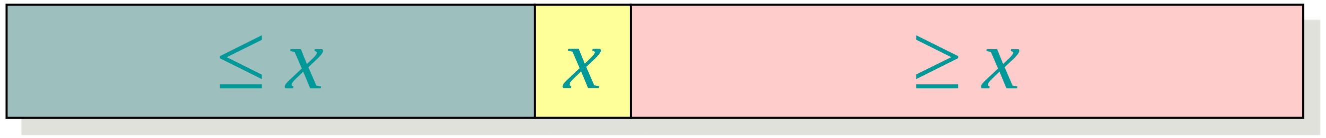
Lecture 4: QuickSort

DIVIDE AND CONQUER

Quicksort an n-element array:

1. ***Divide:*** Partition the array into two subarrays around a ***pivot*** x such that elements in lower subarray $\leq x \leq$ elements in upper subarray.



2. ***Conquer:*** Recursively sort the two subarrays.
3. ***Combine:*** Trivial.

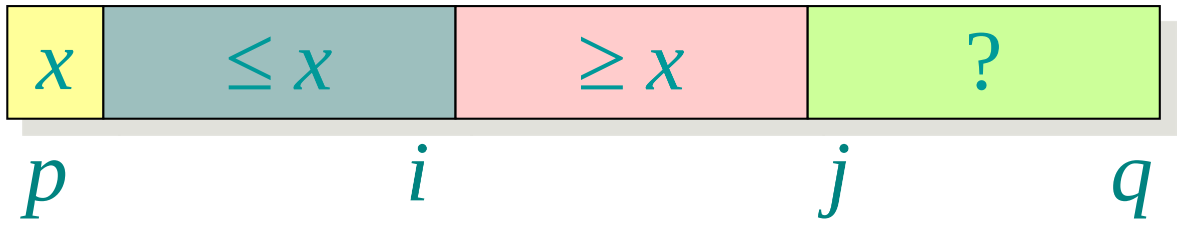
Key: *Linear-time partitioning subroutine.*

PARTITIONING SUBROUTINE

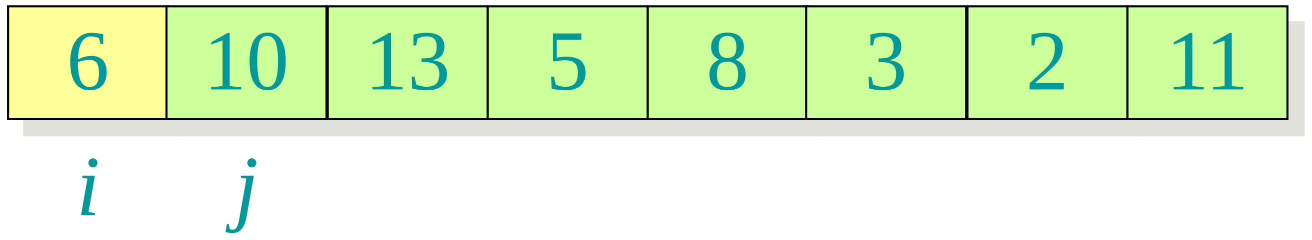
```
PARTITION($A, p, q$)  ⊳ $A[p\,.\,.\,q]$
    $x \leftarrow A[p]$          ⊳ pivot $= A[p]$
    $i \leftarrow p$
    for $j \leftarrow p + 1$ to $q$
        do if $A[j] \leq x$
               then $i \leftarrow i + 1$
                    exchange $A[i] \leftrightarrow A[j]$
    exchange $A[p] \leftrightarrow A[i]$
    return $i$
```

Running time $= O(n)$ for n elements.

Invariant:



EXAMPLE OF PARTITIONING



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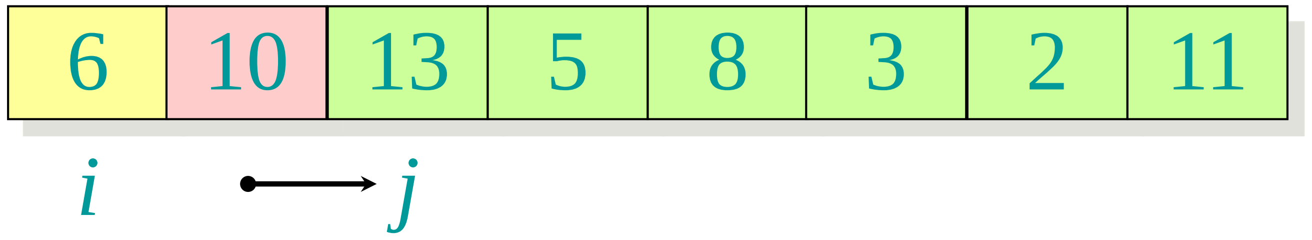

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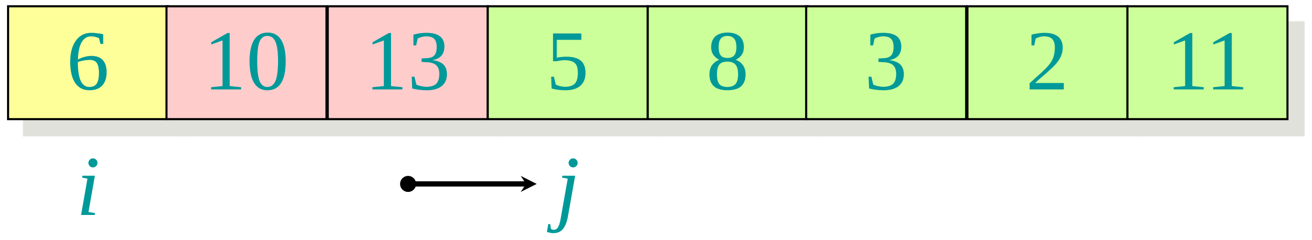

