

FIG. 1. Spherical coordinates.

## I. INTRODUCTION

### A. A viscous drop in a uniform streaming flow

The drop is assumed to be located at the center of a spherical coordinate system, with the meridian angle  $\theta \in [0, \pi]$  and the azimuthal angle  $\phi \in [0, 2\pi]$  defined in figure (1). For a drop in a uniform flow streaming along the  $x_3$ -axis ( $\theta = 0$ ), the velocity field (both inside and outside the drop) may be assumed to be independent of  $\phi$  with an axi-symmetric drop shape close to a sphere of radius  $r_0$ . Denoting the velocity outside the viscous drop as  $\mathbf{U} = (U_R(R, \theta), U_\theta(R, \theta))$ , the axi-symmetric Stokes equations in spherical coordinates are

$$-\frac{\partial P}{\partial R} + \mu_e \left( \nabla^2 U_R - \frac{2U_R}{R^2} - \frac{2}{R^2} \frac{\partial U_\theta}{\partial \theta} - \frac{2}{R^2} U_\theta \cot \theta \right) = 0, \quad (1)$$

$$-\frac{1}{R} \frac{\partial P}{\partial \theta} + \mu_e \left( \nabla^2 U_\theta + \frac{2}{R^2} \frac{\partial U_R}{\partial \theta} - \frac{U_\theta}{R^2 \sin^2 \theta} \right) = 0, \quad (2)$$

$$\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 U_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (U_\theta \sin \theta) = 0, \quad (3)$$

with the Laplace operator

$$\nabla^2 = \frac{1}{R^2} \frac{\partial}{\partial R} \left( R^2 \frac{\partial}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right). \quad (4)$$

In the far field the external velocity  $U_R \rightarrow -U_0 \cos \theta$  and  $U_\theta \rightarrow U_0 \sin \theta$  for  $R \rightarrow \infty$ . Thus we seek a solution for the velocity in the external fluid of the form

$$U_R(R, \theta) = F(R) \cos \theta, \quad (5)$$

$$U_\theta(R, \theta) = G(R) \sin \theta, \quad (6)$$

$$P(R, \theta) = H(R) \cos \theta. \quad (7)$$

Substituting the above functions into the governing equations we obtain

$$2F + 2G + RF' = 0, \quad (8)$$

$$-4\mu_e F - 4\mu_e G + R(2\mu_e F' - RH' + R\mu_e F'') = 0, \quad (9)$$

$$-2\mu_e F - 2\mu_e G + R(H + 2\mu_e G' + R\mu_e G'') = 0, \quad (10)$$

which can be combined into an equation for  $F$

$$R^4 F^{(4)} + 8R^3 F^{(3)} + 8R^2 F^{(2)} - 8RF' = 0. \quad (11)$$

The general solution for  $F$  is

$$F(R) = \frac{A_1}{R^3} + \frac{A_2}{R} + A_3 + A_4 R^2 = \frac{A_1}{R^3} + \frac{A_2}{R} - U_0 \quad (12)$$

due to the far field boundary condition. The other two functions are related to  $F$  and its derivatives as

$$G(R) = \frac{A_1}{2R^3} - \frac{A_2}{2R} + U_0, \quad H(R) = \frac{A_2}{R^2} \mu_e. \quad (13)$$

If the interior fluid is a Newtonian viscous fluid with a flow field denoted as  $\mathbf{u} = (u_r(r, \theta), u_\theta(r, \theta))$ , with

$$u_r(r, \theta) = f(r) \cos \theta, \quad (14)$$

$$u_\theta(r, \theta) = g(r) \sin \theta, \quad (15)$$

$$p(r, \theta) = h(r) \cos \theta, \quad (16)$$

where

$$f(r) = a_3 + a_4 r^2, \quad (17)$$

$$g(r) = -a_3 - 2a_4 r^2, \quad (18)$$

$$h(r) = 10\mu_i a_4 r^2. \quad (19)$$

For the viscous drop to maintain the fixed spherical shape the surface tension must be sufficiently large (Capillary number  $Ca \ll 1$ ) so the normal velocity at the fluid interface is zero. Another consequence of large Capillary number is that the normal stress is dominated by the pressure jump balancing the capillary pressure. Thus the boundary conditions are

$$U_\theta(r_0, \theta) = u_\theta(r_0, \theta), \quad U_R(r_0, \theta) = u_r(r_0, \theta) = 0, \quad \mu_e e_{R\theta}(r_0, \theta) = \mu_i e_{r\theta}(r_0, \theta). \quad (20)$$

We found that

$$A_1 = -\frac{r_0^3 U_0 \mu_i}{2(\mu_e + \mu_i)}, \quad (21)$$

$$A_2 = -\frac{r_0 U_0 (2\mu_e + 3\mu_i)}{2(\mu_e + \mu_i)}, \quad (22)$$

$$a_3 = \frac{U_0 \mu_e}{2(\mu_e + \mu_i)}, \quad (23)$$

$$a_4 = -\frac{U_0 \mu_e}{2r_0^2 (\mu_e + \mu_i)}. \quad (24)$$

Next we consider small deformation of order  $Ca$  with  $Ca \ll 1$ . In this case the shape of the nearly spherical drop is in the form

$$\rho(\mathbf{x}, t) = r - r_0(1 + CaB(\mathbf{x}, t)) = 0, \quad (25)$$

The normal velocity on the fluid interface is no longer zero, and the normal stress balance has to be taken into account. The corresponding boundary conditions are

$$U_\theta(r_0, \theta) = u_\theta(r_0, \theta), \quad U_R(r_0, \theta) = u_r(r_0, \theta) = Ca \frac{db}{dt}, \quad (26)$$

$$\mu_e e_{R\theta}(r_0, \theta) = \mu_i e_{r\theta}(r_0, \theta), \quad \mu_e e_{RR}(r_0, \theta) - \mu_i e_{rr}(r_0, \theta) = -\nabla_s^2 B(\mathbf{x}), \quad (27)$$

where  $B(\mathbf{x}) = b(t) \cos(\theta)$ . We found that

$$A_1 = \frac{2r_0^4 \mu_i b(t)}{3\mu_e(2\mu_e + 3\mu_i)}, \quad (28)$$

$$A_2 = -\frac{2r_0^2 b(t)}{3\mu_e}, \quad (29)$$

$$a_3 = \frac{2b(t)}{3r_0(2\mu_e + 3\mu_i)}, \quad (30)$$

$$a_4 = -\frac{3U_0\mu_e(2\mu_e + 3\mu_i) + 2r_0(3\mu_e + 2\mu_i)b(t)}{3\mu_e(2\mu_e + 3\mu_i)}, \quad (31)$$

with  $b(t)$  satisfying the equation

$$Cab' = -\frac{4r_0(\mu_e + \mu_i)}{3\mu_e(2\mu_e + 3\mu_i)}b - U_0 \quad (32)$$

that gives a equilibrium solution

$$b_{eq} = -\frac{3\mu_e(2\mu_e + 3\mu_i)U_0}{4r_0(\mu_e + \mu_i)}. \quad (33)$$

We first note that the deformation amplitude  $b(t)$  evolves monotonically toward the equilibrium  $b_{eq}$ . Furthermore, equations 28-31 evolve towards the coefficients in equations 21-24. The same methodology applies to a linear flow as shown in G. Leal's textbook, and similarly the small-deformation amplitude evolves monotonically towards equilibrium. Fig. (2)(a) shows the streamlines for three values of the viscosity ratio (interior to exterior). It is also good to keep in mind the following flow patterns around a spherical particle or drop. Fig. (2)(b) shows the streamlines of a vesicle going through osmophoresis at a constant velocity  $U$  (from Anderson). Note the drastic different stream lines around the spherical drop.

