Linear Algebra 2 – Practicals

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Contents

Exercise I did on the board: 3, 7.

1 Exercise 1

Exercise 1. Determine the matrix representation of the linear map

$$f : \mathbb{R}_1[x] \to \mathbb{R}_2[x]$$

$$p(x) \mapsto (x-1) \cdot p(x)$$

in regards of bases $B = \{1-x, 1+x\} \subseteq \mathbb{R}_1[x]$ and $C = \left\{1, 1+x, 1+x+x^2\right\} \subseteq \mathbb{R}^2[x]$.

$$\begin{aligned} f &: \mathbb{R}_1[x] \to \mathbb{R}_2[x] \\ f &: p(x) \mapsto (x-1)p(x) \\ B &= \{1-x, 1+x\} =: \{b_1, b_2\} \\ C &= \left\{1, 1+x, 1+x+x^2\right\} =: \{c_1, c_2, c_3\} \end{aligned}$$

Find $A \in \mathbb{K}^{3\times 2} =: M_C^B(f)$.

$$\forall v \in \mathbb{R}_1 : f(v) = w : \Phi_C(w) = A\Phi_B(v)$$

$$\begin{aligned} f(b_1) &= (1-x)(x-1) = -x^2 + 2x - 1 \\ f(b_2) &= (x-1)(x+1) = x^2 - 1 \end{aligned}$$

$$\Phi_C(f(b_1))$$

Coefficient comparison:

$$\begin{aligned} -x^2 + 2x - 1 &= \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2) \\ x^2 :& \lambda_3 = -1 \\ x^1 :& 2 = \lambda_2 + \lambda_3 \Rightarrow \lambda_2 = 3 \\ x^0 :& -1 = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_1 = -3 \end{aligned}$$

$$\Phi_C(f(b_1)) = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$$

$$\Phi_C(f(b_2)) : x^2 = 1 = \lambda_1 \cdot 1 + \lambda_2(1+x) + \lambda_3(1+x+x^2)$$

$$\begin{aligned} x^2 :& \lambda_3 = 1 \\ x^1 :& \lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -1 \\ x^0 :& -1 = \lambda_1 + \lambda_2 + \lambda_3 \\ & -1 = \lambda_1 - 1 + 1 \\ & -1 = \lambda_1 \end{aligned}$$

$$\Phi_C(f(b_2)) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} -3 & -1 \\ 3 & -1 \\ 1 & 1 \end{pmatrix}$$

2 Exercise 3

Exercise 2. Let $A_1, A_2, \dots, A_k$ be quadratic $n \times n$ matrices over the field $\mathbb{K}$. Show that the product $A_1 A_2 \dots A_k$ is invertible if and only if all A_i are invertible.

All A_i are invertible, then $\prod A_i$ is invertible.

A, B invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. Generalize by induction.

If $\prod A_i$ is invertible, then all A_i are invertible.

Sidenote: We know that $\operatorname{rank}(A) = n - \dim \operatorname{kernel}(A)$.

$k = 1$ trivial

$k = 2$ $A_1 A_2$ is invertible. Let $C = (A_1 A_2)^{-1}$. Then $CA_1A_2 = I_n$. Let $x \in \operatorname{kernel}(A_2) \Rightarrow A_2 x = 0 \Rightarrow \underbrace{CA_1}_{I_n} A_2 x =$

$CA_1 0 = 0$.

$$\operatorname{kernel}(A_2) = 0 \Rightarrow \operatorname{rank}(A_2) = n - 0 : n \Rightarrow A_2 \text{ invertible}$$

$$A_1 = \underbrace{A_1 A_2}_{\text{invertible}} \cdot \underbrace{A_2^{-1}}_{\text{invertible}}$$

$k \to k+1$ Let $A_1 \dots A_{k+1}$ is invertible $\Rightarrow$ $(A_1, \dots, A_k) A_{k+1}$ is invertible $\overset{k=2}{\Longrightarrow}$ $A_1, \dots, A_k$ is invertible, A_{k+1} invertible $\xRightarrow{\text{induction base}}$ $A_1, \dots, A_k, A_{k+1}$ is invertible.

Remark: $A, B \in \mathbb{K}^{n \times n}$. B is inverse of A

$$\Leftrightarrow AB = I = BA \Leftrightarrow AB = I \Leftrightarrow BA = I$$

3 Exercise 2

Exercise 3. Let V be a vector space and $f : V \to \mathbb{V}$ is a nilpotent linear map, hence there exists some $k \in \mathbb{N}$ such that $f^k = 0$.

3.1 Part a

Exercise 4. Show that $\mathrm{id}_V - f$ is invertible with $(\mathrm{id}_V - f)^{-1} = \mathrm{id}_V + f + f^2 + \dots + f^{k-1}$.

Show that: $(\mathrm{id}_v - f)^{-1} = \sum_{i=0}^{k-1} f^i$.

$$(\mathrm{id}_V - f) \circ \left(\sum_{i=0}^{k-1} f^i \right) = \mathrm{id}_V \circ \sum_{i=0}^{k-1} f^i - f \circ \sum_{i=0}^{k-1} f^i - \sum_{i=0}^{k-1} f^{i+1} = f^0 + \sum_{i=1}^{k-1} f^i - \sum_{i=1}^{k-1} f^i - f^k = \mathrm{id}_V - 0 = \mathrm{id}_V$$

and $\left(\sum_{i=0}^{k-1} f^i \right) \circ (\mathrm{id}_V - f)$ analogously.

