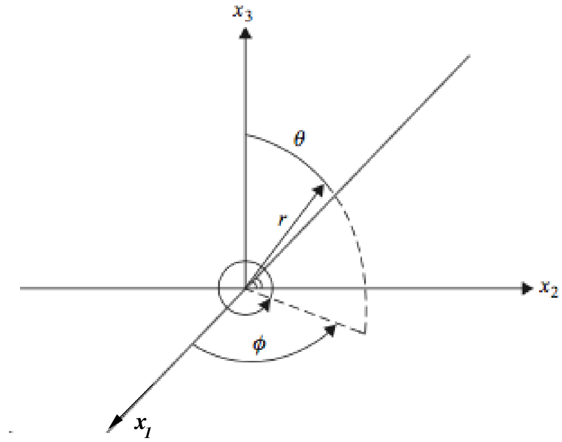

FIG. 1. Spherical coordinates.

I. INTRODUCTION

A. A viscous drop in a uniform streaming flow

The drop is assumed to be loCated at the center of a spherical coordinate system, with the meridian angle $\theta \in [0, \pi]$ and the azimuthal angle $\phi \in [0, 2\pi]$ defined in figure (1). For a drop in a uniform flow streaming along the x_3-axis ($\theta = 0$), the velocity field (both inside and outside the drop) may be assumed to be independent of ϕ with an axi-symmetric drop shape close to a sphere of radius r_0. Denoting the velocity outside the viscous drop as $\mathbf{U} = (U_R(R, \theta), U_\theta(R, \theta))$, the axi-symmetric Stokes equations in spherical coordinates are

$$-\frac{\partial P}{\partial R} + \mu_e \left(\nabla^2 U_R - \frac{2U_R}{R^2} - \frac{2}{R^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2}{R^2} U_\theta \cot\theta \right) = 0, \tag{1}$$

$$-\frac{1}{R}\frac{\partial P}{\partial \theta} + \mu_e \left(\nabla^2 U_\theta + \frac{2}{R^2}\frac{\partial U_R}{\partial \theta} - \frac{U_\theta}{R^2 \sin^2\theta} \right) = 0, \tag{2}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 U_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(U_\theta \sin\theta\right) = 0, \tag{3}$$

with the Laplace operator

$$\nabla^2 = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right). \tag{4}$$

In the far field the external velocity $U_R \to -U_0 \cos\theta$ and $U_\theta \to U_0 \sin\theta$ for $R \to \infty$. Thus we seek a solution for the velocity in the external fluid of the form

$$U_R(R, \theta) = F(R)\cos\theta, \tag{5}$$

$$U_\theta(R, \theta) = G(R)\sin\theta, \tag{6}$$

$$P(R, \theta) = H(R)\cos\theta. \tag{7}$$

Substituting the above functions into the governing equations we obtain

$$2F + 2G + RF' = 0, \tag{8}$$

$$-4\mu_e F - 4\mu_e G + R(2\mu_e F' - RH' + R\mu_e F'') = 0, \tag{9}$$

$$-2\mu_e F - 2\mu_e G + R(H + 2\mu_e G' + R\mu_e G'') = 0, \tag{10}$$

which can be combined into an equation for F

$$R^4 F^{(4)} + 8R^3 F^{(3)} + 8R^2 F^{(2)} - 8RF' = 0. \tag{11}$$

The general solution for F is

$$F(R) = \frac{A_1}{R^3} + \frac{A_2}{R} + A_3 + A_4 R^2 = \frac{A_1}{R^3} + \frac{A_2}{R} - U_0 \tag{12}$$

due to the far field boundary condition. The other two functions are related to F and its derivatives as

$$G(R) = \frac{A_1}{2R^3} - \frac{A_2}{2R} + U_0, \quad H(R) = \frac{A_2}{R^2}\mu_e. \tag{13}$$

If the interior fluid is a Newtonian viscous fluid with a flow field denoted as $\mathbf{u} = (u_r(r,\theta), u_\theta(r,\theta))$, with

$$u_r(r,\theta) = f(r)\cos\theta, \tag{14}$$

$$u_\theta(r,\theta) = g(r)\sin\theta, \tag{15}$$

$$p(r,\theta) = h(r)\cos\theta, \tag{16}$$

where

$$f(r) = a_3 + a_4 r^2, \tag{17}$$

$$g(r) = -a_3 - 2a_4 r^2, \tag{18}$$

$$h(r) = 10\mu_i a_4 r^2. \tag{19}$$

For the viscous drop to maintain the fixed spherical shape the surface tension must be sufficiently large (Capillary number $\mathrm{Ca} \ll 1$) so the normal velocity at the fluid interface is zero. Another consequence of large capillary number is that the normal stress is dominated by the pressure jump balancing the capillary pressure. Thus the boundary conditions are

$$U_\theta(r_0,\theta) = u_\theta(r_0,\theta), \quad U_R(r_0,\theta) = u_r(r_0,\theta) = 0, \quad \mu_e e_{R\theta}(r_0,\theta) = \mu_i e_{r\theta}(r_0,\theta). \tag{20}$$

We found that

$$A_1 = -\frac{r_0^3 U_0 \mu_i}{2(\mu_e + \mu_i)}, \tag{21}$$

$$A_2 = -\frac{r_0 U_0 (2\mu_e + 3\mu_i)}{2(\mu_e + \mu_i)}, \tag{22}$$

$$a_3 = \frac{U_0 \mu_e}{2(\mu_e + \mu_i)}, \tag{23}$$

$$a_4 = -\frac{U_0 \mu_e}{2r_0^2(\mu_e + \mu_i)}. \tag{24}$$

Next we consider small deformation of order Ca with Ca $\ll 1$. In this case the shape of the nearly spherical drop is in the form

$$\rho(\mathbf{x}, t) = r - r_0 \left(1 + \mathrm{Ca}B(\mathbf{x}, t)\right) = 0, \tag{25}$$

The normal velocity on the fluid interface is no longer zero, and the normal stress balance has to be taken into account. The corresponding boundary conditions are

$$U_\theta(r_0, \theta) = u_\theta(r_0, \theta), \quad U_R(r_0, \theta) = u_r(r_0, \theta) = \mathrm{Ca}\frac{db}{dt}, \tag{26}$$

$$\mu_e e_{R\theta}(r_0, \theta) = \mu_i e_{r\theta}(r_0, \theta),\ \mu_e e_{RR}(r_0, \theta) - \mu_i e_{rr}(r_0, \theta) = -\nabla_s^2 B(\mathbf{x}), \tag{27}$$

where $B(\mathbf{x}) = b(t)\cos(\theta)$. We found that

$$A_1 = \frac{2r_0^4 \mu_i b(t)}{3\mu_e(2\mu_e + 3\mu_i)}, \tag{28}$$

$$A_2 = -\frac{2r_0^2 b(t)}{3\mu_e}, \tag{29}$$

$$a_3 = \frac{2b(t)}{3r_0(2\mu_e + 3\mu_i)}, \tag{30}$$

$$a_4 = -\frac{3U_0\mu_e(2\mu_e + 3\mu_i) + 2r_0(3\mu_e + 2\mu_i)b(t)}{3\mu_e(2\mu_e + 3\mu_i)}, \tag{31}$$

with $b(t)$ satisfying the equation

$$\mathrm{Ca}b' = -\frac{4r_0(\mu_e + \mu_i)}{3\mu_e(2\mu_e + 3\mu_i)} b - U_0 \tag{32}$$

that gives a equilibrium solution

$$b_{eq} = -\frac{3\mu_e(2\mu_e + 3\mu_i)U_0}{4r_0\left(\mu_e + \mu_i\right)}. \tag{33}$$

We first note that the deformation amplitude $b(t)$ evolves monotonically toward the equilibrium b_{eq}. Furthermore, equations 28-31 evolve towards the coefficients in equations 21-24. The same methodology applies to a linear flow as shown in G. Leal's textbook, and similarly the small-deformation amplitude evolves monotonically towards equilibrium. Fig. (2)(a) shows the streamlines for three values of the viscosity ratio (interior to exterior). It is also good to keep in mind the following flow patterns around a spherical particle or drop. Fig. (2)(b) shows the streamlines of a vesicle going through osmophoresis at a constant velocity U (from Anderson). Note the drastic different stream lines around the spherical drop.

