Folha de exercícios 6 – soluções

1.

$$\iint_R (x^2y^2+x)\,dA = \int_0^2\int_{-1}^0 (x^2y^2+x)\,dy\,dx = \int_0^2 \left[x^2\cdot\frac{y^3}{3}+xy\right]_{y=-1}^{0} dx$$

$$= \int_0^2 \left[0-\left(-\frac{x^2}{3}-x\right)\right]dx = \int_0^2\left(\frac{x^2}{3}+x\right)dx = \left[\frac{x^3}{9}+\frac{x^2}{2}\right]_{x=0}^{2} = \frac{8}{9}+\frac{4}{2} = \frac{26}{9}$$

2.

a) $$\int_0^3\int_0^4 (4x+y)\,dx\,dy = \int_0^3 \left[2x^2+yx\right]_{x=0}^{4} dy = \int_0^3 (32+4y)\,dy =$$

$$= \left[32y+2y^2\right]_{y=0}^{3} = 114$$

b) $$\int_0^3\int_0^2 6xy\,dy\,dx = 54$$

3.

a) $$\iint_D f\,dA = \int_1^4\int_1^2 f(x,y)\,dy\,dx = \int_1^2\int_1^4 f(x,y)\,dx\,dy$$

b) $$\iint_D f\,dA = \int_0^4\int_{\frac{1}{3}x-\frac{1}{3}}^{2} f(x,y)\,dy\,dx = \int_0^1\int_1^{3y+1} f(x,y)\,dx\,dy + \int_1^2\int_1^4 f(x,y)\,dx\,dy$$

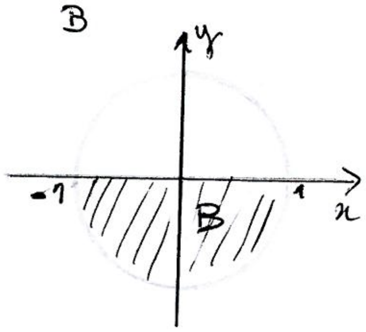
c) $$\iint_D f\,dA = \int_{-1}^3\int_{-2}^{-\frac{3}{4}x+\frac{1}{4}} f(x,y)\,dy\,dx = \int_{-2}^1\int_{-1}^{\frac{1}{3}-\frac{4}{3}y} f(x,y)\,dx\,dy$$

d) $$\iint_D f\,dA = \int_0^3\int_0^{2x} f(x,y)\,dy\,dx + \int_3^5\int_0^{-3x+15} f(x,y)\,dy\,dx = \int_0^6\int_{\frac{y}{2}}^{5-\frac{y}{3}} f(x,y)\,dx\,dy$$

4.

a) $\iint_R dA > 0$, pois representa a área do semi-círculo R.

b) $\iint_B 5x\, dA = 0$, pois o domínio de integração B é simétrico relativamente ao eixo dos yy e a função integranda, $f(x,y) = 5x$, é ímpar com respeito à variável x, isto é, $f(-x,y) = -f(x,y)$.



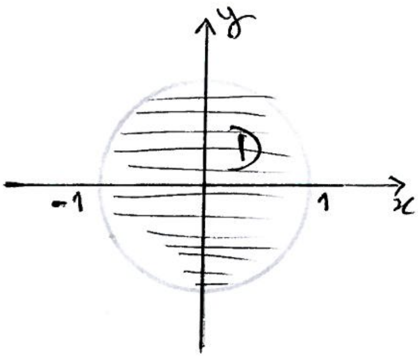
De facto,

$$\iint_B 5x\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0} 5x\, dy\, dx$$

$$= \int_{-1}^{1} \underbrace{5x\sqrt{1-x^2}}_{\text{função ímpar}}\, dx = 0$$

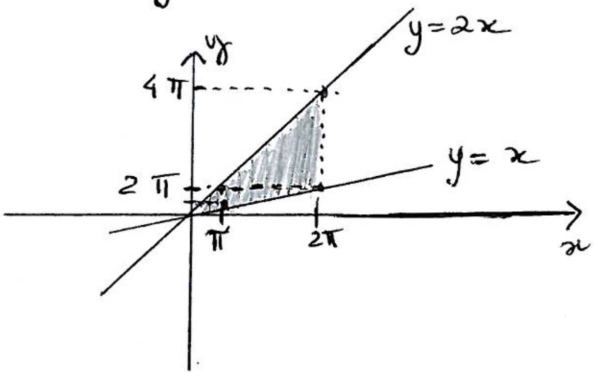
c) $\iint_D 5x\, dA = 0$, pela mesma razão que em b).

