5% and 1% points of F.

v_2 \ v_1	1	2	3	4	5	6	8	12	24	∞
2	18.51	19.00	19.16	19.25	19.30	19.32	19.37	19.41	19.45	19.50
	98.49	99.00	99.17	99.25	99.30	99.33	99.36	99.42	99.46	99.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
	16.26	13.27	12.06	11.39	10.97	10.67	10.27	9.89	9.47	9.02
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
	13.74	10.92	9.78	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
	12.25	9.55	8.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54
	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.01	2.57
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.28	2.08	1.84
	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51
	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.81
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60

* t-distribution is used for testing of comparison of means.
* F-distribution is used for testing of comparison of Variances.

F-test (Test the significance of the difference between population variance)

σ_1^2, σ_2^2 - Population Variances, s_1^2, s_2^2 - Sample Variance

$H_0: \sigma_1^2 = \sigma_2^2$, $H_1: \sigma_1^2 \neq \sigma_2^2$

Let $\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$, $\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$, $v_1 = n_1 - 1$, $v_2 = n_2 - 1$

(i) If $\sigma_1^2 > \sigma_2^2$ then $F = \frac{\sigma_1^2}{\sigma_2^2}$, $(v_1 = n_1 - 1, v_2 = n_2 - 1)$

(ii) If $\sigma_1^2 < \sigma_2^2$ then $F = \frac{\sigma_2^2}{\sigma_1^2}$, $(v_1 = n_2 - 1, v_2 = n_1 - 1)$

1.) A sample of size 13 gave an estimated population variance of 3.0, while another sample of size 15 gave an estimate of 2.5. Could both samples be from populations with the same variances?

Sol: Given $n_1 = 13$, $\sigma_1^2 = 3$, $n_2 = 15$, $\sigma_2^2 = 2.5$

$H_0: \sigma_1^2 = \sigma_2^2$, $H_1: \sigma_1^2 \neq \sigma_2^2$

$F = \frac{\sigma_1^2}{\sigma_2^2}$ If $\sigma_1^2 > \sigma_2^2$

$= \frac{3.0}{2.5} = 1.2$, From F-table, $F_{5\%}(v_1 = 12, v_2 = 14) = 2.53$

$\therefore F < F_{5\%}$, H_0 is accepted.

$\therefore$ The two samples could have come from populations with the same variance.

2.) Two random variables gave the following results $n_1=10$, $\Sigma(x_i-\bar{x})^2=90$, $n_2=12$, $\Sigma(y_i-\bar{y})^2=108$. Test whether the samples came from the populations with same variance.

Sol:

Given $n_1=10$, $n_2=12$, $\Sigma(x_i-\bar{x})^2=90$, $\Sigma(y_i-\bar{y})^2=108$

$$S_1^2=\frac{1}{n_1}\Sigma(x_i-\bar{x})^2=\frac{90}{10}=9, \quad S_2^2=\frac{1}{n_2}\Sigma(y_i-\bar{y})^2=\frac{108}{12}=9$$

$H_0: \sigma_1^2=\sigma_2^2$, $H_1: \sigma_1^2\neq\sigma_2^2$

Now $$\sigma_1^2=\frac{n_1 S_1^2}{n_1-1}=\frac{10\times 9}{9}=10, \quad \sigma_2^2=\frac{n_2 S_2^2}{n_2-1}=\frac{12\times 9}{11}=9.82$$

$$F=\frac{\sigma_1^2}{\sigma_2^2} \text{ If } \sigma_1^2>\sigma_2^2$$

$$=\frac{10}{9.82}=1.02$$, Degrees for freedom $v=(n_1-1, n_2-1)=(9, 11)$

Los $=5\%$ From F-table, $F_{5\%}(v_1=9, v_2=11)=2.955$

$[F_{5\%}(v_1=9, v_2=11) \Rightarrow F(v_1=6, v_2=10)=3.22$

$F(v_1=6, v_2=12)=3.00$

$F(v_1=12, v_2=10)=2.91$

$F(v_1=12, v_2=12)=2.69$

$$F\left(v_1=\frac{6+6+12+12}{4}, v_2=\frac{10+12+10+12}{4}\right)$$

$$=\frac{3.22+3.00+2.91+2.69}{4}$$

$\Rightarrow F(v_1=9, v_2=11)=2.955]$

$\therefore F<F_{5\%}$, H_0 is rejected.

$\therefore$ The two samples came from two populations with same variance.

From the following data, test if the difference between variances is significant at 5% LOS.

Sample	A	B
size	8	10
sum of the squares of deviation from mean	84.4	102.6

Sol:

Given $n_1 = 8$, $n_2 = 10$, $\Sigma(x_i - \bar{x})^2 = 84.4$, $\Sigma(y_i - \bar{y})^2 = 102.6$

$\therefore s_1^2 = \frac{\Sigma(x_i-\bar{x})^2}{n_1} = \frac{84.4}{8} = 10.55$, $s_2^2 = \frac{\Sigma(y_i-\bar{y})^2}{n_2} = \frac{102.6}{10} = 10.26$

$H_0 : \sigma_1^2 = \sigma_2^2$, $H_1 : \sigma_1^2 \neq \sigma_2^2$

Now $\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{8 \times 10.55}{7} = 12.06$ and

$\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{10 \times 10.26}{9} = 11.4$

$\therefore F = \frac{\sigma_1^2}{\sigma_2^2}$ if $\sigma_1^2 > \sigma_2^2$

$= \frac{12.06}{11.4} = 1.058$

degrees of freedom $v = (n_1 - 1, n_2 - 1)$
$= (7, 9)$

From F-table, $F_{5\%}(v_1 = 7, v_2 = 9) = 3.3$

[$F_{5\%}(v_1 = 7, v_2 = 9)$: $F(v_1 = 6, v_2 = 9) = 3.37$
$F(v_1 = 8, v_2 = 9) = 3.23$

$F(v_1 = \frac{6+8}{2}, v_2 = \frac{9+9}{2}) = \frac{3.37 + 3.23}{2}$

$F(v_1 = 7, v_2 = 9) = 3.3$]

$\therefore F < F_{5\%}$, H_0 is accepted.

Hence variances of the two populations are equal.

Two random samples drawn from normal populations are

Sample I	20	16	26	27	23	22	18	24	25	19	–	–
Sample II	27	33	42	35	32	34	38	28	41	43	30	37

obtain estimates of the variances of the populations and test whether the two populations have the same variance.

Sol:

Sample I

x_1	x_1^2
20	400
16	256
26	676
27	729
23	529
22	484
18	324
24	576
25	625
19	361
220	4960

$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{220}{10}$

$= 22$

$s_1^2 = \frac{1}{n_1}\Sigma x_1^2 - (\bar{x}_1)^2$

$= \frac{4960}{10} - (22)^2$

$= 12$

$\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1}$

$= \frac{10 \times 12}{9}$

$= 13.33$

Sample II

x_2	x_2^2
27	729
33	1089
42	1764
35	1225
32	1024
34	1156
38	1444
28	784
41	1681
43	1849
30	900
37	1369
420	15104

$\bar{x}_2 = \frac{\Sigma x_2}{n_2}$

$= \frac{420}{12} = 35$

$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \bar{x}_2^2$

$= \frac{15104}{12} - (35)^2$

$= 26.17$

$\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$

$= \frac{12 \times 26.17}{11}$

$= 28.55$

$H_0: \sigma_1^2 = \sigma_2^2$, $H_1: \sigma_1^2 \neq \sigma_2^2$

$F = \frac{\sigma_2^2}{\sigma_1^2}$ If $\sigma_2^2 > \sigma_1^2$

$= \frac{28.55}{13.33} = 2.14$

degrees of freedom

$v = (n_2 - 1, n_1 - 1)$

$= (12-1, 10-1) = (11, 9)$

From F-table $F_{5\%}(v_1 = 11, v_2 = 9) = 3.11$

[$F_{5\%}(\nu_1 = 11, \nu_2 = 9) = F(\nu_1 = 8, \nu_2 = 9) = 3.23$

$F(\nu_1 = 12, \nu_2 = 9) = 3.07$

$F\left(\nu_1 = \frac{8+12}{2}, \nu_2 = \frac{9+9}{2}\right) = \frac{3.23+3.07}{2}$

$F(\nu_1 = 10, \nu_2 = 9) = 3.15$

$\Rightarrow F(\nu_1 = 10, \nu_2 = 9) = 3.15$

$F(\nu_1 = 12, \nu_2 = 9) = 3.07$

$F\left(\nu_1 = \frac{10+12}{2}, \nu_2 = \frac{9+9}{2}\right) = \frac{3.15+3.07}{2}$

$F(\nu_1 = 11, \nu_2 = 9) = 3.11$]

$\therefore F < F_{5\%}$, H_0 is accepted.

