| | | | |
|---|---|---|---|
| $E(15) = \frac{60 \times 40}{150} = 16$ | $E(5) = \frac{30 \times 40}{150} = 8$ | $E(20) = \frac{60 \times 40}{150} = 16$ | ***40*** |
| $E(20) = \frac{60 \times 50}{150} = 20$ | $E(10) = \frac{30 \times 50}{150} = 10$ | $E(20) = \frac{60 \times 50}{150} = 20$ | ***50*** |
| $E(25) = \frac{60 \times 60}{150} = 24$ | $E(15) = \frac{30 \times 60}{150} = 12$ | $E(20) = \frac{60 \times 60}{150} = 24$ | ***60*** |
| ***60*** | ***30*** | ***60*** | ***150*** |

| $\boldsymbol{O}$ | $\boldsymbol{E}$ | $\boldsymbol{O-E}$ | $(\boldsymbol{O-E})^2$ | $\frac{(\boldsymbol{O-E})^2}{\boldsymbol{E}}$ |
|---|---|---|---|---|
| 15 | 16 | -1 | 1 | 0.0625 |
| 5 | 8 | -3 | 9 | 1.125 |
| 20 | 16 | 4 | 16 | 1 |
| 20 | 20 | 0 | 0 | 0 |
| 10 | 10 | 0 | 0 | 0 |
| 20 | 20 | 0 | 0 | 0 |
| 25 | 24 | 1 | 1 | 0.042 |
| 15 | 12 | 3 | 9 | 0.75 |
| 20 | 24 | -4 | 16 | 0.666 |
| | | | | **3.6458** |

$\boldsymbol{\chi^2} = \sum_{i=1}^{n} \frac{(O_i-E_i)^2}{E_i} = \mathbf{3.6458}$ **,** d.f. = $(r-1)(c-1) = (3-1)(3-1) = 4$ ***at*** 5% LOS = 9.488.

Calculated $\chi^2 <$ tabulated $\chi^2$, $\boldsymbol{H_0}$ **is accepted.** The hair colour and eye colour are independent.

## *UNIT – 4*

## *QUEUEING THEORY*

***Syllabus***

- Introduction to Markovian queueing models
- Single Server Model with Infinite system capacity - $(M/M/1) : (\infty/FIFO)$
- Single Server Model with Finite System Capacity – $(M/M/1) : (K/FIFO)$

### *INTRODUCTION*

***History*** **:** A.K.Erlang (1909) – “The Theory of probabilities and telephone conversations”.

All of us have experienced the annoyance of having to wait in line.

***Example:***1. We wait in line in our cars in traffic jams. 2. We wait in line of barber shops or beauty parlors. 3. We wait in line at supermarket to check out.

***Why then is there waiting?***

There is more demand for service that there is facility for service available.

***Why is this so?***

1. There may be a shortage of available servers.

2. There may be a space limit to the amount of service that can be provided.

***Question :*** 1. How long must a customer wait? 2. How many people will form in the line?

***Answer:*** Queuing theory attempts to answer these questions through detailed mathematical analysis.

***Customer*****:** The term **‘**Customer**’** is used in a general sense and does not imply necessarily a human customer.

**E.g.:** 1. An Air plane waiting in line to take off. 2. A Computer program waiting to be run as a time shared basis.


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***Characteristics of Queuing Process***

1. Arrival pattern of Customers
2. Service pattern of Servers
3. Queue discipline
4. System capacity
5. Number of service channels
6. Number of service stages

***1. Arrival Pattern of Customer :*** (i) Bulk or Batches (ii) Balked (iii) Reneged (iv) Jockey

**(i)** ***Bulk or Batches* :** More than one arrival can be entering the system simultaneously, the input is said to occur in bulk or batches.

**(ii)** ***Balked*** : If customer decides not to enter the queue upon arrival, he is said to have balked.

**(iii)** ***Reneged :*** A Customer may enter the queue, but after a time lose patience and decide to leave. In this case he is said to reneged.

**(iv)** ***Jockey :*** Two or more parallel waiting lines, customers may switch from one to another (i.e) Jockey for position.

