PROBABILITY
&
QUEUEING THEORY

(As per SRM UNIVERSITY Syllabus)

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15MA207 – PROBABILITY AND QUEUEING THEORY

UNIT – I : RANDOM VARIABLES

Syllabus

- Review of Probability Concepts - Types of Events, Axioms
- Conditional Probability, Multiplication Theorem, Applications
- Discrete Random Variable
- Continuous Random Variable
- Expectation and Variance
- Higher Order Moments
- Moment Generating Function
- Function of Random Variable (One Dimensional Only)
- Chebychev's Inequality

PROBABILITY

Probability (or) Chance: Probably, Chances, Likely, Possible - The terms convey the same meaning.

Example:

1. ***Probably*** your method is correct
2. The ***chances*** of getting ranks Ram and Gothai are equal.
3. It is ***likely*** that Ram may not come for taking his classes today.
4. It is ***possible*** to reach the college by 8.30am.

Ordinary Language: The word probability means uncertainty about happening.

Mathematics or Statistics : A numerical measure of uncertainty is practiced by the important branch of statistics is called the **Theory of Probability.**

Day to Day Life:

- ***Certainty*** - Every day the sun rises in the east
- ***Impossibility*** - It is possible to live without water
- ***Uncertainty*** - Probably Raman gets that job.

In the theory of probability, we represent certainty by 1, impossibility by 0 and uncertainty by a positive fraction which lies between 0 and 1.

Applications : There is no area in **social, physical (or) natural sciences** where the probability theory is not used.

- It is the base of the fundamental laws of statistics.
- It gives solutions to betting of games.
- It is extensively used in business situations characterized by uncertainty.
- It is essential tool in statistical inference and forms the basis of the Decision Theory.

Random Experiment (or) Trial and Event (or) Cases:

Consider an experiment of throwing a **coin**. When tossing a coin, we may get a head or tail. Here tossing of a coin is a **trial** and getting a head or tail is an **event.**

Throwing of a **die** is a trial and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

Favourable Events : The number of outcomes favourable to an event in an experiment is the number of outcomes which entail the happening of the event.

Example: In tossing 2 coins the cases favourable to the event of getting a head are HT, TH, and HH.

Exhaustive Events : The total number of possible outcomes in any **trial** is known as exhaustive events.

Example: In tossing a coin the possible outcomes are getting a head or tail. Hence we have 2 exhaustive events in throwing a coin.

Mutually Exclusive Event : Two events are said to be mutually exclusive when the occurrence of one affects the occurrence of the other. In other words, if A & B are mutually exclusive events and if A happens then B will not happen and vice versa. ***Example:*** In tossing a coin the events head or tail are mutually exclusive, since both tail & head cannot appear in the same time.

Equally Likely Events: Two events are said to be equally likely if one of them cannot be expected in preference to the other. ***Example:*** In tossing a coin, head or tail are equally likely events.

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Independent Event : Two events are said to be independent when the actual happening of one does not influence in any way the happening of the other. ***Example :*** In tossing a coin, the event of getting a head in the 1^{st} toss is independent of getting a head in the 2^{nd} toss, 3^{rd} toss, etc.

Mathematical Definition of Probability

If P is the notation for probability of happening of the event, then $P(A) = \frac{Number\ of\ Favourable\ Cases}{Total\ Number\ of\ Exhaustive\ Cases} = \frac{m}{n}$

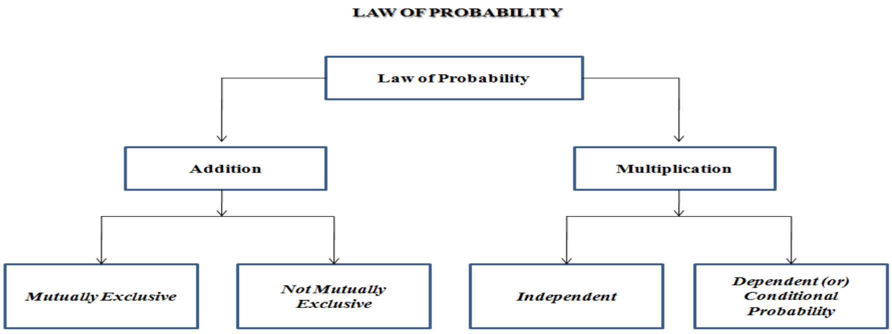
Statistical Definition of Probability

If in n trials, an event E happens m times, then $P(E) = \lim_{n\to\infty} \frac{m}{n}$

Axiomatic Definition of Probability

1. For any event A, $P(A) \geq 0$.
2. $P(S) = 1$
3. If $A_1, A_2, A_3, \dots, A_n$ are finite number of disjoint events of S, then
   $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \cdots = \sum P(A_i)$

LAW OF PROBABILITY



ADDITION LAW OF PROBABILITY

Case (i): When events are mutually exclusive

If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Case (ii): When events are not mutually exclusive

If A and B are any two events, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

THEOREM : ADDITION LAW OF PROBABILITY

If A and B are any two events and are not disjoint, then $\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Proof : $A \cup B = A \cup (\bar{A} \cap B)$ Since A and $(\bar{A} \cap B)$ are disjoint,

$$P(A \cup B) = P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B) = P(A) + P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B)$$

$$= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B) = P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$

$$\boldsymbol{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

MULTIPLICATION LAW OF PROBABILITY

Case (i): When events are independent : The probability that both independent events, A and B will occur is equal to product of the probabilities of each event, then $P(A \cap B) = P(A)\,P(B)$.

Case (ii): When events are dependent (or) conditional probability: If the occurrence of an event A is affected by the occurrence of the another event B, then the events A and B are dependent. $P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)$

THEOREM : MULTIPLICATION LAW OF PROBABILITY

For two events A and B, $\boldsymbol{P(A \cap B) = P(A)\,P(B/A) = P(B)\,P(A/B)}$, $\boldsymbol{P(A) > 0, P(B) > 0}$ where $\boldsymbol{P(B/A)}$ represents the conditional probability of occurrence of B when the event A has already happened and $\boldsymbol{P(A/B)}$ is the conditional probability of happening A, given that B has already happened.

Proof: $P(A) = \frac{n(A)}{n(S)}$, $P(B) = \frac{n(B)}{n(S)}$, $P(A \cap B) = \frac{n(A\cap B)}{n(S)}$

For the conditional event A/B, the favorable outcomes must be one of the sample points of B, that is for the event A/B, the sample space is B and out of the $n(B)$of sample points, $n(A \cap B)$ pertain to the occurrence of the event A.

$P(A \cap B) = \frac{n(B)}{n(S)} \frac{n(A\cap B)}{n(B)} = P(B)\,P(A/B)$. Similarly we can prove, $P(A \cap B) = \frac{n(A)}{n(S)} \frac{n(A\cap B)}{n(A)} = P(A)\,P(B/A)$

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THEOREMS ON PROBABILITY

1. ***The probability of the impossible event is zero, i.e., if $\emptyset$ is the subset containing no sample point, $P(\emptyset) = 0$.***

Proof : The certain event S and the impossible event $\emptyset$ are mutually exclusive.

$P(S \cup \emptyset) = P(S) + P(\emptyset)$ ***(Axiom iii)***

$P(S) = P(S) + P(\emptyset)$ $\because \boldsymbol{S \cup \emptyset = S}$

$P(\emptyset) = P(S) - P(S)$

$\mathbf{P(\emptyset) = 0}$

2. ***If $\bar{A}$ is the complementary event of A, $P(\bar{A}) = 1 - P(A) \le 1$.***

Proof : A and $\bar{A}$ are mutually exclusive events, such that $A \cup \bar{A} = S$.

$P(A \cup \bar{A}) = P(S)$ ***(Axiom iii)***

$P(A) + P(\bar{A}) = P(S)$

$P(A) + P(\bar{A}) = 1$ ***(Axiom ii)***

$P(\bar{A}) = 1 - P(A)$ $\because \mathbf{P(A) \ge 0}$

$\mathbf{P(\bar{A}) \le 1}$

3. ***If $B \subset A, P(B) \le P(A)$***

Proof : B and $A\bar{B}$ are mutually exclusive events, such that $B \cup A\bar{B} = A$.

$P(B \cup A\bar{B}) = P(A)$ ***(Axiom iii)***

$P(B) + P(A\bar{B}) = P(A)$ $\because \boldsymbol{P(A\bar{B}) \ge 0}$

$\boldsymbol{P(B) \le P(A)}$

4. ***If A and B are independent events, prove that***

(i) $\bar{A}$ & B are independent (ii) A & $\bar{B}$ are independent (iii) $\bar{A}$ & $\bar{B}$ are independent.

Proof:

(i) The event $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events, such that $(A \cap B) \cup (\bar{A} \cap B) = B$.

$P\{(A \cap B) \cup (\bar{A} \cap B)\} = P(B)$ ***(Addition Theorem)***

$P(A \cap B) + P(\bar{A} \cap B) = P(B)$

$P(\bar{A} \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\,P(B)$ ***(Multiplication Theorem)***

$= P(B)[1 - P(A)]$

$\mathbf{P(\bar{A} \cap B) = P(\bar{A})\,P(B)}$

(ii) The event $A \cap B$ and $\bar{B} \cap A$ are mutually exclusive events, such that $(A \cap B) \cup (\bar{B} \cap A) = A$.

$P\{(A \cap B) \cup (\bar{B} \cap A)\} = P(A)$ ***(Addition Theorem)***

$P(A \cap B) + P(\bar{B} \cap A) = P(A)$

$P(\bar{B} \cap A) = P(A) - P(A \cap B) = P(A) - P(A)\,P(B)$ ***(Multiplication Theorem)***

$= P(A)[1 - P(B)]$

$\mathbf{P(\bar{B} \cap A) = P(\bar{B})\,P(A)}$

(iii) A and B are mutually exclusive events.

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$ **(Addition Theorem)**

$= 1 - [P(A) + P(B) - P(A \cap B)]$ **(Multiplication Theorem)**

$= 1 - [P(A) + P(B) - P(A)\,P(B)] = 1 - P(A) - P(B) + P(A)\,P(B)$

$= [1 - P(A)] - P(B)[1 - P(A)] = [1 - P(A)]\,[1 - P(B)]$

$\mathbf{P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})}$

PROBLEMS IN PROBABILITY

1. ***A box contains 4 red, 5 white & 6 black balls. What is the probability that 2 balls drawn are red & black?***

Solution : $P(red\ \&\ black) = \frac{4C_1 \times 6C_1}{15C_2} = \frac{8}{35} = 0.2286$

2. ***From a group of 3 Indians, 4 Pakistanis and 5 Americans, a subcommittee of 4 people is selected by lots. Find the probability that the subcommittee will consist of***

(i) 2 Indians and 2 Pakistanis (ii) 1 Indian, 1 Pakistani and 2 Americans (iii) 4 Americans.

Solution :

(i) $P(\text{2 Indians and 2 Pakistanis}) = \frac{3C_2 \times 4C_2}{12C_4}$

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(ii) $\mathrm{P(1\ Indian, 1\ Pakistani\ and\ 2\ Americans\ )} = \frac{3C_1 \times 4C_1 \times 5C_2}{12C_4}$

(iii) $\mathrm{P(4\ Americans)} = \frac{5C_4}{12C_4}$

3. ***A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that (i) Both are good (ii) Both have major defects (iii) At least 1 is good (iv) At most 1 is good (v) Exactly 1 is good (vi) Neither has major defects (vii) Neither is good***

Solution

(i) $\mathrm{P(Both\ are\ good\ )} = \frac{10C_2}{16C_2} = \frac{3}{8}$

(ii) $\mathrm{P(Both\ have\ major\ defects\ )} = \frac{2C_2}{16C_2} = \frac{1}{120}$

(iii) $\mathrm{P(At\ least\ 1\ is\ good)} = \frac{(10C_1 \times 6C_1) + 10C_2}{16C_2} = \frac{7}{8}$

(iv) $\mathrm{P(At\ most\ 1\ is\ good)} = \frac{(10C_0 \times 6C_2) + (10C_1 \times 6C_1)}{16C_2} = \frac{5}{8}$

(v) $\mathrm{P(Exactly\ 1\ is\ good)} = \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$

(vi) $\mathrm{P(Neither\ has\ major\ defects)} = \mathrm{P(both\ are\ non\ major\ defectives)} = \frac{14C_2}{16C_2} = \frac{91}{120}$

(vii) $\mathrm{P(Neither\ is\ good)} = \mathrm{P(both\ are\ defective)} = \frac{6C_2}{16C_2} = \frac{1}{8}$

4. ***Four cards are drawn at random from a well shuffled pack of cards. Find the probability that (i) All the 4 are queens (ii) There is one card from each suit (iii) 3 cards are diamonds and 1 spade (iv) All the 4 cards are hearts, and one of them is a jack.***

Solution

(i) $\mathrm{P}(All\ the\ 4\ are\ queens) = \frac{4C_4}{52C_4} = 0.0000037$

(ii) $\mathrm{P}(one\ card\ from\ each\ suit) = \frac{13C_1 \times 13C_1 \times 13C_1 \times 13C_1}{52C_4} = 0.1055$

(iii) $\mathrm{P}(3\ cards\ are\ diamonds\ and\ 1\ spade) = \frac{13C_3 \times 13C_1}{52C_4} = 0.0137$

(iv) $\mathrm{P}(All\ the\ four\ cards\ are\ hearts, and\ one\ of\ them\ is\ a\ jack.) = \frac{1C_1 \times 12C_3}{52C_4} = 0.00081$

5. ***Two dice are thrown. Find the probability that (i) The total of the numbers on the top faces is 9 (ii) The top face numbers are same (iii) The sum of the numbers on the top faces is less than 7.***

Solution $n(S) = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$

(i) $\mathrm{P(total\ of\ the\ numbers\ on\ the\ top\ faces\ is\ 9\ )} = \frac{4}{36} = 0.111$

(ii) $\mathrm{P(top\ face\ numbers\ are\ same)} = \frac{6}{36} = 0.1667$

(iii) $\mathrm{P(sum\ of\ the\ numbers\ on\ the\ top\ faces\ is} < 7) = \frac{15}{36} = 0.4167$

PROBLEMS IN ADDITION LAW OF PROBABILITY

1. ***The probability that a company director will travel by train is $\frac{1}{5}$ and by bus is $\frac{2}{3}$. What is the probability of his travelling by train or bus?***

Solution: Let A – Travelling by train, B – Travelling by Bus, $P(A) = \frac{1}{5}$, $P(B) = \frac{2}{3}$

$P(A \cup B) = P(A) + P(B)$ ***($\because$ A and B are mutually exclusive)***

$P(train\ or\ bus) = P(A \cup B) = \frac{1}{5} + \frac{2}{3} = \frac{13}{15} = 0.867$

2. ***A is known to hit the target in 2 out of 5 shots. B is known to hit the target in 3 out of 4 shots. Find the probability of the target being hit when both try?***

Solution : $P(A) = \frac{2}{5},\ \ P(B) = \frac{3}{4},\ \ P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***($\because$ A & B are not mutually exclusive)***

$P(A \cap B) = P(A)\ P(B) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$ ***($\because$ A and B are independent)***

$P(A \cup B) = \frac{2}{5} + \frac{3}{4} - \frac{3}{10} = \frac{17}{20} = 0.85$

3. ***If the probability is 0.30 that a teaching job applicant has a P.G. degree, 0.70 for his work experience and 0.2 for both, out of 300 applicants, how many will have either a P.G. degree or work experience?***

Solution : $P(A) = 0.30,\ \ P(B) = 0.70, P(A \cap B) = 0.20$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***($\because$ A and B are not mutually exclusive)***

$P(A \cup B) = 0.3 + 0.7 - 0.2 = 0.8$

4. ***If $A, B, \& C$ are any 3 events such that $\boldsymbol{P(A) = P(B) = P(C) = \frac{1}{4}, P(A \cap B) = P(B \cap C) = P(A \cap B \cap C) = 0, P(C \cap A) = \frac{1}{8}}$. Find the probability that at least 1 of the events $A, B,$ and C occurs.***

Solution : $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8} = 0.625$

PROBLEMS IN MULTIPLICATION LAW OF PROBABILITY

1. ***Two persons A & B appear in an interview for two vacancies for the same post. The probability of A selection is 1/7 & that of B selection is 1/5. What is the probability that (i) Both of them (ii) None of them will be selected.***

Solution: $\mathrm{P(A)} = \frac{1}{7},\ \ \mathrm{P(B)} = \frac{1}{5},\ \ \mathrm{P}(\overline{\mathrm{A}}) = 1 - \frac{1}{7} = \frac{6}{7},\ \ \mathrm{P}(\overline{\mathrm{B}}) = 1 - \frac{1}{5} = \frac{4}{5}$

(i) $P(A \cap B) = P(A)\ P(B) = \frac{1}{7} \times \frac{1}{5} = \frac{1}{35}$ ***($\because$ A and B are independent)***

(ii) $P(\overline{\mathrm{A}} \cap \overline{\mathrm{B}}) = \mathrm{P}(\overline{\mathrm{A}})\ \mathrm{P}(\overline{\mathrm{B}}) = \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$ ***($\because$ $\overline{A}$ and $\overline{B}$ are independent)***

2. ***If the probability that a communication system will have high fidelity is 0.81 and the probability that it will have high fidelity and selectivity is 0.18. What is the prob. that a system with high fidelity will also have selectivity?***

Solution: Let A – selectivity and B – fidelity, $\mathrm{P(B)} = 0.81, \mathrm{P(A \cap B)} = 0.18$

$\mathrm{P(A/B)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(B)}} = \frac{0.18}{0.81} = \frac{2}{9} = 0.222$

3. ***A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the prob. that the other one is also good?***

Solution : Let A − One of the tubes drawn is good & B − Other tube is good

$\mathrm{P(A)} = \frac{6C_1}{10C_1} = \frac{3}{5},\ \ \mathrm{P(A \cap B)} = \frac{6C_2}{10C_2} = \frac{1}{3}$, Using Conditional Probability, $\mathrm{P(B/A)} = \frac{\mathrm{P(A \cap B)}}{\mathrm{P(A)}} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{3}{5}\right)} = \boldsymbol{\frac{5}{9}}$

RANDOM VARIABLE

The outcomes of many random experiments may be non-numerical. It is inconvenient to deal with these descriptive outcomes mathematically.