Hence the two populations have the same Variance.

5) Two random samples gave the following data

	Size	mean	Variance
Sample I	8	9.6	1.2
Sample II	11	16.5	2.5

Can we conclude that the two samples have been drawn from the ~~sample~~ same normal population?

Sol:

Given $n_1 = 8$, $\bar{x}_1 = 9.6$, $s_1^2 = 1.2$, $n_2 = 11$, $\bar{x}_2 = 16.5$, $s_2^2 = 2.5$

$H_0: \bar{x}_1 = \bar{x}_2$, $H_1: \bar{x}_1 \neq \bar{x}_2$ (two tailed test)

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{9.6-16.5}{\sqrt{\left(\frac{(8\times1.2)+(11\times2.5)}{8+11-2}\right)\left(\frac{1}{8}+\frac{1}{11}\right)}} = \frac{-6.9}{0.6864} = 10.05$$

degrees of freedom $\nu = n_1 + n_2 - 2 = 8+11-2 = 17$

From t-table, $t_{5\%}(\nu = 17) = 2.11$

$\therefore |t| > t_{5\%}$, H_0 is rejected.

(6)

Hence the means of two samples differ significantly.

$H_0: \sigma_1^2 = \sigma_2^2, \quad H_1: \sigma_1^2 \neq \sigma_2^2$

Now $\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{8 \times 1.2}{7} = 1.37$ and $\sigma_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{11 \times 2.5}{10} = 2.75$

$F = \frac{\sigma_2^2}{\sigma_1^2}$ If $\sigma_2^2 > \sigma_1^2$

$= \frac{2.75}{1.37} = 2.007$ degrees of freedom $\nu = (n_2 - 1, n_1 - 1)$

$= (11-1, 8-1)$

$= (10, 7)$

From F-table, $F_{5\%}(\nu_1 = 10, \nu_2 = 7) = 3.65$

$[F_{5\%}(\nu_1 = 10, \nu_2 = 7) = F(\nu_1 = 8, \nu_2 = 7) = 3.73$

$F(\nu_1 = 12, \nu_2 = 7) = 3.57$

$F(\nu_1 = \frac{8+12}{2}, \nu_2 = \frac{7+7}{2}) = \frac{3.73 + 3.57}{2}$

$F(\nu_1 = 10, \nu_2 = 7) = 3.65]$

$\therefore F < F_{5\%}$, H_0 is accepted.

ie, the variances of the populations from which samples are drawn may be regarded as equal.

$\therefore$ The two samples could not have been drawn from the same normal population.

6) The nicotine contents in two random samples of tobacco are given below

Sample I	21	24	25	26	27	
Sample II	22	27	28	30	31	36

Can you say that the two samples came from the same population?

Sample I

x_1	x_1^2
21	441
24	576
25	625
26	676
27	729
123	3047

$$\bar{x}_1 = \frac{\Sigma x_1}{n} = \frac{123}{5} = 24.6$$

$$s_1^2 = \frac{\Sigma x_1^2}{n_1} - \bar{x}_1^2 = \frac{3047}{5} - (24.6)^2 = 4.24$$

Sample II

x_2	x_2^2
22	484
27	729
28	784
30	900
31	961
36	1296
174	5154

$$\bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{174}{6} = 29$$

$$s_2^2 = \frac{\Sigma x_2^2}{n_2} - \bar{x}^2 = \frac{5154}{6} - (29)^2 = 18$$

$H_0: \bar{x}_1 = \bar{x}_2$, $H_1: \bar{x}_1 \neq \bar{x}_2$ (two tailed test).

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\left(\frac{n_1 s_1^2 + n_2 s_2^2}{n_1+n_2-2}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} = \frac{24.6-29}{\sqrt{\left(\frac{(5\times 4.24)+(6\times 18)}{5+6-2}\right)\left(\frac{1}{5}+\frac{1}{6}\right)}} = -1.92$$

degrees of freedom $= \nu = n_1+n_2-2 = 5+6-2 = 9$

From t-table, $t_{5\%}(\nu=9) = 2.26$

$\therefore |t| < t_{5\%}$, H_0 is accepted.

Hence the means of two samples donot differ singnificantly.

$H_0: \sigma_1^2 = \sigma_2^2$, $H_1: \sigma_1^2 \neq \sigma_2^2$

Now $\sigma_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{5\times 4.24}{4} = 5.3$, $\sigma_2^2 = \frac{n_2 s_2^2}{n_2-1} = \frac{6\times 18}{5} = 21.6$

$F = \frac{\sigma_2^2}{\sigma_1^2}$, $(\nu_1 = n_2 - 1, \nu_2 = n_1 - 1)$

$= \frac{21.6}{5.3} = 4.07$. From F-table, $F_{5\%}(\nu_1 = 5, \nu_2 = 4) = 6.26$

$F < F_{5\%}$, H_0 is accepted.

$\therefore$ The variance of the two populations can be regarded as equal.

Hence the two samples could have been drawn from the same normal population.

Values of χ^2 with probability P and df ν

P / ν	0.99	0.95	0.50	0.30	0.20	0.10	0.05	0.01
1	0.0002	0.004	0.46	1.07	1.64	2.71	3.84	6.64
2	0.020	0.103	1.39	2.41	3.22	4.60	5.99	9.21
3	0.115	0.35	2.37	3.66	4.64	6.25	7.82	11.34
4	0.30	0.71	3.36	4.88	5.99	7.78	9.49	13.28
5	0.55	1.14	4.35	6.06	7.29	9.24	11.07	15.09
6	0.87	1.64	5.35	7.23	8.56	10.64	12.59	16.81
7	1.24	2.17	6.35	8.38	9.80	12.02	14.07	18.48
8	1.65	2.73	7.34	9.52	11.03	13.36	15.51	20.09
9	2.09	3.32	8.34	10.66	12.24	14.68	16.92	21.67
10	2.56	3.94	9.34	11.78	13.44	15.99	18.31	23.21
11	3.05	4.58	10.34	12.90	14.63	17.28	19.68	24.72
12	3.57	5.23	11.34	14.01	15.81	18.55	21.03	26.22
13	4.11	5.89	12.34	15.12	16.98	19.81	22.36	27.69
14	4.66	6.57	13.34	16.22	18.15	21.06	23.68	29.14
15	5.23	7.26	14.34	17.32	19.31	22.31	25.00	30.58
16	5.81	7.96	15.34	18.42	20.46	23.54	26.30	32.00
17	6.41	8.67	16.34	19.51	21.62	24.77	27.59	33.41
18	7.02	9.39	17.34	20.60	22.76	25.99	28.87	34.80
19	7.63	10.12	18.34	21.69	23.90	27.20	30.14	36.19
20	8.26	10.85	19.34	22.78	25.04	28.41	31.41	37.57
21	8.90	11.59	20.34	23.86	26.17	29.62	32.67	38.93
22	9.54	12.34	21.34	24.94	27.30	30.81	33.92	40.29
23	10.20	13.09	22.34	26.02	28.43	32.01	35.17	41.64
24	10.86	13.85	23.34	27.10	29.55	33.20	36.42	42.98
25	11.52	14.61	24.34	28.17	30.68	34.68	37.65	44.31
26	12.20	15.38	25.34	29.25	31.80	35.56	38.88	45.64
27	12.88	16.15	26.34	30.32	32.91	36.74	40.11	46.96
28	13.56	16.93	27.34	31.39	34.03	37.92	41.34	48.28
29	14.26	17.71	28.34	32.46	35.14	39.09	42.56	49.59
30	14.95	18.49	29.34	33.53	36.25	40.26	43.77	50.89

Chi - Square Test

(1) To test the significant difference between experimental value and theoretical values (we used to χ^2 test of good-ness of fit)

2.) To test whether the given sample is from a hypothetical population (we used to χ^2 test of independence of attributes.

