Impossibility result for Markov Chain Monte Carlo sampling from micro-canonical bipartite graph ensembles

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Markov Chain Monte Carlo (MCMC) algorithms are commonly used to sample from graph ensembles. Two graphs are neighbors in the state space if one can be obtained from the other with only a few modifications, e.g., edge rewirings. For many common ensembles, e.g., those preserving the degree sequences of bipartite graphs, rewiring operations involving two edges are sufficient to create a fully-connected state space, and they can be performed efficiently. We show that, for ensembles of bipartite graphs with fixed degree sequences and number of *butterflies* ($k_{2,2}$ bi-cliques), there is no universal constant c such that a rewiring of at most c edges at every step is sufficient for *any* such ensemble to be fully connected. Our proof relies on an explicit construction of a family of pairs of graphs with the same degree sequences and number of butterflies, with each pair indexed by a natural c, and such that any sequence of rewiring operations transforming one graph into the other *must* include at least one rewiring operation involving at least c edges. Whether rewiring these many edges is *sufficient* to guarantee the full connectivity of the state space of any such ensemble remains an open question. Our result implies the impossibility of developing efficient, graph-agnostic, MCMC algorithms for these ensembles, as the necessity to rewire an impractically large number of edges may hinder taking a step on the state space.

I. INTRODUCTION

Testing the statistical significance of properties of an observed network is a fundamental problem in network science [1]. The significance of the observed value is tested against a null model, an ensemble $\mathcal{H} = (\mathcal{G}, \pi)$ composed of the set $\mathcal{G}$ of possible graphs that can be realized under the null hypothesis and a probability distribution π over $\mathcal{G}$. One typically selects some descriptive characteristics of the observed network, and either defines $\mathcal{G}$ and π in such a way that the *expectations* w.r.t. π of these characteristics over $\mathcal{G}$ are the same as the observed ones (a.k.a. the *canonical* model), or defines $\mathcal{G}$ as the set of all and only the graphs with *exactly the same* values for the characteristics as the observed network, and π can be any distribution, often the uniform. Once the null model is defined, one proceeds by sampling several graphs from this ensemble. These graphs are used to approximate the distribution of the test statistic of interest under the null hypothesis. By comparing the observed statistic to this distribution one can compute an empirical p-value.

For example, the widely used "configuration model" [2] considers the set of graphs with the same degree sequence as the observed network and the uniform distribution. This model has been instrumental in determining that clustering, assortativity, and community structure in real networks are not solely dependent on node degrees, hence highlighting their significance [3]. However, the configuration model fails to generate graphs with a local structure similar to the observed graph [4]. Researchers have thus explored alternative null models that sample from

graph families defined by more complex characteristics of the observed graph, such as joint degree distribution [4–7], core-value sequence [8], and local triangle-count sequence [9].

In this work, we focus on bipartite graphs, i.e., networks whose nodes can be partitioned into two classes such that all edges go from one class to the other. Formally, a bipartite graph is a tuple $G \doteq (L, R, E)$, where L and R are disjoint sets of nodes called *left* and *right* nodes, respectively, and $E \subseteq L \times R$ is a set of edges connecting nodes in L to nodes in R. We consider undirected bipartite graphs, but for ease of presentation, we denote any edge (u, a) so that $u \in L$ and $a \in R$. For any vertex $v \in L \cup R$ we denote with $\Gamma_G(v)$ the set of *neighbors* of v, i.e., the vertices to which v is connected by an edge in G, and we define the *degree* $\mathsf{d}_G(v)$ of v in G as $\mathsf{d}_G(v) \doteq |\Gamma_G(v)|$. Assuming an arbitrary but fixed labeling $u_1, \ldots, u_{|L|}$ (resp. $a_1, \ldots, a_{|R|}$) of the nodes in L (resp. R), the vector $\langle \mathsf{d}_G(u_1), \ldots, \mathsf{d}_G(u_{|L|}) \rangle$ (resp. $\langle \mathsf{d}_G(a_1), \ldots, \mathsf{d}_G(a_{|R|}) \rangle$) is known as the *left (resp. right) degree sequence* of G.

Bipartite networks occur naturally in many applications: when representing words and documents [10], items and itemsets [11], higher-order networks such as hypergraphs and simplicial complexes [12], and many more. Null models and graph ensembles can also be defined on bipartite graphs [11, 13, 14]. For example, Preti *et al.* [11] introduce a null model that preserves the bipartite joint adjacency matrix (i.e., the matrix whose (i, j)-th entry is the number of edges connecting nodes from L with degree i to nodes in R with degree j), of an observed network (thus the degree sequences and the number of *caterpillars*, i.e., paths of length 3), and give Markov Chain Monte Carlo (MCMC) algorithms to sample from this null model. Null models for bipartite graphs are also of particular interest because they align with null models for 0–1 binary matrices [15, Ch. 6]. For example, preserv-

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ing the degree sequences in a bipartite graph corresponds to preserving the row and column marginals of the corresponding bi-adjacency matrix, and several MCMC algorithms have been developed to sample from this null model [16–20].

We consider graph ensembles for which $\mathcal{G}$ is the set of all bipartite graphs $G \doteq (L, R, E)$ that share the same degree sequences and the same number of *butterflies*, i.e., $k_{2,2}$ bi-cliques, defined as follows.

Definition 1 (Butterfly). *Let $G \doteq (L, R, E)$ be a bipartite graph. Two distinct nodes $u, v \in L$ and two distinct nodes $a, b \in R$ belong to the* butterfly $A = \{u, v, a, b\}$ *in G if and only if $\{(u, a), (u, b), (v, a), (v, b)\} \subseteq E$.*

The following result, whose proof is immediate, gives an expression for the number of butterflies to which two nodes both belong.

Fact 1. *Let $G \doteq (L, R, E)$ be a bipartite graph, and let u and v be distinct nodes in L. The number $\mathsf{b}_G(u, v)$ of butterflies in G to which both u and v belong is*

$$\mathsf{b}_G(u, v) = \binom{|\Gamma_G(u) \cap \Gamma_G(v)|}{2},$$

where we assume $\binom{0}{2} = \binom{1}{2} = 0$. A similar result holds for any two distinct nodes in R.

For $u \in L$, we denote with $\mathsf{b}_G(u)$ the number of butterflies in G to which u belongs. It holds

$$\mathsf{b}_G(u) = \sum_{\substack{v \in L \\ v \neq u}} \mathsf{b}_G(u, v). \tag{1}$$

The total number $\mathsf{b}(G)$ of butterflies in G is then

$$\mathsf{b}(G) \doteq \frac{1}{2} \sum_{u \in L} \mathsf{b}_G(u). \tag{2}$$

The butterfly, being the smallest complete subgraph in a bipartite graph, is the most basic building block for composing more complex structures, analogous to the triangle in unipartite graphs. Consequently, preserving the number of butterflies emerges as a natural choice when defining null models that retain more graph properties beyond just the degree sequences. This concept finds applications in studying, e.g., clustering patterns [21].

