Lecture ~~7~~: Order Statistics

ORDER STATISTICS

Select the ith smallest of n elements.

- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: ***median***.

Naive algorithm: Sort and index ith element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$

$= \Theta(n \lg n)$,

using merge sort or heapsort (*not* quicksort).

RANK OF AN ELEMENT

Rank of an element is the position of the element in the sorted array

Given Numbers:

10	13	5	8	3	2	11

Sorted Array:

2	3	5	8	10	11	13

rank (8)=4

Finding the ith smallest of n elements = Finding an element with ***rank*** i.

DIVIDE-AND-CONQUER ALGORITHM

SELECT(A, p, q, i) $\triangleright$ ith smallest of $A[p \,..\, q]$

if $p = q$ **then return** $A[p]$

$r \leftarrow$ PARTITION(A, p, q)

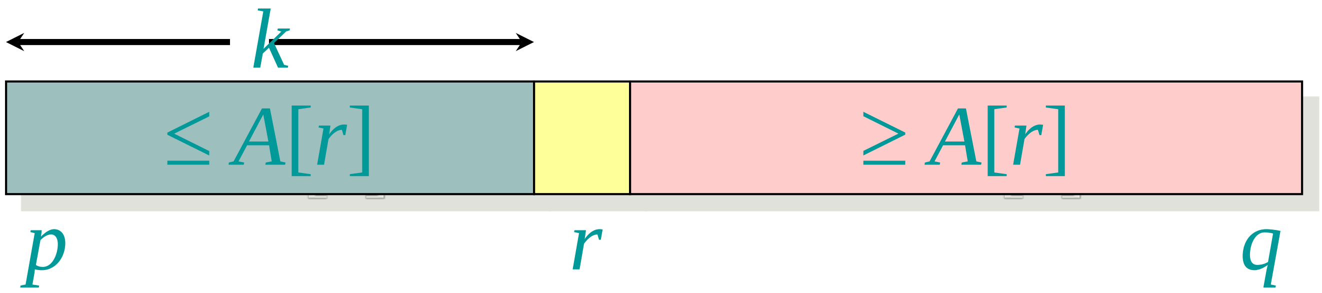
$k \leftarrow r - p + 1$ \\ k=rank of the pivot

if $i = k$ **then return** $A[r]$

if $i < k$

then return SELECT$(A, p, r - 1, i)$

else return SELECT$(A, r + 1, q, i - k)$



EXAMPLE

Select the $i = 7$th smallest:

6	10	13	5	8	3	2	11

$i = 7$

pivot

Partition:

2	5	3	6	8	13	10	11

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.

INTUITION FOR ANALYSIS

(All our analyses today assume that all elements are distinct.)

Lucky:

$T(n) = T(9n/10) + \Theta(n)$ $\quad n^{\log_{10/9} 1} = n^0 = 1$

$= \Theta(n)$ $\quad$ CASE 3

Unlucky:

$T(n) = T(n - 1) + \Theta(n)$ $\quad$ arithmetic series

$= \Theta(n^2)$

Worse than sorting!

RANDOMIZED DIVIDE-AND-CONQUER ALGORITHM

RAND-SELECT(A, p, q, i) ⊳ ith smallest of $A[p \,..\, q]$

    if $p = q$ **then return** $A[p]$

    $r \leftarrow$ RAND-PARTITION(A, p, q)

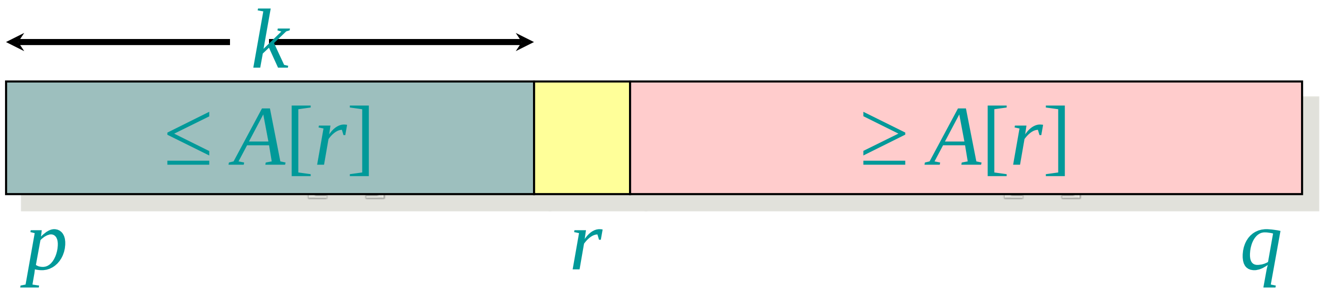
    $k \leftarrow r - p + 1$

    if $i = k$ **then return** $A[r]$

    if $i < k$

        then return RAND-SELECT$(A, p, r - 1, i)$

        else return RAND-SELECT$(A, r + 1, q, i - k)$



ANALYSIS OF EXPECTED TIME

The analysis follows that of randomized quicksort, but it's a little different.

