STANDARD DISTRIBUTIONS.

In this topic, we shall consider some important probability distributions, discrete and continuous, which occur frequently in many engineering and other applications.

These probability distributions have standard names and are indexed by fixed numbers, called parameters.

DISCRETE DISTRIBUTIONS:-

1) BINOMIAL DISTRIBUTION

2) POISSON DISTRIBUTION.

3) GEOMETRIC DISTRIBUTION.

CONTINUOUS DISTRIBUTIONS:

1) NORMAL DISTRIBUTION

2) EXPONENTIAL DISTRIBUTION.

3) UNIFORM DISTRIBUTION.

BINOMIAL DISTRIBUTION.

BERNOULLI TRIAL

A bernoulli Trail (or binomial trial) is a randon Experiment in which there are Only two possible outcomes namely Success (s). and failure (f).

The Sample Space of a Bernoulli trial is $S = \{s, f\}$.

BERNOULLI EXPERIMENT.

The experiment Consists of 'n' independent repeated Bernoulli trials.

BERNOULLI DISTRIBUTION.

Let us Consider an experiment Consists of 'n' independent trials which results Success 'S' and failures 's'. of the random form.

S s f S f f .... f f S.

Let x be a random Variable which denotes the no. of Success, In Specific $x \in X$ be no of Success. and hence $n-x$ no of failures.

p - Probability of getting Success

q - Probability of getting failure.

$$P(SSFS \cdots FFS) = P(S) \cdot P(S) \cdot P(F) \cdots P(S) \cdot P(F) \cdot P(S)$$

All are Indedpendant trials.

$$= p \cdot p \cdot q \cdots q\, q\, p.$$

$$= [p\, p \cdots p]_{x \text{ times}} * [q \cdot q \cdots]_{n-x \text{ times}}.$$

$$= p^{x} q^{n-x}.$$

The Probabibility of 'x' Success in 'n' trials → ???

n trials and 'x' Success Can occur in nC_x ways.

∴ Probability of 'x' Success in 'n' trials is

$$nC_x \, p^x q^{n-x} = P(X=x)$$
$$= P(X = x \text{ successes}).$$

where $x = 0, 1, 2, \ldots, n$ with $p+q=1$.

Note $(p+q)^n = nC_0 \, p^0 q^n + nC_1 \, p^1 q^{n-1} + nC_2 \, p^2 q^{n-2} + \cdots + nC_n \, p^n q^0$

$$\sum_{x=0}^{n} P(x) = \sum_{x=0}^{n} nC_x \, p^x q^{n-x} = (p+q)^n = 1^n = 1.$$

DEFINITION

The Random Variable X that Counts the number of Successes, in the 'n' Bernoulli trials. is said to follow a BINOMIAL DISTRIBUTION WITH PARAMETERS 'n' and 'p' written as $B(n,p)$.

Symbolically $X \sim B(n,p)$, $n \in N$, $p \in [0,1]$.

The Probability mass function of the Binomial distributed discrete random variable X is.

$$P(X=x) = P(x) = nC_x \, p^x q^{n-x}, \quad x = 0,1,2,3,\dots,n$$

with $p+q=1$

Assumptions of the Binomial distribution.

1) The Random experiments Corresponds to two Possible Outcomes (Success or failures).

2) Number of trial is finite.

3) Trials are Independent.

4) The Probability of Success is Constant in any trial.

MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION.

Binomial Distribution.

$$P(X=x) = nC_x \, p^x \, q^{n-x}$$

W.K.T M.G.F is $M_X(t) = E[e^{tX}]$

$$M_x(t) = \sum_{n=0}^{\infty} e^{tx} \, p(x)$$

$$= \sum_{n=0}^{\infty} e^{tx} . \, nC_x \, p^x \, q^{n-x}$$

$$= \sum_{n=0}^{\infty} e^{tx} . p^x . \, nC_x \, q^{n-x} = \sum_{n=0}^{\infty} (pe^t)^x . \, nC_x \, q^{n-x}.$$

$$= nC_0 q^n + nC_1 (pe^t)^1 (q)^{n-1} + \cdots + nC_n (pe^t)^n q^0.$$

$$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} * b^2 + \cdots + b^n.$$

$$M_x(t) = (pe^t + q)^n.$$

$$\mu_n' = \left[\frac{d^n}{dt^n} M_x(t)\right]_{t=0} = M_X^n(0)$$ n-th derivative.

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$\frac{d}{dt}\left[M_X(t)\right] = n\,(pe^t+q)^{n-1}\cdot(pe^t).$$

$$\mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = M_X'(0) = n\,(p+q)^{n-1}\,p(e^0)$$

W.K.T $\boxed{p+q=1}$

$$\boxed{\mu_1' = np}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}\left[M_X(t)\right]\right]_{t=0} = \left[\frac{d}{dt}\left[\frac{d}{dt}M_X(t)\right]\right]_{t=0}.$$

$$\frac{d^2}{dt^2}\left[M_X(t)\right] = \frac{d}{dt}\left[n\,(pe^t+q)^{n-1}\,(pe^t)\right]$$

$e^0=1.$

$$= np\,\frac{d}{dt}\left[(pe^t+q)^{n-1}\cdot e^t\right]$$

$$= np\left[(pe^t+q)^{n-1}\cdot e^t + e^t\cdot(n-1)\,(pe^t+q)^{n-2}\times pe^t\right]$$

$$\left[\frac{d^2}{dt^2}\left[M_X(t)\right]\right]_{t=0} = np\left[(p+q)^{n-1} + (n-1)\,(p+q)^{n-2}\cdot p\right]$$

$$\mu_2' = np\left[1+(n-1)p\right] = np\left[np-p+1\right]$$

$$\boxed{\mu_2' = n^2p^2 - np^2 + np.}$$

Variance $V(x) = \mu_2' - (\mu_1')^2$

$$V(x) = \mu_2' - (\mu_1')^2 = n^2p^2 - np^2 + np - (np)^2.$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p) = npq.$$

[W.K.T $p+q=1$, $q = 1-p$]

$\therefore$ The mean of the Binomial distribution is

$$\boxed{\mu = np}$$

The Variance of the Binomial distribution is

$$\boxed{V(x) = npq}$$

n - number of trials

X = Random Variable (Discrete), represents number of Successes, which follows Binomial distribution.

x = value of the random Variable X.

p = Probability of Success in Single trial.

q = Probability of failure in Single trial ($q = 1-p$)

N = number of times 'n' trials are repeated (or) Total number of Sets.

* Mean of Bionomial Distribution = np
* Variance of Binomial Distribution = npq
* Standard deviation of Binomial Dt. = $\sqrt{npq}$
* $X \sim B(n,p)$ $\quad P(X=x) = nC_x \, p^x q^{n-x}$.
* Mean > Variance always.

PROBLEM 1:

Given mean 4 and Variance 3 of a Binomial distribution, find the Parameters.

The Parameters are p, q $\quad np = 4 = \text{mean}$

$npq = 3 = \text{Variance}$.

$npq = 3$

$(4)q = 3$

$q = 3/4$

$p = 1-q = 1 - 3/4 = 1/4$.

Problem 2: 6 dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or six.

Let X denote the number of Success when 6 dice are thrown.

x	1	2	3	4	5	6
$P(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Success is getting 5 or 6.

X is a binomial R.V with Parameter (n, P).

$$P(X=x) = nC_x \, p^x \, q^{n-x}$$

Here $n=6$, $p = P(5 \text{ or } 6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ //.

$q = 1-P = 1-1/3 = 2/3$.

$$\therefore P(X=x) = 6C_x \left(1/3\right)^x \left(\frac{2}{3}\right)^{6-x}.$$

'given' atleast 3 dice to show 5 or 6.

objective

$$P(X \geq 3) = 1 - P(X<3) = 1 - \{P(x=0) + P(x=1) + P(x=2)\}$$

$$= 1 - \left\{6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^6 + 6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5 + 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4\right\}$$

$$= 1 - \frac{1}{3^6}\{64 + 192 + 240\} = 1 - \frac{496}{729} = \frac{233}{729}.$$

Finally ~~7(29)~~, '6' dices are thrown 729 times.

$\therefore$ For 729 times $729 * \frac{233}{729} = 233.$

The expectation of atleast 3 dice to show 5 or 6 in 729 throws are 233//.

Problem : 3 Out of 800 families with 4 Children each how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) at most 2 girls (iv) Children of both sexes ?

