RANDOM VARIABLES.

A Random Variable (R.V) is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the Sample Space Corresponding to a random ~~Variable~~ Experiment E.

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* The Out Come of random Experiments may be Numerical (or) non-numerical in nature.

Range Space: The set of all values taken by a random Variable X is called its range space and is denoted by R_X. Thus $R_X = \{X(s) \mid s \in S\}$

Example Two Coins are tossed.

Sample Space $S = \{HH, HT, TH, TT\}$

Let X denotes 'no of Heads'.

Then
$$\left.\begin{aligned} X(HH) &= 2 \\ X(HT) &= 1 \\ X(TH) &= 1 \\ X(TT) &= 0 \end{aligned}\right\} \rightarrow \text{Values of } X \text{ is } 0, 1, 2.$$

$R_X = \{0, 1, 2\}$. $\boxed{R_X \text{ is a finite set.}}$

Random Variable

Discrete Random Variable → Probability mass function (PMF)

Continuous random Variable → Probability Density function (PDF).

DISCRETE RANDOM VARIABLE.

A random Variable X is said to be discrete if it takes a finite number of values (or) Countably infinite number of Values.

ie. its Range R_x is finite (or) Countably Infinite.

Example

1. The number of Priniting mistakes in a book.
2. The number of telephones recieved in a office.

A discrete Random Variable assumes each of its Value with Certain Probability.

These Probabilites could be used to define a Probability function (see below).

PROBABILITY FUNCTION (or) PROBABILITY MASS FUNCTION.

Let X be a discrete random Variable which takes Values $x_1, x_2, x_3 \ldots$ Let $P(X=x_i) = p(x_i)$ be Probability of x_i. Then the function p is called the Probability Mass function of X. If the numbers $P(x_i)$ satisfy the Conditions.

(i) $p(x_i) \geq 0 \quad \forall \ i = 1,2,3,\ldots$

(ii) $\sum_{i=1}^{\infty} P(x_i) = 1.$

PROBABILITY DISTRIBUTION:

The set of ordered pairs of numbers $(x_i, P(x_i))$ is called the Probability distribution of the R.V.X.

The Probability distribution is usually displayed in the table form.

x_i	x_1	x_2	x_3	...
$P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	...

COMMULATIVE DISTRIBUTION FUNCTION. (CDF).

(or) DISTRIBUTION FUNCTION OF A RANDOM VARIABLE.

If X is a random Variable, (discrete), then the function $F: R \rightarrow [0,1]$ defined by $F(x) = P(X \leq x)$ is called the Cummulative distribution function of X.

If X is a discrete random Variable

$x_1 < x_2 < x_3 \ldots x_i$ then

$$F(x) = \sum_{x_i \leq x} P(x_i)$$

Properties of distribution function $F(x)$

1) $0 \leq F(x) \leq 1$, $-\infty < x < \infty$

- $F(-\infty) = 0$, $F(\infty) = 1$.

2) If $x_1 < x_2 \Rightarrow F(x_1) < F(x_2)$.

3) If X is discrete R.V. with values $x_1 < x_2 < x_3 \cdots$ then $p(x_i) = F(x_i) - F(x_{i-1})$.

Some Results

* $P(X > x) = 1 - P(X \leq x)$
* $P(X \geq x) = 1 - P(X < x)$
* $P(X \leq x) = 1 - P(X > x)$
* $P(X < x) = 1 - P(X \geq x)$.
* $P(\bar{A}) = 1 - P(A)$.

Conditional Probability

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

If A & B are dependent events.

EXPECTATION :- Expectation of a discrete random Variable.

$$E(X) = \sum_{i=1}^{\infty} x_i \, P(x_i).$$

[Expectation of X (or) Mean Value of X]

Let X be a discrete random Variable taking values

x_i	x_1	x_2	x_3	....
$P(x_i)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	....

Properties :

1) C is constant then $E(C) = C$

2) If a, b are constants, then

$$E(aX + b) = aE(X) + b.$$

3) $E(aX) = aE(X)$, where a is constant

VARIANCE OF X

$$Var(x) = V(x) = E(x^2) - [E(x)]^2$$

$$\sqrt{Var(x)} = \text{Standard Deviation of } X.$$

Properties of Variance :

1) $Var(x) \geq 0$ 2) $Var(a) = 0$, where a is Constant

3) $V(ax \pm b) = a^2 Var(x) + Var(b) = a^2 Var(x)$ //.

where a and b are Constants.

Problem : 1 A Probability distribution of a discrete random Variable X is given by

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

a) find the value of a b) $P(0 < x < 3)$

c) $P(x \geq 3)$ d) find the distribution fn of x

Ans:

a) Since given P(x) is a Probability mass function. of X

$$\therefore \sum P(x) = 1, \quad P(x) \geq 0.$$

$$\therefore a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1.$$

$$81a = 1 \Rightarrow \boxed{a = \frac{1}{81}}$$

b) $P(0 < x < 3) = P(x=1) + P(x=2) = 3a + 5a = 8a$

$$= \frac{8}{81} \text{ //.}$$

c) $P(X \geq 3) = P(X=3) + P(X=4) + \cdots + P(X=8)$

(or)

$P(X \geq 3) = 1 - P(X<3)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$= 1 - \{a + 3a + 5a\} = 1 - 9a$

$= 1 - \frac{9}{81} = \frac{81-9}{81} = \frac{72}{81} = \frac{8}{9}$ //.

d) distribution function (or) cdf. Cummulative Dist. fn.

Cdf of X is $F(x) = P(X \leq x)$.

x:	0	1	2	3	4	5	6	7	8
$p(x)$:	a	3a	5a	7a	9a	11a	13a	15a	17a
$F(x)$:	a	4a	9a	16a	25a	36a	49a	64a	81a.
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	$\frac{81}{81}=1$.

$F(0) = P(X \leq 0) = P(X=0) = a$

$F(1) = P(X \leq 1) = P(X=0) + P(X=1) = a + 3a = 4a$

$F(2) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = 9a$

Simply

$F(3) = P(X \leq 3) = P(X=3) + P(X \leq 2) = 7a + 9a = 16a$

$F(4) = P(X \leq 4) = P(X \leq 3) + P(X=4) = 16a + 9a = 25a$

$F(5) = P(X \leq 5) = P(X \leq 4) + P(X=5) = 25a + 11a = 36a$

etc...

Problem 2: If X is a discrete random Variable with the following Probability function.

x	0	1	2	3	4
$P(x)$	a	$3a$	$5a$	$7a$	$9a$

find 'a'.

a) find $P(X \geq 3)$ b) $P(0 < X < 4)$ c) mean

d) Variance e) $E(3X-4)$ f) $Var(3X-4)$.

