

Contents

1.	Cyclotron	1
1.1	Introduction	1
1.2	Classical cyclotron	2
1.2.1	Closed orbit, time of flight	2
1.2.2	Acceleration	3
1.2.3	Focusing	3
1.2.4	Isochronism	4
1.2.5	Resonant acceleration	5
1.3	Relativistic cyclotron	5
1.3.1	Thomas focusing	6
1.3.2	Orbits	7
1.3.3	Focusing	7
1.3.4	Acceleration	7
1.4	Exercises: hints, help	7
1.5	Bibliography	8

Chapter 1

Cyclotron

1.1 Introduction

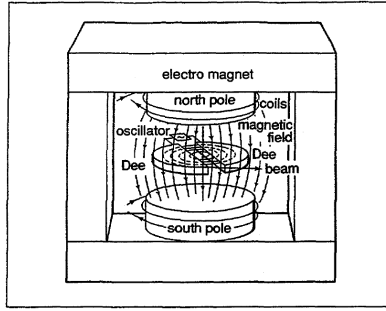


Fig. 1.1 A resonant acceleration device: the cyclotron [1].

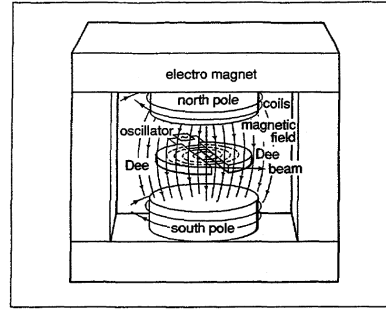


Fig. 1.2 The world greatest beam power cyclotron, at PSI.

In a uniform field B a particle with rest mass m_0 , relativistic mass $m = \gamma m_0$, charge q , momentum $p = mv$, undergoes a circular motion characterized by its radius R , with a revolution period T_{rev} which depends on the field and on the particle charge and mass, but *not* on R neither on the particle velocity. Thus repeated acceleration can be provided by a system (the double-gap formed by the double dee in Fig. 1.1) with fixed frequency f_{RF} in resonance, at some harmonic integer h , with the orbital frequency $f_{\text{rev}} = T_{\text{rev}}^{-1}$ of the particle. These three quantities satisfy, respectively

$$R = p/qB, \quad 2\pi f_{\text{rev}} = v/R = qB/m, \quad f_{\text{RF}} = hf_{\text{rev}} \quad (1.1)$$

These expressions hold in both cases of *classical* cyclotron, which operates in sub-relativistic regime ($\gamma m_0 \sim m_0$), and *isochronous* cyclotron which operates in relativistic regime (γ distant from 1).

Recommended readings, in order to prepare this exercise series:

Cyclotron: bibliography documents listed in page 8.

Zgoubi Users' Guide : (i) applied mathematics methods i sections: 1.2, 1.3.3, 1.4, complete/strengthen with classical Applied Mathematics textbooks. (ii) DIPOLES procedure and ancillaries.

1.2 Classical cyclotron

In the “classical limit”, the relative increase in the revolution period in the uniform field cyclotron, $\Delta T_{\text{rev}}/T_{\text{rev}} = \gamma - 1$, does not exceed 2 – 3%. This translates in a maximum $E - m^2 \approx 25$ MeV kinetic energy for protons, 50 MeV for D and α particles. That is enough energy to transmute all nuclei and, as a matter of fact, the classical cyclotron allowed discovering numerous nuclear reactions and isotopes.

1.2.1 Closed orbit, time of flight

A cyclotron accelerator is a typical case where just putting together drifts and dipoles in a matrix transport works at best at a particular energy and for paraxial motion, yet won't permit designing an accelerator.

The common solution is to use a field map. However, mathematical models for the field are also a good approach in a preliminary design phase due to the flexibility it brings in parameters as orbit excursion, isochronism, focusing, etc.

These two techniques are employed in the exercises, to develop on the properties of particle motion in cyclotrons.

• Exercise 1.2-1

1.2-1.a - Construct a 60-degree sector mid-plane 2-dimensional field map, with constant vertical field $B=0.5$ T, covering $R=25$ to 70 cm (180 keV to 6 MeV (an injector to a higher energy installation for instance, using stripping extraction). Use a uniform mesh in cylindrical coordinates.