3.2 Part b

Exercise 5. Use part a) to determine the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} =: A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - f_A$$

$$f_A = I_n - A = \begin{pmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f_A^2 = f \cdot f = \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^3 = f^2 \cdot f = \begin{pmatrix} 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f^4 = f^3 \cdot f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ f nilpotent.

$$A^{-1} = (\mathrm{id}_v - f)^{-1} = \mathrm{id}_v + f + f^2 + f^3$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -3 & -4 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A \cdot A' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4 Exercise 4

4.1 Part a

Exercise 6. Let A be an invertible $n \times n$ matrix over a field $\mathbb{K}$ and u, v are column vectors (hence $n \times 1$

matrices), such that $\sigma 1 + v^t A^{-1} u \neq 0$. Show that $(A + uv^t)$ is invertible and that

$$(A + uv^t)^{-1} = A^{-1} - \frac{1}{\sigma} A^{-1} u v^t A^{-1}$$

4.2 Part b

Exercise 7. Apply this formula to determine the inverse of the matrix

$$A = \begin{pmatrix} 5 & 3 & 0 & 1 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix}$$

$$B = A + S$$

$$B = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 3 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$$

A is invertible, because it is a block matrix[1].

$$A^{-1} = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -3 & 5 & 0 & 0 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$$\sigma = 1 + \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 + 0 \neq 0$$

$$\Rightarrow B^{-1} = A^{-1} - A^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 2 & -3 & 6 & -4 \\ -3 & 5 & -9 & 6 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

5 Exercise 5

Exercise 8. Show that the linear maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f : (x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2) \quad g : (x_1, x_2) \mapsto (x_1 + x_2, x_1 + x_2) \quad h : (x_1, x_2) \mapsto (x_2, x_1)$$

are linear independent, if they are considered as elements of the vector space $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ of all maps from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Show that

$$\lambda_1 f + \lambda_2 g + \lambda_3 h = 0 \overset{!}{=} \lambda_1 = \lambda_2 = \lambda_3 = 0$$

[1]That's why chose A and S that way

$$f : x \mapsto Ax \qquad A_f = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad A_g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Is an isomorphism, $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) \to \mathbb{R}^{2\times 2}$ with $f \mapsto A_f$.

$$\lambda_1 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \lambda_i = 0 \forall i \in \{1, 2, 3\}$$

6 Exercise 6

Exercise 9. Let V be a vector space with $\dim V = n < \infty$ and $U \subseteq V$ is a subspace with $\dim U = m$.

1. Show that
$$U^\perp = \{v^* \in V^* \mid U \subseteq \mathrm{kernel}(v^*)\}$$
is a subspace of V^*.
2. Determine $\dim U^\perp$.
3. Is $\{v^* \in V^* \mid U = \mathrm{kernel}\, v^*\}$ also a subspace?

$U^\perp$ is called orthogonal space or annihilation of U.

1.
$$U^\perp = \{v^* \in V^* \mid U \subseteq \mathrm{kernel}(v^*)\}$$

$v^* \in \mathrm{Hom}(V, \mathbb{K})$.
$$\mathrm{kernel}(v^*) = \{x \in V \mid v^*(x) = 0\} \supseteq U \Leftrightarrow \forall x \in U : v^*(x) = 0$$

$\mathbf{U^\perp}$ is nonempty
The constant zero-function $u : V \to \mathbb{K}$ with $x \mapsto 0 \in U^\perp$ exists. Hence $U^\perp \neq \emptyset$.

Additivity: $\bigwedge_{\mathbf{u_1}, \mathbf{u_2} \in \mathbf{U}^\perp} \mathbf{u_1} + \mathbf{u_2} \in \mathbf{U}^\perp$
Let $u_1, u_2 \in U^\perp$ be linear. Let $x \in U$.

$$(u_1 + u_2)(x) = \underbrace{u_1(x)}_{\in U^\perp} + \underbrace{u_2(x)}_{\in U^\perp} = 0 + 0 = 0$$

Multiplication: $\bigwedge_{\lambda \in \mathbb{K}} \bigwedge_{\mathbf{u} \in \mathbf{U}^\perp} \lambda \cdot \mathbf{u} \in \mathbf{U}^\perp$
Let $\lambda \in \mathbb{K}$, $u \in U^\perp$ and $x \in U$.

$$(\lambda \cdot u)(x) = \lambda \cdot \underbrace{u(x)}_{\in U^\perp} \Rightarrow \lambda \cdot 0 = 0$$

2.
$$\dim V = n \qquad \dim V^* = n \qquad \dim U = m$$

U is subspace of V, so $m \leq n$.
$$k := \dim U^\perp \leq n = \dim V^*$$

Let $(u_1, \dots, u_m)$ be basis of U.

We apply the *basis extension theorem*: Let $(u_1, \dots, u_m, u_{m+1}, \dots, u_n)$ be a basis of V.