Soln: Since X is discrete R.V. $P(x_i) \geq 0$

$$\therefore \sum P(x) = 1$$

$$\therefore a + 3a + 5a + 7a + 9a = 1 \Rightarrow \boxed{25a = 1}$$

$$\boxed{a = 1/25}$$

a) $P(X \geq 3) = [1 - P(X<3)]$ or $[P(X=3) + P(X=4)]$

$$= 7a + 9a = 16a = \boxed{\frac{16}{25}}.$$

b) $P(0 < X < 4) = P(X=1) + P(X=2) + P(X=3)$

$$= 3a + 5a + 7a = 15a = \boxed{\frac{15}{25}}$$

c) mean of $X = E(X) = \sum x_i P(x_i)$

d) Variance of $X = E(X^2) - [E(X)]^2$.

x	$P(x)$	$x \cdot P(x)$	x^2	$x^2 \cdot P(x)$	
0	$1/25$	0	0	0	
1	$3/25$	$3/25$	1	$3/25$	
2	$5/25$	$10/25$	4	$20/25$	
3	$7/25$	$21/25$	9	$63/25$	
4	$9/25$	$36/25$	16	$144/25$	

c) Mean of $X = E(x) = \sum x P(x)$

$$= 0 + \frac{3}{25} + \frac{10}{25} + \frac{21}{25} + \frac{36}{25} = \boxed{\frac{70}{25}}$$

d) Variance of $X = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum x^2 P(x)$$

$$= 0 + \frac{3}{25} + \frac{20}{25} + \frac{63}{25} + \frac{144}{25} = \frac{230}{25}.$$

$$Var(X) = \left(\frac{230}{25}\right) - \left(\frac{70}{25}\right)^2 = \frac{230}{25} - \frac{4900}{625}$$

$$= \frac{5750 - 4900}{625} = \boxed{\frac{850}{625}} // . = \boxed{\frac{34}{25}}$$

e) $E(3x-4) = E(3x) - E(4) = 3E(x) - 4$

Since $[\, E(a) = a$, a is Constant, $E(ax) = aE(x) \,]$

$$= 3 * \frac{70}{25} - 4 = \frac{210 - 100}{25} = \boxed{\frac{110}{25}}$$

$$= \boxed{\frac{22}{5}}$$

f) $Var(3X-4) = 3^2 Var(X) - Var(4) = 9\,Var(X)$

$$= 9 * \frac{850}{625} = \boxed{\frac{7650}{625}} = \frac{306}{25}$$

Problem : 3 A Random Variable X has the following Probability function.

x	0	1	2	3	4	5	6	7
$P(x)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2+a$.

find (i) a ii) $P(x<4)$ (iii) $P(x\geq4)$ (iv) $P(x\geq6)$

(v) $P(x<6)$ (vi) $P\left(\frac{3}{2}<x\ \frac{9}{2} / x>2\right)$

vii) $P(0\leq x\leq4)$ viii) find cdf.

ix) If $P(x\leq k) > 1/2$, find the least value of K.

Soln: $\Sigma P(x) = 1$.

(i) $0+a+2a+2a+3a+a^2+2a^2+7a^2+a = 1$

$$10a^2+9a-1=0.$$

$$10a^2+9a+a-a-1=0$$

$$10a^2+10a-a-1=0$$

$$10a(a+1)-(a+1)=0$$

$$(10a-1)(a+1)=0.$$

$\therefore$ $10a-1=0$ (or) $a+1=0$

$10a=1$ (or) $\boxed{a=-1}$

$\boxed{a=\frac{1}{10}}$

Since $a\geq0$ (By Condition Pmf) $\therefore$ $\boxed{a=\frac{1}{10}}$

(ii) $P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= 0 + a + 2a + 2a$

$P(x<4) = 5a = \frac{5}{10} = \frac{1}{2}$ w.k.t $a = \frac{1}{10}$

(iii) $P(x \geq 4) = P(x=4) + P(x=5) + P(x=6) + P(x=7)$

(or)

$P(x \geq 4) = 1 - P(x<4) = 1 - \frac{5}{10}$ (from the above)

$$\boxed{P(x \geq 4) = \frac{5}{10} = \frac{1}{2}.}$$

iv) $P(x \geq 6) = P(x=6) + P(x=7) = 2a^2 + 7a^2 + a$

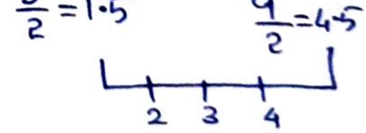
$= 9a^2 + a = 9\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{9}{100} + \frac{1}{10}$.

$= \frac{9+10}{100} = \frac{19}{100}$

v) $P(x<6) = 1 - P(x \geq 6) = 1 - \frac{19}{100} = \boxed{\frac{81}{100}}$

(vi) $P\left(\frac{3}{2} < x < \frac{9}{2} / x>2\right)$

$$\frac{P\left(3/2 < x < \frac{9}{2} \cap x>2\right)}{P(x>2)}$$



$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(x>2) = 1 - P(x \leq 2)$

$= 1 - \{P(x=0) + P(x=1) + P(x=2)\}$

$= 1 - \{0 + a + 2a\}$

$= 1 - 3a = 1 - 3\left(\frac{1}{10}\right) = \frac{7}{10}$//

$P(x>2)$

$= 1 - P(X \leq 2)$

$= 1-$

$$P\left(\frac{3}{2} < x < \frac{9}{2} \cap x > 2\right) = P(1.5 < x < 4.5 \cap x > 2)$$

$$= P(2 < x < 4.5) = P(x=3) + P(x=4)$$

$$= 2a + 3a = 5a = \frac{5}{10}$$

$$P\left(\frac{3}{2} < x < \frac{9}{2} / x > 2\right) = \frac{P\left(\frac{3}{2} < x < 9/2 \cap x > 2\right)}{P(x>2)}$$

$$= \frac{5/10}{7/10} = 5/7.$$

vii) $P(0 \leq x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$$= 0 + a + 2a + 2a + 3a = 8a.$$

$$= \frac{8}{10} = \boxed{\frac{4}{5}}$$

viii) cdf $F(x) = P(X \leq x)$.

$F(0) = P(X \leq 0) = P(x=0) = 0$

$F(1) = P(x \leq 1) = P(x=0) + P(x=1) = 0 + a = \frac{1}{10}$

$F(2) = P(x \leq 2) = P(x \leq 1) + P(x=2) = a + 2a = 3a = \frac{3}{10}$

$F(3) = P(x \leq 3) = P(x \leq 2) + P(x=3) = 3a + 2a = 5a = \frac{5}{10}$

$F(4) = P(x \leq 4) = P(x \leq 3) + P(x=4) = 5a + 3a = 8a = \frac{8}{10}$

$F(5) = P(x \leq 5) = P(x \leq 4) + P(x=5) = 8a + a^2.$

$$= \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$$F(6) = P(X \leq 6) = P(X \leq 5) + P(X=6) = \frac{81}{100} + 2a^2$$

$$= \frac{81}{100} + \frac{2}{100} = \frac{83}{100}$$

$$F(7) = P(X \leq 7) = P(X \leq 6) + P(X=7)$$

$$= \frac{83}{100} + 7a^2 + a = \frac{83}{100} + \frac{7}{100} + \frac{1}{10}$$

$$= \frac{83+7+10}{100} = \frac{100}{100} = 1$$

x	0	1	2	3	4	5	6	7
$P(x)$	0	a	$2a$	$2a$	$3a$	a^2	$2a^2$	$7a^2+a$
$P(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$
$F(x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1.

ix) From the answer Part (viii)

$P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$

$P(X \leq 5) = \frac{81}{100} > 1/2$

$P(X \leq 6) = \frac{83}{100} > 1/2$

$P(X \leq 7) = 1 > 1/2$

the minimum Value of 'k' for $P(X \leq k) > 1/2$.

$\frac{8}{10} = 0.8$, $\frac{81}{100} = 0.81$

$\frac{83}{100} = 0.83$, 1

$\therefore$ k has minimum Value is $\boxed{4}$

CONTINUOUS RANDOM VARIABLE.

Definition A random Variable X is Said to be Continuous if its Range Space R_X is an Un countable set of real numbers. i.e. the random Variable assumes values in the an interval (a,b) (or) in an union of intervals.

