

## Random Variable

Experiment: An operation or process which results in some well defined outcome and generates a set of data is called an Experiment.

Random Experiment: An experiment in which the outcome cannot be predicted with certainty is called a random experiment, even though all possible outcomes are known in advance.

Example: Tossing a fair coin is a random experiment. The possible outcomes are head or tail. But we cannot say with certainty that head will come up.

Similarly the experiments of throwing a die or taking a card from a pack of cards are random experiments.

Trials: Each performance of a random experiment is called a trial.

Example: Tossing a coin 6 times is 6 trials.

Sample Space: The set of all possible outcomes of a random experiment is called its sample space and it is denoted by  $S$ .

Example: (1) Toss a coin twice

$$\text{sample space: } (S) = \{\text{HH, TT, TH, HT}\}$$

(2) Throw a die once

$$S = \{1, 2, 3, 4, 5, 6\}$$

Introduction: (Random Variable)

In many random experiments the outcomes may be numerical or non-numerical.

For example: The outcomes when a die is thrown are numerical whereas when we toss a coin the outcomes are non-numerical.

To deal with outcomes mathematically, we may assign numerical values to them.

i.e., we can assign a real number  $x$  to every element  $s$  of the sample space.

Definition: A random variable (r.v) is a rule that assigns a real number to each possible outcome

of a random experiment. (02)

Let  $S$  be the sample space associated with an experiment  $E$ . A r.v.  $X$  is a real valued function defined on  $S$ ,

i.e.,  $X: S \rightarrow \mathbb{R}$  is a function.

Each  $s \in S$  is associated with a real number  $X(s) = x$ .

Example 1: Toss a coin twice.

The sample space is  $S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$

Let "  $X$  " denote the 'number of heads' then  
 $X$  is a r.v.'s  $X(\text{TT})=0$ ,  $X(\text{HH})=2$ ,  $X(\text{HT})=1$ ,  
 $X(\text{TH})=1$ .

i.e., values of  $X$  are 0, 1, 2.

Note: (1)  $X$  is a 'variable' because it has different possible values. The term 'random' because it takes values with certain probabilities because it takes values with certain probabilities.

(2) The terms 'stochastic variable' and 'chance variable' are also used for 'random variable'. value of random variable depends on probability and hence chance.

Range Space: The set of all values taken by a r.v  $x$  is called its range space and is denoted by  $R_x$ .

$$\text{Thus } R_x = \{x(s) | s \in S\}$$

In the above example, the Range space is

$$R_x = \{0, 1, 2\}$$

Notation: If  $x$  is a real number and  $X$  is a r.v on  $S$ , we write  $X=x$  for the event  $\{s \in S | X(s)=x\}$ .  $P(X=x) = P\{s \in S | X(s)=x\}$

In example 1,

$$P(X=0) = P\{\text{TT}\} = \frac{1}{4}$$

$$P(X=1) = P\{\text{(HT), (TH)}\} = \frac{2}{4} = \frac{1}{2}$$

$$P(X=2) = P\{\text{HH}\} = \frac{1}{4}$$

Types of random Variables:

a) Discrete r.v    b) Continuous r.v

a) Discrete r.v: A r.v  $x$  is said to be discrete if it takes a finite number of values or countably infinite number of values.

i.e., its range  $R_x$  is finite or countably infinite.

Note: The sample space on which a discrete r.v  $x$  is defined need not be discrete.

For example, if  $x$  is defined as the number of telephone calls to an office between 9 A.M and 11 A.M, then it is finite. But the sample space  $S$  is the time interval  $[9, 11]$  which is continuous.

A discrete r.v assumes each of its values with certain probability. These probabilities could be used to define a prob. function as below:

Prob. function (or) Prob. Mass function:

Let  $x$  be a discrete r.v which takes values  $x_1, x_2, x_3, \dots$ . Let  $P(x=x_i) = p(x_i) = p_i$  be prob. of  $x_i$ . Then the function  $p_i$  is called the prob. mass function of  $x$  if the numbers  $p(x_i)$  satisfy the conditions

(i)  $p(x_i) \geq 0$  &  $i=1, 2, 3, \dots$  and

(ii)  $\sum_{i=1}^{\infty} p(x_i) = 1$ .

The set of ordered pairs of numbers  $\{x_i, p(x_i)\}$  is called the prob. distribution of the r.v  $x$ .

The prob. distribution is usually displayed in the form of a table as given below.

$x_1$	$x_1$	$x_2$	$x_3$	...
$p(x_1)$	$p(x_1)$	$p(x_2)$	$p(x_3)$	...

Note: If  $x_1 < x_2 < x_3 < \dots < x_i < \dots$ , then

- $P(X \leq x_i) = P(X = x_1) + P(X = x_2) + \dots + P(X = x_i)$
- $P(X > x_i) = 1 - P(X \leq x_i)$
- $P(X \geq x_i) = 1 - P(X < x_i)$

(b) Continuous r.v.: A r.v.  $X$  is said to be continuous if its range space  $R_X$  is an uncountable set of real numbers.

i.e., the r.v. assume values in an interval  $(a, b)$  or in an union of intervals.

Example: (1) If  $X$  denotes the lifetime of a Transistor then  $X$  is a continuous r.v.

(2) If  $X$  denotes the operating time between two failures of a machine, then  $X$  is a continuous r.v.

Note: (i) To define a continuous r.v., the sample space should be continuous.

(ii) In the case of continuous r.v., we cannot talk of the first value, second value etc., becomes meaningless. So, we define prob. function of a continuous r.v. as below.

Prob. density function: A function  $f$ , defined for all  $x \in (-\infty, \infty)$  is called the pdf of a continuous r.v.  $X$ .

$$(i) f(x) \geq 0 \quad \forall x \in (-\infty, \infty)$$

$$(ii) \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{Note: } (iii) P(X < a) = \int_{-\infty}^a f(x) dx, \quad P(X > a) = \int_a^{\infty} f(x) dx$$

$$(iv) P(a \leq X \leq b) = \int_a^b f(x) dx$$

$$(v) \text{ If } b=a, \quad P(X=a) = \int_a^a f(x) dx = 0$$

(vi) For continuous r.v.

$$P(X \leq a) = P(X < a), \quad P(X > a) = P(X \geq a)$$

$$P(a < X \leq b) = P(a \leq X < b) = P(a \leq X \leq b) = P(a < X \leq b)$$

Now define another function for a r.v which will be suitable for both discrete and continuous r.v.

Cumulative distribution function (CDF) or

Distribution function:

If  $x$  is a r.v, discrete or continuous, then the function  $F: \mathbb{R} \rightarrow [0, 1]$  defined by  $F(x) = P(X \leq x)$  is called the CDF of  $X$ .

(a) If  $X$  is a discrete r.v with values  $x_1 < x_2 < x_3 \dots < x_i < \dots$  then  $F(x) = \sum_{x_i \leq x} P(x_i)$

(b) If  $X$  is a continuous r.v with pdf  $f(x)$  defined for all  $x \in (-\infty, \infty)$ , then  $F(x) = \int_{-\infty}^x f(x) dx$ .

Properties of distribution function  $F(x)$

(1)  $0 \leq F(x) \leq 1$ ,  $-\infty < x < \infty$

(2)  $F(x)$  is an increasing function of  $x$

i.e.,  $a < b \Rightarrow F(a) \leq F(b)$

(3)  $F(-\infty) = \lim_{x \rightarrow -\infty} F(x) = 0$  and

$F(\infty) = \lim_{x \rightarrow \infty} F(x) = 1$

(4) If  $X$  is a continuous r.v then  $\frac{dF(x)}{dx} = f(x)$

Note: (1) If  $X$  is a discrete r.v with values  $x_1 < x_2 < x_3 < \dots$  then  $p(x_i) = F(x_i) - F(x_{i-1})$

Proof:

$$F(x_i) = p(x_1) + p(x_2) + p(x_3) + \dots + p(x_{i-1}) + p(x_i)$$

$$F(x_{i-1}) = p(x_1) + p(x_2) + p(x_3) + \dots + p(x_{i-1})$$

$$\therefore p(x_i) = F(x_i) - F(x_{i-1})$$

(2) If  $a < b$ , then  $P(a < X \leq b) = F(b) - F(a)$

Proof: The event  $X \leq b = (X \leq a) \cup (a < X \leq b)$

$$P(X \leq b) = P(X \leq a \cup a < X \leq b)$$

$$= P(X \leq a) + P(a < X \leq b) \text{ since } X \leq a \text{ and}$$

$a < X \leq b$  are disjoint)

$$F(b) = F(a) + P(a < X \leq b)$$

$$\therefore P(a < X \leq b) = F(b) - F(a)$$

## Problems: [Discrete r.v.]

1) A r.v.  $X$  has the following prob. distribution

$x :$	-2	-1	0	1	2	3
$P(x) :$	$0.1$	$k$	$0.2$	$2k$	$0.3$	$3k$

Find (i)  $k$  (ii)  $P(X < 2)$  (iii)  $P(-2 < X < 2)$

Sol:

(i)  $P(x) \geq 0$  for  $x$ ,  $\therefore k \geq 0$  and  $\sum P(x) = 1$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$\Rightarrow 6k + 0.6 = 1 \Rightarrow k = \frac{0.4}{6} \text{ or } \frac{1}{15}$$

$$(ii) P(X < 2) = 1 - P(X \geq 2) = 1 - [P(X=2) + P(X=3)]$$

$$= 1 - (0.3 + 3k) = 0.7 - 3 \cdot \frac{1}{15} = 0.5$$

$$(iii) P(-2 < X < 2) = P(X=-1) + P(X=0) + P(X=1)$$

$$= k + 0.2 + 2k = 0.2 + 3k$$

$$= 0.2 + 3 \cdot \frac{1}{15} = 0.4$$

2) A r.v.  $X$  has the following probability function

Value of $x=x$	0	1	2	3	4	5	6	7
$P(X=x)$	0	$a$	$2a$	$2a$	$3a$	$a^2$	$2a^2$	$7a^2+a$

Find (i)  $a$ , (ii)  $P(X < 4)$ , (iii)  $P(X \geq 4)$ , (iv)  $P(X \geq 6)$ , (v)  $P(X < 6)$ .

If  $P(X \leq k) > \frac{1}{2}$ , then find the least value of  $k$ .

Find the CDF of  $X$ . Also find  $P(1.5 < X < 4.5 / X > 2)$ ,

$P(X \leq 2)$ ,  $P(X > 3)$  and  $P(1 \leq X \leq 5)$ .

Sol:

(i)  $p(x) \geq 0$  for all  $x$ .  $\therefore a \geq 0$  and  $\sum p(x) = 1$

$$\Rightarrow 0 + a + 2a + 2a + 3a + a^2 + 2a^2 + 7a^2 + a = 1$$

$$\Rightarrow 10a^2 + 9a - 1 = 0 \Rightarrow (10a - 1)(a + 1) = 0$$

$$\Rightarrow a = \frac{1}{10} \text{ or } a = -1$$

Since  $a \geq 0$ ,  $a = \frac{1}{10}$

Probability distribution:

$x$	0	1	2	3	4	5	6	7
$p(x)$	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$$(ii) P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{5}{10} \text{ or } \frac{1}{2}$$

$$(iii) P(X \geq 4) = 1 - P(X \leq 4) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$(iv) P(X \geq 6) = P(X=6) + P(X=7) = \frac{2}{100} + \frac{17}{100} = \frac{19}{100}$$

$$(v) P(X < 6) = 1 - P(X \geq 6) = 1 - \frac{19}{100} = \frac{81}{100}$$

To find the least value of  $k$

$$P(X \leq 1) = \frac{1}{10}, P(X \leq 2) = \frac{3}{10}, P(X \leq 3) = \frac{5}{10} < \frac{1}{2}$$

$$P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$$

$\therefore$  The least value of  $k=4$ .

To find CDF of  $X$

By defn  $F(x) = P(X \leq x)$

$$F(0) = P(X \leq 0) = 0, \quad F(1) = P(X \leq 1) = P(X \leq 0) + P(X=1) = \frac{1}{10}$$

$$F(2) = P(X \leq 2) = P(X \leq 1) + P(X=2) = \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

$$F(3) = P(X \leq 3) = P(X \leq 2) + P(X=3) = \frac{3}{10} + \frac{2}{10} = \frac{5}{10}$$

Similarly

$$F(4) = \frac{8}{10}, \quad F(5) = \frac{81}{100}, \quad F(6) = \frac{83}{100}, \quad F(7) = 1$$

$\therefore$  The CDF of  $F(x)$  is

$x$	0	1	2	3	4	5	6	7
$F(x)$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	$\frac{81}{100}$	$\frac{83}{100}$	1

$$(Q2) F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{10} & \text{for } 1 \leq x < 2 \\ \frac{3}{10} & \text{for } 2 \leq x < 3 \\ \frac{5}{10} & \text{for } 3 \leq x < 4 \\ \frac{8}{10} & \text{for } 4 \leq x < 5 \\ \frac{81}{100} & \text{for } 5 \leq x < 6 \\ \frac{83}{100} & \text{for } 6 \leq x < 7 \\ 1 & \text{for } x \geq 7 \end{cases}$$

$P(1.5 < X < 4.5 | X > 2)$   
 $= \frac{P(1.5 < X < 4.5 \cap X > 2)}{P(X > 2)}$   
 $= \frac{P(2 < X < 4.5)}{1 - P(X \leq 2)}$   
 $= \frac{P(X=3) + P(X=4)}{1 - P(X \leq 2)}$   
 $= \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$

$P(X < 2) = 0 + \frac{1}{10} = \frac{1}{10}$   
 $P(X > 3) = 1 - P(X \leq 3)$   
 $= 1 - [0 + a + 2a + 3a] = 1 - 6a$   
 $= \frac{1}{2}$

$$P(1 < X < 5) = 2a + 2a + 3a = \frac{7}{10}$$

If the discrete r.v  $x$  has the probability function given by the table.

$x$	1	2	3	4
$P(x)$	$\frac{k}{3}$	$\frac{k}{6}$	$\frac{k}{3}$	$\frac{k}{6}$

Find  $k$  and the CDF of  $x$ .

Sol:

$$(i) \sum p(x) = 1 \Rightarrow \frac{k}{3} + \frac{k}{6} + \frac{k}{3} + \frac{k}{6} = 1 \\ \Rightarrow k = 1$$

$x$	1	2	3	4
$F(x)$	$\frac{1}{3}$	$\frac{3}{6}$	$\frac{5}{6}$	1

A discrete r.v  $x$  has the following prob. distribution

$x$	0	1	2	3	4	5	6	7	8
$p(x)$	$a$	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Find the value of  $a$  (ii)  $P(0 < x < 3)$

(iii)  $P(x \geq 3)$  (iv) Find the distribution fn of  $x$ .