Here each family has 4 Childrens $\boxed{n=4}$

Let Boy be 'Success'. 'p' be the Probability of getting boy.

$p = 1/2 \Rightarrow q = 1-p \Rightarrow q = 1/2$.

Let X denotes the number of boys in 4.

$n=4$, $p=1/2$, $q=1/2$. It follows binomial distribution.

$$P(x=x) = nC_x \, p^x q^{n-x}, \quad x = 0,1,2,\dots,n$$

$$P(x=x) = 4C_x \left(\tfrac{1}{2}\right)^x \cdot \left(\tfrac{1}{2}\right)^{4-x} = 4C_x \left(\tfrac{1}{2}\right)^4. \quad x = 0,1,2,3,4.$$

(i) P(2 boys and 2 girls)

$$P(x=2) = 4C_2 \left(\tfrac{1}{2}\right)^4 = 6 \times \frac{1}{2^4} = \frac{6}{16} = 3/8.$$

∴ The number of families with 2 boys and 2 girls

$$= 800 \times \frac{3}{8} = \boxed{300} \text{ families.}$$

(ii) P (at least one boy) = $P(X \geq 1) = 1 - P(X<1)$

$$P(X \geq 1) = 1 - P(X=0) = 1 - {}^4C_0 \left(\tfrac{1}{2}\right)^4 = 1 - \frac{1}{16} = \frac{15}{16}.$$

No. of families with atleast one boy $= 800 * \dfrac{15}{16} = 750$.

(iii) P (at most 2 girls) = P (atleast 2 boys).

$$P(X \geq 2) = 1 - \{P(X=0) + P(X=1)\}$$

$$= 1 - \left\{ {}^4C_0 \left(\tfrac{1}{2}\right)^4 + {}^4C_1 \left(\tfrac{1}{2}\right)^4 \right\}$$

$$= 1 - \left\{ \frac{1}{2^4} + 4 \cdot \frac{1}{2^4} \right\} = 1 - \frac{1}{16}(1+4) = 1 - \frac{5}{16}$$

$$P(X \geq 2) = \frac{11}{16}.$$

$\therefore$ The number of families with at most 2 girls $= 800 * \dfrac{11}{16} = 550$.

(iv) P (children of both sexes)

$= 1 -$ P (children with same sex)

$= 1 - \{$P (all girls) + P (all boys)$\}$

$$= 1 - \{P(X=0) + P(X=4)\} = 1 - \left\{ {}^4C_0 \left(\tfrac{1}{2}\right)^4 + {}^4C_4 \left(\tfrac{1}{2}\right)^4 \right\}$$

$$= 1 - \left\{ \frac{1}{2^4} + \frac{1}{2^4} \right\} = 1 - \left\{ \frac{1}{16} + \frac{1}{16} \right\} = 1 - \frac{2}{16} = 1 - \frac{1}{8} = \frac{7}{8}$$

$\therefore$ The number of families with children of both sexes

$$= 800 * \frac{7}{8} = \boxed{700}.$$

Problem: 4 Assuming 20% of the population of a town are literate if 100 investigators choose 10 individuals each to see whether they are literate, how many investigators would you expect to report that three people or less were literate?

Given p = probability of literate.

Let X denote the number of literates in 10.

X is a Binomial R.V with parameters (n, p)

$$P(X=x) = nC_x \, p^x q^{n-x}, \quad x = 0,1,2,\ldots,n$$

Here $n = 10$ [10 individuals are going to checked]

$$p = 20\% = \frac{20}{100} = 0.2 = \frac{1}{5}, \quad q = 1-p = 1-\frac{1}{5} = 4/5$$

$$P(X=x) = 10C_x \left(1/5\right)^x \left(\frac{4}{5}\right)^{10-x} \quad \text{where } x = 0,1,2,\ldots,10.$$

Investigators would expect to report that three people or less were literate.

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 10C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} + \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 10C_1 + 10C_2 \left(\frac{1}{5}\right)^2 \left(4/5\right)^8$$

$$+ 10C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7.$$

$$= \frac{1}{5^{10}} \left[4^{10} + (4^9)\,10 + \frac{10 \times 9}{2 \times 1} (4)^8 + \frac{10 \times 9 \times 8}{3 \times 2 \times 1} (4)^7 \right]$$

$$= \frac{1}{5^{10}}\left[4^{10} + 10(4^{9}) + 45(4^{8}) + 120(4^{7})\right]$$

$$= \frac{4^7}{5^{10}}\left[4^3 + 10(4^2) + 45(4) + 120\right]$$

$$= \frac{4^7}{5^{10}}\left[64 + 160 + 180 + 120\right] = \frac{4^7}{5^{10}}\left[524\right] = \frac{4^7 \times 524}{5^8 \times 5^2}$$

For 100 investigators $100 \times \frac{4^7 \times 524}{5^8 \times 5^2} = \frac{4 \times 4^7 \times 524}{5^8} = 87.9$

Out of 100 investigators, 88 investigators would report 3 or less literates in a group of 10 People selected.

POISSON DISTRIBUTION

Poisson distribution is a limiting form of binomial distribution when the Probability of Success 'p' is very Small and Bernoulli trials n is Very large So that $\boxed{np = \lambda}$ is a Constant of moderate Value.

Thus the Poisson distribution is also associated with Bernoulli trails and it relates to rare events. It is discovered by French mathematician Poisson.

DEFINITION:

A random Variable X is said to follow a Poisson distribution with Parameter $\lambda > 0$, if its Probability mass function is given by

$$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!} \qquad x = 0, 1, 2, \ldots \infty$$

X is called a poisson random Variable.

PMF

Clearly $P(x) \geq 0$ and

$$\sum P(x) = \sum \frac{e^{-\lambda} \lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda} \left[\frac{1}{0!} + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \left[e^{\lambda} \right] = 1 //.$$

EXAMPLES OF POISSON RANDOM VARIABLES.

1. The number of death in a rare disease
2. The number of defective item Produced by a Company in a day.
3. The number of wrong telephone calls in a day.
4. The number of earth quakes in a given time Interval.

MOMENT GENERATING FUNCTION OF POISSON DISTRIBUTION.

Let X be a Poisson Random Variable with Parameter λ

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,\ldots$$

W.K.T

$e^{ab} = (e^a)^b$

M.G.F of X is

$$M(t) = M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\, e^{\lambda e^t} = e^{(\lambda e^t - \lambda)} = e^{\lambda(e^t-1)}$$

W.K.T $\quad e^x = \sum_{x=0}^{\infty} \frac{1}{x!} \Rightarrow e^{\lambda e t} = \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$$\boxed{M_X(t) = e^{\lambda(e^t-1)}}$$

MEAN AND VARIANCE.

$$\mu_r' = \left[\frac{d^r}{dt^r}\left[M_X(t)\right]\right]_{t=0}.$$

$$\therefore \mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} \qquad \mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$\frac{d}{dt} M_x(t) = \frac{d}{dt}\left[e^{\lambda(e^t-1)}\right] = e^{\lambda(e^t-1)} \lambda e^t$$

$$\left[\frac{d}{dt} M_x(t)\right]_{t=0} = e^{\lambda(e^0-1)} \lambda e^0 = \lambda = \mu_1'$$

$$\therefore \boxed{\mu_1' = \lambda} \leftarrow \text{mean}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]\right]_{t=0}$$

$$\frac{d}{dt}\left[e^{\lambda(e^t-1)} \cdot \lambda e^t\right] = \lambda \frac{d}{dt}\left[e^t \cdot e^{\lambda(e^t-1)}\right]$$

$$= \lambda\left[e^t\, e^{\lambda(e^t-1)} \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot e^t\right]$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \lambda\left[e^0\, e^{\lambda(e^0-1)} \lambda e^0 + e^{\lambda(e^0-1)} \cdot e^0\right]$$

$$\mu_2' = \lambda\left[\lambda + 1\right] = \lambda^2 + \lambda.$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \lambda^2 + \lambda - (\lambda)^2 = \lambda$$

$$\boxed{\text{Variance} = \lambda.}$$

$\therefore$ In Poisson distribution $\boxed{\text{Mean} = \text{Variance}}$

PROBLEM 1: Find the Probability that at most 4 Defective fuses will be found in a box of 200 fuses if experience shows that 2% of such fuses are defective. ∴ ($e^{4} = 0.0183$)

X denotes the number of defectives in a box of 200 fuses

$p =$ Probability of defective fuses. $= 2\% = \frac{2}{100}$

$p = 0.02$

given $n = 200$, and p is very small

X is a Poisson Random Variable with Parameter λ

$$np = \lambda = 200 \times \frac{2}{100} = 4.$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots,\infty$$

$$P(X=4) = \frac{e^{-4}\lambda^{x}}{x!}$$

Atmost 4 defective

$$P(X\leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{e^{-4}\cdot 4^{0}}{0!} + \frac{e^{-4}4^{1}}{1!} + \frac{e^{-4}4^{2}}{2!} + \frac{e^{-4}4^{3}}{3!} + \frac{e^{-4}4^{4}}{4!}$$

$$= e^{-4}\left[1 + 4 + \frac{4^{2}}{2!} + \frac{4^{3}}{3!} + \frac{4^{4}}{4!}\right] = e^{-4}\left[1+4+8+\frac{32}{3}+\frac{32}{3}\right]$$

$$= e^{-4}\left[13 + \frac{64}{3}\right] = 0.0183 * \frac{103}{3} = 0.628.$$

The mean of Poisson Distribution $= \lambda$

The Variance of Poisson Distribution $= \lambda$.

