3 Dezembro

① $\displaystyle\int \frac{-7}{\sqrt{1-5x}}\,dx = -7\int (1-5x)^{-1/2}\,dx =$

$f(x) = 1-5x$
$\alpha = -1/2$
$f'(x) = -5$

$\displaystyle = \frac{-7}{-5}\int \underbrace{-5}_{f'}\,\underbrace{(1-5x)^{-1/2}}_{f^{-1/2}}\,dx = \frac{7}{5}\,\frac{(1-5x)^{1/2}}{\frac{1}{2}} + C$

$\displaystyle = \frac{14}{5}(1-5x)^{1/2} + C, \quad C\in\mathbb{R}$

② $\displaystyle\int x^2 e^{x^3}\,dx = \frac{1}{3}\int \underbrace{3x^2}_{f'}\,\underbrace{e^{x^3}}_{e^{f}}\,dx = \frac{1}{3}e^{x^3} + C, \quad C\in\mathbb{R}$

$f(x) = x^3$
$f'(x) = 3x^2$

③ $\displaystyle\int \frac{x}{\sqrt{2-3x}}\,dx$, efetuando a substituição $\sqrt{2-3x} = t$

(i) <u>Substituição</u>

Fazendo $\sqrt{2-3x} = t$ tem-se $x = \frac{2}{3} - \frac{1}{3}t^2$

$\varphi(t) = \frac{2}{3} - \frac{1}{3}t^2$, $\varphi'(t) = -\frac{2}{3}t$

(ii) <u>Cálculo da nova primitiva</u>

$$\int \frac{\frac{2}{3} - \frac{1}{3}t^2}{t} \cdot \underbrace{-\frac{2}{3}t}_{\varphi'(t)}\,dt = \int -\frac{4}{9} + \frac{2}{9}t^2\,dt$$

$$= -\frac{4}{9}t + \frac{2}{27}t^3 + C, \quad C \in \mathbb{R}$$

(iii) <u>Regresso à variável inicial x</u>

$$\int \frac{x}{\sqrt{2-3x}}\,dx = -\frac{4}{9}\sqrt{2-3x} + \frac{2}{27}\sqrt{(2-3x)^3} + C, \quad C \in \mathbb{R}$$

④ $$\int \frac{x^2+x+10}{(x-1)^2(x+3)}\,dx$$

(i) <u>Zeros de $D(x)$</u> $= (x-1)^2(x+3)$

- $\boxed{x=1}$ real de multiplicidade ②

  ⤳ contribui com ② frações simples

- $\boxed{x=-3}$ real de multiplicidade ①

  ⤳ contribui com ① fração simples

(ii) <u>Decomposição em frações simples</u>

$$\frac{x^2+x+10}{(x-1)^2(x+3)} = \frac{A_1}{(x-1)^2} + \frac{A_2}{x-1} + \frac{B}{x+3}$$

Reduzindo ao mesmo denominador, vem que

$$x^2+x+10 = A_1(x+3) + A_2(x-1)(x+3) + B(x-1)^2$$

$x=1 \rightsquigarrow 12 = 4A_1 \Leftrightarrow A_1 = 3$

$x=-3 \rightsquigarrow 16 = 16B \Leftrightarrow B = 1$

$x=0 \rightsquigarrow 10 = 9 - 3A_2 + 1 \Leftrightarrow A_2 = 0$

(iii) Cálculo das primitivas

$$\int \frac{x^2+x+10}{(x-1)^2(x+3)}\,dx = \int \frac{3}{(x-1)^2}\,dx + \int \frac{1}{x+3}\,dx$$

$$= -\frac{3}{x-1} + \ln|x+3| + C, \quad C \in \mathbb{R}$$

Observação.

