# Analysis 2

Lecture notes, University (of Technology) Graz
based on the lecture by Wolfgang Ring

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## Contents

*This lecture took place on 2018/03/06.*

# Lecture

# Mathematical Redux

## Topological fundamentals

### General, abstract concepts

**Definition 2.1.** *Let $X \neq \emptyset$ be a set. We define a map $d : X \times X \to [0, \infty)$. $d$ should behave like a geometrical distance. We require $\forall x, y, z \in X$:*

- $d(x, y) = d(y, x)$ *[called* symmetry*]*
- $d(x, y) = 0 \iff x = y$ *[called* positive definiteness*]*
- $\forall x, y, z \in X : d(x, z) \leq d(x, y) + d(y, z)$ *[called* triangle inequality*]*

*Then $d$ is called* metric *or* distance function *on $X$. $(X, d)$ is called* metric space*.*

**Example 2.1.**

- $X \subseteq \mathbb{C}$, $d(x, y) = \left|x - y\right|$. *It satisfies* $|x - z| \leq \left|x - y\right| + \left|y - z\right|$
- $X \subseteq \mathbb{R}^n$, $\left\|x - y\right\| = \langle x - y, x - y\rangle^{\frac{1}{2}}$

**Claim.**

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

*It holds that* $\left\|x + y\right\| \leq \|x\| + \left\|y\right\|$ *[triangle inequality].*

*Proof.*

$$
\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 && \text{[see Cauchy-Schwarz inequality]} \\
&= (\|x\| + \|y\|)^2 \\
\|x-y\|^2 &= \langle x-y, x-y\rangle \\
&= \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 \\
\|x+y\|^2 + \|x-y\|^2 &= 2\left(\|x\|^2 + \|y\|^2\right)
\end{aligned}
$$

□

**Theorem 2.1** (Cauchy-Schwarz inequality)**.**

$$\left|\langle x,y\rangle\right| \le \|x\|\|y\|$$

*Proof.*

$$0 \le \langle x-\lambda y, x-\lambda y\rangle = \|x\|^2 - 2\lambda\langle x,y\rangle + \lambda^2\|y\|^2 \qquad \forall\lambda\in\mathbb{R}$$

Let $\lambda = \frac{\langle x,y\rangle}{\|y\|^2}$. Then,

$$0 \le \|x\|^2 - 2\frac{\left|\langle x,y\rangle\right|^2}{\|y\|^2} + \frac{\left|\langle x,y\rangle\right|^2}{\|y\|^4}\cdot\|y\|^2$$

$$\implies 0 \le \|x\|^2 - \frac{\left|\langle x,y\rangle\right|^2}{\|y\|^2}$$

$$\implies \left|\langle x,y\rangle\right|^2 \le \|x\|^2\cdot\|y\|^2$$

□

**Definition 2.2.** $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ *is called* Euclidean norm *(length) of vector* $x\in\mathbb{R}^n$*.* $\|x\| = \langle x,x\rangle^{\frac{1}{2}}$ *It holds that*

1. $\|\lambda x\| = |\lambda|\,\|x\|\ \forall x\in\mathbb{R}^n, \lambda\in\mathbb{R}$
2. $\|x\| = 0 \iff x = 0$ *in* $\mathbb{R}^n$

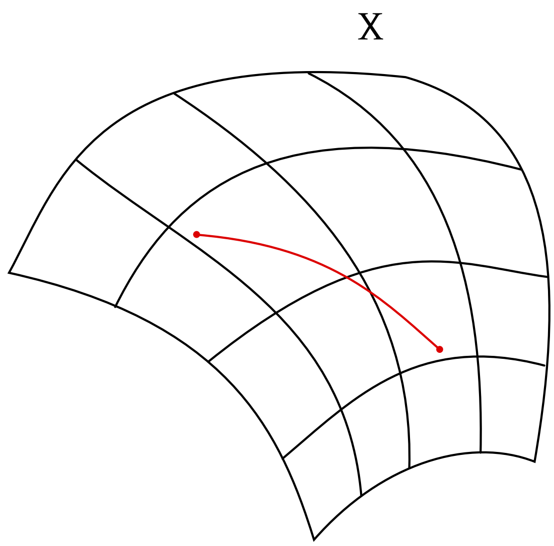



Figure 1: Example in $\mathbb{R}^3$. The red line illustrates the shortest path

3. $\left\|x+y\right\| \le \|x\| + \left\|y\right\|$

*In general: Let $V$ be a vector space over $\mathbb{R}$. A map $\|\cdot\|$, which assigns every vector $x$ a non-negative real number satisfying the properties above, is called* norm on $V$*. Then* $(V, \|\cdot\|)$ *is called a* normed vector space*.*

Let $X \subseteq \mathbb{R}^n$ ($V$ is a normed vector space), then $d(x,y) = \left\|x-y\right\|$ is a metric on $X$.

$$\left\|y-x\right\| = \left\|(-1)(x-y)\right\| = |-1| \cdot \left\|x-y\right\| = \left\|x-y\right\|$$

$$d(x,y) = 0 \iff \left\|x-y\right\| = 0 \iff x-y=0 \iff x=y$$

$$d(x,z) = \|z-x\| = \left\|z-y+y-x\right\| \le \left\|z-y\right\| + \left\|y-x\right\| = d(z,y) + d(y,x)$$

**Example 2.2** (metric space)**.** *Metric space, distance is not a norm. Consider an area in* $\mathbb{R}^3$*.*

$d(x,y)$ *is the shortest path, connecting $x$ and $y$ in $X$. See Figure 1*

**Example 2.3** (French railway)**.** *All connections between two cities pass through Paris except one city is Paris.*

**Example 2.4.** $X = \mathbb{R}^2$*. Let $p \in \mathbb{R}^2$ be fixed.*

$$d(x,y) = \begin{cases} \left|x-y\right| & \text{if } x,y,p \text{ are on one line} \\ \left|x-p\right| + \left|p-y\right| & \text{if } x,y,p \text{ are not on one line} \end{cases}$$

Now we put some terminology into the context of a metric space. $(X, d)$ is a metric space.

**Definition 2.3.** *Let $x \in X$, $r \geq 0$.*

$$K_r(x) = \{z \in X \mid d(x,z) < r\}$$

*Is an* open sphere *with radius r and center x.*

**Definition 2.4.**

$$\overline{K_r(x)} = \{z \in X \mid d(x,z) \leq r\}$$

*Closed sphere with center x and radius r.*

**Definition 2.5** (Sequences in $X$)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X (hence, $x_n \in X \forall n \in \mathbb{N}$)*

1. $(x_n)_{n\in\mathbb{N}}$ *is called* convergent *and limit $x \in X$ if*

   $$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies d(x_n, x) < \varepsilon$$

   *Denoted as* $\lim_{n\to\infty} x_n = x$*.*

2. $(x_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence if*

   $$\forall \varepsilon > 0 \exists N \in \mathbb{N} : n, m \geq N \implies d(x_n, x_m) < \varepsilon$$

*Every convergent sequence is also a Cauchy sequence.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be convergent with limit $x$. Let $\varepsilon > 0$ be arbitrary. Because $(x_n)_{n\in\mathbb{N}}$ is convergent, there exists $N \in \mathbb{N}$ such that $n \geq N \implies d(x_n, x) < \frac{\varepsilon}{2}$. Now let $n, m \geq N$. Then it holds that

$$d(x_n, x_m) \leq \underbrace{d(x_n, x)}_{<\frac{\varepsilon}{2}} + \underbrace{d(x, x_m)}_{<\frac{\varepsilon}{2}} < \varepsilon$$

□

**Definition 2.6.** *$(X, d)$ is called* complete metric space *if every Cauchy sequence in X is also convergent (has a limit).*

$\mathbb{R}$ is complete. $\mathbb{R}^n$ is also complete. $\mathbb{Q} \subseteq \mathbb{R}$ is incomplete.

**Definition 2.7.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of X is called "accumulation point" (dt. Häufungspunkt) of the sequence. $\forall \varepsilon > 0$, it holds that $K_\varepsilon(x)$ contains infinitely many sequence elements.*

*This lecture took place on 2018/03/08.*

TODO

$$d(x,y) = 0 \iff x = y$$
$$\forall x, y \in X : d(x,y) = d(y,x)$$
$$d(x,z) \leq d(x,y) + d(y,z) \forall x, y, z \in X$$

Let $V$ be a vector space. $\|\cdot\|$ is called *norm on $V$*.

$$\|x\| = 0 \iff x = 0$$

$$\forall \lambda \in \mathbb{R}, \mathbb{C} : \forall x \in V : \|\lambda x\| = |\lambda| \, \|x\|$$
$$\forall x, y, z \in V : \left\|x + y\right\| \leq \|x\| + \left\|y\right\|$$

Let $X \subseteq V$ be a subset of normed vector space $V$. Then $X$ is a metric space with $d(x,y) = \left\|x - y\right\|$.

For $V = \mathbb{R}^n$. Then

$$\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$$

is a norm on $\mathbb{R}^n$. $\|x\|_2$ is called *Euclidean norm on* $\mathbb{R}^n$.

Other norms in $\mathbb{R}^n$:

$$\|x\|_\infty = \max\{|x_i| \mid i = 1, \dots, n\}$$
$$\|x\|_1 = \sum_{i=1}^n |x_i|$$

for $1 \leq p < \infty$.

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

e.g. $\|x\|_1$ in $\mathbb{R}^2$

$$\left\|x - y\right\| = \left|x_1 - y_1\right| + \left|x_1 - y_2\right|$$

is the so-called *Manhattan metric*.

The concepts "subsequence", "final element of a sequence", "reordering of a sequence" correspond one-by-one to metric spaces.

**Definition 2.8** (Accumulation point)**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $X$. $x \in X$ is called* accumulation point of sequence $X$ *if* $\forall \varepsilon > 0$ *the sphere $K_\varepsilon(x)$ contains infinitely many elements.*

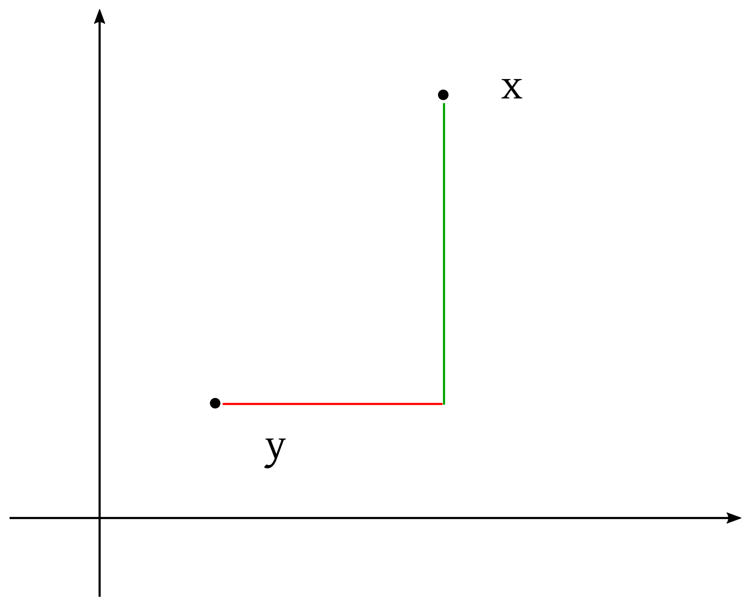



Figure 2: Visualizing $\|x\|_1$

**Lemma 2.1.** *$x \in X$ is accumulation point of sequence $(x_n)_{n\in\mathbb{N}}$ if and only iff there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x = \lim_{k\to\infty} x_{n_k}$.*

*Proof.* See Analysis 1 course $\square$

# Sets in metric spaces

Let $B \subseteq X$, $X$ is a metric space. Then $B$ with $d$ is a metric space itself.

**Definition 3.1.** *Let $B \subseteq X$ and $x \in X$. We say, $x$ is a* contact point of $B$ *if* $\forall \varepsilon > 0 : K_\varepsilon(x) \cap B \neq \emptyset$.

*[ $y \in X$ is not a contact point of $B \iff \exists \varepsilon > 0 : K_\varepsilon(y) \cap B = \emptyset$ ]*

*See Figure 3.*

We let $\overline{B} = \left\{x \in X \,\middle|\, x \text{ is contact point of } B\right\}$.

$\overline{B}$ is called closed hull of $B$.

$B$ is called closed if $B = \overline{B}$, hence, every contact point is also element of $B$.

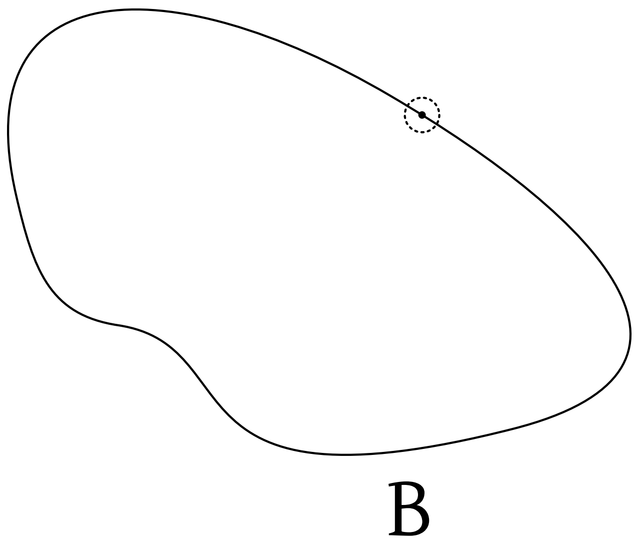


Figure 3: Contact points in set $B$

**Remark 3.1.** *Because* $\forall x \in B$ *holds* $K_r(x) \cap B \supseteq \{x\}\, \forall r > 0$ *is* $x$ *always contact point of* $B$*. Also* $B \subseteq \overline{B}$ *(always)*

**Lemma 3.1.** $x$ *is contact point of* $B \iff \exists (x_n)_{n\in\mathbb{N}}$ *with* $x_n \in B$ *and* $\lim_{n\to\infty} x_n = x$*.*

*Proof.* Let $x$ be a contact point of $B$.

Direction $\Rightarrow$: Because $K_{\frac{1}{n}}(x) \cap B \neq \emptyset$, choose $X_n \in K_{\frac{1}{n}}(x) \cap B$. The sequence $(x_n)_{n\in\mathbb{N}}$ has property $d(x_n, x) < \frac{1}{n}$. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ sch that $N > \frac{1}{\varepsilon}$ (consider the Archimedean axiom). Then for $n \geq N$, $d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$, hence $\lim_{n\to\infty} x_n = x$.

Direction $\Leftarrow$: Let $x = \lim_{n\to\infty} x_n$ and $x_n \in B$. Let $\varepsilon > 0$ be arbitrary and $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon \forall n \geq N$. Then $d(x_N, x) < \varepsilon$, hence

$$x_N \in \underbrace{K_\varepsilon(x) \cap B}_{\neq \emptyset}$$

So $x$ is contact point of $B$. $\square$

**Lemma 3.2.** *It holds that* $\forall B \subseteq X : \overline{B} = \overline{\overline{B}}$*, hence* $\overline{B}$ *itself is closed.*

*Proof.* Show that $x \in \overline{B}$. Let $x \in \overline{\overline{B}}$.

$$\iff \forall \varepsilon > 0 : K_\varepsilon(x) \cap \overline{B} \neq \emptyset$$

Therefore let $\varepsilon > 0$ be arbitrary and $x \in \overline{\overline{B}}$.

Show that $K_\varepsilon(x) \cap B \neq \emptyset$.

Because $x \in \overline{\overline{B}} : \exists y \in \overline{B} : y \in K_{\frac{\varepsilon}{2}}(x)$. Because $y \in \overline{B} : \exists z \in B : z \in K_{\frac{\varepsilon}{2}}(y)$. Hence,

$$d(z,x) \leq \underbrace{d(z,y)}_{<\frac{\varepsilon}{2}} + \underbrace{d(y,x)}_{<\frac{\varepsilon}{2}} < \varepsilon$$

so $z \in K(x, \varepsilon) \cap B$. So $x$ is contact point of $B \implies x \in \overline{B}$. □

**Lemma 3.3.** *Let X be a metric space.*

- *$A_i \subseteq X$ be closed $\forall iinI$. Then $A = \bigcap_{i \in I} A_i = \{x \in X | x \in A_i \forall i \in I\}$ is closed itself.*
- *$A_1, \ldots, A_n \subseteq X$ are closed. Then $\bigcup_{k=1}^{n} A_k$ is closed in X.*
- *$\varphi$ is closed, X is closed.*

*Proof.* See Analysis 1 course. □

**Definition 3.2.** *Let $x \in X$ is called accumulation point of set $B \subseteq X$ if $\forall \varepsilon > 0 : (K_\varepsilon(x) \setminus \{x\}) \cap B \neq \emptyset$.*

**Remark 3.2.** *Accumulation* points *only exist in the context of* sets*. Accumulation* values *only exist in the context of* sequences*.*

*For example* $(+1, -1, +1, -1, +1, \ldots)$ *has accumulation* values $+1$ *and* $-1$*.*

**Lemma 3.4.** *Let $x \in X$ is accumulation point on B $\iff$ every sphere $K_\varepsilon(x)$ contains infinitely many points of B.*

*Proof.* Direction $\Leftarrow$ is trivial.

Direction $\Rightarrow$: Choose $x_1 \in (K_1(x) \setminus \{x\}) \cap B$, hence $x_1 \neq x$, $x_1 \in B$ and $d(x_1, x) < 1$. Let $r_1 = 1$.

Inductive: choose $r_n = \min(\frac{1}{n}, d(x_{n-1}, x))$ and $x_n \in (K_{r_n}(x) \setminus \{x\}) \cap B$. Then $d(x_n, x) > 0$ (because $x_n \neq x$) where $d(x_n, x) < r_n < \frac{1}{n}$.

$$0 < d(x_n, x) < \frac{1}{n}$$

Furthermore, $d(x_n, x) < r_n \leq d(x_{n-1}, x)$. So $x_n \neq x_{n-1}$.

Inductive: $x_n \neq x_{n-1} \neq x_{n-2} \neq \cdots \neq x_1$. Now consider arbitrary $\varepsilon > 0$ and $N$ large enough such that $\frac{1}{N} < \varepsilon$.

Then it holds that $\forall n \geq N : 0 < d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. So $K_\varepsilon(x) \cap B$ contains infinitely many points $x_N, x_{N+1}, x_{N+2}, \ldots$. □

**Definition 3.3.** *Let $U \subseteq X$ and $x \in U$. We say $x$ is an* inner point of $U$ *if* $\exists r > 0 : K_r(x) \subseteq U$. *We let* $\mathring{U} = \left\{ x \in U \,\middle|\, x \text{ is inner point of } U \right\}$ *and call it* interior of *$U$ (offenen Kern von $U$ or das Innere von $U$). $O \subseteq X$ is called* open *(open set), if every point $x \in O$ is also an inner point of $O$. Hence $\mathring{O} = O$.*

**Example 3.1.** *Let $K_r(x)$ with $r > 0$ be an open sphere in $X$. Then $K_r(x)$ is an open set in $X$.*

Why? Let $y \in K_r(x)$. Show that $y$ is an inner point of the sphere. $d(y, x) = s < r$. Define $r' = r - s > 0$. Claim: $K_r'(y) \subseteq K_r(x)$.

TODO drawing

TODO

So it holds that $z \in K_r(x)$ and therefore $K_{r'}(y) \subseteq K_r(x)$.

**Lemma 3.5.** *Let $U \subseteq X$ be arbitrary. Then $\mathring{U} \subseteq X$ be an open set in $X$.*

*Proof.* Let $x \in \mathring{U}$, hence $x$ is an inner point of $U$. Show that $x$ is an inner point of $\mathring{U}$, also $\exists r > 0 : K_r(x) \subseteq \mathring{U}$.

Because $x \in \mathring{U}$, $r > 0 exists$: $K_r(x) \subseteq U$. Claim: Every point $y \in K_r(x)$ is also an inner point of $U$. Obvious (previous example), because $r' > 0$ exists such that $K_{r'}(y) \subseteq K_r(x) \subseteq U$ so $y \in \mathring{U}$ and $K_r(x) \subseteq \mathring{U}$. □

**Theorem 3.1.** *Let $X$ be a metric space.*

$$A \subseteq X \text{ is closed in } X \iff O = X \setminus A = A^C \text{ is open}$$

*Proof.* Direction $\Leftarrow$. Let $A$ be closed and $O + A^C$. We choose $x \in O$ and show that $x$ is in the interior of $O$.