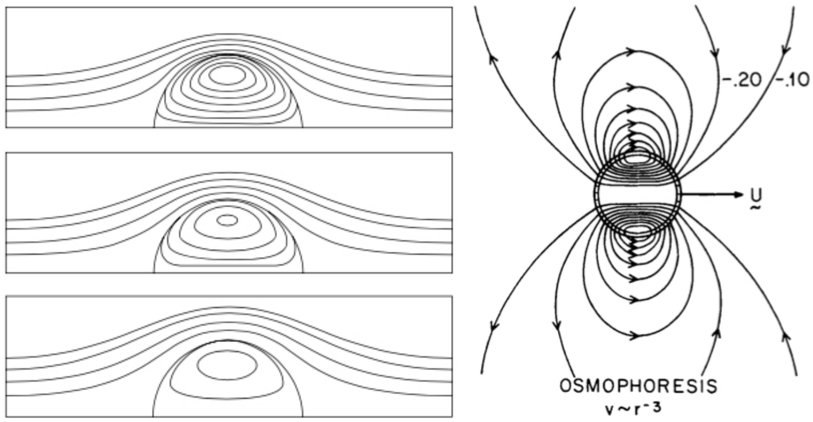

FIG. 2. Left: Streamlines around a viscous drop in a streaming flow (uniform flow). $\mu_r = 0.1$, 1, and 10 from top to bottom. Right: Streamlines around a viscous drop in osmophoresis.

B. Darcy model for the fibrous network inside a spherical drop

For flow inside the porous particle the Darcy equation gives the fluid flow $\mathbf{u}$ inside the spherical particle

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta = -\frac{k}{\mu} \nabla p_i, \;\; \nabla \cdot \mathbf{u} = 0, \tag{34}$$

$$u_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \psi_i}{\partial \theta}, \;\; u_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi_i}{\partial r}, \tag{35}$$

where ψ_i is the streamfunction for the incompressilbe axi-symmetric flow field inside the porous particle. For flow outside the porous particle the Stokes equation gives the uniform streaming flow field driven by a constant pressure gradient

$$-\nabla P_e + \mu_e \nabla^2 \mathbf{U} = 0, \;\; \nabla \cdot \mathbf{U} = 0, \quad \text{with } \mathbf{U} \to U \mathbf{e}_3 \text{ at } r \to \infty \tag{36}$$

$$U_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \psi_e}{\partial \theta}, \;\; U_\theta = \frac{1}{r \sin\theta} \frac{\partial \psi_e}{\partial r}, \tag{37}$$

where ψ_e is the streamfunction for the incompressilbe axi-symmetric flow field outside the porous particle. The boundary conditions for a Darcy porous drop are (1) the continuity of the normal velocity components on the interface, while (2) the tangential velocities are allowed to slip by an amount proportional to the tangential stress exerted on the surface of the sphere by the external fluid, and finally (3) the continuity of pressure on the porous sphere surface:

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \;\; r = a, \tag{38}$$

$$(\mathbf{U} - \mathbf{u}) \cdot \mathbf{t} = \beta \mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t}, \;\; r = a, \tag{39}$$

$$P_e = p_i, \;\; r = a. \tag{40}$$

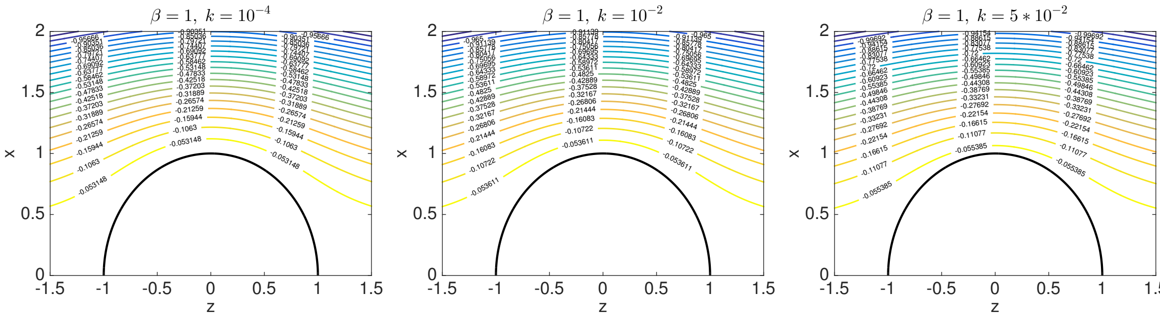

FIG. 3. Streamlines around a nearly spherical Darcy drop with $\mu_e = 1$, $\mu = 1$ and $\beta = 1$. From left to right $k = 10^{-4}$, 10^{-2} and 5×10^{-2}.

The external flow is solved to be

$$U_r = \left(U - \frac{C_2}{r} - \frac{C_1}{3r^3}\right)\cos\theta, \tag{41}$$

$$U_\theta = \left(-U + \frac{C_2}{2r} - \frac{C_1}{6r^3}\right)\sin\theta, \tag{42}$$

$$P_e = \frac{C_2\mu_e}{r^2}\cos\theta. \tag{43}$$

The total force exerted on the porous particle by the external fluid flow can be calculated as

$$\mathbf{F} = \int \mathbf{T}_e\mathbf{n}a^2\sin\theta d\theta d\phi, \tag{44}$$

$$= \frac{12U\pi\mu\mu_e a(\beta+2\mu_e)}{6\mu_e(\mu+k\mu_e)+\beta(2\mu+3k\mu_e)}\mathbf{e}_z. \tag{45}$$

In the limit of small k,

$$\mathbf{F} \approx \left[\frac{6U\pi\mu_e a(\beta+2\mu_e)}{\beta+3\mu_e} - \frac{9(U\pi\mu_e^2(\beta+2\mu_e)^2a)}{\mu(\beta+3\mu_e)^2}k + mboxCalO(k^2)\right]\mathbf{e}_z. \tag{46}$$

In the limit of large β,

$$\mathbf{F} \approx \left[\frac{12U\pi\mu\mu_e a}{2\mu+3k\mu_e} - \frac{24(U\pi\mu^2\mu_e^2)a}{(2\mu+3k\mu_e)^2}\frac{1}{\beta} + mboxCalO\left(\frac{1}{\beta^2}\right)\right]\mathbf{e}_z. \tag{47}$$

Inside the porous spherical particle the pressure is

$$p_i = -a_0 r\cos\theta, \tag{48}$$

$$a_0 = \frac{3U\mu\mu_e(\beta+2\mu_e)}{6\mu_e(\mu+k\mu_e)+\beta(2\mu+3k\mu_e)}, \tag{49}$$

with $\mathbf{u} = -k/\mu\nabla p_i$. Figure 3 and figure 4 show the k-dependence of the streamlines. We observe that as k increases the streamline pattern inside the drop transitions from a non-permeable drop to a permeable drop.

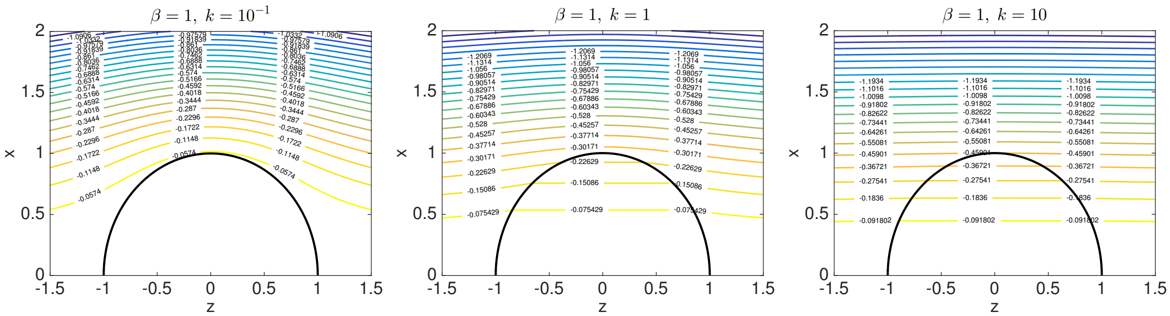

FIG. 4. Streamlines around a nearly spherical Darcy drop with $\mu_e = 1$, $\mu = 1$ and $\beta = 1$. From left to right $k = 0.1$, 1 and 10.

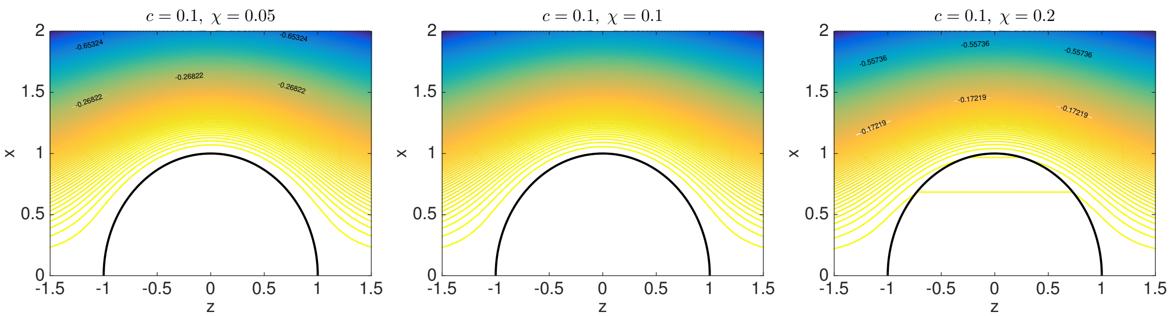

FIG. 5. Streamlines around a nearly spherical Darcy drop with $\mu_e = 1$, $\mu = 1$ and $c = 0.1$. From left to right $\chi = 0.05$, 0.1 and 0.2.