MCMC methods are a popular approach to sample from an ensemble $\mathcal{H} = (\mathcal{G}, \pi)$. They define a suitable Markov chain on the space $\mathcal{G}$ of all possible graphs, such that, after a sufficient burn-in period, the state of the Markov chain is approximately distributed according to π. The correctness of this process requires the Markov chain to be finite, irreducible, and aperiodic [2]. Efficient sampling requires not only that the Markov chain has a fast mixing time, but also that the space can be explored quickly, i.e., that obtaining a neighbor from the current state is efficient. The double edge swap technique, also known as degree-preserving rewiring [22], checkerboard

swap [23], or tetrad [17], is a simple yet fundamental randomization technique used to generate a new graph with the same degree sequence as a given graph. Its efficiency stems from the fact that it involves the rewiring of a *small* number of edges. In bipartite graphs, the most basic rewiring technique is known as the *bipartite swap operation* (BSO).

Definition 2 (BSO). *Let $G \doteq (L, R, E)$ be a bipartite graph and $u \neq v \in L$, $a \neq b \in R$ such that $(u, a), (v, b) \in E$ and $(u, b), (v, a) \notin E$. The BSO involving (u, a) and (v, b) removes (u, a) and (v, b) from E, and adds (u, b) and (v, a) to E. The resulting bipartite graph $G' = (L, R, (E \smallsetminus \{(u, a), (v, b)\}) \cup \{(u, b), (v, a)\})$ has the same left and right degree sequence of G.*

A more sophisticated operation is the *q-edge bipartite swap operation* (q-BSO), which may involve the simultaneous swapping of multiple edges, potentially between a large set of nodes, similar to the *q*-switch operation defined by Tabourier *et al.* [24].

Definition 3 (q-BSO). *Let $G \doteq (L, R, E)$ be a bipartite graph and $q \in \mathbb{N}^+$. A q-BSO is a pair $\mathsf{sw}^q \doteq (S, \sigma)$ with $S = \langle e_1, \dots, e_q \rangle$ being a vector of q distinct edges $e_i \doteq (u_i, a_i) \in E$, and σ being a derangement of $[q]$, i.e., a permutation of $[q]$ with no element in its original position, s.t. $(u_j, a_{\sigma(j)}) \notin E$ for each $j \in [1, q]$. Replacing each e_j with $(u_j, a_{\sigma(j)})$ generates a bipartite graph $G' = \left(L, R, (E \smallsetminus S) \cup \left\{(u_j, a_{\sigma(j)}) \text{ for } j \in [1, q]\right\}\right)$ with the same left and right degree sequence as G.*

According to this definition, a BSO involving (u, a) and (v, b) can be seen as the 2-BSO $(\langle (u, a), (v, b) \rangle, (2\ 1))$. Algorithms such as Verhelst's [17] and Curveball [20] aim to speed up the sampling from the ensemble of bipartite graphs with fixed degree sequences. They execute multiple BSO operations at each step by selecting nodes u and v from L (or R) and exchanging multiple edges originating from u with edges originating from v. Conversely, a q-BSO may involve the simultaneous swapping of multiple edges originating from *multiple* source nodes. Thus, the moves considered by Curveball and Verhelst's can be expressed as q-BSOs, but q-BSOs are more expressive, in the sense that there are q-BSOs that do not correspond to possible moves for these algorithms.

II. CONNECTIVITY OF THE STATE SPACE

A key requirement to use an MCMC method for sampling from a graph ensemble is that the state space, where each state corresponds to a graph in the ensemble, is strongly connected, i.e., for any two states G' and G'' there is a sequence $\langle \rho_1, \rho_2, \dots, \rho_\ell \rangle$ of graph-transforming operations for some ℓ (which may depend on the chosen G' and G''), such that ρ_1 transforms G' into some G_1 that belongs to the state space, ρ_i for $1 < i < \ell$ transforms G_i into G_{i+1} that also belongs to the state space,

and ρ_ℓ transforms $G_{\ell-1}$ into G''. In other words, a class $\mathcal{C}$ of graph-transforming operations defines a neighborhood structure of the state space as follows: given any G in the state space, a neighbor of G is any state that can be obtained by applying a single operation from $\mathcal{C}$, provided that the operation is applicable to G. With this neighborhood structure, the state space is strongly connected if there is a path from any state to any other state.

We can immediately see that the state space $\mathcal{G}$ we consider is strongly connected by sequences of q-BSOs when q is large enough, for any left and right degree sequences, and any number of butterflies (see also [24, Sect. 3.2.2]). In fact, there is always a $|E' \smallsetminus E''|$-BSO that transforms any bipartite graph $G' \doteq (L, R, E')$ into another bipartite graph $G'' \doteq (L, R, E'')$ with the same left and right degree sequences, and number of butterflies (see Supplementary Material [25] for details). While this fact ensures the strong connectivity of the state space via the union of all q-BSOs for $q = 2, \ldots, |E'|$, it has little practical relevance, as we now explain. If we use all these q-BSOs to define the neighborhood structure of the state space, the resulting space would be a *complete* graph, i.e., a clique. Consequently, drawing, according to any distribution, a neighbor of a given state would require a procedure to build an entirely new bipartite graph with the same degree sequences and the same number of butterflies *from scratch*. Developing such a procedure seems even harder than the problem we are attempting to solve, in the same way as devising algorithms for building a bipartite graph with prescribed degree sequences from scratch [26–30] is much harder than devising algorithms for sampling such a graph using MCMC approaches starting from an existing one [16–20].

The correct question to ask is therefore the following: is there a *fixed, universal, constant* q^* such that, for any left and right degree sequences, and any number of butterflies, any two bipartite graphs with those left and right degree sequences, and that number of butterflies, are connected by sequences $\langle \mathsf{sw}_1^{p_1} = (S_1, \sigma_1), \ldots, \mathsf{sw}_z^{p_z} = (S_z, \sigma_z) \rangle$ of p_i-BSOs, with $p_i \leq q^*$, $i = 1, \ldots, z$, where z may depend on the two graphs? By "universal", we mean a quantity that in no way depends on properties of the ensemble $(\mathcal{G}, \pi)$, including properties of the observed network.

Asking this question is reasonable: it is known that $q^* = 2$ when one is only interested in preserving the degree sequences [15, Ch. 6], and it is also known that $q^* = 2$ for the case of preserving the degree sequences and the number of paths of length 3 (a.k.a. caterpillars), in which case the rewiring operations slightly differ from the traditional BSOs [11, 31].

Additionally, we would like q^* to be small, because sampling a q-BSO is not necessarily efficient: the naïve approach of independently sampling q edges and then verifying whether they form a valid q-BSO has an increasing probability of failure as q increases [24]. As a result, the Markov chain would exhibit a high probability of staying in the same state for many consecutive steps, greatly increasing the mixing time.