EXAMPLE:

(i) X denotes the life time of a transistor :

(ii) X denotes the Operating time between two failures of a machine.

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* To define a Continuous random Variable, the Sample Space Should be Continuous.

* In the Case of Continuous Random Variable, we Cannot Speak about first value, second value, and thus $P(x_1)$, $P(x_2)$ etc., becomes meaningless. So the Probability function of a Continuous random Variable is defined as below.

PROBABILITY DENSITY FUNCTION OF A CONTINUOUS RANDOM VARIABLE.

A function 'f', defined for all $x \in (-\infty, \infty)$ is Called the Probability density function of a C.R.V 'X' if. (i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

for all $\forall \; x \in (-\infty, \infty)$.

<u>Note</u>: 1) $P(X<a) = \int_{-\infty}^{a} f(x)\, dx$.

2) $P(X>a) = \int_{a}^{\infty} f(x)\, dx$.

3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$.

If $b=a$, $\int_{a}^{a} f(x)\, dx = 0$.

Since X is a Continuous Random Variable.

$$P(X \leq a) = P(X<a), \quad P(X>a) = P(X \geq a)$$

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b)$$
$$= P(a < X < b).$$

CUMMULATIVE DISTRIBUTION FUNCTION (CDF)

(or) DISTRIBUTION FUNCTION of a random Variable

If X is a Random Variable, discrete or Continuous, then the function $F: R \to [0,1]$ defined by $F(x) = P(X \leq x)$ is called the Cummulative distribution function of X.

If X is a Continuous random Variable with P.d.f $f(x)$ defined for all $x \in (-\infty, \infty)$

then

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\, dx.}$$

Properties of distribution function $F(x)$.

(i) $0 \leq F(x) \leq 1$. $\quad -\infty < x < \infty$

(ii) $F(x)$ is an increasing function of x.

$$a < b \Rightarrow F(a) < F(b).$$

(iii) $\lim_{x \to -\infty} F(x) = 0$ $\quad$ (iv) $\lim_{x \to \infty} F(x) = 1$.

$\therefore F(-\infty) = 0$ $\qquad F(\infty) = 1$.

Problem 1 If X is a Continuous random Variable with Pdf

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{3}{2}(x-1)^2 & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find the Cummulative distribution function $F(x)$ of X and use it to find $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$.

The c.d.f of X is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx.$

If $x<0$ then $f(x)=0$ $\therefore$ $F(x)=0$.

If $0\le x<1$ then $F(x)=\int_0^x f(x)\,dx=\int_0^x x\,dx=\left[\frac{x^2}{2}\right]_0^x=\frac{x^2}{2}$

If $1\le x<2$ then $F(x)=\int_0^x f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx.$

$$=\int_0^1 x\,dx+\int_1^x \frac{3}{2}(x-1)^2\,dx.$$

$$=\left[\frac{x^2}{2}\right]_0^1+\frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_1^x=\frac{1}{2}+\frac{3}{2}\left[\frac{(x-1)^3}{3}\right]$$

$$=\frac{1}{2}+\frac{1}{2}(x-1)^3.$$

If $x\ge 2$, then $F(x)=1$.

$\therefore$ The c.d.f of X is $F(x)=\begin{cases}0 & \text{if } x<0\\ \frac{x^2}{2} & \text{if } 0\le x<1\\ \frac{1}{2}+\frac{1}{2}(x-1)^3 & \text{if } 1\le x<2\\ 1. & x\ge 2\end{cases}$

$$P\left(\frac{3}{2}<X<\frac{5}{2}\right)=\int_{3/2}^{5/2} f(x)\,dx=F\left(\frac{5}{2}\right)-F\left(\frac{3}{2}\right)$$

$$=1-\left[\frac{1}{2}+\frac{1}{2}\left(\frac{3}{2}-1\right)^3\right]=1-\left[\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^3\right]$$

$$=1-\left[\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{8}\right]=1-\left[\frac{8+1}{16}\right]=\frac{16-9}{16}=\frac{7}{16}$$

2) The amount of bread (in hundreds of kgs) that a certain bakery is able to sell in a day is a random variable X with a pdf given by

$$f(x) = \begin{cases} Ax & \text{if } 0 \leq x < 5 \\ A(10-x) & \text{if } 5 \leq x < 10 \\ 0 & \text{otherwise.} \end{cases}$$

i) find the value of A

ii) Find the Probability of that in a day sales is

a) more than 500 kgs

b) Less than 500 kgs c) between 250 and 750 kgs

(iii) $P(X>5 / X<5)$, and $P(X>5 / 2.5 < X < 7.5)$

(iv) find c.d.f, v) mean vi) variance.

The Random Variable X denotes the amount of Sales in 100's of kgs. ∴ So, ∴ Sales 500 kgs mean

$\boxed{X=5}$

(i) find A. Since $f(x)$ is the p.d.f of X, ∴ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^5 \overset{Ax}{f(x)}\,dx + \int_5^{10} A(10-x)\,dx = 1.$$

$$\left[A x^2/2\right]_0^5 + \left[A\left(10x - x^2/2\right)\right]_5^{10} = 1$$

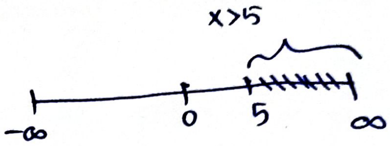
$$A\left[5^2/2\right] + A\left[\left(10(10) - \frac{10^2}{2}\right) - \left(10(5) - \frac{5^2}{2}\right)\right] = 1$$

$$A\left[\frac{25}{2}\right] + A\left[\left(100 - \frac{100}{2}\right) - \left(50 - \frac{25}{2}\right)\right] = 1$$

$$A\left(\frac{25}{2}\right) + A\left[\frac{100}{2} - \frac{75}{2}\right] = 1 \Rightarrow \frac{25A}{2} + \frac{25A}{2} = 1$$

$$25A = 1 \Rightarrow \boxed{A = 1/25}$$

(ii) a) Probability more than 500 Kgs.



$$P(x > 5) = \int_5^\infty f(x)\,dx. = \int_5^{10} A(10-x)\,dx.$$

$$= -A\int_5^{10}(x-10)\,dx = -A\left[\frac{(x-10)^2}{2}\right]_5^{10}$$

$$= -\frac{A}{2}\left[0 - 5^2\right] = \frac{25A}{2} = \frac{25}{2}\left(\frac{1}{25}\right) = 1/2.$$

(iii) b) $P(x<5)$

$$P(x<5) = 1 - P(x \geq 5) = 1 - P(x>5)$$

since C.R.V.

$$= 1 - 1/2 \quad \text{(from the above)}$$

$$P(x<5) = 1/2.$$

(iv) $P(x>5 / x<5)$

$$P(x>5/x<5) = \frac{P(x>5 \cap x<5)}{P(x<5)}$$

There no common 'x' between $x>5$ & $x<5$

$$P(x>5 \cap x<5) = 0 \,//.$$

$$\therefore P(x>5/x<5) = 0 \,/.$$

c) Probability of Sales between 250 to 750 kgs.