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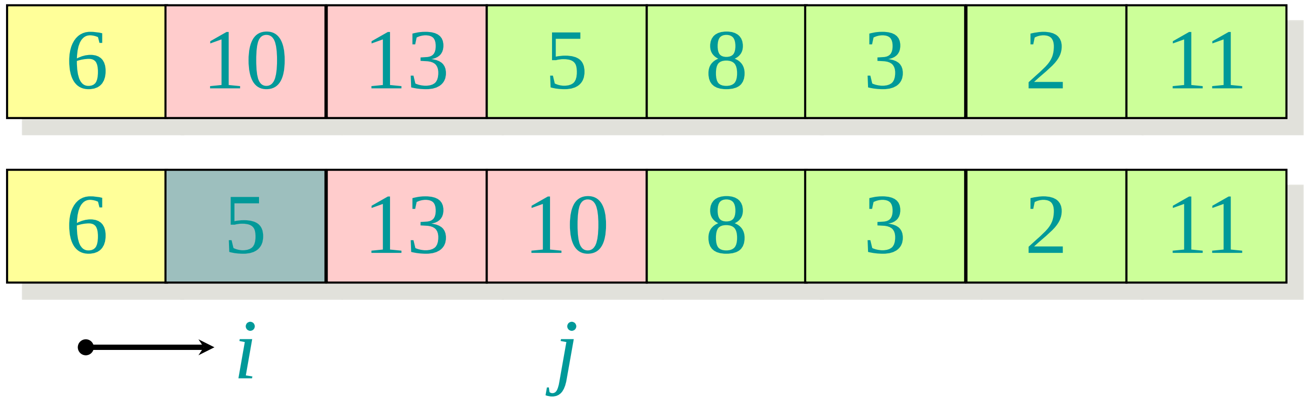

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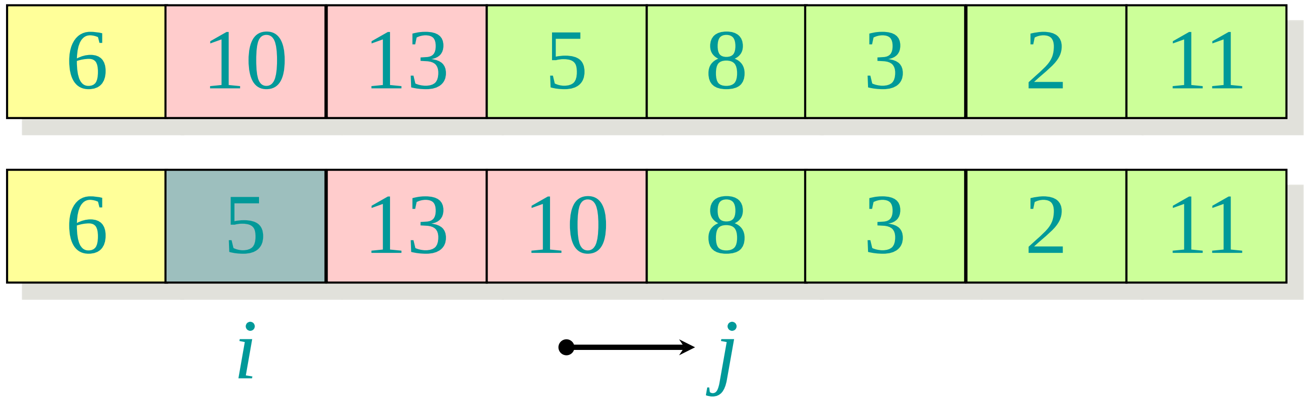

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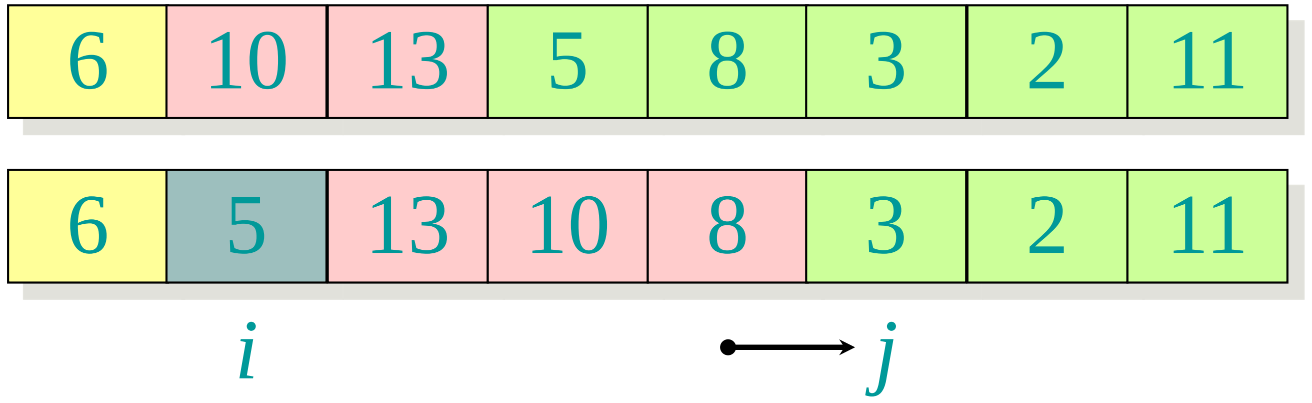

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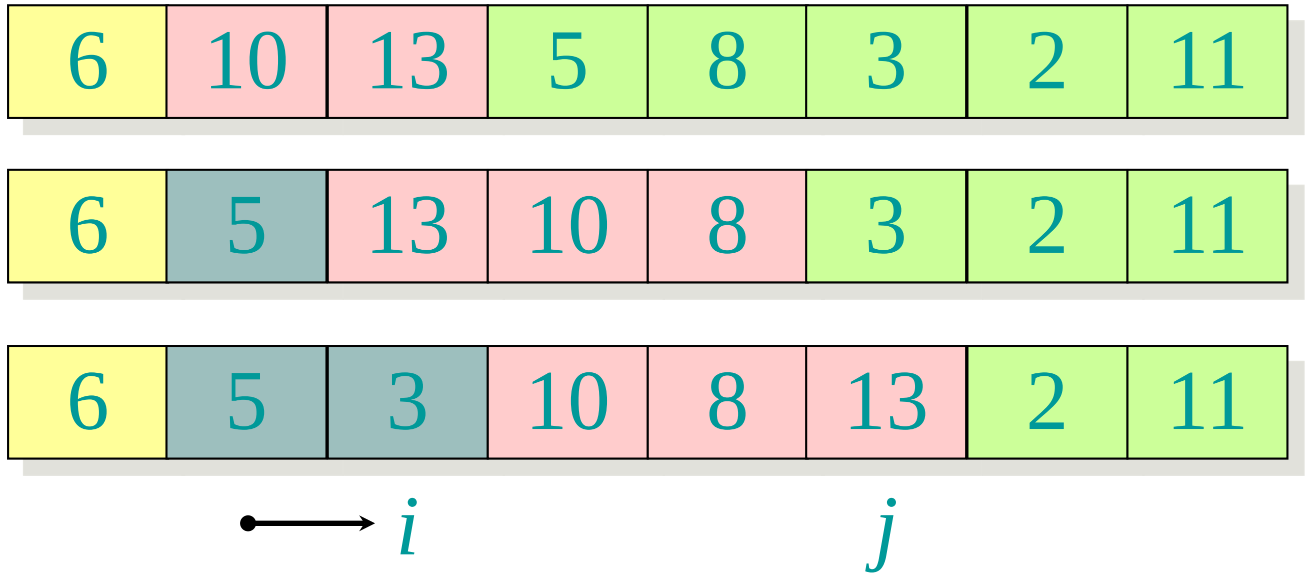

