Vectors, Lines & Planes

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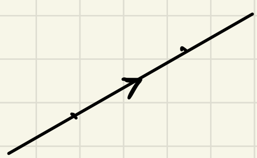
Notes

Basic Concepts of Vector

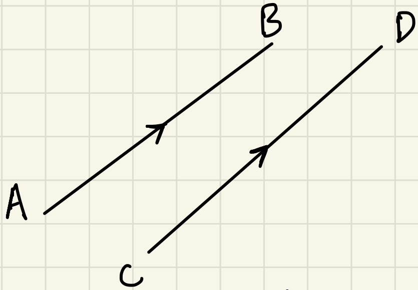
Two types of physical quantities
- Scalars: distance, work, mass, temperature, energy
- Vectors: displacement, force, acceleration, vol of petrol tank

Scalars → described by just a number which can be +ve, -ve or even 0

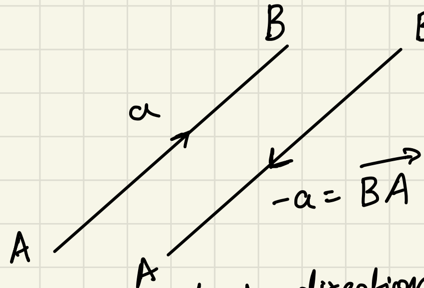
Vectors → They require a magnitude and direction which can be easily illustrated through a line.



scale: —— $= 1\,ms^{-1}$



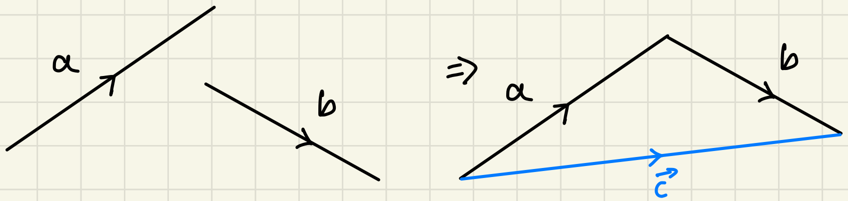
Equal vectors



defined only by direction and magnitude and not the space

Vector addition

join the head of a vector to the tail of the other and complete the end points by forming a triangle. The 3rd line goes from the tail of the initial 'head' vector to the tip of the other 'tailed' vector



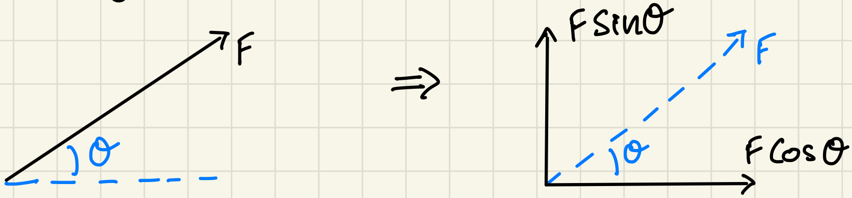
In vector addition

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

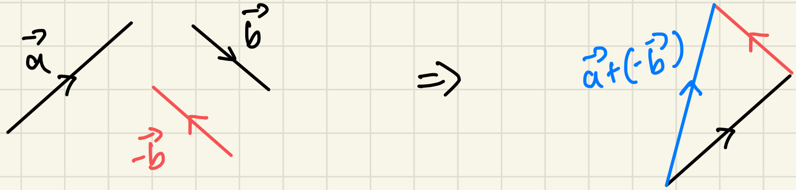
It is commutative and associative

Resolving Forces into two components:



Subtraction of vectors

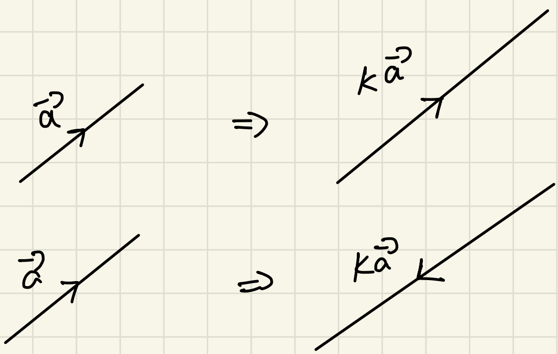
$\vec{a} - \vec{b} \Rightarrow \vec{a} + (-\vec{b})$



Multiplying Vectors:

usually multiplied with a +ve scalar

$\therefore k\vec{a}$



if k is +ve

if k is -ve

$$k(\vec{a}+\vec{b}) = k\vec{a} + k\vec{b}$$

$$(k+l)\vec{a} = k\vec{a} + l\vec{a}$$

$$k(l)\vec{a} = (kl)\vec{a}$$

Unit Vectors:

described as the vector that has magnitude 1.
if $\vec{a}$ has magnitude 3; unit vector in the direction of $\vec{a}$ is $\frac{1}{3}\vec{a}$

$$a = \xrightarrow{\quad\quad\quad a \quad\quad\quad}$$

$$\hat{a} = \overrightarrow{\tfrac{1}{3}\vec{a}}$$

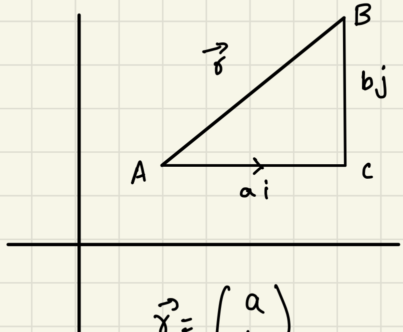
unit vector in direction of the vector $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

unit vector in the +ve direction of the

x-axis $\rightarrow$ $\hat{i}$

y-axis $\rightarrow$ $\hat{j}$

z-axis $\rightarrow$ $\hat{k}$



$$\vec{r} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\vec{r} = a\hat{i} + b\hat{j}$$

if we have two vector $\vec{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\vec{q} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$$\vec{p} + \vec{q} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

Position Vectors

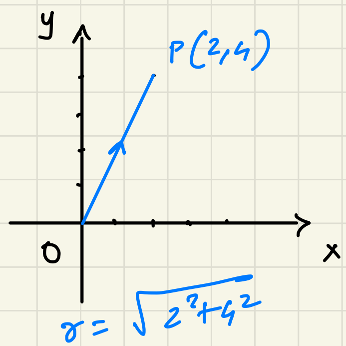
Representing a vector from the origin to that specific point $P(a, b)$ which are its coordinates represented as $a\hat{i} + b\hat{j}$ or $\begin{pmatrix} a \\ b \end{pmatrix}$

eg: $P(2,4) \Rightarrow 2\hat{i} + 4\hat{j}$ or $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

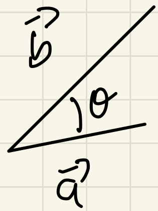
Since they start at origin:

their length can be defined as:

$$r = \sqrt{a^2 + b^2}$$



Scalar Product (Dot Product)



$$a \cdot b = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\hat{i} \cdot \hat{i} = |\hat{i}|\,|\hat{i}| \cos 0 = 1$$

$$\hat{i} \cdot \hat{j} = |\hat{i}|\,|\hat{j}| \cos 90 = 0$$

$i \cdot i = 1$ $\quad$ $i \cdot j = 0$

$j \cdot j = 1$ $\quad$ $j \cdot k = 0$

$k \cdot k = 1$ $\quad$ $k \cdot i = 0$

scalar product of perpendicular vectors are 0

if $\vec{a} = a_1 \hat{i} + b_1 \hat{j}$ ie $\vec{a} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$

& $\vec{b} = a_2 \hat{i} + b_2 \hat{j}$ ie $\vec{b} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

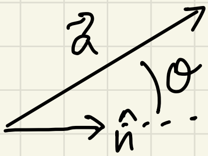
$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$= a_1 a_2 + b_1 b_2$$

result is a scalar.