Let $(v_1^*, \dots, v_n^*)$ the dual basis to $(v_1, \dots, v_n)$ to V^*. Hence

$$v_1^*(v_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Claim: $U^\perp = L(\{v_{m+1}^*, \dots, v_n^*\}) \Rightarrow (v_{m+1}^*, \dots, v_n^*)$ is basis of $U^\perp \Rightarrow \dim U^\perp = n - m$.
Let $v \in V^*$ be arbitrary, $v = \lambda_1 v_1^* + \dots + \lambda_n v_n^*$.

$$v \in U^\perp \Leftrightarrow \forall x \in U : v(x) = 0 \Leftrightarrow v|_U = 0 \xLeftrightarrow{(u_1,\dots,u_m) \text{ is basis of } U} v(u_i) = 0 \quad i = 1, \dots, m$$

$$\begin{aligned}
&\Leftrightarrow \forall i \in \{1, \dots, m\} \, (\lambda_1 v_1^* + \dots + \lambda_n v_n^*)(v_i) = 0 \\
&\Leftrightarrow \forall i \in \{1, \dots, m\} \, v_1 v_1^*(v_i) + \dots + \lambda_n v_n^*(v_i) = 0 \\
&\Leftrightarrow v^k \in L(v_{m+1}^*, \dots, v_n^*) \\
&\Leftrightarrow \forall i \in \{1, \dots, m\} \, \lambda_i = 0
\end{aligned}$$

$$\begin{aligned}
\pi : V &\to {}^V\!/_U \\
x &\mapsto v + U \\
\pi^t : ({}^V\!/_U)^* &\to V^* \\
w &\to w \circ \pi
\end{aligned}$$

π surjective, then π^t is injective and

$$\text{image}(\pi^t) = U^t \Rightarrow {{}^V\!/_U}^k \to U^\perp$$

3. Is $\{v^* \in V^* \mid U = \text{kernel}\, v^*\}$ also a subspace?

   Counterexample: Let $u = \{0\}$ and $V \neq \{0\}$.

   $$\text{kernel}(v^*) = \{x \in V \mid x^*(x) = 0\} = \{0\} = U$$

   If it is a subspace, then the constant null function (which is the zero element of this set) must be contained. This is a contradiction to "only $x = 0$ maps to 0".

7 Exercise 8

Exercise 10. Let $\mathbb{R}[x]$ be the vector space of real polynomials. Show that the dimension of the dual space $\mathbb{R}[x]^*$ is overcountable.

Hint: Show that linear functionals $(\delta_t)_{t \in \mathbb{R}}$ defined as $\langle \delta_t, p(x) \rangle = p(t)$ (function application) is linear independent.

"In welchem Vektorraum leben wir?" (Florian Kainrath)

δ_t are linear maps.

$$\begin{aligned}
\forall p \in \mathbb{R}[x] &: \sum_{i=1}^n \lambda_t \delta_{t_i}(p(x)) = 0 \Rightarrow \lambda_i = 0 \forall i \in \{1, \dots, n\} \\
\forall p \in \mathbb{R}[x] &: \sum_{i=1}^n \lambda_t p(t_i) = 0 \Rightarrow \lambda_i = 0
\end{aligned}$$

Consider the polynomial $(x - t_1)(x - t_2) \dots (x - \hat{t}_j)(x - t_{j+1}) \dots (x - t_n) = p(x)$.

$$\Rightarrow \sum_{i=1}^n \lambda_i p_j(t_i) = 0 \Leftrightarrow \lambda_j p_j(t_j) = 0 = \lambda_j = 0$$

8 Exercise 9

Exercise 11. Let $f \in \mathrm{Hom}(V,W)$ be a linear map between two finite-fimensional vector spaces with bases $B \subseteq V$ and $C \subseteq W$. Show that the matrix representation of the transposed map

$$f^t : W^* \to V^*$$

$$w^* \mapsto w^* \circ f$$

in regards of the dual basis C^* and B^* has the matrix representation

$$\Phi_{B^*}^{C^*}(f^t) = \Phi_C^B(f)^t$$

Show that $f \in \mathrm{Hom}(V,W)$ and $B = (b_1, \dots, b_m)$ is basis of V with dual basis $B^* = (b_1^*, \dots, b_m^*)$. $C = (c_1, \dots, c_n)$ is basis of W with dual basis $C^* = (c_1^*, \dots, c_n^*)$.

$$\Phi_{B^*}^{C^*}(f^t) = \Phi_C^B(f)^t$$

$$A := \Phi_C^B(f)$$

$\Phi_{B^*}^{C^*}(f^t) = P = A^t \forall i \in \{1, \dots, n\}\ j \in \{1, \dots, m\}$ and $a_{ij} = p_{ji}$. $A \in \mathbb{K}^{n \times m}$ and $P \in \mathbb{K}^{m \times n}$.

$$(a_{ij}) = A = \Phi_C^B(f) \Leftrightarrow \forall j \in \{1, \dots, m\}$$

$$\Phi_C(f(b_j)) = A\Phi_B(b_j) = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \Leftrightarrow A = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \Phi_C^{-1}$$

$$f(b_j) = \sum_{i=1}^{n} a_{ij} c_i \qquad \forall j \in \{1, \dots, m\}$$

$$(p_{ij}) = p = \Phi_{B^*}^{C^*}(f^t) \Leftrightarrow f^t(c_j^*) = \sum_{i=1}^{m} p_{ij} b_i^* \forall j \in \{1, \dots, n\}$$

$$\Leftrightarrow f^t(c_j^*) \text{ with } j \in \{1, \dots, n\} = \sum_{i=1}^{m} p_{ij} b_i^* \overset{w}{\Leftrightarrow} c_i \circ f = \sum_{i=1}^{m} p_{ij} b_i^* \forall j \in \{1, \dots, n\}$$

Show that $a_{kj} = p_{ik}$ with $k \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.

$$a_{kj} = C_k^* \left(\sum_{i=1}^{n} a_{ij} c_i \right) = c_k^* \left(f(b_j) \right) = \left(f^t(c_k^*)(b_j) \right) = \left(\sum_{i=1}^{m} p_{ik} b_i^* \right)(b_i) = p_{jk}$$

9 Exercise 10

Exercise 12.