$$P(2.5 < X < 7.5) = \int_{2.5}^{7.5} f(x)\,dx = \int_{2.5}^{5} f(x)\,dx + \int_{5}^{7.5} f(x)\,dx.$$

$$= \int_{2.5}^{5} Ax\,dx + \int_{5}^{7.5} A(10-x)\,dx.$$

$$= A\left[\frac{x^2}{2}\right]_{2.5}^{5} + (-A)\int_{5}^{7.5} (x-10)\,dx.$$

$$= \frac{A}{2}[25 - 6.25] - A\left[\frac{(x-10)^2}{2}\right]_{5}^{7.5}$$

$$= \frac{A}{2}[25-6.25] - \frac{A}{2}[6.25 - 25]$$

$$= \frac{A}{2}[25-6.25 - 6.25 + 25] = \frac{A}{2}[50 - 12.5]$$

$$= \frac{A}{2}\left[50 - \frac{25}{2}\right] = \frac{A}{2}\left[\frac{75}{2}\right] = \frac{A}{4} \times 75 = \frac{75}{4}\left(\frac{1}{25}\right)$$

$$\therefore P(2.5 < X < 7.5) = 3/4.$$

---

iv) $P(X>5 / 2.5 < X < 7.5)$ $\quad P(B/A) = \dfrac{P(A\cap B)}{P(A)}$

$$P(A\cap B) = P(X>5 \cap 2.5 < X < 7.5) = P(5 < X < 7.5)$$

from the above.

$$P(5 < X < 7.5) = -A\int_{5}^{7.5} (x-10)\,dx = -\frac{A}{2}[6.25 - 25]$$

$$= -\frac{A}{2}\left[\frac{25}{4} - 25\right] = -\frac{A}{2}\left[-\frac{75}{4}\right] = \frac{75A}{8}$$

$$P(5 < X < 7.5) = \frac{75}{8} \times \frac{1}{25} = 3/8.$$

$$P(2.5 < X < 7.5) = 3/4 \quad \text{(from the above.)}$$

$$P(x>5 / 2.5<x<7.5) = \frac{3/8}{3/4} = \frac{3}{8} \times \frac{4}{3} = \frac{4}{8} = 1/2.$$

iv) c.d.f

$$F(x) = \int_{-\infty}^{x} f(x)\, dx.$$

If $x<0$ $\quad F(x) = 0.$

If $x \in (0,5)$, then $F(x) = \int_0^x Ax\, dx = A\left[\frac{x^2}{2}\right]_0^x = \frac{Ax^2}{2}$

$$= \frac{x^2}{50} \;//.$$

If $x \in (5,10)$, then $F(x) = \int_{-\infty}^{x} f(x)\, dx.$

$$= \int_0^5 f(x)\, dx + \int_5^x f(x)\, dx.$$

$$= \int_0^5 Ax\, dx + \int_5^x A(10-x)\, dx$$

$$= A\left[\frac{x^2}{2}\right]_0^5 + (-A)\int_5^x (x-10)\, dx = \frac{25A}{2} - A\left[\frac{(x-10)^2}{2}\right]_5^x$$

$$= \frac{25}{2}\left(\frac{1}{25}\right) - \left(\frac{1}{25}\right)\left[\frac{(x-10)^2}{2} - \frac{5^2}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{50}\left[(x-10)^2 - 25\right] = \frac{1}{2} - \frac{1}{50}(x-10)^2 + \frac{25}{50}$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{1}{50}(x-10)^2 = 1 - \frac{1}{50}(x-10)^2 \;//.$$

If $x \geq 10$ $\quad F(x) = 1.$

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2/50 & \text{if } 0 \leq x < 5 \\ 1 - \frac{1}{50}(x-10)^2 & \text{if } 5 \leq x < 10 \\ 1 & \text{if } x \geq 10. \end{cases}$$

C.d.f.

V) Mean

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x \, f(x) \, dx.$$

$$= \int_0^5 x \cdot f(x) \, dx + \int_5^{10} x \cdot f(x) \, dx.$$

$$= \int_0^5 x \cdot Ax \, dx + \int_5^{10} x \cdot A(10-x) \, dx.$$

$$= A\int_0^5 x^2 \, dx + A\int_5^{10} (10x - x^2) \, dx$$

$$= A\left[\frac{x^3}{3}\right]_0^5 + A\left[\frac{10x^2}{2} - \frac{x^3}{3}\right]_5^{10}$$

$$= A \times \frac{125}{3} - A\left[\left(\frac{10 \times 100}{2} - \frac{1000}{3}\right) - \left(\frac{10 \times 25}{2} - \frac{125}{3}\right)\right]$$

$$= \frac{125}{3} \times \frac{1}{25} + \frac{1}{25}\left[1000\left(\frac{1}{2} - \frac{1}{3}\right) - 125\left(\frac{2}{2} - \frac{1}{3}\right)\right]$$

$$= \frac{3}{3} + \frac{1}{25}\left[1000\left(\frac{3-2}{6}\right) - 125\left(\frac{2}{3}\right)\right]$$

$$= 1 + \frac{1}{25}\left[\frac{1000}{6} - \frac{250}{3}\right] = 1 - \frac{1}{25}\left[\frac{1000 - 500}{6}\right]$$

$$= 1 + \frac{1}{25}\left[\frac{500}{6}\right] = 1 + \frac{20}{6} = \frac{26}{6} = \frac{13}{3}.$$

(vi) Variance

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \int_{-\infty}^{\infty} f(x) \cdot x^2 \, dx = \int_0^5 x^2 f(x)\, dx + \int_5^{10} x^2 f(x)\, dx.$$

$$= A \int_0^5 x^2 \cdot x \, dx + A \int_5^{10} x^2 (10-x)\, dx.$$

$$= A \left[\frac{x^3}{3}\right]_0^5 + A \int_5^{10} (10x^2 - x^3)\, dx.$$

$$= A \left[\frac{125}{3}\right] + A \left[10\frac{x^3}{3} - \frac{x^4}{4}\right]_5^{10}$$

$$= \frac{1}{25}\left[\frac{125}{3}\right] + \frac{1}{25}\left[\left(10\left(\frac{1000}{3}\right) - \left(\frac{10000}{4}\right)\right) - \left(\frac{10.5^3}{3} - \frac{5^4}{4}\right)\right]$$

$$= \frac{5}{3} + \frac{1}{25}\left[10000\left(\frac{1}{3} - \frac{1}{4}\right) - 5^4\left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= 1 + \frac{1}{25}\left[\frac{10000}{12} - 5^4\left(\frac{5}{12}\right)\right]$$

$$= 1 + \frac{1}{25}\left[\frac{10000 - 3125}{12}\right] = 1 + \frac{1}{25} * \frac{6875}{12}$$

$$= 1 + \frac{275}{12} = \frac{287}{12}$$

$$V(x) = E(x^2) - [E(x)]^2 = \frac{287}{12} - \left(\frac{13}{3}\right)^2$$

$$= \frac{287}{12} - \frac{169}{9} = \frac{9(287) - 169(12)}{12 \times 9}$$

$$= \frac{2583 - 2028}{108} = \frac{555}{108} \;//$$

RELATION BETWEEN CDF & PDF.

* If X is a Continuous R.V. with P.d.f then

$$F(x) = \int_{-\infty}^{x} f(x)\, dx, \quad \text{for each } x \in (-\infty, \infty).$$

By fundamental theorem of integral Calculus, we get.

$$F'(x) = f(x) \geq 0, \quad \forall x \text{ at the points where}$$

F is differentiable.

