Análise

folha de exercícios 5 — 2017/2018

Soluções

- **Funções vetoriais**

1. (a) $\mathbf{r}(t) = (t-2, 2t+3) = (-2,3) + t(1,2),\ t \in \mathbb{R}$.
Reta que passa no ponto $(-2,3)$ e que tem a direção do vetor $(1,2)$.

(b) Elipse de equação $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$, percorrida uma única vez no sentido direto a partir do ponto $(2,0)$.

(c) Parte superior da elipse de equação $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$, percorrida uma única vez no sentido direto a partir do ponto $(2,0)$.

(d) Parábola de equação $y = x^2$, percorrida no sentido de x crescente.

(e) Curva de equação $y = |x|$, percorrida no sentido de x crescente.

(f) Circunferência de equação $x^2 + y^2 = 4$, percorrida um número infinito de vezes, o sentido direto.

(g) Curva no espaço definida pelas equações $x^2 + y^2 = 9$ e $z = 2$, ou seja, uma circunferência de centro em $(0,0,0)$ e raio 3 contida no plano horizontal $z = 2$, percorrida uma única vez no sentido direto a partir do ponto $(3,0,2)$.

(h) Hélice contida na superfície cilíndrica de equação $x^2 + y^2 = 1$, do ponto $(1,0,0)$ ao ponto $(-3,0,5\pi)$ (duas voltas e meia).

2. (a) $\mathbf{r}(t) = (t, 2t+1),\ t \in \mathbb{R}$;
(b) $\mathbf{r}(t) = (t, t^3+1),\ t \in \mathbb{R}$;
(c) $\mathbf{r}(t) = (t^2, t),\ t \in \mathbb{R}$;
(d) $\mathbf{r}(t) = (2\cos t, 3\,\mathrm{sen}\, t),\ t \in [0, 2\pi]$;
(e) $\mathbf{r}(t) = (t, \mathrm{e}^t),\ t \in \mathbb{R}$;
(f) $\mathbf{r}(t) = (5\cos t, 5\,\mathrm{sen}\, t, 2),\ t \in [0, 2\pi]$.

3. $\mathbf{r}(t) = (\cos t, \mathrm{sen}\, t, 2 - \mathrm{sen}\, t),\ t \in [0, 2\pi]$.

4. (a) $(-2,3)$; (b) $(2,0)$; (c) $(2,0)$; (d) $(0,0)$; (e) $(0,0)$; (f) $(2,0)$; (g) $(3,0,2)$; (h) $(1,0,0)$.

5. (a) Vetor velocidade: $\mathbf{v}'(t) = (1, 2t)$; versor tangente: $T(t) = \left(\dfrac{1}{\sqrt{1+4t^2}}, \dfrac{2t}{\sqrt{1+4t^2}}\right)$.
Por exemplo, em $t = 1$: $\mathbf{v}'(1) = (1,2)$; $T(1) = \left(\dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}}\right)$.

(b) Vetor velocidade: $\mathbf{v}'(t) = (\mathrm{e}^t, -\mathrm{sen}\, t)$; versor tangente: $T(t) = \left(\dfrac{\mathrm{e}^t}{\sqrt{\mathrm{e}^{2t} + \mathrm{sen}^2 t}}, -\dfrac{\mathrm{sen}\, t}{\sqrt{\mathrm{e}^{2t} + \mathrm{sen}^2 t}}\right)$.
Por exemplo, em $t = 0$: $\mathbf{v}'(0) = (1,0)$; $T(0) = (1,0)$.

(c) Vetor velocidade: $\mathbf{v}'(t) = \left(3\cos t, -3\,\mathrm{sen}\, t, \frac{3}{2}t^{1/2}\right)$;
versor tangente: $T(t) = \dfrac{1}{3\sqrt{1 + \frac{1}{4}t}}\left(3\cos t, -3\,\mathrm{sen}\, t, \frac{3}{2}t^{1/2}\right)$.
Por exemplo, em $t = 0$, $\mathbf{v}'(0) = (3,0,0)$; $T(0) = (1,0,0)$.

