October 2015 with lecturer Wolfgang Ring.

Literature recommendation:

- "Analysis 1 (Mathematik für das Lehramt)", Oliver Deiser

Let A and B be statements.

- Logical equivalence is given iff the truthtable of both expressions is the same.
- $\neg(\neg A) \iff A$
- $(A \vee B) \iff (B \vee A)$
- $(A \wedge B) \iff (B \wedge A)$
- $a \implies b$: implication

Boolean Laws:

$$\neg(A \implies B) \iff A \wedge \neg B \tag{1}$$
$$A \iff B \implies (A \implies B) \wedge (B \implies A) \tag{2}$$

"contraposition" or "indirect proof"

$$\neg B \implies \neg A \tag{3}$$
$$A \implies B \iff (\neg B \implies \neg A) \tag{4}$$
$$(A \iff B) \iff (\neg A \iff \neg B) \tag{5}$$
$$\neg(A \wedge B) \iff \neg A \vee \neg B \tag{6}$$
$$\neg(A \vee B) \iff \neg A \wedge \neg B \tag{7}$$
$$\neg(A \implies B) \iff (A \wedge \neg B) \tag{8}$$
$$A \wedge (B \vee C) \iff ((A \wedge B) \vee (A \wedge C)) \tag{9}$$
$$A \vee (B \wedge C) \iff ((A \vee B) \wedge (A \vee C)) \tag{10}$$
$$(A \implies B) \iff (\neg A \vee B) \tag{11}$$

"proof by contradiction"

$$((A \implies B) \wedge (A \implies \neg B)) \implies \neg A \tag{12}$$

"conclusion"

$$((A \implies B) \wedge (B \implies C)) \implies (A \implies C) \tag{13}$$

$$A \vee B \Leftrightarrow \neg(\neg A) \vee \neg(\neg B) \Leftrightarrow \neg(\neg A \wedge \neg B)$$
$$\neg(A \vee B) \Leftrightarrow \neg(\neg(\neg A) \vee (\neg B))$$

Distributive laws:

- $(A \vee B) \wedge C \Leftrightarrow (A \wedge C) \vee (A \wedge C)$
- $(A \wedge B) \vee C \Leftrightarrow (A \vee C) \wedge (B \vee C)$

2.1 Tautologies

A *tautology* is the composition of statements, which always yields the truth value true, independent of the truth value of its subexpressions.

Examples of tautologies:

"Law of excluded middle" $A \vee \neg A$

equivalences are always tautologies $A \leftrightarrow \neg(\neg A)$

implication of itself $A \to A$

Tautology with multiple statements:

implication with or and not $(A \to B) \leftrightarrow (\neg A \vee B)$

proof by contradiction $[(A \to B) \wedge (A \to \neg B)] \to \neg A$

chain inference $[(A \to B) \wedge (B \to C)] \to (A \to C)$

This lecture took place on 14th of Oct 2015 with lecturer Wolfgang Ring.

Proof. We prove, $[(A \to B) \wedge (A \to \neg B)] \to \neg A$.

$$\begin{aligned}(A \to B) \wedge (A \to \neg B) &\iff (\neg A \vee B) \wedge (\neg A \vee \neg B)\\ &\iff \underbrace{(B \wedge \neg B)}_{\perp} \vee \neg A\\ &\iff \neg A\end{aligned}$$

Special case: $A = B$.

$$(A \to A) \wedge (A \to \neg A) \to \neg A$$

$$(A \to \neg A) \to \neg A$$

□

2.2 Negation of a tautology

- is called *contradiction*.
- has always truth value false.

Proof.
$$(A \vee B) \to C \Leftrightarrow \neg(A \vee B) \vee C \Leftrightarrow (\neg A \wedge \neg B) \vee C$$
$$(\neg A \vee C) \wedge (\neg B \vee C) \Leftrightarrow (A \to C) \wedge (B \to C)$$

$$(A \vee B) \to C \Leftrightarrow (A \to C) \wedge (B \to C)$$
$$(A \wedge B) \to C \Leftrightarrow (A \to C) \vee (B \to C)$$
$$A \to (B \wedge C) \Leftrightarrow (A \to B) \wedge (A \to C)$$
$$A \to (B \vee C) \Leftrightarrow (A \to B) \vee (A \to C)$$

□

Example proof by contradiction: Number of prime numbers. We prove a statement by Euklid of Alexandria, 300 BC:

> The number of prime numbers is infinite.

Assume the number of prime numbers is finite. Then there exists some $N \in \mathbb{N}$ such that $\mathbb{P} = \{P_1, P_2, \dots, P_n\}$ is the set of all prime numbers.

Every integer can be represented as product of prime numbers. Therefore for every integer there exists at least one prime number that divides this number (without remainder).

Let $m = p_1 \cdot p_2 \cdot \ldots \cdot p_N + 1$. Let a be a prime number that divides m.

It holds that: Every $p_i \in \mathbb{P}$ is not a divisor of m. Because when dividing $\frac{m}{p_i}$, the remainder is always one.

So $q \in \mathbb{P}$, so there exists more than N prime numbers (at least $N+1$). This contradicts with our assumption, that only N prime numbers exist.

Therefore always one more prime number exists. So the number of prime numbers is infinite. □

2.3 Quantifiers

Quantified statements are statements, in which objects of a set occur.

Example: Let $P(x) = (x > 0)$. Its truth value cannot be determined if the set X is not defined.

Definition 1. *Let M be a set, $x \in M$ and $P(x)$ a predicate.*

The composed statement: for every $x \in M$, it holds that $P(x)$ is true, if the truth value of $P(x)$ is always true independent of the selection of $x \in M$.

Example 1. *Let $M = \mathbb{R}$ and $P(x) = (x^2 + 1 > 0)$.*

This is true for all $x \in M$. We denote: $\forall x \in M : P(x)$.

Example 2. *Let $M = \mathbb{R}$ and $P(x) = (x^2 - 1 > 0)$.*

This is not *true for all $x \in M$. We denote: $\exists x \in M : \neg P(x)$.*

Definition 2. $\forall x \in M : P(x)$ *does not hold if and only if* $\exists x \in M : \neg P(x)$.

$\forall$ *is called* all quantifier. $\exists$ *is called* existence quantifier.

Negation works as follows:

$$\neg (\forall x \in M : P(x)) \Leftrightarrow \exists x \in M : \neg P(x)$$

$$\neg (\exists x \in M : P(x)) \Leftrightarrow \forall x \in M : \neg P(x)$$

This lecture took place on 15th of Oct 2015 with lecturer Wolfgang Ring.

$$\forall x \in M : (P(x) \wedge Q(x)) \iff (\forall x \in X : P(x)) \wedge (\forall y \in M : Q(y)))$$

$$\forall x \in M : (P(x) \vee Q(x)) \leftrightarrow (\forall x \in M : P(x)) \wedge (\forall x \in M : Q(x))$$

Counterexample:

$$M = \mathbb{R} \quad P(x) := (x > 0)$$

A statement B is stronger than C if C

2.4 Composition of several quantifiers

1. The order of quantfiers matters.
2. For every real number x, there exists an integer $n \in \mathbb{N}$ with the property $n > x$:

   $$\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$$

   The statement does not hold if the order is changed.

   $$\exists n \in \mathbb{N} \forall x \in \mathbb{R} : n > x$$

3 Sets

We consider objects, which we call *sets*. For every set M and every element x, it holds that

$$x \in M \vee \neg(x \in M)$$

Consider the set $L = \{M : M \text{ is a set and } M \notin M\}$. Does $L \notin L$ or $L \in L$ hold?

If $L \notin L$, then L satisfies the definition and therefore $L \in L$. If $L \in L$, then elements of L satisfy the property; therefore $L \notin L$.

Set operations:

- union
- intersection
- subsets

- $\forall S : \emptyset \subseteq S$
- complete induction

Theorem 1. *(Pythagoreans, 450 BC)*

$$\forall n \in \mathbb{N}_+ : \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. **Induction base $\mathbf{n = 1}$**

$$P(1) : 1 = \frac{1(1+1)}{2} \quad \checkmark$$

Induction step $\mathbf{n \to n+1}$

Assume $P(n)$ is true. So $(1 + 2 + \cdots + n) = \frac{n(n+1)}{2}$.

$$[(1 + 2 + \cdots + n) + (n+1)] = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(\frac{n}{2} + 1\right)$$

$$= (n+1) \cdot \frac{(n+2)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark$$

So, it simply holds that:

$$s = 1 + 2 + 3 + \cdots + n$$

$$2 \cdot s = \underbrace{n}_{\text{number of items}} \cdot \underbrace{(n+1)}_{\text{sum}} \Rightarrow s = \frac{n \cdot (n+1)}{2}$$

□

This lecture took place on 21st of October 2015 with lecturer Ring Wolfgang.

- Let X be a set. $M = \{x \in X : P(x)\}$.
- $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$ …“enumerating set representation”
- $M = \{x \in X \mid P(x)\}$, $N = \{x \in X \mid Q(x)\}$
- $M \cup N = \{x \in X \mid P(x) \vee Q(x)\}$

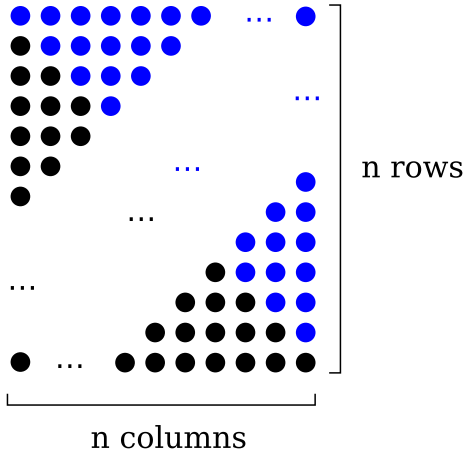

Figure 1: Illustration of the triangular number (illustrative proof)

- Let X be a set. $A_0 \subseteq X$, $A_1 \subseteq X$, $A_2 \subseteq X$, etc
- $\forall n \in \mathbb{N} : A_n \subseteq X$
- $A_0 \cup A_1 \cup A_2 \cup \cdots = \bigcup_{n=1}^{\infty} A_n = \{x \in X \mid (x \in A_0) \vee (x \in A_1) \vee \dots\} = \{x \in X \mid \exists n \in \mathbb{N} : x \in A_n\}$
- $A_0 \cap A_1 \cap A_2 \cap \cdots = \bigcap_{n=1}^{\infty} A_n = \{x \in X \mid \forall n \in \mathbb{N} : x \in A_n\}$

3.1 Cartesian product

Definition 3. *Let A and B sets. The* cartesian product of A and B *is given as:*

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

This operation is not commutative!

Definition 4. *We denote $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.*

Example 3.

$$A = \{a, b, c, d, e, f, g, h\}$$
$$B = \{1, 2, 3, 4, 5, 6, 7, 8\}$$
$$A \times B = \{(a, 1), (a, 2), (a, 3), \dots, (a, 8), (b, 1), (b, 2), \dots\}$$

Example 4.

$$\mathbb{R} \times \mathbb{R} = \{(x, y) \,|\, x, y \in \mathbb{R}\}$$

e.g. $(1, \frac{9}{8}) \in \mathbb{R} \times \mathbb{R}$.

Definition 5. *Let $A_1, A_2, \dots, A_n$ be sets.*

$$A_n = A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \,|\, a_i \in A_i \text{ for } i = 1, 2, \dots, n\}$$

instead of $\underbrace{A \times A \times \dots \times A}_{n \text{ times}} = A^n$.

3.2 Power set

Definition 6. *Let X be a set. Then $\mathcal{P}(X)$ is the* power set *of x.*

$$\mathcal{P}(X) = \{A \,|\, A \subseteq X\}$$

4 Mappings and functions

Definition 7. *Let A and B be sets. A* mapping *f from A to B (denoted $f : A \to B$) is an assignment, such that for every $x \in A$ one $y \in B$ is assigned. We denote the corresponding $y \in B$ for some $x \in A$ with $y = f(x)$. A is called* domain, *B is called* co-domain.

Definition 8 (Alternative definition of mappings)**.** *A mapping f is a subset of $A \times B$ which fulfills the following properties:*

- $\forall x \in A : (\exists y \in B : (x, y) \in f)$
- $\forall x \in A \wedge (y_1, y_2 \in B) : [(x_1, y_1) \in f \wedge (x_1, y_2) \in f] \implies y_1 = y_2$

Notation:

$$(x, y) \notin f \Leftrightarrow y = f(x)$$
$$\{(x, f(x)) \in \,|\, x \in A\} \Rightarrow \textit{graph from } f$$

Definition 9. *Let $f : A \to B$ be a mapping.*

- *The mapping f is called* surjective, *if $\forall y \in B : \exists x \in A : y = f(x)$.*
- *The mapping f is called* injective, *if*

  $$\forall x_1, x_2 \in A : (f(x_1) = f(x_2) \Rightarrow x_1 = x_2).$$

- *Let $B' \subseteq B$. Then we denote $f^{-1}(B') = \{x \in A \,|\, f(x) \in B'\}$ as the* preimage of f.

  Attention! The preimage distinguishes itself from the domain (it is a subset) and the inverse function f^{-1} (a function must not be invertible to have a preimage)!

- *Let $A' \subseteq A$. Then we call $f(A') = \{f(x) \,|\, x \in A\} \subseteq B$ the* image of A' *under f.*

  Special case: $A' = A$, then $f(A) \subseteq B$ is the image of A under f.

  Let $f : A \to B$ be a mapping. We define $f : A \to f(A) \subseteq B$ with $\tilde{f}(x) = f(x)$ for all $x \in A$. The mapping $\tilde{f}$ is surjective $\forall y \in f(A)$ there exists one $x \in A$ such that $y = f(x)$.

- *A mapping is called* bijective *iff the mapping is surjective and injective.*

4.1 Bernoulli's inequality

Definition 10 (Bernoulli's inequality)**.** *Let $x \in \mathbb{R}$ with $x > -1$ and $x \neq 0$. Let $n \in \mathbb{N}$ with $n > 1$. Then it holds that*

$$(1 + x)^n > 1 + nx$$

Proof. Proof by complete induction.

Induction base $\mathbf{n = 2}$

$$(1+x)^2 = 1 + 2x + x^2 > 1 + 2x \quad \checkmark$$

because $x^2 > 0$ for $x \neq 0$.

Induction step $\mathbf{n \to n+1}$
Assume $(1+x)^2 > 1+n$, then $x > -1$ and $x \neq 0$.

$$(1+x)^{n+1} = (1+x)^n \cdot \underbrace{(1+x)}_{>0} > (1+nx) \cdot (1+x)$$

$$= (1 + nx + x + nx^2) = (1 + (n+1) \cdot x + \underbrace{nx^2}_{>0}) > 1 + (n+1) \cdot x$$

□

Back to sets and functions (notes missing):

- injective, surjective, bijective function
- composition of functions: Let $f : X \to Y$ and $g : Y \to Z$. $g \circ f : X \to Z$ is defined as $g(f(x))$ ("g after f").
- Let f and g be mappings. If f and g are injective, $f \circ g$ is injective. If f and g are surjective, $f \circ g$ is surjective. If f and g are bijective, $f \circ g$ is bijective.
- Identity function, $f \circ \mathrm{id} = \mathrm{id} \circ f = f$
- properties of an inverse function, $f \circ f^{-1} : X \to X$, $f^{-1} \circ f : X \to X$

5 About sums of integers

This lecture took place on 21st of Oct 2015 with lecturer Wolfgang Ring.

Definition 11. *The summation notation is defined as,*

$$\sum_{k=h}^{l} a_k$$

Iteration over all values from l to h (inclusive) and evaluation of the enclosed expression with k as iteration value. The resulting terms are added up and the sum gives the result of the summation expression.

Laws:

$$\sum_{k=l}^{h} a_k = \sum_{i=l}^{h} a_i \tag{14}$$

$$\sum_{k=l}^{h} (a_k + b_k) = \left(\sum_{k=l}^{h} a_k\right) + \left(\sum_{k=l}^{h} b_k\right) \tag{15}$$

$$\sum_{k=0}^{h} a_k = a_0 + \sum_{k=1}^{h} a_k \qquad \text{"Extraction of the initial value"} \tag{16}$$

$$\sum_{k=0}^{h} a_k = a_h + \sum_{k=0}^{h-1} a_k \qquad \text{"Extraction of the final value"} \tag{17}$$

$$\sum_{k=u+n}^{h+n} a_k = \sum_{k=u}^{h} a_{k+n} \qquad \text{"index shifting"} \tag{18}$$

$$\sum_{k=l}^{h} \lambda \cdot a_k = \lambda \cdot \sum_{k=l}^{h} a_k \qquad \text{"extraction of a constant } \lambda\text{"} \tag{19}$$

$$\sum_{k=0}^{n} n = \frac{n(n+1)}{2} \qquad \text{"triangular sum"} \tag{20}$$

We consider $S_n = \{(a_1, a_2, \dots, a_n) : a_i \in M_n \forall i = 1, \dots, n \text{ with } a_i \neq a_j\} \subseteq M_n \times M_n \times \cdots \times M_n$. S_n is the set of all arrangements of the numbers $1, \dots, n$.

Example: $\{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

5.1 Factorials

Theorem 2. *It holds that $|\, S_n \,| = n!$ for all $n \in \mathbb{N}$*

Proof. Proof by induction over n.

Induction base $n = 1$: $M_1 = \{1\}, S_1 = \{(1)\} \Rightarrow |S_1| = 1 = 1!$ ✓

Induction step $n \to n+1$:

$$S_{n+1} = \{(a_1, a_2, \ldots, a_n) : a_i \in M_{n+1} \forall i \in M_{n+1}, a_i \neq a_j \text{ for } i \neq j\}$$

For $l \in M_{n+1}$:

$$W_l = \{(a_1, \ldots, a_{n+1}) \in S_{n+1} : a_l = n+1\}$$

It holds that $W_l \cap W_j = \emptyset$ for $l \neq j$ and $S_{n+1} = W_i \cup W_l \cup W_j \cup \ldots \cup W_{n+1}$. Then it holds that $|S_{n+1}| = |W_1| + |W_2| + \ldots + |W_{n+1}| = \sum_{l=1}^{n+1} |W_l|$

Theorem 3. *Claim: For every $l \in M_{n+1}$ it holds that $|W_l| = |S_n| = n!$.*

Proof. We build a bijective map $\phi_l : W_l \to S_n$.

$$W_l = \{(a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}\}$$

$$: a_i \in M_n, \forall i \neq l, a_i \neq a_j \forall i \neq j$$

$$\phi\left((a_1, a_2, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1})\right)$$
$$= (a_1, a_2, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) \in S_n$$

S_n is surjective. Let $(b_1, \ldots, b_n) \in S_n$, then it holds that $(b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n) \in W_l$

$$\phi_l((b_1, \ldots, b_{l-1}, n+1, b_l, \ldots, b_n)) = (b_1, \ldots, b_n)$$

S_n is injective.

$$\phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$= \phi_l((a_1, \ldots, a_{l-1}, n+1, a_{l+1}, \ldots, a_{n+1}))$$

$$\Rightarrow (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1}) = (a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{n+1})$$

ϕ is bijective. □

Therefore $|W_l| = |S_n| = n!$. Therefore $|S_{n+1}| = \sum_{l=1}^{n+1} |S_n| = \sum_{l=1}^{n+1} n! = (n+1)n! = (n+1)!$

□

Remark 1. *Let $f : M_n \to M_n$. f is represented as*

$$(1, 2, 3, 4, \ldots, n-1, n) \to (f(1), f(2), f(3), f(4), \ldots, f(n-1), f(n))$$

Therefore $(f(1), f(2), \ldots, f(n)) \in S_n$. Analogously every $(a_1, \ldots, a_n) \in S_n$ defined by $f(k) = a_k$ for $k = 1, \ldots, n$ is a bijective mapping $f : M_n \to M_n$. Therefore we set $S_n = \{f : M_n \to M_n : f \text{ is bijective}\}$. S_n is called symmetric group of n elements.

5.2 Binomial coefficients

Definition 12. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}$ with $k \leq n$. We define*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad \text{“binomial coefficient } n \text{ choose } k\text{”}$$

It holds that

$$\binom{n}{k} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{(1 \cdot 2 \cdot \ldots \cdot k)(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))}$$
$$= \frac{n(n-1) \cdot \ldots \cdot (k+1)}{(1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-k))}$$

Factorial laws:

$$\binom{1}{0} = \frac{n!}{0!(n-0)!} = 1 \qquad \forall n \in \mathbb{N}$$
$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = 1$$
$$\binom{n}{n-k} = \frac{n!}{(n-k!)(n-n+k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} \qquad \text{“symmetrical”}$$

A recursive definition is given by

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad n \geq 1, 1 \leq k \leq n-1$$

Proof.

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n+1}{k} &= \frac{(n-1)!}{(n-1)!(n-1-(k-1))!} \\
&= \frac{(n-1)!}{k!(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \left(\frac{(n-1)!}{k!(n-1-k)!}\right) \\
&= \frac{k \cdot (n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k!(n-1)!} = \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

□

5.3 Arrangement in Pascal's triangle

Theorem 4. *Let $T_n^k = \{A \subseteq M_n : |A| = k\}$. Then it holds that $|T_n^k| = \binom{n}{k}$.*

Example: $T_3^2 = \{\{1,2\}, \{1,3\}, \{2,3\}\}$.

$|T_3^2| = \binom{3}{2} = \frac{3!}{2!1!} = \frac{6}{2} = 3$

Proof. Let n be fixed. Induction for k.

Induction base $k = 0$

$$T_n^0 = \{\emptyset\}$$

$$|T_n^0| = 1 = \binom{n}{0}$$

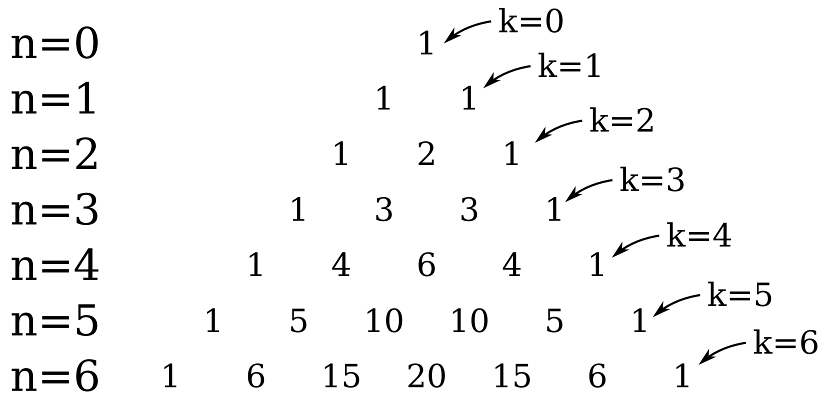

Figure 2: Pascal's triangle describes binomial coefficients. It's structure is given by adding up the two numbers above a number. The margins are defined by 1. For example 6 is given by $\binom{4}{2}$.