- Determine the dual basis of $(\mathbb{R}^4)^*$ to the basis.

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- Determine the matrix of the unique (why?) projection map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ with $\mathrm{image}(\varphi) = \mathcal{L}\left\{(1,2,1,0)^t, (1,0,-1,1)^t\right\}$ and $\mathrm{kernel}(\varphi) = \mathcal{L}\left\{(-1,-2,2,-1)^t, (2,-1,1,1)^t\right\}$.

9.1 Exercise 10.a

$$\begin{pmatrix} 1 & 1 & -1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -3 & 1 & 2 & 5 \\ 0 & 1 & 0 & 0 & -9 & 2 & 5 & 15 \\ 0 & 0 & 1 & 0 & -5 & 1 & 3 & 8 \\ 0 & 0 & 0 & 1 & 4 & -1 & -2 & -6 \end{pmatrix}$$

So

$$b_1^* = \begin{pmatrix} -3 \\ 1 \\ 2 \\ 5 \end{pmatrix} \quad b_2^* = \begin{pmatrix} -9 \\ 2 \\ 5 \\ 15 \end{pmatrix} \quad b_3^* = \begin{pmatrix} -5 \\ 1 \\ 3 \\ 8 \end{pmatrix} \quad b_4^* = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -6 \end{pmatrix}$$

$$B^* = \begin{pmatrix} -3 & 1 & 2 & 5 \\ -9 & 2 & 5 & 15 \\ -5 & 1 & 3 & 8 \\ 4 & -1 & -2 & -6 \end{pmatrix}$$

$$(\mathbb{R}^n)^* \cong \mathbb{R}^{1\times 4}$$
$$b_i^*(b_j) = \delta_{ij}$$

9.2 Exercise 10.b

Find a projective map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ such that $U_1 = \varphi(\mathbb{R}^4)$. So image$(\varphi) = \mathcal{L}(U_1)$ and kernel$(\varphi) = U_2$.

$$U_1 = \mathcal{L}\left\{(1,2,1,0)^t, (1,0,-1,1)^t\right\}$$
$$U_2 = \mathcal{L}\left\{(-1,-2,2,-1)^t, (2,-1,1,1)^t\right\}$$

Why do we get a unique map?

φ is a projection map iff φ is linear and $\varphi \circ \varphi = \varphi$. Consider $b_1 \in U_1 = \varphi(\mathbb{R}^4)$ and $b_1 = \varphi(x) \quad x \in \mathbb{R}^4$. $\varphi(b_1) = \varphi(\varphi(x)) = \varphi(x) = b_1$. This isomorphism ensures that the solution is unique.

Because $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$, the linear map will be represented by a 4×4 matrix.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 \\ -1 & -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 & -12 & -6 & 6 & -9 \\ 0 & 1 & 0 & 0 & 3 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 & 7 & 4 & -3 & 5 \\ 0 & 0 & 0 & 1 & 20 & 10 & -10 & 15 \end{pmatrix}$$

$$\begin{pmatrix} -12 & 3 & 7 & 20 \\ -6 & 2 & 4 & 10 \\ 6 & -1 & -3 & -10 \\ 9 & 2 & 5 & 15 \end{pmatrix}$$

10 Exercise 11

Exercise 13. Given the permutation

$$\pi = \left(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}\right)$$

- Determine π^{-1} and π^k for some $k \in \mathbb{N}$.
- Determine all inversions of π and determine sign(π).

- Decompose π in a product of transpositions.

10.1 Exercise 11.a

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}$$

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$$

We give a recursive definition:

$$\pi^k_{(i)} = \begin{cases} \pi^{k \bmod 4}_{(i)} & i \in \{1,2,3,5\} \\ \pi^{k \bmod 3}_{(i)} & i \in \{4,6,7\} \end{cases}$$

10.2 Exercise 11.b

Inversions are:

$$f_\pi = \{(i,j) \mid i < j \wedge \pi(i) > \pi(j)\}$$
$$F_\pi = \{(1,3),(2,3),(2,5),(2,7),(4,5),(4,7),(6,7)\}$$

$$\operatorname{sign}(\pi) = (-1)^f_\pi = -1$$

10.3 Exercise 11.c

$$\pi \circ \tau_{1,3} = (1\ 5\ 2\ 6\ 3\ 7\ 4)$$
$$\pi \circ \tau_{1,3} \circ \tau_{2,3} \circ \tau_{3,5} \circ \tau_{4,7} \circ \tau_{6,7} = \text{ id}$$
$$\pi = \tau_{6,7} \circ \tau_{4,7} \circ \tau_{3,5} \circ \tau_{2,3} \circ \tau_{1,3}$$

In terms of notation, remember:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix} \circ \tau_{i,j} = \begin{pmatrix} 1 & i & j & n \\ & \pi(j) & \pi(i) & \end{pmatrix}$$

11 Exercise 12

Exercise 14. A permutation $\pi \in \mathfrak{S}_n$ is called cyclic, if there exists some $k \geq 1$ and a sequence $i_1, i_2, \dots, i_k$ such that $\pi(i_j) = i_{j+1}$ for $1 \leq j \leq k-1$, $\pi(i_k) = i_1$ and $\pi(i) = i$ for $i \notin \{i_1, i_2, \dots, i_k\}$, hence

$$i_1 \to i_2 \to \dots \to i_k \to i_1$$

and all other i are fixed. Common notation: $\pi = (i_1, i_2, \dots, i_k)$.