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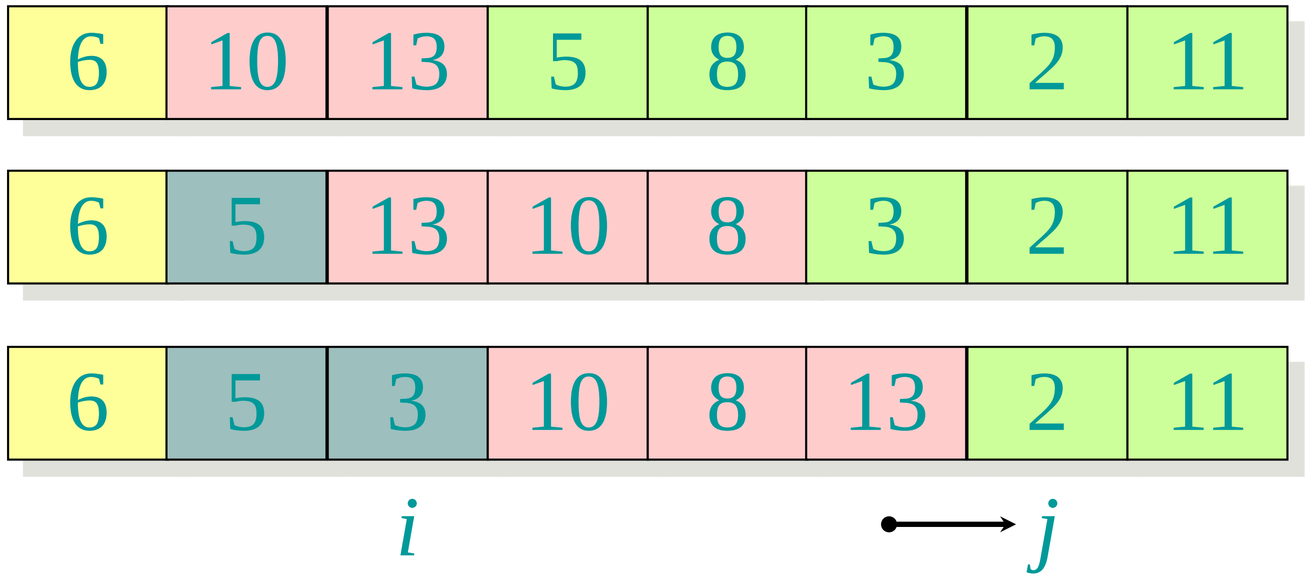

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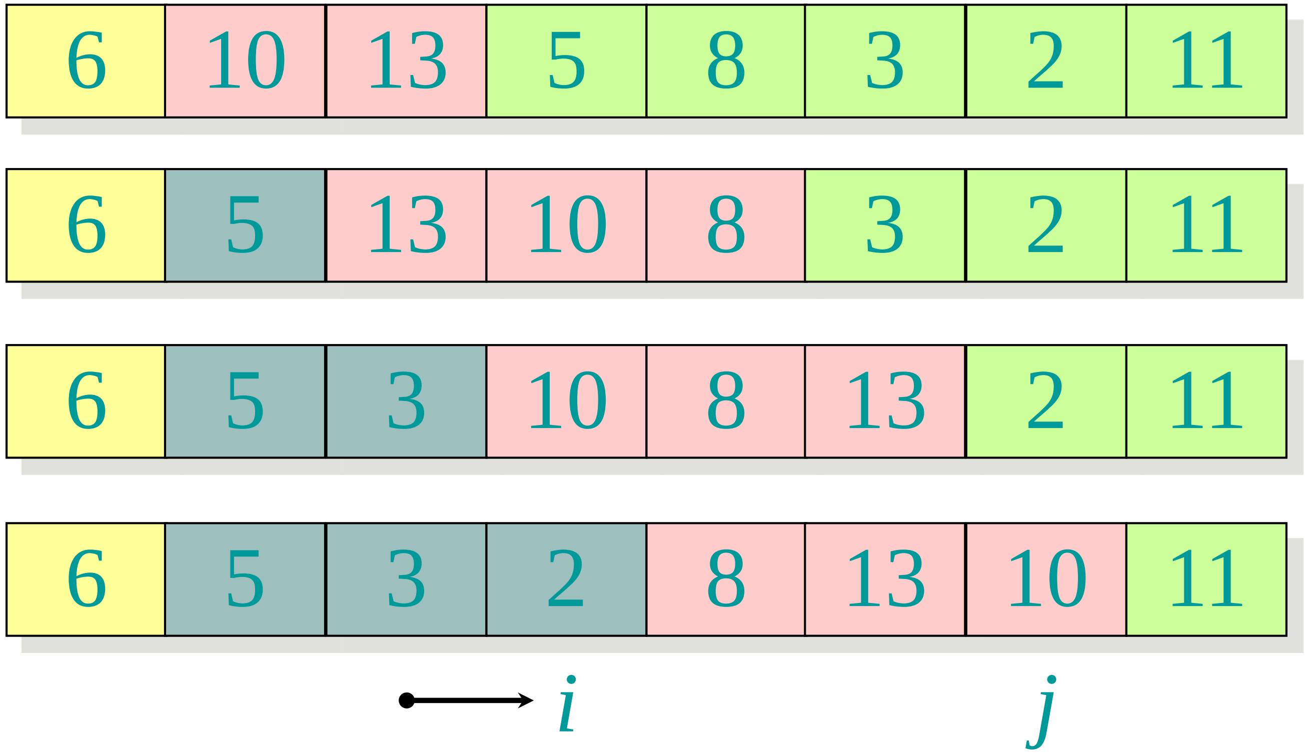