Assume the opoosite.

$$\forall \varepsilon > 0 : \underbrace{\neg \left(K_\varepsilon(x) \subseteq O\right)}_{\iff K_\varepsilon(x) \cap O^C \neq \emptyset}$$

where $O^C = A$. So $x$ is contact point of $A$. Because $A$ is closed, it holds that $x \in A$. This contradicts with $x \in O = A^C$.

Direction $\Rightarrow$. TODO $K_r(x) \cap \underbrace{A}_{=O^C} = \emptyset$. Hence $x$ is not a contact point of $A$.

So every contact point of $A$ is also an element of $A$ and $A$ is closed. □

**Theorem 3.2.** *Let X be a metric space. Then it holds that*

- *If $O_i \subseteq X$ is open in $X$ $\forall i \in I$. Then also $O = \bigcup_{i \in I} O_i$ is open in X.*
- *If $O_1, O_2, \dots, O_n$ is open in X, then $\bigcap_{k=1}^{n} O_k$ is open in X.*
- *X is open, $\emptyset$ is open.*

*Proof.* By Lemma 3.3, Theorem 3.1 and De Morgan's Laws:

$$\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{iinI} A_i^C$$

□

**Definition 3.4.** *Given a set X. If a subset $T \subseteq \mathcal{P}(X)$ is defined such that the elements $O \in T$ (hence $O \subseteq X$) satisfy the conditions of Theorem 3.2, then T is called* topology *on X. $(X, T)$ is called topological space.*

*The sets $O \in T$ are called* open sets *in terms of T. The complements $A = O^C$ for $O \in T$ are called* closed sets*.*

**Definition 3.5.** *Let $x \in U \subseteq X$. We claim that U is a neighborhood of x, if $r > 0$ exists such that $x \in K_r(X) \subseteq U$*

*See Figure 4*

**Remark 3.3.** *$O \subseteq X$ is open iff O is neighborhood of every point $x \in O$.*

**Definition 3.6.** *Let X and Y be metric spaces and $x_0 \in X$. Let $f : X \to Y$ be given. We say f is continuous in $x_0$ if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x \in X : d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$$

*Here, $d_X$ is a metric on X and $d_Y$ is a metric on Y.*

*This lecture took place on 2018/03/13.*

TODO I missed the first twenty minutes (including Satz 3 and 4)

*Proof.* Direction ⇒.

Let $f$ be continuous in $X$ and let $O \subseteq Y$ be open. Let $U = f^{-1}(O)$ and choose $x_0 \in U$. Then $f(x_0) \in O$, hence $O$ is a neighborhood of $f(x_0)$. By Theorem 7.2 (b), it follows that $U = f^{-1}(O)$ is a neighborhood of $x_0$.

Hence, $U$ is neighborhood of every of its points, hence open in $X$.

Direction ⇐.

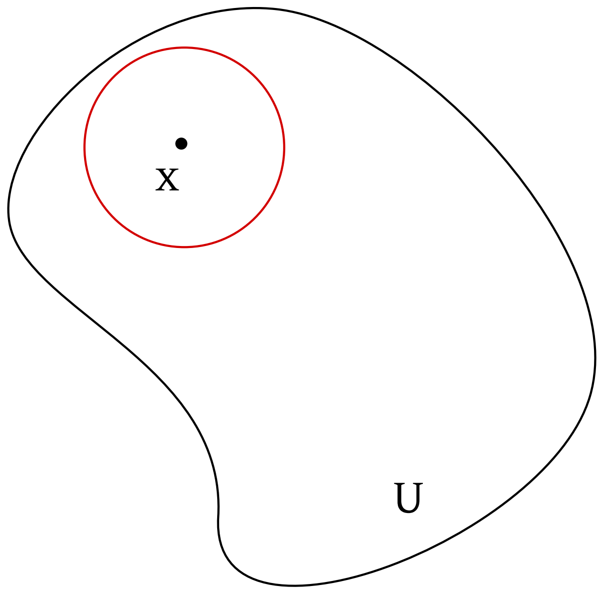



Figure 4: Neighborhood of $x$

Let the preimages of open sets be open and $x_0 \in X$ and $y_0 = f(x_0)$. Let $V$ be a neighborhood of $y_0 = f(x_0)$, hence $\exists \varepsilon > 0 : K_\varepsilon(f(x_0)) \subseteq V$. Because $K_\varepsilon(f(x_0))$ is an open set, it holds that $f^{-1}(K_\varepsilon(f(x_0))) \in x_0$ is open in $X$.

Therefore, there exists $\delta > 0$ such that $K_\delta(x_0) \subseteq f^{-1}(K_\varepsilon(f(x_0))) \subseteq f^{-1}(V)$. Hence, $f^{-1}(V)$ is a neighborhood of $x_0$. Then by Theorem 7.2 (b), it follows that $f$ is continuous in $x_0$ (chosen arbitrarily). Hence $f$ is continuous on $X$. $\square$

## Variations of continuity notions

**Definition 4.1.** *Let $f : X \to Y$ be given. We call "f uniformly continuous on X" if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X \wedge d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

**Remark 4.1.** *Compare it with the definition of "continuous in X":*

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : \forall y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

*The difference is the location of the $\forall x \in X$ quantifier.*

*Every uniformly continuous map is continuous.*

*Example: $f : (0, \infty) \to (0, \infty)$ with $f(x) = \frac{1}{x}$ is continuous, but not continuously continuous.*

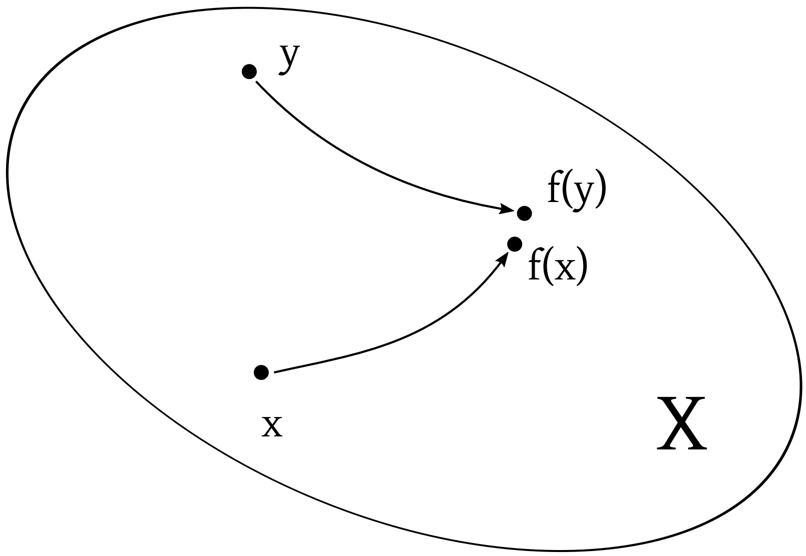



Figure 5: A contraction maps to points closer to each other

**Definition 4.2.** $f : X \to Y$ *is called* Lipschitz continuous *with* Lipschitz constant $L \geq 0$ *if* $\forall x, y \in X : d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$.

*Rudolf Lipschitz [1832–1903], University of Bonn*

**Theorem 4.1.** *Every Lipschitz continuous function is uniformly continuous.*

*Proof.* For $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{L+1}$. Then it holds that $d_X(x, y) < \delta = \frac{\varepsilon}{L+1} \implies d_Y(f(x), f(y)) \leq L \cdot d_X(x, y) < \frac{L}{L+1} \cdot \varepsilon < \varepsilon$. □

- Most often $X \subseteq V$, $Y \subseteq W$. $V$ and $W$ are normed vector spaces and $d(x, y) = \|x - y\|$

**Definition 4.3.** *A Lipschitz continuous map $f : X \to X$ with Lipschitz constant $L < 1$ is called* contraction on $X$. *Compare with Figure 5*

**Theorem 4.2** (Banach fixed-point theorem)**.** *Let $f : X \to X$ be a contraction and $X$ be complete. Then there exists a uniquely defined $\hat{x} \in X$ such that $\hat{x} = f(\hat{x})$. $\hat{x}$ is called fixed point on $f$. Furthermore it holds that $x_0 \in X$ is arbitrary and $x_n = f(x_{n-1})$ for all $n \geq 1$.*

$$\lim_{n \to \infty} x_n = \hat{x}$$

TODO drawing Banach's fixed point theorem

*Proof.* Let $x_0 \in X$ be arbitrary. $x_n$ is constructed inductively by $x_n = f(x_{n-1})$ for all $n \geq 1$.

**Claim.** *$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$.*

$d(x_n, x_{n+k}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-1}, x_{n+k})$

by triangle inequality

$$\begin{aligned}
&= d(x_n, x_{n+1}) + d(f(x_n), f(x_{n+1})) + d(f(x_{n+1}), f(x_n + 2)) + \cdots + d(f(x_{n+k-2}), f(x_{n+k-1})) \\
&\leq d(x_n, x_{n+1}) + L\left(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+k-2}, x_{n+k-1})\right)
\end{aligned}$$

this inequality is given by contraction

$$\begin{aligned}
&= d(x_n, x_{n+1})(1 + L) + L\left(d(f(x_n), f(x_{n+1})) + \cdots + d(f(x_{n+k-3}), f(x_{n+k-2}))\right) \\
&\leq d(x_n, x_{n+1})(1 + L) + L^2\left[d(x_n, x_{n+1} + \cdots + d(x_{n+k-3}, x_{n+k-2})\right] \\
&\leq \cdots \leq d(x_n, x_{n+1})(1 + L + L^2 + \cdots + L^{k-1}) \\
&= d(f(x_{n-1}, f(x_n))\left(\sum_{j=0}^{k-1} L^j\right) \leq L d(x_{n-1}, x_n) \cdot \left(\sum_{j=0}^{k-1} L^j\right) \\
&\leq L^n d(x_0, x_1) \cdot \underbrace{\left(\sum_{j=1}^{k-1} L^j\right)}_{\leq \sum_{j=0}^{\infty} L^j = \frac{1}{1-L}} \\
&\leq \frac{L^n}{1-L} d(x_0, x_1)
\end{aligned}$$

$d(x_n, x_{n+k}) \leq \dfrac{L^n}{1-L} d(x_0, x_1) \forall n \in \mathbb{N} \forall k \in \mathbb{N}_0$

with $0 \leq L < 1$.

$$\underbrace{\frac{L^n}{1-L}}_{>0} d(x_0, x_1) < \varepsilon \impliedby$$

$$\begin{aligned}
L^n &< \frac{\varepsilon}{d(x_0, x_1) + 1}(1 - L) \qquad (L > 0) \\
\iff n \underbrace{\ln L}_{<0} &< \ln \frac{\varepsilon}{d(x_0, x_1) + 1}(1 - L) \\
\iff n &> \frac{1}{\ln L} \ln \frac{\varepsilon}{d(x_0, x_1) + 1}(1 - L)
\end{aligned}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$. $X$ is complete, hence $\exists \hat{x} \in X$ : $\hat{x} = \lim_{n\to\infty} x_n$. Because $\hat{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\hat{x})$ where the last equality is given by continuity of $f$. Therefore $\hat{x} = f(\hat{x})$ is a fixed point on $f$.

It remains to prove uniqueness:

Let $\tilde{x} = f(\tilde{x})$. Then it holds that $d(\hat{x}, \tilde{x}) = d(f(\hat{x}), f(\tilde{x})) \leq L d(\hat{x}, \tilde{x})$ with $L < 1$. If $d(\hat{x}, \tilde{x}) > 0$, then $1 \leq L$. This is a contradiction. Hence $d(\hat{x}, \tilde{x}) = 0$ must hold, hence $\hat{x} = \tilde{x}$. $\square$

**Remark 4.2.**

- *The Fixed Point Theorem provides an algorithm for numeric computation of $\hat{x}$.*
- *It can reformulate problems $f(x) = 0$ (in $\mathbb{R}^n$) to*

$$f(x) + x = g(x) = x$$

- *Attention: The conditions of the Fixed Point Theorem cannot be changed to the structure*

$$d(f(x), f(y)) < L \cdot d(x, y) \wedge L \leq 1$$

  *or*

$$d(f(x), f(y)) \leq L \cdot d(x, y) \wedge L < 1$$

  *This will be discussed in the practicals.*

**Lemma 4.1.** *Let $X$ be a complete metric space. Let $A \subseteq X$ be closed. Then $(A, d)$ is itself a complete, metric space.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $A$ $(x_n \in A)$. Then $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $X$. Because $X$ is complete, there exists $\hat{x} = \lim_{n\to\infty} x_n$. Therefore $\hat{x}$ is a contact point of $A$. Because $A$ is closed, it holds that $\hat{x} \in A$.

Therefore every Cauchy sequence in $A$ has a limit point in $A$, hence $A$ is complete. $\square$

# Compactness

**Definition 5.1.** *A metric space $(X, d)$ is called* compact *if every sequence $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence.*

*Specifically, this definition is called sequence compactness. The other definition defines compactness as closed and bounded subset of an Euclidean space. The latter definition only works for a subset of branches in mathematics. Therefore the generalization is recommended to be remembered.*

**Lemma 5.1.** *Let $X$ be a compact, metric space. Then $X$ is complete.*

*Proof.* Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $X$. By compactness, it follows that $\exists (x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = \hat{x}$. Choose $\varepsilon > 0$ arbitrary and $L$ large enough such that $k \geq L \implies d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2}$. Furthermore choose $N \in \mathbb{N}$ large enough such that $n, m \geq N \implies d(x_n, x_m) < \frac{\varepsilon}{2}$ (satisfied, because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence). Choose $K \geq L$ and $n_k \geq N$. Let $n_k$ be fixed this way. Then it holds $\forall n \geq N : d(x_n, \hat{x}) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \hat{x}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. The first summand $\frac{\varepsilon}{2}$ results from the Cauchy sequence property, the second summand $\frac{\varepsilon}{2}$ results by convergence of $(x_{n_k})$. Hence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit $\hat{x}$. □

**Definition 5.2.** *A metric space $X$ is called* bounded *if there exists $M \geq 0$, such that $d(x, y) \leq M \forall x, y \in X$.*

It holds for arbitrary $x \in X$ that $\forall y \in X : y \in K_M(x)$. So, $X \subseteq K_M(x)$. On the contrary, let $X \subseteq \overline{K_M(x)}$ and let $y \in X$ and $z \in X$ be arbitrary. Then it holds that $d(y, z) \leq d(y, x) + d(x, z) \leq M + M = 2M$. Hence, $X$ is bounded.

So, $X$ is bounded $\iff \exists x \in X \wedge M \geq 0 : X \subseteq \overline{K_M(x)}$.

**Lemma 5.2.** *Every compact, metric space is also bounded.*

*Proof.* Assume $X$ is unbounded.

We construct a sequence of points $(x_n)_{n\in\mathbb{N}}$ with $d(x_n, x_m) \geq 1 \forall n, m \in \mathbb{N}$ with $n \neq m$.

We use the following auxiliary result: Let $B = \bigcup_{j=1}^{n} K_1(z_j)$ for arbitrary $n \in \mathbb{N}$ and arbitrary $z_j \in X$. Then $B$ is bounded. This result will be part of the practicals.

We construct $(x_n)_{n\in\mathbb{N}}$ inductively. Choose arbitrary $x_0 \in X$. Assume $(x_1, \ldots, x_{n-1})$ are already found. Then it holds that

$$\underbrace{X}_{\text{unbounded}} \not\subseteq \underbrace{\bigcup_{j=1}^{n-1} K_1(x_j)}_{\text{bounded}}$$

hence $\exists x_n \in X \setminus \bigcup_{j=1}^{n-1} K_1(x_j)$. Because $x_n \notin K_1(x_j)$ for $j = 0, \ldots, n-1$ it holds that $d(x_n, x_j) \geq 1 \forall j < n$. We get $(x_n)_{n\in\mathbb{N}}$ with $d(x_n, x_m) \geq 1 \forall n \in \mathbb{N} \forall m < n$, hence $m \neq n$. Because $d(x_n, x_m) \geq 1$, i.e. $(x_n)_{n\in\mathbb{N}}$ does not contain any Cauchy sequence as subsequence, $(x_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence. Therefore $X$ is not compact. □

*This lecture took place on 2018/03/15.*

Every compact metric space is bounded. Every compact metric space is complete. In $\mathbb{C}$ ($\mathbb{R}^n$) it holds that $A \subseteq \mathbb{C}$ is closed. Then $A$ with metric $d(x, y) = |x - y|$ is complete as metric space.

If $A$ is additionally bounded, then $A$ is compact (see course Analysis 1, Bolzano-Weierstrass).

Attention! Let $V$ be an infinite-dimensional, complete, normed vector space. For example, $V = C([a,b],\mathbb{R}) = \left\{ f : [a,b] \to \mathbb{R} \,\middle|\, f \text{ is continuous in } [a,b] \right\}$ with norm $\left\|f\right\|_\infty = \max\left\{ \left|f(x)\right| : x \in [a,b] \right\}$ and metric $\left\|f-g\right\|_\infty = \max\left\{ \left|f(x)-g(x)\right| : x \in [a,b] \right\}$. $C([a,b],\mathbb{R})$ is a complete, normed vector space. It holds that $\overline{K_1(0)}$ is not compact in $C([a,b],\mathbb{R})$ (i.e. $V$, for every infinite-dimensional vector space).

Again: do not remember "compactness" not as closed and bounded, as this only holds in the finite-dimensional case.

In the last proof, we have shown: If a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X$ and $d(x_n, x_m) \geq 1$ (or $\geq \varepsilon$) $\forall n \neq m \implies X$ is not compact.

**Definition 5.3.** *$X$ is called totally bounded, if for every $\varepsilon > 0$, finitely many points $X_1^\varepsilon, X_2^\varepsilon, \ldots, X_{N(\varepsilon)}^\varepsilon$ such that $X \subseteq \bigcup_{i=1}^{N(\varepsilon)} K_\varepsilon(X_i^\varepsilon)$.*

*Hence, for every $x \in X$, there exists some $X_j^\varepsilon$ such that $d(X, X_j^\varepsilon) < \varepsilon$.*

**Remark 5.1** (For the practicals)**.** *Let $X$ be totally bounded, then there does not exist some sequence $(x_n)_{n\in\mathbb{N}}$ with $d(x_n, x_m) \geq \varepsilon \forall n \neq m$. It holds, that $X$ is compact if and only if $X$ is totally bounded and complete.*

**Theorem 5.1.** *Let $f : X \to Y$ be continuous. Let $X$ be compact. Then image $f(X) \subseteq Y$ is also compact.*

Be aware, that this proof is a common exam question and students often begin with the wrong order.

*Proof.* Let $(y_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $f(X)$. Show that $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Because $y_n \in f(X)$, there exists at least one $x_n$ with $y_n = f(x_n)$. Then $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$, $X$ is compact, hence there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_{n_k} = \hat{x} \in X$. Because $f$ is continuous, it holds that $\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = f(\hat{x}) =: \hat{y}$. So $(y_n)_{n\in\mathbb{N}}$ has a convergent subsequence. Hence $f(X) \subseteq Y$ is compact. $\square$

**Theorem 5.2** (Conclusion)**.** *Let $X$ be compact, $f : X \to \mathbb{R}$ continuous on $X$. Then there exists $\underline{x}$ and $\overline{x} \in X$, such that*

$$f(\underline{x}) \leq f(x) \leq f(\overline{x}) \qquad \forall x \in X$$

*Hence, $f$ has a maximum and a minimum.*

*Proof.* $f(X) \subseteq \mathbb{R}$ is compact (Theorem 5.1), hence $f(X)$ is bounded and complete, hence closed in $\mathbb{R}$. There exists $\xi \in \mathbb{R}$ with $\xi = \sup f(X)$, because $f(X)$ is complete and $\xi$ is a contact point of $f(X)$, it holds that $\xi \in f(X)$, hence $\exists \overline{x} \in X : \xi = f(\overline{x})$. Furthermore, $\xi$ is an upper bound of $f(X) \to f(x) \leq \xi = f(\overline{x}) \forall x \in X$.

