Curve Sketching

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Find the points where the function is crossing the zero level on the x axis by factorizing or dividing & define the domain.

Find where the function is monotonous and has extremas
For this find the derivative and equate to 0 to find intervals within the domain which shows areas of monotonicity

Also with the intervals create the sign table cross checking with the factorized derivatives for levels of increasing or decreasing slopes.

Now Find the maxima and minimas
For that find second derivative and create similar sign chart. but with critical values from the first derivative plugged into the second derivatives function and also find the values to its nearest left & right number eg if Critical number (CN) is 2 then $f''(1)$ and $f''(3)$

Given: $\begin{cases} f(x) = 3x^4 - 16x^3 + 18x^2 \\ \text{on } [-1, 4] \text{ find monotonicity \& extremas} \end{cases}$

Finding zero crossing is optional when just behavior is asked.

$$f'(x) = 12x^3 - 16\cdot(3)\,x^2 + 18\cdot(2)x = 0$$

$$= 12x^3 - 48x^2 + 36x = 0$$

$$= 12x(x^2 - 4x + 3) = 0$$

$$= 12x(x - 3x - x + 3) = 0$$

$$= 12x(x(x-3) - 1(x-3)) = 0$$

$$f'(x) = 12x(x-3)(x-1) = 0$$

$$x = 0 \quad x = 3 \quad x = 1$$

split domain using critical nos and find periods increasing or decreasing by using sign chart.

sign → +ve increasing
sign → -ve decreasing.

Intervals: $[-1, 0),\ (0, 1),\ (1, 3),\ (3, 4]$

plug in the CN's into 2nd derivative to find the global extremas.

sign → -ve → local max
sign → +ve → local min

slope: as +ve to -ve
+ve -ve

as -ve to +ve
-ve +ve

$$f''(x) = 12(3)x^2 - 48(2)x + 36$$

$$f''(x) = 36x^2 - 96x + 36$$

You dont have to factorize $f''(x)$ just plug CN into them to find local max & min.

Sign Chart: $f'(x)$ 1 SC1

interval	$12x$	$x-3$	$x-1$	sign	
$[-1, 0)$	−	−	−	−	decreasing
$(0, 1)$	+	−	−	+	increasing
$(1, 3)$	+	−	+	−	decrease
$(3, 4]$	+	+	+	+	increase

Sign Chart $f''(x)$ 2 SC2

CN's	$f''(x)$	
$x=0$	+	local min
$x=1$	−	local max
$x=3$	+	local min

Now create function value table with CN's and its neighbors as x values to find global max & min.

Function Value Chart (FVC)

within domain & only f(x) & f'(x)

x	f(x)	f'(x)
-1	37	-96
0	0	0
1	5	0
3	-27	0
4	32	144

global maximum

global minimum.

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

$$f(-1) = 3 + 16 + 18 = 37$$

$$f(0) = 0 - 0 + 0 = 0$$

$$f(1) = 3 - 16 + 18 = 5$$

$$f(3) = 3 - 16(27) + 18(9)$$
$$= 243 - 432 + 162$$
$$= 405 - 432 = -27$$

$$f(4) = 3(4)^4 - 16(4)^3 + 18(16)$$
$$= 768 - 1024 + 288$$
$$= 1056 - 1024$$
$$= 32$$

$27 \times 3 = 81$, $81 \times 3 = 243$, $243 + 162 = 405$

$27 \times 16 = 432$, $18 \times 9 = 162$

$16 \times 16 = 256$, $256 \times 3 = 768$, $18 \times 16 = 288$, $64 \times 16 = 1024$, $288 + 768 = 1056$

$f'(x) = 12x(x-3)(x-1)$

$f'(-1) = -12(-4)(-2) = -12(8) = -96$

$f'(0) = 0$

$f'(1) = 12(-2)(0) = 0$

$f'(3) = 36(0)(2) = 0$

$f'(4) = 48(1)(3) = 144$

Now we find the points where the curve inflects. ie we identify the intervals of concavity

Create a sign table with $f''(x)$ factors.

ie $f''(c) = \begin{cases} \text{defined everywhere} \\ x = \text{value from quadratic form.} \end{cases}$

Find different intervals for $f''(x)$ and create sign chart for those intervals.

$f''(x) = 36x^2 - 96x + 36$

$f''(x) = 0$

$= 6(6x^2 - 16x + 6) = 0$

$= 6 \cdot 2(3x^2 - 8x + 3)$

$= 12(3x^2 - 8x + 3) = 0$

16
6) 96
6
36

$$\frac{-(-8) \pm \sqrt{64-36}}{2(3)}$$

$2 \overline{)28}$
$2 \overline{)14}$
7

$$= \frac{8 \pm \sqrt{28}}{6}$$

$$\alpha = \frac{8+2\sqrt{7}}{6} \qquad \beta = \frac{8-2\sqrt{7}}{6}$$

$\sqrt{9} = 3$
$\sqrt{4} = 2$
$\therefore \sqrt{7} \approx 2.\ldots$

$3\overline{)4}$ = 1.33: 3, 10, 9, 10
$3\overline{)20}$ = 0.66: 18, 20

$$= \frac{4}{3} + \frac{\sqrt{7}}{3} \qquad = \frac{4}{3} - \frac{\sqrt{7}}{3}$$

$$\approx 1.33 + 0.66 \qquad \approx 1.33 - 0.66$$

$\alpha > 2$ $\qquad \beta < 1$

$\alpha \approx 2$ $\qquad \beta \approx 0.5$

if rational; do not factorize; just take num & denom and check signs.

By checking where denom = 0 functn is undefined.

Sign Chart 3 SC3

$x-2$

Sign chart	12	$(x-\alpha)$	$(x-\beta)$		
$[-1, \overset{0.5}{\beta})$	+	−	−	+	◡
$(\overset{0.5}{\beta}, \overset{2}{\alpha})$	+	−	+	−	⌒
$[\overset{2}{\alpha}, 4]$	+	+	+	+	◡

inflection point

inflection point

To make complete curve sketching the zero crossing levels have to be found. also check whats happening at domain edges.

$$\therefore \quad f(x) = 0 \quad \Rightarrow \quad 3x^4 - 16x^3 + 18x^2 = 0$$

$$x^2\left(3x^2 - 16x + 18\right) = 0$$

$$x^2\left(x - \frac{16 \pm \sqrt{16^2 - 4(18)(3}}{6}\right) = 0$$

$x = 0 \qquad x \approx 1.6 \qquad x \approx 3.7$

Domain given $[-1, 4]$

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

$$\lim_{x \to -1} f(x) = 3(-1)^4 - 16(-1) + 18(-1)^2$$

$$= 3 + 16 + 18 = 37$$

$$\lim_{x \to 4} f(x) = 3(4)^4 - 16(4)^3 + 18(4)^2$$

$$= 3(256) - 16(64) + 18(16)$$

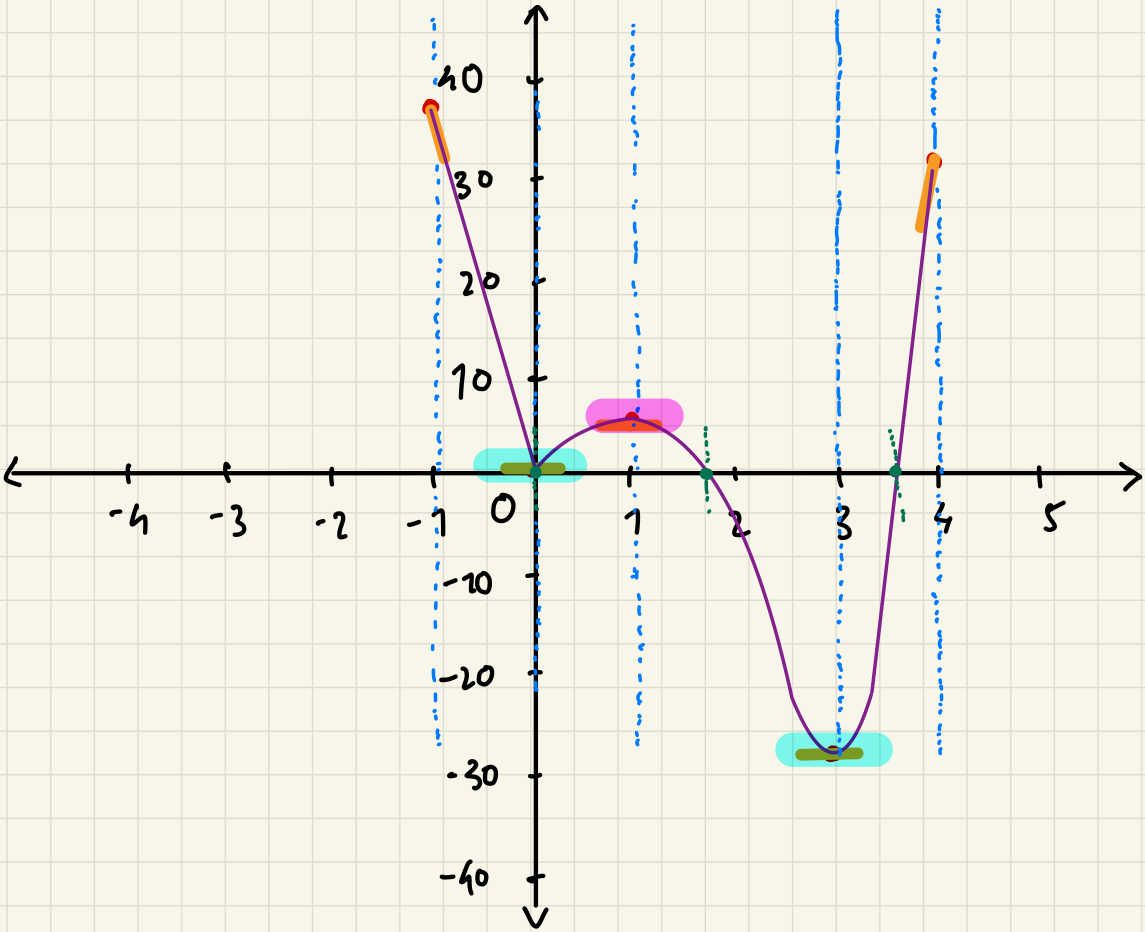
$$= 768 - 1024 + 288$$

$$= 1056 - 1024$$

$$= 32$$

$$\begin{array}{r} {}^{4}18 \\ \underline{16} \\ 108 \\ \underline{18} \\ {}^{1}288 \\ \underline{768} \\ 1056 \end{array}$$

which is what you got in FVC limits applied to domain edges for complicated functions & places where asymptotes can occur.