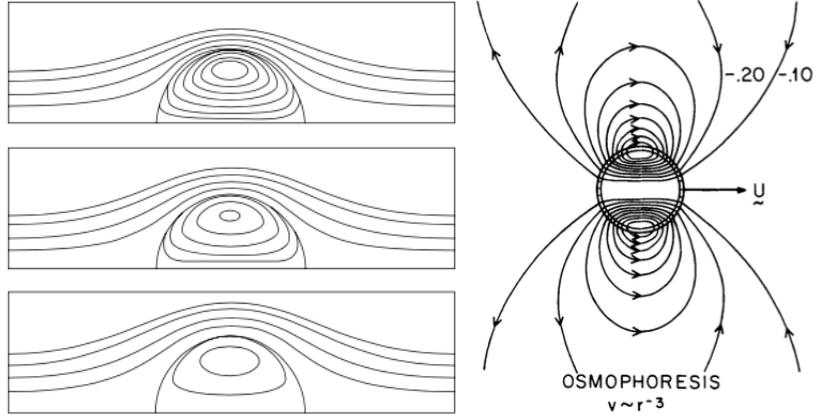


FIG. 2. Left: Streamlines around a viscous drop in a streaming flow (uniform flow).  $\mu_r = 0.1, 1$ , and  $10$  from top to bottom. Right: Streamlines around a viscous drop in osmophoresis.

### B. Darcy model for the fibrous network inside a spherical drop

For flow inside the porous particle the Darcy equation gives the fluid flow  $\mathbf{u}$  inside the spherical particle

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta = -\frac{k}{\mu} \nabla p_i, \quad \nabla \cdot \mathbf{u} = 0, \quad (34)$$

$$u_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi_i}{\partial \theta}, \quad u_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi_i}{\partial r}, \quad (35)$$

where  $\psi_i$  is the streamfunction for the incompressible axi-symmetric flow field inside the porous particle. For flow outside the porous particle the Stokes equation gives the uniform streaming flow field driven by a constant pressure gradient

$$-\nabla P_e + \mu_e \nabla^2 \mathbf{U} = 0, \quad \nabla \cdot \mathbf{U} = 0, \quad \text{with } \mathbf{U} \rightarrow U \mathbf{e}_3 \text{ at } r \rightarrow \infty \quad (36)$$

$$U_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi_e}{\partial \theta}, \quad U_\theta = \frac{1}{r \sin \theta} \frac{\partial \psi_e}{\partial r}, \quad (37)$$

where  $\psi_e$  is the streamfunction for the incompressible axi-symmetric flow field outside the porous particle. The boundary conditions for a Darcy porous drop are (1) the continuity of the normal velocity components on the interface, while (2) the tangential velocities are allowed to slip by an amount proportional to the tangential stress exerted on the surface of the sphere by the external fluid, and finally (3) the continuity of pressure on the porous sphere surface:

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a, \quad (38)$$

$$(\mathbf{U} - \mathbf{u}) \cdot \mathbf{t} = \beta \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t}, \quad r = a, \quad (39)$$

$$P_e = p_i, \quad r = a. \quad (40)$$

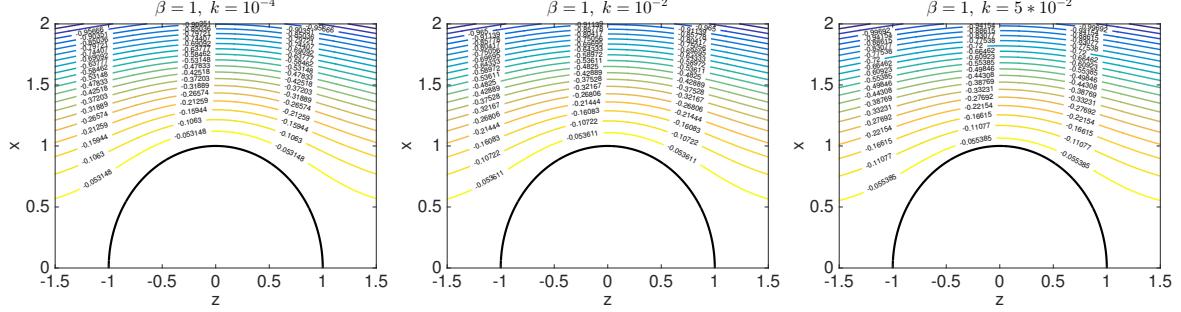


FIG. 3. Streamlines around a nearly spherical Darcy drop with  $\mu_e = 1$ ,  $\mu = 1$  and  $\beta = 1$ . From left to right  $k = 10^{-4}$ ,  $10^{-2}$  and  $5 \times 10^{-2}$ .

The external flow is solved to be

$$U_r = \left( U - \frac{C_2}{r} - \frac{C_1}{3r^3} \right) \cos \theta, \quad (41)$$

$$U_\theta = \left( -U + \frac{C_2}{2r} - \frac{C_1}{6r^3} \right) \sin \theta, \quad (42)$$

$$P_e = \frac{C_2 \mu_e}{r^2} \cos \theta. \quad (43)$$

The total force exerted on the porous particle by the external fluid flow can be calculated as

$$\mathbf{F} = \int \mathbf{T}_e \mathbf{n} a^2 \sin \theta d\theta d\phi, \quad (44)$$

$$= \frac{12U\pi\mu\mu_e a(\beta + 2\mu_e)}{6\mu_e(\mu + k\mu_e) + \beta(2\mu + 3k\mu_e)} \mathbf{e}_z. \quad (45)$$

In the limit of small  $k$ ,

$$\mathbf{F} \approx \left[ \frac{6U\pi\mu_e a(\beta + 2\mu_e)}{\beta + 3\mu_e} - \frac{9(U\pi\mu_e^2(\beta + 2\mu_e)^2 a)}{\mu(\beta + 3\mu_e)^2} k + \mathcal{O}(k^2) \right] \mathbf{e}_z. \quad (46)$$

In the limit of large  $\beta$ ,

$$\mathbf{F} \approx \left[ \frac{12U\pi\mu\mu_e a}{2\mu + 3k\mu_e} - \frac{24(U\pi\mu_e^2 a)}{(2\mu + 3k\mu_e)^2} \frac{1}{\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \right] \mathbf{e}_z. \quad (47)$$

Inside the porous spherical particle the pressure is

$$p_i = -a_0 r \cos \theta, \quad (48)$$

$$a_0 = \frac{3U\mu\mu_e(\beta + 2\mu_e)}{6\mu_e(\mu + k\mu_e) + \beta(2\mu + 3k\mu_e)}, \quad (49)$$

with  $\mathbf{u} = -k/\mu \nabla p_i$ . Figure 3 and figure 4 show the  $k$ -dependence of the streamlines. We observe that as  $k$  increases the streamline pattern inside the drop transitions from a non-permeable drop to a permeable drop.

