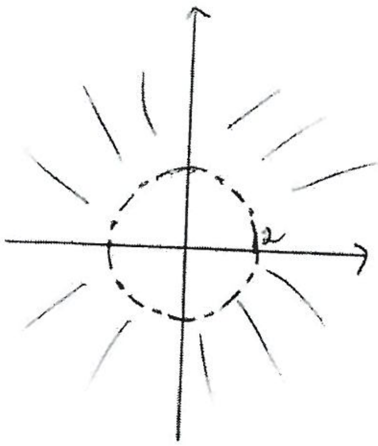
Exame de recurso - modelo

1.

a) $D_f = \{(x,y) \in \mathbb{R}^2 : y^2 + x^2 - 4 > 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 4\}$



Pontos do plano exteriores ao círculo de centro na origem e raio 2.

b) $f(x,y) = 0 \Leftrightarrow x^2 - 4 = 0 \Leftrightarrow x = \pm 2$

Por exemplo, $(-2,1)$ e $(2,1)$ pertencem à curva de nível $f(x,y) = 0$.

2.

- f é contínua em $\mathbb{R}^2 \setminus \{(0,0)\}$ por ser o quociente de duas funções polinomiais que são funções contínuas.
- Dado que

$$\lim_{(x,y)\to(0,0)} y^3 = 0 \quad \text{e} \quad \frac{2x^4}{x^4+y^2} \leq \frac{2x^4}{x^4} = 2,$$

(função limitada)

podemos concluir que

$$\lim_{(x,y)\to(0,0)} \frac{2x^4y^3}{x^4+y^2} = 0.$$

Logo, f é também contínua em $(0,0)$ uma vez que

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

3.

$$\frac{\partial u}{\partial x_1} = a_1 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$\frac{\partial^2 u}{\partial x_1^2} = a_1^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$\frac{\partial u}{\partial x_2} = a_2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$\frac{\partial^2 u}{\partial x_2^2} = a_2^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$\vdots$$

$$\frac{\partial u}{\partial x_n} = a_n e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$\frac{\partial^2 u}{\partial x_n^2} = a_n^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

Assim,

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} =$$

$$= (a_1^2 + a_2^2 + \cdots + a_n^2)\, e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$$

$$= 1 .\, e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = u$$

4.

a) $$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{Q-P}{\|Q-P\|}$$

$$Q-P = (2,1,1) - (1,0,1) = (1,1,0)$$

$$\|Q-P\| = \sqrt{2}$$

$$\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$\vec{\nabla}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = \left(20zxe^{-y}, -10zx^2e^{-y}, 10x^2e^{-y}\right)$$

$$\vec{\nabla}T(P) = \vec{\nabla}T(1,0,1) = (20, -10, 10)$$

$$D_{\vec{u}}T(P) = \nabla T(P) \cdot \vec{u} = (20,-10,10) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$= \frac{10}{\sqrt{2}} = 5\sqrt{2}$$

b) A temperatura aumenta mais rapidamente em P segundo a direção de $\vec{\nabla}T(P) = (20,-10,10)$ e a taxa máxima de crescimento é igual a $\|\vec{\nabla}T(P)\| = \sqrt{600} = 10\sqrt{6}$.

5.

a) Pontos críticos:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 6xy - 6x = 0 \\ 3x^2 + 3y^2 - 6y = 0 \end{cases} \Leftrightarrow \begin{cases} 6x(y-1) = 0 \\ \text{---} \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \vee y = 1 \\ \text{---} \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ 3y^2 - 6y = 0 \end{cases} \vee \begin{cases} y = 1 \\ 3x^2 + 3 - 6 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ 3y(y-2) = 0 \end{cases} \vee \begin{cases} y = 1 \\ 3x^2 = 3 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \vee y = 2 \end{cases} \vee \begin{cases} y = 1 \\ x = \pm 1 \end{cases}$$

$(0,0)$, $(0,2)$, $(\pm 1, 1)$

b) • Discriminante:

