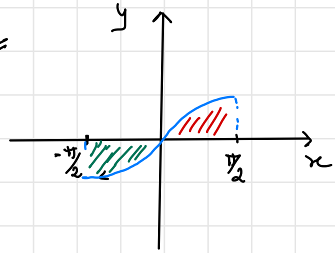
10 Dezembro

① Calcule $\int_0^3 x^2\,dx$

$$\int_0^3 x^2\,dx = \left[\frac{x^3}{3}\right]_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 27$$

② $$\int_{-\pi/2}^{\pi/2} \operatorname{sen} x\,dx = \left[-\cos x\right]_{-\pi/2}^{\pi/2} = -\cos\frac{\pi}{2} - \left(-\cos\left(-\frac{\pi}{2}\right)\right) = 0$$

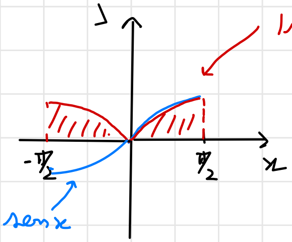
ou



$$\int_{-\pi/2}^{\pi/2} \operatorname{sen} x\,dx = \text{área vermelha} - \text{área verde}$$
$$= 0$$

Em geral, se f é uma função ímpar, então $\int_{-a}^{a} f(x)\,dx = 0$

③ $$\int_{-\pi/2}^{\pi/2} |\operatorname{sen} x|\,dx = 2\int_0^{\pi/2} \operatorname{sen} x\,dx = 2\left[-\cos x\right]_0^{\pi/2}$$
$$= 2\left(-\cos\frac{\pi}{2} - (-\cos 0)\right) = 2$$



Em geral, se f é uma função par, então $\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

ou $$\int_{-\pi/2}^{\pi/2} |\operatorname{sen} x|\, dx = \int_{-\pi/2}^{0} (-\operatorname{sen} x)\, dx + \int_{0}^{\pi/2} \operatorname{sen} x\, dx$$

$$= \left[\cos x\right]_{-\pi/2}^{0} + \left[-\cos x\right]_{0}^{\pi/2} =$$

$$= \left(\cos 0 - \cos\left(-\tfrac{\pi}{2}\right)\right) + \left(-\cos\frac{\pi}{2} - (-\cos 0)\right) = 1 + 1 = 2$$

④ Calcule $\displaystyle\int_{-3}^{5} |x-1|\, dx$

$$|x-1| = \begin{cases} x-1 & \text{se } x-1 \geq 0 \\ -(x-1) & \text{se } x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{se } x \geq 1 \\ -x+1 & \text{se } x < 1 \end{cases}$$

$$\int_{-3}^{5} |x-1|\, dx = \int_{-3}^{1} -(x-1)\, dx + \int_{1}^{5} (x-1)\, dx$$

$$= \left[-\frac{x^2}{2} + x\right]_{-3}^{1} + \left[\frac{x^2}{2} - x\right]_{1}^{5}$$

$$= \left(-\frac{1}{2} + 1\right) - \left(-\frac{(-3)^2}{2} - 3\right) + \left(\frac{5^2}{2} - 5\right) - \left(\frac{1}{2} - 1\right)$$

$$= 16$$

⑤ Calcule $\displaystyle\int_{\sqrt{3}/3}^{\sqrt{3}} \operatorname{arctg}\left(\frac{1}{x}\right) dx$

Sejam

$f'(x) = 1$ $\qquad$ $f(x) = x$

$$g(x) = \operatorname{arctg}\left(\frac{1}{x}\right) \qquad g'(x) = \frac{\left(\frac{1}{x}\right)'}{1+\frac{1}{x^2}} = \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}} = -\frac{1}{x^2+1}$$