d) $\iint_D \operatorname{sen} y\, dA = 0$, por razão análoga à descrita em b) mas considerando agora a simetria do domínio de integração D em relação ao eixo dos xx e o facto da função integranda ser uma função ímpar com respeito à variável y.



5. a) $\pi \leq x \leq 2\pi$

$x \leq y \leq 2x$



$$\int_{\pi}^{2\pi}\int_{x}^{2x} \operatorname{sen} x \, dy\, dx = \int_{\pi}^{2\pi} \operatorname{sen} x \, [y]_{y=x}^{2x} \, dx = \int_{\pi}^{2\pi} x \operatorname{sen} x \, dx \underset{(*)}{=}$$

$$= x \cdot (-\cos x)\Big]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} -\cos x \, dx = (\pi \cos \pi - 2\pi \cos 2\pi) + [\operatorname{sen} x]_{\pi}^{2\pi}$$

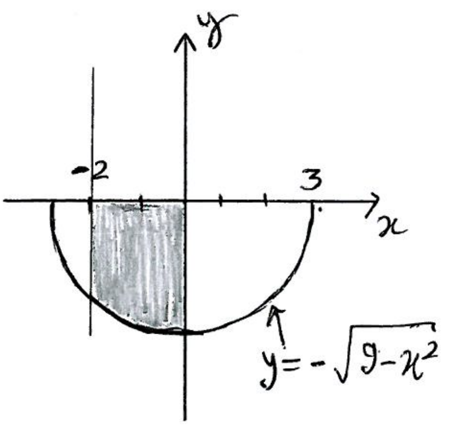
$$= -\pi - 2\pi + 0 = -3\pi$$

Obs.: em (*) foi usada a técnica de primitivação por partes:

$$\int f \cdot g' = f \cdot g - \int f' \cdot g$$

b) $-2 \leq x \leq 0$

$-\sqrt{9-x^2} \leq y \leq 0$



$y = -\sqrt{9-x^2} \Rightarrow y^2 + x^2 = 9$

$$\int_{-2}^{0}\int_{-\sqrt{9-x^2}}^{0} 2xy \, dy\, dx = \int_{-2}^{0} x \, [y^2]_{y=-\sqrt{9-x^2}}^{0} \, dx =$$

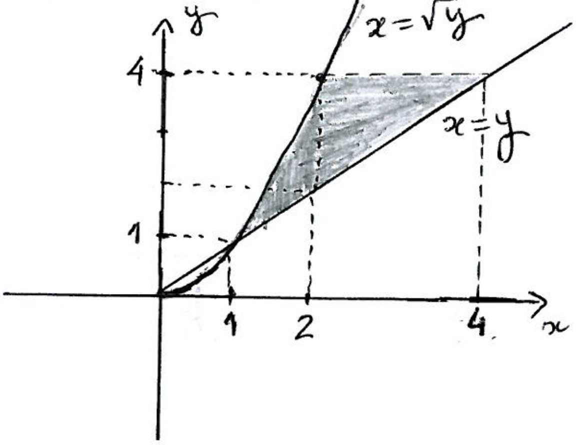
$$= \int_{-2}^{0} -x\left(-\sqrt{9-x^2}\right)^2 dx = \int_{-2}^{0} -x(9-x^2)\, dx =$$

$$= \int_{-2}^{0} (x^3 - 9x)\, dx = \left[\frac{x^4}{4} - \frac{9}{2}x^2\right]_{x=-2}^{0} =$$

$$= -\frac{16}{4} + 18 = 14$$

e) $1 \leq y \leq 4$

$\sqrt{y} \leq x \leq y$



$x = \sqrt{y} \Rightarrow y = x^2$

$$\int_1^4 \int_{\sqrt{y}}^{y} x \, dx \, dy = \int_1^4 \left[\frac{x^2}{2}\right]_{x=\sqrt{y}}^{y} dy =$$