For unipartite graphs, it has been proved that $q^* = 2$ is not always sufficient to ensure strong connectivity of spaces of graphs that share more complex properties [24, 31]. In this work, we demonstrate the nonexistence of a *fixed, universal, constant* q^* for the ensemble of bipartite graphs with the same left and right degree sequences and the same number of butterflies.

Let us give some intuition with an example, which shows that it cannot be $q^* < 4$. Figure 1 shows two bipartite graphs G_1 (upper) and G_2 (lower) with the same left and right degree sequences, and the same number of butterflies $\mathsf{b}(G_1) = \mathsf{b}(G_2) = 10$. There is no sequence of q-BSOs for $q < 4$ that, when applied to G_1, generates a graph isomorphic to G_2: any q-BSO for $q < 4$ applied to G_1 either generates a graph with a different number of butterflies, or generates a graph isomorphic to G_1. On the other hand, the 4-BSO $([(x_1, y_5), (x_5, y_1), (x_6, y_{10}), (x_{10}, y_6)], \sigma)$ with $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$, and $\sigma(4) = 2$ ensures that the two butterflies $\{x_1, x_5, y_1, y_5\}$ and $\{x_6, x_{10}, y_6, y_{10}\}$ disappear, while the two new butterflies $\{x_1, x_{10}, y_1, y_{10}\}$ and $\{x_5, x_6, y_5, y_6\}$ appear, hence preserving the total count $\mathsf{b}(G_1)$.

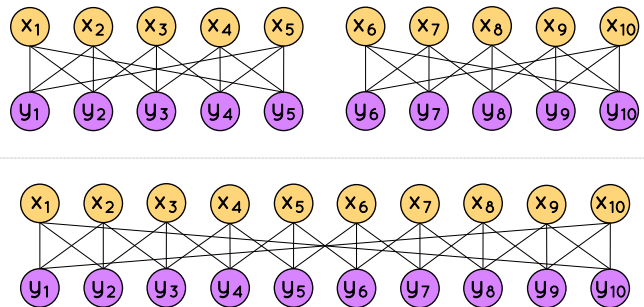

FIG. 1. Graphs that are not connected by q-BSOs for $q < 4$.

Our main result is the following theorem (proof in Supplementary Material [25]).

Theorem 1. *For any $\bar{q} \in \mathbb{N}$ with $\bar{q} > 1$, there exist two non-isomorphic bipartite graphs G_b and G_e with the same left and right degree sequences, and $\mathsf{b}(G_\mathrm{b}) = \mathsf{b}(G_\mathrm{e})$, such that for any sequence $\langle \mathsf{sw}_1^{p_1} = (S_1, \sigma_1), \ldots, \mathsf{sw}_z^{p_z} = (S_z, \sigma_z) \rangle$ of p_i-BSOs with $p_i \in \mathbb{N}^+$, $i = 1, \ldots, z$, that transforms G_b into G_e, there exists $\ell \in \{1, \ldots, z\}$ with $p_\ell \geq \bar{q}$.*

Our proof consists of two parts. First, we construct two bipartite graphs G_b and G_e with the same left and right degree sequences (which will depend on $\bar{q}$ as the second largest left degree will be greater than $\bar{q}$), and the same number of butterflies. Second, we demonstrate that any sequence of q-BSOs applied to G_b to obtain a graph isomorphic to G_e must involve at least one q-BSO for $q > \bar{q}$. Since $\bar{q}$ can be arbitrarily large, a *universal constant* q^* as above cannot exist.

This theorem proves that it is impossible to design efficient MCMC algorithms that sample from ensembles
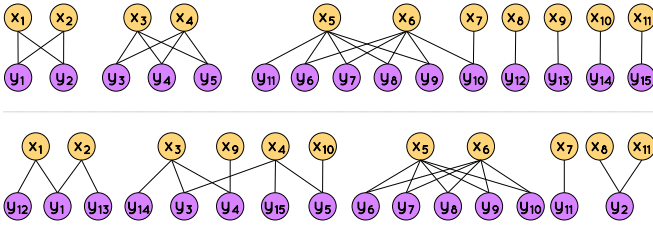
FIG. 2. Bipartite graphs generated by our algorithm (see Supplementary Material [25]) for $s = 2$ and $t = 3$.

$\mathcal{H} = (\mathcal{G}, \pi)$ of bipartite graphs with the same degree sequences and the same number of butterflies, because the state space is not strongly connected by edge swap operations that involve only up to a fixed, universal, number of edges, as is instead the case for simpler null models. Rather, the minimum number of edges that *must* be involved depends on properties of the state space $\mathcal{G}$, not just of the observed network. These may not be easily computable, as they may not depend *just* on the observed network, if any.

This result has profound implications for the design of network null models and for network science in general. If it is unfeasible to preserve the occurrences of the simplest building block of bipartite graphs (the butterfly), it becomes unfeasible to preserve larger structures. When dealing with bipartite graphs and complex observed characteristics, ensembles with "soft" constraints, where the constraints are retained on average over all the graphs sampled from the ensemble [32], might be the only viable option.

The algorithm to construct the graphs G_b and G_e is delineated in the Supplementary Material [25]. We now describe the main characteristics of such graphs.

Let s and t be two naturals with $s > t \geq 2$ and $2(s-1) > k$. We define $n \doteq \binom{s}{2} + \binom{t}{2}$ and $a \doteq ((s+1) \bmod 2) + ((t+1) \bmod 2)$. The graphs G_b and G_e output by the algorithm with inputs s and t have the following properties:

1. G_b and G_e have $7 + a$ left nodes (denoted with the letter x) and $s + t + n + 2 + a$ right nodes (denoted with the letter y);

2. G_b and G_e have the same left and right degree sequences. In particular, x_1 and x_2 have degree s, x_3 and x_4 have degree t, x_5 and x_6 have degree $n + 1$, y_i has degree 2 for $1 \leq i \leq s + t + n + 1$, and all the other left and right nodes have degree 1;

3. $b(G_b) = b(G_e) = n + \binom{n}{2}$;

4. $|\Gamma_{G_e}(x_5) \cap \Gamma_{G_e}(x_6)| = n + 1$, which, with the previous point, implies that x_5 and x_6 belong to *all* butterflies in G_e, and every other node $y \in R$ belongs to no butterfly;

5. $|\Gamma_{G_b}(x_5) \cap \Gamma_{G_b}(x_6)| = n$, $|\Gamma_{G_b}(x_1) \cap \Gamma_{G_b}(x_2)| = s$, $|\Gamma_{G_b}(x_3) \cap \Gamma_{G_b}(x_4)| = t$, which implies, with point

2, that x_5 and x_6 do *not* belong to *all* butterflies in G_b.

Figure 2 shows the bipartite graphs generated for $s = 2$ and $t = 3$.

III. DISCUSSION

MCMC approaches based on swaps of pairs of edges can efficiently sample graphs from simple ensembles, such as those including graphs with prescribed degree sequences, or fixed number of paths of length up to three [11]. The correctness of these approaches relies on the fact that pairwise edge swaps create a strongly connected state space. They are efficient because proposing a neighbor to move to is relatively easy, requiring only to be able to efficiently sample pairs of edges.