For any vector $\vec{r}$, $|\vec{r}|^2 = r \cdot r$

Scalar projection of one vector along the other:



scalar projection

$= a.\hat{n} = |\vec{a}||\hat{n}|\cos\theta$

$= \vec{a}\cos\theta \quad (\hat{n} = 1)$

example point A on the plane with a normal vector ⊥'r to plane ; the ⊥'r distance l from the origin to the plane is simply the projection of $\vec{OA} = \vec{a}$ vector on to the unit vector $\hat{n}$ ie ⊥'r to the plane.

ie the scalar product $\vec{a}\cdot\hat{n}$

Scalar Proj of b on a: $\text{comp}_a b = \dfrac{a.b}{|a|}$

Vector Proj of b on a: $\left(\dfrac{a.b}{|a|}\right)\cdot\dfrac{a}{|a|}$

Find the angle b/w vectors:

$a = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} \qquad b = \begin{pmatrix} 8 \\ -9 \\ 11 \end{pmatrix}$

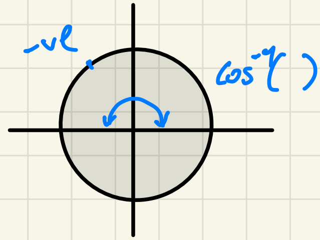
$a \cdot b = |a||b|\cos\theta$

$\cos\theta = \dfrac{a.b}{|a||b|}$

$$a \cdot b = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} \begin{pmatrix} 8 \\ -9 \\ 11 \end{pmatrix} = 40 + (-27) + (-22)$$
$$= -9$$

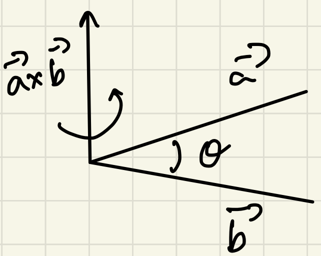
$$|a| = \sqrt{25 + 9 + 4} = \sqrt{38}$$
$$|b| = \sqrt{64 + 81 + 121} = \sqrt{266}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{38}\sqrt{266}} \right)$$



$$= 90^\circ + 5.14^\circ$$
$$= 95.14^\circ$$

Vector Product

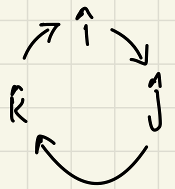


$$\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin\theta \, \hat{e}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

(goes in the opposite direction)

$$\hat{i} \times \hat{i} = 0$$
$$\hat{j} \times \hat{j} = 0$$
$$\hat{k} \times \hat{k} = 0$$



$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$
$$\hat{j} \times \hat{i} = -\hat{k}$$

in anticlockwise signs would be -ve

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\hat{i} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

eg:

$$\vec{p} = 3\hat{i} + 5\hat{j} \qquad \vec{q} = 2\hat{i} - \hat{j}$$

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 0 \\ 2 & -1 & 0 \end{vmatrix}$$

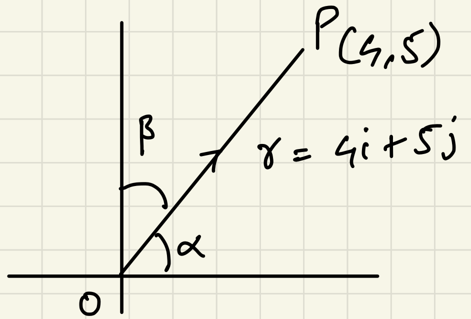
k- components are zero as they are not defined.

$$= \hat{i} \begin{vmatrix} 5 & 0 \\ -1 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} + k \begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix}$$

$= 0\hat{i} - \hat{j}\,0 + \hat{k}\,|-3-10|$

$= -13\hat{k}$

The direction ratio and direction cosines



direction ratio:

$4:5$

direction cosine:

$l;\ \cos\alpha = \dfrac{4}{\sqrt{41}}$

$m;\ \cos\beta = \dfrac{5}{\sqrt{41}}$

For any vector $r = a\hat{i} + b\hat{j} + c\hat{k}$; its direction ratio is $a:b:c$

direction cosines are:

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} \qquad m = \frac{b}{\sqrt{a^2+b^2+c^2}} \qquad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

and $l^2 + m^2 + n^2 = 1$

A line is inclined at 60° to the x-axis 45° to the y axis. Find the inclination to the z axis:

$$l^2 + m^2 + n^2 = 1$$

$$\cos^2 60 + \cos^2 45 + \cos^2(\gamma) = 1$$

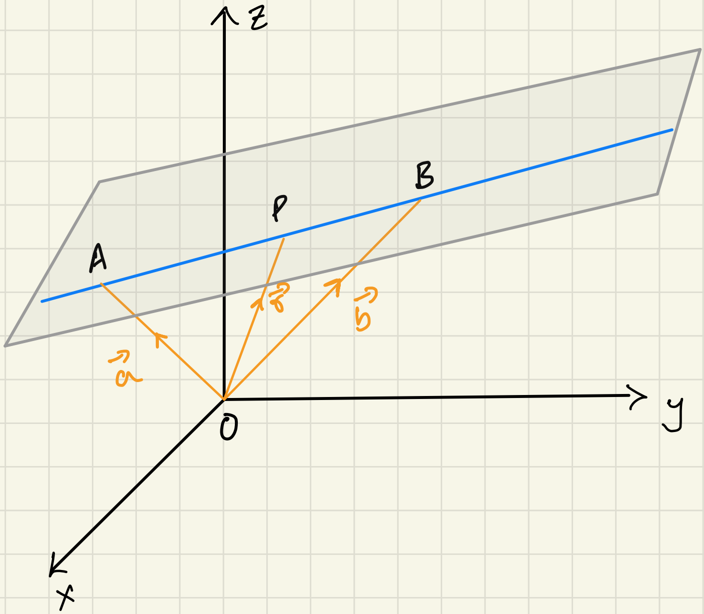
$$\frac{1}{4} + \frac{2}{4} + \cos^2(\gamma) = 1$$

$$\cos^2(\gamma) = 1 - \frac{3}{4}$$

$$\cos^2(\gamma) = \frac{1}{4}$$

$$\cos \gamma = \pm \frac{1}{2}$$

Vector Equation of a line



$\vec{OA} = \vec{a}$

$\vec{OP} = \vec{r}$

$\vec{OB} = \vec{b}$

$\vec{AP} = t\,\vec{AB}$

If we have a line APB on a 3-D plane we can derive the position vectors for the line $\vec{OA}$, $\vec{OP}$ and $\vec{OB}$ and we can also derive that $\vec{AP}$ is a scalar multiple of $\vec{AB}$

$\therefore$ expression for $\vec{AB}$ in terms of $\vec{a}$ & $\vec{b}$

$$\vec{AB} = (-\vec{a}) + \vec{b}$$
$$= \vec{b} - \vec{a}$$

expression for r in terms of $\vec{a}, \vec{b}$ & t

$$\vec{r} = \vec{OP}$$

$$\vec{r} = \vec{OA} + \vec{AP}$$

$$= \vec{a} + t\,\vec{AB}$$

$$\boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a})}$$

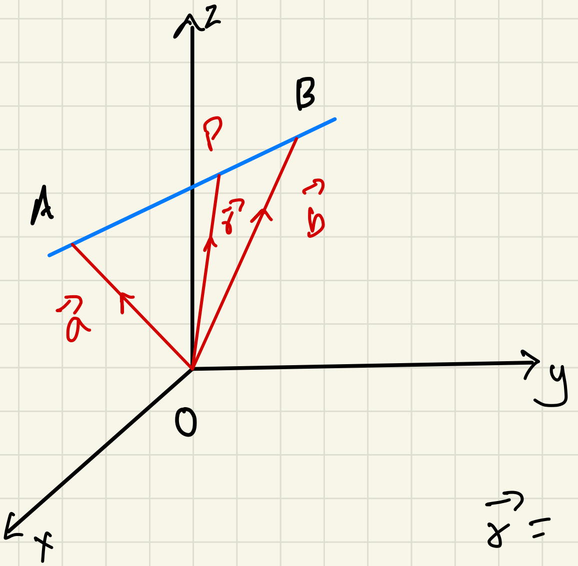
This is the vector equation of a line through A and B. It gives the position vector $\vec{r}$; a point on the line in terms of given vector $\vec{a}$ and $\vec{b}$

t is the parameter which we can vary to move the point through the line.

ie when $t = 0$; $\vec{r} = \vec{a}$ locates point A

when $t = 1$; $\vec{r} = \vec{b}$ locates point B

Write down the vector equation of the line which passes through the points with position vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} + 5\underset{\sim}{j}$. Also express the equation in column vector form.



$$\vec{r} = \vec{OA} + \vec{AP}$$
$$= \vec{a} + t\,\vec{AB}$$
$$= \vec{a} + t(\vec{b} - \vec{a})$$

$$\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$\vec{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\vec{r} = (3\hat{i} + 2\hat{j}) + t(4\hat{i} + 3\hat{j})$$

$\vec{OP}$ $\begin{pmatrix} x \\ y \end{pmatrix}$

Cartesian Form

On occassions it is useful to convert the vector form to cartesian. that means we split the $\vec{r}$ vectors position vectors 3-D points into 3 equations for simplicity & relate them all together with the common parameter t.

ie we have $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

and $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ the coordinates of the position vector

and $\vec{a}$ & $\vec{b}$ Position vectors has its coordinates $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ & $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\therefore \quad \vec{r} = \vec{a} + t(\vec{b} - \vec{a})$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$$\therefore \quad t = \frac{x - a_1}{b_1 - a_1}$$

$$t = \frac{y - a_2}{b_2 - a_2}$$

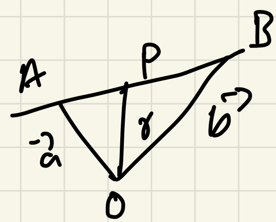
$$t = \frac{z - a_3}{b_3 - a_3}$$

$$\therefore \quad \frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_1} = \frac{z-a_3}{b_3-a_3}$$

(a) Write down the Cartesian form of the equation of the straight line which passes through the two points $(9, 3, -2)$ and $(4, 5, -1)$.