For $\underline{x}$, it works the same way. $\square$

**Theorem 5.3.** *Let $f : X \to Y$ is continuous on $X$ and $X$ is compact. Then $f$ is uniformly continuous on $X$.*

*Indirect proof.* Assume $X$ is compact, $f : X \to Y$ is continuous, but not uniformly continuous. Uniform continuity:

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall x, y \in X : d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon$$

Not uniformly continuous:

$$\exists \varepsilon > 0 \forall \delta_n = \frac{1}{n}(n \in \mathbb{N}) \exists x_n, y_n \in X : d_X(x_n, y_n) < \frac{1}{n} \wedge d_Y(f(x_n), f(y_n)) \geq \varepsilon$$

Now choose some $(x_n)$ and $(y_n)$. We will use a specific $\varepsilon$ later. Because $X$ is compact, there exists a convergent subsequence of $(x_n)_{n\in\mathbb{N}}$, hence $\lim_{k\to\infty} x_{n_k} = \hat{x}$. The sequence $(y_{n_k})_{k\in\mathbb{N}}$ has a convergent subsequence itself:

$$\lim_{l\to\infty} y_{(n_k)_l} = \hat{y}$$

Because $(x_{n_k})_{n\in\mathbb{N}}$ is convergent, the subsequence $(x_{(n_k)_l})_{l\in\mathbb{N}}$ converges towards the same limit $\hat{x}$.

$$\tilde{x}_l := x_{n_{k_l}} \qquad \tilde{y}_l := y_{n_{k_l}}$$

because $l \leq x_{n_l}$ and

$$d_X(\tilde{x}_l, \tilde{y}_l) = d_X(x_{n_{k_l}}, y_{n_{k_l}}) \underbrace{<}_{\text{by assumption}} \frac{1}{n_{k_l}} \leq \frac{1}{l}$$

**Claim.** *For $\hat{x} = \lim_{l\to\infty} \tilde{x}_l$ and $\hat{y} = \lim_{l\to\infty} \tilde{y}_l$, it holds that $\hat{x} = \hat{y}$. Let $\varepsilon' > 0$ be arbitrary, $l$ large enough such that*

- $\frac{1}{l} < \frac{\varepsilon'}{3}$
- $d_X(\tilde{x}_l, \hat{x}) < \frac{\varepsilon'}{3}$
- $d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3}$

Therefore it holds that

$$d_X(\hat{x}, \hat{y}) \leq d_X(\hat{x}, \tilde{x}_l) + d_X(\tilde{x}_l, \tilde{y}_l) + d_X(\tilde{y}_l, \hat{y}) < \frac{\varepsilon'}{3} + \frac{1}{l} + \frac{\varepsilon'}{3} < \varepsilon'$$

Therefore it holds that $d_X(\hat{x}, \hat{y}) = 0$, hence $\hat{x} = \hat{y}$. Because $f$ is continuous and $\tilde{x}_l \to \hat{x}$ and $\tilde{y}_l \to \hat{x}$, there exists $l \in \mathbb{N}$ such that

$$d_Y(f(\tilde{x}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$

and also

$$d_Y(f(\tilde{y}_l), f(\hat{x})) < \frac{\varepsilon}{2}$$

where $\varepsilon$ is the epsilon from the very beginning of the proof.

$$\implies d_Y(f(\tilde{x}_l), f(\hat{x})) + d_Y(f(\tilde{y}_l), f(\hat{x})) < \varepsilon$$

This contradicts to

$$d_Y(f(\tilde{x}_l), f(\tilde{y}_l)) = d_Y(f(x_{n_{k_l}}), f(y_{n_{k_l}})) \geq \varepsilon$$

Hence, $f$ is uniformly continuous. □

Subsets of $(\mathbb{R}^n, \|\cdot\|)$ (or $(V, \|\cdot\|)$) as metric spaces.

We consider $\Omega \subseteq V$ where $V$ is a normed vector space. $(\Omega, d)$ is $d(x, y) = \left\|x - y\right\|$ is a metric space.

$$K_r^{\Omega}(x) = \left\{ y \in \Omega \,\middle|\, \left\|y - x\right\| < r \right\}$$

is a sphere with center $x$ and radius $r$ in $\Omega$.

$$K_r^V(x) = \left\{ y \in V \,\middle|\, \left\|y - x\right\| < r \right\}$$

obvious: $K_r^{\Omega}(x) = \Omega \cap K_r^V(x)$.

TODO drawing 08

**Lemma 5.3.** *Let $O' \subseteq \Omega \subseteq V$.*

*Then it holds that $O'$ is open in $\Omega$ $\iff$ there exists $O \subseteq V$ is open in $V$ such that $O' = O \cap \Omega$.*

*Proof.* $\Rightarrow$ Let $O' \subseteq \Omega$ be open in $\Omega$ and $x \in O'$ be arbitrary. Then there exists $r(x) > 0 : x \in K_{r(x)}^{\Omega}(x) = K_{r(x)}^V(x) \cap \Omega \subseteq O'$. Then it holds that

$$O' = \bigcup_{x \in O'} = \{x\} \subseteq \bigcup_{x \in O'} K_{r(x)}^{\Omega}(x) = \left( \bigcup_{x \in O'} (K_{r(x)}^V(x) \right) \cap \Omega) = \underbrace{\left( \bigcup_{x \in O'} K_{r(x)}^V(x) \right)}_{= O \subseteq V \text{ is open in } V} \cap \Omega \subseteq O'$$

So every $\subseteq$ in this inclusion chain is actually an equality. So $O' = O \cap \Omega$.

$\Leftarrow$ Let $O' = O \cap \Omega$ and $x \in O'$ be chosen arbitrarily. Because $x \in O$ and $O$ is open in $V$.

$$\exists r > 0 : K_r^V(x) \subseteq O \implies \underbrace{K_r^V(x) \cap \Omega}_{= K_r^{\Omega}(x)} \subseteq O \cap \Omega = O'$$

So $O'$ is open in $\Omega$.

□

**Remark 5.2.** *$A' \subseteq \Omega$ is closed in $\Omega \iff \exists A \subseteq V$ closed in $V$ with $A' = A \cap \Omega$.*

**Remark 5.3.** *Let $T$ be an arbitrary topological space with topology $\tau$ on $T$ (a system of open sets). Furthermore let $\Omega \subseteq T$.*

*Then $\Omega$ itself is a topological space with $O' \subseteq \Omega$ is open $\iff \exists O \subset T$ open in $T$ with $O' = O \cap \Omega$.*

*Also called "subspace topology", "trace topology" or "relative topology".*

Attention!

$$O' \subseteq \Omega \text{ open in } \Omega \implies O' \text{ open in } V$$

does *not* hold in general.

**Example 5.1.**

$$\Omega = [0, 1] \cap [0, 1)$$

*$K_{\frac{1}{2}}(p) \cap \Omega$ is open in $\Omega$ but not open in $\mathbb{R}^2$.*

Analogously,

$$A' \subseteq \Omega \text{ is closed} \implies A' \text{ closed in } V$$

does *not* hold in general.

**Remark 5.4.** *$K$ is compact in $\Omega \implies K$ is compact in $V$*

Let $(x_n)_{n\in\mathbb{N}}$ is a sequence in $K$. Compactness $\implies \exists (x_{n_k})_{k\in\mathbb{N}} : x_{n_k} \to \hat{x}$ for $k \to \infty$ and $K \subseteq \Omega \subseteq V$.

Then $(x_n)_{n\in\mathbb{N}}$ also has a convergent subsequence in $V$.

## Normed vector spaces

**Definition 5.4.** *Let $V$ be a vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ are normed on $V$. We say, $\|\cdot\|_1$ is equivalent to norm $\|\cdot\|_2$, if $0 < m \leq M$ exist such that*

$$m \|v\|_1 \leq \|v\|_2 \leq M \|v\|_1 \, \forall v \in V$$

**Remark 5.5.** *Equivalence of norms is an equivalence relation.*

**reflexivity** *Let $m = M = 1$. TODO*

**symmetry**

$$m \|v\|_1 \leq \|v\|_2 \implies \|v\|_1 \leq \frac{1}{m} \|v\|_2 \wedge \|v\|_2 \leq M \cdot \|v\|_1 \implies \frac{1}{M} \|v\|_2 \leq \|v\|_1$$

$$\implies \underbrace{\frac{1}{M}}_{m'} \|v\|_2 \leq \|v_1\| \leq \underbrace{\frac{1}{m}}_{M'} \|v\|_2$$

*hence the equivalence relations of norms are symmetrical.*

**transitivity** *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent. Let $\|\cdot\|_2$ and $\|\cdot\|_3$ be equivalent.*

$$m \cdot \|v\|_1 \leq \|v\|_2 \leq M \|v\|_1 \; \forall v \in V$$
$$m' \cdot \|v\|_2 \leq \|v\|_3 \leq M' \|v\|_2 \; \forall v \in V$$
$$\implies m \cdot m' \|v\|_1 \leq m' \|v\|_2 \leq \|v\|_3 \leq M' \|v\|_2 \leq M \cdot M' \|v\|_1$$

*This lecture took place on 2018/03/20.*

Addendum:

- Let $(x_n)_{n\in\mathbb{N}}$ be in $(X, d)$, then it holds that

$$\underbrace{x = \lim_{n\to\infty} x_n}_{\text{in } X} \iff \underbrace{\lim_{n\to\infty} d(x_n, x) = 0}_{\text{in } \mathbb{R}}$$

$$(\iff \lim_{n\to\infty} \|x_n - x\| = 0 \text{ in normed vector spaces } V)$$

- Inversed triangle inequality: Let $V$ be a normed vector space. Let $x, y \in V$.

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$$

  Hence,
$$\|x\| - \|y\| \leq \|x - y\|$$
  By exchanging $x$ and $y$,
$$\|y\| - \|x\| \leq \|x - y\|$$
  Hence, it holds that
$$\left|\|x\| - \|y\|\right| \leq \|x - y\|$$

- Define the map $n : V \to [0, \infty)$ on $(V, \|\cdot\|)$ with $n(x) = \|x\|$. Then $n$ is continuous on $V$ because

$$|n(x_1) - n(x_2)| = |\|x_1\| - \|x_2\|| \leq \|x_1 - x_2\|$$

  Hence, $n$ is Lipschitz continuous with constant 1.

Regarding the equivalence of norms:

**Lemma 5.4.** *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent norms on $V$. Then it holds that*

1. $\lim_{n\to\infty} \|x_n - x\|_1 = 0 \iff \lim_{n\to\infty} \|x_n - x\|_2 = 0$, *hence $(x_n)_{n\in\mathbb{N}}$ is convergent with limit $x$ in regards of $\|\cdot\|_1 \iff (x_n)_{n\in\mathbb{N}}$ is convergent with limit $x$ in regards of $\|\cdot\|_2$.*

2. *$O \subseteq V$ is open in regards of $\|\cdot\|_1 \iff O$ is open in regards of $\|\cdot\|_2$, hence $\tau_1 = \tau_2$ (topologies are equivalent).*
3. *$K \subseteq V$ is compact in regards of $\|\cdot\|_1 \iff K$ is compact in regards of $\|\cdot\|_2$.*

*Proof.* Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, hence $\exists m, M > 0 : m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \ \forall x \in V$.

1. Let $\varepsilon > 0$ and $\lim_{n\to\infty} \|x_n - x\|_1 = 0$. Choose $N \in \mathbb{N}$ such that $n \geq N \implies \|x_n - x\|_1 < \frac{\varepsilon}{M}$. For those $n$ it holds that
$$\|x_n - x\|_2 \leq M\|x_n - x\|_1 < \frac{\varepsilon}{M} \cdot M = \varepsilon$$
Hence, $\lim_{n\to\infty} \|x_n - x\|_2 = 0$.
2. $K_r^2(x) = \left\{ y \in V \,\middle|\, \left\|y - x\right\|_2 < r \right\}$. For $y \in K_r^2(x)$ it holds that
$$m\left\|y - x\right\|_1 \leq \left\|y - x\right\|_2 < r$$
hence,
$$\left\|y - x\right\|_1 < \frac{r}{m} \implies y \in K^1_{\frac{r}{m}}(x)$$
hence $K_r^2(x) \subseteq K^1_{\frac{r}{m}}(x)$. Let $y \in K^1_{\frac{r}{M}}(x)$. Then it holds that,
$$\left\|y - x\right\|_2 \leq M\left\|y - x\right\|_1 < M \cdot \frac{r}{M} = r$$
hence $y \in K_r^2(x)$. $\implies K^1_{\frac{r}{M}}(x) \subseteq K_r^2(x)$. Now let $O$ be open in regards of $\|\cdot\|_2$, hence
$$\forall x \in O \exists r > 0 : K_r^2(x) \subseteq O \implies K^1_{\frac{r}{M}}(x) \subseteq K_r^2(x) \subseteq O$$
so $O$ is open in regards of $\|\cdot\|_1 \implies O$ is open in regards of $\|\cdot\|_2$ analogously.
3. Let $K$ be compact in regards of $\|\cdot\|_1$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $K$. Then there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with $\left\|x_{n_k} - x\right\|_1 \to 0$ for $k \to \infty$ $\xRightarrow{\text{by the first property}} \left\|x_{n_k} - x\right\|_2 \to 0$. Hence $(x_{n_k})_{k\in\mathbb{N}}$ is also a convergent subsequence in regards of $\|\cdot\|_2$.

□

**Remark 5.6** (Proven in the practicals)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^k$*
$$\|x\|_\infty = \max\left\{ \left|x^i\right| \,\middle|\, i = 1, \dots, n \right\}$$
$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{bmatrix}$$
*It holds that* $\lim_{n\to\infty} \|x_n - x\|_\infty = 0 \iff \lim_{n\to\infty} \left|x_n^i - x^i\right| = 0$ *for all $i \in \{1, \dots, k\}$.*

**Theorem 5.4** (Bolzano-Weierstrass theorem in $\mathbb{R}^k$)**.** *Let $K \subseteq \mathbb{R}^k$ be closed and bounded. Then $K$ is compact in $(\mathbb{R}^k, \|\cdot\|_\infty)$.*

*Proof.* Let $\|x\|_\infty \leq M \forall x \in K \iff \left|x^i\right| \leq M \forall x \in K$ and $i \in \{1, \ldots, k\}$. Choose $(x_n)_{n\in\mathbb{N}}$ an arbitrary sequence in $K$ $(x_n^i)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. Because $(x_n^1)_{n\in\mathbb{N}}$ is bounded, there exists a convergent subsequence $\left(x^1_{n_{l_1}}\right)_{l_1\in\mathbb{N}}$

$$\lim_{l_1\to\infty} x^1_{n_{l_1}} = x^1$$

Consider $(x^2_{n_{l_1}})_{l_1\in\mathbb{N}}$, a subsequence of a bounded sequence, hence bounded itself. By the Bolzano-Weierstrass theorem in $\mathbb{R}$, there exists a convergent subsequence $(x^2_{n_{l_{1_{l_2}}}})_{l_2\in\mathbb{N}}$ with $\lim_{l_2\to\infty} x^2_{n_{l_{1_{l_2}}}} = x^2$. Consider $x^1_{n_{l_{1_{l_2}}}}$ as subsequence of $x^1_{n_{l_1}}$ is already convergent, hence $\lim_{l_2\to\infty} x^1_{n_{l_{1_{l_2}}}} = x^1$. Furthermore, up to index $i$, it holds that:

$$\lim_{l_k\to\infty} x_{n_{l_{1_{l_{2\ldots l_k}}}}} = x^i \qquad \text{for } i = 1, \ldots, k$$

Hence, with $\tilde{x_{l_k}} = x_{n_{l_{1_{l_{2\ldots l_k}}}}}$ gives a subsequence of $x_n$, converging by each coordinate. Thus,

$$\lim_{l_k\to\infty} \left\|\tilde{x}_{l_k} - x\right\|_\infty = 0$$

Because $\tilde{x}_{l_n} \in K$ and $K$ be closed, it holds that $x \in K$. Hence $K$ is compact. $\square$

**Theorem 5.5** (Norm equivalence in $\mathbb{R}^k$)**.** *In $\mathbb{R}^k$, all norms are equivalent.*

*Proof.* We show: Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Then $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$. By transitivity of norm equivalence, two arbitrary norms are equivalent to each other.

1. Let $(e_1, e_2, \ldots, e_k)$ be the canonical basis in $\mathbb{R}^k$.

$$x = \begin{bmatrix} x^1 \\ \vdots \\ x^k \end{bmatrix} = \sum_{j=1}^k x^j e_j$$

   Furthermore let $M' = \max\left\{\left\|e_j\right\| : j = 1, \ldots, k\right\}$ with $\left\|e_j\right\| \neq 0$ and $M' > 0$. Then it holds that

$$\|x\| = \left\|\sum_{j=1}^k x^j e_j\right\| \leq \sum_{j=1}^k \left\|x^j e_j\right\| = \sum_{j=1}^k \left|x^j\right| \left\|e_j\right\| \leq M' \sum_{j=1}^k \underbrace{\left|x_j\right|}_{\leq\|x\|_\infty} \leq \underbrace{M' \cdot k}_{M} \|x\|_\infty = M \|x\|_\infty$$

2. We consider $\nu : \mathbb{R}^k \to [0, \infty)$. $\nu(x) = \|x\|$ as map on $(\mathbb{R}^k, \|\cdot\|_\infty)$.

   **Claim.** *$\nu$ is continuous on $(\mathbb{R}^k, \|\cdot\|_\infty)$.*

*Proof.* Show that,

$$\left|\nu(x)-\nu(y)\right| = \left|\|x\| - \left\|y\right\|\right| \underbrace{\leq}_{\text{inversed triangle ineq.}} \left\|x-y\right\| \underbrace{\leq}_{\text{because of (1)}} M\left\|x-y\right\|$$

Hence $\nu$ is Lipschitz continuous. □

We consider $S_\infty^{k-1} = \left\{x \in \mathbb{R}^k\right\} \|x\|_\infty = 1 = \text{boundary}(K_1^\infty(0)$. $S_\infty^{k-1}$ is bounded. Let $(x_n)_{n\in\mathbb{N}}$ is a sequence in $S_\infty^{k-1}$ with $x = \lim_{n\to\infty} x_n$. Because $n(x) = \|x\|_\infty$ is continuous, it holds that

$$\lim_{n\to\infty} \underbrace{\|x_n\|_\infty}_{=1} = \underbrace{\|x\|}_{=1}$$

Hence $x \in S_\infty^{k-1}$. Hence, $S_\infty^{k-1}$ is closed in $(\mathbb{R}^k, \|\cdot\|_\infty)$. Hence $S_\infty^{k-1}$ is compact in $(\mathbb{R}^k, \|\cdot\|_\infty)$, $\nu : S_\infty^{k-1} \to [0,\infty)$, with $S_\infty^{k-1}$ compact, is continuous. Has a minimum $n$ on $S_\infty^{k-1}$. Thus there exists $\bar{x} \in S_\infty^{k-1}$ : $\underbrace{m}_{>0} = \left\|\underbrace{\bar{x}}_{\neq 0}\right\| \leq \|x\| \, \forall x \in S_\infty^{-1}$. Let $x \in \mathbb{R}^k$ be arbitrary with $x \neq 0$. Then it holds that $\frac{x}{\|x\|_\infty} \in S_\infty^{k-1}$ and it holds that

$$m \leq \left\|\frac{x}{\|x\|_\infty}\right\| = \frac{1}{\|x_\infty\|}\|x\| \implies m\|x\|_\infty \leq \|x\|$$

Inequality also holds true for $x = 0$.