* If Cdf is given $F(x) \Rightarrow$ To find p.d.f

$$\boxed{F'(x) = f(x)}$$

* If X is a discrete R.V. with Values. $x_1 < x_2 < x_3 < \cdots$, then.

$$P(x_i) = F(x_i) - F(x_{i-1}).$$

PROBLEM: 1 Find the Pdf for given Cummulative distribution function of a R.V X is

$$F(x) = \begin{cases} 1 - \dfrac{4}{x^2}, & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$$

given C.D.F

$$F(x) = \begin{cases} 1 - \frac{4}{x^2}, & x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-n}{x^{n+1}}$$

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}.$$

The p.d.f of X is $f(x) = F'(x) = \begin{cases} -4\left(\frac{-2}{x^3}\right), & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

$\therefore$ The pdf is $f(x) = \begin{cases} 8/x^3, & \text{if } x > 2 \\ 0, & \text{if } x \leq 2. \end{cases}$

PROBLEM 2 The cdf of a discrete random Variable X is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{6} & \text{if } 0 \leq x < 2 \\ 1/2 & \text{if } 2 \leq x < 4 \\ 5/8 & \text{if } 4 \leq x < 6 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Find the Probability distribution.

To find the Probability distribution of X, we have to find the probabilities at the Changing Points 0, 2, 4, 6.

$P(x=0) = \frac{1}{6}$, $P(x=2) = F(2) - F(0) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$P(x=4) = F(4) - F(2) = \frac{5}{8} - \frac{1}{2} = \frac{1}{8}$

$P(x=6) = 1 - F(4) = 1 - \frac{5}{8} = \frac{3}{8}.$

The probability distribution of X is

x	0	2	4	6
$P(x)$	1/6	1/3	1/8	3/8

MOMENTS AND MOMENT GENERATING FUNCTIONS

MOMENT The Expected Value of an Integral Power of a random Variable is called its moment.

Moments are classified into two types.

Moment about mean (Central moments)

$$\boxed{\mu_r = E(x-\mu)^r}$$

Moment about any Point (a)

When $a = 0$.

Moment about orgin [Raw moments].

$$\boxed{\mu_r' = E(x-a)^r}$$

The four moments about the mean μ are

$$\mu_1 = E(x-\mu)^1 = E(x) - E(\mu) = \mu - \mu = 0 \quad \therefore \boxed{\mu_1 = 0}$$

$$\mu_2 = E(x-\mu)^2 = E(x^2) - [E(x)]^2 = \text{Variance of } x.$$

$$\mu_3 = E(x-\mu)^3$$

$$\mu_4 = E(x-\mu)^4.$$

MOMENT ABOUT ANY POINT

The r^{th} moment about any point (a) for a random variable X is defined as

$$\mu_r' = E(x-a)^r$$

First four moments about the point a

$$\mu_1' = E(x-a) = E(x) - a$$

$$\Rightarrow \mu_1' = \mu - a \Rightarrow \boxed{\mu = \mu_1' + a}$$

$$\mu_2' = E(x-a)^2$$

$$\mu_3' = E(x-a)^3$$

$$\mu_4' = E(x-a)^4$$

First four moments about the origin.

$$\mu_1' = E(x)$$

$$\mu_2' = E(x^2)$$

$$\mu_3' = E(x^3)$$

$$\mu_4' = E(x^4)$$

RELATION BETWEEN MOMENTS ABOUT THE MEAN AND MOMENTS ABOUT ANY POINT 'a'.

$$\mu_r = E[x-\mu]^r = E[x-a+a-\mu]^r$$

$$= E\left[(x-a)-(\mu-a)\right]^r \quad \text{W.K.T} \quad \mu_1' = \mu - a$$

$$= E\left[(x-a) - \mu_1'\right]^r$$

$$= E\left[(x-a)^r - {}^rC_1 (x-a)^{r-1} \mu_1' + {}^rC_2 (x-a)^{r-2} (\mu_1')^2 + \cdots + (-1)^r (\mu_1')^r\right]$$

$$= E(x-a)^r - {}^rC_1 E(x-a)^{r-1} \mu_1' + {}^rC_2 E(x-a)^{r-2} (\mu_1')^2 + \cdots + (-1)^r (\mu_1')^r.$$

$$\mu_r = \mu_r' - {}^rC_1 \mu_{r-1}' \mu_1' + {}^rC_2 \mu_{r-2}' (\mu_1')^2 + \cdots (-1)^r (\mu_1')^r$$

W.K.T $\boxed{\mu_0' = 1}$ $\quad \mu_0' = E(x^0) = 1.$

$$\mu_1 = \mu_1' - \mu_0 \mu_1'$$

$$\therefore \mu_1 = \mu_1' - \mu_1' = 0 \qquad \therefore \boxed{\mu_1 = 0}$$

$$\mu_2 = \mu_2' - {}^2C_1 \mu_1' \mu_1' + {}^2C_2 \mu_0' (\mu_1')^2$$

$$\mu_2 = \mu_2' - 2(\mu_1')^2 + (\mu_1')^2 (1) \qquad [\because \mu_0 = 1]$$

$$\boxed{\mu_2 = \mu_2' - (\mu_1')^2} = E(x^2) - [E(x)]^2 = \text{VARIANCE}(x).$$

$\therefore$ Similarly $\boxed{\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2(\mu_1')^3}.$

$$\boxed{\mu_4 = \mu_4' - 4\mu_3' \mu_1 + 6\mu_2' (\mu_1')^2 - 3(\mu_1')^4.}$$

Since moments of a random Variable are important in Characterising or determining its distribution, it will be (helpful) useful if a function could be found that would give all the moments.

Such a function is called a $\boxed{\text{moment generating function.}}$

MOMENT GENERATING FUNCTION (MGF)

The moment generating function of a random Variable X is defined as $E(e^{tX})$ for all $x \in (-\infty, \infty)$

It is denoted by $M(t)$ (or) $M_X(t)$.

$$M(t) \text{ (or) } M_X(t) = \begin{cases} \sum e^{tx_i} P(x_i) & \rightarrow \text{D.R.V} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx & \rightarrow \text{C.R.V.} \end{cases}$$

FINDING MOMENTS FROM THE EXPANSION OF MGF.

$$M_x(t) = E(e^{tx})$$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^r}{r!} + \cdots\right]$$

$$= 1 + \frac{E(tx)}{1!} + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots + \frac{t^r}{r!}E(x^r)$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots$$

μ_1' = Coefficient of $\frac{t}{1!}$

μ_2' = Coefficient of $\frac{t^2}{2!}$

$\vdots \qquad \vdots$

μ_r' = coefficient of $\frac{t^r}{r!}$

FINDING RAW MOMENTS (FROM) BY DIFFERENTIATION of MGF without EXPANSION.

$$\mu_r' = M_x^{(r)}(0)$$

$$\mu_1' = \left[M_x'(t)\right]_{t=0} = \left[\frac{d}{dt}M_x(t)\right]_{t=0} = E(x)$$

$$\mu_2' = \left[M_x''(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0} = E(x^2)$$

$\vdots \qquad \vdots$

$$\mu_r' = E(x^r) = \left[M_x^{''\cdots r \text{ times}}(t)\right]_{t=0}$$

$$= \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}.$$

Proof

$$M_x(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots \frac{t^r}{r!}\mu_r' + \cdots$$

$$M_x'(t) = \mu_1' + \frac{2t}{2}\mu_2' + \frac{3t^2}{3!}\mu_3' + \cdots + \frac{r\,t^{r-1}}{r!}\mu_r' + \cdots$$

$$M_x''(t) = \mu_2' + \frac{3\times 2t}{3!}\mu_3' + \cdots + \frac{r(r-1)}{r!}t^{r-2}\mu_r' + \cdots$$

$$\vdots$$

$$M_x^{(r)}(t) = \mu_r' + \text{terms containing } t.$$

When $t=0$. we get.

$$\left[M_x'(t)\right]_{t=0} = \mu_1' = M_x'(0)$$

$$\left[M_x''(t)\right]_{t=0} = \mu_2' = M_x''(0)$$

$$\vdots \quad = \quad \vdots \quad = \quad \vdots$$

$$\left[M_x^{(r)}(t)\right]_{t=0} = \mu_r' = M_x^{(r)}(0)$$

PROPERTIES MOMENT GENERATING FUNCTION (MGF).