(b) State the equivalent vector equation.

$$\vec{a} = \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$$



$$r = \vec{a} + t(\vec{b} - \vec{a})$$

$$r = OA + AB$$

$$r = \vec{a} + t(b - a)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$$

$$x = 9 - 5t$$

$$y = 3 + 2t$$

$$z = -2 + t$$

$$\frac{x-9}{-5} = \frac{y-3}{+2} = \frac{z+2}{+1}$$

Cartesian form.

Distance of point to the line

→ Write line in vector form.

$$r = a + t(b-a)$$

→ $(b-a)$ is the direction vector

→ we have a point given ⊥r to the direction velor

→ write vector form for that point by finding its position vector $OP = \vec{P} - \vec{O}$

we have $l_2 = \vec{OP} + t_2\begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

→ Since P is ⊥r to initial line's direction vector

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \cdot \begin{pmatrix} b-a \end{pmatrix} = 0$$

→ equate them to find a_2, b_2 & rewrite the vector form of the point line. l_2.

Finding intersection of two lines

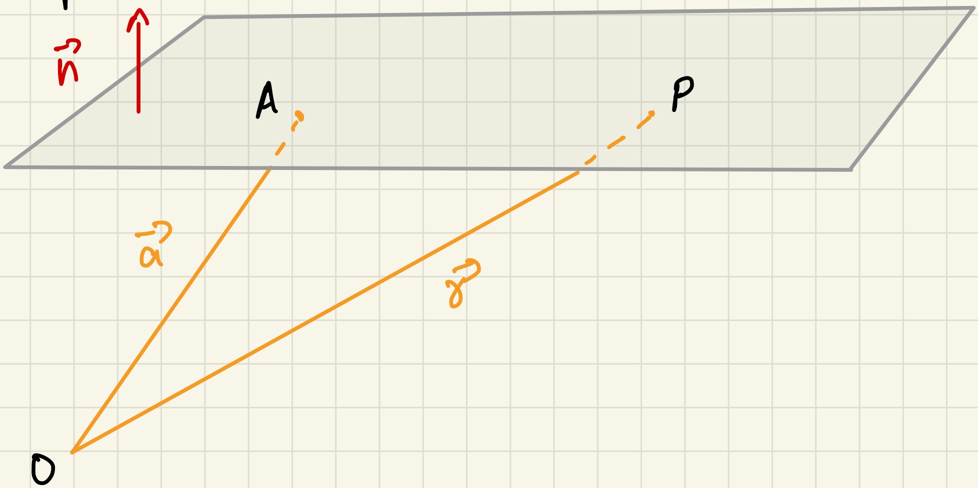
→ set them equal (vector form)

→ get two equation in t, r params

→ solve the SLE for t, r

→ substitute t back in equs to get x, y to find the intersection points.

Equation of a Plane



The vector equation of AP in terms of $\vec{a}$ & $\vec{r}$:

$$\vec{AP} = (-\vec{a}) + \vec{r}$$

$$\vec{AP} = \vec{r} - \vec{a}$$

Relation b/w $\vec{AP}$ and normal vector $\vec{n}$:

They both are perpendicular; $\therefore$ their dot product will be equal to 0.

ie $\quad \vec{AP} \cdot \vec{n} = 0$

$$(\vec{r} - \vec{a})\,\vec{n} = 0$$

$$\boxed{\vec{n}\,\vec{r} = \vec{n}\,\vec{a}}$$ Equation of the plane

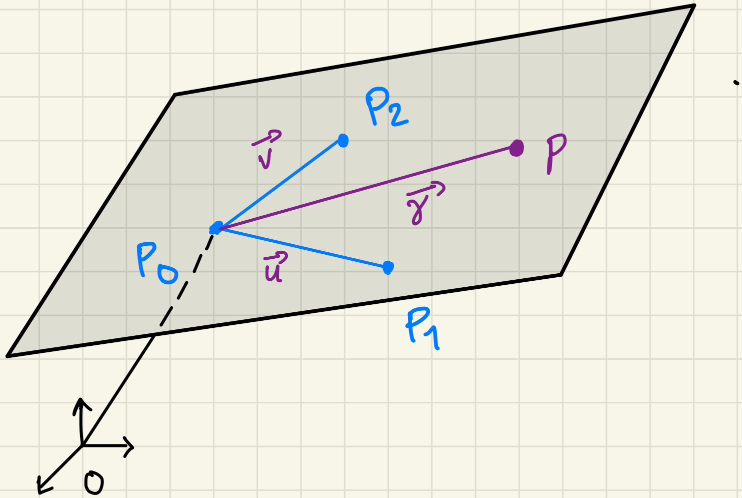
when $\vec{n}$ is a unit vector; $\hat{n}\vec{a}$ denotes the perpendicular distance of plane from the origin; denoted by d

$\therefore$ when $\vec{n} \Rightarrow \hat{n}.a = d$

$$\therefore \quad \boxed{\vec{n}\,\vec{r} = d}$$

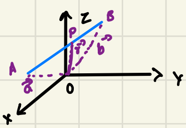
where d is the $\perp$r distance d from Origin.

Planes :



In vector form of a line we know:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$



In a plane we have 3 points and a reference point P ; which would be the $\vec{r}$ here when connected to one of the initial points. this would end up as the main vector for the planar map to the point P .

Thus we need to formulate a relation with P_0 and r . For this we use other two points P_1 & P_2 to form two directional vectors which relates to P_0

direction vectors from:

P_0 to P_1 : $\vec{u}$ which is $\overrightarrow{P_0P_1} = \vec{P_1} - \vec{P_0}$

P_0 to P_2 : $\vec{v}$ which is $\overrightarrow{P_0P_2} = \vec{P_2} - \vec{P_0}$

In equation of line we only had one directional vector ∴ one parameter λ
Here we have 2 ∴ we introduce params: t, s

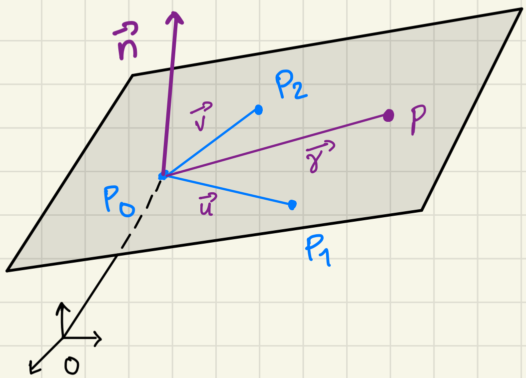
∴ Parametric form of a plane is:

$$\vec{r} = \vec{P_0} + t\vec{u} + s\vec{v} \; ; \quad s, t \in R$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{P_0} + t\vec{u} + s\vec{v} \; ; \quad s, t \in R}$$

Normal form of plane :

Here we introduce a normal vector by utilizing the cross product of the two directional vectors
∴ This normal vector is ⊥ to the planar mapping vector $\vec{r}$ to the point P.