40
30
20
10
-10
-20
-30
-40
-4
-3
-2
-1
0
1
2
3
4
5

Summary 1

Domain & zero crossing → factorize $f(x)$ & find points $x_1, x_2, x_3 \ldots$

Find what's going on at the edges of domain points by applying limits
(Helps find asymptodes if they exist)

monotonicity and its intervals → first derivative $f'(x)$
↳ equate $f'(x) = 0$
↳ CN's

Use critical Nums (CN's) to split domain.
↳ (,) (,) (,)

Apply limit on the point x is not defined.

→ find second derivative $f''(x)$

SC-1

Sign table 1
intervals
$f'(x)$ factors
↓
increasing or decreasing

SC-2

Sign table 2
intervals
$f''(x)$ values
↓
local max
local min

FVC

Function value chart

$f(x)$ $f'(x)$ values with CN's & its domain end points.

Concavity → equate $f''(x) = 0$

↳ find new intervals

↳ create sign chart for the intervals

↳ Apply smiley faces

Summary 2

Domain & zero cross

→ Define intervals.

→ factorize

→ Apply limits

Apply limit at the point x is not defined

Monotonicity & extremas.

$f'(x) \rightarrow f'(x) = 0 \rightarrow$ CN's → split domain with CN's → SC-1 CN intervals & $f'(x)$ factors

$f''(x) \rightarrow f''(CN's) \rightarrow$ SC-2 CN's $f''(x)$ sign → local max or, mins

FVC → with $f(x)$ & $f'(x)$ Function values.

Concavity

$f''(x) = 0 \rightarrow f''(x)$ { undefined at points where rational's denom becomes = 0; defined at $x = \dots$ & value; if none defined nowhere

create SC-3

functions representations :

$f'(x)$ & $f''(x)$ $\Rightarrow$ $\begin{cases} \text{undefined / defined everywhere} \\ \text{undefined at } x = \ldots \\ \text{value of } x \text{ at defined x axis value.} \end{cases}$

Example 2

Find the critical numbers of the following function:

$$f(x) = x^{3/5} \cdot (4-x)$$

Monotonicity & extremas $\quad f'(x) = 0 \rightarrow$ CN's $\rightarrow$ split domain

$\downarrow$

SC1
split intervals
f(x) factors

$\downarrow$

function values

$$f(x) = x^{3/5} \cdot (4-x)$$

$$f'(x) = 4x^{3/5} - x^{3/5+1}$$

$$= \frac{d}{dx}\left(4x^{3/5} - x^{8/5}\right)$$

$$= 4 \cdot \frac{3}{5} x^{3/5-1} - \frac{8}{5} x^{8/5-1}$$

$$= \frac{12}{5} x^{-2/5} - \frac{8}{5} x^{3/5}$$

$$= \frac{1}{5}\left(\frac{12}{x^{2/5}} - 8x^{3/5}\right)$$

$$= \frac{1}{5}\left(\frac{12 - 8x^{5/5}}{x^{2/5}}\right)$$

$$= \frac{12 - 8x}{5x^{2/5}} = \frac{12 - 8x}{5\sqrt[5]{x^2}}$$

$$f'(x) = \frac{12 - 8x}{5\sqrt[5]{x^2}}$$

$$f'(x) = 0$$

$$(12 - 8x)\cdot\left(\frac{1}{5\sqrt[5]{x^2}}\right) = 0$$

$$12 - 8x = 0 \qquad x = 0$$

$$8x = 12$$

$$x = \frac{3}{2}$$

Critical nos: $x = 0$ & $x = \frac{3}{\cancel{4}\,2}$

The domain of the polynomial function is $(-\infty, \infty)$

$\therefore$ The split intervals are

$$(-\infty, 0) \cup \left(0, \frac{3}{\cancel{4}\,2}\right) \cup \left(\frac{3}{\cancel{4}\,2}, \infty\right)$$

	$12-8x$	$\frac{1}{5\cdot\sqrt[5]{x^2}}$	$f'(x)$	
$(-\infty, 0)$	+	+	+	increasing
$(0, \frac{3}{2})$	+	+	+	increasing
$(\frac{3}{2}, \infty)$	−	+	−	decreasing.

Function values
neighbors & CN's

selected 4

x	$f(x)$	$f'(x)$
-1	3	$4/5$
0	0	Not Def
1	-5	$13/3$
1.5	≈ 3	0
2		

$$f(x) = x^{3/5}(4-x) \qquad f'(x) = \frac{12-8x}{5\sqrt[5]{x^2}}$$

$$f(-1) = 1(4-1) = 3 \qquad f'(-1) = \frac{12-8}{5} = \frac{4}{5}$$

$f(0) = 0$

$f(-1) = -1(4+1)$
$= -5$

$f(1.5) = \left(\frac{3}{2}\right)^{3/5}(4-x)$

$f'(0) = \cancel{0}$ $\frac{12-0}{0} = \frac{12}{0} =$ Not defined

apply limit there!

$f'(-1) = \frac{12+8}{5(1)}$
$= \frac{13}{5}$

$f'\left(\frac{3}{2}\right) = 0$

Example 3

Find the absolute extrem values of

$f(x) = x^3 - 3x^2 + 1 \qquad x \in [-\frac{1}{2}, 4]$

Domain & zero crossing

$[-\frac{1}{2}, 4] \qquad f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

$f'(x) = 0 \Rightarrow 3x^2 - 6x = 0$

$x(3x-6) = 0$

$3x(x-2) = 0$

$3x = 0 \qquad x - 2 = 0$

CN's: $x = 0 \qquad x = 2$

$\therefore$ we have to inspect monotonicity at these intervals. with sign chart $[-\frac{1}{2}, 0) \cup (0, 2) \cup (2, 4]$

Sign Chart 1

interval	$(3x)$	$(x-2)$	$f'(x)$	
$[-\frac{1}{2}, 0)$	$-$	$-$	$+$	increasing
$(0, 2)$	$+$	$-$	$-$	decreasing
$(2, 4]$	$+$	$+$	$+$	increasing

$$f''(x) = f'(f'(x))$$
$$= f'(3x^2 - 6x)$$
$$= 6x - 6$$

Sign chart 2:

CN's	$f''(x)$	sign	
0	$6(0)-6 = -6$	$-$ve	local maximum
2	$6(2)-6 = 6$	$+$ve	local minimum

function values

x	$f(x)$	$f'(x)$
$-\frac{1}{2}$	$-\frac{1}{8}$	$15/4$
0	$\cancel{0}$ 1	0
1	-1	-3
2	-3	0
4	15	24

global min

global max

$$f(x) = x^3 - 3x^2 + 1$$

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3 \cdot \frac{1}{4} + 1$$

$$= -\frac{1}{8} - \frac{3}{4} + 1$$

$$= \frac{-4 - 24}{32} + 1$$

$$= \frac{-28 + 32}{32}$$

$$= -\frac{4}{32} = \frac{-1}{8}$$

$$f'(x) = 3x^2 - 6x$$

$$f'\left(-\frac{1}{2}\right) = 3\,\frac{1}{4} - 6\left(-\frac{1}{2}\right)$$

$$= \frac{3}{4} + \frac{6}{2}$$

$$= \frac{6 + 24}{8}$$

$$= \frac{30}{8} = \frac{15}{4}$$

$f(0) = 0 - 0 + 1$
$= 1$

$f'(0) = 0 - 0$
$= 0$

$f(1) = 1 - 3 + 1$
$= -1$

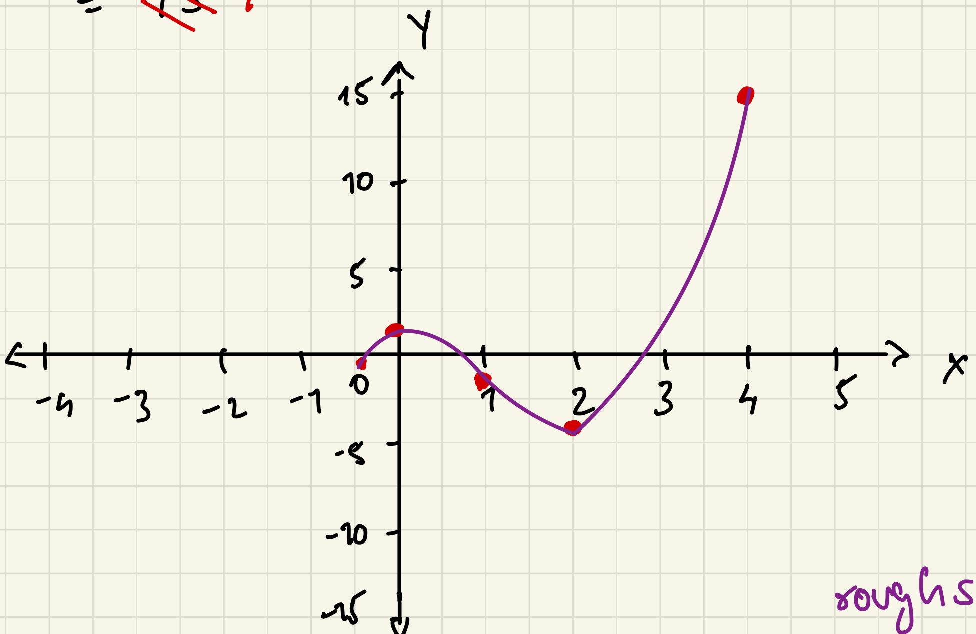
$f'(1) = 3 - 6$
$= -3$

$f(2) = 8 - 12 + 1$
$= -3$

$f'(2) = 12 - 12$
$= 0$

$f(4) = 64 - 48 + 1$
$= \cancel{15}\ 17$

$f'(4) = 48 - 24$
$= 24$



rough sketch as concavity and other parts have not been asked in the question

Example

$f(x) = x - 2\sin x \qquad 0 \le x \le 2\pi$

Domain: $[0, 2\pi]$

$f(x) = x^4 - 4x^3$

Domain : $(-\infty \quad \infty)$

$f'(x) = 4x^3 - 4 \cdot 3x^2$

$= 4x^3 - 12x^2$

$= 4x^2(x-3)$

$f'(x) = 0$

$\cancel{4}x^2 = 0 \qquad x - 3 = 0$

critical nos : CN's $\qquad x = 0 \qquad x = 3$

$(-\infty, 0) \quad (0, 3) \quad (3, \infty)$

Sign Chart 1

interval	4	x^2	$x-3$	
$(-\infty, 0)$	+	+	−	decreasing
$(0, 3)$	+	+	−	decreasing
$(3, \infty)$	+	+	+	increasing

