$c_1n^2 + c_2n \leq c_3n$ for $n \leq 98$ and $c_1n^2 + c_2n > c_3n$ for $n > 98$. If $c_1 = 1$, $c_2 = 2$, and $c_3 = 1000$, then $c_1n^2 + c_2n \leq c_3n$ for $n \leq 998$.

No matter what the values of c_1, c_2, and c_3, there will be an n beyond which the algorithm with complexity c_3n will be faster than the one with complexity $c_1n^2 + c_2n$. This value of n will be called the *break-even point*. If the break-even point is zero, then the algorithm with complexity c_3n is always faster (or at least as fast). The exact break-even point cannot be determined analytically. The algorithms have to be run on a computer in order to determine the break-even point. To know that there is a break-even point, it is sufficient to know that one algorithm has complexity $c_1n^2 + c_2n$ and the other c_3n for some constants c_1, c_2, and c_3. There is little advantage in determining the exact values of c_1, c_2, and c_3.

1.3.3 Asymptotic Notation (O, Ω, Θ)

With the previous discussion as motivation, we introduce some terminology that enables us to make meaningful (but inexact) statements about the time and space complexities of an algorithm. In the remainder of this chapter, the functions f and g are nonnegative functions.

Definition 1.4 [Big "oh"] The function $f(n) = O(g(n))$ (read as "f of n is big oh of g of n") iff (if and only if) there exist positive constants c and n_0 such that $f(n) \leq c * g(n)$ for all n, $n \geq n_0$. □

Example 1.11 The function $3n + 2 = O(n)$ as $3n + 2 \leq 4n$ for all $n \geq 2$. $3n + 3 = O(n)$ as $3n + 3 \leq 4n$ for all $n \geq 3$. $100n + 6 = O(n)$ as $100n + 6 \leq 101n$ for all $n \geq 6$. $10n^2 + 4n + 2 = O(n^2)$ as $10n^2 + 4n + 2 \leq 11n^2$ for all $n \geq 5$. $1000n^2 + 100n - 6 = O(n^2)$ as $1000n^2 + 100n - 6 \leq 1001n^2$ for $n \geq 100$. $6 * 2^n + n^2 = O(2^n)$ as $6 * 2^n + n^2 \leq 7 * 2^n$ for $n \geq 4$. $3n + 3 = O(n^2)$ as $3n + 3 \leq 3n^2$ for $n \geq 2$. $10n^2 + 4n + 2 = O(n^4)$ as $10n^2 + 4n + 2 \leq 10n^4$ for $n \geq 2$. $3n + 2 \neq O(1)$ as $3n + 2$ is not less than or equal to c for any constant c and all $n \geq n_0$. $10n^2 + 4n + 2 \neq O(n)$. □

We write $O(1)$ to mean a computing time that is a constant. $O(n)$ is called *linear*, $O(n^2)$ is called *quadratic*, $O(n^3)$ is called *cubic*, and $O(2^n)$ is called *exponential*. If an algorithm takes time $O(\log n)$, it is faster, for sufficiently large n, than if it had taken $O(n)$. Similarly, $O(n \log n)$ is better than $O(n^2)$ but not as good as $O(n)$. These seven computing times–$O(1)$, $O(\log n)$, $O(n)$, $O(n \log n)$, $O(n^2)$, $O(n^3)$, and $O(2^n)$–are the ones we see most often in this book.

As illustrated by the previous example, the statement $f(n) = O(g(n))$ states only that $g(n)$ is an upper bound on the value of $f(n)$ for all n, $n \geq n_0$. It does not say anything about how good this bound is. Notice that $n = O(2^n)$, $n = O(n^{2.5})$, $n = O(n^3)$, $n = O(2^n)$, and so on. For the statement $f(n) = O(g(n))$ to be informative, $g(n)$ should be as small a function of n as one can come up with for which $f(n) = O(g(n))$. So, while we often say that $3n + 3 = O(n)$, we almost never say that $3n + 3 = O(n^2)$, even though this latter statement is correct.

From the definition of O, it should be clear that $f(n) = O(g(n))$ is not the same as $O(g(n)) = f(n)$. In fact, it is meaningless to say that $O(g(n)) = f(n)$. The use of the symbol $=$ is unfortunate because this symbol commonly denotes the equals relation. Some of the confusion that results from the use of this symbol (which is standard terminology) can be avoided by reading the symbol $=$ as "is" and not as "equals."

Theorem 1.2 obtains a very useful result concerning the order of $f(n)$ (that is, the $g(n)$ in $f(n) = O(g(n))$) when $f(n)$ is a polynomial in n.

Theorem 1.2 If $f(n) = a_m n^m + \cdots + a_1 n + a_0$, then $f(n) = O(n^m)$.