Thus we have:

$$\vec{n} = \vec{u} \times \vec{v}$$

and $\vec{n} \cdot \vec{r} = 0$

$$\vec{n}\,(\vec{P} - \vec{P_0}) = 0$$ here $\vec{P}$ is arbitrary points

pos. vector: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\boxed{\therefore \quad \vec{n} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \vec{P_0} \right) = 0}$$

Co ordinat form of a Plane:

Opening up the brackets fro the vector form; we have

$$\vec{n}\,(\vec{P} - \vec{P_0}) = 0$$

$$\vec{n}\,\vec{P} - \vec{n}\,P_0 = 0$$

If we denote normal vector as $\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$; the dot product $\vec{n}\,P_0$ will return a constant value 'd'

$$\therefore \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} - d = 0$$

$$\boxed{ax + by + cz = d}$$

Changing Forms:

Coordinate to Parametric:

we will have the form $ax + by + cz = d$

Apply SLE with pivots and introduce parameters t, s & group equations in terms of x, y & z to form vector eqn.

ie $y = s \approx 1s$

$z = t \approx 1t$

Example:

$ax = d - by + cz$

$$x = \frac{d - bs + ct}{a} = \frac{d}{a} - \frac{b}{a}s + \frac{c}{a}t$$

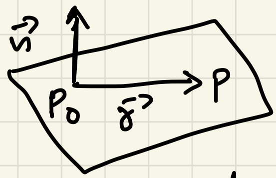
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{d}{a} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} \frac{c}{a} \\ 0 \\ 1 \end{pmatrix} t$$

Parametric to Normal:

in parametric form we have the form

$$r = \vec{P}_0 + s\vec{u} + t\vec{v}$$

To find normal vector find cross product b/w $\vec{u}$ & $\vec{v}$



& then $\vec{r} \cdot \vec{n} = 0$

where $r = (P - P_0)$; P being arbitrary $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\therefore \quad \vec{u} \times \vec{v} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P_0 \right) = 0$$

$$= \quad \vec{n} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - P_0 \right) = 0$$

Normal to Coordinate:

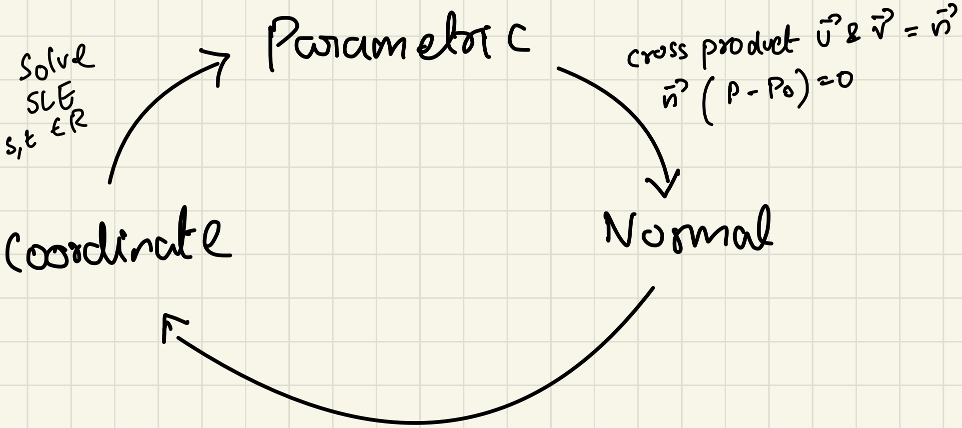
open up brackets

$$\vec{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \therefore \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} - d = 0$$

$$ax + by + cz = 0$$

∴ Parameter to Coordinate

→ First turn to normal form & then turn normal to coordinate form.



open up brackets $\vec{n}(\vec{p}-\vec{p_0})=0$

group $\underbrace{\vec{n}\vec{p}}_{\begin{pmatrix}a\\b\\c\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}\text{(arb)}} \cdot \underbrace{\vec{n}\vec{p_0}}_{\text{scalar}}$

Finding the intersection of two planes.

Get the two planes in coordinate form

solve the SLE

look at the pivots:

if just 1 pivot in whole system its a plane; introduce s, t params and write the param form eqn.

if just 2 pivots on LHS; its a line; introduce 1 parameter λ or t for z & write the other eqns in terms of λ for x & y. & write vector form eqn of line.

if pivot appears on rhs; the system is not solvable.

Find the height of a point above a plane.

$Q(1,0,0)$ $\quad 10x + 2y - z = 5$

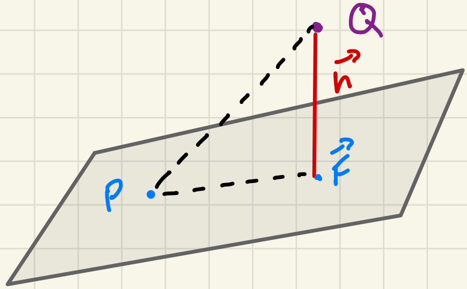
we have normal vector $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ -1 \end{pmatrix}$

We need to find a point on the plane $\therefore$ we can set $x, y = 0$ & find the z intercept here which is easier to solve

$$0 + 0 - z = 5$$
$$z = -5$$

$\therefore$ P; a point is; z intercept $= \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$

given point is Q & it is above the plane. which is essentially making a normal vector to the plane $\vec{n}$ which we alredy have from the coordinate form eqn.



$\therefore$ The distance is essentially the vector projection of $\vec{PQ}$ on $\vec{n}$

Homework

1. Name five physical/abstract quantities that needs to be expressed as vectors.[for example: velocity and Force are vectors. Temperature or mass of an object is not a vector. Why not?]
2. What does the 'Direction' of a vector mean?[slide 03a page 2]
3. Imagine you are hired to create a database for Edeka where you want to list the price of all different items. There are 20 different ice cream products. Suddenly, one day the main freezer stops working. your boss wants you to change the prices of all the ice creams by 50 percent. How could vectors be useful in this scenario? [hint: vectors(tensors) are a mathematical way of organizing data]

1) scalars:
volume of a petrol tank
distance measured in m
work done by a force
temperature
pressure

vectors:
angular velocity
velocity
acceleration
force
momentum of a particle

2) Represents the direction in which the magnitude of the vector is acting based on the coordinate system.

3) Assuming the complete prices are in a data matrix $\begin{bmatrix} a \\ b \\ \vdots \end{bmatrix}$; we multiply the dataset by 0.5 to generate the new prices.

4. Add the following vectors algebraically:

   (a) [-4,3] + [1,-2]

   (b) [2,3,0] + [-3,0] [It is not a typo]

   (c) [-4,-3,8,9] + [4,3,-9,-8]

5. Graphically add the following pairs of vectors:

   (a) [4,3] and [-1,2]

   (b) [0,3] and [-3,0]

   (c) [-4,-3] and [4,3]

6. Graphically subtract the second vector from the first vector:

   (a) [4,3]-[-1,2]

   (b) [0,3]-[-3,0]

   (c) [-4,-3]-[4,3]

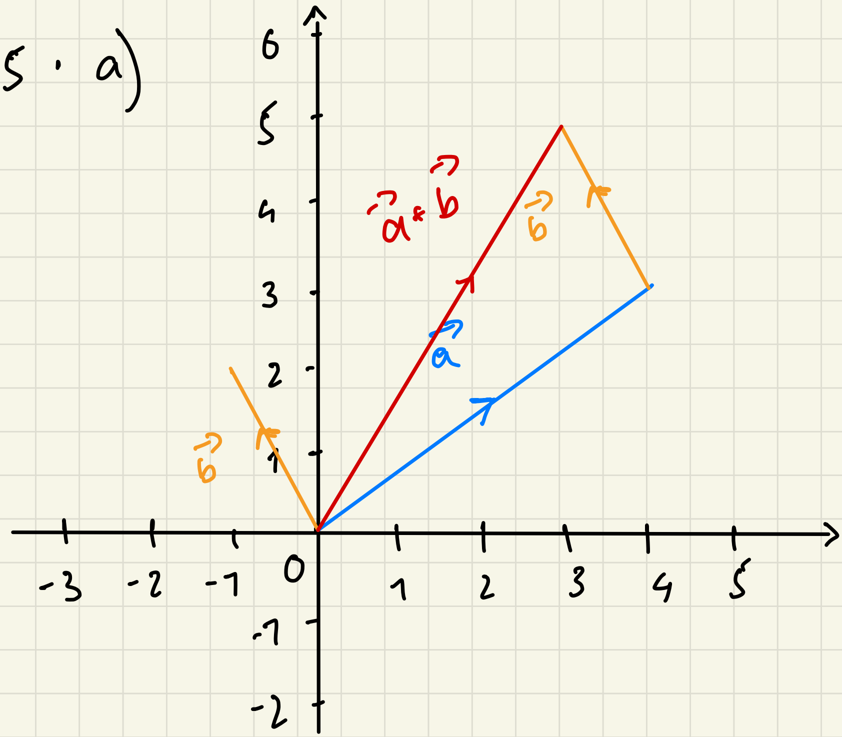
7. Find the angles the following vectors make with their basis vectors:[hint: draw a diagram for each vector. Also, slide 03a page 12]