6. (a) $\|\gamma(t)\| = \sqrt{(a\cos t)^2 + (a\,\mathrm{sen}\, t)^2} = \sqrt{a^2(\cos^2 t + \mathrm{sen}^2 t)} = \sqrt{a^2} = |a| = a$.
Temos que $\gamma'(t)$ é ortogonal a $\gamma(t)$ uma vez que

$$\gamma(t) \cdot \gamma'(t) = (a\cos t, a\,\mathrm{sen}\, t) \cdot (-a\,\mathrm{sen}\, t, a\cos t) = -a^2\cos t\,\mathrm{sen}\, t + a^2\,\mathrm{sen}\, t\cos t = 0.$$

(b) Seja $\|\gamma(t)\| = c$, com $c > 0$ constante. Temos, então,

$$\begin{aligned}\sqrt{\gamma(t)\cdot\gamma(t)} = c &\Longrightarrow \gamma(t)\cdot\gamma(t) = c^2\\ &\Longrightarrow \big(\gamma(t)\cdot\gamma(t)\big)' = (c^2)'\\ &\Longrightarrow \gamma(t)'\cdot\gamma(t)+\gamma(t)\cdot\gamma'(t) = 0\\ &\Longrightarrow 2\gamma(t)'\cdot\gamma(t) = 0\\ &\Longrightarrow \gamma(t)'\cdot\gamma(t) = 0,\end{aligned}$$

ou seja, $\gamma'(t)$ e $\gamma(t)$ são ortogonais.

7.
$$\begin{cases} x = t\\ y = 1\\ z = 0\end{cases},\quad t\in\mathbb{R}$$

8.
$$\begin{cases} x = -2t\\ y = 1\\ z = \frac{\pi}{2}+t\end{cases},\quad t\in\mathbb{R}$$

9. $\gamma(t) = \left(\frac{t^2}{2}, \mathrm{e}^t-6, \frac{t^3}{3}+1\right)$, $t\in\mathbb{R}$.

10. $\mathbf{r}(t) = \left(\frac{2}{3}t^3+t+1, t^3-t, \frac{t^2}{2}+t\right)$, $t\geq 0$, e $\mathbf{v}(t) = \left(2t^2+1, 3t^2-1, t+1\right)$, $t\geq 0$.

11. Vetor posição: $\mathbf{s}(t) = \left(\cos t, -\mathrm{e}^{2t}, \left(\frac{t}{5}-1\right)^5\right)$;

Vetor velocidade: $\mathbf{v}(t) = \mathbf{s}'(t) = \left(-\operatorname{sen} t, -2\,\mathrm{e}^{2t}, \left(\frac{t}{5}-1\right)^4\right)$;

Vetor aceleração: $\mathbf{a}(t) = \mathbf{s}''(t) = \left(-\cos t, -4\,\mathrm{e}^{2t}, \frac{4}{5}\left(\frac{t}{5}-1\right)^3\right)$;

$\mathbf{s}(0) = (1,-1,-1)$; $\mathbf{v}(0) = (0,-2,1)$; $\mathbf{a}(0) = \left(-1,-4,-\frac{4}{5}\right)$.

12. Vetor velocidade: $\mathbf{r}'(t) = (3t^2, 2t)$;

Velocidade escalar: $\|\mathbf{r}'(t)\| = \sqrt{9t^4+4t^2}$;

Vetor aceleração: $\mathbf{r}''(t) = (6t, 2)$;

$\mathbf{r}'(1) = (3,2)$; $\|\mathbf{r}'(1)\| = \sqrt{13}$; $\mathbf{r}''(1) = (6,2)$.

13. Vetor velocidade: $\mathbf{r}'(t) = (2t, \mathrm{e}^t, \mathrm{e}^t + t\,\mathrm{e}^t)$;

Velocidade escalar: $\|\mathbf{r}'(t)\| = \sqrt{4t^2+\mathrm{e}^{2t}(2+2t+t^2)}$;

Vetor aceleração: $\mathbf{r}''(t) = (2, \mathrm{e}^t, 2\,\mathrm{e}^t + t\,\mathrm{e}^t)$.