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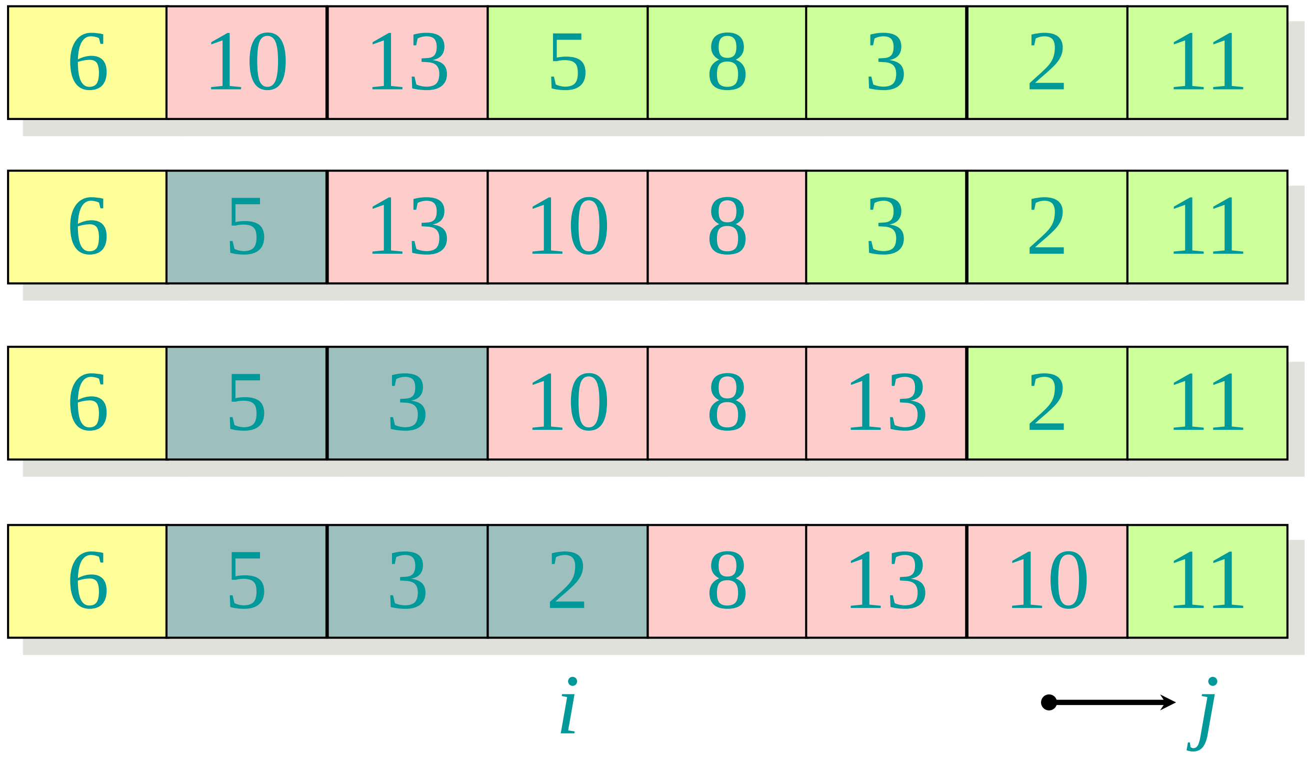

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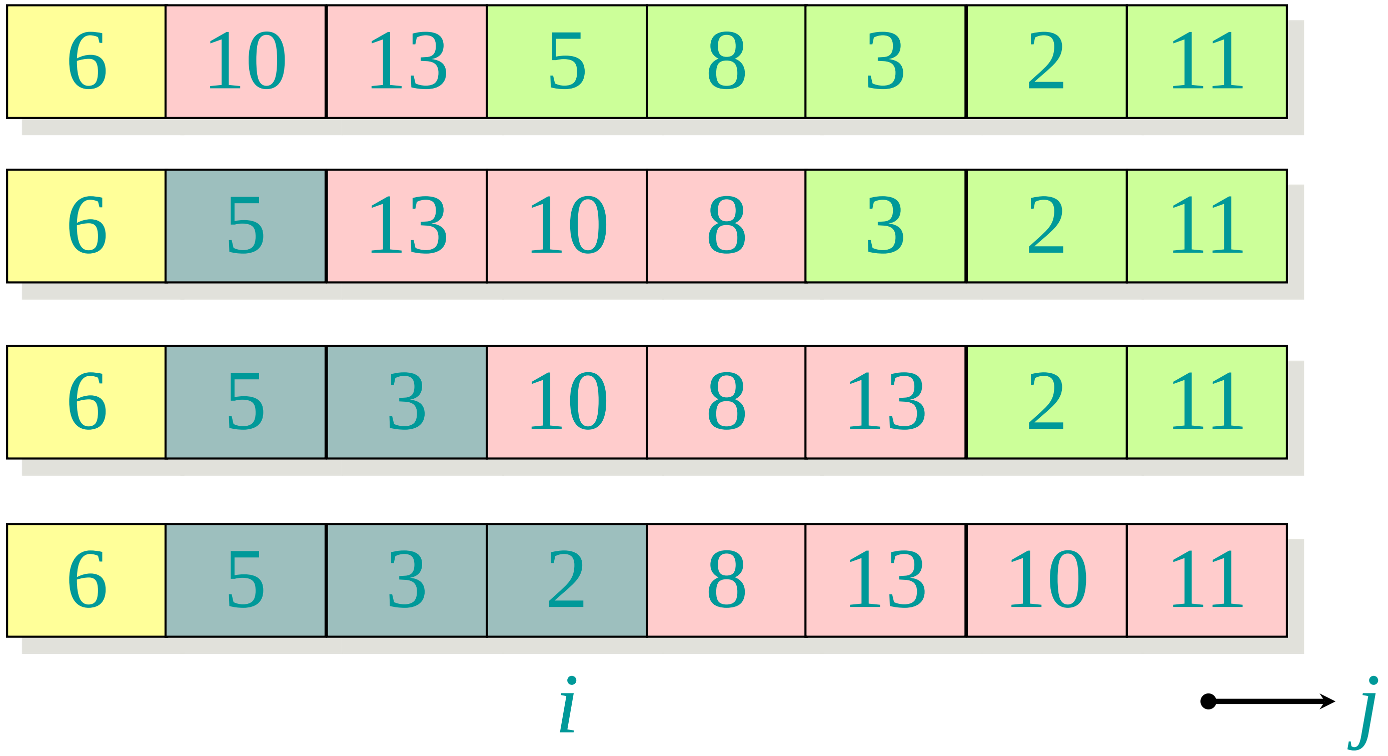