4.a) $\begin{pmatrix} -4 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

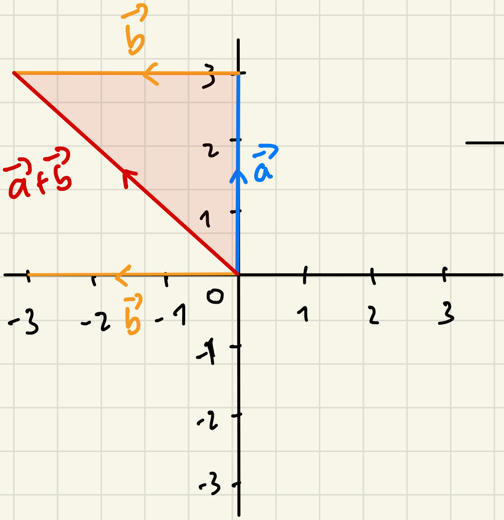
b) $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$

c) $\begin{pmatrix} -4 \\ -3 \\ 8 \\ 9 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \\ -9 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

5. a)

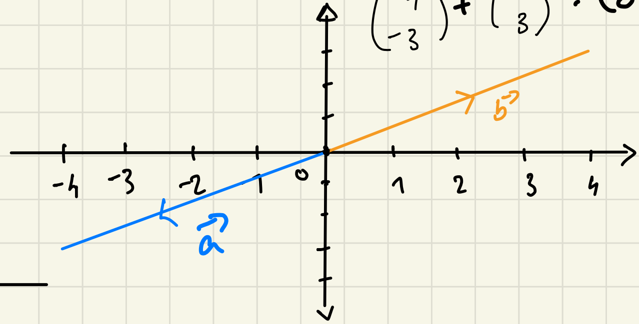


b) $\begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}$

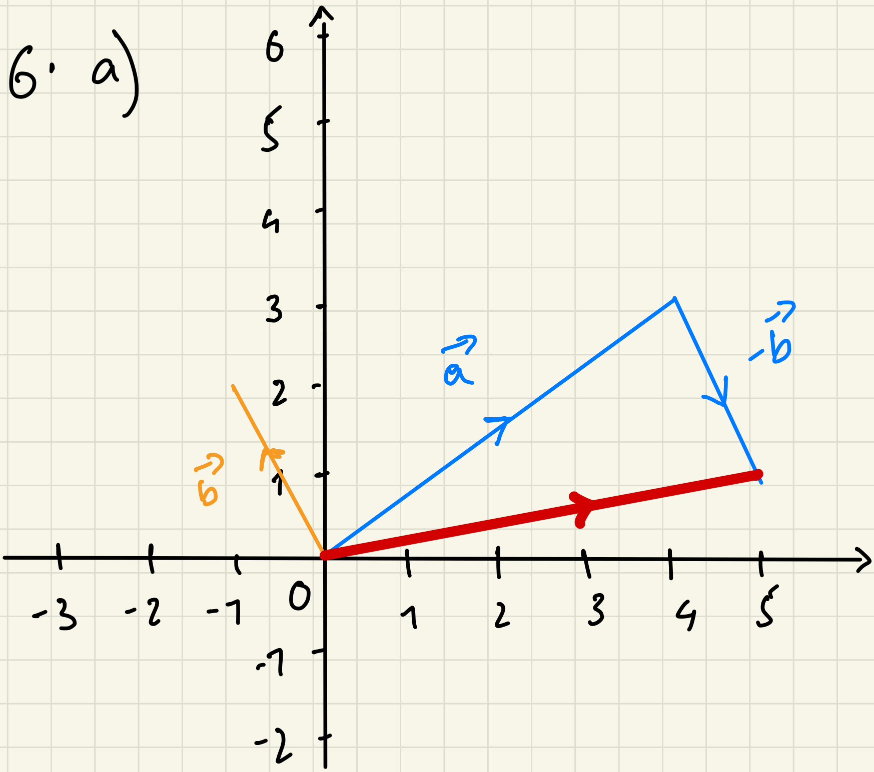


c)

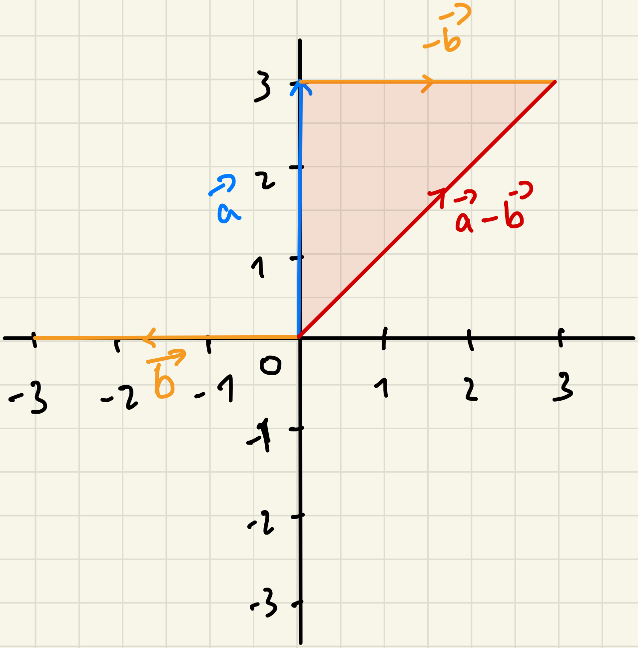
$\begin{pmatrix} -4 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



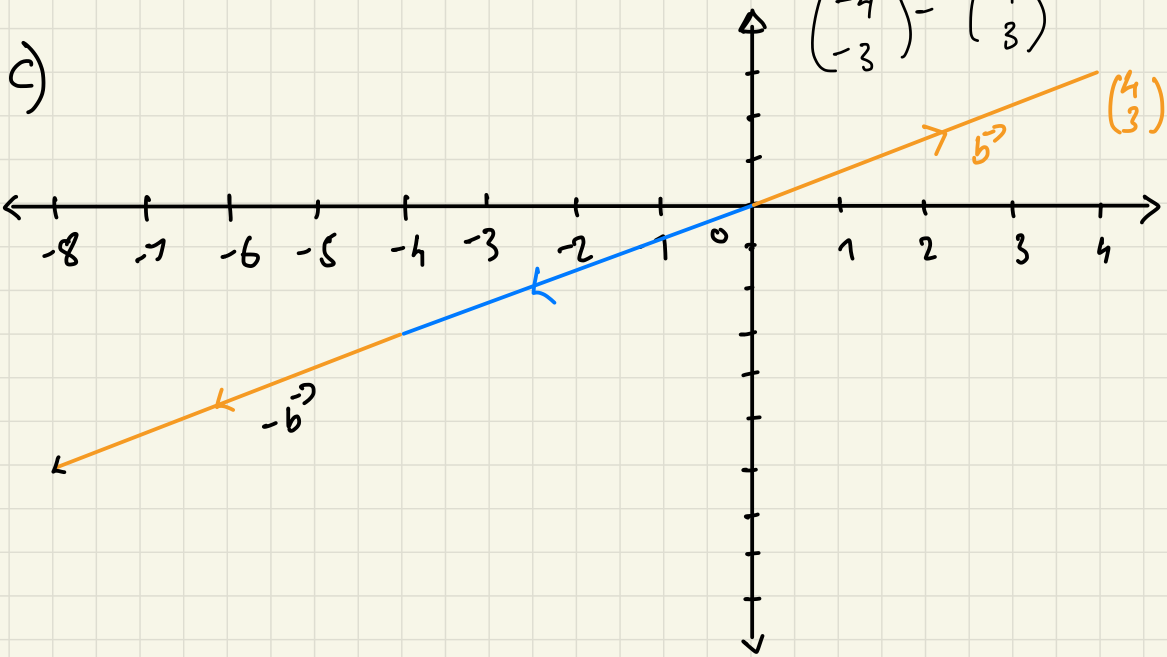
6. a)



b) $\begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix}$



c) $\begin{pmatrix} -4 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



7. Find the angles the following vectors make with their basis vectors:[hint: draw a diagram for each vector. Also, slide 03a page 12]