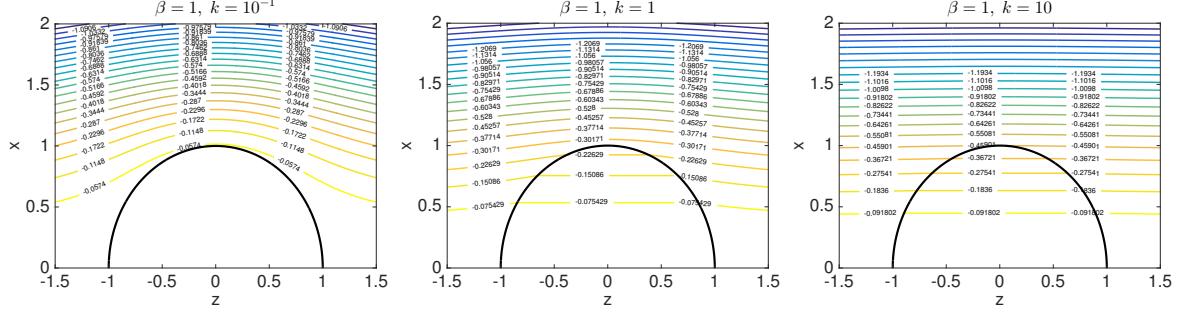


FIG. 4. Streamlines around a nearly spherical Darcy drop with  $\mu_e = 1$ ,  $\mu = 1$  and  $\beta = 1$ . From left to right  $k = 0.1, 1$  and  $10$ .

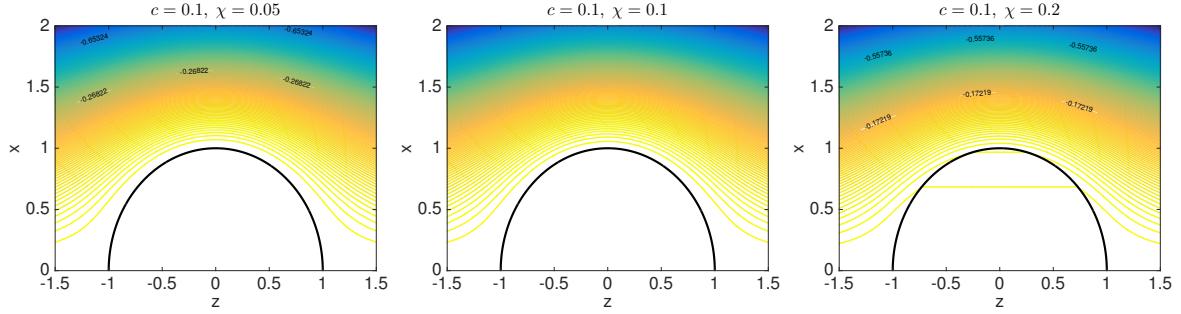


FIG. 5. Streamlines around a nearly spherical Darcy drop with  $\mu_e = 1$ ,  $\mu = 1$  and  $c = 0.1$ . From left to right  $\chi = 0.05, 0.1$  and  $0.2$ .

Bars and Worster (JFM 2006) found that homogenization over the pore length scales gives rise to boundary conditions that are consistent with Beaver and Joseph's boundary conditions (equation 38-40):

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a(1 - \delta), \quad (50)$$

$$\mathbf{t} \cdot \mathbf{U} = \mathbf{t} \cdot \mathbf{u}, \quad r = a(1 - \delta), \quad (51)$$

$$P_e = p_i, \quad r = a(1 - \delta), \quad (52)$$

where the slip length  $\delta$  is computed as a function of the fluid volume fraction  $\chi_f$  if the permeability  $k$  is assumed a certain dependence on  $\chi_f$ . For example,

$$k(\chi_f) = k_0 \frac{\chi_f^3}{(1 - \chi_f)^2}, \quad (53)$$

$$\delta \equiv c \sqrt{\frac{k(\chi_f)}{\chi_f}} = ck_0^{1/2} \frac{\chi_f}{1 - \chi_f}, \quad (54)$$

where  $k_0$  is a reference permeability and  $c$  is an undetermined parameter that is usually assumed to be of order one. In the following we set  $k_0 = 1$  and adjust the slip length by changing the value of  $c$ . Figures 5-6 show the streamlines for various values of  $\chi_f$ .

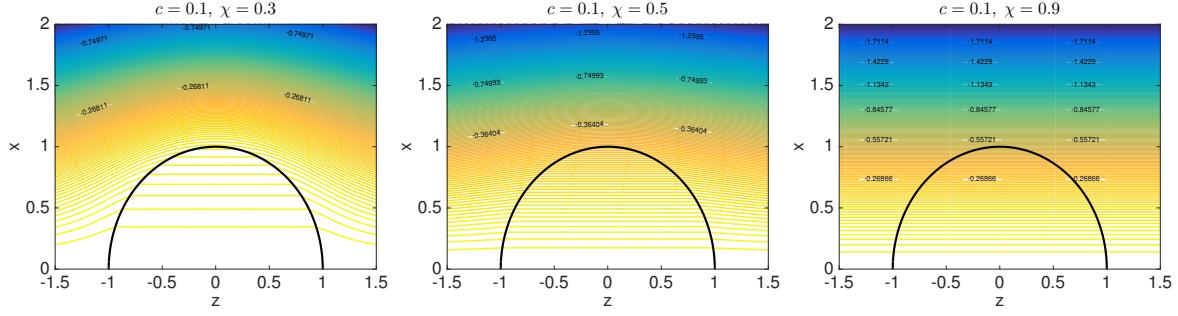


FIG. 6. Streamlines around a nearly spherical Darcy drop with  $\mu_e = 1$ ,  $\mu = 1$  and  $c = 0.1$ . From left to right  $\chi = 0.3, 0.5$  and  $0.9$ .