- Show that two cyclic permutations $\pi = (i_1, i_2, \dots, i_k)$ and $\rho = (j_1, j_2, \dots, j_l)$ commute ($\pi \circ \rho = \rho \circ \pi$) if $\{i_1, \dots, i_k\} \cap \{j_1, \dots, j_l\} = \emptyset$.
- Decompose the cycle into a product of transpositions and show that for a cyclic permutation it holds that $\operatorname{sign}(\pi) = (-1)^{k-1}$.

11.1 Exercise 12.a

Case 1: $m \in \{i_1, i_2, \ldots, i_k\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = \pi(m)$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = \pi(m)$$

Case 2: $m \in \{j_1, j_2, \ldots, j_l\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = \rho(m)$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = \rho(m)$$

Case 3: $m \notin \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_l\}$

$$\pi \circ \rho(m) = \pi(\rho(m)) = m$$
$$\rho \circ \pi(m) = \rho(\pi(m)) = m$$

11.2 Exercise 12.b

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_2 & i_3 \ldots & i_1 & \ldots & n \end{pmatrix}$$
$$\pi \circ \tau_{i_1, i_k} = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_1 & i_3 \ldots & i_2 & \ldots & n \end{pmatrix}$$
$$\pi \circ \tau_{i_1, i_k} \circ \tau_{i_2, i_k} = \begin{pmatrix} 1 & 2 & \ldots & i_1 & i_2 & i_3 & \ldots & i_k & \ldots & n \\ 1 & 2 & \ldots & i_1 & i_2 & i_4 & \ldots & i_3 & \ldots & n \end{pmatrix}$$
$$\tau \circ \tau_{i_1, i_k} \circ \tau_{i_2, i_k} \circ \ldots \circ \tau_{i_{k-1}, i_k} = \mathrm{id}$$
$$\pi = \tau_{i_{k-1}, i_k} \circ \ldots \circ \tau_{i_l, i_{l+1}} \circ \ldots \circ \tau_{i_1, i_k}$$

11.3 Exercise 13

Exercise 15. Let $\pi \in \mathfrak{S}_n$ be a permutation and $i \in \{1, 2, \ldots, n\}$.

- Show that the sequence i, $\pi(i)$, $\pi^2(i)$, … is periodic and the first number which occurs twice is i.
- The sequence $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$ where k is the smallest exponent such that $\pi^k(i) = i$, is called cycle of i. Show that the relation, $i \sim j :\Leftrightarrow j$ is in cycle of i, is a equivalence relation in $\{1, 2, \ldots, n\}$.
- Show that every permutation can be represented as product of commutative cycles.
- Apply this decomposition for the permutation π from exercise 11.

11.4 Exercise 13.a

- i, $\pi(i)$, …, $\pi^k(i)$ is periodic.
- the first element which occurs twice is i
- $$\left\{ \pi^k(i) \,\middle|\, k \in \{1, \ldots, n+1\} \right\}$$
  at least one elemtn must have occured twice.
- $$\pi^k(i) = \pi^l(i)$$
  wlog. $k > l$
  $$\pi^{k-l}(i) = i \qquad k - l < k$$

$$\pi^{k-l}(i) = (\pi^l)^{-1} \left(\pi^k(\tau) \right) = (\pi^e)^{-1} (\pi^e(i))$$

11.5 Exercise 13.b

reflexive

$$i \sim i \quad \Leftrightarrow \quad \exists k : \pi^k(i) = i$$

symmetrical

$$i \sim j \Rightarrow j \sim i \qquad \exists l : \pi^l(i) = j \quad \pi^k(i) = i \quad \pi^{k-l}(i) = i$$

transitive

$$i \sim j \land j \sim m \Rightarrow i \sim m \qquad (\exists l_1 : \pi^{l_1}(i) = j) \land (\exists l_2 : \pi^{l_2}(j) = m)$$
$$\Rightarrow \exists l_3 = l_1 + l_2 : \pi^{l_3}(i) = m$$

11.6 Exercise 13.c

Lengthy and therefore skipped.

11.7 Exercise 13.d

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 6 & 3 & 7 & 4 \end{pmatrix}$$
$$\pi = (1\ 2\ 5\ 3)(4\ 6\ 7)$$

12 Exercise 14

Exercise 16. Determine the determinant of the following matrix using three different methods (Leibniz, Laplace, Gauß-Jordan).

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

Using Leibniz' definition:

$$\det(A) = 1 \cdot (-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + (-1)^{2+1} \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} + 2(-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}$$

Using Gauß' definition:

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -5 & -4 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = -1$$

Using Leibniz' definition:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 2 & -1 & 2 \end{vmatrix} = 1 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 2 + 3 \cdot 1 \cdot (-1) - 2 \cdot 1 \cdot 3 - (-1) \cdot 2 \cdot 1 - 2 \cdot 1 \cdot 2 = -1$$

13 Exercise 15

Exercise 17. The numbers 18984, 10962, 40026, 17976 and 14994 are divisible by 42. Show that the determinant of A is divisible by 42 without explicitly computing it.

$$A = \begin{pmatrix} 1 & 8 & 9 & 8 & 4 \\ 1 & 0 & 9 & 6 & 2 \\ 4 & 0 & 0 & 2 & 6 \\ 1 & 7 & 9 & 7 & 6 \\ 1 & 4 & 9 & 9 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 8 & 9 & 8 & 4 \\ 1 & 0 & 9 & 6 & 2 \\ 4 & 0 & 0 & 2 & 6 \\ 1 & 7 & 9 & 7 & 6 \\ 1 & 4 & 9 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 8 & 9 & 8 & 18984 \\ 1 & 0 & 9 & 6 & 10962 \\ 4 & 0 & 0 & 2 & 40026 \\ 1 & 7 & 9 & 7 & 17976 \\ 1 & 4 & 9 & 9 & 14994 \end{vmatrix} = 42 \cdot B$$

where B is some matrix with modified 5-th column.

