Detection of Almost-Cyclostationarity: An Approach Based on a Multiple Hypothesis Test

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***Abstract*—This work presents a technique to detect whether a signal is almost cyclostationary (ACS) or wide-sense stationary (WSS). Commonly, ACS (and also CS) detectors require a priori knowledge of the cycle period, which in the ACS case is not an integer. To tackle the case of unknown cycle period, we propose an approach that combines a resampling technique, which handles the fractional part of the cycle period and allows the use of the generalized likelihood ratio test (GLRT), with a multiple hypothesis test, which handles the integer part of the cycle period. We control the probability of false alarm based on the known distribution of the individual GLRT statistic, results from order statistics, and the Holm multiple test procedure. To evaluate the performance of the proposed detector we consider a communications example, where simulation results show that the proposed technique outperforms state-of-the-art competitors.**

I. Introduction

Many natural and manmade processes have periodic statistical characteristics. These processes are called cyclostationary (CS) and they are commonly encountered in various fields of science and technology, such as climatology, mechanics, astronomy, and communications [1]. For instance, in communications, periodicity is typically induced by modulation, sampling, and multiplexing operations [2].

Continuous-time signals are typically sampled before further processing. However, sampling a continuous-time CS signal with a sampling interval smaller than the period of the CS process generally yields an almost CS (ACS) rather than a CS discrete-time signal [3]. In this paper, we consider discrete-time zero-mean second-order ACS processes, where the covariance function is an almost-periodic function in the sense of Bohr [4].

The detection of ACS signals is an important problem with many applications. For instance, it is particularly relevant in the context of spectrum sensing for cognitive radio, where we need to detect vacant communication channels. Some cyclostationarity detectors test for non-zero cyclic autocorrelation function, such as the detector in [5], which extends [6] to multivariate signals. Others [7], [8] test for correlation between the process and its frequency-shifted version. A third class of ACS detectors determines whether the Loève spectrum has support on lines parallel to the stationary manifold [9]. However, all these detectors require a priori knowledge of the cycle period. Since existing cycle period estimators are either ad-hoc estimators [9], [10] or are only able to detect integer-valued cycle periods [10], [11], it is desirable to build a detector that does not require this information.

To this end, we propose a combination of a resampling technique and a multiple hypothesis test to detect the presence of ACS signals with unknown cycle period. Given a possible range of potential cycle periods we divide the problem into handling the integer part and the fractional part of the cycle period separately. For a given set of candidate fractional parts, the optimal resampling rate yields a process with integer-valued cycle period (or a sufficiently small fractional part), which allows us to apply the GLRT derived in [12]. In order to control the probability of false alarm we use Holm's multiple test procedure [13]. To apply this we use the distribution of an individual GLRT statististic derived from [14] combined with results from order statistics.

II. Problem Formulation

A continuous-time zero-mean scalar-valued process $u(t)$ is said to be (second-order) CS if its covariance function is periodic with period T_0, i.e.

$$r_{uu}(t,\tau) = \mathrm{E}\left[u(t)u^*(t-\tau)\right] = r_{uu}(t+T_0,\tau).$$

Sampling $u(t)$ with period $T_s < T_0$ yields a discrete-time ACS process $u[n]$ with cycle period T_0/T_s [3]. Hence, the covariance function $r_{uu}[n,m] = \mathrm{E}\left[u[n]u^*[n-m]\right]$ is an almost-periodic function in n in the sense of Bohr [4].

Now let us consider a continuous-time multivariate zero-mean CS process $\mathbf{u}(t) \in \mathbb{C}^L$ with *unknown* cycle period T_0. Sampling $\mathbf{u}(t)$ with period T_s yields the discrete-time process $\mathbf{u}[n]$. Since T_0 is unknown, the sampled process is generally ACS with non-integer cycle period

$$P = \frac{T_0}{T_s} = P_{\mathrm{int}} + \epsilon,$$

where $P_{\mathrm{int}} \in \{2, 3, \dots\}$ is the integer part, and $\epsilon \in [-0.5, 0.5)$ is the fractional part of the cycle period. Here and in the following, we assume that $T_0 \geq 1.5T_s$.

