Análise

Teste 2 - Proposta de resolução

1. • Pontos críticos

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + 2xy = 0 \\ y + x^2 = 0 \end{cases} \Leftrightarrow \begin{cases} x(1+y) = 0 \\ y = -x^2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee$$

$$\vee \begin{cases} y = -1 \\ x^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \vee \begin{cases} x = 1 \\ y = -1 \end{cases} \vee \begin{cases} x = -1 \\ y = -1 \end{cases}$$

Pontos críticos: $(0,0), (1,-1), (-1,-1)$

• Discriminante

$$\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) \cdot \frac{\partial^2 f}{\partial y^2}(x,y) - \left[\frac{\partial^2 f}{\partial y \partial x}(x,y)\right]^2$$

$$= (2 + 2y) \cdot 1 - (2x)^2$$

$$= 2 + 2y - 4x^2$$

• Classificação dos pontos críticos

$$\left.\begin{array}{l} \Delta f(0,0) = 2 > 0 \\ \frac{\partial^2 f}{\partial x^2}(0,0) = 2 > 0 \end{array}\right\} \Rightarrow (0,0) \text{ é minimizante local.}$$

$$\Delta f(-1,-1) = -4 < 0 \Rightarrow (-1,-1) \text{ é ponto de sela.}$$

$$\Delta f(1,-1) = -4 < 0 \Rightarrow (1,-1) \text{ é ponto de sela.}$$

2.

$$\begin{cases} \nabla f(x,y,z) = \lambda g(x,y,z) \\ g(x,y,z) = 4 \end{cases}, \quad \text{com } g(x,y,z) = x+y-z$$

$$\begin{cases} \left(2x,\ 2(y-2) + 2(z-1)\right) = \lambda\,(1,1,-1) \\ x+y-z = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x = \lambda \\ 2(y-2) = \lambda \\ 2(z-1) = -\lambda \\ x+y-z = 4 \end{cases} \Leftrightarrow \begin{cases} \overline{} \\ 2(y-2) = 2x \\ 2(z-1) = -2x \\ \overline{} \end{cases} \Leftrightarrow \begin{cases} \overline{} \\ y = x+2 \\ z = -x+1 \\ x+x+2-(-x+1) = 4 \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{} \\ \overline{} \\ \overline{} \\ 3x = 3 \end{cases} \Leftrightarrow \begin{cases} \overline{} \\ y = 3 \\ z = 0 \\ x = 1 \end{cases}$$

Soluçaõ: $(1,3,0)$

Mínimo: $f(1,3,0) = 1 + (3-2)^2 + (0-1)^2 = 6$

- Método de reduçaõ de dimensaõ

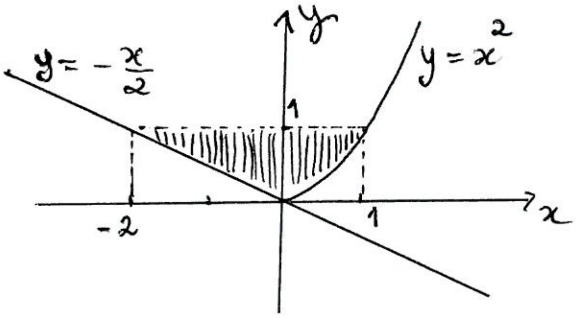
$$x+y-z=4 \Leftrightarrow z = x+y-4$$

$$\begin{aligned} f(x,y,z) = f(x,y,x+y-4) &= x^2 + (y-2)^2 + (x+y-4-1)^2 \\ &= x^2 + (y-2)^2 + (x+y-5)^2 \\ &= h(x,y) \end{aligned}$$

Problema: Determinar o mínimo de h.
Começar por calcular os pontos críticos de h e aplicar o critério do discriminante.

3.

a)



b) $y = x^2 \Rightarrow x = \pm\sqrt{y}$

$y = -\frac{x}{2} \Rightarrow x = -2y$

$$I = \int_0^1 \int_{-2y}^{\sqrt{y}} f(x,y)\, dx\, dy$$

c) Quando $f(x,y) \geq 0$, I representa o volume do sólido compreendido entre a região de integração e a superfície de equação $z = f(x,y)$.

