Teste 2 - modelo

Proposta de resolução

1. a) $(x,y)=(0,0)$ é um ponto crítico de f para $K\in\mathbb{R}$ uma vez que

$$\frac{\partial f}{\partial x}(0,0)=\left(2Kx-4y\right)\Big|_{(0,0)}=0 \qquad e$$

$$\frac{\partial f}{\partial y}(0,0)=\left(2y-4x\right)\Big|_{(0,0)}=0.$$

b) $$\Delta f(x,y)=\frac{\partial^2 f}{\partial x^2}(x,y)\cdot\frac{\partial^2 f}{\partial y^2}(x,y)-\left[\frac{\partial^2 f}{\partial x\partial y}(x,y)\right]^2$$
$$=2k\times 2-(-4)^2$$
$$=4k-16$$

i. Se $\Delta f(0,0)<0 \iff 4k-16<0 \iff k<4$, o ponto $(0,0)$ é ponto de sela.

ii. Se $k>4$, tem-se $\Delta f(0,0)>0$ e $\frac{\partial^2 f}{\partial x^2}(0,0)=2k>0$. Neste caso, $(0,0)$ é ponto minimizante.

Se $k=4$, temos

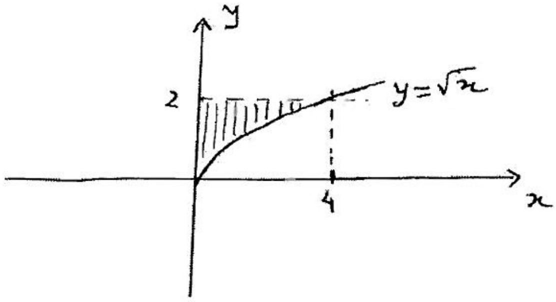
$$f(x,y)=4x^2+y^2-4xy=(2x+y)^2\geq 0,\ \forall (x,y)\in\mathbb{R}^2.$$

Ou seja, $(0,0)$ é também um ponto minimizante.

Logo, não existem valores de k para os quais $(0,0)$ é ponto maximizante.

iii. Para $k>4$, $(0,0)$ é ponto minimizante, como já observado em ii.

2.



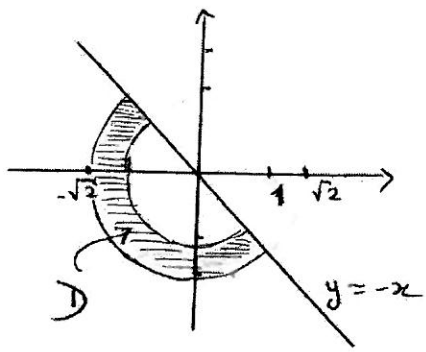
$$I=\int_0^4\int_{\sqrt{x}}^2 (1+y^3)^{1/2}\,dy\,dx=\int_0^2\int_0^{y^2}(1+y^3)^{1/2}\,dx\,dy=\int_0^2\left[(1+y^3)^{1/2}x\right]_{x=0}^{y^2}dy$$

$$=\int_0^2(1+y^3)^{1/2}y^2\,dy=\frac{1}{3}\int_0^2 3y^2(1+y^3)^{1/2}\,dy=\left[\frac{1}{3}\cdot\frac{(1+y^3)^{1/2+1}}{1/2+1}\right]_0^2$$

$$=\left[\frac{2}{9}(1+y^3)^{3/2}\right]_0^2=\frac{2}{9}\left[(1+8)^{3/2}-1\right]=\frac{2}{9}\cdot\left(\sqrt{9^3}-1\right)=\frac{2}{9}\times 26=\frac{52}{9}$$

3.