- $$\int \frac{1}{(x-1)^2}\,dx = \int \underbrace{1}_{f'}\,\underbrace{(x-1)^{-2}}_{f^{-2}}\,dx \underset{R_2}{=} \frac{(x-1)^{-1}}{-1} + C$$

$$= -\frac{1}{x-1} + C, \quad C \in \mathbb{R}$$

$f(x) = x-1$

$\alpha = -2$

$f'(x) = 1$

- $$\int \frac{\overset{f'}{1}}{\underset{f}{x+3}}\,dx \underset{R_3}{=} \ln|x+3| + C, \quad C \in \mathbb{R}$$

⑤ $\displaystyle\int \frac{x^2+2x-3}{(x-2)(x^2+1)}\,dx$

(i) <u>Zeros de $D(x)$</u> $= (x-2)(x^2+1)$

- $\boxed{x=2}$ real de multiplicidade 1

  ⇝ contribui com 1 fração simples

- $\boxed{x=\pm i}$ par de complexos conjugados de multiplicidade 1

  ⇝ contribui com 1 fração simples

(ii) <u>Decomposição em frações simples</u>

$$\frac{x^2+2x-3}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

Reduzindo ao mesmo denominador, vem que:

$$x^2+2x-3 = A(x^2+1) + (Bx+C)(x-2)$$

$$\Leftrightarrow \quad x^2+2x-3 = Ax^2 + A + Bx^2 - 2Bx + Cx - 2C$$

$$\Leftrightarrow \quad \underline{\underline{1}}\,x^2 + \underline{2}\,x \,\underline{-3} = (A+B)x^2 + (C-2B)x + (A-2C)$$

$$\Leftrightarrow \begin{cases} 1 = A+B \\ 2 = C-2B \\ -3 = A-2C \end{cases} \quad \Leftrightarrow \begin{cases} A = 1 \\ B = 0 \\ C = 2 \end{cases}$$

(iii) Cálculo das primitivas

$$\int \frac{x^2+2x-3}{(x-2)(x^2+1)}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{2}{x^2+1}\,dx$$

$$= \ln|x-2| + 2\,\operatorname{arctg} x + C,\quad C\in\mathbb{R}$$

Observação:

- $\displaystyle\int \frac{\overset{f'}{1}}{\underset{f}{x-2}}\,dx \underset{R_3}{=} \ln|x-2| + C,\quad C\in\mathbb{R}$

- $\displaystyle\int \frac{1}{x^2+1}\,dx \underset{R_{15}}{=} \operatorname{arctg} x + C,\quad C\in\mathbb{R}$

⑥ $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$, efetuando a substituição $x = \operatorname{sen} t$
$t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

(i) Substituição

Fazendo $x = \operatorname{sen} t$, $t \in \left]-\pi/2, \pi/2\right[$, tem-se

$\varphi(t) = \operatorname{sen} t$, $\varphi'(t) = \cos t$, $t = \operatorname{arcsen} x$

(ii) Cálculo da nova primitiva

$$\int \frac{\operatorname{sen}^2 t}{\underbrace{\sqrt{1-\operatorname{sen}^2 t}}} \cos t \, dt = \int \operatorname{sen}^2 t \, dt = \int \frac{1-\cos(2t)}{2}\, dt$$

$\sqrt{\cos^2 t} = |\cos t| = \cos t$
$t \in \left]-\pi/2, \pi/2\right[$

$$= \frac{1}{2}\int 1\, dt - \frac{1}{2}\int \cos(2t)\, dt = \frac{1}{2}\int 1\, dt - \frac{1}{2}\,\frac{1}{2}\int \underbrace{2}_{f'}\, \underbrace{\cos(2t)}_{\cos f}\, dt$$

$$= \frac{1}{2}t - \frac{1}{4}\operatorname{sen}(2t) + C, \quad C \in \mathbb{R}$$

$\boxed{\operatorname{sen}(2t) = 2\operatorname{sen} t \cos t}$

$$= \frac{t}{2} - \frac{1}{2}\operatorname{sen} t \cos t + C, \quad C \in \mathbb{R}$$