Example: When toss a coin we get two outcomes, namely head or tail. We can assign numerical values; say 1 to head and 0 to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

Example: Three students sat for an examination & X denotes the number of students who passed. Describe the RV X.

Sample Space S	None	S_1	S_2	S_3	S_1S_2	S_2S_3	S_3S_1	$S_1S_2S_3$
No. of Students who passed X	0	1	1	1	2	2	2	3

$n(S) = 8,\quad P(X = 0) = \frac{1}{8},\quad P(X = 1) = \frac{3}{8},\quad P(X = 2) = \frac{3}{8},\quad P(X = 3) = \frac{1}{8}$

TYPES OF RANDOM VARIABLE

Random Variable

Discrete Random Variable | **Continuous Random Variable**

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DISCRETE RANDOM VARIABLE

A random variable X is discrete, if it assumes only finite number or countably infinite number of values.

Example

- *The mark obtained by a student in an examination. It's possible values are 0, 85 or 100.*
- *The number of students who are absent for a particular period.*

1. Probability Mass Function (p.m.f.) $\sum_{i=1}^{\infty} P(x_i) = 1$
2. Mean $E(X) = \sum_{i=1}^{\infty} x_i\, P(x_i)$, $E(X^2) = \sum_{i=1}^{\infty} {x_i}^2\, P(x_i)$
3. Variance $V(X) = E(X^2) - [E(X)]^2$
4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \sum_{i=1}^{x} P(x_i)$

CONTINUOUS RANDOM VARIABLE

A random variable X is continuous, if it takes all possible values between certain limits or in an interval which may be finite or infinite.

Example:

- *The density of milk taken for testing at a farm.*
- *The operating time between two failures of a computer.*

1. Probability Density Function (p.d.f.) $\int_{-\infty}^{\infty} f(x)dx = 1$
2. Mean $E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx$
3. Variance $V(X) = E(X^2) - [E(X)]^2$
4. Cumulative Distribution Function (c.d.f.) $F(X) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

PROPERTIES OF EXPECTATION

If X and Y are random variables and a, b are constants, then

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$
4. $E(X - \bar{X}) = 0$
5. $|E(X)| \le E(|X|)$
6. $E(X) \ge 0, if\ X \ge 0$
7. $E(X + Y) = E(X) + E(Y)$ ***(Additive Theorem)***
8. $E(XY) = E(X)E(Y)$ ***(∵ A and B are independent)***
9. $E\big(a\, g(X)\big) = aE\big(g(X)\big)$
10. $E(g(X) + a) = E\big(g(X)\big) + a$
11. $(E[g(X)\,]) = g[E(X)]$ $[\boldsymbol{g(X)\ is\ linear\ in\ X}]$
12. $P(X \ge a) \le \frac{E(X)}{a}, a > 0$ ***(Markov Inequality)***
13. $P\{|X - E(X)| \ge k\} \ge \frac{\sigma_X^2}{\epsilon^2}$ ***(Chebyshev's Inequality)***

PROPERTIES OF VARIANCE

1. $Var(X) \ge 0$
2. $E(X^2) \ge [E(X)]^2$
3. $Var(b) = 0, b\ constant$
4. If X is a random variables, a is constants then $Var(aX) = a^2 Var(X)$
5. If a and b are constants, $Var(aX \pm b) = a^2 Var(X)$
6. If X and Y are two independent RV, a and b are constants then $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$

PROPERTIES OF CUMULATIVE DISTRIBUTION FUNCTION

1. If F is the distribution function of the RV X and if $a < b$, then $P(a < X \le b) = F(b) - F(a)$
2. If F is the distribution function of one dimensional RV X, then (i) $0 \le F(X) \le 1$ (ii) $F(X) \le F(Y)$, if $x < y$
   In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.
3. If F is the distribution function of one dimensional random variable X, then
   $F(-\infty) = \lim_{x \to -\infty} F(X) = 0$ and $F(\infty) = \lim_{x \to \infty} F(X) = 1$

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4. $P(a < X \leq b) = F(b) - F(a)$
5. $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$
6. $P(a < X < b) = F(b) - F(a) - P(X = b)$
7. $P(a \leq X < b) = P(a < X < b) + P(X = a)$
8. $f(x) = \frac{d}{dx}(F(x))$

MOMENTS

Definition: The n^{th} moment about origin of a RV X is defined as the expected value of the n^{th} power of X.

Moments about Origin (Raw Moments)

For discrete, $\mu_n' = E(X^n) = \sum_i x_i^n p_i$, $\quad n \geq 1$

For continuous, $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$, $\quad n \geq 1$

Moment about Mean (Central Moments)

For discrete, $\mu_n = E[(X - \bar{X})^n] = \sum_i (x_i - \bar{X})^n p_i$, $\quad n \geq 1$

For continuous, $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f(x)dx$, $\quad n \geq 1$

Relationship between moments about origin and moment about mean

$$\mu_r = \mu_r' - rC_1 \, \mu \, \mu_{r-1}^1 + rC_2 \, \mu^2 \, \mu_{r-2}' - \cdots$$

Hence, $\mu_1 = 0$

$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$\mu_3 = \mu_3' - 3\mu_2' \, \mu_1' + 2(\mu_1')^3$$
$$\mu_4 = \mu_4' - 4\mu_3' \, \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

MOMENT GENERATING FUNCTION

Definition : Moment generating function of a random variable about the origin is defined as

Discrete : $M_X(t) = E(e^{tX}) = \sum_x e^{tx} \, p(x)$,

Continuous : $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \, f(x)dx$

Where the integration or summation is taken over the entire range of X, t being a real parameter, assuming that integration or summation is absolutely convergent.

$$M_X(t) = 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r', \quad \text{Where } \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

Note:

1. $\mu_r' = \frac{d^r}{dt^r}[M_X(t)]_{t=0}$
2. $M_{CX}(t) = M_X(Ct)$, C being a constant.
3. If $X_1, X_2, \ldots X_n$ are n independent RVs, then $M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t).M_{X_2}(t) \ldots M_{X_n}(t)$
4. $M_{X=a}(t) = e^{-at} \, M_X(t)$

PROBLEMS IN DISCRETE RANDOM VARIABLE

1. *A discrete RV X has the following probability distribution*

x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of a (ii) $P(X < 3)$ (iii) $P(X \geq 3)$ (iv) $P(0 < X < 3)$ (v) Find the distribution function of X.

Solution

(i) $\sum_{x=0}^{8} P(x) = 1 \Rightarrow P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1 \Rightarrow 81a = 1 \Rightarrow \boldsymbol{a = \frac{1}{81}}$

(ii) $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2) = a + 3a + 5a = 9a = 9 \times \frac{1}{81} = \frac{1}{9}$

(iii) $P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{1}{9} = \frac{8}{9}$

(iv) $P(0 < X < 3) = P(X = 1) + P(X = 2) = 3a + 5a = 8a = \frac{8}{81}$

(v)

x	0	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

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2. ***A discrete random variable X has the probability function given below:***

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
$p(x)$	*0*	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) The value of K (ii) $P(1.5 < X < 4.5/X > 2)$ ***(iii) The smallest value of*** λ ***for which*** $P(X \le \lambda) > 1/2$.

Solution:

(i) $\sum_{x=0}^{7} P(x) = 1 \Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$

$$0+K+2K+2K+3K+K^2+2K^2+7K^2+K = 1 \Rightarrow 10K^2+9K = 1$$

$$(10K-1)(K+1) = 0 \Rightarrow K = \frac{1}{10}, -1 \Rightarrow \boldsymbol{K = \frac{1}{10}} \quad (\because \boldsymbol{K = -1}, \textit{which is meaningless})$$

(ii) $P(1.5 < X < 4.5/X > 2) = \frac{P[(1.5<X<4.5)\cap(X>2)]}{P(X>2)}$ $\qquad \because \boldsymbol{P(A/B) = \frac{P(A\cap B)}{P(B)}}$

$$P(1.5 < X < 4.5/X > 2) = \frac{P(3)+P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)} = \frac{\left(\frac{5}{10}\right)}{\left(\frac{7}{10}\right)} = \boldsymbol{\frac{5}{7}}$$

(iii) $P(X \le \lambda) > \frac{1}{2}$, $\lambda = 0,\ P(X \le 0) = 0 \ngtr \frac{1}{2}$; $\lambda = 1,\ P(X \le 1) = \frac{1}{10} \ngtr \frac{1}{2}$;

$\lambda = 2,\ P(X \le 2) = \frac{3}{10} \ngtr \frac{1}{2}$; $\lambda = 3,\ P(X \le 3) = \frac{5}{10} \ngtr \frac{1}{2}$; $\lambda = 4,\ P(X \le 4) = \frac{8}{10} > \frac{1}{2}$

The smallest value of λ for which $P(X \le \lambda) > 1/2$ is 4.

3. ***If the RV X takes the values 1, 2, 3 & 4 such that*** $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, ***find the probability distribution and cumulative distribution function of X.***

Solution: Let $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = 30K$

x	1	2	3	4
$p(x)$	$15K$	$10K$	$30K$	$6K$

$\sum_{x=1}^{4} P(x) = 1 \Rightarrow P(1)+P(2)+P(3)+P(4) = 1 \Rightarrow 15K+10K+30K+6K = 1 \Rightarrow 61K = 1 \Rightarrow \boldsymbol{K = \frac{1}{61}}$

Cumulative distribution function of X

x	1	2	3	4
$p(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. ***A discrete RV X has the following probability distribution***

x	*-2*	*-1*	*0*	*1*	*2*	*3*
$p(x)$	*0.1*	K	*0.2*	$2K$	*0.3*	$3K$

Find (i) K (ii) $P(X < 2)$ ***(iii)*** $P(-2 < X < 2)$ ***(iv) the cdf of X (v) the mean of X.***

Solution

(i) $\sum_{x=-2}^{3} P(x) = 1 \Rightarrow P(-2)+P(-1)+P(0)+P(1)+P(2)+P(3) = 1 \Rightarrow 6K+0.6 = 1 \Rightarrow \boldsymbol{K = \frac{1}{15}}$

	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$

(ii) $P(X < 2) = P(-2)+P(-1)+P(0)+P(1) = \frac{1}{10}+\frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{1}{2}$

(iii) $P(-2 < X < 2) = P(-1)+P(0)+P(1) = \frac{1}{15}+\frac{2}{10}+\frac{2}{15} = \frac{2}{5}$

(iv)

x	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{2}{10}$	$\frac{2}{15}$	$\frac{3}{10}$	$\frac{3}{15}$
$F(X)$	$\frac{1}{10}$	$\frac{1}{6}$	$\frac{11}{30}$	$\frac{1}{2}$	$\frac{4}{5}$	1

(v) ***Mean of X***

$$E(X) = \sum_{x=-2}^{3} x\,P(x) = (-2)P(-2)+(-1)P(-1)+0\,P(0)+1\,P(1)+2\,P(2)+3\,P(3)$$

$$= \left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{2}{10}\right)+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{3}{15}\right) = \frac{16}{15}$$

5. ***If X is RV having the density function*** $f(x)=\begin{cases}\frac{x}{6} \; for\; x=1,2,3\\ 0,otherwise\end{cases}$. ***Find*** $E(X^3+2X+7)$ ***and*** $Var\,(4X+5)$.

Solution

x	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$

$$E(X)=\sum_{x=1}^{3} x\,P(x)=\left(1\times\frac{1}{6}\right)+\left(2\times\frac{2}{6}\right)+\left(3\times\frac{3}{6}\right)=\frac{7}{3}$$

$$E(X^2)=\sum_{x=1}^{3} x^2\,P(x)=\left(1\times\frac{1}{6}\right)+\left(4\times\frac{2}{6}\right)+\left(9\times\frac{3}{6}\right)=6$$

$$E(X^3)=\sum_{x=1}^{3} x^3\,P(x)=\left(1\times\frac{1}{6}\right)+\left(8\times\frac{2}{6}\right)+\left(27\times\frac{3}{6}\right)=\frac{49}{3}$$

$$E(X^3+2X+7)=E(X^3)+2E(X)+7=\frac{84}{3}$$

$$Var(X)=E(X^2)-[E(X)]^2=\frac{5}{9},\qquad Var\,(4X+5)=4^2Var(X)=16\times\frac{5}{9}=\frac{80}{9}$$

6. ***If X has the distribution function*** $F(x)=\begin{cases}\mathbf{0}, & x<1\\ \frac{1}{3}, & 1\le x<4\\ \frac{1}{2}, & 4\le x<6\\ \frac{5}{6}, & 6\le x<10\\ \mathbf{1}, & x\ge 10\end{cases}$

Find (i) The probability distribution of X (ii) $P(2<X<6)$ ***(iii) Mean of X (iv) Variance of X.***

Solution

(i) For the given c.d.f., the probability distribution of X is

x	0	1	4	6	10
$p(x)$	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

(ii) $P(2<X<6)=P(X=4)=\frac{1}{6}$

(iii) $E(X)=\sum_i x_i\,P(x_i)=(0\times 0)+\left(1\times\frac{1}{3}\right)+\left(4\times\frac{1}{6}\right)+\left(6\times\frac{2}{6}\right)+\left(10\times\frac{1}{6}\right)=\frac{14}{3}$

$$E(X^2)=\sum_i x_i^2\,P(x_i)=(0\times 0)+\left(1\times\frac{1}{3}\right)+\left(16\times\frac{1}{6}\right)+\left(36\times\frac{2}{6}\right)+\left(100\times\frac{1}{6}\right)=\frac{95}{3}$$

$$Var(X)=E(X^2)-[E(X)]^2=\frac{89}{3}$$

7. ***When a die is thrown, X denotes the number that turns up. Find*** $E(X),E(X^2),Var(X)$***and standard deviation.***

Solution: $p=\frac{1}{6}$, X = 1, 2, 3, 4, 5, 6 Here X is a discrete RV

$$E(X)=\sum_i x_i\,P(x_i)=\left(1\times\frac{1}{6}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(5\times\frac{1}{6}\right)+\left(6\times\frac{1}{6}\right)=3.5$$

$$E(X^2)=\sum_i x_i^2\,P(x_i)=\left(1\times\frac{1}{6}\right)+\left(4\times\frac{1}{6}\right)+\left(9\times\frac{1}{6}\right)+\left(16\times\frac{1}{6}\right)+\left(25\times\frac{1}{6}\right)+\left(36\times\frac{1}{6}\right)=\frac{91}{6}=15.167$$

$$Var(X)=E(X^2)-[E(X)]^2=2.9166\,,\qquad S.D.=\sigma_X=\sqrt{Var(X)}=1.7078$$

8. ***A coin is tossed until a head appears. What is the expectation of the number of tosses required?***
Solution: Let X – No. of tosses required to get the 1st head. The 1st head may appear in the 1st or 2nd … and so on. The events are H, TH, TTH, TTTH, ... $p=\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3},\ldots$

x	1	2	3	4	5	…
$p(x)$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^5}$	…

$$E(X)=\sum_i x_i\,P(x_i)=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\cdots\right]=\frac{1}{2}\left(1-\frac{1}{2}\right)^{-2}=2\qquad [\because \mathbf{(1-x)^{-2}=1+2x+3x^2+\cdots}]$$

9. ***By throwing a fair dice, a player gains Rs. 20 if 2 turns up, gains Rs. 40 if 4 turns up and loses Rs. 30 if 6 turns up. He never loses or gains if any other number turns up. Find the expected value of money he gains.***
Solution: Let X – money won on an trial. x_i = Amount of money won, if the faces show $i=1,2,3,4,5,6$.