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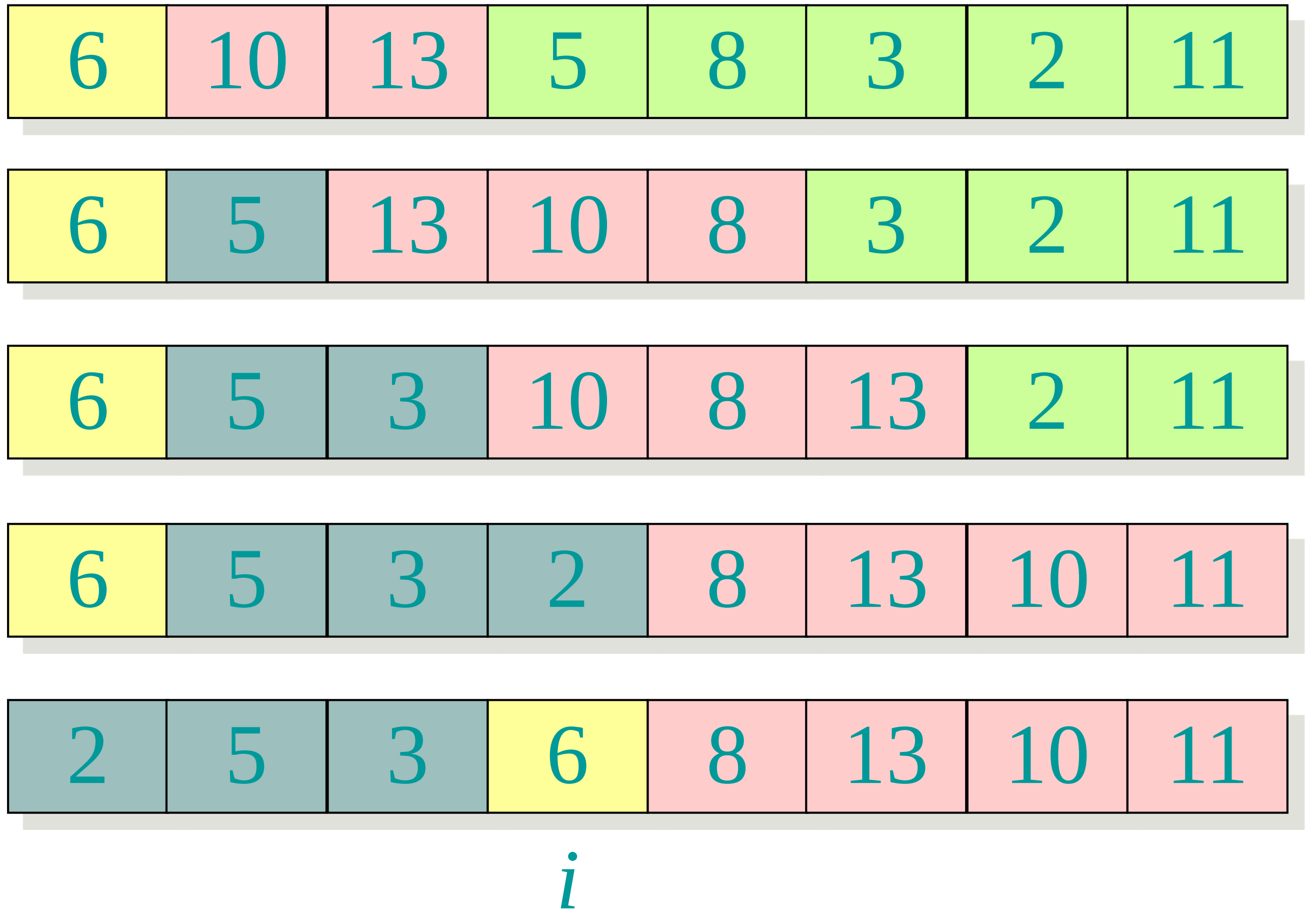

PSEUDOCODE FOR QUICKSORT

```
QUICKSORT($A$, $p$, $r$)
    if $p < r$
        then $q \leftarrow$ PARTITION($A$, $p$, $r$)
            QUICKSORT($A$, $p$, $q$)
            QUICKSORT($A$, $q+1$, $r$)
```

Initial call: QUICKSORT(A, 1, n)

ANALYSIS OF QUICKSORT

- Assume all input elements are distinct.
- In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n) =$ worst-case running time on an array of n elements.

WORST-CASE OF QUICKSORT

- Input sorted or reverse sorted.
- Partition around min or max element.
- One side of partition always has no elements.

$$\begin{aligned} T(n) &= T(0) + T(n-1) + \Theta(n) \\ &= \Theta(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n^2) \qquad \textit{\textbf{(arithmetic series)}} \end{aligned}$$

WORST-CASE RECURSION TREE

$$T(n) = T(0) + T(n-1) + cn$$

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$$T(n)$$

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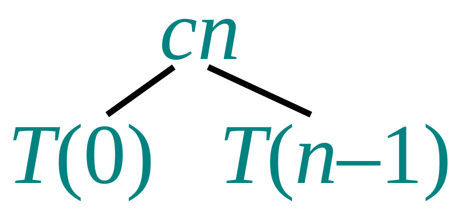


