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Nonmath analogies in teaching mathematics

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Abstract

Way too often, students find some concepts too abstract to comprehend. One of the strategies used to assist students with building conceptual knowledge is to use analogies. We investigate the place of nonmath analogies in teaching school mathematics. First, we demonstrate the widespread use of analogies by drawing examples through context analysis of tutoring websites, textbooks, and teaching experiences. Second, we argue that analogies reflect the grounded nature of mathematical concepts in common life experiences and, thus, have an essential place in instruction. To support our argument we offer a theoretical rationale based on research literature and historical sources.

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1. Introductions Examples from Practice

"Life is good for only two things, discovering mathematics and teaching mathematics.

Siméon-Denis Poisson.

NCTM Standards and Principals strongly emphasize the importance of learning with understanding. "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge."(NCTM 2000, p.19). But way too often, students find mathematical concepts too abstract to comprehend. One of the strategies that teachers use to assist students with building conceptual knowledge is to use analogies. Richland, Holyoak, & Stigler (2004) point out that that math and distant (nonmath) analogies drawn from non mathematics contexts are routinely generated by teachers. When we use nonmath analogies in our mathematics classes, we observe in students a lowered level of math anxiety, deeper understanding of concepts being taught, deeper understanding of what we are doing when doing math, better memorization of the skills taught, and ease in building connections between different strands of math.

In this paper we, a practicing teacher and university educator, will investigate the place of nonmath analogies (here-in-after "analogies") in teaching school mathematics. First, we demonstrate the widespread use of analogies in teaching mathematics by drawing examples through context analysis of tutoring websites, selected textbooks, and

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from our own teaching experiences. These examples are specific to algebra because it is during algebra lessons when students usually first encounter pure abstract concepts. Second, we argue that analogies reflect the grounded nature of mathematical concepts in common life experiences and, thus, have an essential place in school mathematics instruction. To support our argument we offer a theoretical rationale based on research literature and historical sources.

2. Examples from practice

The sites of our content analysis were professional magazines such as *Mathematics Teacher (MT)*, *Mathematics in Middle School*, *Ontario Mathematics Gazette*, math textbooks of pre-algebra and algebra courses of intermediate level, and math help websites such as mathforum.org, math.com or purplemath.com.

Textbooks proved to be the scarcest source of analogies. Many textbooks, despite their inclusion of real life contexts and pictures, shy away from employing nonmath contexts, especially analogies, as an explanation tool when handling more abstract or advanced topics. We have found only few analogies that are prominently presented in textbooks. The one is when an equation is compared to a scale (e.g. MathPower 8, McGraw-Hill Ryerson, 1990, p. 396), another one is when a function is compared to a machine that has input and output slots (e.g. Mathematics 9, Addison-Wesley, 1999, p.147).

The analogies that we find in professional magazines tend to be of a linguistic kind (general meaning of a mathematical term). For instance, “Since many secondary students have used the word *limit* in their daily lives, the concept will not be novel for them. ...a discussion about the meaning of limit could be invaluable at the beginning of class.” (MT, Vol. 100, No. 8, p.551) or “The process of substituting one expression for another is an abstract concept. What may seem like a simple mathematical procedure to a mathematics teacher can pose a conceptual barrier for many of our students. In this instance, help came when I considered what it generally means to substitute one item for another.” (MT, Vol. 100, No. 2, p.135).

Math help websites are an excellent source of anecdotal evidence of specific math concepts that students find hard to comprehend, and the strategies they can use to facilitate their comprehension. In the wealth of on-line math help we found that analogies are used routinely in responses to students’ or teacher’s questions. What surprised us is that most of the analogies are exactly the same as the ones we thought we had independently invented to use in our own math classrooms! The table below is a collection of some of analogies most frequently used to explain basic algebra concepts. We are sure that most of readers with teaching experience would recognize one or two analogies.