***2. Service Pattern of Services***

If the system is empty, the service facility is idle. Service may also be deterministic (or) probabilistic. Service may also be single (or) batch one generally thinks of one customer being served at a time by a given server, but there are many situations where customer may be served simultaneously by the same server. **E.g.:** 1. Computer with parallel processing. 2. People boarding a train.

The service rate may depend on the number of customer waiting for service. A server may work faster if sees that the queue is building up (or) conversely, he may get flustered and became less efficient. The situation in which service depends on the no. of customers waiting is referred to as state dependent service.

***3. Queue Discipline***

(i) First Come First Served (FCFS) or First In First Out (FIFO)
(ii) Last Come First Served (LCFS) or Last In First Out (LCFO)
(iii) Random Selection for Services (RSS)
(iv) Priority (a) Preemptive (b) Non- Preemptive

***(a) Preemptive :*** The customer with the highest priority is allowed to enter service immediately even if a customer with lower priority is already in service when the higher priority customer enters system.

***(b) Non - preemptive :*** The highest priority customer goes to the head the queue but cannot get into service until the customer presently in service is completely, even through this customer has a lower priority.

**4.** ***System Capacity :*** (i) Finite (ii) Infinite

**(i) Finite:** A queue with limited waiting room, so that when the time reaches a certain length, no further customer are allowed to enter until space becomes available by a service completion.

**(ii) Infinite:** A queue with unlimited waiting room.

**5.** ***Number of Service Channels***

(i) Single channel system
(ii) Multiple channel systems.

**(i)** ***Single Channel System***



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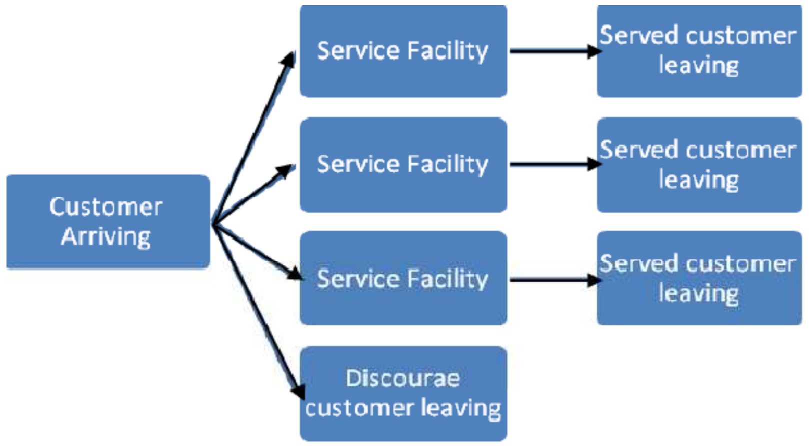
**(ii) Multiple Channel System**



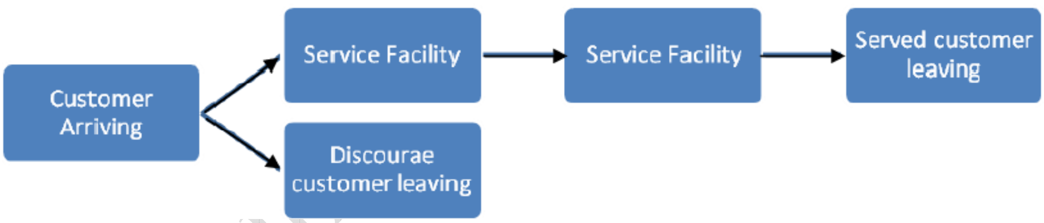

**Eg:** 1. Barber Shop 2. Supermarket 3. Ticket Countrs.

6. **Number of Service Stages :** (i) Single stage (ii) Multiple stage

**(i) Single Stage**


**Eg:** 1. Barber Shop 2. Supermarket 3. Theater.

**(ii) Multiple Stage**




**Eg:** 1. Medical History 2. Bank A/c opening 3.Canteen

***Kendall Notation (A/B/X/Y/Z)***

| | | |
|---|---|---|
| A - Inter arrival time | : | M - Exponential, D - Deterministic, Ek - Erlang type k, Hk – Hyper exponential type k, Ph - phase type, G – General. |
| B - Service time | : | M - Exponential, D - Deterministic, Ek - Erlang type k, Hk - Hyper exponential type k, Ph – phase type, G – General. |
| X - No. of parallel servers | : | 1, 2 …∞ |
| Y - System capacity | : | 1, 2 …∞ |
| Z - Queue discipline | : | FCFS, LCFS, RSS, PR, GD |