- **Comprimento de arco e curvatura**

14. (a) $\displaystyle\int_0^{\pi}\|\mathbf{r}'(t)\|\,dt = \int_0^{\pi} 6\,dt = 6\pi$;

(b) $\displaystyle\int_0^1 \sqrt{2}\,\mathrm{e}^t\;dt = \sqrt{2}(\mathrm{e}-1)$;

(c) $\displaystyle\int_a^b \sqrt{5}\,dt = \sqrt{5}(b-a)$;

(d) $\displaystyle\int_0^{2\pi} \sqrt{3}\,\mathrm{e}^t\;dt = \sqrt{3}(\mathrm{e}^{2\pi}-1)$;

(e) $\displaystyle\int_0^1 6\left(1+t^2\right)\,dt = \frac{24}{3}$;

(f) $\displaystyle\int_1^{\mathrm{e}}\left(\frac{1}{t}+2t\right)\,dt = \mathrm{e}^2$.

15. (a) Comprimento de arco: $s = \displaystyle\int_0^t \|\mathbf{r}'(u)\|\,du = \int_0^t \sqrt{2}\,\mathrm{e}^u\;du = \sqrt{2}(\mathrm{e}^t-1)$.

Assim, $t = \ln\left(s/\sqrt{2}+1\right)$ e a reparametrização por comprimento de arco é dada por:

$\mathbf{r}\left(\ln\left(s/\sqrt{2}+1\right)\right) = \left(\mathrm{e}^{\ln\left(s/\sqrt{2}+1\right)}\operatorname{sen}\left(\ln\left(s/\sqrt{2}+1\right)\right), \mathrm{e}^{\left(\ln\left(s/\sqrt{2}+1\right)\right)}\cos\left(\ln\left(s/\sqrt{2}+1\right)\right)\right)$

(b) Comprimento de arco: $s = \displaystyle\int_0^t \sqrt{30}\,du = \sqrt{30}t$.

Reparametrização por comprimento de arco: $\mathbf{r}\left(s/\sqrt{30}\right) = \left(1+2s/\sqrt{30}, 3+s/\sqrt{30}, -5s/\sqrt{30}\right)$.

(c) Comprimento de arco: $s = 5t$.

Reparametrização por comprimento de arco: $\mathbf{r}(s/5) = \left(3\operatorname{sen}(s/5), 4s/5, 3\cos(s/5)\right)$.

16. (a) $\mathbf{T}(t) = \left(\frac{4}{5}\cos 4t, \frac{3}{5}, -\frac{4}{5}\operatorname{sen} 4t\right)$; $\kappa(t) = \dfrac{16}{25}$.

(b) $\mathbf{T}(t) = \left(\dfrac{1}{1+t^2}, \dfrac{\sqrt{2}t}{1+t^2}, \dfrac{t^2}{1+t^2}\right)$; $\kappa(t) = \dfrac{\sqrt{2}}{6\left(1+t^2\right)^2}$.

(c) $\mathbf{T}(t) = \left(-\operatorname{sen} t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\cos t\right)$; $\kappa(t) = \dfrac{\sqrt{2}}{2}$.

(d) $\mathbf{T}(t) = \left(\dfrac{2t^2}{1+2t^2}, \dfrac{2t}{1+2t^2}, \dfrac{1}{1+2t^2}\right)$, $\kappa(t) = \dfrac{2t}{\left(1+2t^2\right)^2}$.

17. Determine a curvatura das curvas parametrizadas por

(a) $\kappa(t) = \dfrac{1}{a}$;

(b) $\kappa(t) = \dfrac{2}{\left(1+4t^2\right)^{3/2}}$;

(c) $\kappa(t) = \dfrac{\sqrt{6}}{2\left(5-4t+2t^2\right)^{3/2}}$;

(d) $\kappa(t) = \dfrac{3}{\left(1+18t^2\right)^{3/2}}$;

(e) $\kappa(t) = \dfrac{\sqrt{2}}{\left(1+(\cos t)^2\right)^{3/2}}$;

(f) $\kappa(t) = \dfrac{6|t|}{\left(1+9t^4\right)^{3/2}}$.