$f''(x) = 12x^2 - 24x$

Sign chart 2

x	$f''(x)$		
0	0	$-$	
3	36	+ve	local minimum

$12(9) - 24(3)$

$108 - 72 = 36$

FVC Function value .

x	$f(x)$	$f'(x)$		
-1	5	-16	\	
0	0	0	$-$	
1	-3	-8	\	
2	-16	-16	\	
3	-27	0	$-$	global min
4				

$f(x) = x^4 - 4x^3$

$f(-1) = 1 + 4 = 5$

$f(0) = 0$

$f(1) = 1 - 4$
$= -3$

$f(2) = 16 - 32$
$= \cancel{-32}\ -16$

$f(3) = 81 - 27(4)$
$= 81 - 108$
$= -27$

$f'(x) = 4x^2(x-3)$

$f'(-1) = 4(-4) = -16$

$f'(0) = 0$

$f'(1) = 4(1-3)$
$= -8$

$f'(2) = 16(2-3)$
$= -16$

$f'(3) = 36(0)$
$= 0$

Concavity

$f''(x) = 0$

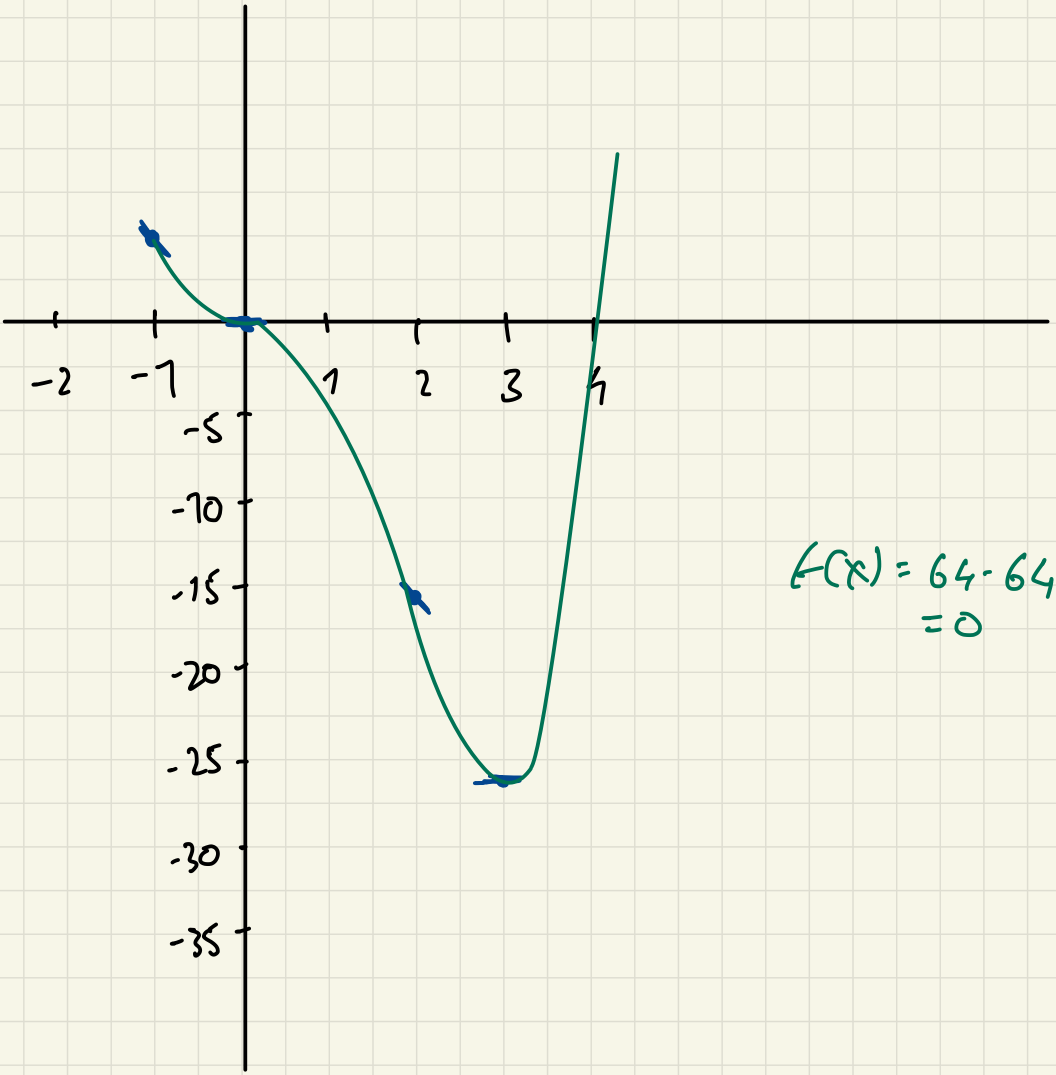
$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0$
$x = 2$

interval	$12x$	$(x-2)$	$f''(x)$	
$(-\infty, 0)$	$-$	$-$	+ve	
				inflecti
$(0, 2)$	$+$	$-$	-ve	
				inflectn
$(2, \infty)$	$+$	$+$	+v	

x	-1	0	1	2	3	4
$f(x)$	5	0	-3	-16	-27	0



$f(x) = 64 - 64$
$= 0$

Example:

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f''(x) = 12 \cdot 3x^2 - 12 \cdot 2x - 24$$

$$= 36x^2 - 24x - 24$$

Domain $(-\infty, \infty)$ $\quad f(x) = \{$ defined everywhere

$$f'(x) = 0$$

$$12x(x^2 - x - 2) = 0$$

$1 \cdot -2 = -2$

$1 + -2 = -1$

$$12x(x^2 + x - 2x - 2) = 0$$

$$12x(x(x+1) - 2(x+1)) = 0$$

$$12x((x+1)(x-2)) = 0$$

$$12x(x+1)(x-2) = 0$$

Critical nos: $x = 0 \quad x = -1 \quad x = 2$

Intervals to inspect monotonicity $(-\infty, 0)\ (0, 1)\ (1, 2)\ (2, \infty)$

$$f'(x) = \begin{cases} \text{defined everywhere} \\ x=0 \quad x=-1 \quad x=2 \end{cases}$$

Sign Chart 1

interval	$12x$	$(x+1)$	$(x+2)$	slope	
$(-\infty, 0)$	$-$	$-$	$-$	$-$	decreasing
$(0, 1)$	$+$	$+$	$+$	$+$	increase
$(1, 2)$	$+$	$+$	$+$	$+$	increase
$(2, \infty)$	$+$	$+$	$+$	$+$	increase

Sign Chart 2

x	$f''(x)$	
0	-24	local maximum
-1	36	local minimum
2	82	local minimum.

$$f''(x) = 36x^2 - 24x - 24$$
$$= 12(3x^2 - 2x - 2)$$

$$f''(0) = 12\,(0 - 0 - 2) = -24$$

$$f''(-1) = 12\,(3 + 2 - 2) = 36$$

$$f''(2) = 12\,(12 - 4 - 2) = 12 \cdot 6 = 82$$

Function Value Chart

x	$f(x)$	$f'(x)$	Slope
-2	37	-96	\
-1	0	0	—
0	5	0	—
1	-8	-24	\
2	-27	0	—
3	62	144	/
4	325	480	/

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

we also have to find zero crossings:

$$f(-2) = 3(16) - 4(-8) - 12(4) + 5$$

$$= 48 + 32 - 48 + 5 = 37$$

$$f(-1) = 3(1) - 4(-1) - 12(1) + 5$$
$$= 3 + 4 - 12 + 5 = 0$$

$(x+1)$ is a factor

$$f(0) = 3(0) - 4(0) - 12(0) + 5$$
$$= 5$$

$$f(1) = 3(1) - 4(1) - 12(1) + 5$$
$$= 3 - 4 - 12 + 5$$
$$= 8 - 16 = -8$$

$$f(2) = 3(16) - 4(8) - 12(4) + 5$$
$$= 48 - 32 - 48 + 5$$
$$= -27$$

$$f(3) = 3(81) - 4(27) - 12(9) + 5$$
$$= 273 - 108 - 108 + 5$$
$$= 278 - 216 = 62$$

$$f(4) = 3(256) - 4(64) - 12(16) + 5$$
$$= 768 - 256 - 192 + 5$$
$$= 773 - 448 = 325$$