1.2-1.b - Track H- ions on their periodic circular orbit across this 60-degree sector, assuming the orbit centered at the center of the field map, for a few different momenta ranging from 1 MeV to 5 MeV. Plot, as a function of momentum, their equilibrium orbit radius R and the time of flight T_{rev} on that orbit, from both ray-tracing and theory on a same graphic. Establish that the slight TOF increase with momentum observed is as expected from relativity. Check evolution of these quantities with field map mesh density and with step size.

- Exercise 1.2-2 CYCLOTRON or DIPOLE are two procedures amongst others, that provide an analytical magnetic field model $B_y(r, \theta)|_{y=0}$ in the median plane of a dipole magnet. Use it to produce a uniform field and re-derive the results above. From the two series of results, comment on various pros and cons of the two methods, analytical field models and fieldmaps.

1.2.2 Acceleration

The RF gap provides a voltage

$$V_{\text{RF}}(t) = \hat{V} \cos \dots \quad (1.2)$$

Particles are accelerated as long as they belong in the $[-90, +90]$ degree phase interval, the closer to $\phi = 90$ deg, the smaller the number of turns (the time interval) necessary to reach the extraction radius of the cyclotron. A deviation of the field B from the isochronous value $2\pi m f_{\text{rev}}/q$ will result in a shift in the arrival phase of the particle at the RF gap amounting to

$$\Delta(\sin \phi) = 2\pi h n \Delta B/B \quad (1.3)$$

***** prendre de valeurs R , B , etc. realistes, e.g. in ../biblio/22047216.pdf, 23001796.pdf*****

- Exercise 1.2.2-1 Assume an accelerating double-gap configuration as in Fig. 1.1. What is the minimum number of turns expected from 5 keV to 10 MeV ? Track a particle over that range, play with the RF phase, conclude on the expectations. In a $V(t)$ diagram, plot the position of the particle along the $V(t)$ curve at the accelerating gap, for a magnetic field defect $\Delta B/B = 10^{-4}$, homogeneous, in the previous sector map.

1.2.3 Focusing

Let B_r , B_y be the radial and axial components of the magnetic field, respectively, $x = r - R$ a small radial displacement with respect to the reference circular orbit, $\omega_{\text{rev}} = 2\pi f_{\text{rev}}$ the angular frequency of the circular motion. The radial and axial strengths experienced by a particle moving in the vicinity of that reference orbit write, to the first order in the radial, x , and axial, y , coordinates

$$F_x = m\ddot{x} = -qvB_y + m\frac{v^2}{r} \approx -qv(B_y|_{x=0} + \frac{\partial B_y}{\partial r}x) + m\frac{v^2}{R}(1 - \frac{x}{R}),$$

yielding $\ddot{x} + \omega_r^2 x = 0$

$$F_y = m\ddot{y} = qvB_r \approx qv\frac{\partial B_r}{\partial y}y = qv\frac{\partial B_y}{\partial r}y, \text{ yielding } \ddot{y} - \omega_y^2 y = 0 \quad (1.4)$$

wherein $\omega_r^2 = \omega_{\text{rev}}^2(1 + \frac{R}{B} \frac{\partial B_y}{\partial r})$, $\omega_y^2 = \omega_{\text{rev}}^2 \frac{R}{B} \frac{\partial B_y}{\partial r}$. Focusing by a restoring force appears owing to the use of a magnetic field with radial index

$$k = \frac{R}{B} \frac{\partial B_y}{\partial r} \Big|_{x=0, y=0} \quad (1.5)$$

The two quantities

$$\nu_r = \omega_r / \omega_{\text{rev}} = \sqrt{1+k}, \quad \nu_y = \omega_y / \omega_{\text{rev}} = \sqrt{-k} \quad (1.6)$$

are known respectively as the radial and the axial “wave number” of the oscillatory motion in the neighborhood of the reference circular orbit. Note that $\nu_r^2 + \nu_y^2 = 1$. Vertical motion stability requires k to be negative: B_y (respectively, the magnet gap) is slowly decreasing (increasing) with radius, restoring force toward the median plane. Focussing in both radial and axial motions requires $0 < k < -1$, a condition known as “weak focusing”. Note that at low energy the electric field in the region of the accelerating gap also contributes to the focusing, an aspect omitted here.