Bars and Worster (JFM 2006) found that homogenization over the pore length scales gives rise to boundary conditions that are consistent with Beaver and Joseph's boundary conditions (equation 38-40):

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a(1-\delta), \tag{50}$$

$$\mathbf{t} \cdot \mathbf{U} = \mathbf{t} \cdot \mathbf{u}, \quad r = a(1-\delta), \tag{51}$$

$$P_e = p_i, \quad r = a(1-\delta), \tag{52}$$

where the slip length δ is computed as a function of the fluid volume fraction χ_f if the permeability k is assumed a certain dependence on χ_f. For example,

$$k(\chi_f) = k_0 \frac{\chi_f^3}{(1-\chi_f)^2}, \tag{53}$$

$$\delta \equiv c\sqrt{\frac{k(\chi_f)}{\chi_f}} = c k_0^{1/2} \frac{\chi_f}{1-\chi_f}, \tag{54}$$

where k_0 is a reference permeability and c is an undetermined parameter that is usually assumed to be of order one. In the following we set $k_0 = 1$ and adjust the slip length by changing the value of c. Figures 5-6 show the streamlines for various values of χ_f.

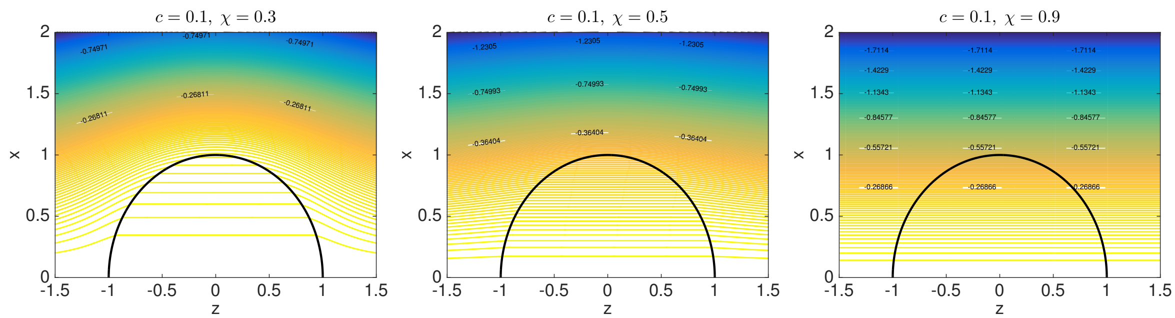

FIG. 6. Streamlines around a nearly spherical Darcy drop with $\mu_e = 1$, $\mu = 1$ and $c = 0.1$. From left to right $\chi = 0.3$, 0.5 and 0.9.

C. A Brinkman drop in a uniform streaming flow

We now extend the previous Calculations to a Brinkman drop in a uniform flow in the limit where $\mathbf{v}_1 = 0$. Denoting the velocity outside the viscous drop as $\mathbf{U} = (U_R(R,\theta), U_\theta(R,\theta))$, the axi-symmetric Stokes equations in spherical coordinates are

$$-\frac{\partial P}{\partial R} + \left(\nabla^2 U_R - \frac{2U_R}{R^2} - \frac{2}{R^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2}{R^2}U_\theta \cot\theta\right) = 0, \tag{55}$$

$$-\frac{1}{R}\frac{\partial P}{\partial \theta} + \left(\nabla^2 U_\theta + \frac{2}{R^2}\frac{\partial U_R}{\partial \theta} - \frac{U_\theta}{R^2 \sin^2\theta}\right) = 0, \tag{56}$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 U_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial \theta}\left(U_\theta \sin\theta\right) = 0. \tag{57}$$

For the velocity inside $\mathbf{u} = (u_r(r,\theta), u_\theta(r,\theta))$ the axi-symmetric Brinkman equations are

$$-\frac{\partial p}{\partial r} + \mu_r\left(\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{2}{r^2}u_\theta \cot\theta\right) - \lambda u_r = 0, \tag{58}$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu_r\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta}\right) - \lambda u_\theta = 0, \tag{59}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) = 0. \tag{60}$$

Again we seek a solution for the velocity in the external fluid of the form

$$U_R(R,\theta) = F(R)\cos\theta,\ U_\theta(R,\theta) = G(R)\sin\theta,\ P(R,\theta) = H(R)\cos\theta, \tag{61}$$

$$u_r(r,\theta) = f(r)\cos\theta,\ u_\theta(r,\theta) = g(r)\sin\theta,\ p(r,\theta) = h(r)\cos\theta. \tag{62}$$

The general solutions for F, G and H are

$$F(R) = \frac{A_1}{R^3} + \frac{A_2}{R} - U_0, \tag{63}$$

$$G(R) = \frac{A_1}{2R^3} - \frac{A_2}{2R} + U_0, \tag{64}$$

$$H(R) = \frac{A_2}{R^2}. \tag{65}$$

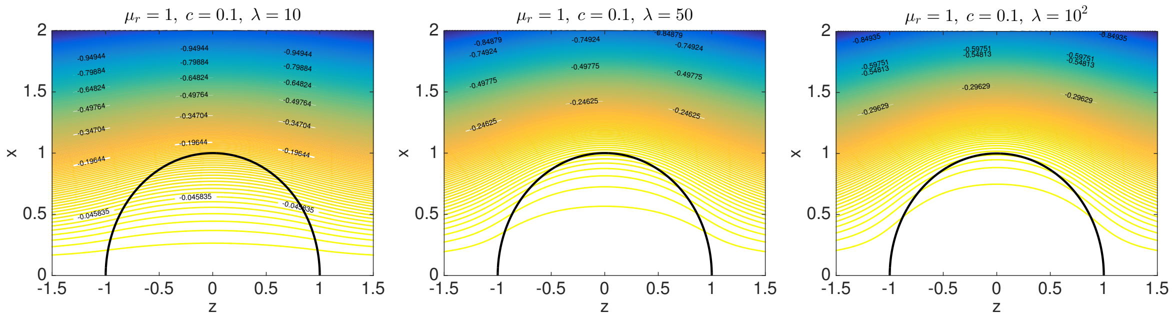

FIG. 7. Streamlines around a spherical Brinkman drop with $\mu_r = 1$ and $c = 0.1$. From left to right $\lambda = 10$, 50 and 100.

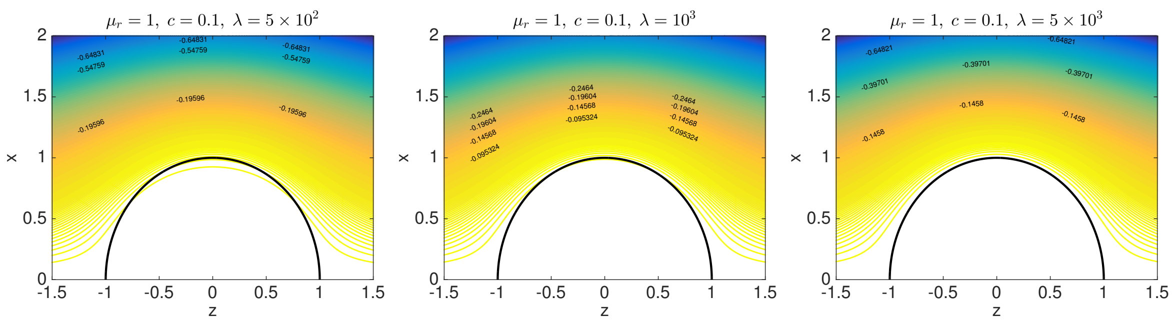

FIG. 8. Streamlines around a spherical Brinkman drop with $\mu_r = 1$ and $c = 0.1$. From left to right $\lambda = 500$, 10^3 and 5×10^3.