Table 1. Analogies for basic algebra concepts

Mathematical Concept	Analogy
Solving equations	Undressing a person vs. Dressing a person
Isolating a variable	Peeling an onion one layer at a time or Shacking corn
Collecting like terms	Sorting out the mix fruits into separate piles
Distributive property	Passing out papers from a stack to each student in a class
Substituting numbers for the variables	Putting a thing in the place of another thing.

Our own teaching experience provided us with the following two analogies. We chose them specifically, because they help to answer questions that teachers of intermediate and senior level of math would recognize as “cursed”—cursed in that they are questions that recur every year one teaches. We want to demonstrate how analogies can help students to “make sense” of the answers to these cursed questions.

First question is: why can we not divide by zero? (Analytic geometry extension is, for instance, why the slope of a vertical line is undefined?) Some students would not bother to ponder over this problem by simply accepting that it is just a rule and even calculators show an error sign when you type in division by zero. But there will always be some curious minds in a class who would be baffled by this “rule”. This is not surprising given the long winded historical evolution of zero. But the source of their bewilderment is actually deeper than just a question of divisibility by zero. What we mean is that every student in intermediate and higher grades would probably know how to divide a number by a smaller number, to share a collection of apples, for example, among a given number of

students, which reflects a partitioning (or sharing) interpretation of division. But the measurement interpretation of division, which is determining how many times one quantity is contained in another, is often not been rooted in students' conceptual framework. And then, while engaged in solving word problems such as, "How many times bigger is the sun at 10^9 m to a giant squid at 10^1 m?" Many students do not know what mathematical operation they have to use to find the answer. The analogy that helps evoke a conceptual metaphor for division is "squeezing objects in a container." Division is used when you want to find out how many smaller things you can put into a larger one. When students understand that, in order to find how many giant squids you can squeeze into the sun you have to employ division, the problem starts to "make sense." This analogy also solves once and for all the problem of "Why can't we divide by zero?" Ask your students what it means to divide a number by zero. Isn't it like trying to figure out how many nothings you can put into anything? How many nothings fit into something? The students almost immediately answer that it is infinity or as much as you want—in a nut shell undefined. And there comes a moment of: "Ah, I get it!"

A second "cursed" question is what a graph has to do with an equation. Students' reluctance to see connection between different representations of a mathematical relation, especially between an equation and graph, constantly draws attention of educators (Kaput, Carraher and Blanton, 2008). In our classroom, while investigating linear relations and analytic geometry, we apply the analogy that compares different representations of a relation to different ways of identifying a person. Within this analogy, a graph of a relation is like a snapshot of a person. What is a graph after all if not a visual representation of a relation? Compare some of the definitions of "a graph": A graph is "a *drawing* that expresses the relationship between two sets of numbers" (Webster's, 1990) or "a *picture which shows* how two sets of information or variable amounts are related, usually by lines or curves" (Cambridge Dictionary, 2005).

With a constant reminder that a graph is just a *visual* ID of a relation, many areas of analytic geometry start to make sense to students. Take for instance, a concept of a domain of the relation. When a student has to graph an equation, let's say $y = 2x - 3$ for the domain of real numbers, a usual mistake is to find a few ordered pairs, plot them and then connect the points with a line segment. You can ask a student, "Can you see the whole relation on this graph?" If the student insists on having represented the whole relation, you show a picture of a person covering the eyes and ask: "What color are this guy's eyes? You can't tell it from a partial picture, can you? So, how does anyone know from your graph the way your relation work beyond what you show to me? If you want to represent that *y-values equal take away 3 from double x-values* on domain of real numbers, you have to extend the line in both directions and put two arrows at the ends."

3. Rationale: Research and Historical connection

We now turn to research on analogies. Specifically, we will explore theoretical distinctions and the structure of analogies in relation to the nature of mathematics concepts, and examples of analogies from historical mathematics sources.