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	1	2	3	4	5	6
x	0	20	0	40	0	-30
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \sum_i x_i P(x_i) = \left(0 \times \tfrac{1}{6}\right) + \left(20 \times \tfrac{1}{6}\right) + \left(0 \times \tfrac{1}{6}\right) + \left(40 \times \tfrac{1}{6}\right) + \left(0 \times \tfrac{1}{6}\right) + \left(-30 \times \tfrac{1}{6}\right) = 5$$

10. Find the first three moments of X if X has the following distribution

x	-2	1	3
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Solution : $\mu_n' = E(X^n) = \sum_i x_i^n p_i\,, \quad n \geq 1$

$n = 1,\ \mu_1' = E(X) = \sum_i x_i P(x_i) = \left(-2 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{4}\right) = 0$

$n = 2,\ \mu_2' = E(X^2) = \sum_i x_i^2 P(x_i) = \left(4 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(9 \times \frac{1}{4}\right) = \frac{9}{2}$

$n = 3,\ \mu_3' = E(X^3) = \sum_i x_i^3 P(x_i) = \left(-8 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(27 \times \frac{1}{4}\right) = 3$

11. ***A RV X has the probability function $f(x) = \frac{1}{2^x}, x = 1, 2, 3 \ldots$ Find the (i) moment generating function (ii) Mean***

Solution :

(i) $M_X(t) = \sum_x e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$

$$= \left(\frac{e^t}{2}\right)^1 + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2 - e^t}$$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0} = \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0} = \frac{(2 - e^0)e^0 - e^0(-e^0)}{(2 - e^0)^2} = 2$

12. ***Find the moment generating function for the following function given by***

x	0	1	2	3	4	5	6
$p(x)$	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Solution :

$M_X(t) = \sum_{x=0}^{6} e^{tx} p(x) = e^{0t} p(0) + e^t p(1) + e^{2t} p(2) + e^{3t} p(3) + e^{4t} p(4) + e^{5t} p(5) + e^{6t} p(6)$

$= \frac{1}{49}[1 + 3e^t + 5e^{2t} + 7e^{3t} + 9e^{4t} + 11e^{5t} + 13e^{6t}]$

13. ***If a RV X has moment generating function $M_X(t) = \frac{3}{3-t}$, obtain the standard deviation of X.***

Solution : $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1 - \frac{t}{3}\right)} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \left(\frac{t}{3}\right) + \left(\frac{t}{3}\right)^2 + \left(\frac{t}{3}\right)^3 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{3}\right) + \frac{t^2}{2!}\left(\frac{2}{9}\right) + \frac{t^3}{3!}\left(\frac{6}{27}\right) + \cdots$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}, \quad \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{3}, \quad \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{2}{9}$

Variance $= \mu_2' - (\mu_1')^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$, Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

PROBLEMS IN CONTINUOUS RANDOM VARIABLE

1. ***If $p(x) = \begin{cases} x\, e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ (i) Show that $p(x)$ is a p.d.f. (ii) Find its distribution function $P(x)$.***

Solution

(i) $\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{0} p(x)dx + \int_0^{\infty} p(x)dx = \int_{-\infty}^{0} 0\, dx + \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx = \int_0^{\infty} x\, e^{-\frac{x^2}{2}} dx$

Put $x^2 = t$, $2x\, dx = dt \Rightarrow x\, dx = \frac{dt}{2}$, $x = 0,\ t = 0$ and $x = \infty,\ t = \infty$

$$\int_{-\infty}^{\infty} p(x)dx = \int_0^{\infty} e^{-\frac{t}{2}} \frac{dt}{2} = \frac{1}{2}\int_0^{\infty} e^{-\frac{t}{2}} dt = \frac{1}{2}\left[\frac{e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = -e^{-\infty} + e^0 = 1 \qquad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$$

$\therefore p(x)$ is a p.d.f. of a RV X.

(ii) $F(X) = P(X \leq x) = \int_0^x p(x)dx = \int_0^x x\, e^{-\frac{x^2}{2}} dx = 1 - e^{-\frac{x^2}{2}},\ x \geq 0$

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2. ***A continuous RV X has a pdf $f(x) = 3x^2, 0 \le x \le 1$. Find a and b such that***

(i) $P(X \le a) = P(X > a)$ (ii) $P(X > b) = 0.05$

Solution:

(i) $P(X \le a) = P(X > a) \Rightarrow \int_{-\infty}^{a} f(x)dx = \int_{a}^{\infty} f(x)dx$

$\int_0^a 3\,x^2 dx = \int_a^1 f(x)dx \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = 3\left[\frac{x^3}{3}\right]_a^1$

$a^3 = 1 - a^3 \Rightarrow 2a^3 = 1 \Rightarrow a^3 = \frac{1}{2} \Rightarrow \boldsymbol{a} = \left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\frac{1}{3}} = \mathbf{0.7937}$

(ii) $P(X > b) = 0.05 \Rightarrow \int_b^1 3\,x^2 dx = 0.05 \Rightarrow 3\left[\frac{x^3}{3}\right]_b^1 = 0.05$

$1 - b^3 = 0.05 \Rightarrow b^3 = 0.95 \Rightarrow \boldsymbol{b} = (\mathbf{0.95})^{\frac{1}{3}} = \mathbf{0.9830}$

3. ***A Continuous RV X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $P(X < 4)$.***

Solution : $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_2^5 k(1+x)dx = 1 \Rightarrow k\left[x + \frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[\left(5 + \frac{25}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{2}}{\mathbf{27}}$

$P(X < 4) = \frac{2}{27}\int_2^4 (1+x)dx = \frac{2}{27}\left[x + \frac{x^2}{2}\right]_2^4 = \frac{2}{27}\left[\left(4 + \frac{16}{2}\right) - \left(2 + \frac{4}{2}\right)\right] = \frac{16}{27}$

4. ***A RV X has a pdf $f(x) = kx^2e^{-x}, x \ge 0$. Find k, mean, variance and $E(3X^2 - 2X)$.***

Solution: $\int_{-\infty}^{\infty} f(x)dx = 1, \quad \int_0^{\infty} kx^2e^{-x}dx = 1$

$\boldsymbol{Differentiation}: u = x^2, \quad u' = 2x, \quad u'' = 2, \quad u''' = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}$ $\quad (\because \int \boldsymbol{uv\,dx} = \boldsymbol{uv_1 - u'v_2 + u''v_3 - \cdots})$

$k\left[x^2\frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 1 \Rightarrow k[(0-0+0)-(0-0+2)] = 1 \Rightarrow \boldsymbol{k} = \frac{\mathbf{1}}{\mathbf{2}}$ $\quad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

Mean of X $E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_0^{\infty} x\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^3e^{-x}dx$

$\boldsymbol{Differentiation}: u = x^3, \quad u' = 3x^2, \quad u'' = 6x, \quad u''' = 6, \quad u^{iv} = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, v_2 = \frac{e^{-x}}{(-1)^2}, v_3 = \frac{e^{-x}}{(-1)^3}, v_4 = \frac{e^{-x}}{(-1)^4}$ $\quad (\because \int \boldsymbol{uv\,dx} = \boldsymbol{uv_1 - u'v_2 + u''v_3 - \cdots})$

$E(X) = \frac{1}{2}\left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty} = 3$ $\quad (\because \boldsymbol{e^{-\infty} = 0,\ e^0 = 1})$

$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)dx = \int_0^{\infty} x^2\left(\frac{1}{2}x^2e^{-x}\right)dx = \frac{1}{2}\int_0^{\infty} x^4e^{-x}dx$

$\boldsymbol{Differentiation}: u = x^4, \quad u' = 4x^3, \quad u'' = 12x^2, \quad u''' = 24x, \quad u^{iv} = 24, \quad u^{v} = 0$

Integration : $v = e^{-x}, v_1 = \frac{e^{-x}}{(-1)}, \quad v_2 = \frac{e^{-x}}{(-1)^2}, \quad v_3 = \frac{e^{-x}}{(-1)^3}, \ v_4 = \frac{e^{-x}}{(-1)^4}, v_5 = \frac{e^{-x}}{(-1)^5}, \ v_6 = \frac{e^{-x}}{(-1)^6}$

$E(X^2) = \frac{1}{2}\left[x^4\frac{e^{-x}}{(-1)} - 4x^3\frac{e^{-x}}{(-1)^2} + 12x^2\frac{e^{-x}}{(-1)^3} - 24x\frac{e^{-x}}{(-1)^4} + 24x\frac{e^{-x}}{(-1)^5}\right]_0^{\infty} = 12$

$V(X) = E(X^2) - [E(X)]^2 = 12 - 9 = 3$

$E(3X^2 - 2X) = 3E(X^2) - 2E(X) = 3(12) - 2(3) = 36 - 6 = 30$

5. ***The prob. distribution function of a RV X is*** $\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x}, & \mathbf{0} < \boldsymbol{x} < 1 \\ \mathbf{2} - \boldsymbol{x}, & \mathbf{1} < \boldsymbol{x} < 2 \\ \mathbf{0}, & \boldsymbol{x} > 2 \end{cases}$ ***. Find the mean and variance.***

Solution

$E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{-\infty}^{0} x\,f(x)dx + \int_0^1 x\,f(x)dx + \int_1^2 x\,f(x)dx + \int_2^{\infty} x\,f(x)dx$

$= \int_{-\infty}^{0} 0\,dx + \int_0^1 x\,(x)dx + \int_1^2 x\,(2-x)dx + \int_2^{\infty} 0\,dx = \int_0^1 x^2dx + \int_1^2 (2x - x^2)dx$

$E(X) = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1$

$E(X^2) = \int_{-\infty}^{\infty} x^2\,f(x)dx = \int_{-\infty}^{0} x^2\,f(x)dx + \int_0^1 x^2\,f(x)dx + \int_1^2 x^2\,f(x)dx + \int_2^{\infty} x^2\,f(x)dx$

$= \int_0^1 x^3x + \int_1^2 (2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 = \frac{1}{4} + \left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$

$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6}$

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6. ***The distribution function of a RV X is given by $F(x) = 1 - (1+x)e^{-x},\ x \ge 0$. Find the density function, mean and variance of X.***

Solution

$f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1-(1+x)e^{-x}] = [0-(1+x)(-e^{-x}) - e^{-x}] = e^{-x} + xe^{-x} - e^{-x} = xe^{-x}, x \ge 0$

$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \int_0^{\infty} x^2 e^{-x} dx = \left[x^2 \frac{e^{-x}}{(-1)} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty} = 2$

$E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \int_0^{\infty} x^3 e^{-x}dx = \left[x^3\frac{e^{-x}}{(-1)} - 3x^2\frac{e^{-x}}{(-1)^2} + 6x^2\frac{e^{-x}}{(-1)^3} - 6\frac{e^{-x}}{(-1)^4}\right]_0^{\infty} = 6$

$V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$

7. ***The cdf of a continuous RV X is given by*** $F(x) = \begin{cases} \mathbf{0}, & x < 0 \\ x^2, & 0 \le x < \frac{1}{2} \\ 1 - \frac{3}{25}(3-x)^2, & \frac{1}{2} \le x < 3 \\ 1, & x \ge 3 \end{cases}$

Find the p.d.f. of X and evaluate $P(|X| \le 1)$ and $P\left(\frac{1}{3} \le X < 4\right)$ using both the pdf and cdf.

Solution: $f(x) = \frac{d}{dx}[F(x)]$

$$f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

pdf : $P(|X| \le 1) = P(-1 \le X \le 1) = \int_{-1}^{0} 0\, dx + \int_0^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)dx = 2\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = \frac{13}{25}$

cdf: $P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) = \frac{13}{25}$

pdf : $P\left(\frac{1}{3} \le X < 4\right) = \int_{\frac{1}{3}}^{\frac{1}{2}} 2x\, dx + \int_{\frac{1}{2}}^{3} \frac{6}{25}(3-x)dx + \int_3^4 0\, dx = 2\left[\frac{x^2}{2}\right]_{\frac{1}{3}}^{\frac{1}{2}} + \frac{6}{25}\left[3x - \frac{x^2}{2}\right]_{\frac{1}{2}}^{3} = \frac{8}{9}$

cdf: $P\left(\frac{1}{3} \le X < 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \frac{1}{9} = \frac{8}{9}$

8. ***If X has probability density function given by $f(x) = \frac{x+1}{2}, -1 \le x \le 1$. Find the 1st four central moments.***

Solution: $\mu_n' = E(X^n) = \int_{-\infty}^{\infty} x^n f(x)dx$

$n = 1,\ \mu_1' = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$

$n = 2,\ \mu_2' = E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^3 + x^2)dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$

$n = 3,\ \mu_3' = E(X^3) = \int_{-\infty}^{\infty} x^3\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^4 + x^3)dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{5}$

$n = 4,\ \mu_4' = E(X^4) = \int_{-\infty}^{\infty} x^4\, f(x)dx = \frac{1}{2}\int_{-1}^{1}(x^5 + x^4)dx = \frac{1}{2}\left[\frac{x^6}{6} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{1}{5}$

Moment about Mean (Central Moments)

$\mu_r = \mu_r' - rC_1\, \mu\, \mu_{r-1}^1 + rC_2\, \mu^2\, \mu_{r-2}' - rC_3\, \mu^3\, \mu_{r-3}' + rC_4\, \mu^4\, \mu_{r-4}' - \cdots$

$r = 1,\ \mu_1 = 0$

$r = 2,\ \mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{9}$

$r = 3,\ \mu_3 = \mu_3' - 3\mu_2'\, \mu_1' + 2(\mu_1')^3 = -\frac{8}{135}$

$r = 4,\ \mu_4 = \mu_4' - 4\mu_3'\, \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4 = \frac{48}{405}$

9. ***A RV X has density function given by $f(x) = \begin{cases} 2\, e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$. Obtain the (i) moment generating function (ii) Four moments about the origin (iii) Mean (iv) Variance.***

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Solution: $M_X(t) = \int_{x=-\infty}^{\infty} e^{tx} f(x)dx = \int_{x=0}^{\infty} e^{tx}\, 2\, e^{-2x}\, dx = \int_{x=0}^{\infty} 2\, e^{-(2-t)x}\, dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty} = \frac{2}{2-t}$

$$M_X(t) = \frac{2}{2-t} = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \cdots = 1 + \frac{t}{1!}\left(\frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{2}\right) + \frac{t^3}{3!}\left(\frac{3}{4}\right) + \frac{t^3}{4!}\left(\frac{3}{2}\right) + \cdots$$

$\mu_r' = coefficient\ of\ \frac{t^r}{r!}$, $\quad r = 1,\ \mu_1' = coefficient\ of\ \frac{t^1}{1!} = \frac{1}{2}$

$r = 2,\ \mu_2' = coefficient\ of\ \frac{t^2}{2!} = \frac{1}{2}$, $\quad r = 3,\ \mu_3' = coefficient\ of\ \frac{t^3}{3!} = \frac{3}{4}$

$r = 4,\ \mu_4' = coefficient\ of\ \frac{t^4}{4!} = \frac{3}{2}$, Mean $= \mu_1' = \frac{1}{2}$, Variance $= \mu_2' - (\mu_1')^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

FUNCTION OF RANDOM VARIABLE

One to One Transformation of Random Variables:

Consider that a random variable X is linearly transformed into an another random variable Y. Let Y be $T(x)$.
A monotonically increasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 < x_2$. For example, $y = ax, a > 0$
A monotonically decreasing transformation is one where $T(x_1) < T(x_2)$ for all $x_1 > x_2$. For example, $y = ax, a < 0$
If the transformation is monotonically increasing $f_Y(y) = f_X(x)\ \frac{dx}{dy}$
If the transformation is monotonically decreasing $f_Y(y) = f_X(x)\ \left(-\frac{dx}{dy}\right)$
In general, for a linear transformation $\boldsymbol{f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|, where\ x = g^{-1}(y)}$

Non - One to One Transformation of Random Variables:

For a transformation which is non - one to one, the transformation will be broken up into transformations each of which one to one. $\quad \boldsymbol{f_Y(y) = f_X(x_1)\ \left|\frac{dx_1}{dy}\right| + f_X(x_2)\ \left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\ \left|\frac{dx_n}{dy}\right|}$

PROBLEMS IN FUNCTION OF RANDOM VARIABLE

1. ***Consider a RV X with p.d.f. $f(x) = e^{-x}, x \geq 0$ with transformation $y = e^{-x}$. Find the transformed density function.***

   Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right| = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} = \frac{e^{-x}}{|-e^{-x}|} = \frac{y}{y} = 1, 0 < y \leq 1$

2. ***Let X be a RV with p.d.f. $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$. Find the p.d.f. of $Y = 8X^3$.***

   Solution: $f_Y(y) = f_X(x)\ \left|\frac{dx}{dy}\right|$, *Let* $y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \left(\frac{y}{8}\right)^{\frac{1}{3}}$, $\quad \frac{dx}{dy} = \frac{1}{3}\left(\frac{y}{8}\right)^{\frac{1}{3}-1}\frac{1}{8} = \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}}$

   $f_Y(y) = 2x\ \frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = 2\left(\frac{y}{8}\right)^{\frac{1}{3}}\frac{1}{24}\left(\frac{y}{8}\right)^{-\frac{2}{3}} = \frac{1}{12}\left(\frac{y}{8}\right)^{-\frac{1}{3}}, 0 < y < 8$

   Range: $0 < x < 1 \Rightarrow 0 < \left(\frac{y}{8}\right)^{\frac{1}{3}} < 1 \Rightarrow 0 < \frac{y}{8} < 1 \Rightarrow 0 < y < 8$

3. ***If the continuous RV X has p.d.f. $f(x) = \frac{2}{9}(x+1), -1 < x < 2$. Find the p.d.f. of $Y = X^2$.***

   Solution: The transformation function $Y = X^2$ is not monotonic in $(-1, 2)$. So we divide the interval into two parts. i.e., $(-1,\ 1)$ and $(1, 2)$. Since $(-1, 1)$ is a symmetric interval, we have *Let* $y = x^2 \Rightarrow x = y^{\frac{1}{2}}$

   Range: $-1 < x < 0 \Rightarrow -1 < y^{\frac{1}{2}} < 0 \Rightarrow 0 < y < 1$

   $f_Y(y) = \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] = \frac{1}{2\sqrt{y}}\left[\frac{2}{9}(1+\sqrt{y}) + \frac{2}{9}(1-\sqrt{y})\right] = \frac{2}{9\sqrt{y}}, 0 < y < 1$

   Range: $1 < x < 2 \Rightarrow 1 < y^{\frac{1}{2}} < 2 \Rightarrow 1 < y < 4$ strictly increasing

   $f_Y(y) = \frac{2}{9}(x+1)\frac{1}{2}\, y^{-\frac{1}{2}} = \frac{2}{9}\left(y^{\frac{1}{2}}+1\right)\frac{1}{2}\, y^{-\frac{1}{2}} = \frac{1}{9}\left(y^{\frac{1}{2}}+1\right) y^{-\frac{1}{2}} = \frac{1}{9}\left(1 + y^{-\frac{1}{2}}\right), 1 < y < 4$

TCHEBYCHEFF INEQULAITY

Statement :

If X is a RV with $E(X) = \mu$ and $V(X) = \sigma^2$, then $P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$ or $P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}$, $c > 0$.

Alternative Form : **If we put $c = k\sigma$, where $k > 0$ then Tchebycheff inequality takes the form**

$$P\left\{\left|\frac{X-\mu}{k}\right| \geq \sigma\right\} \leq \frac{1}{k^2} \text{ or } P\left\{\left|\frac{X-\mu}{k}\right| \leq \sigma\right\} \geq 1 - \frac{1}{k^2}$$

PROBLEMS IN TCHEBYCHEFF INEQULAITY

1. ***A RV X has mean $\mu = 12$ and variance $\sigma^2 = 9$ and an unknown probability distribution. Find $P(6 < X < 18)$.***

   Solution: Since the probability distribution of X is not known, we can not find the value of the required probability. We can find only a lower bound for the probability using Tchebycheff inequality.

   $$P\{|X-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}, c > 0$$
   $$P\{|X-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}, \ c > 0$$
   $$P\{-c < (X-\mu) < c\} \geq 1 - \frac{\sigma^2}{c^2}$$
   $$P\{\mu - c < X < \mu + c\} \geq 1 - \frac{\sigma^2}{c^2}$$
   $$\mu = 12, \ \sigma^2 = 9, \ P\{12 - c < X < 12 + c\} \geq 1 - \frac{9}{c^2}$$
   $$Put \ c = 6, \ P\{12 - 6 < X < 12 + 6\} \geq 1 - \frac{9}{6^2}$$
   $$\boldsymbol{P\{6 < X < 18\} \geq \frac{3}{4}}$$

2. ***A fair die is tossed 720 times. Use Tchebycheff inequality to find a lower bound for the probability of getting 100 to 140 sixes.***

   Solution: Let X – no. of sixes obtained when a fair die is tossed 720 times. $p = \frac{1}{6}, \ q = \frac{5}{6}, \ n = 720$

   X follows a binomial distribution with mean $np = 120$ and variance $npq = 100$, that is $\mu = 120, \sigma = 10$

   By Tchebycheff inequality $P\{|X-\mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$

   $$P\{|X-120| \leq 10k\} \geq 1 - \frac{1}{k^2}$$
   $$P\{-10k < (X-120) < 10k\} \geq 1 - \frac{1}{k^2}$$
   $$P\{120 - 10k < X < 120 + 10k\} \geq 1 - \frac{1}{k^2}$$
   $$Put \ k = 2, \ P\{100 < X < 140\} \geq 1 - \frac{1}{4}$$
   $$\boldsymbol{P\{100 < X < 140\} \geq \frac{3}{4}}$$

3. ***A discrete RV X takes the values $-1, 0, 1$ with probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P\{|X-\mu| \geq 2\sigma\}$and compare it with the upper bound given by Tchebycheff inequality.***

   Solution:

   $$E(X) = \sum_{x=-1}^{1} x\, P(x) = \left(-1 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{8}\right) = 0$$
   $$E(X^2) = \sum_{x=-1}^{1} x^2\, P(x) = \left(1 \times \frac{1}{8}\right) + \left(0 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{8}\right) = \frac{1}{4}$$
   $$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - 0 = \frac{1}{4}$$
   $$P\{|X-\mu| \geq 2\sigma\} = P\{X \geq 1\} = P(X = -1 \ or \ X = 1) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

   By Tchebycheff inequality, $P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

   $$\boldsymbol{P\{|X-\mu| \geq 2\sigma\} \leq \frac{1}{4}}$$

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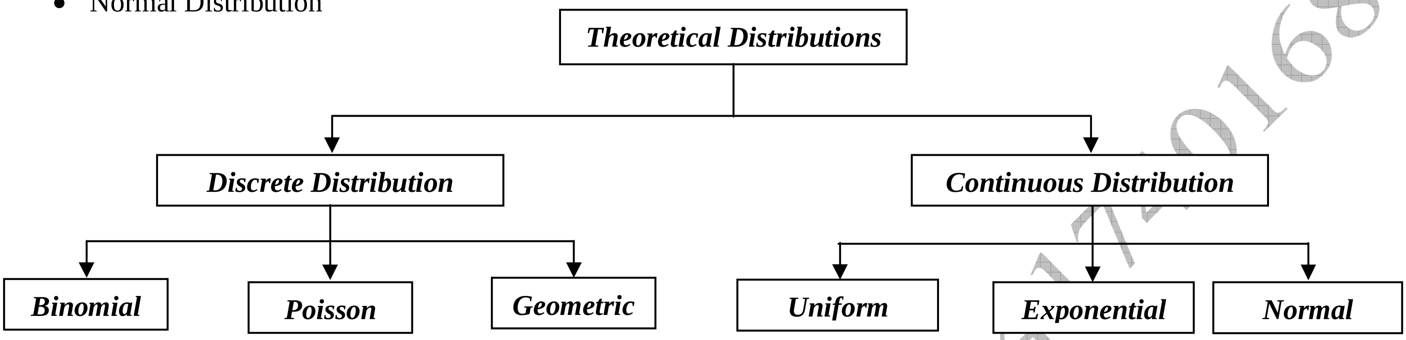
UNIT – II - THEORETICAL DISTRIBUTIONS

Syllabus

- Binomial Distribution
- Poisson Distribution
- Geometric Distribution
- Uniform Distribution
- Exponential Distribution
- Normal Distribution



DISCRETE DISTRIBUTION

Discrete	**Probability Mass Function (p.m.f.)**	**Moment Generating Function (m.g.f.) $M_X(t)$**	**Mean $E(X)$**	**Variance $V(X)$**
Binomial	$P(X=x)=nC_x p^x q^{n-x},\ \ x=0,1,\dots n$	$(q+pe^t)^n$	np	npq
Poisson	$P(X=x)=\dfrac{e^{-\lambda}\lambda^x}{x!},\ \ x=0,1,\dots\infty$	$e^{\lambda(e^t-1)}$	λ	λ
Geometric	$P(X=x)=p\,q^{x-1},\ \ x=1,\dots\infty$	$\dfrac{p\,e^t}{1-q\,e^t}$	$\dfrac{1}{p}$	$\dfrac{q}{p^2}$

CONTINOUS DISTRIBUTION

Continuous	**Probability Density Function (p.d.f.)**	**Moment Generating Function (m.g.f.) $M_X(t)$**	**Mean $E(X)$**	**Variance $V(X)$**
Uniform	$f(x)=\dfrac{1}{b-a},\qquad a<x<b$	$\dfrac{e^{bt}-e^{at}}{t(b-a)}$	$\dfrac{b+a}{2}$	$\dfrac{(b-a)^2}{12}$
Exponential	$f(x)=\lambda\,e^{-\lambda x},\qquad x\ge 0$	$\dfrac{\lambda}{\lambda-t}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Normal	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty<x<\infty, -\infty<\mu<\infty, \sigma>0$	$e^{\left(\mu t+\frac{t^2\sigma^2}{2}\right)}$	μ	σ^2

BINOMIAL DISTRIBUTION

Bernoulli Distribution : A Bernoulli distribution is one having the following properties

(i) The experiment consists of n repeated trials.

(ii) Each trial results in an outcome that may be classified under two mutually exclusive categories as a success or as a failure.

(iii) The probability of success denoted by p, remains constant from trial to trial.

(iv) The repeated trials are independent.

Binomial Distribution

A Bernoulli trial can result in a success with probability p and a failure with probability $q=1-p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials

$$\boldsymbol{P(X=x)=nC_x p^x q^{n-x},\ \ x=0,1,\dots n}$$

The quantities n & p are called the parameters of binomial distribution.

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Areas of Application

1. Quality control measures and Sampling processes in industries to classify items as defective or non defective.
2. Medical applications as success or failure of a surgery, cure or no cure of a patient.

Moment Generating Function (m.g.f.) in Binomial Distribution

$M_X(t) = \sum_{x=0}^{n} e^{tx}\, p(x) = \sum_{x=0}^{n} e^{tx}\, nC_x p^x q^{n-x} = \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$

$= nC_0(pe^t)^0 q^{n-0} + nC_1(pe^t)^1 q^{n-1} + nC_2(pe^t)^2 q^{n-2} + \cdots + nC_n(pe^t)^n q^{n-n}$

$= q^n + nC_1(pe^t)^1 q^{n-1} + nC_2(pe^t)^2 q^{n-2} + \cdots + (pe^t)^n$

$\boldsymbol{M_X(t) = (q + pe^t)^n}$

Mean and Variance using Moment Generating Function in Binomial Distribution

$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}(q+pe^t)^n\right]_{t=0} = [n(q+pe^t)^{n-1}\, pe^t]_{t=0} = n(q+pe^0)^{n-1}\, pe^0 = \boldsymbol{np}$

$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt} np\,(q+pe^t)^{n-1}\, e^t\right]_{t=0} = np[(n-1)(q+pe^t)^{n-2}\, pe^t + (q+pe^t)^{n-1} e^t]_{t=0}$

$= np[(n-1)(q+pe^0)^{n-2}\, pe^0 + (q+pe^0)^{n-1} e^0] = np[(n-1)p+1] = \boldsymbol{n^2p^2 - np^2 + np}$

$V(X) = E(X^2) - [E(X)]^2 = n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np = np(1-q) = \boldsymbol{npq}$

<u>Problems in Binomial Distribution</u>

1. ***Four coins are tossed simultaneously. What is the probability of getting 2 heads and at least 2 heads?***

Solution : $n = 4,\ p = \frac{1}{2},\quad q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2},\ P(X = x) = nC_x p^x q^{n-x},\ \ x = 0,1,\ldots n.$

$P(X = x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x},\ \ x = 0,1,\ldots n$

(i) $P(2\ heads) = P(X = 2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{3}{8}$

(ii) $P(atleast\ 2\ heads) = P(X \ge 2) = P(X = 2) + P(X = 3) + P(X = 4)$

$= 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} + 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{4-3} + 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4} = \frac{3}{8} + \frac{1}{4} + \frac{1}{16} = \frac{11}{16}$

2. ***The probability that a patient recovers from a disease is 0.3. If 18 people are affected from this disease, What is the probability that (i) At least 10 survive (ii) Exactly 6 survive (iii) 4 to 7 survive***

Solution: $n = 18,\ p = 0.3,\quad q = 1 - p = 1 - 0.3 = 0.7,\ P(X = x) = 18C_x (0.3)^x (0.7)^{18-x}$

(i) $P(X \ge 10) = 1 - P(X < 10) = 1 - [P(0) + P(1) + \cdots + P(9)] = 1 - 0.9790 = 0.021$

(ii) $P(X = 6) = 18C_6 (0.3)^6 (0.7)^{18-6} = 0.1873$

(iii) $P(4\ to\ 7\ survice) = P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$

$= 18C_4(0.3)^4(0.7)^{18-4} + 18C_5(0.3)^5(0.7)^{18-5} + 18C_6(0.3)^6(0.7)^{18-6} + 18C_7(0.3)^7(0.7)^{18-7} = 0.6947$

3. ***In a large consignment of electric bulbs 10% are defective. A random sample of 20 is taken for inspection. Find the probability that (i) All are good (ii) At most there are 3 defective (iii) Exactly there are 3 defective bulbs.***

Solution : $n = 20,\ p = 10\% = \frac{10}{100} = 0.1,\quad q = 1 - p = 1 - 0.1 = 0.9,\ P(X = x) = 20C_x (0.1)^x (0.9)^{20-x}$

(i) $P(X = 0) = 20C_0 (0.1)^0 (0.9)^{20-0} = 0.1216$

(ii) $P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 20C_0(0.1)^0(0.9)^{20-0} + 20C_1(0.1)^1(0.9)^{20-1} + 20C_2(0.1)^2(0.9)^{20-2} + 20C_3(0.1)^3(0.9)^{20-3}$

$= 0.8671$

(iii) $P(X = 3) = 20C_3 (0.1)^3 (0.9)^{20-3} = 0.1901$

4. ***It is known that the probability of an item produced by a certain machine will be defective is 5%. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) At least 2 defective items (ii) At most 2 defective items in a consignment of 1000 packets using Binomial distribution.***

Solution : $n = 20,\ p = 5\% = \frac{5}{100} = 0.05,\quad q = 1 - p = 0.95\ ,\ P(X = x) = 20C_x (0.05)^x (0.95)^{20-x}$

(i) $P(X \ge 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)]$

$= 1 - 20C_0(0.05)^0(0.95)^{20-0} - 20C_1(0.05)^1(0.95)^{20-1} = 1 - 0.3585 - 0.3774 = 0.2641$

$N\ P(X \ge 2) = 1000 \times 0.2641 = 264$

(ii) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= 20C_0(0.05)^0(0.95)^{20-0} + 20C_1(0.05)^1(0.95)^{20-1} + 20C_2(0.05)^2(0.95)^{20-2} = 0.9246$

$N\ P(X \le 2) = 1000 \times 0.9246\ \ = 925$

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5. ***Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) At least 1 boy (iii) At most 2 girls (iv) Children of both genders. Assume equal prob. for boys and girls.***

Solution : Considering each child as a trial $n = 4$. Assuming that birth of a boy is a success, $p = \frac{1}{2}$, $q = \frac{1}{2}$.