The general solution for f is

$$f(r) = a_4 + a_3 \frac{-\sqrt{\frac{\lambda}{\mu_r}} r \cosh\left(\sqrt{\frac{\lambda}{\mu_r}} r\right) + \sinh\left(\sqrt{\frac{\lambda}{\mu_r}} r\right)}{r^3}. \tag{66}$$

Based on Bars and Worster (JFM 2006) here we adopt the following boundary conditions for the Brinkman drop immersed in a Stokes flow:

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a(1-\delta), \tag{67}$$
$$\mathbf{t} \cdot \mathbf{U} = \mathbf{t} \cdot \mathbf{u}, \quad r = a(1-\delta), \tag{68}$$
$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{n}, \quad r = a(1-\delta), \tag{69}$$
$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{t}, \quad r = a(1-\delta), \tag{70}$$

with the slip length defined as

$$\delta \equiv c\sqrt{\frac{\mu_r}{\lambda}}. \tag{71}$$

Recently Angot, Boyeau and Ochoa-Tapia (PRE 2017) conducted asymptotic analysis on the viscous fluid flow at the interface between a fluid and a porous layer described by a

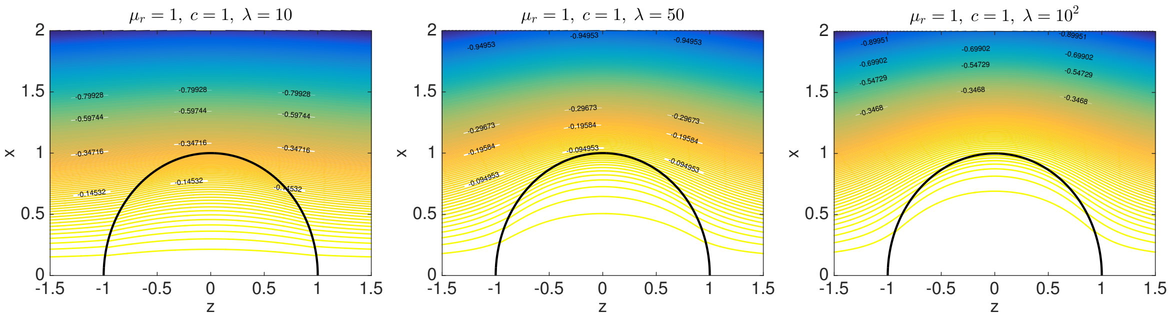

FIG. 9. Streamlines around a spherical Brinkman drop with $\mu_r = 1$ and $c = 1$. From left to right $\lambda = 10$, 50 and 100.

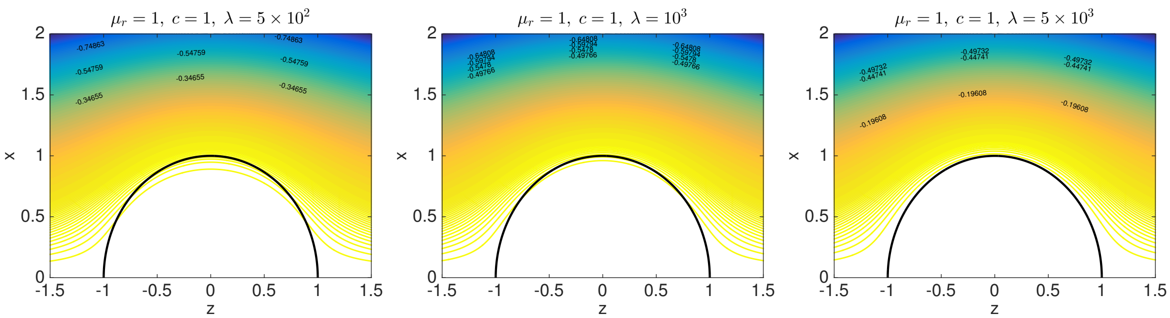

FIG. 10. Streamlines around a spherical Brinkman drop with $\mu_r = 1$ and $c = 1$. From left to right $\lambda = 500$, 10^3 and 5×10^3.

Darcy-Brinkman medium. They derived the following boundary conditions at the interface

$$\mathbf{n} \cdot \mathbf{U} = \mathbf{n} \cdot \mathbf{u}, \quad r = a, \tag{72}$$

$$\mathbf{t} \cdot (\mathbf{U} - \mathbf{u}) = \frac{\beta}{2} \left(\mathbf{t} T_e \mathbf{n} + \mathbf{t} T_i \mathbf{n} \right), \quad r = a, \tag{73}$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{n}, \quad r = a, \tag{74}$$

$$\mathbf{n} \cdot \mathbf{T}_e \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{T}_i \cdot \mathbf{t}, \quad r = a, \tag{75}$$

with the coefficient β related to the permeability $1/\lambda$, viscosity μ_i and the boundary layer thickness d (see Angot *et al.*).

II. PROBLEM FORMULATION I

The deformable soft porous material immersed in fluid has been modeled by MacMinn Dufresne and Wettlaufer (PhysiCal Review Applied, 2016). Starting from the definition of total volume flux $\mathbf{q}$ as

$$\mathbf{q} \equiv \phi_f \mathbf{v}_f + (1 - \phi_f) \mathbf{v}_s, \tag{76}$$

where ϕ_f is the loCal volume fraction of fluid ($1 - \phi_f$ is the loCal volume fraction of solid). $\mathbf{v}_f$ is the fluid velocity field and $\mathbf{v}_s$ is the solid velocity field. The volume conservation is

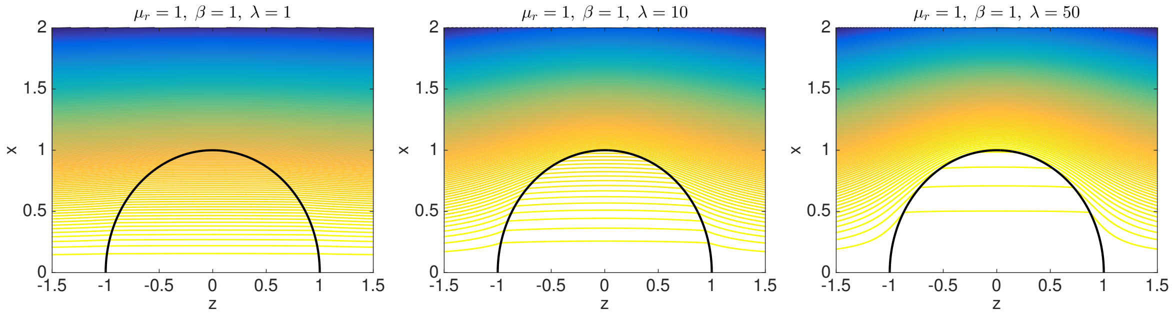

FIG. 11. Streamlines around a spherical Brinkman drop with $\mu_r = 1$, From left to right $\lambda = 1$, 10 and 50.

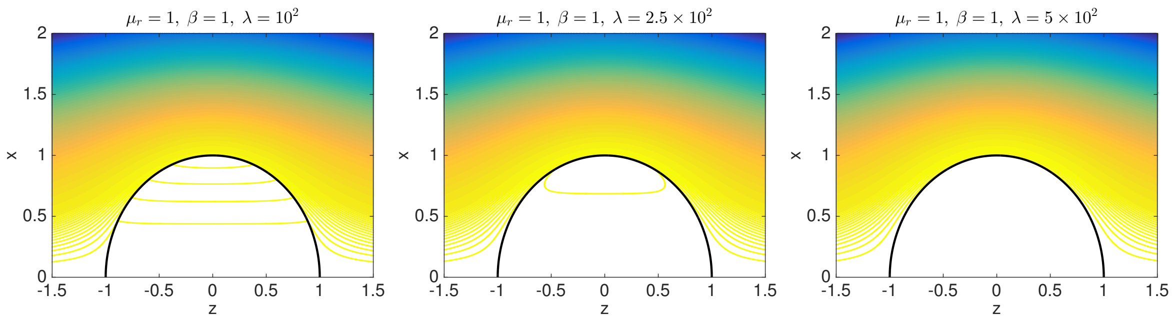

FIG. 12. Streamlines around a spherical Brinkman drop with $\mu_r = 1$. From left to right $\lambda = 100$, 250 and 500.

given below in terms of the porosity ϕ_f as

$$\frac{\partial \phi_f}{\partial t} + \nabla \cdot \left[\phi_f \mathbf{q} - (1-\phi_f) \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}) \right] = 0, \tag{77}$$

$$\nabla \cdot \mathbf{q} = 0, \tag{78}$$

$$\mathbf{v}_f = \mathbf{q} - \left(\frac{1-\phi_f}{\phi_f} \right) \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}), \tag{79}$$

$$\mathbf{v}_s = \mathbf{q} + \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}). \tag{80}$$

Equations (79-80) Can be combined to give the constitutive laws for fluid flow

$$\phi_f (\mathbf{v}_f - \mathbf{v}_s) = -\frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}), \tag{81}$$

which is the Darcy's law for the fluid flow relative to the solid skeleton. The loCal average velocity $\mathbf{q}$ Can be Calculated as

$$\mathbf{q} = \frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{u}_s - \frac{k(\phi_f)}{\mu} (\nabla p - \rho_f \mathbf{g}). \tag{82}$$

In the following we will ignore the gravitational force by setting $\rho_f \mathbf{g} = 0$. In the absence of the external force, the pressure gradient is balanced by stress in the strain of the porous medium as

$$\nabla p = \nabla \cdot \sigma', \tag{83}$$

$$\sigma' \equiv \mathbf{\Lambda} \mathrm{tr}(\mathbf{H})\mathbf{I} + (\mathcal{M} - \mathbf{\Lambda})\mathbf{H}, \tag{84}$$

$$\mathbf{H} \equiv \frac{1}{2} \ln \left(\mathbf{F}\mathbf{F}^{\mathbf{T}}\right), \tag{85}$$

where the deformation-gradient tensor $\mathbf{F}$ is defined as

$$\mathbf{F}^{-1} = \nabla \mathbf{X} = \mathbf{I} - \nabla \mathbf{u_s}. \tag{86}$$