Let $T(n)$ = the random variable for the running time of RAND-SELECT on an input of size n, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

ANALYSIS (CONTINUED)

To obtain an upper bound, assume that the ith element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n) \right).$$

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

Take expectations of both sides.

CALCULATING EXPECTATION

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

Linearity of expectation.

CALCULATING EXPECTATION

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E[X_k(T(\max\{k, n-k-1\}) + \Theta(n))] \\
&= \sum_{k=0}^{n-1} E[X_k]\cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]
\end{aligned}
$$

Independence of X_k from other random choices.

CALCULATING EXPECTATION

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k\big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)
\end{aligned}
$$

Linearity of expectation; $E[X_k] = 1/n$.

CALCULATING EXPECTATION

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k \big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k \big(T(\max\{k, n-k-1\}) + \Theta(n)\big)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)] \\
&= \frac{1}{n}\sum_{k=0}^{n-1} E[T(\max\{k, n-k-1\})] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n) \\
&\le \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Upper terms appear twice.

HAIRY RECURRENCE

(But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Prove: $E[T(n)] \le cn$ for constant $c > 0$.

- The constant c can be chosen large enough so that $E[T(n)] \le cn$ for the base cases.

Use fact: $\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \le \frac{3}{8} n^2$ (exercise).

SUBSTITUTION METHOD

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.

SUBSTITUTION METHOD

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

Use fact.

SUBSTITUTION METHOD

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n)\right)
\end{aligned}
$$

Express as ***desired – residual***.

SUBSTITUTION METHOD

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n)\right) \\
&\le cn,
\end{aligned}
$$

if c is chosen large enough so that $cn/4$ dominates the $\Theta(n)$.

SUMMARY OF RANDOMIZED ORDER-STATISTIC SELECTION

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is ***very*** bad: $\Theta(n^2)$.

Q. Is there an algorithm that runs in linear time in the worst case?

A. Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

IDEA: Generate a good pivot recursively.