$f'(x) = 12x(x+1)(x-2)$

$f'(-2) = 12(-2)(-2+1)(-2-2)$

$= -24(-1)(-4)$

$= -24 \cdot 4 = -96$

$f'(-1) = -12(0)(-1-2) = 0$

$f'(0) = 0$

$f'(1) = 12(2)(-1) = -24$

$f'(2) = 24(3)(0) = 0$

$f'(3) = 36(4)(1) = 144$

$f'(4) = 12(4)(5)(2)$

$= 48(10) = 480$

Sign Chart 3 Concavity

$f''(x) = 0$

$12(3x^2 - 2x - 2)$

$12\left(x - \frac{2 \pm \sqrt{4+24}}{6}\right)$

$2 \,|\, 28$

$2 \,|\, 14$

7

$$12\left(x - \frac{2 \pm \sqrt{28}}{6}\right)$$

$$12\left(x - \frac{2 \pm 2\sqrt{7}}{6}\right)$$

$$12\left(x - \frac{1+\sqrt{7}}{3}\right)\left(x - \frac{1-\sqrt{7}}{3}\right)$$

$$x_1 = \frac{1+\sqrt{7}}{3} \qquad x_2 = \frac{1-\sqrt{7}}{3}$$

$\sqrt{7} \approx 2.6$

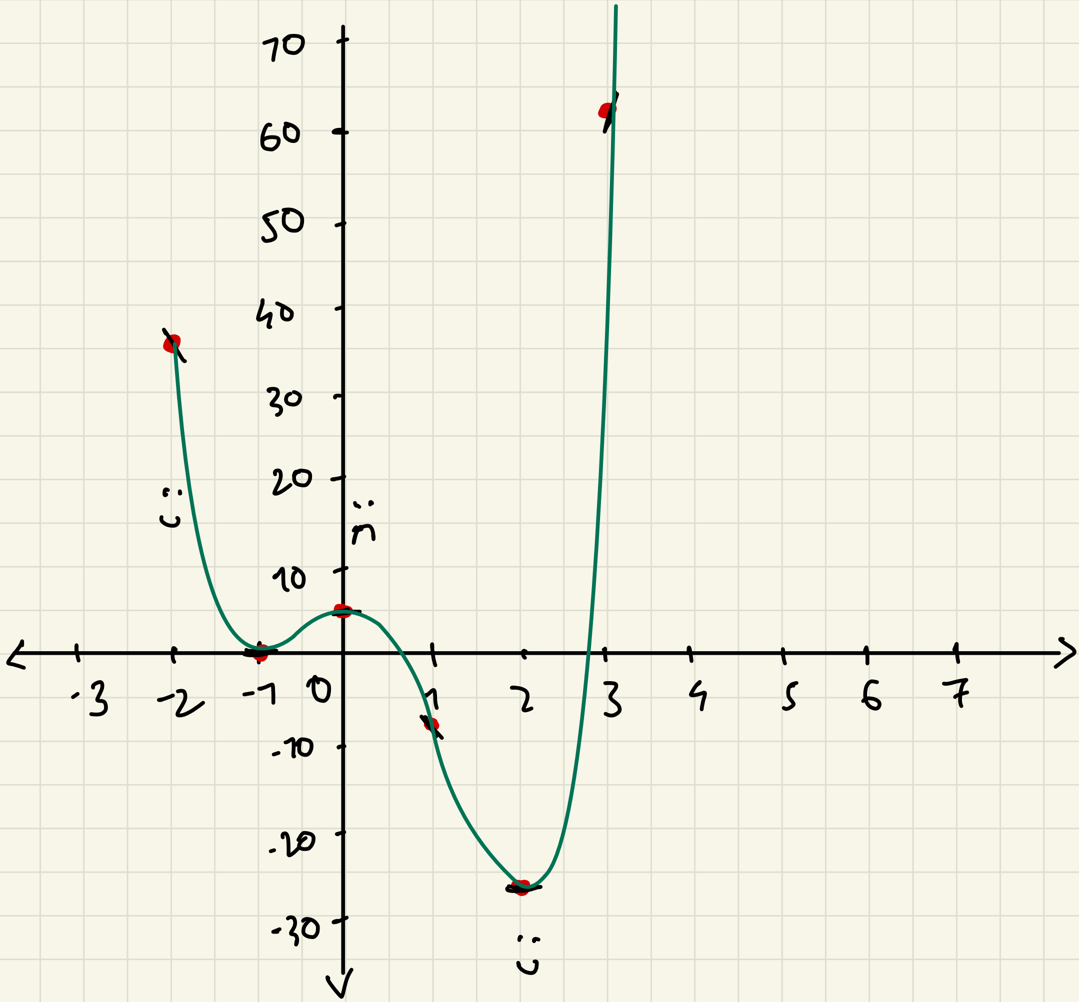
$$x_1 = \frac{3.6}{3} \qquad x_2 = \frac{1-3.6}{3} = \frac{-2.6}{3}$$

$$x_1 = 1.1 \qquad x_2 = -0.8 \approx$$

Sign Chart3

interval	12	$(x-1.1)$	$(x+0.8)$	$f''(x)$
$(-\infty, -0.8)$	+	−	−	+
$(-0.8, 1.1)$	+	−	+	−
$(1.1, \infty)$	+	+	+	+

x	-2	-1	0	1	2	3	4
$f(x)$	37	0	5	-8	-27	62	325



example

$f(x)\ 2\cos x + \cos^2 x$

Domain $(-\infty, \infty)$

$f(x) = 2\cos x + \cos^2 x$

$f(x) = 0$

$\cos x\,(2 + \cos x) = 0$

$\cos x = 0 \qquad \cos x = -2$

$x = \cos^{-1}(0) \qquad x = \cos^{-1}(-2)$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ No zeros crossings here

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
Cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

Monotonicity

$f'(x) = 0$

$\frac{d}{dx}\left(2\cos x + \cos^2 x\right)$

$= -2\sin x + -\sin x \cdot 2\cos x$

$= -2\sin x - 2\sin x\cos x$

$f'(x) = 0$

$-2\sin x\,(1 + \cos x) = 0$

$\sin x = 0 \qquad \cos x = -1$

$x = 0, \pi, 2\pi \ldots \qquad x = \pi, 3\pi$

main intervals to check monotonicity: $(0, \pi)\ (\pi, 3\pi)$

splitting intervals for simplicity

$\left(0, \frac{\pi}{2}\right)\ \left(\frac{\pi}{2}, \pi\right)\ \left(\pi, \frac{3\pi}{2}\right)\ \left(\frac{3\pi}{2}, 2\pi\right)$

$-2\sin x\,(1 + \cos x) = 0$

intervals	$-2\sin x$	$1+\cos x$	slope	
$(0, \frac{\pi}{2})$	$-$	$+$	$-$	decrease
$(\frac{\pi}{2}, \pi)$	$-$	$+$	$-$	decrease
$(\pi, \frac{3\pi}{2})$	$+$	$+$	$+$	increase
$(\frac{3\pi}{2}, 2\pi)$	$+$	$+$	$+$	increase

$$f''(x) = \frac{d}{dx}\left(-2\sin x - 2\sin x\cos x\right)$$

$$= -2\cos x - 2\left[\sin x(-\sin x) + \cos x(\cos x)\right]$$

$$= -2\cos x - 2\left[\cos^2 x - \sin^2 x\right]$$

$$= -2\cos x - 2\cos^2 x + 2\sin^2 x$$

$$= -2\cos x - 2\cos^2 x + 2(1 - \cos^2 x)$$

$$= -2\cos x - 2\cos^2 x + 2 - 2\cos^2 x$$

$$= 2\left(-\cos x - \cos^2 x + 1 - \cos^2 x\right)$$

$$= 2\cdot 2\left(-\frac{\cos x}{2} - \frac{\cos^2 x}{2} + \frac{1}{2} - \frac{\cos^2 x}{2}\right)$$

$$= 4\left(-\frac{\cos x}{2} - 2\,\frac{\cos^2 x}{2} + \frac{1}{2}\right)$$

$$= 4\left(\frac{-1}{2}\cos x - \cos^2 x + \frac{1}{2}\right)$$

$$= -4\left(\cos^2 x + \frac{1}{2}\cos x - \frac{1}{2}\right) = 0$$

Let $x = \cos x$ $\Rightarrow$ $x^2 + \frac{1}{2}x - \frac{1}{2} = 0$

$$\frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 4\cdot\frac{1}{2}}}{2}$$

$$= \frac{-\frac{1}{2} \pm \sqrt{\frac{9}{4}}}{2} = \frac{-\frac{1}{2} \pm \frac{3}{2}}{2}$$

$$x_{1,2} = -\frac{1}{4} \pm \frac{3}{4}$$

$$x_{1,2} = \frac{-1+3}{4} \quad \text{or} \quad \frac{-1-3}{4}$$

$$x_{1,2} = \frac{2}{4} = \frac{1}{2} \quad \text{or} \quad -1$$

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$

$$\text{ie} \quad \left(\cos x - \frac{1}{2}\right)\left(\cos x + 1\right) = 0$$

$$f''(x) = 0$$

$$= -4\left(\cos x - \frac{1}{2}\right)\left(\cos x + 1\right) = 0$$

$$x = \frac{\pi}{3} \qquad x = \frac{\pi}{2}, \frac{3\pi}{2}$$

intervals	-4	$\left(\cos x - \frac{1}{2}\right)$	$(\cos x + 1)$	$f''(x)$
$0, \frac{\pi}{3}$	$-$	$+$	$+$	$-$
				- PoI -
$\frac{\pi}{3}, \frac{\pi}{2}$	$-$	$-$	$+$	$+$
$\frac{\pi}{2}, \frac{3\pi}{2}$	$-$	$-$	$+$	$+$

x	$f(x)$	$f'(x)$
0	3	
$\frac{\pi}{3}$	≈ 2.45	
$\frac{\pi}{2}$	0	
π	-1	
$\frac{3\pi}{2}$	0	