A. Linear Poroelasticity

When the strain of the porous medium is small

$$\frac{\phi_f - \phi_{f,0}}{1 - \phi_{f,0}} \approx \nabla \cdot \mathbf{u}_s \sim \epsilon \ll 1. \tag{87}$$

In this limit, the porosity satisfies the well-known linear poroelastic flow equation:

$$\frac{\partial \phi_f}{\partial t} - \nabla \cdot \left[(1 - \phi_{f,0}) \frac{k_0}{\mu} \nabla p\right] = 0, \tag{88}$$

where $k_0 = k(\phi_{f,0})$ is the relaxed or undeformed permeability. In the small-strain limit, the linear elasticity applies and we have

$$\sigma' = \mathbf{\Lambda} \mathrm{tr}(\varepsilon)\mathbf{I} + (\mathcal{M} - \mathbf{\Lambda})\varepsilon, \tag{89}$$

$$\varepsilon = \frac{1}{2} \left[\nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T\right], \tag{90}$$

and the pressure gradient is equal to the divergence of the effective solid stress:

$$\nabla p = \nabla \cdot \sigma'. \tag{91}$$

Following the scaling in MacMinn *et al.*, the resultant dimensionless equations for the linear poroelasticity problem are (dropping the prime in σ')

$$\frac{\partial \phi_f}{\partial t} - \nabla \cdot [(1 - \phi_{f,0})\nabla p] = 0, \tag{92}$$

$$\nabla p = \nabla \cdot \sigma, \tag{93}$$

$$\sigma = \mathbf{\Lambda} \mathrm{tr}(\varepsilon)\mathbf{I} + (1 - \mathbf{\Lambda})\varepsilon, \tag{94}$$

$$\varepsilon = \frac{1}{2} \left[\nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T\right], \tag{95}$$

$$\mathbf{v}_f = \mathbf{q} - \left(\frac{1 - \phi_f}{\phi_f}\right) k(\phi_f)\nabla p, \tag{96}$$

$$\mathbf{v}_s = \mathbf{q} + k(\phi_f)\nabla p, \tag{97}$$

where $k(\phi_f)$ is the dimensionless permeability scaled to the relaxed value $k_0 = k(\phi_{f,0})$.

Our goal is to apply this two-phase model to a flow problem such as a porous drop in a uniform streaming flow assumed to be in the z-axis direction. In an axi-symmetric configuration with no azimuthal ϕ dependence

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \tag{98}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \tag{99}$$

$$\varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta = \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta, \tag{100}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \tag{101}$$

$$\varepsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) = \frac{1}{2}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right), \tag{102}$$

$$\varepsilon_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{\cot\theta}{r}u_\phi\right) = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{\cot\theta}{r}u_\phi\right). \tag{103}$$

Accordingly the components of σ are

$$\sigma_{rr} = \varepsilon_{rr} + \mathbf{\Lambda}\left(\varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}\right), \tag{104}$$

$$\sigma_{\theta\theta} = \varepsilon_{\theta\theta} + \mathbf{\Lambda}\left(\varepsilon_{rr} + \varepsilon_{\phi\phi}\right), \tag{105}$$

$$\sigma_{\phi\phi} = \varepsilon_{\phi\phi} + \mathbf{\Lambda}\left(\varepsilon_{rr} + \varepsilon_{\theta\theta}\right), \tag{106}$$

$$\sigma_{r\theta} = \left(1 - \mathbf{\Lambda}\right)\varepsilon_{r\theta}, \tag{107}$$

$$\sigma_{r\phi} = \left(1 - \mathbf{\Lambda}\right)\varepsilon_{r\phi}, \tag{108}$$

$$\sigma_{\theta\phi} = \left(1 - \mathbf{\Lambda}\right)\varepsilon_{\theta\phi}. \tag{109}$$

The balance of stress $\nabla p = \nabla \cdot \sigma$ with

$$\begin{aligned}\nabla \cdot \sigma = &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{r\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r}\right]\mathbf{e_r} + \\ &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r}\sigma_{r\theta} - \cot\theta\frac{\sigma_{\phi\phi}}{r}\right]\mathbf{e}_\theta + \\ &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\phi}\right) + \frac{\sin\theta}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos\theta\frac{\sigma_{\theta\phi}}{r} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{r\phi} + \sigma_{\theta\phi}}{r}\right]\mathbf{e}_\phi,\end{aligned} \tag{110}$$

which in the absence of ϕ dependence takes the following form

$$\begin{aligned}\nabla \cdot \sigma = &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{r\theta}\sin\theta\right) - \frac{\sigma_{\theta\theta} + \sigma_{\phi\phi}}{r}\right]\mathbf{e_r} + \\ &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r}\sigma_{r\theta} - \cot\theta\frac{\sigma_{\phi\phi}}{r}\right]\mathbf{e}_\theta + \\ &\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\phi}\right) + \frac{\sin\theta}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos\theta\frac{\sigma_{\theta\phi}}{r} + \frac{\sigma_{r\phi} + \sigma_{\theta\phi}}{r}\right]\mathbf{e}_\phi.\end{aligned} \tag{111}$$

B. spherical porous drop in a uniform streaming flow: Steady state

To study the flow around and in the porous drop immersed in a uniform streaming flow in the z-direction, we assume axi-symmetry (no ϕ-dependence) and the following forms for the displacement field

$$u_r = f(r)\cos\theta, \quad u_\theta = g(r)\sin\theta, \quad u_\phi = h(r)\sin\theta. \tag{112}$$

First we examine the equilibrium state, when $\mathbf{v}_s = 0$ and the total flow inside the drop $\mathbf{q} = -k\nabla p$ in the dimensionless form. Assuming that the scaled permeability is constant by setting $k = 1$, we obtain the following equations for $f(r)$, $g(r)$, and $h(r)$:

$$r^3 f^{(3)} + 4r^2 f^{(2)} + 2r^2 g^{(2)} - 2rf' - 4f - 4g = 0, \tag{113}$$
$$r^3 g^{(3)} + 3r^2 g^{(2)} + r^2 f^{(2)} - 2rg' - 2g - 2f = 0, \tag{114}$$
$$r^2 h'' + 2rh' - 2h = 0, \tag{115}$$

where equation 113 is derived from the divergence free condition for $\mathbf{q}$, equation 114 is derived from the consistence condition for $\partial^2 p/\partial\theta\partial r = \partial^2 p/\partial r\partial\theta$, and equation 115 is from the assumption that there is no ϕ component in the pressure gradient due to a uniform streaming flow in the z-direction. The general solutions are

$$f(r) = f_0 + f_2 r^2, \quad g(r) = -f_0 + g_2 r^2, \quad h(r) = h_1 r, \tag{116}$$

with constants f_0, f_2, g_2, and h_1 to be determined from the boundary conditions. Two additional constants A_1 and A_2 for the fluid velocity field outside the porous drop need to be determined. Among all six coefficients, h_1 is decoupled from the rest and is set to zero if we assume that $u_\phi = 0$ based on the axi-symmetric assumption for the deformation field. Thus, we have the following five conditions to determine the remaining five unknowns

$$u_r(1,\theta) = \delta\cos\theta = (-f_0 + f_2)\cos\theta, \tag{117}$$
$$\sigma_{rr}(1,\theta) = 2\sigma_0\cos\theta = 2\left(f_2 + f_2\Lambda + g_2\Lambda\right)\cos\theta, \tag{118}$$
$$\mathbf{U}\cdot\mathbf{e_r} = \mathbf{q}\cdot\mathbf{e_r} \;\text{ at } r = 1, \tag{119}$$
$$\beta\left(\mathbf{U}\cdot\mathbf{e}_\theta - \mathbf{q}\cdot\mathbf{e}_\theta\right) = \bar{\sigma}_{r\theta} \;\text{ at } r = 1, \tag{120}$$
$$-P + \Sigma_{rr} = -p \;\text{ at } r = 1, \tag{121}$$

where $\bar{\sigma}_{r\theta}$ is the mean tangential stress at the spherical interface.

A different set of boundary conditions are often used (see Davis and Stone):

$$u_r(1,\theta) = \delta\cos\theta = (-f_0 + f_2)\cos\theta, \tag{122}$$
$$\sigma_{rr}(1,\theta) = 2\sigma_0\cos\theta = 2\left(f_2 + f_2\Lambda + g_2\Lambda\right)\cos\theta, \tag{123}$$
$$\mathbf{U}\cdot\mathbf{e_r} = \mathbf{q}\cdot\mathbf{e_r} \;\text{ at } r = 1, \tag{124}$$
$$\beta\left(\mathbf{U}\cdot\mathbf{e}_\theta - \mathbf{q}\cdot\mathbf{e}_\theta\right) = \bar{\sigma}_{r\theta} \;\text{ at } r = 1, \tag{125}$$
$$-P = -p \;\text{ at } r = 1, \tag{126}$$

where $\bar{\sigma}_{r\theta}$ is the mean tangential stress at the spherical interface.