Let X be a Random Variable with MGF $M_X(t)$

and c is a Constant then

(i) $M_{cX}(t) = M_X(ct)$

(ii) $M_{X+c}(t) = e^{ct} M_X(t)$.

(iii) $M_{aX+b}(t) = e^{bt} M_X(at)$.

(iv) If X and Y are independent random Variables then

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t).$$

PROBLEM: 1 The first four moments of a distribution about $X=4$. are $1, 4, 10, 45$. Show that the mean is 5, Variance 3, $\mu_3 = 0$, $\mu_4 = 0$.

Let
$$\left.\begin{aligned} \mu_1' &= 1 \\ \mu_2' &= 4 \\ \mu_3' &= 10 \\ \mu_4' &= 25 \end{aligned}\right\}$$
are the first four moments. (given).

Moment about $X=4$.

$$E(x-a)^r = \mu_r'$$

$\mu_1' = E(x-a)^1 \Rightarrow 1 = E(x-4)$

$1 = E(x) - 4 \Rightarrow \boxed{E(x) = 5}$

$\therefore$ Mean $= E(x) = 5$.

Varriance $\mu_2 = \mu_2' - (\mu_1')^2 = 4 - (1)^2 = 4 - 1 = 3$ //.

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$\mu_3 = 10 - 3(4)(1) + 2(1) = 10 - 12 + 2 = 0$ //.

$\mu_4 = \mu_4' - 4\mu_3'\mu_1 + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 45 - 4(10)(1) + 6(4)(1) - 3(1)$

$= 45 - 40 + 24 - 3 = 26$ //.

PROBLEM : 2 The first two moments about 3 are 1 and 8. Find the mean and Variance.

Let μ_1', μ_2' be the first two moments of X about 3.

$\mu_1' = 1, \ \mu_2' = 8., \ a = 3$

$E(x-a)^r = \mu_r' \Rightarrow \mu_1' = E(x-3) = E(x) - 3$

$\therefore \mu_1' = E(x) - 3. \Rightarrow 1 = E(x) - 3 \Rightarrow \boxed{E(x) = 4}$

$\mu_2 = \mu_2' - (\mu_1')^2 = 8 - (1)^2 = 7$ //.

Problem: 3 A Continuous random Variable X has pdf $f(x) = k(1-x)$ for $0 < x < 1$. Find the r^{th} moment about the origin. Hence find mean and Variance.

Given p.d.f of X is $f(x) = k(1-x)$, $0 < x < 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1, \quad f(x) \geq 0.$$

$$\int_{-\infty}^{\infty} k(1-x)\,dx = 1 \quad [\Rightarrow \text{given } 0 < x < 1.]$$

$$\therefore \int_{0}^{1} k(1-x)\,dx = 1 \Rightarrow k\left[x - \frac{x^2}{2}\right]_0^1 = 1$$

$$\Rightarrow k\left[\left(1 - \tfrac{1}{2}\right) - 0\right] = 1$$

$$\Rightarrow \boxed{k = 2}$$

$$\therefore f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$\mu_r' = r^{th}$ moment about the origin.

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx. = \int_0^1 x^r 2(1-x)\,dx.$$

$$= 2\int_0^1 (x^r - x^{r+1})\,dx$$

$$= 2\left[\frac{x^{r+1}}{(r+1)} - \frac{x^{r+2}}{(r+2)}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right] = 2\left[\frac{(r+2)-(r+1)}{(r+1)(r+2)}\right] = \frac{2}{(r+1)(r+2)}$$

$$\therefore \boxed{\mu_r' = \frac{2}{(r+1)(r+2)}}$$

$$\mu_1' = \frac{2}{2\times 3} = 1/3 = E(X) = \text{Mean}.$$

$$\mu_2' = \frac{2}{3\times 4} = \frac{1}{6}$$

$$\text{Variance} = E(x-\mu)^2 = \mu_2 = \mu_2' - (\mu_1')^2.$$

$$\mu_2 = \frac{1}{6} - (1/3)^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

$\therefore$ Mean of $X = \mu_1' = 1/3$

Variance of $X = \mu_2 = \frac{1}{18}$.

Problem 4: Find the moment generating function of the RVX, whose Probability function $P(x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$ Hence find its mean and Variance. ⑥

Given X is a discrete R.V.

$$P(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \ldots$$

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}.$$

$$= e^t \frac{1}{2} + \frac{(e^t)^2}{2^2} + \frac{(e^t)^3}{2^3} + \cdots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^3 + \left(\frac{e^t}{2}\right)^4 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} \qquad (1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$= \cancel{\frac{e^t}{2}} / \times \cancel{(1-e^t/2)} = \frac{e^t}{2} \frac{1}{\left[1 - \frac{e^t}{2}\right]} = \frac{e^t}{2} * \frac{1}{\left[\frac{2-e^t}{2}\right]} = \frac{2e^t}{2(2-e^t)}.$$

$$\therefore M_X(t) = \frac{e^t}{2 - e^t}.$$

$$\text{Now } [M_X'(t)] = \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right] = \frac{(2-e^t)(e^t) - e^t(-e^t)}{(2-e^t)^2}$$

$$= \frac{[2 - e^t + e^t]e^t}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

$$\therefore M_X'(t) = \frac{2e^t}{(2-e^t)^2}.$$

$$M_x''(t) = 2\left[\frac{(2-e^t)^2\cdot(e^t) - e^t\, 2(2-e^t)\cdot(-e^t)}{(2-e^t)^4}\right]$$

$$= 2\left[\frac{(2-e^t)^2 e^t + 2e^t\cdot e^t(2-e^t)}{(2-e^t)^4}\right]$$

$$= \frac{2(2-e^t)\,e^t\left[(2-e^t)+2e^t\right]}{(2-e^t)^4}$$

$$M_x''(t) = \frac{2e^t\left[2+e^t\right]}{(2-e^t)^3}$$

$$\mu_2' = M_x''(0) = \frac{2\times 3}{(2-1)^3} = 6.$$

$$\mu_1' = M_x'(0) = \frac{2}{(2-1)^2} = 2.$$

Mean of $X = \mu_1' = 2$

Variance of $X = \mu_2' - (\mu_1')^2 = 6 - 2^2 = 2$ //

PROBLEM: 5 Let the random Variable X have P.d.f $f(x) = \frac{1}{2} e^{-x/2}$, $x>0$. Find the moment generating fn and hence find the mean and variance of X.