In the case of ensembles preserving more complex properties of the networks, the strong connectivity of the state space may require more than two edges to be swapped at every step, i.e., performing q-switches [24], or q-BSOs, for some value of q.

In this work, we consider the ensemble of bipartite graphs with fixed degree sequences and fixed number of butterflies ($k_{2,2}$ bi-cliques), for its important role in a variety of applications, e.g., investigating clustering patterns [21]. We show that the state space is not strongly connected by sequences of q-BSOs for any *fixed, universal,* constant q. In other words, the number of edges to be rewired at each step is upper bounded by a quantity that depends on properties of the graphs in the ensemble. This result is in strong contrast with the cases for the space of bipartite graphs with fixed degree sequences, and for that of bipartite graphs with fixed degree sequences and fixed number of paths of length three (a.k.a. caterpillars), where $q = 2$ is sufficient for all ensembles [11, 31].

This discovery has far-reaching implications for network science. First and foremost, we rule out the possibility of designing *efficient* MCMC algorithms for sampling from the space of bipartite graphs with fixed degree sequences and fixed number of butterflies, specifically, from the micro-canonical ensemble that maintains these properties exactly. In fact, we demonstrate the necessity of swaps with size dependent on the characteristics of the graph space $\mathcal{G}$, not necessarily just on the observed network. Finding what this size $q^*_{\mathcal{G}}$ is may not even be feasible. It may perhaps be possible to develop an efficient procedure to find this quantity, but then one also needs an efficient procedure to, at each step of the Markov chain, generate a q-BSO for $q \leq q^*_{\mathcal{G}}$, to propose a neighbor to move to. Solving both these algorithmic questiosn seem challenging. Moreover, the lower bound $\bar{q}$ to the size of the BSOs needed to connect the two graphs G_b and G_e from Thm. 1 gives only a *necessary* condition for the strong connectivity of the graph space, not a *sufficient* one: we only know that one of the BSOs to connect these two graphs must contain more than $\bar{q}$ edges, but not how exactly how many. Even if we knew exactly

this number $\hat{q}$, there may be other pairs of graphs in the same ensemble (i.e., with the same degree sequences and number of butterflies) such that any sequence of BSOs connecting these two graphs must have size even greater than $\hat{q}$. Therefore the situation might be even more dire than our findings suggest.

Given that sampling from a null model that preserves the number of butterflies is impractical, preserving larger structures seems an even more unattainable task. A butterfly is at the same time the smallest cycle and the smallest non-trivial bi-clique, hence it is a basic building block of bipartite graphs. Thus, our findings present a large obstacle to developing efficient algorithms to sample from more complex ensembles, and therefore to testing network properties under more descriptive null models.

What other options are then available, if any? If one wishes to maintain the number of butterflies as a hard constraint (i.e., to sample from the micro-canonical ensemble), one potential approach involves avoiding MCMC algorithms and opting for a direct-sampling algorithm like stub matching [33]. However, such algorithms are already limited to small graph instances, for the case of sampling from the space of graphs with the same degree sequence, due to their complexity scaling quadrati-cally or cubically with the number of nodes, depending on the graph density [34, 35]. The straightforward application of existing stub-matching techniques may also suffer from generating graphs with a different number of butter-flies, thus leading to a high rejection rate. Thus, we need to explore alternative methodologies or refine existing stub-matching algorithms to better accommodate these more complex constraints. Finally, implementations of canonical methods such as the Chung-Lu model [36] offer a more efficient alternative, albeit at the cost of imposing a *soft* constraint. Indeed, while the ensemble average aligns precisely with the desired value of each constraint, individual graph instances may lie far from the desired constraints. The canonical ensemble brings other challenges, including difficulties in generating graphs that closely match the desired expectations for certain degree distributions (*degeneracy* problem) [37–41].

Overall, our findings represent a strong negative result that the network science community needs to reckon with. By showing that this research avenue is not fruitful, we hope to spur alternative and innovative approaches to designing null models for graphs, and algorithms for sampling from them.

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Impossibility result for Markov Chain Monte Carlo sampling from micro-canonical bipartite graph ensembles: Supplementary Material

I. PROOFS

Our main result, Theorem 1, relies on the following lemma.

Lemma 1. *For any bipartite graph $G \doteq (L, R, E)$ with $\mathsf{d}_G(a) \le 2$, for each $a \in R$, and any $u \in L$, it holds*

$$\mathsf{b}_G(u) \le \binom{\mathsf{d}_G(u)}{2},$$

with equality if and only if there exists $v \in L \setminus \{u\}$, s.t. $\Gamma_G(u) \subseteq \Gamma_G(v)$.

The meaning of this Lemma is that, when *all* nodes in R have degree *at most 2* , each node $u \in L$ can be part of *at most one butterfly for each unordered pair (w, z) of u's neighbors.* Specifically, if either w or v has degree 1, then there can be no butterfly involving u, w, z and another node in L. If both w and z have degree 2, there may be at most one butterfly involving u, w, z and another node in L. If any node in R has degree $d > 2$, a pair of neighbors of $u \in L$ could be part of more than one butterfly together with u (at most $d - 1$).

We use the following technical lemma in the proof of Lemma 1.

Lemma 2. *Let $d \in \mathbb{N}$, $d \ge 2$. For any sequence $a_1, \ldots, a_z$ of $1 < z \le d$ strictly positive naturals s.t. $\sum_{i=1}^{z} a_i \le d$, it holds*

$$\binom{d}{2} > \sum_{i=1}^{z} \binom{a_i}{2} .$$

Proof. Assume by contradiction that there exists a sequence $a_1, \ldots, a_z$ of $1 < z \le d$ strictly positive naturals s.t. $\sum_{i=1}^{z} a_i \le d$ and for which

$$\binom{d}{2} \le \sum_{i=1}^{z} \binom{a_i}{2} = \frac{1}{2} \sum_{i=1}^{z} (a_i^2 - a_i) . \tag{3}$$

It holds

$$2\binom{d}{2} = d^2 - d \ge \left(\sum_{i=1}^{z} a_i \right)^2 - \sum_{i=1}^{z} a_i,$$

where the inequality comes from the fact that $q^2 - q \ge g^2 - g$ for any $q \ge 1$, $0 \le g \le q$. Expanding the r.h.s. of the last inequality, we obtain

$$2\binom{d}{2} \ge \sum_{i=1}^{z} a_i^2 + 2 \sum_{i=1}^{z} \sum_{h=i+1}^{z} a_i a_h - \sum_{i=1}^{z} a_i$$

$$= \sum_{i=1}^{z} (a_i^2 - a_i) + 2 \sum_{i=1}^{z} \sum_{h=i+1}^{z} a_i a_h . \tag{4}$$