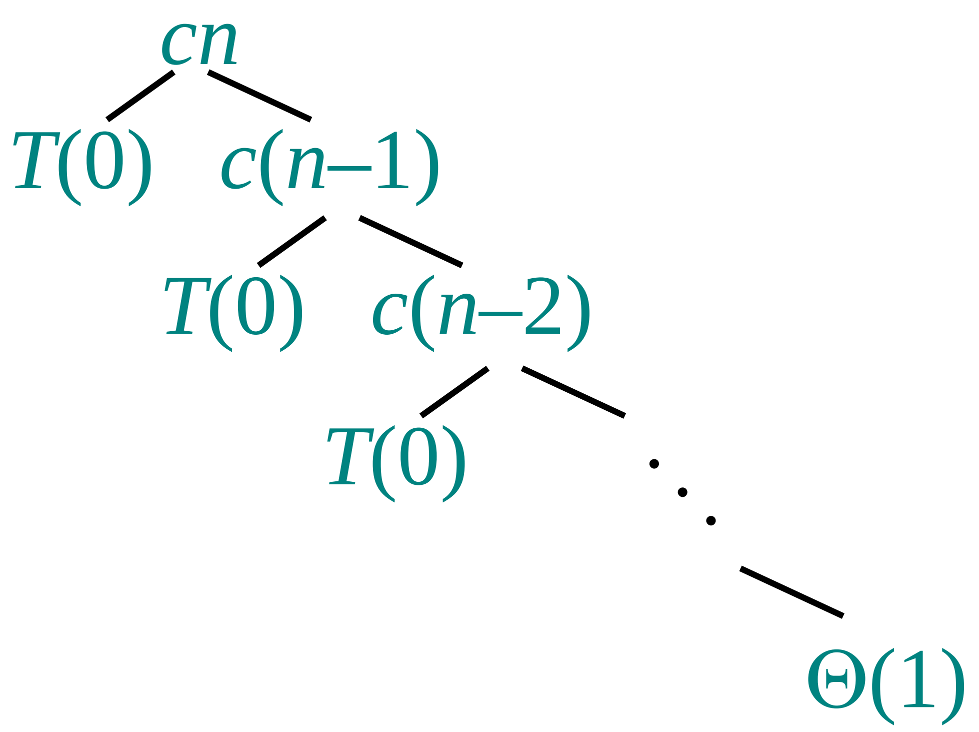
WORST-CASE RECURSION TREE

$$T(n) = T(0) + T(n\text{–}1) + cn$$

$$\begin{tikzcd}
 & cn \arrow[dl, dash] \arrow[dr, dash] & & \\
T(0) & & c(n\text{–}1) \arrow[dl, dash] \arrow[dr, dash] & \\
 & T(0) & & T(n\text{–}2)
\end{tikzcd}$$

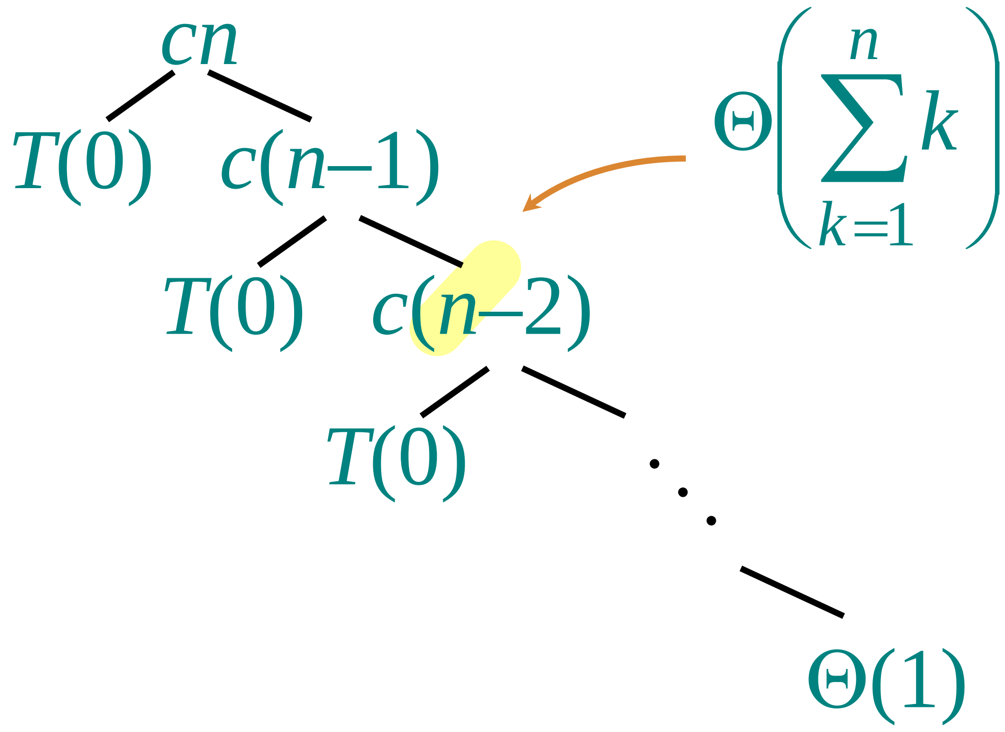
WORST-CASE RECURSION TREE

$$T(n) = T(0) + T(n-1) + cn$$



WORST-CASE RECURSION TREE

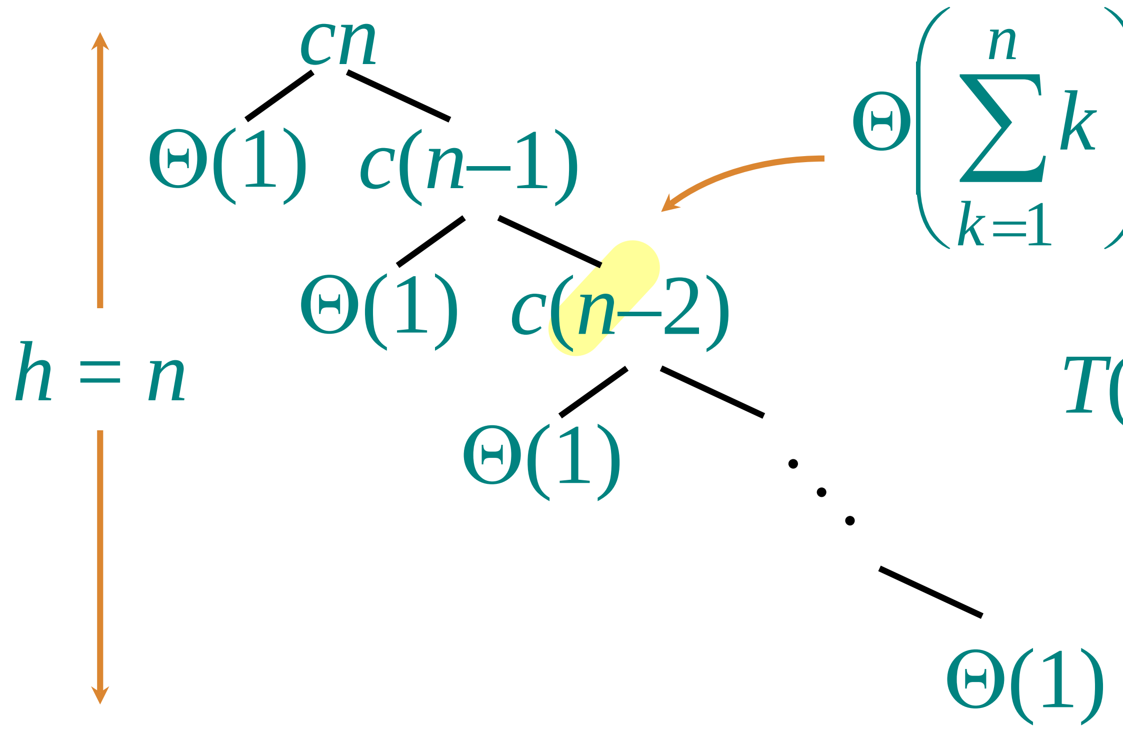
$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

