Markov chain

If $P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \cdots X_0 = a_0]$

$$\Rightarrow P[X_n = a_n / X_{n-1} = a_{n-1}] \quad \forall \; n.$$

then the process $\{X_n\}$, $n = 0, 1, 2 \cdots$ is called Markov chain.

1) $a_1, a_2 \cdots a_n$ are states of Markov chain.

2) Conditional probability:

$P[X_n = a_j / X_{n-1} = a_i] = P_{ij}((n-1), n)$
is called one step transition probability from state a_i to state a_j at the n^{th} step.

3) If one step transition prob. does not depend on the step (ie) if $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, then Markov chain is called homogeneous Markov chain (or) chain is said to have stationary transition probabilities.

4) When M.C is homogeneous, the one step transition probability is denoted by P_{ij}.

The matrix $\boxed{P = P_{ij}}$ is called One - step transition prob. matrix (tpm).

5) The tpm of M.C is a stochastic Matrix

(1) $P_{ij} \geq 0$.

(2) $\Sigma P_{ij} = 1$ (sum of elts of any row is 1).

6) The conditional probability that the process is in the state a_j at the step n, given that it was in state a_i at step 0.

$P[X_n = a_j / X_0 = a_i] = P_{ij}(n)$. is called n-step transition probability

7) Let P_i = prob. that the process is in state a_i at any step $(i = 1, 2 \dots k)$. Row vector $P = (P_1, P_2 \dots P_k)$ is called prob. dis/: of process at that time.

8) $P^0 = (P_1^{(0)}, P_2^{(0)} \dots P_k^{(0)})$ is called initial prob. dist: where $P_1^{(0)} = P(X_0 = 1)$, $P_2^{(0)} = P(X_0 = 2) \dots$ are initial probability for states $1, 2 \dots$

9) If P is the tpm of the regular chain & π (a row vector) is the steady state dis/: then

$$\boxed{\pi P = \pi}$$

Chapman - Kolmogorov theorem.

If P is tpm of homogeneous Markov chain and n^{th} step tpm $P^{(n)} = P^n$, then it is n^{th} power of tpm.

$\therefore$ $P^{(2)} = P \times P$.

$P^{(3)} = P^{(2)} \times P$

$P^{(4)} = P^{(3)} \times P$ and so on.

(ie) $\boxed{P_{ij}^{(n)} = (P_{ij})^n}$

Regular Matrix.

A stochastic Matrix P is said to be a regular matrix, if all the entries of P^m (for some +ve integer m) are positive.

If tpm is Regular $\Rightarrow$ Hom. M.C is regular.

classification of states of a M.C.

1) Irreducible.

A M.C is irreducible if every state can be reached from every other state where $(P_{ij})^n > 0$ for some n & $\forall$ i & j.

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Note : Tpm of an irreducible chain is an irregular Matrix.
Otherwise, it is reducible.

2) Return state :

If $P_{ij}^{(n)} > 0$, for some $n > 1$, then we call state i of M.C as return state.

3) Period :

Let $P_{ii}^{(m)} > 0 \; \forall m$. Let i be a return state

Then, period $d_i = GCD\{m : P_{ij}^{(m)} > 0\}$

state i is said to be periodic with

$\rightarrow \begin{cases} \text{period } d_i & \text{if } d_i > 1 \text{ and} \\ \text{aperiodic} & \text{if } d_i = 1 \end{cases}$

4) Recurrence time probability

The prob. that chain returns to state i, starting from state i for the first time at the n^{th} step is called recurrence time probability or the first return time probability and is denoted by $f_{ii}^{(n)}$

Recurrent state :

1) If $F_{ii} = \sum_{i=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain and state 'i' is said to be persistent (or) recurrent. Otherwise, it is said to be transient.

2) $\mu_{ii} = \sum_{i=1}^{\infty} n \cdot f_{ii}^{(n)}$ is called Mean Recurrence time of state i.

3) The state i is said to be non-null persistent if its mean recurrence time μ_{ii} is finite & null persistent if $\mu_{ii} = \infty$

4) A Non-null persistent & aperiodic state is called <u>ergodic</u>.

Note :

1) If M.C is irreducible then all its state are of the same type. They are all transient, all null persistent (or) all non-null persistent. All its states are either aperiodic (or) periodic with same period.

2) If a M.C is finite, irreducible, all its states are non-null persistant.

Problems :

1) The initial process of Markov tpm is given by

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$ with initial probability

$p_1^{(0)} = 0.4$, $p_2^{(0)} = 0.3$, $p_3^{(0)} = 0.3$. Find $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}$

Let $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}$ be all entries corresponding to matrix $p^{(1)}$

(ie) $p^{(1)} = p^{(0)} . P$.