Sol: (i)  $\sum p(x) = 1$

$$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$\Rightarrow 81a = 1 \Rightarrow a = \frac{1}{81}$$

$$(ii) P(0 < x < 3) = P(x=1) + P(x=2) = 3a + 5a = 8a = \frac{8}{81}$$

$$\begin{aligned}
 \text{(iii)} \quad P(X \geq 3) &= 1 - P(X < 3) \\
 &= 1 - [P(X=0) + P(X=1) + P(X=2)] \\
 &= 1 - (a + 3a + 5a) = 1 - 9a = 1 - \frac{9}{81} = \frac{8}{9}
 \end{aligned}$$

$$\text{(iv)} \quad F(x) = P(X \leq x)$$

$x$	0	1	2	3	4	5	6	7	8
$F(x)$	$\frac{1}{81}$	$\frac{4}{81}$	$\frac{9}{81}$	$\frac{16}{81}$	$\frac{25}{81}$	$\frac{36}{81}$	$\frac{49}{81}$	$\frac{64}{81}$	1

A r.v  $X$  has the following prob. function:

$x:$	0	1	2	3	4
$p(x):$	$k$	$2k$	$5k$	$7k$	$9k$

$$\text{Find } k, \quad P(X \geq 3) \text{ and } P(0 < X < 4)$$

Sol:

$$\text{(i)} \quad k + 2k + 5k + 7k + 9k = 1$$

$$\Rightarrow 24k = 1 \Rightarrow k = \frac{1}{24}$$

$$\text{(ii)} \quad P(X \geq 3) = P(X=3) + P(X=4)$$

$$= 7k + 9k = 16k = \frac{16}{24} \text{ or } \frac{2}{3}$$

$$\text{(iii)} \quad P(0 < X < 4) = P(X=1) + P(X=2) + P(X=3)$$

$$= 2k + 5k + 7k = 14k = 14 \cdot \frac{1}{24} = \frac{7}{12}$$

6) The prob. function of a rv  $x$  is given by

$$P(x=x) = \frac{1}{2^x}, x=1, 2, 3 \dots \text{Find (i) } P(x \text{ is even}),$$

(ii)  $P(x \text{ is odd})$ , (viii)  $P(x \geq 4)$  (iv)  $P(x \text{ is a multiple of 3})$

Sol:

$$(i) P(x \text{ is even}) = P(x=2) + P(x=4) + P(x=6) + \dots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + \dots$$

$$= \frac{1}{2^2} \left[ 1 + \frac{1}{2^2} + \frac{1}{2^4} + \dots \right]$$

$$= \frac{1}{2^2} \left[ 1 - \frac{1}{2^2} \right]^{-1} = \frac{1}{3}$$

$$(ii) P(x \text{ is odd}) = P - P(x \text{ is even}) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$(iii) P(x \geq 4) = 1 - P(x < 4)$$

$$= 1 - [P(x=1) + P(x=2) + P(x=3)]$$

$$= 1 - \left[ \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \right] = 1 - \frac{7}{8} = \frac{1}{8}$$

$$(iv) P(x \text{ is a multiple of 3}) = P(x=3) + P(x=6) + P(x=9) + \dots$$

$$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \frac{1}{2^{12}} + \dots$$

$$= \frac{1}{2^3} \left[ 1 + \frac{1}{2^3} + \frac{1}{(2^3)^2} + \dots \right]$$

$$= \frac{1}{8} \left[ 1 - \frac{1}{2^3} \right]^{-1} = \frac{1}{7}$$

Two cards are drawn successively with replacement from a well shuffled deck of 52 cards. Find the prob. dist. of number of kings.

Sol:

Let  $x$  denote the number of kings.

$\therefore$  The values of  $x$  are 0, 1, 2.

$$P(\text{a king}) = \frac{4C_1}{52C_2} = \frac{4}{52} \text{ or } \frac{1}{13}$$

$$P(\text{not a king}) = 1 - \frac{1}{13} = \frac{12}{13}$$

$$P(x=0) = P(\text{both are not kings}) = \frac{12}{13} \cdot \frac{12}{13} = \frac{144}{169}$$

$$\begin{aligned} P(x=1) &= P(\text{one king and one not a king}) \\ &= \left( \frac{1}{13} \times \frac{12}{13} \right) + \left( \frac{12}{13} \times \frac{1}{13} \right) = \frac{24}{169} \end{aligned}$$

$$P(x=2) = P(\text{two kings}) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

The prob. dist. of  $x$  is given below

$x$	0	1	2
$P(x)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

- 8) A shipment of 6 television sets contains 2 defective sets. A hotel makes a random purchase of 3 sets. If  $X$  is the number of defective sets purchased by the hotel, find the prob. dist. of  $X$ .

Sol:

Let  $X$  denote the number of defective sets  
Then the values of  $X$  are 0, 1, 2.

$$P(X=0) = \frac{4C_3}{6C_3} = \frac{1}{5}, \quad P(X=1) = \frac{2C_1 \times 4C_2}{6C_3} = \frac{3}{5}$$

$$P(X=2) = \frac{2C_2 \times 4C_1}{6C_3} = \frac{1}{5}$$

The prob. distribution of  $X$  is given below:

$x$	0	1	2
$P(x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

- 9) A r.v  $X$  takes values 1, 2, 3, 4 such that  
 $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ . Find the prob. dist. function and hence find the CDF.

Sol: Let  $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$

$$\therefore P(X=1) = \frac{k}{2}, \quad P(X=2) = \frac{k}{3}, \quad P(X=3) = k, \quad P(X=4) = \frac{k}{5}$$

$x$	1	2	3	4
$P(x)$	$\frac{k}{2}$	$\frac{k}{3}$	$k$	$\frac{k}{5}$

$$\sum p(x) = 1 \Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \\ \Rightarrow k = \frac{30}{61}$$

$x=x$	1	2	3	4
prob. dist.	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
CDF	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

- 10) Let  $X$  be a r.v. such that  $P(X=-2) = P(X=-1) = P(X=1) = P(X=2)$  and  $P(X<0) = P(X=0) = P(X>0)$ . Determine the prob. mass function of  $X$  and dist. fn. of  $X$ .

Sol:

$$\text{Let } P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = a \\ \therefore P(X<0) = P(X=-2) + P(X=-1) = a + a = 2a \\ \Rightarrow P(X<0) = P(X=0) = P(X>0) = 2a$$

$x$	-2	-1	0	1	2
$P(X=x)$	$a$	$a$	$2a$	$a$	$a$

$$\text{Since } \sum p(x) = 1 \Rightarrow a + a + 2a + a + a = 1$$

$$\Rightarrow a = \frac{1}{6}$$

$x$	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

11) Let  $X$  be a discrete r.v. whose CDF  $F(x)$  is given by

$$F(x) = \begin{cases} 0, & x < -3 \\ \frac{1}{6}, & -3 \leq x < 6 \\ \frac{1}{2}, & 6 \leq x < 10 \\ 1, & 10 \leq x \end{cases}$$

Find (i)  $P(X \leq 4)$  (ii)  $P(-5 < X \leq 4)$  (iii) the prob. dist.

Sol:

$$(i) P(X \leq 4) = F(4) = \frac{1}{6}$$

$$(ii) P(-5 < X \leq 4) = F(4) - F(-5) \quad \because P(a < X \leq b) = F(b) - F(a)$$

$$= \frac{1}{6} - 0 = \frac{1}{6}$$

(iii) Given  $X$  is a discrete r.v. and  $X$  take the values are  $-3, 6, 10$ .

$X=x$	-3	6	10
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$\text{Since } P(X \leq -3) = P(X = -3) + P(X < -3)$$

$$\Rightarrow P(X = -3) = P(X \leq -3) - P(X < -3)$$

$$= \frac{1}{6} - 0 = \frac{1}{6}$$

$$P(X \leq 6) = P(X = 6) + P(X < 6)$$

$$\Rightarrow P(X = 6) = P(X \leq 6) - P(X < 6) = \frac{1}{2} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

$$\text{and } P(X \leq 10) = P(X = 10) + P(X < 10)$$

$$\Rightarrow P(X = 10) = P(X \leq 10) - P(X < 10) = 1 - \frac{1}{2} = \frac{1}{2}$$

Problems: [continuous r.v.]

- 1.) If the density function of a continuous r.v  $x$  is given by

$$f(x) = \begin{cases} ax, & 0 \leq x \leq 1 \\ a, & 1 \leq x \leq 2 \\ 3a - ax, & 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

- (i) Find  $a$  (ii) Find the CDF of  $x$ .

Sol:

(i) Given  $f(x)$  is the pdf of a continuous r.v  $x$ .

$$\therefore \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^1 ax dx + \int_1^2 a dx + \int_2^3 (3a - ax) dx = 1$$

$$\Rightarrow a \left[ \frac{x^2}{2} \right]_0^1 + a[x]_1^2 + \left[ 3a \cdot x - a \cdot \frac{x^2}{2} \right]_2^3 = 1$$

$$\Rightarrow a(\frac{1}{2} - 0) + a(2 - 1) + \left[ 3a \cdot 3 - a \cdot \frac{9}{2} - 3a \cdot 2 + a \cdot \frac{4}{2} \right] = 1$$

$$\Rightarrow \frac{a}{2} + a + a \left[ 3a - \frac{5a}{2} \right] = 1 \Rightarrow a = \frac{1}{2}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 1 \\ \frac{1}{2}, & 1 \leq x \leq 2 \\ \frac{1}{2}(3-x), & 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

(ii) To find the CDF of  $x$

$$\text{W.K.T } F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$$

If  $x < 0$ , then  $F(x) = 0$ .

If  $0 \leq x < 1$ , then  $F(x) = \int_0^x f(x) dx$

$$\Rightarrow F(x) = \int_0^x x dx = \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^x = \frac{x^2}{4}$$

If  $1 \leq x < 2$ , then  $F(x) = \int_0^1 f(x) dx + \int_1^x f(x) dx$

$$= \int_0^1 \frac{x}{2} dx + \int_1^x \frac{1}{2} dx$$

$$= \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^1 + \frac{1}{2} [x]_1^x = \frac{1}{4} + \frac{1}{2}(x-1)$$

If  $2 \leq x < 3$ , then  $F(x) = \int_0^2 f(x) dx + \int_2^x f(x) dx$

$$\Rightarrow F(x) = \int_0^2 \frac{x}{2} dx + \int_2^x \frac{1}{2} dx + \int_2^x \frac{1}{2}(3-x) dx$$

$$= \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^2 + \frac{1}{2} [x]_2^x + \frac{1}{2} \left[ 3x - \frac{x^2}{2} \right]_2^x$$

$$= \frac{1}{4} (-x^2 + 6x - 5)$$

If  $x \geq 3$ , then  $F(x) = 1$

$$F(x) = \begin{cases} 0 & \text{If } x < 0 \\ \frac{x^2}{4} & \text{If } 0 \leq x < 1 \\ \frac{1}{4}(2x-1) & \text{If } 1 \leq x < 2 \\ \frac{1}{4}(-x^2 + 6x - 5) & \text{If } 2 \leq x < 3 \\ 1 & \text{If } x \geq 3 \end{cases}$$

- 2) If  $x$  is a continuous r.v whose pdf is given by  
 $f(x) = \begin{cases} C(4x - 2x^2), & 0 < x < 2 \\ 0, & \text{otherwise,} \end{cases}$  Find (i)  $C$  (ii)  $P(X > 1)$ .

Sol:

(i) Given  $f(x)$  is a pdf.

$$\therefore f(x) \geq 0 \text{ and } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^2 C(4x - 2x^2) dx = 1 \Rightarrow C \left[ \frac{4x^2}{2} - \frac{2x^3}{3} \right]_0^2 = 1$$

$$\Rightarrow C \left[ 2 \cdot 4 - \frac{2}{3} \cdot 8 \right] = 1$$

$$\Rightarrow 8C \left( 1 - \frac{2}{3} \right) = 1 \Rightarrow C = \frac{3}{8}$$

$$(ii) P(X > 1) = \int_1^2 f(x) dx = \frac{3}{8} \int_1^2 (4x - 2x^2) dx$$

$$= \frac{3}{8} \left[ \frac{4x^2}{2} - \frac{2x^3}{3} \right]_1^2$$

$$= \frac{3}{8} \left[ 2 \cdot 4 - \frac{2}{3} \cdot 8 - 2 + \frac{2}{3} \right] = \frac{1}{2}$$

- 3) A r.v  $X$  has the pdf  $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$  Find

- (i)  $P(X < \frac{1}{2})$  (ii)  $P(\frac{1}{4} < X < \frac{1}{2})$  (iii)  $P(X > \frac{3}{4} | X > \frac{1}{2})$   
 (iv)  $P(X < \frac{3}{4} | X > \frac{1}{2})$ .

Sol:

$$(i) P(X < \frac{1}{2}) = \int_{-\infty}^{\frac{1}{2}} f(x) dx = \int_0^{\frac{1}{2}} 2x dx = 2 \left( \frac{x^2}{2} \right)_0^{\frac{1}{2}} = \frac{1}{4}$$

$$(ii) P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(x) dx = \int_{\frac{1}{4}}^{\frac{1}{2}} 2x dx$$

$$= 2 \left[\frac{x^2}{2}\right]_{\frac{1}{4}}^{\frac{1}{2}} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

$$(iii) P(x > \frac{3}{4} | x > \frac{1}{2}) = \frac{P(x > \frac{3}{4} \cap x > \frac{1}{2})}{P(x > \frac{1}{2})}$$

$$= \frac{P(x > \frac{3}{4})}{P(x > \frac{1}{2})}$$

$$= \frac{\int_{\frac{3}{4}}^{\frac{1}{2}} 2x dx}{\int_{\frac{1}{2}}^{\frac{1}{2}} 2x dx} = \frac{\left(\frac{x^2}{2}\right)_{\frac{3}{4}}^{\frac{1}{2}}}{\left(\frac{x^2}{2}\right)_{\frac{1}{2}}^{\frac{1}{2}}} = \frac{\frac{1}{16} - \frac{9}{16}}{1 - \frac{1}{4}} = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}$$

$$(iv) P(x < \frac{3}{4} | x > \frac{1}{2})$$

$$= \frac{P(x < \frac{3}{4} \cap x > \frac{1}{2})}{P(x > \frac{1}{2})} = \frac{P(\frac{1}{2} < x < \frac{3}{4})}{P(x > \frac{1}{2})}$$

$$= \frac{\int_{\frac{1}{2}}^{\frac{3}{4}} 2x dx}{\int_{\frac{1}{2}}^{\frac{1}{2}} 2x dx} = \frac{\left(\frac{x^2}{2}\right)_{\frac{1}{2}}^{\frac{3}{4}}}{\left(\frac{x^2}{2}\right)_{\frac{1}{2}}^{\frac{1}{2}}} = \frac{\frac{9}{16} - \frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{\frac{5}{16}}{\frac{3}{4}} = \frac{5}{12}$$

4) Verify whether  $f(x)$  is a pdf

$$(i) f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty \quad (ii) f(x) = \begin{cases} |x|, & -1 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Sol:

(i) To prove:  $\int_{-\infty}^{\infty} f(x) dx = 1$  since  $f(x)$  is a pdf.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{dx}{1+x^2} = 1 \Rightarrow \frac{1}{\pi} [\tan^{-1} x]_{-\infty}^{\infty} = 1$$

$$\Rightarrow \frac{1}{\pi} [\tan^{-1} \infty - \tan^{-1}(-\infty)] = 1$$

$$\Rightarrow \frac{1}{\pi} \left[ \frac{\pi}{2} + \frac{\pi}{2} \right] = 1 \Rightarrow 1 = 1$$

∴ Given function is a pdf.

$$(ii) \int_{-1}^1 f(x) dx = \int_{-1}^0 -x dx + \int_0^1 x dx \\ = -\left(\frac{x^2}{2}\right)_{-1}^0 + \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

∴ Given function is a pdf.