Aplicando a fórmula de integração por partes, temos que

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \operatorname{arctg}\left(\frac{1}{x}\right) dx = \left[x \operatorname{arctg}\left(\frac{1}{x}\right)\right]_{\sqrt{3}/3}^{\sqrt{3}} - \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{-x}{x^2+1} dx$$

$$= \left[x \operatorname{arctg}\left(\frac{1}{x}\right)\right]_{\sqrt{3}/3}^{\sqrt{3}} + \int_{\sqrt{3}/3}^{\sqrt{3}} \frac{x}{x^2+1} dx$$

$$= \left[x \operatorname{arctg}\left(\frac{1}{x}\right)\right]_{\sqrt{3}/3}^{\sqrt{3}} + \frac{1}{2}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{2x}{x^2+1} dx$$

$$= \left[x \operatorname{arctg}\left(\frac{1}{x}\right)\right]_{\sqrt{3}/3}^{\sqrt{3}} + \frac{1}{2}\left[\ln(x^2+1)\right]_{\sqrt{3}/3}^{\sqrt{3}}$$

$$= \sqrt{3}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{3}{\sqrt{3}}\right) + \frac{1}{2}\ln 4 - \frac{1}{2}\ln\frac{4}{3}$$

$$= \sqrt{3}\operatorname{arctg}\left(\frac{\sqrt{3}}{3}\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}(\sqrt{3}) + \frac{1}{2}\ln 4 - \frac{1}{2}\ln\frac{4}{3}$$

$$= \sqrt{3}\,\frac{\pi}{6} - \frac{\sqrt{3}}{3}\,\frac{\pi}{3} + \frac{1}{2}\ln 4 - \frac{1}{2}\ln\frac{4}{3}$$

(6) $\displaystyle\int_{1/2}^{3/4} \frac{1}{\sqrt{x}\,\sqrt{1-x}}$, efetuando a substituição $x = \operatorname{sen}^2 t$

(i) <u>Substituição</u>

Fazendo $x = \operatorname{sen}^2 t$, tem-se

$$\varphi(t) = \operatorname{sen}^2 t, \quad \varphi'(t) = 2 \operatorname{sen} t \cos t, \quad \varphi(\pi/4) = \frac{1}{2}, \quad \varphi(\pi/3) = \frac{3}{4}$$

(ii) <u>Cálculo do novo integral</u>

$$\int_{1/2}^{3/4} \frac{1}{\sqrt{x}\,\sqrt{1-x}}\, dx = \int_{\pi/4}^{\pi/3} \frac{1}{\underbrace{\sqrt{\operatorname{sen}^2 t}}_{\substack{= |\operatorname{sen} t| \\ = \operatorname{sen} t \\ t \in [\pi/4,\, \pi/3]}}\ \underbrace{\sqrt{1-\operatorname{sen}^2 t}}_{\substack{= \sqrt{\cos^2 t} = |\cos t| \\ = \cos t \\ t \in [\pi/4,\, \pi/3]}}} \underbrace{2 \operatorname{sen} t \cos t}_{\varphi'(t)}\, dt$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\operatorname{sen} t \cos t}\, 2 \operatorname{sen} t \cos t\, dt = \int_{\pi/4}^{\pi/3} 2\, dt =$$

$$= \Big[2t \Big]_{\pi/4}^{\pi/3} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

(7) Estabeleça um integral (ou soma de integrais) que dê a área da região

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : x - y \geq -2 \quad \wedge \quad 1 \leq y \leq 2 - x^2\}$$

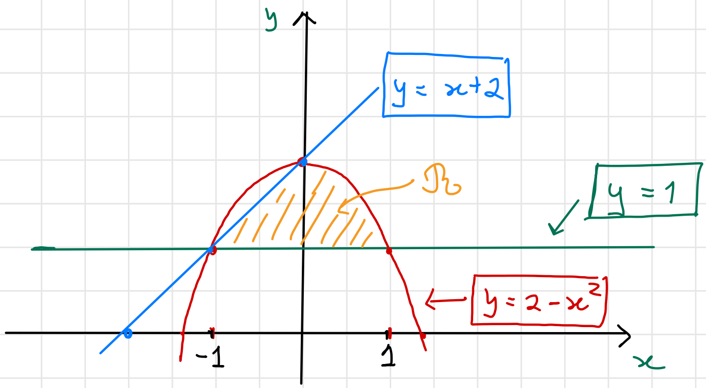
$x - y = -2$ $\Leftrightarrow$ $y = x + 2$ $\qquad y = 1 \qquad y = 2 - x^2$