$$f(x) = 2\cos x + \cos^2 x$$

$$f'(x) = -2\sin x - 2\cos x \sin x$$

$$f(0) = 2 + 1 = 3$$

$$f\left(\frac{\pi}{3}\right) = 2\cos\frac{\pi}{3} + \cos^2\frac{\pi}{3}$$

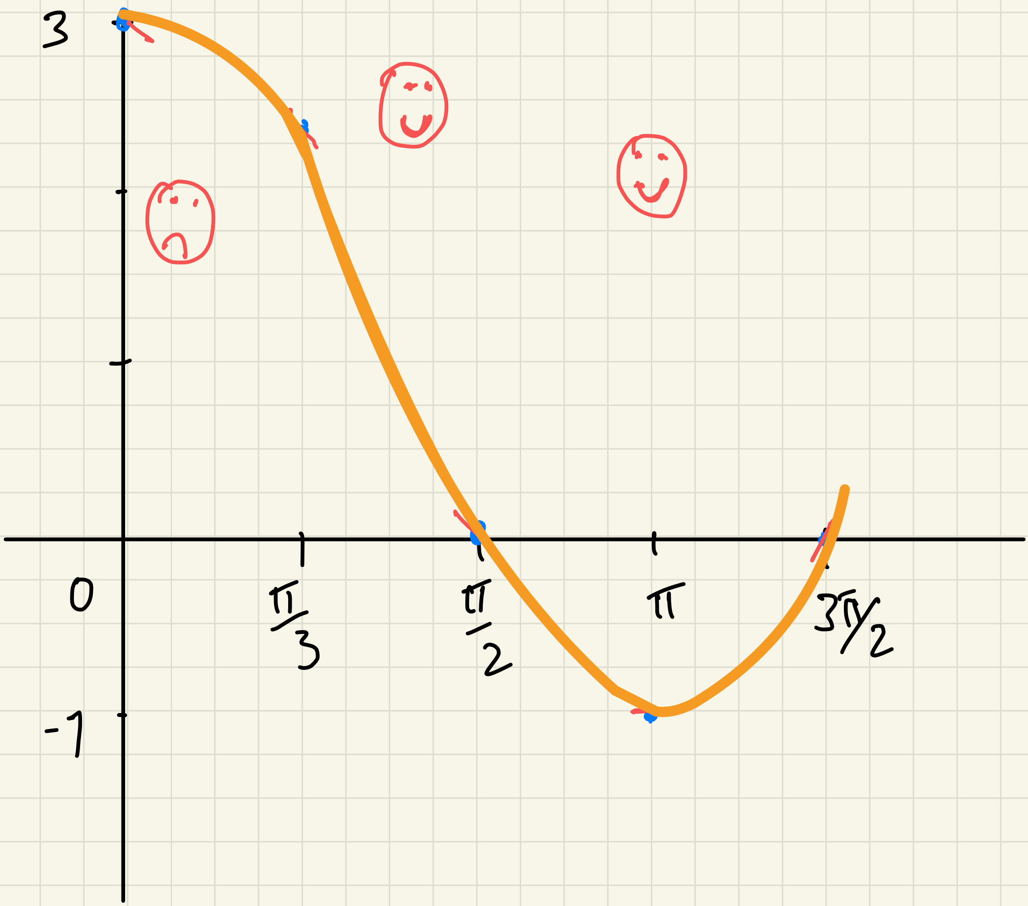
$$= 2 \cdot \frac{\sqrt{3}}{2} + \frac{3}{4} = \sqrt{3} + 0.75$$

$$=$$

$f\left(\frac{\pi}{2}\right) = 0$

$f(\pi) = 2(-1) + (-1)^2 = -2 + 1 = -1$

$f\left(\frac{3\pi}{2}\right) = 2(0) + 0 = 0$



1. $f(x) = x^4 - 6x^2$

Domain $(-\infty, \infty)$

8:38

$f(x) = 0$

$x^2(x^2 - 6) = 0$

$x = 0 \qquad x^2 = 6$

$\qquad\qquad x = \pm\sqrt{6}$ } zero crossings

$f'(x) = 4x^3 - 12x$

$f'(x) = 0$

$4x(x^2 - 3) = 0$

$x = 0 \qquad x = \pm\sqrt{3}$

monotonicity intervals: $(-\infty, -\sqrt{3})$ $(-\sqrt{3}, 0)$ $(0, \sqrt{3})$ $(\sqrt{3}, \infty)$

interval	$4x$	x^2-3	$f'(x)$	
$(-\infty, -\sqrt{3})$	$-$	$+$	$-$	decreasing
$(-\sqrt{3}, 0)$	$-$	$-$	$+$	increase
$(0, \sqrt{3})$	$+$	$-$	$-$	decrease
$(\sqrt{3}, \infty)$	$+$	$+$	$+$	increase

Maxima & Minima

$$f''(x) = \frac{d}{dx}(4x^3 - 12x)$$

$$= 12x^2 - 12 = 12(x^2-1)$$

points to check at: $-\sqrt{3}$ 0 $\sqrt{3}$

value	$f''(x)$	
-2	$12(4-1) = 36$	
$-\sqrt{3}$	$12(3-1) = 24$	
-1	$12(1-1) = 0$	
0	$12(0-1) = -12$	-ve to +ve local max
1	$12(1-1) = 0$	
$\sqrt{3}$	$12(3-1) = 24$	
2	$12(2-1) = 12$	

Concavity:

$f''(x) = 0$

$$12(x^2 - 1) = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

interval	12	$(x-1)$	$(x+1)$	$f''(x)$		
$(-2, -1)$	+	−	−	+	☺	
$(-1, 0)$	+	−	+	−	☹	POI
$(0, 1)$	+	−	+	−	☹	
$(1, 2)$	+	+	+	+	☺	POI

Function Value Chart.

x	$f(x)$	$f'(x)$
-3	27	
-2	-8	
$-\sqrt{3}$	-9	
-1	-5	
0	0	
1	-5	
$\sqrt{3}$	-9	
2	-8	
3	27	

$f(x) = x^4 - 6x^2$

$f(-3) = 81 - 54 = 27$

$f(-2) = 16 - 24 = -8$

$f(-\sqrt{3}) = 9 - 18 = -9$

$f(-1) = 1 - 6 = -5$

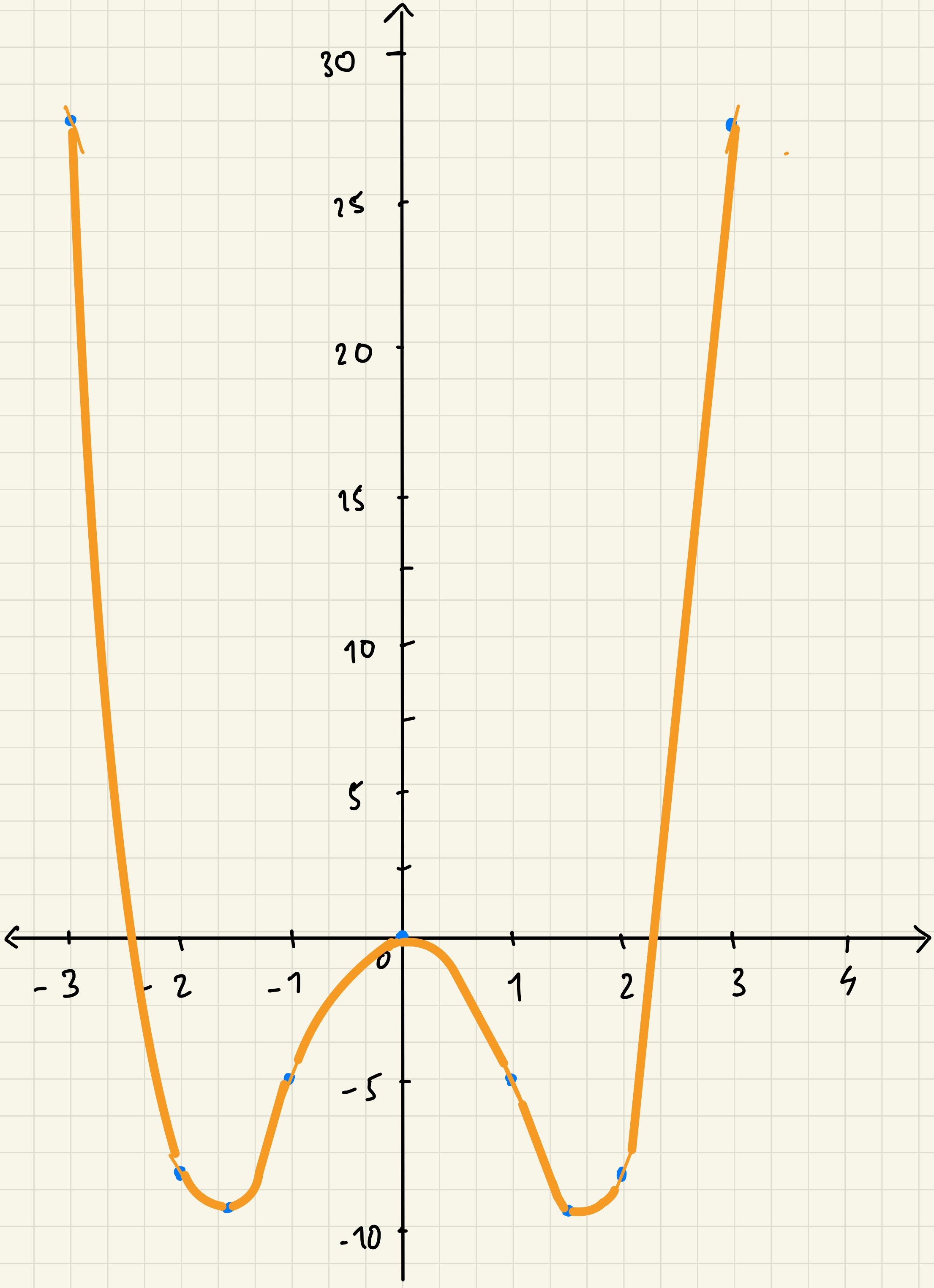
$f(0) = 0 - 6(0) = 0$

$f(1) = 1 - 6 = -5$

$f(\sqrt{3}) = 9 - 18 = -9$

$f(2) = 16 - 24 = -8$

$f(3) = 81 - 54 = 27$


30
25
20
15
10
5
-5
-10
-3
-2
-1
0
1
2
3
4

11:07

② $f(x) = (x^2 - 1)^3$

zero crossing & Domain

$f(x) = 0$

$(x^2-1)^3 = 0$

$x^2 = 1$

$x = \pm 1$

Domain $(-\infty, \infty)$

zero crossing at:

$x = -1$ & $x = 0$

Monotonicity & Extremas

$f'(x) = 2x \cdot 3(x^2-1)^2$

$= 6x(x^2-1)^2$

$f''(x) = 6x \cdot 2x \cdot (x^2-1) + (x^2-1)^2 \cdot 6$

$= 12x^2(x^2-1) + (x^2-1)^2 \cdot 6$

$f'(x) = 0$

$6x(x^2-1)^2 = 0$

$6x = 0$

$x = 0$

$x^2 - 1 = 0$

$x = \pm 1$

monotonicity intervals $(-\infty, -1)$ $(-1, 0)$

$(0, 1)$ $(1, \infty)$

interval	$6x$	$(x^2-1)^2$	$f(x)$	
$(-\infty, -1)$	$-$	$+$	$-$	decreasing
$(-1, 0)$	$-$	$+$	$-$	decreasing
$(0, 1)$	$+$	$+$	$+$	increasing
$(1, \infty)$	$+$	$+$	$+$	increasing