5) A continuous r.v.  $X$  has a p.d.f  $f(x) = 6x(1-x)$ ,  $0 \leq x \leq 1$ . Determine  $b$  if  $P(X < b) = P(X > b)$

Sol: Given  $P(X < b) = P(X > b)$

$$= 1 - P(X \leq b)$$

$$\Rightarrow P(X < b) + P(X \leq b) = 1$$

$$\Rightarrow 2P(X < b) = 1 \Rightarrow P(X < b) = \frac{1}{2}$$

$$\Rightarrow \int_0^b f(x) dx = \frac{1}{2}$$

$$\Rightarrow \int_0^b 6x(1-x)dx = \frac{1}{2} \Rightarrow 6 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^b = \frac{1}{2}$$

$$\Rightarrow [3x^2 - 2x^3]_0^b = \frac{1}{2} \Rightarrow 3b^2 - 2b^3 = \frac{1}{2}$$

$$\Rightarrow 4b^3 - 6b^2 + 1 = 0 \Rightarrow 4b^3 - 2b^2 - 4b^2 + 1 = 0$$

$$\Rightarrow 2b^2(2b-1) - (4b^2-1) = 0$$

$$\Rightarrow 2b^2(2b-1) - (2b+1)(2b-1) = 0$$

$$\Rightarrow (2b-1)(2b^2-2b-1) = 0$$

$$2b-1 = 0 \quad \text{or} \quad 2b^2-2b-1 = 0$$

$$b = \frac{1}{2}$$

$$b = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

$$= \frac{1+\sqrt{3}}{2} > 1 \quad \text{or} \quad \frac{1-\sqrt{3}}{2} < 0$$

These values are not possible.

$$\therefore b = \frac{1}{2}$$

- 6) A continuous r.v  $x$  has a p.d.f.  $f(x) = 3x^2$ ,  $0 \leq x \leq 1$ .  
 Find 'a' and 'b' such that  $P(X \leq a) = P(X > a)$  and  
 $P(X > b) = 0.05$ .

Sol: Given  $P(X \leq a) = P(X > a)$

$$= 1 - P(X \leq a)$$

$$\Rightarrow 2P(X \leq a) = 1 \Rightarrow P(X \leq a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)dx = \frac{1}{2} \Rightarrow \int_0^a 3x^2 dx = \frac{1}{2}$$

$$\Rightarrow (x^3)_0^a = \frac{1}{2} \Rightarrow a = (\frac{1}{2})^{\frac{1}{3}}$$

$$\text{Given } P(X > b) = 0.05 \Rightarrow P(X > b) = \frac{5}{100} = \frac{1}{20}$$

$$\Rightarrow \int_b^1 f(x) dx = \frac{1}{20} \Rightarrow \int_b^1 3x^2 dx = \frac{1}{20}$$

$$\Rightarrow (x^3) \Big|_b^1 = \frac{1}{20} \Rightarrow 1 - b^3 = \frac{1}{20}$$

$$\Rightarrow b = \left(\frac{19}{20}\right)^{1/3}$$

7) If  $f(x) = k(1+x)$  in  $2 < x < 5$  is the p.d.f. of a continuous r.v.  $X$ , find (i)  $P(X < 4)$ , (ii)  $P(3 < X < 4)$ .

Sol:

Given  $f(x)$  is a p.d.f.  $\therefore \int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_2^5 k(1+x) dx = 1 \Rightarrow k \left[ \frac{(1+x)^2}{2} \right]_2^5 = 1$$

$$\Rightarrow \frac{k}{2} (6^2 - 3^2) = 1 \Rightarrow \frac{k}{2} (36 - 9) = 1$$

$$\Rightarrow k = \frac{2}{27}$$

$$(i) P(X < 4) = \int_2^4 \frac{2}{27} (1+x) dx = \frac{2}{27} \left[ \frac{(1+x)^2}{2} \right]_2^4$$

$$= \frac{1}{27} (5^2 - 3^2) = \frac{16}{27}$$

$$(ii) P(3 < X < 4) = \int_3^4 \frac{2}{27} (1+x) dx = \frac{2}{27} \left[ \frac{(1+x)^2}{2} \right]_3^4$$

$$= \frac{1}{27} (5^2 - 4^2) = \frac{1}{3}$$

8.) The pdf of a continuous r.v  $x$  is given by  
 $f(x) = kx(2-x)$ ,  $0 < x < 2$ . Find the cumulative distribution function.

Sol: Given function  $f(x)$  is the pdf.

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_0^2 kx(2-x) dx = 1 \\ \Rightarrow k \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = 1 \Rightarrow k \left[ 4 - \frac{8}{3} \right] = 1 \\ \Rightarrow k \left( \frac{12-8}{3} \right) = 1 \Rightarrow k = \frac{3}{4} \end{aligned}$$

To find CDF

$$\begin{aligned} F(x) &= P(X \leq x) = \int_{-\infty}^x f(x) dx \\ &= \int_0^x \frac{3}{4} x(2-x) dx \\ &= \frac{3}{4} \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^x = \frac{x^2}{4}(3-x) \end{aligned}$$

$$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{4}(3-x), & 0 \leq x < 2 \\ 1 & , x \geq 2 \end{cases}$$

9.) If  $X$  is a continuous r.v with pdf

$$f(x) = \begin{cases} x & , 0 \leq x < 1 \\ \frac{3}{2}(x-1)^2, & 1 \leq x < 2 \\ 0 & , \text{otherwise} \end{cases} \quad \text{find the CDF } F(x)$$

and use it to find  $P\left(\frac{3}{2} < X < \frac{5}{2}\right)$

If  $x < 0$  then  $F(x) = 0$

$$\text{If } 0 \leq x < 1, \text{ then } F(x) = \int_0^x x dx = \left[ \frac{x^2}{2} \right]_0^x = \frac{x^2}{2}$$

$$\text{If } 1 \leq x < 2, \text{ then } F(x) = \int_0^1 f(x) dx + \int_1^x f(x) dx$$

$$\Rightarrow F(x) = \int_0^1 x dx + \int_1^x \frac{3}{2}(x-1)^2 dx$$

$$= \left[ \frac{x^2}{2} \right]_0^1 + \frac{3}{2} \left[ \frac{(x-1)^3}{3} \right]_1^x$$

$$= \frac{1}{2} + \frac{1}{2} (x-1)^3$$

If  $x \geq 2$  then  $F(x) = 1$

$$\therefore F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x^2}{2} & \text{if } 0 \leq x < 1 \\ \frac{1}{2} + \frac{1}{2} (x-1)^3 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Now

$$P\left(\frac{3}{2} < x < \frac{5}{2}\right) = F\left(\frac{5}{2}\right) - F\left(\frac{3}{2}\right)$$

$$= 1 - \left[ \frac{1}{2} + \frac{1}{2} \left( \frac{3}{2} - 1 \right)^3 \right] = \frac{7}{16}$$

(10) If  $f(x) = \begin{cases} xe^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$  then (i) Show that  $f(x)$  is a pdf (ii) find  $F(x)$ .

Sol:

(i) To prove  $f(x)$  is the pdf, we have to prove  $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} xe^{-x^2/2} dx$$

$$\text{put } \frac{x^2}{2} = t \Rightarrow x^2 = 2t$$

$$2x dx = 2dt \Rightarrow x dx = dt$$

when  $x=0 \Rightarrow t=0, x=\infty \Rightarrow t=\infty$

$$\therefore \int_0^{\infty} xe^{-x^2/2} dx = \int_0^{\infty} e^{-t} dt \\ = [-e^{-t}]_0^{\infty} = 1$$

$\Rightarrow f(x)$  is a pdf

(ii) To find CDF ( $F(x)$ ):

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$$

If  $x < 0$  then  $F(x) = 0$

$$\text{If } x \geq 0 \text{ then } F(x) = \int_0^x xe^{-t^2/2} dt \\ = \int_0^{x^2/2} e^{-t} dt = [-e^{-t}]_0^{x^2/2} = 1 - e^{-x^2/2}$$

11. The pdf of a continuous r.v is  $f(x) = ce^{-|x|}$ ,  $-\infty < x < \infty$ , find value of c, the CDF of x.

Sol:

$$(i) \int_{-\infty}^{\infty} ce^{-|x|} dx = 1$$

$$\Rightarrow c \left[ \int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx \right] = 1$$

$$\Rightarrow c \left[ (e^x) \Big|_{-\infty}^0 + (-e^{-x}) \Big|_0^{\infty} \right] = 1$$

$$\Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$$

(ii) To find CDF:

$$F(x) = \int_{-\infty}^x f(x) dx \text{ where } f(x) = \begin{cases} \frac{1}{2}e^x & \text{if } x < 0 \\ \frac{1}{2}e^{-x} & \text{if } x \geq 0 \end{cases}$$

$$\text{If } x < 0 \text{ then } F(x) = \int_{-\infty}^x \frac{1}{2}e^x dx$$

$$= \frac{1}{2} [e^x] \Big|_{-\infty}^x = \frac{e^x}{2}$$

$$\text{If } x \geq 0 \text{ then } F(x) = \int_{-\infty}^0 f(x) dx + \int_0^x f(x) dx$$

$$= \int_{-\infty}^0 \frac{1}{2}e^x dx + \int_0^x \frac{1}{2}e^{-x} dx$$

$$= \frac{1}{2} [e^x] \Big|_{-\infty}^0 + \frac{1}{2} [-e^{-x}] \Big|_0^x$$

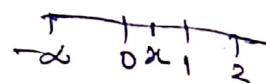
$$= \frac{1}{2} - \frac{1}{2}(e^{-x} - 1) = 1 - \frac{e^{-x}}{2}$$

$$\text{Sol: } F(x) = \int_{-\infty}^x f(c) dc$$

If  $x < 0$  then  $F(x) = 0$

$$\text{If } 0 \leq x < 1 \text{ then } F(x) = \int_0^x x dx$$

$$\Rightarrow F(x) = \left[ \frac{x^2}{2} \right]_0^x = \frac{x^2}{2}$$



$$\text{If } 1 \leq x < 2 \text{ then } F(x) = \int_0^1 x dx + \int_1^x (2-x) dx$$

$$\begin{aligned}\Rightarrow F(x) &= \left[ \frac{x^2}{2} \right]_0^1 + \left[ 2x - \frac{x^2}{2} \right]_1^x \\ &= \frac{1}{2} + 2x - \frac{x^2}{2} - \left( 2 - \frac{1}{2} \right) \\ &= 2x - \frac{x^2}{2} - 1\end{aligned}$$

$$\therefore F(x) = \begin{cases} 0 & \text{If } x < 0 \\ \frac{x^2}{2} & \text{If } 0 \leq x < 1 \\ 2x - \frac{x^2}{2} - 1 & \text{If } 1 \leq x < 2 \\ 1 & \text{If } x \geq 2 \end{cases}$$

A Continuous Variable  $X$  follows the probability law  $f(x) = Ax^2$ ,  $0 \leq x \leq 1$ , determine  $A$  and find the probability that  $X$  lies between 0.2 and 0.5 and  $F(x)$ .

Soln (i)  $\int_0^1 Ax^2 dx = 1 \Rightarrow A \left[ \frac{x^3}{3} \right]_0^1 = 1 \Rightarrow A = 3$

$$P(0.2 < X < 0.5) = \int_{0.2}^{0.5} 3x^2 dx = 3 \left[ \frac{x^3}{3} \right]_{0.2}^{0.5} = (0.5)^3 - (0.2)^3 = 0.117$$

(iii)  $F(x) = P(X \leq x) = \int_{-\infty}^x f(x) dx$

If  $x < 0$  then  $F(x) = 0$

If  $0 \leq x < 1$ , then  $F(x) = \int_0^x 3x^2 dx = x^3$

If  $x \geq 1$  then  $F(x) = 1$

$$\therefore F(x) = \begin{cases} 0 & \text{If } x < 0 \\ x^3 & \text{If } 0 \leq x < 1 \\ 1 & \text{If } x \geq 1 \end{cases}$$

If the r.v  $X$  has the following cumulative distribution function  $F(x) = 1 - (1+x)e^{-x}$  for  $x \geq 0$ , then find the probability that it assume the value less than 1.

Sol:

$$f(x) = \frac{dF}{dx} = 0 - 1 \cdot e^{-x} - (1+x)(-e^{-x}) \\ = -e^{-x} + e^{-x} + xe^{-x} = xe^{-x}$$

$$P(X < 1) = \int_0^1 xe^{-x} dx = \left[ -xe^{-x} - e^{-x} \right]_0^1 = \frac{e-2}{e}$$

- (b) If the cumulative distribution function of a r.v  $x$  is given by  $F(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{If } x > 2 \\ 0 & \text{If } x \leq 2 \end{cases}$  Find (i)  $P(X < 3)$   
 (ii)  $P(4 < X < 5)$  (iii)  $P(X \geq 3)$

Sol:  $f(x) = \frac{dF(x)}{dx} = \begin{cases} 0 & \text{If } x \leq 2 \\ \frac{8}{x^3} & \text{If } x > 2 \end{cases}$

$$(i) P(X < 3) = \int_2^3 \frac{8}{x^3} dx = 8 \left[ \frac{x^{-2}}{-2} \right]_2^3 = -4 \left[ \frac{1}{9} - \frac{1}{4} \right] = \frac{5}{9}$$

$$(ii) P(4 < X < 5) = \int_4^5 \frac{8}{x^3} dx = 8 \left[ \frac{x^{-2}}{-2} \right]_4^5 = -4 \left[ \frac{1}{5^2} - \frac{1}{4^2} \right] = \frac{9}{100}$$

$$(iii) P(X \geq 3) = 1 - P(X < 3) = 1 - \frac{5}{9} = \frac{4}{9}$$

- 17) The diameter of an electric cable  $x$  is a continuous random variable with pdf  $f(x) = kx^2(1-x)$ ,  $0 \leq x \leq 1$ . Find the value of  $k$ ;  $F(x)$  and  $P(X \leq Y_2 | Y_3 < X < Y_3)$

Sol:  $\int_0^1 kx^2(1-x) dx = 1 \Rightarrow k \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = 1 \Rightarrow k = 6$

$$\therefore f(x) = 6x^2(1-x)$$

$$F(x) = P(X \leq x) = \int_0^x 6x^2(1-x) dx = 6 \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^x = 3x^2 - 2x^3$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2(2-2x) & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

$$\begin{aligned} P(X < \frac{1}{2} \mid \frac{1}{3} < X < \frac{2}{3}) &= \frac{P(X < \frac{1}{2} \cap \frac{1}{3} < X < \frac{2}{3})}{P(\frac{1}{3} < X < \frac{2}{3})} \\ &= \frac{P(\frac{1}{3} < X < \frac{1}{2})}{P(\frac{1}{3} < X < \frac{2}{3})} \\ &= \frac{\int_{\frac{1}{3}}^{\frac{1}{2}} 6x(1-x) dx}{\int_{\frac{1}{3}}^{\frac{2}{3}} 6x(1-x) dx} = \frac{\frac{13}{54}}{\frac{13}{27}} = \frac{1}{2} \end{aligned}$$

18.) For a continuous r.v.  $X$ , the CDF is given by

$$F(x) = \begin{cases} 0 & \text{If } x < 2 \\ K(x-2) & \text{If } 2 \leq x < 6 \\ 1 & \text{If } x \geq 6 \end{cases}$$

Find (i) the pdf of  $X$

(ii) the value of  $K$  (iii)  $P(X > 4)$  (iv)  $P(3 < X < 5)$ .

Sol:

$$(i) f(x) = \frac{d}{dx} F(x) = \begin{cases} K, & 2 \leq x < 6 \\ 0, & \text{otherwise} \end{cases}$$

$$(ii) \int_2^6 f(x) dx = 1 \Rightarrow \int_2^6 K dx = 1 \Rightarrow K \cdot 4 = 1 \Rightarrow K = \frac{1}{4}$$

$$(iii) P(X > 4) = \int_4^6 \frac{1}{4} dx = \frac{1}{4}(x) \Big|_4^6 = \frac{1}{2}$$

$$(iv) P(3 < X \leq 5) = \int_3^5 \frac{1}{4} dx = \frac{1}{4}(x) \Big|_3^5 = \frac{1}{2}$$

$\leftarrow x \rightarrow$

## Expectation and Variance

An average of a prob. dist. of a r.v  $x$  is called the expectation or the expected value or Mathematical expectation of  $x$  and is denoted by  $E(x)$ .