PROBLEM 2: SUPPOSE THAT the number of telephones calls coming into a telephone exchange between 9 Am and 10 Am is a Poisson (distribution) random Variable with Parameter 2, and the number of telephone calls coming between 10 Am and 11 Am is a poisson random Variable with Parameter 6. If these two random variables are independent, then what is the Probability that more than 5 calls come in between 9 Am and 11 Am ?

Let X_1 denote number of calls between 9 Am to 10 Am. and X_2 denote the number of calls between 10 Am and 11 AM.

Given X_1 and X_2 are poisson R.V.s with Parameter 2 and 6.

By additive Property $X = X_1 + X_2$. is a P.R.V.

with $\lambda = 2 + \cancel{6} = (\lambda_1 + \lambda_2) = 8$.

X represents the number of calls between 9 Am and 11 AM

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-8}8^{x}}{x!} \quad x=0,1,2,\dots\infty$$

Required $P(X>5)$ = More than 5 Calls between 9 Am to 11 Am.

$$P(X>5) = 1-P(X\leq 5)$$

$$= 1-\left\{ \begin{array}{l} P(X=0) + P(X=1) + P(X=2) \\ + P(X=3) + P(X=4) + P(X=5) \end{array} \right]$$

$$= 1-e^{-8}\left[\frac{8^0}{0!}+\frac{8^1}{1!}+\frac{8^2}{2!}+\frac{8^3}{3!}+\frac{8^4}{4!}+\frac{8^5}{5!}\right]$$

$$= 1-e^{-8}\left[1+8+32+\frac{256}{3}+\frac{512}{3}+\frac{4096}{15}\right\}$$

$$= 1-0.1912 = 0.8088 //.$$

PROBLEM: 3 There are 500 Boxes each Containing 1000 ballot Papers for election. The Chance of that a ballot Paper is defective is 0.02. Assuming Poisson distribution for the number of defective ballot Papers, find the number of boxes Containing (i) atleast (ii) atmost and (iii) exactly one defective ballot Paper.

'X' reperents the ballot Paper for election to be defective.

number of ballot Papers $n = 1000$

$p = 0.002$

This follows Poisson distribution.

$\lambda = np = 1000 \times 0.002 = 2.$

$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(2)^x e^{-2}}{x!} = e^{-2}\left[\frac{2^x}{x!}\right]$

(i) At most One defective

$P(X \leq 1) = P(x=0) + P(x=1)$

$= e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!}\right] = e^{-2}[1+2] = 3e^{-2}$

$= 3(0.1353) = 0.4059$

for 500 boxes $= 0.4059 * 500 = 202.95 \cong 203$

(ii) Atleast One defective

$P(X \geq 1) = 1 - P(x<1) = 1 - P(x=0)$

$= 1 - e^{-2}\left[\frac{2^0}{0!}\right] = 1 - e^{-2} = 1 - 0.1353 = 0.8647.$

For 500 boxes $= 0.8647 * 500 = 432.35 = 432$

(iii) Exactly one

$P(x=1) = e^{-2}\left[\frac{2^1}{1!}\right] = e^{-2} * 2 = 2 * 0.1353 = 0.2706$

For 500 boxes $0.2706 * 500 = 135.3 \cong 135$.

$$= \frac{1}{5^{10}}\left[4^{10} + 10(4^9) + 45\cdot(4^8) + 120\cdot(4^7)\right]$$

$$= \frac{4^7}{5^{10}}\left[4^3 + 10(4^2) + 45(4) + 120\right] = \frac{4^7}{5^{10}}\left[64 + 160 + 180 + 120\right]$$

$$= \frac{4^7}{5^{10}}\left[524\right] = \frac{4^7 . 524 \times 4}{5^8 \times 5^2}$$

For 100 investigators $100 \times \frac{4^7 \times 524}{5^8 \times 5^2} = \frac{4^7 \times 524 \times 4}{5^8} = 87.9$

Out of 100 Investigators, 88 investigators would report 3 or less literates in a group of 10 People selected.

— x —

GEOMETRIC DISTRIBUTIONS.

We know that Binomial distribution gives the Probability of exactly x Success in a fixed number of 'n' trials. Suppose we ~~t~~ are interested in finding the first Success in a sequence of Bernoulli trials, we cannot show how many trails are required.

So the number of trials required cannot be fixed in advance and infact it is a random number.

— x —

If X denotes the number of trails required for the first Success in repeated Bernoulli trails with probability of Success 'p', then

$$\boxed{P(X=x) = q^{x-1} \cdot p,} \quad x = 1, 2, 3, \ldots$$

i.e., The x^{th} trial is the first Success and the Preceeding '$x-1$' trials are failures.

DEFINITION:

A random Variable X is Said to follow a geometric distribution with Parameter p if its Probability mass function is given by

$$P(X=x) = \begin{cases} q^{x-1} p, & x=1,2,3,\dots\infty \\ 0, & \text{otherwise.} \end{cases}$$

X is called a geometric (or Pascal) random Variable.

NOTE The name geometric is due to the fact that the Probabilities $p, qp, q^2p, \dots,$ for $x=1,2,3,\dots$ form a geometric Progression.

ANOTHER FORM Sometimes, the Probability mass function of a geometric random Variable Y is given by

$$P(Y=y) = q^y p, \quad y=0,1,2,\dots$$

In this form Y respresents the number of failures Preceeding the first Success. Before the first Success there may not be any failure. i.e., 0 failure (or) 1 failure (or) 2 failures and So on.

From this form to Pascal form we Can go by the transformation $Y = X-1$ (or) $X = Y+1$.

* The geometric random Variable is important in queueing theory. A geometric random Variable is known as waiting time random Variable. It represents how long one has to wait for a "Success".

MOMENT GENERATING FUNCTION OF GEOMETRIC DISTRIBUTION:

The geometric distribution with parameter 'p' is

$$P(X=x) = q^{x-1} \cdot p, \quad x = 1, 2, 3, \dots$$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} p = \sum_{x=1}^{\infty} e^{t} \cdot e^{-t} \cdot e^{tx} \cdot q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{t} \, e^{t(x-1)} \cdot q^{x-1} p = \sum_{x=1}^{\infty} e^{t} (e^{t} q)^{x-1} p$$

$$= pe^{t} \sum_{x=1}^{\infty} (qe^{t})^{x-1} = pe^{t} \left[1 + (qe^{t}) + (qe^{t})^{2} + \dots\right]$$

$$= pe^{t} \frac{1}{1-qe^{t}} \quad \text{if } qe^{t} < 1.$$

$$M_X(t) = \frac{pe^{t}}{1-qe^{t}}$$

Mean and Variance

Mean $= \mu_1'$ Variance $= \mu_2' - (\mu_1')^2$

$$\boxed{M_x(t) = \frac{pe^t}{1-qe^t}.}$$

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}.$$

$$\boxed{p+q=1}$$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$\frac{d}{dt} M_x(t) = p\left[\frac{(1-qe^t)(e^t) - e^t(-qe^t)}{(1-qe^t)^2}\right]$$

$$= p\left[\frac{e^t(1-qe^t+qe^t)}{(1-qe^t)^2}\right] = \frac{pe^t}{(1-qe^t)^2}$$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2}$$

$$\boxed{\mu_1' = \text{Mean} = \frac{1}{p}.}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]_{t=0} = \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$\frac{d^2}{dt^2} M_x(t) = p\left[\frac{(1-qe^t)^2 e^t - e^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = p\left[\frac{(1-q)^2 - 2(1-q)(-q)}{(1-q)^4}\right]$$

$$= p\left[\frac{(1-q)[1-q+2q]}{(1-q)^4}\right] = p\frac{(1+q)}{(1-q)^3} = \frac{p(1+q)}{p^3}$$

$$\mu_2' = \frac{1+q}{p^2} = \frac{1}{p^2} + \frac{q}{p^2}$$

$$Var(X) = \mu_2' - (\mu_1')^2 = \frac{1}{p^2} + \frac{q}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2}$$

$$\boxed{Var(X) = \frac{q}{p^2}}$$

Problem 1 If the Probability that a target is destroyed by anyone shot is 0.6, what is the probability that it should be destroyed on the fifth attempt?