Extremas

value	$f''(x)$
-2	$+218$
-1	0
0	$+6$
1	0
2	$+218$

$$f''(x) = 12x^2(x^2-1) + (x^2-1)^2 \cdot 6$$

$$f''(x) = 0$$

$$(x^2-1)\left[12x^2 + 6(x^2-1)\right]$$

$$(x^2-1)\left[12x^2 + 6x^2 - 6\right]$$

$$f''(-2) = (4-1)\left[48 + 24 - 6\right]$$

$$= 3\left[66\right] = 218$$

$$f''(-1) = 0$$

$$f''(0) = -1\left[0 + 0 - 6\right] = +6$$

$$f''(1) = 0$$

$$f''(2) = 218$$

$$f''(x) = 0$$

$$(x^2-1)\left[12x^2 + 6x^2 - 6\right] = 0$$

$$(x^2-1)(18x^2-6) = 0$$

$$(x^2-1)\,6\,(3x^2-1) = 0$$

$$6(x^2-1)(3x^2-1) = 0$$

$$x^2 - 1 = 0 \qquad\qquad 3x^2 = 1$$

$$x = \pm 1 \qquad\qquad x^2 = \frac{1}{3}$$

$$x = \pm \frac{1}{\sqrt{3}}$$

concavity intervals

$$(-\infty, -1), \left(-1, -\frac{1}{\sqrt{3}}\right) \left(-\frac{1}{\sqrt{3}}, 0\right)$$

$$\left(0, \frac{1}{\sqrt{3}}\right) \left(\frac{1}{\sqrt{3}}, 1\right) \left(1, \infty\right)$$

interval	6	(x^2-1)	$(3x^2-1)$	$f''(x)$	
$(-\infty, -1)$	$+$	$+$	$+$	$+$	☺
$\left(-1, -\frac{1}{\sqrt{3}}\right)$	$+$	$-$	$+$	$-$	☹
$\left(-\frac{1}{\sqrt{3}}, 0\right)$	$+$	$-$	$-$	$+$	☺

$(0, \frac{1}{\sqrt{3}})$	+	−	−	+	☺
$(\frac{1}{\sqrt{3}}, 1)$	+	−	+	−	☹
$(1, \infty)$	+	+	+	+	☺

Function value chart

x	$f(x)$
-3	452
-2	27
-1	0
$-\frac{1}{\sqrt{3}}$	≈ -0.3
0	-1
$\frac{1}{\sqrt{3}}$	≈ -0.3
1	0
2	27
3	452

$f(x) = (x^2 - 1)^3$

$f(-3) = 8^3 = 452$

$f(-2) = 3^3 = 27$

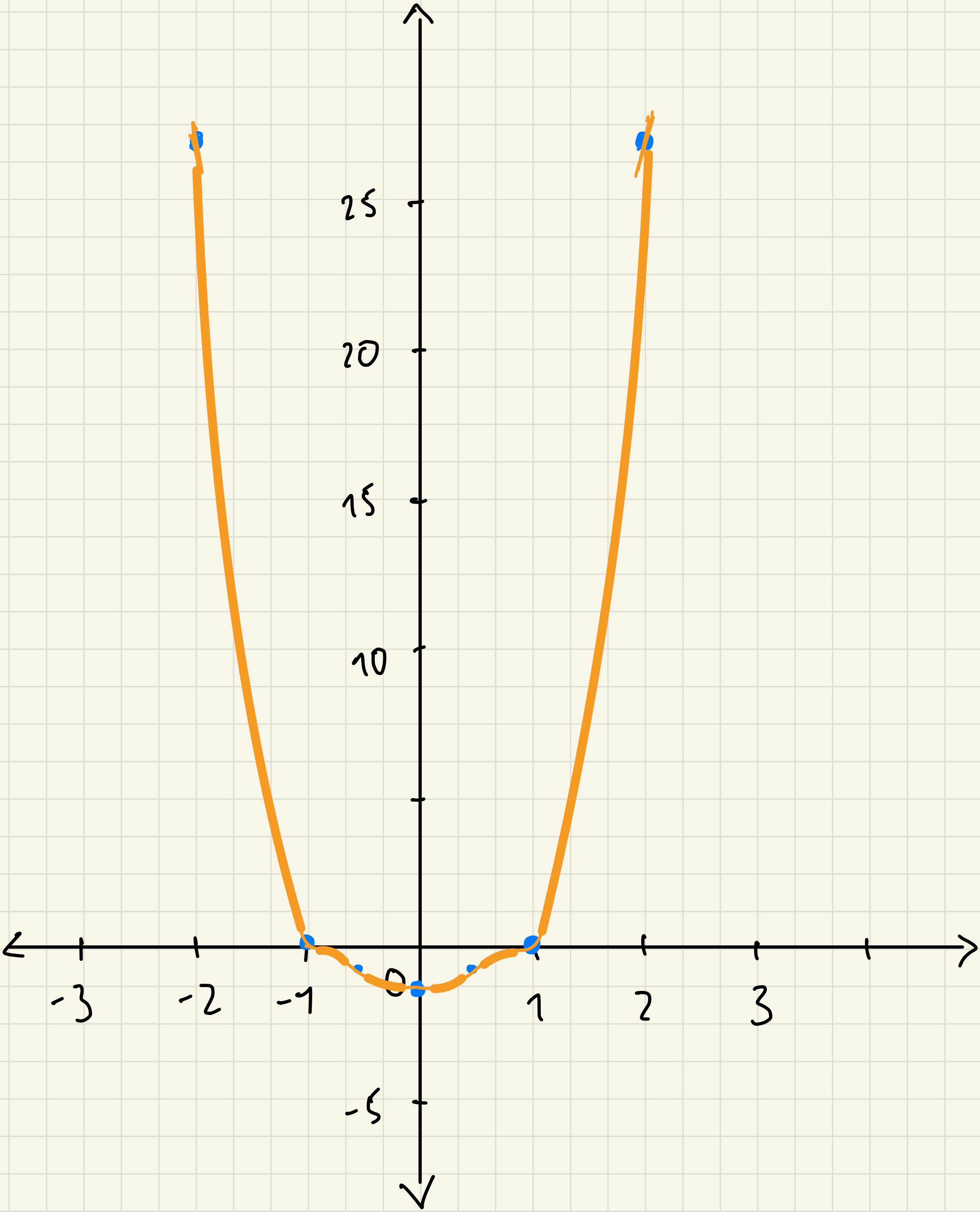
$f(-1) = 0$

$f(-\frac{1}{\sqrt{3}}) = (\frac{1}{3} - 1)^3$

$= \frac{-2}{3} = \approx -0.3$

$f(0) = -1$

$f(\frac{1}{\sqrt{3}}) = \approx -0.3$


25
20
15
10
-5
-3
-2
-1
0
1
2
3

③ $f(x) = x\sqrt{x^2+1}$

Domain $(-\infty, \infty)$

$f(x) = 0$

$x\left(\sqrt{x^2+1}\right) = 0$

$x = 0 \qquad x^2+1 = 0$

$x^2 = -1$

$x = \sqrt{-1} = \pm i$

complex number is not possible as function has real values

$$f'(x) = x \cdot 2x \cdot \frac{1}{2\sqrt{x^2+1}} + \sqrt{x^2+1} \cdot 1$$

$$= \frac{2x^2}{2\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{x^2 + x^2+1}{\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}$$

$f'(x) = 0 \qquad \dfrac{2x^2+1}{\sqrt{x^2+1}} = 0$

$$(2x^2+1) = 0 \qquad \frac{1}{\sqrt{x^2+1}} = 0$$

$$x^2 = -\frac{1}{2}$$

$$x = \sqrt{-\frac{1}{2}}$$

complex roots not possible as function has Real values.

$\therefore$ No critical nos.

The function derivative $f'(x) = \frac{2x^2+1}{\sqrt{x^2+1}}$ is positive throughout the comple domain hence it is increasing throughout.

$$f''(x) = \frac{d}{dx}\left(\frac{2x^2+1}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}\,(4x) - 2x^2+1 \cdot \frac{2x}{2\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{4x\sqrt{x^2+1} - \frac{x(2x^2+1)}{\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{4x(x^2+1) - x(2x^2+1)}{\sqrt{x^2+1}\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{4x^3+4x-2x^3-x}{(x^2+1)^{3/2}}$$

$$= \frac{2x^3+3x}{(x^2+1)^{3/2}} = 0$$

$$= x(2x^2+3) = 0$$

$$x = 0 \qquad \underbrace{x^2 = \frac{-3}{2}}_{\text{complex}}$$

$\therefore$ not on the actual domain.

interval for concavity $(-\infty, 0)$ & $(0, \infty)$

	x	$(2x^2+3)$	$\frac{1}{(x^2+1)^{3/2}}$	$f''(x)$
$(-\infty, 0)$	$-$	$+$	$+$	$-$
$(0, \infty)$	$+$	$+$	$+$	$+$

- POI -

Extremas also does not exist as the function does not have any critical numbers in the defined domain to plug into the 2nd derivative.