Discrete r.v: Let  $x$  be a discrete r.v taking values  $x_1, x_2, \dots, x_n, \dots$  with probabilities  $p(x_1), p(x_2), \dots, p(x_i)$ , then the expected value of  $x$  is defined as

$$E(x) = \sum_{i=1}^{\infty} x_i p(x_i)$$

Continuous r.v: Let  $x$  be a continuous r.v with pdf  $f$  defined in  $(-\infty, \infty)$ , then the expected value of  $x$  is defined as

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

Note: (1)  $E(x)$  is called the mean of the distribution or mean of  $x$  and is denoted by  $\bar{x}$  or  $\mu$ .

$$(2) E(x^2) = \sum x_i^2 p(x_i) \text{ If } x \text{ is discrete} \\ = \int_{-\infty}^{\infty} x^2 f(x) dx \text{ If } x \text{ is continuous}$$

### Properties of Expectation:

(i) If  $c$  is constant, then  $E(c) = c$

(ii) If  $a, b$  are constants, then  $E(ax+b) = aE(x)+b$

Variance: Let  $X$  be a r.v. with mean  $E(X)$ , then the variance of  $X$  is defined as

$$E[X - E(X)]^2$$

It is denoted by  $\text{Var}(X)$  or  $\sigma_X^2$

$$\text{ie., } \text{Var}(X) = E[X - E(X)]^2$$

For Discrete r.v.: If  $X$  is a discrete r.v. taking values  $x_1, x_2, x_3, \dots$  with probabilities  $p(x_1), p(x_2), p(x_3), \dots$  and  $E(X) = \mu$  then

$$\text{Var}(X) = \sum_i (x_i - \mu)^2 p(x_i)$$

For Continuous r.v.: If  $X$  is a continuous r.v. with pdf  $f(x)$ ,  $x \in (-\infty, \infty)$  then

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Prove that  $\text{Var}(X) = E(X^2) - [E(X)]^2$

Proof: Let  $X$  be a r.v. then mean of  $X = E(X) = \mu$  (say)

$$\begin{aligned}\therefore \text{Var}(X) &= E[X - E(X)]^2 = E[X - \mu]^2 \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E[E(X^2)] - 2\mu E(X) + \mu^2 \\ &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2 \\ &= E(X^2) - [E(X)]^2\end{aligned}$$

## Properties of Variance

If  $a$  and  $b$  are constants, then

$$(i) \text{Var}(ax) = a^2 \text{Var}(x) \quad (ii) \text{Var}(ax+b) = a^2 \text{Var}(x)$$

$$(iii) \text{Var}(ax-b) = a^2 \text{Var}(x)$$

Note:  $\text{Var}(b) = 0$

i. If  $X$  has the probability distribution

$x$	-1	0	1	2
$p(x)$	0.3	0.1	0.4	0.2

Find  $E(x)$ ,  $E(x^2)$ ,  $\text{Var}(x)$ ,

$E(2x+1)$ ,  $\text{Var}(2x+1)$ .

Sol:

$$(i) E(x) = \sum x_i p(x_i) = (-1) \times 0.3 + (0 \times 0.1) + (1 \times 0.4) + (2 \times 0.2) \\ = -0.3 + 0.4 + 0.4 = 0.5$$

$$(ii) E(x^2) = \sum x_i^2 p(x_i) \\ = [(-1)^2 \times 0.3] + [0 \times 0.1] + [1^2 \times 0.4] + [2^2 \times 0.2] \\ = 0.3 + 0.4 + 0.8 = 1.5$$

$$(iii) \text{Var}(x) = E(x^2) - [E(x)]^2 = 1.5 - (0.5)^2 = 1.25$$

$$(iv) E(2x+1) = 2E(x) + 1 = (2 \times 0.5) + 1 = 2 \quad \therefore E(ax+b) \\ = aE(x) + b$$

$$(v) \text{Var}(2x+1) = 4 \text{Var}(x) = 4 \times 1.25 = 5$$

$$\therefore \text{Var}(ax+b) \\ = a^2 \text{Var}(x)$$

2) If  $X$  has the probability distribution

$x$	0	1	2	3	4
$P(x)$	$\frac{1}{25}$	$\frac{3}{25}$	$\frac{5}{25}$	$\frac{7}{25}$	$\frac{9}{25}$

Find mean, Variance,  $E(3x - 4)$  and  $Var(3x - 4)$ .

Sol:

$$\begin{aligned} \text{(i) Mean} &= E(x) = \sum x_i p(x_i) \\ &= \left(0 \times \frac{1}{25}\right) + \left(1 \times \frac{3}{25}\right) + \left(2 \times \frac{5}{25}\right) + \left(3 \times \frac{7}{25}\right) + \left(4 \times \frac{9}{25}\right) \\ &= \frac{70}{25} = \frac{14}{5} \end{aligned}$$

$$\text{(ii) Variance} = E(x^2) - [E(x)]^2$$

$$\begin{aligned} \text{Now } E(x^2) &= \sum x_i^2 p(x_i) \\ &= \left(0^2 \times \frac{1}{25}\right) + \left(1^2 \times \frac{3}{25}\right) + \left(2^2 \times \frac{5}{25}\right) + \left(3^2 \times \frac{7}{25}\right) + \left(4^2 \times \frac{9}{25}\right) \\ &= \frac{230}{25} = \frac{46}{5} \end{aligned}$$

$$\therefore Var(x) = \frac{46}{5} - \left(\frac{14}{5}\right)^2 = \frac{230 - 196}{25} = \frac{34}{25}$$

$$\text{(iii) } E(3x - 4) = 3E(x) - 4 = 3 \times \frac{14}{5} - 4 = \frac{22}{5}$$

$$\text{(iv) } Var(3x - 4) = 9Var(x) = 9 \times \frac{34}{25} = \frac{306}{25}$$

3) In a gambling game of tossing 4 coins once, a man gets Rs. 10 if he gets all heads or all tails and loses Rs. 5 if he gets 1 or 2 or 3 heads.

What is his expected gain?

Sol: Let  $x$  denote the amount gained.

The possible values of  $x$  are  $10, -5$ .

$$P(x=10) = P[\text{getting 4 heads or 4 tails}]$$

$$= \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$

$$P(x=-5) = P(1H \text{ and } 3T \text{ or } 2H \text{ and } 2T \text{ or } 3H \text{ and } 1T)$$

$$= 4C_1 \times \frac{1}{16} + 4C_2 \times \frac{1}{16} + 4C_3 \times \frac{1}{16}$$

$$= \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{7}{8}$$

The prob. dist. of  $x$  is

$x$	-5	10
$P(x)$	$\frac{1}{8}$	$\frac{7}{8}$

$$E(x) = \sum x_i p(x_i) = (-5 \times \frac{1}{8}) + (10 \times \frac{7}{8}) = \frac{25}{8}$$

4.) When a die is thrown find the expectation of the number on the die.

Sol: Let  $x$  denote the number on a die.

∴ The possible values of  $x$  are  $1, 2, 3, 4, 5, 6$

The prob. dist. of  $x$  is

$x$	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore E(x) = \sum x_i p(x_i)$$

$$= (1 \times \frac{1}{6}) + (2 \times \frac{1}{6}) + (3 \times \frac{1}{6}) + (4 \times \frac{1}{6}) + (5 \times \frac{1}{6}) + (6 \times \frac{1}{6})$$
$$= \frac{7}{2}$$

5) A coin is tossed until a head appears. What is the expected value of the number of tosses?

Sol:

Let  $x$  denote the number of tosses required to get the first success (head).

$\therefore$  The head can appear in the first toss or second toss or third toss and so on.

i.e., the events are  $\{H, TH, TTH, TTTH, \dots\}$

$\therefore$  The probabilities are  $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\}$

$$E(x) = \sum x_i p(x_i)$$

$$= \frac{1}{2} + 2 \left(\frac{1}{2}\right)^2 + 3 \left(\frac{1}{2}\right)^3 + \dots \quad [\because x \text{ takes the values } 1, 2, 3, \dots]$$

$$= \frac{1}{2} [1 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2^2} + \dots]$$

$$= \frac{1}{2} \left(1 - \frac{1}{2}\right)^{-2} \quad [\because (1-x)^{-2} = 1 + 2x + 3x^2 + \dots]$$

$$= 2$$

A man draws 3 balls from an urn containing 5 white and 7 black balls. He gets Rs. 10 for each white ball and Rs. 5 for each black ball. Find his expectation.

Sol:

Let  $x$  denote the amount gained.

3 balls drawn may be

(i) 3 black (ii) 2 black & 1 white

(iii) 1 black and 2 white (iv) 3 white

$\therefore$  the values of  $x$  are 15, 20, 25, 30

$$P(x=15) = P(3 \text{ black balls}) = \frac{7C_3}{12C_3} = \frac{7}{44}$$

$$P(x=20) = P(2 \text{ black and 1 white}) = \frac{7C_2 \times 5C_1}{12C_3} = \frac{21}{44}$$

$$P(x=25) = P(1 \text{ black and 2 white}) = \frac{7C_1 \times 5C_2}{12C_3} = \frac{14}{44}$$

$$P(x=30) = P(3 \text{ white balls}) = \frac{5C_3}{12C_3} = \frac{2}{44}$$

The prob. distribution of  $x$  is

$x$	15	20	25	30
$P(x)$	$\frac{7}{44}$	$\frac{21}{44}$	$\frac{14}{44}$	$\frac{2}{44}$

$$E(x) = (15 \times \frac{7}{44}) + (20 \times \frac{21}{44}) + (25 \times \frac{14}{44}) + (30 \times \frac{2}{44})$$

$$= \frac{935}{44}$$

- 7.) Find the expectation of the number of failures proceeding the first success in an infinite series of independent trials with constant probability  $p$  of success in each trial.

Sol:

Let  $x$  denote the number of failures proceeding first success.

Then events are  $\{S, FS, FFS, \dots\}$

Hence  $x$  takes the values  $0, 1, 2, 3, \dots$

and the corresponding probabilities are

$$P, qP, q^2P, q^3P, \dots$$

$$E(x) = 0 \cdot p + 1 \cdot pq + 2 \cdot q^2p + 3 \cdot q^3p + \dots$$

$$= qp [1 + 2q + 3q^2 + 4q^3 + \dots]$$

$$= qp(1-q)^{-2} = qp \cdot \frac{1}{p^2} = \frac{q}{p} \quad \because 1-p=q$$

- 8.) The monthly demand for Allwyn watches is known to have the following distribution

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Determine the expected demand for watches. Also compute the variance.

Sol: Let  $X$  denote the demand for Allwyn watches

$$\begin{aligned} E(X) &= \sum x_i p(x_i) \\ &= (1 \times 0.08) + (2 \times 0.12) + (3 \times 0.19) + (4 \times 0.24) + (5 \times 0.16) \\ &\quad + (6 \times 0.10) + (7 \times 0.07) + (8 \times 0.04) = 4.06 \end{aligned}$$

$$\begin{aligned} E(X^2) &= \sum x_i^2 p(x_i) \\ &= (1 \times 0.08) + (4 \times 0.12) + (9 \times 0.19) + (16 \times 0.24) + (25 \times 0.16) \\ &\quad + (36 \times 0.10) + (49 \times 0.07) + (64 \times 0.04) = 19.7 \end{aligned}$$

$$Var(X) = E(X^2) - [E(X)]^2 = 19.7 - (4.06)^2 = 3.2164$$

Given the following prob. dist. of  $X$  compute

- (i)  $E(X)$  (ii)  $E(X^2)$  (iii)  $E[2X+3]$  (iv)  $Var(2X+3)$

$X=x$	-3	-2	-1	0	1	2	3
$p(X=x)$	0.05	0.10	0.30	0	0.30	0.15	0.1

Sol:

$$\begin{aligned} (i) E(X) &= \sum x_i p(x_i) \\ &= (-3 \times 0.05) + (-2 \times 0.10) + (-1 \times 0.30) + 0 + (1 \times 0.3) \\ &\quad + (2 \times 0.15) + (3 \times 0.1) = 0.25 \end{aligned}$$

$$\begin{aligned} (ii) E(X^2) &= \sum x_i^2 p(x_i) \\ &= (9 \times 0.05) + (4 \times 0.10) + (1 \times 0.30) + 0 + (1 \times 0.3) \\ &\quad + (4 \times 0.15) + (9 \times 0.1) = 2.95 \end{aligned}$$

$$(iii) E(2X+3) = 2E(X) + 3 = 2(0.25) + 3 = -2.5, 3.5$$

$$(iv) \text{Var}(2x \pm 3) = 2^2 \text{Var}(x) = 4 \text{Var}(x)$$

$$\text{Now } \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$= 2.95 - (0.25)^2 = 2.8875$$

$$\therefore \text{Var}(2x \pm 3) = 4(2.8875) = 11.55$$

- 10) Find the mean and variance of the following density function  $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \leq x < 2 \\ 0, & \text{otherwise.} \end{cases}$

$$\text{Sol: } E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$\Rightarrow E(x) = \int_0^1 x \cdot x dx + \int_1^2 x(2-x) dx$$

$$= \left[ \frac{x^3}{3} \right]_0^1 + \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_1^2$$

$$= \frac{1}{3} + \left( 4 - \frac{8}{3} \right) - \left( 1 - \frac{1}{3} \right) = 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$\Rightarrow E(x^2) = \int_0^1 x^2 dx + \int_1^2 x^2(2-x) dx$$

$$= \left( \frac{x^4}{4} \right)_0^1 + \left( \frac{2x^3}{3} - \frac{x^4}{4} \right)_1^2$$

$$= \frac{1}{4} + \left( \frac{2^4}{3} - \frac{2^4}{4} \right) - \left( \frac{2}{3} - \frac{1}{4} \right) = \frac{1}{4} + 2^4 \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{8-3}{12} \right)$$

$$= \frac{1}{4} + \frac{2^4}{12} - \frac{5}{12} = \frac{14}{12} \text{ or } \frac{7}{6}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

- 11) The density function of a continuous r.v  $x$  is given by  $f(x) = \begin{cases} x/2 & \text{If } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$  Find (i) the expectation of  $x$  (ii) Variance of  $x$  (iii)  $E(3x^2 - 2x)$ .