Proof:

$$\begin{aligned} f(n) &\leq \textstyle\sum_{i=0}^{m} |a_i| n^i \\ &\leq n^m \textstyle\sum_{i=0}^{m} |a_i| n^{i-m} \\ &\leq n^m \textstyle\sum_{i=0}^{m} |a_i| \qquad \text{for } n \geq 1 \end{aligned}$$

So, $f(n) = O(n^m)$ (assuming that m is fixed). □

Definition 1.5 [Omega] The function $f(n) = \Omega(g(n))$ (read as "f of n is omega of g of n") iff there exist positive constants c and n_0 such that $f(n) \geq c * g(n)$ for all n, $n \geq n_0$. □

Example 1.12 The function $3n + 2 = \Omega(n)$ as $3n + 2 \geq 3n$ for $n \geq 1$ (the inequality holds for $n \geq 0$, but the definition of Ω requires an $n_0 > 0$). $3n+3 = \Omega(n)$ as $3n+3 \geq 3n$ for $n \geq 1$. $100n+6 = \Omega(n)$ as $100n+6 \geq 100n$ for $n \geq 1$. $10n^2 + 4n + 2 = \Omega(n^2)$ as $10n^2 + 4n + 2 \geq n^2$ for $n \geq 1$. $6 * 2^n + n^2 = \Omega(2^n)$ as $6 * 2^n + n^2 \geq 2^n$ for $n \geq 1$. Observe also that $3n + 3 = \Omega(1)$, $10n^2 + 4n + 2 = \Omega(n)$, $10n^2 + 4n + 2 = \Omega(1)$, $6 * 2^n + n^2 = \Omega(n^{100})$, $6 * 2^n + n^2 = \Omega(n^{50.2})$, $6 * 2^n + n^2 = \Omega(n^2)$, $6 * 2^n + n^2 = \Omega(n)$, and $6 * 2^n + n^2 = \Omega(1)$. □

As in the case of the big oh notation, there are several functions $g(n)$ for which $f(n) = \Omega(g(n))$. The function $g(n)$ is only a lower bound on $f(n)$. For the statement $f(n) = \Omega(g(n))$ to be informative, $g(n)$ should be as large a function of n as possible for which the statement $f(n) = \Omega(g(n))$ is true. So, while we say that $3n + 3 = \Omega(n)$ and $6 * 2^n + n^2 = \Omega(2^n)$, we almost never say that $3n + 3 = \Omega(1)$ or $6 * 2^n + n^2 = \Omega(1)$, even though both of these statements are correct.

Theorem 1.3 is the analogue of Theorem 1.2 for the omega notation.

Theorem 1.3 If $f(n) = a_m n^m + \cdots + a_1 n + a_0$ and $a_m > 0$, then $f(n) = \Omega(n^m)$.

Proof: Left as an exercise. □

Definition 1.6 [Theta] The function $f(n) = \Theta(g(n))$ (read as "f of n is theta of g of n") iff there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all n, $n \geq n_0$. □

Example 1.13 The function $3n + 2 = \Theta(n)$ as $3n + 2 \geq 3n$ for all $n \geq 2$ and $3n + 2 \leq 4n$ for all $n \geq 2$, so $c_1 = 3$, $c_2 = 4$, and $n_0 = 2$. $3n + 3 = \Theta(n)$, $10n^2 + 4n + 2 = \Theta(n^2)$, $6 * 2^n + n^2 = \Theta(2^n)$, and $10 * \log n + 4 = \Theta(\log n)$. $3n + 2 \neq \Theta(1)$, $3n + 3 \neq \Theta(n^2)$, $10n^2 + 4n + 2 \neq \Theta(n)$, $10n^2 + 4n + 2 \neq \Theta(1)$, $6 * 2^n + n^2 \neq \Theta(n^2)$, $6 * 2^n + n^2 \neq \Theta(n^{100})$, and $6 * 2^n + n^2 \neq \Theta(1)$. □

The theta notation is more precise than both the the big oh and omega notations. The function $f(n) = \Theta(g(n))$ iff $g(n)$ is both an upper and lower bound on $f(n)$.

Notice that the coefficients in all of the $g(n)$'s used in the preceding three examples have been 1. This is in accordance with practice. We almost never find ourselves saying that $3n + 3 = O(3n)$, that $10 = O(100)$, that $10n^2 + 4n + 2 = \Omega(4n^2)$, that $6 * 2^n + n^2 = O(6 * 2^n)$, or that $6 * 2^n + n^2 = \Theta(4 * 2^n)$, even though each of these statements is true.

Theorem 1.4 If $f(n) = a_m n^m + \cdots + a_1 n + a_0$ and $a_m > 0$, then $f(n) = \Theta(n^m)$.

Proof: Left as an exercise. □

Definition 1.7 [Little "oh"] The function $f(n) = o(g(n))$ (read as "f of n is little oh of g of n") iff

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

□

Example 1.14 The function $3n + 2 = o(n^2)$ since $\lim_{n \to \infty} \frac{3n+2}{n^2} = 0$. $3n + 2 = o(n \log n)$. $3n + 2 = o(n \log \log n)$. $6 * 2^n + n^2 = o(3^n)$. $6 * 2^n + n^2 = o(2^n \log n)$. $3n + 2 \neq o(n)$. $6 * 2^n + n^2 \neq o(2^n)$. □

Analogous to o is the notation ω defined as follows.