

1. Given $A \in \mathbb{R}^{1000 \times 2}$, $B \in \mathbb{R}^{2 \times 1000}$ and $C \in \mathbb{R}^{1000 \times 1}$, write down the number of real number multiplications and additions needed to compute the product ABC by the two procedures indicated below:

- $(AB)C$
- $A(BC)$

2. We build a computer, where the real numbers are represented using 5 digits as explained below:

S	A	B	C	E
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where

- S is the sign bit; 0 for positive and 1 for negative
- A,B,C: First three significant digits in decimal expansion with the decimal point occurring between A and B
- E is the exponent in base 10 with a bias of 5
- All digits after the third significant digit are chopped off
- $+0$ is represented by setting $S = 0$ and $A = 0$ (B,C,E) can be anything
- -0 is represented by setting $S = 1$ and $A = 0$ (B,C,E) can be anything
- $+\infty$ is represented by $S = 0$, $A = B = C = E = 9$
- $-\infty$ is represented by $S = 1$, $A = B = C = E = 9$
- Not A Number is represented by setting S other than 0 and 1.

For example, the number $\pi = 3.14159\dots$ is represented as follows. Chopping off after the third significant digit, we have $\pi = +3.14 \times 10^0$. Hence, the representation of π in our system is:

0	3	1	4	5
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The number -0.001259 is represented as follows. Chopping off after the third significant digit, we have the number as -1.25×10^{-3} . Hence, the representation in our system is:

1	1	2	5	2
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Answer the following questions:

- (a) How many non-zero floating point numbers (from now on abbreviated as FPN) can be represented by our machine (both positive and negative)?
- (b) How many FPNs are in the following intervals?
 - $(9, 10)$
 - $(10, 11)$
 - $(0, 1)$
- (c) Identify the smallest positive and largest positive FPN on this machine.
- (d) Identify the machine precision.
- (e) What is the smallest positive integer not representable exactly on this machine?
- (f) Consider the recurrence:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

with $a_1 = a_2 = 2.93$. Note that $a_1, a_2, 5$ and 4 are exactly represented on our machine. Compute a_n for $n \in \{3, 4, \dots, 10\}$ in our machine (Work out the values by hand). Note that at each step in the recurrence $5a_n$ and $4a_{n-1}$ will be both chopped down to the first three significant digits before the subtraction is performed.

3. Consider the following differential equation:

$$\frac{d^2u}{dt^2} = 0 \text{ with } u(0) = 2.95 \text{ and } \frac{du}{dt}(t=0) = 0$$

- Solve the differential equation analytically.
- Using Taylor series show that

$$\frac{u(t+3\delta t) - 3u(t+\delta t) + 2u(t)}{3(\delta t)^2} = \frac{d^2u}{dt^2} + \mathcal{O}(\delta t)$$

- Discretize the differential equation by using the above finite difference for u_{tt} , i.e., setting $u_n = u(n\delta t)$, obtain

$$u_{n+3} = 3u_{n+1} - 2u_n$$

where $n \geq 0$.

- Take $u_0 = 2.95$, $u_1 = 2.95$ and $u_2 = 2.95$ (Note that we have satisfied $\frac{du}{dt}(t=0) = 0$ by taking the forward and central difference). Solve for u_n for all n manually (without using a computer). Does this match with the analytic solution?
- Solve the recurrence obtained above (use Octave/MATLAB) and display $u_0, u_1, u_2, \dots, u_{69}$ till the first 16 digits after the decimal in successive lines. Does this match with the analytic solution or the solution of the discretized equation? Explain your observation.
- Rewrite the equations in matrix form, i.e.,

$$Au = b$$

- Plot the condition number of the matrix A as a function of n .
- Comment on how the condition number scales with n .
- Comment on the relationship of the condition number and accuracy of the solution u_n obtained.

4. Recall that $\text{fl}(x \oplus y) = (x \oplus y)(1 + \delta)$, where $x, y \in \mathbb{R}$, $\oplus \in \{+, -, \times, \div\}$, $|\delta| \leq \mu$, μ is the machine precision and $\text{fl}(z)$ is the floating point representation of z . If $a, b \in \mathbb{R}^{n \times 1}$ and $\frac{|\text{fl}(a^T b) - a^T b|}{\|a\|_2 \|b\|_2} = \phi_\mu(n)$, obtain an upper bound $\phi_\mu(n)$ in terms of n and the machine precision μ .