Given tpm is $P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$

Initial prob. dis is

$$p^{(0)} = \begin{bmatrix} p_1^{(0)} & p_2^{(0)} & p_3^{(0)} \end{bmatrix}$$
$$= \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix}$$

To find $p^{(1)}$:

$$p^{(1)} = p^{(0)} . P.$$

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$$P^{(1)} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.2 & 0.7 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.08+0.03+0.18 & 0.12+0.06+0.09 & 0.2+0.21+0.03 \end{bmatrix}$$

$$= \begin{bmatrix} 0.29 & 0.27 & 0.44 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} P_1^{(1)} & P_2^{(1)} & P_3^{(1)} \end{bmatrix}$$

Hence
$$\boxed{\begin{aligned} P_1^{(1)} &= 0.29 \\ P_2^{(1)} &= 0.27 \\ P_3^{(1)} &= 0.44 \end{aligned}}$$

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2) The tpm of a Markov chain $\{X_n\}$, $n = 1, 2, 3.$ having 3 states 1, 2, 3 is
$$P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

and initial distribution is $P^{(0)} = (0.7 \quad 0.2 \quad 0.1)$.

Find (1) $P[X_2 = 3]$ & (2) $P\{X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2\}$

Given TPM, $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$

& initial distribution.

$$P^{(0)} = (P_1^{(0)} \quad P_2^{(0)} \quad P_3^{(0)})$$
$$= (0.7 \quad 0.2 \quad 0.1)$$

(ie) $P[X_0 = 1] = P_1^{(0)} = 0.7$

$P[X_0 = 2] = P_2^{(0)} = 0.2$

$P[X_0 = 3] = P_3^{(0)} = 0.2$

1) To find $P[X_2 = 3]$.

$$P^{(2)} = P \cdot P.$$

$$= \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.01+0.3+0.12 & 0.05+0.10+0.16 & 0.04+0.1+0.12 \\ 0.06+0.12+0.06 & 0.3+0.04+0.08 & 0.24+0.04+0.06 \\ 0.03+0.24+0.09 & 0.15+0.08+0.12 & 0.12+0.08+0.09 \end{bmatrix}$$

$$P^{(2)} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

Hence $P_{13}^{(2)} = 0.26$

$P_{23}^{(2)} = 0.34$

$P_{33}^{(2)} = 0.29$

$$P\{X_2 = 3\} = \sum_{i=1}^{3} P\{X_2 = 3 / X_0 = i\} \times P\{X_0 = i\}$$

(ie) $P(A/B) = \dfrac{P(A,B)}{P(B)}$

$\Rightarrow P(A,B) = P(A/B) \cdot P(B)$.

$$P\{X_2 = 3\} = P[X_2 = 3 / X_0 = 1] \cdot P[X_0 = 1] + P[X_2 = 3 / X_0 = 2] \cdot P[X_0 = 2] + P[X_2 = 3 / X_0 = 3] \cdot P[X_0 = 3]$$

$$= P_{13}^{(2)} P[X_0 = 1] + P_{23}^{(2)} P[X_0 = 2] + P_{33}^{(2)} P[X_0 = 3]$$

$$= (0.26 \times 0.7) + (0.34 \times 0.2) + (0.29 \times 0.1)$$

$$= 0.182 + 0.068 + 0.029$$

$$= 0.279 .$$

(2) $P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$

$$= P[X_3 = 2 / X_2 = 3, X_1 = 3, X_0 = 2] \cdot P[X_2 = 3, X_1 = 3, X_0 = 2]$$

$$= P[X_3 = 2 / X_2 = 3] \cdot P[X_2 = 3, X_1 = 3, X_0 = 2]$$

Since by Markov chain,

$$P[X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \dots X_0 = a_0]$$
$$= P[X_n = a_n / X_{n-1} = a_{n-1}]$$

$$P[X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 2]$$
$$= P_{32}^{3-2} \cdot P[X_2 = 3 / X_1 = 3, X_0 = 2] \cdot P[X_1 = 3, X_0 = 2]$$
$$= P_{32}^{(1)} \cdot P[X_2 = 3 / X_1 = 3] \cdot P[X_1 = 3, X_0 = 2]$$
$$= P_{32}^{(1)} \cdot P_{33}^{(2-1)} \cdot P[X_1 = 3 / X_0 = 2] \cdot P[X_0 = 2]$$
$$= P_{32}^{(1)} \; P_{33}^{(1)} \cdot \cancel{P[} \cdot P_{23}^{(1-0)} \; P[X_0 = 2]$$
$$= P_{32}^{(1)} \; P_{33}^{(1)} \; P_{23}^{(1)} \; P[X_0 = 2]$$
$$= (0.4)(0.3)(0.2)(0.2)$$
$$= 0.0048 .$$

---

3) A gambler has Rs 2. He bets Rs 1 at a time & wins Rs. 1 with prob ½. He stops playing if he losses Rs 2 or wins Rs 4

1) what is the tpm of related Markov chain?