□

# Integral calculus

**Definition 6.1.** *Let $a < b$ with $a, b \in \mathbb{R}$. We consider functions of $[a,b]$. We call $(x_j)_{j=0}^n$ a* partition *of $[a,b]$ if $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. $x_j$ decomposes $[a,b]$ in subintervals $(x_{j-1}, x_j)$. $\varphi : [a,b] \to \mathbb{R}$ is called* step function *in $[a,b]$ in regards of partition $(x_j)_{j=0}^n$ if $\varphi|_{(x_{j-1},x_j)} = c_j$, so constant for $j = 1, \ldots, n$.*

*$\varphi$ is called* step function *in $[a,b]$ if there exists a partition such that $\varphi$ is a subsequence.*

$$\tau[a,b] = \{\varphi : [a,b] \to \mathbb{R} : \varphi \textit{ is subsequence}\}$$

- *Let $(\xi_i)_{i=0}^m$ be a partition of $[a,b]$ and $(x_j)_{j=0}^n$ is a partition as well. Then we call $(\xi_i)_{i=0}^m$ a* refinement *of $[a,b]$ and $(x_j)_{j=1}^n$ as well. Then $(\xi_i)_{i=0}^n$ is a refinement of $(x_j)_{j=0}^k$ if $\{x_0, x_1, \ldots, x_n\} \subseteq \{\xi_0, \xi_1, \ldots, \xi_m\}$*

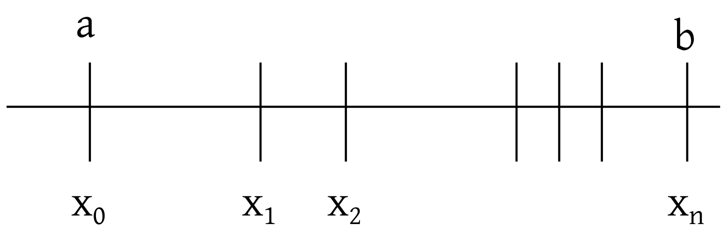



Figure 6: Illustration of a partition

*TODO drawing*

*Functions values in boundaries $x_{j-1}$ and $x_j$ do not have any constraints and will be relevant for an integral. A $\varphi$ can be a step function in terms of many, various partitions.*

**Lemma 6.1.** *Let $\varphi \in \tau[a,b]$ be a step function in terms of partition $(x_j)_{j=0}^n$ and let $(x_i)_{i=0}^n$ be a refinement of $(x_j)_{j=0}^n$ in terms of $(x_i)_{i=0}^m$.*

*Proof.* Refinement: For every $j \in \{0, \dots, n\}$ there exists $i_j \in \{0, \dots, m\}$ such that $X_j = \xi_{i_j}$. $i_0 = 0, i_n = m$. $i_{j-1} < i_j$.

Let $i \in \{1, \dots, m\}$. Then there exists a uniquely determined $j \in \{1, \dots, n\}$ such that $\xi_{i_{j-1}} < \xi_i \leq \xi_j$

TODO drawing

Then it holds that $(\xi_{i-1}, \xi_i) \subseteq \underbrace{(\xi_{i_{j-1}})}_{=(x_{j-1},x_j)}, \xi_{i_j})$ and $\varphi|_{(\xi_{i-1},\xi_j)} = c_j = \text{const}$. So $\varphi$ is a subsequence in regards of $(\xi_i)_{i=0}^m$. □

**Definition 6.2.** *Let $\varphi \in \tau[a,b]$ in terms of partition $(X_j)_{j=0}^n$ with $\varphi|_{(X_{j-1},X_j)} = c_j$ and $\triangle X_j = X_j - X_{j-1} > 0$ for $g = 1, \dots, n$. Then we define ...*

$$\int_a^b \varphi \, dx = \sum_{j=1}^n c_j \triangle x_j$$

*is called* integral of $\varphi$ *in terms of partition* $(x_j)_{j=0}^n$

*This lecture took place on 2018/03/22.*

Step function $\varphi$. $\varphi|_{x_{j-1},x_j} = c_j$

$$\delta x_j = x_j - x_{j-1}$$

$$\int_a^b \varphi \, dx = \sum_{j=1}^{n} c_j \cdot \delta x_j$$

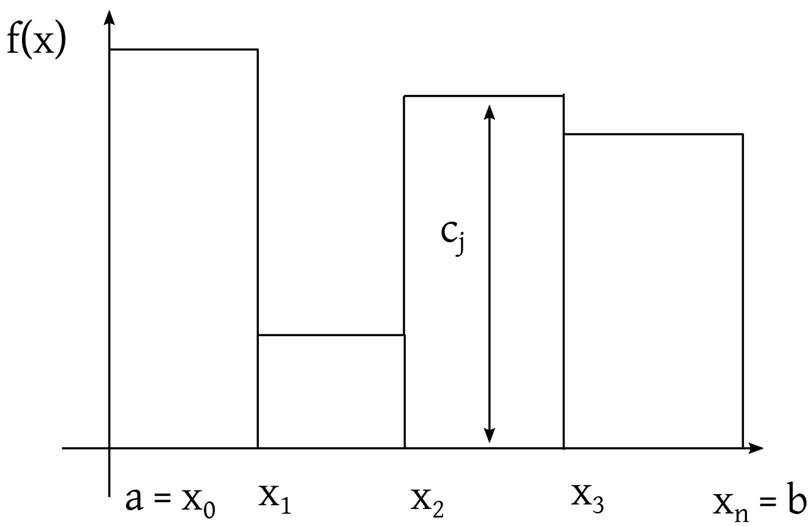



Figure 7: Integral of a step function as sum of areas of rectangles

**Lemma 6.2.** *Let $(x_i)_{j=0}^n$ be a partition of $[a, b]$ and $(\xi_i)_{i=0}^m$ be a refinement of $(x_j)_{j=0}^n$. Furthermore let $\varphi$ be a subsequence with respect to $(x_j)_{j=0}^n$ (so also with respect to $(\xi_j)_{i=0}^m$). Then the integrals of $\varphi$ with respect to $(x_j)_{j=0}^n$ and $(\xi_i)_{i=0}^m$ are equal.*

*Proof.* There exist indices $i_j$ for $j = 0, n$ such that $x_j = \xi_{ij}$.

$$i_0 = 0 \qquad i_n = m \qquad i_{j-1} < i_j$$

$$\delta x_j = x_j - x_{j-1} = \xi_{i_j} - \xi_{i_{j-1}} = \xi_{i_j} - \xi_{i_{j-1}} = \underbrace{\sum_{i=i_{j-1}+1}^{i_j}}_{\text{telescoping sum}} (\xi_i - \xi_{i-1}) = \sum_{i=i_{j-1}+1}^{i_j} \delta \xi_i$$

$$\varphi|_{x_{j-1}, x_j} = c_j \implies \varphi|_{(\xi_{i-1}, \xi_i)} = c_j \text{ for } i = i_{j-1} + 1, \ldots, i_j$$

$$\tilde{c}_i = \varphi|_{(\xi_{i-1}, \xi_i)}$$

$$\underbrace{\sum_{i=1}^{m} \tilde{c}_i \delta\xi_i}_{\text{integral of } \varphi \text{ w.r.t } (\xi_i)_{i=0}^m} = \sum_{j=1}^{n} \sum_{i=i_{j-1}+1}^{i_j} \tilde{c}_i \delta\xi_i = \sum_{j=1}^{n} c_j \underbrace{\sum_{i=i_{j-1}+1}^{i_j} \delta\xi_i}_{=x_j} = \sum_{j=1}^{n} c_j \delta x_j$$

This is the integral of $\varphi$ with respect to $(x_j)_{j=0}^n$. □

**Lemma 6.3.** *Let $\varphi$ be a step function with respect to $(x_j)_{j=0}^n$ and $(w_i)_{i=0}^L$. Then the integrals of $\varphi$ with respect to $(x_j)_{j=0}^n$ and with respect to $(w_l)_{l=0}^L$ equal.*

*Proof.* Let $\{\xi_i \mid i = 1, \ldots, m\} = \left\{x_j \,\middle|\, j = 0, \ldots, n\right\} \cup \{w_l \mid l = 0, \ldots, L\}$ with $\xi_0 = a$, $\xi_m = x_n = w_L = b$ and $\xi_{i-1} < \xi_i$ for $i = 1, \ldots, m$. Then $(\xi_i)_{i=0}^m$ is a refinement of $(x_j)_{j=0}^n$ as well as $(w_l)_{l=0}^L$. By Lemma 6.2, the integral of $\varphi$ with respect to $(x_j)_{j=0}^n =$ integral of $\varphi$ with respect to $(\xi_i)_{i=1}^m =$ integral of $\varphi$ with respect to $(w_l)_{l=0}^L$. Here we discard the statement "with respect to $(x_j)_{j=0}^n$". □

**Lemma 6.4.** *Let $f, g$ be step functions on $[a, b]$. $f, g \in \tau[a, b]$.*

- *for $\alpha, \beta \in \mathbb{R}$, let $\alpha f + \beta g \in \tau[a, b]$ and*

  $$\int_a^b (\alpha f + \beta g)\, dx = \alpha \int_a^b f\, dx + \beta \int_a^b g\, dx$$

  *Hence, the integral is linear on $[a, b]$. $\tau[a, b]$ is a vector space.*

- *$f \leq g$ in $[a, b]$, then $\int_a^b f\, dx \leq \int_a^b g\, dx$ (monotonicity).*

- *$\left|\int_a^b f\, dx\right| \leq \int_a^b \left|f\right|\, dx$ ($\left|f(x)\right|$ s also a step function)*

*Proof.* 1. Let $f, g \in \tau[a, b]$. Let $(\xi_i)_{i=0}^m$ be a partition such that $f|_{(\xi_{i-1}, \xi_i)} = c_i$ and $g|_{(\xi_{i-1}, \xi_i)} = d_i$. Then

$$\int_a^b (\alpha f + \beta g)\, dx = \sum_{i=1}^{m} (\alpha c_i + \beta d_i) \delta\xi_i = \alpha \sum_{i=1}^{m} c_i \delta\xi_i + \beta \sum_{i=1}^{m} d_i \delta\xi_i = \alpha \int_a^b f\, dx + \beta \int_a^b g\, dx$$

Furthermore,

$$(\alpha f + \beta g)|_{(\xi_{i-1}, \xi_i)} = \alpha c_i + \beta d_i = \text{ const.}$$

Thus,

$$\alpha f + \beta g \in \tau[a, b]$$

2. Let $h \in \tau[a, b]$ with $h(x) \geq 0 \forall x \in [a, b]$ be a step function and $\int_a^b h\, dx = \sum_{i=1}^m \underbrace{h_i}_{\geq 0} \delta\xi_i \geq 0$ TODO Hence, it holds that $0 \leq \int_a^b h\, dx = \int_a^b (g - f)\, dx = \int_a^b g\, dx - \int_a^b f\, dx$.

3. $f \leq \left|f\right|$, hence $\int_a^b f\,dx \leq \int_a^b \left|f\right|\,dx$ and also $-f \leq \left|f\right|$, so

$$\int_a^b (-f)\,dx = -\int_a^b f\,dx \leq \int_a^b \left|f\right|\,dx$$

$$\implies \left|\int_a^b f\,dx\right| \leq \int_a^b \left|f\right|\,dx$$

It is left to prove: $\left|f\right| \in \tau[a,b]$ (i.e. $\left|f\right|$ is a step function)

Let $f|_{(\xi_{i-1},\xi_i)} = c_i \implies \left|f\right||_{(\xi_{i-1},\xi_i)} = |c_i| =$ constant. Hence $\left|f\right| \in \tau[a,b]$.

□

**Definition 6.3.** *Let $a \subseteq \mathbb{R}^k$. We call $\chi_A : \mathbb{R}^n \to \mathbb{R}$ with*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

*a* characteristic function *(indicator function) of set A. Often denoted as $\chi_A = \mathbb{1}$.*

**Remark 6.1.** *TODO drawings*

*Let $A = (a', b')$ with $a \leq a' < b' \leq b$. Then $\chi_{(a',b')} \in \tau[a,b]$. Also for $x \in [a,b]$, it holds that $\chi_{\{x\}} = \tau[a,b]$. Therefore every linear combination of characteristic functions of open subintervals $(a', b')$ of $[a,b]$ as characteristic functions of one-point sets $\chi_{\{x\}}, x \in [a,b]$ a step function on $[a,b]$.*

$$\sum_{j=1}^{n} \alpha_j \chi_{(a_j,b_j)} + \sum_{k=1}^{m} \beta_k \chi_{\{x_k\}} \in \tau[a,b]$$

*On the opposite, $f \in \tau[a,b]$, hence*

$$f|_{(x_{j-1},x_j)} \underbrace{=}_{j=1,\ldots,n} c_j \text{ and } f(x_j) \underbrace{=}_{j=0,\ldots,n} d_j$$

$$f = \sum_{j=1}^{n} c_j \chi_{(x_{j-1},x_j)} + \sum_{j=0}^{n} d_j \chi_{\{x_j\}} = (*)$$

*for $x \in (x_{j-1}, x_j)$ it holds that $\xi_{(x_{j-1},x_j)}(x) = 1$.*

$$\chi_{(x_{l-1},x_l)}(x) = 0 \text{ for } l \neq j$$

$$\chi_{\{x_l\}}(x) = 0 \text{ for } l = 0, \ldots, n$$

*i.e. $\sum_{j=1}^{n} c_l \chi_{(x_{l-1},x_l)}(x) + \sum_{l=0}^{n} d_j \chi_{\{x_l\}}(x) = c_j \cdot 1 + 0 = c_j$ hence $(*) = c_j$ on $(x_{j-1}, x_j)$. Therefore $f \in \tau[a,b] \iff f$ is linear combination of characteristic functions of open intervals or one-pointed sets.*

## Regulated functions

**Definition 6.4.** *Let X be a metric space $A \subseteq X$ and $x \in X$ is an accumulating point*[1] *of A. Let $f : A \to \mathbb{R}$. We say, f has limit $c \in \mathbb{R}$ in x ($\lim_{\xi \to x} f(\xi) = c$) if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi \neq x \text{ and } d(\xi, x) < \delta : \left| f(\xi) - c \right| < \varepsilon$$

**Remark 6.2.** *$x \in A$ and $c = f(x) \implies f$ is continuous in x.*

*We usually consider $A = [a, b] \subseteq \mathbb{R}$, $x \in [a, b]$.*

*It is possible, that f in x has a limit, $x \in A$ and $c = \lim_{\xi \to x} f(\xi) \neq f(x)$.*

TODO drawing

**Definition 6.5.** *Now let $A \subseteq \mathbb{R}$ and x is a accumulation point of A. Let $f : A \to \mathbb{R}$ be given. We say f has a right-sided limit c in x with $c = \lim_{\xi \to x^+} f(\xi) = c$ if $\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi \in A, \xi > x$*

$$\wedge |\xi - x| = \xi - x < \delta \implies \left| f(\xi) - c \right| < \varepsilon$$

*The left-sided limit follows analogously.*

$$c = \lim_{\xi \to x^-} f(\xi)$$

$$c = \lim_{\xi \to x^+} f(\xi) \qquad d = \lim_{\xi \to x^-} f(\xi)$$

*TODO drawing*

**Lemma 6.5** (Sequence criterion for limits of functions)**.** *Let $f : A \subseteq X \to \mathbb{R}$ be given. x is an accumulation point of A. Then it holds that*

$$\lim_{\xi \to x} f(\xi) = c \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi_n \in A, \xi_n \neq x \text{ and } \lim_{n \to \infty} \xi_n = x \text{ it holds that } \lim_{n \to \infty} f(\xi_n) = c$$

*For one-sided limits $A \subseteq \mathbb{R}$ it holds that*

$$c = \lim_{\xi \to x^+} f(\xi) \iff \forall (\xi_n)_{n \in \mathbb{N}} : \xi \in A \qquad \xi_n > x \text{ with } \lim_{n \to \infty} \xi_n = x \text{ it holds that } \lim_{n \to \infty} f(\xi_n) = c$$

**Remark 6.3.** *Attention! We, therefore, use two different definitions of limits.*

**Lemma 6.6** (Cauchy criterion of limits of functions)**.** *Let $f : A \subseteq X \to \mathbb{R}$. Let x be an accumulation point of A. Let X be a metric space. Then it holds that f has a limit in x if and only if*

$$\forall \varepsilon > 0 \exists \delta > 0 : \forall \xi, \eta \in A : \xi \neq x_i : \eta \neq x :$$

[1] An accumulation point has 3 equivalent definitions (sequence, intersection, infinitely many elements in sphere).

*with $d(\xi, x) < \delta$ and $d(\eta, x) < \delta$ it holds that $\left|f(\xi) - f(\eta)\right| < \varepsilon$. Analogously for one-sided limits with $A \subseteq \mathbb{R}$. Additionally, we need the constraint that $\xi > X$ and $\eta > x$ for $\lim_{\xi\to x^+} f(\xi)$ or equivalently, $\xi < x$ and $\eta < x$ for $\lim_{\xi\to x^-} f(\xi)$.*

*TODO normalize and visualize equivalent statements for left-sided and right-sided limit (using Ring's notes)*

*Proof.* $\Leftarrow$ Let $c = \lim_{\xi\to x} f(\xi)$ and let $\varepsilon > 0$ be chosen arbitrarily. Then there exists $\delta > 0$ such that $d(\xi, x) < \delta$ and $\xi \neq x$

$$\implies \left|f(\xi) - c\right| < \frac{\varepsilon}{2}$$

For $\xi, \eta$: $d(\xi, x) < \delta$ and $d(\eta, x) < \delta$ with $\xi, \eta \neq x$ is therefore

$$\left|f(\xi) - f(\eta)\right| = \left|f(\xi) - c + c - f(\eta)\right| \leq \left|f(\xi) - c\right| + \left|f(\eta) - c\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{=}\varepsilon$$

$\Rightarrow$ Assume the Cauchy criterion holds. We show that

1. for every sequence $(\xi_n)_{n\in\mathbb{N}}$, $\xi_n \in A \setminus \{x\}$ with $\lim_{n\to\infty} \xi_n = x$ it holds that $(f(\xi_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and therefore convergent in $\mathbb{R}$.
2. all Cauchy sequences have the *same* limit $c$.

We prove (1.)

Let $(\xi_n)_{n\in\mathbb{N}}$ be as above. Let $\varepsilon > 0$ be arbitrary. and $N_\varepsilon$ large enough such that $\forall n \in N_\varepsilon$ it holds that $d(\xi_n, x) < \delta$ ($\delta$ chosen appropriately to $\varepsilon$ according to the Cauchy criterion).

By the Cauchy criterion, $\left|f(\xi_n) - f(\xi_m)\right| < \varepsilon$ for all $m, n \geq N_\varepsilon$. Therefore $(f(\xi_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. If $\mathbb{R}$ is complete, then there exists $c = \lim_{n\to\infty} f(\xi_n)$. QED.

We prove (2.)

Let $\xi_n \to x$ as above and $\xi'_n \to x$ as above and $c = \lim_{n\to\infty} f(\xi_n)$ as well as $c' = \lim_{n\to\infty} f(\xi'_n)$. Let $\varepsilon > 0$ be arbitrary, $N_\varepsilon$ such that $n \geq N_\varepsilon \implies \left|f(\xi_n) - c\right| < \frac{\varepsilon}{3}$ and $N'_\varepsilon \in \mathbb{N}$ such that $n \geq N'_\varepsilon \implies \left|f(\xi'_n) - c'\right| < \frac{\varepsilon}{3}$.

Furthermore choose $\delta > 0$ such that

$$d(\xi, x) < \delta \wedge d(\eta, x) < \delta \implies \left|f(\xi) - f(\eta)\right| < \frac{\varepsilon}{3}$$

(because of the Cauchy criterion). $M_\varepsilon$ such that

$$n \geq M_\varepsilon \implies d(\xi_n, x) < \delta \wedge M'_\varepsilon : n \geq M'_\varepsilon \implies d(\xi'_n, x) < \delta$$

Let $n \geq \max\{N_\varepsilon, N'_\varepsilon, M_\varepsilon, M'_\varepsilon\}$.

*This lecture took place on 2018/04/10.*

Then it holds that

$$|c - c'| \leq \underbrace{\left|c - f(\xi_n)\right|}_{<\frac{\varepsilon}{3}} + \underbrace{\left|f(\xi_n) - f(\xi'_n)\right|}_{<\frac{\varepsilon}{3}} + \underbrace{\left|f(\xi'_n) - c'\right|}_{<\frac{\varepsilon}{3}} \qquad \forall \varepsilon > 0$$

Hence, $c = c'$. We have shown that $\exists c \in \mathbb{R} : \forall (\xi_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \xi_n = x$ it holds that $\lim_{n\to\infty} f(\xi_n) = c$. So $\lim_{\xi\to\infty} f(\xi) = c$ because of Lemma 6.5. QED.