χ^2 test of goodness of fit:

On the basis of the hypothesis assumed about the population, we find the expected frequencies E_i corresponding to the observed frequencies O_i ~~such~~ such that $\Sigma E_i = \Sigma O_i$

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i}, \quad v = n-1$$

χ^2 test of independence of attributes:

If the population has two characteristics, then χ^2 distribution is used to test whether the characteristics are associated or independent, based on a sample drawn from the population.

H_0: The assumption that the two attributes A and B are independent.

We computed the expected frequencies E_{ij}, using the following formula

$$E_{ij} = \frac{\left\{ \begin{array}{l} (\text{Total of observed frequencies in the } i^{th} \text{ row}) \times \\ (\text{Total of observed frequencies in the } j^{th} \text{ column}) \end{array} \right\}}{\text{Total of all cell frequencies.}}$$

$$\chi^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}, \quad \nu = (m-1)(n-1)$$

Conditions for the validity of χ^2-Test

1) The number of observations N in the sample must be large. (ie, $N \geq 50$, N - total frequency)

2) The number of classes n must be neither too small nor too large. ie, $4 \leq n \leq 16$

3) The sample observations (experimental data) must be independent of each other.

4) Individual frequencies must not be too small (≥ 10). In case < 10, it is combined with the neighbouring frequencies, so that the combined frequency is ≥ 10.

1.) The following table gives the number of accidents that took place in an industry during various days of the week. Test whether the accidents are uniformly distributed over the week.

Days	Mon	Tue	Wed	Thurs	Fri	Sat
Number of accidents	14	18	12	11	15	14

Sol:

H_0: The accidents are uniformly distributed over the 6 days.

Under H_0, the expected frequencies for each day $= \frac{84}{6} = 14$

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
14	14	0	0	0
18	14	4	16	1.143
12	14	-2	4	0.286
11	14	-3	9	0.643
15	14	1	1	0.071
14	14	0	0	0.000
				2.143

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i} = 2.143$$

since $\Sigma E_i = \Sigma O_i$, $\nu = 6-1 = 5$

From the χ^2 table, $\chi^2_{5\%}(\nu=5) = 11.07$.

$\therefore \chi^2 < \chi^2_{5\%}$, H_0 is accepted.

The accidents are uniformly distributed over the 6 days

2.) The following tables gives the number of air-craft accidents that occurred during the various days of a week. Test whether the accidents are uniformly distributed over the week.

Days	Mon	Tue	Wed	Thu	Fri	Sat
No. of accidents	15	19	13	12	16	15

Sol:

H_0: The accidents are uniformly distributed over the 6 days.

Under H_0, the expected frequency for each day $= \frac{90}{6} = 15$

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
15	15	0	0	0
19	15	4	16	1.0667
13	15	-2	4	0.2667
12	15	-3	9	0.6
16	15	1	1	0.0667
15	15	0	0	0
				2.0001

Since $\Sigma E_i = \Sigma O_i$, $\nu = 6 - 1 = 5$

From the χ^2-table, $\chi^2_{5\%}(\nu = 5) = 11.07$.

since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

Hence the accidents are uniformly distributed over the 6 days.

3.) 200 digits were chosen at random from a set of tables. The frequencies of the digits were given below

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	18	19	23	21	16	25	22	20	21	15

Test whether the digits may be taken to occur equally frequently in the table.

Sol:

H_0: The digits occur equally frequently.

Under H_0, the expected frequency $= \frac{200}{10} = 20$

O_i	E_i	$O_i - E_i$	$(O_i-E_i)^2$	$(O_i-E_i)^2/E_i$
18	20	−2	4	0.20
19	20	−1	1	0.05
23	20	3	9	0.45
21	20	1	1	0.05
16	20	−4	16	0.8
25	20	5	25	1.25
22	20	2	4	0.20
20	20	0	0	0
21	20	1	1	0.05
15	20	−5	25	1.25
				4.3

Since $\Sigma O_i = \Sigma E_i$, $\nu = 10-1 = 9$

From the χ^2 table, $\chi^2_{5\%} = 16.919$

$\therefore \chi^2 < \chi^2_{5\%}$, H_0 is accepted.

$\therefore$ the digits were equally distributed in the table

Scanned by CamScanner

4) The following table shows the distribution of digits in the numbers chosen at random from a telephone directory:

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	1026	1107	997	966	1075	933	1107	972	964	853

Test whether the digits may be taken to occur equally frequently in the directory.

Sol:

H_0: The digits occur equally frequently

Under H_0, the expected frequency $= \frac{10000}{10} = 1000$

O_i	1026	1107	997	966	1075	933	1107	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000
$O_i - E_i$	26	107	-3	-34	75	-67	107	-28	-36	-147
$(O_i-E_i)^2$	676	11449	9	1156	5625	4489	11449	784	1296	21609
$\frac{(O_i-E_i)^2}{E_i}$	0.676	11.449	0.009	1.156	5.625	4.489	11.449	0.784	1.296	21.609

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 58.542, \quad \text{since } \sum O_i = \sum E_i$$

$$\nu = 10 - 1 = 9$$

χ^2 table, $\chi^2_{5\%} = 16.919$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

$\therefore$ The digits do not occur uniformly in the directory.

5) The following data shows defective articles produced by four machines:

Machine	A	B	C	D
Production time	1 hour	1 hour	2 hours	3 hours
No. of defective	12	30	63	98

Do the figures indicate a significant difference in the performance of the machine?

Sol: H_0: Production rates of the machines are the same.

Total number of defectives = 203.

Under H_0, the expected numbers of defectives are

E_i: $\frac{1}{7} \times 203$, $\frac{1}{7} \times 203$, $\frac{2}{7} \times 203$, $\frac{3}{7} \times 203$

= 29, 29, 58, 87

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
12	29	−17	289	9.9656
30	29	1	1	0.0345
63	58	5	25	0.4310
98	87	11	121	1.3908
				11.8219

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 11.8219$$

Since $\Sigma O_i = \Sigma E_i$; $\nu = 4 - 1 = 3$

From χ^2 table, $\chi^2_{5\%} = 7.815$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

∴ There is significant difference in the performance of machines.

6. Theorey predicts that the proportion of beans in four groups A, B, C, D should be 9:3:3:1. In an experiment among 1600 beans, the numbers in the four groups were 882, 313, 287 and 118. Does the experiment support the theory?

Sol:

Ho: The experiment supports the theory.

Under Ho, the expected numbers of beans are

E_i: $\frac{9}{16} \times 1600$, $\frac{3}{16} \times 1600$, $\frac{3}{16} \times 1600$, $\frac{1}{16} \times 1600$

E_i: 900, 300, 300, 100

O_i	E_i	$(O_i - E_i)$	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
882	900	-18	324	0.36
313	300	13	169	0.563
287	300	-13	169	0.563
118	100	18	324	3.24
				4.726

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i} = 4.726$$

Since $\Sigma O_i = \Sigma E_i$, $v = n-1 = 4-1 = 3$

From χ^2 table, $\chi^2_{5\%} = 7.82$

Since $\chi^2 < \chi^2_{5\%}$, Ho is accepted.

∴ the experimental data support the theory.

Scanned by CamScanner

7.) A sample analysis of examination results of 1000 students were made and it was found that 260 failed, 110 first class, 420 second class and rest obtained third class. Applying χ^2 test, whether the general examination result is in the ratio 2:1:4:3.

