# Linear Algebra 2 – Practicals

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# 1 Exercise 1

**Exercise 1.** Determine the matrix representation of the linear map

$$f : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$$

$$p(x) \mapsto (x-1) \cdot p(x)$$

in regards of bases $B = \{1-x, 1+x\} \subseteq \mathbb{R}_1[x]$ and $C = \left\{1, 1+x, 1+x+x^2\right\} \subseteq \mathbb{R}^2[x]$.

$$\begin{aligned} f &: \mathbb{R}_1[x] \to \mathbb{R}_2[x] \\ f &: p(x) \mapsto (x-1)p(x) \\ B &= \{1-x, 1+x\} =: \{b_1, b_2\} \\ C &= \left\{1, 1+x, 1+x+x^2\right\} =: \{c_1, c_2, c_3\} \end{aligned}$$

Find $A \in \mathbb{K}^{3\times 2} =: M_C^B(f)$.

$$\forall v \in \mathbb{R}_1 : f(v) = w : \Phi_C(w) = A\Phi_B(v)$$

$$\begin{aligned} f(b_1) &= (1-x)(x-1) = -x^2 + 2x - 1 \\ f(b_2) &= (x-1)(x+1) = x^2 - 1 \end{aligned}$$

$$\Phi_C(f(b_1))$$

Coefficient comparison:

$$\begin{aligned} -x^2 + 2x - 1 &= \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2) \\ x^2 :& \lambda_3 = -1 \\ x^1 :& 2 = \lambda_2 + \lambda_3 \Rightarrow \lambda_2 = 3 \\ x^0 :& -1 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_1 = -3 \end{aligned}$$

$$\Phi_C(f(b_1)) = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\Phi_C(f(b_2)) : x^2 = 1 = \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2)$$

$$\begin{aligned} x^2 :& \lambda_3 = 1 \\ x^1 :& \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -1 \\ x^0 :& -1 = \lambda_1 + \lambda_2 + \lambda_3 \\ & -1 = \lambda_1 - 1 + 1 \\ & -1 = \lambda_1 \end{aligned}$$

$$\Phi_C(f(b_2)) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & -1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$$

## 2 Exercise 3

**Exercise 2.** Let $A_1, A_2, \ldots, A_k$ be quadratic $n \times n$ matrices over the field $\mathbb{K}$. Show that the product $A_1 A_2 \ldots A_k$ is invertible if and only if all $A_i$ are invertible.

All $A_i$ are invertible, then $\prod A_i$ is invertible.

$A, B$ invertible, then $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Generalize by induction.

If $\prod A_i$ is invertible, then all $A_i$ are invertible.

Sidenote: We know that $\operatorname{rank}(A) = n - \dim \operatorname{kernel}(A)$.

$k = 1$ trivial

$k = 2$ $A_1 A_2$ is invertible. Let $C = (A_1 A_2)^{-1}$. Then $C A_1 A_2 = I_n$. Let $x \in \operatorname{kernel}(A_2) \Rightarrow A_2 x = 0 \Rightarrow \underbrace{C A_1}_{I_n} A_2 x = C A_1 0 = 0$.

$$\operatorname{kernel}(A_2) = 0 \Rightarrow \operatorname{rank}(A_2) = n - 0 : n \Rightarrow A_2 \text{ invertible}$$

$$A_1 = \underbrace{A_1 A_2}_{\text{invertible}} \cdot \underbrace{A_2^{-1}}_{\text{invertible}}$$

$k \to k+1$ Let $A_1 \ldots A_{k+1}$ is invertible $\Rightarrow$ $(A_1, \ldots, A_k) A_{k+1}$ is invertible $\xRightarrow{k=2}$ $A_1, \ldots, A_k$ is invertible, $A_{k+1}$ invertible $\xRightarrow{\text{induction base}}$ $A_1, \ldots, A_k, A_{k+1}$ is invertible.

Remark: $A, B \in \mathbb{K}^{n \times n}$. $B$ is inverse of $A$

$$\Leftrightarrow AB = I = BA \Leftrightarrow AB = I \Leftrightarrow BA = I$$

## 3 Exercise 2

**Exercise 3.** Let $V$ be a vector space and $f : V \to \mathbb{V}$ is a nilpotent linear map, hence there exists some $k \in \mathbb{N}$ such that $f^k = 0$.

### 3.1 Part a

**Exercise 4.** Show that $\mathrm{id}_V - f$ is invertible with $(\mathrm{id}_V - f)^{-1} = \mathrm{id}_V + f + f^2 + \ldots + f^{k-1}$.

Show that: $(\mathrm{id}_v - f)^{-1} = \sum_{i=0}^{k-1} f^i$.

$$(\mathrm{id}_V - f) \circ \left( \sum_{i=0}^{k-1} f^i \right) = \mathrm{id}_V \circ \sum_{i=0}^{k-1} f^i - f \circ \sum_{i=0}^{k-1} f^i - \sum_{i=0}^{k-1} f^{i+1} = f^0 + \sum_{i=1}^{k-1} f^i - \sum_{i=1}^{k-1} f^i - f^k = \mathrm{id}_V - 0 = \mathrm{id}_V$$

and $\left( \sum_{i=0}^{k-1} f^i \right) \circ (\mathrm{id}_V - f)$ analogously.

### 3.2 Part b

**Exercise 5.** Use part a) to determine the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1\end{pmatrix} =: A = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} - f_A$$

$$f_A = I_n - A = \begin{pmatrix}0&-2&-3&-4\\0&0&-2&-3\\0&0&0&-2\\0&0&0&0\end{pmatrix}$$

$$f_A^2 = f \cdot f = \begin{pmatrix}0&0&4&12\\0&0&0&4\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$f^3 = f^2 \cdot f = \begin{pmatrix}0&0&0&-8\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$f^4 = f^3 \cdot f = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$\Rightarrow$ f nilpotent.

$$A^{-1} = (\mathrm{id}_v - f)^{-1} = \mathrm{id}_v + f + f^2 + f^3$$

$$= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} + \begin{pmatrix}0&-2&-3&-4\\0&0&-2&-3\\0&0&0&-2\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&4&12\\0&0&0&4\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&-8\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&-2&1&0\\0&1&-2&1\\0&0&1&-2\\0&0&0&1\end{pmatrix}$$

$$A \cdot A' = \begin{pmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1\end{pmatrix} \cdot \begin{pmatrix}1&-2&1&0\\0&1&-2&1\\0&0&1&-2\\0&0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

# 4 Exercise 4

## 4.1 Part a

**Exercise 6.** Let $A$ be an invertible $n \times n$ matrix over a field $\mathbb{K}$ and $u, v$ are column vectors (hence $n \times 1$ matrices), such that $\sigma 1 + v^t A^{-1} u \neq 0$. Show that $(A + uv^t)$ is invertible and that

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} uv^t A^{-1}$$

## 4.2 Part b

**Exercise 7.** Apply this formula to determine the inverse of the matrix

$$A = \begin{pmatrix} 5 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

$$B = A + S$$

$$B = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

$A$ is invertible, because it is a block matrix[1].

$$A^{-1} = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$$\sigma = 1 + \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 \neq 0$$

$$\Rightarrow B^{-1} = A^{-1} - A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 2 & -3 & 6 & -4 \\ -3 & 5 & -9 & 6 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

[1]That's why chose $A$ and $S$ that way