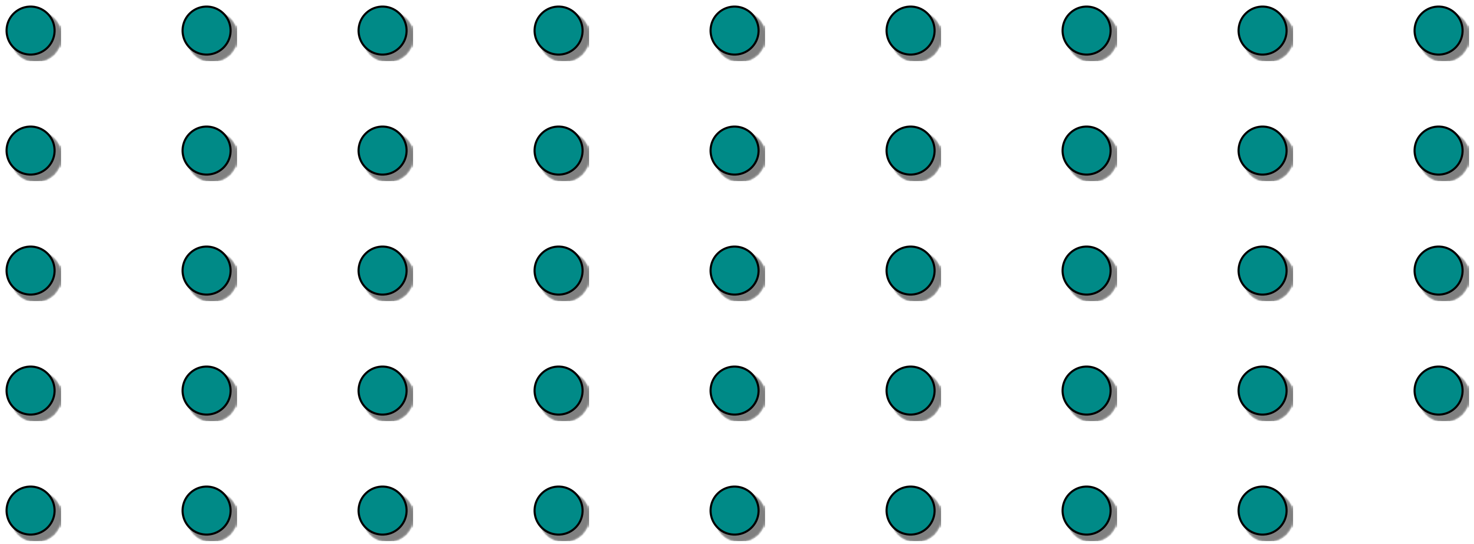
WORST-CASE LINEAR-TIME ORDER STATISTICS

SELECT(i, n)

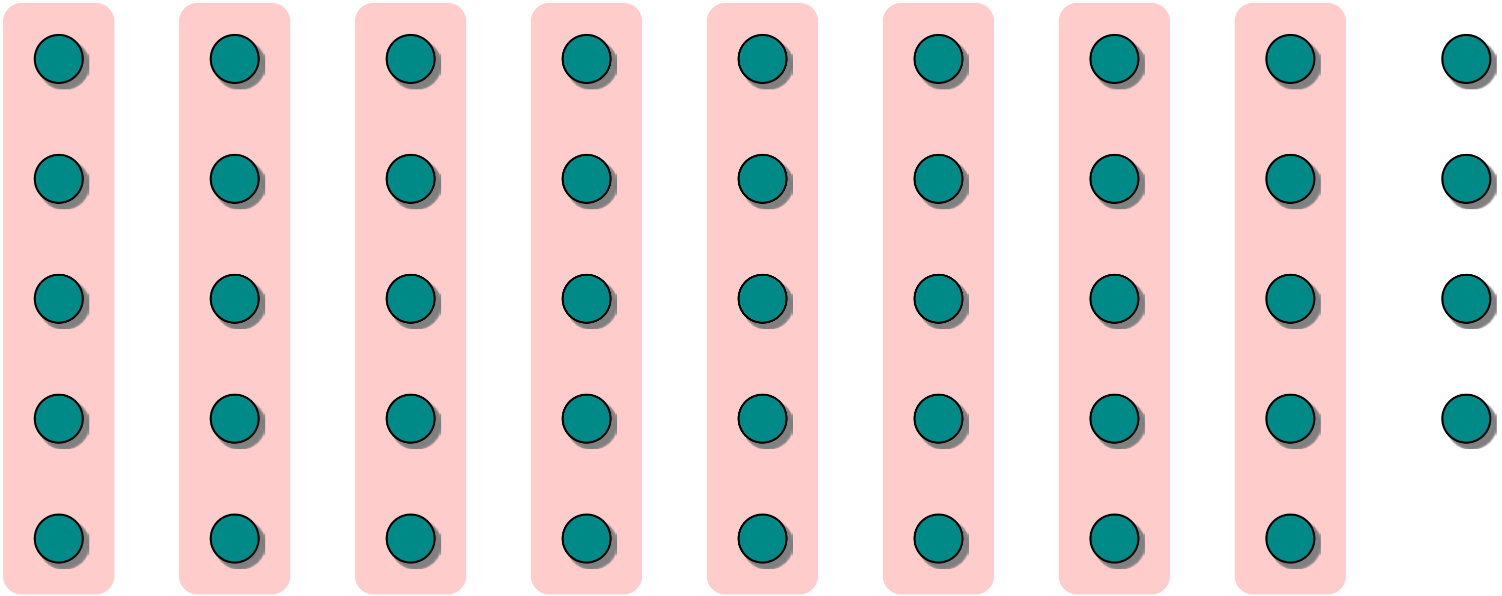
1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
3. Partition around the pivot x. Let $k = \text{rank}(x)$.
4. **if** $i = k$ **then return** x
   elseif $i < k$
   then recursively SELECT the ith smallest element in the lower part
   else recursively SELECT the $(i–k)$th smallest element in the upper part