***Queuing Models***

1. Probabilistic or stochastic models 2. Deterministic models 3. Mixed models

***Probabilistic model:*** When there is uncertainty in both arrival rate and service rate (i.e. not treated a customer or not know) and are assumed to be random variables.

***Deterministic model***: Both arrival rate and service rate are constants (exactly known).

***Mixed model:*** When either the arrival rate or the service rate is exactly known and the other is not known.

***We look for probabilistic model only***

1. $(M/M/1) : (\infty/FIFO)$

2. $(M/M/1) : (K/FIFO)$



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## ***SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1):(\infty/FIFO)$***

1. $P_0 = 1 - \frac{\lambda}{\mu}$ and $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$ , $\lambda = arrival\ rate,\ \mu = service\ rate$
2. Probability that the system is busy $= 1 - P_0 = \frac{\lambda}{\mu}$
3. Expected number of customers in the system : $L_s = \frac{\lambda}{\mu-\lambda}$
4. Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)}$
5. Expected number of customers in non empty queues: $L_n = \frac{\mu}{\mu-\lambda}$
6. Expected waiting time a customer in the system: $W_s = \frac{1}{\mu-\lambda}$
7. Average waiting time that a customer in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)}$
8. Probability that the number of customers in the system exceeds k : $P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$
9. Probability that the number of customers in the system greater than or equal to k : $P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^{k}$
10. Probability that the waiting time of a customer in the system exceeds t : $P(w > t) = e^{-(\mu-\lambda)t}$
11. Probability that the waiting time of a customer in the queue exceeds t : $P(w > t) = \frac{\lambda}{\mu} e^{-(\mu-\lambda)t}$
12. Probability density function of the waiting time in the system: $f(w) = (\mu - \lambda)e^{-(\mu-\lambda)\,w}$
    Which is the probability density function of an exponential distribution with parameter $\mu - \lambda$.
13. Probability density function of the waiting time in the queue: $g(w) = \begin{cases} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)\,w}, & w > 0 \\ 1 - \frac{\lambda}{\mu}, & w = 0 \end{cases}$

***Little's formula:*** $L_s = \lambda W_s$ , $L_q = \lambda W_q$ , $W_s = W_q + \frac{1}{\mu}$, $L_s = L_q + \frac{\lambda}{\mu}$

## ***SINGLE SERVER MODEL WITH INFINITE SYSTEM CAPACITY - $(M/M/1):(\infty/FIFO)$***

1. ***Customers arrive at a one man barber shop according to a Poisson process with a mean inter arrival time of 20 minutes. Customers spend an average of 15 minutes in the barber chair. If an hour is used as the unit of time, then***
   - ***(i) What is the probability that a customer need not wait for a hair cut?***
   - ***(ii) What is the expected number of customers in the barber shop and in the queue?***
   - ***(iii) How much time can a customer expect to spend in the barbershop?***
   - ***(iv) Find the average time that a customer spends in the queue?***
   - ***(v) What is the probability that there will be more than 6 customers?***
   - ***(vi) What is the probability that there will be 6 or more customers waiting for service?***
   - ***(vii) What is the probability that the waiting time in the (a) system (b) queue, is > 12 minutes?***

***Solution:*** $\frac{1}{\lambda} = \frac{20}{30} = \frac{1}{3} \Longrightarrow \boldsymbol{\lambda = 3}$ customers/hour, $\frac{1}{\mu} = \frac{15}{60} = \frac{1}{4} \Longrightarrow \boldsymbol{\mu = 4}$ customers/hour.