(a) [4,3]

(b) [0,3]

(c) [-4,-3]

8. Find the magnitude of the following vectors:

(a) [4,3]

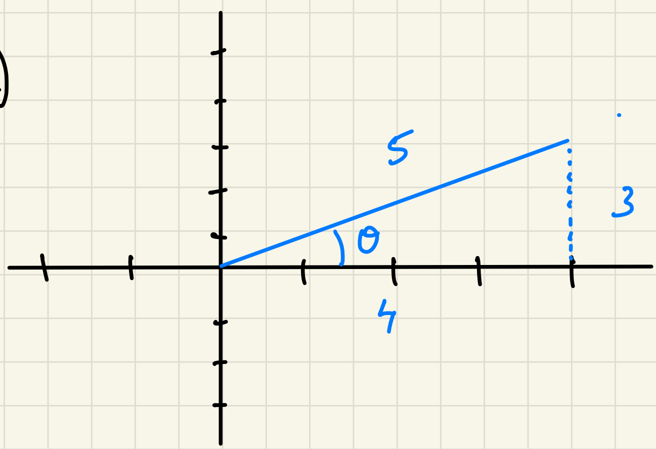
(b) [3,4]

(c) [3,4,5]

(d) [-4,-3,3,4,-4,-3]

9. Find the direction cosines of the vector [4,4,3] [hint: slide 03b page 8]

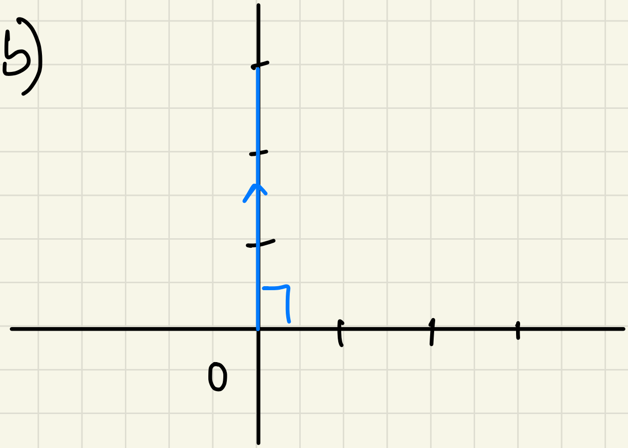
7) a)



$$\cos\theta = \frac{4}{5}$$

$$\theta = \cos^{-1}\left(\frac{4}{5}\right)$$

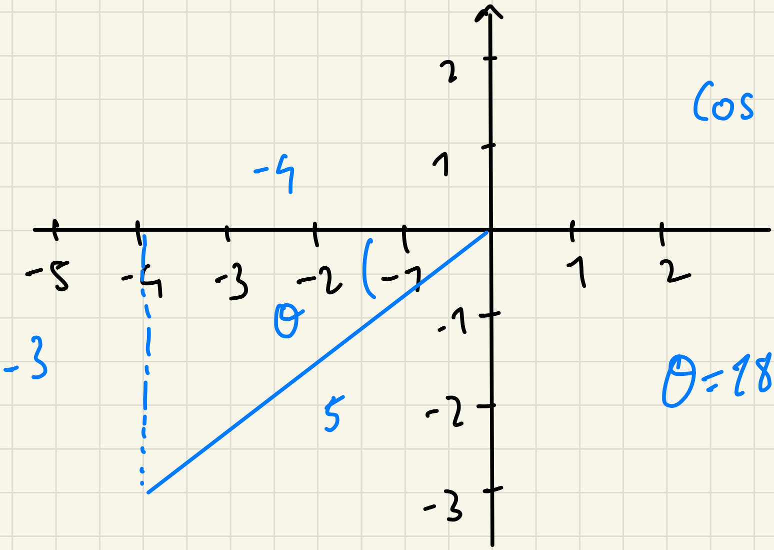
b)



vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

90°

c)



$\cos\theta = \frac{-4}{5}$

$\theta = \cos^{-1}\left(-\frac{4}{5}\right)$

$\theta = 180^\circ - \cos^{-1}\left(\frac{4}{5}\right)$

8) a) $\sqrt{4^2+3^2} = \sqrt{25} = 5$

b) $\sqrt{3^2+4^2} = \sqrt{25} = 5$

c) $\sqrt{3^2+4^2+5^2} = \sqrt{50}$

d) $\sqrt{(-4)^2+(-3)^2+(3)^2+(4)^2+(-4)^2+(-3)^2}$

$= \sqrt{16+9+9+16+16+9}$

$= \sqrt{48+27} = \sqrt{75}$

9) direction cosines $l = \cos\alpha = \dfrac{a}{\sqrt{a^2+b^2+c^2}}$

$m = \cos\beta = \dfrac{b}{\sqrt{a^2+b^2+c^2}}$

$n = \cos\gamma = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

$l = \dfrac{4}{\sqrt{16+16+9}} = \dfrac{4}{\sqrt{41}}$

$m = \dfrac{4}{\sqrt{44}}$

$n = \dfrac{4}{\sqrt{41}}$

1. Perform the following dot products

   (a) $[-2, 4, 6] \cdot [4, -2, -6]$

   (b) $(2i + 2j - 4k) \cdot (-4k + 2j)$

   (c) $[-a, 0, a, a^2] \cdot [b, b^3, b, 1/a^2]$

   (d) $(i - j + k) \cdot [-1, 1, -1]$

2. If I give you the coordinates of the vertices of a triangle, can you find out if it is a right-angle triangle? How does dot product help here? What about the area of the triangle?

1 a) $\begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ -6 \end{pmatrix}$

dot product or scalar product
result is scalar

$= (-2)(4) + (4)(-2) + (6)(-6)$

$= -8 - 8 - 36$

$= -14 - 36 = -50$

b) $\begin{pmatrix} 2\hat{i} \\ 2\hat{j} \\ -4\hat{k} \end{pmatrix} \begin{pmatrix} 0\hat{i} \\ 2\hat{j} \\ -4\hat{k} \end{pmatrix}$

$= 0 + 4 + 16 = 20$

c) $\begin{pmatrix} -a \\ 0 \\ a \\ a^2 \end{pmatrix} \begin{pmatrix} b \\ b^3 \\ b \\ 1/a^2 \end{pmatrix}$

$= -ab + 0 + ab + 1 = 1$

d) $$\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \begin{pmatrix} \hat{i} \\ -\hat{j} \\ \hat{k} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \begin{array}{c} \hat{i} \\ \hat{j} \\ \hat{k} \end{array} \quad ?$$

$$= \begin{pmatrix} 1\hat{i} \\ -1\hat{j} \\ 1\hat{k} \end{pmatrix} \begin{pmatrix} -1\hat{i} \\ 1\hat{j} \\ -1\hat{k} \end{pmatrix}$$

$$= -1 - 1 - 1 = -3$$

② We can find if its is a right triangle by calculating the dot product b/w the vectors.

By checking if for 3 pairs and dot product of 0 indicates $\perp$'s $\triangle$

3. perform the following cross products:

(a) $[4, 3, 0] \times [3, 2, 0]$

(b) $[-4, -8, -2] \times [2, 4, 1]$

(c) $[3, 2, 0] \times [4, 3, 0]$

(d) $(2i - 3j + 7k) \times [3, -1, 2]$

4. Find the result of $[1, 2, 3] \times [4, 5, 6]$ and prove that both vectors are orthogonal to the result vector.

5. Prove that [1.4.-7],[2,-1,4] and [0,-9,18] are coplanar.

a) $$\begin{vmatrix} \hat{i} & \hat{j} & k \\ 4 & 3 & 0 \\ 3 & 2 & 0 \end{vmatrix}$$

$$\hat{i}\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix}$$

$$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(8-9)$$

$$= -\hat{k}$$

b) $\begin{vmatrix} \hat{i} & \hat{j} & k \\ -4 & -8 & -2 \\ 2 & 4 & 1 \end{vmatrix}$

$= \hat{i} \begin{vmatrix} -8 & -2 \\ 4 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} -4 & -2 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} -4 & -8 \\ 2 & 4 \end{vmatrix}$

$= \hat{i}(-8+8) - \hat{j}(-4+4) + \hat{k}(-16+16)$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= 0$

c) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 4 & 3 & 0 \end{vmatrix}$

$\hat{i} \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & 0 \\ 4 & 0 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}$

$\hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(9-8)$

$= 1\hat{k}$

d) $$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 2 & -3 & 7 \\ 3 & -1 & 2 \end{vmatrix}$$

$$\hat{\imath}\begin{vmatrix} -3 & 7 \\ -1 & 2 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 7 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -3 \\ 3 & -1 \end{vmatrix}$$

$$= \hat{\imath}(-6+7) - \hat{\jmath}(4-21) + \hat{k}(-2+9)$$

$$= \hat{\imath} + 17\hat{\jmath} + 7\hat{k}$$

5 $$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$\hat{\imath}\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= \hat{\imath}(12-15) - \hat{\jmath}(6-12) + \hat{k}(5-8)$$

$$= -3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}$$

To check if they are orthogonal find dot products b/w original vectors & normal vector

$$= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = -3 + 12 - 9$$

$$= 0$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix} = -12 + 30 - 18$$

$$= 0$$

$\therefore$ the vector is orthogonal.