### C. A Brinkman drop in a uniform streaming flow

We now extend the previous calculations to a Brinkman drop in a uniform flow in the limit where  $\mathbf{v}_1 = 0$ . Denoting the velocity outside the viscous drop as  $\mathbf{U} = (U_R(R, \theta), U_\theta(R, \theta))$ , the axi-symmetric Stokes equations in spherical coordinates are

$$-\frac{\partial P}{\partial R} + \left( \nabla^2 U_R - \frac{2U_R}{R^2} - \frac{2}{R^2} \frac{\partial U_\theta}{\partial \theta} - \frac{2}{R^2} U_\theta \cot \theta \right) = 0, \quad (55)$$

$$-\frac{1}{R} \frac{\partial P}{\partial \theta} + \left( \nabla^2 U_\theta + \frac{2}{R^2} \frac{\partial U_R}{\partial \theta} - \frac{U_\theta}{R^2 \sin^2 \theta} \right) = 0, \quad (56)$$

$$\frac{1}{R^2} \frac{\partial}{\partial R} (R^2 U_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (U_\theta \sin \theta) = 0. \quad (57)$$

For the velocity inside  $\mathbf{u} = (u_r(r, \theta), u_\theta(r, \theta))$  the axi-symmetric Brinkman equations are

$$-\frac{\partial p}{\partial r} + \mu_r \left( \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{2}{r^2} u_\theta \cot \theta \right) - \lambda u_r = 0, \quad (58)$$

$$-\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu_r \left( \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} \right) - \lambda u_\theta = 0, \quad (59)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) = 0. \quad (60)$$

Again we seek a solution for the velocity in the external fluid of the form

$$U_R(R, \theta) = F(R) \cos \theta, \quad U_\theta(R, \theta) = G(R) \sin \theta, \quad P(R, \theta) = H(R) \cos \theta, \quad (61)$$

$$u_r(r, \theta) = f(r) \cos \theta, \quad u_\theta(r, \theta) = g(r) \sin \theta, \quad p(r, \theta) = h(r) \cos \theta. \quad (62)$$

The general solutions for  $F$ ,  $G$  and  $H$  are

$$F(R) = \frac{A_1}{R^3} + \frac{A_2}{R} - U_0, \quad (63)$$

$$G(R) = \frac{A_1}{2R^3} - \frac{A_2}{2R} + U_0, \quad (64)$$

$$H(R) = \frac{A_2}{R^2}. \quad (65)$$

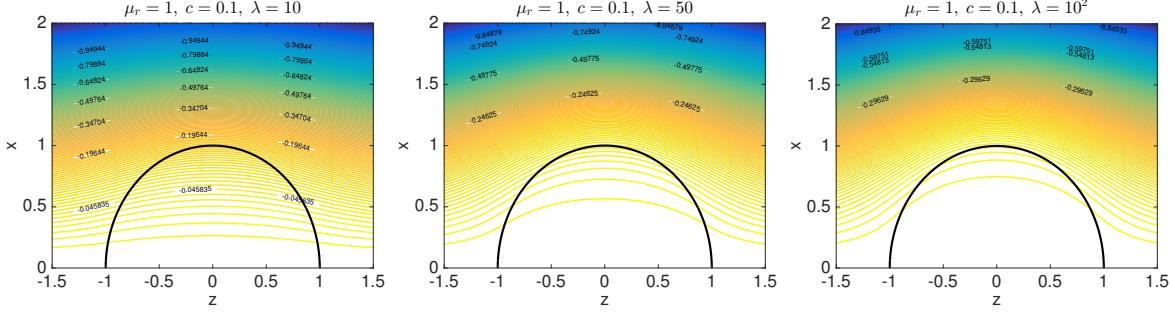


FIG. 7. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$  and  $c = 0.1$ . From left to right  $\lambda = 10, 50$  and  $100$ .

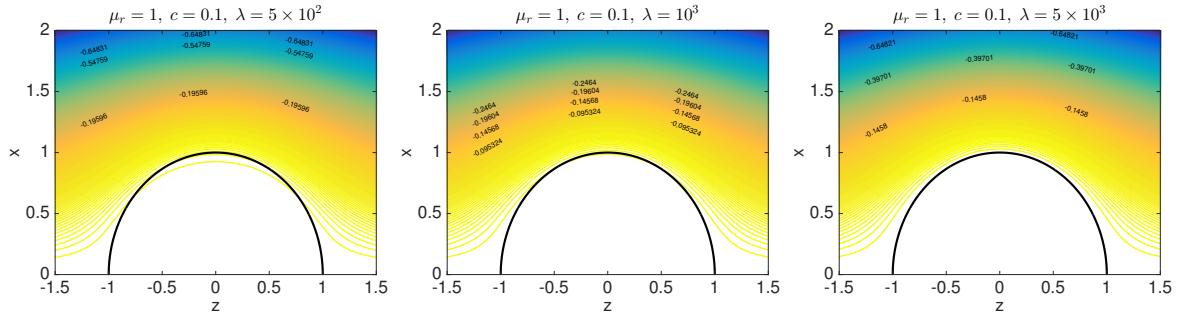


FIG. 8. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$  and  $c = 0.1$ . From left to right  $\lambda = 500, 10^3$  and  $5 \times 10^3$ .

The general solution for  $f$  is

$$f(r) = a_4 + a_3 \frac{-\sqrt{\frac{\lambda}{\mu_r}} r \cosh \left( \sqrt{\frac{\lambda}{\mu_r}} r \right) + \sinh \left( \sqrt{\frac{\lambda}{\mu_r}} r \right)}{r^3}. \quad (66)$$

Based on Bars and Worster (JFM 2006) here we adopt the following boundary conditions for the Brinkman drop immersed in a Stokes flow:

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a(1 - \delta), \quad (67)$$

$$\mathbf{t} \cdot \mathbf{U} = \mathbf{t} \cdot \mathbf{u}, \quad r = a(1 - \delta), \quad (68)$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{n}, \quad r = a(1 - \delta), \quad (69)$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{t}, \quad r = a(1 - \delta), \quad (70)$$

with the slip length defined as

$$\delta \equiv c \sqrt{\frac{\mu_r}{\lambda}}. \quad (71)$$

Recently Angot, Boyeau and Ochoa-Tapia (PRE 2017) conducted asymptotic analysis on the viscous fluid flow at the interface between a fluid and a porous layer described by a

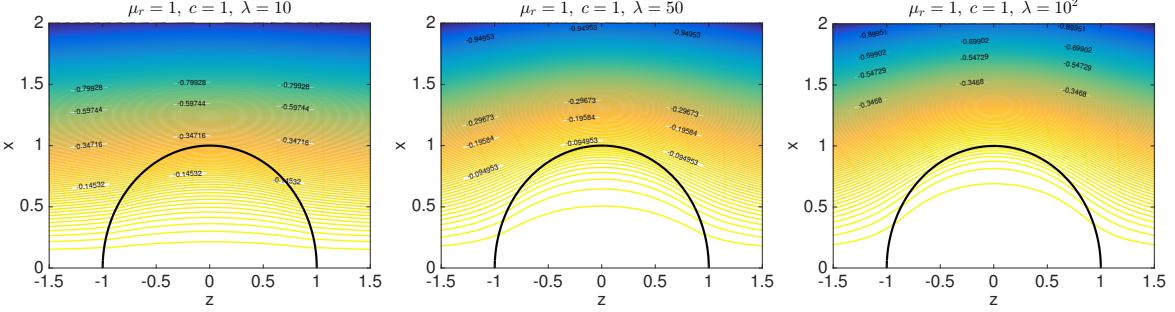


FIG. 9. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$  and  $c = 1$ . From left to right  $\lambda = 10, 50$  and  $100$ .