Observemos que:

$x - y \geq -2$

$\Leftrightarrow -y \geq -2 - x$

$\Leftrightarrow y \leq 2 + x$



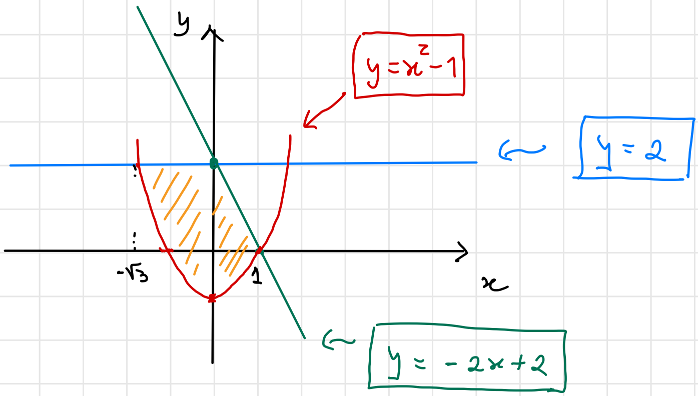
$$\text{Área}(\mathcal{R}) = \int_{-1}^{0} (2 + x - 1)\, dx + \int_{0}^{1} (2 - x^2 - 1)\, dx$$

⑧ Estabeleça um integral (ou soma de integrais) que dê a área da região

$$\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : y \leq -2x+2 \ , \ y \leq 2 \ , \ y \geq x^2 - 1\},$$

$y = -2x+2$ $\quad$ $y = 2$ $\quad$ $y = x^2 - 1$

fazendo previamente um esboço da região $\mathcal{R}$



$$\text{Área}(\mathcal{R}) = \int_{-\sqrt{3}}^{0} \left(2 - (x^2 - 1)\right) dx + \int_{0}^{1} \left((-2x+2) - (x^2-1)\right) dx$$

(9) Calcule $\int_0^1 x^2 \ln(x^2+1)\,dx$

Sejam

$f'(x) = x^2 \qquad f(x) = \frac{x^3}{3}$

$g(x) = \ln(x^2+1) \qquad g'(x) = \frac{2x}{x^2+1}$

Aplicando a fórmula de integração por partes, temos que

$$\int_0^1 x^2 \ln(x^2+1)\,dx = \left[\frac{x^3}{3}\ln(x^2+1)\right]_0^1 - \int_0^1 \frac{2}{3}\frac{x^4}{x^2+1}\,dx$$

$$= \left[\frac{x^3}{3}\ln(x^2+1)\right]_0^1 - \frac{2}{3}\int_0^1 \frac{x^4}{x^2+1}\,dx$$

Observação.

(1)
$$\begin{array}{r|l} x^4 & x^2+1 \\ \cline{2-2} -x^4-x^2 & x^2-1 \\ \hline -x^2 & \\ x^2+1 & \\ \hline 1 & \end{array}$$