□

**Definition 6.6** (Regulated function)**.** *Let $a < b$, $f : [a, b] \to \mathbb{R}$. We call $f$ a* regulated function on $[a, b]$ *if*

1. *$\forall x \in (a, b)$, $f$ in $x$ has a right-sided and a left-sided limit.*
2. *in $x = a$, $f$ has a right-sided limit.*
3. *in $x = b$, $f$ has a left-sided limit.*

$$\mathcal{R}[a, b] = \left\{ f : [a, b] \to \mathbb{R} \,\middle|\, f \text{ is a regulated function} \right\}$$

**Definition 6.7** (Equivalent definition)**.**

1. *$\forall x \in [a, b)$, $f$ has a right-sided limit in $x$*
2. *$\forall x \in (a, b]$, $f$ has a left-sided limit in $x$*

**Example 6.1.** *Let $f$ be continuous in $[a, b]$. Let $\varphi \in \tau[a, b]$ be a regulated function. Then $\varphi \in \mathcal{R}[a, b]$.*

*Rationale:*

*Let $x_0 = a < x_1 < \cdots < x_n = b$ and $\varphi|_{(x_{j-1}, x_j)} = c_j$.*

*Let $x \in [a, b]$ be chosen arbitrarily.*

**Case 1** *Let $x \in (x_{j-1}, x_j)$ for some $j \in \{1, \ldots, n\}$*

$$\implies \lim_{\xi\to x} \varphi(\xi) = c_j$$

*Choose $\delta$ small enough such that $(x-\delta, x+\delta) \subseteq (x_{j-1}, x_j)$. $\forall \xi$ with $\xi \in (x-\delta, x+\delta)$ it holds that*

$$\left|\varphi(\xi) - c_j\right| = 0$$

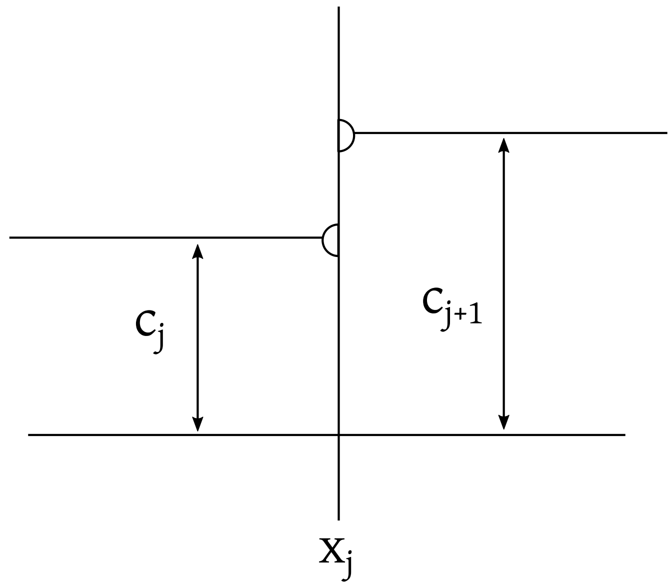



Figure 8: Regulated function

**Case 2** *Let $x = x_j$ for $j = 1, \dots, n-1$.*

$$\implies \lim_{\xi \to x_j^+} \varphi(\xi) = c_{j+1}$$

$$\lim_{\xi \to x_j^-} \varphi(\xi) = c_j$$

*Compare with Figure 8.*

**Case 3** *Let $x = x_0 = a \implies \lim_{\xi \to a^+} \varphi(\xi) = c_1$.*

$$x = x_n = b \implies \lim_{\xi \to b^-} \varphi(\xi) = c_n$$

*Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing oder monotonically decreasing. Then $f \in \mathcal{R}[a, b]$. The proof will be done in the practicals.*

**Definition 6.8** (Boundedness)**.** *Let $X \neq \emptyset$ be a set. $f : X \to \mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We say: $f$ is bounded on $X$, if $f(X) \subseteq \mathbb{K}$ is a bounded set in $\mathbb{K}$. Hence, $\exists m \geq 0 : \left|f(x)\right| \leq m \forall x \in X$. We let,*

$$\mathcal{B}(X) = \left\{ f : X \to \mathbb{K} \,\middle|\, f \text{ is bounded} \right\}$$

*$\mathcal{B}(X)$ has vector space structure. $f, g \in \mathcal{B}(X), \lambda \in \mathbb{K}$.*

$$(f+g)(x) = f(x) + g(x)$$
$$(\lambda \cdot f)(x) = \lambda \cdot f(x)$$

*$f + g \in \mathcal{B}(X)$ and $\lambda f \in \mathcal{B}(X)$. Let $\left|f(x)\right| \leq m \forall x \in X$ and $\left|g(x)\right| \leq m' \forall x \in X$. Then it holds that*

$$\left|(f+g)(x)\right| = \left|f(x) + g(x)\right| \leq \left|f(x)\right| + \left|g(x)\right| \leq m + m'$$

**Remark 6.4.** *It is very interesting, that $X$ does not require any kind of algebraic structure.*

We let

$$\left\|f\right\|_\infty = \sup \underbrace{\left\{ \left|f(x)\right| \,\middle|\, x \in X \right\}}_{\text{bounded in } \mathbb{R}} = \min \left\{ m \geq 0 \,\middle|\, \left|f(x)\right| \leq m \forall x \in X \right\}$$

Some work is required to show that $\|\cdot\|_\infty$ is a norm on $\mathcal{B}(X)$.

Hence, $(\mathcal{B}(X), \|\cdot\|_\infty)$ is a normed vector space. Convergence in $\mathcal{B}(X)$: It holds that $f_n \to f$ in $(\mathcal{B}(X), \|\cdot\|_\infty)$ if and only if $\forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies \left\|f_n - f\right\|_\infty < \varepsilon$.

$$\begin{aligned} \left\|f_n - f\right\|_\infty < \varepsilon &\iff \sup\left\{ \left|f_n(x) - f(x)\right| : x \in X \right\} \\ &\iff \left|f_n(x) - f(x)\right| \leq \varepsilon \forall x \in X \end{aligned}$$

Hence, $f_n \to f$ in $(\mathcal{B}(X), \|\cdot\|_\infty) \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} : n \geq N \implies \left|f_n(x) - f(x)\right| \leq \varepsilon \forall x \in X$. We say "$f_n$ converges *uniformly* to $f$ on $X$".

**Theorem 6.1** (Approximation theorem for regulated function)**.** *Let $f : [a,b] \to \mathbb{R}$. Then it holds that $f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0$ there exists some step function $\varphi \in \tau[a,b]$ such that $\left|\varphi(x) - f(x)\right| < \varepsilon \forall x \in [a,b]$ ($\left\|\varphi - f\right\|_\infty < \varepsilon$).*

*Especially $\varepsilon_n = \frac{1}{n}$ and $\varphi_n$ as above. Then it holds that $\left\|\varphi_n - f\right\|_\infty < \frac{1}{n}$, hence $f = \lim_{n\to\infty} \varphi_n$ uniformly on $[a,b]$.*

*Proof.* Direction $\Rightarrow$. Let $f \in \mathcal{R}[a,b]$.

Proof by contradiction. We negate our hypothesis:

$$\exists \varepsilon > 0 : \forall \varphi \in \tau[a,b] \exists x \in [a,b] : \left|\varphi(x) - f(x)\right| \geq \varepsilon \tag{1}$$

Assume (1) holds for $f \in [a,b]$. We construct nested intervals $[a_n, b_n]$ with $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and $b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$ and (1) holds on $[a_n, b_n] \forall n \in \mathbb{N}$. Hence $\forall \varphi \in \tau[a_n, b_n] \exists x \in [a_n, b_n]$ such that $\left|\varphi(x) - f(x)\right| \geq \varepsilon$. This is what we want to show.

Let $a_0 = a$ and $b_0 = b$. Then (1) holds on $[a_0, b_0]$ by assumption. $n \to n+1$: Construction of $[a_{n+1}, b_{n+1}]$. Let $m_n = \frac{1}{2}(a_n + b_n)$. We need to prove: (1) holds either on $[a_n, m_n]$ or on $[m_n, b_n]$.

Because if the opposite of (1) holds on $[a_n, m_n]$ as well as $[m_n, b_n]$, then there exists $\varphi_1^n \in \tau[a_n, m_n]$ with $\left|\varphi_n^1(x) - f(x)\right| < \varepsilon \forall x \in [a_n, m_n]$ and if the opposite of (1) holds on $[m_n, b_n]$:

$$\exists \varphi_n^2 \in \tau[m_n, b_n] : \left|\varphi_n^2(x) - f(x)\right| < \varepsilon \forall x \in [m_n, b_n]$$

Let

$$\varphi^n(x) = \begin{cases} \varphi_n^1(x) & \text{if } x \in [a_n, m_n] \\ \varphi_n^2(x) & \text{if } x \in (m_n, b_n] \end{cases}$$

Then $\varphi^n$ is piecewise constant, hence $\varphi^n \in \tau[a_n, b_n]$ and it holds that

$$\left|\varphi^n(x) - f(x)\right| = \begin{cases} \underbrace{\left|\varphi_1^n(x) - f(x)\right|}_{<\varepsilon} & \text{for } x \in [a_n, m_n] \\ \underbrace{\left|\varphi_2^n(x) - f(x)\right|}_{<\varepsilon} & \text{for } x \in [m_n, b_n] \end{cases} < \varepsilon$$

This contradicts with (1) on $[a_n, b_n]$.

Hence: (1) holds on $[a_n, m_n]$ or on $[m_n, b_n]$.

Choose $[a_{n+1}, b_{n+1}]$ as one of the subintervals in which (1) holds. $\square$

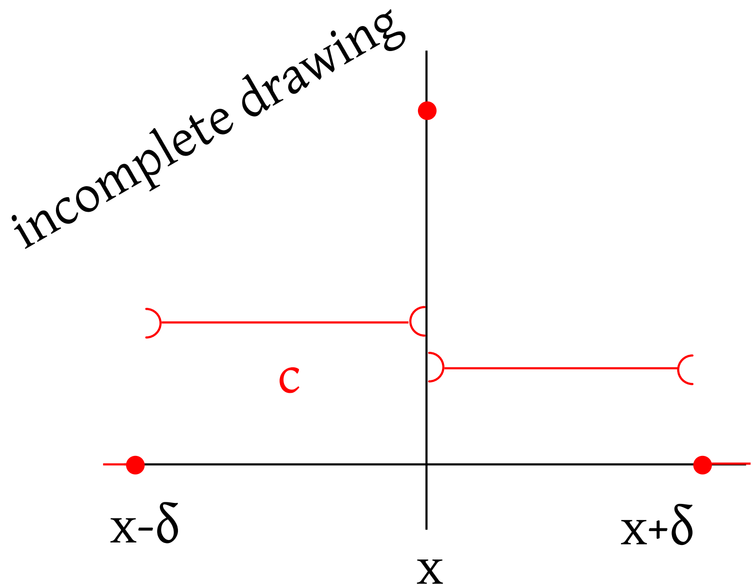
Let $X \in \bigcap_{n\in\mathbb{N}} [a_n, b_n]$ (by completeness of $\mathbb{R}$).

1. Let $x \in (a, b)$. Let $\varepsilon$ as above such that (1) holds on every interval $[a_n, b_n]$. Let $c_+ = \lim_{\xi\to x^+} f(\xi)$ and $c_- = \lim_{\xi\to x^-} f(\xi)$ (possible, because $f \in \mathcal{R}[a, b]$).

   Limes property: $\exists \delta > 0 : |\xi - x| < \delta$ and $\xi < x$, then $\left|f(\xi) - c_-\right| < \varepsilon$ and $|\xi - x| < \delta$ and $x < \delta$ then $\left|f(\xi) - c_+\right| < \varepsilon$.

   Additionally, choose $\delta$ sufficiently small enough such that $(x - \delta, x + \delta) \subseteq [a, b]$. Let

   $$\hat{\varphi}(\xi) = \begin{cases} 0 & \text{for } \xi \in [a, b] \setminus (x - \delta, x + \delta) \\ c_- & \text{for } \xi \in (x - \delta, x) \\ c_+ & \text{for } \xi \in (x, x + \delta) \\ f(x) & \text{for } \xi = x \end{cases}$$




$\hat{\varphi} \in \tau[a,b]$ and it holds that

$$\forall \xi \in (x-\delta, x+\delta) : \left|\hat{\varphi}(\xi) - f(\xi)\right| = \begin{cases} \underbrace{\left|c_- - f(\xi)\right|}_{<\varepsilon} & \text{for } \delta \in (x-\delta, x) \\ \underbrace{\left|f(x) - f(x)\right|}_{=0} & \text{for } \xi = x \\ \underbrace{\left|c_+ - f(\xi)\right|}_{<\varepsilon} & \text{for } \xi \in (x, x+\delta) \end{cases} < \varepsilon$$

Now let $N$ be sufficiently large enough such that $[a_N, b_N] \subseteq (x-\delta, x+\delta)$ (possible because $([a_n, b_n])_{n\in\mathbb{N}}$ gives nested intervals tightening on $x$). Then it holds on $[a_N, b_N]$ that:

$$\hat{\varphi}|_{[a_N, b_N]} \in \tau[a_N, b_N]$$

and $\forall \xi \in [a_N, b_N] \subseteq (x-\delta, x+\delta)$ it holds that $\left|\hat{\varphi}(\xi) - f(\xi)\right| < \varepsilon$. This contradicts with (1) on $[a_N, b_N]$.

We also need to cover the special cases $x = a$ and $x = b$. But this works analogously with one-sided limits.

Direction $\Leftarrow$: Let $f = \lim_{n\to\infty} \varphi_n$ uniform on $[a,b]$. Show that $\forall x \in [a,b)$ there exists a right-sided limit of $f$ in $x$.

Let $\varepsilon > 0$ be arbitrary. $N \in \mathbb{N}$ sufficiently large such that $\left|f(\xi) - \varphi_N(\xi)\right| < \frac{\varepsilon}{2} \forall \xi \in [a, b]$. $\varphi_N$ is piecewise constant. Choose $\delta > 0$ such that $\varphi_N|_{(x,x+\delta)} = c$. Now let $\xi, \eta \in (x, x + \delta)$ be chosen arbitrarily. Then it holds that

$$\left|f(\xi) - f(\eta)\right| \leq \left|f(\xi) - \underbrace{c}_{=\varphi_N(\xi)}\right| + \left|\underbrace{c}_{=\varphi_N(\eta)} - f(\eta)\right|$$

$$= \left|\underbrace{f(\xi) - \varphi_N(\xi)}_{<\frac{\varepsilon}{2}}\right| + \left|\underbrace{\varphi_N(\eta) - f(\eta)}_{<\frac{\varepsilon}{2}}\right| < \varepsilon$$

Therefore $f$ has a right-sided limit in $x$ by the Cauchy criterion. $f$ has left-sided limit in every point $x \in (a, b]$ analogously.

**Corollary.** *Every regulated function $f \in \mathcal{R}[a, b]$ is bounded. Let $\varphi \in \tau[a, b]$ with $\left\|f - \varphi\right\|_\infty < 1$. $\varphi$ is bounded, hence $\exists m \in [0, \infty)$: $\left|\varphi(x)\right| \leq m \forall x \in [a, b]$. Then it holds that $\left|f(x)\right| \leq \left|f(x) - \varphi(x)\right| + \left|\varphi(x)\right| < 1 + m \forall x \in [a, b]$, hence $f \in \mathcal{B}[a, b]$.*

$$\mathcal{R}[a, b] \subseteq \mathcal{B}[a, b]$$

**Corollary.** *Let $f \in \mathcal{R}[a, b] \iff f = \sum_{j=0}^{\infty} \psi_j$ with $\psi_j \in \tau[a, b]$ and the series converges uniformly on $[a, b]$.*

*Proof.* Direction $\Longleftarrow$.

Let $f = \sum_{j=0}^{\infty} \psi_j$ with uniform convergence. Let $\varphi_n = \sum_{j=0}^{\infty} \psi_j \in \tau[a, b]$ and $f = \lim_{n\to\infty} \phi_n$ uniform on $[a, b] \underbrace{\Longrightarrow}_{\text{Satz 1?!}} f \in \mathcal{R}[a, b]$.

Direction $\Longrightarrow$.

Let $f \in \mathcal{R}[a, b]$ and $f = \lim_{n\to\infty} \varphi_n$ with $\varphi_n \in \tau[a, b]$ (by Satz 1?!).

$$\begin{aligned}
\psi_0 &= \varphi_0 \\
\psi_j &= \varphi_j - \varphi_{j-1} \qquad \text{for } j \geq 1 \\
\sum_{j=0}^{n} \psi_j &= \varphi_0 + \sum_{j=1}^{n} (\varphi_j - \varphi_{j-1}) = \varphi_0 + \sum_{j=1}^{n} \varphi_j - \sum_{j=0}^{n-1} \varphi_j = \varphi_n
\end{aligned}$$

converges uniformly to $f$. □

# Integration of regulated functions

**Definition 7.1** (Definition with a theorem)**.** *Let $f \in \mathcal{R}[a,b]$ and $\varphi_n \in \tau[a,b]$ with $f = \lim_{n\to\infty} \varphi_n$ is uniform on $[a,b]$. We let*

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b \varphi_n\,dx$$

*for the integral of $f$ on $[a,b]$.*

*Theorem: This limit (on the right-hand side) always exists and is independent of the particular choice of the approximating sequence.*

*Proof.* $\varphi_n$ is chosen as above.

$$i_n = \int_a^b \varphi_n\,dx$$

Show: $i_n$ is cauchy sequence in $\mathbb{R}$.

*This lecture took place on 2018/04/12.*

Let $\varepsilon > 0$ be chosen arbitrary. Choose $N \in \mathbb{N}$ such that

$$n \geq N \implies \left\|f - \varphi_n\right\|_\infty < \frac{\varepsilon}{2(b-a)}$$

For $n, m \geq N$ it holds for $x \in [a,b]$ that

$$\begin{aligned}\left|\varphi_n(x) - \varphi_m(x)\right| &\leq \left|\varphi_n(x) - f(x)\right| + \left|f(x) - \varphi_m(x)\right| \\ &\leq \left\|\varphi_n - f\right\|_\infty + \left\|f - \varphi_m\right\|_\infty < \frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon}{b-a}\end{aligned}$$

$\left|\varphi_n - \varphi_m\right|$ is a step function.

$$\left|\varphi_n - \varphi_m\right| \leq \frac{\varepsilon}{b-a} \cdot \underbrace{\chi_{[a,b]}}_{\in\tau[a,b]}$$

Integral for subsequence is monotonous:

$$\begin{aligned}|i_n - i_m| &= \left|\int_a^b \varphi_n\,dx - \int_a^b \varphi_m\,dx\right| = \left|\int_a^b (\varphi_n - \varphi_m)\,dx\right| \leq \int_a^b \left|\varphi_n - \varphi_m\right|\,dx \\ &\underbrace{<}_{\text{by monotonicity}} \int_a^b \frac{\varepsilon}{b-a} \cdot \chi_{[a,b]}\,dx = \frac{\varepsilon}{b-a} \underbrace{\int_a^b \chi_{[a,b]}\,dx}_{1\cdot(b-a)} = \varepsilon\end{aligned}$$

So $(i_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. $\mathbb{R}$ is complete, hence $i = \lim_{n\to\infty} i_n$ exists.

Uniqueness: (dt. mithilfe des Reissverschlussprinzips)

Let $(\varphi_n)_{n\in\mathbb{N}}$, $(\Phi_n)_{n\in\mathbb{N}}$ be two sequences of step functions, converging uniformly towards $f$.

$$i_n = \int_a^b \varphi_n \, dx \quad \text{and} \quad j_n = \int_a^b \Phi_n \, dx$$

$$i = \lim_{n\to\infty} i_n \qquad j = \lim_{n\to\infty} j_n$$

Show that $i = j$.

Now we construct a sequence $(\mu_n)_{n\in\mathbb{N}}$ of step functions.

$$\underbrace{(\varphi_1, \Phi_1, \varphi_2, \Phi_2, \dots)}_{(\mu_n)_{n\in\mathbb{N}}}$$

$\mu_n$ is a sequence of step functions converging uniformly towards $f$ (the proof is left as an exercise to the reader).

Because of part 1 of the proof:

$$m_n = \int_a^b \mu_n \, dx \text{ converges with limit } m$$

$(i_n)_{n\in\mathbb{N}}$ as well as $(j_n)_{n\in\mathbb{N}}$ are subsequences of $(m_n)_{n\in\mathbb{N}}$. Hence it holds that $i = \lim_{n\to\infty} i_n = m = \lim_{n\to\infty} j_n = j$. □

**Theorem 7.1** (Elementary properties of an integral)**.** *Let $f, g \in \mathcal{R}[a,b]$, $\lambda, \mu \in \mathbb{R}$. Then it holds that*

**Linearity**

$$\lambda f + \mu g \in \mathcal{R}[a,b] \text{ and } \int_a^b (\lambda f + \mu g) \, dx = \lambda \int_a^b f \, dx + \mu \int_a^b g \, dx$$

**Monotonicity** *If $f(x) \leq g(x) \forall x \in [a,b]$ ($f \leq g$) it holds that*

$$\int_a^b f \, dx \leq \int_a^b g \, dx$$

**Boundedness** $\left|f\right| \in \mathcal{R}[a,b]$ *and*

$$\left|\int_a^b f \, dx\right| \leq \int_a^b \left|f\right| \, dx$$

*Proof.* We prove linearity.