Induction step $k \to k+1$

$$T_n^k = \underbrace{\{\{a_1, \dots, a_k\} : a_1 \in M_n, (i = 1, \dots, k), a_i \neq a_j \text{ for } i \neq j\}}_{A_1}$$

$$\cup \underbrace{\{\{a_1, \dots, a_{k-1}\} \cup [n] \in M_{n-1}\}}_{A_2}$$

$$|T_N^k| = |A_1| + |A_2|$$

□

This lecture took place on 28th of October 2015 with lecturer Ring Wolfgang.

Let A, B be sets and define

$$A \setminus B = \{x : x \in A \wedge x \notin B\}$$

Then the domain of $A \setminus B$ is "A without B".

Theorem 5.

$$T_n^x = \{x \subseteq M_x : |\,X\,| = x\}$$

Let $k \in \mathbb{N}$ and $0 \leq k \leq 1$.

$$|\,T_n^x\,| = \binom{1}{k}$$

There are exactly $\binom{n}{k}$ k-ary subsets of M_n.

Proof.

$$M_0 = \emptyset \qquad T_0^0 = \{\emptyset\} \qquad |\,T_n^0\,| = 1 = \binom{0}{n}$$

Proof by complete induction over n of the following statement:

$$\forall n \in \mathbb{N} : \forall k \in \mathbb{N} \text{ with } 0 \leq k \leq n : |\,T_n^k\,| = \binom{n}{k}$$

Induction base $n = 0$ is fine. For $n = 1$ there are two cases: $k = 0$ or $k = 1$.

$$M_1 = \{1\}$$

$$T_1^0 = \{\emptyset\} \qquad |\,T_1^0\,| = 1 = \binom{1}{0}$$

$$T_1^1 = \{\{1\}\} \qquad |\,T_1^1\,| = 1 = \binom{1}{1}$$

Is also fine.

Induction step The hypothesis is our assumption:

$$\forall 0 \leq k \leq 1 : |\,T_n^k\,| = \binom{n}{k}$$

Consider M_{n+1}. Special case $k = 0$:

$$T_{n+1}^0 = \{\emptyset\} \qquad \left|\,T_{n+1}^0 = 1 = \binom{n+1}{0}\,\right|$$

Special case $k = n + 1$:

$$T_n = \{M_{n+1}\} \qquad |\,T_N n + 1^{n+1}\,| = 1 = \binom{n+1}{n+1}$$

Let $1 \leq k \leq n$.

$$T_{n+1}^x TODO$$

Union is disjoint $\Rightarrow |\,T_{n+1}^k\,| = |\,R_{n+1}^k\,| + |\,S_{n+1}^k\,|$

$$R_{n+1}^k = \{A \subseteq M_n : |\,A\,| = k\} = T_n^k$$

$$|\,R_{n+1}^k\,| = |\,T_n^k\,| = \binom{n}{k}$$

by induction hypothesis.

$$S_{n+1}^k = \{A \subseteq M_{n+1} : A = A' \cup \{n+1\} : A' \subseteq M_n : |\,A'\,| = k - 1\}$$

We prove $|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,|$.

$$f : S_{n+1}^k \to T_n^{k-1}$$

$$f(A) = f(A' \cup \{n+1\}) = A'$$

f is bijective. f is surjective: Let $A' \in T_n^k$ define $A = A' \cup \{n+1\} \in S_{n+1}^n$ and $f(A) = A'$. f is injective: Let $f(A) = f(B)$ and $A = A' \cup \{n+1\} \in S_{n+1}^k$.

$B = B' \cup \{n+1\} \in S_{n+1}^k$. $A', B' \in T_n^{k-1}$.

$$f(A) = f(B) \Rightarrow A' = B' \Rightarrow A' \cup \{n+1\} = B' \cup \{n+1\} \Rightarrow A = B$$

$$|\,S_{n+1}^k\,| = |\,T_n^{k-1}\,| \overset{\text{ind. hypo.}}{=} \binom{n}{k-1}$$

Therefore $|\,T_{n+1}^k\,| = \binom{n}{n} = \binom{n}{k-1} = \binom{n+1}{k}$. The last equation follows from the recursive definition of binomial coefficients.

□

5.4 Binomial theorem

Theorem 6 (Binomial theorem)**.** *Let $a, b \in \mathbb{R}$ (or $a, b \in \mathbb{C}$). Then it holds that*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof. 1. Proof by induction over n.

Induction step $n = 0$: $(a+b)^0 = 1$

$$\sum_{k=0}^{0} \binom{0}{k} a^k b^{0-k} = \binom{0}{0} a^0 b^0 = 1$$

Induction step $n \to n+1$

$$(a+b)^{n+1} = (a+b)^n \cdot (a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^n b^{n-k} \right) (a+b)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1}$$

$$= \underbrace{\sum_{n=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k}}_{\substack{\text{index shift} \\ h+1=j, h=0 \\ \Rightarrow j=1, h=j-1, h=n-1 \\ \Rightarrow j=n}} + \underbrace{\binom{n}{k} a^{n+1} \cdot b^0}_{a^{n-1}}$$

$$+ \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} + \binom{n}{0} a^0 b^{n+1}$$

$$\sum_{j=1}^{n} \binom{n}{j-1} a^j b^{n-(j-1)} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k}$$

$$+ \binom{n+1}{n+1} a^{n+1} + \binom{n+1}{0} b^{n+1}$$

Renaming j to k:

$$= \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k-1} + \binom{n}{k} \right]}_{\binom{n+1}{k} \text{ by recursive definition}} a^k b^{n+1-k}$$

$$+ \binom{n+1}{n+1} a^{n+1} b^0 + \binom{n+1}{0} a^0 b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$$

Therefore the binomial theorem holds for $n+1$.

□

This lecture took place on 29th of October 2015 with lecturer Ring Wolfgang.

$$\forall a, b \in \mathbb{R}, n \in \mathbb{N} : (a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k}$$

Induction base $n = 0, n = 1$ follows immediately

Induction step

$$(a+b)^n = \underbrace{(a+b)(a+b)(a+b)(a+b)\dots(a+b)}_{n \text{ times}}$$

When multiplying the products $a^n b^{n-k}$ are created $(0 \le k \le n)$. $a^n b^{n-k}$ are created iff a is the factor resulting from k parenthesis groups and b originates from the remaining $(n-k)$ groups. There are exactly $\binom{n}{k}$ possibilities to select from n groups. $a^k b^{n-k}$ occurs $\binom{n}{k}$ times. Therefore

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

This is a rather informal proof, but suffices at this point.

6 Arithmetics of numbers

We consider two fundamental arithmetic operators and determine fundamental properties.

Definition 13. *Let K be a set where two arithmetic operators are defined: Therefore $\forall a, b \in K$ let $a + b \in K$ and $a \cdot b \in K$.*

We require the following properties:

A1 $\forall a, b \in K : a + b = b + a$

A2 $\forall a, b, c \in K : (a + b) + c = a + (b + c)$

A3 $\exists 0 \in K \forall a \in K : a + 0 = a$

A4 $\forall a \in K \exists \tilde{a} : a + \tilde{a} = 0$

Then $(K, +)$ is a commutative group ("abelian group"). In general we denote $\tilde{a}$ as $-a$. We define $a - b = a + (-b)$ ("subtraction").

M1 $\forall a, b \in K : a \cdot b = b \cdot a$

M2 $\forall a, b, c \in K : a \cdot (b \cdot c) = (a \cdot b) \cdot c$

M3 $\exists 1 \in K : a \cdot 1 = a \forall a \in K$ *(neutral element)*

M4 $\forall a \in K \setminus \{0\} \exists \hat{a} : \hat{a} \cdot a = 1$

In general we denote $\hat{a}$ as a^{-1}.

We set $\frac{a}{b} = a \cdot b^{-1}$.

$$\frac{1}{b} = 1 \cdot b^{-1} \text{ for } b \neq 0$$

Definition 14 (Composition)**.** *Compatibility of + and $\cdot$:*

D $\forall a, b, c \in K : a \cdot (b + c) = a \cdot b + a \cdot c$

Under these conditions K is called a field.

Example 5. *Examples for fields: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.*

In every field it holds that

- *the inverse element of a is unique ($\tilde{a}$ is unique). Let $-a$ be the inverse element of a and $a + b = 0 \Rightarrow b = -a$*

  Proof. TODO
  $$(a + (-a)) + (b + 0) = a + b =$$
  □

- $0 \cdot a = 0$

  Proof.
  $$0 = 0 + 0$$
  follows from **D**.
  $$0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$$
  $$0 \cdot a + (-0 \cdot a) = 0 \cdot a + [0 \cdot a + (-0 \cdot a)]$$
  $$0 = 0 \cdot a$$
  □

- $-a = (-1) \cdot a$

  Proof.
  $$\begin{aligned} a + (-1) \cdot a = (1 + (-1))a &= 0 \\ a + (-1) \cdot a &= 0 \\ -a &= (-1) \cdot a \end{aligned}$$
  □

6.1 Integers and the field of rational numbers $\mathbb{Q}$

For $\mathbb{N}$, **A1**, **A2** and **A3**. If $n \geq m$, then also $n - m \in \mathbb{N}$. $n - m = k \in \mathbb{N}$ is defined in such a way that $n = m + k$.

Corollary 1. *Extension:*

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, \ldots\} = \mathbb{N}_+ \cup \{0\} \cup \{-n : n \leq \mathbb{N}_0\}$$

We define $-0 := 0$ and $\forall n \in \mathbb{N}_+$ let $n + (-n) := 0$.

Therefore for every $z \in \mathbb{Z}$ exists some $\tilde{z}$ such that $z + \tilde{z} = 0$.

- $z \in \mathbb{Z}_+ \Rightarrow \tilde{z} = -z$
- $z = 0 \Rightarrow \tilde{z} = 0$
- $z = -n$ *for* $n \in \mathbb{N}_+$
- $\tilde{z} = n$

$$\forall z \in \mathbb{Z} \exists \tilde{z} \in \mathbb{Z} : z + \tilde{z} = 0$$

In general we denote $\tilde{z} = (-z)$. Also $-(-z) = z$.

For $z, w \in \mathbb{Z}$:

$$z + w = \begin{cases} z + w & z, w \in \mathbb{N} \\ (-z) + (-w) & -z, -w \in \mathbb{N} \\ z - (-w) & z, -w \in \mathbb{N} \text{ and } z > (-w) \\ -((-w) - z) & z, -w \in \mathbb{N} \text{ and } (-w) > z \end{cases}$$

$$z \cdot w = \begin{cases} z \cdot w & z, w \in \mathbb{N} \\ (-z)(-w) & -z, -w \in \mathbb{N} \\ -((-z) \cdot w) & -z \in \mathbb{N}, w \in \mathbb{N} \end{cases}$$

*In $\mathbb{Z}$ the properties **A1**, **A2**, **A3**, **A4**, **M1**, **M2**, **M3** and **D** hold.*

Definition 15.

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

where $\frac{m}{n} = \frac{m'}{n'} \Leftrightarrow m \cdot n' = n \cdot m'$. $\mathbb{Q}$ is called the set of rational numbers.

We define

$$\frac{m}{n} + \frac{k}{l} := \frac{ml + nk}{nl}$$
$$\frac{m}{n} \cdot \frac{k}{l} = \frac{mk}{nl}$$

Show that

$$\frac{m}{n} = \frac{m'}{n'} \text{ and } \frac{k}{l} = \frac{k'}{l'}$$
$$\Rightarrow \frac{ml + nk}{nl} = \frac{m'l' + n'k'}{n'l'}$$
$$\Rightarrow (ml + nk)(n'l') = (m'l' + n'k')$$
$$\Leftrightarrow mn' \cdot ll' + nn' \cdot kl = m'n \cdot ll' + nn' \cdot k'l$$

Analogously for $\frac{m}{n} \cdot \frac{k}{l}$.

***A1**–**A4**, **M1**–**M4** and **D** hold for $\mathbb{Q}$.*

For $z \in \mathbb{Z}$ we set $z = \frac{z}{1}$. Therefore it holds that $\mathbb{Z} \subseteq \mathbb{Q}$. $0 = \frac{0}{1}$ and $\frac{m}{n} + 0 = \frac{m}{n} + \frac{0}{1} = \frac{m \cdot 1 + n \cdot 0}{n \cdot 1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$. 0 is neutral in regards of addition in $\mathbb{Q}$.

Inverse element in regards of addition:

$$\frac{m}{n} + \frac{-m}{n} = \frac{mn + (-m)n}{n^2} = \frac{(m + (-m))n}{n \cdot n} = \frac{0n}{n^2} = \frac{0}{1}$$

because $0 \cdot 1 = 0 \cdot n^2$.

Concerning multiplication:

$$1 = \frac{1}{1} \qquad \frac{m}{n} \cdot \frac{1}{1} = \frac{m \cdot 1}{n \cdot 1} = \frac{m}{n}$$

1 is a neutral element in regards of multiplication in $\mathbb{Q}$.

Let $\frac{m}{n} \in \mathbb{Q} \setminus \{0\} \Rightarrow m \neq 0 \Rightarrow \frac{n}{m} \in \mathbb{Q}$ and $\frac{m}{n}\frac{n}{m} = \frac{mn}{mn} = \frac{1}{1}$. TODO: verify because $m \cdot n \cdot 1 = 1 \cdot m \cdot n$.

Corollary 2.

$$\forall \frac{m}{n} \in \mathbb{Q} : -\frac{m}{n} = \frac{-m}{n}$$

$$\forall \frac{m}{n} \in \mathbb{Q} \setminus \{0\} : \left(\frac{m}{n}\right)^{-1} = \left(\frac{n}{m}\right)$$

Therefore $\mathbb{Q}$ is a field.

This lecture took place on 30th of October 2015 with lecturer Ring Wolfgang.

Literature:

- Eblinghaus et al., "Zahlen", Springer Verlag
- E. Landau: "Grundlagen der Analysis", uses Peano axioms to build calculus

6.2 Ordered fields

Definition 16. *Let K be a field. We assume that K is taken from two sets: $K = K_+ \cup \{0\} \cup K_-$ with $0 \notin K_+, 0 \notin K_-$. It holds that*

- *$\forall a \in K$ it holds that either $a \in K_+$ or $a = 0$ or $a \in K_+$*
  $a \in K_+ \Leftrightarrow -a \in K_-$
- *$\forall a, b \in K_+$: $a + b \in K \wedge a \cdot b \in K$*

If those properties are satisfied, such a field is called an ordered field. *Instead of $a \in K_+$ we write $a > 0$ (namely "positive numbers") and $a < 0$ for $a \in K_-$ correspondingly (namely "negative numbers").*

For arbitrary $a, b \in K$ we define

$$a > b \Leftrightarrow a - b > 0$$

It holds that $a > b \Leftrightarrow b < a$.

$$a \geq b \Leftrightarrow a > b \vee a = b$$

Lemma 1. *Let K be an ordered field. Then it holds that*

1. *$a \in K_+ \wedge b \in K_- \Rightarrow a \cdot b \in K_-$*
   $a \in K_- \wedge b \in K_- \Rightarrow a \cdot b \in K_+$
2. *$\forall a, b \in K$ one of the following relations hold:*

   $$a > b \vee a = b \vee a < b$$

   Therefore $<$ defines a total order on K.
3. *$\forall a, b, c \in K : [(a < b) \wedge (b < c) \implies a < c]$*
   Therefore $<$ is transitive.
4. *If $a > b > 0$ then $\frac{1}{a} < \frac{1}{b}$ If $a > 0$ holds, then also $a^{-1} = \frac{1}{a} > 0$.*
5. *$\forall a, b, c \in K : a < b \implies a + c < b + c$*
6. *$\forall a, b \in K : \forall c > 0 : [a > b \implies ac > bc]$*
   $\forall a, b \in K : \forall c < 0 : [a > b \implies ac < bc]$
7. *$\forall a \in K \setminus \{0\} : a^2 = a \cdot a > 0$*

Proof. 1. We know from the practicals: $\forall a, b \in K : (-a)(-b) = ab$

$$(-a)b = -(ab)$$

Let $a \in K_+, b \in K_-$, therefore $a \in K_+$, $(-b) \in K_-$, then it holds that $ab = (-a)(-b) = -(a(-b)) \in K_-$. Let $a \in K_-$ and $b \in K_-$ therefore $(-0) \in K_+ \wedge (-b) \in K_+ \implies ab = (-a)(-b) \in K_+$.

2. Let $a, b \in K$. Then one of the following properties hold:

$$a - b > 0 \vee a - b = 0 \vee a - b < 0$$

Equivalently,

$$a > b \vee a = b \vee a < b$$

3. Let $a > b$ and $b > c$. Therefore $a - b > 0$ and $b - c > 0$.

$$\Rightarrow (a - b) + (b - c) > 0$$

$$a(-b + b) - c > 0$$

$$a - c > 0 \Leftrightarrow a > c$$

4. Let $a > 0 \Rightarrow a^{-1} \neq 0$. Assume $\frac{1}{a} = a^{-1} < 0 \Rightarrow a^{-1} \cdot a = 1 < 0$. Otherwise it holds that $1 = 1 \cdot 1 = 1^2 > 0$`

5. Let $a > b > 0$. Then it holds that

$$a^{-1}b^{-1}(b-a) = a^{-1}b^{-1}b - a^{-1}b^{-1}a = -a^{-1} \cdot b^{-1} = \frac{1}{a} \cdot \frac{1}{b} \Rightarrow a^{-1} < b^{-1}$$

6. $a < b$ therefore $a - b < 0 \Rightarrow a + c - c - b < 0 \Rightarrow (a+c) - (b+c) < 0$

$$\Leftrightarrow a + c < b + c$$

7. Let $a > b, c > 0 \Rightarrow (a-b) > 0 \Rightarrow (a-b) \cdot c > 0 \Rightarrow ac - bc > 0 \Rightarrow ac > bc$. For the second statement, it holds analogously: $a < b, c < 0 \Rightarrow (a-b) < 0 \Rightarrow (a-b) \cdot c < 0 \Rightarrow ac - bc < 0 \Rightarrow ac < bc$

8. $a > 0 \Rightarrow a \cdot a > 0$. Let $a < 0 \Rightarrow (-a) > 0$. It holds $a \cdot a = (-a)(-a) > 0$. Therefore the square of two numbers is always positive.

□

6.3 Remarks about some common fields

Remark 2. *$\mathbb{C}$ is not an ordered ordered field. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are ordered.*

Remark 3. *Let $q \in \mathbb{Q}$.*

a) Let $m, n \in \mathbb{N}_+$ such that $q = \frac{m}{n}$ then $q > 0$.

b) Let $m, n \in \mathbb{N}_+$ such that $q = -\frac{m}{n}$ then $q < 0$.

We show that $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$. Every $q \in \mathbb{Q}$ has a representation of either a) or b), but not both. $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$.

$$q \neq 0 \Rightarrow q = \begin{cases} \frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ -\frac{m}{n} & m, n \in \mathbb{N}_+ \\ \frac{-m}{-n} & m, n \in \mathbb{N}_+ \end{cases}$$

$$q = \frac{n}{-m} = \frac{-n}{m}$$

because $nm = (-n)(-m)$.

$$q = \frac{-m}{-n} = \frac{m}{n}$$

because $(-m) \cdot n = m \cdot (-n)$.

Remark 4. *We want to show that $\mathbb{Q}_+ \cap \mathbb{Q}_- = \emptyset$. Let $q \in \mathbb{Q}_+ \cap \mathbb{Q}_-$.*

$$q = \frac{m}{n} = -\frac{m'}{n'} \qquad m, n, m', n' \in \mathbb{N}_+$$

$$\Rightarrow n \cdot n' = (-m')n$$

$$\Rightarrow \underbrace{\underbrace{mn'}_{\in \mathbb{N}_+} + \underbrace{m'n}_{\in \mathbb{N}_+}}_{\in \mathbb{N}_+} = 0 \qquad ↯$$

Furthermore $p \in \mathbb{Q}_+ \land q \in \mathbb{Q}_+$

$$\Rightarrow p + q \in \mathbb{Q}_+ \land pq \in \mathbb{Q}_+$$

$$\Rightarrow p = \frac{k}{l} \qquad q = \frac{m}{n} \qquad k, l, m, n \in \mathbb{N}_+$$

$$p + q = \frac{\overbrace{kn + ml}^{\in \mathbb{N}_+}}{nm} \in \mathbb{Q}_+$$

$$pq = \frac{k}{l} \cdot \frac{m}{n} = \frac{\overbrace{km}^{\in \mathbb{N}_+}}{\underbrace{ln}_{\in \mathbb{N}_+}} \in \mathbb{Q}_+$$

Definition 17. *Let K be an ordered field $a \in K$. The absolute value of a is defined as*

$$|a| = \begin{cases} a & \text{if } a \in K_+ \\ 0 & \text{if } a = 0 \\ -a & \text{if } a \in K_- \end{cases}$$

Remark 5. *Let K be an ordered field. Then it holds that*

$$\mathbb{Q} \subseteq K \subseteq \mathbb{R}$$

except for isomorphism.

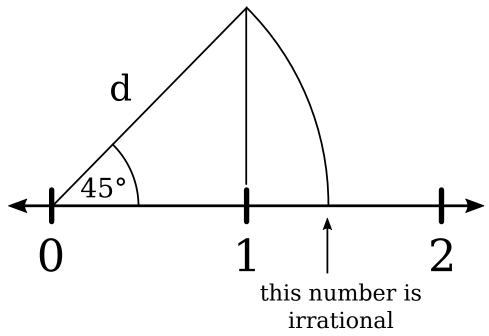

Figure 3: Illustration of an irrational number

6.4 Triangle inequality

Theorem 7.

$$\forall a, b \in K : |a+b| \leq |a| + |b| \qquad \text{“Triangle inequality”}$$

Proof. **Case 1**

$$a \cdot b > 0 \Rightarrow a \cdot b > 0 : |ab| = ab \qquad |a| \cdot |b| = ab$$

Case 2

$$a > 0, b < 0 : a \cdot b < 0 : |ab| = -ab \qquad |a| \cdot |b| = a \cdot (-b)$$

$$b < 0 \Rightarrow -b > 0 \Rightarrow b < -b \Rightarrow \underbrace{a+b}_{|a+b|} < \underbrace{a-b}_{|a|+|b|}$$

Case 3

$$a < 0, b < 0 : a \cdot b > 0 : |ab| = ab \qquad |a| = -a \qquad |b| = -b$$

$$|a| \cdot |b| = -a \cdot -b = ab$$

Case 4

$$a > 0, b < 0 : a + b < 0$$

$$|a| = a \qquad |b| = b \qquad |a+b| = -(a+b) = -a-b$$

$$a > 0 \Rightarrow -a < 0 \qquad -a-b < a-b$$

$$-(a+b) = |a+b|$$

□

This lecture took place on 4th of November 2015 with lecturer Wolfgang Ring.

6.5 Laws for absolute values

Theorem 8. *Let $y \geq 0$. Then it holds that $|x| \leq y \Leftrightarrow -y \leq x \wedge x \leq y$*

Proof. First direction $\Rightarrow$:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

Case 1 Let $x \geq 0$. Then

$$|x| \leq y \Rightarrow x \leq y \Rightarrow -y \leq x$$

because $-y \leq 0 \wedge x \geq 0$ anyways.