Sol:

$$\begin{aligned} \text{(i)} \quad E(x) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2} \left(\frac{x^3}{3}\right)_0^2 = \frac{4}{3} \end{aligned}$$

$$\text{(ii)} \quad \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$\begin{aligned} \text{Now } E(x^2) &= \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^2 x^2 \cdot \frac{x}{2} dx = \frac{1}{2} \left(\frac{x^4}{4}\right)_0^2 = 2 \end{aligned}$$

$$\therefore \text{Var}(x) = 2 - \left(\frac{4}{3}\right)^2 = \frac{18-16}{9} = \frac{2}{9}$$

$$\text{(iii)} \quad E(3x^2 - 2x) = 3E(x^2) - 2E(x)$$

$$= 6 - \frac{8}{3} = \frac{10}{3}$$

- 12) Find the expectation and Variance of the r.v  $x$  whose pdf is given by  $f(x) = \begin{cases} 2e^{-2x} & \text{If } x > 0 \\ 0 & \text{otherwise} \end{cases}$

Sol:

$$(i) E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot 2e^{-2x} dx = 2 \int_0^{\infty} x e^{-2x} dx$$

With T Sudvi =  $uv - u'v_1 + u''v_2 - \dots$

$$\begin{aligned} u &= x & dv &= e^{-2x} dx \\ u' &= 1 & v &= \frac{e^{-2x}}{-2} \\ && v_1 &= \frac{e^{-2x}}{2^2} \end{aligned}$$

$$E(x) = 2 \left[ -\frac{x e^{-2x}}{2} - \frac{e^{-2x}}{2^2} \right]_0^{\infty} = 2 \cdot \frac{1}{2^2} = \frac{1}{2}$$

$$(ii) \text{Var}(x) = E(x^2) - [E(x)]^2$$

$$\text{Now } E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= 2 \int_0^{\infty} x^2 e^{-2x} dx$$

$$\begin{aligned} E(x^2) &= 2 \left[ x^2 \cdot \left( \frac{e^{-2x}}{-2} \right) - (2x) \left( \frac{e^{-2x}}{2^2} \right) + 2 \cdot \left( \frac{e^{-2x}}{-2^3} \right) \right]_0^{\infty} \\ &= 2 \cdot \frac{1}{2^2} = \frac{1}{2} \end{aligned}$$

$$\therefore \text{Var}(x) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

(3) Find mean and standard deviation of the distribution

$$f(x) = \begin{cases} kx(2-x), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

Sol:

$$\text{W.K.T} \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^2 kx(2-x) dx = 1 \Rightarrow k \left[ \frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = 1$$

$$\Rightarrow k \left[ 4 - \frac{8}{3} \right] = 1 \Rightarrow k = \frac{3}{4}$$

Mean:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^2 x \cdot \frac{3}{4} x(2-x) dx$$

$$= \frac{3}{4} \left[ 2 \frac{x^3}{3} - \frac{x^4}{4} \right]_0^2 = \frac{3}{4} \cdot 2^4 \left( \frac{1}{3} - \frac{1}{4} \right)$$

$$= \frac{3}{4} \cdot 2^4 \cdot \frac{1}{12} = 1$$

$$\text{Now } E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^2 x^2 \cdot \frac{3}{4} x(2-x) dx$$

$$= \frac{3}{4} \left[ 2 \frac{x^4}{4} - \frac{x^5}{5} \right]_0^2 = \frac{3}{4} \cdot 2^5 \left( \frac{1}{4} - \frac{1}{5} \right) = \frac{6}{5}$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\Rightarrow S.D = \sqrt{\sigma} = \frac{1}{\sqrt{5}}$$

A test engineer found that the dist. function of the life time  $x$  in years of an equipment is given by

$$F(x) = \begin{cases} 0 & \text{If } x < 0 \\ 1 - e^{-x/5} & \text{If } x \geq 0 \end{cases}$$

Find i) the pdf of  $X$

ii) the expected life time of the equipment

iii) the variance of the life time "

Sol:

(i)  $f(x) = \frac{d}{dx} F(x)$  or  $F'(x)$

$$= \begin{cases} \frac{1}{5} e^{-x/5}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(ii)  $E(X) = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{5} \int_0^{\infty} x e^{-x/5} dx$

$$= \frac{1}{5} \left[ x \cdot \frac{-e^{-x/5}}{-1/5} - 1 \cdot \frac{e^{-x/5}}{(-1/5)^2} \right]_0^{\infty}$$
$$= \frac{1}{5} \left[ \frac{1}{Y_5^2} \right] = 5$$

(iii)  $\text{Var}(X) = E(X^2) - [E(X)]^2$

Now  $E(X^2) = \int_0^{\infty} x^2 f(x) dx$

$$= \frac{1}{5} \int_0^{\infty} x^2 e^{-x/5} dx$$
$$= \frac{1}{5} \left[ x^2 \cdot \frac{-e^{-x/5}}{-1/5} - 2x \cdot \frac{e^{-x/5}}{(-1/5)^2} + 2 \cdot \frac{e^{-x/5}}{(-1/5)^3} \right]_0^{\infty}$$
$$= \frac{1}{5} \left[ 2 \cdot \frac{1}{Y_5^3} \right] = 50$$

$$\therefore \text{Var}(X) = 50 - 25 = 25$$

A continuous r.v has pdf for  $x = \begin{cases} abx, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

If the mean of the distribution is  $\frac{1}{2}$  then find  
a, b. Also evaluate  $\text{Var}(X)$ .

Sol:

$$\text{W.K.T} \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^1 (ax + bx^2) dx = 1 \Rightarrow \left( ax + \frac{bx^2}{2} \right)_0^1 = 1$$

$$\Rightarrow 2a + b = 2 \rightarrow (1)$$

Given  $E(X) = 2$

$$\Rightarrow \int_{-\infty}^{\infty} x f(x) dx = 2$$

$$\Rightarrow \int_0^1 x(ax + bx^2) dx = 2$$

$$\Rightarrow \left[ \frac{ax^2}{2} + \frac{bx^3}{3} \right]_0^1 = 2$$

$$\Rightarrow 3a + 2b = 3 \rightarrow (2)$$

Solving (1) and (2), we get,  $a=1, b=0$

$$\text{Now } E(X^2) = \int_0^1 x^2 f(x) dx$$

$$= \int_0^1 x^2 (ax + bx^2) dx = \int_0^1 x^2 dx$$

$$= \left[ \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

H.W

(b) Find the value of (i)  $k$  (ii) mean of the following dist.

a)  $f(x) = \begin{cases} k(x-x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Ans: (a)  $k=6, E(X)=1/2$

(b)  $k=1, E(X)=1$

H.W (17) A continuous r.v  $x$  has the pdf  $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find the mean and Variance of  $X$ . Ans:  $\frac{1}{3}, \frac{2}{9}$

(8) Let  $x$  be a r.v with  $E(x)=10$  and  $V(x)=25$ . Find the tve values of 'a' and 'b' such that  $y=ax+b$  has expectation  $o$  and Variance 1.

Sol:

Given  $E(x)=10, V(x)=25, E(y)=o, V(y)=1$

$$\text{Now } E(y)=E(ax+b)=aE(x)+b$$

$$\Rightarrow o=10a+b \quad \rightarrow (1)$$

$$\text{and } V(y)=V(a+b)=a^2V(x) \Rightarrow 1=a^2 \cdot 25 \Rightarrow a^2=\frac{1}{25} \rightarrow (2)$$

Solving (1) and (2), we get

$$a=\frac{1}{5}, b=2$$

(9) If  $X$  has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } 1 \leq x < 4 \\ \frac{1}{2} & \text{for } 4 \leq x < 6 \\ \frac{5}{6} & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x > 10 \end{cases}$$

Find (i) the probability distribution of  $X$ .

(ii)  $P(2 < X < 6)$  (iii) Mean of  $X$  (iv) Variance of  $X$ .

Sol:

(i) W.K.T  $P(X=x_i) = F(x_i) - F(x_{i-1})$

$$P(X=1) = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}$$

$$P(X=4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$P(X=6) = F(6) - F(4) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$$

$$P(X=10) = F(10) - F(6) = 1 - \frac{5}{6} = \frac{1}{6}$$

$\therefore$  the prob. dist. of  $X$  is

$x$	1	4	6	10
$p(x)$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$

(ii)  $P(2 < X < 6) = P(X=4) = \frac{1}{3}$

(iii) Mean of  $X = E(X) = \sum x p(x)$

$$= (1 \times \frac{1}{3}) + (4 \times \frac{1}{6}) + (6 \times \frac{1}{3}) + (10 \times \frac{1}{6}) = \frac{14}{3}$$

(iv)  $Vari(X) = E(X^2) - [E(X)]^2$

Now  $E(X^2) = \sum x^2 p(x)$

$$= (1 \times \frac{1}{3}) + (16 \times \frac{1}{6}) + (36 \times \frac{1}{3}) + (100 \times \frac{1}{6}) = \frac{95}{3}$$

$$\therefore Vari(X) = \frac{95}{3} - \left(\frac{14}{3}\right)^2 = \frac{89}{9}$$

20) If the cumulative distribution function of a r.v.  $x$  is  $F(x) = 1 - (1+x)e^{-x}$ ,  $x > 0$ . Find the probability density function of  $x$ , mean and variance of  $x$ .

Sol:

$$\begin{aligned} \text{pdf } f(x) &= \frac{d}{dx} F(x) = \frac{d}{dx} [1 - (1+x)e^{-x}] \\ &= \frac{d}{dx} [1 - e^{-x} - xe^{-x}] \\ &= 0 + e^{-x} - [xe^{-x} + e^{-x}] = xe^{-x}, \quad x > 0 \end{aligned}$$

$$\text{Mean} = E(x) = \int_0^\infty x f(x) dx = \int_0^\infty x^2 e^{-x} dx$$

$$= \left[ x^2 \left( \frac{e^{-x}}{-1} \right) - 2x \left( \frac{e^{-x}}{-1} \right) + 2 \left( \frac{e^{-x}}{-1} \right) \right]_0^\infty = 2$$

$$E(x^2) = \int_0^\infty x^2 \cdot x e^{-x} dx = \int_0^\infty x^3 e^{-x} dx$$

$$= \left[ x^3 \left( \frac{e^{-x}}{-1} \right) - 3x^2 \left( \frac{e^{-x}}{-1} \right) + 6x \left( \frac{e^{-x}}{-1} \right) - 6 \left( \frac{e^{-x}}{-1} \right) \right]_0^\infty = 6$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2 = 6 - 4 = 2$$

### Home Work

- i) Find the mean and variance of the discrete r.v.  $x$  with the following prob. function

$$p(x) = \begin{cases} 2/5 & \text{for } x=3 \\ 3/5 & \text{for } x=6 \end{cases} \quad \text{Ans: } \frac{24}{5}, \frac{54}{25}$$

2) If the r.v.  $x$  has the pdf  $f(x) = ax^3$ ,  $0 < x < 1$  then find (i)  $a$  (ii)  $E(x)$  (iii)  $\text{Var}(x)$  (iv) the value of  $m$  so that  $P(X \leq m) = \frac{1}{2}$ .

$$\text{Ans: } a=4, E(x)=\frac{4}{5}, \text{Var}(x)=\frac{2}{75}, m=\sqrt[4]{\frac{1}{2}}$$

3) A test engineer proposed, after a series of tests, that the life time  $x$  of component is a r.v. with pdf  $f(x) = \begin{cases} \frac{x}{100} e^{-x/10}, & x \geq 10 \\ 0, & \text{otherwise} \end{cases}$  find (i)  $P(X > 20)$  (ii)  $E(x)$  (iii)  $\text{Var}(x)$ .

$$\text{Ans: } P(X > 20) = 0.406, E(x) = 20, \text{Var}(x) = 200.$$

4) If  $x$  has the distribution function

$$F(x) = \begin{cases} 0 & \text{If } x < 0 \\ 1/6 & \text{If } 0 \leq x < 2 \\ 1/2 & \text{If } 2 \leq x < 4 \\ 5/8 & \text{If } 4 \leq x < 6 \\ 1 & \text{If } x \geq 6 \end{cases}$$

Find (i) the prob. dist. of  $x$  (ii)  $P(1 < x < 5)$   
 (iii) Mean and Variance of  $x$ .

Ans:	$x$	0	2	4	6
(i)	$P(x)$	$1/6$	$1/3$	$1/8$	$3/8$

$$(ii) \frac{11}{24}$$

$$(iii) E(x) = \frac{37}{12}, E(x^2) = \frac{101}{6}$$

$$\text{Var}(x) = 7.32$$

## Moments

The expected value of an integral power of a random variable is called its moment.

### Moments about the origin or raw moments.

The  $r^{\text{th}}$  moment of a r.v  $X$  about the origin is defined as  $E(X^r)$  and is denoted by  $\mu_r'$ .

$$\therefore \mu_1' = E(X^1)$$

The first four moments about the origin are

$$\mu_1' = E(X) = \text{Mean}$$

$$\mu_2' = E(X^2), \mu_3' = E(X^3), \mu_4' = E(X^4)$$

### Moments about the mean or central moments

The  $r^{\text{th}}$  moment of a r.v  $X$  about the mean  $\mu$  is defined as  $E[(X-\mu)^r]$  and is denoted by  $\mu_r$ .

$$\therefore \mu_1 = E(X-\mu)$$

The first four moments about the mean are

$$(i) \mu_1 = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$$

$\therefore$  For every distribution discrete or continuous, the first moment about its mean is zero

$$(ii) \mu_2 = E(X-\mu)^2 = \text{Variance of } X$$

$$(iii) \mu_3 = E(X-\mu)^3 \quad (iv) E(X-\mu)^4 = \mu_4$$

The moments about any point 'a'

The  $2^{\text{nd}}$  moment of a rv  $x$  about a point 'a' is defined as  $E(x-a)^2$  and is also denoted by  $\mu_2'$  and is given as:  $\mu_2' = E(x-a)^2$

The first four moments about the point 'a' are

$$\mu_1' = E(x-a)$$

$$\Rightarrow \mu_1' = \mu - a$$

$$\therefore \mu = \mu_1' + a$$

$$\mu_2' = E(x-a)^2, \mu_3' = E(x-a)^3, \mu_4' = E(x-a)^4$$

Relation between moments about the mean and moments about any point 'a'

$$\mu_1' = 0$$

$$\mu_2' = \mu_2' - \mu_1'^2$$

$$\mu_3' = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4' = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

- 1.) The first four moments of a distribution about  $x=4$  are 1, 4, 10, 45. Show that the mean is 5, variance is 3,  $\mu_3=0$ ,  $\mu_4=26$ .

Sol: Let  $\mu_1^1, \mu_2^1, \mu_3^1, \mu_4^1$  be the first four moments about  $x=4$ . Then all these were 1, 4, 10, 45 about  $x=4$ .

Given  $\mu_1^1 = 1, \mu_2^1 = 4, \mu_3^1 = 10, \mu_4^1 = 45$  about  $x=4$ .

$$\mu_1^1 = E(x-4)^1 = 1 \text{ (by definition)}$$

$$\Rightarrow 1 = E(x) - 4 \Rightarrow E(x) = 5$$

$$\text{Variance } (\mu_2) = \mu_2^1 - \mu_1^1$$

$$\mu_3 = \mu_3^1 - 3\mu_2^1\mu_1^1 + 2\mu_1^{1^3} = 10 - 3(4)(1) + 2(1) = 0$$

$$\mu_4 = \mu_4^1 - 4\mu_3^1\mu_1^1 + 6\mu_2^1\mu_1^{1^2} - 3\mu_1^{1^4}$$

$$= 45 - (4 \times 10 \times 1) + (6 \times 4 \times 1) - (3 \times 1) = 26$$

Q) The first two moments about 3 are 1 and 8. Find the mean and variance.

Sol: Let  $\mu_1^1, \mu_2^1$  be the first two moments of  $x$  about 3.

Given  $\mu_1^1 = 1, \mu_2^1 = 8$ .

$$\text{Mean} = \mu_1^1 + a = 1 + 3 = 4$$

$$\mu_2 = \mu_2^1 - \mu_1^{1^2} = 8 - 1 = 7$$

$$\therefore \text{Variance} = \mu_2 = 7$$

3) The first three moments about the origin are 5, 26, 78. Show that the first three moments about the value  $x=3$  are 2, 5, -48.