We can combine (3) and (4) as

$$\sum_{i=1}^{z} (a_i^2 - a_i) + 2 \sum_{i=1}^{z} \sum_{h=i+1}^{z} a_i a_h \le 2\binom{d}{2} \le \sum_{i=1}^{z} (a_i^2 - a_i),$$

which is clearly impossible because the a_i's are strictly positive, thus the second term on the leftmost side is strictly positive. Thus we reached a contradiction, and the sequence $a_1, \ldots, a_z$ cannot exists. $\square$

Proof of Lemma 1. We start by showing that if v as in the thesis exists, then $\mathsf{b}_G(u) = \binom{\mathsf{d}_G(u)}{2}$. Such a v must be unique, due to the restrictions on the degree of the nodes in R. It must then hold that $\Gamma_G(u) \cap \Gamma_G(w) = \varnothing$ for any $w \in L \setminus \{v, u\}$, as all $a \in \Gamma_G(u)$ have degree exactly two and $\{(u, a), (v, a)\} \subseteq E$. From this fact and Fact 1, we get that $\mathsf{b}_G(u, w) = 0$ for any $w \in L \setminus \{v, u\}$, and that $\mathsf{b}_G(u, v) = \binom{\mathsf{d}_G(u)}{2}$, as $|\Gamma_G(u) \cap \Gamma_G(v)| = |\Gamma_G(u)| = \mathsf{d}_G(u)$ from the hypothesis. The desired equality follows from these facts and the definition of $\mathsf{b}_G(u)$ from Equation (1) in the main text.

We now show that if v as in the thesis does *not* exists, it must be $\mathsf{b}_G(u) < \binom{\mathsf{d}_G(u)}{2}$. Assume that there is exactly one $v' \in L \setminus \{u\}$ s.t. $\Gamma_G(u) \cap \Gamma_G(v') \ne \varnothing$. Then it must be $|\Gamma_G(u) \cap \Gamma_G(v')| < \mathsf{d}_G(u)$, otherwise v' would be as v in the thesis, which we just said can not happen. Then, from Fact 1, we get $\mathsf{b}_G(u, v') < \binom{\mathsf{d}_G(u)}{2}$. From this fact, the fact that $\mathsf{b}_G(u, w) = 0$ for any $w \in L \setminus \{v', u\}$, and Eq. (1) from the text, we obtain the desired result.

Assume instead that there are $1 < z \le d$ distinct nodes $v_1, \ldots, v_z \in L \setminus \{u\}$ s.t. $\Gamma_G(u) \cap \Gamma_G(v_i) \ne \varnothing$. The desired result $\mathsf{b}_G(u) < \binom{\mathsf{d}_G(u)}{2}$ follows from Lemma 2 using $a_i = |\Gamma_G(u) \cap \Gamma_G(v_i)|$, $1 \le i \le z$. $\square$

Proof of Theorem 1. Let G_{b} and G_{e} be the graphs output by Alg. 1 with inputs s and t. For completeness, let us recall the five properties satisfied by such graphs.

1. G_{b} and G_{e} have $7 + a$ left nodes (denoted with the letter x) and $s + t + n + 2 + a$ right nodes (denoted with the letter y);

2. G_{b} and G_{e} have the same left and right degree sequences. In particular, x_1 and x_2 have degree s, x_3 and x_4 have degree t, x_5 and x_6 have degree $n + 1$, y_i has degree 2 for $1 \le i \le s + t + n + 1$, and all the other left and right nodes have degree 1;

3. $\mathsf{b}(G_{\mathrm{b}}) = \mathsf{b}(G_{\mathrm{e}}) = n + \binom{n}{2}$;

4. $|\Gamma_{G_{\mathrm{e}}}(x_5) \cap \Gamma_{G_{\mathrm{e}}}(x_6)| = n + 1$, which, with the previous point, implies that x_5 and x_6 belong to *all* butterflies in G_{e}, and every other node $y \in R$ belongs to no butterfly;

5. $|\Gamma_{G_{\mathrm{b}}}(x_5) \cap \Gamma_{G_{\mathrm{b}}}(x_6)| = n$, $|\Gamma_{G_{\mathrm{b}}}(x_1) \cap \Gamma_{G_{\mathrm{b}}}(x_2)| = s$, $|\Gamma_{G_{\mathrm{b}}}(x_3) \cap \Gamma_{G_{\mathrm{b}}}(x_4)| = t$, which implies, with point 2, that x_5 and x_6 do *not* belong to *all* butterflies in G_{b}.

For each $i \in [1, z]$, we denote with G^i the graph obtained from the application of $\mathsf{sw}_1^{p_1}, \dots, \mathsf{sw}_i^{p_i}$ to G_{b}. By definition of q-BSO, each G^i has the same left and right degree sequences, and $\mathsf{b}(G^i) = \mathsf{b}(G_{\mathrm{b}}) = \mathsf{b}(G_{\mathrm{e}})$. We set $G^0 = G_{\mathrm{b}}$.

From properties 4 and 5 of G_{b} and G_{e} listed above, there must be an index $\ell \in [1, z]$ s.t. in the graph G^ℓ, x_5 and x_6 share $n + 1$ neighbors. We now show that x_5 and x_6 must share exactly n neighbors in $G^{\ell-1}$, i.e., $|\Gamma_{G^{\ell-1}}(x_5) \cap \Gamma_{G^{\ell-1}}(x_6)| = n$.

Assume by contradiction that $|\Gamma_{G^{\ell-1}}(x_5) \cap \Gamma_{G^{\ell-1}}(x_6)| = r < n$. Then, $\mathsf{b}_{G^{\ell-1}}(x_5, x_6) = \binom{r}{2}$. From Equation (2) in the main text, given the degree sequence of the nodes in L, we can write

$$\mathsf{b}(G^{\ell-1}) = \sum_{i=1}^{5} \sum_{j=i+1}^{6} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) = n + \binom{n}{2} . \qquad (5)$$

It then must be

$$\sum_{i=1}^{4} \sum_{j=i+1}^{6} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) = n + \binom{n}{2} - \binom{r}{2} . \qquad (6)$$

It holds

$$\sum_{i=1}^{4} \sum_{j=i+1}^{6} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) \le \sum_{i=1}^{4} \mathsf{b}_{G^{\ell-1}}(i),$$

as some butterflies may be counted twice in the sum on the r.h.s.. From Lemma 1 applied to each of x_1, x_2, x_3, and x_4, it holds that

$$\sum_{i=1}^{4} \mathsf{b}_{G^{\ell-1}}(i) \le 2\binom{s}{2} + 2\binom{t}{2} = 2n . \qquad (7)$$

Consider for now the case $r < n - 1$. If we can show that

$$2n < n + \binom{n}{2} - \binom{r}{2} \qquad (8)$$

then we would have reached a contradiction, because this inequality, together with Eq. (7), implies that Eq. (6) cannot be true. The r.h.s. of Eq. (8) decreases as r increases, so if we can show that Eq. (8) holds for the maximum value of $r = n - 2$, then it would hold for all $r \le n - 2$. For $r = n - 2$, we can rewrite Eq. (8) as

$$2n < n + \binom{n}{2} - \binom{n-2}{2}, \qquad (9)$$

which is true for any $n > 3$, i.e., for all possible values of s and t. Thus we reached a contradiction and it cannot be $r < n - 1$.