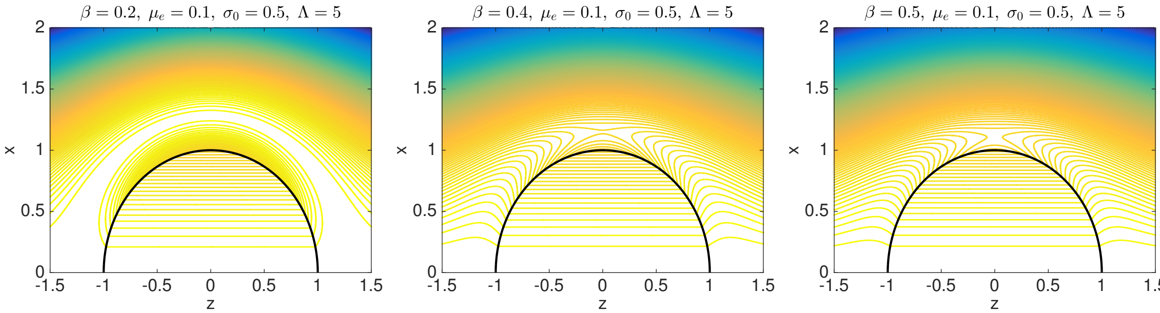

FIG. 13. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 0.1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 0.2$, 0.4 and 0.5.

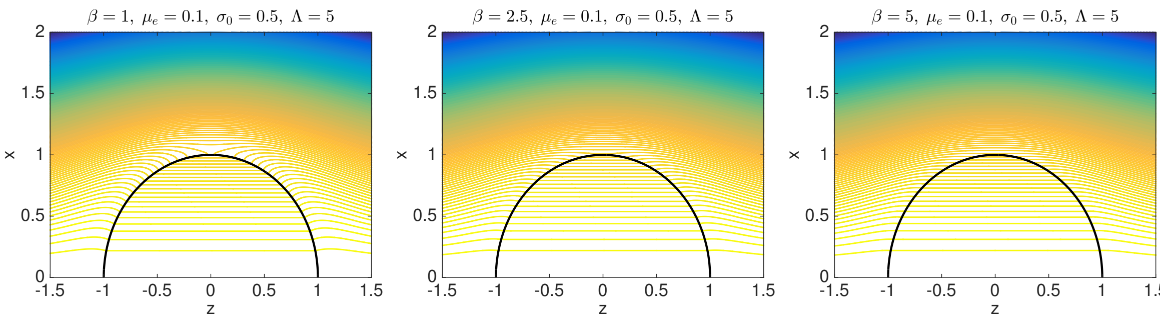

FIG. 14. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 0.1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 1$, 2.5 and 5.

C. spherical porous drop in a uniform streaming flow: Dynamics with deformation

MacMinn's model also allows for dynamics and deformation in the porous medium. Within the linear poroelastic model for small-deformation, we write the solid displacement field as

$$u_r = f(r,t)\cos\theta,\ u_\theta = g(r,t)\sin\theta,\ u_\phi = h(r,t)\sin\theta. \tag{127}$$

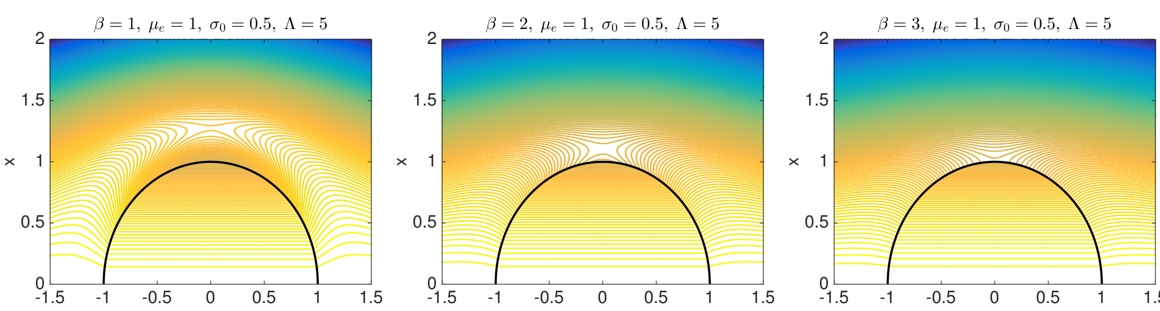

FIG. 15. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 1$, 2 and 3.

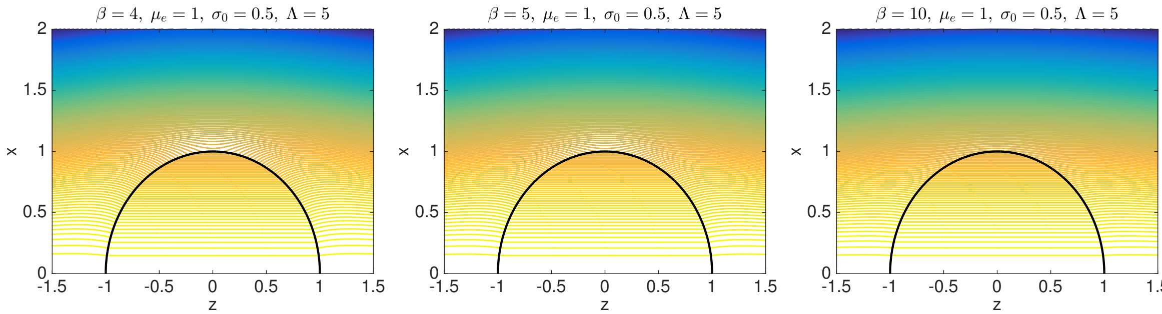

FIG. 16. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 4$, 5 and 10.

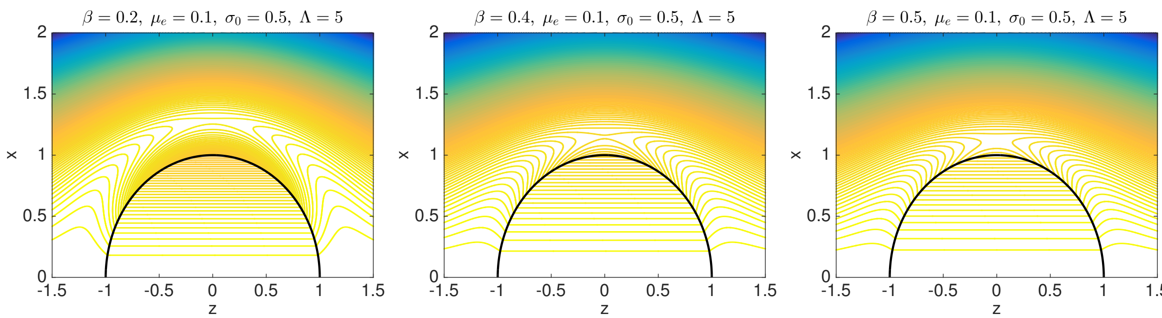

FIG. 17. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 0.1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 0.2$, 0.4 and 0.5.

With the assumption of asial-symmetry for a nearly spherical drop in a uniform streaming flow, $h(r, t) = 0$ and the porosity is approximated as

$$\phi_f - \phi_{f,0} = (1 - \phi_{f,0})\nabla \cdot \mathbf{u}_s = (1 - \phi_{f,0}) \left[\frac{\partial u_r}{\partial r} + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) + \frac{u_r}{r} + \frac{\cot\theta}{r} u_\theta \right], \quad (128)$$

and it satisfies the transport equation

$$\frac{\partial \phi_f}{\partial t} - \nabla \cdot [(1 - \phi_{f,0})\nabla p] = 0, \quad (129)$$

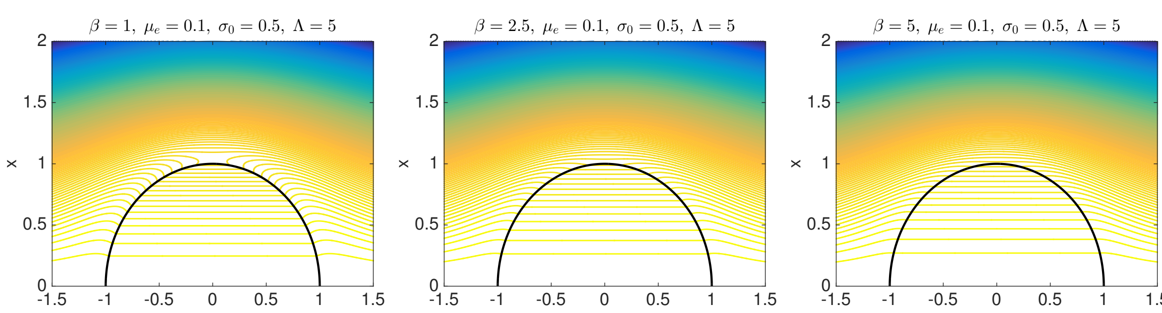

FIG. 18. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 0.1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 1$, 2.5 and 5.