Então, $\frac{x^4}{x^2+1} = x^2-1 + \frac{1}{x^2+1}$

ou

(2)
$$\frac{x^4}{x^2+1} = \frac{x^4-1+1}{x^2+1} = \frac{x^4-1}{x^2+1} + \frac{1}{x^2+1} = \frac{(x^2-1)\cancel{(x^2+1)}}{\cancel{x^2+1}} + \frac{1}{x^2+1} =$$

$$= x^2-1 + \frac{1}{x^2+1}$$

Então,

$$\int_0^1 x^2 \ln(x^2+1)\,dx = \left[\frac{x^3}{3}\ln(x^2+1)\right]_0^1 - \frac{2}{3}\int_0^1 \frac{x^4}{x^2+1}\,dx$$

$$= \left[\frac{x^3}{3}\ln(x^2+1)\right]_0^1 - \frac{2}{3}\int_0^1 \left(x^2-1+\frac{1}{x^2+1}\right)dx$$

$$= \left[\frac{x^3}{3}\ln(x^2+1)\right]_0^1 - \frac{2}{3}\left[\frac{x^3}{3} - x + \operatorname{arctg} x\right]_0^1$$

$$= \frac{1}{3}\ln 2 - \frac{2}{3}\left(\frac{1}{3} - 1 + \operatorname{arctg} 1\right)$$

(10) Calcule $\displaystyle\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{x^2+1}}\,dx$, efectuando a substituição $x = \dfrac{1}{t}$

(i) <u>Substituição</u>

Fazendo $x = \dfrac{1}{t}$ tem-se

$$\varphi(t) = \frac{1}{t}, \quad \varphi'(t) = -\frac{1}{t^2}, \quad \varphi\left(\frac{4}{3}\right) = \frac{3}{4}, \quad \varphi\left(\frac{3}{4}\right) = \frac{4}{3}$$

(ii) <u>Cálculo do novo integral</u>

$$\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int_{4/3}^{3/4} \frac{1}{\frac{1}{t^2}\sqrt{\frac{1}{t^2}+1}} \cdot \underbrace{\left(-\frac{1}{t^2}\right)}_{\varphi'(t)} dt =$$

$$= \int_{3/4}^{4/3} \frac{1}{\cancel{\frac{1}{t^2}}\sqrt{\frac{1}{t^2}+1}} \, \cancel{\frac{1}{t^2}}\, dt = \int_{3/4}^{4/3} \frac{1}{\sqrt{\frac{1}{t^2}+1}}\, dt =$$

$$= \int_{3/4}^{4/3} \frac{1}{\sqrt{\frac{1+t^2}{t^2}}}\, dt = \int_{3/4}^{4/3} \sqrt{\frac{t^2}{1+t^2}}\, dt = \int_{3/4}^{4/3} \frac{t}{\sqrt{1+t^2}}\, dt$$

$$= \int_{3/4}^{4/3} t\,(1+t^2)^{-1/2}\, dt = \frac{1}{2}\int_{3/4}^{4/3} 2t\,(1+t^2)^{-1/2}\, dt = \frac{\cancel{1}}{2}\left[\frac{(1+t^2)^{1/2}}{\frac{1}{\cancel{2}}}\right]_{3/4}^{4/3}$$

$$= \left[(1+t^2)^{1/2}\right]_{3/4}^{4/3} = \sqrt{1+\frac{16}{9}} - \sqrt{1+\frac{9}{16}} = \frac{5}{3} - \frac{5}{4} = \frac{5}{12}$$

(11) Calcule $\int_1^2 x\sqrt{x-1}\,dx$, efetuando a substituição $x-1=t^2$

(i) Substituição

Fazendo $x = t^2+1$, tem-se

$\varphi(t) = t^2+1$, $\varphi'(t) = 2t$, $\varphi(0) = 1$, $\varphi(1) = 2$

(ii) Cálculo do novo integral

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (t^2+1)\sqrt{t^2+1-1}\cdot \underbrace{2t}_{\varphi'(t)}\,dt =$$

$$= \int_0^1 (t^2+1)\,|t|\,2t\,dt = \int_0^1 (t^2+1)\,t\,2t\,dt =$$
$$t \geqslant 0$$

$$= \int_0^1 (2t^4+2t^2)\,dt = \left[2\frac{t^5}{5} + 2\frac{t^3}{3}\right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$$