4. Structure, Nature and Distinctions

Analogy, Latin *analogia* and French *analogie*, originates from Greek mathematics *analogos* for likeness of two ratios: proportion (ate) (Oxford English Dictionary 1989). Many take analogies to mean "metaphors." Both involve understanding one phenomenon in terms of another. Analogies and similes are overtly evoked when explaining something. They highlight a similarity between the explained phenomenon and another phenomenon (Aubusson, Harrison, and Ritchie 2006). Using analogy we speak of A is like, looks like or is similar to B, and using similes speak of as "white as snow." Metaphors and metonyms on the other hand are covertly used. We speak about concepts, like anger or love, in terms of a more tangible concepts, such as heat (Lakoff 1987). In the expressions *eye* of a needle and *head* count, objects are used to metonymically stand for others. All these concepts involve relations between two phenomena, one used to explain and the other one explained.

Analogies such as "equations are like a balanced scale" are structural; they highlight various similarities: two sides of a scale, state of balancing, actions that keep the scale balancing; respectively for two sides of an equation - equivalence, equivalent actions. Thinking of division as fitting things into a container is a visual analogy. Explaining multiplication in terms of directed movements on a number line combines a math and nonmath analogy. Some

analogies such as “to isolate terms is like what you do to turn a complicated sentence into a simpler one” are propositional (Aubusson et al. 2006) and literal; they involve a few surface similarities (Lee, Kim, Na, Han, and Song, 2007).

That teachers generate non math analogies on a “regular basis ... raises empirical questions about the nature of children’s learning from analogies” (Richland et al. 2004, p. 56). Teachers use analogies as teaching aids, serving communicational intent of vividly illustrating. At many times teachers invoke analogies to assist students deal with *curved* concepts (Wilbers and Duit 2006; Richland et al. 2004). Teachers package difficult concepts into a conceptual analogy (Wilbers and Duit 2006) to aid students to bridge “the epistemological gap between already know and the yet unknown” (Wilbers and Duit 2006, p.39; Yilmaz, Eryilmaz, and Geban 2006). English (1997) refers to the leap afforded by analogies as an abstraction shift, and Sfard (1994) and Mason (1989) have referred to it as reification. But analogies are also used by mathematicians and by students as mental tools to think with (English 1997). Cognitive psychologists study analogical reasoning as a general thinking skill to be developed among learners.

By highlighting structural commonalities, analogies unpack relations and properties (English and Sharry 1996). Lakoff and Núñez (2001), and associates explain that general cognitive skills in understanding are metaphorical in nature. Even abstract mathematical ideas such as infinity and continuity are better explained using more basic, every day contexts such as arithmetic, the number line and perpetual action. Pirie and Kieren (1994) observe that a current state of a student’s understanding contains outer and inner levels of understanding. Accordingly, students *fold back* to (refer to) previous levels of understanding—the analogies, metaphors and images—as they encounter outer-level abstractions.

5. Ancient Uses of Analogies

Analogies, such as scales and polynomial tiles, adopted in textbooks today date back to ancient mathematics traditions (Heeffer 2007). The specific historical example elaborated on here is from the Babylonian tablet.

- i. I have *heaped* the *surface* and my *confrontation* [side length]: it is $\frac{3}{4}$. [In today’s symbols $x^2 + x = \frac{3}{4}$]
- ii. 1 the *projection* [line] you *posit*, [A square on which a unit length and x units wide rectangle is joined; $x^2 + 1.x$]
- iii. half-part you *break*, [Rectangle $1x$ split into two equal parts, $\frac{1}{2}$ length and x width; $\Leftrightarrow x^2 + \frac{1}{2}x + \frac{1}{2}x$]
- iv. make $\frac{1}{2}$ and $\frac{1}{2}$ *hold*, [outer half moved to the adjacent side of x^2 to make a frame that encloses a square of length and width $\frac{1}{2}$ i.e., area $\frac{1}{4}$; $x^2 + \frac{1}{2}x + \frac{1}{2}x + (\frac{1}{2})^2$]
- v. $\frac{1}{4}$ to $\frac{3}{4}$ you *join*: [Now with a square with dimensions $x + \frac{1}{2}$; $x^2 + 1.x = \frac{3}{4} \Leftrightarrow x^2 + 1.x + (\frac{1}{2})^2 = \frac{3}{4} + \frac{1}{4} \Leftrightarrow (x + \frac{1}{2})^2 = 1$]
- vi. ... $\frac{1}{2}$ which you have made *hold* from the *body* of 1 you *tear out*: [$x = 1 - \frac{1}{2}$]
- vii. $\frac{1}{2}$ is the *confrontation*. [$x = \frac{1}{2}$] (Høyrup, 2007, p.266-268)