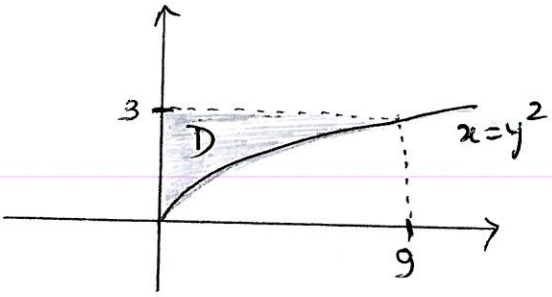
$$= \int_1^4 \frac{1}{2}(y^2 - y) \, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^2}{2}\right]_{y=1}^{4}$$

$$= \frac{1}{2}\left(\frac{64}{3} - 8 - \frac{1}{3} + \frac{1}{2}\right) = \frac{27}{4}$$

6. $0 \leq y \leq 3$

$0 \leq x \leq y^2$



$$\text{Volume } (U) = \iint_D f(x,y) \, dA$$

U - região de $\mathbb{R}^3$ compreendida entre D e a superfície de equação $z = f(x,y)$.

$$\iint_D f(x,y) \, dA = \int_0^3 \int_0^{y^2} (x+y) \, dx \, dy = \int_0^3 \left[\frac{x^2}{2} + yx\right]_{x=0}^{y^2} dy$$

$$= \int_0^3 \left(\frac{y^4}{2} + y^3\right) dy = \left[\frac{y^5}{10} + \frac{y^4}{4}\right]_{y=0}^{3} = \frac{3^5}{10} + \frac{3^4}{4}$$

7.

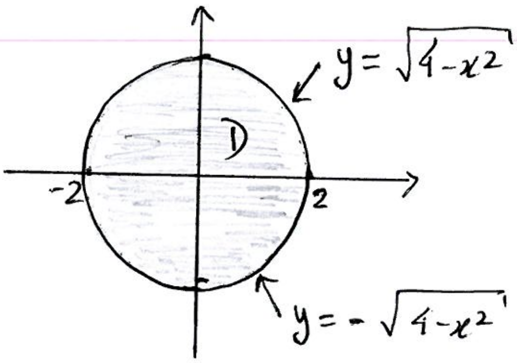
a) Note-se que $0 < f(x,y) = \dfrac{2}{e^{(x-1)^2+y^2}} \leq 2 \ , \ \forall (x,y) \in \mathbb{R}^2.$

Linhas (ou curvas) de nível $k \in]0, 2]$:

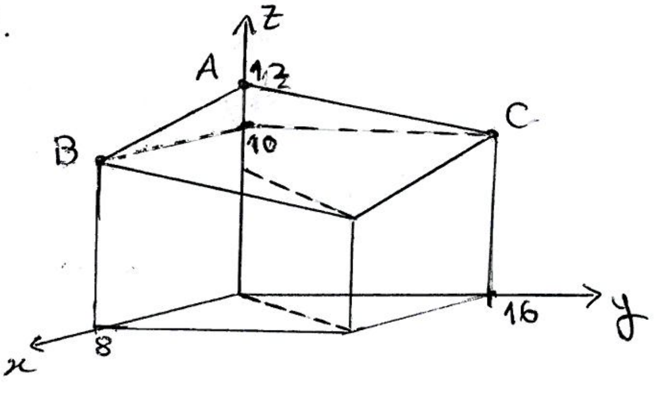
$$f(x,y) = k \iff 2e^{-(x-1)^2-y^2} = k$$
$$\iff e^{-(x-1)^2-y^2} = \frac{k}{2}$$
$$\iff -(x-1)^2 - y^2 = \ln \frac{k}{2}$$
$$\iff (x-1)^2 + y^2 = -\ln \frac{k}{2}$$

As curvas de nível de f são circunferências de centro $(1,0)$ e raio $\sqrt{-\ln\frac{k}{2}}$, $0 < k \leq 2$. Observe-se que $\frac{k}{2} \leq 1$ e, portanto, $\ln \frac{k}{2} \leq \ln 1 = 0$.

b) $$\text{Volume}(R) = \iint_D f(x,y)\, dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2e^{-(x-1)^2-y^2}\, dy\, dx$$



8.