(i) $P(a\ customer\ need\ not\ wait) = P(no\ customer\ in\ the\ system)$: $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{3}{4} = 0.25$

(ii) Expected number of customers in the barber shop : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{3}{4-3} = 3$ customers

Expected number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{3^2}{4(4-3)} = \frac{9}{4} = 2.25$ customers

(iii) Expected time a customer spend in the barbershop: $W_s = \frac{1}{\mu-\lambda} = \frac{1}{4-3} = 1$ hour

(iv) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{3}{4(4-3)} = \frac{3}{4} = 0.75$ hour

(v) The probability that there will be more than 6 customers:

$$P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1} \Rightarrow P(n > 6) = \left(\frac{3}{4}\right)^{6+1} = 0.1335$$





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(vi) The probability that there will be 6 or more customers waiting for service:

$$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k \Rightarrow P(n \geq 6) = \left(\frac{3}{4}\right)^6 = 0.1779$$

(vii) The probability that the waiting time in the system is greater than 12 minutes?

$$P(w > t) = e^{-(\mu-\lambda)t} \Rightarrow P(w > 12) = e^{-(4-3)\times\frac{12}{60}} = e^{-0.2} = 0.8187$$

(viii) The probability that the waiting time in the queue, is greater than 12 minutes?

$$P(w > t) = \frac{\lambda}{\mu}\, e^{-(\mu-\lambda)t} \Rightarrow P(w > 12) = \left(\frac{3}{4}\right) e^{-(4-3)\times\frac{12}{60}} = \left(\frac{3}{4}\right) e^{-0.2} = 0.61405$$

**2.** ***If People arrive to purchase cinema tickets at the average rate of 6 per minute at a one man counter, and it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket,***

***(i) can he expect to be seated for the start of the picture?***

***(ii) What is the probability that he will be seated for the start of the picture?***

***(iii) How early must he arrive in order to be 99% sure of being seated for the start of the picture?***

***Solution:*** $\lambda = 6$ /minute, $\mu = 8$ /minute

(i) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{8-6} = \frac{1}{2} = 0.5$ minute

E(total time required to purchase the ticket and to reach the seat) $= 0.5 + 1.5 = 2$ min

(ii) $P(\text{total time } < 2 \text{ minute}) = P(w < t) = 1 - P(w > t) = 1 - e^{-(\mu-\lambda)t}$

$$P\left(w < \frac{1}{2}\right) = 1 - e^{-(8-6)\times\frac{1}{2}} = 1 - e^{-1} = 0.63$$

(iii) $P(w < t) = 99\% = 0.99 \Rightarrow 1 - P(w > t) = 0.99 \Rightarrow P(w > t) = 0.01 \Rightarrow e^{-(\mu-\lambda)t} = 0.01 \Rightarrow$

$e^{-(8-6)\,t} = 0.01 \Rightarrow e^{-2\,t} = 0.01 \Rightarrow -2t = \ln(0.01) \Rightarrow -2t = -4.6 \Rightarrow t = 2.3$ minute

$P(\text{ticket purchasing time} < 2.3) = 0.99$

$P[\text{total time to get the ticket and to go to the seat} < (2.3 + 1.5)] = 0.99$

∴ The person must arrive at least 2.64 minutes early so as to be 99% sure of seeing the start of the picture.

**3.** ***The arrivals at the counter in a bank occur in accordance with a Poisson process at an average rate of 8 per hour. The duration of service of a customer has an exponential distribution with a mean of 6 minutes. Find the probability that an arriving customer (i) Has to wait (ii) Finds 4 customers in the system (iii) Has to spend less than 15 minutes in the bank.***

***Solution:*** $\lambda = 8$ / hour, $\mu = \frac{1}{6}$/minute $= 10$/hour

(i) Probability that a customer has to wait = Probability that the system is busy $= \frac{\lambda}{\mu} = \frac{8}{10} = 0.8$

(ii) Probability that there are 4 customers in the system $= P_4 = \left(\frac{8}{10}\right)^4 \left(1 - \frac{8}{10}\right) = 0.08192$

(iii) Probability that a customer has to spend less than 15 minutes in the bank: $P(w > t) = e^{-(\mu-\lambda)t}$

$$P(W_s < 15 \text{ minutes}) = P\left(W_s < \frac{1}{4}\ \text{hour}\right) = 1 - P\left(W_s > \frac{1}{4}\ \text{hour}\right) = 1 - e^{-(10-8)\left(\frac{1}{4}\right)} = 0.3935$$