2) what is the prob. that he lost his money at the end of 5 plays?

3) what is the prob that game lasts more than 7 plays?

Let X_n represents the amount with the player at the end of n^{th} round of the play.

State space of $\{X_n\}$ = (0, 1, 2, 3, 4, 5, 6) as the game ends, if the player loses all the money ($X_n = 0$) (or) wins Re 4 (ie) he has Rs 6. ($X_n = 6$).

(2+4 = 6)

1) Tpm of Markov chain,

$$P = \begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

states of X_n (present)

States of X_{n-1} (Previous)

The initial dist: of $\{X_n\}$ is

$$p^{(0)} = (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0)$$

as the player has got Rs 2. to start with.

$$p^{(1)} = p^{(0)} \cdot P$$

$$= (0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0) \cdot P$$

$$= (0 \ 1/2 \ 0 \ 1/2 \ 0 \ 0 \ 0)$$

$$p^{(2)} = p^{(1)} \cdot P$$

$$= (0 \ 1/2 \ 0 \ 1/2 \ 0 \ 0 \ 0) \cdot P$$

$$= (1/4 \ 0 \ 1/2 \ 0 \ 1/4 \ 0 \ 0)$$

$$p^{(3)} = p^{(2)} \cdot P$$

$$= (1/4 \ 1/4 \ 0 \ 3/8 \ 0 \ 1/8 \ 0) \cdot \cancel{P}$$

$$p^{(4)} = p^{(3)} \cdot P$$

$$= (3/8 \ 0 \ 5/16 \ 0 \ 1/4 \ 0 \ 1/16)$$

$$p^{(5)} = p^{(4)} \cdot P$$

$$= (3/8 \ 5/32 \ 0 \ 9/32 \ 0 \ 1/8 \ 1/16)$$

$p^{(6)} = p^{(5)} \cdot P$

$= \left(29/64 \quad 0 \quad 7/32 \quad 0 \quad 13/64 \quad 0 \quad 1/8 \right)$

$p^{(7)} = p^{(6)} \cdot P$

$= \left(29/64 \quad 7/64 \quad 0 \quad 27/128 \quad 0 \quad 13/128 \quad 1/8 \right)$

(2) P[Man has lost money at the end of 5 plays]

$= P[X_5 = 0]$

= (Entry corresponding to the first row first column is $p^{(5)}$)

$= 3/8$

(3) P[game last more than 7 plays]

= P[system is neither in state 0 nor in 6 at the end of seventh round]

$= P[X_7 = 1, 2, 3, 4 \text{ (or) } 5]$

$= P[X_7 = 1] + P[X_7 = 2] + P[X_7 = 3] + P[X_7 = 4] + P[X_7 = 5]$

$= \frac{7}{64} + 0 + \frac{27}{128} + 0 + \frac{13}{128}$

$= \frac{14 + 27 + 13}{128}$

$= \frac{54}{128}$

$= \frac{27}{64}$

---

4) A salesman territory consists of 3 cities A, B, C. He never sells in the same city on successive days. If he sells in A, then the next day, he sells in city B. However if he sells in either B or C,

the next day he is twice as likely to sell in city A as in other city. In the long run, how often does he sell in each of the cities ?

Formation of tpm :

Let $\{X_n\}$ represents salesman's sales patten.
The process $\{X_n\}$ has 3 states A, B, C.

$$P = \begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \; (X_n) \\ X_{n-1} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

Steady state distribution :

Let $\pi = (\pi_1 \ \pi_2 \ \pi_3)$ be steady state prob. dis of MC.

By steady state, $\pi P = \pi$

$$(\pi_1 \ \pi_2 \ \pi_3) \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = (\pi_1 \ \pi_2 \ \pi_3)$$

$$\left(0 + \frac{2\pi_2}{3} + \frac{2\pi_3}{3} \quad \pi_1 + \frac{\pi_3}{3} \quad \frac{\pi_2}{3}\right) = (\pi_1 \ \pi_2 \ \pi_3).$$

$$\Rightarrow \frac{2\pi_2}{3} + \frac{2}{3}\pi_3 = \pi_1 \longrightarrow 2\pi_2 + 2\pi_3 = 3\pi_1$$

$$\pi_1 + \frac{\pi_3}{3} = \pi_2 \longrightarrow 3\pi_1 + \pi_3 = 3\pi_2$$

$$\frac{\pi_2}{3} = \pi_3. \longrightarrow \boxed{\pi_2 = 3\pi_3}$$

$$3\pi_1 + \pi_3 = 3(3\pi_3).$$

$$3\pi_1 + \pi_3 - 9\pi_3 = 0.$$

$$3\pi_1 = 8\pi_3.$$

$$\boxed{\pi_1 = \frac{8}{3}\pi_3}$$

Since π is steady state dist:

$\pi_1 + \pi_2 + \pi_3 = 1$ ($\because$ sum of Row elts $= 1$)

$\frac{8}{3}\pi_3 + 3\pi_3 + \pi_3 = 1$

$8\pi_3 + 9\pi_3 + 3\pi_3 = 3$.