Let X denote the no. of successes (boys). $P(X = x) = nC_x p^x q^{n-x}$, $x = 0,1,\ldots n$. $P(X = x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

(i) $P(2\ boys\ and\ 2\ girls) = P(X = 2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{3}{8}$

Number of families having 2 boys and 2 girls $= N\ P(X = 2) = 800\left(\frac{3}{8}\right) = 300$

(ii) $P(\text{At least 1 boy}) = P(X \geq 1) = 1 - P(X = 0) = 1 - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} = 1 - \frac{1}{16} = \frac{15}{16}$

Number of families having At least 1 boy $= N\ P(X \geq 1) = 800\left(\frac{15}{16}\right) = 750$

(iii) $P(\text{At most 2 girls}) = P(\text{exactly 0 , 1 or 2 girls}) = P(4) + P(3) + P(2) = 1 - [P(0) + P(1)]$

$$= 1 - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} - 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{4-1} = \frac{11}{16}$$

Number of families having At most 2 girls $= 800\left(\frac{11}{16}\right) = 550$

(iv) $P(\text{Children of both sexes}) = 1 - P(\text{children of the same sex})$

$$= 1 - [P(\text{all are boys}) + P(\text{all are girls})] = 1 - P(4) - P(0)$$

$$= 1 - 4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{4-4} - 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{4-0} = \frac{7}{8}$$

Number of families having Children of both sexes $= 800\left(\frac{7}{8}\right) = 700$

6. ***For a binomial distribution the mean is 6 and variance is 2. Find the distribution and find $P(X = 1)$.***

Solution : Mean $= np = 6$, Variance $= npq = 2$, $\frac{npq}{np} = \frac{2}{6} \Rightarrow \boldsymbol{q = \frac{1}{3}}$, $p = 1 - q = 1 - \frac{1}{3} = \boldsymbol{\frac{2}{3}}$

$np = 6 \Rightarrow n = \frac{6}{p} \Rightarrow n = \frac{6}{\left(\frac{2}{3}\right)} = \frac{18}{2} = 9, n = 9,\ p = \frac{2}{3}, q = \frac{1}{3}$, $P(X = 1) = 9C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^{9-1} = 0.0012$

7. ***A Binomial variable X satisfies the relation $9P(X = 4) = P(X = 2)$ when $n = 6$. Find the parameter p of Binomial distribution.***

Solution : $n = 6$, $P(X = x) = nC_x p^x q^{n-x}$, $x = 0,1,\ldots n$

$9P(X = 4) = P(X = 2) \Rightarrow 9 \times 6C_4 p^4 q^{6-4} = 6C_2 p^2 q^{6-2} \Rightarrow 9 \times 6C_4 p^4 q^2 = 6C_2 p^2 q^4$

$9 \times 6C_4 p^2 = 6C_2 q^2 \Rightarrow 135p^2 = 15q^2 \Rightarrow 9p^2 - q^2 = 0 \Rightarrow 9p^2 - (1-p)^2 = 0 \Rightarrow 8p^2 + 2p - 1 = 0$

$p = -\frac{1}{2}\ (or)\ \frac{1}{4}$, $\boldsymbol{p = \frac{1}{4}}$ ($\because$ ***p cannot be negative***)

8. ***Fit a binomial distribution for the following data. Find the parameters of the distribution.***

x	***0***	***1***	***2***	***3***	***4***	***5***	***6***	***Total***
f	***5***	***18***	***28***	***12***	***7***	***6***	***4***	***80***

Solution : Fitting a binomial distribution means assuming that the given distribution is approximately binomial and hence finding the probability mass function and the finding the theoretical frequencies.

To find the binomial frequency distribution $N(q + p)^n$, which fits the given data, we require $N, n\ and\ p$.

We assume N = total frequency = 80 and n = no. of trials = 6 from the given data.

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*Total*
f	*5*	*18*	*28*	*12*	*7*	*6*	*4*	*80*
fx	*0*	*18*	*56*	*36*	*28*	*30*	*24*	*192*

$\bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{192}{80} = 2.4$, $np = 2.4 \Rightarrow 6p = 2.4$, $p = 0.4$, $q = 1 - p = 0.6$

Theoretical frequencies are given by N $P(X = x) = N\ nC_x p^x q^{n-x}$, $x = 0,1,\ldots n$

$80\ P(X = 0) = 80 \times 6C_0\,(0.4)^0(0.6)^{6-0} = 3.73$, $80\ P(X = 1) = 80 \times 6C_1\,(0.4)^1(0.6)^{6-1} = 14.93$

$80\ P(X = 2) = 80 \times 6C_2\,(0.4)^2(0.6)^{6-2} = 24.88$, $80\ P(X = 3) = 80 \times 6C_3\,(0.4)^3(0.6)^{6-3} = 22.12$

$80\ P(X = 4) = 80 \times 6C_4\,(0.4)^4(0.6)^{6-4} = 11.06$, $80\ P(X = 5) = 80 \times 6C_5\,(0.4)^5(0.6)^{6-5} = 2.95$

$80\ P(X = 6) = 80 \times 6C_6\,(0.4)^6(0.6)^{6-6} = 0.33$

x	0	1	2	3	4	5	6	*Total*
Theoretical f	4	15	25	22	11	3	0	*80*

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9. ***A discrete RV X has moment generating function*** $M_X(t) = \left(\frac{1}{4} + \frac{3}{4}e^t\right)^5$. ***Find*** $E(X), Var(X)$ ***and*** $P(X=2)$.

Solution : $M_X(t) = (q+pe^t)^n$, $p = \frac{3}{4}$, $q = \frac{1}{4}$, $n = 5$, $E(X) = np = \frac{15}{4}$, $V(X) = npq = \frac{15}{16}$

$$P(X=2) = 5C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{5-2} = \frac{45}{512}$$

POISSON DISTRIBUTION

Definition : The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region represented as t, is $\boldsymbol{P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}}$, $\boldsymbol{x = 0, 1, \ldots \infty}$. where λ is the average number of outcomes per unit time or region.

Poisson distribution as Limiting Form of Binomial Distribution

Poisson distribution is a limiting case of binomial distribution under the following conditions

(i) n, the number of trials is indefinitely large, i.e., $n \to \infty$.

(ii) p, the constant probability of success in each trial is very small, i.e., $p \to 0$.

(iii) $\lambda = np$ is finite or $p = \frac{\lambda}{n}$ and $q = 1 - \frac{\lambda}{n}$, where λ is a positive real number.

Areas of Application

1. The number of misprints on a page of a book.
2. The number of deaths due to accidents in a month on national highway 47.
3. The number of break downs of a printing machine in a day.
4. The number of vacancies occurring during a year in a particular department.

Moment Generating Function (m.g.f.) in Poisson Distribution

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}\,p(x) = \sum_{x=0}^{\infty} e^{tx}\,\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty} e^{tx}\,\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} \qquad \left(\because \boldsymbol{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots}\right)$$

$$= e^{-\lambda}\left[\frac{(\lambda e^t)^0}{0!} + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right] = e^{-\lambda}\left[1 + \frac{(\lambda e^t)^1}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right] = e^{-\lambda}\,e^{\lambda e^t} = \boldsymbol{e^{\lambda(e^t-1)}}$$

Mean and Variance Using Moment Generating Function in Poisson Distribution

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}e^{\lambda(e^t-1)}\right]_{t=0} = \left[e^{\lambda(e^t-1)}\,\lambda e^t\right]_{t=0} = e^{\lambda(e^0-1)}\,\lambda e^0 = \boldsymbol{\lambda} \qquad (\because \boldsymbol{e^0 = 1})$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\,\lambda e^t\,e^{\lambda(e^t-1)}\right]_{t=0} = \lambda\left[e^t\,e^{\lambda(e^t-1)}\,\lambda e^t + e^t\,e^{\lambda(e^t-1)}\right]_{t=0} = \boldsymbol{\lambda(\lambda+1)}$$

$$V(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \boldsymbol{\lambda}$$

Problems in Poisson distribution

1. ***On an average a typist makes 2 mistakes per page. What is the prob. that she will make (i) No errors on a page (ii) 4 or more errors on a particular page?***

Solution : $\lambda = 2$, Let X represent the number of errors on a page. $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0,1,\ldots\infty$

(i) $P(X=0) = \frac{e^{-2}\,2^0}{0!} = e^{-2} = 0.1353$

(ii) $P(X \geq 4) = 1 - P(X<4) = 1 - [P(0) + P(1) + P(2) + P(3)] = 1 - \left(\frac{e^{-2}\,2^0}{0!} + \frac{e^{-2}\,2^1}{1!} + \frac{e^{-2}\,2^2}{2!} + \frac{e^{-2}\,2^3}{3!}\right)$

$$= 1 - e^{-2}\left(1 + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = 0.1434$$

2. ***The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (i) Without a breakdown (ii) With only 1 breakdown (iii) With at least 1 breakdown***

Solution : $\lambda = 1.8$, Let X denote the number of breakdown of the computer in a month.

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,\ldots\infty$$

(i) $P(X=0) = \frac{e^{-1.8}\,(1.8)^0}{0!} = e^{-1.8} = 0.1653$

(ii) $P(X=1) = \frac{e^{-1.8}\,(1.8)^1}{1!} = e^{-1.8}(1.8) = 0.2975$

(iii) $P(X \geq 1) = 1 - P(X=0) = 0.8347$

3. ***It is known that the probability of an item produced by a certain machine will be defective is 5%. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) At least 2 defective items (ii) At most 2 defective items in a consignment of 1000 packets using Poisson distribution***

Solution : $p = 5\% = \frac{5}{100} = 0.05,\ q = 0.95,\ n = 20,\ \lambda = np = 20 \times \frac{5}{100} = 1$

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots\infty$

(i) $P(X \geq 2) = 1 - P(X < 2) = 1 - \left[\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!}\right] = 1 - [e^{-1} + e^{-1}] = 0.2642$

$NP(X \geq 2) = 1000 \times 0.2642 = 264$

(ii) $P(X \leq 2) = P(0) + P(1) + P(2) = \frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} = e^{-1} + e^{-1} + \frac{e^{-1}}{2} = 0.9197$

$N\,P(X \leq 2) = 1000 \times 0.9197 = 920$

4. ***A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is approximate probability that a box will fail to meet the guaranteed quality?***

Solution : $p = 5\% = \frac{5}{100} = 0.05,\ q = 0.95,\ n = 100,\ \lambda = np = 100 \times \frac{5}{100} = 5$

$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots\infty$

$P(X > 10) = 1 - P(X \leq 10)$

$= 1 - [P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) + P(9) + P(10)]$

$= 1 - e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!} + \frac{5^8}{8!} + \frac{5^9}{9!} + \frac{5^{10}}{10!}\right] = 0.014$

5. ***In a book of 520 pages, 390 typographical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.***

Solution : $\lambda = \frac{390}{520} = 0.75,\qquad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots\infty$

$P(1\ page\ contains\ no\ error) = P(X = 0) = \frac{e^{-0.75}(0.75)^0}{0!} = e^{-0.75}$

$P(5\ pages\ contains\ no\ error) = (e^{-0.75})^5 = 0.0235$

6. ***Let X be a RV following Poisson distribution such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$. Find the mean and standard deviation of X.***

Solution : $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots\infty$

$P(X = 2) = 9P(X = 4) + 90P(X = 6)\ \Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = 9\frac{e^{-\lambda}\lambda^4}{4!} + 90\frac{e^{-\lambda}\lambda^6}{6!}$

$Dividing\ by\ e^{-\lambda}\ \lambda^2$

$\frac{1}{2!} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!} \Rightarrow \frac{1}{2} = \frac{3\lambda^2}{8} + \frac{\lambda^4}{8} \Rightarrow \frac{\lambda^4}{8} + \frac{3\lambda^2}{8} - \frac{1}{2} = 0$

$\lambda^4 + 3\lambda^2 - 4 = 0$

$\lambda = 1, -4$

Mean = $\lambda = 1$, Variance = $\lambda = 1$, S.D. = $\sqrt{Variance} = 1$

7. ***Fit a Poisson distribution for the following data:***

x	***0***	***1***	***2***	***3***	***4***	***5***	***Total***
f	***142***	***156***	***69***	***27***	***5***	***1***	***400***

Finding the probability mass function and then finding the theoretical frequencies.

Solution : $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0,1,\dots\infty$

x	0	1	2	3	4	5	Total
f	142	156	69	27	5	1	400
fx	0	156	138	81	20	5	400

$\lambda = \bar{x} = \frac{\sum fx}{\sum f} = \frac{400}{400} = 1,\ N = 400$

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Theoretical frequencies are given by $N\,P(X=x)=\frac{N\,e^{-\lambda}\,\lambda^x}{x!}=\frac{400\,e^{-1}\,1^x}{x!}, x=0,1,\dots\infty$

$400\,P(X=0)=\frac{400\,e^{-1}\,1^0}{0!}=147.15$, $400\,P(X=1)=\frac{400\,e^{-1}\,1^1}{1!}=147.15$

$400\,P(X=2)=\frac{400\,e^{-1}\,1^2}{2!}=73.58$, $400\,P(X=3)=\frac{400\,e^{-1}\,1^3}{3!}=24.53$

$400\,P(X=4)=\frac{400\,e^{-1}\,1^4}{4!}=6.13$, $400\,P(X=5)=\frac{400\,e^{-1}\,1^5}{5!}=1.23$

x	0	1	2	3	4	5	Total
Theoretical f	147	147	74	25	6	1	400

GEOMETRIC DISTRIBUTION

Definition: If repeated independent trials can result in a success with probability p and a failure with probability $q=1-p$, then the probability distribution of the random variable X, the number of trials on which the first success occurs, is $\boldsymbol{P(X=x)=p\,q^{x-1}}$, $\boldsymbol{x=1,2,\dots\infty}$ **.**

Application: Geometric distribution has important application in queueing theory, related to the number of units which are being served or waiting to be served at any given time.