(iii) Regresso à variável inicial x

Para desfazer a substituição, recorde-se que:

$\operatorname{sen} t = x$, $t \in \left]-\pi/2, \pi/2\right[$, $t = \operatorname{arcsen} x$

$\operatorname{sen}^2 t + \cos^2 t = 1 \Leftrightarrow \cos^2 t = 1 - \operatorname{sen}^2 t \Leftrightarrow \cos t = \cancel{\pm} \sqrt{1-\operatorname{sen}^2 x}$

porque $t \in \left]-\pi/2, \pi/2\right[$

Então,

$$\int \frac{x^2}{\sqrt{1-x^2}}\, dx = \frac{\arcsen x}{2} - \frac{1}{2} x \sqrt{1-x^2} + C, \quad C \in \mathbb{R}$$

(7) $\displaystyle\int x \operatorname{arcsen}(x^2)\,dx = \frac{x^2}{2}\operatorname{arcsen}(x^2) - \int \frac{x^2}{2}\,\frac{2x}{\sqrt{1-x^4}}\,dx$

$f'(x) = x \qquad f(x) = \dfrac{x^2}{2}$

$g(x) = \operatorname{arcsen}(x^2) \qquad g'(x) = \dfrac{2x}{\sqrt{1-(x^2)^2}} = \dfrac{2x}{\sqrt{1-x^4}}$

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$$

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) - \int x^3(1-x^4)^{-1/2}\,dx$$

$f(x) = 1-x^4$
$\alpha = -1/2$
$f'(x) = -4x^3$

R2

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) + \int -x^3(1-x^4)^{-1/2}\,dx$$

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) + \frac{1}{4}\int \underbrace{-4x^3}_{f'}\,\underbrace{(1-x^4)^{-1/2}}_{f^{-1/2}}\,dx$$

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) + \frac{1}{4}\,\frac{(1-x^4)^{1/2}}{\frac{1}{2}} + C$$

$$= \frac{x^2}{2}\operatorname{arcsen}(x^2) + \frac{1}{2}\sqrt{1-x^4} + C, \quad C \in \mathbb{R}$$

⑧ $$\int \frac{2+\sqrt{\operatorname{arctg}(2x)}}{1+4x^2}\,dx = \int \frac{2}{1+4x^2}\,dx + \int \frac{\sqrt{\operatorname{arctg}(2x)}}{1+4x^2}\,dx$$

$$= \int \frac{2}{1+(2x)^2}\,dx + \int \frac{1}{1+4x^2}\,(\operatorname{arctg}(2x))^{1/2}\,dx$$

$$= \int \frac{2}{1+(2x)^2}\,dx + \frac{1}{2}\int \frac{2}{1+(2x)^2}\,(\operatorname{arctg}(2x))^{1/2}\,dx$$

$$= \operatorname{arctg}(2x) + \frac{1}{2}\,\frac{(\operatorname{arctg}(2x))^{3/2}}{\frac{3}{2}} + C$$

$$= \operatorname{arctg}(2x) + \frac{1}{3}(\operatorname{arctg}(2x))^{3/2} + C,\quad C \in \mathbb{R}$$

Observação.

- $$\int \frac{2}{1+4x^2}\,dx = \int \frac{2}{1+(2x)^2}\,dx \underset{R_{15}}{=} \operatorname{arctg}(2x) + C,\quad C\in\mathbb{R}$$

$f(x) = 2x$
$f'(x) = 2$

- $$\int \frac{1}{1+4x^2}\,(\operatorname{arctg}(2x))^{1/2}\,dx = \int \frac{1}{1+(2x)^2}\,(\operatorname{arctg}(2x))^{1/2}\,dx$$