WORST-CASE RECURSION TREE

$$T(n) = T(0) + T(n-1) + cn$$



$$\Theta\left(\sum_{k=1}^{n} k\right) = \Theta(n^2)$$

$$T(n) = \Theta(n) + \Theta(n^2)$$
$$= \Theta(n^2)$$

BEST-CASE ANALYSIS

If we're lucky, PARTITION splits the array evenly:

$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

(same as merge sort)

What if the split is always $\frac{1}{10} : \frac{9}{10}$?

$$T(n) = T\left(\tfrac{1}{10} n\right) + T\left(\tfrac{9}{10} n\right) + \Theta(n)$$

What is the solution to this recurrence?

ANALYSIS OF “ALMOST-BEST” CASE

$$T(n)$$

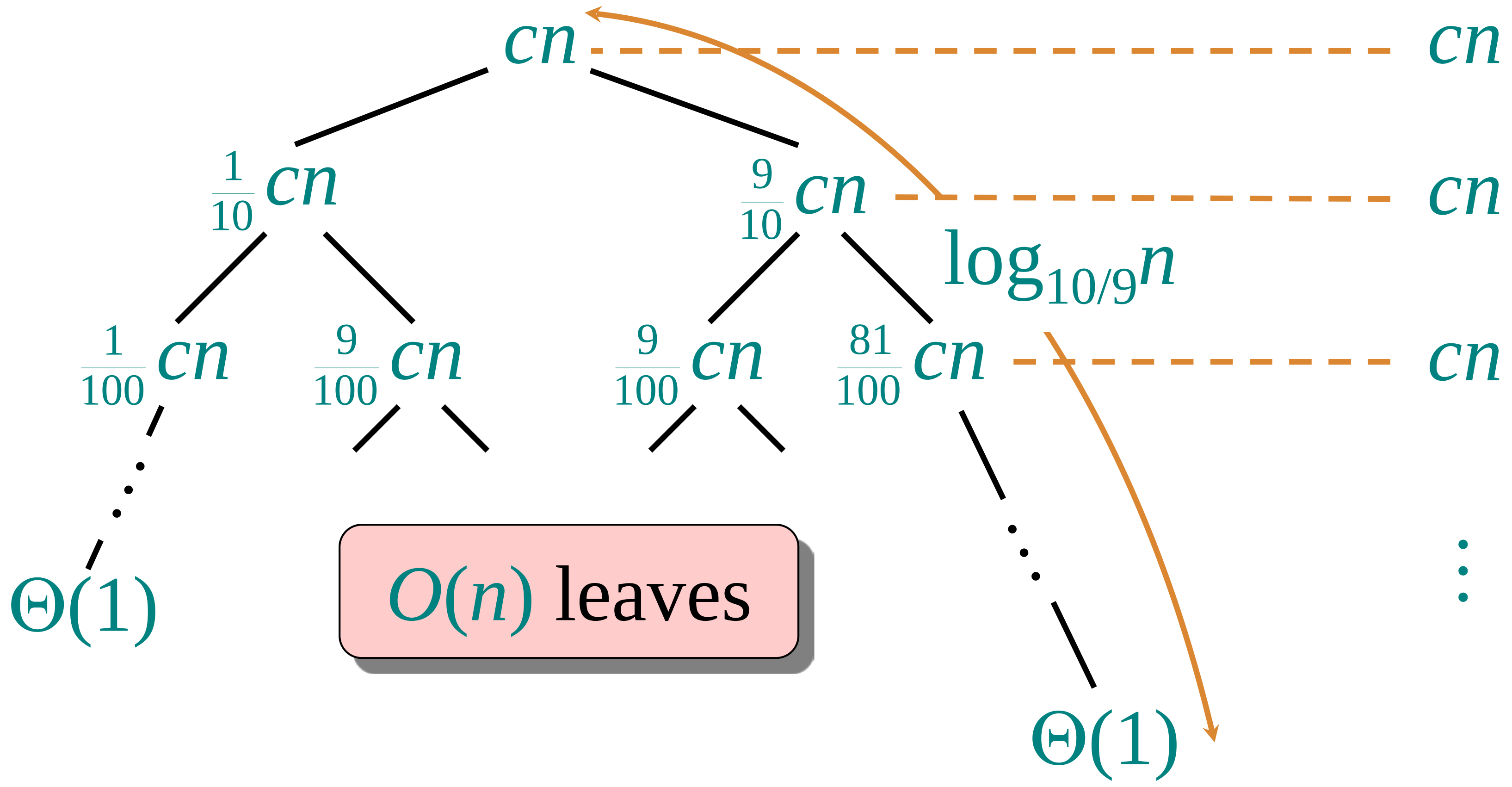
ANALYSIS OF "ALMOST-BEST" CASE

$$\begin{tikzcd}
 & cn \arrow[dl, no head] \arrow[dr, no head] & \\
T\left(\frac{1}{10}n\right) & & T\left(\frac{9}{10}n\right)
\end{tikzcd}$$