Our problem is to detect whether $\mathbf{u}[n]$ is wide-sense stationary (WSS) or ACS that is,

$$\mathcal{H}: \mathbf{u}[n] \text{ is WSS,}$$
$$\mathcal{A}: \mathbf{u}[n] \text{ is ACS,}$$

where we assume that $\mathbf{u}[n]$ is proper complex Gaussian. Furthermore, we assume to observe M i.i.d. realizations of $\mathbf{u}[n]$ of length N. The paper [12] presented the generalized likelihood ratio test (GLRT) for testing WSS vs. CS with known (integer-valued) cycle period. This GLRT can be applied by resampling $\mathbf{u}[n]$ such that the resampled process $\tilde{\mathbf{u}}[n]$ has an integer-valued cycle period (or at least a sufficiently small fractional part). The difficulty, however, is that 1) the integer part of the cycle period is unknown and 2) the required resampling rate Δ (i.e., the fractional part of the cycle period) is also unknown. We deal with these two issues separately.

III. Resampling and Multiple-Hypothesis-Test Procedure

We first deal with the unknown fractional part. For this, we use a resampling approach, which allows us to apply the GLRT from [12]. The issue of unknown integer part of the cycle period is addressed afterwards in Section III-B, where we propose a multiple hypothesis test.

A. Resampling Technique

To address the unknown fractional part of the cycle period let us first assume that $\mathbf{u}[n]$ is ACS with known P_{int}. Our goal is to find the factor Δ such that the cycle period of the resampled process $\tilde{\mathbf{u}}[n]$ is P_{int}. To this end, we specify a set of D candidate resampling rates

$$\Delta_d = \frac{P_{\text{int}}}{P_{\text{int}} + \epsilon_d}, \quad d = 1, \dots, D,$$

where $\epsilon_d = -0.5 + (d-1)/D$, and obtain the resampled signal $\tilde{\mathbf{u}}[n]$ for each d. The maximum likelihood estimate of ϵ (or, equivalently, Δ) is obtained by maximizing the likelihood under $\mathcal{A}$ for given integer part P_{int}. Equivalently, we can minimize the GLRT statistic for the test CS vs. WSS derived in [12] for each candidate Δ_d.

We briefly outline the computation of the statistic. More details can be found in [12]. The GLRT statistic for each candidate Δ_d is

$$\mathcal{G}(\Delta_d) = \prod_{k=1}^{N/P_{\text{int}}} \det\left(\hat{\mathbf{C}}_k\right), \tag{1}$$

where $\hat{\mathbf{C}}_k$ is the kth $LP_{\text{int}} \times LP_{\text{int}}$ diagonal block of

$$\hat{\mathbf{C}} = \left[\operatorname{diag}_L(\hat{\mathbf{S}})\right]^{-1/2} \operatorname{diag}_{LP_{\text{int}}}(\hat{\mathbf{S}}) \left[\operatorname{diag}_L(\hat{\mathbf{S}})\right]^{-1/2}.$$

Here, $\operatorname{diag}_L(\hat{\mathbf{S}})$ denotes a block-diagonal matrix obtained from the $L \times L$ blocks on the diagonal of

$$\hat{\mathbf{S}} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{z}_i \mathbf{z}_i^H,$$

where

$$\mathbf{z}_i = (\mathbf{L}_{N,N/P_{\text{int}}} \otimes \mathbf{I}_L)(\mathbf{F}_N \otimes \mathbf{I}_L)^H \mathbf{y}_i,$$

$\mathbf{L}_{N,N/P_{\text{int}}}$ is the commutation matrix, $\mathbf{F}_N$ is the DFT matrix, and $\mathbf{y}_i = [\tilde{\mathbf{u}}_i^T[0] \cdots \tilde{\mathbf{u}}_i^T[N-1]]^T$. Finally, the maximum likelihood estimate of Δ under $\mathcal{A}$ is found by minimizing the GLR, i.e.

$$\Delta_{\min} = \underset{\Delta_{d=1,\dots,D}}{\arg\min}\; \mathcal{G}(\Delta_d).$$