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

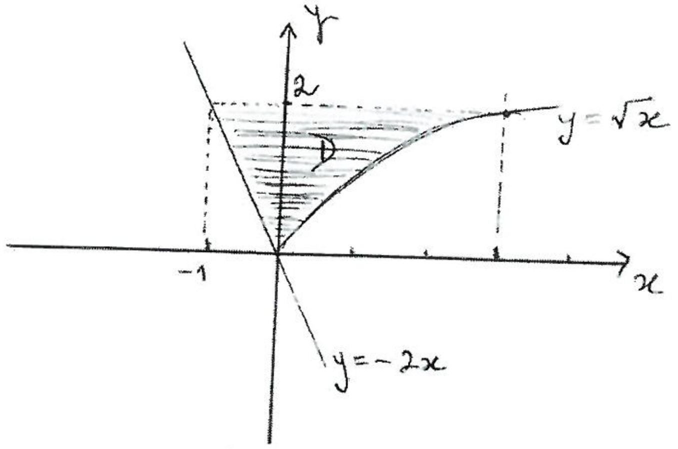
$$= (6y-6) \cdot (6y-6) - (6x)^2 = 36\left[(y-1)^2 - x^2\right]$$

• Classificação dos pontos críticos:

- $D(0,0) = 36 > 0$ Logo, $(0,0)$ é maximizante local.
  $\dfrac{\partial^2 f}{\partial x^2}(0,0) = -6$
- $D(0,2) = 36 > 0$ Logo, $(0,2)$ é minimizante local.
  $\dfrac{\partial^2 f}{\partial x^2}(0,2) = 6 > 0$
- $D(\pm 1, 1) = -36$ Logo, $(\pm 1, 1)$ são pontos de sela.

6.

a)



b)

$$I = \int_0^2 \int_{-\frac{y}{2}}^{y^2} f(x,y)\, dx\, dy$$

$y = -2x \Leftrightarrow x = -\frac{y}{2}$

$y = \sqrt{x} \Rightarrow x = y^2$

c) I dá-nos o volume do sólido que fica entre a região D no plano xOy e a superfície de equação $z = f(x,y)$ quando $(x,y) \in D$ (gráfico de f restrita a D).

d)

$$I = \int_0^2 \int_{-\frac{y}{2}}^{y^2} (x+y)\, dx\, dy = \int_0^2 \left[\frac{x^2}{2} + yx\right]_{x=-\frac{y}{2}}^{y^2} dy$$

$$= \int_0^2 \left(\frac{y^4}{2} + y^3 - \frac{y^2}{8} + \frac{y^2}{2}\right) dy$$

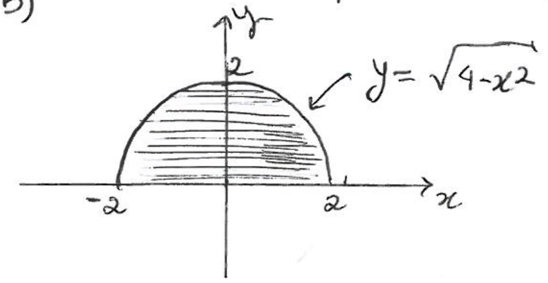
$$= \int_0^2 \left(\frac{y^4}{2} + y^3 + \frac{3}{8}y^2\right) dy$$

$$= \left[\frac{y^5}{10} + \frac{y^4}{4} + \frac{y^3}{8}\right]_{y=0}^{2} = \frac{2^5}{10} + \frac{2^4}{4} + \frac{2^3}{8} = \frac{2^5}{10} + 4 + 1$$

$$= \frac{2^5 + 50}{10} = \frac{82}{10} = \frac{41}{5}$$

7.

a) e b) Projeção no plano xOy :



Coordenadas cilíndricas:

$x = r\cos\theta$

$y = r \operatorname{sen}\theta$

$z = z$

$y=\sqrt{4-x^2} \Rightarrow x^2+y^2=4$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{x^2+y^2+1}} 2xz \; dz\,dy\,dx =$$

$$= \int_{0}^{2}\int_{0}^{\pi}\int_{0}^{\sqrt{r^2+1}} 2(r\cos\theta)z \cdot r \; dz\,d\theta\,dr = \int_{0}^{2}\int_{0}^{\pi}\int_{0}^{\sqrt{r^2+1}} 2r^2(\cos\theta)z \; dz\,d\theta\,dr =$$

$$= \int_{0}^{2}\int_{0}^{\pi} r^2\cos\theta\left[z^2\right]_{z=0}^{\sqrt{r^2+1}} d\theta\,dr = \int_{0}^{2}\int_{0}^{\pi} r^2(r^2+1)\cos\theta \; d\theta\,dr$$

$$= \int_{0}^{2} r^2(r^2+1)\left[\operatorname{sen}\theta\right]_{\theta=0}^{\pi} dr = \int_{0}^{2} 0 \; dr = 0$$