$20\pi_3 = 3$.

$$\boxed{\pi_3 = \frac{3}{20}}$$

$\therefore$ $\pi_1 = \frac{8}{3}\left(\frac{3}{20}\right)$; $\boxed{\pi_1 = 2/5}$

$\pi_2 = 3\left(\frac{3}{20}\right)$; $\boxed{\pi_2 = \frac{9}{20}}$

$\pi = (\pi_1 \quad \pi_2 \quad \pi_3)$

$\pi = \left(\frac{2}{5} \quad \frac{9}{20} \quad \frac{3}{20}\right)$

$\pi = (0.4 \quad 0.45 \quad 0.15)$

In the long run, he sells $\begin{cases} 40\% \text{ of time in city A} \\ 45\% \text{ of time in city B} \\ 15\% \text{ of time in city C.} \end{cases}$

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2) A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, Man tossed a fair die and drove to work iff a 6 appeared. Find

i) Prob that he takes a train on the 3rd day.

ii) Prob that he drives to work in the long run.

Tpm of Markov chain.

Travel pattern is a M.c with state space = (train car).

$$P = \begin{array}{c c} & \begin{array}{cc} T & C \end{array} \\ \begin{array}{c} T \\ C \end{array} & \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

$p^{(1)}$ = initial state prob. dis.

$$p^{(1)} = \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{bmatrix} 5/6 & 1/6 \end{bmatrix} \end{array}$$

P[travelling by car] = P[getting 6 in tossing the die]

$= 1/6$

P[travelling by train] $= 5/6$.

$$p^{(2)} = p^{(1)} \cdot P = (5/6 \quad 1/6) \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = (1/12 \quad 11/12)$$

$$p^{(3)} = p^{(2)} \cdot P = (1/12 \quad 11/12) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (11/24 \quad 13/24)$$

P[Man travels by train on third day] $= 11/24$

Let $\pi = (\pi_1 \quad \pi_2)$ be limiting form of steady state prob. dis (or) stationary state distr. of M.c.

By steady state property, $\pi P = \pi$ &

$\pi_1 + \pi_2 = 1$

$$(\pi_1 \quad \pi_2) \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1 \quad \pi_2)$$

$\frac{\pi_2}{2} = \pi_1 \longrightarrow (1)$ | $\pi_1 + \frac{\pi_2}{2} = \pi_2 \longrightarrow (2)$ and $\pi_1 + \pi_2 = 1 \longrightarrow (3)$

From (3) $\Rightarrow \pi_2 = 1 - \pi_1 \longrightarrow (4)$

sub (1) & (4) in (2),

$$\pi_1 + \pi_1 = 1 - \pi_1 .$$

$$3\pi_1 = 1$$

$$\boxed{\pi_1 = 1/3} \quad (\text{Train}).$$

(By ③), $\boxed{\pi_2 = 2/3}$ (car)

P[Man Travels by car in a long run] $= 2/3$.

---

b) Suppose that the prob. of dry day following a rainy day is $1/3$ and that the prob of a rainy day following a dry day is $1/2$. Given that May 1 is a dry day. Find the prob that May 3 is dry day and also May 5 is a dry day.

State space $\rightarrow \{D, R\}$ the tpm of M.C is

$$P = \begin{array}{c|cc} & D & R \\ \hline D & 1/2 & 1/2 \rightarrow \text{giv.} \\ R & 1/3 \rightarrow \text{giv.} & 2/3 \end{array}$$

Given May 1 is a dry day

$$p^{(1)} = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad (D \quad R)$$

$$p^{(2)} = p^{(1)} \cdot P = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$$

$$p^{(3)} = p^{(2)} \cdot P = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5/12 & 7/12 \end{pmatrix}$$

(1) P[May 3 is a dry day] $= 5/12$

$$p^{(4)} = p^{(3)} \cdot P = \begin{pmatrix} 5/12 & 7/12 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 29/72 & 43/72 \end{pmatrix}$$

$$P^{(5)} = P^{(4)} \cdot P = \left(\tfrac{29}{72} \quad \tfrac{43}{72} \right) \begin{pmatrix} \tfrac{1}{2} & \tfrac{1}{2} \\ \tfrac{1}{3} & \tfrac{2}{3} \end{pmatrix}$$

$$= \left(\tfrac{173}{432} \quad \tfrac{259}{432} \right)$$

(2) $P[\text{May 5 is a dry day}] = \dfrac{173}{432}$

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7) The tpm of a Markov chain with three states 0,1,2 is $P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$ and initial state distribution of the chain is $P[X_0 = i] = 1/3$, $i = 0,1,2$.