Let X be the number of attempts for the first success.

X is said to follow geometric distribution.

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, \ldots$$

$p = 0.6$, $q = 1-p = 0.4$

Here the first success is 5^{th} attempt. $\boxed{\therefore x = 5}$

$$P(X=5) = (0.4)^{5-1}(0.6) = (0.4)^4(0.6) = 0.01536.$$

Problem 2

A die is thrown until 6 appears. What is the Probability that it must be thrown more than four times?

Getting 6 is the Success. for the first Success it should be thrown more than 4 times.

So geometric distribution is used.

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, \ldots$$

p = Probability of 6 $= \frac{1}{6}$ $\boxed{q = 5/6}$ $\boxed{P = 1/6}$

Required $P(X>4) = 1 - P(X \leq 4)$

$$= 1 - \left\{ P(X=1) + P(X=2) + P(X=3) + P(X=4) \right\}$$

$$= 1 - \left\{ p + pq + q^2 p + q^3 p \right\} = 1 - p\left\{1 + q + q^2 + q^3\right\}$$

$$= 1 - p\frac{(1-q^4)}{1-q}$$

G·P $1 + a + a^2 + \cdots + a^{n-1} = \frac{(1-a^n)}{(1-a)}$

$$= 1 - \frac{p(1-q^4)}{p}$$

(or) $a, ar, ar^2 \ldots$

$$= 1 - 1 + q^4 = q^4$$

$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad (r \neq 1)$

$$= (5/6)^4 = \frac{625}{1296} = 0.48.$$

PROBLEM: 3 The Probability that an applicant for a driver's licence will Pass the road test on any given trial is 0.7. Find the Probability that he will pass the test (i) on the third trial

(ii) Before the fourth trial.

Let X denotes the number of trails required for. Pass

X follows geometric distribution with Probability function.

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, 3, \ldots$$

given $p = 0.7$, $q = 0.3 = 1-p$

(i) $P(X=3) = (0.3)^2 (0.7) = 0.063$

(ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= p + qp + q^2 p$

$= (0.7) + (0.3)(0.7) + (0.3)^2 (0.7).$

$= 0.7 + 0.21 + 0.063 = 0.973$ //

Problem 4

A Couple decide to have Children until they have a male child. What is the Probability distribution of the number of Children they would have? If the Probability of a male child in their family is $\frac{1}{3}$; how many Children are they expect to have before the first male Child is born?

Let X denote the number of Children until the first male Child. So X follows the geometric distribution.

$$P(X=x) = q^{x-1} p, \quad x = 1, 2, 3, \cdots$$

$$p = \frac{1}{3}, \quad q = \frac{2}{3}$$

$$\text{Required } E(x) = \sum_{x=1}^{\infty} x \cdot P(x) = \sum_{x=1}^{\infty} x \cdot q^{x-1} \cdot p.$$

$$= p + 2pq + 3q^2 p + 4q^3 p + \cdots$$

$$= p(1 + 2q + 3q^2 + 4q^3 + \cdots) = p(1-q)^{-2}$$

$$= \frac{p}{(1-q)^2} = \frac{p}{(p)^2} = \frac{1}{p}$$

$$E(x) = \frac{1}{1/3} = 3$$

$\therefore$ The expected number of children before the first male Child $= E(x) - 1 = 3 - 1 = 2$ //.

CONTINUOUS DISTRIBUTION

1) EXPONENTIAL

2) NORMAL DISTRIBUTION.

3) UNIFORM DISTRIBUTION.

EXPONENTIAL DISTRIBUTION

A Continuous random Variable X is said to follow an exponential distribution with Parameter $\lambda > 0$, if its Probability density function is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

X is called an exponential random Variable.

* Exponential distribution is also known as negative exponential distribution.

Mean of the exponential distribution $\frac{1}{\lambda}$ .. Sometimes θ

Variance of the exponential distribution $\frac{1}{\lambda^2}$.

Problem 1: The daily consumption of milk in excess of 20,000 gallons is approximately exponentially distributed with $\theta = 3000$. The City has a daily stock of 35,000 gallons. What is the probability that of two days selected at random, the stock is insufficient for both days?

X denotes the consumption of milk.

Let $Y = X - 20000$, the excess

Given Y is exponentially distributed with $\theta = 3000$

$$\text{Mean} = 3000 \Rightarrow \frac{1}{\lambda} = 3000 \Rightarrow \boxed{\lambda = \frac{1}{3000}}$$

The probability density function of Y is

$$f(y) = \lambda e^{-\lambda y}, \quad y \geq 0$$

$$= \frac{1}{3000} e^{-\frac{y}{3000}}, \quad y \geq 0.$$

Probability of stock insufficient for a day

$$= P(X > 35000) = P(Y > 15000)$$

$$= \frac{1}{3000} \int_{15000}^{\infty} e^{-y/3000} \, dy$$

$$= \frac{1}{3000} \left[\frac{e^{-\frac{y}{3000}}}{-\frac{1}{3000}} \right]_{15000}^{\infty} =$$

$$= -(e^{-\infty} - e^{-5}) = -(0 - e^{-5}) = e^{-5}.$$

The Probability for insufficient stock for 2 days

$$= e^{-5} \cdot e^{-5} = e^{-10} = 0.000045.$$

— x —

Problem: 2 Suppose that the amount of waiting time a customer spends at a restaurant has an exponential distribution with a mean value of 6 minutes. Find the probability that a customer will spend more than 12 minutes in the restaurant.

Let X represent the waiting time in the restaurant.

Given X follows exponential distribution withe Parameter λ

$$\text{mean} = \frac{1}{\lambda} = 6 \Rightarrow \boxed{\lambda = \frac{1}{6}}$$

∴ The exponential distribution is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

$$= \frac{1}{6} e^{-x/6}, \quad x \geq 0$$

Required $P(X > 12) = \int_{12}^{\infty} f(x)\,dx = \frac{1}{6} \int_{12}^{\infty} e^{-x/6}\,dx.$

$$= \frac{1}{6} \left[\frac{e^{-x/6}}{-1/6} \right]_{12}^{\infty} = -\left[e^{-\infty} - e^{-12/6} \right] = -\left(0 - e^{-12/6}\right)$$

$$= e^{-2} = 0.1353 //$$

NORMAL DISTRIBUTION.

Normal distribution is the most important continuous distribution in Statistics both from practical and theoritical point of views.

Many random phenomena that occur in nature, industry and scientific studies are found to follow that normal law very closely.

The normal distribution was first discovered by De-Moivre in 1733 as a limiting form of binomial distribution. It is also known as Gaussian distribution.

NORMAL DISTRIBUTION.

DEFINITION:

A Continuous random Variable X with Parameters μ and σ^2 is Said to follow a normal distribution if its Probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$\sigma > 0.$$

X is Called a normal random Variable.

* The notation $X \sim N(\mu, \sigma^2)$ means X is normally distributed with mean μ and Variance σ^2.

* The graph of the normal distribution is Called the normal Curve. It is bell Shaped Curve, It is Symmetric about $x = \mu$.

* In the binomial distribution with Parameters n and p, when n is very large and p is nearly to $1/2$, the binomial approaches normal.

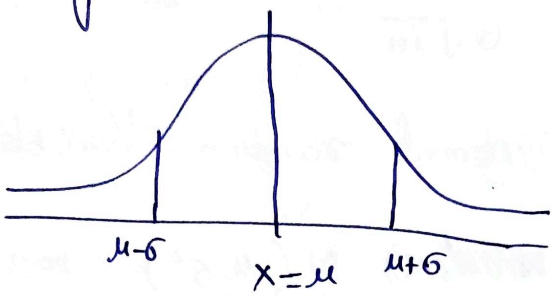
Mean $= \mu$, Variance $= \sigma^2$.