Consider now the case $r = n - 1$. In this case,

$$n + \binom{n}{2} - \binom{r}{2} = 2n - 1 .$$

Using this fact and Eq. (6), we can write

$$2n - 1 = \sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) + \\ \sum_{i=1}^{4} \left(\mathsf{b}_{G^{\ell-1}}(x_i, x_5) + \mathsf{b}_{G^{\ell-1}}(x_i, x_6) \right) . \qquad (10)$$

Now, since $r = n - 1$ and $\mathsf{d}_{G^{\ell-1}}(x_5) = \mathsf{d}_{G^{\ell-1}}(x_6) = n + 1$, it must hold

$$\sum_{i=1}^{4} \left(\mathsf{b}_{G^{\ell-1}}(x_i, x_5) + \mathsf{b}_{G^{\ell-1}}(x_i, x_6) \right) \le 2 \qquad (11)$$

because x_5 and x_6 can share at most two neighbors each in R with one of x_1, x_2, x_3, and x_4. Due to the limitations on the degree of the nodes in R, if any of x_i, $i = 5, 6$ shares a neighbor with any x_j, $j = 1, 2, 3, 4$, then x_i cannot share the same neighbor with any other of $\{x_1, x_2, x_3, x_4\} \smallsetminus \{x_j\}$. Thus, combining Eq. (10) and Eq. (11), we get that

$$2n - 1 \le \sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) + 2 . \qquad (12)$$

It holds

$$\sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) = \frac{1}{2} \sum_{i=1}^{4} \sum_{\substack{j=1 \\ j \ne i}}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j)$$

$$\le \frac{1}{2} \sum_{i=1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i) \le n,$$

where the last inequality comes from Eq. (7). Combining the above with Eq. (12) we obtain

$$2n - 1 \le n + 2$$

which is only true for $n \le 3$. But from our hypothesis on s and t, it must be $n > 3$, so we reached a contradiction, and it cannot be $r = n - 1$.

Thus, it must be $|\Gamma_{G^{\ell-1}}(x_5) \cap \Gamma_{G^{\ell-1}}(x_6)| = n$, i.e., $\mathsf{b}_{G^{\ell-1}}(x_5, x_6) = \binom{n}{2}$. We now show that the remaining $n = \binom{s}{2} + \binom{t}{2}$ butterflies in $G^{\ell-1}$ are s.t. x_1 and x_2 both belong to $\binom{s}{2}$ of them, and x_3 and x_4 both belong to $\binom{t}{2}$ of them. In other words $\mathsf{b}_{G^{\ell-1}}(x_1, x_2) = \binom{s}{2}$, $\mathsf{b}_{G^{\ell-1}}(x_3, x_4) = \binom{t}{2}$, and $\mathsf{b}_{G^{\ell-1}}(x_i, x_j) = 0$ for any other $(i, j) \in \{(i, j) : 1 \le i \le 4, i < j \le 6\} \smallsetminus \{(1, 2), (3, 4)\}$.

Clearly, it must hold $\mathsf{b}_{G^{\ell-1}}(x_i, x_5) = \mathsf{b}_{G^{\ell-1}}(x_i, x_6) = 0$ because x_5 and x_6 can each at most share one neighbor with any of x_i, $1 \le i \le 4$, which is not sufficient to obtain any butterfly to which both x_i and x_5 or both x_i and x_6 may belong. Thus, it must hold

$$\sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) = n .$$

We also have

$$\sum_{i=1}^{3} \sum_{j=i+1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i, x_j) = \frac{1}{2} \sum_{i=1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i) \le n,$$

where the last inequality comes from Eq. (7). To obtain equality, it must hold

$$\sum_{i=1}^{4} \mathsf{b}_{G^{\ell-1}}(x_i) = 2\binom{s}{2} + 2\binom{t}{2} \ .$$

From Lemma 1, we have that

$$\mathsf{b}_{G^{\ell-1}}(x_i) \leq \begin{cases} \binom{s}{2} & \text{if } i = 1, 2 \\ \\ \binom{t}{2} & \text{if } i = 3, 4 \end{cases},$$

with equality only if each of x_1, x_2, x_3, and x_4 shares *all* its neighbors with another one of them. Thus, it must be that x_1 shares all its neighbors with x_2, and that x_3 shares all its neighbors with x_4, leading to the desired results that $\mathsf{b}_{G^{\ell-1}}(x_1, x_2) = \binom{s}{2}$, $\mathsf{b}_{G^{\ell-1}}(x_3, x_4) = \binom{t}{2}$, $\mathsf{b}_{G^{\ell-1}}(x_5, x_6) = \binom{n}{2}$, and $\mathsf{b}_{G^{\ell-1}}(x_i, x_j) = 0$ for any other $(i, j) \in \{(i, j) : 1 \leq i \leq 5, i < j \leq 6\} \setminus \{(1, 2), (3, 4), (5, 6)\}$.

At step ℓ, the number of common neighbors between x_5 and x_6 increases to $n + 1$, meaning that the number of butterflies involving the pair (x_5, x_6) increases by $\binom{n+1}{2} - \binom{n}{2} = n$. Since $n = \binom{s}{2} + \binom{t}{2}$, the p_ℓ-BSO $\mathsf{sw}_\ell^{p_\ell} \doteq (S_\ell, \sigma)$ must transform all the butterflies in $G^{\ell-1}$ to which x_5 and x_6 do not already belong, into an equal number of butterflies in G^ℓ to which both x_5 and x_6 belong.

To this end, $\mathsf{sw}_\ell^{p_\ell}$ must swap at least

- 1 edge involving x_5 (or x_6) to connect x_5 (or x_6) to a node in $\Gamma_{G^{\ell-1}}(x_6) \setminus \Gamma_{G^{\ell-1}}(x_5)$ (or in $\Gamma_{G^{\ell-1}}(x_5) \setminus \Gamma_{G^{\ell-1}}(x_6)$): this way, $|\Gamma_{G^{\ell-1}}(x_5) \cap \Gamma_{G^{\ell-1}}(x_6)| = n + 1$, i.e., x_5 and x_6 will share $n + 1$ neighbors in G^ℓ;

- $s - 1$ edges involving either x_1 or x_2 to reduce the number of neighbors shared by these two nodes by at least $s - 1$: this way $|\Gamma_{G^\ell}(x_1) \cap \Gamma_{G^\ell}(x_2)| \leq 1$, implying $\mathsf{b}_{G^\ell}(x_1, x_2) = 0$; and

- $t - 1$ edges involving either x_3 or x_4 to reduce the number of neighbors shared by these two nodes by at least $t - 1$: this way $|\Gamma_{G^\ell}(x_3) \cap \Gamma_{G^\ell}(x_4)| \leq 1$ and thus $\mathsf{b}_{G^\ell}(x_3, x_4) = 0$.