- Exercise 1.2.3-1 Plot two particle trajectories that demonstrate the value of the radial wave number in the uniform field of Sec. 1.2. Conclude on orbit and horizontal motion stability. Derive the vertical transport matrix from ray-tracing, conclude on the stability of the vertical motion in a uniform field.
- Exercise 1.2.3-2 Back to the field map of exercise 1.2-1, or to the analytical model of exercise 1.2-2: introduce a field index $-1 < k < 0$. Plot the radial and vertical phase space of a 5 MeV ion on a $1\mu\text{m}$ normalized invariant. Compute its radial and axial motion wave numbers, ν_r and ν_y , using two different methods, namely, 1-turn mapping and Fourier analysis of multi-turn motion. From multiturn tracking, generate the envelope of a 5 MeV beam around the ring cyclotron.
- Exercise 1.2.3-3 Using either the field map or the analytical model devised in the exercise 1.2.3-2, plot the energy dependence of the reference orbit radius, $R(E)$. Plot $\nu_r^2 + \nu_y^2$ as a function of radius, compare with the value of the field index. On a common graphic, plot the horizontal phase space of the 1, 10, 20, 50 MeV particle motion, assuming the latter on a $1\mu\text{m}$ normalized invariant in each case. Plot the vertical phase space motion for these very energies. Plot the components of the field vector experienced by a particle as a function of azimuthal angle, over a few turns.

1.2.4 Isochronism

The condition for vertical focusing, $-1 < k < 0$ spoils the isochronism: the guiding field B and thus $\omega_{\text{rev}} = qB/m$ decrease with R (with increasing

momentum). As a consequence, the arrival time of a particle at the RF gap (by extension the “RF phase”) is not constant (ABCDE path)

• Exercise 1.2.4-1

bunching : particle beam injected into the cyclotron necessarily gets bunched, at the frequency of the RF (the time interval between two bunches is an RF period)

Classical cyclotron $0 < n < 1$

Isochronous cyclotron, $0 < n < 1$ cannot be satisfied as $B \propto \gamma$. Introduce Thomas focusing, using sector magnets, no longer slow r -decrease, weak focusing uniform field.

1.2.5 Resonant acceleration

A tight tuning between particle time-of-flight, which means in practice the field B , and the RF frequency is necessary in order to reach the resonance condition, $f_{\text{RF}} = hf_{\text{rev}}$.

The accelerating gap in the simulation provide $V_{\text{RF}}(t) = \hat{V} \sin(\omega_{\text{RF}} t) \dots \dots$

(i) Check the evolution of the particle acceleration when getting away from the resonant frequency:

- Plot the particle position along the $V_{\text{RF}}(t)$ curve in a $V_{\text{RF}}(t)$ diagram.
- Plot the final phase ϕ_{final} as a function of $\Delta B/B$.
- Find the tolerable deviation from the isochronous field, $\Delta B/B$, if particles are required to be dispersed in momentum by less than $dp/p = 10^{-3}$ at the end of the acceleration cycle.

1.3 Relativistic cyclotron

The bad news with relativistic energies, is, the cyclotron resonance $\omega_0 = qB/\gamma m_0$, with $R = \beta c/\omega_0$ yields

$$k = \frac{R}{B} \frac{\partial B}{\partial R} = \frac{\beta}{\gamma} \frac{\partial \gamma}{\partial \beta} = \beta^2 \gamma^2$$

k is positive and increases with energy, the weak focussing condition $-1 < k < 0$ is not satisfied.

The time of flight on the equilibrium orbit, momentum $p = qB\rho$ and circumference \mathcal{C} , is $T = \mathcal{C}/\beta c = 2\pi\gamma m_0/qB$. Isochronism requires p -invariant time of flight, $dT/dp = 0$. Differentiating the previous expression, this requirement yields

$$B = \frac{B_0}{\gamma_0} \gamma$$

with B_0 and γ_0 some reference conditions and time of flight $T_0 = (2\pi m_0/q)(\gamma_0/B_0)$. In other words, isochronism requires $B(r) \propto \gamma$, which yields vertical defocusing.