In the procedure above, analogies are used when referring to procedures, attributes and variables: *heap* and *join* for add, *break* and *tear out* for subtract, *hold* for enclose, and *alongside* for square root or factor. Two distinct analogies are used to refer to what would be symbolically the same: *Heap* in the i. to refer to addition of values of different kinds” whereas *join* in the v. to addition of values of the same kind. Even without visual drawings on the tablets, Babylonian algebra utilized visual, geometrical analogies (see Phrase iv--“make $\frac{1}{2}$ and $\frac{1}{2}$ *hold*” for multiplying $\frac{1}{2}$ by $\frac{1}{2}$) akin to current use of polynomial tiles and completion of squares (Heeffer 2007; Namukasa, Tuchtie and Stanley, 2009). The well documented curse questions “why can’t $7a + 5b$ equal $12ab$ ” could use the distinction *heaping not joining*. Prior to symbolic algebra and birth of the concept of an equation in the 16th century (Heeffer 2007), algebra procedures were narrated. And what do teachers do when they explain mathematics? They mainly narrate.

Ancient narrated algebra has great pedagogical relevance. *Pedagogy of analogies* (Swetz 1995) is also evidenced in 9th-17th century Arabic and European mathematics textbooks in which for instance various algorithms such multiplication algorithms are configured in a form that visually looks like a familiar real life object and, in turn, are named after these objects. Cases in point are multiplication by the method of a pipe organ or little stairs (for our common algorithm of multiplying) or division by the boat method (Berg 2001; NCTM 1969; Smith 1952; Swetz 1995).

6. Conclusion

Of course analogies have to be used very carefully, thoughtfully and always, always, always as a side dish to the main course of mathematical reasoning. They may pose limitations when over generalized. It is important that learners see the correspondence between the analogy and the structure of the concept Hall (1998) - otherwise, analogies generated by a teacher are not necessarily transparent to students.

When we decide which analogy to keep in our “teaching toolbox”, its extendibility is one of the criteria for selection. We also look for students’ positive reactions to analogies that we use. We check if these analogies can work as memory clues. And, the most important criterion is whether the analogy seems meaningful to our own understanding of the mathematical concept in question.

“Learning with understanding also makes subsequent learning easier. Mathematics makes more sense and is easier to remember and to apply when students connect new knowledge to existing knowledge in meaningful ways” (NCTM 2000, p.19). Analogies help students connect new mathematical knowledge to their existing experiential knowledge and, thus, facilitate understanding and memorization of mathematical concepts.

Rubenstein (1996) identifies the use of analogies as a way to facilitate learning the mathematics language. And Stavy and Tirosh (2000), as a means to eliminate common misunderstandings of some mathematics concepts. Analogies trigger metaphorical thinking that helps conceptualize abstract concepts in concrete terms. A visitor can hear in our classes remarks strange to an outsider’s ear, like “Cause the negative sign is not in the building!” (Response to “Why is -2^2 equal -4 and not $+4$), or “Because you can put an infinity of nothings into something.” (Response to “Why can we not divide a number by a zero?”). Analogies make math a friendlier subject and less detached from students’ experiential base. Everyday logic and out-of-school experiences start playing an active role in students’ mathematical reasoning.

Analogies are not only beneficial to students but also to teachers. When, as a teacher, you start looking for analogies and deciphering the metaphors behind your math, you achieve a different level of understanding and ease in your teaching. As shown above in the example of divisibility by zero, you see the connections between different topics more clearly. You feel more connected to your students and, ultimately, have more fun in your classroom. Your students like math because it is no longer a detached, abstract discipline of symbols and symbolic notations, but rather a new language to speak about the world around us.

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