Auto correlation Function and its Properties:

If the process $\{x(t)\}$ is stationary either in the strict sense or in the wide sense, then $E\{x(t)x(t-\tau)\}$ is a function of τ, denoted by $R_{xx}(\tau)$ or $R(\tau)$ or $R_x(\tau)$. This function $R(\tau)$ is called the autocorrelation function of the process $\{x(t)\}$.

$R(\tau)$ is an even function of τ. ie, $R(\tau) = R(-\tau)$

Proof:

$$R(\tau) = E\{x(t)\,x(t-\tau)\}$$

$$\Rightarrow R(-\tau) = E\{x(t)\,x(t+\tau)\}$$

put $t_1 = t+\tau$ then $t = t_1 - \tau$

$$\therefore R(-\tau) = E\{x(t_1-\tau)\,x(t_1)\}$$

$$= E\{x(t_1)\,x(t_1-\tau)\} = R(\tau)$$

$R(\tau)$ is maximum at $\tau = 0$. ie, $|R(\tau)| \leq R(0)$

Proof:

Cauchy schwarz inequality is

$$\{E(xy)\}^2 \leq E(x^2)\,E(y^2)$$

put $x = x(t)$ and $y = x(t-\tau)$

Then $\{E(x(t)\,x(t-\tau))\}^2 \leq E\{x^2(t)\}\,E\{x^2(t-\tau)\}$

$$\{R(\tau)\}^2 \le [E\{x^2(t)\}]^2$$

since $E\{x(t)\}$ and $var\{x(t)\}$ are constant for a stationary process

$$\Rightarrow \{R(\tau)\}^2 \le \{R(0)\}^2 \qquad \because R(\tau) = E\{x(t)\,x(t-\tau)\} \quad R(0) = E\{x^2(t)\}$$

Taking square root on both sides

$$|R(\tau)| \le R(0) \qquad \because R(0) \text{ is positive.}$$

If the autocorrelation function $R(\tau)$ of a real stationary process $\{x(t)\}$ is continuous at $\tau=0$, it is continuous at every other point.

Proof:

Consider $E[\{x(t)-x(t-\tau)\}^2]$

$$= E\{x^2(t)\} + E\{x^2(t-\tau)\} - 2E\{x(t)x(t-\tau)\}$$

$$= R(0) + R(0) - 2R(\tau)$$

$$= 2[R(0) - R(\tau)]$$

Since $R(\tau)$ is continuous at $\tau=0$, $\lim_{\tau\to 0} R(\tau) = R(0)$

$$\Rightarrow E[\{x(t)-x(t-\tau)\}^2] = 0$$

$$\therefore \lim_{\tau\to 0} x(t-\tau) = x(t)$$

i.e., $x(t)$ is continuous for all t.

Consider $R(\tau+h) - R(\tau)$

$$= E\{x(t)\,x(t-(\tau+h))\} - E\{x(t)x(t-\tau)\}$$

$= E[X(t)\{X(t-\tau-h)-X(t-\tau)\}]$

Now $\lim_{h\to 0}[X(t-\tau-h)-X(t-\tau)]=0$

$\Rightarrow \lim_{h\to 0}[R(\tau+h)-R(\tau)]=0$

ie, $\lim_{h\to 0} R(\tau+h)-R(\tau)$

Hence $R(\tau)$ is continuous for all τ.

If $R(\tau)$ is the autocorrelation function of a stationary process $\{X(t)\}$ with no periodic component, then $\lim_{\tau\to\infty} R(\tau)=\mu_x^2$, provided the limit exists.

ie, $\mu_x=\sqrt{\lim_{\tau\to\infty} R(\tau)}$

Proof:

$R(\tau)=E\{X(t)X(t-\tau)\}$

When τ is very large, $X(t)$ and $X(t-\tau)$ are two sample functions of the process $\{X(t)\}$ observed at a very long interval of time.

$\therefore$ $X(t)$ and $X(t-\tau)$ tend to become independent [$X(t)$ and $X(t-\tau)$ may be dependent, when $X(t)$ contains contains a periodic component, which is not true].