**4.** ***Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes and the service time is an exponential random variable with mean 8 minutes.***

***(i) Find the average number of customers in the shop.***

***(ii) Find the average time a customer spends in the shop.***

***(iii) Find the average number of customers in the queue.***

***(iv) What is the probability that the server is idle?***

***Solution:*** $\lambda = 6$ /hour, $\mu = \frac{60}{8} = \frac{15}{2}$ /hour.

(i) The average number of customers in the shop : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{6}{\frac{15}{2}-6} = 4$ customers

(ii) The average time a customer spends in the shop : $W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{15}{2}-6} = \frac{2}{3}$ hour



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(iii) The average number of customers in the queue: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{6^2}{\frac{15}{2}\left(\frac{15}{2}-6\right)} = \frac{16}{5}$

(iv) $P(\text{system is empty})$: $P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{\left(\frac{15}{2}\right)} = \frac{1}{5}$

***5. A departmental store has a single cashier. During the rush hours, customers arrive at the rate of 20 customers per hour. The average number of customers that can be processed by the cashier is 24 per hour. Calculate the following***

***(i) What is the probability that the cashier is idle?***

***(ii) What is the average number of customers in the queueing system.***

***(iii) What is the average time a customer spends in the system.***

***(iv) What is the average number of customers in the queue?***

***(v) What is the average time a customer spends in the queue, waiting for service?***

***Solution:*** $\lambda = 20$ /hour, $\mu = 24$ /hour.

(i) $P(\text{cashier is idle}) = P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{20}{24} = 0.1674$

(ii) Average number of customers in the queueing system : $L_s = \frac{\lambda}{\mu-\lambda} = \frac{20}{24-20} = 5$ customers

(iii) Average time a customer spend in the system: $W_s = \frac{1}{\mu-\lambda} = \frac{1}{24-20} = \frac{1}{4}$ hour

(iv) Average number of customers in the queueing system : $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{20^2}{24(24-20)} = 4.167$ customers

(v) Average time that a customer spends in the queue: $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{20}{24(24-20)} = \frac{20}{96}$ hour

## ***SINGLE SERVER MODEL WITH FINITE SYSTEM CAPACITY - $(M/M/1):(k/FIFO)$***

$$P_0 = \begin{cases} \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \frac{1}{k+1}, & \lambda = \mu \end{cases} \quad \textit{and} \quad P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0, & \lambda \neq \mu \\ \frac{1}{k+1}, & \lambda = \mu \end{cases}$$

Average number of customers in the system : $L_s = \begin{cases} \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}}, & \lambda \neq \mu \\ \frac{k}{2}, & \lambda = \mu \end{cases}$

Average number of customers in the queue : $L_q = L_s - \frac{\lambda'}{\mu}$

Effective arrival rate : $\lambda' = \mu(1 - P_0)$

Average waiting time of a customers in the system : $W_s = \frac{L_s}{\lambda'}$

Average waiting time of a customers in the queue : $W_q = \frac{L_q}{\lambda'}$

## ***SINGLE SERVER MODEL WITH FINITE SYSTEM CAPACITY - $(M/M/1):(k/FIFO)$***

***1. Patients arrive at a clinic according to Poisson distribution at a rate of 60 patients per hour. The waiting room does not accommodate more than 14 patients. Investigation time per patient is exponential with mean rate of 40 per hour.***

***(i) Determine the effective arrival rate at the clinic.***

***(ii) What is the probability that an arriving patient will not wait?***

***(iii) What is the expected waiting time until a patient is discharged from the clinic?***

***Solution:*** $\lambda = 60$ patients/hr, $\mu = 40$ patients/hr, $k = 14 + 1 = 15$ (14 waiting patients + 1 patient under investigation)

(i) Effective arrival rate $\lambda' = \mu(1 - P_0)$, Where $P_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]}$, $\lambda \neq \mu$

$P_0 = \frac{\left(1-\frac{60}{40}\right)}{1-\left(\frac{60}{40}\right)^{15+1}} = 0.0007624$ , $\lambda' = 40(1 - 0.0007624) = 39.9695$ per hour.