Same as RAND-SELECT

CHOOSING THE PIVOT

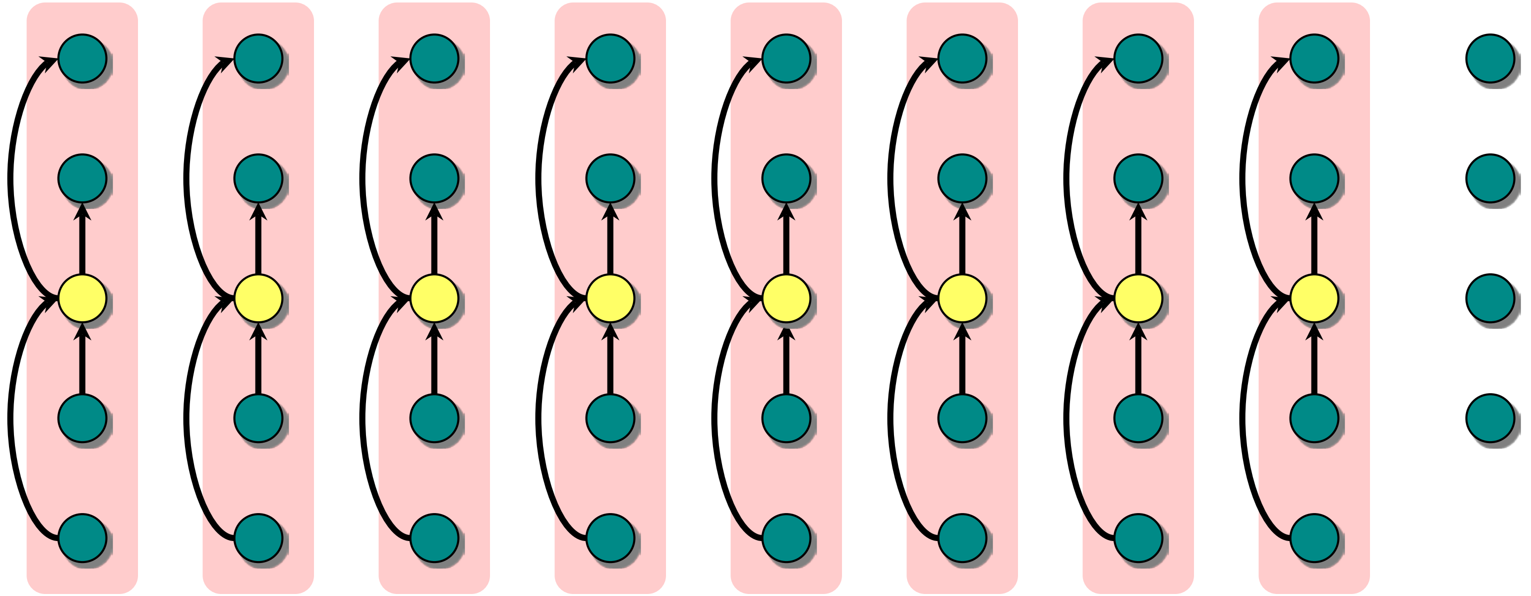


CHOOSING THE PIVOT

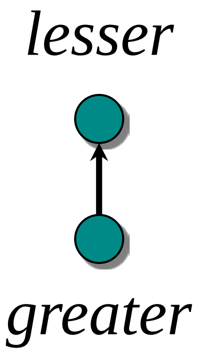


1. Divide the n elements into groups of 5.

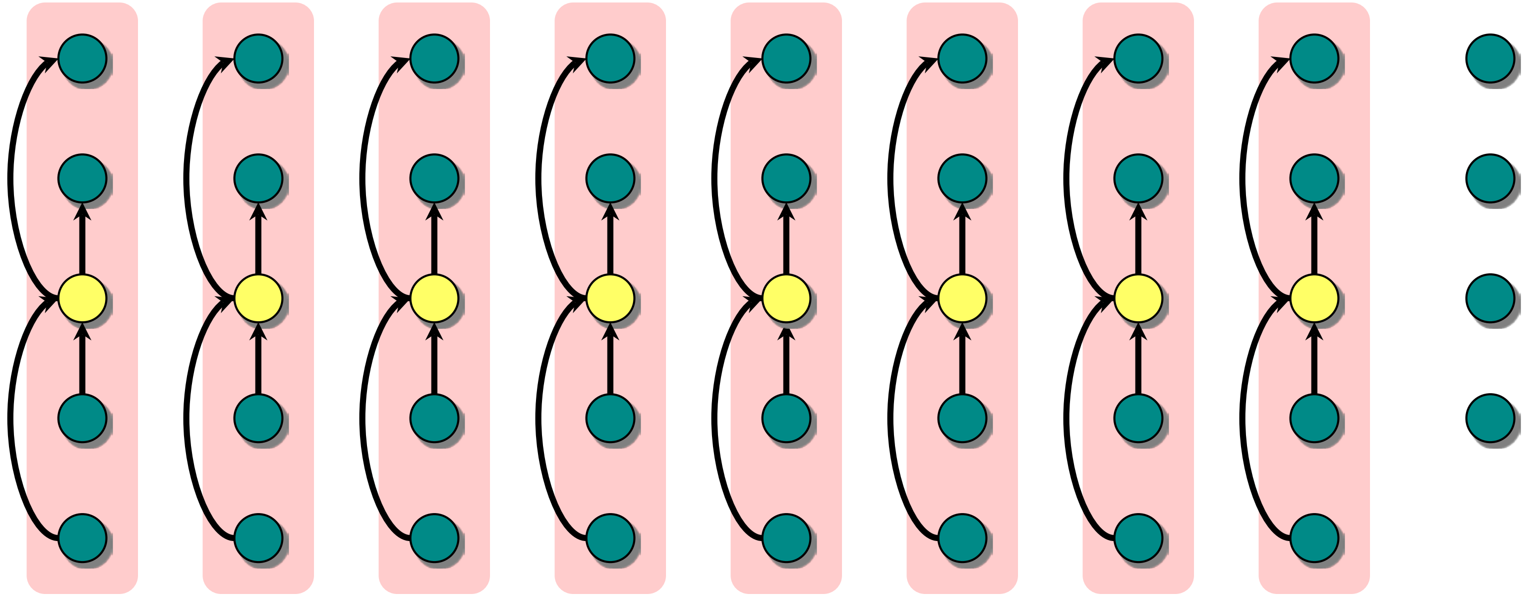