Thus, $S_\ell \doteq \{(u_1, a_1), \ldots, (u_{p_\ell}, a_{p_\ell})\}$ must have the following properties:

- S_ℓ must contain at least $s - 1$ edges (u_i, a_i) with $u_i \in \{x_1, x_2\}$ and s.t. the edge $(u_{\sigma(i)}, a_{\sigma(i)}) \in S_\ell$ has $u_{\sigma(i)} \notin \{x_1, x_2\}$. Indeed if that was not the case, the edge $(u_i, a_{\sigma(i)})$ would already exist in $G^{\ell-1}$, because, as previously discussed, x_1 and x_2 share *all* their neighbors in $G^{\ell-1}$. Thus, S_ℓ must contain, in addition to the $s-1$ edges as above, *another* $s-1$ edges whose endpoint in L is neither x_1 nor x_2. Additionally, of these $s-1$ edges, only at most 4 may have either x_3 or x_4 as endpoint in L, as if there were more, then there would be $x_i \in \{x_1, x_2\}$ and $x_j \in \{x_3, x_4\}$ that would share at least two neighbors in G^ℓ, and therefore it would be $\mathsf{b}_{G^\ell}(x_i, x_j) > 0$, which cannot be. S_ℓ is only required to have $t - 1$ edges in the form (u_i, a_i) with

$u_i \in \{x_3, x_4\}$, thus really only at most $\min\{4, t - 1\}$ can be as above.

- Similarly, S_ℓ must contain at least $t - 1$ edges (u_i, a_i) with $u_i \in \{x_3, x_4\}$ and s.t. the edge $(u_{\sigma(i)}, a_{\sigma(i)}) \in S_\ell$ has $u_{\sigma(i)} \notin \{x_3, x_4\}$. Indeed if that was not the case, the edge $(u_i, a_{\sigma(i)})$ would already exist in $G^{\ell-1}$, because, as previously discussed, x_3 and x_4 share *all* their neighbors in $G^{\ell-1}$; Thus, S_ℓ must contain, in addition to the $t-1$ edges as above, *another* $t-1$ edges whose node in L is neither x_3 nor x_4. Additionally, of these $t-1$ edges, only at most $\min\{4, t-1\}$ may have either x_1 or x_2 as endpoint in L, as if there were more, then there would be $x_i \in \{x_1, x_2\}$ and $x_j \in \{x_3, x_4\}$ that would share at least two neighbors in G^ℓ, and therefore it would be $\mathsf{b}_{G^\ell}(x_i, x_j) > 0$, which cannot be.

Thus, S_ℓ must contain at least $(2(s-1) - \min(4, t-1)) + (2(t-1) - \min(4, t-1))$ edges. For any value of t, this quantity is at least $2(s-1)$. We then have $p_\ell \geq 2(s-1) > \bar{q}$, where the last inequality comes from the definition of s. This fact concludes the proof. $\square$

II. BIPARTITE GRAPH GENERATOR

We present our algorithm to generate the two bipartite graphs G_b and G_e used in the proof of our main result.

The algorithm (pseudocode in Alg. 1) receives in input two naturals $s \neq t \in \mathbb{N} \wedge s, t \geq 2$, and generates two bipartite $G_\mathrm{b} = (L, R, E_\mathrm{b})$ and $G_\mathrm{e} = (L, R, E_\mathrm{e})$ with the same left and right degree sequence and each with $\binom{n+1}{2}$ butterflies, where $n = \binom{s}{2} + \binom{t}{2}$.

The algorithm starts with the creation of the set of left nodes L (any node in this set will be denoted as x_w, for some w) and of right nodes R (any node in this set will be denoted as y_w, for some w), equal for both graphs. Then, it populates the edge set E_b of G_b and the edge set E_e of G_e. In G_b, the butterflies involve three pairs of left nodes: (x_1, x_2), (x_3, x_4), and (x_5, x_6). Nodes x_1 and x_2 have degree s and share s neighbors (line 5); nodes x_3 and x_4 have degree t and share t neighbors (line 6); and nodes x_5 and x_6 have degree $n + 1$ and share n neighbors (line 7). In G_e, the butterflies involve only the left nodes x_5 and x_6, which share $n + 1$ neighbors (line 10).

We will construct the two edge sets so that any other pair of left nodes share at most one neighbor. Thus, it hols that $\mathsf{b}(G_\mathrm{b}) = \binom{s}{2} + \binom{t}{2} + \binom{n}{2} = \binom{n+1}{2} = \mathsf{b}(G_\mathrm{e})$. Nodes x_5 and x_6 have degree $n + 1$ in both graphs, but, so far, in G_b we have added only n edges to them. Similarly, node $y_{s+t+n+1}$ have degree 2 in G_e, but, so far, in G_b it is connected to only one left node. Therefore, we insert in E_b one edge involving x_5, one edge involving x_6 (we connect them to different right nodes to avoid creating other butterflies between them), and one edge involving $y_{s+t+n+1}$ (line 8). The construction of E_b is completed the addition of up to $s + t$ isolated edges, i.e., edges not connected with any other edge (line 9). These edges do not

Algorithm 1: Bipartite Graph Constructor

Input: Naturals s and t with $s \neq t \geq 2$
Output: Two bipartite graphs with the same degree sequences and $\binom{n+1}{2}$ butterflies