PROPERTIES OF NORMAL DISTRIBUTION.

The normal distribution $X \sim N(\mu, \sigma^2)$ has the following Properties.

1. The mean, median, mode Coincide at $x = \mu$

2. It is Symmetric about $\mu = x$.



3. The maximum Value of $f(x)$ is at $x = \mu$.

$$= \frac{1}{\sigma\sqrt{2\pi}}$$

4. The Points of inflection are at $x = \mu - \sigma$ and $x = \mu + \sigma$.

The curve is Concave down in the interval $(\mu-\sigma, \mu+\sigma)$ Concave upwards (or Convex) Otherwise.

5. The total area under the Curve is 1.

6. The Curve approaches the horizontal axis asymptotically On either Side of $x = \mu$.

7. Coefficient of Skewness is 0 and Coefficient of Kurthosis is 3.

Z is a Standard Normal random Variable

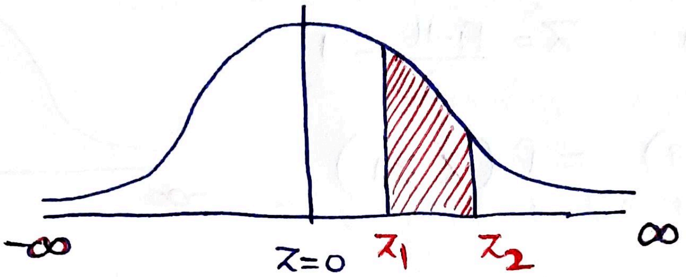
It's pdf is given by $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \; -\infty < z < \infty$

$$P(x_1 < X < x_2) = P(z_1 < Z < z_2)$$

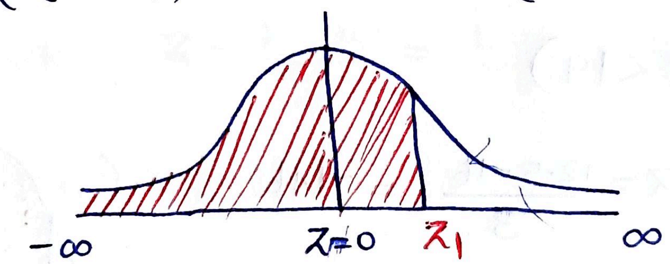
$$z_1 = \frac{x_1 - \mu}{\sigma}, \quad z_2 = \frac{x_2 - \mu}{\sigma} \qquad \boxed{X \sim N(0,1)}$$

Models

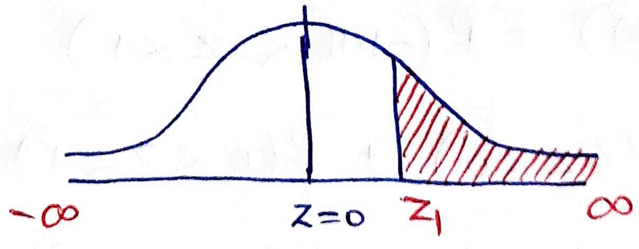
1) $P(z_1 < Z < z_2) = P(0 < Z < z_2) - P(0 < Z < z_1)$



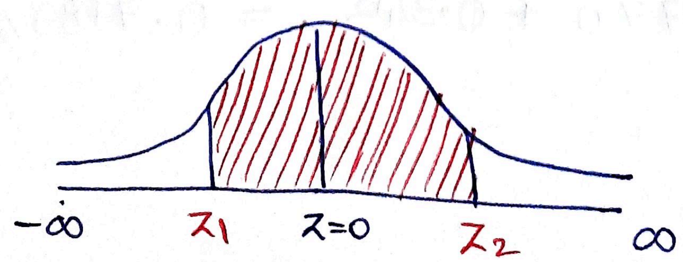
2) $P(Z < z_1) = 0.5 + P(0 < Z < z_1)$



3) $P(Z > z_1) = 0.5 - P(0 < Z < z_1)$



4) $P(z_1 < Z < z_2) = P(0 < Z < z_1) + P(0 < Z < z_2)$



PROBLEM 1 X is a normal distribution with mean 16 and standard deviation 3.

find (i) $P(X \geq 19)$ (ii) $P(12.5 < X < 19)$

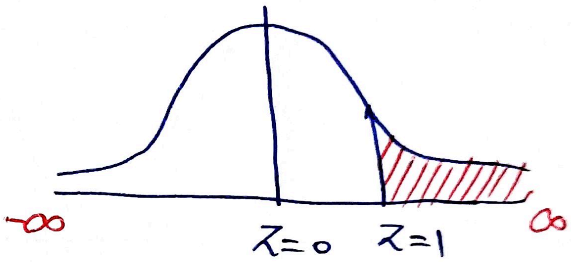
(iii) $P(10 < X < 25)$ (iv) K if $P(X > K) = 0.24$

Given $\mu = 16$, $\sigma = 3$, $Z = \dfrac{X - \mu}{\sigma}$ $\boxed{Z = \dfrac{X-16}{3}}$

(i) $P(X \geq 19)$

At $X = 19$ $\quad Z = \dfrac{19-16}{3} = 1$

$\therefore P(X \geq 19) = P(Z \geq 1)$

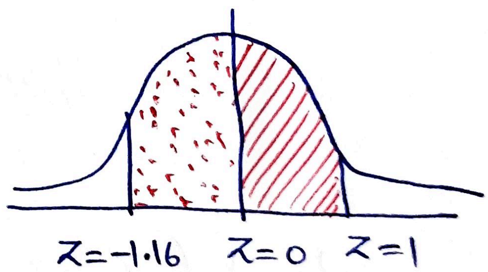


$P(Z \geq 1) = 0.5 - P(0 < Z < 1) = 0.5 - 0.3413 = 0.1587.$

(ii) $P(12.5 < X < 19)$

At $X = 12.5$, $\quad Z = \dfrac{12.5-16}{3} = -1.16$

$X = 19$, $\quad Z = \dfrac{19-16}{3} = 1.$



$P(12.5 < X < 19) = P(-1.16 < Z < 1)$

$= P(-1.16 < Z < 0) + P(0 < Z < 1)$ [By Symmetry Property]

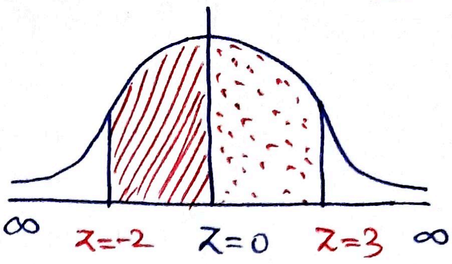
$= P(0 < Z < 1.16) + P(0 < Z < 1)$

$= 0.3770 + 0.3413 = 0.7183$//.

(iii) $P(10 < x < 25)$

At $x = 10$ $\quad z = \dfrac{10-16}{3} = -2.$

$x = 25$ $\quad z = \dfrac{25-16}{3} = \dfrac{9}{3} = 3.$



$$P(10 < x < 25) = P(-2 < z < 3)$$

$$= P(-2 < z < 0) + P(0 < z < 3)$$

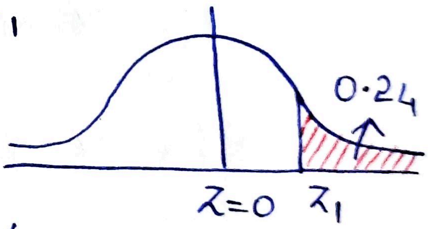
$$= P(0 < z < 2) + P(0 < z < 3)$$

By Symmetry Property.

$$= 0.4772 + 0.4987. = 0.9759.$$

(iv) $P(x > k) = 0.24$

If $x = k$ $\quad z = \dfrac{k-\mu}{\sigma} = \dfrac{k-16}{3} = z_1$



$P(z > z_1) = 0.24$

$P(0 < z < z_1) = 0.5 - 0.24 = 0.26.$

z_1 is the value of z corresponding to the area 0.26

$\boxed{z_1 = 0.7}$ Using Normal distribution table

$\dfrac{k-16}{3} = 0.7 \Rightarrow k-16 = (0.7)(3) \Rightarrow k = 16 + 2.1$

$\boxed{k = 18.1}$

<u>Note</u> $P(\mu-\sigma < X < \mu+\sigma) = P(-1<Z<1) = 0.6826$

$P(\mu-2\sigma < X < \mu+2\sigma) = P(-2<Z<2) = 0.9545$

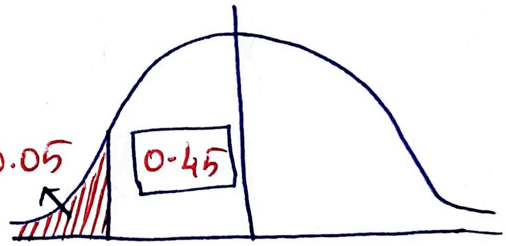
$P(\mu-3\sigma < X < \mu+3\sigma) = P(-3<Z<3) = 0.9973$

Outside the interval $(\mu-3\sigma, \mu+3\sigma)$ the area is

0.0027 //.

