EML Assignment 1 Problem 4

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Question 1

Begin by expanding the right-side of the equation:

$$
\begin{aligned}
\operatorname{argmin}_c \mathbb{E}\left[(Y-c)^2\right] &= \operatorname{argmin}_c \mathbb{E}\left[Y^2 + (c^2 - 2cY)\right] \\
&= \operatorname{argmin}_c \left[\mathbb{E}(Y^2) + (c^2 - 2c\,\mathbb{E}(Y))\right] \quad \mathbb{E}(Y^2) \text{ is independent on } c \\
&= \operatorname{argmin}_c \left[c^2 - 2c\,\mathbb{E}(Y)\right]
\end{aligned}
$$

Then, we just need to find the value of c that minimizes the function $g(c) = c^2 - 2c\,\mathbb{E}(Y)$ which is a parabola. We can differentiate it to find its critical points.

$$
\begin{aligned}
\frac{dg}{dc}(c) &= 0 \\
\frac{d}{dc}(c^2 - 2c\,\mathbb{E}(Y)) &= 0 \\
2c - 2\,\mathbb{E}(Y) &= 0 \\
c &= E(Y)
\end{aligned}
$$

Which gives the result:

$$\operatorname{argmin}_c \mathbb{E}\left[(Y-c)^2\right] = \mathbb{E}(Y)$$

Question 2

Univariant case

We begin by breaking down the total sum of squares TSS term.

$$\begin{aligned} TSS &= \sum_i (y_i - \bar{y}_i)^2 \\ &= \sum_i ((y_i - \hat{y}_i) + (\hat{y}_i - \bar{y}_i))^2 \\ &= \sum_i (y_i - \hat{y}_i)^2 + \sum_i (\hat{y}_i - \bar{y}_i)^2 + 2\sum_i (y_i - \hat{y}_i)(\hat{y}_i - \bar{y}_i) \end{aligned}$$

The first term is residual sum of squares RSS and the second term we will label explained sum of squares ESS, which is also equal to $\mathrm{Var}(\hat{Y})$. The third term is always zero, which we will prove at the end.

Additionally, notice that

$$TSS = \sum (y_i - \bar{y})^2 = \mathrm{Var}(Y)$$

.

Finally, we plug these result into the general rule.

$$\begin{aligned} R^2 &= \frac{TSS - RSS}{TSS} \\ &= \frac{ESS}{TSS} = \frac{\mathrm{Var}(\hat{Y})}{\mathrm{Var}(Y)} \\ &= \frac{\mathrm{Var}(\hat{\beta}_0 + \hat{\beta}_1 X)}{\mathrm{Var}(Y)} \\ &= \hat{\beta_1}^2 \frac{\mathrm{Var}(X)}{\mathrm{Var}(Y)} \end{aligned}$$

We know that

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x - \bar{x})} = \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}$$

.

Which means that:

$$\begin{aligned} R^2 &= \frac{\mathrm{Cov}(X, Y)^2}{\mathrm{Var}(X)^2} \frac{\mathrm{Var}(X)}{\mathrm{Var}(Y)} \\ &= \frac{\mathrm{Cov}(X, Y)^2}{\mathrm{Var}(X)\mathrm{Var}(Y)} \\ &= Cor(X, Y) \end{aligned}$$

Multivariate case