Sol:

H_0: The result is in the ratio 2:1:4:3

Based on H_0, the expected frequencies are

E_i : $\frac{2}{10} \times 1000$, $\frac{1}{10} \times 1000$, $\frac{4}{10} \times 1000$, $\frac{3}{10} \times 1000$

= 200, 100, 400, 300

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2 / E_i$
260	200	60	3600	18
110	100	10	100	1
420	400	20	400	1
210	300	-90	8100	27
				47

$\chi^2 = \Sigma (O_i - E_i)^2 / E_i = 47$

Since $\Sigma O_i = \Sigma E_i$, $\nu = 4 - 1 = 3$

From χ^2 table, $\chi^2_{5\%} = 7.82$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

The examination results are not in the ratio 2:1:4:3

8) A sample analysis of examination results of 500 students was made. It was found that 220 students have failed, 170 have secured a 3^{rd} class, 90 have secured a 2^{nd} class and the rest a first class. So do these figures support the general belief that the above categories are in the ratio 4:3:2:1 respectively?

Sol:

H_0: The results in the four categories are in the ratio 4:3:2:1

Under H_0, the expected frequencies are

$$E_i : \frac{4}{10} \times 500, \quad \frac{3}{10} \times 500, \quad \frac{2}{10} \times 500, \quad \frac{1}{10} \times 500$$

$$: 200, \quad 150, \quad 100, \quad 50$$

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
220	200	20	400	2
170	150	20	400	2.667
90	100	−10	100	1
20	50	−30	900	18
				23.667

Since $\Sigma O_i = \Sigma E_i$, $\nu = 4 - 1 = 3$

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i} = 23.667$$

From the χ^2 table, $\chi^2_{5\%} = 7.815$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

$\therefore$ The results of the four categories are not in the ratio 4:3:2:1.

9) The following data gives the number of male and female births in 1000 familes having five children

Male child	0	1	2	3	4	5
Female child	5	4	3	2	1	0
frequency	40	300	250	200	130	80

Test whether the given data is consistent with the hypothesis that the binomial law holds with even chance.

Sol:

H_0: Let the given data follow a binomial distribution with $p = \frac{1}{2}$.

Based on H_0, the expected frequencies (number of families) are

$$= N\,P(X=x) = 1000 \times 5C_x \times \frac{1}{2^5}, \quad x = 0, 1, \ldots 5$$

E_i : 31.25, 156, 312.5, 312.5, 156, 31.25

: 31, 156, 313, 313, 156, 31

Since $\Sigma O_i = \Sigma E_i$, $\nu = n-1 = 6-1 = 5$

From χ^2 table, $\chi^2_{5\%} (\nu=5) = 11.07$

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i - E_i)^2/E_i$
40	31	49	81	2.613
300	156	144	20736	132.923
250	313	-63	3969	12.681
200	313	-113	12769	40.796
130	156	-26	676	4.333
80	31	49	2401	77.451
				270.797

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 270.797.$$

Since $\chi^2 > \chi^2_{5\%}$, Ho is rejected.

$\therefore$ The binomial fit is not good.

10. A survey of 320 families with five children each revealed the following distribution

No. of boys:	0	1	2	3	4	5
No. of girls:	5	4	3	2	1	0
No. of families:	12	40	88	110	56	14

Is this result consistent with the hypothesis that male and female births are equally probable?

Ans:

E_i 10, 50, 100, 100, 50, 10

$\chi^2 = 7.16$, $\chi^2_{5\%}(v=5) = 11.07$, Ho is accepted.

Scanned by CamScanner

11) Fit a binomial distribution for the following data and also test the goodness of fit.

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80

Sol:

x	0	1	2	3	4	5	6	Total
f	5	18	28	12	7	6	4	80
fx	0	18	56	36	28	30	24	192

$\therefore \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{80} = 2.4$

Mean of the binomial distribution $= np$

ie, $np = 2.4 \Rightarrow 6p = 2.4 \Rightarrow p = 0.4$ and $q = 1-p = 0.6$

The expected frequencies are $NP(X=x)$

$= 80 \times 6C_x (0.4)^x (0.6)^{6-x}$, $x = 0, 1, 2 \ldots 6$

E_i : 3.73, 14.93, 24.88, 22.12, 11.06, 2.95, 0.33

: 4, 15, 25, 22, 11, 3, 0

O_i : 5, 18, 28, 12, 7, 6, 4

H_0 : There is no significant difference between expected frequencies and observed frequencies.

Since $\Sigma O_i = \Sigma E_i$,

The first class is combined with the second and the last two classes are combined with the last. After regrouping, we have

E_i :	19	25	22	14
O_i :	23	28	12	17

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{4^2}{19} + \frac{3^2}{25} + \frac{(-10)^2}{22} + \frac{3^2}{14} = 6.39$$

degrees of freedom $\nu = N-k = 4-2 = 2$

From χ^2 table, $\chi^2_{5\%}\ (\nu=2) = 5.99$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

$\therefore$ The fit is not good.

12) Fit a binomial distribution for the following data and also test the goodness of fit.

x :	0	1	2	3	4
f :	5	29	36	25	5

Sol:

x :	0	1	2	3	4	Total
f :	5	29	36	25	5	100
fx :	0	29	72	75	20	196

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{196}{100} = 1.96$$

$\Rightarrow np = 1.96 \Rightarrow 4p = 1.96 \Rightarrow p = 0.49$ and $q = 1-p = 0.51$

The expected frequencies are

$$= N \times P(X=x) = 100 \times 4C_x\,(0.49)^x\,(0.51)^{4-x},$$

$x = 0, 1, 2, 3, 4$

E_i : 6.77, 25.99, 37.47, 24, 5.76

= 7, 26, 37, 24, 6

$\therefore \Sigma E_i = \Sigma O_i$

H_0: There is no significant difference between expected frequencies and observed frequencies.

The new expected frequencies are

E_i :	33	37	30
O_i :	34	36	30

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i} = \frac{1^2}{33} + \frac{(-1)^2}{37} + 0 = 0.057$$

From χ^2 table, $\chi^2_{5\%}\ (\nu = N-K = 3-2 = 1) = 3.841$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

$\therefore$ The binomial fit is good.

<u>Note</u>: In fitting binomial distribution, If we find $\bar{x}$ and using $\bar{x}$, we find p and q and hence the probabilities, then $\nu = N-2$.

13.) Fit a Poisson distribution for the following data and test the goods of fit.

Value of x	0	1	2	3	4
Frequencies	122	60	15	2	1

Sol: To find mean

x	0	1	2	3	4	Total
f	122	60	15	2	1	200
fx	0	60	30	6	4	100

$$\bar{x} = \frac{\Sigma fx}{N} = \frac{100}{200} = 0.5$$

Mean of the Poisson distribution = λ

$\therefore \lambda = 0.5$

The expected frequencies are $N \times P(X=x)$

$$= 200 \times \frac{e^{-0.5}(0.5)^x}{x!}, \quad x = 0, 1, 2, 3, 4$$

E_i : 121.31, 60.65, 15.16, 2.53, 0.316

= 121, 61, 15, 3, 0

The last three classes are combined into one.

Thus, after regrouping, we whave

E_i :	121	61	18
O_i	122	60	18

H_0: There is no significant difference between obsersed frequencies and expected frequencies.

$$\chi^2 = \Sigma \frac{(O_i - E_i)^2}{E_i} = \frac{1^2}{122} + \frac{(-1)^2}{60} + 0 = 0.0249$$

Since $\Sigma O_i = \Sigma E_i$, $\nu = n - k = 3 - 2 = 1$

From χ^2 table, $\chi^2_{5\%}(\nu = 1) = 3.841$. Since $\chi^2 < \chi^2_{5\%}$

Hence ~~Since~~ H_0 is accepted.

$\therefore$ The Poisson fit is best.

14) Fit a Poisson distribution for the following distribution and also test the goodness of fit.

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400

Sol:

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

$\therefore \lambda = \frac{400}{400} = 1$

The expected frequencies are $NP(X=x)$

$$= 400 \times \frac{e^{-1}}{x!}, \quad x = 0, 1, \ldots 5$$

E_i : 147.15, 147.15, 73.58, 24.53, 6.13, 1.23

: 147, 147, 74, 25, 6, 1

The last three classes are combined into one so that the new expected frequency are

E_i	142	156	69	33
O_i	147	147	74	32

H_0: There is no significant difference between O_i & E_i

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = \frac{5^2}{147} + \frac{9^2}{147} + \frac{5^2}{74} + \frac{1^2}{32} = 1.09$$

since $\sum O_i = \sum E_i$, $\quad \nu = N - K = 4 - 2 = 2$

From χ^2 table, $\chi^2_{5\%} = 5.99$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

$\therefore$ The Poisson fit is good.