Sol: Given  $E(x)=5$ ,  $E(x^2)=26$ ,  $E(x^3)=78$

Let  $\mu'_1, \mu'_2, \mu'_3$  be the first three moments about  $x=3$ .

$$\therefore \mu'_1 = E(x-3) = E(x)-3 = 5-3 = 2$$

$$\begin{aligned} \mu'_2 &= E(x-3)^2 = E(x^2 - 6x + 9) \\ &= E(x^2) - 6E(x) + 9 = 26 - (6 \times 5) + 9 = 5 \end{aligned}$$

$$\begin{aligned} \mu'_3 &= E(x-3)^3 = E(x^3 - 9x^2 + 27x - 27) \\ &= E(x^3) - 9E(x^2) + 27E(x) - 27 \\ &= 78 - (9 \times 26) + (27 \times 5) - 27 = -48 \end{aligned}$$

4) A continuous r.v.  $X$  has the p.d.f  $f(x)=kx^2e^{-x}$ ,  $x \geq 0$ . Find the  $n^{\text{th}}$  moment of  $X$  about the origin. Hence find the mean and variance.

Sol:

$$\text{W.K.T. } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} kx^2 e^{-x} dx = 1$$

$$k \left[ x^2 (-e^{-x}) - 2x (e^{-x}) + 2 (-e^{-x}) \right]_0^{\infty} = 1$$

$$\Rightarrow k = 1/2$$

$\mu_2' = 2^{\text{th}}$  moment about the origin

$$= E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^{\infty} x^2 \cdot \frac{1}{2} x^2 e^{-x} dx = \frac{1}{2} \int_0^{\infty} x^{2+2} e^{-x} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-x} x^{r+3-1} dx \quad \because \Gamma_n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= \frac{1}{2} \Gamma(r+3) = \frac{1}{2} (r+2)! \quad \therefore \Gamma_n = (n-1)!$$

$$\text{Mean} = \mu_1' = \frac{1}{2} \cdot 3! = 3$$

$$\mu_2' = \frac{1}{2} \cdot 4! = 12$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = 12 - 9 = 3$$

5) A continuous r.v.  $x$  has pdf  $f(x) = k(1-x)$  for  $0 < x < 1$ .

Find the  $r^{\text{th}}$  moment about the origin. Hence find mean and variance.

$$\text{Sol: } \text{W.K.T. } \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^1 k(1-x) dx = 1 \Rightarrow k \left( x - \frac{x^2}{2} \right) \Big|_0^1 = 1 \Rightarrow k = 2$$

$\mu_2' = 2^{\text{th}}$  moment about the origin

$$= E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^1 x^2 \cdot 2(1-x) dx$$

$$= 2 \left[ \frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2} \right]_0^1 = \frac{2}{(r+1)(r+2)}$$

$$\mu'_1 = \text{Mean} = \frac{2}{2 \times 3} = \frac{1}{3}$$

$$\mu'_2 = \frac{2}{3 \times 4} = \frac{1}{6}$$

$$\therefore \mu_2 = \mu'_2 - \mu'_1 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}.$$

- 6) Find the first three moments of  $x$  if  $x$  has the following distribution:

Sol:

$x$	-2	1	3
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Let  $\mu'_1, \mu'_2, \mu'_3$  be the first three moments about the origin.

$$\therefore \mu'_1 = E(x) = \sum x p(x)$$

$$= (-2 \times \frac{1}{2}) + (1 \times \frac{1}{4}) + (3 \times \frac{1}{4}) = 0$$

$$\mu'_2 = E(x^2) = \sum x^2 p(x)$$

$$= (-2)^2 \times \frac{1}{2} + (1^2 \times \frac{1}{4}) + (3^2 \times \frac{1}{4}) = \frac{9}{2}$$

$$\mu'_3 = E(x^3) = \sum x^3 p(x)$$

$$= (-2)^3 \times \frac{1}{2} + (1^3 \times \frac{1}{4}) + (3^3 \times \frac{1}{4}) = 3$$

- 7) Find the first four moments about the origin for a r.v  $x$  having the pdf

$$f(x) = \begin{cases} \frac{4x(9-x^2)}{81}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

Sol:

Let  $\mu_2'$  be the 2<sup>th</sup> moment about the origin.

$$\begin{aligned}\therefore \mu_2' &= E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx \\ &= \int_0^3 x^2 \cdot \frac{4x(9-x^2)}{81} dx \\ &= \frac{4}{81} \int_0^3 (9x^{2+1} - x^{2+3}) dx \\ &= \frac{4}{81} \left[ 9 \cdot \frac{x^{2+2}}{2+2} - \frac{x^{2+3}}{2+4} \right]_0^3 \\ &= \frac{4}{81} \cdot 3^{2+4} \left[ \frac{1}{2+2} - \frac{1}{2+4} \right]\end{aligned}$$

If  $\gamma=1, 2, 3, 4$  we get the first moments about the origin.

$$\therefore \mu_1' = \frac{4}{81} \cdot 3^5 \left[ \frac{1}{3} - \frac{1}{5} \right] = \frac{8}{5}$$

$$\mu_2' = \frac{4}{81} \cdot 3^6 \left[ \frac{1}{4} - \frac{1}{6} \right] = 3$$

$$\mu_3' = \frac{4}{81} \cdot 3^7 \left( \frac{1}{5} - \frac{1}{7} \right) = 6.17$$

$$\mu_4' = \frac{4}{81} \cdot 3^8 \left( \frac{1}{6} - \frac{1}{8} \right) = 13.5$$

H.W

- 8) Find the first four raw moments and central moments of  $f(x) = kx(2-x)$ ,  $0 \leq x \leq 2$ .

$$\text{Ans: } k = \frac{3}{4}, \mu_1' = 1, \mu_2' = \frac{6}{5}, \mu_3' = \frac{8}{5}, \mu_4' = \frac{16}{7}, \mu_1 = 0, \mu_2 = \frac{1}{5}$$

$$\mu_3 = 0, \mu_4 = \frac{3}{35}$$

9) If  $f(x) = ce^{-ax}$ ,  $a > 0$ ,  $x \geq 0$  is a pdf then find  $c$  and the first three moments about mean.

Sol:

$$\text{W.K.T} \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_0^{\infty} ce^{-ax} dx = 1 \Rightarrow c \left[ \frac{e^{-ax}}{-a} \right]_0^{\infty} = 1$$

$$\Rightarrow c \left( \frac{1}{a} \right) = 1 \Rightarrow c = a$$

$$\mu'_1 = \int_0^{\infty} ax e^{-ax} dx = a \left[ x \left( -\frac{e^{-ax}}{a} \right) - 1 \cdot \left( \frac{e^{-ax}}{a^2} \right) \right]_0^{\infty} = \frac{1}{a}$$

$$\mu'_2 = a \int_0^{\infty} x^2 e^{-ax} dx = a \left[ x^2 \left( \frac{e^{-ax}}{-a} \right) - 2x \left( \frac{e^{-ax}}{a^2} \right) + 2 \left( \frac{e^{-ax}}{-a^3} \right) \right]_0^{\infty}$$

$$\text{Similarly } \mu'_1 = \frac{2}{a^2}$$

$$\mu'_3 = \frac{6}{a^3}$$

$$\therefore \mu_1 = 0, \mu_2 = \mu'_2 - \mu'_1^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

$$\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2\mu'_1^3 = \frac{6}{a^3} - 3 \cdot \frac{2}{a^2} + \frac{2}{a^3} = \frac{2}{a^3}$$

— x —

## Moments Generating Function : (MGF)

MGF of a r.v  $x$  is defined as  $E(e^{tx})$  where ' $t$ ' is a real variable. It is denoted by  $M_x(t)$  or  $M(t)$ .

$$\text{i.e., } M_x(t) = E(e^{tx})$$

$$\text{If } x \text{ is discrete, then } M_x(t) = \sum_{x_i} e^{t x_i} p_i$$

where  $x$  takes the values  $x_1, x_2, x_3, \dots$  with probabilities  $p_1, p_2, p_3, \dots$

If  $x$  is a continuous r.v with density function,  $f(x)$ , then  $M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$ .

## Properties of MGF

(1) Let  $x$  be a RV with MGF  $M_x(t)$  and  $c$  is a constant. Then (a)  $M_{cx}(t) = M_x(ct)$  (b)  $M_{x+c}(t) = e^{ct} M_x(t)$ .

(2) If  $x$  and  $y$  are two independent r.v's then  $M_{x+y}(t) = M_x(t) \cdot M_y(t)$ .

(3) If the MGF of  $x$  is  $M_x(t)$  and If  $y = ax + b$ , then  $M_y(t) = e^{bt} M_x(at)$

(4) MGF of a r.v  $x$  generates all the moments about the origin.

Proof: (5)  $E(x^n) = \mu'_n$  is the co-efficient of  $\frac{t^n}{n!}$  in

the expansion of  $M_x(t)$  in series of powers of  $t$

$$(6) \mu_0^1 = E(X^r) = \left[ \frac{d^r}{dt^r} M_x(t) \right]_{t=0}$$

Property 4, 5: (proof)

$$\begin{aligned} M_x(t) &= E(e^{tx}) \\ &= E \left[ 1 + \frac{(tx)}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots + \frac{(tx)^r}{r!} + \dots \right] \\ &= 1 + \frac{t}{1!} E(x) + \frac{t^2}{2!} E(x^2) + \frac{t^3}{3!} E(x^3) + \dots + \frac{t^r}{r!} E(x^r) + \dots \\ &= 1 + \frac{t}{1!} \mu_1^1 + \frac{t^2}{2!} \mu_2^1 + \frac{t^3}{3!} \mu_3^1 + \dots + \frac{t^r}{r!} \mu_r^1 + \dots \rightarrow ① \end{aligned}$$

$\therefore \mu_1^1$  = co-eff. of  $t$  in the expansion of  $M_x(t)$

$$\mu_2^1 = " \quad \frac{t^2}{2!} \quad " \quad " \quad "$$

$$\vdots \quad \mu_r^1 = " \quad \frac{t^r}{r!} \quad " \quad " \quad "$$

$\therefore M_x(t)$  generates all moments about the origin  
and hence we call it as MGF.

Property 6: (proof)

Equation ①, Diff. w.r.t 't' we get

$$M_x'(t) = \mu_1^1 + t \mu_2^1 + \dots + \frac{r t^{r-1}}{r!} \mu_r^1 + \dots$$

$$M_X''(t) = \mu_2^1 + t\mu_3^1 + \dots + \frac{r(r-1)t^{r-2}}{r!} \mu_r^1 + \dots$$

$$M_X'''(t) = \mu_3^1 + t\mu_4^1 + \dots + \frac{r(r-1)(r-2)t^{r-3}}{r!} \mu_r^1 + \dots$$

:

putting  $t=0$ , we get

$$M_X^1(0) = \mu_1^1, M_X^2(0) = \mu_2^1, M_X^3(0) = \mu_3^1 \dots$$

In general  $\mu_r^1 = M_X^{(r)}(0)$

A coin is tossed three times. If  $X$  denotes the number of heads that appear, find the MGF of  $X$  and hence find the mean and variance.

Sol:

$S = \{HHH, HHT, THT, HTH, HTT, THH, TTH, TTT\}$   
and no. of heads  $\times$

$X$  is a discrete r.v. taking values  $0, 1, 2, 3$

$X=x$	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^3 e^{tx} p(x)$$

$$= \frac{1}{8} [1 + 3e^t + 3e^{2t} + e^{3t}]$$

$$M_X^1(t) = \frac{1}{8} [3e^t + 6e^{2t} + 3e^{3t}]$$

$$\therefore \text{Mean} = E(X) = M_X^1(0) = \frac{1}{8} (3 + 6 + 3) = \frac{12}{8} = \frac{3}{2}$$

$$M_X(t) = \frac{3}{8} [e^t + 4e^{2t} + 3e^{3t}]$$

$$M_X(0) = E(X^2) = \frac{3}{8} (1+4+3) = 3$$

$$\therefore \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{3}{4}$$

Find the MGF of a r.v with the probability function  
 $P(X=x) = q^{x-1} p$ ,  $x=1, 2, 3, \dots$ . Find the mean and variance.

Sol:  $M_X(t) = E(e^{tX})$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q} [(qe^t) + (qe^t)^2 + (qe^t)^3 + (qe^t)^4 + \dots]$$

$$= \frac{p}{q} \cdot qe^t [1 + qe^t + (qe^t)^2 + (qe^t)^3 + \dots]$$

$$= pe^t (1 - qe^t)^{-1}$$

Diff. w.r.t. 't', we get

$$M'_X(t) = p \left\{ e^t (1 - qe^t)^{-1} + e^t (-1)(1 - qe^t)^{-2} (-qe^t) \right\}$$

$$= pe^t (1 - qe^t)^{-1} \cdot [1 + qe^t (1 - qe^t)^{-1}]$$

$$= pe^t (1 - qe^t)^{-1} \left[ \frac{1 - qe^t + qe^t}{1 - qe^t} \right]$$

$$= pe^t (1 - qe^t)^{-2}$$

$$\mu_1 = M'_X(0) = p(1-q)^2 = \frac{p}{p^2} = \frac{1}{p} \quad \because p+q=1$$

$$\therefore \text{Mean} = \frac{1}{p}$$

$$q=1-p \\ \text{or } p=1-q$$

Again diff. w.r.t. 't', we get

$$M_x''(t) = p \left\{ e^t (1-qe^t)^{-2} + e^t (-2)(1-qe^t)^{-3} (-qe^t) \right\}$$

$$= pe^t (1-qe^t)^{-2} [1+2qe^t (1-qe^t)^{-1}]$$

$$\mu_2' = M_x''(0) = p (1-q)^{-2} (1+2q(1-q)^{-1})$$

$$= \frac{p}{p^2} (1+\frac{2q}{p})$$

$$\therefore \mu_2 = \text{Variance} = \mu_2' - \mu_1'^2$$

$$= \frac{1}{p^2} (p+2q) - \frac{1}{p^2}$$

$$= \underbrace{\frac{p+2q-1}{p^2}}_{=} = \frac{2q-(1-p)}{p^2} = \frac{q}{p^2}$$

- 3) Find the MGF for the distribution where

$$f(x) = \begin{cases} 2/3, & \text{at } x=1 \\ 1/3, & \text{at } x=2 \\ 0 & \text{otherwise} \end{cases}$$

Sol:

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^2 e^{tx} f(x) = \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$

- 4) Find the MGF of the rv whose moments are

$$\mu_r' = (r+1)! 2^r$$

Sol: W.K.T.  $M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)! 2^r$$

$$M_X(t) = \sum_{r=0}^{\infty} (r+1)(2t)^r = 1 + 2(2t) + 3(2t)^2 + \dots$$

$$= (1 - 2t)^{-2}$$

$\therefore (1-x)^{-2} = 1 + 2x + 3x^2 + \dots$

If a r.v 'x' has the MGF  $M_X(t) = \frac{3}{3-t}$ , find the S.D.

Sol: Method: 1

$$M_X(t) = \frac{3}{3-t} = 3(3-t)^{-1}$$

$$M_X'(t) = 3(3-t)^{-2}, \quad M_X''(t) = 6(3-t)^{-3}$$

$$\Rightarrow E(x) = M_X'(0) = \frac{1}{3}, \quad E(x^2) = M_X''(0) = \frac{2}{9}$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$\therefore S.D = \sqrt{\text{Var } x} = \frac{1}{3}$$

Method: 2

$$M_X(t) = (1 - \frac{t}{3})^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \dots$$

$$E(x) = \text{Co-eff. of } \frac{t^1}{1!} = \frac{1}{3}$$

$$E(x^2) = \text{Co-eff. of } \frac{t^2}{2!} = \frac{2}{9}$$

$$\therefore \text{Var}(x) = E(x^2) - [E(x)]^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

Find the MGF of a r.v 'x' whose prob. function is  $P(x) = \frac{1}{2^x}$ ,  $x=1, 2, 3, \dots$ . Hence find its mean.