B. Multiple-Hypothesis-Test Procedure

Since the integer part of the cycle period is unknown as well, we propose a multiple hypothesis test for all possible periods. The test is implemented as a set of binary tests with a common null hypothesis. Hence, we test

$$\mathcal{H}: \; \mathbf{u}[n] \text{ is WSS}$$

versus the following set of alternatives

$$\begin{aligned}
\mathcal{A}_1 &: \; \mathbf{u}[n] \text{ is ACS with } P_{\text{int}} = 2 \\
\mathcal{A}_2 &: \; \mathbf{u}[n] \text{ is ACS with } P_{\text{int}} = 3 \\
&\vdots \\
\mathcal{A}_K &: \; \mathbf{u}[n] \text{ is ACS with } P_{\text{int}} = P_{\max}
\end{aligned}$$

where $P_{\max} = K + 1$ is the largest integer cycle period to be tested.

In order to reach a decision in the overall test $\mathcal{H}$ (WSS) vs. $\mathcal{A}$ (ACS), we reject $\mathcal{H}$ if it is rejected in at least one of the binary tests $\mathcal{H}$ vs. $\mathcal{A}_i$. Therefore, the probability of false alarm in the overall test is the probability of at least one false rejection, which is the family-wise error rate (FWER). In order to control the FWER we use Holm's sequentially rejective Bonferroni test [13]. Let $p_1, \; p_2, \; \dots, p_K$ be the p-values of the K individual tests, and let $p_{(1)} \leq p_{(2)} \leq \cdots \leq p_{(K)}$ be the ordered p-values with corresponding hypotheses $\mathcal{A}_{(1)}, \; \mathcal{A}_{(2)}, \; \dots, \mathcal{A}_{(K)}$. The individual tests are performed sequentially: if in the first step $p_{(1)} \geq \alpha/K$, we fail to reject $\mathcal{H}$ and stop. If $p_{(1)} < \alpha/K$, we reject $\mathcal{H}$, and we could continue testing the remaining $K-1$ hypotheses. However, since we want to reach a decision in the overall test $\mathcal{H}$ vs. $\mathcal{A}$, we can stop after the first step since we either reject $\mathcal{H}$ (and we infer that the signal is ACS) or fail to reject $\mathcal{H}$ and Holm's procedure stops (and the signal is said to be WSS).

In order to obtain the p-values, we need the distribution under $\mathcal{H}$. For each binary hypothesis test $\mathcal{H}$ vs. $\mathcal{A}_i, \; i = 1, \dots K$, we compute $\mathcal{G}(\Delta_{\min})$ for $P_{\text{int}} = i + 1$. To determine the p-value for each of these tests, we need the distribution of $\mathcal{G}(\Delta_{\min})$. For a fixed Δ_d, the cumulative

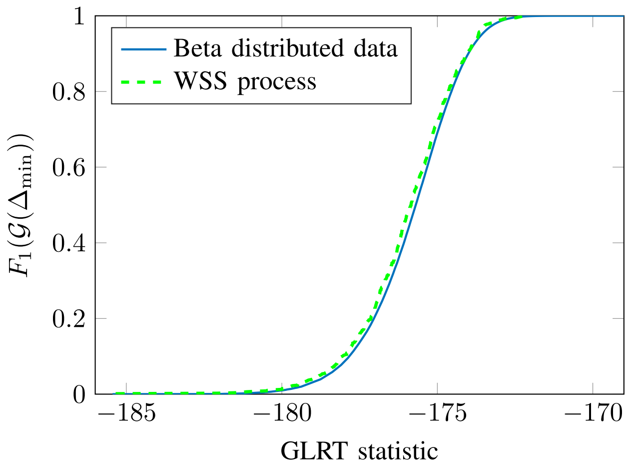

Fig. 1: Comparison of cdfs under the null hypothesis, i.e. the WSS process and its approximated distribution shown in blue for the following parameters: $P_{\text{int}} = 4$, $N = 768$, $L = 2$, $M = 25$ and $D = 20$

distribution function $F(\mathcal{G}(\Delta_d))$ may be approximated as the distribution of a product of independent Beta random variables [14]. After applying the results of [14] to the GLR in (1) we obtain the following distribution