Independence of attributes

1.) From the following table, test the independence of skilled fathers having intelligent son.

Father \ Son	Intelligent son	Unintelligent son
Skilled father	24	12
Unskilled father	32	32

Sol:

H_0: The skill of father and intelligence of son are independent.

	Intelligent Son	Unintelligent Son	Total
Skilled father	24	12	36
Unskilled father	32	32	64
Total	56	44	100

O_i :	24	12	32	32
E_i :	$\frac{56\times36}{100}$	$\frac{44\times36}{100}$	$\frac{56\times64}{100}$	$\frac{44\times64}{100}$
:	20.16	15.84	35.84	28.16

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2$	$(O_i-E_i)^2/E_i$
24	20	4	16	0.8
12	16	-4	16	1
32	36	-4	16	0.444
32	28	4	16	0.571
				2.815

$\chi^2 = 2.815$, $\nu = (m-1)(n-1) = (2-1)(2-1) = 1$

Since From χ^2- table, $\chi^2_{5\%}(\nu=1) = 3.84$

Since $\chi^2 < \chi^2_{5\%}$, Ho is accepted.

$\therefore$ there is independence between skill of fathers and intelligence of sons.

2) In a prepoll survey out of 1000 rural voters, 620 favoured A and the rest B. Out of 1000 urban voters 450 favoured B and the rest A. Examine whether the nature of the area is related to voting preference.

Sol:

H_0: Voting preference is independent of the nature of the area.

		Voting pattern		
		Favoured A	Favoured B	Total
Area	Rural	620	380	1000
	Urban	550	450	1000
	Total	1170	830	2000

O_i	E_i	$O_i - E_i$	$(O_i - E_i)^2/E_i$
620	$\frac{1000 \times 1170}{2000} = 585$	35	2.0944
380	$\frac{1000 \times 830}{2000} = 415$	-35	2.95
550	$\frac{1000 \times 1170}{2000} = 585$	-35	2.0944
450	$\frac{1000 \times 830}{2000} = 415$	35	2.95
			10.088

$$\chi^2 = \sum \frac{(O_i - E_i)^2}{E_i} = 10.088, \quad \nu = (m-1)(n-1) = 1$$

From χ^2 table, $\chi^2_{5\%} = 3.84$,

since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

$\therefore$ The nature of the area is not related to voting preference.

3.) The following data are collected on two characters

	Smokers	Non-Smokers
Literates	83	57
Illiterates	45	68

Based on this, can you say that there is no relation between smoking and literacy?

Sol:

H_0: Literacy and smoking habit are independent.

	Smokers	Non-Smokers	Total
Literates	83	57	140
Illiterates	45	68	113
Total	128	125	253

O_i : 83   57   45   68

E_i : $\frac{128\times140}{253}$ $\frac{125\times113}{253}$ $\frac{128\times113}{253}$ $\frac{125\times113}{253}$

: 70.83   69.17   57.17   55.83

O_i	E_i	$O_i - E_i$	$(O_i-E_i)^2/E_i$
83	71	12	~~20.57~~ 2.0282
57	69	12	2.0870
45	57	12	2.5263
68	56	12	2.5714
			9.2129

$$\chi^2 = \sum \frac{(O_i-E_i)^2}{E_i} = 9.2129, \quad \nu = (m-1)(n-1) = 1$$

From χ^2 table, $\chi^2_{5\%}(\nu=1) = 3.84$

Since $\chi^2 > \chi^2_{5\%}$, H_0 is rejected.

Hence there is some association between literacy and smoking.

4.) Given the following contingency table for hair colour and eye colour, find the value of χ^2. Is there a good association between the two?

		Hair colour		
		Fair	brown	black
Eye colour	blue	15	5	20
	gray	20	10	20
	brown	25	15	20

Sol:

Ho: The two attributes hair colour and eye colour are independent.

	Fair	brown	black	Total
blue	15	5	20	40
Gray	20	10	20	50
brown	25	15	20	60
Total	60	30	60	150

Expected frequencies

$\frac{60\times40}{150}=16$	$\frac{30\times40}{150}=8$	$\frac{60\times40}{150}=16$	40
$\frac{60\times50}{150}=20$	$\frac{30\times50}{150}=10$	$\frac{60\times50}{150}=20$	50
$\frac{60\times60}{150}=24$	$\frac{30\times60}{150}=12$	$\frac{60\times60}{150}=24$	60
60	30	60	150

O_i	E_i	$O_i - E_i$	$(O_i-E_i)^2$	$(O_i-E_i)^2/E_i$
15	16	-1	1	0.0625
5	8	-3	9	1.125
20	16	4	16	1
20	20	0	0	0
10	10	0	0	0
20	20	0	0	0
25	24	1	1	0.0625
15	12	3	9	0.75
20	24	-4	16	0.67
				3.6458

$$\chi^2 = \sum \frac{(O_i-E_i)^2}{E_i} = 3.6458, \quad \vartheta = (m-1)(n-1) = (3-1)(3-1) = 4$$

From χ^2 table, $\chi^2_{5\%} = 9.488$

Since $\chi^2 < \chi^2_{5\%}$, H_0 is accepted.

ie, there is no association between the two.

Example 12 *Prove that the value of* χ^2 *for the* 2×2 *contingency table*

a	b
c	d

is given by

$$\chi^2 = \frac{N(ad - bc)^2}{(a+b)(c+d)(a+c)(b+d)}, \text{ where } N = a + b + c + d.$$

A			B				Row Total
	B_1	B_2	-	B_j	-	B_n	
A_1	O_{11}	O_{12}	-	O_{1j}	-	O_{1n}	O_{1*}
A_2	O_{21}	O_{22}	-	O_{2j}	-	O_{2n}	O_{2*}
:	-	-	-	-	-	-	-
A_i	O_{i1}	O_{i2}	-	O_{ij}	-	O_{in}	O_{i*}
:	-	-	-	-	-	-	-
A_m	O_{m1}	O_{m2}		O_{mj}	-	O_{mn}	O_{m*}
Column Total	O_{*1}	O_{*2}	-	O_{*j}	-	O_{*n}	N

Now, based on the null hypothesis H_0, i.e. the assumption that the two attributes A and B are independent, we compute the expected frequencies E_{ij} for various cells, using the following formula:

$$E_{ij} = \frac{O_{i*} \times O_{*j}}{N}, i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$$

i.e. $$E_{ij} = \left(\frac{\text{Total of observed frequencies in the } ith \text{ row} \times \text{total of a observed frequencies in the } j\text{th column}}{\text{Total of all cell frequencies}} \right)$$

Queueing Theory

Introduction: We find queues everywhere in our daily life - in banks, post-offices, restaurants, ticket Counters, hospitals etc. Queues are formed by car waiting at a traffic signal, machines waiting to be repaired, cars waiting for service and so on.

Queues are formed if the demand for service is more than the capacity to provide the service.

A queueing system can be described as customers arriving for service, waiting if service is not available immediately and leaving the system after having been served.

The basic characteristic of a queueing system are

1.) Arrival pattern 2) Service pattern
3.) Number of servers 4) System capacity
5.) Queue discipline.

Arrival pattern: In usual queueing situations, the process of arrivals is stochastic and it is thus necessary to know the prob. dist. describing the times between successive customer arrivals (inter arrival times). It is also necessary to know whether customers can arrive simultaneously (batch or bulk arrivals) and if so, the prob. dis. describing the size of the batch.

The arrival pattern is measured by the mean arrival rate or inter-arrival time. Usually arrival process is assumed to be a Poisson process. The arrival rate follows a Poisson dist. and hence the inter-arrival time follows an exponential distribution.