Moment Generating Function (M.G.F.) in Geometric Distribution

$$M_X(t)=\sum_{x=0}^{\infty}e^{tx}\,p(x)=\sum_{x=1}^{\infty}e^{tx}\,p\,q^{x-1}=\frac{p}{q}\sum_{x=1}^{\infty}(q\,e^t)^x=\frac{p}{q}[(q\,e^t)^1+(q\,e^t)^2+(q\,e^t)^3+\cdots]$$

$$=\frac{p}{q}(q\,e^t)[1+(q\,e^t)^1+(q\,e^t)^2+\cdots]=pe^t[1-q\,e^t]^{-1}=\boldsymbol{\frac{pe^t}{1-q\,e^t}}\qquad [\because \boldsymbol{(1-x)^{-1}=1+x+x^2+\cdots}]$$

Mean and Variance using Moment Generating Function in Geometric Distribution

$$E(X)=\left[\frac{d}{dt}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\left(\frac{pe^t}{1-q\,e^t}\right)\right]_{t=0}=\left[\frac{(1-q\,e^t)pe^t-pe^t(-q\,e^t)}{(1-q\,e^t)^2}\right]_{t=0}\qquad \left[\because \boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}}\right]$$

$$=\left[\frac{pe^t-p\,q\,e^{2t}+p\,q\,e^{2t}}{(1-q\,e^t)^2}\right]_{t=0}=\left[\frac{pe^t}{(1-q\,e^t)^2}\right]_{t=0}=\frac{pe^0}{(1-q\,e^0)^2}=\frac{p}{p^2}=\boldsymbol{\frac{1}{p}}\qquad (\because \boldsymbol{p+q=1})$$

$$E(X^2)=\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\,\frac{pe^t}{(1-q\,e^t)^2}\right]_{t=0}=\left[\frac{(1-q\,e^t)^2pe^t-pe^t2(1-q\,e^t)^1(-q\,e^t)}{(1-q\,e^t)^4}\right]_{t=0}\qquad \left[\because \boldsymbol{\frac{d}{dx}(u\,v)=u\,v'+v\,u'}\right]$$

$$=\left[\frac{(1-q\,e^0)^2pe^0-pe^02(1-q\,e^0)^1(-q\,e^0)}{(1-q\,e^0)^4}\right]=\frac{p^3+2p^2q}{p^4}=p^2\left(\frac{p+2q}{p^4}\right)=\boldsymbol{\frac{1+q}{p^2}}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{1+q}{p^2}-\frac{1}{p^2}=\frac{1+q-1}{p^2}=\boldsymbol{\frac{q}{p^2}}$$

ANOTHER FORM OF GEOMETRIC DISTRIBUTION

Definition: If X denotes the number of failure before the first success, then $\boldsymbol{P(X=x)=p\,q^x}$, $\boldsymbol{x=0,1,2,\dots\infty}$ **.**

Moment Generating Function (M.G.F.) in Geometric Distribution

$$M_X(t)=\sum_{x=0}^{\infty}e^{tx}\,p(x)=\sum_{x=0}^{\infty}e^{tx}\,p\,q^x=p\sum_{x=0}^{\infty}(q\,e^t)^x=p[1+(q\,e^t)^1+(q\,e^t)^2+(q\,e^t)^3+\cdots]$$

$$=p[1-q\,e^t]^{-1}=\boldsymbol{\frac{p}{1-q\,e^t}}\qquad [\because \boldsymbol{(1-x)^{-1}=1+x+x^2+\cdots}]$$

Mean and Variance using Moment Generating Function in Geometric Distribution

$$E(X)=\left[\frac{d}{dt}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\left(\frac{p}{1-q\,e^t}\right)\right]_{t=0}=\left[p\,\frac{d}{dt}(1-q\,e^t)^{-1}\right]_{t=0}=[p(-1)(1-q\,e^t)^{-1-1}(-q\,e^t)]_{t=0}$$

$$=[pqe^t\,(1-q\,e^t)^{-2}]_{t=0}=pqe^0\,(1-q\,e^0)^{-2}=\boldsymbol{\frac{q}{p}}\qquad (\because \boldsymbol{p+q=1})$$

$$E(X^2)=\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\,\frac{p\,q\,e^t}{(1-q\,e^t)^2}\right]_{t=0}=\left[\frac{(1-q\,e^t)^2p\,q\,e^t-p\,q\,e^t2(1-q\,e^t)^1(-q\,e^t)}{(1-q\,e^t)^4}\right]_{t=0}\qquad \left[\because \boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{vu'-uv'}{v^2}}\right]$$

$$E(X^2)=\left[\frac{(1-q\,e^0)^2p\,q\,e^0-p\,q\,e^02(1-q\,e^0)^1(-q\,e^0)}{(1-q\,e^0)^4}\right]=\frac{p^3q+2p^2q^2}{p^4}=p^2q\left(\frac{p+2q}{p^4}\right)=\boldsymbol{\frac{q^2+q}{p^2}}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{q^2+q}{p^2}-\frac{q^2}{p^2}=\frac{q^2+q-q^2}{p^2}=\boldsymbol{\frac{q}{p^2}}$$

Memory less Property of Geometric Distribution

Statement: ***If X is a random variable with geormetic distribution, then X lacks memory, in the sense that*** $\boldsymbol{P(X>s+t/X>s)=P(X>t)}$.

Proof : $P(X=x)=p\,q^{x-1}$, $x=1,\dots\infty$

$$P(X>s+t/X>s)=\frac{P(X>s+t\cap X>s)}{P(X>s)}=\frac{P(X>s+t)}{P(X>s)}$$

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$$P(X > k) = \sum_{x=k+1}^{\infty} p\, q^{x-1} = p\, q^{k+1-1} + p\, q^{k+2-1} + p\, q^{k+3-1} + \cdots = p\, q^{k} + p\, q^{k+1} + p\, q^{k+2} + \cdots$$

$$= p\, q^{k}(1 + q^1 + q^2 + \cdots) = p\, q^{k}(1-q)^{-1} = \frac{p\, q^{k}}{p} = \boldsymbol{q^{k}}$$

Hence $P(X > s + t) = q^{s+t}$ *and* $P(X > s) = q^{s}$

$$P(X > s + t / X > s) = \frac{q^{s+t}}{q^{s}} = \frac{q^{s}\, q^{t}}{q^{s}} = q^{t} = \boldsymbol{P(X > t)}$$

Problems in Geometric Distribution

1. ***If the probability that an applicant for a drivers license will pass the road test on any given trial is 0.8. What is the probability that he will finally pass the test (i) On the fourth trial (ii) In less than 4 trials?***

Solution : $p = 0.8,\ q = 1 - p = 0.2,\ P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

(i) $P(X = 4) = (0.8)\,(0.2)^{4-1} = 0.0064$

(ii) $P(X < 4) = P(1) + P(2) + P(3) = (0.8)\,[(0.2)^{1-1} + (0.2)^{2-1} + (0.2)^{3-1}] = 0.992$

2. ***A typist types 2 letters erroneously for every 100 letters. What is the probability that the 10th letter typed is the 1st erroneous letter?***

Solution : $p = \frac{2}{100} = 0.02,\ q = 1 - p = 0.98,\ P(X = 10) = (0.02)\,(0.98)^{10-1} = 0.0167$

3. ***A die is tossed until 6 appears. What is the probability that it must be tossed more than 5 times?***

Solution : $p = \frac{1}{6},\ q = 1 - p = \frac{5}{6},\ P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

$$P(X > 5) = 1 - P(X \le 5) = 1 - [P(1) + P(2) + P(3) + P(4) + P(5)]$$

$$= 1 - \frac{1}{6}\left[\left(\frac{5}{6}\right)^{1-1} + \left(\frac{5}{6}\right)^{2-1} + \left(\frac{5}{6}\right)^{3-1} + \left(\frac{5}{6}\right)^{4-1} + \left(\frac{5}{6}\right)^{5-1}\right] = 0.4019$$

4. ***A trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8. (i) What is the probability that the target would be first hit at the 6th attempt? (ii) What is the probability that it takes less than 5 shots?***

Solution : $p = 0.8,\ q = 1 - p = 0.2,\ P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

(i) $P(X = 6) = (0.8)\,(0.2)^{6-1} = 0.00026$

(ii) $P(X < 5) = P(1) + P(2) + P(3) + P(4) = (0.8)\,[(0.2)^{1-1} + (0.2)^{2-1} + (0.2)^{3-1} + (0.2)^{4-1}] = 0.9984$

5. ***The probability that a candidate can pass in an exam is 0.6. (i) What is the probability that he pass in the 3rd trial (ii) What is the probability that he pass before the 3rd trial?***

Solution : : $p = 0.6,\ q = 1 - p = 0.4,\ P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

(i) $P(X = 3) = (0.6)\,(0.4)^{3-1} = 0.096$

(ii) $P(X < 3) = P(1) + P(2) = (0.6)\,[(0.4)^{1-1} + (0.4)^{2-1}] = 0.84$

6. ***A discrete RV X has moment generating function $\boldsymbol{M_X(t) = (5 - 4\, e^{t})^{-1}}$ find $\boldsymbol{P(X = 5\ or\ 6)}$.***

Solution : $P(X = x) = p\, q^{x},\ x = 0, 1, \ldots \infty,\qquad M_X(t) = p(1 - q\, e^{t})^{-1}$

$$M_X(t) = (5 - 4\, e^{t})^{-1} = \frac{1}{5}\left(1 - \frac{4}{5}\, e^{t}\right)^{-1},\ p = \frac{1}{5},\ q = \frac{4}{5}$$

$$P(X = 5\ or\ 6) = P(x = 5) + P(X = 6) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{5} + \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{6} = 0.118$$

7. ***A coin is tossed until the first head occurs. Assuming that the tosses are independent and the probability of a head occurring is p, find the value of p so that the probability that an odd number of tosses is required is equal to 0.6. Can you find a value of p so that the probability is 0.5 that an odd number of tosses is required?***

Solution : $P(X = x) = p\, q^{x-1},\ x = 1, \ldots \infty$

$$P(X = odd\ number) = P(1) + P(3) + P(5) + \cdots = p\,(q^{1-1} + q^{3-1} + q^{5-1} + \cdots)$$

$$= p(1 + q^2 + q^4 + \ldots) = p[1 + (q^2) + (q^2)^2 + \ldots]$$

$$= p(1 - q^2)^{-1} = \frac{p}{1 - q^2} = \frac{p}{(1-q)(1+q)} = \frac{p}{p(1+q)} = \frac{1}{1+q}$$

Now $\frac{1}{1+q} = 0.6 \Rightarrow \frac{1}{2-p} = 0.6 \Rightarrow 0.6(2 - p) = 1 \Rightarrow 0.6p = 0.2 \Rightarrow p = \frac{1}{3}$

Now $\frac{1}{1+q} = 0.5 \Rightarrow \frac{1}{2-p} = 0.5 \Rightarrow 0.5(2 - p) = 1 \Rightarrow 0.5p = 0 \Rightarrow p = 0$

Though we get $p = 0$ it is meaningless. Hence the value of p cannot be found out.

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8. ***A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials required to open the door, (i) If unsuccessful keys are not eliminated from further selection (ii) If they are eliminated from further selection.***

Solution : (i) Let the 1st success be got at the x^{th} trial and not before. Then the RV X (denoting the number of trials required to open the door) follows geometric distribution $P(X=x)=p\,q^{x-1}, x=1,\dots\infty,\ p=\frac{1}{n},\ q=1-\frac{1}{n}$

Mean = $E(X)=\frac{1}{p}=\frac{1}{\left(\frac{1}{n}\right)}=n$, Variance = $V(X)=\frac{q}{p^2}=\frac{\left(1-\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^2}=n(n-1)$

(ii) Let X be the number of trials required to open the door. If unsuccessful keys are eliminated then X will take values $1,2,\dots n$. $\text{P(Success in } 1^{st} \text{ trial)}=\frac{1}{n}$

$\text{P}(1^{st}\text{ Success in }2^{nd}\text{ trial}) = \text{P(failure in }1^{st}\text{trial)}\times\text{P}(\text{Success in }2^{nd}\text{trial})=\left(1-\frac{1}{n}\right)\left(\frac{1}{n-1}\right)=\frac{1}{n}$

$\text{P}(1^{st}\text{ Success in }3^{nd}\text{ trial}) = \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)=\frac{1}{n}$ and so on($\because$ *in* r^{th} trial there are $n-r+1$ keys

$\text{P}(X=x)=\text{P}(1^{st}\text{ Success in }x^{th}\text{ trial})$

Mean = $E(X)=\sum_{x=1}^{n}x\,p(x)=\sum_{x=1}^{n}x\left(\frac{1}{n}\right)=\frac{1}{n}\sum_{x=1}^{n}x=\frac{1}{n}(1+2+\dots+n)=\frac{1}{n}\left[\frac{n(n+1)}{2}\right]=\frac{n+1}{2}$

$E(X^2)=\sum_{x=1}^{n}x^2\,p(x)=\sum_{x=1}^{n}x^2\left(\frac{1}{n}\right)=\frac{1}{n}\sum_{x=1}^{n}x^2=\frac{1}{n}(1^2+2^2+\dots+n^2)=\frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]=\frac{(n+1)(2n+1)}{6}$

Variance = $V(X)=E(X^2)-[E(X)]^2=\frac{(n+1)(2n+1)}{6}-\left(\frac{n+1}{2}\right)^2=\frac{(n+1)(2n+1)}{6}-\frac{n^2+2n+1}{4}=\frac{n^2-1}{12}$

9. ***A and B shoot independently until each has hit his own target and they have probabilities of $\frac{3}{5},\frac{5}{7}$ of hitting the targets at each shot respectively. Find the probability that B will require more shots than A.***

Solution : Let X and Y are the number of shots fired by A and B respectively. $\because$ X and Y are independent geometric variates. $P(X=x,\ Y=y)=P(X=x)\,P(Y=y)=\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{x-1}\left(\frac{5}{7}\right)\left(\frac{2}{7}\right)^{y-1},\ x,y=1,2,3\dots$

$P(Y>X)=\sum_{x=1}^{\infty}P(X=x,\ Y>X)=\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{x-1}\left(\frac{5}{7}\right)\left(\frac{2}{7}\right)^{y-1}=\left(\frac{3}{7}\right)\sum_{x=1}^{\infty}\sum_{y=x+1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left(\frac{2}{7}\right)^{y-1}$

$=\left(\frac{3}{7}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\sum_{y=x+1}^{\infty}\left(\frac{2}{7}\right)^{y-1}=\left(\frac{3}{7}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left[\left(\frac{2}{7}\right)^{x}+\left(\frac{2}{7}\right)^{x+1}+\left(\frac{2}{7}\right)^{x+2}+\cdots\right]$

$=\left(\frac{3}{7}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left(\frac{2}{7}\right)^{x}\left[1+\left(\frac{2}{7}\right)^{1}+\left(\frac{2}{7}\right)^{2}+\cdots\right]=\left(\frac{3}{7}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left(\frac{2}{7}\right)^{x}\left(1-\frac{2}{7}\right)^{-1}=\left(\frac{3}{5}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x-1}\left(\frac{2}{7}\right)^{x}$

$=\left(\frac{3}{2}\right)\sum_{x=1}^{\infty}\left(\frac{2}{5}\right)^{x}\left(\frac{2}{7}\right)^{x}=\left(\frac{3}{2}\right)\sum_{x=1}^{\infty}\left(\frac{4}{35}\right)^{x}=\left(\frac{3}{2}\right)\left[\left(\frac{4}{35}\right)^{1}+\left(\frac{4}{35}\right)^{2}+\left(\frac{4}{35}\right)^{3}+\cdots\right]$

$=\left(\frac{3}{2}\right)\left(\frac{4}{35}\right)\left[1+\left(\frac{4}{35}\right)^{1}+\left(\frac{4}{35}\right)^{2}+\cdots\right]=\left(\frac{6}{35}\right)\left(1-\frac{4}{35}\right)^{-1}=\frac{6}{31}$

UNIFORM DISTRIBUTION

Definition : A continuous RV X with parameters a and b is uniform, if it has the p.d.f is $\boldsymbol{f(x)}=\begin{cases}\frac{\mathbf{1}}{\boldsymbol{b-a}}, & \boldsymbol{a<x<b}\\ \mathbf{0}, & \boldsymbol{otherwise}\end{cases}$

Moment Generating Function (M.G.F.) in Negative Binomial Distribution

$M_X(t)=\int_{-\infty}^{\infty}e^{tx}\,f(x)\,dx=\int_a^b e^{tx}\,\frac{1}{b-a}\,dx=\frac{1}{b-a}\int_a^b e^{tx}\,dx=\frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b=\boldsymbol{\frac{e^{bt}-e^{at}}{t(b-a)}}$

Mean and Variance using Moment Generating Function in Negative Binomial Distribution

$E(X)=\int_{-\infty}^{\infty}x\,f(x)\,dx=\int_a^b x\,\frac{1}{b-a}\,dx=\frac{1}{b-a}\int_a^b x\,dx=\frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b=\frac{b^2-a^2}{2(b-a)}=\frac{(b-a)(b+a)}{2(b-a)}=\boldsymbol{\frac{b+a}{2}}$

$E(X^2)=\int_{-\infty}^{\infty}x^2\,f(x)\,dx=\int_a^b x^2\,\frac{1}{b-a}\,dx=\frac{1}{b-a}\int_a^b x^2\,dx=\frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b=\frac{b^3-a^3}{3(b-a)}=\frac{(b-a)(b^2+ab+a^2)}{3(b-a)}=\boldsymbol{\frac{b^2+ab+a^2}{3}}$

$V(X)=E(X^2)-[E(X)]^2=\frac{b^2+ab+a^2}{3}-\left(\frac{b+a}{2}\right)^2=\frac{b^2+ab+a^2}{3}-\frac{b^2+2ab+a^2}{4}=\boldsymbol{\frac{(b-a)^2}{12}}$

Problems in Uniform Distribution

1. ***If the MGF of a uniform distribution for a random variable X is $\frac{1}{t}\left(e^{5t}-e^{4t}\right)$, find $E(X)$.***

Solution: $M_X(t)=\frac{e^{bt}-e^{at}}{t(b-a)},\ b=5,\ a=4,\quad E(X)=\frac{b+a}{2}=\frac{5+4}{2}=4.5$

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2. A random variable X has a uniform distribution over $(-3, 3)$.