Why does this work? Well, this can be proven using Leibniz' definition of the determinant.

$$\det((a_{ij})) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_1 \dots$$

14 Exercise 16

Exercise 18. Compute the $n \times n$-determinants:

1.
$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & 0 & 3 & 4 & \dots & n-1 & n \\ -1 & -2 & 0 & 4 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & -4 & \dots & 0 & n \\ -1 & -2 & -3 & -4 & \dots & -n+1 & 0 \end{pmatrix}$$

2.
$$\begin{pmatrix} 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & \dots & a_{n-1} & * \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & a_2 & * & \dots & * \\ a_1 & * & \dots & & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ -1 & 0 & 3 & 4 & \dots & n-1 & n \\ -1 & -2 & 0 & 4 & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & -4 & \dots & 0 & n \\ -1 & -2 & -3 & -4 & \dots & -n+1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 2 & * & * & \dots & n-1 & n \\ 0 & 0 & 3 & * & \dots & n-1 & n \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & 0 & n \end{pmatrix} = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n = n!$$

$$\begin{vmatrix} 0 & 0 & \dots & 0 & a_n \\ 0 & 0 & \dots & a_{n-1} & * \\ \vdots & & \vdots & \vdots & * \\ 0 & a_2 & * & \dots & * \\ a_1 & * & \dots & & * \end{vmatrix} = (-1)^k \begin{vmatrix} a_1 & * & \dots & * & a_n \\ 0 & a_2 & \dots & \ddots & * \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & & a_{n-1} & * \\ 0 & 0 & \dots & 0 & a_n \end{vmatrix} = \left(\prod_{k=1}^{n} a_k \right) (-1)^k$$

where $k = \frac{n}{2}$ is n is even or $k = \frac{n-1}{2}$ is odd.

15 Exercise 17

Exercise 19. Let $A \in \mathbb{K}_{m\times m}$, $B \in \mathbb{K}_{m\times n}$, $D \in \mathbb{K}_{n\times n}$ matrices. Show that,

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \cdot \det D$$

Let $T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$

If A is singular, the rows are linear dependent. So $\det T = 0$. The same applies to D.

We apply row operations to A to retrieve an upper triangular matrix A_1. If we do the same operations on T, we get B_1. We apply row oeprations to D to retrieve an upper triangular matrix D_1.

$$\hat{T} = \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}$$

Let a be the product of diagonal elements of A_1. Let d be the product of diagonal elements of D_1.

So $a \cdot d$ is the product of diagonal elements of $\hat{T}$.

Let p be the number of swaps in A_1. Let q be the number of swaps in A_2.

$$p + q = \hat{T}$$

Then

$$\det A = (-1)^p a \qquad \det D = (-1)^q b$$
$$\det T = (-1)^{p+q} a \cdot b$$

16 Exercise 18

Exercise 20. Compute the entry $(A^{-1})_{4,3}$ of the inverse matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2 & -1 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

We compute the inverse matrix A^{-1}.

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 2 & -1 & -2 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 & -2 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

But we can also use the Theorem from the lecture.

Use the adjoint matrix $\hat{A}$ of A where $\hat{a}_{kl} = (-1)^{k+l} \det A_{lk}$. Then $A^{-1} = \frac{1}{\det A} \cdot \hat{A}$.

$$A^{-1} = \frac{1}{\det A} \cdot \hat{A}$$
$$A^{-1}_{43} = \frac{1}{\det A}(-1)^{3+4} \det A_{3,4} = -1$$

But we can also determine it more easily. $(A^{-1})_{4,3}$ is the element in the 4th row and 3rd column. It is also the element in the 4-th row of $A^{-1}e_3$.

So

$$A_{e_4} = -e_3$$

So −1.

17 Exercise 19

Exercise 21. Let $\mathbb{K}$ be a field and $a_1, a_2, \dots, a_n \in \mathbb{K}$. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix} = \prod_{i<j} (a_j - a_i)$$

Proof by complete induction over n.

Induction base: $n = 0$ Empty product.

$$\begin{vmatrix} 1 \end{vmatrix} = 1$$

Is true.

Induction step: $n \to n+1$ We start from the last column and add it to the second from last row. This goes on for all columns.