$$-\,\mathcal{G}(\Delta_d)|\mathcal{H} \overset{\mathcal{D}}{=} \prod_{k=1}^{N/P_{\text{int}}} \prod_{i=1}^{P_{\text{int}}-1} \prod_{j=1}^{L} Y_{ij}^{(k)}, \tag{2}$$

where $Y_{ij}^{(k)} \sim \text{Beta}(\alpha_{ij}, \beta_i)$ with the parameters

$$\alpha_{ij} = M - iL - (j-1),$$
$$\beta_i = iL.$$

However, in our case, we need the distribution of $\mathcal{G}(\Delta_{\min})$. Simplifying the problem by assuming independence among the GLRTs yields a cumulative distribution given by the first-order statistic [15]

$$F_1(\mathcal{G}(\Delta_{\min})) = 1 - [1 - F(\mathcal{G}(\Delta_{\min}))]^D.$$

Based on $F_1(\mathcal{G}(\Delta_{\min}))$ we estimate the p-value p_i for each binary hypothesis test $\mathcal{H}$ vs. $\mathcal{A}_i$ for $i = 1, \ldots, K$. Figure 1 evaluates the validity of this approximation. Two empirical cumulative distribution functions are displayed. The dashed plot is obtained by generating a signal under the null hypothesis, i.e. a WSS process, and computing the GLRT $\mathcal{G}(\Delta_{\min})$ as shown in Section III-A. The solid plot is generated by drawing samples from a Beta distribution with corresponding parameters and forming the product (2). The distributions agree quite well, even though the dashed plot is a little left of the blue one, which will result in slightly underestimated p-values. Further simulations (not shown here) confirm these findings. The complete test procedure is summarized in Algorithm 1.

Algorithm 1 Multiple-Hypothesis-Test Procedure

Input: Discrete-time process $\mathbf{u}[n]$, maximum integer part $P_{\max}$, number D of grid points of fractional parts, desired probability of false alarm α.

Resampling

$k \leftarrow 1$
for $k \leq P_{\max} - 1$ **do**
  $P_{\text{int}} = k + 1$
  $d \leftarrow 1$
  for $d \leq D$ **do**
    $\tilde{\mathbf{u}}[n] \leftarrow$ resample $\mathbf{u}[n]$ by Δ_d
    Compute $\mathcal{G}(\Delta_d)$ with (1)
    $d \leftarrow d + 1$
  end for
  Obtain $\Delta_{\min} = \underset{\Delta_{d=1,\ldots,D}}{\arg\min}\ \mathcal{G}(\Delta_d)$
  Determine p-value p_k of $\mathcal{G}(\Delta_{\min})$
  $k \leftarrow k + 1$
end for

Multiple Hypothesis Test

$p_{\min} = \min_{k=1,\ldots,P_{\max}-1} p_k$
if $p_{\min} \geq \alpha/(P_{\max} - 1)$ **then**
  Fail to reject $\mathcal{H}$: $\mathbf{u}[n]$ is WSS
else
  Reject $\mathcal{H}$: $\mathbf{u}[n]$ is ACS
end if

IV. Numerical Simulations

The performance of the proposed technique is evaluated in this section using Monte Carlo simulations in a communications scenario, where we want to detect a signal in noise. Concretely, we consider the following model

$$\mathcal{H}: \ \mathbf{u}[n] = \mathbf{w}[n],$$
$$\mathcal{A}: \ \mathbf{u}[n] = \mathbf{H}[n] * \mathbf{s}[n] + \mathbf{w}[n],$$

where $\mathbf{w}[n] \in \mathbb{C}^L$ is colored Gaussian noise generated with a moving average filter of order 20 and $\mathbf{H}[n] \in \mathbb{C}^{L\times L}$ is a Rayleigh fading channel with 10 taps at the symbol rate and a sampling frequency of 1.2 MHz. The transmitted ACS signal $\mathbf{s}[n] \in \mathbb{C}^L$ is obtained by subsampling a long QPSK-signal with raised-cosine pulse shaping and roll-off factor 1. In order to obtain M realizations we generate one long sequence that is cut into M pieces. Figure 2 compares the empirically obtained probability of false alarm with the desired FWER $= \alpha$. As mentioned in Section III-B, we tend to underestimate the p-values, which leads to a probability of false alarm that is slightly higher than the specified significance level for typical values of α. However, for larger values of α and $K > 1$