18. Escolhendo x como parâmetro, temos a parametrização $\mathbf{r}(x) = (x, f(x), 0)$, considerando a mesma curva mas no espaço, no plano $z = 0$.

Assim, $\mathbf{r}'(x) = \big(1, f'(x), 0\big)$, $\mathbf{r}''(x) = \big(0, f''(x), 0\big)$ e $\mathbf{r}'(x) \times \mathbf{r}''(x) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = \big(0, 0, f''(x)\big)$.

Temos $\|\mathbf{r}'(x) \times \mathbf{r}''(x)\| = |f''(x)|$ e $\|\mathbf{r}'(x)\| = \Big[1 + [f'(x)]^2\Big]^{1/2}$. Logo,

$$\kappa(x) = \frac{\|\mathbf{r}'(x) \times \mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} = \frac{|f''(x)|}{\big[1 + [f'(x)]^2\big]^{3/2}}.$$

19. De acordo com a alínea anterior, $\kappa(x) = \dfrac{2}{\left(1+4x^2\right)^{3/2}}$.

Assim, nos pontos $(-1,1)$, $(0,0)$, $(1,1)$ e $(2,4)$ as curvaturas são, respetivamente, $\kappa(-1) = \dfrac{2}{5^{3/2}}$, $\kappa(0) = 2$, $\kappa(1) = \dfrac{2}{5^{3/2}}$ e $\kappa(2) = \dfrac{2}{17^{3/2}}$. A curvatura é máxima em $(0,0)$.

20. Considere-se uma parábola de equação $y = ax^2$, com $a \neq 0$. Temos $\kappa(x) = \dfrac{|2a|}{\big[1 + (4a^2x^2)^2\big]^{3/2}}$. Assim, $\kappa(0) = 4$ quando $|2a| = 4$, ou seja, quando $a = 2$ ou $a = -2$.

21. Determine a curvatura das curvas em $\mathbb{R}^2$ de equações:

(a) $\kappa(x) = \dfrac{|\operatorname{sen}(x)|}{\left(1+\cos^2 x\right)^{3/2}}$;

(b) $\kappa(x) = \dfrac{12x^2}{\left(1+16x^9\right)^{3/2}}$;

(c) $\kappa(x) = \dfrac{\mathrm{e}^x}{\left(1+\mathrm{e}^{2x}\right)^{3/2}}$.

22. Considere-se a parametrização $\mathbf{r}(t) = (f(t), g(t), 0)$, da mesma curva mas no plano $z = 0$ em $\mathbb{R}^3$.
Assim, $\mathbf{r}'(t) = \big(f'(t), g'(t), 0\big)$, $\mathbf{r}''(t) = \big(f''(t), g''(t), 0\big)$ e

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f'(t) & g'(t) & 0 \\ f''(t) & g''(t) & 0 \end{vmatrix} = \big(0, 0, f'(t)g''(t) - g'(t)f''(t)\big).$$

Temos $\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \big|f'(t)g''(t) - g'(t)f''(t)\big|$ e $\|\mathbf{r}'(t)\| = \Big[[f'(t)]^2 + [g'(t)]^2\Big]^{1/2}$. Logo,

$$\kappa(t) = \frac{\big|f'(t)g''(t) - g'(t)f''(t)\big|}{\big[[f'(t)]^2 + [g'(t)]^2\big]^{3/2}} = \frac{|x'y'' - y'x''|}{\big[(x')^2 + (y')^2\big]^{3/2}}.$$

23. (a) $\kappa(t) = \dfrac{6}{|t|\left(9t^2+4\right)^{3/2}}$; (b) $\kappa(t) = \dfrac{2+t^2}{\left(1+t^2\right)^{3/2}}$.

- **Vetores tangente, normal e binormal**

24. Para as funções vetoriais seguintes determine os vetores unitários tangente, normal e binormal, $\mathbf{T}$, $\mathbf{N}$ e $\mathbf{B}$.