1 $n \leftarrow \binom{s}{2} + \binom{t}{2}$; add $\leftarrow s + t - 2$
2 **if** s *is even* **then** add $\leftarrow$ add $+ 1$
3 **if** t *is even* **then** add $\leftarrow$ add $+ 1$
4 $L \leftarrow [x_1, \ldots, x_{7+\text{add}}]$; $R \leftarrow [y_1, \ldots, y_{s+t+n+2+\text{add}}]$
    /* butterflies btw x_1 and x_2 */
5 $E_b \leftarrow \{(x_1, y_l), (x_2, y_l)\}$ for $l \in [1, s]$
    /* butterflies btw x_3 and x_4 */
6 $E_b \leftarrow E_b \cup \{(x_3, y_{s+l}), (x_4, y_{s+l})\}$ for $l \in [1, t]$
    /* butterflies btw x_5 and x_6 */
7 $E_b \leftarrow E_b \cup \{(x_5, y_{s+t+l}), (x_6, y_{s+t+l})\}$ for $l \in [1, n]$
    /* x_5 and x_6 have deg $n+1$; $y_{s+t+n+1}$ has deg 2
    */
8 $E_b \leftarrow E_b \cup \{(x_5, y_{s+t+n+1}), (x_6, y_{s+t+n+2}), (x_7, y_{s+t+n+1})\}$
    /* auxiliary isolated edges   */
9 $E_b \leftarrow E_b \cup \{(x_{7+l}, y_{s+t+n+2+l})\}$ for $l \in [1, \text{add}]$
    /* butterflies btw x_5 and x_6 */
10 $E_e \leftarrow \{(x_5, y_{s+t+l}), (x_6, y_{s+t+l})\}$ for $l \in [1, n+1]$
    /* x_1, x_2, x_3 and x_4 share at most 1 neighbor */
11 $E_e \leftarrow E_e \cup \{(x_1, y_1), (x_2, y_1), (x_3, y_{s+1}), (x_4, y_{s+1})\}$
12 $h_1 \leftarrow \lfloor (s-1)/2 \rfloor$; $h_2 \leftarrow \lfloor (t-1)/2 \rfloor$
    /* we split $s-1$ right nodes btw x_1 and x_2 */
13 $E_e \leftarrow E_e \cup \{(x_1, y_{1+l}), (x_2, y_{1+h_1+l})\}$ for $l \in [1, h_1]$
    /* we split $t-1$ right nodes btw x_3 and x_4 */
14 $E_e \leftarrow E_e \cup \{(x_3, y_{s+1+l}), (x_4, y_{s+1+h_2+l})\}$ for $l \in [1, h_2]$
    /* x_1, x_2 have deg s and x_3, x_4 have deg t */
15 $a_1 \leftarrow s - (1 + h_1)$; $a_2 \leftarrow t - (1 + h_2)$
16 $E_e \leftarrow E_e \cup \{(x_1, y_{s+t+n+2+l}), (x_2, y_{s+t+n+2+a_1+l})\}$ for
    $l \in [1, a_1]$
17 $E_e \leftarrow E_e \cup \{(x_3, y_{s+t+n+2+2a_1+l}), (x_4, y_{s+t+n+2+2a_1+a_2+l})\}$
    for $l \in [1, a_2]$
    /* x_7 and $y_{s+t+n+2}$ have deg 1   */
18 $E_e \leftarrow E_e \cup \{(x_7, y_{s+t+n+2})\}$
    /* the first $(s+t)$ right nodes have deg 2   */
19 $E_e \leftarrow E_e \cup \{(x_{7+l}, y_{1+l})\}$ for $l \in [1, s-1]$
20 $E_e \leftarrow E_e \cup \{(x_{7+s-1+l}, y_{s+1+l})\}$ for $l \in [1, t-1]$
21 add $\leftarrow 0$
22 **if** s *is even* **then** $E_e \leftarrow E_e \cup \{(x_{7+s+t-1}, y_s)\}$;
    add $\leftarrow$ add $+ 1$
23 **if** t *is even* **then** $E_e \leftarrow E_e \cup \{(x_{7+s+t-1+\text{add}}, y_{s+t})\}$
24 $G_b \leftarrow (L, R, E_b)$; $G_e \leftarrow (L, R, E_e)$
25 **return** G_b, G_e

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participate in any butterfly, because they involve nodes with degree 1. They are needed to guarantee that G_b and G_e have the same degree sequences, as it will become evident as we outline the other edges included in E_e.

We add edges to E_e in such a way that the pairs of vertices (x_1, x_2) and (x_3, x_4) have no more than one neighbor in common, so they do not belong to any butterfly. The idea is to **(i)** connect both x_1 and x_2 to y_1 (line 11), **(ii)** divide the remaining nodes to which x_1 is connected in G_b into two groups $(y_2, \ldots, y_{h_1})$ and

$(y_{h_1+1}, \ldots, y_{2h_1+1})$ with $h_1 = \lfloor (s-1)/2 \rfloor$, and **(iii)** insert into E_e one edge between x_1 and each node in the first group, and one edge between x_2 and each node in the second group (line 13). Similarly, to ensure that x_3 and x_4 only have one neighbor in common, we **(i)** connect both x_3 and x_4 to y_{s+1} (line 11), **(ii)** divide the remaining nodes to which x_3 is connected in G_b into two groups $(y_{s+2}, \ldots, y_{s+2+h_2})$ and $(y_{s+3+h_2}, \ldots, y_{s+1+2h_2})$ with $h_2 = \lfloor (t-1)/2 \rfloor$, and **(iii)** insert into E_e one edge between x_3 and each node in the first group, and one edge between x_4 and each node in the second group (line 14). So far, in E_e we have added only $(1 + h_1)$ of the s neighbors that x_1 and x_2 must have, and only $(1 + h_2)$ of the t neighbors that x_3 and x_4 must have. Thus, we include $s - (1 + h_1)$ edges to different right nodes (to avoid creating butterflies between them) for x_1 and x_2 (line 16), and $t - (1 + h_2)$ edges to different right nodes for x_3 and x_4 (line 17). Similarly, so far we have added only one edge to the right nodes $y_2, \ldots, y_{s-1}, y_{s+2}, \ldots, y_{s+t-1}$, and up to one edge to the right nodes y_s and y_{s+t}. In fact, depending on the value of h_1 and h_2, such nodes are included/excluded from the group of nodes connected to x_2 and x_4, respectively. Since all these nodes have degree 2 in G_b, we add the missing edges to E_e (lines 19–23). Lastly, nodes x_7 and $y_{s+t+n+2}$ have degree 1 in G_b, and thus they must be connected to one neighbor also in G_e (line 18). Finally, the two graphs $G_b \doteq (L, R, E_b)$ and $G_e \doteq (L, R, E_e)$ are returned.

III. CONNECTING ARBITRARY BIPARTITE GRAPHS IN THE STATE SPACE VIA q-BSO

This section shows how to build a q-BSO that transforms an arbitrary bipartite graph $G' \doteq (L, R, E')$ into another arbitrary graph $G'' \doteq (L, R, E'')$ with the same left and right degree sequences and number of butterflies. The q-BSO needs to swap all links except those that are in common between the two graphs, i.e., $q = |E' \smallsetminus E''|$. Let $E' \cap E'' = \langle e'_1, \ldots, e'_c \rangle = \langle e''_1, \ldots, e''_c \rangle$ be the list of c edges in common to the two graphs, $E' = \langle e'_1, \ldots, e'_{|E'|} \rangle$ the list of edges of G', $E'' = \langle e''_1, \ldots, e''_{|E'|} \rangle$ the list of edges of G'', and $q = |E'| - c$ the number of edges unique to G'. We now show a q-BSO operation $\mathsf{sw} \doteq (\langle e'_{c+1}, \ldots, e'_{|E'|} \rangle, \sigma)$ that transforms G' into G''. We build the derangement σ incrementally, denoting with $\hat{\mathrm{Im}}(\sigma)$ the *current* (i.e., as σ is being built) image of σ. At the beginning of the construction process, $\hat{\mathrm{Im}}(\sigma) = \varnothing$, and at the end, $\hat{\mathrm{Im}}(\sigma) = \mathrm{Im}(\sigma) = \{1, \ldots, q\}$. For each $i \in \{1, \ldots, q\}$, let $e'_{c+i} \doteq (u, v)$ and $k \in \{1, \ldots, q\}$ be any index such that $k \notin \hat{\mathrm{Im}}(\sigma)$ and $e''_{c+k} \doteq (u, w)$. Then, we set $\sigma(i) \doteq k$. The index k must always exist because G' and G'' have the same degree sequences.