Case 2 Let $x < 0$, therefore $|x| = -x$. Because

$$-x \leq y \Rightarrow x \geq -y$$

$x \leq y$ holds anyways because $x < 0$ and $y \geq 0$.

Second direction $\Leftarrow$:

Let $-y \leq x \leq y$.

Case 1 $x \geq 0 : |x| = x \leq y$ because of the second inequality.

Case 2 $x < 0 : |x| = -x$

$$-(-1) \Rightarrow -(-y) \geq -x \text{ or equivalently } y \geq -x = |x|$$

□

Theorem 9.

$$|x| = 0 \Leftrightarrow x = 0$$
$$\forall a \in K : |a| = |-a|$$
$$\forall \varepsilon > 0 : |x - y| \leq \varepsilon \Rightarrow x = y$$

Proof. **First direction** $\Rightarrow$ Without loss of generality: $x \geq y$.

$$x \neq y \Rightarrow \exists \varepsilon > 0 : |x - y| > \varepsilon$$

Let $x \neq y$. Because $x \geq y$ holds, so does $x > y$. Therefore $x - y > 0$. We define $\varepsilon = \frac{x-y}{2} < x - y$

$$2 = 1 + 1 > 1$$
$$2^{-1} = \frac{1}{2} < 1 = 1^{-1}$$

Therefore it holds that $\varepsilon : |x - y| = x - y > \frac{1}{2}(x - y) = \varepsilon > 0$.

Second direction $\Leftarrow$ $x = y \Rightarrow |x - y| = 0 \leq \varepsilon \forall \varepsilon > 0$

□

Theorem 10 (Inversed triangle inequality)**.** *Let $a, b \in K$. Then it holds that*

$$||a| - |b|| \leq |a - b|$$

Proof. Show that $-|a - b| \leq |a| - |b| \leq |a - b|$.

First inequality

$$|b| = |b - a + a| \leq |b - a| + |a| \Rightarrow -|a - b| \leq |a| - |b|$$

Second inequality

$$|a| = |a - b + b| \leq |a - b| + |b| \Rightarrow |a| - |b| \leq |a - b|$$

□

6.6 Intervals

This lecture took place on 5th of November 2015 with lecturer Wolfgang Ring.

Definition 18 (Intervals)**.** *Let $a, b \in K$.*

$$(a, b) = \{x \in K \mid (x > a) \wedge (x < b)\}$$
$$[a, b) = \{x \in K \mid (x \geq a) \wedge (x < b)\}$$
$$(a, b] = \{x \in K \mid (x > a) \wedge (x \leq b)\}$$
$$[a, b] = \{x \in K \mid (x \geq a) \wedge (x \leq b)\}$$

Theorem 11 (Laws for intervals)**.**

$$(a, b) = \emptyset \text{ if } b \leq a \tag{21}$$
$$[a, b] = \emptyset \text{ if } b < a \tag{22}$$
$$[a, a] = \{a\} \tag{23}$$

If I is an non-empty interval (hence $I \neq \emptyset$), then $|I| = b - a$ is called length of the interval. *Furthermore*

$$(a, \infty) = \{x \in K \mid x > a\} \tag{24}$$
$$[a, \infty) = \{x \in K \mid x \geq a\} \tag{25}$$
$$(-\infty, a) = \{x \in K \mid x < a\} \tag{26}$$
$$(-\infty, a] = \{x \in K \mid x \leq a\} \tag{27}$$

Theorem 12. *$\mathbb{Q}$ is arithmetically incomplete.*

Proof. We define a mapping from $\mathbb{N}_+$ to $\mathbb{N}$: Let $n \in \mathbb{N}_+$ then we know that n can be represented distinctly as product of prime numbers. Let $\mathrm{Z}(n)$ be the number of twos in the prime product representation.

Examples:

$$\mathrm{Z}(14) = \mathrm{Z}(2 \cdot 7) = 1$$
$$\mathrm{Z}(15) = \mathrm{Z}(3 \cdot 5) = 0$$
$$\mathrm{Z}(24) = \mathrm{Z}(2 \cdot 2 \cdot 2 \cdot 3) = 3$$

It holds that $\mathrm{Z}(2n) = \mathrm{Z}(n) + 1 \forall n \in \mathbb{N}_+$ and $\mathrm{Z}(n^2) = \mathrm{Z}(n) \cdot 2 \forall n \in \mathbb{N}_+$.

We claim,

$$\nexists q : q = \frac{m}{n} \text{ with } q^2 = 2$$

Proof by contradiction:

1. Assume $\left(\frac{m}{n}\right)^2 = 2$.
2. Then $\frac{m^2}{n^2} = 2$.
3. Then $m^2 = 2 \cdot n^2$.
4. With $Z(m^2) = 2 \cdot \mathrm{Z}(n)$.
5. With $\mathrm{Z}(2 \cdot n^2) = \mathrm{Z}(n^2) + 1 = 2 \cdot \mathrm{Z}(n) + 1$.
6. If $m^2 = 2n^2$, then $\mathrm{Z}(m^2)$ must be even and $\mathrm{Z}(2 \cdot n^2)$ must be odd.
7. Then equality cannot be satisfied ↯

□

6.7 Archimedean property and Completeness axiom

Theorem 13. *$\mathbb{Q}$ is geometrically incomplete.*

We consider an infinite straight number line. We define $\mathbb{R}$ as ordered field with properties:

Archimedean property $\mathbb{N} \subseteq \mathbb{R}$ *with* $\forall x \in \mathbb{R} : \exists n \in \mathbb{N} : x < n$

$$\mathbb{N} \subseteq \mathbb{R} \forall n \in \mathbb{N} : -n \in \mathbb{N}$$
$$\Rightarrow \forall n \in \mathbb{N}_+ : n^{-1} \in \mathbb{R}$$
$$\Rightarrow \mathbb{Z} \subseteq \mathbb{R}$$

Therefore $\forall m \in \mathbb{N} : m \cdot \frac{1}{n} = \frac{m}{n} \in \mathbb{R} \Rightarrow \mathbb{Q} \subseteq \mathbb{R}$.

Definition 19. *Let $I_0, I_1, \ldots, I_z$. $(I_n)_{n\in\mathbb{N}}$ is a sequence of closed intervals with*

1. $\forall a \in \mathbb{N} : I_{n+1} \subseteq I_n$

2. $\forall \varepsilon > 0 \exists n \in \mathbb{N} : n \geq N \Rightarrow |I_n| < \varepsilon$

Completeness axiom *Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals in $\mathbb{R}$. Then there exists some $x \in \mathbb{R} : x \in I_n : \forall n \in \mathbb{N}_+$.*

Be aware, there exists only one *$x \in \mathbb{R}$ with the property: $x \in I_n \forall n \in \mathbb{N}$.*

Assume $x \in I_n$ and $y \in I_n \forall n \in \mathbb{N}$ and $x \neq y$.

$$|\beta - \alpha| \leq b - a = |I|$$

Proof. Without loss of generality: $\alpha \leq \beta$. Then it holds that $|\beta - \alpha| = \beta - \alpha \leq \beta + (-\alpha) \leq b + (-\alpha) = b - a = |I|$.

$$a \leq \alpha \Rightarrow -a \geq -\alpha$$

Consider arbitrary small $\varepsilon > 0$ and $N \in \mathbb{N}$ sufficiently large, such that $|I_n| < \varepsilon$. Because $x, y \in I_n \Rightarrow |x - y| < \varepsilon \Rightarrow x = y$. □

Corollary 3. *From the Archimedean property it follows that,*

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} < \varepsilon$$

Proof. Let $x > \frac{1}{\varepsilon} \in \mathbb{R}$. Archimedean property: $\exists N \in \mathbb{N} : N > x$.

For $n \geq \mathbb{N}$ it holds that $n > x > 0 \Rightarrow \frac{1}{n} < \frac{1}{x} = \varepsilon$. □

Corollary 4. *Let $p \in \mathbb{R}, p > 1 \forall x \in \mathbb{R} : n \geq N \Rightarrow p^n > x$.*

Proof. $p > 1 + u$ with $u = p - 1$

$$p^n = (1+u)^n \underbrace{>}_{\text{Bernoulli}} 1 - nu = 1 + n(p-1)$$

Let $x \in \mathbb{R}$ arbitrary, select $N \in \mathbb{N} : \frac{x-1}{p-1} < N$.

Then it holds for $n \geq N$:

$$\frac{x-1}{\underbrace{p-1}_{>0}} \Leftrightarrow x - 1 < n \cdot (p-1) \Leftrightarrow x < 1 + n(p-1) < p^n$$

□

Theorem 14. *Let $q \in \mathbb{R}$ with $|q| < 1$. Then it holds that*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |q^n| = |q|^n < \varepsilon$$

Proof. Let $s = |q| \geq 0$. Consider $q > 0$. Then

$$q^n = 0$$
$$|q^n| = 0$$
$$|q|^n < \varepsilon \forall \varepsilon > 0 \forall n \in \mathbb{N}$$

Let $q \neq 0$, then $0 < s < 1$. Let $p = \frac{1}{s} \Rightarrow p > 1$. Choose arbitrary $\varepsilon > 0$ and $x = \frac{1}{\varepsilon}$. Because of the Completeness axiom

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow p^n > X$$

So it holds that

$$\frac{1}{p^n} = S^n < \frac{1}{x} = \varepsilon \forall n \geq N$$
$$\Rightarrow (|q|)^n = |q^n|$$

□

Theorem 15. *Let $x \in \mathbb{R}, x > 0$ and let $k \in \mathbb{N}_+$. Then there exists a distinct $y \in \mathbb{R}$ with $y \geq 0$ such that*

$$y^k = x$$

We denote $y = \sqrt[k]{x}$ and conclude there exists k-th root numbers.

Proof. Idea: Construct nested intervals.

$(I_n)_{n\in\mathbb{N}}$ such that $y \in \bigcap_{n\in\mathbb{N}} I_n$ satisfies the property that $y^k = x$.

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k$$

We define $J_0 = [a_0, b_0]$ with $a_0 = 0$ and $b_0 = 1 + x$. Then it holds that

$$a_0^k = 0^k = 0 \leq x$$

$$b_0^k = (1+x)^k = 1 + k_n + \binom{k}{2} x^2 + \cdots + x^k \geq 1 + kx > 0$$

□

This lecture took place on 6th of November 2015 with lecturer Wolfgang Ring.

Theorem 16. *We prove:*

$$0 \leq y_1 < y_2 \Rightarrow y_1^k \leq y_2^k$$

Proof. A short proof by a student:

k = 2

$$y^{k+1} = y^k \cdot y < y_2^k x < y_2^k y_2 = y^{k+1}$$

k → k + 1

$$y_1^2 < y_2^2$$

□

Theorem 17. *Let $a, b \in K$ and $k \in \mathbb{N}$. Then it holds that*

$$a^k - b^k = (a-b)\left(\sum_{j=0}^{k-1} a^{k-1-j} b^j\right)$$

$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Proof.

$$(a-b)\left(\sum_{j=0}^{k-1} a^{k-j-1} b^j\right) = \sum_{j=0}^{j-1} a^{k-j} b^j - \sum_{j=0}^{k-1} a^{k-j-1} b^{j+1}$$
$$= a^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \underbrace{b^{k-1}}_{j=k-1} - \sum_{j=0}^{k-2} a^{k-j-1} b^{j+1}$$
$$= a^k - b^k + \sum_{j=1}^{k-1} a^{k-j} b^j - \sum_{l=1}^{k-1} a^{k-l} b^l$$
$$= a^k$$

□

Theorem 18. *Let $y_2 > y_1$ then*

$$y_2^k - y_1^k = \underbrace{(y_2 - y_1)}_{>0} \underbrace{\left(\sum_{j=0}^{k-1} y_2^{k-j-1} y_1^j\right)}_{>0}$$

$$\Rightarrow y_2^k - y_1^k > 0$$

Proof.

$$\forall x \geq 0 \in \mathbb{R} : \exists y \geq 0 \in \mathbb{R} : y^k = x \text{ with } k \in \mathbb{N}_+$$

Special case $x = 0$ and $y = 0$ is the solution.

Let $x > 0$: We construct y with $y \in \bigcap_{k=0}^{\infty} I_n$ where I_n are nested intervals. Specifically I_n must have the properties:

- $I_n = [a_1, b_n]$ with $a^k \leq x, b_n^k \geq x \quad \forall n \in \mathbb{N}$
- $I_{n+1} \subseteq I_n : |I_n| = \frac{1}{2}|I_{n+1}| = \left(\frac{1}{2}\right)^n |I_0|$

$$n = 0 \qquad I_0 = [0, x-1]$$
$$a_0 = b \qquad b_0 = x+1$$
$$a_0^k = 0 < x \qquad \checkmark$$
$$b_0^k = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \cdots + x^k > 1 + kx > x \text{ for } k \geq 1$$

Let I_n be given: $I_n = [a_n, b_n]$. Define $m_n = \frac{1}{2}(a_n + b_n)$

Case 1

$$m_n^k \geq x \Rightarrow \text{ let } a_{n+1} = a_n, b_{n+1} = m$$
$$I_{n+1} = [a_n, m_n] \subseteq [a_n, b_n] = I_n$$
$$|I_{n+1}| = m_n - a_n = \frac{1}{2}a_n + \frac{1}{2}b_n - a_n$$
$$\frac{1}{2}(b_n - a_n) = \frac{1}{2}|I_n|$$
$$a_{n+1}^k = a^k \leq x \quad \checkmark$$

All conditions are satisfied.

Case 2 $m_n^k < x$: Let $a_{n+1} = m_1, b_{n+1} = b_n$. It holds that $a_{n+1} = m_n < x, b_{n+1} = b_n \geq x \quad \checkmark$. Furthermore it holds that $I_{n+1} \subseteq I$ and $|I_{n+1}| = \frac{1}{2}|I_n|$.

I_n is set of nested intervals. Let $\varepsilon > 0$ be arbitrary. Then

$$\exists N \in \mathbb{N} : n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{1+x}$$

For those $n \geq N$ it holds that

$$|I_n| = \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^n (x+1) < \frac{\varepsilon}{1+x} \cdot (1+x)$$

Let $y \in I_n \forall n \in \mathbb{N}$. Further nesting of intervals:

$$(I_n)_{n \in \mathbb{N}} \text{ with } I_n = [a_n^k, b_n^k]$$

It holds that

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n \text{ because } I_{n+1} \subseteq I_n \Rightarrow a_n^b \leq a_{n+1}^k < b_{n+1}^k \leq b_n^k$$

Length of I_n:

$$I_n = b_n^k - a_n^k = (b_n - a_n) \sum_{j=0}^{k-1} a_n^{k-1-j} b_n^j$$

Because $I_n \leq I_0 \Rightarrow a_n < b_0 \Rightarrow b_n \leq b_0$,

$$< (b_n - b_0) \sum_{j=0}^{k-1} b_0^{k-1-j} b_0^j$$

$$= (b_n - a_n) k b_0^k = (b_n - a_n) k (1+x)^k$$

Let $\varepsilon > 0$ be arbitrary. Find some $N \in \mathbb{N}$ with $n \geq N$:

$$|I_n| = (b_n - a_n) < \frac{\varepsilon}{k(1+x)^k}$$

For those n it holds that

$$|I_n| < |I_n| \cdot k(1-x)^k < \frac{\varepsilon}{k(1+x)^k} k(1+x)^k = \varepsilon$$

Therefore $(I_n)_{n\in\mathbb{N}}$ a set of nested intervals.

$\exists z \in \mathbb{R}$ with $z \in \lfloor a_n^k, b_n^k \rfloor : \forall n \in \mathbb{N}$ and z is unique. By construction of I_n it holds that $a_n^k \leq x \leq b_n^k$

$$\Rightarrow x \in I_n \forall n \in \mathbb{N} \Rightarrow x = z \in \bigcap_{n\in\mathbb{N}} I_n.$$

On the opposite side it holds that $y \in I_n$ (hence $a_n \leq y \leq b_n \Rightarrow a_n^k \leq y^k \leq b_n^k$). So $y^k \in I_n \forall n \in \mathbb{N} \Rightarrow y^k = z = x$. So we have found some y^k which is x. But is $y \geq 0$ with $y^k = x$ unique?

Let $y_1 \neq y_2$ with $y_1^k = y_2^k = x$ and without loss of generality,

$$0 \leq y_1 < y_2 \Rightarrow y_1^k < y_2^k \quad \grave{}$$

So, y is unique.

□

7 Supremum property of $\mathbb{R}$

7.1 Boundedness in $\mathbb{R}$

Definition 20. *Let $A \subseteq \mathbb{R}$.*

- *We call A to be* bounded above *if there exists some $u \in \mathbb{R}$ such that $\forall a \in A : a \leq u$.*
- *A number u with that property is called* upper bound of *A.*
- *We call A to be* bounded below *if there exists some $l \in \mathbb{R}$ such that $\forall a \in A : a \geq l$.*
- *A number l with that property is called* lower bound of *A.*
- *A is called* bounded *if there exists a lower and upper bound of A.*

Corollary 5. *Let (a, b) be bounded. Let u be its upper bound and let $v \geq u$. Then v is also an upper bound of (a, b).*

This lecture took place on 11th of November 2015 with lecturer Wolfgang Ring.

7.2 Supremum and infimum in $\mathbb{R}$

Definition 21. *Let A be bounded above. Assume $s \in \mathbb{R}$ has the properties*

1. *s is an upper bound for A*
2. *$\forall \sigma \in \mathbb{R} : \sigma < S$: σ is not an upper bound for A.*

If those properties are satisfied, we call s supremum *of A. A supremum s is always the smallest upper bound of A. We denote $s = \sup A$.*

There exists at most one supremum for A. Let s_1 and s_2 be two suprema, then $s_1 \neq s_2$. So wlog. $\sigma_1 < \sigma_2$. This invalidates the supremum property of $s_2 \Rightarrow s_1$ is not a supremum of A ↯.

Analogously an infimum *of A is the greatest lower bound of A. Let A be bounded below. $t \in \mathbb{R}$ is called* infimum *of A if*

1. *$\forall a \in A : t \leq a$ (t is a lower bound of A)*
2. *$\forall x > t$ so x is no lower bound of A*

$$\Leftrightarrow \exists a \in A : a < x$$

We denote $t = \inf A$.

Definition 22. *Let $A \subseteq \mathbb{R}$. We denote $u = \max A$ for the* maximum *of A if*

1. *$u \in A$ (is element of A)*
2. *$\forall a \in A : a \leq u$ (is an upper bound)*

$l \in \mathbb{R}$ denoted $l = \min A$ is called minimum of A if

1. *$l \in A$ (is element of A)*
2. *$\forall a \in A : l \leq a$ (l is a lower bound)*

Theorem 19. *Let $A \subseteq R$ and u be the maximum of A. Then it holds that $u = \sup A$. If $l = \min A \Rightarrow l = \inf A$.*

Proof. We need to show, that l is an upper bound of A. This follows by definition. For $x < u$ it holds that x not an upper bound.

Let $x < u$, because $u \in A$ there exists some element y in A with $y > x$. Therefore x is not an upper bound of A. □

Example 6.

$$A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} = \left\{\frac{1}{n} : n \in \mathbb{N}_+\right\}$$

Then it holds that $1 \in A$ and $1 \geq \frac{1}{n} \forall n \in \mathbb{N}_+$. Therefore $1 = \max A = \sup A$.

$0 = \inf A$, because 0 is a lower bound of A ($\frac{1}{n} > 0 \forall n \in \mathbb{N}_+$). Let $\varepsilon > 0$, then $\exists N \in \mathbb{N} : n \geq N \Rightarrow \frac{1}{n} \leq \varepsilon$. Therefore ε is not a lower bound of A.

So A does not have a minimum, because otherwise $l = \max A = \inf A = 0$.

Theorem 20. *Let $A \neq \emptyset$ and $A \subseteq \mathbb{R}$ be bounded above. So some $s = \sup A \in \mathbb{R}$ exists (therefore $\mathbb{R}$ has a supremum property).*

Proof. We construct nested intervals $(I_n)n \in \mathbb{N}$ such that for $s \in \bigcap_{n\in\mathbb{N}} I_n$ gilt $s = \sup A$. We construct I_{n+1} inductively using I_n

Case $\mathbf{n = 0}$

Because $A \neq 0$, we select $a_0 \in A$. Because A is bounded above, $\exists b_0 \in \mathbb{R}$ such that b_0 is an upper bound of A. We define $I_0 = [a_0, b_0]$.

Case $\mathbf{n \to n+1}$

Let $a_0 = b_0$, then it holds that b_0 is upper bound and $b_0 \in A$. We call that terminating condition. Therefore $b_0 = \max A = \sup A$ and the supremum was found. Instead of n we use $n+1$. Let $I_0 = [a_n, b_n]$ with $a_n \neq b_n$ and $a_n \in A$, b_n is an upper bound of A. Furthermore it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider I_{n+1} such that the same properties are satisfied. Let $m_1 = \frac{1}{2}(a_1 + b_1)$. It holds that $a_n < m_n < b_n$.

Case $\mathbf{m_n}$ is an upper bound of A Then we set $a_{n+1} = a_n \in A$ and $b_{n+1} = m_n$ is an upper bound of A.

$$|I_{n+1}| = b_{n+1} + a_{n+1} = \frac{1}{2}(b_n + a_n) - a_n$$

$$= \frac{1}{2}b_1 - \frac{1}{2}a_n = \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| = \left(\frac{1}{2}\right)^{n+1} |I_n| \qquad \checkmark$$

Case $\mathbf{m_n}$ is not an upper bound of A Therefore $\exists x \in A$ with $x > m_n$.

Subcase $\mathbf{x = b_1}$ So b_1 is an upper bound. Therefore $x \in A$ and x is upper bound.

$$x = \max A = \sup A$$

We found the supremum.

Subcase $\mathbf{m_n < x < b_n}$ Let $a_{n+1} = x \in A$ and $b_{n+1} = b_n$ is an upper bound and

$$I_{n+1} = b_{n+1} - a_{n+1} - b_n - x < b_n - m_n - b_n - \frac{1}{2}(b_n + a_n) + \frac{1}{2}(b_n - a_n)$$

$$= \frac{1}{2}|I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

We have found supremum $s = \sup A$.

If in any case the terminating condition holds, then we have found the supremum.

The remaining case is $\forall n \in :a_n < b_n, a_n \in A, b_n$ is upper bound of A.

$$|I_n| = b_n - a_n \leq \left(\frac{1}{2}\right)^n |I_0|$$

Consider $\varepsilon > 0$ and N such that $n \geq N \Rightarrow \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{|I_n|}$. For those n it holds that

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0| < \frac{\varepsilon}{|I_0|} |I_0| = \varepsilon$$

Therefore $(I_n)_{n\in\mathbb{N}}$ are nested intervals.

□

What remains for completeness: $s \in \mathbb{R}, s \in I_n : \forall n \in N$. We need to show that $s = \sup A$.

This lecture took place on 12th of November 2015 with lecturer Wolfgang Ring.

Theorem 21. *Completeness of* $\mathbb{R}$*:*

$$\exists s \in \mathbb{R} : s \in I_n \forall n \in \mathbb{N}$$

Proof *cont.* Every set with an upper bound has a supremum.