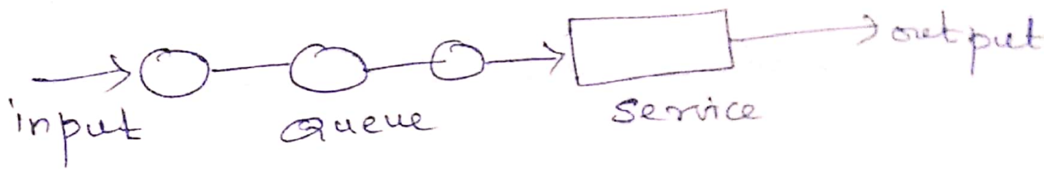
Arrival rate is usually denoted by λ.

The <u>Service pattern</u>: The mode of service is represented by means of the prob. dist. of the number of customers serviced per unit of time or of the inter-service time.

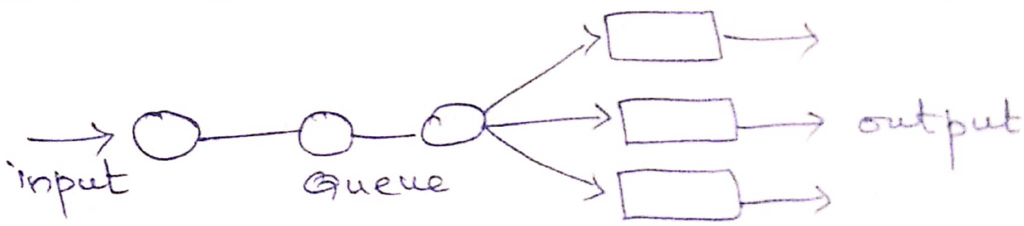
We shall mostly deal with only those queueing systems in which the number of customers serviced per unit of time has a Poisson dist. with mean μ or the inter service time has exponential dist. with mean $\frac{1}{\mu}$.

Number of servers: There may be one or more servers to provide a service. In multi-server queues there are a number of channels, providing identical service facilities. We denote the number of service channels by c.

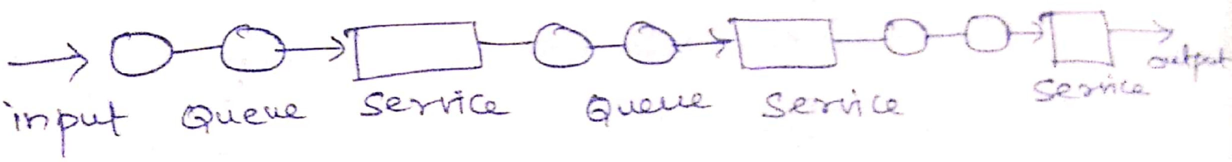
Single server queueing system



Multiple servers (in parallel) queueing system



Multiple servers (in series) queueing system



Capacity of the system:

Number of customers allowed in the system can be either infinite or finite. In some facilities, only a limited number of customers are allowed in the system. But in general, number of customers in the system is infinity.

Queue discipline: It is the rule according to which customers are selected for service when a queue has been formed. The common queue disciplines are

(i) FCFS - First come, first served
(or FIFO - First in First out)

(ii) LIFO - Last in first out

(iii) SIRO - selection in random order

In the queueing systems, we shall assume that service is provided on the FCFS basis.

Kendall's Notation: Generally a queueing process is specified in symbolic form as

$$(a/b/c):(d/e).$$

where a - Arrival distribution

b - service distribution

c - Number of servers

d - Capacity of the system

e - queue discipline.

<u>Note</u>: A queue with Poisson arrival (service) has exponential inter-arrival time (service time) and this is denoted by M which specifies the Markovian or memoryless property of the exponential dist.

Some of the queueing models are

$(M/M/1) : (\infty/FIFO)$

$(M/M/c) : (\infty/FIFO)$

$(M/M/1) : (K/FIFO)$

$(M/M/c) : (K/FIFO)$, etc

<u>Notation</u>:

L_s - Average length of the system = Average number of customers in the system

L_q - Average Length of the queue

W_s - Average waiting time of a customer in the system

W_q - Average waiting time of a customer in the queue.

Characteristic of Infinite Capacity, Single server Poisson queue:

Model I: $(M/M/1):(\infty/FIFO)$ [single server, infinite capacity queue], when $\lambda_n = \lambda$ and $\mu_n = \mu$ $(\lambda < \mu)$

(i) Average number of customers in the system (L_s)

$$L_s = \frac{\lambda}{\mu - \lambda}$$

(ii) Average number of customers in the queue (or) Average Length of the queue (L_q)

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

(iii) Average number of customers in nonempty queues (L_w)

$$L_w = \frac{\mu}{\mu - \lambda}$$

(iv) Prob. that the number of customers in the system exceeds k

$$P(N > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$$ where N = number of customers in the system.

(v) Prob. that the system is busy

$$= 1 - P_0 = \frac{\lambda}{\mu} \qquad \because P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

(vi) Prob. that the system is empty

$$= P_0 = 1 - \frac{\lambda}{\mu}$$

(vii) Average waiting time in the queue

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$$

(viii) Average waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu-\lambda}$$

(ix) Prob. density function of the waiting time in the system

$f(w) = (\mu-\lambda)\, e^{-(\mu-\lambda)w}$ which is the p.d.f of an exponential dist. with parameter $\mu-\lambda$.

(x) Prob. that the waiting time of a customer in the system exceeds t

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

(xi) Prob. that the waiting time in the queue exceeds $t = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$

(xii) Average waiting time of a customer in the queue, if he has to wait

$$E(W_q / W_q > 0) = \frac{1}{\mu-\lambda}$$

Relation among $E(N_s)$, $E(N_q)$, $E(W_s)$ and $E(W_q)$

(i) $E(N_s) = \lambda E(W_s) = \dfrac{\lambda}{\mu-\lambda}$ $\quad \because E(N_s) = L_s$

(ii) $E(N_q) = \lambda E(W_q) = \dfrac{\lambda^2}{\mu(\mu-\lambda)}$ $\quad \because E(N_q) = L_q$

(iii) $E(W_s) = E(W_q) + \dfrac{1}{\mu}$

(iv) $E(N_s) = E(N_q) + \dfrac{\lambda}{\mu}$

The above relations, called Little's formulas, hold good for the models with infinite capacity.

Characteristics of Finite Capacity, Single server Poisson Queue

Model II $(M/M/1):(K/FIFO)$ Model

Values of P_0 and P_n

$$P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}} & \text{If } \lambda \neq \mu \\ \dfrac{1}{K+1} & \text{If } \lambda = \mu \end{cases}$$

$$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n \left[\dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{K+1}}\right] & \text{If } \lambda \neq \mu \\ \dfrac{1}{K+1} & \text{If } \lambda = \mu \end{cases}$$

2.) Average number of customers in the system

$$L_s = \begin{cases} \dfrac{\lambda}{\mu-\lambda} - \dfrac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \text{If } \lambda \neq \mu \\ \dfrac{k}{2} & \text{If } \lambda = \mu \end{cases}$$

3.) Average number of customers in the queue

$$L_q = L_s - \frac{\lambda'}{\mu}$$ where λ' = effective arrival rate

4.) Average waiting times in the system and in the queue

$$W_s = \frac{L_s}{\lambda'} \text{ and } W_q = \frac{L_q}{\lambda'}$$

5.) The effective arrival rate

$$\lambda' \text{ or } \lambda_{eff} = \mu(1-P_0)$$

Model 1 (M|M|1) : (∞/FIFO)

1) Arrivals at a telephone booth are considered to be Poisson with an average time of 12 min. between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(a) Find the average number of persons waiting in the system.

(b) What is the prob. that a person arriving at the booth will have to wait in the queue?

(c) What is the prob. that it will take him more than 10 min. altogether to wait for the phone and complete his call?

(d) Estimate the fraction of the day when the phone will be in use

(e) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for at least 3 min. for phone. By how much the flow of arrivals should increase in order to justify a second booth?

(f) What is the average length of the queue that forms from time to time?