$\therefore$ Function value chart

x	$f(x)$
-3	≈ -12
-2	-6
-1	≈ -1.41
0	0
1	≈ 1.41

$$f(x) = x\sqrt{x^2+1}$$

$$f(-3) = -3\sqrt{9+1}$$
$$= -3\sqrt{10}$$

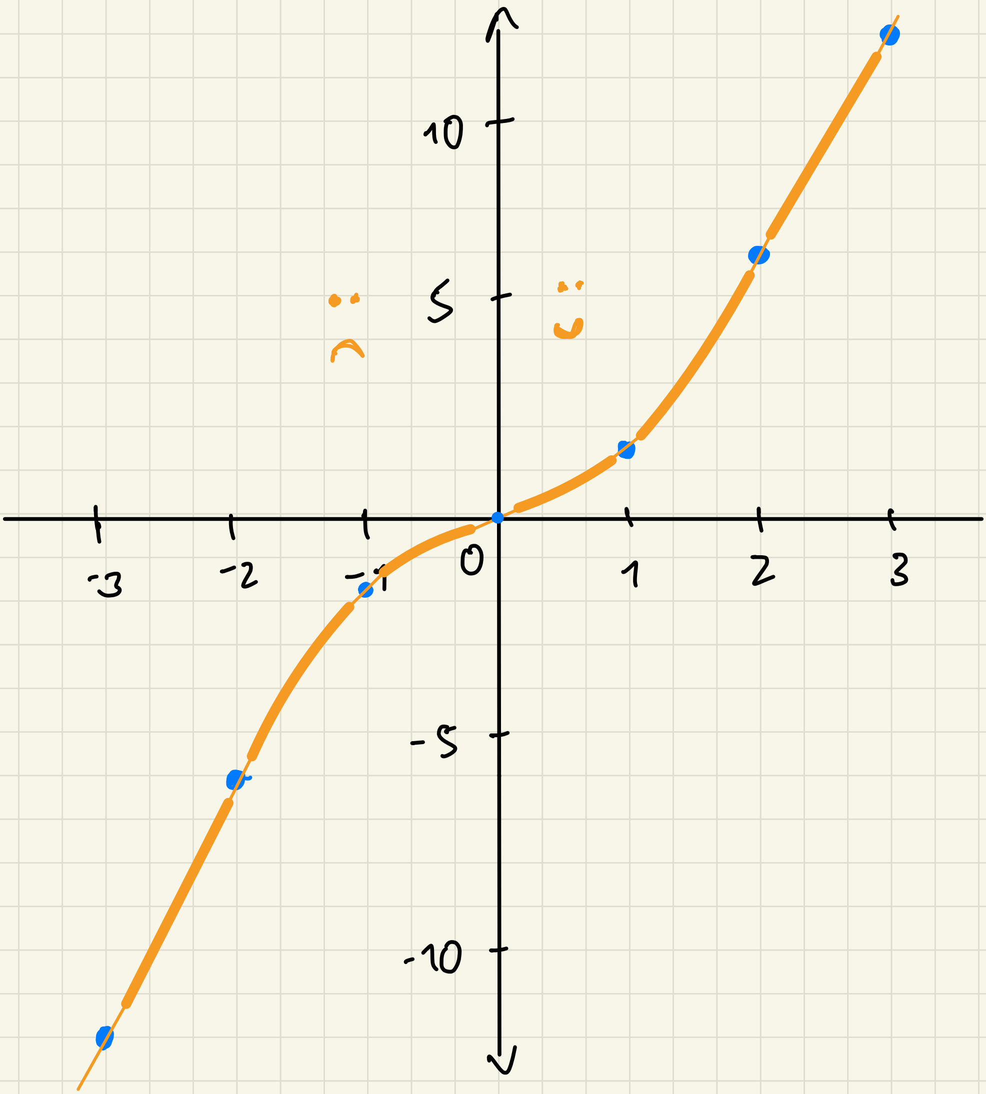
$$f(-2) = -2\sqrt{4+5}$$
$$= -6$$

$$f(-1) = -1\sqrt{2}$$

$$f(0) = 0$$

2 6

3 ≈ 12



12:38

(4) $f(x) = \dfrac{x}{(x-1)^2}$

Domain $f(x) = \left\{ \begin{array}{l} \text{defined every where} \\ \text{except at } x = 1 \end{array} \right.$

Function zero crossing: $f(x) = 0$

$x = 0$

Limit of function as it approaches $x = 1$

$$\lim_{x \to 1} f(x) = \frac{x}{(x-1)^2} = \frac{1}{0}$$

case 2 $\dfrac{\text{num}}{0}$ form $\therefore$ one sided limits have to be checked

$$\lim_{x \to 1^-} f(x) = \frac{x}{(x-1)^2} = \frac{1}{(1^- - 1)^2} = \frac{1}{(0^-)^2} = +\infty$$

$$\lim_{x \to 1^+} f(x) = \frac{1}{(1^+ - 1)^2} = \frac{1}{(0^+)^2} = +\infty$$

One sided limits are the same

$\therefore$ Two sided limits exist

$$\lim_{x \to 1} f(x) = +\infty$$

$$f'(x) = \frac{(x-1)^2 \cdot 1 - x \cdot 1 \cdot 2(x-1)}{((x-1)^2)^2}$$

$$= \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4}$$

$$= \frac{(x-1)(x-1-2x)}{(x-1)^4}$$

$$= \frac{(x-1)(-x-1)}{(x-1)^4}$$

$$= \frac{-1(x+1)(x-1)}{(x-1)^4} = 0$$

Critical nos: $x+1=0$
$x=-1$

undefined at $x=1$

$\therefore$ intervals selected

$(-\infty,-1)\ (-1,0)$

$(1,0)\ (1,\infty)$

interval	-1	$(x+1)$	$(x-1)$	$f(x)$	inc/dec
$(-\infty,-1)$	$-$	$-$	$-$	$-$	$\downarrow$ \
$(-1,0)$	$-$	$+$	$-$	$+$	$\uparrow$ /
$(1,0)$	$-$	$+$	$-$	$+$	$\uparrow$ /
$(1,\infty)$	$-$	$+$	$+$	$-$	$\downarrow$ \

$$f''(x) = \frac{d}{dx}\left(\frac{(x-1)(-x-1)}{(x-1)^4}\right)$$

$$(x-1)(-x-1) = -x^2 - x + x + 1$$
$$= -x^2 + 1 = 1 - x^2$$

$$f''(x) = \frac{(x-1)^4 \frac{d}{dx}(1-x^2) - (1-x^2)\frac{d}{dx}((x-1)^4)}{(x-1)^4}$$

$$= \frac{(x-1)^4(-2x) - (1-x^2)\,4\,(x-1)^3}{(x-1)^4}$$

$$= \frac{(x-1)^3\left[(x-1)(-2x) - (1-x^2)4\right]}{(x-1)^4}$$

$$= \frac{(x-1)^3\left[-2x^2 + 2x - 4 + 4x^2\right]}{(x-1)^4}$$

$$= \frac{2x^2 + 2x - 4}{(x-1)} = \frac{x^2 + x - 2}{(x-1)}$$

$-\underline{1}\ x\ \underline{2} = -2$

$\underline{-1} + \underline{2} = 1$

$$= \frac{x^2 - x + 2x - 2}{x-1}$$

$$f''(x) = 0 = \frac{x(x-1) + 2(x-1)}{x-1}$$

$$= \frac{(x-1)(x+2)}{(x-1)} = 0$$

$$= (x+2) = 0$$

$$x = -2$$

Extremas:

	$f''(x)$
-2	0
-1	1
0	2
1	3
2	4

Concavity:

interval	$(x+2)$	1	
$(-\infty, -2)$	$-$	$+$	
			$\leftarrow$ POI point of inflection
$(-2, -1)$	$+$	$+$	
$(-1, 0)$	$+$	$+$	
$(0, 1)$	$+$	$+$	
$(1, \infty)$	$+$	$+$	

Function Value Chart

x	$f(x)$
-3	≈ 0.11
-2	≈ -0.22
-1	-0.25
0	0

$$f(x) = \frac{x}{(x-1)^2}$$

1	–	(limit)
2	2	
3	0.75	

$f(-3) = \frac{-3}{(-4)^2}$

$= \frac{-3}{16}$

$f(-2) = \frac{-2}{9}$

$f(-1) = \frac{-1}{4}$

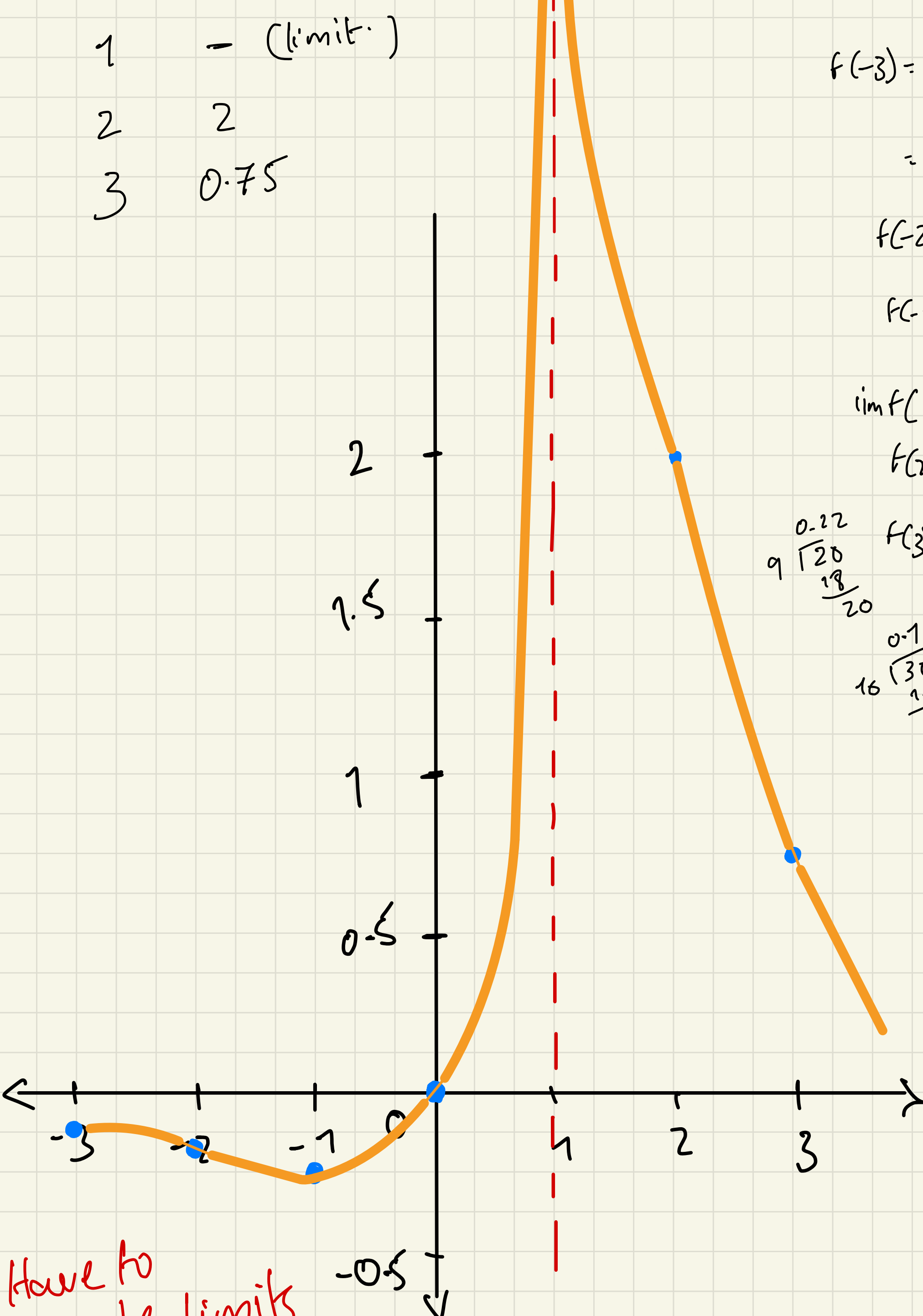
$\lim f(1) = \infty$

$f(2) = \frac{2}{1}$

$f(3) = \frac{3}{4}$

$$\begin{array}{r} 0.22 \\ 9\overline{)20} \\ \underline{18} \\ 20 \end{array}$$

$$\begin{array}{r} 0.109 \\ 16\overline{)30} \\ \underline{16} \\ 140 \end{array}$$



Have to compute limits at infinity.