PROBLEM 2 In a normal distribution of a large group of men 5% are under 60" in height and 40% are between 60 and 65. Find the mean height and Standard deviation.

Let $X \sim N(\mu, \sigma^2)$



$P(X<60) = 5\% = \frac{5}{100} = 0.05$

$z = \frac{X-\mu}{\sigma} = \frac{60-\mu}{\sigma} = -z_1 \Rightarrow 60-\mu = -\sigma z_1$

$\therefore \boxed{\mu - \sigma z_1 = 60} \longrightarrow ①$

$P(x<60) = P(Z<-z_1) = 0.5 - P(-z_1 < Z < 0)$

$= 0.5 - P(0<Z<z_1) \rightarrow (*)$

[Symmetry Property]

[Substitute from (*)]

W.K.T $P(Z < -z_1) = 0.05$

$\therefore P(0 < Z < z_1) = 0.5 - 0.05$.

$\therefore P(0 < Z < z_1) = 0.45$

Using normal distribution Table we find $\boxed{z_1 = 1.645}$

$\therefore \boxed{\mu - 1.645\sigma = 60} \rightarrow (1)$

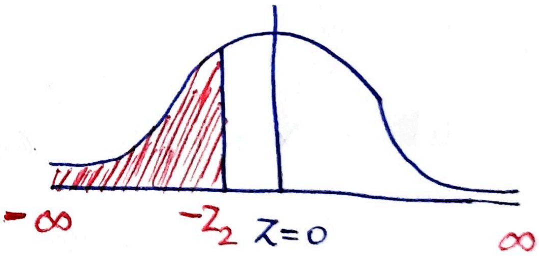
When data between 60 to 65 is 40%.

$P(60 < x < 65) = 40\% = \frac{40}{100} = 0.4.$

$P(x < 65) = P(x < 60) + P(60 < x < 65)$

$= 0.05 + 0.4$

$\boxed{\therefore P(x < 65) = 0.45}$



$P(x < 65) = P(Z < -z_2) = 0.5 - P(-z_2 < Z < 0)$

$0.45 = 0.5 - P(-z_2 < Z < 0)$

$P(-z_2 < Z < 0) = 0.5 - 0.45$

By Symmetry $P(0 < Z < z_2) = 0.05$

from Normal distribution table we get $\boxed{z_2 = 0.13}$

$$\frac{65-\mu}{\sigma} = -z_2 \qquad \boxed{z = \frac{x-\mu}{\sigma}}$$

$$65-\mu = -\sigma z_2 \Rightarrow \boxed{65 = \mu - 0.13\sigma} \rightarrow (2)$$

Solving Equations (1) & (2) we get

$\mu = 65.42$, $\sigma = 3.29$.

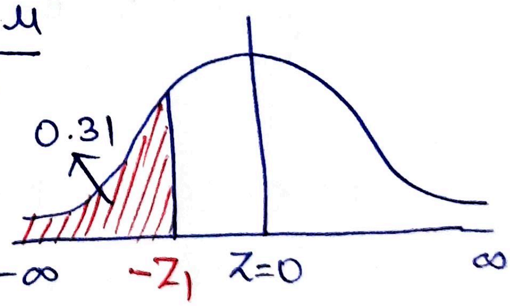
Mean $= 65.42$, Standard deviation $= 3.29$.

PROBLEM:3 In a normal distribution 31% items are under 45 and 8% over 64. Find mean and S.D.

$P(x < 45) = 0.31 = \frac{31}{100} = 31\%$ | $P(x > 64) = 8\%$

$P(x < 45) = 0.31$ | $P(x > 64) = 0.08$

When $x = 45$ $\quad z = \frac{x-\mu}{\sigma}$ $\quad z = \frac{45-\mu}{\sigma}$



$$\frac{45-\mu}{\sigma} = -z_1 \Rightarrow 45-\mu = -\sigma z_1$$

$$P(z < -z_1) = 0.5 - P(-z_1 < Z < 0) = P(x < 45)$$

$$0.31 = 0.5 - P(0 < Z < z_1) \quad [\text{Symmetry}]$$

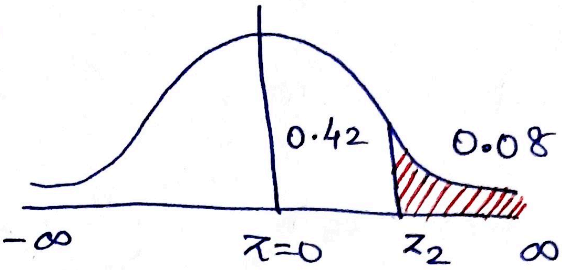
$$\therefore P(0 < z < z_1) = 0.5 - 0.31 = 0.19.$$

Using the normal distribution table $\boxed{z_1 = 0.5}$

So $45 - \mu = -\sigma z_1 \Rightarrow 45 - \mu = -0.5\sigma$

$\therefore \boxed{\mu - 0.5\sigma = 45} \longrightarrow (1)$

When $X = 64$ $\quad Z = \dfrac{64-\mu}{\sigma} = Z_2$



$64 - \mu = \sigma Z_2$

$64 - \mu = 1.41\sigma$

$\boxed{\mu + 1.41\sigma = 64} \rightarrow (2)$

$P(Z > Z_2) = 0.5 - P(0 < Z < Z_2)$

$0.08 = 0.5 - P(0 < Z < Z_2)$

$P(0 < Z < Z_2) = 0.42.$

from table $\boxed{Z_2 = 1.41}$

Solving (1) & (2) weget $\mu = 49.97$, $\sigma = 9.947$.

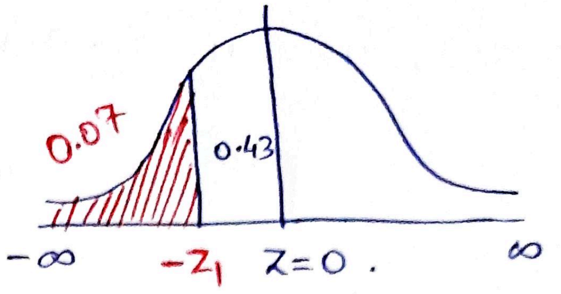
PROBLEM:4 In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and Standard deviation of the distribution.

Let $X \sim N(\mu, \sigma^2)$

$P(X < 35) = 7\% = 0.07$ $\qquad$ Under $35 \rightarrow 7\%$

$P(X < 63) = 89\% = 0.89$ $\qquad$ Under $63 \rightarrow 89\%$

$$z = \frac{x-\mu}{\sigma}$$



When $x = 35$, $\quad z = \frac{35-\mu}{\sigma} = -z_1$

$$P(x<35) = P(z<-z_1) = 0.07.$$

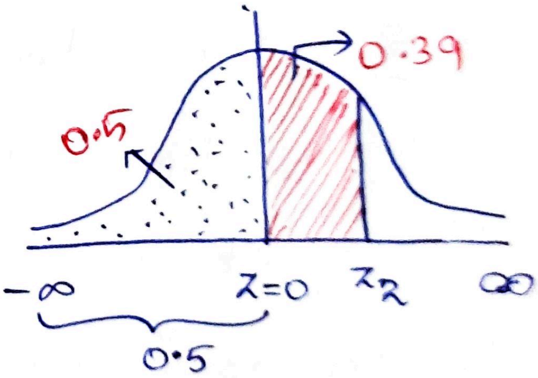
$$0.07 = 0.5 - P(-z_1 < z < 0) = 0.5 - P(0<z<z_1)$$

$$\therefore \; P(0<z<z_1) = 0.5 - 0.07 = 0.43.$$

from the normal dist. table $\boxed{z_1 = 1.48}$

$$\therefore \; \frac{35-\mu}{\sigma} = -z_1 \Rightarrow 35-\mu = -\sigma(1.48)$$

$$\boxed{\mu - 1.48\sigma = 35} \rightarrow (1)$$



When $x = 63$, $\quad z = \frac{63-\mu}{\sigma} = z_2$

$$P(x<63) = P(z<z_2).$$

$$0.89. = 0.5 + P(0<z<z_2)$$

$P(0<z<z_2) = 0.39$ from the table

$$\boxed{z_2 = 1.23}$$

$$\frac{63-\mu}{\sigma} = z_2 \Rightarrow 63-\mu = 1.23\sigma \Rightarrow \boxed{\mu + 1.23\sigma = 63} \rightarrow (2)$$

Solving (1) & (2) we get $\boxed{\mu = 50.3}$ $\boxed{\sigma = 10.33.}$

UNIFORM DISTRIBUTION

(or)

CONTINUOUS UNIFORM DISTRIBUTION

(or)

RECTANGULAR DISTRIBUTION.