$A = (0,0,12)$

$B = (8,0,10)$

$C = (0,16,10)$

$\vec{AB} \times \vec{AC} = (32, 16, 128)$

6

Equação do plano que contém os pontos A, B e C:

$$(2,1,8)\cdot(x,y,z-12)=0 \iff z = 12-\frac{x}{4}-\frac{y}{8}$$

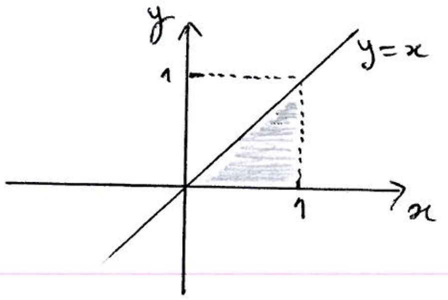
Volume do edifício:

$$\int_0^8\int_0^{16}\left(12-\frac{x}{4}-\frac{y}{8}\right)dy\,dx = \int_0^8\left[12y-\frac{x}{4}y-\frac{y^2}{16}\right]_{y=0}^{16}dx =$$

$$= \int_0^8(176-4x)\,dx = \left[176x-2x^2\right]_{x=0}^{8} = 1280$$

9. a) $0 \leq y \leq 1$

$y \leq x \leq 1$



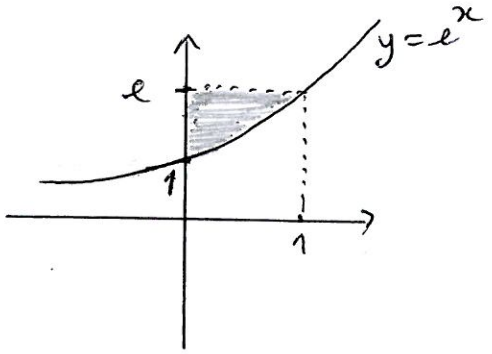
$$\int_0^1\int_y^1 e^{x^2}dx\,dy = \int_0^1\int_0^x e^{x^2}dy\,dx = \int_0^1\left[e^{x^2}\cdot y\right]_{y=0}^{x}dx =$$

$$= \int_0^1 e^{x^2}\cdot x\,dx = \left[\frac{e^{x^2}}{2}\right]_{x=0}^{1} =$$

$$= \frac{1}{2}(e-1)$$

b) $0 \leq x \leq 1$

$e^x \leq y \leq e$



$1 \leq y \leq e$

$0 \leq x \leq \ln y$

$$\int_0^1\int_{e^x}^{e}\frac{y}{\ln y}\,dy\,dx = \int_1^e\int_0^{\ln y}\frac{y}{\ln y}\,dx\,dy =$$

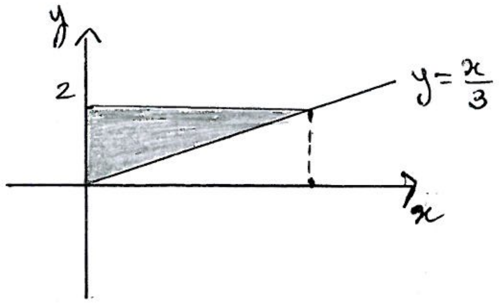
$$= \int_1^e\left[\frac{y}{\ln y}\cdot x\right]_{x=0}^{\ln y}dy = \int_1^e y\,dy =$$

$$= \left[\frac{y^2}{2}\right]_{y=1}^{e} = \frac{1}{2}(e^2-1)$$

10.