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(ii) $P(\text{a patient will not wait}) = P_0 = 0.0007624$

(iii) Expected waiting time until a patient is discharged from the clinic: $W_s = \frac{L_s}{\lambda'}$

$$L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{60}{40-60}\right) - \frac{(15+1)\left(\frac{60}{40}\right)^{15+1}}{1-\left(\frac{60}{40}\right)^{15+1}} = 13 \text{ patients}, \qquad W_s = \frac{14}{39.9695} = 0.3203 \text{ hour}$$

2. ***At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady state probabilities for the number of trains in the system. Also find the average waiting time of a new train coming into the yard. If the handling rate is reduced to half, what is the effect of the above results?***

***Solution: Case (i):*** $\lambda = 6$ trains/hr, $\mu = 12$ trains/hr, $k = 2 + 1 = 3$

Steady state probabilities for the number of trains in the system $= P_1, P_2 \text{ and } P_3$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0, \text{ Where } P_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1-\frac{6}{12}\right)}{1-\left(\frac{6}{12}\right)^4} = 0.5333, \ \lambda \neq \mu$$

$$P_1 = \left(\frac{6}{12}\right)^1 (0.5333) = 0.2667, \ \ P_2 = \left(\frac{6}{12}\right)^2 (0.5333) = 0.1333, \ \ P_3 = \left(\frac{6}{12}\right)^3 (0.5333) = 0.0667$$

Average waiting time of a new train coming in the yard : $W_s = \frac{L_s}{\lambda'}$

$\lambda' = \mu(1 - P_0) = 12(1 - 0.5333) = 5.6004$

$$L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{6}{12-6}\right) - \frac{(3+1)\left(\frac{6}{12}\right)^{3+1}}{1-\left(\frac{6}{12}\right)^{3+1}} = 0.7333 \text{ train}, \ W_s = \frac{0.7333}{5.6004} = 0.1309 \text{ hour}$$

***Case (ii):*** If the handling rate is reduced to half, then $\lambda = 6$ trains/hr, $\mu = 6$ trains/hr, $k = 2 + 1 = 3$

Steady state probabilities for the number of trains in the system $= P_1, P_2 \text{ and } P_3$

$$P_n = \frac{1}{k+1}, \ \ P_1 = \frac{1}{3+1} = \frac{1}{4}, \ \ P_2 = \frac{1}{4}, \ \ P_3 = \frac{1}{4}$$

Average waiting time of a new train coming in the yard : $W_s = \frac{L_s}{\lambda'}$

$$P_0 = \frac{1}{k+1} = \frac{1}{4}, \ \lambda' = \mu(1 - P_0) = 6\left(1 - \frac{1}{4}\right) = 4.5, \ L_s = \frac{k}{2} = \frac{3}{2} = 1.5 \text{ train}, \ W_s = \frac{1.5}{4.5} = 0.3333 \text{ hours}$$

3. ***A petrol pump with only one pump can accommodate 5 cars. The arrival of cars is Poisson with a mean rate of 10 per hour. The service time is exponentially distributed with a mean 2 minutes. How many cars are in the petrol pump on an average? What is the probability of a newly arriving customer finding the system full and leaving without availing service?***

***Solution:*** $\lambda = 10$/hr, $\mu = 30$/hr, $k = 5$ , $P_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1-\frac{10}{30}\right)}{\left[1-\left(\frac{10}{30}\right)^{5+1}\right]} = 0.667, \ \lambda \neq \mu$

$$L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{10}{30-10}\right) - \frac{(5+1)\left(\frac{10}{30}\right)^{5+1}}{1-\left(\frac{10}{30}\right)^{5+1}} = 0.492$$

$P(\text{System full}) = P(5 \text{ cars in the system}) = \left(\frac{\boldsymbol{\lambda}}{\boldsymbol{\mu}}\right)^{\mathbf{5}} \boldsymbol{P_0} = \left(\frac{10}{30}\right)^{\mathbf{5}} (0.667) = 0.00274.$