5. To prove they are coplanar take the scalar triple product

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix}$$

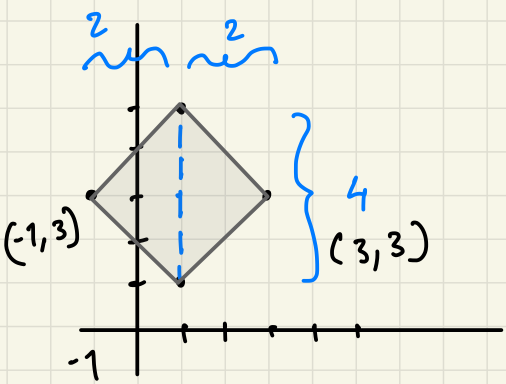
$$\left|\begin{array}{ccc|cc} 1 & 4 & -7 & 1 & 4 \\ 2 & -1 & 4 & 2 & -1 \\ 0 & -9 & 18 & 0 & -9 \end{array}\right|$$

$$\left[-18 + 0 + (14)(9)\right] - \left[0 - 36 + (36)4\right]$$

$$\left[126 - 18\right] - \left[154 - 36\right]$$

$$= 108 - 108 = 0$$

1. Let's say you are given 2 opposing points of square i.e. the coordinates of the diagonal's vertices on a Cartesian plane. Assume they are [1,1] and [1,5]. What are the other 2 points of the square? Hint: use what you know about dot products and the distance between 2 points in a Cartesian plane. This problem is to demonstrate how to do geometry with algebra.



$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{matrix} 0 \\ 4 \end{matrix}$$

$$\text{distance} = \sqrt{0^2 + 4^2} = 4$$

From the mid point of line we take two steps to the right & left to get the coordinates

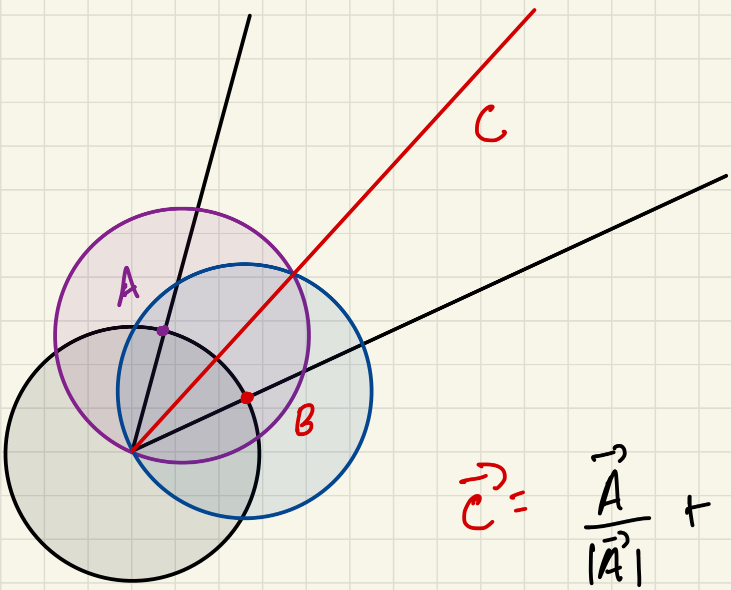
coordinates are : $(-1, 3)$ & $(3, 3)$

2. Find the vector that has a length of 5 and bisects the angle between the vectors $i + j$ and $i + k$

3. Find the unit vector(s)(?) orthogonal to both [2,0,-3] and [-1,4,2]

② Angle b/w 2 vectors

The vector that bisects the angle b/w two vectors is the sum of the two vectors unit vectors



$$\vec{C} = \frac{\vec{A}}{|\vec{A}|} + \frac{\vec{B}}{|\vec{B}|}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1\,\hat{i} \\ 1\,\hat{j} \\ 0\,k \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1\,\hat{i} \\ 0\,\hat{j} \\ 1\,k \end{pmatrix} = \frac{2}{\sqrt{2}}\begin{pmatrix} 2\,\hat{i} \\ 1\,\hat{j} \\ 1\,\hat{k} \end{pmatrix}$$

$\hat{c}$, bisecting unit vector $= \sqrt{2}\begin{pmatrix} 2\,\hat{i} \\ 1\,\hat{j} \\ 1\,\hat{k} \end{pmatrix}$

$\therefore \quad \hat{c} = \dfrac{\vec{c}}{|c|}$; given $|c| = 5$

$$\therefore \quad \vec{c} = \hat{c} \cdot |c|$$

$$= 5 \cdot \sqrt{2}\begin{pmatrix} 2\,\hat{i} \\ 1\,\hat{j} \\ 1\,k \end{pmatrix}$$

$$= \begin{pmatrix} 10\sqrt{2}\,\hat{i} \\ 5\sqrt{2}\,\hat{j} \\ 5\sqrt{2}\,k \end{pmatrix}$$

③ $\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ -1 & 4 & 2 \end{bmatrix}$

$$\hat{i} \begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix}$$

$$= \hat{i}\,(0 + 12) - \hat{j}\,(4 + 3) + \hat{k}\,(8 + 0)$$

$$= 12\hat{i} - 7\hat{j} + 8\hat{k}$$

$$= \begin{pmatrix} 12 \\ -7 \\ 8 \end{pmatrix}$$

4. You are given 2 vectors **a** and **b** where you don't know the exact component values for neither of the vectors. After tinkering with them a bit, you see that $a \times b = 0$ and $a \cdot b = 0$. What can you say about the component elements of these two vectors?

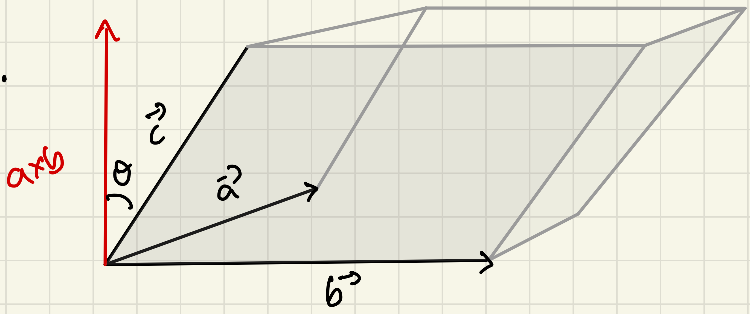
5. Look up **"scalar triple product"** and how it gives the volume of a **'parallelepiped'**

6. Let's say **a,b and c** are 3 non-zero vectors. Given $a \times b = a \times c$ **and** $a \cdot b = a \cdot c$, is it true that $\mathbf{b} = \mathbf{c}$?

4. given $a \cdot b = 0$ ie $|a||b| \cos\theta = 0$
θ has to be $\frac{\pi}{2}$ for the dot product to be zero

$a \times b$ cross product becomes zero when the vectors lie along same or opposite direction ie $|a \times b| = ab \sin\theta$
when $\theta = 0$; $|a \times b| = 0$
If both cases happen at the same time; that means it's a null vector

5.



Scalar triple product $(a \times b) \cdot c =$ volume of parallelepiped.

6. given $a \cdot b = a \cdot c$

$a \cdot b - a \cdot c = 0$

$a(b-c) = 0$

$\vec{b} = \vec{c}$

$\underbrace{a \times b = a \times c}$

normal vector from cross product is same in same direction.