The MGF of X is $M_X(t)$

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{2}\int_{0}^{\infty} e^{-x/2}.e^{tx}.dx$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-\frac{1}{2}(1-t(2))x} dx \qquad \int e^{ax} dx = \frac{e^{ax}}{a}$$

$$= \frac{1}{2}\left[\frac{e^{-1/2(1-2t)x}}{-1/2(1-2t)}\right]_0^{\infty} \qquad \text{if } 1-2t>0 \Rightarrow t<1/2.$$

$$= \frac{1}{2}\left[0 - \frac{1}{-1/2(1-2t)}\right] = \frac{1}{2}\times 2 \times \frac{1}{(1-2t)}$$

$$M_X(t) = \frac{1}{1-2t} \quad \text{if } t<1/2.$$

Mean and Variance:

$$M_X(t) = \frac{1}{1-2t} = (1-2t)^{-1}$$

W.K.T $(1-x)^{-1} = 1+x+x^2+\cdots\cdots$

$$(1-2t)^{-1} = 1+(2t)+(2t)^2+(2t)^3+\cdots\cdots$$

$$\therefore M_X(t) \;\; 1+2t+(2t)^2+(2t)^3+\cdots\cdots \rightarrow (1)$$

Differentiating w.r.t to 't':

$$M_x'(t) = 2 + 8t + 24t^2 + \cdots$$

$$M_x''(t) = 8 + 48t + \cdots$$

Therefore: Mean of $X = E(X) = \mu_1' = M_x'(0) = 2.$

$$\mu_2' = M_x''(0) = 8$$

Variance $V(x) = \mu_2' - (\mu_1')^2 = 8 - (2)^2 = 8-4 = 4$

PROBLEM 5.1 A PERFECT COIN IS TOSSED THREE TIMES. If X denotes the number of heads that appear, find the MGF of X and hence find the mean and Variance.

Coin is tossed three times.

Given X Represents the number of heads that appear in 3 Tosses.

Values of X are 0, 1, 2, 3.

$$S = \{ \overset{3}{HHH}, \overset{2}{HHT}, \overset{2}{HTH}, \overset{2}{THH}, \overset{1}{HTT}, \overset{1}{THT}, \overset{1}{TTH}, \overset{0}{TTT} \}$$

$n(S) = 8.$

$$p(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0 \\ \frac{3}{8} & \text{for } x = 1 \\ \frac{3}{8} & \text{for } x = 2 \\ \frac{1}{8} & \text{for } x = 3 \end{cases}$$

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\, p(x)$$

$$= \frac{1}{8} \times e^0 + \frac{3}{8} \times e^t + \frac{3}{8} e^{2t} + \frac{1}{8} e^{3t}$$

$$= \frac{1}{8}\left[1+3e^t + 3e^{2t} + e^{3t} \right] = \frac{1}{8}(1+e^t)^3$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Mean & Variance

$$M_x(t) = \frac{1}{8}\left[1 + 3e^t + 3e^{2t} + e^{3t}\right]$$

$$M_x'(t) = \frac{1}{8}\left[3e^t + 6e^{2t} + 3e^{3t}\right]$$

$$M_x''(t) = \frac{1}{8}\left[3e^t + 12e^{2t} + 9e^{3t}\right]$$

$$\mu_1' = M_x'(0) = \frac{1}{8}[3+6+3] = \frac{12}{8} = 3/2.$$

$$\mu_2' = M_x''(0) = \frac{1}{8}[3+12+9] = \frac{24}{8} = 3.$$

$$\text{Mean of } x = \mu_1' = M_x'(0) = 3/2.$$

$$\text{Variance of } x = \mu_2' - (\mu_1')^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4}$$

$$\therefore \text{ Mean} = \frac{3}{2}$$

$$\text{Variance} = \frac{3}{4}.$$

Tchebycheff Inequality

If X is a Random Variable with $E(x) = \mu$ and Variance $Var(x) = \sigma^2$, then

$$P\{|x-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}, \quad \text{here } c>0.$$

Another form of Tchebycheff Inequality

$$P\{|x-\mu| < c\} \geq 1 - \frac{\sigma^2}{c^2}, \quad \text{here } c>0$$

Alternative forms

If we put $c = k\sigma$, where $k>0$, then Tchebycheff inequality takes the form

$$P\left\{\left|\frac{x-\mu}{k}\right| \geq \sigma\right\} \leq \frac{1}{k^2}$$

$$P\left\{\left|\frac{x-\mu}{k}\right| < c\right\} \geq 1 - \frac{1}{k^2}.$$

(or)

$$P\{|x-\mu| \geq k\sigma\} \leq 1/k^2 \text{ (upper bound)}$$

$$P\{|x-\mu| < k\sigma\} \geq 1 - \frac{1}{k^2} \text{ (lower bound)}.$$

Tchebycheff Inequality

Problem 1: Let X be a random Variable with mean 12, and Variance 9. If the probability function is not known, find $P(6 < x < 18)$

given $\mu = 12$, $\sigma^2 = 9$, $\sigma = 3$.

By Tchebycheff's Inequality

$$P(|x-\mu| < K\sigma) \geq 1 - \frac{1}{K^2}$$

$$P(|x-12| < 3K) \geq 1 - \frac{1}{K^2}$$

$$P(-3K < X-12 < 3K) \geq 1 - \frac{1}{K^2}$$

$$P(-3K+12 < x < 3K+12) \geq 1 - \frac{1}{K^2}$$

To find $P(6 < x < 18)$

$-3K+12 = 6 \Rightarrow \boxed{K = 2.}$

$3K = 12-6$

$\therefore P(6 < x < 8) \geq 1 - \frac{1}{4} \Rightarrow P(6 < x < 8) \geq 3/4.$

The $P(6 < x < 8)$ has minimum 75%.

Problem 2:

If X denotes the sum of the numbers obtained when 2 dice are thrown, obtain an upper bound for $P\{|X-7| \geq 3\}$. Compare with exact probability.

X	occurance.	P(x)
2	(1,1)	1/36
3	(1,2),(2,1)	2/36
4	(2,2),(1,3),(3,1)	3/36
5	(1,4),(4,1),(3,2) (2,3)	4/36
6	(1,5),(5,1),(2,4),(4,2) (3,3)	5/36
7	(1,6),(6,1),(2,5),(5,2),(4,3),(3,4)	6/36
8	(4,4), ~~[illegible]~~, (2,6), (6,2), (5,3),(3,5)	5/36
9	(6,3),(3,6),(4,5),(5,4).	4/36
10	(4,6) (6,4) (5,5)	3/36
11	(6,5) (5,6)	2/36
12	(6,6).	1/36.

To find the Bound we need $E(x)$, $Var(x)$

$$E(x) = \sum_{x=2}^{12} x \cdot P(x)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + \cdots + 12\left(\frac{1}{36}\right)$$

$$E(x) = \frac{252}{36} = 7. = \mu.$$

$\boxed{\text{Mean is } 7}$

Variance $\sigma^2 = E(x^2) - [E(x)]^2$

$$E(x^2) = \sum_{x=2}^{12} x^2 P(x)$$

$$= 2^2\left(\frac{1}{36}\right) + 3^2\left(\frac{2}{36}\right) + 4^2\left(\frac{3}{36}\right) + \cdots + 12^2\left(\frac{1}{36}\right)$$

$$E(x^2) = \frac{1974}{36}$$

$$Var(x) = E(x^2) - [E(x)]^2 = \frac{1974}{36} - (7)^2 = \frac{105}{18}. = \frac{35}{6}$$

By chebychev's inequality

$$P\{|x-\mu| \geq K\sigma\} \leq 1/K^2$$

$$P\{|x-\mu| \geq c\} \leq \frac{\sigma^2}{c^2}$$

from the Problem $C = 3$

$$P\{|x-\mu| \geq c\} \leq \frac{35/6}{c^2}.$$

$$P\{|x-\mu| \geq 3\} \leq \frac{35/6}{9}$$

$P(|X-7| \geq 3) \leq \frac{35}{54} \cong 0.6481$ (64.81 %)

Actual Probability

$P(|X-7| \geq 3) = 1 - P(|X-7| < 3)$

$= 1 - P(-3 < X-7 < 3)$

$= 1 - P(4 < X < 10)$

$= 1 - \{P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)\}$

$= 1 - \frac{1}{36}[4+5+6+5+4] = 1 - \frac{24}{36} = 1/3 = 0.33$

33%

There is much difference between the actual value and the upper bound given Tchebycheff inequality

———x———

Problem 3: If X is a random Variable

A fair die is thrown 720 times, use Tchebycheff inequality to find a lower bound for Probability of getting 100 to 140 sixes.