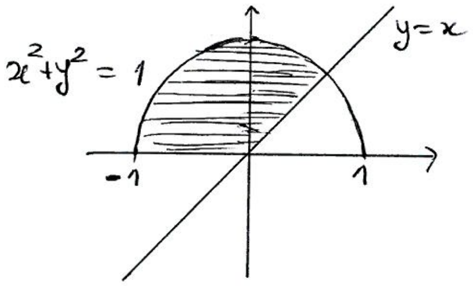
d)

$$I = \int_0^1 \int_{-2y}^{\sqrt{y}} 2xy\, dx\, dy = \int_0^1 \left[x^2 y\right]_{x=-2y}^{\sqrt{y}} dy =$$

$$= \int_0^1 (y^2 - 4y^3)\, dy = \left[\frac{y^3}{3} - y^4\right]_{y=0}^1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

4.

a)



$$D=\left\{(r,\theta):\ 0\leq r\leq 1\ ,\ \frac{\pi}{4}\leq\theta\leq\pi\right\}$$

$$x=r\cos\theta$$
$$y=r\,\mathrm{sen}\,\theta$$

b) $$\iint_D \cos(x^2+y^2)\,dx\,dy=\int_0^1\int_{\frac{\pi}{4}}^{\pi}\cos(r^2\cos^2\theta+r^2\mathrm{sen}^2\theta)\cdot r\,d\theta\,dr=$$

$$=\int_0^1\int_{\frac{\pi}{4}}^{\pi} r\cos(r^2)\,d\theta\,dr=\int_0^1 r\cos(r^2)\big[\theta\big]_{\theta=\frac{\pi}{4}}^{\pi}\,dr=\int_0^1\frac{3}{4}\pi r\cos(r^2)\,dr=$$

$$=\frac{3}{8}\pi\int_0^1 2r\cos(r^2)\,dr=\frac{3}{8}\pi\Big[\mathrm{sen}(r^2)\Big]_{r=0}^{1}=\frac{3}{8}\pi\,\mathrm{sen}(1).$$

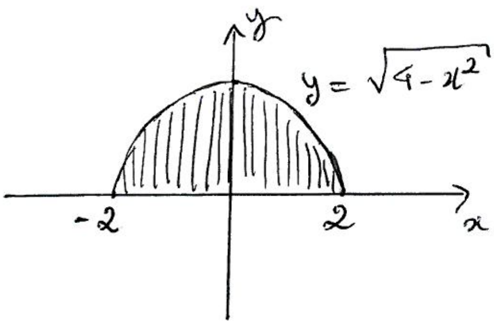
5.

$$-2\leq x\leq 2$$
$$0\leq y\leq\sqrt{4-x^2}$$
$$0\leq z\leq x^2+y^2$$

$$y=\sqrt{4-x^2}\Rightarrow x^2+y^2=4$$

a)

Projeças ao plano xOy:



Coordenadas cilíndricas:

$$\begin{cases}x=r\cos\theta\\ y=r\,\mathrm{sen}\,\theta\\ z=z\end{cases}$$

$$x^2+y^2=r^2$$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{x^2+y^2} y\,dz\,dy\,dx = \int_0^2\int_0^{\pi}\int_0^{r^2} r\,\text{sen}\theta \cdot r\,dz\,d\theta\,dr$$

$$= \int_0^2\int_0^{\pi}\int_0^{r^2} r^2\,\text{sen}\theta\,dz\,d\theta\,dr$$

b) $$= \int_0^2\int_0^{\pi} r^2\,\text{sen}\theta\,[z]_{z=0}^{r^2}\,d\theta\,dr = \int_0^2\int_0^{\pi} r^4\,\text{sen}\theta\,d\theta\,dr$$

$$= \int_0^2 r^4\,[-\cos\theta]_{\theta=0}^{\pi}\,dr = \int_0^2 r^4\,(-\cos\pi + \cos 0)\,dr$$

$$= \int_0^2 2r^4\,dr = \left[2\frac{r^5}{5}\right]_{r=0}^{2} = \frac{64}{5}$$

Alternativa à alínea b) :