$\therefore \lim_{\tau\to\infty}\{R(\tau)\}=E\{X(t)\}E\{X(t-\tau)\}$

Since $X(t)$ is stationary, $E\{X(t)\}$ is a

Constant $= \mu_x$.

$$\therefore \lim_{\tau \to \infty} R(\tau) = \mu_x^2$$

$$\Rightarrow \mu_x = \sqrt{\lim_{\tau \to \infty} R(\tau)}$$

Cross Correlation Function and its Properties

If the processes $\{X(t)\}$ and $\{Y(t)\}$ are jointly wide-sense stationary, then $E\{X(t)Y(t-\tau)\}$ is a function of τ, denoted by $R_{xy}(\tau)$. This function $R_{xy}(\tau)$ is called the cross-correlation function of the processes $\{X(t)\}$ and $\{Y(t)\}$.

$$R_{yx}(\tau) = R_{xy}(-\tau)$$

Proof:

$$R_{xy}(\tau) = E[X(t)Y(t+\tau)]$$

$$R_{xy}(-\tau) = E[X(t)\,Y(t-\tau)]$$

put $t_1 = t - \tau$

$$\Rightarrow R_{xy}(-\tau) = E[Y(t_1)X(t_1+\tau)]$$

$$= R_{yx}(\tau)$$

$$|R_{xy}(\tau)| \le \sqrt{R_{xx}(0)\,R_{yy}(0)}$$

This means that the maximum of $R_{xy}(\tau)$ can occur anywhere, but it cannot exceed $\sqrt{R_{xx}(0) R_{yy}(0)}$

Proof:

For any real number α, we know that

$$E[\alpha x(t) + y(t+\tau)]^2 \geq 0$$

$$\Rightarrow E[\alpha^2 x^2(t) + y^2(t+\tau) + 2\alpha x(t) y(t+\tau)] \geq 0$$

$$\Rightarrow \alpha^2 E[x^2(t)] + E[y^2(t+\tau)] + 2\alpha E[x(t) y(t+\tau)] \geq 0$$

Since $\{x(t)\}$ and ~~y&x~~ $\{y(t)\}$ are jointly WSS, each is a WSS process.

Hence the second order moments are constants.

But $E[x^2(t)] = R_{xx}(0)$ by the property of auto correlation function and $E\{y^2(t+\tau)\} = R_{yy}(0)$

$$\alpha^2 R_{xx}(0) + 2\alpha R_{xy}(\tau) + R_{yy}(0) \geq 0 \quad \forall \alpha$$

Since $R_{xx}(0) > 0$ and α is any real number, the discriminant is ≤ 0.

$$4[R_{xy}(\tau)]^2 - 4 R_{xx}(0) R_{yy}(0) \leq 0$$

$$[R_{xy}(\tau)]^2 \leq R_{xx}(0) R_{yy}(0)$$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}$$

$$|R_{xy}(\tau)| \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$$

Proof:

W.K.T $R_{xx}(0)$ and $R_{yy}(0)$ are positive numbers

So their A.M $\geq$ G.M

$$\frac{R_{xx}(0)+R_{yy}(0)}{2} \geq \sqrt{R_{xx}(0)R_{yy}(0)}$$

By property 2, $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)}$

$$\Rightarrow |R_{xy}(\tau)| \leq \sqrt{R_{xx}(0)R_{yy}(0)} \leq \frac{R_{xx}(0)+R_{yy}(0)}{2}$$

$$\therefore |R_{xy}(\tau)| \leq \frac{R_{xx}(0)+R_{yy}(0)}{2}$$

If the processes $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal, then $R_{xy}(\tau)=0$.

If the processes $\{X(t)\}$ and $\{Y(t)\}$ are independent, then $R_{xy}(\tau)=\mu_x \times \mu_y$.

Given that the auto correlation function for a stationary ergodic process with no periodic components is $R_{XX}(\tau)=25+\frac{4}{1+6\tau^2}$. Find the mean value and variance of the process.

Sol:

By property of autocorrelation function

$$\mu_x^2 = \lim_{\tau\to\infty} R_{XX}(\tau)$$

$$= \lim_{\tau\to\infty}\left(25+\frac{4}{1+6\tau^2}\right) = 25$$

$\therefore \mu_x = 5$

$\Rightarrow E[x(t)] = 5$

Now, $var[x(t)] = E[x^2(t)] - [E[x(t)]]^2$

We know that $E[x^2(t)] = R_{xx}(0) = 29$

$[\because R_{xx}(0) = 25+4]$

$\therefore Var[x(t)] = 29 - 5^2 = 4$

2.) The autocorrelation function of a stationary process is given by $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$. Find the mean value of the random variable $y = \int_0^2 x(t)\,dt$ and variance of $x(t)$.