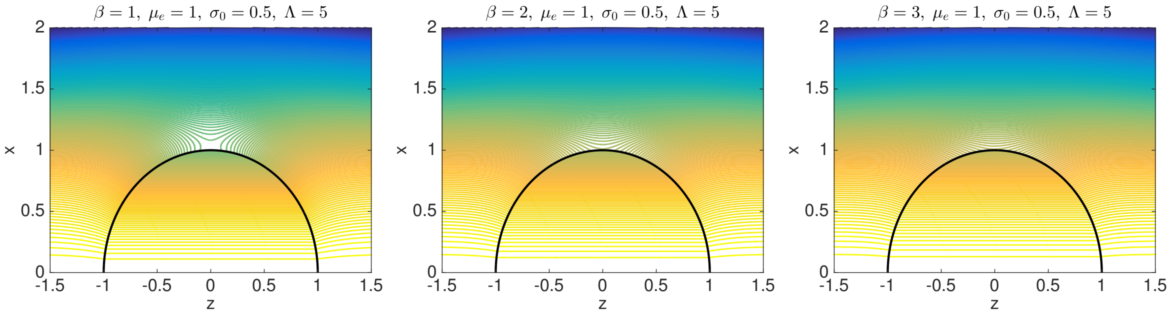

FIG. 19. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 1$, 2 and 3.

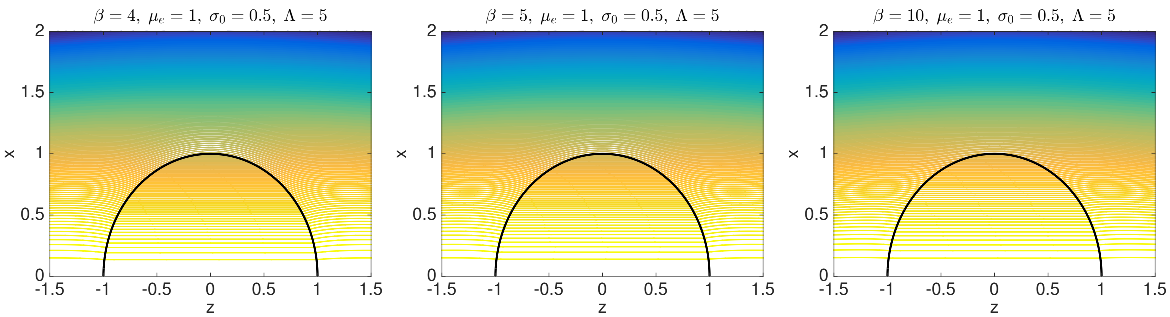

FIG. 20. Streamlines around a spherical Darcy drop with linear poroelasticity, $\mu_e = 1$, $\sigma_0 = 0.5$ and $\Lambda = 5$. From left to right $\beta = 4$, 5 and 10.

with $\nabla p = \nabla \cdot \sigma$. Equation (129) Can be Cast in terms of displacement field as

$$r^3 f_{rrr} + 2r^2 g_{rr} + 4r^2 f_{rr} - r^3 f_{rt} - 2r f_r - 2r^2(f_t + g_t) - 4(f + g) = 0. \tag{130}$$

The consistency condition ($\partial^2 p/\partial r \partial\theta = \partial^2 p/\partial\theta\partial r$) gives another equation

$$r^3 g_{rrr} + 3r^2 g_{rr} + r^2 f_{rr} - 2r g_r - 2(f + g) = 0. \tag{131}$$

Assuming a time dependence of $e^{-\lambda^2 t}$ with $\lambda \in R$, we Can solve for f and g by first observing that equation (130) Can be reCast in terms of $\Phi(r)e^{-\lambda^2 t} \equiv r f_r + 2(f + g)$ as

$$r^2 \Phi'' - (2 - \lambda^2 r)\Phi = 0, \tag{132}$$

with a regular solution (at $r = 0$) as

$$\Phi_1 = -\cos(\lambda r) + \frac{\sin(\lambda r)}{\lambda r}. \tag{133}$$

The two functions f and g are expressed in terms of Φ_1 as

$$f_\lambda = e^{-\lambda^2 t}\left[a_\lambda + b_\lambda r^2 - \frac{c_\lambda}{\lambda^2 r^2}\left(2\cos(\lambda r) + \frac{(-2 + \lambda^2 r^2)\sin(\lambda r)}{\lambda r}\right)\right], \tag{134}$$

$$g_\lambda = e^{-\lambda^2 t}\left[-a_\lambda - 2b_\lambda r^2 - \frac{c_\lambda}{\lambda^2 r^2}\left(\cos(\lambda r) - \frac{\sin(\lambda r)}{\lambda r}\right)\right]. \tag{135}$$

We assume that the deformation of the porous drop shape takes the form $\Gamma = r - (1 + \mathrm{Ca}b(t)\cos\theta) = 0$, where Ca is the capillary number, and $b(t)$ is the magnitude of small deformation. Further assume that $b(t) \to b_{eq}$ as $t \to \infty$. For convenience in the following derivation we decompose the functions f and g in the form

$$f(r,t) = f^*(r,t) + \sum_\lambda f_\lambda(r,t), \quad g(r,t) = g^*(r,t) + \sum_\lambda g_\lambda(r,t),$$

where the functions $f^*(r,t)$ and $g^*(r,t)$ satisfy the steady state equations (113-114) with boundary conditions

$$f^*(1,t) = \mathrm{Ca}b(t), \tag{136}$$

$$\sigma^*_{rr}(1,t) = 2\sigma_0(t)\cos\theta. \tag{137}$$

The boundary conditions for f_λ and g_λ are

$$f_\lambda(0,t) = f_\lambda(1,t) = 0, \tag{138}$$

$$f'_\lambda + 2\Lambda(f_\lambda + g_\lambda) = 0 \text{ at } r = 0 \text{ and } r = 1. \tag{139}$$

The condition $f_\lambda(0,t) = 0$ gives $a_\lambda = c_\lambda/3$. The condition $f_\lambda(1,t) = 0$ gives

$$b_\lambda = \left[-\frac{1}{3} + \frac{1}{\lambda^2}\left(2\cos\lambda + (-2+\lambda)\frac{\sin\lambda}{\lambda}\right)\right] c_\lambda. \tag{140}$$

$f'_\lambda + 2\Lambda(f_\lambda + g_\lambda) = 0$ at $r = 0$ is automatically satisfied. The last condition is

$$2b_\lambda\lambda^3(-1+\Lambda) + c_\lambda\left[\lambda\left(-6+\lambda^2+6\Lambda\right)\cos(\lambda) + \left(6 - 6\Lambda + \lambda^2(-3+2\Lambda)\right)\sin\lambda\right] = 0. \tag{141}$$

To satisfy the above condition with non-trivial solution, the eigenvalue λ must satisfy the characteristic equation

$$\frac{2\lambda^3}{3}(-1+\Lambda) - \lambda\left(\lambda^2 + 10(-1+\Lambda)\right)\cos\lambda - \left((5-4\Lambda)\lambda^2 + 10(-1+\Lambda)\right)\sin\lambda = 0. \tag{142}$$

The eigenvalue λ can be obtained by numerically finding roots of the characteristic equation. Figure (21)(a) shows the characteristic equation versus λ for different values of Λ with $\Lambda = 0.5$ for the curve at the bottom and $\Lambda = 20$ at the top. We note that no real eigenvalues can be found for large Λ, and instead complex roots are expected as shown in figure (**??**)(b) for $\Lambda = 10$, with the corresponding real part in figure (**??**(c). Figure (22)(a) shows the eigenfunction for the first non-zero eigenvalue with $\Lambda = 0.6$. Figure (22)(b) shows the radiation of the radial stress of the strain σ_{rr} for $\lambda = 3.7045$ with $\Lambda = 0.6$.