4. ***A one person barber shop has 6 chairs to accommodate people waiting for a hair cut. Assume that customers who arrive when all the 6 chairs are full leave without entering the barber shop. Customers arrive at the average rate of 3 per hour and spend an average of 15 minutes in the barber chair.***
   ***(i) What is the probability that a customer can get directly into the barber chair upon arrival?***
   ***(ii) What is the expected number of customers waiting for a hair cut?***
   ***(iii) How much time can a customer expect to spend in the barber shop?***
   ***(iv) What fraction of potential customers are turned away?***

   ***Solution:*** $\lambda = 3$/hr, $\mu = 4$/hr, $k = 6 + 1 = 7$





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$$P_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} = \frac{\left(1-\frac{3}{4}\right)}{\left[1-\left(\frac{3}{4}\right)^{7+1}\right]} = 0.2778, \;\; \lambda \neq \mu$$

Effective arrival rate $\lambda' = \mu(1 - P_0) = 4(1 - 0.2778) = 2.89/hour$

(i) Probability of empty system = $P_0 = 0.2778$

(ii) $L_s = \left(\frac{\lambda}{\mu-\lambda}\right) - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} = \left(\frac{3}{4-3}\right) - \frac{(7+1)\left(\frac{3}{4}\right)^{7+1}}{1-\left(\frac{3}{4}\right)^{7+1}} = 2.11$

$L_q = L_s - \frac{\lambda'}{\mu} = 2.11 - \frac{2.89}{4} = 1.3875$

(iv) $W_s = \frac{L_s}{\lambda'} = \frac{2.11}{2.89} = 0.7301/hour$

(v) $P(a\ customer\ is\ turned\ away) = P(system\ is\ full) = P_7 = \left(\frac{\lambda}{\mu}\right)^7 P_0 = \left(\frac{3}{4}\right)^7 (0.2778) = 0.037$

Hence 3.7% of potential customers are turned away.

# ***UNIT – 5***
# ***MARKOV CHAINS***

***Syllabus***

- Introduction to Stochastic process, Markov process, Markov chain one step & n-step Transition Probability.
- Transition Probability Matrix and Applications
- Chapman Kolmogorov theorem (Statement only) – Applications.
- Classification of states of a Markov chain – Applications

## ***INTRODUCTION***

***Random Processes or Stochastic Processes :*** A random process is a collection of random variables $\{X(s,\ t)\}$ which are functions of a real variable $t$ (time). Here $s \in S$ (sample space) and $t \in T$ (index set) and each $\{X(s,\ t)\}$ is a real valued function. The set of possible values of any individual member is called state space.

***Classification :*** Random processes can be classified into 4 types depending on the continuous or discrete nature of the state space S and index set T.

1. Discrete random sequence : If both S and T are discrete
2. Discrete random process : If S is discrete and T is continuous
3. Continuous random sequence : If S is continuous and T is discrete
4. Continuous random process : If both S and T are continuous.

***Markov Process***

If, for $t_1 < t_2 < t_3 < \cdots < t_n$, we have $P\{X(t) \leq x/X(t_1) = x_1, X(t_2) = x_2 \ldots X(t_n) = x_n\} = P\{X(t) \leq x/X(t_n) = x_n\}$ then the process $\{X(t)\}$ is called a Markov process. That is, if the future behaviour of the process depends only on the present state and not on the past, then the random process is called a Markov process.

***Markov Chain :*** If, for all $n$, $P\{X_n = a_n/X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2} \ldots X_0 = a_0\} = P\{X_n = a_n/X_{n-1} = a_{n-1}$ then the process $X_n; n=0,1,2\ldots$ is called a Markov chain.

***One Step Transition Probability :*** The conditional probability $P_{ij}(n-1, n) = P\left(X_n = a_j/X_0 = a_i\right)$ is called one step transition probability from state $a_i$ to state $a_j$ in the $n^{th}$ step.

***Homogeneous Markov Chain :*** If the one step transition probability does not depend on the step. That is, $\boldsymbol{P_{ij}(n-1, n) = P_{ij}(m-1, m)}$ the Markov chain is called a homogeneous markov chain or the chain is said to have stationary transition probabilities**.**





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