Let $x \in [a,b)$ and $c_+ = \lim_{\xi \to x_+} f(\xi)$ as well as $d_+ = \lim_{\xi \to x_+} g(\xi)$ $(f, g \in \mathcal{R}[a,b])$. Then it holds that

$$\lim_{\xi \to x^+} (\lambda f(\xi) + \mu g(\xi)) = \lambda \lim_{\xi \to x^+} f(\xi) + \mu \lim_{\xi \to x^+} g(\xi) = \lambda c_+ + \mu d_+$$

exists. Analogously for the left side, hence $\lambda f + \mu g \in \mathcal{R}[a,b]$.

Let $\varphi_n, \Phi_n \in \tau[a,b]$ with $\varphi_n \to f$ and $\Phi_n \to g$ is uniform on $[a,b]$. Hence $\lambda\varphi_n + \mu\Phi_n \to \lambda f + \mu g$ is continuous on $[a,b]$.

Proof of this:

Let $\varepsilon > 0$ be arbitrary, $N$ such that $n \geq N \implies \left\|\varphi_n - f\right\|_\infty < \frac{\varepsilon}{2(|\lambda|+1)}$ and $M$ such that $n \geq M \implies \left\|\Phi_n - g\right\|_\infty < \frac{\varepsilon}{2(\left|\mu\right|+1)}$.

Then it holds that

$$\left\|\lambda\varphi_n + \mu\Phi_n - \lambda f - \mu g\right\|_\infty \leq |\lambda| \left\|\varphi_n - f\right\|_\infty + \left|\mu\right| \left\|\Phi_n - g\right\|_\infty$$

$$< \frac{|\lambda|}{2(|\lambda| + 1)} \cdot \varepsilon + \frac{\left|\mu\right|}{2(\left|\mu\right| + 1)} \cdot \varepsilon < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We continue:

$$\int_a^b (\lambda f + \mu g)\, dx = \lim_{n\to\infty} \int_a^b (\lambda\varphi_n + \mu\Phi_n)\, dx = \lim_{n\to\infty} (\lambda \int_a^b \varphi_n\, dx + \mu \int_a^b \Phi_n\, dx)$$

$$= \lambda \underbrace{\lim_{n\to\infty} \int_a^b \varphi_n\, dx}_{\text{exists}} + \mu \underbrace{\lim_{n\to\infty} \int_a^b \Phi_n\, dx}_{\text{exists}}$$

$$= \lambda \int_a^b f\, dx + \mu \int_a^b g\, dx$$

We prove monotonicity.

Show: Let $h \in \mathcal{R}[a,b]$ with $h \geq 0$ in $[a,b]$. Then it holds that $\int_a^b h\, dx \geq 0$.

We will show that $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ exists with $\tilde{\varphi}_n \to h$ uniform on $[a,b]$ and $\tilde{\varphi}_n \geq 0$.

Therefore we prove: Let $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in \tau[a,b]$ with $\varphi_n \to h$ uniform on $[a,b]$.

Define $\tilde{\varphi}_n$ such that

$$\varphi_n = \sum_{j=1}^{m_n} c_j \chi_{(x_{j-1},x_j)} + \sum_{j=0}^{m_n} d_j \chi_{\{x_j\}}$$

Let

$$\tilde{\varphi}_n = \sum_{j=1}^{m_n} \underbrace{\tilde{c}_j}_{\geq 0} \chi_{(x_{j-1},x_j)} + \sum_{j=0}^{m_n} \underbrace{h(x_j)}_{\geq 0} \chi_{\{x_j\}}$$

and $\tilde{c}_j := \max c_j, 0 \geq 0$. So it holds that $\tilde{\varphi}_n \geq 0$.
For $x = x_l$ ($l \in \{0, \dots, m_n\}$) it holds that

$$\left|\tilde{\varphi}_n(x_l) - h(x_l)\right| = \left|\sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1},x_j)}(x_l)}_{=0 \text{ bc. } x_l \notin (x_{j-1},x_j)} + \sum_{j=0}^{m_n} h(x_j) \underbrace{\chi_{\{x_j\}}(x_l)}_{=\delta_{j,l}} - h(x_l)\right|$$

$$= |h(x_l) - h(x_l)| = 0 \leq \left|\varphi_n(x_l) - h(x_l)\right|$$

For $x \in (x_{j-1}, x_j)$ it holds that

$$\left|\tilde{\varphi}_n(x) - h(x)\right| = \left|\sum_{j=1}^{m_n} \tilde{c}_j \underbrace{\chi_{(x_{j-1},x_j)}(x)}_{\delta_{l,j}} + \sum_{j=0}^{m_n} h(x) \cdot \underbrace{\chi_{\{x_j\}}(x)}_{=0 \text{ bc. } x \neq x_j} - h(x)\right|$$

$$= |\tilde{c}_l - h(x)| = \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ |h(x)| = h(x) & \text{if } c_l < 0 \end{cases}$$

$$\leq \begin{cases} |c_l - h(x)| & \text{if } c_l \geq 0 \\ h(x) - c_l & \text{if } c_l < 0 \end{cases}$$

$$= \begin{cases} \left|\varphi_n(x) - h(x)\right| & \text{if } c_l = \varphi_n(x) \geq 0 \\ \left|h(x) - \varphi_n(x)\right| & \text{if } c_l = \varphi_n(x) < 0 \end{cases}$$

$$= \left|\varphi_n(x) - h(x)\right|$$

hence, $\left|\tilde{\varphi}_n(x) - h(x)\right| \leq \left|\varphi_n(x) - h(x)\right|$ for $x \in (x_{l-1}, x_l)$ as well as $x = x_i$, hence

$$\left\|\tilde{\varphi}_n - h\right\|_\infty \leq \underbrace{\left\|\varphi_n - h\right\|_\infty}_{\to 0 \text{ for } n \to \infty}$$

Hence $\left\|\tilde{\varphi}_n - h\right\|_\infty \to 0$ for $n \to \infty$, hence $\tilde{\varphi}_n$ converges uniformly to $h$. There exists

$$\int_a^b h\,dx = \underbrace{\lim_{n\to\infty} \int_a^b \underbrace{\tilde{\varphi}_n}_{\geq 0}\,dx}_{\geq 0} \geq 0$$

Monotonicity: Let $f \leq g$ in $[a, b]$, hence $h = g - f \geq 0$ in $[a, b]$

$$\implies 0 \leq \int_a^b h\,dx = \int_a^b g\,dx - \int_a^b f\,dx$$

$$\implies \int_a^b f\,dx \leq \int_a^b g\,dx$$

And finally, boundedness is left.

Consider $\left|f\right| \in \mathbb{R}[a,b]$. Proving this is left as an exercise. $f \leq \left|f\right|$ in $[a,b] \implies \int_a^b f\,dx \leq \int_a^b \left|f\right|\,dx$.

TODO

$$-f \leq \left|f\right| \text{ in } [a,b] \implies \int_a^b (-f)\,dx = -\int_a^b f\,dx \leq \int_a^b \left|f\right|\,dx \implies \left|\int_a^b f\,dx\right| TODO$$

□

**Remark 7.1.** *$\mathcal{R}[a,b]$ is a vector space.*

1. *$f,g \in \mathbb{R}[a,b] \implies \lambda f + \mu g \in \mathcal{R}[a,b]$. $\|\cdot\|_\infty$ is a norm on $\mathcal{R}[a,b]$. ($\mathcal{R}[a,b], \|\cdot\|_\infty$) is a normed vector space. Subspace of ($\mathcal{B}[a,b], \|\cdot\|_\infty$). We will show in the practicals that ($\mathcal{R}[a,b], \|\cdot\|_\infty$) is complete.*

**Theorem 7.2** (Mean value theorem of integral calculus)**.** *Let $f$ be continuous on $[a,b]$ and $p \in \mathcal{R}[a,b]$ and $p \geq 0$ in $[a,b]$. Then $f \cdot p \in \mathcal{R}[a,b]$ and there exists $\xi \in [a,b]$ such that*

$$\int_a^b f \cdot p\,dx = f(\xi) \cdot \int_a^b p\,dx$$

*Proof.* Let $m = \min\{f(z) : z \in [a,b]\}$ (exists because $f$ is continuous and $[a,b]$ is compact).

$$M = \max\{f(z) : z \in [a,b]\}$$

$$f([a,b]) = [m,M] \text{ (by the mean value theorem)}$$

It holds that

$$m \cdot \underbrace{p(x)}_{\geq 0} \leq f(x) \cdot p(x) \leq M \cdot p(x)$$

By monotonicity,

$$m \int_a^b p(x)\,dx \leq \int_a^b fp\,dx \leq M \int_a^b p\,dx$$

Therefore, there exists $\eta \in [m,M]$.

$$\eta \cdot \int_a^b p(x)\,dx = \int_a^b fp\,dx$$

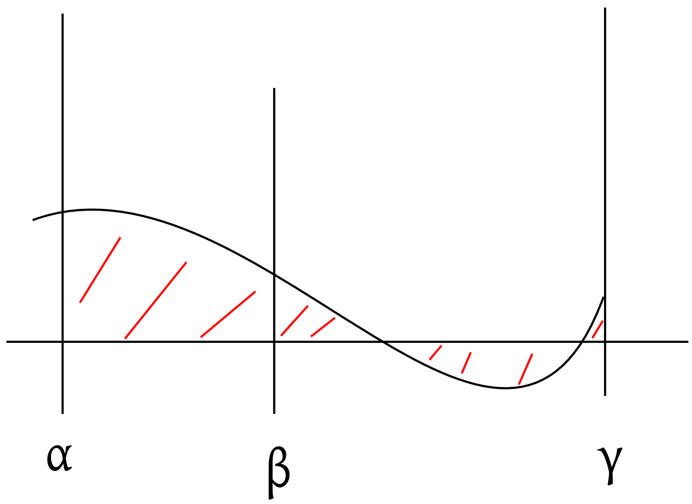



Figure 9: Positive and negative area covered by the integral

Mean value theorem: For $\eta \in [m, M]$ there exists $\xi \in [a, b]$ such that

$$\eta = f(\xi) \text{ (f is continuous!)}$$

Hence,

$$f(\xi) \int_a^b p \, dx = \int_a^b f \cdot p \, dx$$

$f \cdot p$ is regulated function (over one-sided limits). □

**Lemma 7.1.** *Let $f \in \mathcal{R}[a, b]$ and $a \leq \alpha < \beta < \gamma \leq b$. Then*

$$f|_{[\alpha,\beta]} \in \mathcal{R}[\alpha, \beta], f|_{\beta,\gamma} \in \mathcal{R}[\beta, \gamma]$$

$$f|_{[\alpha,\gamma]} \in \mathcal{R}[\alpha, \gamma] \text{ (immediate over onesided limit)}$$

*and it holds that*

$$\int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx$$

*Compare with Figure 9.*

*Proof.* Show that this statement holds for $\varphi \in \tau[a,b]$. Without loss of generality, $\alpha = a, \gamma = b$.

$$\gamma = \sum_{j=1}^{m} c_j \chi_{(x_{j-1},x_j)} + \sum_{j=0}^{m} \underbrace{0}_{\text{it does not matter for the integral}} \cdot \chi_{x_j}$$

**Case 1** $\beta = x_l$ for some $l \in \{1, \dots, m-1\}$

$$\int_{\alpha}^{\gamma} \varphi\, dx = \sum_{j=1}^{m} c_j (x_j - x_{j-1})$$

$$\int_{\alpha}^{\beta} \varphi\, dx = \int_{\alpha}^{x_l} \varphi\, dx = \sum_{j=1}^{l} c_j (x_j - x_{j-1})$$

$$\int_{\beta}^{\gamma} \varphi\, dx = \int_{x_l}^{\gamma} \varphi\, dx = \sum_{j=l+1}^{m} c_j (x_j - x_{j-1})$$

And now,

$$\sum_{j=l+1}^{m} c_j (x_j - x_{j-1}) + \sum_{j=1}^{l} c_j (x_j - x_{j-1}) = \sum_{j=1}^{m} c_j (x_j - x_{j-1})$$

**Case 2** $\beta \in (x_{l-1}, x_l)$ for some $l \in \{1, \dots, m\}$.

$$\int_{\beta}^{\gamma} \varphi\, dx = c_l (x_l - \beta) + \sum_{j=l+1}^{m} c_j (x_j - x_{j-1})$$

$$\int_{\alpha}^{\beta} \varphi\, dx + \int_{\beta}^{\gamma} \varphi\, dx = \sum_{j=1}^{l-1} c_j (x_j - x_{j-1})$$

$$+ c_l(\beta - x_{l-1}) + c_l(x_l - \beta) + \sum_{j=l+1}^{m} c_j (x_j - x_{j-1})$$

$$= \sum_{j=1}^{m} c_j (x_j - x_{j-1}) = \int_{\alpha}^{\gamma} \varphi\, dx$$

TODO verify previous lines Let $\varphi_n \in \tau[\alpha, \beta]$ with $\varphi_n \to f$ uniform on $[\alpha, \beta] \implies \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]}$ uniform on $[\alpha, \beta]$ and also $\varphi_n|_{[\beta,\gamma]} \to f|_{[\beta,\gamma]}$ uniform on $[\beta, \gamma]$.

$$\int_{\alpha}^{\gamma} f\, dx = \lim_{n\to\infty} \int_{\alpha}^{\gamma} \varphi_n\, dx = \lim_{n\to\infty} (\int_{\alpha}^{\beta} \varphi_n\, dx + \int_{\beta}^{\gamma} \varphi_n\, dx)$$

$$= \underbrace{\lim_{n\to\infty} \int_\alpha^\beta \varphi_n \, dx}_{\text{exists because } \varphi_n|_{[\alpha,\beta]} \to f|_{[\alpha,\beta]} \text{uniform}} + \lim_{n\to\infty} \int_\beta^\gamma \varphi_n \, dx$$

$$= \int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx$$

□

**Remark 7.2** (Notation). *Let $\alpha < \beta$, $\alpha, \beta \in [a, b]$ and $f \in \mathcal{R}[a, b]$. We let*

$$\int_\beta^\alpha f \, dx := -\int_\alpha^\beta f \, dx$$

*By this convention, it holds that*

$$\int_\alpha^\alpha f \, dx = -\int_\alpha^\alpha f \, dx \implies \int_\alpha^\alpha f \, dx = 0$$

**Lemma 7.2.** *Let $f \in \mathcal{R}[a, b]$ and $\alpha, \beta, \gamma \in [a, b]$ (without particular order). Then it holds that*

$$\int_\alpha^\gamma f \, dx = \int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx$$

*Proof.* Special case: 2 points are equal

$$\alpha = \gamma \implies \int_a^\alpha f \, dx = 0$$

$$\int_\alpha^\beta f \, dx + \int_\beta^\alpha f \, dx = \int_\alpha^\beta f \, dx - \int_\alpha^\beta f \, dx = 0$$

$$\beta = \gamma \qquad \beta = \alpha$$

Case: $\alpha < \beta < \gamma$ follows immediately

And just as a representative other case: $\alpha < \gamma < \beta$

$$\int_\alpha^\beta f \, dx \underbrace{=}_{\text{by Lemma 4.1}} \int_\alpha^\gamma f \, dx + \underbrace{\int_\gamma^\beta f \, dx}_{-\int_\beta^\gamma f \, dx}$$

$$\int_\alpha^\beta f \, dx + \int_\beta^\gamma f \, dx = \int_\alpha^\gamma f \, dx$$

□

*This lecture took place on 2018/04/17.*

**Lemma 7.3.** *Let $f \in \mathcal{R}[a,b]$. Then there exists an at most countable set $A \subseteq [a,b]$ such that $f$ is continuous in every point $x \in [a,b] \setminus A$.*

*Proof.* Let $f \in \mathcal{R}[a,b]$ and $(\varphi_n)_{n\in\mathbb{N}}$ with $\varphi_n \in \tau[a,b]$ and $\varphi \to f$ converging uniformly on $[a,b]$.

$$\varphi_n = \sum_{j=1}^{m_n} c_j^n \chi_{(X_{j-1}^n, X_j^n)} + \sum_{j=0}^{m_n} d_j^n \chi_{\left\{x_j^n\right\}}$$

$$x_0^n = a < x_1^n < \ldots < x_{m_n}^n = b$$

are separating points for $\varphi_n$

$$A = \left\{X_j^n : n \in \mathbb{N}, j \in \{0, \ldots, m_n\}\right\}$$

$A$ is a countable union of finite sets $A_n = \left\{x_0^n, x_{m_n}^n\right\}$. A is countable (as unions of finite sets are).

Now we show: $f$ is continuous in every point $x \in [a,b] : x \notin A$. Let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ sufficiently large such that $\left\|\varphi_N - f\right\|_\infty < \frac{\varepsilon}{2}$. Because $x \in A$, there exists $j \in \{1, \ldots, m_N\}$ such that $x \in (x_{j-1}^N, X_j^N)$ is open. Choose $\delta > 0$ such that $(x-\delta, x+\delta) \subset (x_{j-1}^N, x_j^n)$, hence $\forall \xi \in (x-\delta, x+\delta)$ it holds that $\varphi_N(\xi) = c_j^N$. Now consider $\xi \in (x-\delta, x+\delta)$, hence $|\xi - x| < \delta$. Then it holds that

$$\left|f(\xi) - f(x)\right| = \left|f(\xi) - \underbrace{\varphi_N(x)}_{c_j^N = \varphi_N(\xi)} + \varphi_N(x) - f(x)\right|$$

$$\leq \underbrace{\left|f(\xi) - \varphi_N(\xi)\right|}_{\leq \|f-\varphi_N\|_\infty} + \underbrace{\left|\varphi_N(x) - f(x)\right|}_{\leq \|\varphi_N - f\|_\infty} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $f$ is continuous in $x$. ◻

**Remark 7.3** (Notation)**.** *Let $f \in \mathcal{R}[a,b]$. For $x \in [a,b)$, there exists $f_+(x) := \lim_{\xi \to x_+} f(\xi)$. For $x \in (a,b]$, there exists $f_-(x) := \lim_{\xi \to x_-} f(\xi)$. Because of Lemma 7.3, it holds that $f_+(x) = f_-(x) = f(x)$ for all $x \in [a,b] \setminus A$ and $A$ is at most countable.*

**Definition 7.2** (One-sided derivatives)**.** *Let $g : [a,b] \to \mathbb{R}$ and $x \in [a,b)$. We say $g$ has the* right-sided derivative *$g'_+(x)$ if*

$$\lim_{\xi \to x_+} \frac{g(\xi) - g(x)}{\xi - x} =: g'_+(x)$$

*exists. Analogously we define the left-sided derivative*

$$g'_-(x) = \lim_{\xi \to x_-} \frac{g(\xi) - g(x)}{\xi - x}$$

*for $x \in (a, b]$. Compare with Figure 10.*

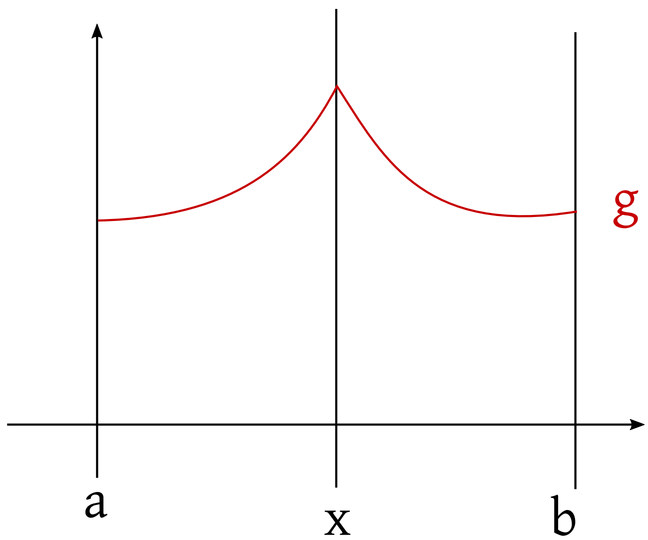



Figure 10: In this example, the left- and right-sided derivatives are not equal. $f'_+(x) \neq f'_-(x)$

**Remark 7.4.** *If $g$ in $x$ has a one-sided derivative, then it holds that*

$$\lim_{\xi \to x_\pm} (g(\xi) - g(x)) = 0$$

*Hence $g$ is continuous in $x$.*

**Remark 7.5.** *$g : [a, b] \to \mathbb{R}$ is differentiable in point $x \in (a, b)$ with derivative $g'(x)$ $\iff$ $g$ has a left- and right-sided derivative in $x$ and it holds that $g'_-(x) = g'_+(x)$ $(= g'(x))$.*

**Theorem 7.3** (Fundamental theorem of differential/integration calculus, variation 1)**.** *Isaac Barrow (1630–1677), Isaac Newton (1642–1726), Gottfried Wilhelm von Leibniz (1646–1716).*

*Let $f \in \mathcal{R}[a, b]$, $\alpha \in [a, b]$ and we define*

$$F(x) = \int_\alpha^x f \, d\xi$$

*Then $F$ is right-sided differentiable in every point $x \in [a,b)$ and in every $x \in (a,b]$ left-sided differentiable. Furthermore it holds that*

$$F'_+(x) = f_+(x) \forall x \in [a,b) \tag{2}$$
$$F'_-(x) = f_-(x) \forall x \in (a,b] \tag{3}$$

**Remark 7.6.**

$$\frac{d}{dx}\left(\int_\alpha^x f\, d\xi\right) = f(x)$$

*for all $x$ such that $f$ is continuous in $x$. For those $x$, $F'(x)$ is differentiable in $x$ with $F'(x) = f(x)$.*

**Definition 7.3.** *Let $f \in \mathcal{R}[a,b]$ and $\varphi : [a,b] \to \mathbb{R}$ such that $\varphi$ is one-sided differentiable on $[a,b]$. If $\Phi'_+(x) = f_+(x) \forall x \in [a,b)$ and $\Phi'_-(x) = f_-(x) \forall x \in (a,b]$ then we call $\Phi$ an antiderivative of regulated function $f$.*

*Proof of the Theorem 7.3.* Let $x_1, x_2 \in [a,b]$ be arbitrary. Let $F$ be defined as above. Then it holds that

$$\begin{aligned} |F(x_2) - F(x_1)| &= \left|\int_\alpha^{x_2} f\, d\xi - \int_\alpha^{x_1} f\, d\xi\right| \\ &= \left|\int_\alpha^{x_2} f\, d\xi + \int_{x_1}^{\alpha} f\, d\xi\right| = \left|\int_{x_1}^{x_2} f\, d\xi\right| \\ \leq \int_{x_1}^{x_2} \left|f\right| d\xi &\leq \int_{x_1}^{x_2} \underbrace{\left\|f\right\|_\infty}_{\text{const independent of } \xi} d\xi = \left\|f\right\|_\infty \cdot |x_2 - x_1| \end{aligned}$$

Hence $F$ is Lipschitz continuous with Lipschitz constant $\left\|f\right\|_\infty$. So $F$ is continuous in $[a,b]$.