We construct $(I_n)_{n \in \mathbb{N}}$ with $I_n = [a_n, b_n]$ and $I_{n+1} \subseteq I_n$. $\forall n \in \mathbb{N} : a_n \in A$, b_n is the upper bound of A.

$$|I_{n+1}| \leq \frac{1}{2} |I_n| \leq \left(\frac{1}{2}\right)^{n+1} |I_0|$$

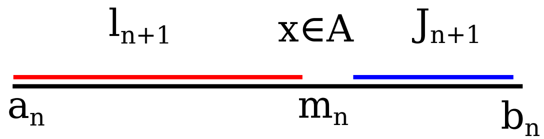

Figure 4: Relation of a_n and b_n and J_{n+1}

Consider $I_{n+1} \subseteq I_n$ with $a_n < b_n \forall n \in \mathbb{N}$.

$$|I_n| \leq \left(\frac{1}{2}\right)^n |I_0|$$

1. Claim: s is $\sup A$.

   We need to show (by contradiction): S is upper bound of A. Assume $a \in A$ and $a > s$. Let $\varepsilon = a - s > 0$ and choose N sufficiently large such that

   $$|I_n| < \varepsilon = a - s$$

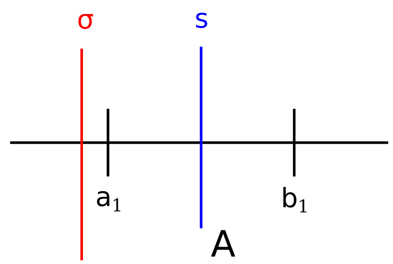

Figure 5: Illustration of s between a_n and b_n

   Then it holds that

   $$b_N = \underbrace{b_n - a_n}_{\varepsilon} \cancel{+} \underbrace{a_N}_{<s} < s + \varepsilon = a$$

   $$\Rightarrow b_N < a \in A$$

   Because b_n is an upper bound.

2. $\forall \sigma < s$ it holds that σ is not an upper bound of A. Let $\sigma < s$ and $\varepsilon = s - \sigma > 0$ and choose $n \in N$ large enough such that $b_N - a_N < \varepsilon$. Then it holds that

   $$\begin{aligned} a_N &= a_N - b_N + b_N \\ &> -\varepsilon + s \\ &= -s + \sigma + s = \sigma \qquad \checkmark \end{aligned}$$

   Therefore it holds that s is smallest upper bound of A and therefore supremum.

□

Theorem 22. *Every set with a lower bound in* $\mathbb{R}$ *has an infimum. Every set with an upper bound in* $\mathbb{R}$ *has an supremum.*

Theorem 23. *Remember that M has the same cardinality like A if $\varphi : M \to A$. φ is bijective, M is called countably infinite if M has the same cardinality like $\mathbb{N}$.*

Let $\varphi : \mathbb{N} \to M$ be bijective therefore $M = \{\varphi(1), \varphi(2), \varphi(3), \ldots\} = \{\varphi(n) \,|\, n \in \mathbb{N}\}$ and $\varphi(i) \neq \varphi(j)$ for $i \neq j$.

Notation. *$\varphi(n) = m_n$.*

$M = \{m_0, m_1, m_2, \ldots\}$ with $m_i \neq m_j$ for $i \neq j$. φ is a complete enumeration of all elements of M.

Therefore every element of M has the structure: m_n with $i \in \mathbb{N}$.

Theorem 24.

$$\mathbb{Q}^+ = \left\{\frac{m}{n}, m \in \mathbb{N}, n \in \mathbb{N}_+\right\}$$

The set $\mathbb{Q}^+$ is countably infinite.

Proof. We enumerate the elements of $\mathbb{Q}^+$.

$$\mathbb{Q}_+ = \{q_0, q_1, q_2, \ldots\}$$

$$\mathbb{Q}_- = \{-q_0, -q_1, -q_2, \ldots\}$$

$$\mathbb{Q} = \{0, q_0, -q_0, q_1, -q_1, \ldots\}$$

An enumeration exists. So $\mathbb{Q}$ is countably infinite. □

Theorem 25. *There is no bijective relation $\varphi : \mathbb{N} \to \mathbb{R}$. Therefore we call $\mathbb{R}$* uncountable.

Proof. We provide a proof by contradiction. Assume $\mathbb{R} = \{x_0, x_1, x_2, x_3, \ldots\}$ is countable.

We construct nested intervals.

Case $\mathbf{n = 0}$

$$I_0 = [x_0 + 1, x_0 + 2]$$

Let $|I_0| = 1$ and $x_0 \notin I_0$.

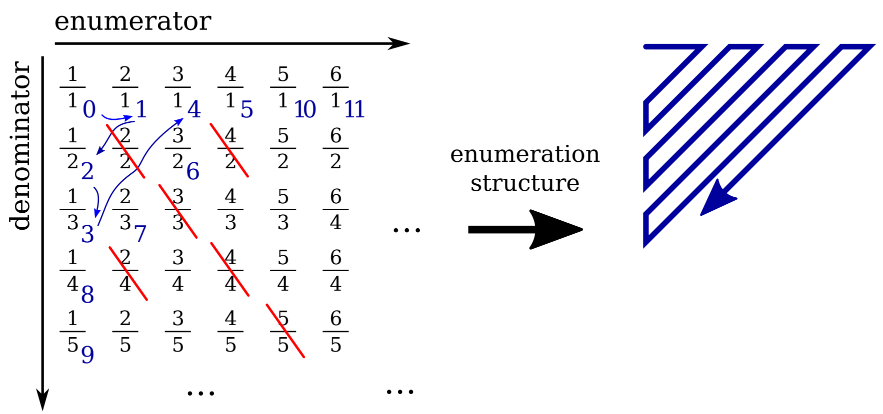

Figure 6: A complete enumeration of $\mathbb{Q}^+$ (diagonalization argument). We traverse the whole matrix diagonally. The blue numbers indicate the enumeration and red lines cross out values already enumerated. On the right-hand side the general order of the enumeration is illustrated.

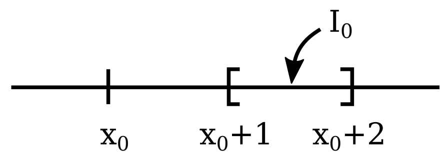

Figure 7: Construction of a nested interval and its I_0

Case $\mathbf{n \to n+1}$ Assume $I_0 \ldots I_n$ were already defined with $x_k \notin I_k$ for $0 \leq k \leq n$.

$$I_{k+1} \leq I_k \text{ for } k = 0, \ldots, n-1$$

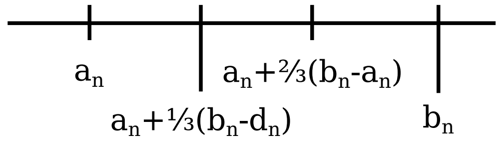

Figure 8: Construction of a nested interval and its I_n

$$|I_k| = \left(\frac{1}{3}\right)^k$$

We construct I_{n+1}. Let $I_n = [a_n, b_n]$.

$$I_n^1 = \left[a_n, \frac{2}{3}a_n + \frac{1}{3}b_n\right]$$
$$I_n^2 = \left[\frac{2}{3}a_n + \frac{1}{3}b_n, \frac{1}{3}a_n + \frac{2}{3}b_n\right]$$
$$I_n^3 = \left[\frac{1}{3}a_n + \frac{2}{3}b_n, b_n\right]$$

So x_n certainly is not contained in all three intervals I_n^1, I_n^2 and I_n^3 because $I_n^1 \cap I_N^2 \cap I_N^3 = \emptyset$. Choose I_{n+1} as one of the three intervals I_n^l with $x_{n+1} \notin I_n^l = I_{n+1}$. $I_{n+1} < I_n$.

$$|I_{n+1}| = \frac{1}{3}I_n = \left(\frac{1}{3}\right)^{n+1}$$

For $\varepsilon > 0$ it holds that there exists some $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |I_1| = \left(\frac{1}{3}\right)^n < \varepsilon$. Therefore nested intervals I_n are given.

Let $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} : X \in I_n$ (because of completeness law). Then it holds that $\forall x_n : x \neq x_n$. $x \in I_n$ and $x_n \notin I_n$. Therefore $x \in \{x_0, x_1, x_2, \ldots\} = \mathbb{R}$.

This contradicts with the assumption that $\mathbb{R}$ is countable.

□

8 Complex numbers $\mathbb{C}$

We introduce a new arithmetic unit denoted i, which extends the field $\mathbb{R}$. Elements of $\mathbb{C}$ are represented as $a + bi$ with $a, b \in \mathbb{R}$.

$$\forall a, b \in \mathbb{R} : a + bi = 0 \Leftrightarrow a = 0 \wedge b = 0 \quad (28)$$
$$i^2 = -1 \quad (29)$$
$$\text{associativity, commutativity etc holds} \quad (30)$$

This lecture took place on 13th of November 2015 with lecturer Wolfgang Ring.

Definition 23. *We consider an "arithmetic element" i extending $\mathbb{R}$ ("adjungiert"). Arithmetic operations are well-defined for i. Associativity and commutativity holds. It holds that*

- $a + ib = 0$ *with* $a, b \in \mathbb{R} \Leftrightarrow a = 0 \wedge b = 0$
- $i^2 = -1$ *i.e.* $i^2 + 1 = 0$.
- *Arithmetic operations still hold.*

By the first law,

$a + ib = a' + ib' \Leftrightarrow (a - a') + i(b - b') = 0 \Leftrightarrow a - a' = 0 \wedge b - b' = 0$ *therefore* $a = a' \wedge b = b'$

By the second law, i is the solution of the quadratic equation $i^2 + 1 = 0$.

Let $z = a + ib$ a complex number. We call i the "imaginary unit".

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}\}$$

$\mathbb{C}$ is the field of complex numbers with the following properties:

- *For addition, it holds that*

$$(a + ib) + (c + id) = (a + b) + i(b + d) \subseteq \mathbb{C}$$

and

$$(a + ib) + (-a - ib) = (a - a) + i(b - b) = 0 + i \cdot 0 = 0$$

- *For multiplication, it holds that*

$$(a+ib)\cdot(c+id) = (ac + \underbrace{(i)^2}_{=-1} bd) + i(bc+ad)$$

$$(ac-bd) + i(bc+ad)$$

- *Laws* $\mathbf{A_n}$ *to* $\mathbf{A_4}$, $\mathbf{M_1}$ *to* $\mathbf{M_3}$ *and* $\mathbf{D}$ *hold.*
- *The one element exists:*

$$1 = 1 + 0\cdot i$$

$$(a+i\cdot b)(1+i\cdot 0) = (a+(i)^2\cdot 0) + i(b+0) = a+ib$$

- ***M4*** *holds: Let* $z \in \mathbb{C}\backslash\{0\}$. *Let* $z = a+ib$ *and* $\neg(a=0 \wedge b=0) \Leftrightarrow a^2+b^2>0$.

We define

$$w = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

$$z\cdot w = (a+ib)\left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right)$$

$$= \left(\underbrace{\frac{a^2}{a^2+b^2} - \frac{b\cdot(-b)}{a^2+b^2}}_{=1}\right) + i\cdot\left(\underbrace{\frac{ba}{a^2+b^2} - \frac{a\cdot b}{a^2+b^2}}_{=0}\right)$$

$$= 1 + i\cdot 0 = 1$$

Therefore $w = z^{-1} = \frac{1}{z}$.

Therefore $\mathbb{C}$ *is a field.*

We denote

$$a = \Re(z)$$

$$b = \Im(z)$$

$$\overline{z} = a - ib$$

$$|z| = \sqrt{a^2+b^2}$$

a is called *real part of* z. b is called *imaginary part of* z. z is called complex conjugate. $|z|$ is called absolute value of z.

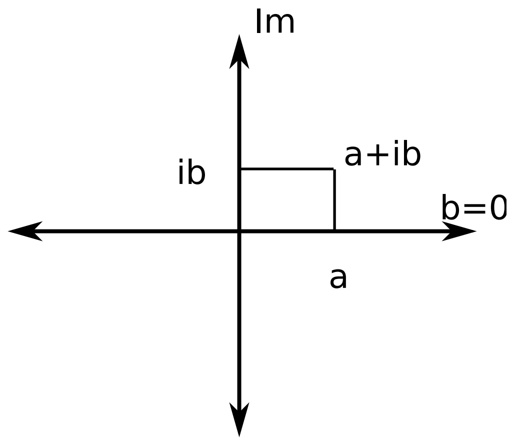

Figure 9: Illustration of complex numbers

Theorem 26.

$$\overline{(\overline{z})} = z$$

Proof.

$$\overline{(\overline{z})} = \overline{(a-ib)} = (a-(-ib)) = a+ib = z$$

□

Theorem 27.

$$\Re(z) = \frac{1}{2}(z+\overline{z})$$

Theorem 28.

$$\frac{1}{2}(z+\overline{z}) = \frac{1}{2}(a+ib+a-ib) = \frac{1}{2}(2a) = a\checkmark$$

Theorem 29.

$$\Im(z) = \frac{1}{2i}(z-\overline{z})$$

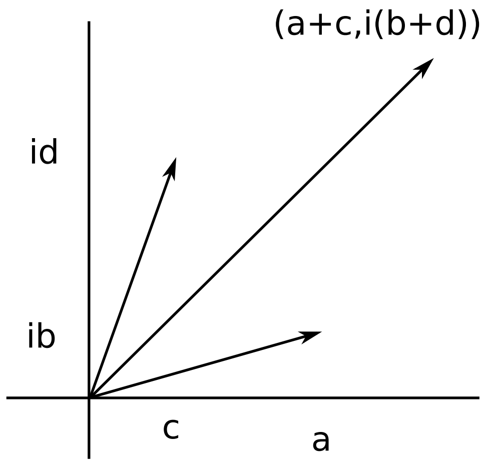

Figure 10: Illustration of complex number addition

Proof.

$$\frac{1}{2i}(a + ib - (a - ib)) = \frac{1}{2i}(2ib) = b\checkmark$$

□

Theorem 30.

$$z \in \mathbb{R} \Leftrightarrow z = \overline{z}$$

Proof.

$$z = a \in \mathbb{R} \Rightarrow \overline{z} = a = z$$

On the opposite, let $z = \overline{z}$ therefore

$$a = ib = a - ib \Rightarrow 2ib = 0 \Rightarrow b = 0$$

Therefore $z = a \in \mathbb{R}$. □

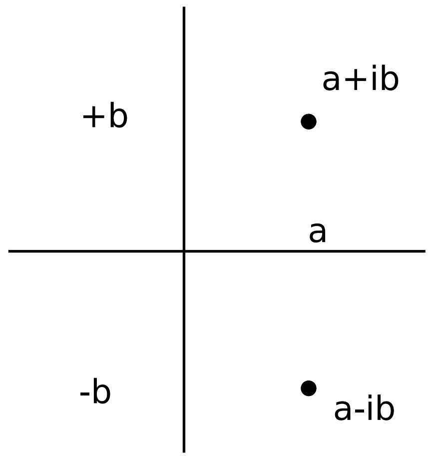

Figure 11: Illustration of the complex conjugate

Theorem 31.

$$z \in i\mathbb{R} = \{ib : b \in \mathbb{R}\} \Leftrightarrow z = -\overline{z}$$

Proof follows analogously.

Theorem 32. *It holds that* $|z| = \sqrt{z \cdot \overline{z}}$.

Proof.

$$\begin{aligned}\sqrt{z \cdot \overline{z}} &= ((a + ib)(a - ib))^{\frac{1}{2}} \\ &= (a^2 - (ib)^2)^{\frac{1}{2}} = (a^2 - i^2b^2)^{\frac{1}{2}} \\ &= (a^2 + b^2)^{\frac{1}{2}} = |z| \quad \checkmark\end{aligned}$$

□

Theorem 33. *Let* $z, w \in \mathbb{C}$*:*

$$\overline{(zw)} = \overline{z} \cdot \overline{w}$$

Proof.

$$z = a + ib \qquad w = c + id$$
$$zw = (ac - bd) + i(bc + ad)$$
$$\overline{zw} = (ac - bd) - i(bc + ad)$$
$$\overline{zw} = a - ib \qquad \overline{w} = c - id$$
$$\overline{z} \cdot \overline{w} = (ac - (-b)(-d)) + i(-bc + a(-d)) = (ac - bd) - i(bc + ad)$$

□

Corollary 6.

$$\overline{z + w} = \overline{z} + \overline{w}$$

Theorem 34.

$$|zw| = |z| \cdot |w|$$

Proof.

$$|z \cdot w| = (zw) \cdot (\overline{z \cdot w})^{\frac{1}{2}}$$
$$= (z \cdot \overline{z} \cdot w \cdot \overline{w})^{\frac{1}{2}} = (z \cdot \overline{z})^{\frac{1}{2}} \cdot (w \cdot \overline{w})^{\frac{1}{2}} = |z| \cdot |w|$$

□

Theorem 35.

$$z = 0 \Leftrightarrow |z| = 0 \in \mathbb{R}$$

Proof.

$$z = 0 = 0 + i0 \Rightarrow |z| = \sqrt{0^2 + 0^2} = 0$$

Let $|z| = \sqrt{a^2 + b^2} = 0 \Rightarrow a^2 + b^2 = 0$.

$$\Rightarrow a = 0 \land b = 0$$

□

Theorem 36.

$$|\Re(z)| = |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$
$$|\Im(z)| = |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2} = |z| =$$

Theorem 37. *The triangle inequality holds:*

$$\forall z, w \in \mathbb{C} : |z + w| \leq |z| + |w|$$

Remark 6. *Let $0 \leq y_1 < y_2$ with $y_1, y_2 \in \mathbb{R}$. Let $k \in \mathbb{N}_+$. Then it holds that*

$$\sqrt[k]{y_1} < \sqrt[k]{y_2}$$

Proof. Indirect proof: Let $\sqrt[k]{y_1} \geq \sqrt[k]{y_2} \geq 0$.

$$\Rightarrow (\sqrt[k]{y_1})^k \geq (\sqrt[k]{y_2})^k$$

therefore $y_1 \geq y_2$. This is the negation of our assumption. □

***Proof** of the triangle inequality.* We show that $|z + w|^2 \leq (|z| + |w|)^2$.

$$|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = \underbrace{z\overline{z}}_{|z|^2} + w\overline{z} + z\overline{w} + \underbrace{w\overline{w}}_{|w|^2}$$

$$\begin{aligned}
&= 2\Re(w\overline{z}) \\
&= (w\overline{z} + \underbrace{\overline{(w \cdot \overline{z})}}_{\overline{w} \cdot \overline{\overline{z}} = \overline{w} \cdot z} \\
&= |z|^2 + 2\Re(w \cdot \overline{z}) + |w|^2 \\
&\leq |z|^2 + 2\,|\Re(w \cdot \overline{z})| + |w|^2 \\
&\leq |z|^2 + 2 \cdot |w \cdot \overline{z}| + |w|^2 \\
&= |z|^2 + 2 \cdot |w| \cdot |\overline{z}| + |z|^2 \\
&= |z|^2 + 2 \cdot |w| \cdot |z| + |w|^2 \\
&= (|z| + |w|)^2
\end{aligned}$$

□

Theorem 38. *In our previous proof there was a small loop hole: We need to show that*

$$|z| = |\overline{z}|$$

Proof.

$$\sqrt{a^2 + b^2} = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$$

□

8.1 Interpretation of multiplication

Multiplication with i. Let $z = a + ib$.

$$iz = i \cdot a + i^2 \cdot b = (-b) + ia$$

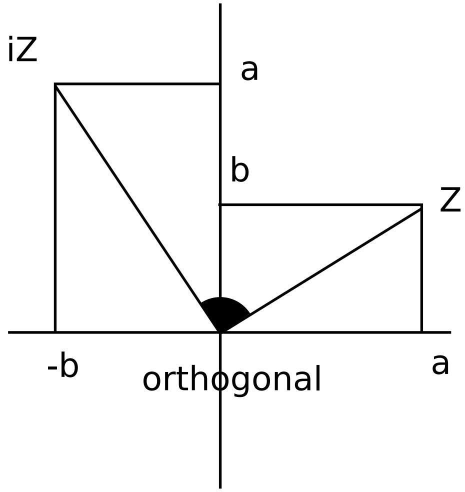

Figure 12: Multiplication corresponds to a rotation by 90°

Multiplication with i rotates z counter-clockwise by 90° in the plane.

Let $z \in \mathbb{C}$ and $w = c + id$.

This lecture took place on 18th of November 2015 with lecturer Wolfgang Ring.

8.2 Taking roots

$$\forall a \in \mathbb{R} : a \geq 0 \forall n \in \mathbb{N}_+ : \exists x \geq 0 \in \mathbb{R} : x^n = a$$

Taking the n-th root only works for positive integers, because $\forall x \geq 0 : x^2 \geq 0$ and no solution in $\mathbb{R}$ exists for the equation $x^2 = -1$.

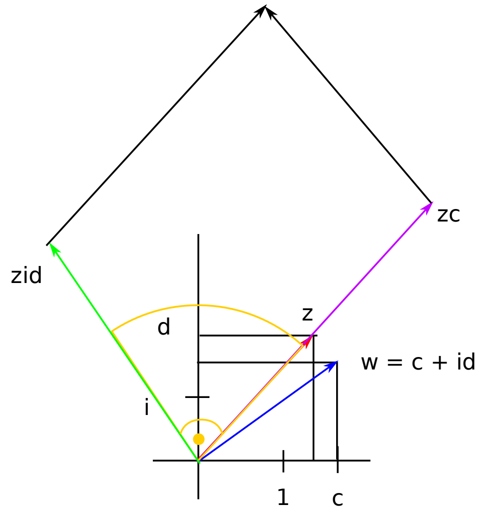

Figure 13: In regards of multiplication with w the complex number z is scaled by $|w|$ and then rotated by an angle which is given between w and the positive real axis.

In $\mathbb{C}$ it holds that $\forall w \in \mathbb{C} \setminus \{0\}$. $\forall n \in \mathbb{N}$ there exist exactly n different solutions of the equation$z^n = w$.

9 Sequences of real and complex numbers

Definition 24. *Let a be a mapping $\mathbb{N} \to \mathbb{R}$ is called* sequence *of real numbers.*

$$\forall n \in \mathbb{N} : a(b) \in \mathbb{R}$$

We denote $a_n := a(n)$. Instead of $a : \mathbb{N} \to \mathbb{C}$ we write $(a_n)_{n \in \mathbb{N}} = (a_0, a_1, \ldots)$. Analogously for the complex numbers $\mathbb{C}$ and general sets X.

Example 7. *$a_n = \sqrt[n]{2}\frac{1}{n+1}$ with $(a_n)_{n\in\mathbb{N}}$. Or simply:*

$$\left(\sqrt[n]{2}\frac{1}{n+1}\right)_{n\in\mathbb{N}}$$

Example 8. *Let $(I_n)_{n\in\mathbb{N}}$ be nested intervals. Therefore $(I_n)_{n\in\mathbb{N}}$ is a sequence of elements in $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$.*

Definition 25. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. $(a_n)_{n\in\mathbb{N}}$ is called* bounded above *if $o \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \leq o$. $(a_n)_{n\in\mathbb{N}}$ is called* bounded below *if $u \in \mathbb{R}$ exists such that $\forall n \in \mathbb{N} : a_n \geq u$.*

$(a_n)_{n\in\mathbb{N}}$ is called bounded, if $(a_n)_{n\in\mathbb{N}}$ is bounded above and below.