Matlab code is available to download from: https://github.com/SSTGroup/Cyclostationary-Signal-Processing

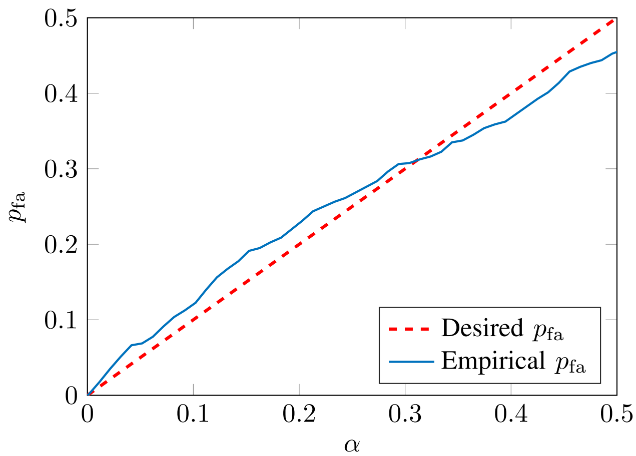

Fig. 2: Probability of false alarm compared to the desired one, i.e. the threshold α for the following scenario: $N = 768$ observations, $L = 2$ antennas, $M = 25$ realizations, $P_{\max} = 10$, and $D = 20$

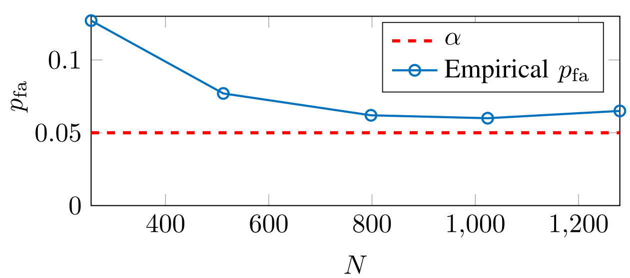

Fig. 3: Probability of false alarm for different observations lengths N for the following scenario: $L = 2$ antennas, $M = 25$ realizations, $P_{\max} = 10$, and $D = 20$.

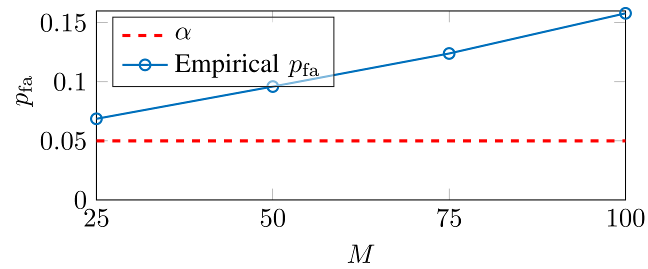

Fig. 4: Probability of false alarm for different number of realizations M for the following scenario: $N = 786$ observations, $L = 2$ antennas, $P_{\max} = 10$, and $D = 20$

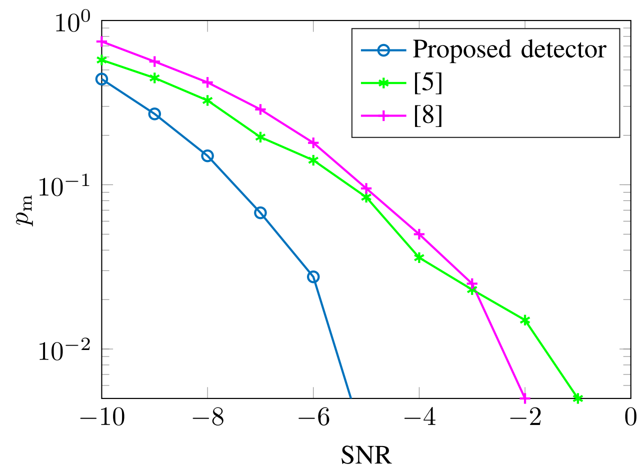

Fig. 5: Probability of missed detection (for $\alpha = 0.05$) for the following scenario: $P = 4.2$, $S = 192$ symbols, $L = 2$ antennas, $M = 25$ realizations, $P_{\max} = 10$, and $D = 20$

the absolute difference between the minimum significance level α/K and the desired probability of false alarm α increases, which results in a decreasing probability of false alarm.