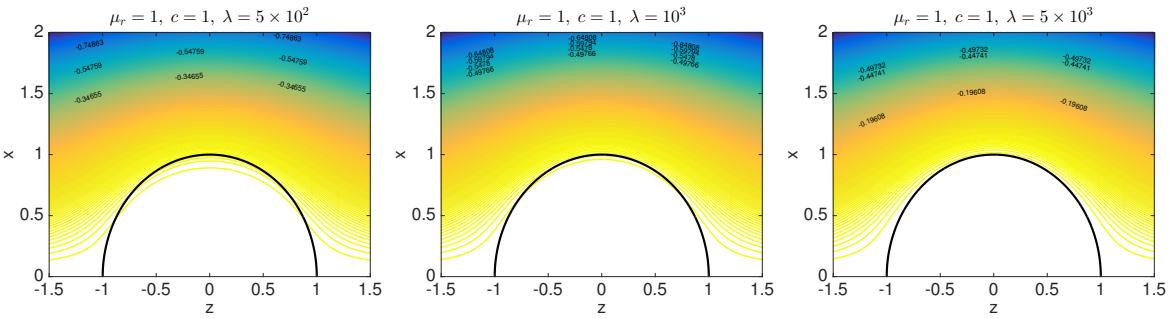


FIG. 10. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$  and  $c = 1$ . From left to right  $\lambda = 500, 10^3$  and  $5 \times 10^3$ .

Darcy-Brinkman medium. They derived the following boundary conditions at the interface

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a, \quad (72)$$

$$\mathbf{t} \cdot (\mathbf{U} - \mathbf{u}) = \frac{\beta}{2} (\mathbf{t} T_e \mathbf{n} + \mathbf{t} T_i \mathbf{n}), \quad r = a, \quad (73)$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{n}, \quad r = a, \quad (74)$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{t}, \quad r = a, \quad (75)$$

with the coefficient  $\beta$  related to the permeability  $1/\lambda$ , viscosity  $\mu_i$  and the boundary layer thickness  $d$  (see Angot *et al.*).

## II. PROBLEM FORMULATION I

The deformable soft porous material immersed in fluid has been modeled by MacMinn Dufresne and Wetlaufer (Physical Review Applied, 2016). Starting from the definition of total volume flux  $\mathbf{q}$  as

$$\mathbf{q} \equiv \phi_f \mathbf{v}_f + (1 - \phi_f) \mathbf{v}_s, \quad (76)$$

where  $\phi_f$  is the local volume fraction of fluid ( $1 - \phi_f$  is the local volume fraction of solid).  $\mathbf{v}_f$  is the fluid velocity field and  $\mathbf{v}_s$  is the solid velocity field. The volume conservation is

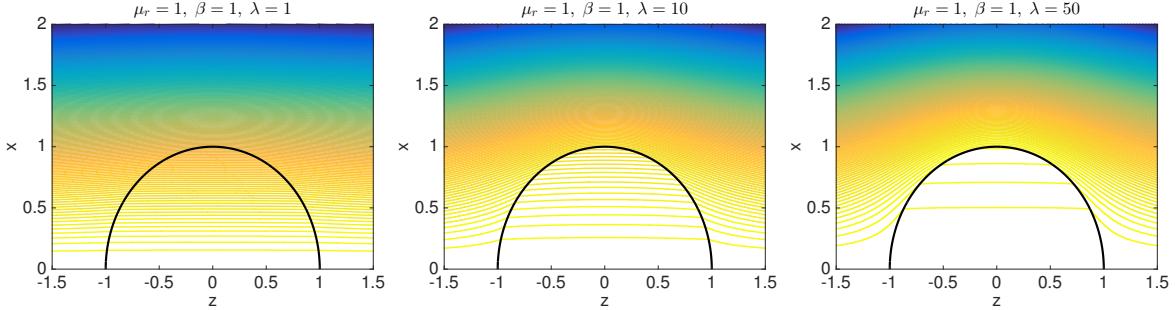


FIG. 11. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$ , From left to right  $\lambda = 1, 10$  and  $50$ .

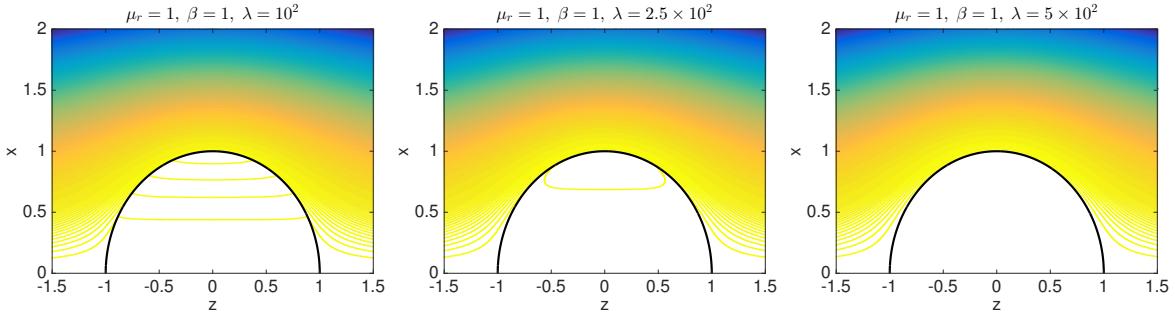


FIG. 12. Streamlines around a spherical Brinkman drop with  $\mu_r = 1$ . From left to right  $\lambda = 100, 250$  and  $500$ .

given below in terms of the porosity  $\phi_f$  as

$$\frac{\partial \phi_f}{\partial t} + \nabla \cdot \left[ \phi_f \mathbf{q} - (1 - \phi_f) \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}) \right] = 0, \quad (77)$$

$$\nabla \cdot \mathbf{q} = 0, \quad (78)$$

$$\mathbf{v}_f = \mathbf{q} - \left( \frac{1 - \phi_f}{\phi_f} \right) \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}), \quad (79)$$

$$\mathbf{v}_q = \mathbf{q} + \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}). \quad (80)$$