Find (1) $P[X_2 = 2]$ & (2) $P[X_3 = 1;\ X_2 = 2;\ X_1 = 1;\ X_0 = 2]$

Given $P[X_0 = 0] = 1/3$

$P[X_0 = 1] = 1/3$ & $P = \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \end{array}$

$P[X_0 = 2] = 1/3$

$$P^{(2)} = P \cdot P = \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{bmatrix}$$

$$= \begin{array}{c|c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 10/16 & 5/16 & 1/16 \\ 5/16 & 8/16 & 3/16 \\ 3/16 & 9/16 & 4/16 \end{bmatrix} \end{array}$$

(1) $P[X_2 = 2] = \sum_{i=0}^{2} P[X_2 = 2 \mid X_0 = i] \cdot P[X_0 = i]$

$= P[X_2 = 2 \mid X_0 = 0] \cdot P[X_0 = 0] + P[X_2 = 2 \mid X_0 = 1]\, P[X_0 = 1]$
$\quad + P[X_2 = 2 \mid X_0 = 2] \cdot P[X_0 = 2]$

$= P_{02}^{(2-0)}\, P[X_0 = 0] + P_{12}^{(2-0)} \cdot P[X_0 = 1] + P_{22}^{(2-0)}\, P[X_0 = 2]$

$= P_{02}^{(2)}\, P[X_0 = 0] + P_{12}^{2}\, P[X_0 = 1] + P_{22}^{2}\, P[X_0 = 2]$

$= \frac{1}{16}(\frac{1}{3}) + \frac{3}{16}(\frac{1}{3}) + \frac{4}{16}(\frac{1}{3})$.

$= \frac{1}{3}[\frac{1}{16} + \frac{3}{16} + \frac{4}{16}]$

$= \frac{1}{3} \cdot \frac{8}{16} = \frac{1}{6}$ //.

(2) $P[X_3 = 1; X_2 = 2; X_1 = 1; X_0 = 2]$

$= P[X_3 = 1 / X_2 = 2, X_1 = 1, X_0 = 2] \, P[X_2 = 2, X_1 = 1, X_0 = 2]$

$= P[X_3 = 1 / X_2 = 2] \, P[X_2 = 2, X_1 = 1, X_0 = 2]$

$= P_{21}^{1} \cdot P[X_2 = 2 / X_1 = 1, X_0 = 2] \cdot P[X_1 = 1, X_0 = 2]$

$= P_{21}^{(1)} \cdot P[X_2 = 2 / X_1 = 1] \, P[X_1 = 1, X_0 = 2]$

$= P_{21}^{(1)} \cdot P_{12}^{(1)} \cdot P[X_1 = 1 / X_0 = 2] \cdot P[X_0 = 2]$

$= P_{21}^{(1)} P_{12}^{(1)} P_{21}^{(1)} P[X_0 = 2]$

$= (\frac{3}{4})(\frac{1}{4})(\frac{3}{4})(\frac{1}{3})$

$= \frac{3}{64}$.

---

8) An Engineer analyzing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal, whereas 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals between. Given that only highly distorted signals are not recognizable. Find the fraction of signals that are highly distorted.

If $n \geq 1$

$X_n = 1$, if n^{th} signal generated is highly distorted.

$X_n = 0$, if n^{th} signal generated is recognizable.

State space $\{0, 1\}$.

Transition probability Matrix is gn by.

$$P_{X/\!\!\#} = \begin{bmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{bmatrix}$$

$\pi_0 \rightarrow$ the fraction of signals that are recognizable

$\pi_1 \rightarrow$ the fraction of signals that are highly distorted

$$(\pi_0 \quad \pi_1) \begin{bmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{bmatrix} = (\pi_0 \quad \pi_1)$$

$$\frac{20}{23}\pi_0 + \frac{20}{23}\pi_1 = \pi_0$$

$$\frac{3}{23}\pi_0 + \frac{1}{15}\pi_1 = \pi_1 .$$

W.K.T $\quad \pi_0 + \pi_1 = 1.$

(ie) $\pi_1 = 1 - \pi_0 .$

$$\frac{20}{23}\pi_0 + \frac{14}{15}(1 - \pi_0) = \pi_0 .$$

$$\frac{20}{23}\pi_0 + \frac{14}{15} - \frac{14}{15}\pi_0 = \pi_0 .$$

$$\left(\frac{20}{23} - \frac{14}{15}\right)\pi_0 + \frac{14}{15} = \pi_0 .$$

$$-\frac{22}{345}\pi_0 + \frac{14}{15} = \pi_0 .$$

$$\frac{14}{15} = \pi_0 + \frac{22}{345}\pi_0 .$$

$$\frac{14}{15} = \frac{367\pi_0}{345}$$

$$\therefore \pi_0 = \frac{322}{367} = 0.877 .$$

$$\therefore \pi_1 = 1 - \pi_0 = 1 - 0.877 = 0.123 .$$

$\therefore$ 12.3% of signals generated by the testing syste[m] are highly distroted.