Sol:

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$M_X(t) = \frac{e^t}{2} \left[ 1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots \right]$$

$$= \frac{e^t}{2} \left( 1 - \frac{e^t}{2} \right)^{-1} = \frac{e^t}{2} \left( \frac{2-e^t}{2} \right)^{-1} = \frac{e^t}{2-e^t}$$

$$M_X'(t) = e^t (2-e^t)^{-1} + e^t (-1)(2-e^t)^{-2}(-e)$$

$$M_X'(0) = e^0 \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$M_X'(0) = (2-1)^{-1} + (2-1)^{-2} = 2$$

$\therefore \text{Mean} = 2$

H.W  
T) If a r.v  $x$  has the MGF  $M_X(t) = \frac{2}{2-t}$ , find the variance. Ans:  $M_X'(0) = \frac{1}{2}$ ,  $M_X''(0) = \frac{1}{2}$ ,  $\text{Var}(x) = \frac{1}{2}$

8) If  $x$  represents the outcome, when a fair die is tossed. Find the MGF of  $x$  and hence find  $E(x)$  and  $\text{Var}(x)$

Sol:

$x$  denotes the number on a die

$X=x$	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$M_X(t) = \sum_{x=1}^6 e^{tx} P(X=x)$$

$$= \frac{1}{6} [e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$$

$$= \frac{1}{6} [e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]$$

$$M_X'(t) = \frac{1}{6} [e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]$$

$$E(x) = M_X'(0) = \frac{1}{6} (1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

$$M_x''(t) = \frac{1}{6} [e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$$

$$E(X^2) = M_x''(0) = \frac{1}{6} [1 + 4 + 9 + 16 + 25 + 36] = \frac{91}{6}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

- 9.) The RV  $X$  has MGF  $M_x(t) = \frac{3}{3-t}$ . Find first four moments about its origin and also find four moments about its mean.

Sol:

$$\text{Given } M_x(t) = 3(3-t)^{-1}$$

$$\Rightarrow M_x'(t) = 3(3-t)^{-2}, M_x''(t) = 6(3-t)^{-3}, M_x'''(t) = 18(3-t)^{-4}$$

$$\text{and } M_x^{(4)}(t) = 72(3-t)^{-5}$$

$$\mu_1 = M_x'(0) = \frac{1}{3}, \mu_2 = M_x''(0) = \frac{6}{27} = \frac{2}{9}, \mu_3 = \frac{18}{3^4} = \frac{2}{9}, \mu_4 = \frac{8}{27}$$

To find four moments about its mean

$$\mu_1 = 0, \mu_2 = \mu_2' - \mu_1^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1^2 + 2\mu_1^3 = \frac{2}{9} - 3 \cdot \frac{2}{9} \cdot \frac{1}{3} + 2 \cdot \frac{1}{27} = \frac{2}{27}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1^2 + 6\mu_2'\mu_1^4 - 3\mu_1^4$$

$$= \frac{8}{27} - 4 \cdot \frac{2}{9} \cdot \frac{1}{3} + 6 \cdot \frac{2}{9} \cdot \frac{1}{9} - 3 \cdot \frac{1}{3^4} = \frac{1}{9}$$

- 10.) If  $x$  is a discrete RV with prob. distribution  $P(x) = \frac{1}{K^x}$ ,  $x=1, 2, \dots, K$  a constant, find the MGF of  $X$  and hence find its mean and variance.

$$\begin{aligned}
 \text{Sol: } M_X(t) &= E(e^{tx}) = \sum_{x=1}^{\infty} \left(\frac{e^t}{k}\right)^x = \frac{e^t}{k} + \left(\frac{e^t}{k}\right)^2 + \left(\frac{e^t}{k}\right)^3 + \dots \\
 &= \frac{e^t}{k} \left[1 + \frac{e^t}{k} + \frac{e^{2t}}{k^2} + \dots\right] = \frac{e^t}{k} \left(1 - \frac{e^t}{k}\right)^{-1} \\
 &= \frac{e^t}{k - e^t}
 \end{aligned}$$

$$\begin{aligned}
 M_X'(t) &= e^t (k - e^t)^{-1} + e^t (-1)(k - e^t)^{-2}(-e^t) \\
 &= e^t (k - e^t)^{-1} \left[1 + \frac{e^t}{(k - e^t)}\right] = \frac{k e^t}{(k - e^t)^2}
 \end{aligned}$$

$$\begin{aligned}
 \$ M_X''(t) &= k \left[ e^t (k - e^t)^{-2} + e^t (-2)(k - e^t)^{-3}(-e^t) \right] \\
 &= k e^t (k - e^t)^{-2} \left[1 + \frac{2e^t}{k - e^t}\right] \\
 &= \frac{k e^t (k + e^t)}{(k - e^t)^3}
 \end{aligned}$$

$$\Rightarrow M_X'(0) = \frac{k}{(k-1)^2}, \quad M_X''(0) = \frac{k(k+1)}{(k-1)^3}$$

$$\begin{aligned}
 \therefore \text{Mean} = \mu_1 &= \frac{k}{(k-1)^2}, \quad \text{Variance} = \frac{k(k+1)}{(k-1)^3} - \frac{k^2}{(k-1)^4} = \frac{k(k+k^2)}{(k-1)^4} \\
 &= \frac{k(k+1) - k(k-1)}{(k-1)^3}
 \end{aligned}$$

ii) A r.v  $x$  has the pdf given by  $f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Find a) MGF b) the first four moments about the origin.

Sol:

$$M_X(t) = 2 \int_0^\infty e^{tx} e^{-2x} dx = 2 \int_0^\infty e^{-(2-t)x} dx$$

$$= 2 \left[ \frac{e^{-(2-t)x}}{-(2-t)} \right]_0^\infty = \frac{2}{2-t}$$

$$\Rightarrow M'_X(t) = 2(2-t)^{-2}, M''_X(t) = 4(2-t)^{-3}, M'''_X(t) = 12(2-t)^{-4}$$

$$M''''_X(t) = 48(2-t)^{-5}$$

$$\therefore \mu'_1 = M'_X(0) = \frac{1}{2}, \mu'_2 = M''_X(0) = \frac{1}{2}, \mu'_3 = \frac{3}{4}, \mu'_4 = \frac{3}{2}$$

12) Find the MGF of the rv 'x' having pdf

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Sol:

$$M_X(t) = \int_0^1 e^{tx} \cdot x dx + \int_1^2 e^{tx} (2-x) dx$$

$$\Rightarrow M_X(t) = \left[ x \cdot \frac{e^{tx}}{t} - 1 \cdot \frac{e^{tx}}{t^2} \right]_0^1 + \left[ (2-x) \frac{e^{tx}}{t} - (-1) \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} = \frac{(e^t - 1)^2}{t^2}$$

13) Let x be a rv with pdf  $f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

Find a)  $P(X > 3)$  b) MGF of x c)  $E(x)$  and  $\text{Var}(x)$ .

Sol:

$$\text{a) } P(X > 3) = \frac{1}{3} \int_3^\infty e^{-\frac{x}{3}} dx = \frac{1}{3} \left[ \frac{-e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_3^\infty = e^{-1} = \frac{1}{e}$$

$$(b) M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \cdot \frac{1}{3} e^{-x/3} dx$$

$$= \frac{1}{3} \int_0^{\infty} e^{-(\frac{1}{3}-t)x} dx = \frac{1}{3} \left[ \frac{e^{-(\frac{1}{3}-t)x}}{-(\frac{1}{3}-t)} \right]_0^{\infty}$$

$$= \frac{1}{3} \left[ \frac{1}{(\frac{1}{3}-t)} \right] = \frac{1}{1-3t}$$

$$(c) \text{ Mean} = M'_x(0), \text{ Variance} = M''_x(0) - [M'_x(0)]^2$$

$$= E(X^2) - [E(X)]^2$$

$$M'_x(t) = (-1)(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$$

$$M''_x(t) = 3(-2)(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$$

$$\Rightarrow M'_x(0) = 3, M''_x(0) = 18$$

$$\therefore \text{Mean} = 3, \text{ Variance} = 18 - 9 = 9$$

Find the MGF of the dist. given by  $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

and hence find fourth moment.

Sol:

$$M_x(t) = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx = \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx$$

$$= \lambda \left[ \frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty} = \frac{\lambda}{\lambda-t}$$

$$\Rightarrow M'_x(t) = \lambda(-1)(\lambda-t)^{-2}(-1) = \lambda(\lambda-t)^{-2}$$

$$M''_x(t) = \lambda(-2)(\lambda-t)^{-3}(-1) = 2\lambda(\lambda-t)^{-3}, M'''_x(t) = 6\lambda(\lambda-t)^{-4}$$

$$M''''_x(t) = 24\lambda(\lambda-t)^{-5}, \therefore M''''_x(0) = 24/\lambda^4$$

Find the MGF of a r.v 'x' having the density function  
 $f(x) = \begin{cases} 2x/2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$  and also find first three moments about origin.

Sol:

$$M_x(t) = \int_0^2 e^{tx} \cdot \frac{x}{2} dx = \frac{1}{2} \left[ x \cdot \frac{e^{tx}}{t} - \frac{1}{t} \cdot \frac{e^{tx}}{t^2} \right]_0^2$$

$$= \frac{1}{2} \left[ \frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2} \right] = \frac{1}{2t^2} [2te^{2t} - e^{2t} + 1]$$

$$E(x) = \int_0^2 x \cdot \frac{x}{2} dx = \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{4}{3}$$

$$E(x^2) = \int_0^2 x^2 \cdot \frac{x}{2} dx = \frac{1}{2} \left[ \frac{x^4}{4} \right]_0^2 = 2$$

$$E(x^3) = \int_0^2 x^3 \cdot \frac{x}{2} dx = \frac{1}{2} \left[ \frac{x^5}{5} \right]_0^2 = \frac{16}{5}$$

A r.v x has density function given by

$$f(x) = \begin{cases} 1/K, & \text{for } 0 < x < K \\ 0, & \text{otherwise} \end{cases}$$

Find (i) MGF (ii)  $x^{th}$  moment

(iii) Mean (iv) Variance.

Sol:

$$(i) M_x(t) = E(e^{tx}) = \int_0^K \frac{1}{K} e^{tx} dx = \frac{1}{K} \left( \frac{e^{tx}}{t} \right)_0^K = \frac{e^{Kt} - 1}{K}$$

$$(ii) \mu_x^r = \int_0^K x^r \cdot \frac{1}{K} dx = \frac{1}{K} \left[ \frac{x^{r+1}}{r+1} \right]_0^K = \frac{K^r}{r+1}$$

$$(iii) r=1 \text{ in (ii)} \Rightarrow \mu_1^1 = \frac{k}{2} \quad \therefore \text{Mean} = \frac{k}{2}$$

$$(iv) r=2 \text{ in (ii)} \Rightarrow \mu_2^1 = \frac{k^2}{3}$$

$$\begin{aligned}\text{Variance} &= \mu_2^1 - \mu_1^{1^2} = \frac{k^2}{3} - \frac{k^2}{4} \\ &= \frac{k^2}{12}.\end{aligned}$$

→ —

Note: If MGF does not exist. ~~It~~ MGF is valid for t lying in an interval containing zero

$$\begin{aligned}2) M_X(t) &= \sum_x e^{tx} p(x) \\ &= \int_{-\infty}^{\infty} e^{tx} f(x) dx\end{aligned}$$

t being the real parameter and it is being assumed that the RHS is absolutely convergent for some positive number h such that  $-h < t < h$ .

## Function of One Random Variable

Let  $X$  be an RV with the associated Sample space  $S_x$  and a known probability distribution. Let  $g$  be a scalar function that maps each  $x \in S_x$  into  $y = g(x)$ . The expression  $y = g(x)$  defines a new RV  $Y$ . For a given outcome,  $x(s)$  is a number of  $x$  and  $g[x(s)]$  is another number specified by  $g(x)$ . This number is the value of The RV  $Y$ . i.e.,  $y(s) = y = g(x)$ .

The sample space  $S_y$  of  $Y$  is the set

$$S_y = \{y = g(x) : x \in S_x\}$$

To Find  $f_y(y)$ , when  $f_x(x)$  is known

Let us now derive a procedure to find  $f_y(y)$ , the pdf of  $Y$ , when  $y = g(x)$ , where  $x$  is a continuous RV with pdf  $f_x(x)$  and  $g(x)$  is a strictly monotonic function of  $x$ .

Case (i):  $g(x)$  is a strictly increasing function of  $x$ .

$$F_y(y) = P(Y \leq y) \text{ where } F_y(y) \text{ is the CDF of } Y$$

$$= P[g(x) \leq y]$$

$$= P[x \leq g^{-1}(y)] = F_x(g^{-1}(y))$$

Diff. both sides w.r.t  $y$ , we get

$$f_y(y) = f_x(x) \cdot \frac{dx}{dy} \quad \text{where } x = g^{-1}(y). \rightarrow ①$$

Case iii  $g(x)$  is strictly decreasing function of  $x$

$$\begin{aligned} F_y(y) &= P(Y \leq y) \\ &= P[g(x) \leq y] \\ &= P[x \geq g^{-1}(y)] \\ &= 1 - P[x \leq g^{-1}(y)] \\ &= 1 - F_x[g^{-1}(y)] \end{aligned}$$

$$\therefore F_y(y) = -f_x(x) \cdot \frac{dx}{dy} \rightarrow ②$$

From ① & ②, we get

$$f_y(y) = f_x(x) \left| \frac{dx}{dy} \right| \quad \text{or} \quad f_y(y) = \frac{f_x(x)}{\left| \frac{dx}{dy} \right|}$$

$$\text{or} \quad f_y(y) = f_x(x) / |g'(x)|$$

Note: 1) The above formula for  $f_y(y)$  can be used only when  $x = g^{-1}(y)$  is single valued.

2) When  $x = g^{-1}(y)$  takes finitely many values  $x_1, x_2, \dots, x_n$  or  $g(x) = y$  has finitely many values to find  $f_y(y)$  we use the extended formula:

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + \frac{f_x(x_2)}{|g'(x_2)|} + \dots + \frac{f_x(x_n)}{|g'(x_n)|} \quad \text{or}$$

$$f_y(y) = f_x(x_1) \left| \frac{dx_1}{dy} \right| + f_x(x_2) \left| \frac{dx_2}{dy} \right| + \dots + f_x(x_n) \left| \frac{dx_n}{dy} \right|$$

Procedure to find  $f_y(y)$

1) Write  $y$  in terms of  $x$ .

2) Write  $x_i$  as a function of  $y$ .

3) Find  $\left| \frac{dx}{dy} \right|$ .

4) Write  $f_y(y) = f_x(x) \left| \frac{dx}{dy} \right|$ .

5) Using the range of  $x$ , find the range of  $y$ .

1) Let  $X$  be a continuous RV with pdf

$f(x) = \begin{cases} \frac{x}{2}, & 1 < x < 5 \\ 0, & \text{otherwise} \end{cases}$ . Find the pdf of  $Y = 2x - 3$ .