Example 9. *$(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{n}{n+1}$ is bounded below by 0 and bounded above by 1: $n \leq n+1 \Rightarrow n\frac{1}{n+1} < \frac{n+1}{n+1} = 1\checkmark$.*

9.1 Monotonicity

Definition 26.

- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically increasing *if $\forall n \in \mathbb{N} : a_{n+1} \geq a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically decreasing *if $\forall n \in \mathbb{N} : a_{n+1} \leq a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically strictly increasing *if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.*
- *$(a_n)_{n\in\mathbb{N}}$ is called* monotonically strictly decreasing *if $\forall n \in \mathbb{N} : a_{n+1} > a_n$.*

In $\mathbb{C}$, elements are not ordered, hence no complex sequences can be given. Let $(a_n)_{n\in\mathbb{N}}$ a complex sequence. We define:

- *$(a_n)_{n\in\mathbb{N}}$ is called bounded if $(|a_n|)_{n\in\mathbb{N}}$ is a bounded real sequence. Hence $\exists o \in \mathbb{R} : \forall n \in \mathbb{N} : |a_n| \leq o$.*
- *The lower bound is implicitly given by 0.*

Example 10. *$a_n \coloneqq i^n$ and $(a_n)_{n\in\mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1, i, -1, \dots)$*

$$|1| = 1 \qquad |-1| = 1 \qquad |i| = \sqrt{0^2 + 1^2} = 1 \qquad |-i| = \sqrt{0^2 + (-1)^2} = 1$$

So $(|a_n|)_{n\in\mathbb{N}} = (1,1,1,1,1,\dots)$. It holds that

$$|z| = |-z| = |\overline{z}|$$

Definition 27. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{C}$ and let $a \in \mathbb{C}$. We state: $(a_n)_{n\in\mathbb{N}}$ has a limit (lat. limes) a if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$$

We denote

$$\lim_{n\to\infty} a_n = a$$

The distance $|a_n - a|$ becomes arbitrary small, if n is sufficiently large.

A sequence, which has a limit, is called convergent*. A sequence, which does not have a limit, is called* divergent*.*

Remark 7. *Sometimes we consider mappings $a : \mathbb{N}_+ \to \mathbb{C}$, which we also call sequences:*

$$a \leftrightarrow (a_1, a_2, \dots)$$

Example 11.

$$a_n = \frac{1}{n}$$

We know:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \to \frac{1}{n} < \varepsilon$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Let $q \in \mathbb{C}$, $|q| < 1$.

We know $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \to |q^n - 0| < \varepsilon$.

$$\lim_{n\to\infty} q^n = 0$$

This lecture took place on 19th of November 2015 with lecturer Wolfgang Ring.

Remark 8. *Consider* $\forall\varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \implies |a_n - a| < \varepsilon]$ *as a circle with radius* ε*. So if* n *is sufficiently large, all new sequence numbers are located inside the circle.*

Lemma 2. *A sequence* $(a_n)_{n\in\mathbb{N}}$ *with* $a_n \in \mathbb{C}$ *can have at most one limit.*

Proof. Assume a and b are limes of $(a_n)_{n\in\mathbb{N}}$. Then we prove:

$$\forall\varepsilon > 0 : |a - b| < \varepsilon$$

$$\Rightarrow a = b$$

Let $\varepsilon > 0$ arbitrary: Because $a = \lim_{n\to\infty} a_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\right]$$

Because $b = \lim_{n\to\infty} b_n$ there exists

$$N_1 \in \mathbb{N} : \left[n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}\right]$$

Let $N = \max(N_1, N_2)$, hence $N \geq N_1 \wedge N \geq N \geq N_2$.

$$\Rightarrow |a_N - a| < \frac{\varepsilon}{2} \wedge |a_N - b| < \frac{\varepsilon}{2}$$

$$|a - b| = |a \underbrace{-a_N + a_N}_{0} - b| \leq \underbrace{|a - a_N|}_{<\frac{\varepsilon}{2}} + \underbrace{|a_N - b|}_{<\frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

□

Theorem 39 (Well-known convergent sequences.)**.**

1. *Let* $s = \frac{p}{q} \in \mathbb{Q}_+$ *and* $n \in \mathbb{N}_+$*. Consider* $\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$.

$$n^s = n^{\frac{p}{q}} := \sqrt[q]{n^q}$$

It holds that

$$\lim_{n\to\infty} \frac{1}{n^s} = 0$$

2. *Let* $q \in \mathbb{C}, |q| < 1$*. Then it holds that*

$$\lim_{n\to\infty} q^n = 0$$

3. *Let* $a \in \mathbb{R}, a > 0, n \in \mathbb{N}_+$*. Then it holds that*

$$\lim_{n\to\infty} \sqrt[n]{a} = 0$$

4. *It holds that* $(n \in \mathbb{N}_+)$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

5. *Let* $z \in \mathbb{C} : |z| > 1$*. Let* $k \in \mathbb{N}$*. Then it holds that*

$$\lim_{n\to\infty} \frac{n^k}{z^n} = 0$$

Remark 9 (Remark to sequence 5)**.** $|z^n|$ *grows faster then* n^k*.*

Proof of sequence 1. Let $0 \leq x_n < x_2$.

$$\Rightarrow 0 \leq x_1^p < x_2^p \Rightarrow \sqrt[q]{x_1^p} < \sqrt[q]{x_2^p}$$

Therefore $f(x) = x^s$ is strongly monotonic rising for $x \in (0, \infty)$. Let $\varepsilon > 0$ arbitrary and $N > \frac{1}{\varepsilon^{\frac{1}{s}}} = \varepsilon^{\frac{1}{s}} = \varepsilon^{-\frac{q}{p}}$. Then it holds that $n \geq N$:

$$\left|\frac{1}{n^s} - 0\right| = \frac{1}{n^s} \leq \frac{1}{N^s}$$

$$\frac{1}{n^s} < \frac{1}{N^s} \implies n^s \geq N^s$$

$$\frac{1}{n^s} \leq \frac{1}{N^s} < \frac{1}{\left(\frac{1}{\varepsilon^{\frac{1}{s}}}\right)^s} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

□

Proof of sequence 2. Already done. □

Proof of sequence 3. **Case $\mathbf{a > 1}$** Let $a > 1$. Consider $\varepsilon > 0$. Show that $|\sqrt[n]{a} - 1| < \varepsilon$ for sufficiently large n.

$$x_n = \sqrt[n]{a} - 1 = \left|\sqrt[n]{a} - 1\right|$$

$$a > 1 \implies \sqrt[n]{a} > \sqrt[n]{1} = 1 \implies \sqrt[n]{a} - 1 > 0$$

It holds that $x_n + 1 = \sqrt[n]{q}$, i.e. $(x_n + 1)^n = a$.

$$a = (\underbrace{x_1}_{>0} + 1)^n \underbrace{>}_{\text{Bernoulli}} 1 + n \cdot x_n$$

$$\Rightarrow x_n < \frac{a-1}{n}$$

$$N > \frac{a-1}{\varepsilon} \xRightarrow{\text{for } x \geq N} \left|\sqrt[n]{a} - 1\right| = x_n$$

$$< \frac{a-1}{n} \leq \frac{a-1}{N} < \frac{a-1}{\frac{a-1}{2}} = \varepsilon$$

Case $\mathbf{a = 1}$

$$\sqrt[n]{a} = \sqrt[n]{1} = 1$$

$$\left(\sqrt[n]{a}\right)_{n \in \mathbb{N}} = (1, 1, 1, 1, \dots)$$

has the limit 1.

Case $\mathbf{0 < a < 1}$ Let $0 < a < 1 \Rightarrow 0 < \sqrt[n]{q} < \sqrt[n]{1} = 1$.

$$x_n = 1 - \sqrt[n]{a} > 0$$

Show that $\forall \varepsilon > 0 \exists N \in \mathbb{N} : [n \geq N \Rightarrow x_n < \varepsilon]$.

$$x_n = 1 - \sqrt[n]{a} = \sqrt[n]{a}\left(\frac{1}{\sqrt[n]{a}} - 1\right) = \sqrt[n]{a}\left(\sqrt[n]{\frac{1}{a}} - 1\right) < \left(\sqrt[n]{a'} - 1\right)$$

with $a' = \frac{1}{a} > 1$. From case $a > 1$ we already know

$$\exists N \in \mathbb{N} : \left[n \geq N \Rightarrow \left|\sqrt[n]{a'} - 1\right| = \sqrt[n]{a'} - 1 < \varepsilon\right]$$

$$\Rightarrow x_n < \varepsilon$$

□

Proof of sequence 4. This proof works similar to the proof of sequence 3.

$$x_n = \sqrt[n]{n} - 1 > 0 \text{ for } n \geq 2$$

Therefore $|x_n| = x_n$. Let $\varepsilon > 0$ be arbitrary.

$$x_n + 1 = \sqrt[n]{n} \quad \text{i.e.} \quad (x_n + 1)^n = n$$

$$n = (1 + x_n)^n = 1 + \underbrace{n x_n}_{>0} + \underbrace{\binom{n}{2} x_n^2}_{>0} + \underbrace{\binom{n}{3} x_n^3}_{>0} + \underbrace{\dots + x_n^n}_{>0} > 1 + \binom{n}{2} x_n^2$$

All expressions we remove are positive (but we don't remove all positive expressions).

$$x_n^2 < \frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2 \cdot 1}} = \frac{2}{n}$$

$$x_n < \sqrt{\frac{2}{n}}$$

Choose $N > \frac{2}{\varepsilon^2}$. Then it holds for $n \geq N$ that

$$x_n < \sqrt{\frac{2}{n}} < \sqrt{\frac{2}{N}} < \sqrt{\frac{2}{\frac{2}{\varepsilon^2}}} = \varepsilon$$

Consider $\sqrt{\frac{2}{n}} < \varepsilon$ hence $\frac{2}{n} < \varepsilon^2$ hence $n > \frac{2}{\varepsilon^2}$. □

Proof of sequence 5.

$$|z| > 1 \text{ thus } x = |z| - 1 > 0 \text{ it holds that } |z| = 1 + x$$

We show that for $\varepsilon > 0$ arbitrary, there exists $N \in \mathbb{N}$:

$$n \geq N \implies \left|\frac{n^k}{z^n} - 0\right| = \left|\frac{n^k}{z^n}\right| = \frac{n^k}{|z|^n} < \varepsilon$$

Let $\varepsilon > 0$ be given,

- For $n > 2k$ it holds that $n - k > n - \frac{n}{2} = \frac{n}{2}$.

$$|z|^n = (1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j > \underbrace{\binom{n}{k+1} x^{k+1}}_{j=k+1}$$

$$n > 2k \geq k+1$$

$$\underbrace{\binom{n}{k+1}}_{j=k+1} x^{k+1} = \frac{\overbrace{n}^{>\frac{n}{2}}\overbrace{(n-1)}^{>\frac{n}{2}}\overbrace{(n-2)}^{>\frac{n}{2}}\dots\overbrace{(n-k)}^{>\frac{n}{2}}}{(k+1)!} x^{k+1} > \frac{\frac{n^{k+1}}{2^{n+1}}}{(k+1)!} x^{n+1}$$

Therefore $|z|^n > \frac{n^{k+1}}{2^{k+1}(k+1)!} x^{k+1}$. So,

$$\frac{n^k}{|z|^n} < \frac{n^k \cdot 2^{k+1}(k+1)!}{n^{k+1} \cdot x^{k+1}} = \underbrace{\frac{2^{k+1}(k+1)!}{x^{n+1}}}_{\substack{= \text{constant} \wedge >0 \\ =: M}} \cdot \frac{1}{n} = M \cdot \frac{1}{n}$$

$$\frac{n^k}{|z|^n} < M \cdot \frac{1}{n} \text{ for } n > 2k$$

Consider N such that $N > \frac{M}{\varepsilon}$ and $N > 2k$. Then it holds that

$$\frac{n^k}{|z|^n} < M\frac{1}{n} \leq \frac{M}{N} < \frac{M}{\frac{M}{\varepsilon}} = \varepsilon$$

$\square$

Lemma 3. *Every convergent sequence is bounded (in $\mathbb{C}$).*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. This means especially e.g. $\varepsilon = 13$.

$$\exists N \in \mathbb{N} \text{ s.t. } [n \geq N \implies |a_n - a| < 13]$$

Consider $O > 0$ such that

$$O = \max\{|a_0|, |a_1|, |a_2|, \dots, |a_{N-1}|, |a| + 13\}$$

So $O \geq |a_n|$ for $n \in \{0, \dots, N\}$. Then for $0 \leq n < N$ it holds that $|a_n| < O$. ✓

For $n \geq N$ it holds that

$$|a_n| = |a_n - a + a| \leq \underbrace{|a_n - a|}_{<13} + |a| < \underbrace{13 + |a|}_{\leq O}$$

Therefore $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and followingly $(|a_n|)_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}$. $\square$

Theorem 40. *Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then the following laws hold:*

1. *$\lim_{n\to\infty}(a_n + b_n)$ is convergent with limes $a + b$*
2. *$\lim_{n\to\infty}(a_n \cdot b_n)$ is convergent with limes $a \cdot b$*
3. *$\lim_{n\to\infty} \frac{a_n}{b_n}$ is convergent with limes $\frac{a}{b}$ if $\forall n \in \mathbb{N} : b_n \neq 0 \wedge b \neq 0$.*

Proof. 1. Let $\varepsilon > 0$ arbitrary. Because $(a_n)_{n\in\mathbb{N}}$ is convergent,

$$\exists N_1 : \left[n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\right]$$

(b_n) is convergent hence

$$\exists N_2 : \left[n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}\right]$$

$N = \max\{N_1, N_2\}$, hence for $n \geq N$ both statements above hold. Let $n \geq N$, then the triangle inequality holds:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

2. $(a_n)_{n\in\mathbb{N}}$ is convergent and therefore also bounded. Therefore,

$$\exists m \geq 0 : \forall n \in \mathbb{N} : |a_n| \leq m$$

$(b_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_1 : n \geq N_1 :\Rightarrow |b_n - b| < \frac{\varepsilon}{2} \cdot \frac{1}{m+1}$$

$(a_n)_{n\in\mathbb{N}}$ is convergent, hence

$$\exists N_2 \leq N : n \geq N_2 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}\frac{1}{|b|+1}$$

$N = \max\{N_1, N_2\}$. For $n \geq N$ both relations above hold. Let $n \geq N$:

$$|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$$
$$\leq |a_n(b_n - b)| + |b(a_n - a)| = |a_n|\,|b_n - b| + |b|\,|a_n - a|$$
$$\leq m\frac{\varepsilon}{2}\frac{1}{m+1} + |b|\,\frac{\varepsilon}{2}\frac{1}{|b|+1} < \frac{\varepsilon}{2}\cdot 1 + \frac{\varepsilon}{2}\cdot 1 = \varepsilon$$

3. Left for the practicals.

□

9.2 Laws for convergent complex sequences

Theorem 41. *Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a, $(a_n \to a)$. Then it holds that*

- *$(\Re(a_n))_{n\in\mathbb{N}}$ is convergent.*
$$\lim_{n\to\infty}(\Re(a_n)) = \Re(a)$$
- *$(\Im(a_n))_{n\in\mathbb{N}}$ is convergent.*
$$\lim_{n\to\infty}(\Im(a_n)) = \Im(a)$$
- *$(|a_n|)_{n\in\mathbb{N}}$ is a convergent real sequence.*
$$\lim_{n\to\infty}|a_n| = |a|$$
- *$(\overline{a_n})_{n\in\mathbb{N}}$ is convergent with*
$$\lim_{n\to\infty}\overline{a_n} = \overline{a}$$

On the opposite, let $(a_n)_{n\in\mathbb{N}}$ with $a_n = \alpha_n + i\beta_n$ a sequence of complex numbers. Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be convergent with limes α i.e. β. Then $(a_n)_{n\in\mathbb{N}}$ is a convergent complex sequence with limes $a = \alpha + \beta i$.

Proof. Let $\varepsilon > 0$. Consider N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$.

$$\underbrace{|a_n - a|}_{(\alpha_n-\alpha)+(\beta_n-\beta)i} = \sqrt{(\alpha_n-\alpha)^2 + (\beta_n-\beta)^2}$$

TODO

Therefore $(\alpha_n) = (\Re(a_n))_{n\in\mathbb{N}}$ is convergent. $(\beta_n) = (\Im(a_n))_{n\in\mathbb{N}}$ is convergent.

Let $\varepsilon > 0$. Consider N such that $n \geq N \Rightarrow |a_n - a| < \varepsilon$.

$$||a_n| - |a|| \underbrace{\leq}_{\text{inverse triangular inequality}} |a_n - a| < \varepsilon \text{ for } n \geq N$$

Now we need to show $\alpha_n \to \alpha$ and $\beta_n \to \beta$

$$\Rightarrow a_n \to a$$

Let $\varepsilon > 0$ be arbitrary. Because $(\alpha_n)_{n\in\mathbb{N}}$ be convergent, there exists $N_1 \in \mathbb{N}$:

$$n \geq N_1 \Rightarrow |\alpha_1 - \alpha| < \frac{\varepsilon}{\sqrt{2}}$$

$(\beta_n)_{n\in\mathbb{N}}$ is convergent. So,

$$\exists N_2 \in \mathbb{N} : n \geq N_2$$
$$|\beta_n - \beta| < \frac{\varepsilon}{\sqrt{2}}$$

For $N = \max\{N_1, N_2\}$ and $n \geq N$ both relations hold.

Let $n \geq N$:

$$|a_n - a| = |(\alpha_n - \alpha) + i(\beta_n - \beta)|$$
$$= \sqrt{(\alpha_n-\alpha)^2 + (\beta_n-\beta)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \sqrt{\varepsilon^2} = \varepsilon$$

Let $a_n = \alpha_n + i\beta_n$ is convergent with limes $\alpha + i\beta$ which is a.

$$\Rightarrow \lim_{n\to\infty} \alpha_n = \alpha \wedge \lim_{n\to\infty} \beta_n = \beta$$

$$\Rightarrow \lim_{n\to\infty} (-\beta_n) = -\beta \qquad \text{"multiplication rule"}$$

$$\Rightarrow (\overline{a_n})_{n\in\mathbb{N}} = (\underbrace{\alpha_n}_{\text{convergent}} - \underbrace{i\beta_n}_{\text{convergent}})_{n\in\mathbb{N}}$$

$$\Rightarrow \lim_{n\to\infty} \overline{a_n} = \alpha - i\beta = \overline{a}$$

□

9.3 Further laws for complex sequences

Theorem 42. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{R}$ with limes a (i.e. b) and it must hold that $\forall n \in \mathbb{N} : a_n \leq b_n$. Then also $a \leq b$.*

Proof. Consider $a - b = \varepsilon > 0$.

$$\exists N_1 \in \mathbb{N} : n \geq N_1 \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$$

$$\exists N_2 \in \mathbb{N} : n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{2}$$

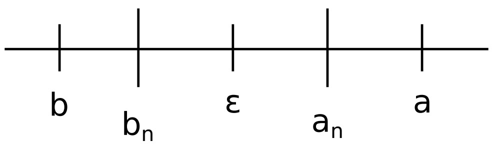

Figure 14: the sequences a_n, b_n and limes a, b and ε in relation

For $N = \max\{N_1, N_2\}$:

$$b_N = b_N - b + b \leq b + |b_N - b| < b\frac{\varepsilon}{2} = b + \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$a_N = \underbrace{a_N - a}_{\geq -|a_n - a|} + a \geq a - |a_n - a| > a - \frac{\varepsilon}{2} = a - \frac{a-b}{2} = \frac{1}{2}(a+b)$$

$$b_N < \frac{1}{2}(a+b) < d_N$$

Attention:

$$a_n < b_n \not\Rightarrow a < b$$

Example: $a_n = 0$, $b_n = \frac{1}{n}$. □

9.4 Convergence criteria

Are there criteria such that if they have a specific structure, they are obivously convergent?

9.4.1 Squeeze theorem

Theorem 43. *Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be convergent real sequences with $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = A$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $M \in \mathbb{N}$ such that*

$$\forall n \geq M : A_n \leq a_n \leq B_n$$

Then it holds that $(a_n)_{n\in\mathbb{N}}$ is also convergent and $\lim a_n = A$.

Proof. Let $\varepsilon > 0$ be arbitrary. Consider N such that,

- $M \geq M$
- $n \geq N \Rightarrow |A_n - A| < \varepsilon$
- $n \geq N \Rightarrow |B_n - A| < \varepsilon$

Then it holds that for $n \geq N$:

$$\left.\begin{array}{l} A - a_n \leq A - A_n \leq |A - A_n| < \varepsilon \\ a_n - A \leq B_n - A \leq |B_n - A| < \varepsilon \end{array}\right\} = 1$$

$$\Rightarrow |a_n - A| < \varepsilon$$

$$\lim_{n\to\infty} a_N = A$$

$\square$

Example 12. *Let $s \in \mathbb{Q}_+$. Then it holds that*

$$\lim_{n\to\infty} \left(\sqrt[n]{n^s}\right) = 1$$

We apply the squeeze theorem:

$$n^2 \geq 1 \forall n \in \mathbb{N}$$

$$\Rightarrow \sqrt[n]{n^s} \geq 1$$

Let $k \in \mathbb{N}_+$. Then it holds that

$$\lim_{n\to\infty} \sqrt[n]{n^k} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}\sqrt[n]{n}\ldots\sqrt[n]{n}}_{k \text{ times}}$$

$$= 1 \cdot 1 \cdot 1 \cdots = 1$$

For the last two lines we actually need to read them from right to left.

Let $s = \frac{p}{q}$.

$$\Rightarrow n^s = n^{\frac{p}{q}} \leq q \cdot \left(n\frac{p}{q}\right)^q = n^p$$

$$q \geq 1 \Rightarrow \sqrt[n]{n^s} \leq \underbrace{\sqrt[n]{n^p}}_{\text{convergent with limes } 1} \qquad p \in \mathbb{N}$$

Then it holds that $\lim_{n\to\infty} \sqrt[n]{n^s} = 1$ with the squeezing theorem.

Remark 10. *Let $A \subseteq \mathbb{R}$ be bounded above. Then it holds that*

$$S = \sup A \Leftrightarrow s \text{ is upper bound of } A \land \forall \varepsilon > 0 \exists a \in A : a > s - \varepsilon$$

Proof. Implication from left to right: Let $s = \sup A$. Then it holds that s is upper bound of A and $s - \varepsilon < s$ is not an upper bound. Therefore $\exists a \in A : a > s - \varepsilon$.