$$\int_0^2\int_0^y\int_0^{x^2+y^2} y\,dz\,dx\,dy = \int_0^2\int_0^y y[z]_{z=0}^{x^2+y^2}\,dx\,dy$$

$$= \int_0^2\int_0^y (yx^2+y^3)\,dx\,dy = \int_0^2\left[y\frac{x^3}{3} + xy^3\right]_{x=0}^{y} dy$$

$$= \int_0^2\left(\frac{y^4}{3} + y^4\right)dy = \int_0^2 \frac{4}{3}y^4\,dy = \frac{4}{3}\left[\frac{y^5}{5}\right]_{y=0}^{2} = \frac{4}{3}\cdot\frac{2^5}{5} = \frac{2^7}{15}$$

6.

$$\iiint_E 1\,dx\,dy\,dz = \int_0^2 \int_0^{2\pi} \int_0^{\pi/2} \rho^2 \operatorname{sen}\varphi \, d\varphi \, d\theta \, d\rho =$$

$$= \int_0^2 \int_0^{2\pi} \rho^2 \left[-\cos\varphi\right]_{\varphi=0}^{\pi/2} d\theta \, d\rho = \int_0^2 \int_0^{2\pi} \rho^2 \left[-\cos\frac{\pi}{2} - (-\cos 0)\right] d\theta \, d\rho$$

$$= \int_0^2 \int_0^{2\pi} \rho^2 \, d\theta \, d\rho = \int_0^2 \rho^2 \left[\theta\right]_{\theta=0}^{2\pi} d\rho = \int_0^2 2\pi\rho^2 \, d\rho = 2\pi\left[\frac{\rho^3}{3}\right]_{\rho=0}^{2} =$$

$$= \frac{16\pi}{3}$$

7.

a) $v(t) = r'(t) = (-\operatorname{sen}t, \cos t, 3)$

$v(0) = (0, 1, 3)$

$a(t) = r''(t) = (-\cos t, -\operatorname{sen}t, 0)$

$a(0) = (1, 0, 0)$

b)

$$k(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3} = \frac{\| (3\operatorname{sen}t, -3\cos t, 1) \|}{\| (-\operatorname{sen}t, \cos t, 3) \|^3}$$

$$= \frac{\sqrt{9\operatorname{sen}^2 t + 9\cos^2 t + 1}}{\left(\sqrt{\operatorname{sen}^2 t + \cos^2 t + 9}\right)^3} = \frac{\sqrt{10}}{\left(\sqrt{10}\right)^3} = \frac{1}{10}$$

c) $T(t) = \dfrac{r'(t)}{\|r'(t)\|} = \dfrac{1}{\sqrt{10}}(-\text{sen}t, \cos t, 3)$

$T'(t) = \dfrac{1}{\sqrt{10}}(-\cos t, -\text{sen}t, 0)$; $\|T'(t)\| = \dfrac{1}{\sqrt{10}}$

$N(t) = \dfrac{T'(t)}{\|T'(t)\|} = (-\cos t, -\text{sen}t, 0)$

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\text{sen}t}{\sqrt{10}} & \frac{\cos t}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\cos t & -\text{sen}t & 0 \end{vmatrix} =$$

$$= \left(\frac{3}{\sqrt{10}}\text{sen}t, -\frac{3}{\sqrt{10}}\cos t, \frac{1}{\sqrt{10}}\right)$$

d) $r(t) = (1,0,1) \Rightarrow t = 0$

$B(0) = \left(0, -\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

Plano osculador: $(0,-3,1) \cdot (x-1, y-0, z-1) = 0$

$\iff -3y + z = 1$

7. A curva está parametrizada por comprimento de arco se $\|r'(t)\| = 1, \forall t \in \mathbb{R}$.

Assim,
$\|r'(t)\| = \|(-\text{sen}t, f'(t), \cos t)\| = \sqrt{1+[f'(t)]^2} = 1$

se e só se $[f'(t)]^2 = 0$, ou seja, se e só se $f'(t) = 0$, ou ainda, se e só se $f(t) = K$, com K constante.

Neste caso, para $t \in [2,5]$, o comprimento da curva é
$L = \int_2^5 \|r'(t)\|\,dt = \int_2^5 1\,dt = [t]_2^5 = 5-2 = 3.$