Sol:

Mean inter-arrival time $= \frac{1}{\lambda} = 12$ min

$\therefore$ Mean arrival rate $= \lambda = \frac{1}{12}$ per min

Mean service time $= \frac{1}{\mu} = 4$ min

$\therefore$ mean Service rate $= \mu = \frac{1}{4}$ per min

(a) $L_s = E(N) = \frac{\lambda}{\mu - \lambda} = \frac{1/12}{\frac{1}{4} - 1/12} = 0.5$ customer

(b) $P(W > 0) = 1 - P(W = 0)$

$= 1 - P$(no customer in the system)

$= 1 - P_0 = 1 - (1 - \lambda/\mu) = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = 1/3$

(c) $P(W > 10) = e^{-(\mu - \lambda) \times 10}$ $\quad \because P(W_s > t) = e^{-(\mu - \lambda)t}$

$= e^{-(\frac{1}{4} - \frac{1}{12}) \times 10}$

$= e^{-5/3}$ or 0.1889

(d) P(the phone will be in use)

$= 1 - P$(the phone will be idle)

$= 1 - P_0 = 1 - 2/3 = 1/3$

or the fraction of the day when the phone will be in use $= 1/3$.

(e) The second phone will be installed, if $E(W_q) > 3$

i.e., If $\dfrac{\lambda}{\mu(\mu-\lambda)} > 3$

i.e., If $\dfrac{\lambda_R}{\frac{1}{4}\left(\frac{1}{4}-\lambda_R\right)}$ where λ_R is the required arrival rate

$\Rightarrow \lambda_R > \dfrac{3}{4}\left(\dfrac{1}{4}-\lambda_R\right)$

$\Rightarrow \lambda_R > 3/28$

Hence, the arrival rate should increase by $\frac{3}{28} - \frac{1}{12} = \frac{1}{42}$ per minute, to justify a second phone.

(f) $E(N_q$ / the queue is always available)

$= E(N_q / N_q > 0)$

$= E(N_q / N > 1) = \dfrac{E(N_q)}{P(N>1)} = \dfrac{E(N_q)}{1-P_0-P_1}$

$= \dfrac{\lambda^2}{\mu(\mu-\lambda)} \cdot \dfrac{1}{1-\left(1+\frac{\lambda}{\mu}\right)P_0}$

$= \dfrac{\lambda^2}{\mu(\mu-\lambda)} \cdot \dfrac{1}{1-\left(1+\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)} = \dfrac{\lambda^2}{\mu(\mu-\lambda)} \cdot \dfrac{\mu^2}{\lambda^2}$

$= \dfrac{\mu}{\mu-\lambda} = \dfrac{1/4}{1/4 - 1/12} = 1.5$ persons

2) Customers arrive at a one-man barber shop according to a Poisson process with a mean interarrival time of 12 min. Customers spend an average of 10 min in the barber's chair.

(a) What is the expected number of customers in the barber shop and in the queue?

(b) Calculate the percentage of time an arrival can walk straight into the barber's chair without having to wait.

(c) How much time can a customer except to spend in the barber's shop?

(d) What is the average time customers spend in the queue?

(e) What is the prob. that the waiting time in the system is greater than 30 min?

(f) Calculate the percentage of customers who have to wait prior to getting into the barber's chair

(g) What is the prob. that more than 3 customers are in the system?

Sol:

Given $\frac{1}{\lambda} = 12$, $\frac{1}{\mu} = 10$; $\therefore \lambda = \frac{1}{12}$ per min, $\mu = \frac{1}{10}$ per min

(a) $E(N_s) = \frac{\lambda}{\mu-\lambda} = \frac{1/12}{1/10 - 1/12} = 5$ customers

$E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\frac{1}{144}}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)} = 4.17$ customers

(b) P(a customer straight goes to the barber's chair)

$= P(\text{No customer in the system})$

$= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{1/12}{1/10} = \frac{1}{6}$

$\therefore$ percentage of time an arrival need not wait $= 16.7$

(c) $E(W) = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{10}-\frac{1}{12}} = 60$ min or 1h

(d) $E(W_q) = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1/12}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{12}\right)} = 50$ min

(e) $P(W > 30) = e^{-\left(\frac{1}{10}-\frac{1}{12}\right)\times 30} = e^{-0.5}$ or 0.6065

(f) $P(\text{a customer has to wait}) = P(W > 0)$

$= 1 - P(W=0) = 1 - P(N=0) = 1 - P_0$

$= \frac{\lambda}{\mu} = \frac{1/12}{1/10} = \frac{5}{6}$

$\therefore$ percentage of customers who have to wait

$= \frac{5}{6} \times 100 = 83.33$

(g) $P(N>3) = P_4 + P_5 + P_6 + \cdots$

$= 1 - [P_0 + P_1 + P_2 + P_3]$

$= 1 - \left(1-\frac{\lambda}{\mu}\right)\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3\right]$

$\because P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$

$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{5}{6}\right)^4 = 0.4823$

Model II :

5.) In a single server queueing system with Poisson input and exponential service times; if the mean arrival rate is 3 calling units per hour, the expected service time is 0.25h and the maximum possible number of calling units in the system is 2, find P_n $(n \geq 0)$, average number of calling units in the system and in the queue and average waiting time in the system and in the queue.

Sol:

$\lambda = 3, \; \mu = 4, \; K = 2$

$$P_0 = \frac{1-\lambda/\mu}{1-\left(\lambda/\mu\right)^{K+1}}, \quad P_n = \left(\frac{\lambda}{\mu}\right)^n\left[\frac{1-\lambda/\mu}{1-\left(\lambda/\mu\right)^{K+1}}\right], \quad \lambda \neq \mu$$

$$\therefore \quad P_0 = \frac{1-3/4}{1-\left(\frac{3}{4}\right)^3} = \frac{16}{37} \text{ or } 0.4324$$

and $$P_n = \left(\frac{3}{4}\right)^n \left[\frac{1-3/4}{1-\left(\frac{3}{4}\right)^3}\right] = (0.4324)(0.75)^n$$

$$E(N) = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{3}{4-3} - \frac{3\times\left(3/4\right)^3}{1-\left(3/4\right)^3} = 3 - \frac{81}{37} = \frac{30}{37}$$

$$E(N_q) = E(N) - (1-P_0) = \frac{30}{37} - \left(1-\frac{16}{37}\right) = \frac{9}{37}$$

$$E(W_s) = \frac{1}{\lambda'}E(N), \quad \lambda' = \mu(1-P_0), \quad E(W_q) = \frac{1}{\lambda'}E(N_q)$$

$$\therefore \lambda' = 4\left(1-\frac{16}{37}\right) = \frac{84}{37}, \quad E(W_s) = \frac{37}{84}\times\frac{30}{37} = \frac{5}{14}\text{ h}$$

$$E(W_q) = \frac{37}{84}\times\frac{9}{37} = \frac{3}{28}\text{ h}$$

2.) The local one-person barber shop can accommodate a maximum of 5 people at a time (4 waiting and 1 getting hair cut). Customers arrive according to a Poisson dist. with mean 5 per hour. The barber cuts hair at an average rate of 4 per hour (Exponential service time).

(a) What percentage of time is the barber idle?

(b) What fraction of the potential customers are turned away?

(c) What is the expected number of customers waiting for a hair-cut?

(d) How much time can a customer expect to spend in the barber shop?

Sol:

$$\lambda = 5, \quad \mu = 4, \quad K = 5$$

(a) P(the barber is idle) = P(N=0)

$$= P_0 = \frac{1 - \lambda/\mu}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}}$$

$$= \frac{1 - 5/4}{1 - (5/4)^6} = 0.0888$$

∴ Percentage of time when the barber is idle = 9

b) P(a customer is turned away) = P(N > 5)

$$= P_5 = \left(\frac{\lambda}{\mu}\right)^5 \left[\frac{1 - \lambda/\mu}{1 - \left(\lambda/\mu\right)^{K+1}}\right]$$

$$= \left(\frac{5}{4}\right)^5 \left[\frac{1 - 5/4}{1 - (5/4)^6}\right] = 0.2711$$

∴ 0.2711 × potential customers are turned away.

