UChicago Math

Part 1 — Individual

Multiple Choice

In this section, all answers will be in the set $\{1, 2, 3, 4, 5\}$. For each question answered incorrectly, you are assigned 0 points. For each question left blank, you are assigned 1 point. And for each question answered correctly, you are assigned the number of points listed next to the question.

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1. (a) (3 points) While 2021 is not a prime, if we insert a 0, between any two of the digits of 2021, we get a prime number. How many 2 digit numbers have this property?

   (b) (3 points) For any positive integer n, let $p(n)$ be the number of distinct prime divisors of n. What is the most common value of $p(n)$ for $1 \leq n \leq 2021$?

   (c) (4 points) There exists a positive integer n such that the number of integer solutions to $\frac{1}{n+5} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ with $1 \leq a \leq b \leq c \leq d$ is 2021. What is n?

2. (a) (3 points) Let the roots of the quadratic $(bx + x)^2 + bx + \frac{bc}{2}$ be (b, c, i, j, k are integers)

$$\frac{i \pm \sqrt{2021j}}{k}$$

   What is a possible value for j?

   (b) (3 points) Let n be a positive integer. What is a possible value for

$$9 \cdot \frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + \ldots + n} - 10$$

   (c) (4 points) How many integers $n > 1$ are there such that for all integers a, n is a divisor of $a^{n+1} - a$?

3. (a) (3 points) In this problem, we call a cube a 3-cube, and a square a 2-cube. Given a k-cube, we call the boundary count of the k-cube the sum of the number of vertices, edges, faces, etc, of the k-cube. For example, a 2-cube has 4 edges and 4 vertices, so it has a boundary count of 8. A 3-cube has 6 vertices, 12 edges, and 8 faces, so it has a boundary count of 26. What is the last digit of the boundary count of the 2021-cube?

   (b) (3 points) We say a die is a fair six-sided cube with 6 integers written on its sides. A set of dice is said to be comparable if no two dice share an integer on their sides. Given two comparable dice A, B, A beats B if the probability of rolling both dice and getting a higher roll on A than B is more than 0.5. A set of comparable dice D is k-winning if for any dice $d_1, \ldots, d_k$ in D, there exists a dice x in D that beats all of them. There is a set of 3 dice that is 1-winning. How much bigger is the smallest set of 2-winning dice?

   (c) (4 points) There exists a positive integer n such that for any rational number r, there exists rational numbers $r_1, \ldots, r_n$ that have the following properties:

$$r_1 + \ldots + r_n = 0$$

$$r_1 \cdot \ldots \cdot r_n = r$$

   What is the smallest possible value of n?

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