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & & & \ddots & \vdots \\ 1 & a_{n+1} & \dots & \dots & a_{n+1}^n \end{vmatrix} \overset{!}{=} \prod_{\substack{i,j=1 \\ j>i}} (a_j - a_i) \rightsquigarrow \begin{vmatrix} 1 & (a_1 - a_{n+1}) & \dots & a_1^{n-1}(a_1 - a_{n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (a_n - a_{n+1}) & \dots & a_n^{n-1}(a_n - a_{n+1}) \\ 1 & (a_{n+1} - a_{n+1}) & \dots & a_n^{n-1}(a_{n+1} - a_{n+1}) \end{vmatrix}$$

$$= (-1)^{n+1+1}(a_1 - a_{n+1}) \cdot (a_2 - a_{n+1}) \dots (a_n - a_{n+1}) \cdot \begin{vmatrix} 1 & a_1 & \dots & a_1^{n-1} \\ \vdots & & \ddots & \vdots \\ 1 & a_n & \dots & a_n^{n-1} \end{vmatrix}$$

$$\overset{\text{induction hypothesis}}{=} (a_{n+1} - a_1) \dots (a_{n+1} - a_n) \cdot \prod_{\substack{i,j=1 \\ j>i}} (a_j - a_i) = \prod_{j,i=1}^{n+1} (a_j - a_i)$$

18 Exercise 20

Exercise 22. Let $A, B \in \mathbb{K}^{n \times n}$. Show that, using elementary row and column transformations, the following identity holds for block matrices.

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = \begin{vmatrix} I & B \\ 0 & AB \end{vmatrix}$$

Use this to derive an alternative proof for the multiplicity of the determinant.

$$\det(AB) = \det(A) \cdot \det(B)$$

18.1 Exercise 20.a

$$\begin{vmatrix} 1 & 0 & \dots & 0 & b_{11} & b_{12} & \dots & b_{1n} \\ 0 & 1 & \dots & 0 & b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & b_{n,1} & b_{n,2} & \dots & b_{n,n} \\ -a_{11} & -a_{12} & \dots & -a_{1n} & 0 & 0 & \dots & 0 \\ -a_{21} & -a_{22} & \dots & -a_{2n} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & -a_{n,n} & 0 & 0 & \dots & 0 \end{vmatrix}$$

Add the a_{11}-multiple of the first row to the $n+1$-th row. Add the a_{21}-multiple of the first row to the $n+1$-th row. Add the a_{n1}-multiple of the first row to the $2n$-th row.

$$\begin{vmatrix} 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & B \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & \\ & 0 & & & A \cdot B \end{vmatrix}$$

18.2 Exercise 20.b

$$\begin{vmatrix} I & B \\ -A & 0 \end{vmatrix} = (-1)^n \begin{vmatrix} B & I \\ 0 & -A \end{vmatrix} = (-1)^n \cdot \det B \cdot \det -A$$

We multiply n rows by -1,

$$= (-1)^n \cdot (-1)^n \cdot \det B \cdot \det A = \det A \cdot \det B$$

19 Exercise 21

Exercise 23. Let $A, B, C, D \in \mathbb{K}_{n \times n}$ be matrices where D is invertible. Let M be a $2n \times 2n$ block matrix.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

1. Show: M is invertible iff $A - BD^{-1}C) \det D$.
2. Show: $\det M = \det(A - BD^{-1}C) \cdot \det D$.

19.1 Exercise 21.b

$$\det M = \det(A - BD^{-1}C) \cdot \det(D)$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \det \left[\begin{pmatrix} 1 & B \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} A - BD^{-1} & 0 \\ D^{-1}C & 1 \end{pmatrix} \right] = \begin{vmatrix} 1 & B \\ 0 & D \end{vmatrix} \cdot \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{vmatrix}$$

$$= \det(1) \cdot \det(D) \cdot \det(A - BD^{-1}C)$$

$$= \det(D) \cdot \det(A - BD^{-1}C)$$

19.2 Exercise 21.b

M is invertible, so $A - BD^{-1}C$ is invertible. $\det(D) \neq 0$.

$$\begin{aligned} \det M \neq 0 &\Leftrightarrow \det(A - BD^1C) \cdot \det(D) \neq 0 \\ &\Leftrightarrow \det(A - BD^{-1}C) \neq 0 \end{aligned}$$

Corollary of this exercise:

$$\det(AD - BC) = \begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

20 Exercise 22

Exercise 24. Let V be an n-dimensional vector space over a field $\mathbb{K}$ and $\triangle : V^n \to \mathbb{K}$ is a non-trivial determinant form. Furthermore let $a_1, a_2, \dots, a_{n-1} \in V$ vectors. Show that

- the following element is a linear functional with $\mathcal{L}(a_1, a_2, \dots, a_{n-1}) \subseteq \text{kernel}\, v^*$

$$\begin{aligned} v^* : V &\to \mathbb{K} \\ x &\mapsto \triangle(a_1, a_2, \dots, a_{n-1}, x) \end{aligned}$$

- $\mathcal{L}(a_1, a_2, \dots, a_{n-1}) = \text{kernel}\, v^*$ iff $a_1, a_2, \dots, a_{n-1}$ is linear independent.
- Determine the equation (hence, a linear functional v^* such that $\text{kernel}\, v^* = U$)

$$U = \mathcal{L}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \right)$$

20.1 Exercise 22.a

1. Firstly,

$$\begin{aligned} v^*(x_1 + x_2) = v^*(x_1) + v^*(x_2) : v^*(x + 1 + x_2) &= \triangle(a_1, a_2, \dots, a_{n-1}, x_1 + x_2) \\ &= \triangle(a_1, \dots, a_{n-1}, x_1) + \triangle(a_1, \dots, a_{n-1}, x_2) \\ &= v^*(x_1) + v^*(x_2) \end{aligned}$$

Secondly,

$$\begin{aligned} v^*(\lambda x_1) = \lambda v^*(x_1) : v^*(\lambda x_1) &= \triangle(a_1, a_2, \dots, a_{n-1}, \lambda x_1) \\ &= \lambda \triangle(a_1, \dots, a_{n-1}, x_1) \end{aligned}$$

$\mathcal{L}(a_1, \dots, a_{n-1}) \subseteq \text{kernel}(v^*)$ is by definition $\triangle(a_1, \dots, a_n) = 0$ if $i, j \in \{1, \dots, n\}$ and $i \neq j$ and a_1 and a_j are linear independent.

$$\forall i \in \{1, \dots, n-1\} : \triangle(a_1, \dots, a_{n-1}, a_i) = 0$$

20.2 Exercise 22.b

First we show $\Leftarrow$.