$\vec{r} = \vec{a} + \vec{AP}$

$\vec{r} = \vec{a} + t\, AB$

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$\alpha x + \beta y = \gamma$

$\beta y = \gamma - \alpha x$

$y = \frac{\gamma}{\beta}$ $\quad r$

$\alpha x + \beta y = \gamma$

$a = \vec{p} + \lambda u$

$-x + 2y = 2$

$$\begin{array}{cc|c} x & y & \\ -1 & 2 & 2 \end{array}$$

$x = t$

$-1t + 2y = 2$

$2y = t + 2$

$y = \frac{1}{2}t + 1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Examples:

a) Find the plane through the point $P = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ with normal vector $\vec{n} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

b) Find the intercept of the three axes:

a) we know

$$\vec{r} = P_0 + s\vec{u} + t\vec{v}$$

$$(\vec{u} \times \vec{v}) \cdot \vec{r} = 0 \quad \& \quad \vec{r} = (\vec{P} - \vec{P_0})$$

$$\therefore \quad \vec{n} \cdot (\vec{P} - \vec{P_0}) = 0 \quad \text{where } P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ arb;}$$

we have here: $\vec{n} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$$P_0 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \right) = 0$$

This is the normal form

Coordinate form:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \begin{pmatrix} x-2 \\ y-4 \\ z+1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2x-4 \\ 3y-12 \\ 4z+4 \end{pmatrix} \Rightarrow$$

$2x - 4 + 3y - 12 + 4z + 4$

$2x + 3y + 4z - 12 = 0$

$2x + 3y + 4z = 12$

This is the coordinate form.

$$\begin{array}{ccc|c} x & y & z & \\ 2 & 3 & 4 & 12 \end{array}$$

Let $y, z = s, t \in R$

$2x + 3s + 4t = 12$

$2x = 12 - 3s - 4t$

$x = \frac{12}{2} - \frac{3}{2}s - \frac{4}{2}t$

$y = 0 + 1s + 0$

$z = 0 + 0 + 1t$

param form:
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12/2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -4/2 \\ 0 \\ 1 \end{pmatrix} t$$

b) Finding the intercepts.

→ Use the coordinat form.

$ax + by + cz = d$

→ In line, to find y intercept we set $x = 0$
Similarly to find intercepts in plane:

to find x intercept set $y, z = 0$

to find y intercept set $x, z = 0$

to find z intercept set $x, y = 0$

we have $2x + 3y + 4z = 12$

For x intercept; $y, z = 0$

$$2x + 0 + 0 = 12$$
$$x = 6$$

x intercept $= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

For y intercept; $x, z = 0$

$$0 + 3y + 0 = 12$$
$$y = 4$$

y intercept $= \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

For z intercept; $x, y = 0$

$$0 + 0 + 4z = 12$$
$$z = 3$$

z intercept $= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

Find the intersection of planes:

$$x + y + z = 1$$

$$x - 2y + 3z = 1$$

$$\begin{array}{ccc|c} x & y & z & \\ 1 & 1 & 1 & 1 \\ 1 & -2 & 3 & 1 \end{array} \quad II - \frac{1}{1} I$$

$$\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & 0 \end{array}$$

$\underline{II} - \underline{I}$

$1 - 1 = 0$

$-2 - 1 = -3$

$3 - 1 = 2$

$1 - 1 = 0$

Two pivots

$\therefore$ it is a line

introduce 1 parameter.

$$z = \lambda$$

$$-3y + 2z = 0$$

$$3y = 2\lambda$$

$$y = \frac{2\lambda}{3}$$

$$x + y + z = 1$$

$$x + \frac{2\lambda}{3} + \lambda = 1$$

$$x + \frac{5\lambda}{3} = 1$$

$$x = 1 - \frac{5}{3}\lambda$$

$$x = 1 + \frac{-5}{3}\lambda$$

$$y = 0 + \frac{2}{3}\lambda$$

$$z = 0 + \lambda$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{5}{3} \\ 2/3 \\ 1 \end{pmatrix} \lambda$$

$$\operatorname{Proj}_{\vec{n}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|} \cdot \frac{\vec{n}}{|\vec{n}|}$$

we have $n = \begin{pmatrix} 10 \\ 2 \\ -1 \end{pmatrix}$ $P = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$ $Q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$PQ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ +5 \end{pmatrix}$$

$$\operatorname{proj}_n PQ = \frac{\vec{PQ} \cdot \vec{n}}{|n|} \cdot \frac{\vec{n}}{|n|} \quad \text{where } \frac{\vec{n}}{|n|} = \vec{u}_n$$

$$PQ \cdot n = \begin{pmatrix} 1 \\ 0 \\ +5 \end{pmatrix} \begin{pmatrix} 10 \\ 2 \\ -1 \end{pmatrix} = 10 - 0 \overset{-}{+} 5 = \cancel{15}\ 10$$

$$|n| = \sqrt{100+4+1} = \sqrt{105}$$

$$\vec{QF} = \frac{10}{\sqrt{105}} \cdot \frac{\vec{n}}{|n|}$$

$$= \frac{10}{\sqrt{105}} \vec{u}_n$$

Task 3

For the following objects, check if they intersect. If they do, find the intersection. Is it a point, a line or a plane?

a) $l_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad$ and

$l_2 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} + r \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad r \in \mathbb{R}$

b) $l_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}, \quad$ and

$l_2 : 2x + 7y = 8$

a) $l_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$l_2 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} + r \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

for point of intersection; set $l_1 = l_2$
& solve SLE

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} + r \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$9 + 2t = -5 + 2r$$

$$2t - 2r = -5 - 9$$

$$2t - 2r = -14$$

$$4 - t = -5 + 3\gamma$$

$$4 + 5 = t + 3\gamma$$

$$1t + 3\gamma = 9$$

t	γ	
2	-2	-14
1	3	9
2	-2	-14
0	4	16

$\underline{\text{III}} - \frac{1}{2}\,\underline{\text{II}}$

$1 - \frac{1}{2}(2)$

$3 - \frac{1}{2}(-2)$

$9 - \frac{1}{2}(-14)$

$$4\gamma = 16$$

$$\boxed{\gamma = 4}$$

$$2t - 2(4) = -14$$

$$2t - 8 = -14$$

$$2t = -14 + 8$$

$$2t = -6$$

$$\boxed{t = -3}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$x = 9 + -6 = 3$$

$$y = 4 + 3 = 7$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix} + r\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$x = -5 + 4(2) = 3$$

$$y = -5 + 4(3) = 7$$

point of intersection $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

b) $l_1 : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$l_2 = 2x + 7y = 8$

For l_2: we have coordinate form which we have to convert to parametric form.

x	y	
2	7	8

set $x = \gamma$

$$2\gamma + 7y = 8$$

$$7y = -2\gamma + 8$$

$$y = -\frac{2}{7}\gamma + \frac{8}{7}$$

$$x = 0 + \gamma$$

$$y = -\frac{2}{7}\gamma + \frac{8}{7}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{8}{7} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{2}{7} \end{pmatrix}\gamma$$

$$\begin{pmatrix} 0 \\ \frac{8}{7} \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{2}{7} \end{pmatrix}\gamma = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}t$$

$$0 + 1t = 2 + 3\gamma$$

$$t - 3\gamma = 2 \rightarrow (1)$$

$$\frac{8}{7} - \frac{2}{7}\gamma = 1 - t$$

$$8 - 2\gamma = 7 - 7t$$

$$7t - 2\gamma = 7 - 8$$

$$7t - 2\gamma = -1 \longrightarrow 2$$

t	γ	
1	-3	2
7	-2	-1

$\mathrm{II} - \frac{7}{1}\mathrm{I}$

$\mathrm{II} - 7\,\mathrm{I}$

$$7 - 7(1) = 0 = 0$$

$$-2 - 7(-3) = -2 + 21 = 19$$

$$-1 - 7(2) = -1 - 14 = -15$$

t	γ	
1	-3	2
0	19	15

reduced solvable
one pivot
one unique solution

$$19r = 15$$

$$r = \frac{15}{19}$$

$$1t - 3 \cdot \frac{15}{19} = 2$$

$$19t - 45 = 2 \cdot 19$$

$$19t = 38 + 45$$

$$t = \frac{83}{19}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{83}{19} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$x = 2 + \frac{83 \cdot 3}{19} = \frac{38 + 249}{19}$$

$$= \frac{287}{19}$$