In the following we will ignore the gravitational force by setting  $\rho_f \mathbf{g} = 0$ . In the absence of the external force, the pressure gradient is balanced by stress in the strain of the porous medium as

$$\nabla p = \nabla \cdot \sigma', \quad (81)$$

$$\sigma' \equiv \boldsymbol{\Lambda} \text{tr}(\mathbf{H}) \mathbf{I} + (\mathcal{M} - \boldsymbol{\Lambda}) \mathbf{H}, \quad (82)$$

$$\mathbf{H} \equiv \frac{1}{2} \ln (\mathbf{F} \mathbf{F}^T), \quad (83)$$

where the deformation-gradient tensor  $\mathbf{F}$  is defined as

$$\mathbf{F}^{-1} = \nabla \mathbf{X} = \mathbf{I} - \nabla \mathbf{u}_s. \quad (84)$$

## A. Linear Poroelasticity

When the strain of the porous medium is small

$$\frac{\phi_f - \phi_{f,0}}{1 - \phi_{f,0}} \approx \nabla \cdot \mathbf{u}_s \sim \epsilon \ll 1. \quad (85)$$

In this limit, the porosity satisfies the well-known linear poroelastic flow equation:

$$\frac{\partial \phi_f}{\partial t} - \nabla \cdot \left[ (1 - \phi_{f,0}) \frac{k_0}{\mu} \nabla p \right] = 0, \quad (86)$$

where  $k_0 = k(\phi_{f,0})$  is the relaxed or undeformed permeability. In the small-strain limit, the linear elasticity applies and we have

$$\sigma' = \Lambda \text{tr}(\varepsilon) \mathbf{I} + (\mathcal{M} - \Lambda) \varepsilon, \quad (87)$$

$$\varepsilon = \frac{1}{2} \left[ \nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T \right], \quad (88)$$

and the pressure gradient is equal to the divergence of the effective solid stress:

$$\nabla p = \nabla \cdot \sigma'. \quad (89)$$

Following the scaling in MacMinn *et al.*, the resultant dimensionless equations for the linear poroelasticity problem are (dropping the prime in  $\sigma'$ )

$$\frac{\partial \phi_f}{\partial t} - \nabla \cdot [(1 - \phi_{f,0}) \nabla p] = 0, \quad (90)$$

$$\nabla p = \nabla \cdot \sigma, \quad (91)$$

$$\sigma = \Lambda \text{tr}(\varepsilon) \mathbf{I} + (1 - \Lambda) \varepsilon, \quad (92)$$

$$\varepsilon = \frac{1}{2} \left[ \nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T \right], \quad (93)$$

$$\mathbf{v}_f = \mathbf{q} - \left( \frac{1 - \phi_f}{\phi_f} \right) k(\phi_f) \nabla p, \quad (94)$$

$$\mathbf{v}_s = \mathbf{q} + k(\phi_f) \nabla p, \quad (95)$$

where  $k(\phi_f)$  is the dimensionless permeability scaled to the relaxed value  $k_0 = k(\phi_{f,0})$ .

Our goal is to apply this two-phase model to a flow problem such as a porous drop in a uniform streaming flow assumed to be in the  $z$ -axis direction. In an axi-symmetric

configuration with no azimuthal  $\phi$  dependence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad (96)$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad (97)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot \theta}{r} u_\theta = \frac{u_r}{r} + \frac{\cot \theta}{r} u_\theta, \quad (98)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \quad (99)$$

$$\varepsilon_{r\phi} = \frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) = \frac{1}{2} \left( \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right), \quad (100)$$

$$\varepsilon_{\theta\phi} = \frac{1}{2} \left( \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{\cot \theta}{r} u_\phi \right) = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{\cot \theta}{r} u_\phi \right). \quad (101)$$

Accordingly the components of  $\sigma$  are

$$\sigma_{rr} = \varepsilon_{rr} + \Lambda (\varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}), \quad (102)$$

$$\sigma_{\theta\theta} = \varepsilon_{\theta\theta} + \Lambda (\varepsilon_{rr} + \varepsilon_{\phi\phi}), \quad (103)$$

$$\sigma_{\phi\phi} = \varepsilon_{\phi\phi} + \Lambda (\varepsilon_{rr} + \varepsilon_{\theta\theta}), \quad (104)$$

$$\sigma_{r\theta} = (1 - \Lambda) \varepsilon_{r\theta}, \quad (105)$$

$$\sigma_{r\phi} = (1 - \Lambda) \varepsilon_{r\phi}, \quad (106)$$

$$\sigma_{\theta\phi} = (1 - \Lambda) \varepsilon_{\theta\phi}. \quad (107)$$

The balance of stress  $\nabla p = \nabla \cdot \sigma$  with

$$\begin{aligned} \nabla \cdot \sigma &= \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma_{r\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r} \right] \mathbf{e}_r + \\ &\quad \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} \sigma_{r\theta} - \cot \theta \frac{\sigma_{\phi\phi}}{r} \right] \mathbf{e}_\theta + \\ &\quad \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r\phi}) + \frac{\sin \theta}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos \theta \frac{\sigma_{\theta\phi}}{r} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{r\phi} + \sigma_{\theta\phi}}{r} \right] \mathbf{e}_\phi, \end{aligned} \quad (108)$$

which in the absence of  $\phi$  dependence takes the following form

$$\begin{aligned} \nabla \cdot \sigma &= \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma_{r\theta} \sin \theta) - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r} \right] \mathbf{e}_r + \\ &\quad \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sigma_{\theta\theta} \sin \theta) + \frac{1}{r} \sigma_{r\theta} - \cot \theta \frac{\sigma_{\phi\phi}}{r} \right] \mathbf{e}_\theta + \\ &\quad \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r\phi}) + \frac{\sin \theta}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos \theta \frac{\sigma_{\theta\phi}}{r} + \frac{\sigma_{r\phi} + \sigma_{\theta\phi}}{r} \right] \mathbf{e}_\phi. \end{aligned} \quad (109)$$

## B. Spherical porous drop in a uniform streaming flow

To study the flow around and in the porous drop immersed in a uniform streaming flow in the  $z$ -direction, we assume axi-symmetry (no  $\phi$ -dependence) and the following forms for

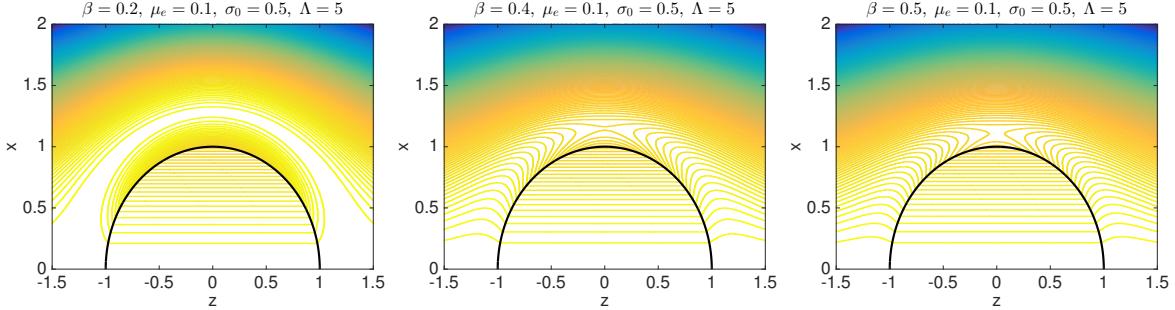


FIG. 13. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 0.1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 0.2$ ,  $0.4$  and  $0.5$ .