CHOOSING THE PIVOT



1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

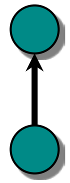


CHOOSING THE PIVOT



1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

lesser



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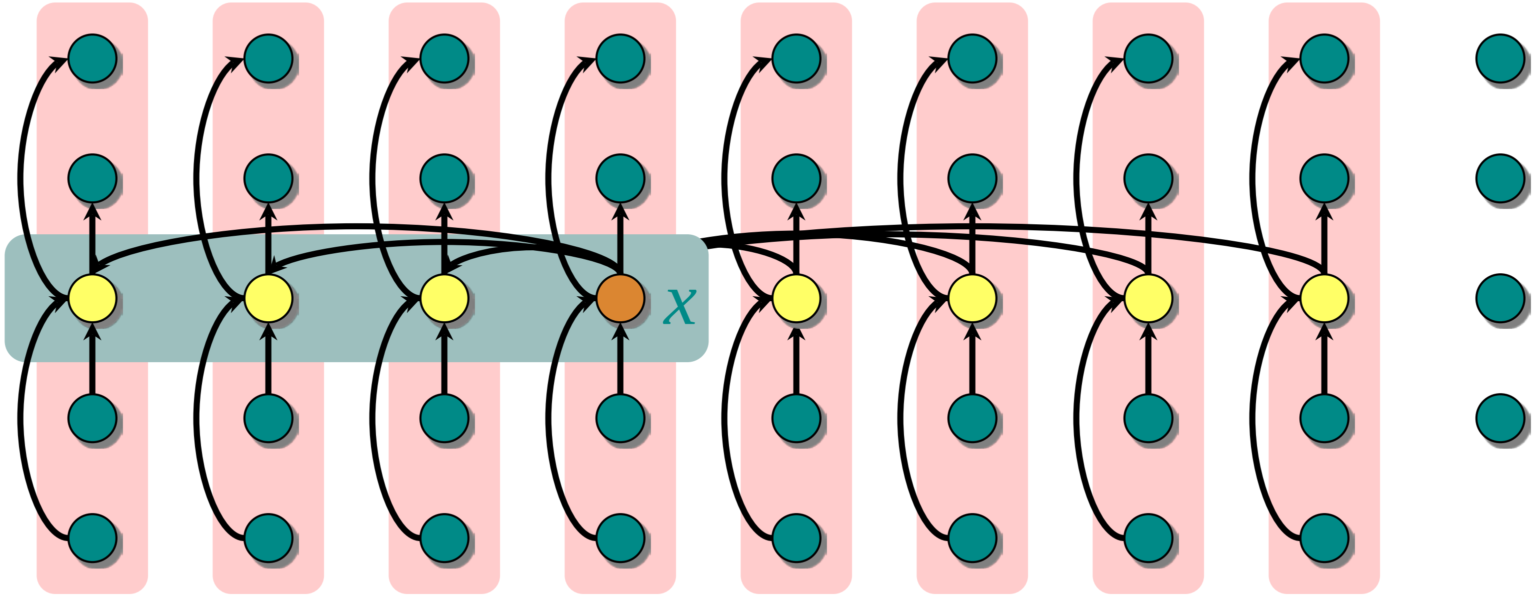
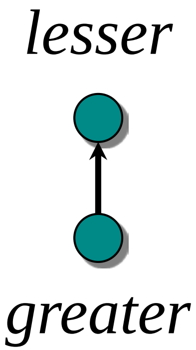
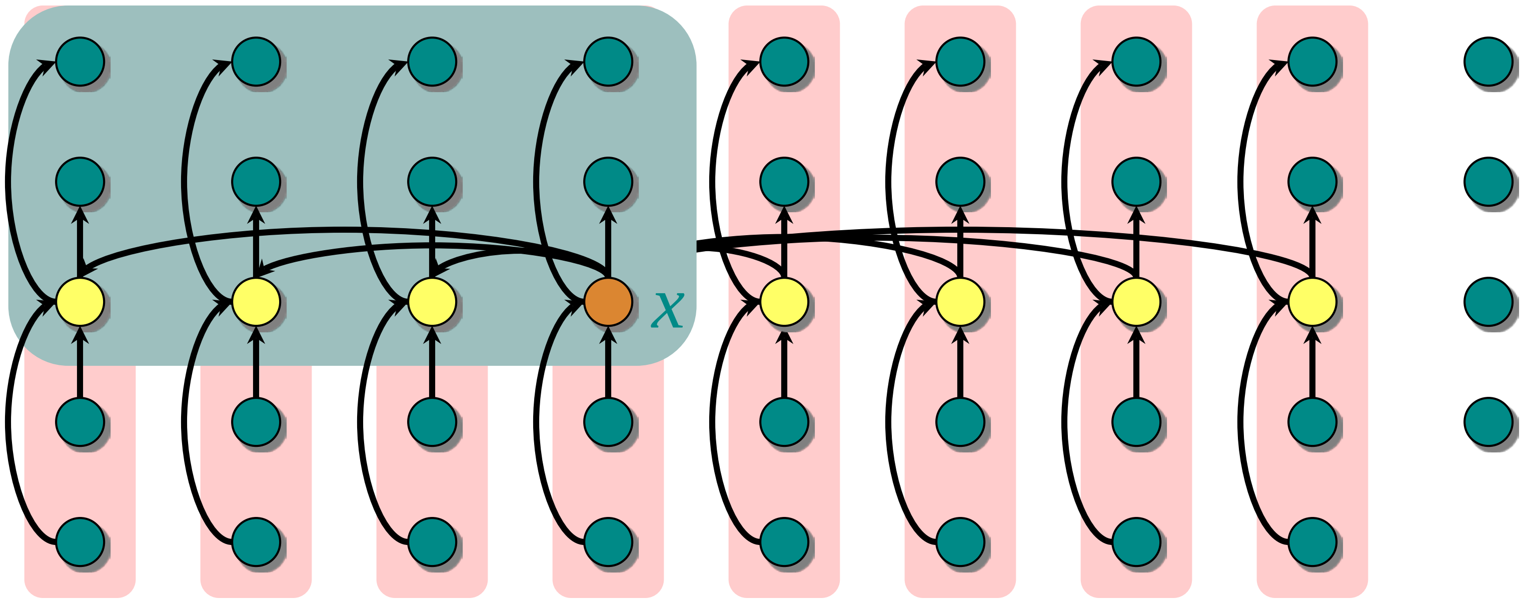
ANALYSIS



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.