Implication from right to left: Consider that both statements on the RHS hold. So s is an upper bound. We need to show that any t is not an upper bound with $t > s$. Let $t < s, s - t = \varepsilon > 0$. Therefore $t = s - \varepsilon$. Because of the right statement $\exists a \in A : a > s - \varepsilon = t$ therefore t is not an upper bound. $\square$

Remark 11. *Analogously:*

$$\sigma = \inf A \Leftrightarrow \sigma \text{ is lower bound} \land \forall \varepsilon > 0 \exists a \in A : a < \sigma + \varepsilon$$

Theorem 44. *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded monotonic sequence. Then $(a_n)_{n\in\mathbb{N}}$ has a limes a with*

- *$a = \sup\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing.*
- *$a = \inf\{a_n : n \in \mathbb{N}\}$ if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be monotonically increasing. Let $a = \sup\{a_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ be arbitrary. Because a is a supremum, there exists $a_N \in \{a_n : n \in \mathbb{N}\}$ such that $a_N > a - \varepsilon$.

$$\Rightarrow \underbrace{a - a_N}_{\geq 0} < \varepsilon$$

because a is an upper bound. Therefore

$$|a - a_N| < \varepsilon$$

Let $n \geq N$ then it holds that

$$|a - a_n| \underbrace{=}_{a \text{ is upper bound}} a - a_n \leq a - a_N$$

because $a_N \leq a_n$ is increasing:

$$a - a_N < \varepsilon$$

Therefore $\lim_{n\to\infty} a_n = a$. $\square$

This lecture took place on 25th of November 2015 with lecturer Wolfgang Ring.

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. If $(a_n)_{n\in\mathbb{N}}$ is bounded and monotonous. Then $(a_n)_{n\in\mathbb{N}} \in \mathbb{N}$ is convergent.

Example: Wallis product $\boxed{\text{John Wallis (1616–1703)}}$

$$p_n = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)} = \prod_{k=1}^{n} \frac{2k}{2k-1}$$

Consider

$$\alpha_n = \frac{p_n}{\sqrt{n}} \qquad \beta_n = \frac{p_n}{\sqrt{n+1}}$$

We need to show that

- (α_n) is monotonously decreasing
- (β_n) is monotonously increasing

$$\forall n \in \mathbb{N} : n \geq 1 : \alpha_n > \beta_n$$

Both are convergent.

1. Show that,

$$\alpha_{n+1} < \alpha_n \Leftrightarrow \frac{\alpha_{n+1}}{\alpha_n} < 1 \Leftrightarrow \frac{(\alpha_{n+1})^2}{(\alpha_n)^2} < 1$$

$$\left(\frac{\alpha_{n+1}}{\alpha_n}\right)^2 = \left(\frac{\frac{2\cdot4\cdot6\cdot\ldots\cdot(2n+2)}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)\cdot(2n+1)}}{\frac{2\cdot4\cdot\ldots\cdot2n}{1\cdot3\cdot5\cdot\ldots\cdot(2n+1)}} \cdot \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n+1}}}\right)^2$$

$$= \frac{(2n+2)^2 \cdot n}{(2n+1)^2(n+1)} = \frac{4n^3+8n^2+4n}{(4n^2+4n+1)\cdot(n+1)} = \frac{4n^3+8n^2+4n}{4n^3+8n^2+5n+1} < 1$$

2. We show,

$$\left(\frac{\beta_{n+1}}{\beta_n}\right)^2 = \frac{(2n+2)^2 \cdot (n+1)}{(2n+1)^2 \cdot (n+2)} = \frac{(4n^2+8n+4)(n+1)}{(4n^2+2n+1)(n+2)}$$

$$= \frac{4n^3+12n^2+12n+4}{4n^3+12n^2+9n+2} > 1 \Rightarrow \beta_{n+1} > \beta_n \Rightarrow \beta_n \text{ is monotonically increasing}$$

Let $p = \lim_{n\to\infty} a_n$ and $p' = \lim_{n\to\infty} b_n$.

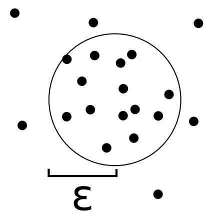

Figure 15: Illustration of a limit point in the Euclidean plane. The point is represented as circle with radius ε. Finitely many points lie outside the limit point; infinitely many inside.

$$\beta_n = \frac{p_n}{\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = \alpha_n \cdot \sqrt{\frac{n}{n+1}}$$

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \alpha_n \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \alpha_n \cdot \underbrace{\lim_{n\to\infty} \sqrt{\frac{n}{n+1}}}_{=1}$$

$$\Rightarrow \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} a_n \Rightarrow p = p'$$

It holds that $p = \lim_{n\to\infty} \frac{p_n}{\sqrt{n}} = \sqrt{n}$.

9.5 On limit points and subsequences

Definition 28. *Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence. The complex number a is called* limit point *(german "Häufungspunkt") of $(a)_{n\in\mathbb{N}}$ if $\forall \varepsilon > 0 : |a_n - a| < \varepsilon$ for infinitely many indices $n \in \mathbb{N}$. Hence infinitely many numbers of the sequence lie within a circle with center a and radius ε.*

Remark 12. *Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limit a. Then it holds that a is the only limit point of the sequence $(a_n)_{n\in\mathbb{N}}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent. Let

$$\varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a|$$

Therefore $\forall n \in \{N, N+1, N+2, \dots\}$ it holds that $|a_n - a| < \varepsilon$. Assume $a' \in \mathbb{C}$ is another limit point with $a \neq a'$. Let

$$\varepsilon = \frac{|a - a'|}{2} > 0$$

Let $N \in \mathbb{N}$ such that $\forall n \geq N : |a_n - a| < \varepsilon$.

$$\Rightarrow n \in \mathbb{N} : |a' - a_n| = |a' - a + a - a_n| = |a' - a - (a_n - a)| \geq |a' - a| - |a_n - a|$$

$$= 2\varepsilon - |a_n - a| > 2\varepsilon - \varepsilon = \varepsilon$$

At most for $n \in \{1, \dots, N-1\}$ it is possible that $|a_n - a'| < \varepsilon$. □

Remark 13. *$a_n = (-1)^n$ has the limit points $+1$ and -1.*

The lecture on 26th of November 2015 got cancelled.

This lecture took place on 27th of November 2015 with lecturer Wolfgang Ring.

Definition 29. *Let $a \in \mathbb{C}$ and $r > 0$ and*

$$B(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$$

and we call $B(a, r)$ an open *circle with center a and radius r. So the circle itself is not part of the set, unlike the following set:*

$$B'(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}$$

Let a be a limit point of $(a_n)_{n\in\mathbb{N}} \Leftrightarrow \forall \varepsilon > 0$. $B(a, \varepsilon)$ contains infinitely many sequence numbers.

Example 13.

$$a_n = \frac{1}{2}\left[1 + (-1)^n \left(\frac{1-n}{n}\right)\right] \qquad n \geq 1$$

$$\Rightarrow a_1 = \frac{1}{2} \quad a_2 = \frac{1}{4} \quad a_3 = \frac{5}{6}$$

$$a_4 = \frac{1}{8} \quad a_5 = \frac{9}{10} \quad a_6 = \frac{1}{12} \quad a_7 = \frac{13}{14}$$

"$\frac{5}{6}$? Ah, passt ma eh bessa." (Wolfgang Ring)

Estimated limit points: $a = 0, b = 1$.

Proof. Let $\varepsilon > 0$ and $a = 0$. We consider sequence numbers with even index. So for indices it holds that $n = 2k$.

$$\begin{aligned} |a_{2k} - 0| &= \left|\frac{1}{2}(1 + \underbrace{(-1)^{2k}}_{+1}\left(\frac{1-2k}{2k}\right))\right| \\ &= \frac{1}{2}\left|1 + \frac{1-2k}{2k}\right| \\ &= \frac{1}{2}\left|\frac{2k+1-2k}{2k}\right| \\ &= \frac{1}{4k} < \varepsilon \text{ if } \quad \underbrace{k > \frac{1}{4\varepsilon}}_{\substack{\text{infinitely many ks} \\ \text{satisfy the relation}}} \end{aligned}$$

Let $\varepsilon > 0$ and $b = 1$. We consider sequence numbers of structure $n = 2k + 1$.

$$\begin{aligned} |a_{2k+1} - 1| &= \left|\frac{1}{2}\left[1 + \underbrace{(-1)^{2k+1}}_{=-1}\left[\frac{1-(2k+1)}{2k+1}\right]\right] - 1\right| \\ &= \left|\frac{1}{2}\left[1 - \frac{-2k}{2k+1}\right] - 1\right| \\ &= \left|\frac{1}{2}\frac{2k+1+2k}{2k+1} - 1\right| \\ &= \left|\frac{4k+1}{4k+2} - 1\right| \\ &= \left|\frac{4k+1-4k-2}{4k+2}\right| \\ &= \frac{1}{4k+2} \\ &< \varepsilon \end{aligned}$$

$$\text{if } 4k + 2 > \frac{1}{\varepsilon} \Rightarrow \underbrace{k}_{\substack{\text{infinitely many} \\ \text{indices}}} > \frac{1}{4}\left(\frac{1}{\varepsilon} - 2\right)$$

Example 14. *$(c_n)_{n\in\mathbb{N}}$ is defined with $c_n = i^n$.*

$$(c_n)_{n\in\mathbb{N}} = (1, i, -1, -i, 1, i, -1, -i, 1 \dots)$$

What are its limit points?

Definition 30. *Let $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \mathbb{C}$. For example,*

$$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$$

We remove some elements

$$(1, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \dots)$$

A subsequence *is created. We also reenumerate the numbers:*

$$(\underbrace{1}_{n_0}, \underbrace{\frac{1}{3}}_{n_1}, \underbrace{\frac{1}{4}}_{n_2}, \underbrace{\frac{1}{6}}_{n_3}, \dots)$$

Let $n : \mathbb{N} \to \mathbb{N}$ be strictly monotonically increasing. Therefore

$$\forall k \in \mathbb{N} : n(k+1) > n(k) \Rightarrow n_{k+1} > n_k$$

We call $(n_k)_{n\in\mathbb{N}}$ an index subsequence *and $(a_{n_k})_{k\in\mathbb{N}}$ is called subsequence of $(a_n)_{n\in\mathbb{N}}$.*

Lemma 4. *Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a and $(a_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(a_n)_{n\in\mathbb{N}}$. Then also the subsequence is convergent and has the same limes a.*

Proof. For every subsequence index n_k with $k \in \mathbb{N}$ it holds that $n_k \geq k$.

Proof by induction: $k = 0$

$\mathbf{n_0} \in \mathbb{N}$

$$n_0 \geq 0 = k \qquad \checkmark$$

$\mathbf{n_k \geq k}$ Because $\underbrace{n_{k+1}}_{\in\mathbb{N}} > n_k$ (strictly monotonic). Therefore,

$$n_{k+1} \geq n_k + 1 > k + 1$$

□ Proof of limes: $\lim_{k\to\infty} a_{n_k} = a$. Let $\varepsilon > 0$. Because $(a_n)_{n\in\mathbb{N}}$ is convergent, it holds that $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \varepsilon$. Let $k \geq N$. This holds because $n_k \geq k \geq N : |a_{n_k} - a| < \varepsilon$. Therefore $(a_{n_k})_{k\in\mathbb{N}}$ has limes a. □

Lemma 5. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. Then it holds that $a \in \mathbb{C}$ is limit point if and only if there exists some subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a$.*

Proof. We first prove direction $\Leftarrow$.

Assume $(a_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(a_n)_{n\in\mathbb{N}}$ with limes a. Let $\varepsilon > 0$.

$$\exists N \in \mathbb{N} : k \geq N \Rightarrow |a_{n_k} - a| < \varepsilon$$

Therefore $B(a, \varepsilon)$ has infinitely many sequence numbers of $(a_{n_k})_{k\in\mathbb{N}}$ and therefore also infinitely many sequence numbers of $(a_n)_{n\in\mathbb{N}}$.

We prove direction $\Rightarrow$.

We build a convergent subsequence. Consider $k \in \mathbb{N}$ with $k \geq 1$.

$$\varepsilon_k = \frac{1}{k}$$

We define $n_0 = 0$ and $a_{n_0} = a_0$. Assume $a_{n_0}, a_{n_1}, \dots, a_{n_{k-1}}$ are already defined.

Definition of a_{n_k}: In $B(a, \varepsilon_k)$ there are infinitely many sequence numbers of $(a_n)_{n\in\mathbb{N}}$. We consider $n_k > n_{k-1}$ and $a_{n_k} \in B(a, \varepsilon_k)$.

Then it holds that $\lim_{k\to\infty} a_{n_k} = a$. Let $\varepsilon > 0$ be arbitrary. Consider $K > \frac{1}{\varepsilon}$. Hence $\varepsilon > \frac{1}{K} = \varepsilon_K$ for all $k \geq K$ it holds that $n_k \geq n_K$ and $|a_{n_k} - a| < \varepsilon_k = \frac{1}{k} \leq \frac{1}{K} < \varepsilon$. □

9.6 Bolzano-Weierstrass theorem

$\boxed{\text{Bernard Bolzano (1781–1848), Karl Weierstrass (1815–1897)}}$

Theorem 45. *Every bounded sequence of real numbers has a limit point in $\mathbb{R}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$, hence $\exists M > 0$ such that all sequence numbers a_n in $I_0 = [-M, M]$ and let $F_0 = \{n \in \mathbb{N} \,|\, a_n \in I_0\} = \mathbb{N}$ (index set). F_0 is infinite. We build nested intervals with the properties:

- $I_{n+1} \subseteq I_n$
- $|I_{n+1}| = \frac{1}{2}|I_n|$
- $F_n = \{k \in \mathbb{N} \,|\, a_k \in I_n\}$ is infinite.

This construction is inductive:

induction base I_0 ✓

induction step Let $I_n = [A_n, B_n]$ be given and $M_n = \frac{1}{2}(A_n + B_n)$. Let $J_n = [A_n, M_n]$ and $L_n = [M_n, B_n]$. It holds that $J_n \subseteq I_n \wedge L_n \subseteq I_n$ and $|J_n| = \frac{1}{2}|I_n| \wedge |L_n| = \frac{1}{2}|I_n|$. Because there are infinitely many sequence numbers of $(a_n)_{n\in\mathbb{N}}$ in I_n and $I_n = J_n \cup L_n$, in at least one subinterval there have to be infinitely many sequence numbers.

Therefore select $I_{n+1} = J_n$ if J_n contains infinitely many sequence numbers and consider $I_{n+1} = L_n$ if J_n contains only finitely many sequence numbers. Therefore I_{n+1} contains infinitely many sequence numbers.

$$F_{n+1} = \{k \in \mathbb{N} \,|\, a_k \in I_{n+1}\}$$

is infinite. So $(I_n)_{n\in\mathbb{N}}$ is a nested interval.

Let $a \in \bigcap_{n\in\mathbb{N}} I_n$ (completeness of $\mathbb{R}$).

Claim: a is limit point of $(a_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be given and n sufficiently large, such that $|I_n| = B_n - A_n < \varepsilon$. Then it holds that for every $x \in I_n$ that $|x - a| \leq B_n - A_n < \varepsilon$ (with $x \in I_n, a \in I_n$). Because I_n contains infinitely many sequence numbers of $(a_n)_{n\in\mathbb{N}}$, it holds that infinitely many sequence numbers a_k satisfy the relation $|a_n - a| < \varepsilon$. Therefore a is limit point of $(a_n)_{n\in\mathbb{N}}$. □

Corollary 7 (typical definition of the Bolzano-Weierstrass theorem)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Theorem 46 (Bolzano-Weierstrass theorem in $\mathbb{C}$)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{C}$. Then $(a_n)_{n\in\mathbb{N}}$ has a convergent subsequence and therefore also at least one limit point in $\mathbb{C}$.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be bounded. $a_n = \alpha_n + i\beta_n$. So $(\alpha_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$ as well as $(\beta_n)_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}$.

Consider a convergent subsequence of $(\alpha_n)_{n\in\mathbb{N}}$, $(\alpha_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \alpha_{n_k} = \alpha$. Now consider bounded $(\beta_{n_k})_{k\in\mathbb{N}}$. From the Bolzano-Weierstrass theorem it follows that there exists a convergent subsequence $\left(\beta_{n_{k_l}}\right)_{l\in\mathbb{N}}$ with $\beta = \lim_{l\to\infty} \beta_{n_{k_l}}$.

$\left(\alpha_{n_{k_l}}\right)_{l\in\mathbb{N}}$ is subsequence of $(\alpha_{n_k})_{k\in\mathbb{N}}$ convergent with limit point α.

Let $a_{n_{k_l}} = \alpha_{n_{k_l}} + i\beta_{n_{k_l}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Real and imaginary parts are convergent, therefore $\lim_{l\to\infty} a_{n_{k_l}} = a = \alpha + i\beta$. Therefore $(a_n)_{n\in\mathbb{N}}$ contains a convergent subsequence. □

This lecture took place on 2nd of December 2015 with lecturer Wolfgang Ring.

Theorem 47 (Weierstrass-Bolzano theorem)**.** *Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Theorem 48 (Convergence)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be convergent in $\mathbb{C}$ with limes x.*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : |x_n - x| < \varepsilon$$

Definition 31 (Metric space)**.** *Let X be a set. We call $d : X \times X \to \mathbb{R}$ a* distance function *(or* metric*) on X if,*

- $\forall x \in X : d(x, x) = 0$
- $\forall x, y \in X : d(x, y) = d(y, x)$ *(symmetry)*
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$ *(triangle inequality)*

(X, d) *is called* metric space.

Example 15. $X = \mathbb{C}$, $d(x, y) = |x - y|$.

Definition 32 (Convergence with metric spaces)**.** *Let X be a metric space. $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements in X. Let $x \in X$. We call $(x_n)_{n\in\mathbb{N}}$ convergent with limes x if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : d(x_n, x) < \varepsilon$$

Definition 33. *Let $K \subseteq X$ be a subset of the metrical space X. We call K* pre-compact *if every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in K$ has a convergent subsequence. K is called compact if the limes a of the convergent subsequence is also in K.*

Definition 34. *In $\mathbb{C}$ it holds that every bounded set is pre-compact.*

9.7 Cauchy sequences in $\mathbb{R}$ and $\mathbb{C}$

Augustin-Louis Cauchy (1789–1857)

Definition 35. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We call $(a_n)_{n\in\mathbb{N}}$ a* Cauchy sequence *(fundamental sequence) if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow |a_n - a_m| < \varepsilon$$

Definition 36 (Cauchy sequence in a metric space)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. We call $(a_n)_{n\in\mathbb{N}}$ a* Cauchy sequence *(fundamental sequence) if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \wedge m \geq N \Rightarrow d(a_n, a_m) < \varepsilon$$

Lemma 6. *Every convergent sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is a Cauchy sequence.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be convergent with limes a. Let $\varepsilon > 0$ be arbitrary.

Convergence implies that $\exists N \in \mathbb{N} : n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$. For $m, n \geq N$ it holds that

$$|a_n - a_m| = |a_n - a + a - a_m| \leq \underbrace{|a_n - a|}_{<\frac{\varepsilon}{2} \text{ because } n \geq N} + \underbrace{|a - a_m|}_{<\frac{\varepsilon}{2} \text{ because } m \geq N}$$

□

Lemma 7. *Every Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is bounded.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$. The Cauchy condition for $\varepsilon = 1$ states:

$$\exists N \in \mathbb{N} : \forall m, n \geq N : |a_n - a_m| < 1$$

specifically $m = N : \forall n \geq N$

$$|a_n - a_N| < 1$$

Therefore $|a_n| = |a_n - a_N + a_N| \leq \underbrace{|a_n - a_N|}_{<1} + |A_N| < |a_N| + 1$.

Let $m = \max\{|a_0|, |a_1|, \dots, |a_{N-1}|\}$ and $M = \max\{m, |A_N| + 1\}$.

Then for $n \leq N - 1$ it holds that

$$|a_n| \leq m \leq M$$

and for $n \geq N$ it holds that

$$|a_n| \leq |a_N| + 1 \leq M$$

Therefore $\forall n \in \mathbb{N} : |a_n| \leq M$. Therefore $(a_n)_{n\in\mathbb{N}}$ is bounded. □

9.8 Is $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ complete?

Theorem 49 (Cauchy sequences and limes)**.** *Every Cauchy sequence in $\mathbb{C}$ has a limes and is therefore convergent. Followingly we call $\mathbb{C}$ to be* complete.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$. We know that $(a_n)_{n\in\mathbb{N}}$ is bounded. From the Bolzano-Weierstrass theorem it follows that a limit point a of $(a_n)_{n\in\mathbb{N}}$ exists. Let $\varepsilon > 0$ be arbitrary.

1. We choose $N \leq \mathbb{N}$ sufficiently large such that
$$n, m \geq N \Rightarrow |a_n - a_m| < \frac{\varepsilon}{2}$$
2. Because $B(a, \frac{\varepsilon}{2})$ contains infinitely many sequence numbers (a is limit point), $K \geq N$ exists with $|a - a_K| < \frac{\varepsilon}{2}$.

Let $n \geq N$. Then

$$|a_n - a| = |a_n - a_K + a_K - a| \leq \underbrace{|a_n - a_K|}_{<\frac{\varepsilon}{2} \text{ (Cauchy seq.)}} + \underbrace{|a_K - a|}_{<\frac{\varepsilon}{2} \text{ (limit point } a)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is convergent with limes a.

We have proven that if $(a_n)_{n\in\mathbb{N}}$ has a limit point, this limit point is also its limes.

We concluded: nested intervals $\Rightarrow$ compactness / Bolzano-Weierstrass theorem $\Rightarrow$ completeness.

Actually nested intervals are equivalent to completeness. □

This lecture took place on 3rd of December 2015 with lecturer

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Corollary 8. *$\mathbb{C}$ is complete.*

Proof. Let $(z_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{C}$.

$$z_n = a_n + ib_n$$

Then $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $\mathbb{R}$.

Show that this property: Let $\varepsilon > 0$. Because $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it holds that

$$\exists N \in \mathbb{N} : n, m \geq N \Rightarrow |z_n - z_m| < \varepsilon$$

Because $|a_n - a_m| \leq |z_n - z_m|$ and $|b_n - b_m| \leq |z_n - z_m|$ hold, it follows that for $n, m \geq N : |a_n - a_m| < \varepsilon \wedge |b_n - b_m| < \varepsilon$. Therefore $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences.

Because $\mathbb{R}$ is complete, it follows that $\exists a \in \mathbb{R}$ such that

$$a = \lim_{n\to\infty} a_n \text{ and } \exists b \in \mathbb{R}$$

with $b = \lim_{n\to\infty} b_n$. Because $\lim_{n\to\infty} z_n = z = a + ib$,

$$\Leftrightarrow a = \lim_{n\to\infty} a_n \wedge b = \lim_{n\to\infty} b_n$$

□

Example 16. *We show a counterexample for the completeness of $\mathbb{Q}$. So we have Cauchy sequences with limes, which lies outside $\mathbb{Q}$.*

We define a recursion:

$$a_n = \begin{cases} 2 & \text{if } n = 0 \\ \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) & \text{if } n > 0 \end{cases}$$

We observe, $\forall n \in \mathbb{N} : a_n > 0 \wedge a_n \in \mathbb{Q}$.