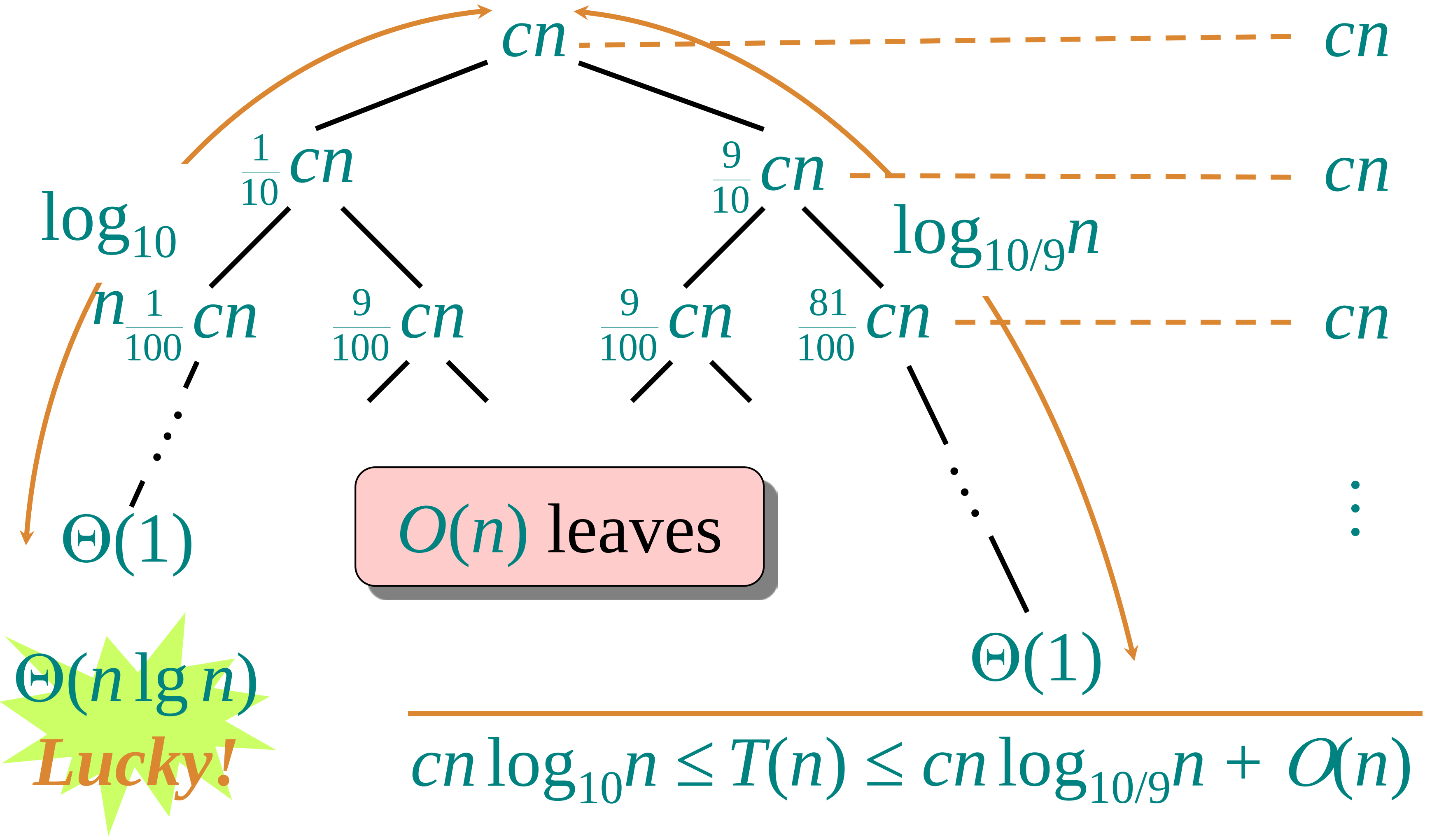
ANALYSIS OF “ALMOST-BEST” CASE

$$\begin{tikzcd}
 & & & cn \arrow[dll, dash] \arrow[drr, dash] & & & \\
 & \frac{1}{10}cn \arrow[dl, dash] \arrow[dr, dash] & & & & \frac{9}{10}cn \arrow[dl, dash] \arrow[dr, dash] & \\
T\left(\frac{1}{100}n\right) & & T\left(\frac{9}{100}n\right) & & T\left(\frac{9}{100}n\right) & & T\left(\frac{81}{100}n\right)
\end{tikzcd}$$

ANALYSIS OF "ALMOST-BEST" CASE



Analysis of "Almost-Best" Case



$$cn \log_{10} n \le T(n) \le cn \log_{10/9} n + O(n)$$

MORE INTUITION

Suppose we alternate lucky, unlucky, lucky, unlucky, lucky, ….

$$L(n) = 2U(n/2) + \Theta(n) \quad \textit{lucky}$$

$$U(n) = L(n-1) + \Theta(n) \quad \textit{unlucky}$$

Solving:

$$\begin{aligned} L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2-1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

Lucky!

How can we make sure we are usually lucky?

RANDOMIZED QUICKSORT

IDEA: Partition around a ***random*** element.

- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

RANDOMIZED QUICKSORT ANALYSIS

Let $T(n)$ = the random variable for the running time of randomized quicksort on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n–1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if } \textsc{Partition} \text{ generates a } k : n–k–1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_k] = \Pr\{X_k = 1\} = 1/n$, since all splits are equally likely, assuming elements are distinct.

ANALYSIS (CONTINUED)

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right).$$

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

Take expectations of both sides.

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

Linearity of expectation.

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

Independence of X_k from other random choices.

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(k) + T(n-k-1) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$.

CALCULATING EXPECTATION

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k\big(T(k)+T(n-k-1)+\Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(k)+T(n-k-1)+\Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n) \\
&= \frac{2}{n}\sum_{k=1}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Summations have identical terms.

HAIRY RECURRENCE

$$E[T(n)] = \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

(The $k = 0, 1$ terms can be absorbed in the $\Theta(n)$.)

Prove: $E[T(n)] \le a n \lg n$ for constant $a > 0$.

- Choose a large enough so that $a n \lg n$ dominates $E[T(n)]$ for sufficiently small $n \ge 2$.

Use fact: $\sum_{k=2}^{n-1} k \lg k \le \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$ (exercise).

SUBSTITUTION METHOD

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

Substitute inductive hypothesis.

SUBSTITUTION METHOD

$$
\begin{aligned}
E[T(n)] &\leq \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\
&\leq \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)
\end{aligned}
$$

Use fact.

SUBSTITUTION METHOD

$$E[T(n)] \le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\le \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n)$$

$$= an\lg n - \left(\frac{an}{4} - \Theta(n)\right)$$

Express as ***desired* – *residual***.

SUBSTITUTION METHOD

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\
&= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n) \\
&= an\lg n - \left(\frac{an}{4} - \Theta(n)\right) \\
&\le an\lg n\,,
\end{aligned}
$$

if a is chosen large enough so that $an/4$ dominates the $\Theta(n)$.

QUICKSORT IN PRACTICE

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is typically over twice as fast as merge sort.
- Quicksort can benefit substantially from ***code tuning***.
- Quicksort behaves well even with caching and virtual memory.