That was the end of the story in the late 1930s. Hans Bethe (1937) : “... it seems useless to build cyclotrons of larger proportions than the existing ones... an accelerating chamber of 37 cm radius will suffice to produce deuterons of 11 MeV energy which is the highest possible...”. Frank Cole : “If you went to graduate school in the 1940s, this inequality ($1 < k < 0$) was the end of the discussion of accelerator theory.”

Until...

1.3.1 *Thomas focusing*

- 1938, L.H. Thomas, “The Paths of Ions in the Cyclotron”, introduces the “Thomas focusing”, based on separate sector bending, namely, “edge-focusing”,
- 1954, Kerst, spiral edges increase vertical focusing further $\nu_z = \sqrt{-k + F^2(1 + 2 \tan^2 \xi)}$, $F = F_{lutter} = \frac{\langle B^2 \rangle - \langle B \rangle^2}{\langle B \rangle^2}$
- That allowed having $B(r)$ increase in proportion to γ , so to ensure constant RF frequency ($\omega_0 = qB/\gamma m$), while *preserving vertical focusing*.
- Modern cyclotrons still rely on these principles
- Cyclotron is limited in energy by its field strength and magnet size.

Exercise

On RF harmonic 1, which RF frequency applied on dees is needed to accelerate a proton in a \sim uniform, 1 Tesla magnetic field ?

What is the energy gained by a proton when it reaches $r = 0.3$ m ?

Exercise

On RF harmonic 1, which RF frequency applied on dees is needed to accelerate a proton in a \sim uniform, 1 Tesla magnetic field ?

What is the energy gained by a proton when it reaches $r = 0.3$ m ?

Answers :

1/ $f_{rev.} = eB/2\pi m = c^2 B/2\pi(mc^2/e) = 9 \cdot 10^{16} \times 1/2\pi 10^9 \approx 15 \text{ MHz}$
 Two accelerating gaps at 180 degrees $\Rightarrow f_{RF} = f_{rev.}$

2/ From $r = \frac{mv}{eB}$ one gets $\frac{1}{2}mv^2 \equiv E = \frac{1}{2} \frac{(eBr)^2}{m} = \frac{1}{2} \frac{eB^2 r^2 c^2}{(mc^2/e)}$ hence
 $\frac{E}{e} (eV) = \frac{1}{2} \frac{B^2 r^2 c^2}{(mc^2/e)}$
 $\frac{E}{e} (eV) = \frac{1}{2} \frac{1^2 (0.3)^2 (3 \cdot 10^8)^2}{10^9} \approx 4.3 \text{ MeV}.$

1.3.2 Orbits

- find the orbits, energy dependence of radius, time of flight.

Plot these quantities against theoretical expectations

1.3.3 Focusing

Check $Q_x^2 + Q_y^2$. Plot radial dependence of field index $n(r)$, check $0 < n(r) < 1$.

1.3.4 Acceleration

Introduce a double-Dee simulation. Hint: use CAVITE

Accelerate a complete cycle, on $h=1$, $h=3$. In $V(t)$ space, plot $V(t)$ path of reference particle (use PRINT in CAVITE).

1.4 Exercises: hints, help

• Exercise 1.2.2-1

The field map has to be in a uniform mesh, preferably in cylindrical coordinates due to the symmetry of the problem. Namely, $B(R, \theta)$ on nodes at fixed ΔR , $\Delta \theta$ intervals. Aim $\approx 1 \text{ mm}$ intervals $R \times \Delta \theta$ at $R = R_{\min}$. Amongst other possibilities, possible ways generate that field map :

- a short computer program,
- use DIPOLE-M procedure. Given the option “ $IL = 2$ ”, the field map that it generates will be stored in a dedicated file,
- use DIPOLE, or CYCLOTRON, together with OPTIONS, option “CONSTY”.

8 **Learning Particle Accelerators - A Computer Game**

1.5 Bibliography

Stambach CAS

R Baartman Cyclotrons: Classic to FFAG