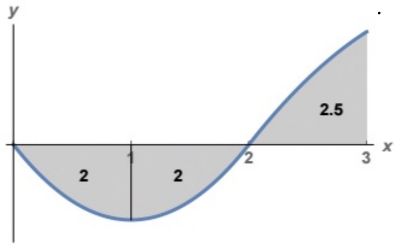
Aula 20

17 Dezembro

Exercício 1.

Na figura estão assinaladas três regiões limitadas entre o gráfico de uma função $f : [0,3] \longrightarrow \mathbb{R}$, derivável, e o eixo das abcissas, que correspondem às abcissas dos intervalos $[0,1]$, $[1,2]$ e $[2,3]$, respetivamente. A área de cada uma destas regiões vem inscrita no seu interior.



Nestas condições, considere a função $F : [-3,6] \longrightarrow \mathbb{R}$ definida por $F(x) = \int_1^{\frac{3+x}{3}} f(t)\,dt$.

(a) Determine os valores de $F(-3)$, $F(0)$, $F(3)$ e $F(6)$.

(b) Determine expressões para $F'(x)$ e $F''(x)$.

(c) Represente F graficamente.

a) $F(-3) = \int_1^0 f(t)\,dt = -\int_0^1 f(t)\,dt = -(-2) = 2$

$F(0) = \int_1^1 f(t)\,dt = 0$

$F(3) = \int_1^2 f(t)\,dt = -2$

$F(6) = \int_1^3 f(t)\,dt = 0{,}5 \quad (= -2{,}0 + 2{,}5)$

b) Como f é contínua, aplicando o Primeiro Teorema Fundamental do Cálculo, a função F é derivável e

$$F'(x) = f\left(\frac{3+x}{3}\right)\cdot \underbrace{\left(\frac{3+x}{3}\right)'}_{=\frac{1}{3}} = \frac{1}{3}\, f\left(\frac{3+x}{3}\right)$$

Então,

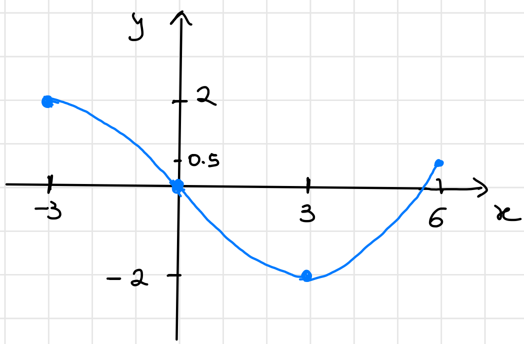
$$F''(x) = \left(\frac{1}{3}\, f\left(\frac{3+x}{3}\right)\right)' = \frac{1}{3}\left(f\left(\frac{3+x}{3}\right)\right)' = \frac{1}{3}\, f'\left(\frac{3+x}{3}\right)\underbrace{\left(\frac{3+x}{3}\right)'}_{=\frac{1}{3}}$$

$$= \frac{1}{9}\, f'\left(\frac{3+x}{3}\right)$$

c) Observemos que:

- $0 < \frac{3+x}{3} < 1 \iff 0 < 3+x < 3 \iff -3 < x < 0 \rightsquigarrow F'(x) < 0,\ F''(x) < 0$
- $1 < \frac{3+x}{3} < 2 \iff 3 < 3+x < 6 \iff 0 < x < 3 \rightsquigarrow F'(x) < 0,\ F''(x) > 0$
- $2 < \frac{3+x}{3} < 3 \iff 6 < 3+x < 9 \iff 3 < x < 6 \rightsquigarrow F'(x) > 0,\ F''(x) > 0$

	-3 … 0	0 … 3	3 … 6
$F'(x)$	− ↘	− ↘	+ ↗
$F''(x)$	− ∩	+ ∪	+ ∪



Exercício 2. Calcule $\displaystyle\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}$, efetuando a substituição $x = \operatorname{sen} t$

(i) Substituição

$x = \varphi(t)$

Fazendo $x = \operatorname{sen} t$, tem-se

$$\varphi(t) = \operatorname{sen} t, \quad \varphi'(t) = \cos t, \quad \varphi(\pi/6) = \frac{1}{2}, \quad \varphi(\pi/2) = 1$$

(ii) Cálculo do novo integral

$$\int_{1/2}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx = \int_{\pi/6}^{\pi/2} \frac{\sqrt{1-\operatorname{sen}^2 t}}{\operatorname{sen}^2 t} \cdot \cos t\,dt =$$

$$= \int_{\pi/6}^{\pi/2} \frac{\sqrt{\cos^2 t}}{\operatorname{sen}^2 t} \cdot \cos t\,dt = \int_{\pi/6}^{\pi/2} \frac{|\cos t|}{\operatorname{sen}^2 t} \cos t\,dt =$$

$t \in [\pi/6, \pi/2]$

$$= \int_{\pi/6}^{\pi/2} \frac{\cos t}{\operatorname{sen}^2 t} \cos t\,dt = \int_{\pi/6}^{\pi/2} \frac{\cos^2 t}{\operatorname{sen}^2 t}\,dt =$$

$$= \int_{\pi/6}^{\pi/2} \frac{1-\operatorname{sen}^2 t}{\operatorname{sen}^2 t}\,dt = \int_{\pi/6}^{\pi/2} \left(\frac{1}{\operatorname{sen}^2 t} - 1\right) dt$$

$$= \int_{\pi/6}^{\pi/2} \frac{1}{\operatorname{sen}^2 t}\,dt - \int_{\pi/6}^{\pi/2} 1\,dt = \Big[-\operatorname{cotg} t\Big]_{\pi/6}^{\pi/2} - \Big[t\Big]_{\pi/6}^{\pi/2}$$

$$= \left(-\operatorname{cotg}\frac{\pi}{2} - \left(-\operatorname{cotg}\frac{\pi}{6}\right)\right) - \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$$

Observação: $\cot g\, t = \dfrac{\cos t}{\operatorname{sen} t}$

- $\cot g \dfrac{\pi}{2} = \dfrac{\cos \pi/2}{\operatorname{sen} \pi/2} = \dfrac{0}{1} = 0$

- $\cot g \dfrac{\pi}{6} = \dfrac{\cos \pi/6}{\operatorname{sen} \pi/6} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$

Exercício 3. Calcule $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+2\,\text{sen}\,x}}\,dx$

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+2\,\text{sen}\,x}}\,dx = \int_0^{\pi/2} \cos x\,(1+2\,\text{sen}\,x)^{-1/2}\,dx$$

$f(x) = 1+2\,\text{sen}\,x$

$\alpha = -1/2$

$f'(x) = 2\cos x$

$$= \frac{1}{2}\int_0^{\pi/2} \underbrace{2\cos x}_{f'}\,\underbrace{(1+2\,\text{sen}\,x)^{-1/2}}_{f^{-1/2}}\,dx =$$

$$= \frac{1}{\cancel{2}}\left[\frac{(1+2\,\text{sen}\,x)^{1/2}}{\cancel{1/2}}\right]_0^{\pi/2} = \left[\sqrt{1+2\,\text{sen}\,x}\right]_0^{\pi/2}$$

$$= \sqrt{1+2\,\text{sen}\,\pi/2} - \sqrt{1+2\,\text{sen}\,0} = \sqrt{3}-1$$

Um Feliz Natal

e um

Excelente Ano Novo!