Sol: Given $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$

We know that $\mu_x = \sqrt{\lim_{\tau \to \infty} R_{xx}(\tau)}$

$\Rightarrow \mu_x = \sqrt{\lim_{\tau \to \infty} [9 + 2e^{-|\tau|}]} = 3$

$\therefore E[x(t)] = 3$

$Var[x(t)] = E[x^2(t)] - [E(x(t))]^2$

W.K.T $E[x^2(t)] = R_{xx}(0) = 9+2 = 11$

$\therefore Var[x(t)] = 11 - 9 = 2$

Given $y = \int_0^2 x(t)\,dt$

$\therefore E(y) = \int_0^2 E[x(t)]\,dt = \int_0^2 3\,dt = 3[t]_0^2 = 6$

3) A stationary random process has an autocorrelation function given by $R_{XX}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$. Find the mean and variance of the process.

Sol:

We know that $\mu_x^2 = \lim_{\tau\to\infty} R_{XX}(\tau)$

$$= \lim_{\tau\to\infty} \left(\frac{25\tau^2+36}{6.25\tau^2+4}\right)$$

$$= \lim_{\tau\to\infty} \left(\frac{25+36/\tau^2}{6.25+4/\tau^2}\right)$$

$$= \frac{25}{6.25} = 4$$

$\therefore \mu_x = E[X(t)] = 2$

$Var[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2$

W.K.T $E[X^2(t)] = R_{XX}(0) = \frac{36}{4} = 9$

$\therefore Var[X(t)] = 9 - 4 = 5$

4) If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{XX}(T)$ and If $Y(t) = X(t+\alpha) - X(t-\alpha)$ show that $R_{YY}(T) = 2R_{XX}(T) - R_{XX}(T+2\alpha) - R_{XX}(T-2\alpha)$

Proof:

Given $X(t)$ is a WSS Process

$\therefore E[X(t)]$ is constant and $R_{XX}(t, t+T) = R_{XX}(T)$

Given $Y(t) = X(t+\alpha) - X(t-\alpha)$

$\therefore R_{YY}(t, t+T) = E[Y(t)Y(t+T)]$

$= E\{[X(t+\alpha) - X(t-\alpha)][X(t+T+\alpha) - X(t+T-\alpha)]\}$

$= E\{X(t+\alpha)X(t+T+\alpha) - X(t+\alpha)X(t+T-\alpha)$

$- X(t-\alpha)X(t+T+\alpha) + X(t-\alpha)X(t+T-\alpha)\}$

$= E[X(t+\alpha)X(t+T+\alpha)] - E[X(t+\alpha)X(t+T-\alpha)]$

$- E[X(t-\alpha)X(t+T+\alpha)] + E[X(t-\alpha)X(t+T-\alpha)]$

$= E[X(t+\alpha)X((t+\alpha)+T)] - E[X(t+\alpha)X((t+\alpha)+T-2\alpha)]$

$- E[X(t-\alpha)X((t-\alpha)+T+2\alpha)] + E[X(t-\alpha)X((t-\alpha)+T)]$

$= R_{XX}(T) - R_{XX}(T-2\alpha) - R_{XX}(T+2\alpha) + R_{XX}(T)$

$= 2R_{XX}(T) - R_{XX}(T-2\alpha) - R_{XX}(T+2\alpha)$

5.) Given that $X(t)$ is a random process with mean $\mu(t) = 3$ and auto correlation function $R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$. Determine the mean, variance and covariance of the random variables $Y = X(5)$ and $Z = X(8)$.

Sol:

Given $\mu(t) = 3 \Rightarrow E[X(t)] = 3$ for any t.

$R(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$

and $Y = X(5)$, $Z = X(8)$

$\therefore E(Y) = E[X(5)] = 3$ and $E(Z) = E[X(8)] = 3$

$E[Y^2] = E[X^2(5)] = R(5,5)$

$= 9 + 4e^{-0.2(0)} = 9 + 4 = 13$

$\therefore Var(Y) = E[X^2(5)] - (E[X(5)])^2$

$= 13 - 9 = 4$

Similarly $E[Z^2] = E[X^2(8)] = R(8,8) = 13$ & $Var(Z) = 4$

$Cov(Y,Z) = E[YZ] - E[Y]E[Z]$

Now $E[YZ] = E[X(5)X(8)]$

$= R_{XX}(5,8) = 9 + e^{-0.2|5-8|}$

$= 9 + 4e^{-0.6}$

$\therefore Cov(Y,Z) = 9 + 4e^{-0.6} - 9 = 2.195$

If $\{X(t)\}$ and $\{Y(t)\}$ are independent WSS with zero means find the autocorrelation function of $\{Z(t)\}$ where $Z(t) = a + bX(t) + cY(t)$.