Proof by complete induction:

Induction base: $n = 0$

$$a_0 = 2 > 0 \wedge 2 \in \mathbb{Q} \quad \checkmark$$

Induction step: $n \to n+1$ *Let $a_n > 0$ and $a_n \in \mathbb{Q}$.*

$$a_{n+1} = \frac{1}{2}\left(\underbrace{a_n}_{>0} + \underbrace{\frac{2}{a_n}}_{>0}\right) > 0$$

and $a_{n+1} \in \mathbb{Q}$.

We prove by induction: $\forall n \in \mathbb{N} : a_n^2 > 2$.

Induction base: $n = 0$

$$a_0 = 2 \qquad a_0^2 = 4 > 2 \quad \checkmark$$

Induction step: $n \to n+1$ *It holds that $a_n^2 - 2 > 0$.*

$$a_{n+1}^2 - 2 = \frac{1}{4}\left(a_n^2 + 4 + \frac{4}{a_n^2}\right) - 2 = \frac{1}{4a_n^2}\left(a_n^4 + 4a_n^2 + 4 - 8a_n^2\right)$$
$$= \frac{1}{4a_n^2}\left(a_n^4 - 4a_n^2 + 4\right) = \frac{1}{4a_n^2}\underbrace{\left(a_n^2 - 2\right)^2}_{>0} > 0$$

Furthermore it holds that $a_{n+1} < a_n$.

$$2a_{n+1} = a_n + \frac{2}{a_n} \Rightarrow 2(a_{n+1} - a_n) = -a_n + \frac{2}{a_n} = \frac{\overbrace{2 - a_n^2}^{<0}}{a_n} < 0$$
$$\Rightarrow a_{n+1} - a_n < 0 \Rightarrow a_{n+1} < a_n$$

Therefore the sequence $(a_n)_{n\in\mathbb{N}}$ is strictly monotonically decreasing and is bound by below. Therefore some $a \in \mathbb{R}$ exists with $a = \lim_{n\to\infty} a_n$.

Monotonicity really depends on the completeness of $\mathbb{R}$. We cannot argue equivalently to Theorem 44 with the supremum.

For this example we know that $(a_n)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$. $(a_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{R}$. So $(a_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathbb{Q}$.

For the limes a it holds that,

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$
$$= \frac{1}{2}\lim_{n\to\infty} a_n + \frac{1}{2}\frac{2}{\lim_{n\to\infty} a_n} = \frac{1}{2}a + \frac{1}{a}$$

$$a = \frac{1}{2}a + \frac{1}{a} \Rightarrow \frac{1}{2}a = \frac{1}{a}$$
$$a^2 = 2 \Rightarrow a = +\sqrt{2} \notin \mathbb{Q}$$

Therefore $(a_n)_{n\in\mathbb{N}}$ is not *convergent in $\mathbb{Q}$. We found a convergent Cauchy sequence whose limes is not in $\mathbb{Q}$ which immediately means that $\mathbb{Q}$ is incomplete.*

Definition 37 (Tending towards infinity)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.*

- *We state $(a_n)_{n\in\mathbb{N}}$* tends to infinity *with limes $+\infty$:*

$$\lim_{n\to\infty} a_n = +\infty$$

$$\text{if } \forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n > M$$

- *We state $(a_n)_{n\in\mathbb{N}}$* tends to negative infinity *with limes $-\infty$:*

$$\lim_{n\to\infty} a_n = -\infty$$

$$\forall M > 0 \exists N \in \mathbb{N} : n \geq N \Rightarrow a_n < -M$$

Example 17.

$$a_n = \frac{n^2+2}{n+1}$$

has limes $+\infty$. The proof is given in the practicals. We show that ...

$$\frac{n^2+2}{n+1} > M \Leftrightarrow \dots$$

Definition 38 (Limes superior, Limes inferior)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence which is bounded above and*

$$H = \left\{\xi \in \mathbb{R} \,|\, \xi \text{ is limit point of } (a_n)_{n\in\mathbb{N}}\right\} \neq \emptyset$$

Then H is also bounded by above and we call $S^* = \sup H$ a limes superior *of the sequence $(a_n)_{n\in\mathbb{N}}$. We denote:*

$$S^* = \limsup_{n\to\infty} a_n$$

Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence which is bounded below and

$$H = \left\{\xi \in \mathbb{R} \,|\, \xi \text{ is limit point of } (a_n)_{n\in\mathbb{N}}\right\} \neq \emptyset$$

Then H is also bounded by below and we call $S^* = \inf H$ a limes inferior *of the sequence $(a_n)_{n\in\mathbb{N}}$. We denote:*

$$S_* = \liminf_{n\to\infty} a_n$$

Theorem 50. *If $(a_n)_{n\in\mathbb{N}}$ is bounded by above by M, $H \neq \emptyset$, then M is also an upper bound of H.*

Proof. Assume $\exists s \in H$ with $s > M$. Choose $\varepsilon = S - M > 0$. Because S is a limit point of $(a_n)_{n\in\mathbb{N}}$ it holds that $(s-\varepsilon, s+\varepsilon)$ contains infinitely many sequence numbers. So for infinitely many indices n it holds that,

$$a_n > s - \varepsilon = s - (s - M) = M$$

This contradicts with M being the upper bound of the sequence. ☐

Lemma 8. *Let $(a_n)_{n\in\mathbb{N}}$ be bounded by above. $a_2 \in \mathbb{R}$. Let $H \neq \emptyset$ be defined as above. Then it holds that*

$$s^* = \limsup_{n\to\infty}(a_n) = \max H$$

ie. S^* is a limit point itself of the sequence.

Proof. Show that S^* itself is limit point of the sequence. Let $\varepsilon > 0$: Choose $\xi \in H$ such that

$$\xi > S^* - \frac{\varepsilon}{2} \Rightarrow S^* - \xi = |S^* - \xi| < \frac{\varepsilon}{2}$$

Because ξ is a limit point of the sequence, in $(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2})$ there are infinitely many sequence numbers.

Let $x \in (\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}) \Leftrightarrow |x - \xi| < \frac{\varepsilon}{2}$. Then it holds that

$$|x - s^*| = |x - \xi + \xi - s^*| \leq \underbrace{|x - \xi|}_{<\frac{\varepsilon}{2}} + \underbrace{|\xi - s^*|}_{=S^* - \xi < \frac{\varepsilon}{2}}$$

$$\Rightarrow x \in (S^* - \varepsilon, S^* + \varepsilon)$$

Followingly,

$$\underbrace{\left(\xi - \frac{\varepsilon}{2}, \xi + \frac{\varepsilon}{2}\right)}_{\text{contains infinitely many sequence numbers}} \subseteq \underbrace{(S^* - \varepsilon, S^* + \varepsilon)}_{\text{contains infinitely many sequence numbers}}.$$

□

Remark 14. *The analogous statement holds for the limes inferior.*

$$S^* = \limsup_{n\to\infty} a_n \Leftrightarrow$$

1. *$S^* \in H$, therefore S^* is limit point of $(a_n)_{n\in\mathbb{N}}$.*
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

Proof. Let $S^* = \limsup_{n\to\infty} a_n$.

1. The first property holds immediately.
2. We use an indirect proof.

$$\Rightarrow \exists \varepsilon > 0 : \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Therefore infinitely many sequence numbers a_n exist with $a_n \geq S^* + \varepsilon$. We sort the sequence numbers in a subsequence $(a_{n_k})_{k\in\mathbb{N}}$. It holds that

$$S^* + \varepsilon \leq a_{n_k} \leq M$$

$(a_{n_k})_{k\in\mathbb{N}}$ is bounded and has a limit point S with $S^* + \varepsilon < S \Rightarrow S > S^*$. S is also a limit point of the original sequence $(a_n)_{n\in\mathbb{N}}$ with $S > S^* = \max H$. This is a contradiction.

□

This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

Theorem 51 (Repetition of the theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be bounded above and let $(a_n)_{n\in\mathbb{N}}$ has a limit point. Then it holds that $S^* = \limsup_{n\to\infty} a_n \Leftrightarrow$*

1. *S^* is limit point of $(a_n)_{n\in\mathbb{N}}$.*
2. $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n < S^* + \varepsilon$

Therefore above $S^* + \varepsilon$ there are only finitely many sequence numbers.

Proof. We prove the first direction $\Rightarrow$.

Let $S^* = \limsup_{n\to\infty} a_n$. Let $\varepsilon > 0$ be arbitrary. The first property follows immediately. The second property needs to be shown.

Proof by contradiction for the second property.

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} : \exists n \geq N : a_n \geq S^* + \varepsilon$$

Then we build a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ from $(a_n)_{n\in\mathbb{N}}$ with $a_{n_k} \geq s^* + \varepsilon$.

The subsequence is built inductively:

$n = 0$ then (because the second property holds negated) there exists $x_n \geq 0$: $a_{n_0} \geq s^* + \varepsilon$.

$k \to k + 1$ Let $a_{n_0}, a_{n_1}, \dots, a_{n_k}$ be found with $a_{n_l} \geq S^* + \varepsilon$ with $l = 0, \dots, n$ and $n_l < n_{l+1}$. Let $N = n_k + 1$. Because the second property holds negated, $n_{k+1} \geq N > n_k$ such that $a_{n_{k+1}} \geq S^* + \varepsilon$.

The subsequence numbers have the properties:

- $a_{n_k} \geq S^* + \varepsilon \qquad \forall k \in \mathbb{N}$

- Because $(a_n)_{n\in\mathbb{N}}$ is bounded above, also $(a_{n_k})_{k\in\mathbb{N}}$ is bounded above

From the Bolzano-Weierstrass theorem it follows that $(a_{n_k})_{k\in\mathbb{N}}$ has a limit point $S \geq S^* + \varepsilon$. Because every limit point of $(a_{n_k})_{k\in\mathbb{N}}$ is a limit point of $(a_n)_{n\in\mathbb{N}}$, it holds that S is limit point of $(a_n)_{n\in\mathbb{N}}$ and $S > S^* + \varepsilon > S^*$. This is a contradiction. $\square$

We prove the second direction $\Leftarrow$.

Assume properties 1 and 2 hold. It remains to show that S^* is the largest limit point. Assume $S > S^*$. We need to show that S cannot be a limit point.

$$\varepsilon = \frac{S - S^*}{2} > 0 \Rightarrow 2\varepsilon = S - S^* \Rightarrow S^* + \varepsilon = S - \varepsilon$$

Because the second property holds, there exists some $N \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow a_n < S^* + \varepsilon$. Therefore only finitely many sequence numbers are larger than $S^* + \varepsilon = S - \varepsilon$. Therefore at most finitely many sequence numbers $(S - \varepsilon, S + \varepsilon)$. Followingly S is not a limit point. $\square$

Theorem 52 (Analogous result for limes inferior)**.**

$$S_* = \liminf_{n\to\infty} a_n \Leftrightarrow$$

1. *S_* is limit point of $(a_n)_{n\in\mathbb{N}}$.*

2. *$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N : a_n > S_* - \varepsilon$*

Theorem 53. *Let $(a_n)_{n\in\mathbb{N}}$ be bounded above and $(a_n)_{n\in\mathbb{N}}$ has a limit point.*

- *Let $k \in \mathbb{N}$. We define*

$$A_k = \{a_k, a_{k+1}, a_{k+2}, \dots\} = \{a_j : j \geq k\}$$

- *It holds that $A_{k+1} \subseteq A_k$ and A_k is bounded above*[1]*.*

[1] Obviously.

We define $S_k = \sup A_k$. Then $(S_k)_{k\in\mathbb{N}}$ is a monotonically decreasing sequence in $\mathbb{R}$ and $(S_k)_{k\in\mathbb{N}}$ is bounded below. Therefore $(S_k)_{k\in\mathbb{N}}$ is convergent and it holds that

$$\lim_{n\to\infty} S_k = \inf\{S_k : k \in \mathbb{N}\} = S^*$$

It turns out that

$$S^* = \limsup_{n\to\infty} a_n$$

We denote

$$\lim_{k\to\infty} \sup A_k = \lim_{k\to\infty} \sup\{a_j : j \geq k\} = \inf\{\sup A_k : k \in \mathbb{N}\} = \limsup_{n\to\infty} a_n$$

Proof.

$$A_{k+1} \subseteq A_k \Rightarrow \sup A_{k+1} \leq \sup A_k \Rightarrow S_{k+1} \leq S_k$$

$(S_k)_{k\in\mathbb{N}}$ is bounded below. Choose $\xi \in H$ and ξ is limit point of $(a_n)_{n\in\mathbb{N}}$. Then $\xi - 1$ is a lower bound for $(S_k)_{k\in\mathbb{N}}$ because infinitely many sequence numbers are in $(\xi - 1, \xi + 1)$. Therefore,

$$\forall k \in \mathbb{N} : \exists n \geq k : a_n > \xi - 1 \quad \Rightarrow S_k = \sup A_k > \xi - 1 \quad \checkmark$$

We know that $(S_k)_{k\in\mathbb{N}}$ is convergent. Let $S^* = \lim_{n\to\infty} S_k$. We show the first property:

S^* is limit point of $(a_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be given. We need to show that infinitely many sequence numbers are in $(S^* - \varepsilon, S^* + \varepsilon)$.

Because $\lim_{k\to\infty} S_k = S^*$ there exists some

$$N \in \mathbb{N} : k \geq N \Rightarrow \underbrace{|S_k - S^*|}_{-S^*} < \frac{\varepsilon}{2}.$$

We build a subsequence of $(a_n)_{n\in\mathbb{N}}$ inductively, which is entirely inside $(S^* - \varepsilon, S^* + \varepsilon)$. Because $S_N = \sup\{a_N, a_{N+1}, a_{N+2}, \dots\}$ exists, there exists $a_j \geq S_N - \frac{\varepsilon}{2}$ with $j \geq N$.

$$\Rightarrow \underbrace{S_N - a_j}_{=|S_N - a_j|} \leq \frac{\varepsilon}{2}$$

$k = 0$ Choose $n_0 = j \geq N$ (j from above), therefore it holds that

$$|S^* - a_{n_0}| = |S^* - S_N + S_N - a_{n_0}| \leq |S^* - S_N| + |S_N - a_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $a_{n_0} \in (S^* - \varepsilon, S^* + \varepsilon)$.

$k \to k+1$ Consider $a_{n_0}, a_{n_1}, \dots, a_{n_k}$ such that $n_k > n_{k-1} > \cdots > n_0 \geq N$ holds and $|a_{n_l} - S^*| < \varepsilon$. Because $n_k + 1 > N$ holds

$$|S^* - S_{n_k+1}| < \frac{\varepsilon}{2}$$

because $S_{n_k+1} = \sup\{a_{n_k+1}, a_{n_k+2}, \dots\}$, exists $j' \geq n_k + 1 > n_k$ such that

$$|S_{n_k+1} - a_{j'}| = S_{n_k+1} - a_{j'} < \frac{\varepsilon}{2}$$

Choose $n_{k+1} = j'$ from above.

$$n_{k+1} \geq n_k + 1 > n_k \text{ and } \left|S^* - a_{n_{k+1}}\right| = |S^* - S_{n_k+1} + S_{n_k+1} - a_{j'}|$$

$$\leq |S^* - S_{n_k+1}| + |S_{n_k+1} - a_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}1 = \varepsilon$$

Therefore we have found a subsequence $(a_n)_{n\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N} : a_{n_k} \in (S^* - \varepsilon, S^* + \varepsilon)$$

$\Rightarrow S^*$ is limit point of the sequence.

We show that S^* is the largest limit point. Let $S < S^*$. We show that S is not a limit point.

Let $\varepsilon = \frac{1}{2}(S - S^*) > 0$ such that $S^* + \varepsilon = S - \varepsilon$. Choose $k \in \mathbb{N}$ such that $S_k - S^* = |S_k - S^*| < \varepsilon$. $\forall n \geq K$ it holds that $a_n \leq S_k < S^* + \varepsilon = S - \varepsilon$. Therefore there are at most finitely many sequence numbers in $(S-\varepsilon, S+\varepsilon)$. Therefore S is not a limit point.

□

The analogous result for the limes inferior also holds and is given in the practicals.

10 Infinite series

Definition 39. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers. We define*

- $S_0 = a_0$
- $S_1 = a_0 + a_1$
- $S_2 = a_0 + a_1 + a_2$
- ...
- $S_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k$

We call $(S_n)_{n\in\mathbb{N}}$ an infinite series *with a_k sequence numbers. We call S_n the* n-th partial sum *of the series. The series is called* convergent *if $(S_n)_{n\in\mathbb{N}}$ is a convergent series in $\mathbb{C}$. For a convergent series instead of*

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \underbrace{\sum_{k=0}^{n} a_k}_{=S_n}$$

we denote

$$S = \sum_{k=0}^{\infty} a_k$$

Actually a series must be denoted like a sequence with $(S_n)_{n\in\mathbb{N}}$. But we also say "let $\sum_{k=0}^{\infty} a_k$ be a series" (but actually the sum of partial sums is meant). So this an ambiguous definition (per default always assume that the sum of partial sums is considered).

10.1 The geometric series

Theorem 54. *Let $q \in \mathbb{C}$ with $q \neq 1$. Consider $\sum_{k=0}^{\infty} q^k$ hence $S_n = \sum_{k=0}^{n} q^n$. The limes of this series is given with $\frac{1-q^{n+1}}{1-q}$ for $|q| < 0$.*

Proof. We find a simple equation for S_n:

$$S_n - q \cdot S_n = (1-q)S_n$$

$$\begin{aligned}
&(1+q+q^2+\cdots+q^n) - q(1+q+q^2+\cdots+q^n) \\
&= (1+q+q^2+\cdots+q^n) - (q+q^2+\cdots+q^n+q^{n+1}) \\
&= (1-q^{n+1})
\end{aligned}$$

Therefore $(1-q)\cdot S_n = 1-q$. That is,

$$S_n = \frac{1-q^{n+1}}{1-q}$$

If $|q| < 1$ it holds that

$$\lim_{n\to\infty} q^{n+1} = q \lim_{n\to\infty} q^n = q\cdot 0 = 0$$

$$\lim_{n\to\infty} S_n = \frac{1-\lim_{n\to\infty} q^{n+1}}{1-q} = \frac{1}{1-q}$$

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

If $|q| > 1$ it holds that

$$|S_n| = \frac{1}{|1-q|}\cdot\left|1-q^{n+1}\right| \geq \frac{1}{|1-q|}\left(\left|q^{n+1}\right|-1\right)$$

This is the inversed triangle inequality.

$$= \frac{1}{|1-q|}\left(\underbrace{|q|^{n+1}}_{\to\infty} - 1\right)$$

Hence $(S_n)_{n\in\mathbb{N}}$ is unbounded and therefore not convergent. □

Theorem 55. *Let* $a_n = \frac{1}{n}$ *hence* $\sum_{k=1}^{\infty}\frac{1}{k}$.

$$\sum_{k=1}^{\infty}\frac{1}{n} \text{ is divergent}$$

Proof. Consider

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\dots \\
&> \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{6}+\frac{1}{6}+\frac{1}{8}+\frac{1}{8}+\dots \\
&= \frac{1}{2}+2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\dots\right) \\
&= \frac{1}{2}+2\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right) \\
&= \frac{1}{2}+\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots\right) \\
&= \frac{1}{2}+\sum_{k=1}^{\infty}\frac{1}{n}
\end{aligned}$$

This is a contradiction. □

This lecture took place on 9th of December 2015 with lecturer Wolfgang Ring.

TODO

Case 1

$$n = 2^k - 1$$

$$S_{\alpha,2^1-1} = \underbrace{1}_{<\frac{1}{2^\alpha}} + \underbrace{\frac{1}{2^\alpha}+\frac{1}{3^\alpha}}_{2^1 \text{ terms}} + \underbrace{\frac{1}{4^\alpha}+\frac{1}{5^\alpha}+\frac{1}{6^\alpha}+\frac{1}{7^\alpha}}_{2^2 \text{ terms}} + \underbrace{\frac{1}{8^\alpha}+\dots}_{2^3 \text{ terms}} TODO$$

$$\begin{aligned}
&< 1 + 2\frac{1}{2^\alpha} + 4\frac{1}{4^\alpha} + 8\frac{1}{8^\alpha} + \dots + 2^{k-1}\frac{1}{\left(2^{k-1}\right)^\alpha} \\
&= 1 + \frac{1}{2^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \frac{1}{8^{\alpha-1}} + \dots + \frac{1}{\left(2^{n-1}\right)^{\alpha-1}}
\end{aligned}$$

$$= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \left(\frac{1}{3^{\alpha-1}}\right)^3 + \dots$$

$$= \underbrace{\sum_{j=0}^{k-1} \left(\frac{1}{2^{\alpha-1}}\right)^2}_{\substack{\text{geometric}\\\text{series}}} = \frac{1 - \left(\frac{1}{2^{\alpha-1}}\right)^2}{1 - \frac{1}{2^{\alpha-1}}}$$

Therefore $(S_{\alpha,2^k-1})$ is bounded. Let $n \in \mathbb{N}$ be arbitrary and choose a sufficiently large K such that $2^k > n+1$. Therefore $2^k - 1 > n$. Because $\frac{1}{j^\alpha} > 0$ for all $j \geq 1$, it holds that $S_{2^k-1} > S_n$. At the same time $S_{2^k-1} < \frac{2^{\alpha-1}}{2^{\alpha-1}-1}$. So $(S_n)_{n\in\mathbb{N}}$ is bounded. Hence $\sum_{k=1}^\infty \frac{1}{k^\alpha}$ is convergent.

Case 2: $\alpha \leq 1$ Then it holds that $k^\alpha \leq k$ and therefore $\frac{1}{k^\alpha} \geq \frac{1}{k}$. Because $S_{\alpha,n} \geq S_{1,n}$ and because $S_{1,n}$ is unbounded, it holds that $(S_{\alpha,n})_{n\in\mathbb{N}}$ is unbounded and followingly $\sum_{k=0}^\infty \frac{1}{k^\alpha}$ is divergent.

Remark 15. *$\alpha \in \mathbb{Q}_+$ can be replaced by $\alpha \in \mathbb{R}_+$. It is even possible to choose $\alpha \in \mathbb{C}$. Then we can define $\zeta : M \subseteq \mathbb{C} \to \mathbb{C}$ with $\xi(z) = \sum_{k=1}^\infty \frac{1}{k^z}$. This is Rieman's Zeta function.*

Definition 40. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence with $a_n \geq 0$. Then we call $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n = (-1)^n a_n$, or equivalently $\alpha_n = (-1)^{n+1} a_n$, an .*

A series of structure $\sum_{k=0}^\infty (-1)^k a_k$ with $a_k \geq 0$ is called alternating series.