a)



$$D=\left\{(r,\theta): 1\le r\le\sqrt{2},\ \frac{3\pi}{4}\le\theta\le\frac{7}{4}\pi\right\}$$

b)

$$\text{Area }(D)=\iint_D dx\,dy=\int_1^{\sqrt{2}}\int_{\frac{3\pi}{4}}^{\frac{7\pi}{4}} r\,d\theta\,dr=\int_1^{\sqrt{2}}\left[r\,\theta\right]_{\theta=\frac{3\pi}{4}}^{\frac{7\pi}{4}}dr$$

$$=\int_1^{\sqrt{2}}\pi r\,dr=\left[\pi\frac{r^2}{2}\right]_1^{\sqrt{2}}=\pi-\frac{\pi}{2}=\frac{\pi}{2}$$

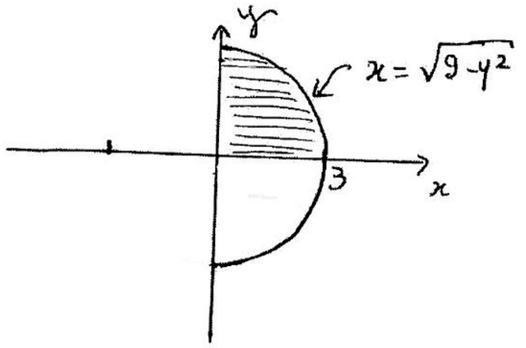
4.

a)

$$0 \leq y \leq 3$$

$$0 \leq x \leq \sqrt{9-y^2}$$

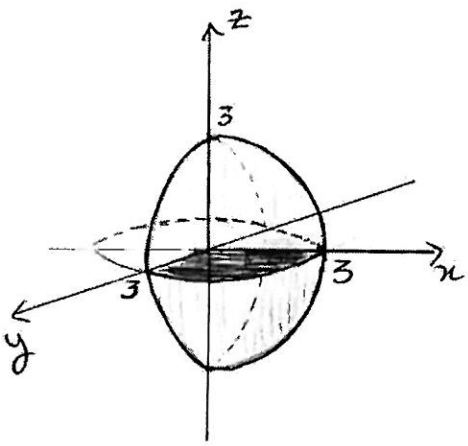
$$x = \sqrt{9-y^2} \Rightarrow x^2+y^2 = 9$$



$$-\sqrt{9-x^2-y^2} \leq z \leq \sqrt{9-x^2-y^2}$$

$$z = \sqrt{9-x^2-y^2} \Rightarrow x^2+y^2+z^2 = 9$$

Domínio de integração:



Um quarto de uma esfera de centro na origem e raio 3

Coordenadas esféricas:

$$\{(\rho, \theta, \varphi) : 0 \leq \rho \leq 3, \; 0 \leq \theta \leq \pi/2, \; 0 \leq \varphi \leq \pi\}$$

$$\rho^2 = x^2+y^2+z^2$$

b)

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy =$$

$$= \int_0^3 \int_0^{\pi} \int_0^{\pi/2} \rho . \rho^2 \operatorname{sen}\varphi \, d\theta \, d\varphi \, d\rho$$

$$= \int_0^3 \int_0^{\pi} \rho^3 \operatorname{sen}\varphi \cdot [\theta]_0^{\pi/2} d\varphi\, d\rho = \int_0^3 \int_0^{\pi} \frac{\pi}{2}\rho^3 \operatorname{sen}\varphi \, d\varphi\, d\rho$$

$$= \int_0^3 \frac{\pi}{2}\rho^3 [-\cos\varphi]_0^{\pi} d\rho = \frac{\pi}{2}\int_0^3 \rho^3 \, [-\cos\pi - (-\cos 0)]\, d\rho$$

$$= \pi \int_0^3 \rho^3 \, d\rho = \pi \left[\frac{\rho^4}{4}\right]_0^3 = \frac{\pi}{4} \times 3^4$$

5.