Sol:

Given  $g(x) = y = 2x - 3$

$$\therefore x = \frac{y+3}{2} \Rightarrow \frac{dx}{dy} = \frac{1}{2}$$

$$f_y(y) = f_x(x) \cdot \left| \frac{dx}{dy} \right| = \frac{x}{2} \cdot \frac{1}{2} = \frac{(y+3)}{8}$$

$$1 < x < 5 \Rightarrow 1 < \frac{y+3}{2} < 5 \Rightarrow 2 < y + 3 < 10 \\ \Rightarrow -1 < y < 7$$

$$\therefore f_y(y) = \begin{cases} \frac{y+3}{8}, & -1 < y < 7 \\ 0, & \text{otherwise} \end{cases}$$

- 2) If the pdf of a RV  $X$  is  $f_X(x) = 2x$ ,  $0 < x < 1$ , find the pdf of  $y = e^{-x}$

Sol:

$$\text{Given } g(x) = y = e^{-x}$$

$$\frac{dy}{dx} = -e^{-x} = -y$$

$$f_Y(y) = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|} = \frac{2x}{| -y |} = \frac{2 \log y}{y}$$

$$\begin{aligned} &\text{Taking log on both sides} \\ &\log y = -x \\ &\text{or } x = -\log y \\ &= \log \frac{1}{y} \end{aligned}$$

$$0 < x < 1 \Rightarrow -1 < -x < 0 \Rightarrow e^{-1} < e^{-x} < e^0$$

as  $e^x$  is an increasing function of  $x$ .

$$\text{i.e., } \frac{1}{e} < y < 1$$

$$\therefore f_Y(y) = \frac{2}{y} \log \frac{1}{y}, \quad \frac{1}{e} < y < 1.$$

- 3) If  $x$  has an exponential distribution with parameter 1, find the pdf of  $y = \sqrt{x}$ .

Sol:

$$y = \sqrt{x} \Rightarrow y = \sqrt{x} \Rightarrow y^2 = x$$

$$2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$x$  follows an exponential distribution.

$$\text{pdf: } f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\therefore f(x) = e^{-x}$$

$$f_y(y) = \frac{f_x(x)}{\left| \frac{dy}{dx} \right|} = \frac{e^{-x^2}}{\left| \frac{1}{2y} \right|} = 2y e^{-y^2}$$

Since  $x > 0 \Rightarrow y > 0$

$$\therefore f_y(y) = 2y e^{-y^2}, y > 0$$

- 4) If  $x$  is uniformly distributed RV in  $(-\pi/2, \pi/2)$ , find the pdf of  $y = \tan x$ .

Sol:

$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \quad \because x = \tan^{-1} y$$

$x$  follows uniform distribution

$$\text{pdf: } f(x) = \begin{cases} \frac{1}{\pi}, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore f(x) = \frac{1}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{1}{\pi}$$

$$f_y(y) = f_x(x) \cdot \frac{dx}{dy} = \frac{1}{\pi} \cdot \frac{1}{1+y^2}$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow \tan(-\pi/2) < \tan x < \tan \pi/2 \Rightarrow -\infty < y < \infty$$

$$\therefore f_y(y) = \frac{1}{\pi(1+y^2)}, -\infty < y < \infty$$

- 6) Given the Rv  $X$  with pdf  $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$   
 find the pdf of  $y = 8x^3$ .

Sol:

$$g(x) = y = 8x^3 \quad \text{or} \quad x = \frac{1}{2}y^{1/3}$$

$8x^3 = g(x)$  is strictly increasing in  $(0, 1)$

$$\frac{dy}{dx} = 24x^2 = 24 \cdot \frac{1}{4}y^{2/3} = 6y^{2/3}$$

$$f_y(y) = \frac{2x}{6y^{2/3}} = \frac{y^{1/3}}{6y^{2/3}} = \frac{y^{-1/3}}{6}$$

$$0 < x < 1 \Rightarrow 0 < x^3 < 1 \Rightarrow 0 < 8x^3 < 8 \quad \text{or} \quad 0 < y < 8$$

$$\therefore f_y(y) = \begin{cases} \frac{1}{6y^{1/3}}, & 0 < y < 8 \\ 0, & \text{otherwise.} \end{cases}$$

- 7) If the continuous Rv  $X$  has pdf  $f_x(x) = \frac{2}{9}(x+1)$ , in  $-1 < x < 2$  and 0, elsewhere, find the pdf of  $y = x^2$ .

Sol:

The transformation function  $y = x^2$  is not monotonic in  $(-1, 2)$ . So we divide the interval into two parts.

i.e.,  $(-1, 1)$  and  $(1, 2)$

In  $(-1, 0)$ ,  $x = -\sqrt{y}$

$$f_y(y) = \frac{f_x(x)}{\left| \frac{dy}{dx} \right|} = \frac{1}{2\sqrt{y}} \cdot \frac{2}{9} [1 - \sqrt{y}]$$

$$\because y = x^2$$

$$\frac{dy}{dx} = 2x$$

In  $(0, 1)$ ,  $x = \sqrt{y}$

$$f_y(y) = \frac{f_x(x)}{2\sqrt{y}} = \frac{\frac{1}{2}}{2\sqrt{y}} \cdot \frac{2}{9} [1 + \sqrt{y}]$$

In  $(-1, 1)$ ,  $f_y(y) = \frac{1}{2\sqrt{y}} \left[ \frac{2}{9}(1 + \sqrt{y}) + \frac{2}{9}(1 - \sqrt{y}) \right]$

$$= \frac{2}{9\sqrt{y}}$$

In  $-1 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 < y < 1$

In  $1 < x < 2$ ,  $x = \sqrt{y}$

$$\therefore f_y(y) = \frac{2}{9} [1 + \sqrt{y}] \cdot \frac{1}{2\sqrt{y}} = \frac{1}{9} \left[ 1 + \frac{1}{\sqrt{y}} \right]$$

In  $1 < x < 2$ ,  $1 < x^2 < 4$  or  $1 < y < 4$

$$\therefore f_y(y) = \begin{cases} \frac{2}{9\sqrt{y}}, & 0 < y < 1 \\ \frac{1}{9} \left[ 1 + \frac{1}{\sqrt{y}} \right], & 1 < y < 4. \end{cases}$$

$\rightarrow x \rightarrow$

## Tchebycheff Inequality:

Introduction: If we know the prob. distribution of a r.v  $x$ , we may compute  $E(x)$  and  $\text{Var}(x)$ . Conversely, if  $E(x)$  and  $\text{Var}(x)$  are known, we cannot construct the prob. distribution of  $x$  and hence compute quantities such as  $P\{|x - E(x)| \leq k\}$ . Although we cannot evaluate such probabilities from a knowledge of  $E(x)$  and  $\text{Var}(x)$ , several approximation techniques have been developed to upper and/or lower bounds to such probabilities. The most important of such techniques is Tchebycheff inequality.

Statement: If  $x$  is a RV with  $E(x) = \mu$  and  $\text{Var}(x) = \sigma^2$ , then  $P\{|x - \mu| > c\} \leq \frac{\sigma^2}{c^2}$  where  $c > 0$ .

$$\text{or } P\{|x - \mu| \leq c\} \geq 1 - \frac{\sigma^2}{c^2} \text{ where } c > 0.$$

Alternative forms: If we put  $c = k\sigma$ , where  $k > 0$ , then Tchebycheff inequality takes the form

$$P\left\{\left|\frac{x-\mu}{\sigma}\right| > k\right\} \leq \frac{1}{k^2}$$

$$\text{or } P\left\{\left|\frac{x-\mu}{\sigma}\right| \leq k\right\} \geq 1 - \frac{1}{k^2}$$

- 1) A r.v.  $x$  has mean  $\mu=12$  and variance  $\sigma^2=9$  and an unknown prob. distribution. Find  $P(6 < x < 18)$ .

Sol:

By Tchebycheff's inequality

$$P\{|x-\mu| \leq c\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P\{-c \leq x-\mu \leq c\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P\{\mu-c \leq x \leq \mu+c\} \geq 1 - \frac{\sigma^2}{c^2}$$

Taking Given  $\mu=12$ ,  $\sigma^2=9$

$$P\{12-c < x < 12+c\} \geq 1 - \frac{9}{c^2}$$

putting,  $c=6$ , we get

$$P\{6 < x < 18\} \geq 1 - \frac{9}{36}$$

$$\Rightarrow P\{6 < x < 18\} \geq \frac{3}{4}$$

- 2) If the RV  $x$  is uniformly distributed over  $(-\sqrt{3}, \sqrt{3})$ , compare  $P\{|x-\mu| \geq \frac{3\sqrt{2}}{2}\}$  and compare it with the upper bound obtained by Tchebycheff's inequality.

Sol:

Uniform distribution}:  $f(x) = \frac{1}{b-a}$ ,  $a < x < b$   
pdf:

$$\text{Mean} = \frac{1}{2}(b+a)$$

$$\text{Variance} = \frac{1}{12}(b-a)^2$$

$$\text{Mean} = \mu = \frac{\sqrt{3} - \sqrt{3}}{2} = 0, \text{ Variance} = \sigma^2 = \frac{1}{12} (\sqrt{3} + \sqrt{3})^2 = 1$$

$$\therefore \mu = 0, \sigma^2 = 1$$

$$\begin{aligned}\therefore P\left\{|X-\mu| \geq \frac{3\sigma}{2}\right\} &= P\left\{|X| \geq \frac{3}{2}\right\} \\ &= 1 - P\left\{-\frac{3}{2} \leq X \leq \frac{3}{2}\right\} \\ &= 1 - \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{2\sqrt{3}} dx \quad \because f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b \\ &\quad -\frac{3}{2} \quad \frac{3}{2} \\ &= 1 - \frac{1}{2\sqrt{3}} \left[ x \right]_{-\frac{3}{2}}^{\frac{3}{2}} \\ &= 1 - \frac{1}{2\sqrt{3}} \left[ \frac{3}{2} + \frac{3}{2} \right] \\ &= 1 - \frac{\sqrt{3}}{2} = 0.134.\end{aligned}$$

By Tchebycheff's inequality

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\therefore P\left\{|X-\mu| \geq \frac{3}{2}\sigma\right\} \leq \frac{4}{9} = 0.444$$

which is a poor upper bound.

3) Can we find a RV  $X$  for which  $P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = 0.6$ ?

Sol:

$$P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} = P\{|X - \mu| \leq 2\sigma\}$$

By Tchebycheff's inequality,  $P\{|X - \mu| \leq c\} \geq 1 - \frac{\sigma^2}{c^2}$

$$\text{or } P\left\{\left|\frac{X-\mu}{\sigma}\right| \leq \frac{2}{\sigma}\right\} \geq 1 - \frac{1}{\frac{4}{\sigma^2}} = 1 - \frac{\sigma^2}{4}$$

$$\Rightarrow P\{\mu - 2\sigma \leq X \leq \mu + 2\sigma\} \geq 1 - \frac{1}{\frac{4}{\sigma^2}} \quad \because \sigma = \sqrt{3} \quad \geq 1 - \frac{1}{\frac{4}{3}} = 0.75$$

$\therefore$  there does not exist a RV  $X$  satisfying the given condition.

A discrete RV  $X$  takes the values  $-1, 0, 1$  with probabilities  $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$  respectively. Evaluate  $P\{|X-\mu| \geq 2\sigma\}$  and compare it with the upper bound given by Tchebycheff's inequality.

Sol:

Given

X	-1	0	1
P(X=x)	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{8}$

$$E(X) = (-1 \times \frac{1}{8}) + (0 \times \frac{3}{4}) + (1 \times \frac{1}{8}) = 0$$

$$E(X^2) = (-1)^2 \times \frac{1}{8} + (0^2 \times \frac{3}{4}) + (1^2 \times \frac{1}{8}) = \frac{2}{8} = \frac{1}{4}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - 0 = \frac{1}{4}$$

$$\begin{aligned} \Rightarrow P\{|X-\mu| \geq 2\sigma\} &= P\{|X-0| \geq 2 \cdot \frac{1}{2}\} \\ &= P\{|X| \geq 1\} \\ &\leq 1 - P\{|X| < 1\} \\ &= 1 - P\{-1 \leq X \leq 1\} = 1 - P\{X=0\} \\ &= 1 - \frac{3}{4} = \frac{1}{4} \end{aligned}$$

By Tchebycheff's inequality

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\Rightarrow P\{|X-\mu| \geq 2\sigma\} \leq \frac{1}{2^2} = \frac{1}{4}$$

The two values coincide.

5.) A fair die is tossed 720 times. Use Tchebycheff's inequality to find a lower bound for the prob. of getting 100 to 140 sixes.

Sol: Let  $X$  be the number of sixes

$$p = P\{\text{getting '6' in a single toss}\} = \frac{1}{6}$$

$$q = \frac{5}{6} \text{ and } n = 720$$

$X$  follows a binomial distribution

$$\therefore \text{Mean} = np \text{ and Variance} = npq$$

$$\Rightarrow \text{Mean} = 720 \times \frac{1}{6} \text{ and Variance} = 720 \times \frac{1}{6} \times \frac{5}{6} = 100$$

$$\therefore \mu = 120 \text{ and } \sigma = 10.$$

By Tchebycheff's inequality

$$P\{|X - \mu| \leq k\sigma\} \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow P\{|X - 120| \leq 10k\} \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow P\{-10k \leq X - 120 \leq 10k\} \geq 1 - \frac{1}{k^2}$$

$$\Rightarrow P\{120 - 10k \leq X \leq 120 + 10k\} \geq 1 - \frac{1}{k^2}$$

Putting  $k=2$ , we get

$$P\{100 \leq X \leq 140\} \geq 1 - \frac{1}{4} = \frac{3}{4}$$

$\therefore$  the lower bound for the prob. = 0.75

G) Use Tchebycheff's inequality to find how many times a fair coin must be tossed in order that the prob. that <sup>the</sup> ratio of the number of heads to the number of tosses will lie between 0.45 and 0.55 will be at least 0.95.

Sol: Given  $P\left\{0.45 \leq \frac{x}{n} \leq 0.55\right\} \geq 0.95 \rightarrow ①$

Let  $x$  be the number of heads

Then  $x$  follows a binomial distribution.

i.e.,  $x \sim B(n, p)$ , where  $p = q = \frac{1}{2}$

$$\therefore \frac{x}{n} \sim B\left(\frac{1}{2}, \frac{1}{2\sqrt{n}}\right)$$

By Tchebycheff's inequality

$$P\left\{\left|\frac{x}{n} - \frac{1}{2}\right| \leq c\right\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P\left\{0.5 - c \leq \frac{x}{n} \leq 0.5 + c\right\} \geq 1 - \frac{1}{4nc^2} \quad [\because \sigma^2 = \frac{1}{4n}]$$

Putting  $c = 0.05$ , we get

$$P\left\{0.45 \leq \frac{x}{n} \leq 0.55\right\} \geq 1 - \frac{100}{n} \rightarrow ②$$

Comparing ① and ②, we get

$$1 - \frac{100}{n} = 0.95$$

$$\Rightarrow 1 - 0.95 = \frac{100}{n}$$

$$\Rightarrow n = \frac{100}{0.05}$$

$$\boxed{n = 2000}$$

7) A RV  $X$  is exponentially distributed with parameter

1. Use Tchebycheff's inequality to show that

$P(-1 \leq X \leq 3) > 3/4$ . Find the actual probability also.

Sol:

For Exponential distribution:

$$\text{pdf: } f(x) = \lambda e^{-\lambda x}, x \geq 0$$

$$\text{Mean} = \frac{1}{\lambda} \text{ and Variance} = \frac{1}{\lambda^2}$$

$$\text{Here } \lambda = 1$$

$$\text{r.e., } \mu = 1 \text{ and } \sigma = 1$$

By Tchebycheff's inequality,

$$P\{|X - \mu| \leq 2\} \geq 1 - \frac{1}{4}$$

$$\Rightarrow P\{-1 \leq X \leq 3\} > 3/4$$

$$\begin{aligned} \therefore P\{|X - \mu| \leq 3\} \\ = P\{|X - \mu| \leq k\sigma\} \\ \geq 1 - \frac{1}{k^2} \end{aligned}$$

To find Actual probability:

$$P(-1 \leq X \leq 3) = \int_{-1}^3 e^{-x} dx$$

$$= (-e^{-x}) \Big|_{-1}^3 = 1 - e^{-3} = 0.9502$$