EML Assignment 4 - Problem 2

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Question 1

Let:

$$f_a(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$
$$f_b(x) = b_3(x-\zeta)^3 + b_2(x-\zeta)^2 + b_1(x-\zeta) + b_0.$$

The following constraints must be satisfied:

- Firstly, $f_b(\zeta) = f_a(\zeta)$. Thus:

$$b_0 = a_3\zeta^3 + a_2\zeta^2 + a_1\zeta + a_0$$

- Secondly, $f_b^{'}(\zeta) = f_a^{'}(\zeta)$. Thus:

$$b_1 = 3a_3\zeta^2 + 2a_2\zeta + a_1$$

- Finally, $f_b^{''}(\zeta) = f_a^{''}(\zeta)$. Thus:

$$b_2 = 6a_3\zeta + 2a_2$$

b_0, b_1, b_2 are fixed by the selection of a_0, a_1, a_2, a_3, but b_3 remains a free parameter. Thus the number of degrees of freedom of this model is 5.

Question 2

$$f(x) = \begin{cases} a_2x^2 + a_1x + a_0 & x \leq \zeta \\ b_2(x-\zeta)^2 + b_1(x-\zeta) + b_0 & x > \zeta \end{cases}$$

Similarly, let:

$$f_a(x) = a_2x^2 + a_1x + a_0$$
$$f_b(x) = b_2(x-\zeta)^2 + b_1(x-\zeta) + b_0$$

Assuming a_2, a_1, a_0 are given, and using a similar process to **1**:

$$f_a(\zeta) = f_b(\zeta) \implies b_0 = a_2\zeta^2 + a_1\zeta$$
$$f_a^{'}(\zeta) = f_b^{'}(\zeta) \implies b_1 = 2a_2\zeta + a_1$$

Again, b_0, b_1 are fixed by the selection of a_0, a_1, a_2, but b_2 remains a free parameter. Thus the number of degrees of freedom of this model is 4.

Question 3

The difference is one extra free parameter in cubic splines compared to quadratic splines. It is exactly as I would intuitively expect, given that there is also an extra constraint on the second derivatives.

Question 4

In the original data, the curves are not differentiable at the knots. Since splines of the kth degree are $k-1$ times differentiable it cannot exactly fit the data. However, the higher the degree of the splines, the closer it can get to the actual data at the boundaries.