a) $v(t) = \int a(t)\, dt = \int (-\operatorname{sen} t, -\cos t, 0)\, dt$

$= \left(\int -\operatorname{sen} t\, dt, \int -\cos t\, dt, k_1 \right)$

$= (\cos t + k_2, -\operatorname{sen} t + k_3, k_1)$, k_1, k_2, k_3 constante

$$V(0) = (1, 0, 2) \Rightarrow \begin{cases} \cos 0 + k_2 = 1 \\ -\operatorname{sen} 0 + k_3 = 0 \\ k_1 = 2 \end{cases} \Rightarrow \begin{cases} k_2 = 0 \\ k_3 = 0 \\ k_1 = 2 \end{cases}$$

Assim,

$$v(t) = (\cos t, -\operatorname{sen} t, 2)$$

$$r(t) = \int v(t)\,dt = \int (\cos t, -\operatorname{sen} t, 2)\,dt$$
$$= \left(\int \cos t\,dt, \int -\operatorname{sen} t\,dt, \int 2\,dt \right)$$
$$= (\operatorname{sen} t + k_1, \cos t + k_2, 2t + k_3), \quad k_1, k_2, k_3 \text{ Constante}$$

$$r(0) = (0,1,1) \Rightarrow \begin{cases} \operatorname{sen} 0 + k_1 = 0 \\ \cos 0 + k_2 = 1 \\ 2\times 0 + k_3 = 1 \end{cases} \Rightarrow \begin{cases} k_1 = 0 \\ k_2 = 0 \\ k_3 = 1 \end{cases}$$

Logo,

$$r(t) = (\operatorname{sen} t, \cos t, 2t+1)$$

b)

$$K(t) = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

$$r'(t) \times r''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & -\operatorname{sen} t & 2 \\ -\operatorname{sen} t & -\cos t & 0 \end{vmatrix} = (-2\cos t, -2\operatorname{sen} t, -\cos^2 t - \operatorname{sen}^2 t)$$
$$= (-2\cos t, -2\operatorname{sen} t, -1)$$

$$\| r'(t) \times r''(t) \| = (4\cos^2 t + 4\operatorname{sen}^2 t + 1)^{1/2} = \sqrt{5}$$
$$\| r'(t) \|^3 = (\cos^2 t + \operatorname{sen}^2 t + 4)^{3/2} = 5^{3/2} = 5\sqrt{5}$$

Assim,

$$K(t) = \frac{\sqrt{5}}{5\sqrt{5}} = \frac{1}{5}$$

c) Plano normal a r em $P = r(\pi) = (\operatorname{sen}\pi, \cos\pi, 2\pi+1) = (0, -1, 2\pi+1)$

$$r'(\pi) \cdot (x-0, y+1, z-(2\pi+1)) = 0$$
$$\Leftrightarrow (-1, 0, 2) \cdot (x, y+1, z-2\pi-1) = 0 \Leftrightarrow -x + 2z = 4\pi + 2$$

6.

Seja $r(t) = (x(t), y(t), z(t))$

(a)

$$r(t) \times r'(t) =$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = (yz' - y'z, -xz' + zx', xy' - x'y)$$

$$[r(t) \times r'(t)]' = \Big(\cancel{y'z'} + yz'' - y''z - \cancel{y'z'},$$
$$\cancel{-x'z'} - xz'' + \cancel{z'x'} + zx'',$$
$$\cancel{x'y'} + xy'' - x''y - \cancel{x'y'} \Big)$$
$$= (yz'' - y''z, -xz'' + zx'', xy'' - x''y)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x'' & y'' & z'' \end{vmatrix} = r(t) \times r''(t)$$

b)

$$u(t) = r(t) \cdot [r'(t) \times r''(t)]$$

$$u'(t) = r'(t) \cdot [r'(t) \times r''(t)] + r(t) \cdot [r'(t) \times r''(t)]'$$
$$= r'(t) \cdot [r'(t) \times r''(t)] + r(t) \cdot [r'(t) \times r'''(t)]$$
$$= \begin{vmatrix} x' & y' & z' \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} + r(t) \cdot [r'(t) \times r'''(t)]$$
$$= r(t) \cdot [r'(t) \times r'''(t)].$$