classification of states of a Markov chain

Consider the discrete time Markov chain $\{X_n\}$ over the state space $\{0,1,2,3\dots\}$. The states of the chain can be classified in general, depending on the nature of movements of the state.

1) Accessible or Reachable state :

State j is said to be reachable or accessible from state i if the chain (or system) can reach j in finite no/. of transitions. (ie) $P_{ij}^{(n)} > 0$ for some integer $n \geq 0$.

Note :

If $P_{ij}^{(n)} > 0$ $\forall$ i & j and some n then every state is accessible from every other state.

2) Communicating States :

If two states i and j are accessible to each other, then they are said to communicate each other.

(ie) $i \leftrightarrow j$ if $i \rightarrow j$ and $j \rightarrow i$.

The relation of communication of states is an equivalence relation.

(i) Reflexivity : Any state communicates with itself (ie) $i \leftrightarrow i$ $\forall$ i

(ii) Symmetry : $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

(iii) Transitivity : $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

The set of all states communicating with each other form a class, called communicating class. Two classes are either identical or disjoint.

3) Irreducible Markov chain :

A Markov chain is said to be irreducible if every state is reachable from every other state. (ie) $P_{ij}^{(n)} > 0$ for every i and j and some integer $n \geq 0$.

In otherwords, if states of a Markov chain form only one class then the chain is irreducible, because all states communicate with each other in this case.

Otherwise the chain is said to be reducible or non-irreducible.

4) Return state :

State i of a Markov chain is called a return state if $P_{ii}^{(n)} > 0$ for some $n > 1$ (ie) system comes back to i starting from i)

5) Absorbing state :

A state i of a Markov chain is called an absorbing state if no other state is accessible from it. (ie) for an absorbing state i, $P_{ii} = 1$ and $P_{ij} = 0$ if $i \neq j$

~~Note:~~

6) Period of a return state :

The period d_i of a return state i is the g.c.d of all integers m such that $P_{ii}^{(m)} > 0$

(ie) $d_i = g.c.d \{ m \mid P_{ii}^{(m)} > 0 \}$

State i is periodic with period d_i if $d_i > 1$

State i is aperiodic (with no such integer $d_i > 1$ exists) (ie) if $d_i = 1$

Note : clearly state i is aperiodic if $P_{ii} \neq 0$.

(ie) state i for which $P_{ii} > 0$ has period 1 and hence it is aperiodic

A Markov chain is aperiodic if every state is aperiodic.

7) First return probability :

The probability that the process (ie) the chain having started from i returns to i for the first time at the n^{th} step is called first return probability and is denoted by $f_{ii}^{(n)}$

$$f_{ii}^{(n)} = P[X_n = i, \quad X_m \neq i, \quad m = 1, 2 \ldots n-1 / X_0 = i]$$

$f_{ii}^{(n)}$ – first return time probability or recurrence time probability.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$ the return to state i is certain

The Mean recurrence time to the state i is

$$\mu_{ii} = \sum_{n=1}^{\infty} n \cdot f_{ii}^{(n)} .$$

8) Recurrent or Persistent state :

A state i is said to be persistent or recurrent if the eventual return to state i is certain. if $F_{ii} = 1$.

If $F_{ii} < 1$ (ie) if return to the state i is uncertain, then state i is said to be transient or non-recurrent.

9) Non-null and null persistent state.

A persistent state i ((ie) $F_{ii} = 1$) is called a null state if its mean recurrence time is not finite (ie) $\mu_{ii} = \infty$. and non-null if its mean recurrence time μ_{ii} is finite (ie) $\mu_{ii} < \infty$.

10) **Ergodic state :**

A persistent, non null and aperiodic state i is called an ergodic state.

11) **Ergodic chain :**

A Markov chain is said to be an ergodic chain if all its states are ergodic.

12) **Essential state :**

A state i is called an essential state if it communicates with any state from which it is accessible.

Note : 1

A state i is persistent iff $\sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$. and

transient iff $\sum_{n=0}^{\infty} P_{ii}^{(n)}$ is finite

where $P_{ij}^{(n)}$ are transition probabilities.

Note : 2

If a M.c is irreducible then all its states are of the same type.