(i) $\boldsymbol{P(X<2)}$ ***(ii)*** $\boldsymbol{P(|X|<2)}$ ***(iii)*** $\boldsymbol{P(|X-2|<2)}$ ***(iv) Find K for which*** $\boldsymbol{P(X>K)=\frac{1}{3}}$

Solution: $f(x)=\frac{1}{b-a},\ a<x<b,\quad f(x)=\frac{1}{6},\ -3<x<3$

(i) $P(X<2)=\int_{-3}^{2}\frac{1}{6}\,dx=\frac{1}{6}[x]_{-3}^{2}=\frac{5}{6}$

(ii) $P(|X|<2)=\int_{-2}^{2}\frac{1}{6}\,dx=\frac{1}{6}[x]_{-2}^{2}=\frac{2}{3}$

(iii) $P(|X-2|<2)=P(-2<(x-2)<2)=P(0<x<4)=\int_{0}^{3}\frac{1}{6}\,dx=\frac{1}{6}[x]_{0}^{3}=\frac{1}{2}$

(iv) $P(X>k)=\frac{1}{3}\Rightarrow\int_{k}^{3}\frac{1}{6}\,dx=\frac{1}{3}\Rightarrow\frac{1}{6}[x]_{k}^{3}=\frac{1}{3}\Rightarrow 3-k=2\Rightarrow \boldsymbol{k=1}$

3. *Busses arrive at a specified stop at 15 min intervals starting at 7 am this is they arrive at 7, 7.15, 7.30, 7.45 and so on. If a passenger arrives at the stop at a random time that is uniformly distributed between 7 and 7.30 am find the probability that he waits (i) Less than 5 min for a bus. (ii) At least 12 min for a bus.*

Solution: $f(x)=\frac{1}{b-a},\ a<x<b,\quad f(x)=\frac{1}{30},\ 0<x<30$

(i) $P(\textit{Less than } 5 \textit{ minutes})=P(10<X<15)+P(25<X<30)$

$$=\int_{10}^{15}\frac{1}{30}\,dx+\int_{25}^{30}\frac{1}{30}dx=\frac{1}{30}[x]_{10}^{15}+\frac{1}{30}[x]_{25}^{30}=\frac{1}{3}$$

(ii) $P(\textit{Atleast } 12 \textit{ minutes})=P(0<X<3)+P(15<X<18)$

$$=\int_{0}^{3}\frac{1}{30}\,dx+\int_{15}^{18}\frac{1}{30}dx=\frac{1}{30}[x]_{0}^{3}+\frac{1}{30}[x]_{15}^{18}=\frac{1}{5}$$

4. *Trains arrive at a station at 15 minutes intervals starting at 4 am. If a passenger arrive to the station at a time that is uniformly distributed between 9.00 and 9.30, find the probability that he has to wait for the train for*

(i) Less than 6 minutes (ii) More than 10 minutes.

Solution: $f(x)=\frac{1}{b-a},\ a<x<b,\quad f(x)=\frac{1}{30},\ 0<x<30$

(i) $P(\textit{Less than } 6 \textit{ minutes})=P(9<X<15)+P(24<X<30)$

$$=\int_{9}^{15}\frac{1}{30}\,dx+\int_{24}^{30}\frac{1}{30}dx=\frac{1}{30}[x]_{9}^{15}+\frac{1}{30}[x]_{24}^{30}=\frac{1}{30}(15-9)+\frac{1}{30}(30-24)=\frac{2}{5}$$

(ii) $P(\textit{More than } 10 \textit{ minutes})=P(0<X<5)+P(15<X<20)$

$$=\int_{0}^{5}\frac{1}{30}\,dx+\int_{15}^{20}\frac{1}{30}dx=\frac{1}{30}[x]_{0}^{5}+\frac{1}{30}[x]_{15}^{20}=\frac{1}{3}$$

5. *If X is a RV with a continuous distribution function* $\boldsymbol{F(x)}$***, prove that*** $\boldsymbol{Y=F(X)}$ ***has a uniform distribution in*** $\boldsymbol{(0,1)}$***. Further if*** $\boldsymbol{f(x)=\begin{cases}\frac{1}{2}(x-1), & 1\le x\le 3\\ 0, & otherwise\end{cases}}$ ***find the range of Y corresponding to the range*** $\boldsymbol{1.1\le X\le 2.9}$***.***

Solution: The distribution function of Y is given by

$G_Y(y)=P(Y\le y)=P(F(X)\le y)=P[X\le F^{-1}(y)]=F[F^{-1}(y)]=y$,

$g(y)=\frac{d}{dy}G_Y(y)=1$

Also the range of Y is $0\le y\le 1$, since the range of F(x) is (0, 1).

EXPONENTIAL DISTRIBUTION

Definition: A continuous RV X defined in $(0,\ \infty)$ is said to follow an exponential distribution if the probability density function is $\boldsymbol{f(x)=\lambda\, e^{-\lambda x},\ x\ge 0}$.

Application: Exponential distribution is useful in queueing theory and reliability theory. Time to failure of a component and time between arrivals can be modeled using exponential distribution.

Moment Generating Function (M.G.F.) in Negative Binomial Distribution

$$M_X(t)=\int_{-\infty}^{\infty}e^{tx}f(x)\,dx=\int_{0}^{\infty}e^{tx}\lambda\,e^{-\lambda x}\,dx=\lambda\int_{0}^{\infty}e^{-(\lambda-t)x}\,dx=\lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_{0}^{\infty}=\boldsymbol{\frac{\lambda}{\lambda-t}}$$

Mean and Variance using Moment Generating Function in Negative Binomial Distribution

$$E(X)=\left[\frac{d}{dt}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right]_{t=0}=\lambda\left[\frac{d}{dt}(\lambda-t)^{-1}\right]_{t=0}=\lambda[(-1)(\lambda-t)^{-2}(-1)]_{t=0}=\boldsymbol{\frac{1}{\lambda}}$$

$$E(X^2)=\left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}=\left[\frac{d}{dt}\lambda\,(\lambda-t)^{-2}\right]_{t=0}=\lambda\,[(-2)(\lambda-t)^{-3}(-1)]_{t=0}=\boldsymbol{\frac{2}{\lambda^2}}$$

$$V(X)=E(X^2)-[E(X)]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\boldsymbol{\frac{1}{\lambda^2}}$$

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<u>Memory less Property of Exponential Distribution</u>

***Statement:* If X is exponential distributed with parameter λ, then for any 2 positive integers s and t $\boldsymbol{P(X > s + t/X > s) = P(X > t)}$.**

Proof : $P(X > s + t/X > s) = \frac{P(X>s+t \cap X>s)}{P(X>s)} = \frac{P(X>s+t)}{P(X>s)}$, $\quad f(x) = \lambda\, e^{-\lambda x},\ x \ge 0$

$$P(X > k) = \int_k^\infty f(x)\, dx = \int_k^\infty \lambda\, e^{-\lambda x}\, dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^\infty = \boldsymbol{e^{-\lambda k}}$$

Hence $P(X > s + t) = e^{-\lambda(s+t)}$ *and* $P(X > s) = e^{-\lambda s}$

$$P(X > s + t/X > s) = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \boldsymbol{P(X > t)}$$

<u>Problems in Exponential Distribution</u>

1. *The time (in hours) required to repairs a machine is exponential, distributed with parameter $\lambda = \frac{1}{2}$. (i) What is the probability that the repair time exceeds 2 hours? (ii) What is the conditional prob. that a repair takes at least 10h given that its duration exceeds 9h?*

Solution: $f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{2}\, e^{-\frac{x}{2}},\ x > 0$

(i) $P(X > 2) = \int_2^\infty f(x)dx = \int_2^\infty \frac{1}{2}\, e^{-\frac{x}{2}}dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_2^\infty = \frac{e^{-\infty} - e^{-1}}{-1} = e^{-1} = 0.3679$

(ii) $P(X \ge 10/X > 9) = P(X > 1) = \int_1^\infty f(x)dx = \int_1^\infty \frac{1}{2}\, e^{-\frac{x}{2}}dx = \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-\frac{1}{2}}\right]_1^\infty = \frac{e^{-\infty} - e^{-\frac{1}{2}}}{-1} = e^{-\frac{1}{2}} = 0.6065$

2. *The mileage which car owners get with a certain kind of radial tire is a RV having an exponential distribution with mean 40,000 km. Find the prob. that one of these tires will last (i) Atleast 20,000 km (ii) At most 30,000 km*

Solution: $Mean = \frac{1}{\lambda} = 40{,}000\ km,\ \lambda = \frac{1}{40{,}000},\ \ f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}},\ x > 0$

(i) $P(X \ge 20{,}000) = \int_{20{,}000}^\infty f(x)dx = \int_{20{,}000}^\infty \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}}dx = \frac{1}{40{,}000}\left[\frac{e^{-\frac{x}{40{,}000}}}{-\frac{1}{40{,}000}}\right]_{20{,}000}^\infty = e^{-\frac{1}{2}} = 0.6065$

(ii) $P(X \le 30{,}000) = \int_0^{30{,}000} f(x)dx = \int_0^{30{,}000} \frac{1}{40{,}000}\, e^{-\frac{x}{40{,}000}}dx = \frac{1}{40{,}000}\left[\frac{e^{-\frac{x}{40{,}000}}}{-\frac{1}{40{,}000}}\right]_0^{30{,}000} = 1 - e^{-\frac{3}{4}} = 0.527$

3. *The length of time a person speaks over phone follows exponential distribution with mean 6. What is the probability that the person will talk for (i) More than 8 min (ii) Less than 4 min (iii) Between 4 and 8 min*

Solution: $Mean = \frac{1}{\lambda} = 6,\ \lambda = \frac{1}{6},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0, \quad f(x) = \frac{1}{6}\, e^{-\frac{x}{6}},\ x > 0$

(i) $P(X > 8) = \int_8^\infty f(x)dx = \int_8^\infty \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_8^\infty = e^{-\frac{4}{3}} = 0.2635$

(ii) $P(X < 4) = \int_0^4 f(x)dx = \int_0^4 \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_0^4 = 1 - e^{-\frac{2}{3}} = 0.4865$

(iii) $P(4 \le X \le 8) = \int_4^8 \frac{1}{6}\, e^{-\frac{x}{6}}\, dx = \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{-\frac{1}{6}}\right]_4^8 = e^{-\frac{2}{3}} - e^{-\frac{4}{3}} = 0.5134 - 0.2635 = 0.2499$

4. *The amount of time that a watch can run without having to be reset is a random variable having exponential distribution, with mean 120 days. Find the prob. that such a watch will have to be reset in less than 24 days.*

Solution:, $Mean = \frac{1}{\lambda} = 120\ days, \lambda = \frac{1}{120},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ f(x) = \frac{1}{120}\, e^{-\frac{x}{120}},\ x > 0$

$$P(X < 24) = \int_0^{24} f(x)dx = \int_0^{24} \frac{1}{120}\, e^{-\frac{x}{120}}dx = \frac{1}{120}\left[\frac{e^{-\frac{x}{120}}}{-\frac{1}{120}}\right]_0^{24} = 1 - e^{-\frac{1}{5}} = 0.1813$$

5. *The number of kilo meters that a car can run before its battery has to be replaced is exponentially distributed with an average of 10,000 kms. If the owner desires to take a tour consisting of 8000 kms, what is the probability that he will be able to complete is his tour with our replacing the battery?*

Solution:, $Mean = \frac{1}{\lambda} = 10{,}000, \lambda = \frac{1}{10{,}000},\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ f(x) = \frac{1}{10{,}000}\, e^{-\frac{x}{10{,}000}},\ x > 0$

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$$P(X > 8000) = \int_{8000}^{\infty} f(x)dx = \int_{8000}^{\infty} \frac{1}{10{,}000}\, e^{-\frac{x}{10{,}000}}dx = \frac{1}{10{,}000}\left[\frac{e^{-\frac{x}{10{,}000}}}{-\frac{1}{10{,}000}}\right]_{8000}^{\infty} = e^{-\frac{4}{5}} = 0.4493$$

6. ***In a construction site, 3 lorries unload materials per hour, on an average. What is the probability that the time between arrival of successive lorries will be (i) at least 30 minutes (ii) less than 10 minutes?***

Solution: $\lambda = 3,\ f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = 3\, e^{-3x},\ x > 0$

(i) Probability that the time between arrival of successive lorries equal to 30 minutes or $\frac{1}{2}$ hour

$$P\left(X \geq \frac{1}{2}\right) = \int_{\frac{1}{2}}^{\infty} f(x)dx = \int_{\frac{1}{2}}^{\infty} 3\, e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_{\frac{1}{2}}^{\infty} = e^{-\frac{3}{2}} = 0.223$$

(ii) Probability that the time between arrival of successive lorries equal to 10 minutes or $\frac{1}{6}$ hour

$$P\left(X < \frac{1}{6}\right) = \int_{0}^{\frac{1}{6}} f(x)dx = \int_{0}^{\frac{1}{6}} 3\, e^{-3x}dx = 3\left[\frac{e^{-3x}}{-3}\right]_{0}^{\frac{1}{6}} = 1 - e^{-\frac{1}{2}} = 0.393$$

7. ***The life length X of an electronic component follows an exponential distribution. There are 2 processes by which the component may be manufactured. The expected life length of the component is 100h. if process I is used to manufacture, while it is 150 h if process II is used. The cost of manufacturing a single component by process I is Rs. 10, while it is Rs. 20 for process II. Moreover if the component lasts less than the guaranteed life of 200 h, a loss of Rs. 50 is to be borne by the manufacturer. Which process is advantageous to the manufacturer?***

Solution: If process I is used, the density function of X is given by

$Mean = \frac{1}{\lambda} = 100,\ \lambda = \frac{1}{100}, f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{100}\, e^{-\frac{x}{100}},\ x > 0$

$$P(X \geq 200) = \int_{200}^{\infty} \frac{1}{100}\, e^{-\frac{x}{100}}dx = \frac{1}{100}\left[\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right]_{200}^{\infty} = e^{-2}$$

$$P(X < 200) = \int_{0}^{200} \frac{1}{100}\, e^{-\frac{x}{100}}dx = \frac{1}{100}\left[\frac{e^{-\frac{x}{100}}}{-\frac{1}{100}}\right]_{0}^{200} = 1 - e^{-2}$$

Similarly, if process II is used, the density function of X is given by

$Mean = \frac{1}{\lambda} = 150,\ \lambda = \frac{1}{150}, f(x) = \lambda\, e^{-\lambda x},\ x > 0,\ \ f(x) = \frac{1}{150}\, e^{-\frac{x}{150}}\ \ x > 0$

$$P(X \geq 200) = \int_{200}^{\infty} \frac{1}{150}\, e^{-\frac{x}{150}}dx = \frac{1}{150}\left[\frac{e^{-\frac{x}{150}}}{-\frac{1}{150}}\right]_{200}^{\infty} = e^{-\frac{4}{3}}$$

$$P(X < 200) = \int_{0}^{200} \frac{1}{150}\, e^{-\frac{x}{150}}dx = \frac{1}{150}\left[\frac{e^{-\frac{x}{150}}}{-\frac{1}{150}}\right]_{0}^{200} = 1 - e^{-\frac{4}{3}}$$

Let C_1 and C_2 be the costs per component corresponding to the processes I and II respectively.

Then $C_1 = \begin{cases} 10, & X \geq 200 \\ 60, & X < 200 \end{cases}$

$$E(C_1) = 10 \times P(X \geq 200) + 60 \times P(X < 200) = 10 \times e^{-2} + 60 \times (1 - e^{-2}) = 60 - 50e^{-2} = 53.235$$

Now $C_2 = \begin{cases} 20, & X \geq 200 \\ 70, & X < 200 \end{cases}$

$$E(C_2) = 20 \times P(X \geq 200) + 70 \times P(X < 200) = 20 \times e^{-\frac{4}{3}} + 70 \times \left(1 - e^{-\frac{4}{3}}\right) = 70 - 50e^{-\frac{4}{3}} = 56.765$$

Since $E(C_1) < E(C_2)$, process I is advantageous to the manufacturer.

NORMAL DISTRIBUTION

The Normal distribution was first described by De Moive in 1933 as the limiting form of Binomial distribution as the number of trials becomes infinite. This discovery came into limelight after its discovery by both Laplace and Gauss half a century later. So this distribution is also called Gaussion distribution.