A Continuous Variable X is Said to follow a uniform distribution over an interval (a,b) if its Probability density function is given by

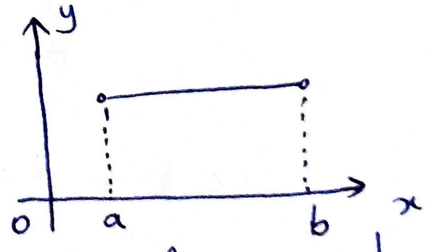
$$f(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}, \quad \text{where } k \text{ is a constant.}$$

a and b are Parameters of this distribution

$\therefore$ Since $f(x)$ is a pdf, $\int_a^b f(x)\,dx = 1.$

$$\Rightarrow \int_a^b k\,dx = 1 \Rightarrow k[x]_a^b = 1 \Rightarrow k(b-a) = 1.$$

$$\therefore \boxed{k = \frac{1}{b-a}.}$$



$\therefore$ The uniform distribution on an interval (a,b) has

Pdf

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

MOMENT GENERATING FUNCTION, MEAN AND VARIANCE.

$$M_X(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$\boxed{M_X(t) = \frac{1}{t(b-a)}\left[e^{bt} - e^{at}\right]} \quad t \neq 0.$$

$$\text{Mean} = E(X) = \int_a^b x f(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx.$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$$

$$E(X^2) = \int_a^b x^2 f(x)\, dx = \int_a^b \frac{1}{b-a} x^2\, dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right] = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}.$$

$$\boxed{E(X^2) = \frac{(a^2+ab+b^2)}{3}} \qquad \boxed{E(X) = \frac{a+b}{2}.}$$

$$\therefore \text{Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{4(a^2+ab+b^2) - 3(a+b)^2}{12}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}.$$

$$= \frac{1}{12}\left[a^2+b^2-2ab\right] = \frac{(a-b)^2}{12}.$$

$$\therefore \boxed{\text{Variance} = \frac{(a-b)^2}{12}}, \quad \boxed{\text{Mean} = \frac{a+b}{2}.}$$

$$M_x(t) = \frac{e^{bt}-e^{at}}{t(b-a)} = \left[\left\{1+\frac{bt}{1!}+\frac{(bt)^2}{2!}+\cdots\cdots\right\} - \left\{1+\frac{at}{1!}+\frac{(at)^2}{2!}+\cdots\cdots\right\}\right] * \frac{1}{t(b-a)}$$

$$= \frac{1}{t(b-a)}\left[\frac{t(b-a)}{1!}+\frac{t^2(b^2-a^2)}{2!}+\frac{t^3(b^3-a^3)}{3!}\cdot\right]$$

$$= 1+\frac{t^2(b-a)(b+a)}{t(b-a)\,2!}+\frac{t^3(b^3-a^3)}{t(b-a)\,3!}+\cdots\cdots$$

$$= 1+\frac{t}{2!}(b+a)+\frac{t^2}{3!}\frac{(b-a)(a^2+ab+b^2)}{(b-a)}+\cdots$$

$$= 1+\frac{t}{2!}(b+a)+\frac{t^2}{3!}(a^2+ab+b^2)+\cdots$$

$$\mu_r' = \text{Coeff of } \frac{t^r}{r!} \text{ in } M_x(t)$$

$M_x(t)$ = M.G.F

$$\mu_1' = \text{Coeff of } \frac{t}{1!} = \frac{(b+a)}{2}$$

$$\mu_2' = \text{Coeff of } \frac{t^2}{2!} = \frac{(a^2+ba+b^2)}{3}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{1}{12}\left[4a^2+4ab+4b^2-(3a^2+3b^2+6ab)\right]$$

$$= \frac{1}{12}\left[a^2-2ab+b^2\right] = \frac{(a-b)^2}{12}.$$

$$\boxed{\text{Variance} = \frac{(a-b)^2}{12}} \qquad \boxed{\text{Mean} = \frac{a+b}{2}}$$

PROBLEM 1 If X is Uniformally distributed over $(0,10)$

Find (i) $P(X<4)$ (ii) $P(X>6)$ (iii) $P(2<X<5)$

X is Uniformly distributed over $(0,10)$

$$\text{P.d.f is } f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$

$$\text{P.df is } f(x) = \begin{cases} \frac{1}{10}, & 0<x<10 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X<4) = \int_0^4 \frac{1}{10}\,dx = \frac{1}{10}[x]_0^4 = \frac{4}{10} = \frac{2}{5}$$

$$P(X>6) = \int_6^{10} \frac{1}{10}\,dx = \frac{1}{10}[x]_6^{10} = \frac{1}{10}(10-6) = \frac{2}{5}$$

$$P(2<X<5) = \int_2^5 f(x)\,dx = \int_2^5 \frac{1}{10}\,dx = \frac{1}{10}(5-2) = \frac{3}{10}.$$

Problem 2 A Random Variable 'X' has Uniform distribution over -3 to 3. Compute

(i) $P(X<2)$ (ii) $P(|X|<2)$ (iii) $P(|X-2|<2)$

(iv) Find k for which $P(X>k) = \frac{1}{3}$.

X is a Uniform distribution $f(x) = \frac{1}{b-a} \quad a<x<b.$

$$f(x) = \frac{1}{3-(-3)} \quad -3<x<3.$$

$$\boxed{f(x) = \frac{1}{6}}$$

(i) $P(X<2) = \int_{-3}^{2} f(x)\,dx = \int_{-3}^{2} \frac{1}{6}\,dx = \frac{1}{6}[2-(-3)] = \frac{1}{6}[5] = \frac{5}{6}$

(ii) $P(|X|<2) = P(-2<X<2) = \int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \frac{1}{6}\,dx$

$$= \frac{1}{6}[x]_{-2}^{2} = \frac{1}{6}[2-(-2)] = \frac{4}{6} = \frac{2}{3}$$

(iii) $P(|X-2|<2) = P(-2<x-2<2)$

$$= P(-2+2<x<2+2) = P(0<x<4)$$

But Upper limit is 3.

$$= \int_{0}^{3} \frac{1}{6}\,dx = \frac{1}{6}[x]_{0}^{3} = \frac{3}{6} = \frac{1}{2}.$$

(iv) $P(X > K) = \frac{1}{3}$

$$\int_K^3 f(x)\,dx = \frac{1}{3} \Rightarrow \int_K^3 \frac{1}{6}\,dx = \frac{1}{3}$$

$$\Rightarrow \frac{1}{6}\left[x\right]_K^3 = \frac{1}{3} \Rightarrow \frac{1}{6}(3-K) = \frac{1}{3}$$

$$\Rightarrow \frac{1}{2}(3-K) = 1 \Rightarrow (3-K) = 2 \Rightarrow K = 3-2.$$

$$\boxed{\therefore K = 1}$$

PROBLEM 3: Buses arrive at a Specified Stop @ 15 minutes interval Starting at 6 am. i.e., they arrive at 6 AM, 6.15 AM, 6.30 AM. and So on. If a Passenger arrive at the Stop at a time that is Uniformly distributed between 6 and 6.30 AM., find the Probability that he waits

(i) less than 5 minutes for a bus

(ii) More than 10 minutes for a bus.

Let X denotes the number of minutes after 6 AM that a Passanger arrives at the Stop.

X is a Uniform R.V over the interval (0,30) 18

$\therefore$ its pdf is $f(x) = \frac{1}{30}, \quad 0 < x < 30.$

(i) Since the bus arrives at 15 minutes interval Starting with 6AM, a Passanger has to wait less than 5 minutes if he comes to the Stop between 6.10 to 6.15 (or) 6.25 and 6.30 Am.

