1. Given $A \in \mathbb{R}^{1000\times 2}$, $B \in \mathbb{R}^{2\times 1000}$ and $C \in \mathbb{R}^{1000\times 1}$, write down the number of real number multiplications and additions needed to compute the product $ABC$ by the two procedures indicated below:
   - $(AB)C$
   - $A(BC)$

2. We build a computer, where the real numbers are represented using 5 digits as explained below:

| | | | | |
|---|---|---|---|---|
| S | A | B | C | E |

where

- S is the sign bit; 0 for positive and 1 for negative
- A,B,C: First three significant digits in decimal expansion with the decimal point occuring between A and B
- E is the exponent in base 10 with a bias of 5
- All digits after the third significant digit are chopped off
- $+0$ is represented by setting $S = 0$ and $A = 0$ (B,C,E) can be anything
- $-0$ is represented by setting $S = 1$ and $A = 0$ (B,C,E) can be anything
- $+\infty$ is represented by $S = 0$, $A = B = C = E = 9$
- $-\infty$ is represented by $S = 1$, $A = B = C = E = 9$
- Not A Number is represented by setting $S$ other than 0 and 1.

For example, the number $\pi = 3.14159\ldots$ is represented as follows. Chopping off after the third significant digit, we have $\pi = +3.14 \times 10^0$. Hence, the represention of $\pi$ in our system is:

| | | | | |
|---|---|---|---|---|
| 0 | 3 | 1 | 4 | 5 |

The number $-0.001259$ is represented as follows. Chopping off after the third significant digit, we have the number as $-1.25 \times 10^{-3}$. Hence, the represention in our system is:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 2 |

Answer the following questions:

(a) How many non-zero floating point numbers (from now on abbreviated as FPN) can be represented by our machine (both positive and negative)?

(b) How many FPNs are in the following intervals?
   - $(9, 10)$
   - $(10, 11)$
   - $(0, 1)$

(c) Identify the smallest positive and largest positive FPN on this machine.

(d) Identify the machine precision.

(e) What is the smallest positive integer not representable exactly on this machine?

(f) Consider the recurrence:

$$a_{n+1} = 5a_n - 4a_{n-1}$$

with $a_1 = a_2 = 2.93$. Note that $a_1$, $a_2$, 5 and 4 are exactly represnted on our machine. Compute $a_n$ for $n \in \{3, 4, \ldots, 10\}$ in our machine (Work out the values by hand). Note that at each step in the recurrence $5a_n$ and $4a_{n-1}$ will be both chopped down to the first three significant digits before the subtraction is performed.

3. Consider the following differential equation:

$$\frac{d^2u}{dt^2} = 0 \text{ with } u(0) = 2.95 \text{ and } \frac{du}{dt}(t=0) = 0$$

- Solve the differential equation analytically.
- Using Taylor series show that
$$\frac{u(t+3\delta t) - 3u(t+\delta t) + 2u(t)}{3\left(\delta t\right)^2} = \frac{d^2u}{dt^2} + \mathcal{O}\left(\delta t\right)$$
- Discretize the differential equation by using the above finite difference for $u_{tt}$, i.e., setting $u_n = u(n\delta t)$, obtain
$$u_{n+3} = 3u_{n+1} - 2u_n$$
where $n \geq 0$.
- Take $u_0 = 2.95$, $u_1 = 2.95$ and $u_2 = 2.95$ (Note that we have satisfied $\frac{du}{dt}(t=0) = 0$ by taking the forward and central difference). Solve for $u_n$ for all $n$ manually (without using a computer). Does this match with the analytic solution?
- Solve the recurrence obtained above (use Octave/MATLAB) and display $u_0, u_1, u_2, \ldots, u_{69}$ till the first 16 digits after the decimal in successive lines. Does this match with the analytic solution or the solution of the discretized equation? Explain your observation.
- Rewrite the equations in matrix form, i.e.,
$$Au = b$$
- Plot the condition number of the matrix $A$ as a function of $n$.
- Comment on how the condition number scales with $n$.
- Comment on the relationship of the condition number and accuracy of the solution $u_n$ obtained.

4. Recall that $\text{fl}\left(x \bigoplus y\right) = \left(x \bigoplus y\right)(1+\delta)$, where $x, y \in \mathbb{R}$, $\bigoplus \in \{+, -, \times, \div\}$, $|\delta| \leq \mu$, $\mu$ is the machine precision and $\text{fl}\left(z\right)$ is the floating point representation of $z$. If $a, b \in \mathbb{R}^{n\times 1}$ and $\dfrac{\left|\text{fl}\left(a^Tb\right) - a^Tb\right|}{\|a\|_2\|b\|_2} = \phi_\mu\left(n\right)$, obtain an upper bound $\phi_\mu(n)$ interms of $n$ and the machine precision $\mu$.