We investigate the behavior of the empirical probability of false alarm for different observation lengths N and number of realizations M in Figures 3 and 4, respectively. We can observe that for an increasing observation length, the desired p_{fa} is almost approached. However, if we increase N beyond a certain limit, p_{fa} rises again. In order to prevent this it would be necessary to increase M at the same time. However, M cannot be chosen too large as we can observe in Figure 4. Moreover, decreasing M is not an option because the probability of detection will decrease as well. For these reasons M and N have to be set jointly to ensure the correct p_{fa}. Since both M and N are user defined parameters, they can be chosen appropriately in advance.

We compare the performance of the proposed detector with two competing techniques. The work in [5] tests for nonzero cyclic covariance functions, whereas [8] tests for correlation between the signal and its frequency-shifted version. Both competitors require knowledge of the cycle period and the selection of the lags of the cyclic covariance function that are included in the test. Regarding the lags, we select those that provide good performance. Regarding the cycle period, we sweep the same grid of values that we use for our method; however, instead of performing a multiple hypothesis test we simply choose the maximum test statistic.

Let us first investigate the performance if the true ϵ lies on the grid of candidate fractional parts. As an example for this scenario, we let $P = 4.2$ and use a set of $D = 20$ equally spaced resampling rates for each integer cycle period. Figure 5 shows the probability of missed detection for different SNRs for a fixed probability of false alarm $p_{\mathrm{fa}} = 0.05$. As can be seen, the proposed detector outperforms the two competing techniques.

If the true ϵ does not lie on the grid of candidate fractional parts, the performance of our detector, but also of the competing techniques, decreases rapidly. An example of such a case is shown in Figure 6. Even though the distance of the fractional part of the cycle period to the nearest

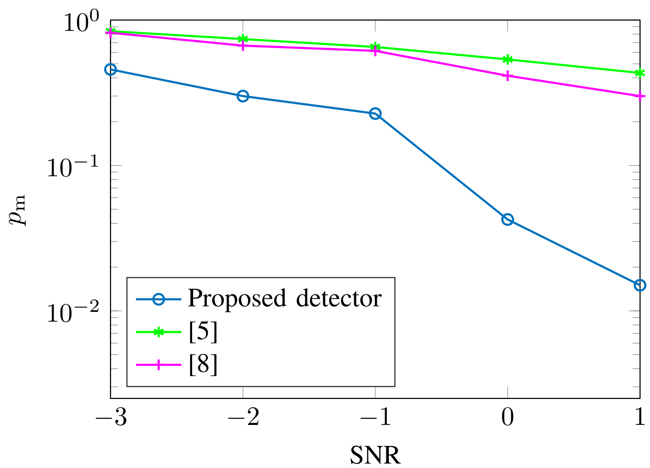

Fig. 6: Probability of missed detection (for $\alpha = 0.05$) for the following scenario: $P = 2.84$, $S = 64$ symbols, $L = 2$ antennas, $M = 25$ realizations, $P_{\max} = 10$, and $D = 85$

grid point is $1.2 \cdot 10^{-3}$, the performance has suffered substantially. This shows that it is critical to choose a fine enough grid, i.e., a large enough D. The price to pay for this, however, is computational complexity.

V. Conclusion

We have presented a detector of almost-cyclostationarity for unknown cycle period. To address the problem of unknown cycle period we have combined a resampling technique with a multiple hypothesis test to handle the fractional and integer part of the cycle period separately. Simulation results have shown that the detector performs well as long as the grid of resampling rates is chosen fine enough. Because a fine grid comes at the price of high computational complexity, future research should focus on how to make this approach less computationally complex.

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