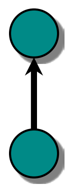
ANALYSIS



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.

lesser



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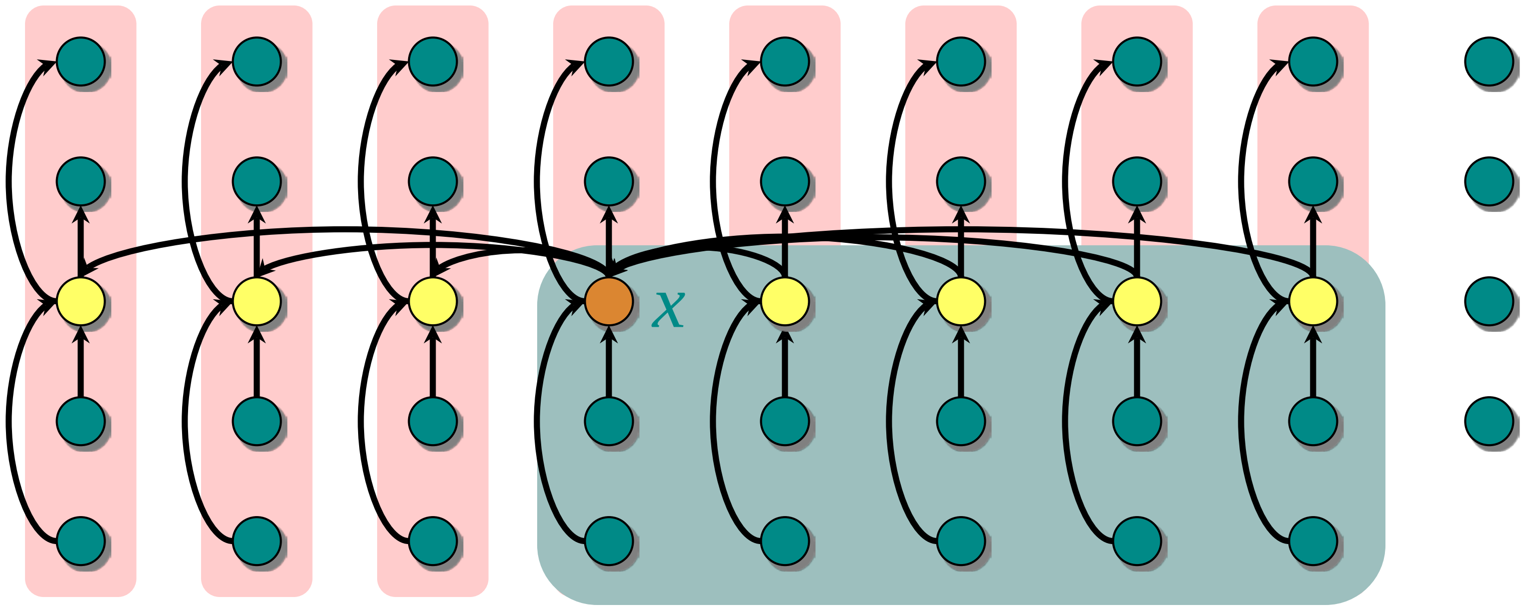
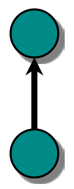
ANALYSIS (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3\lfloor n/10 \rfloor$ elements are $\leq x$.
- Similarly, at least $3\lfloor n/10 \rfloor$ elements are $\geq x$.



DEVELOPING THE RECURRENCE

$T(n)$	SELECT(i, n)
$\Theta(n)$	1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.
$T(n/5)$	2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.
$\Theta(n)$	3. Partition around the pivot x. Let $k = \text{rank}(x)$.
$T(7n/10)$	4. **if** $i = k$ **then return** x **elseif** $i < k$ **then** recursively SELECT the ith smallest element in the lower part **else** recursively SELECT the $(i–k)$th smallest element in the upper part

SOLVING THE RECURRENCE

$$T(n) = T(n/5) + T(7n/10) + \Theta(n)$$

Substitution:

$T(n) \le cn$

$$\begin{aligned} T(n) &\le cn/5 + 7cn/10 + \Theta(n) \\ &= 9cn/10 + \Theta(n) \\ &= cn - \left(cn/10 - \Theta(n)\right) \\ &\le cn \end{aligned}$$

,

if c is chosen large enough to handle both the $\Theta(n)$ and the initial conditions. $7n/10$

CONCLUSIONS

- Since the work at each level of recursion is a constant fraction (9/10) smaller, the work per level is a geometric series dominated by the linear work at the root.
- In practice, this algorithm runs slowly, because the constant in front of n is large.
- The randomized algorithm is far more practical.