One-sided derivatives: Let $x \in [a,b)$ and $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\forall \xi \in [x, x+\delta)$ it holds that $\left|f(\xi) - f_+(x)\right| < \varepsilon$. For $\xi \in (x, x+\delta)$ it holds that

$$\left|\frac{F(\xi) - F(x)}{\xi - x} - f_+(x)\right| = \frac{1}{|\xi - x|}\left|\underbrace{\int_x^\xi f\, dy}_{F(\xi)-F(x)} - \underbrace{f_+(x)(\xi - x)}_{\int_x^\xi \underbrace{f_+(x)}_{\text{const.}} dy}\right| = \frac{1}{|\xi - x|}\left|\int_x^\xi (f - f_+(x))\, dy\right| \leq \frac{1}{|\xi - x|}\int_x^\xi \underbrace{\left|f(y) - f_+(x)\right|}_{<\varepsilon} dy$$

$$y \in (x, \xi) \subseteq (x, x+\delta)$$

$$< \frac{1}{\xi - x}\varepsilon \cdot \underbrace{\int_x^\xi 1\, dy}_{|\xi - x|} = \varepsilon$$

Hence, $F'_+(x) = f_+(x)$. Analogously, $F'_-(x) = f_-(x)$ for $x \in (a,b]$. □

**Theorem 7.4** (Fundamental theorem of differential/integration calculus, variation 2)**.** *Let $f \in \mathcal{R}[a,b]$ and $\phi$ is an arbitrary antiderviative of $f$ according to Definition 7.3. For $\alpha, \beta \in [a,b]$ arbitrary, it holds that*

$$\int_\alpha^\beta f\,dx = \phi(\beta) - \phi(\alpha)$$

**Remark 7.7.** *Let $f$ be continuous and $\phi$ be an antiderivative of $f$. Hence, $\Phi'(x) = f(x) \forall x \in [a,b]$. Then it holds that*

$$\int_\alpha^\beta \Phi'\,dx = \Phi(\beta) - \Phi(\alpha)$$

*"Integral of a derivative of $\Phi$ gives $\Phi(\beta) - \Phi(\alpha)$".*

**Lemma 7.4.** *Let $A \subseteq [a,b]$ countable. $f : [a,b] \to \mathbb{R}$ is continuous and $f$ is differentiable in every point $x \in [a,b] \setminus A$. Furthermore let $\left|f'(x)\right| \leq L$ $(L \geq 0)$ for all $x \in [a,b] \setminus A$. Then $f$ is Lipschitz continuous on $[a,b]$ with constant $L$, hence*

$$\left|f(x_2) - f(x_1)\right| \leq L\,|x_2 - x_1|\,\forall x_1, x_2 \in [a,b]$$

**Remark 7.8.** *Some people call it* differentiable almost everywhere, *but this expression collides with a different definition pronounced the same way from measure theory.*

*Proof.* Let $x_1, x_2 \in [a,b]$, wlog. $x_1 < x_2$. Let $\varepsilon > 0$ be arbitrary. We define

$$F_\varepsilon(x) = \left|f(x) - f(x_1)\right| - (L + \varepsilon)(x - x_1)$$

for $x \in [x_1, b]$.

Let $\varepsilon > 0$ be arbitrary. We prove: $F_\varepsilon(x) \leq 0 \forall x \in [x_1, b]$. In particular: $F_\varepsilon(x_2) \leq 0$. Hence,

$$\left|f(x_2) - f(x_1)\right| \leq (L + \varepsilon)\underbrace{(x_2 - x_1)}_{|x_2 - x_1|}$$

We prove by contradiction: Assume there exists $\varepsilon > 0$ and $x_\varepsilon > x_1$ such that $F_\varepsilon(x_\varepsilon) > 0$.

We recognize: Let $A' = [x_1, b] \cap A$ be countable.

1. hence $F_\varepsilon(A') \subseteq \mathbb{R}$ is countable
2. $F_\varepsilon(x_1) = 0, F_\varepsilon(x_\varepsilon) > 0 \implies x_\varepsilon > x_1$
3. $F_\varepsilon$ is continuous on $[x_1, b]$. It holds that $0 \in F_\varepsilon([x_1, x_\varepsilon])$ and because $0 = F_\varepsilon(x_1)$ and $\varepsilon \in F_\varepsilon([x_1, x_\varepsilon])$ because $\varepsilon = F_\varepsilon(x_\varepsilon)$.

By the Intermediate Value Theorem, it follows that $[0, \varepsilon] \subseteq$ TODO By the Intermediate Value Theorem, it follows that $\underbrace{[0, \eta]}_{\text{uncountable}} \subseteq F_\varepsilon([x_1, x_\varepsilon])$.

$F_\varepsilon(A')$ is countable, hence there exists $\gamma \in (0, \eta]$ such that $\gamma = F_\varepsilon(y)$ and $\gamma \notin A'$ $(\gamma > 0)$ [2]. Hence, $y \notin A'$. So $f$ in $y$ is differentiable. Let $B := F_\varepsilon^{-1}(\{\gamma\}) \cap ([x_1, x_\varepsilon] \setminus A')$. Then $B \neq \emptyset$.

$B \subseteq [x_1, x_\varepsilon]$ is therefore bounded, $B \neq 0$. Hence, $B$ has a supremum. Let $x = \sup B$. Choose $(y_n)_{n \in \mathbb{N}}$ with $y_n \in B$ and $y_n \to x$ for $n \to \infty$. Because $F_\varepsilon$ is continuous, it holds that

$$\lim_{n \to \infty} \underbrace{F_\varepsilon(y_n)}_{\gamma} = F_\varepsilon(x)$$

hence $F_\varepsilon(x) = \gamma$. This implies $x \notin A$.

Furthermore it holds for $w \in (x, x_\varepsilon]$ that $F_\varepsilon(w) > \gamma$. Because assume the opposite ($F_\varepsilon(w) \leq \gamma$ for $w > x$). Furthermore it holds that $F_\varepsilon(x_\varepsilon) = \eta \geq \gamma$. Because of the Intermediate Value Theorem, $\exists y \geq w$ with $F_\varepsilon(y) = \gamma$. This contradicts with the supremum property of $x$.

Now let $y \in (x, x_\varepsilon]$.

$$\begin{aligned}
\varphi(y) &= \frac{F_\varepsilon(y) - F_\varepsilon(x)}{y - x} \\
&\underbrace{=}_{\substack{\text{definition of} \\ F_\varepsilon}} \frac{\left|f(y) - f(x_1)\right| - \left|f(x) - f(x_1)\right|}{y - x} - \frac{(L + \varepsilon)(y - x_1 - x + x_1)}{y - x} \\
&\underbrace{\leq}_{\text{inversed triangle ineq.}} \frac{f(y) - f(x)}{y - x} - (L + \varepsilon)
\end{aligned}$$

Because $F_\varepsilon(y) > \gamma = F_\varepsilon(x)$ it holds that $\varphi(y) > 0$ for $y > x$. So,

$$\frac{\left|f(y) - f(x)\right|}{y - x} \geq L + \varepsilon$$

$$\left|f'(x)\right| = \lim_{y \to x_+} \left|\frac{f(y) - f(x)}{y - x}\right| \geq L + \varepsilon$$

This contradicts with the boundedness of the derivative by $L$ and $f$ is in $x \notin A$ differentiable.

So, equations 2 do not hold. Therefore $\forall x_1, x_2$ with $x_1 < x_2$ in $[a, b]$ and $\forall \varepsilon > 0$,

$$\begin{aligned}
&\left|f(x_2) - f(x_1)\right| \leq (L + \varepsilon)\,|x_2 - x_1| \\
\implies &\left|f(x_2) - f(x_1)\right| \leq L\,|x_2 - x_1|
\end{aligned}$$

[2] remember this as reference (*)

□

**Corollary** (Corollary to Lemma 7.4)**.** *Let $f, g : [a, b] \to \mathbb{R}$ differentiable for all points $x \in [a, b] \setminus A$ and $A$ is countable. Furthermore let $f'(x) = g'(x) \forall x \notin A$. Then there exists $K \in \mathbb{R}$ such that $f(x) = g(x) + K \forall x \in [a, b]$.*

*Proof.* Let $h = f - g$. Then it holds that

$$h'(x) = f'(x) - g'(x) = 0 \forall x \in [a, b] \setminus A$$

By Lemma 7.4 with $L = 0$, it follows that

$$\left|h(x_1) - f(x_2)\right| \leq 0 \cdot |x_1 - x_2| = 0$$

$$\implies h(x_1) = h(x_2) \forall x_1, x_2 \in [a, b]$$

Hence, $h(x) = K \in \mathbb{R}$.

$$\implies f(x) = g(x) + h(x) = g(x) + K$$

□

*This lecture took place on 2018/04/19.*

By reference (*), $\gamma \in [0, \eta)$ (uncountable) and $\gamma \notin f(A)$ (countable).

$$\implies \forall u \in [x_1, b) \text{ with } F_\varepsilon(u) = \gamma$$

it holds that $u \notin A$, hence $f$ is differentiable in $u$.

*Proof of Theorem 7.4.* Let $f \in \mathcal{R}[a, b]$, $\phi$ is an antiderivative of $f$, hence $\phi'_+ = f_+$, $\phi'_- = f_-$. Let $\alpha \in [a, b]$ be arbitrary. By the Theorem variant 1, $F(x) = \int_\alpha^x f \, d\xi$ is also an antiderivative of $f$. By Lemma **??**, $\exists K \in \mathbb{R} : F(x) = \int_\alpha^x f \, d\xi = \phi(x) + K$. Determine $K$: Let $x = \alpha \implies F(\alpha) = \int_\alpha^\alpha f \, dx = 0 = \phi(\alpha) - K$ hence $K = \phi(\alpha)$. Hence,

$$\int_\alpha^x f \, d\xi = \phi(x) - \phi(\alpha)$$

Let $x = \beta$. □

**Remark 7.9** (Remark for the previous corollary)**.** *$F$, $\phi$ are differentiable on all points $x$ for which $f$ is continuous (all of them except for countable many). For those $x$, it holds that $F'(x) = \varphi'(x) = f(x)$.*

**Remark 7.10** (Notation)**.** *Let $f \in \mathcal{R}[a, b]$. Then*

$$\int f \, dx$$

- *is some particular antiderivative of $f$ (usually some arbitrary chosen)*
- *the set of* all *antiderivatives of $f$*

$$\int f\,dx = \{F : F \text{ is antiderivative of } f\}$$

*If $F_0$ is some fixed antiderivative, then*

$$\int f\,dx = \{F_0 + K : K \in \mathbb{R}\}$$

*Then $\int f\,dx$ is the so-called* indefinite integral of $f$. *Notation:*

$$\int x^k\,dx = \frac{x^{k+1}}{k+1} + c \qquad (k \neq -1)$$

| $f$ | $F$ | remark |
|---|---|---|
| $x^\alpha$ | $\frac{x^{\alpha+1}}{\alpha+1} + c$ | $\alpha \in \mathbb{R} \setminus \{-1\}$; restrict $x$ such that $x^\alpha$ and $x^{\alpha+1}$ are defined |
| $x^{-1}$ | $\ln x + c$ $(x > 0)$ | |
| $\left(\frac{1}{-x}\right) \cdot (-1) = x^{-1}$ | $\ln -x + c$ $(x < 0)$ | |
| $e^x$ | $e^x$ | |
| $\sin x$ | $-\cos x$ | |
| $\cos x$ | $\sin x$ | |
| $\sinh x$ | $\cosh x$ | |
| $\cosh x$ | $\sinh x$ | |
| $\frac{1}{1+x^2}$ | $\arctan x$ | |
| $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin x$ | $\|x\| < 1$ |
| $-\frac{1}{\sqrt{1-x^2}}$ | $\arccos x$ | |

Table 1: Table of antiderivatives

## Integration methods

In this chapter, we discuss how to determine the antiderivative of a function. Usually they are composites of basic functions. Some of these are given in Table 1.

**Remark 7.11.** *Let $F, G : [a,b] \to \mathbb{R}$ in $x \in [a,b)$ right-sided differentiable. Then also $F \cdot G$ in $x$ is right-sided differentiable and it holds that*

$$(F \cdot G)'_+(x) = F'_+(x) \cdot G(x) + F(x) \cdot G'_+(x)$$

*hence the product law holds.*

*Analogously, the same holds for the left-sided derivative.*

*Look up the proof in the course Analysis 1.*

**Partial integration**

**Definition 7.4** (Partial integration)**.** *Let $f, g$ be given. Let $F, G$ be its antiderivatives respectively. Then $F \cdot G$ is an antiderivative of $F \cdot g + f \cdot G$.*

*This is immediate, because*

$$(F \cdot G)'_+ = F'_+ \cdot G + F \cdot G'_+ = f_+ \cdot G + F \cdot g_+ = f_+ G_+ + F_+ \cdot g_+$$

*Hence, it holds that*

$$\int_a^b (Fg + fG)\, dx = \underbrace{F(b) \cdot G(b) - F(a)G(a)}_{=: F \cdot G|_a^b}$$

*Usually, this is rewritten as*

$$\int_a^b F \cdot g\, dx = F \cdot G|_a^b - \int_a^b fG\, dx$$

*If $F = u$ is continuously differentiable and $G = v$ as well, then $f = u'$ and $g = v'$ and the law has the structure*

$$\boxed{\int_a^b uv'\, dx = u \cdot v|_a^b - \int_a^b u'v\, dx}$$

**Example 7.1.** *Let $a \neq -1$ and $x > 0$.*

$$\int \underbrace{x^a}_{v'} \cdot \underbrace{\ln x}_{u}\, dx = \underbrace{\begin{vmatrix} u = \ln x & u' = \frac{1}{x} \\ v' = x^\alpha & v = \frac{x^{\alpha+1}}{\alpha+1} \end{vmatrix}}_{\text{scribble notes}} \quad \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \int \frac{1}{x} \cdot \frac{x^{\alpha+1}}{\alpha+1}\, dx$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{\alpha+1} \int x^\alpha\, dx = \frac{x^{\alpha+1}}{\alpha+1} \cdot \ln x - \frac{1}{(\alpha+1)^2} x^{\alpha+1}$$

**Example 7.2.** *Let $k \in \{2, 3, 4, \ldots\}$.*

$$\int \cos^k(x)\, dx = \begin{vmatrix} u = \cos^{k-1}(x) & u' = (k-1) \cdot \cos^{k-2}(x) \cdot (-\sin x) \\ v' = \cos x & v = \sin x \end{vmatrix}$$

$$\cos^{k-1}(x) \sin x + (k-1) \int \cos^{k-2}(x) \cdot \underbrace{\sin^2(x)}_{(1-\cos^2 x)}\, dx$$

$$= \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x)\, dx - (k-1) \int \cos^k(x)\, dx$$

*Then we can use the following identity:*

$$k \int \cos^k(x)\,dx = \cos^{k-1}(x) \cdot \sin(x) + (k-1) \int \cos^{k-2}(x)\,dx$$

*This gives a recursive formula:*

$$\int \cos^k(x)\,dx = \frac{1}{k} \cos^{k-1}(x) \cdot \frac{k-1}{k} \sin(x) + (k-1) \int \cos^{k-2}(x)\,dx$$

*Analogously,*

$$\int \sin^k(x)\,dx = -\frac{1}{k} \sin^{k-1}(x) \cdot \cos(x) + \frac{k-1}{k} \int \sin^{k-2}(x)\,dx$$

*Let* $c_m = \int_0^{\frac{\pi}{2}} \cos^m(x)\,dx$. *Then the following formula holds:*

$$\begin{aligned} c_{2n} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \frac{2n-5}{2n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \prod_{k=1}^{n} \frac{2k-1}{2k} \cdot \frac{\pi}{2} \\ c_{2n+1} &= \prod_{k=1}^{n} \frac{2k}{2k+1} \end{aligned}$$

*Proof by induction.* Let $n = 1$.

$$\begin{aligned} c_2 &= \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \frac{1}{2} \cos x \sin x \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\,dx = 0 - 0 + \frac{\pi}{4} \\ &= \underbrace{\prod_{k=1}^{1} \frac{2k-1}{2k}}_{\frac{1}{2}} \cdot \frac{\pi}{2} \\ c_1 &= \int_0^{\frac{\pi}{2}} \cos x\,dx = \sin x\big|_0^{\frac{\pi}{2}} = 1 - 0 = 1 \end{aligned}$$

$$\underbrace{\prod_{k=1}^{0} \frac{2k}{2k+1}}_{\text{empty product}} = 1$$

We make the induction step $n \to n+1$:

$$c_{2(n+1)} = \frac{1}{2n+2} \cdot \underbrace{\cos^{2n+1}(x)}_{=0 \text{ for } x=\frac{\pi}{2}} \cdot \underbrace{\sin(x)}_{=0 \text{ for } x=0}\Bigg|_0^{\frac{\pi}{2}} + \frac{2n+1}{2n+2}\int_0^{\frac{\pi}{2}} \cos^{2n}(x)\,dx$$

$$= \frac{2n+1}{2n+2}\prod_{k=1}^{n}\frac{2k-1}{2k}\cdot\frac{\pi}{2} = \prod_{k=1}^{n+1}\frac{2k-1}{2k}\cdot\frac{\pi}{2}$$

$c_{2(n+1)+1}$ analogously. □

**Theorem 7.5** (Wallis product)**.** *John Wallis (1616–1703), result from 1655*

*Let* $w_n = \prod_{k=1}^{n} \frac{(2k)^2}{(2k-1)(2k+1)} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\ldots$*. Then it holds that* $\lim_{n\to\infty} w_n = \frac{\pi}{2}$.