Gottfried Wilhelm Leibniz (1646–1716

Theorem 56 (Leibniz convergence criterion)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a monotonic zero sequence with $a_n \geq a_{n+1} \geq 0 \quad \forall n \in \mathbb{N}$. Then $\sum_{k=0}^\infty (-1)^k a_k$ is convergent.*

Proof.

$$\begin{aligned}
S_{2n-1} &= \sum_{k=0}^{2n-1} (-1)^k a_k \\
S_{2n} &= \sum_{k=0}^{2n-1} (-1)^k a_k + (-1)^{2n} a_{2n} \\
&= S_{2n-1} + a_{2n} \\
S_{2n+1} &= S_{2n-1} + \underbrace{a_{2n} - a_{2n-1}}_{\geq 0} \\
S_{2n+2} &= \underbrace{S_{2n-1} + a_{2n}}_{S_{2n}} \underbrace{-a_{2n+1} + a_{2n+2}}_{=-(a_{2n+1}-a_{2n+2})\geq 0}
\end{aligned}$$

Therefore it holds that $S_{2n+1} \geq S_{2n-1}$, $S_{2n+2} \leq S_{2n}$ and $S_{2n} \geq S_{2n-1}$.

$(S_{2n})_{n\in\mathbb{N}}$ is monotonically decreasing. $(S_{2n+1})_{n\in\mathbb{N}}$ is monotonically increasing.

It holds that: $\forall m, n \in \mathbb{N} : S_{2n} \geq S_{2m-1}$.

Proof. **Case 1: $\mathbf{m > n}$**

$$S_{2m+1} \leq S_{2n} \leq S_{2n} \qquad \checkmark$$

Case 2: $\mathbf{m \leq n}$

$$S_{2m+1} \leq S_{2n+1} \underbrace{\leq}_{\alpha<1} S_{2n}$$

So $(S_{2n})_{n\in\mathbb{N}}$ is monotonically decreasing and bounded by below (for example by S_1). Therefore $S_{2n} \to S^*$ for $n \to \infty$ (S_{2n+1}) is monotonically increasing and bounded by above by S_*:

$$S_{2n+1} \to S_* \text{ for } n \to \infty$$

It holds that $S_* \leq S^*$ because $S_{2n+1} \leq S_{2n}$. ☐

This lecture took place on 10th of December 2015 with lecturer

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Given $S_* \leq S^*$, we show that $S^* = S_*$ and we prove that $\forall \varepsilon > 0 : S^* - S_* < \varepsilon$.

Let $\varepsilon > 0$ and choose N sufficiently large, such that $a_{2N} < \varepsilon$.

$$a_{2N} = S_{2N} - S_{2N-1} > S^* - S_*$$

$$a_{2N} < \varepsilon$$

So $\forall \varepsilon > 0$, it holds that

$$S^* - S_* = |\, S^* - S_* \,| < \varepsilon$$

$$\Rightarrow S^* = S_* = S$$

So it holds that,

$$\lim_{n\to\infty} S_n = S^* = S_* = S$$

and the series converges. □

Example 18.

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{k} \text{ is convergent}$$

10.2 Series in $\mathbb{C}$ and absolute convergence

Theorem 57 (Cauchy convergence criterion)**.** *The complex series $\sum_{k=0}^{\infty} a_k$ is convergent if and only if the partial sums $(s_n)_{n\in\mathbb{N}}$ are a Cauchy sequence in $\mathbb{C}$.*

Remark 16. *Therefore*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : \forall n, m > N$$

$$\Rightarrow |S_n - S_m| < \varepsilon$$

Therefore without loss of generality, $n \geq m$.

$$S_n - S_m = \sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k = \sum_{k=m+1}^{n} a_k$$

Hence $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq m \geq N$.

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$$

Equivalently, with $m + 1 = n$ and $n - m = l$.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N \text{ and } l \in \mathbb{N}$$

$$\left| \sum_{k=0}^{l} a_{n+k} \right| < \varepsilon$$

Proof by $(S_n)_{n\in\mathbb{N}}$ being convergent.

$$(S_n)_{n\in\mathbb{N}} \Leftrightarrow \text{Cauchy sequence}$$

Lemma 9. *Let $\sum_{k=0}^{\infty} a_n$ be convergent in $\mathbb{C}$. Then $(a_n)_{n\in\mathbb{N}}$ is a zero sequence.*

Proof. Follows directly from the Cauchy criterion for $l = 0$.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n > N : \underbrace{\left| \sum_{n=0}^{0} a_{n+k} \right|}_{|a_n|} < \varepsilon \quad \text{hence } a_n \to 0$$

□

□

Definition 41. *The complex series $\sum_{k=0}^{\infty} a_k$ is called* absolute convergent *if the real series $\sum_{k=0}^{\infty} |a_k|$ is convergent.*

Example 19.

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{n^2} \qquad \text{absolute convergent}$$

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{n} \qquad \cancel{\text{absolute}} \text{ convergent (Leibniz)}$$

Lemma 10. *Let $\sum_{k=0}^{\infty} a_k$ be absolute convergent. Then $\sum_{k=0}^{\infty} a_k$ is also convergent.*

Proof. Let $\sum_{k=0}^{\infty}|a_k|$ be convergent. From the Cauchy criterion it follows that,

$$\forall\varepsilon>0\exists N\in\mathbb{N}:n\geq m\geq N:$$

$$\left|\sum_{k=m+1}^{n}|a_k|\right|=\sum_{k=m+1}^{n}|a_k|\geq\left|\sum_{k=m+1}^{n}a_k\right|<\varepsilon$$

$\Rightarrow\sum_{k=0}^{\infty}a_k$ is convergent according to Cauchy criterion. □

Theorem 58 (Direct comparison test (dt. Majorantenkriterium)). *1. Let $\sum_{k=0}^{\infty}a_k$ and $\sum_{k=0}^{\infty}b_k$ be complex series. Let $\sum_{k=0}^{\infty}b_k$ be absolute convergent and $\exists N\in\mathbb{N}:k\geq N\Rightarrow|a_k|\leq|b_k|$.*

Then $\sum_{k=0}^{\infty}a_k$ is absolute convergent. $\sum_{k=0}^{\infty}b_k$ is called majorant *of $\sum_{k=0}^{\infty}a_k$.*

2. Let $\sum_{k=0}^{\infty}a_k$ and $\sum_{k=0}^{\infty}b_k$ be complex series. Let $\sum_{k=0}^{\infty}a_k$ be divergent. Assume $\exists N\in\mathbb{N}:k\geq N\Rightarrow|a_k|\leq|b_k|$. Then also $\sum_{k=0}^{\infty}b_k$ is divergent. $\sum_{k=0}^{\infty}a_k$ is minorant *of $\sum_{k=0}^{\infty}b_k$.*

Proof. 1. We need to show that $\sum_{n=0}^{\infty}\underbrace{|a_k|}_{\geq 0}$ is convergent. It suffices to show that

$$\sum_{k=0}^{n}|a_k|=\sigma_n$$

$(\sigma_n)_{n\in\mathbb{N}}$ is bounded. Let $n\geq N$.

$$\begin{aligned}\sigma_n&=\sum_{n=0}^{n}|a_k|\\&=|a_0|+|a_1|+\dots+|a_{N-1}|+\sum_{k=N}^{n}|a_k|\\&\leq\underbrace{|a_0|+\dots+|a_{N-1}|+\underbrace{\sum_{k=N}^{\infty}|b_k|}_{s\geq 0}}_{M}\end{aligned}$$

Therefore $(\sigma_n)_{n\in\mathbb{N}}$ is bounded and therefore $\sum_{n=0}^{\infty}a_n$ is absolute convergent.

2. Let $\sum_{k=0}^{\infty}a_k$ be divergent. Then also $\sum_{k=0}^{\infty}|a_k|$ is divergent. Otherwise $\sum_{k=0}^{\infty}a_k$ is absolute convergent and therefore convergent.

$$\Rightarrow\sigma_n=\sum_{k=0}^{n}|a_k|$$

$(\sigma_n)_{n\in\mathbb{N}}$ is unbounded. Because

$$\begin{aligned}\sum_{k=0}^{n}|b_k|&=|b_0|+\dots+|b_{N-1}|+\sum_{k=N}^{n}|b_k|\\&\geq|b_0|+\dots+|b_{N-1}|+\sum_{k=N}^{N}|a_k|\\&=\underbrace{|b_0|+\dots+|b_{N-1}|-(|a_0|+\dots+|N-1|)+\sum_{k=0}^{n}|a_k|}_{z}\\&=z+\sigma_n\end{aligned}$$

$z+\sigma_n$ is unbounded. Therefore $\sum_{k=0}^{\infty}|b_k|$ is not convergent. Therefore $\sum_{k=0}^{\infty}b_k$ is not absolute convergent.

□

Theorem 59 (Ratio test (dt. Quotientenkriterium)). *1. Let $\sum_{k=0}^{\infty}a_k$ be a complex series. Assume $\exists q\in[0,1)$ with $(0\leq q<1)$ and $N\in\mathbb{N}$ such that*

- $\frac{|a_{n+1}|}{|a_n|}<q\quad\forall n\geq N$ *with $|a_n|\neq 0$, or*
- $\sqrt[n]{|a_n|}<q\quad\forall n\geq N$

Then the series $\sum_{k=0}^{\infty}a_k$ is absolute convergent.

"Ratio test"

2. Assume there exists $q>1$ and $N\in\mathbb{N}$ such that

- $\frac{|a_{n+1}|}{|a_n|}\geq q\quad\forall n\geq N$

- $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$

Then $\sum_{k=0}^{\infty} a_k$ is divergent.

"Square root test"

Proof. This follows from the direct comparison criterion. Compare with geometric series $\sum_{k=0}^{\infty} q^k$.

1. Assume the two statement of the ratio test holds. Therefore $\forall n \geq N$ it holds that $\sqrt[n]{|a_n|} \leq q \Leftrightarrow |a_n| \leq q^n$. $|a_n| \leq q^n$. Due to the direct comparison test, $\sum_{k=0}^{\infty} q^k \checkmark$.

   Assume the first statement of the ratio test does not hold.

$$\frac{|a_{n+1}|}{|a_n|} \leq q (< 1)$$

   Then it holds that $\forall k \in \mathbb{N}$:

$$|a_{k+N}| \leq |a_N| \cdot q^k$$

   Proof by induction over k:

   $\mathbf{k = 0}$

$$|a_N| \leq |a_N| \cdot q^0 \qquad \checkmark$$

   $\mathbf{k \to k + 1}$ Assume $|a_{N+k}| \leq |a_N| \cdot q^k$. Because

$$\frac{|a_{N+k+1}|}{|a_{N+k}|} \leq q \Rightarrow |a_{N+K-1}| \leq q\,|a_{N+k}| \leq q \cdot |a_N| \cdot q^k = |a_N|\, q^{k+1} \qquad \checkmark$$

   We set

$$b_K = \begin{cases} 0 & \text{for } k = 0, 1, 2, \dots, N-1 \\ |a_N| \cdot q^{K-n} & \text{for } n \geq N \end{cases}$$

$$\sum_{k=0}^{\infty} b_n = 0 + 0 + 0 + \dots + 0 + |a_N|\, TODO$$

$$= |a_N| \sum_{j=0}^{\infty} q_j \text{ is absolute convergent}$$

   $\sum_{k=0}^{\infty} b_k$ is an absolute convergent majorant. for $\sum_{k=0}^{\infty} a_k \Rightarrow \sum_{k=0}^{\infty} a_k$ is convergent.

2. Assume the second statement (square root test) holds. TODO Therefore $(a_n)_{n\in\mathbb{N}}$ is not a zero sequence. Therefore $\sum_{k=1}^{\infty} a_k$ is divergent.

   Assume the first statement holds.

$$\Rightarrow |a_{N+k}| \geq |a_N| \cdot q^k$$

   Because $|a_N| \cdot q^k$ is unbounded, $|a_{N+k}|$ is unbounded. $(a_k)_{k\in\mathbb{N}}$ are not zero sequences.

□

Remark 17. *Assume $\frac{|a_{n+1}|}{|a_n|}$ is bounded and $q = \limsup_{n\to\infty} \left(\frac{|a_{n+1}|}{a_n}\right) < 1$. Let $2\varepsilon = 1 - q > 0$.*

$$\Rightarrow \exists N \in \mathbb{N} : n \geq N : \frac{|a_{n+1}|}{|a_n|} < q + \varepsilon$$

$$= q + \frac{1}{2}(1-q) = \frac{1}{2}(1+q) = 1 - \varepsilon < 1$$

Due to the ratio test, the series $\sum_{k=0}^{\infty} a_k$ is absolute convergent.

Lemma 11. *Let $\sum_{k=0}^{\infty} a_k$ be a complex series with $a_k \neq 0 \forall k \in \mathbb{N}$. Furthermore it holds that*

$$q = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

Then $\sum_{k=0}^{\infty} a_k$ is absolute convergent.

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = q$$

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10.3 Revision

So $\sum_{k=0}^{\infty}$ is absolute convergent if $\exists q \in [0, 1) \exists N \in \mathbb{N}$.

- $\frac{|a_{n+1}|}{|a_n|} \leq q \quad \forall n \geq N$

- $\sqrt[n]{|a_n|} \leq q \quad \forall n \geq N$

If $q > 1$ and either $\frac{|a_{n+1}|}{|a_n|} \geq q \quad \forall n \geq N$ or $\sqrt[n]{|a_n|} \geq q \quad \forall n \geq N$, then this series is convergent.

Corollary 9. *Let $q = \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{k=0}^{\infty} a_k$ is absolute convergent. Let $q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then $\sum_{k=0}^{\infty} a_k$ is absolute convergent.*

Let $q = \limsup \sqrt[n]{|a_n|} > 1$. Then $\sum_{k=0}^{\infty} a_k$ is divergent.

Proof. Let $q = \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$$2\varepsilon = 1 - q > 0$$

Then there exists some $N \in \mathbb{N} : n \geq N$

$$\Rightarrow \sqrt[n]{|a_n|} \leq q + \varepsilon = 1 - \varepsilon < 1$$

Is absolute convergent according to the square root theorem.

We also need to show divergence: Let $q > 1$ be limit point of $\sqrt[n]{|a_n|}$. So there exists some subsequence $\left(\sqrt[n_k]{|a_{n_k}|}\right)_{k\in\mathbb{N}}$ with $\lim_{n\to\infty} \sqrt[n_k]{|a_{n_k}|} = q > 1 \Rightarrow \varepsilon = \frac{1}{2}(q-1) > 0$.

$$\sqrt[n_k]{|a_{n_k}|} > q - \varepsilon \qquad \forall k \geq K$$
$$\Rightarrow |a_{n_k}| > (q-\varepsilon)^{n_k} = (1+\varepsilon)^{n_k} > 1$$
$$\Rightarrow (|a_{n_k}|)_{k\in\mathbb{N}} \text{ is not a zero sequence}$$
$$\Rightarrow (|a_n|)_{n\in\mathbb{N}} \text{ is also not a zero sequence}$$
$$\Rightarrow \sum_{k=0}^{\infty} a_k \text{ is divergent}$$

□

Example 20 (Binomial series)**.** *Let $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \dots, n\}$.*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1\cdot 2\cdot \ldots \cdot (n-k)(n-k+1)\cdot \cdots \cdot n}{k! \cdot 1 \cdot 2 \cdot \ldots \cdot (n-k)}$$

$$= \frac{n\cdot(n-1)\cdot \ldots \cdot (n-k+1)}{k!}$$

Let $s \in \mathbb{C}$. We define the binomial coefficient $\binom{s}{k} = \frac{s\cdot(s-1)\cdot(s-2)\cdot\ldots\cdot(s-k+1)}{k!}$. Also let $\binom{s}{0} = 1$ and $\binom{s}{1} = s$. Let $k > n$ and $n \in \mathbb{N}$, then

$$\binom{n}{k} = \frac{n(n-1)\cdot\ldots\cdot\overbrace{(n-n)}^{0}\cdot\ldots\cdot(n-k+1)}{k!} = 0$$

Example 21. *We define the binomial series for $s, z \in \mathbb{C}$ with*

$$B_S(z) = \sum_{k=0}^{\infty} \underbrace{\binom{s}{k} z^k}_{:=a_k}$$

What about convergence? Well,

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\frac{s\cdot(s-1)\cdot\ldots\cdot(s-(k+1)+1)}{(k+1)!} z^{k+1}\right|}{\left|\frac{s(s-1)(s-2)\cdot\ldots\cdot(s-k+1)}{k!} z^k\right|}$$

$$\frac{|a_{k+1}|}{|a_k|} = \left|\frac{(s-k)z}{k+1}\right| = \left|\frac{\left(\overbrace{\frac{s}{k}}^{\to 0} - 1\right)\cdot z}{1 + \underbrace{\frac{1}{k}}_{\to 0}}\right| \to |z|$$

Therefore $B_S(z)$ is convergent for $|z| < 1$ and divergent for $|z| > 1$. So geometrically, it is convergent within a circle of radius 1 or i (at center $(0,0)$) and divergent outside.

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

We know, for $s \in \mathbb{N}$:

$$B_S(z) = \sum_{k=0}^{\infty} \binom{n}{k} z^k = \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n$$

Remind that $\binom{n}{k} = 0$ *for* $k > n$.

Therefore

$$(1+z)^s := \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

This is the definition of a power function i.e.

$$z = \xi - 1 \qquad 1 + z = \xi$$

$$\xi^S = \sum_{k=0}^{\infty} \binom{s}{k} (\xi - 1)^k$$

is convergent for $|\xi - 1| < 1$.

Geometrically, this is a circle of radius 1 *or* i *(at center* $(1,0)$*).*

11 Power series

Definition 42. *A power series (in one variable) is an infinite series of the form*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

So we have one free variable. Coefficients of the series contains a variable.

- *In* $\sum_{k=1}^{\infty} \frac{1}{k^2}$ *all summands are fixed.*
- *However* $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ *with* $|z| < 1$ *is variable with the variable* z.

Example 22.

$$f : B(0,1) \to \mathbb{C}$$

$$B_S(z) = \sum_{k=0}^{\infty} \binom{s}{k} z^k$$

Mapping:

$$B_S : B(0,1) \to \mathbb{C}$$

$$\varepsilon(z) = \sum_{k=0}^{\infty} \overline{k!}$$

$$\varepsilon : \mathbb{C} \to \mathbb{C}$$

Let $z \in \mathbb{C}$ *arbitrary.*

$$\frac{|a_{k+1}|}{|a_k|} = \frac{\left|\frac{z^{k+1}}{(k+1)!}\right|}{\left|\frac{z^k}{k!}\right|} = \left|\frac{z}{k+1}\right| \to 0$$

$\Rightarrow \varepsilon(z)$ *is convergent for all* $z \in \mathbb{C}$.

$$\varepsilon : \mathbb{C} \to \mathbb{C}$$

Corollary 10. *Using series sum we can define mappings (functions).*

Definition 43. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a complex sequence and let* $z \in \mathbb{C}$. *Then* $\sum_{k=0}^{\infty} a_k \cdot z^k$ *is called* power series with coefficient sequence $(a_k)_{k\in\mathbb{N}}$.

Its convergence property depends on z. *For* $z = 0$ *every power series is convergent.*

$$\sum_{k=0}^{\infty} a_k \cdot 0^k$$

Because we define $0^0 := 1$ *here, the constant series* a_0 *is given.*

Lemma 12. *Let* $\sum_{k=0}^{\infty} a_k z^k$ *is a power series in* $\mathbb{C}$ *and* $z_0 \in \mathbb{C} \setminus \{0\}$ *such that* $\sum_{k=0}^{\infty} a_k z_0^k$ *is convergent. Then the power series is absolute convergent for all* z *with* $|z| < |z_0|$.

Geometrically, if the series is convergent at one point z_0 *at the circle, it is convergent in all points of the circle.*

Proof. Direct comparison test: Because $\sum_{k=0}^{\infty} a_k z_0^k$ is convergent, it holds that $\lim_{k\to\infty} a_k z_0^k = 0$. Therefore $(a_k z_0^k)_{n\in\mathbb{N}}$ is also bounded and there exists some $m \geq 0$ such that $\left|a_k z_0^k\right| \leq m \quad \forall k \in \mathbb{N}$.

Let $|z| < |z_0|$. Then,

$$\left|a_k z^k\right| = \left|a_k \frac{z^k}{z_0^k} \cdot z_0^k\right| = \left|a_k z_0^k\right| = \underbrace{\left|a_k z_0^k\right|}_{\leq m} \underbrace{\left|\frac{z}{z_0}\right|^k}_{:=q} \leq m \cdot q^k$$

with $0 \leq q < 1$. Therefore $\sum_{k=0}^{\infty} a_k z^k$ is convergent because of the direct comparison test with $\sum_{k=0}^{\infty} m \cdot q^k = m \cdot \sum_{k=0}^{\infty} q^k$. □

Definition 44. *Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series in $\mathbb{C}$. We define*

$$\rho(P) = \sup\{r \geq 0, r \in \mathbb{R} : P(r) \text{ is convergent}\}$$

$\rho(P)$ is called convergence radius of P. If $\{r \geq 0 : P(r) \text{ is convergent}\}$ is unbounded, then we define $P(r) = \infty$.

Lemma 13. *Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series in $\mathbb{C}$ and let $\rho(P)$ be its convergence radius of P. Then $P(z)$ is absolute convergent for all $z \in \mathbb{C}$ with $|z| < \rho(P)$.*

Proof. For $\rho(P) = 0$, nothing has to be shown.

Let $\rho(P) > 0$ and $|z| < \rho(P)$, then $\varepsilon := \rho(P) - |z|$. Because $\rho(P) = \sup\{r \geq 0 : P(r) \text{ is convergent}\}$, there exists some $r \in \mathbb{R}$ such that $\rho(P) - \varepsilon < r \leq \rho(P)$ and $P(r)$ is convergent. $\rho(P) - \varepsilon = |z| < r$. So $P(z)$ is absolute convergent according to Lemma 12.

Geometrically, $\rho(P)$ is a circle and its interior is convergent. On the outside the power series is divergent. The convergence property at the circle itself is unknown (not generally uniform). □

Lemma 14. *Let $z \in \mathbb{C}$, P is a power series and $|z| > \rho(P)$. Then $\sum_{k=0}^{\infty} a_k z^k$ is divergent for this point.*

Proof. Proof by contradiction. Assume $P(z)$ is convergent and $|z| > \rho(P)$. Let $\varepsilon = 2(|z| - \rho(P))$. Then $\rho(P) + \varepsilon < |z|$ with $\rho(P) + \varepsilon > \rho(P)$. From the previous lemma it follows that $P(\rho(P) + \varepsilon)$ is convergent. But this contradicts with $\rho(P) = \sup\{r \geq 0 : P(r) \text{ is convergent}\}$. □

Remark 18. *$B(0, \rho(P))$ is called* convergence circle of P.

Theorem 60 (Formulas to compute $\rho(P)$)**.** *Let $P(z) = \sum_{k=0}^{\infty} a_k z^k$ be a power series. Then it holds in every case that,*

- *$\rho(P) = \frac{1}{L}$ with $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (for $L = \infty$ if $\left(\sqrt[n]{|a_n|}\right)_{n\in\mathbb{N}}$ is unbounded and $\frac{1}{\infty} := 0$) (Cauchy & Hadamard)*
- *If $q := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then the convergence circle of this power series is $\frac{1}{q}$:*

$$\rho(P) = \frac{1}{q}$$

with $\frac{1}{0} := \infty$ and $\frac{1}{\infty} = 0$.

German keywords

English keywords