a) $0 \leq x \leq 6$

$\frac{x}{3} \leq y \leq 2$



$0 \leq y \leq 2$

$0 \leq x \leq 3y$

b) Invertendo a ordem de integração, Temos

$$I = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\,dx\,dy = \int_0^2 \sqrt{y^3+1}\cdot\left[\frac{x^2}{2}\right]_{x=0}^{3y} dy$$

$$= \int_0^2 \frac{9}{2} y^2 \sqrt{y^3+1}\,dy = \frac{3}{2}\int_0^2 3y^2 (y^3+1)^{1/2} dy$$

$$= \frac{3}{2}\left[(y^3+1)^{3/2}\right]_{y=0}^{2} = \frac{3}{2}\left(9^{3/2}-1\right) = \frac{78}{2}$$

11.

$$\iiint_P xyz\,dV = \int_0^1 \int_1^2 \int_2^3 xyz\,dz\,dy\,dx = \int_0^1 \int_1^2 xy\left[\frac{z^2}{2}\right]_2^3 dy\,dx$$

$$= \frac{5}{2}\int_0^1 \int_1^2 xy\,dy\,dx = \frac{5}{2}\int_0^1 x\left[\frac{y^2}{2}\right]_{y=1}^{2} dx = \frac{15}{2}\int_0^1 x\,dx = \frac{15}{8}$$

12.

$$\iiint_U f\,dV = \int_0^2 \int_{-1}^1 \int_2^3 (x^2+5y^2-z)\,dz\,dy\,dx$$

$$= \int_0^2 \int_{-1}^1 \left[x^2z + 5y^2z - \frac{z^2}{2}\right]_{z=2}^3 =$$

$$= \int_0^2\int_{-1}^1 \left(x^2+5y^2-\frac{5}{2}\right)dy\,dx = \int_0^2 \left[x^2y + \frac{5}{3}y^3 - \frac{5}{2}y\right]_{y=-1}^1 dx =$$

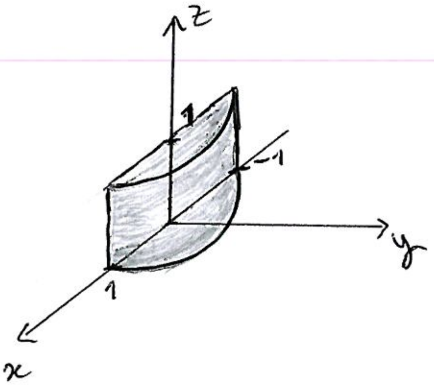
$$= \int_0^2 \left(2x^2 - \frac{5}{3}\right)dx = \left[\frac{2}{3}x^3 - \frac{5}{3}x\right]_{x=0}^2 = 2$$

13.

a) $0 \le z \le 1$

$-1 \le x \le 1$

$0 \le y \le \sqrt{1-x^2}$



$-1 \le x \le 1$

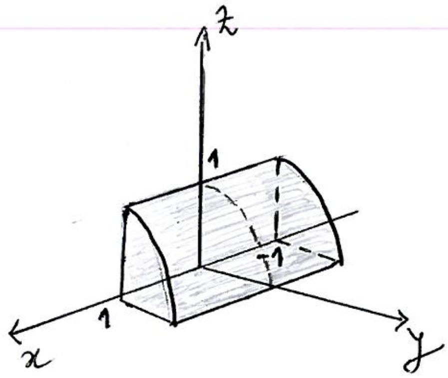
$0 \le y \le \sqrt{1-x^2}$

$0 \le z \le 1$

b) $0 \le z \le 1$

$-1 \le x \le 1$

$0 \le y \le \sqrt{1-z^2}$



$0 \le z \le 1$

$0 \le y \le \sqrt{1-z^2}$

$-1 \le x \le 1$

14.