The boundary conditions for the dynamics of a deformable porous drop under a uniform streaming flow are

$$f^*(1,t) + f_\lambda(1,t) = f^*(1,t) = \mathrm{Ca}b(t), \tag{143}$$

$$\sigma_{rr}(1,\theta,t) = 2\sigma_0\cos\theta, \tag{144}$$

$$\mathbf{U}\cdot\mathbf{e}_r = \phi_f\mathbf{v}_f\cdot\mathbf{e}_r \text{ at } r = 1, \tag{145}$$

$$\beta\left(\mathbf{U}\cdot\mathbf{e}_\theta - \phi_f\mathbf{v}_f\cdot\mathbf{e}_\theta\right) = \frac{\tau_{r\theta} + \sigma_{r\theta}}{2} \text{ at } r = 1, \tag{146}$$

$$\tau_{rr} = -P_e + \mu_e\left(\nabla U + (\nabla U)^T\right)_{rr} = -p_i + 2b(t)\cos\theta \text{ at } r = 1, \tag{147}$$

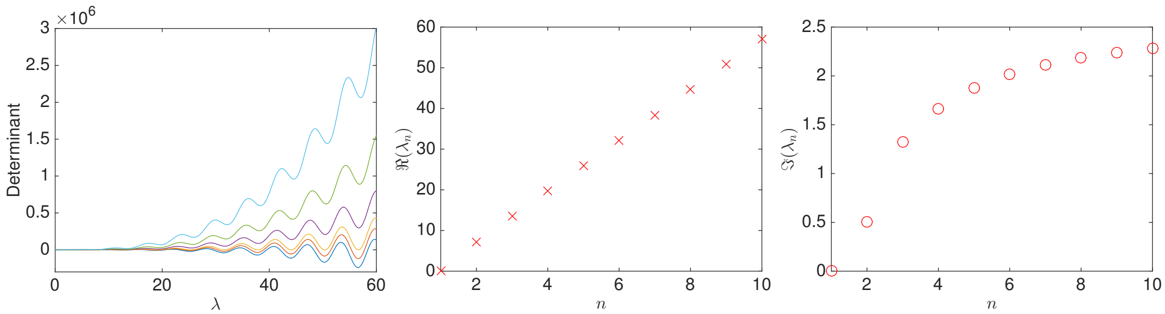

FIG. 21. (a) Characteristic equation for $\Lambda = 0.5$, 1.5, 2.5, 5.0, 10, and 20 from bottom to top curves. (b) Imaginary part of the complex eigenvalue when $\Lambda = 10$. (c) Real part of the complex eigenvalue when $\Lambda = 10$.

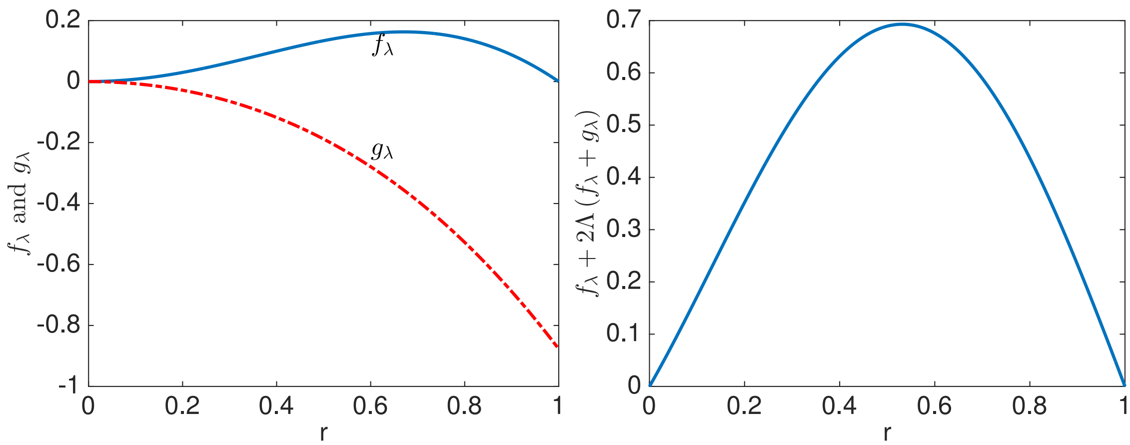

FIG. 22. (a) Spatial variation of the eigen functions f_λ and g_λ for the first non-zero eigenvalue with $\Lambda = 0.6$. (b) Radial stress of the strain σ_{rr} for $\Lambda = 0.6$ and $\lambda = 3.70448$.

Recall that the solid and fluid velocity fields are

$$\mathbf{v}_s = \mathbf{q} + k\nabla p, \tag{148}$$

$$\mathbf{v}_f = \mathbf{q} - \left(\frac{1-\phi_f}{\phi_f}\right) k\nabla p, \tag{149}$$

with the local volume average velocity field $\mathbf{q}$ to be determined. The interior pressure field

$$p_i = \frac{\cos\theta}{2r}\left[4(f+g) + (1+\Lambda)rf_r + 2(-1+\Lambda)rg_r + 2(-1+\Lambda)r^2 g_{rr}\right]. \tag{150}$$

Thus all the coefficients are expressed of either the physical parameters, $b(t)$ and/or the initial condition. To close the system we need an extra condition for the dynamics of the drop shape deformation $b(t)$. We start from the conservation law that the total volume integral of the solid skeleton is a constant with respect to time:

$$
\begin{aligned}
\frac{d}{dt}\iiint\limits_V (1-\phi_f)\,dV &= \frac{d}{dt}\text{ constant } = 0 \qquad (151)\\
\frac{d}{dt}\left[\frac{4\pi}{3}(1+\mathrm{Ca}^2 b(t)^2)\right] - \frac{d}{dt}\iiint\limits_V \phi_f dV &= 0\\
\frac{d}{dt}\iiint\limits_V \phi_f dV &= \frac{8\pi}{3}\mathrm{Ca}^2 b b',\\
\iiint\limits_V \frac{\partial \phi_f}{\partial t} dV + 2\pi\int_0^\pi \phi_f(1+\mathrm{Ca}b(t),\theta,t)\frac{d}{dt}\mathrm{Ca}b(t)\cos\theta\sin\theta d\theta &= \frac{8\pi}{3}\mathrm{Ca}^2 b b',
\end{aligned}
$$

with

$$
\begin{aligned}
\iiint\limits_V \frac{\partial \phi_f}{\partial t} dV &= \iiint\limits_V -\nabla\cdot\left[\phi_f \mathbf{q} - (1-\phi_f)\frac{k(\phi_f)}{\mu}\nabla p\right] dV, \qquad (152)\\
&= -\oiint\limits_S \left[\phi_f \mathbf{q} - (1-\phi_f)\frac{k(\phi_f)}{\mu}\nabla p\right]\mathbf{n} dS = -\oiint\limits_S \phi_f \mathbf{v}_f \mathbf{n} dS.
\end{aligned}
$$

The surface integral of the local fluid velocity can be calculated with the assumption that $k(\phi_f)/\mu \approx k_0/\mu =$ constant , $f(r,t) = f_0(t) + f_2(t)r^2$ and $g(r,t) = -f_0(t) + g_2(t)r^2$:

$$
\oiint\limits_S \frac{k(\phi_f)}{\mu}\nabla p\cdot\mathbf{n}dS = \frac{8\pi k_0}{3\mu}\mathrm{Ca}b\left[(3+\Lambda)f_2(t) + (-1+3\Lambda)g_2(t)\right]. \qquad (153)
$$

Thus together the evolution of the drop shape deformation is described by

$$
b' = -\frac{k_0}{\mu}\left[(3+\Lambda)f_2(t) + (-1+3\Lambda)g_2(t)\right]. \qquad (154)
$$

In the above derivation we assume that the fluid interface moves at the local velocity of the porous structure

$$
\mathbf{v}_s\cdot\mathbf{e}_r = \frac{d\mathbf{u}_s}{dt}\cdot\mathbf{e}_r = \mathrm{Ca}\frac{db(t)}{dt}\cos\theta \text{ at } r=1. \qquad (155)
$$

In particular, in the small-deformation limit,

$$
\mathbf{v}_s = \frac{d\mathbf{u}_s}{dt} \approx \frac{\partial \mathbf{u}_s}{\partial t},\;\; \mathbf{v}_s\cdot\mathbf{e}_r = \mathrm{Ca}b'\cos\theta \text{ at } r=1, \rightarrow b(t) = \frac{1}{\mathrm{Ca}}f(1,t). \qquad (156)
$$

Recall Ca is the Capillary number, and $b(t)$ is the magnitude of small deformation of the drop shape $\Gamma = r - (1+\mathrm{Ca}b(t)\cos\theta) = 0$.

To solve for the dynamics, we write the functions $f(r,t)$ and $g(r,t)$ as follows based on previous results for the steady solution:

$$
\begin{aligned}
f(r,t) &= f_0(t) + f_2(t)r^2 + \sum_\lambda e^{-\lambda^2 t}\left[a_\lambda + b_\lambda r^2 - \frac{c_\lambda}{\lambda^2 r^2}\left(2\cos(\lambda r) + \frac{(-2+\lambda^2 r^2)\sin(\lambda r)}{\lambda r}\right)\right],\\
g(r,t) &= -f_0(t) + g_2(t)r^2 + \sum_\lambda e^{-\lambda^2 t}\left[-a_\lambda - 2b_\lambda r^2 - \frac{c_\lambda}{\lambda^2 r^2}\left(\cos(\lambda r) - \frac{\sin(\lambda r)}{\lambda r}\right)\right],
\end{aligned}
$$

with a_λ and b_λ related to c_λ from $f_\lambda = 0$ at $r = 0$ and $r = 1$, respectively. The coefficients c_λ are determined from the initial conditions, and the time-dependent coefficients $f_0(t)$, $f_2(t)$, and $g_2(t)$ are determined from the boundary conditions in equations (143-147).

First we obtain f_0, f_2 and g_2 in terms of $b(t)$ from solving equations (143), (144) and (147). Then we substitute these expressions into equations (145) and (146) to express the velocity field outside the porous drop in terms of $b(t)$ and $b'(t)$. The dynamics of $b(t)$ is obtained by solving equation (154).