Sol: Given $Z(t) = a + bX(t) + cY(t)$

By defn $R_{ZZ}(\tau) = E[Z(t).Z(t+\tau)]$

$= E\{[a + bX(t) + cY(t)][a + bX(t+\tau) + cY(t+\tau)]\}$

$= E\{a^2 + abX(t+\tau) + acY(t+\tau) + abX(t)$

$+ b^2 X(t)X(t+\tau) + bcX(t)Y(t+\tau) + acY(t)$

$+ bcY(t)X(t+\tau) + c^2 Y(t)Y(t+\tau)\}$

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$$R_{ZZ}(t) = a^2 + abE[X(t+\tau)] + acE[Y(t+\tau)] + abE[X(t)]$$
$$+ b^2E[X(t)X(t+\tau)] + bcE[X(t)Y(t+\tau)] + acE[Y(t)]$$
$$+ bcE[Y(t)X(t+\tau)] + c^2E[Y(t)Y(t+\tau)].$$

Now $X(t)$ and $Y(t)$ have zero means (ie. constant)

$\therefore X(t) = 0 = X(t+\tau)$ and $Y(t) = 0 = Y(t+\tau)$

$$\therefore R_{ZZ}(\tau) = a^2 + b^2R_{XX}(\tau) + bcE[X(t)Y(t+\tau)] + bcE[Y(t)X(t+\tau)]$$
$$+ c^2R_{YY}(\tau)$$
$$= a^2 + b^2R_{XX}(\tau) + c^2R_{YY}(\tau)$$

[$\because$ $X(t)$ and $Y(t)$ are independent.
$E\{X(t)Y(t+\tau)\} = E\{X(t)\}E\{Y(t+\tau)\}$
$= 0$
$\therefore E[Y(t)X(t+\tau)] = 0$]

7.) If $X(t) = Y\cos t + Z\sin t$ for all 't' where Y and Z are independent binary RV's, each of which assumes the values -1 and 2 with prob. $2/3$ and $1/3$ respectively, prove that $\{X(t)\}$ is a WSS.

Proof:

To prove: $\{X(t)\}$ is a WSS if mean $E[X(t)]$ is a constant and autocorrelation $R_{XX}(t, t+\tau)$ depends only on τ.

Since Y is a discrete r.v which assumes values -1 and 2 with probability $2/3$ and $1/3$ respectively

ie

i.e.,

$Y=y$	-1	2
$P(y)$	$2/3$	$1/3$

Mean $= \sum y\, p(y) = -\frac{2}{3}+\frac{2}{3}=0$ and

$$E(Y^2)=\sum y^2 p(y) = \frac{2}{3}+\frac{4}{3}=2$$

Similarly $E(Z)=0$, $E(Z^2)=2$

Again Y and Z are independent r.v's

$$E(YZ)=E(Y)E(Z)$$

$$\therefore E(YZ)=0$$

Now $X(t)=Y\cos t+Z\sin t$

$$\therefore E[X(t)]=E[Y\cos t+Z\sin t]$$

$$=\cos t\, E[Y]+\sin t\, E(Z)=0$$

$\therefore$ Mean is a Constant.

Now, by defn

$$R_{XX}(t,t+\tau)=E[X(t)X(t+\tau)]$$

$$=E\{[Y\cos t+Z\sin t][Y\cos(t+\tau)+Z\sin(t+\tau)]\}$$

$$=E\{Y^2\cos t\cos(t+\tau)+YZ\cos t\sin(t+\tau)$$
$$+ZY\sin t\cos(t+\tau)+Z^2\sin t\sin(t+\tau)\}$$

$$=\cos t\cos(t+\tau)E(Y^2)+\cos t\sin(t+\tau)E(YZ)$$
$$+\sin t\cos(t+\tau)E(ZY)+\sin t\sin(t+\tau)E(Z^2)$$

$$=2[\cos t\cos(t+\tau)+\sin t\sin(t+\tau)]$$

$\because E(Z^2)=E(Y^2)=2$, $E(YZ)=E(ZY)=0$

$R_{xx}(t, t+\tau) = 2\cos(t+\tau-t) = 2\cos\tau$

$\therefore R_{xx}(t, t+\tau)$ which depends on τ alone

Hence $\{x(t)\}$ is a WSS process.