Definition: A continuous RV X, with parameters μ and σ^2 is normal if it has a probability density function $\boldsymbol{f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}, -\infty < x < \infty$

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Standard Normal distribution:

If X is a RV following normal distribution with parameter μ and σ, then $z = \frac{X-\mu}{\sigma}$ is called a Standard Normal variate and the p.d.f. of the standard variate Z is given by $\boldsymbol{\varphi(z)} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}\,\boldsymbol{\pi}}}\, \boldsymbol{e}^{-\frac{z^2}{2}}, -\infty < z < \infty$

Application:

(i) The most important continuous probability distribution in the statistics field is Normal distribution.
(ii) In nature like rainfall and meteorological studies.
(iii) In industry
(iv) In error calculation of experiments
(v) Statistical quality control

Moment Generating Function (M.G.F.) in Normal Distribution

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\,\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sigma\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma} \Rightarrow dz = \frac{dx}{\sigma},\ \ x = \mu + \sigma z$

$x = -\infty,\ z = -\infty\ and\ x = \infty,\ z = \infty$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{-\frac{z^2}{2}} \sigma\, dz = \frac{e^{\mu t}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} e^{-\frac{z^2}{2}} dz = \frac{e^{\mu t}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2}-t\sigma z\right)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2t\sigma z)} dz = \frac{e^{\mu t}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2-2t\sigma z+\sigma^2 t^2-\sigma^2 t^2)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} e^{\frac{\sigma^2 t^2}{2}} dz = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

Put $u = z - \sigma t \Rightarrow du = dz$

$z = -\infty,\ u = -\infty\ and\ z = \infty,\ u = \infty$

$$M_X(t) = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} du = \frac{2\,e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\,\pi}} \int_{0}^{\infty} e^{-\frac{u^2}{2}} du$$

Put $t^2 = \frac{u^2}{2} \Rightarrow 2t\, dt = u\, du,\ du = \sqrt{2}\, dt,\ u = t\sqrt{2}\ ,\qquad u = 0,\ t = 0\ and\ u = \infty,\ t = \infty$

$$M_X(t) = \frac{2\,e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{2\,\pi}} \int_{0}^{\infty} e^{-t^2} \sqrt{2}\, dt = \frac{2\,e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\ dt$$

Put $x = t^2 \Rightarrow dx = 2t\, dt,\ dt = \frac{dx}{2\sqrt{x}},\ t = \sqrt{x}\ ,\ \ t = 0,\ x = 0\ and\ t = \infty,\ x = \infty\left[\because \boldsymbol{\Gamma(n)} = \int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-x}\, x^{n-1}\ dx}\right]$

$$M_X(t) = \frac{2\,e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x} \frac{dx}{2\sqrt{x}} = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x} x^{-\frac{1}{2}}\ dx = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x} x^{\left(1-\frac{1}{2}\right)}\ dx$$

$$M_X(t) = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \Gamma(1) = \frac{e^{\mu t+\frac{\sigma^2 t^2}{2}}}{\sqrt{\pi}} \sqrt{\pi} = \boldsymbol{e^{\mu t+\frac{\sigma^2 t^2}{2}}}$$

Mean and Variance using Moment Generating Function in Normal Distribution

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(e^{\mu t+\frac{\sigma^2 t^2}{2}}\right)\right]_{t=0} = \left[e^{\mu t+\frac{\sigma^2 t^2}{2}}(\mu+\sigma^2 t)\right]_{t=0} = \boldsymbol{\mu}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\ e^{\mu t+\frac{\sigma^2 t^2}{2}}(\mu+\sigma^2 t)\right]_{t=0} = \left[e^{\mu t+\frac{\sigma^2 t^2}{2}}(\sigma^2) + e^{\mu t+\frac{\sigma^2 t^2}{2}}(\mu+\sigma^2 t)^2\right]_{t=0} = \boldsymbol{\sigma^2+\mu^2}$$

$$V(X) = E(X^2) - [E(X)]^2 = \sigma^2 + \mu^2 - \mu^2 = \boldsymbol{\sigma^2}$$

<u>Problems in Normal Distribution</u>

1. ***If X is normally distributed and the mean X is 12 and the SD is 4. Find out the probability of the following (i) $\boldsymbol{X \ge 20}$ (ii) $\boldsymbol{X \le 20}$ (iii) $\boldsymbol{0 \le X \le 12}$***

Solution: $\mu = 12,\ \sigma = 4$

(i) $P(X \ge 20) = P\left(\frac{X-\mu}{\sigma} \ge \frac{20-\mu}{\sigma}\right) = P\left(Z \ge \frac{20-12}{4}\right) = P(Z \ge 2) = 0.5 - P(0 \le Z \le 2) = 0.5 - 0.4772 = 0.0228$

(ii) $P(X \le 20) = P\left(\frac{X-\mu}{\sigma} \le \frac{20-\mu}{\sigma}\right) = P\left(Z \le \frac{20-12}{4}\right) = P(Z \le 2) = 0.5 + P(0 \le Z \le 2) = 0.5 + 0.4772 = 0.9772$

(iii) $P(0 \le X \le 12) = P\left(\frac{0-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{12-\mu}{\sigma}\right) = P\left(\frac{-12}{4} \le Z \le \frac{12-12}{4}\right) = P(-3 \le Z \le 0) = P(0 \le Z \le 3) = 0.4987$

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2. ***In an examination the marks obtained by the students in Maths, Physics and Chemistry are normally distributed about mean 50, 52, 48 and S.D. 15, 12, 16 respectively. Find the prob. of securing a total mark of 180 or above.***

Solution: Let X, Y, Z be the marks of respective subjects. The total marks $T = X + Y + Z$

$$\mu = E(T) = E(X+Y+Z) = E(X) + E(Y) + E(Z) = 50 + 52 + 48 = 150$$

$$\sigma^2 = V(T) = V(X+Y+Z) = V(X) + V(Y) + V(Z) = 15^2 + 12^2 + 16^2 = 225 + 144 + 256 = 625\,, \quad \sigma = 25$$

$$P(T \geq 180) = P\left(Z \geq \frac{180-150}{25}\right) = P(Z \geq 1.2) = 0.5 - P(0 \leq Z \leq 1.2) = 0.5 - 0.3849 = 0.1151$$

3. ***If the actual amount of instant coffee which a filling machine puts into '6 – ounce' jars is a RV having a normal distribution with S.D. is 0.05 ounce and if only 3% of the jars are to contain less than 6 ounce of coffee, what must be the mean fill of these jars?***

Solution: Let X be the actual amount of coffee put into the jars. Then X follows N(μ, σ), σ = 0.05

$$P(X < 6) = 3\% = \frac{3}{100} = 0.03 \Rightarrow \quad P\left(-\infty < Z < \frac{6-\mu}{0.05}\right) = 0.03 \quad \Rightarrow \quad P\left(0 < Z < \frac{\mu-6}{0.05}\right) = 0.5 - 0.03 = 0.47$$

From the table, $\frac{\mu-6}{0.05} = 1.808$, $\quad \boldsymbol{\mu = 6.0904\ ounces}$

4. ***In an engineering examination, a student is considered to have failed, secured second class, first class and distinction, according as he scores less than 45%, between 45% and 60%, between 60% and 75% and above 75% respectively. In a particular year 10% of the students failed in the examination and 5% of the students got distinction. Find the percentages of students who have got first class and second class.***

Solution: Let X follow the distributionN(μ, σ).

Given: $P(X < 45) = 0.10$ *and* $P(X > 75) = 0.05$

$P\left(-\infty < \frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.1$ *and* $P\left(\frac{75-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \infty\right) = 0.05$

$P\left(-\infty < Z < \frac{45-\mu}{\sigma}\right) = 0.1$ *and* $P\left(\frac{75-\mu}{\sigma} < Z < \infty\right) = 0.05$

$P\left(0 < Z < \frac{\mu-45}{\sigma}\right) = 0.4$ *and* $P\left(0 < Z < \frac{75-\mu}{\sigma}\right) = 0.45$

From the table, $\frac{\mu-45}{\sigma} = 1.28$ *and* $\frac{75-\mu}{\sigma} = 1.64$

$\mu - 1.28\sigma = 45$ (1) *and* $\mu + 1.64\sigma = 75$ (2)

Solving equations (1) and (2), $\mu = 58.15$ *and* $\sigma = 10.28$

$$P(Students\ gets\ first\ class) = P(60 < X < 75) = P\left(\frac{60-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{75-\mu}{\sigma}\right) = P\left(\frac{60-58.15}{10.28} < Z < \frac{75-58.15}{10.28}\right)$$

$$= P(0.18 < Z < 1.64) = P(0 < Z < 1.64) - P(0 < Z < 0.18)$$

$$= 0.4495 - 0.0714 = 0.3781$$

Percentage of students getting first class = 38

Now percentage of students getting second class = 100 – (students who have failed, got 1st class and got distinction)

Percentage of students getting second class $= 100 - (10 + 38 + 5) = 47$.

5. ***The percentage X of a particular compound contained in a rocket fuel follows the distribution $N(33,\ 3)$, through the specification for X is that it should lie between 30 and 35. The manufacturer will get a net profit (per unit of the fuel) of Rs. 100, if $30 < X < 35$, Rs. 50, if $25 < X \leq 30$ or $35 \leq X < 40$ and incur a loss of Rs. 60 per unit of the fuel otherwise. Find the expected profit of the manufacturer. If he wants to increase his expected profit by 50% by increasing the net profit on that category of the fuel that meets the specification, what should be the new net profit per unit of the fuel of this category?***

Solution: $N(\mu,\ \sigma),\ N(33,\ 3),\ \mu = 33,\ \sigma = 3$

$$P(30 < X < 35) = P\left(\frac{30-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{35-\mu}{\sigma}\right) = P\left(\frac{30-33}{3} < Z < \frac{35-33}{2}\right) = P(-1 < Z < 0.67)$$

$$= P(-1 < Z < 0) + P(0 < Z < 0.67) = P(0 < Z < 1) + P(0 < Z < 0.67)$$

$$= 0.3413 + 0.2486 = 0.5899$$

(∵From Normal Table)

$$P(25 < X \leq 30) = P\left(\frac{25-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{30-\mu}{\sigma}\right) = P\left(\frac{25-33}{3} < Z < \frac{30-33}{2}\right) = P(-2.67 < Z < -1)$$

$$= P(1 < Z < 2.67) = P(0 < Z < 2.67) - P(0 < Z < 1) = 0.4962 - 0.3413 = 0.1549$$

$$P(35 \leq X < 40) = P\left(\frac{35-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{40-\mu}{\sigma}\right) = P\left(\frac{35-33}{3} < Z < \frac{40-33}{2}\right) = P(0.67 < Z < 2.33)$$

$$= P(0 < Z < 2.33) - P(0 < Z < 0.67) = 0.4901 - 0..2486 = 0.2415$$

$$P[(25 < X \leq 30)\ or\ (35 \leq X < 40)] = P(25 < X \leq 30) + P(35 \leq X < 40) = 0.1549 + 0.2415 = 0.3964$$

$$P(X < 25\ or\ X > 40) = 1 - (0.5899 + 0.3964) = 0.0137$$

Profit / Unit	***Probability***
Rs. 100	0.5899
Rs. 50	0.3964
Rs. -60	0.0137

$E(Profit\ per\ unit) = Rs.(100 \times 0.5899 + 50 \times 0.3964 - 60 \times 0.0137) = Rs.79$

Let the revised net profit per unit of the first category fuel be k.

$E(Revised\ Profit\ per\ unit) = Rs.(k \times 0.5899 + 50 \times 0.3964 - 60 \times 0.0137) = Rs.(0.5899k + 18.998)$

$E(Revised\ Profit\ per\ unit) = Rs.79 + Rs.39.5$

$0.5899k + 18.998 = 118.5 \Rightarrow k = \frac{118.5-18.998}{0.5899} = 168.68 \approx 169$

6. ***The savings bank account of a customer showed an average balance of Rs. 150 and a S.D. of Rs. 50. Assuming that the account balances are normally distributed (i) What percentage of account is over Rs. 200? (ii) What percentage of account is between Rs. 120 & Rs. 170? (iii) What % of account is less than Rs. 75?***

Solution: $\mu = 150,\ \sigma = 50$

(i) $P(X \geq 200) = P\left(Z \geq \frac{200-150}{50}\right) = P(Z \geq 1) = 0.5 - P(0 \leq Z \leq 1) = 0.5 - 0.3413 = 0.1587$

Percentage of account is over Rs. 200 is 15.87%

(ii) $P(120 < X < 170) = P\left(\frac{120-150}{50} < Z < \frac{170-150}{50}\right) = P(-0.6 < Z < 0.4)$

$= P(0 < Z < 0.6) + P(0 < Z < 0.4) = 0.2257 + 0.1554 = 0.3811$

Percentage of account is between Rs. 120 & Rs. 170 is 38.11%

(iii) $P(X < 75) = P\left(Z < \frac{75-150}{50}\right) = P(Z < -1.5) = 0.5 - P(0 < Z < 1.5) = 0.5 - 0.4332 = 0.0668$

Percentage of account is less than Rs. 75 is 6.68%

7. ***In a newly constructed township, 2000 electric lamps are installed with an average life of 1000 burning hours and standard deviation of 200hours. Assuming the life of the lamps follows normal distribution, find***

(i) The number of lamps expected to fail during the first 700 hours.

(ii) In what period of burning hours 10% of the lamps fail.

Solution: $\mu = 1000,\ \sigma = 200$

(i) $P(X \leq 700) = P\left(\frac{X-\mu}{\sigma} < \frac{700-1000}{200}\right) = P(Z < -1.5) = P(Z > 1.5) = 0.5 - P(0 < Z < 1.5)$

$= 0.5 - 0.4332 = 0.0668$ ***(∵From Normal Table)***

The no. of lamps that fail to burn in the first 700 hours $= 2000 \times 0.0668 = 133.6 \approx 134$

(ii) Let t be the period at which 10% of lamps fail.

$P(X \leq t) = 0.1 \Rightarrow P\left(\frac{X-\mu}{\sigma} \leq \frac{t-1000}{200}\right) = 0.1 \Rightarrow P\left(Z \geq \frac{1000-t}{200}\right) = 0.1$

$P\left(0 \leq Z \leq \frac{1000-t}{200}\right) = 0.5 - 0.1 = 0.4 \Rightarrow \frac{1000-t}{200} = 1.28 \Rightarrow \boldsymbol{t = 744}$ ***(∵From Normal Table)***

9. ***The marks obtained by a number of students in a certain subject are approximately normally distributed with mean 65 and standard deviation 5. If 3 students are selected at random from this group, what is the probability that at least 1 of them would have scored above 75?***

Solution: $\mu = 65,\ \sigma = 5$

$P(X > 75) = P\left(\frac{X-\mu}{\sigma} > \frac{75-\mu}{\sigma}\right) = P\left(Z > \frac{75-65}{5}\right) = P(Z > 2) = 0.5 - P(0 < Z < 2) = 0.5 - 0.4772 = 0.0228$

$p = P(a\ student\ scores > 75) = 0.0228,\ q = 0.9772, n = 3, P(y) = 3C_y(0.0228)^y(0.9772)^{3-y},\ y = 0,1,\dots n$

$P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(0) = 1 - 3C_0(0.0228)^0(0.9772)^{3-0} = 0.0667$

10. ***In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and S.D.***

Solution: Let mean by μ and standard deviation σ

31% of the items are under 45	***and***	***8% are over 64***
$P(X < 45) = 31\%$	***and***	$P(X > 64) = 8\%$
$P\left(\frac{X-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.31$	***and***	$P\left(\frac{X-\mu}{\sigma} > \frac{64-\mu}{\sigma}\right) = 0.08$
$P\left(Z < \frac{45-\mu}{\sigma}\right) = 0.31$	***and***	$P\left(Z > \frac{64-\mu}{\sigma}\right) = 0.08$
$P\left(0 < Z < \frac{\mu-45}{\sigma}\right) = 0.5 - 0.31 = 0.19$	***and***	$P\left(0 < Z < \frac{64-\mu}{\sigma}\right) = 0.5 - 0.08 = 0.42$
$\frac{\mu-45}{\sigma} = 0.5$	***and***	$\frac{64-\mu}{\sigma} = 1.4$
$\mu - 0.5\sigma = 45$	***and***	$\mu + 1.4\sigma = 64$

Solving for μ and σ, we get $\boldsymbol{\mu = 50}$ and $\boldsymbol{\sigma = 10}$.

All the Best

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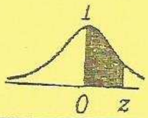
Normal Distribution Table

Area under the Normal curve
from 0 to z



z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993

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