the displacement field

$$u_r = f(r) \cos \theta, \quad u_\theta = g(r) \sin \theta, \quad u_\phi = h(r) \sin \theta. \quad (110)$$

First we examine the equilibrium state, when  $\mathbf{v}_s = 0$  and the total flow inside the drop  $\mathbf{q} = -k\nabla p$  in the dimensionless form. Assuming that the scaled permeability is constant by setting  $k = 1$ , we obtain the following equations for  $f(r)$ ,  $g(r)$ , and  $h(r)$ :

$$r^3 f^{(3)} + 4r^2 f^{(2)} + 2r^2 g^{(2)} - 2rf' - 4f - 4g = 0, \quad (111)$$

$$r^3 g^{(3)} + 3r^2 g^{(2)} + r^2 f^{(2)} - 2rg' - 2g - 2f = 0, \quad (112)$$

$$r^2 h'' + 2rh' - 2h = 0, \quad (113)$$

where equation 111 is derived from the divergence free condition for  $\mathbf{q}$ , equation 112 is derived from the consistence condition for  $\partial^2 p / \partial \theta \partial r = \partial^2 p / \partial r \partial \theta$ , and equation 113 is from the assumption that there is no  $\phi$  component in the pressure gradient due to a uniform streaming flow in the  $z$ -direction. The general solutions are

$$f(r) = f_0 + f_2 r^2, \quad g(r) = -f_0 + g_2 r^2, \quad h(r) = h_1 r, \quad (114)$$

with constants  $f_0$ ,  $f_2$ ,  $g_2$ , and  $h_1$  to be determined from the boundary conditions. Two additional constants  $A_1$  and  $A_2$  for the fluid velocity field outside the porous drop need to be determined. Among all six coefficients,  $h_1$  is decoupled from the rest and is set to zero if we assume that  $u_\phi = 0$  based on the axi-symmetric assumption for the deformation field. Thus, we have the following five conditions to determine the remaining five unknowns

$$u_r(1, \theta) = \delta \cos \theta = (-f_0 + f_2) \cos \theta, \quad (115)$$

$$\sigma_{rr}(1, \theta) = 2\sigma_0 \cos \theta = 2(f_2 + f_2 \Lambda + g_2 \Lambda) \cos \theta, \quad (116)$$

$$\mathbf{U} \cdot \mathbf{e}_r = \mathbf{q} \cdot \mathbf{e}_r \text{ at } r = 1, \quad (117)$$

$$\mathbf{U} \cdot \mathbf{e}_\theta = \mathbf{q} \cdot \mathbf{e}_\theta + \bar{\sigma}_{r\theta} \text{ at } r = 1, \quad (118)$$

$$-P + \Sigma_{rr} = -p \text{ at } r = 1, \quad (119)$$

where  $\bar{\sigma}_{r\theta}$  is the mean tangential stress at the spherical interface.

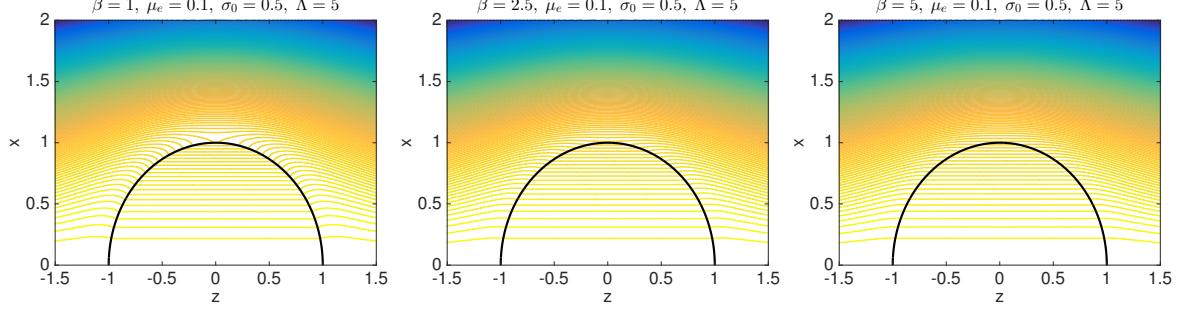


FIG. 14. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 0.1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 1, 2.5$  and  $5$ .

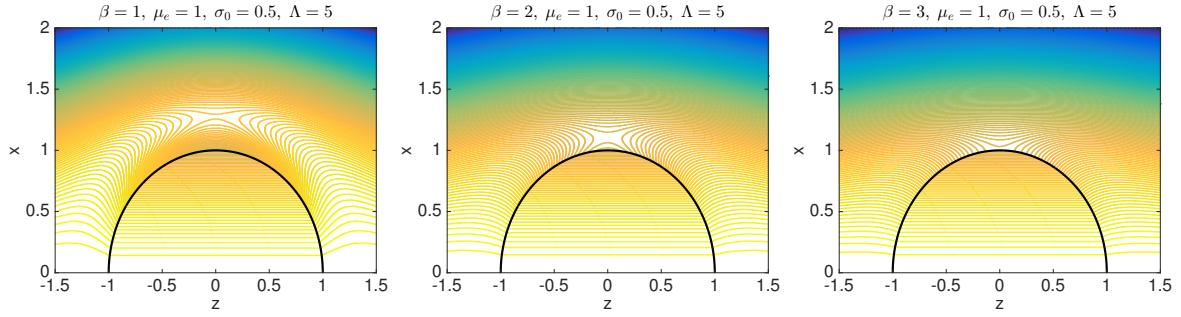


FIG. 15. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 1, 2$  and  $3$ .

A different set of boundary conditions are often used (see Davis and Stone):

$$u_r(1, \theta) = \delta \cos \theta = (-f_0 + f_2) \cos \theta, \quad (120)$$

$$\sigma_{rr}(1, \theta) = 2\sigma_0 \cos \theta = 2(f_2 + f_2 \Lambda + g_2 \Lambda) \cos \theta, \quad (121)$$

$$\mathbf{U} \cdot \mathbf{e}_r = \mathbf{q} \cdot \mathbf{e}_r \text{ at } r = 1, \quad (122)$$

$$\mathbf{U} \cdot \mathbf{e}_\theta = \mathbf{q} \cdot \mathbf{e}_\theta + \bar{\sigma}_{r\theta} \text{ at } r = 1, \quad (123)$$

$$-P = -p \text{ at } r = 1, \quad (124)$$

where  $\bar{\sigma}_{r\theta}$  is the mean tangential stress at the spherical interface.