$P(\text{Success}) = \boxed{1/6 = p}$, $p+q=1$, $q = 1-p$ $\boxed{q = 5/6}$

$n = 720$

$\boxed{\mu = 120}$

mean and Variance

$\boxed{\sigma^2 = 100}$

mean $= np$, Variance $= npq$.

$M = 720 \times 1/6$ $V = 720 \times \frac{5}{6} \times 1/6$

To find lower bound

$$P(|X-\mu| < c) \geq 1-\frac{\sigma^2}{c^2}$$

$$P(|X-120| < c) \geq 1-\frac{100}{c^2}$$

$$P(-c < X-120 < c) \geq 1-\frac{100}{c^2}$$

$$P(-c+120 < X < 120+c) \geq 1-\frac{100}{c^2}$$

given $P(100 < X < 140)$

$\therefore \quad -c+120 = 100 \Rightarrow c = 120-100 \Rightarrow \boxed{c=20}$

$$P(100 < X < 140) \geq 1-\frac{100}{(20)^2} = 1-\frac{100}{400}$$

$$P(100 < X < 140) \geq \frac{3}{4}.$$

$\therefore$ The Probability of 100 to 140 Sixes as has lower bound of at least 75%

Problem : 4 A discrete R.v. X takes the values $-1, 0, 1$ with Probabilities $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$ respectively. Evaluate $P(|X-\mu| \geq 2\sigma)$ and Compare it with the upper bound given by Tchebycheff's inequality.

X	-1	0	1
$P(x)$	$1/8$	$3/4$	$1/8$

$\therefore \quad \sigma^2 = 1/4 \Rightarrow \boxed{\sigma = 1/2}$

$\boxed{\mu = 0}$

$$E(x) = \mu = (-1)\left(\frac{1}{8}\right) + (0)\left(\frac{3}{4}\right) + (1)\left(1/8\right) = 0$$

$$E(x^2) = (-1)^2\left(\frac{1}{8}\right) + (0)\left(\frac{3}{4}\right) + (1)^2\left(\frac{1}{8}\right) = \frac{1}{8} + \frac{1}{8} = 1/4$$

$$Var(x) = [E(x^2)] - [E(x)]^2 = \frac{1}{4} - 0 = 1/4 = \sigma^2$$

$$P(|x-\mu| \geq 2\sigma) = P(|x| \geq 1) = 1 - P(|x| < 1)$$

$$= 1 - \{P(x=0) + P(x=(-1))\}$$

$$= 1 - \left\{\frac{1}{8} + \frac{3}{4}\right\} = 1 - \left\{\frac{7}{8}\right\} = \frac{1}{8}$$

$$P(|x-\mu| \geq k\sigma) \leq 1/k^2$$

$$P(|x-\mu| \geq 2\sigma) \leq 1/4$$

FUNCTIONS OF A RANDOM VARIABLE.

In many Engg. Problems a mathematical model of the System of interest and the input signal will be given and we will be asked to find the characteristics of the output Signals. If the input to a System is random then the Output of System will also be random in general.

If the Input is set of random Variables then the output will be also a set of random Variables.

FUNCTION OF ONE RANDOM VARIABLE.

Let X be a Continuous random Variable with pdf $f_X(x)$ and $Y = g(x)$ be a given Transformation of X, where g is a differentiable function of x.

Case (i): If g is Strictly monotonic i.e., g is a one-to-one function of X. and if $f_Y(y)$ is the Pdf of Y then.

$$f_Y(y) = f_X(x) \cdot \left| \frac{dx}{dy} \right| , \quad \text{where } x = g^{-1}(y).$$

Case (ii) If g is not Strictly monotonic, that is, g is not one-to-one function we find the Sub intervals in which it is strictly increasing or strictly decreasing and in this intervals find their inverse.

$x = g^{-1}(y)$ has Values $x_1, x_2, \ldots, x_n$

then the pdf f_y of Y is given by

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \cdots + f_X(x_n)\left|\frac{dx_n}{dy}\right|$$

PROBLEM 1: Let X be a continuous random Variable with pdf

$$f(x) = \begin{cases} \frac{x}{12}, & 1 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

Find the pdf of $Y = 2X - 3$.

given $f_X(x) = \frac{x}{12}$ $\quad 1 < x < 5$.

The transformation function $y = 2x - 3$ is a one-one fn Since it is strictly increasing as $\frac{dy}{dx} = 2 > 0$.

we Solve for x in terms of y $\quad \therefore 2x = y + 3$

$$\frac{dx}{dy} = \frac{1}{2} \leftarrow \boxed{x = \frac{y+3}{2}}$$

limits. when $x=1$ $y=-1$

$x=5$ $y=7$. $\therefore -1 < y < 7$.

By the formula $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{x}{12}\cdot\frac{1}{2} = \frac{x}{24}$

W.K.T $x = \frac{y+3}{2}$, the pdf of Y is

$$f_Y(y) = \begin{cases} \frac{y+3}{48}, & -1 < y < 7. \\ 0, & \text{otherwise}. \end{cases}$$

Problem: 2 If X has exponential distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

with Parameter λ. find the pdf of $Y = e^{-\lambda x}$

given $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

The transformation function $y = e^{-\lambda x}$ is one-one

$$\frac{dy}{dx} = -\lambda e^{-\lambda x} < 0$$

So y is decreasing strictly.

$$y = e^{-\lambda x}$$

$$\log_e y = -\lambda x \Rightarrow x = -\frac{1}{\lambda}\log_e y$$

$$\frac{dx}{dy} = -\frac{1}{\lambda} \times \frac{1}{y}.$$

When $x = 0$, $y = 1$.

$x \to \infty \to y \to 0$.

$$\boxed{\therefore 0 \leq y \leq 1}$$

By formula $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$$= \lambda e^{-\lambda x}\left|-\frac{1}{\lambda}\cdot\frac{1}{y}\right| = \lambda\cdot y \times \frac{1}{\lambda} \times \frac{1}{y}$$

$$= 1.$$

$\therefore$ The P.d.f of Y is $f_Y(y) = \begin{cases} 1 & \text{if } 0 < y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

Problem 3: If X is uniformly distributed in $(-\frac{\pi}{2}, \pi/2)$ find the pdf of $Y = \tan x$

Uniform distribution $f(x) = \frac{1}{b-a}; a \leq x \leq b$

Here $(a,b) = (-\pi/2, \pi/2)$ $\quad f_X(x) = \frac{1}{\pi/2 - (-\pi/2)} = \frac{1}{\pi}.$

given $f_X(x) = \begin{cases} \frac{1}{\pi}, & -\pi/2 < x < \pi/2 \\ 0, & \text{otherwise} \end{cases}$

The transformation function

$y = \tan x \qquad -\pi/2 < x < \pi/2$ is One-one

$x = \tan^{-1} y \qquad -\infty < y < \infty$

$$\frac{dx}{dy} = \frac{1}{1+y^2}$$

The p.d.f of Y is $f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$

$$= \frac{1}{\pi}\left(\frac{1}{1+y^2}\right)$$

$$\therefore \quad f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty.$$