(a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left(\frac{4}{5}\cos 4t, \frac{3}{5}, -\frac{4}{5}\sin 4t\right)$; $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = (-\operatorname{sen} 4t, 0, -\cos 4t)$

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left(-\frac{3}{5}\cos 4t, \frac{4}{5}, \frac{3}{5}\sin 4t\right)$

(b) $\mathbf{T}(t) = \left(\dfrac{1}{1+t^2}, \dfrac{\sqrt{2}t}{1+t^2}, \dfrac{t^2}{1+t^2}\right)$; $\mathbf{N}(t) = \left(\dfrac{-\sqrt{2}t}{1+t^2}, \dfrac{1-t^2}{1+t^2}, \dfrac{\sqrt{2}t}{1+t^2}\right)$;

$\mathbf{B}(t) = \left(\dfrac{t^2+t^4}{(1+t^2)^2}, \dfrac{-\sqrt{2}(t+t^3)}{(1+t^2)^2}, \dfrac{1+t^2}{(1+t^2)^2}\right)$

(c) $\mathbf{T}(t) = (-\operatorname{sen} t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\cos t)$; $\mathbf{N}(t) = (-\cos t, -\frac{1}{\sqrt{2}}\operatorname{sen} t, -\frac{1}{\sqrt{2}}\operatorname{sen} t)$;

$\mathbf{B}(t) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(d) $\mathbf{T}(t) = \left(\dfrac{2t^2}{1+2t^2}, \dfrac{2t}{1+2t^2}, \dfrac{1}{1+2t^2}\right)$; $\mathbf{N}(t) = \left(\dfrac{2t}{1+2t^2}, \dfrac{-2t^2+1}{1+2t^2}, \dfrac{-2t}{1+2t^2}\right)$

$\mathbf{B}(t) = \left(\dfrac{-2t^2-1}{(1+2t^2)^2}, \dfrac{4t^3+2t}{(1+2t^2)^2}, \dfrac{-4t^4-2t^2}{(1+2t^2)^2}\right)$

25. (a) $\mathbf{T}(1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$; $\mathbf{N}(1) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$; $\mathbf{B}(1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

(b) $\mathbf{T}(0) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; $\mathbf{N}(0) = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; $\mathbf{B}(0) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

(c) $\mathbf{T}(0) = (0,1,0)$; $\mathbf{N}(0) = (0,0,1)$; $\mathbf{B}(0) = (1,0,0)$.

26. (a) Plano osculador: $\mathbf{B}(1) \cdot (x-1, y-\frac{2}{3}, z-1) = 0 \Longleftrightarrow -2x+y+2z = \frac{2}{3}$;

Plano normal: $\mathbf{T}(1) \cdot (x-1, y-\frac{2}{3}, z-1) = 0 \Longleftrightarrow 2x+2y+z = \frac{13}{3}$.

(b) Plano osculador: $-2x+y+z=-2$;

Plano normal: $x+y+z=2$.

(c) Plano osculador: $x=1$;

Plano normal: $y=0$.

27. $(-1,-3,1)$

28. (a) Plano osculador: $x+6y=6\pi$;

Plano normal: $-6x+y=\pi$.

(b) Plano osculador: $-3x+3y-z=-1$;

Plano normal: $x+2y+3z=6$.

29. Considere-se uma curva no espaço parametrizada por $\mathbf{r} = \mathbf{r}(t)$. Recorde-se que $s = s(t)$ e $\dfrac{ds}{dt} = \|\mathbf{r}'(t)\|$.

Para $\mathbf{T} = \mathbf{T}(s)$, usando a regra de derivação da cadeia, vem $\dfrac{d\mathbf{T}}{dt} = \dfrac{d\mathbf{T}}{ds} \cdot \dfrac{ds}{dt}$. Logo,

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \cdot \frac{1}{\frac{ds}{dt}} = \frac{d\mathbf{T}}{dt} \cdot \frac{1}{\|\mathbf{r}'(t)\|} = \mathbf{T}'(t) \cdot \frac{1}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \cdot \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \mathbf{N}(t)\kappa(t).$$

Ou seja, podemos escrever $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.

30. $\tau(t) = \dfrac{12}{9t^4+36t^2+4}$.

31. $\kappa(t) = \dfrac{a}{a^2+b^2}$; $\tau(t) = \dfrac{a^2b}{a^2b^2+a^4}$.