(ie) they are all transient, all null persistent or non null persistent.

All periodic with same period or aperiodic.

Note : 3

If a finite M.c, it is impossible to have all states transient. Further, if it is irreducible then all its states are persistent and non null.

Note : 4

A finite state M.c which is irreducible and aperiodic is ergodic.

1) Find Nature of states of M.C with tpm.

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & 0 & 1 & 0 \end{array}$$

Now we find

$$P^{(2)} = P \cdot P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^{(3)} = P^{(2)} \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = P$$

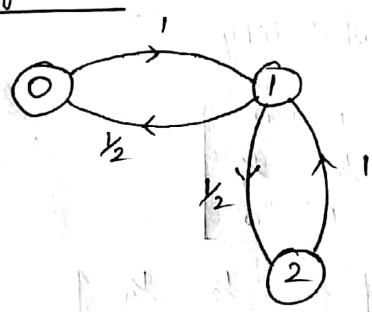
$$P^{(4)} = P^{(3)} \cdot P = P \cdot P = P^{(2)}$$

$$P^{(5)} = P^{(4)} \cdot P = P^{(2)} \cdot P = P^{(3)} = P \ldots$$

and so on.

$$P^{2n} = P^2 \ \& \ P^{(2n+1)} = P, \quad n = 1, 2, 3 \ldots$$

Transition diagram :



To prove : M.C is irreducible

$P_{00}^{(2)} = \frac{1}{2} > 0$. $P_{01}^{(1)} = 1 > 0$ $P_{02}^{(2)} = \frac{1}{2} > 0$.

$P_{10}^{(1)} = \frac{1}{2} > 0$. $P_{11}^{(2)} = 1 > 0$. $P_{12}^{(1)} = \frac{1}{2} > 0$

$P_{20}^{(2)} = \frac{1}{2} > 0$. $P_{21}^{(1)} = 1 > 0$ $P_{22}^{(2)} = \frac{1}{2} > 0$.

All $P_{ij}^{(n)} > 0$, $i, j = 0, 1, 2$. & $n = 1, 2, 3 \ldots$

$\therefore$ M.C is irreducible.

To find period :

$$d_i = GCD\{m : P_{ij}^{(m)} > 0\}.$$

First state, $P_{00} > 0 \longrightarrow \{2, 4, 6 \cdots\}$.

$GCD = 2$.

Sec. state, $P_{11} > 0 \longrightarrow \{2, 4, 6 \cdots\}$.

$GCD = 2$

Third state, $P_{22} > 0 \longrightarrow \{2, 4, 6 \cdots\}$

$GCD = 2$.

Hence period $(d_i) = 2$.

Chain is finite, irreducible, Non-null persistent. but periodic.

$\therefore$ states are not ergodic.

---

2) Let $\{X_n : n = 1, 2 \ldots\}$ be a M.c with state space $S = \{0, 1, 2\}$ and one step TPM. $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$

1) Is chain ergodic ? Explain.

2) Find the invariant probability

(1) Given $P = \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$

$$P^{(2)} = P \cdot P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 3/4 & 1/8 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

All $P_{ij}^{(2)} > 0 \quad \forall\, i, j$.

The chain is irreducible.

Since $P_{11} > 0 \longrightarrow \{1, 2\}$.

$GCD = 1$.

If $d_i = 1$, state 1 is aperiodic.

M.c is ergodic.

(2) Let $\Pi = (\Pi_0, \Pi_1, \Pi_2)$ be steady state prob. dist.

then $\Pi P = \Pi$. where $\Pi_0 + \Pi_1 + \Pi_2 = 1$.

$$[\Pi_0 \; \Pi_1 \; \Pi_2] \begin{bmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} = [\Pi_0 \; \Pi_1 \; \Pi_2]$$

$$[\tfrac{1}{4}\Pi_1 \quad \Pi_0 + \tfrac{1}{2}\Pi_1 + \Pi_2 \quad \tfrac{1}{4}\Pi_1] = [\Pi_0 \; \Pi_1 \; \Pi_2]$$

$$\frac{\Pi_1}{4} = \Pi_0.$$

$$\frac{\Pi_1}{4} = \Pi_2.$$

$$\Pi_0 + \frac{\Pi_1}{2} + \Pi_2 = \Pi_1$$

W.K.T $\Pi_0 + \Pi_1 + \Pi_2 = 1.$

$$\frac{\Pi_1}{4} + \Pi_1 + \frac{\Pi_1}{4} = 1.$$

$$\frac{3}{2}\Pi_1 = 1, \quad \boxed{\Pi_1 = 2/3}$$

and $\boxed{\Pi_0 = 1/6}$, $\boxed{\Pi_2 = 1/6}$

$$\Pi = [1/6 \quad 2/3 \quad 1/6].$$

---

3) consider a M.c with state space [0,1] & tpm, P is

$$P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}.$$

1) Is the state 0 recurrent? Explain.