$f(x) = \operatorname{arctg}(2x)$
$\alpha = 1/2$
$f'(x) = \frac{2}{1+(2x)^2}$

$$= \frac{1}{2}\int \underbrace{\frac{2}{1+(2x)^2}}_{f'}\,\underbrace{(\operatorname{arctg}(2x))^{1/2}}_{f^{1/2}}\,dx \underset{R_2}{=} \frac{1}{2}\,\frac{(\operatorname{arctg}(2x))^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{1}{3}(\operatorname{arctg}(2x))^{3/2} + C,\quad C\in\mathbb{R}$$

⑨ $$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx = \int \frac{1}{2}\,dx + \int \frac{1}{2}\cos(2x)\,dx$$

$$= \int \frac{1}{2}\,dx + \frac{1}{2}\cdot\frac{1}{2}\int \underbrace{2}_{f'}\,\underbrace{\cos(2x)}_{\cos f}\,dx \rightarrow R5$$

$$= \frac{1}{2}x + \frac{1}{4}\operatorname{sen}(2x) + C,\quad C \in \mathbb{R}$$

⑩ $$\int x \operatorname{arctg}(x^2)\,dx = \frac{x^2}{2}\operatorname{arctg}(x^2) - \int \frac{x^2}{2}\,\frac{2x}{1+x^4}\,dx$$

$f'(x) = x$ $\qquad f(x) = \frac{x^2}{2}$

$g(x) = \operatorname{arctg}(x^2)$ $\qquad g'(x) = \frac{2x}{1+(x^2)^2} = \frac{2x}{1+x^4}$

$$= \frac{x^2}{2}\operatorname{arctg}(x^2) - \int \frac{x^3}{1+x^4}\,dx$$

$f(x) = 1+x^4$
$f'(x) = 4x^3$

R3

$$= \frac{x^2}{2}\operatorname{arctg}(x^2) - \frac{1}{4}\int \frac{4x^3}{1+x^4}\,dx$$

$$= \frac{x^2}{2}\operatorname{arctg}(x^2) - \frac{1}{4}\ln(1+x^4) + C,\quad C \in \mathbb{R}$$

(11) $$\int \frac{1}{x^2}\cos\left(\frac{2}{x}\right)dx = -\frac{1}{2}\int \underbrace{\frac{-2}{x^2}}_{f'}\underbrace{\cos\left(\frac{2}{x}\right)}_{\cos f}dx$$

$f(x) = \frac{2}{x}$

$f'(x) = -\frac{2}{x^2}$

$$\underset{R5}{=} -\frac{1}{2}\,\text{sen}\left(\frac{2}{x}\right) + C, \quad C \in \mathbb{R}$$

(12) $$\int \frac{1}{x^3} e^{1/x^2} dx = -\frac{1}{2}\int \underbrace{\frac{-2}{x^3}}_{f'}\underbrace{e^{1/x^2}}_{e^f}dx$$

$f(x) = \frac{1}{x^2}$

$$\underset{R4}{=} -\frac{1}{2}e^{1/x^2} + C, \quad C \in \mathbb{R}$$

$f'(x) = -\frac{2}{x^3}$

(13) $\int e^{\operatorname{sen}x} \operatorname{sen}x \cos x \, dx = e^{\operatorname{sen}x} \operatorname{sen}x - \int e^{\operatorname{sen}x} \cdot \cos x \, dx$

$f'(x) = e^{\operatorname{sen}x} \cos x \qquad f(x) = e^{\operatorname{sen}x}$

$g(x) = \operatorname{sen}x \qquad g'(x) = \cos x$

$= e^{\operatorname{sen}x} \operatorname{sen}x - e^{\operatorname{sen}x} + C, \quad C \in \mathbb{R}$

<u>Observação</u>.

$$\int \underbrace{e^{\operatorname{sen}x}}_{e^{f}} \cdot \underbrace{\cos x}_{f'} \, dx \underset{R_4}{=} e^{\operatorname{sen}x} + C, \quad C \in \mathbb{R}$$

$f(x) = \operatorname{sen}x$
$f'(x) = \cos x$