Let $a_1, \dots, a_{n-1}$ be linear independent.

$$\mathcal{L}(a_1, \dots, a_{n-1}) \subseteq \text{kernel}(v^*)$$

Assume $\text{kernel}(v^*) \supsetneq \mathcal{L}(a_1, \ldots, a_{n-1})$. So there exists $x \in \text{kernel}(v^*)$ with $x \notin \mathcal{L}(a_1, \ldots, a_{n-1})$. So $(a_1, \ldots, a_{n-1}, x)$ are linear independent. This forms a basis of V.

$$\triangle(a_1, \ldots, a_{n-1}, x) \neq 0 \Rightarrow v^*(x) \neq 0$$

This is a contradiction to our assumption that $x \in \text{kernel}(v^*)$.

Second we show $\Rightarrow$.

Proof by contradiction. Assume $\mathcal{L}(a_1, a_2, \ldots, a_{n-1}) = \text{kernel}\, v^*$ and $a_1, a_2, \ldots, a_{n-1}$ linear independent.

$$\triangle(a_1, \ldots, a_{n-1}, x) = 0 \qquad \forall x \in V$$

$\Rightarrow V - \mathcal{L}(a_1, \ldots, a_{n-1})$ is a contradiction to $\dim(K) = n$.

20.3 Exercise 22.c

Use the linear functional from exercise (a).

$$\begin{aligned} v^* &: \mathbb{K}^4 \to \mathbb{K} \\ x &\mapsto \det(a_1, a_2, a_3, x) \end{aligned}$$

$$v^* = \begin{vmatrix} 1 & -1 & 3 & x_1 \\ 2 & 2 & -1 & x_2 \\ 3 & 0 & 2 & x_3 \\ 1 & 0 & 1 & x_4 \end{vmatrix} = 2x_1 + x_2 + x_3 - 7x_4$$

21 Exercise 23

Exercise 25. Let $x, y, u, v \in \mathbb{R}^3$.

1. Show that the identity $\langle x \times y, u \times v \rangle = \langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle$
2. Conclude that
$$\|u\|^2 \|v\|^2 = \|u \times v\|^2 + \langle u, v \rangle^2$$
for arbitrary vectors $u, v \in \mathbb{R}^3$.

21.1 Exercise 23.a

Case 1: $u \times v = 0$ So u and v are linear dependent, so $\exists a \in \mathbb{R} : u = av$ or $v = au$. Without loss of generality: $u = av$ ($v = au$ analogously).

$$\begin{aligned} &\langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle \\ &= \langle x, u \rangle \langle y, av \rangle - \langle x, au \rangle \langle y, v \rangle = a \langle x, u \rangle \langle y, u \rangle - a \langle x, u \rangle \langle y, u \rangle \\ &= 0 = \langle x \times y, 0 \rangle = \langle x \times y, u \times u \rangle \end{aligned}$$

Case 2: $u \times v \neq 0$

$$\langle x \times y, u \times v \rangle \langle u \times v, u \times v \rangle = \det(x|\, y\, |uxv) \cdot \det(u|\, v\, |u \times v) = \det(x|\, y\, |u \times v)^t \cdot \det(u|\, v\, |u \times v)$$

$$= \det \begin{pmatrix} x^t \\ y^t \\ (u \times v)^t \end{pmatrix} \cdot \det(u|\, v\, |u \times v) = \det \begin{pmatrix} x^t \\ y^t \\ (u \times v)^t \end{pmatrix} (u|\, v\, |u \times v)$$

$$= \det\begin{pmatrix} xv & x^t v & x^t(u\times v) \\ yv^t & y^t v & y & t(u\times v) \\ (u\times v)^t \cdot v & (u\times v)^t \cdot v & (u\times v)^t(u\times v) \end{pmatrix} = \det\begin{pmatrix} \langle x,u\rangle & \langle x,v\rangle & \langle x,u\times v\rangle \\ \langle y,u\rangle & \langle y,v\rangle & \langle y,u\times v\rangle \\ \langle u\times v,v\rangle & \langle u\times v,v\rangle & \langle u\times v,u\times v\rangle \end{pmatrix}$$

$$\langle u\times v,u\times v\rangle \cdot (\langle x,u\rangle\langle y,v\rangle - \langle x,v\rangle\langle y,u\rangle)$$

$$\Rightarrow \langle x\times y,u\times v\rangle = \langle x,u\rangle\langle y,v\rangle - \langle x,v\rangle\langle y,u\rangle$$

21.2 Exercise 23.b

$$\begin{aligned} \|u\times v\|^2 = \langle u\times v,u\times v\rangle &= \langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle\langle v,u\rangle \\ &= \|u\|^2\|v\|^2 - \langle u,v\rangle^2 \end{aligned}$$

$$\Rightarrow \|u\times v\|^2 + \langle u,v\rangle^2 = \|u\|^2 \cdot \|v\|^2$$