*Proof.*

$$\frac{\pi}{2}\cdot\frac{c_{2n+1}}{c_{2n}} = \frac{\pi}{2}\cdot\prod_{k=1}^{n}\frac{\frac{2k}{2k+1}}{\prod_{k=1}^{n}\frac{2k-1}{2k}\cdot\frac{\pi}{2}} = \prod_{k=1}^{n}\frac{(2k)^2}{(2k-1)(2k+1)} = w_n$$

It remains to show that $\lim_{n\to\infty}\frac{c_{2n+1}}{c_{2n}} = 1$ in $[0,\frac{\pi}{2}]$ it holds that $0 \le \cos x \le 1$.

$$\implies \cos^{2n+2}(x) \le \cos^{2n+1}(x) \le \cos^{2n}(x)$$

So, $c_{2n+2} \le c_{2n+1} \le c_{2n}$ for $n \ge 1$.

$$1 \ge \frac{c_{2n+1}}{c_{2n}}$$

$$\implies 1 \ge \frac{c_{2n+1}}{c_{2n}} \ge \frac{c_{2n+2}}{c_{2n}} = \frac{\prod_{k=1}^{n+1}\frac{2k-1}{2k}\frac{\pi}{2}}{\prod_{k=1}^{n}\frac{2k-1}{2k}\frac{\pi}{2}}$$

$$= \frac{2n+2-1}{2n+2} \to 1 \text{ for } n\to\infty$$

Because of the sandwich lemma for convergent sequences, the intermediate expression must also converge to 1, hence

$$\lim_{n\to\infty}\frac{c_{2n+1}}{c_{2n}} = 1 \qquad \wedge \qquad \frac{\pi}{2}\cdot\underbrace{\lim_{n\to\infty}\frac{c_{2n+1}}{c_{2n}}}_{=1} = \lim_{n\to\infty} w_n$$

□

**Integration by substitution**

**Definition 7.5** (Integration by substitution)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $t : [\alpha, \beta] \to [a, b]$ be continuously differentiable. Let $F$ be an antiderivative of $f$ ($F$ is therefore continuously differentiable). Then $F \circ t : [\alpha, \beta] \to \mathbb{R}$ is also continuously differentiable and the chain rule holds:*

$$(F \circ t)' = (F' \circ t) \cdot t' = (f \circ t) \cdot t'$$

*Hence $F \circ t$ is an antiderivative of $(f \circ t) \cdot t'$. We apply it to integration:*

$$\int_\alpha^\beta (f \circ t)(u) \cdot t'(u)\, du = (F \circ t)(\beta) - (F \circ t)(\alpha) = F(t(\beta)) - F(t(\alpha)) = \int_{t(\alpha)}^{t(\beta)} f(x)\, dx$$

*Then we get the substitution integration method:*

$$\boxed{\int_{t(\alpha)}^{t(\beta)} f(x)\, dx = \int_\alpha^\beta f(t(u)) \cdot t'(u)\, du}$$

**Remark 7.12** (Mnemonic)**.** *Consider the left-hand side and right-hand side simultaneously. Let $x = t(u)$ (expressions inside parentheses). Then $dx = t'(u) \cdot du$ (expressions on the right). Let $u = \alpha \implies x = t(\alpha)$ and $u = \beta \implies x = t(\beta)$ (interval boundaries).*

**Example 7.3.**

$$\int_0^1 2x\sqrt{1 - x^2}\, dx$$

*Usually we have some expression, we want to substitute with $u$.*

$$1 - x^2 = u \qquad x = \sqrt{1 - u} = t(u)$$

$$x = 0 = t(1) \qquad x = 1 = t(0)$$

$$dx = \frac{1}{2} \cdot \frac{1}{\sqrt{1 - u}} \cdot (-1)\, du$$

$$\int_0^1 2x\sqrt{1 - x^2}\, dx = \int_1^0 2 \cdot \sqrt{1 - u} \cdot u \cdot \frac{1}{2}(-1)\frac{1}{\sqrt{1 - u}}\, du = \int_0^1 \sqrt{u}\, du = \left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_0^1 = \frac{2}{3}$$

$$\int_0^1 2x\sqrt{\underbrace{1 - x^2}_{u}}\, dx = \begin{vmatrix} u = 1 - x^2 & \\ x = 0 & \Leftrightarrow u = 1 \\ x = 1 & \Leftrightarrow xu = 0 \\ 1 \cdot du & = -2x\, dx \end{vmatrix} = -\int_1^0 \sqrt{u}\, du = \int_0^1 \sqrt{u}\, du$$

*In general: we set $h(u) = g(x)$, then it holds that $h'(u)\, du = g'(x)\, dx$.*

**Theorem 7.6.** *Let $f, \tilde{f} \in \mathcal{R}[a,b]$ and $A \subseteq [a,b]$ countable. Furthermore $f(x) = \tilde{f}(x) \forall x \in [a,b] \setminus A$. Then it holds that*

$$\int_a^b \left| f - \tilde{f} \right| dx = 0$$

*Then it follows especially that*

$$\int_a^b f \, dx = \int_a^b \tilde{f} \, dx$$

*This lecture took place on 2018/04/24.*

*Proof.* Show: $r \in \mathcal{R}[a,b], r \geq 0$. $\int_a^b r \, dx = 0$ and $r(x) = 0$ for $x \in [a,b] \setminus A$. Then it holds that $\int_a^b r \, dx = 0$. Let $r$ be as above. First, we show: $r_+(x) = \lim_{\xi \to x_+} r(\xi) = 0 \forall x \in [a,b)$ and also $r_-(x) = 0 \forall x \in (a,b]$.

Proof of that: Let $x \in [a,b)$ and $y = r_+(x)$ (exists because $r \in \mathcal{R}[a,b]$). Choose $\delta_n = \frac{1}{n}$. $(x, x + \frac{1}{n}) \cap [a,b)$ is an open interval with uncountable many points, so there is certainly one point in $A$. So there exists $\xi_n \in ((x, x + \frac{1}{n}) \cap [a,b)) \setminus A$ and $|\xi_n - x| < \delta_n = \frac{1}{n}$. Hence, $\lim_{n \to \infty} \xi_n = x$ and $r(\xi_n) = 0$. Therefore, $\lim_{n \to \infty} r(\xi_n) = 0$ where $r(\xi_n) = y = r_+(x)$.

Analogously, $r_-(x) = 0$ on $(a,b]$.

Let $\varepsilon > 0$ be arbitrary. We let $A_\varepsilon = \{w \in [a,b] \,|\, r(w) > \varepsilon\}$. We show: $A_\varepsilon$ is finite.

Assume $A_\varepsilon$ would have infinitely many points. Choose a sequence $(w_n)_{n \in \mathbb{N}}$ with $w_n \in A_\varepsilon$ and $w_n \neq w_m$ for $n \neq m$ (works because $A_\varepsilon$ is infinite). $(w_n)_{n \in \mathbb{N}}$ is bounded, hence there exists a convergent subsequence $(w_{n_k})_{k \in \mathbb{N}}$ with $x = \lim_{k \to \infty} w_{n_k} \in [a,b]$ and $w_{n_k} \in [a,b]$.

Either $(w_{n_k})$ contains infinitely many sequence element $w_{n_k} < x$ (variant (a)) or infinitely many $w_{n_k} > x$ (variant (b)). Let variant b hold without loss of generality.

Combine all $w_{n_k} > x$ to one subsequence $(w_{n_{k_l}})_{l \in \mathbb{N}}$. This gives $\lim_{l \to \infty} w_{n_{k_l}} = x$ and $w_{n_{k_l}} > x$, thus $\lim_{l \to \infty} \underbrace{r(w_{n_{k_l}})}_{\geq \varepsilon \text{ because } w_{n_{k_l}} \in A_\varepsilon} = r_+(x) = 0$. This gives a contradiction.

$A_\varepsilon$ must be finite.

Consider

$$A_{\frac{1}{n}} = \left\{ w_1^n, \ldots, w_{m_n}^n \right\}$$

finite. Let $\varphi_n = \sum_{k=1}^{m_n} r(w_k^n) \cdot \chi_{\{w_k^n\}} \in \tau[a,b]$.

For $x = w_k^n \in A_{\frac{1}{n}}$ it holds that

$$\varphi_n(w_k^n) = \sum_{k=1}^{m_n} r(w_k^n) \cdot \underbrace{\chi_{\{w_k^n\}}(w_j^n)}_{\delta_{jk}} = r(w_j^n)$$

so $\left|\varphi_n(x) - r(x)\right| = 0 \forall x \in A_{\frac{1}{n}}$. Let $x \in [a,b] \setminus A_{\frac{1}{n}}$. Then it holds $0 \leq r(x) < \frac{1}{n}$ and for $x \notin A_{\frac{1}{n}}$ it holds that $\varphi(x) = 0$. Therefore,

$$\left|r(x) - \varphi(x)\right| = r(x) < \frac{1}{n}$$

hence $\left\|r - \varphi_n\right\|_\infty < \frac{1}{n}$. This means that $\varphi_n \to r$ uniformly on $[a,b]$. Therefore

$$\lim_{n\to\infty} \underbrace{\int_a^b \varphi_n \, dx}_{=0} = \int_a^b r \, dx = 0$$

Now we want to finish the proof of our theorem: Let $r(x) = \left|f(x) - \tilde{f}(x)\right| \geq 0$ and $r(x) = 0$ for $x \notin A$. So, $\int_a^b \left|f - \tilde{f}\right| \, dx = 0$ (first part proven).

$$\left|\int_a^b f \, dx - \int_a^b \tilde{f} \, dx\right| = \left|\int_a^b (f - \tilde{f}) \, dx\right| \leq \int_a^b \left|f - \tilde{f}\right| \, dx = 0$$

$$\implies \int_a^b f \, dx = \int_a^b \tilde{f} \, dx$$

Second part proven. □

**Lemma 7.5.** *Let $f \in \mathcal{R}[a,b]$. Then it holds that $f_+ \in \mathcal{R}[a,b]$ and also $f_- \in \mathcal{R}[a,b]$.*

*Proof.* Only for $f_+$: First, we show: Let $x \in [a,b)$.

$$f_+(x) = \lim_{\xi\to x_+} f(\xi) = \lim_{\xi\to x_+} f_+(x)$$

(the plus is important on the right-hand side!).

Proof of this: Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that $\forall \xi \in (x, x+\delta)$: $\left|f(\xi) - f_+(x)\right| < \frac{\varepsilon}{2}$. Now let $z \in (x, x+\delta)$ be arbitrary chosen. For $z$ there exists $\xi \in (z, x+\delta)$.

$\xi$ sufficiently close enough to $z$ such that $\left|f(\xi) - f_+(z)\right| \leq \frac{\varepsilon}{2}$ because $f_+(z)$ exists.

$$\left|f_+(z) - f_+(x)\right| \leq \left|f_+(z) - f(\xi)\right| + \left|f(\xi) - f_+(x)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

TODO some content missing here

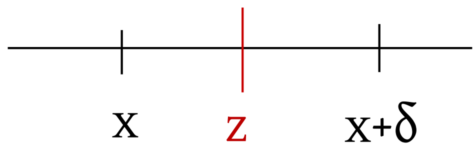



Figure 11: $x$ and $z$

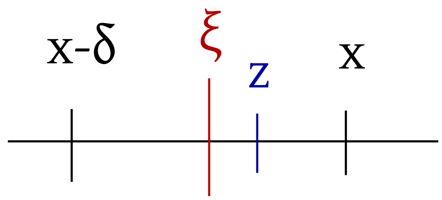



Figure 12: $\xi$ and $z$

It remains to show: $f_+$ has left-sided limits. Let $x \in (a, b]$ be arbitrary and $f_-(x) = \lim_{\xi \to x_-} f(\xi)$. We show: $f_-(x) = \lim_{\xi \to x_-} f_+(x)$ (again: the plus is important).

Let $\varepsilon > 0$ be arbitrary. Choose $\delta > 0$ such that $\forall z \in (x - \delta, x)$ it holds that $\left|f(z) - f_-(x)\right| < \frac{\xi}{2}$.

Now let $\xi \in (x - \delta, x)$ (compare with Figure 12) and choose $x > z > \xi$ with the property that $\left|f(z) - f_+(\xi)\right| < \frac{\xi}{2}$ (possible because $f$ in $\xi$ has a right-sided limit):

$$\left|f_+(\xi) - f_-(x)\right| \leq \underbrace{\left|f_+(\xi) - f(z)\right|}_{<\frac{\xi}{2}} + \underbrace{\left|f(z) - f_-(x)\right|}_{<\frac{\xi}{2}}$$

because of the choice of $\delta$ and $z \in (\xi, x) \subseteq (x - \delta, x)$.

Hence, $\lim_{\xi \to x_-} f_+(\xi) = f_-(x)$. Analogously for $f_-$ □

**Remark 7.13.**

$$\lim_{\xi \to x_+} f_+(\xi) = f_+(x)$$

$$\lim_{\xi \to x_-} f_-(\xi) = f_-(x)$$

*from the proof. So $f_+$ is right-sided continuous and $f_-$ is left-sided continuous.*

**Lemma 7.6.** *Let $f \in \mathcal{R}[a,b]$. Then it holds that*

$$\int_a^b f\,dx = \int_a^b f_+\,dx = \int_a^b f_-\,dx$$

*Proof.* For $f_+$:

$$f, f_+ \in \mathcal{R}[a,b]$$

$\forall x \in [a,b]$ with $f$ is continuous in $x$ it holds that

$$f(x) = \lim_{\xi\to x} f(\xi) = \lim_{\xi\to x_+} f(\xi) = f_+(x)$$

$f$ has at most countable many discontinuity points. By Satz 7.6,

$$\int_a^b \left|f - f_+\right|\,dx = 0 \quad \text{or equivalently} \quad \int_a^b f\,dx = \int_a^b f_+\,dx$$

$\square$

## Improper integrals

Let $I$ be an interval in $\mathbb{R}$ with marginal points $a$ and $b$ with $-\infty \leq a < b \leq +\infty$. Let $f$ be a regulated function on $I$. We define

1. If $I = [a,b)$, $\int_a^b f\,dx = \lim_{\beta\to b_-} \int_a^\beta f\,dx$
2. If $I = (a,b]$, $\int_a^b f\,dx = \lim_{\alpha\to a_+} \int_\alpha^b f\,dx$
3. If $I = (a,b)$, $\int_a^b f\,dx = \lim_{\alpha\to a_+} \int_\alpha^c f\,dx + \lim_{\beta\to b_-} \int_c^\beta f\,dx$

for an arbitrarily chosen $c \in (a,b)$ under the constraint that the corresponding limits in $\mathbb{R}$ exist.

Standard examples will follow:

**Example 7.4.** *Let $s > 1$.*

$$\int_1^\infty x^{-s}\,dx = \lim_{\beta\to\infty} \int_1^\beta x^{-s}\,dx = \lim_{\beta\to\infty} \left(\frac{1}{-s+1} x^{-s+1}\right)\Bigg|_1^\beta$$

$$= \frac{1}{1-s} \cdot \underbrace{\lim_{\beta\to\infty} \frac{1}{\beta^{\underbrace{s-1}_{>0}}}}_{=0} - \frac{1}{1-s} \cdot 1 = \frac{1}{s-1}$$

*TODO drawing*

**Example 7.5.** *Let $s < 1$.*

$$\int_0^1 x^{-s}\,dx = \lim_{\alpha\to 0_+} \int_\alpha^1 x^{-s}\,ds = \lim_{\alpha\to 0_+} \frac{1}{-s+1} x^{-s+1}\Big|_\alpha^1$$

$$= \frac{1}{1-s} - \frac{1}{1-s} \cdot \underbrace{\lim_{\alpha\to 0} \alpha^{\overbrace{1-s}^{>0}}}_{=0} = \frac{1}{1-s}$$

*TODO drawing*

*For $s = 1$, neither $\int_0^1 \frac{1}{x}\,dx$ nor $\int_1^\infty \frac{1}{x}\,dx$ exists.*

**Example 7.6.** *For $c > 0$,*

$$\int_0^\infty e^{-cx}\,dx = \lim_{\beta\to\infty} \int_0^\beta e^{-cx}\,dx = \lim_{\beta\to\infty} \left(-\frac{1}{c}\right) \cdot e^{-cx}\Big|_0^\beta - \frac{1}{c} \cdot \underbrace{\lim_{\beta\to\infty} e^{-c\beta}}_{=0} + \frac{1}{c} = \frac{1}{c}$$

**Theorem 7.7** (Direct comparison test for improper integrals)**.** *In German, "Majorantenkriterium für uneigentliche Intergale".*

*Let $f, g$ be regulated functions on $I$ and it holds that*

$$\left|f(x)\right| \leq g(x) \forall x \in I$$

*Assume $\int_a^b g\,dx$ exists as improper integral. Then also the following improper integrals exist:*

$$\int_a^b \left|f\right|\,dx \text{ and } \int_a^b f\,dx$$

*In German, $g$ is called Majorante of $f$ (there is no equivalent terminology in English).*

*Proof.* Without loss of generality, let $I = [a, b)$. Let $G(\beta) = \int_a^\beta g\,dx$. We know that $\lim_{\beta\to b_-} G(\beta)$ exists. By Lemma 6.6 (Cauchy criterion for existence of limits): Let $\varepsilon > 0$ be arbitrary, then there exists a right-sided neighborhood $U$ of $b$ ($U = (b-\delta, b)$ if $b < \infty$ and $U = (M, \infty)$ if $b = \infty$) with $u, v \in U$, then it holds that $|G(v) - G(u)| < \varepsilon$.

$$|G(v) - G(u)| = \left|\int_a^v g\,dx - \int_a^u g\,dx\right| = \left|\int_u^a g\,dx + \int_a^v g\,dx\right| = \left|\int_u^v g\,dx\right|$$

Let $F(\beta) = \int_a^\beta \left|f\right|\,dx$. Analogously as for $G$, it holds that $F(v) - F(u) = \int_u^v \left|f\right|\,dx$. Let $u, v \in U$. Then it holds that

$$|F(v) - F(u)| = \left|\int_u^v \left|f\right|\,dx\right| \leq \left|\int_u^v g\,dx\right| = |G(v) - G(u)| < \varepsilon$$

hence by the Cauchy criterion for $F$: $\lim_{\beta\to b_-} F(\beta)$ exists, so there exists $\int_a^b \left|f\right| dx$ as improper integral. The same applies for the existence of $\int_a^b f\,dx$. □

**Example 7.7.** *The cardinal sine function is defined as*

$$\operatorname{sinc}(x) = \frac{\sin x}{x}$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \operatorname{sinc}(0) = 1$$

*So* $\operatorname{sinc}(x)$ *is continuous on* $\mathbb{R}$.

$$\int_0^\infty \frac{\sin x}{x}\,dx = \underbrace{\int_0^1 \underbrace{\frac{\sin x}{x}}_{\text{continuous}}\,dx}_{\text{exists}} + \int_1^\infty \frac{\sin x}{x}\,dx$$

*How about* $\int_1^\infty \frac{\sin(x)}{x}\,dx$*?*

$$\lim_{\beta\to\infty} \int_1^\beta \frac{\sin x}{x}\,dx = \begin{vmatrix} u = \frac{1}{x} & u' = -\frac{1}{x^2} \\ v' = \sin x & v = -\cos x \end{vmatrix} = \lim_{\beta\to\infty} \left[-\frac{1}{x}\cos x\right]_1^\beta - \int_1^\beta \frac{\cos x}{x^2}\,dx$$

$$= \cos(1) - \lim_{\beta\to\infty} \int_1^\beta \frac{\cos(x)}{x^2}\,dx$$

$$\left|\frac{\cos(x)}{x^2}\right| \le \frac{1}{x^2} \textit{ on } [1,\beta]$$

*and* $\int_1^\infty \frac{1}{x^2}\,dx$ *exists. So* $g(x) = \frac{1}{x^2}$ *is a majorant of* $\frac{\cos(x)}{x^2}$ *and by Theorem 7.7,* $\lim_{\beta\to\infty} \int_1^\beta \frac{\cos(x)}{x^2}\,dx$ *eixsts.*

*Attention!* $\int_0^\infty \left|\frac{\sin(x)}{x}\right| dx$ *does not exist. Is not Lebesgue integrable.*

**Definition 7.6.** *Let* $x > 0$. *We call* $\Gamma$ Euler's Gamma function.

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\,dx$$

**Remark 7.14.** *The improper integral in the definition of the* $\Gamma$*-function exists for all* $x > 0$.