8.

a) $v(t) = r'(t) = (1, 2t, 1)$; $v(0) = (1, 0, 1)$

$a(t) = v'(t) = (0, 2, 0)$; $a(0) = (0, 2, 0)$

b) $$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{(1, 2t, 1)}{\|(1, 2t, 1)\|} = \left(\frac{1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}}, \frac{1}{\sqrt{2+4t^2}}\right)$$

$$T'(t) = \left(\frac{-4t(2+4t^2)^{-1/2}}{2+4t^2}, \frac{2(2+4t^2)^{1/2} - 8t^2(2+4t^2)^{-1/2}}{2+4t^2}, \frac{-4t(2+4t^2)^{-1/2}}{2+4t^2}\right)$$

$$= \left(-4t(2+4t^2)^{-3/2}, 2(2+4t^2)^{-1/2} - 8t^2(2+4t^2)^{-3/2}, -4t(2+4t^2)^{-3/2}\right)$$

$$\|T'(t)\| = \sqrt{16t^2(2+4t^2)^{-3} + 4(2+4t^2)^{-1} - 32t^2(2+4t^2)^{-3} + 64t^4(2+4t^2)^{-3} + 16t^2(2+4t^2)^{-3}}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

c) $r(1) = (1, 0, 1)$

$r'(1) = (1, 2, 1) \parallel T'(1)$

Plano normal à curva no ponto $(1, 0, 1)$:

$$v'(1) \cdot (x-1, y-0, z-1) = 0$$

$$\Leftrightarrow (1, 2, 1) \cdot (x-1, y, z-1) = 0$$

$$\Leftrightarrow x - 1 + 2y + z - 1 = 0$$

$$\Leftrightarrow x + 2y + z = 2$$

7.

a) Nos pontos em que $\|r(t)\|$ tem um máximo ou mínimo local temos $\|r(t)\|'=0$. Assim,

$$\|r(t)\|'=0 \Leftrightarrow \|(f(t), g(t), h(t))\|'=0$$

$$\Leftrightarrow \left(\sqrt{f^2(t)+g^2(t)+h^2(t)}\right)'=0$$

$$\Leftrightarrow \frac{1}{2}\frac{1}{\sqrt{f^2(t)+g^2(t)+h^2(t)}}\cdot\left(f^2(t)+g^2(t)+h^2(t)\right)'=0$$

$$\Leftrightarrow \left(f^2(t)+g^2(t)+h^2(t)\right)'=0$$

$$\Leftrightarrow 2f(t)\cdot f'(t)+2g(t)\cdot g'(t)+2h(t)\cdot h'(t)=0$$

$$\Leftrightarrow f(t)\cdot f'(t)+g(t)\cdot g'(t)+h(t)\cdot h'(t)=0$$

$$\Leftrightarrow r(t)\cdot r'(t)=0$$

$\Leftrightarrow$ $r(t)$ e $r'(t)$ são ortogonais.

b) $r(t)=(\cos t, \operatorname{sen} t, t)$

$r'(t)=(-\operatorname{sen} t, \cos t, 1)$

$$\|r(t)\|=\sqrt{\cos^2 t+\operatorname{sen}^2 t+t^2}=\sqrt{1+t^2}$$

$$\|r(t)\|'=0 \Leftrightarrow \left(\sqrt{1+t^2}\right)'=0 \Leftrightarrow \frac{1}{2}(1+t^2)^{-1/2}\cdot 2t=0$$

$$\Leftrightarrow t=0$$

$\|r(t)\|$ tem um valor mínimo em $t=0$.

$$r(0)\cdot r'(0)=(1,0,0)\cdot(0,1,1)=0.$$