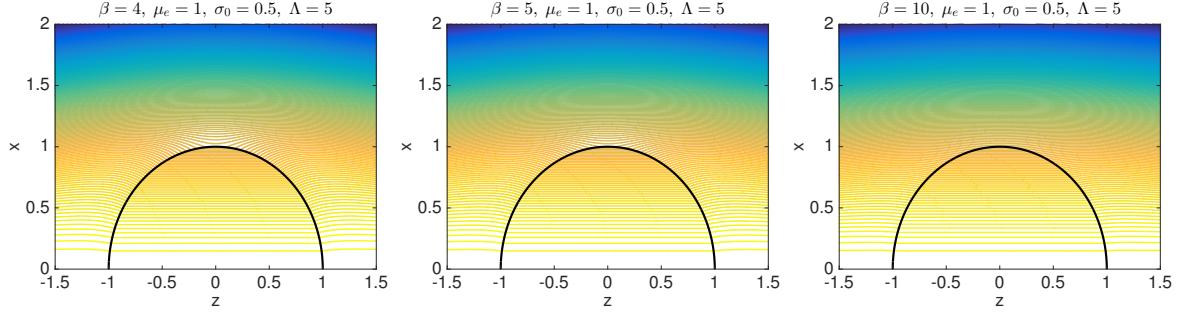


FIG. 16. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 4, 5$  and  $10$ .

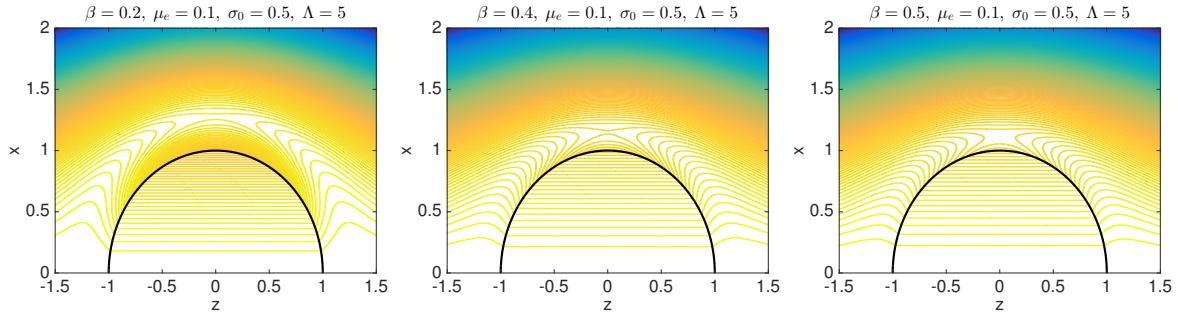


FIG. 17. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 0.1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 0.2, 0.4$  and  $0.5$ .

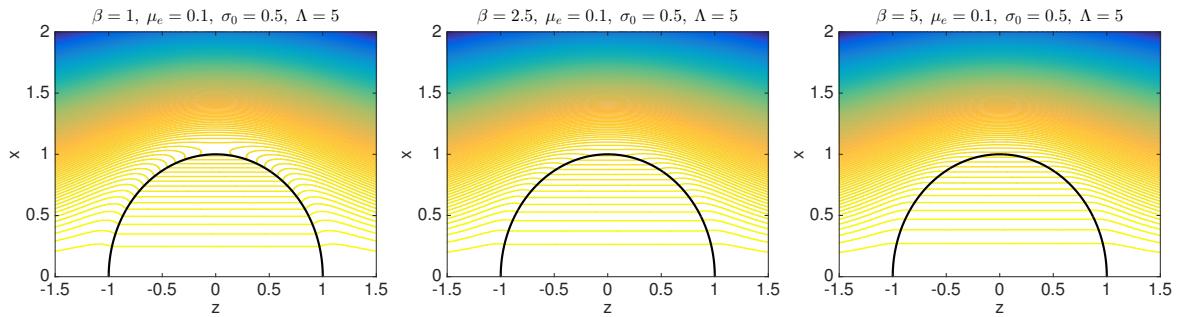


FIG. 18. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 0.1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 1, 2.5$  and  $5$ .

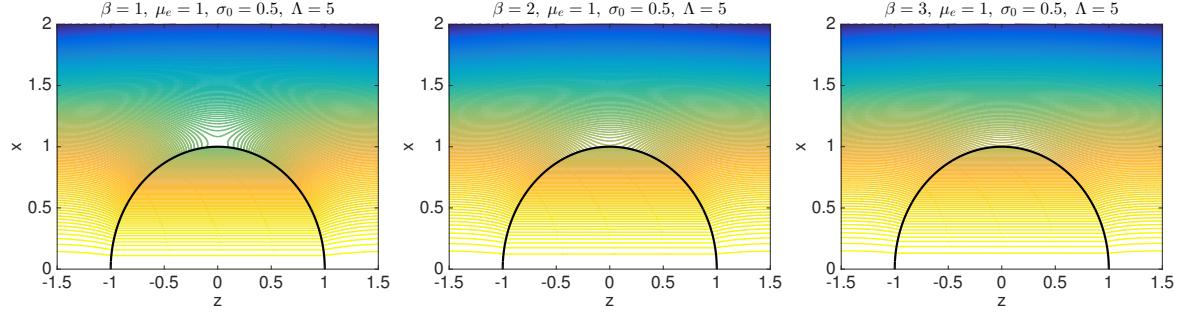


FIG. 19. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 1, 2$  and  $3$ .

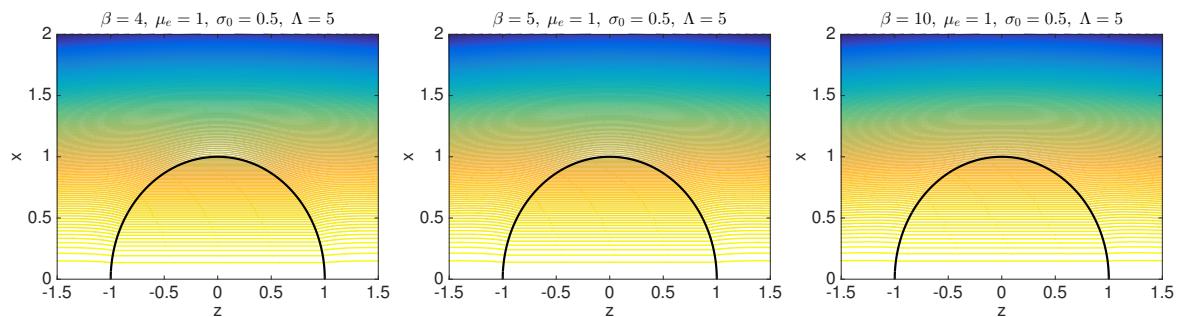


FIG. 20. Streamlines around a spherical Darcy drop with linear poroelasticity,  $\mu_e = 1$ ,  $\sigma_0 = 0.5$  and  $\Lambda = 5$ . From left to right  $\beta = 4, 5$  and  $10$ .