(c) $E(N_q) = E(N) - (1-P_0)$

$$= \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{\left(1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right)} - (1-P_0)$$

$$= -5 - \frac{6 \times \left(\frac{5}{4}\right)^6}{1-\left(\frac{5}{4}\right)^6} - (1-0.0888)$$

$= 2.2$ customers

d) $E(W) = \frac{1}{\lambda'} E(N) = \frac{1}{\mu(1-P_0)} \times E(N)$

$$= \frac{3.1317}{3.6448} = 0.8592h$$

3) Patients arrive at a clinic according to Poisson dist. at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(a) Find the effective arrival rate at the clinic

(b) What is the prob. that an arriving patient will not wait?

(c) What is the expected waiting time until a patient is discharged from the clinic?

Sol:

$\lambda = 30$ per hr, $\mu = 20$ per hr, $k = 14+1 = 15$

Since $\lambda \neq \mu$, $P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{k+1}}$

$$= \frac{1-3/2}{1-(3/2)^{16}} = 0.00076$$

Effective arrival rate $\lambda' = \mu(1-P_0)$

$$= 20(1-0.00076)$$

$$= 19.98 \text{ per hr}$$

(b) P(a patient will not wait) $= P_0 = 0.00076$

(c) $E(N) = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)(\frac{\lambda}{\mu})^{k+1}}{1-(\lambda/\mu)^{k+1}}$

$$= -3 - \frac{16 \times (3/2)^{16}}{1-(3/2)^{16}} = 13 \text{ patients nearly}$$

$$E(W) = \frac{E(N)}{\lambda'} = \frac{13}{19.98} = 0.65h$$

4.) At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 6 per hr. Assuming Poisson arrivals and exponential service dist., find the probabilities for the numbers of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is doubled, how will the above results get modified?

Sol:

(i) $\lambda = 6$ per hour, $\mu = 6$ per hour, $K = 2+1 = 3$

Since $\lambda = \mu$, $\begin{cases} P_0 = \dfrac{1}{K+1} = \dfrac{1}{4} \\ P_n = \dfrac{1}{K+1} = \dfrac{1}{4}, \quad n \geq 1 \end{cases}$

$$E(N) = \frac{K}{2} = 1.5 \text{ trains}$$

$$E(W) = \frac{1}{\lambda'} E(N) = \frac{1.5}{\mu(1-P_0)}$$

$$= \frac{1.5}{6 \times \frac{3}{4}} = \frac{1}{3} h$$

(ii) $\lambda = 6$, $\mu = 12$, $K = 3$

Since $\lambda \neq \mu$, $\begin{cases} P_0 = \dfrac{1 - \lambda/\mu}{1 - \left(\frac{\lambda}{\mu}\right)^{K+1}} = \dfrac{1 - 1/2}{1 - \left(\frac{1}{2}\right)^4} = \dfrac{8}{15} \\ P_n = \left(\dfrac{\lambda}{\mu}\right)^n \left[\dfrac{1 - \lambda/\mu}{1 - \left(\lambda/\mu\right)^{K+1}}\right] = \dfrac{8}{15} \cdot \left(\dfrac{1}{2}\right)^n, \quad n \geq 1 \end{cases}$

$$E(N) = \frac{\lambda}{\mu-\lambda} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= 1 - \frac{4 \times \left(\frac{1}{2}\right)^4}{1-\left(\frac{1}{2}\right)^4} = 1 - \frac{4}{15} = \frac{11}{15} = 0.73 \text{ train}$$

$$E(W) = \frac{1}{\lambda'} E(N) = \frac{1}{\mu(1-P_0)} \times E(N)$$

$$= \frac{11/15}{12\left(1-\frac{8}{15}\right)} = \frac{11}{84} h$$

Model 1:

(1) If people arrive to purchase cinema tickets at the average rate of 6 per minute, it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 min before the picture starts and if it takes exactly 1.5 min to reach the correct seat after purchasing the ticket, (a) Can he except to be seated for the start of the picture?

(b) What is the prob. that he will be seated for the start of the picture?

(c) How early must he arrive in order to be 99% sure of being seated for the start of the picture?

Sol:

Arrival rate $\lambda = 6$ per minute

Service rate $\mu = 7.5$ sec $= \frac{60}{7.5} = 8$ per min

(a) E(total time required to purchase the ticket and to reach the seat)

$=$ waiting time + Time to reach the seat

Now, waiting time of a customer in the system

$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2} \text{ min.}$$

$\therefore$ E(total time ... reach the seat) $= \frac{1}{2} + 1\frac{1}{2} = 2$ min

Hence he can just be seated for the start of the picture.

(b) P(he will be seated for the start of the picture)

$= P(\text{total time} < 2 \text{ minutes})$

$= P(W < \frac{1}{2}) = 1 - P(W > \frac{1}{2})$

$= 1 - e^{-(\mu-\lambda)t} = 1 - e^{-(8-6) \cdot \frac{1}{2}}$

$= 1 - e^{-1} = 0.63$

(c) $P(W < t) = 0.99$

ie, $P(W > t) = 0.01$

$\Rightarrow e^{-(\mu-\lambda)t} = 0.01 \Rightarrow -(\mu-\lambda)t = \log(0.01)$

$\Rightarrow -2t = -4.6$

$\therefore t = 2.3$ minutes

i.e., P[ticket purchasing time < 2.3] = 0.99

$\therefore$ P[total time to get the ticket and to go to the seat < (2.3+1.5)] = 0.99

Therefore, a person must arrive atleast 3.8 minutes early so as to 99% sure of seeing the start of the picture.

2.) A duplicating machine maintained for office use is operated by an office Assistant who earns Rs. 5/- per hour. The time to complete each job varies according to an exponential distribution with mean 6 minutes. Assume a Poisson input with an average arrival rate of 5 jobs per hour. If an 8-hour day is used as a base, determine

(i) the percentage of idle time in the system

(ii) the average time a job is in the system

(iii) the average earning per day of the assistant

Sol:

Arrival rate $\lambda = 5$ per hour

Service rate $\mu = \frac{60}{6} = 10$ per hour

(i) P (idle time in the system) $= P_0 = 1 - \frac{\lambda}{\mu}$

$= 1 - \frac{5}{10} = 1/2$

$\therefore$ Percentage of idle time in the system $= 50\%$.

(ii) Average time of a job in the system is

$W_S = \frac{1}{\mu - \lambda} = \frac{1}{5}$ hr or 12 min

(iii) Average earning per day

= (Average no. of jobs) × (earning per job)

= (Average no of jobs done per day)

× (Average time in hour per job) × (earning per hr)

$= (8 \times 5) \times \frac{1}{5} \times 5 =$ Rs. 40.

3) At what average rate must a clerk in a super market work in order to ensure a prob. of 0.90 that the customer will not wait longer than 12 minutes? It is assumed that there is only one counter at which customers arrive in a Poisson fashion at an average rate of 15 per hr and that the length of the service by he clerk has an exponential distribution

Sol: $\lambda = 15$ per customer/hr $= \frac{15}{60} = \frac{1}{4}$ customers/min

Prob. that a customer will not have to wait longer than 12 minutes $= 0.90$

ie, $P(W_q \geq 12) = 0.90$

$\therefore$ Prob. that a customer will wait longer than 12 minutes $= 1 - 0.90 = 0.10$

Now, $P(W_q \geq 12) = \int_{12}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda) e^{-(\mu-\lambda)t} \, dt$

$$\therefore \quad 0.10 = \frac{\lambda}{\mu}(\mu-\lambda) \int_{12}^{\infty} e^{-(\mu-\lambda)t} \, dt$$

$$= \frac{\lambda}{\mu}(\mu-\lambda) \left[\frac{e^{-(\mu-\lambda)t}}{-(\mu-\lambda)} \right]_{12}^{\infty}$$

$$= \frac{1}{4\mu} e^{-12(\mu-\lambda)}$$

$$\Rightarrow 0.4\mu = e^{-12(\mu-\lambda)}$$

$$\Rightarrow e^{3-12\mu} = 0.4\mu$$

$$\therefore \mu = \frac{1}{2.48} = 0.4 \text{ customers/minute}$$

or 24 customers/hr