Required Probability $= P(10 < X < 15) + P(25 < X < 30)$

$$= \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx = \frac{1}{30}[15-10] + \frac{1}{30}[30-25]$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \frac{1}{3}.$$

(ii) The Passanger has to wait more than 10 minutes if he comes to the Stop between 6 and 6.5 Am (or) between 6.15 and 6.20 Am.

Required Probability $= P(0 < X < 5) + P(15 < X < 20)$

$$= \int_{0}^{5} \frac{1}{30}\,dx + \int_{15}^{20} \frac{1}{30}\,dx = \frac{1}{30}(5-0) + \frac{1}{30}(20-15)$$

$$= \frac{5}{30} + \frac{5}{30} = \frac{10}{30} = \frac{1}{3}.$$

PROBLEM : 4 If X is Uniformly distributed with mean 1 and Variance $4/3$. Find $P(x<0)$

$$\text{Mean} = E(x) = \frac{b+a}{2}$$

$$\text{Variance} = V(x) = \frac{(b-a)^2}{12}$$

given mean $= 1$, Variance $= 4/3$.

$$\frac{b+a}{2} = 1, \quad \frac{(b-a)^2}{12} = \frac{4}{3}$$

$$\boxed{b+a = 2} \rightarrow (1), \quad (b-a)^2 = 16$$

$$\boxed{(b-a) = \pm 4.} \rightarrow (2)$$

Solve (1) & (2)

$b+a = 2$	$b+a = 2$
$b-a = 4$	$b-a = -4$
$2b = 6$	$2b = -2$
$\boxed{b=3}$	$\boxed{b=-1}$
$a+b = 2$	$a+b = 2$
$a = 2-b = 2-3$	$a = 2-b = 2-(-1)$
$\boxed{a=-1}$	$\boxed{a=3}$

for Uniform distribution $a < x < b$

$a = -1$. and $b = 3$.

(19)

The Pdf is $f(x) = \dfrac{1}{b-a}$, $a < x < b$.

$$= \frac{1}{3-(-1)} = \frac{1}{4}, \quad -1 < x < 3$$

$$P(X<0) = \int_{-1}^{0} f(x)\,dx = \int_{-1}^{0} \frac{1}{4}\,dx = \frac{1}{4}\left[x\right]_{-1}^{0} = \frac{1}{4}(0+1)$$

$$P(X<0) = \frac{1}{4}.$$

FITTING DISTRIBUTIONS FOR GIVEN DATA.

Suppose, fitting a binomial distribution means assuming that the given distribution is approximately Binomial and hence finding the Probability mass function and then finding the theoretical frequencies, that is to find the expected frequencies.

PROBLEM 1 Fit a binomial distribution for the following data :

x :	0	1	2	3	4	5	6
f :	5	18	28	12	7	6	4

The binomial distribution is defined as

$$P(X=x) = nC_x \, p^x \, q^{n-x} \quad , \quad x = 0, 1, 2, \ldots, n$$

Find the mean of binomial distribution.

$$\mu = \bar{x} = E(x) = \sum x \cdot P(x)$$

$$\text{Mean} = \bar{x} = \frac{\sum fx}{N} = \frac{\sum fx}{\sum f}$$

x	f	$f * x$
0	5	0
1	18	18
2	28	56
3	12	36
4	7	28
5	6	30
6	4	24
	$\Sigma f = 80$	$\Sigma fx = 192$

Mean $\bar{x} = \dfrac{\Sigma fx}{\Sigma f}$

$\bar{x} = \dfrac{192}{80} = 2.4$

$\therefore \boxed{\bar{x} = 2.4}$

W.K.T $\boxed{\text{Mean} = np}$

$\therefore np = 2.4, \quad \boxed{n = 6}$ largest x value

$P = \dfrac{2.4}{6} = 0.4$

$\therefore \boxed{P = 0.4} \quad \boxed{q = 0.6} \quad q = 1 - P$

The binomial distribution

$$P(X=x) = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n$$

$$P(X=x) = 6C_x (0.4)^x (0.6)^{6-x} \quad x = 0, 1, 2, \ldots, 6$$

$P(X=0) = 6C_0 (0.4)^0 (0.6)^6$

$P(X=1) = 6C_1 (0.4)^1 (0.6)^5$

$P(X=2) = 6C_2 (0.4)^2 (0.6)^4$

$P(X=3) = 6C_3 (0.4)^3 (0.6)^3$

$P(X=4) = 6C_4 (0.4)^4 (0.6)^2$

$P(X=5) = 6C_5 (0.4)^5 (0.6)^1$

$P(X=6) = 6C_6 (0.4)^6 (0.6)^0$

x	$P(x=x)$	Expected Frequency. $N * P(x=x)$
0	0.04665	$80 * 0.04665 = 3.73 \cong 4$
1	0.03110	$80 * 0.03110 = 14.93 \cong 15$
2	0.3105	$80 * 0.3105 = 24.84 \cong 25$
3	0.13824	$80 * 0.13824 = 11.06 \simeq 11$
4	0.2764	$80 * 0.2764 = 22.11 \cong 22$
5	0.03686	$80 * 0.03686 = 2.95 \cong 3$
6	0.004096	$80 * 0.004096 = 0.33 \cong 0$

∴ The expected frequency are given below

x :	0	1	2	3	4	5	6
Expected frequency :	4	15	25	22	11	3	0

Problem 2 Fit a Poisson distribution for the following data:

x :	0	1	2	3	4	5
f :	142	156	69	27	5	1

x	$P(X=x)$	Expected frequency $N * P(X=x)$
0	0.3679	$400 * 0.3679 = 147.15 \cong 147$
1	0.3679	$400 * 0.3679 \simeq 147.15 \cong 147$
2	0.18395	$400 * 0.18395 = 73.58 \cong 74$
3	0.0613	$400 * 0.0613 = 24.53 = 25$
4	0.0153	$400 * 0.0153 = 6.13 \cong 6$
5	0.003306	$400 * 0.0033 = 1.23 = 1.$

The expected frequency are given below.

X	:	0	1	2	3	4	5
{Expected frequency}	:	147	147	74	25	6	1

MEMMORY LESS PROPERTY.

A non-negative random Variable X is memeryless if

$$P(X>s+t \mid X>s) = P(X>t) \quad \text{for all } s,t \geq 0.$$

Note:

This Property is also known as Markov Property and ageless Property.

PROVE THAT GEOMETRIC RANDOM VARIABLE HAS MEMORYLESS PROPERTY.

The geometric distribution is

$$P(X=x) = q^{x-1}p, \quad x=1,2,3,\ldots$$

Consider $P(X>t) = P(X=t+1) + P(X=t+2) + \cdots$

$$= q^t \cdot p + q^{t+1} \cdot p + \cdots$$

$$= q^t p\left[1+q+q^2+\cdots\right] = q^t p \cdot [1-q]^{-1}$$

$$= \frac{q^t p}{(1-q)} = \frac{q^t p}{p} = q^t.$$

$$\therefore \boxed{P(X>t) = q^t.} \rightarrow (1)$$

$$P[x>s+t / x>s] = \frac{P[x>s+t \cap x>s]}{P(x>s)}$$

$$= \frac{P[x>s+t]}{P(x>s)} = \frac{q^{s+t}}{q^s} = \frac{q^s \cdot q^t}{q^s} = q^t. \rightarrow ②$$

$\therefore$ from ① + ② we get.

$$P[x>s+t / x>s] = P(x>t)$$

——— x ——— x ———

MEMORY LESS PROPERTY OF EXPONENTIAL DISTRIBUTION.

If X is a CRV following E.D and s, t are real numbers then

$$P[x>s+t / x>s] = P[x>t]$$

Proof The p.d.f of Exponential distribution is

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

$$P[x>t] = \int_t^\infty \lambda e^{-\lambda x} dx = \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_t^\infty = -\left[e^{-\infty} - e^{-t\lambda}\right]$$

$$= -\left[0 - e^{-t\lambda}\right] = e^{-t\lambda}$$

$$\therefore \boxed{P[x>t] = e^{-t\lambda}.} \rightarrow ①$$

Now we Consider

$$P[x>s+t/x>s] = \frac{P[x>s+t \cap x>s]}{P[x>s]}$$

$$= \frac{P[x>s+t]}{P[x>s]} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t}.$$

$$\boxed{\therefore\ P[x>s+t/x>s] = e^{-t\lambda}.} \rightarrow (2)$$

$\therefore$ from (1) & (2) We get

$$\therefore\ P[x>s+t/x>s] = P[x>t]$$