a) $$\int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}}\int_{-\sqrt{r^2-x^2-y^2}}^{\sqrt{r^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

b) $$\int_{0}^{r}\int_{0}^{\sqrt{r^2-x^2}}\int_{0}^{\sqrt{r^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

c) $$\int_{0}^{r}\int_{0}^{\sqrt{r^2-y^2}}\int_{0}^{1} f(x,y,z)\,dx\,dz\,dy$$

d) $$\int_{0}^{r}\int_{0}^{\sqrt{r^2-x^2}}\int_{0}^{1} f(x,y,z)\,dy\,dz\,dx$$

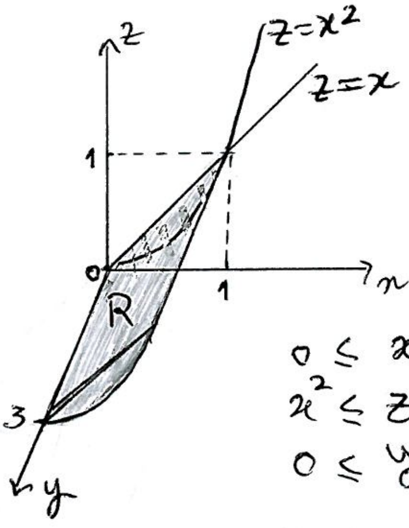
15.

$$\text{Volume} = \int_0^8\int_0^{16}\int_0^{12-\frac{x}{4}-\frac{y}{8}} 1\,dz\,dy\,dx =$$

$$= \int_0^8\int_0^{16}\left[z\right]_{z=0}^{12-\frac{x}{4}-\frac{y}{8}} dy\,dx =$$

$$= \int_0^8\int_0^{16}\left(12-\frac{x}{4}-\frac{y}{8}\right)dy\,dx = 1280$$

16.



$0 \le x \le 1$

$x^2 \le z \le x$

$0 \le y \le 3$

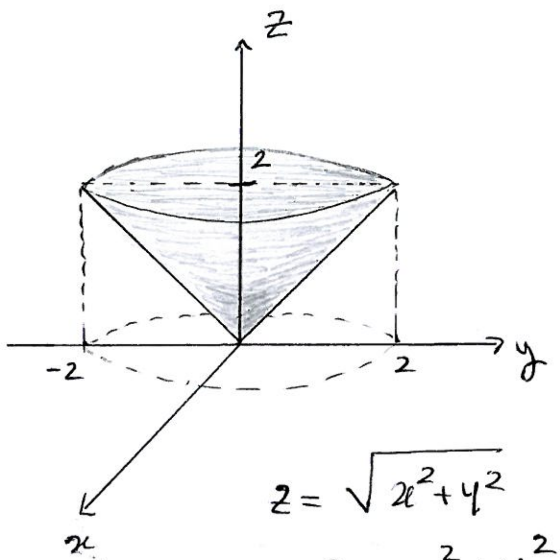
$$\text{Volume}(R) = \int_0^1\int_{x^2}^{x}\int_0^3 1\,dy\,dz\,dx$$

$$= \int_0^1\int_{x^2}^{x} 3\,dz\,dx$$

$$= \int_0^1 3(x-x^2)\,dx = \frac{1}{2}$$

17.

$$\mathcal{U}=\{(x,y,z)\in\mathbb{R}^3 : z \geq \sqrt{x^2+y^2} \text{ e } z \leq 2\}$$



$$-2 \leq x \leq 2$$
$$-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$$
$$\sqrt{x^2+y^2} \leq z \leq 2$$

$z = \sqrt{x^2+y^2}$

$\Rightarrow z^2 = x^2+y^2$

$z=2 \Rightarrow x^2+y^2=4$

a) $\displaystyle\iiint_{\mathcal{U}} \sqrt{x^2+y^2}\, dV > 0$, pois a função integranda, $f(x,y,z)=\sqrt{x^2+y^2}$, assume valores positivos para todo o $(x,y,z)\in\mathcal{U}$.

b) $\displaystyle\iiint_{\mathcal{U}} x\, dV = 0$, pois a funçaõ integranda, $f(x,y,z)=x$, é uma funçaõ ímpar com respeito a x e o domínio de integraçaõ é simétrico relativamente à origem.

c) $\displaystyle\iiint_{\mathcal{U}} z-\sqrt{x^2+y^2}\, dV > 0$, pois a funçaõ integranda, $f(x,y,z)=z-\sqrt{x^2+y^2}$, assume valores positivos para todo o $(x,y,z)\in\mathcal{U}$.