2) Is the state 1 transient? Explain.

Given chain has state space [0,1] . &

$$\text{tpm} \quad P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1/2 & 1/2 \end{array}$$

$$P^{(2)} = P \cdot P = \begin{bmatrix} 1 & 0 \\ 3/4 & 1/4 \end{bmatrix}$$

$$P^{(3)} = P^2 \cdot P = \begin{bmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{bmatrix}$$

Transition diagram :

1) System goes from $0 \to 0$ with prob. 1.

$\therefore$ state 0 is recurrent

2) System goes from $1 \to 1$ with prob $\frac{1}{2} < 1$.

$\therefore$ State 1 is non-recurrent (or) transient.

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4) Three boys A, B, C are throwing a ball to each other, A always throw the ball to B and B always throws to C but C is just as likely to throw the ball to B as to A. S.T the process is Markovian. Find transition Matrix & classify the states.

Let state space be $\{A, B, C\}$.

The tpm P is given by,

$$\begin{array}{ccc|ccc}
 & & & \text{A} & \text{B} & \text{C} \\
 & & & (0) & (1) & (2) \\
\hline
 & & A\ (0) & 0 & 1 & 0 \\
P = & \text{States of } X_{n-1} & B\ (1) & 0 & 0 & 1 \\
 & & C\ (2) & \frac{1}{2} & \frac{1}{2} & 0
\end{array}$$

(columns: States of X_n)

A always throws to B, $P_{01} = 1$.

B always throws to C, $P_{12} = 1$.

C throws to A or B with equal probability,

$$P_{20} = P_{21} = \frac{1}{2}$$

Now $P^2 = P \cdot P = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$P^3 = P^2 \cdot P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$P^4 = P^3 \cdot P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$P^5 = P^4 \cdot P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$

$P^6 = P^5 \cdot P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{8} & \frac{1}{2} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$

<u>To check : M.C is irreducible :</u>

$P_{00}^{(3)} = \frac{1}{2} > 0$ $\quad$ $P_{01}^{(2)} = \frac{1}{2} > 0$ $\quad$ $P_{02}^{(2)} = 1 > 0$.

$P_{10}^{(2)} = \frac{1}{2} > 0$ $\quad$ $P_{11}^{(2)} = \frac{1}{2} > 0$. $\quad$ $P_{12}^{(4)} = \frac{1}{2} > 0$.

$P_{20}^{(3)} = \frac{1}{4} > 0$ $\quad$ $P_{21}^{(2)} = \frac{1}{2} > 0$ $\quad$ $P_{22}^{(2)} = \frac{1}{2} > 0$.

All $P_{ij}^{(n)} > 0$, $\therefore$ M.C is irreducible. & is finite.

To find period :

$P_{11} > 0 \longrightarrow \{2, 3, 4, 5, 6 \ldots\}$

G.C.D is 1.

$P_{22} > 0 \longrightarrow \{2, 3, 4, 5, 6 \ldots\}$

G C D is 1.

∴ Period of state 1 is 1.

∴ State 1 & 2 are aperiodic.

M.c is finite and irreducible and aperiodic.

Hence, All states are ergodic.

All P^5 has only positive probability values.

∴ tpm P is regular Matrix.

⟹ M.c is regular.

=> All regular chains are ergodic.

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5). Let $\{X_n : n = 1, 2, 3 \ldots\}$ be a M.c, the space $S = \{1, 2, 3\}$ with one step transition Matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

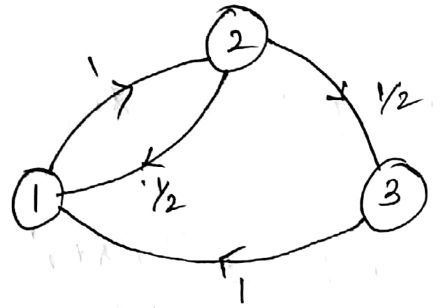
1) Sketch the transition diagram

2) Is the chain irreducible explain?

3) Is the chain ergodic.

$$\text{Given } P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & 1 & 0 & 0 \end{array}$$

1) Transition diagram



2) To decide irreducibility :

$$P^2 = P \cdot P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^4 = P^3 \cdot P = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^5 = P^4 \cdot P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 3/8 & 1/2 & 1/8 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

All $P_{ij}^{(5)} > 0$. Hence M.C is irreducible.

3) To find Period :

$P_{11} > 0 \longrightarrow \{2, 3, 4, 5 \ldots\}$.

G.C.D is 1.

State 1 is aperiodic. & $\{X_n\}$ is finite.

Hence, M.C is ergodic.