$\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$

$\lim_{x \to -\infty} \frac{x}{(x-1)^2} = \frac{\infty}{\infty}$ form.

$\lim_{x \to -\infty} \frac{x}{x^2 + 1^2 - 2x}$

$\lim_{x \to -\infty} \frac{x}{x\left(x + \frac{1}{x} - 2\right)} = \frac{1}{-\infty} = 0$

$\therefore \lim_{x \to \infty} f(x)$

$= \lim_{x \to \infty} \frac{x}{x\left(x + \frac{1}{x} - 2\right)} = \frac{1}{\infty} = 0$

Limit approaches 0 at the domain ends.

Supplementary Problems

14:48

13 $(x-2)(x-6)y = 2x^2$

$$y = \frac{2x^2}{(x-2)(x-6)}$$

$$f(x) = \begin{cases} \text{defined everywhere} \\ \text{except at } x=2 \text{ \& } x=6 \end{cases}$$

$f(x) = 0 \qquad x = 0$

$$(x-2)(x-6) = x^2 - 6x - 2x + 12$$
$$= x^2 - 8x + 12$$

$$f'(x) = \frac{d}{dx}\left(\frac{2x^2}{x^2-8x+12}\right)$$

$$\frac{x^2-8x+12\,(4x) - 2x^2(2x-8)}{((x-2)(x-6))^2}$$

$$= \frac{4x^3 - 32x^2 + 48x - 4x^3 + 16x^2}{(x-2)^2(x-6)^2}$$

$$= \frac{-16x^2 + 48x}{(x-2)^2(x-6)^2}$$

$$= \frac{16x(-x+3)}{(x-2)^2(x-6)^2}$$

$$= \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

$f'(x) = 0$

critical nos $\begin{cases} x = 0 \quad x = 3 \\ \text{undefined at } x = 2, 6 \end{cases}$

intervals $(-\infty, 0)$ $(0, 2)$ $(2, 3)$ $(3, 6)$ $(6, \infty)$

intervals	16	x	$(3-x)$	$(x-2$
$(-\infty, 0)$	$+$	$-$	$+$	
$(0, 2)$	$+$	$+$	$+$	
	$+$	$+$	$-$	

$$f'(x) = \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

$$f''(x) =$$

Sketch the curve for the following function:

$$(x-2)(x-6)\,y = 2x^2$$

15:29

$$f(x) = \frac{2x^2}{(x-2)(x-6)}$$

Domain ; $f(x) = \begin{cases} \text{defined everywhere} \\ \text{except } x = 2, 6 \end{cases}$

zero crossing $\quad f(x) = 0 \qquad x = 0$

Monotonicity and extremas $\rightarrow$ Finding $f'(x)$ & finding critical nos & evaluating its intervals

$$f'(x) = \frac{(x-2)(x-6)(4x) - 2x^2 \frac{d}{dx}\left((x-2)(x-6)\right)}{(x-2)^2(x-6)^2}$$

$$(x-2)(x-6) = x^2 - 6x - 2x + 12$$
$$= x^2 - 8x + 12$$

$$f'\left((x-2)(x-6)\right) = 2x-8$$

$$f'(x) = \frac{\cancel{4x^3} - 32x^2 + 48x - \cancel{4x^3} + 16x^2}{(x-2)^2(x-6)^2}$$

$$= \frac{-16x^2 + 48x}{(x-2)^2(x-6)^2} = \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

critical numbers are = $\begin{cases} x = 0, \; x = 3 \\ \text{undefined at } x = 2, \; x = 6 \end{cases}$

intervals: $(-\infty, 0)$ $(0, 2)$ $(2, 3)$ $(3, 6)$ $(6, \infty)$

interval	$16x$	$(3-x)$	$(x-2)^2$	$(x-6)^2$	$f'(x)$		
$(-\infty, 0)$	−	+	+	+	−	↓	\
$(0, 2)$	+	+	+	+	+	↑	/
$(2, 3)$	+	+	+	+	+	↑	/
$(3, 6)$	+	+	+	+	+	↑	/
$(6, \infty)$	+	+	+	+	+	↑	/

$$f'(x) = \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

For finding 2^{nd} derivative: Let $f'(x) = y$

$$y = \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

$$\ln y = \ln\left|\frac{16x(3-x)}{(x-2)^2(x-6)^2}\right|$$

$$\ln y = \ln|16x(3-x)| - \ln|(x-2)^2(x-6)^2|$$

$$\ln y = \ln|16x| + \ln|3-x| - \left[\ln|(x-2)^2| + \ln|(x-6)^2|\right]$$

$$\ln y = \ln|16| + \ln|x| + \ln|3-x| - \left[2\ln|x-2| + 2\ln|x-6|\right]$$

$$\frac{1}{y}\frac{dy}{dy} = 0 + \frac{1}{x} + \frac{1}{3-x} - \left[\frac{2}{x-2} + \frac{2}{x-6}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3-x+x}{x(3-x)} - \left[\frac{2x-12+2x-4}{(x-2)(x-6)}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x(3-x)} - \frac{4x-16}{(x-2)(x-6)}$$

$$\frac{dy}{dx} = y\left[\frac{3(x-2)(x-6) - 2x(x-4)(3-x)}{x(3-x)(x-2)(x-6)}\right]$$

$$f''(x) = \frac{16\cancel{x}(\cancel{3-x})}{(x-2)^2(x-6)^2}\left[\frac{3(x-2)(x-6) - 2x(x-4)(3-x)}{\cancel{x}(\cancel{3-x})(x-2)(x-6)}\right]$$

$$f''(x) = \frac{48(x-2)(x-6)}{(x-2)^3(x-6)^3} - \frac{32x(x-4)(3-x)}{(x-2)^3(x-6)^3}$$

Numerator:

$$= 48(x-2)(x-6) = (x^2 - 6x - 2x + 12)\,48$$

$$= (x^2 - 8x + 12)\,48$$

$$= 48x^2 - 384x + 576$$

$$= 32x((x-4)(3-x)) = (3x - x^2 - 12 + 4x)\,32x$$

$$= (7x - 12 - x^2)\,32x$$

$$= 224x^2 - 384x - 32x^3$$

$$48x^2 - \cancel{384}x + 576 - 224x^2 + \cancel{384}x + 32x^3$$

$$= 32x^3 - 176x^2 + 576$$

$$f''(x) = \frac{32x^3 - 176x^2 + 576}{(x-2)^3 (x-6)^3}$$

$$\ln f'(x) = \ln \frac{16x(3-x)}{(x-2)^2(x-6)^2}$$

$$= \ln|48x - 16x^2| - \ln|(x-2)^2(x-6)^2|$$

$$= (-32x+48) \cdot \frac{1}{\ln|48x-16x^2|} -$$

$$\frac{d}{dx}\left(\ln|(x-2)^2| + \ln|(x-6)^2|\right)$$

$$= 1 \cdot \frac{2(x-2)}{\ln|x-2|} + \frac{2(x-6)}{\ln|x-6|}$$

$$\ln f'(x) = \frac{-32x+48}{\ln|48x-16x^2|} + \frac{2x-4}{\ln|x-2|} + \frac{2x-12}{\ln|x-6|}$$

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$$f(x) = \frac{4x^2 - 16x - 9}{x^2 - 4x - 5}$$

Denom: $\underline{1} \times \underline{-5} = -5$

$\underline{1} + \underline{-5} = -4$

$x^2 + x - 5x - 5 = 0$

$x(x+1) - 5(x+1) = 0$

$(x+1)(x-5) = 0$

Num: $-1\underline{8} \times \underline{2} = -36$

$-\underline{18} + \underline{2} = -16$

$4x^2 - 18x + 2x - 9$

$2x(2x-9) + 1(2x-9)$

$(2x-9)(2x+1)$

$$f(x) = \frac{(2x-9)(2x+1)}{(x+1)(x-5)}$$

Domain: $\begin{cases} \text{Defined everywhere.} \\ \text{except } x = 1, \quad x = 5 \end{cases}$

zero crossing

$f(x) = 0 \qquad (2x-9)(2x+1) = 0$

$x = \frac{9}{2} \qquad x = -\frac{1}{2}$

at $\Rightarrow \qquad x = 4.5 \qquad x = -0.5$

vertical asymptode $\rightarrow$ at $x = -1$ and $x = 5$

critical points $\rightarrow$ $f'(x) = 0$

$f'(x) =$

$$\frac{(x^2-4x-5)(8x-16)-(4x^2-16x-9)(2x-4)}{(x+1)^2(x-5)^2}$$

$$= \frac{(x+1)(x-5)8(x-2)-(2x-9)(2x+1)2(x-2)}{(x+1)^2(x-5)^2}$$

$$= \frac{(x-2)\left[(x+1)(x-5)8 - 2(2x-9)(2x+1)\right]}{(x+1)^2(x-5)^2}$$

$$= \frac{(x-2)\left[(x^2-5x+x-5)8 - 2(4x^2+2x-18x-9)\right]}{(x+1)^2(x-5)^2}$$

$$= \frac{(x-2)\left[(x^2-4x-5)8 - 2(4x^2-16x-9)\right]}{(x+1)^2(x-5)^2}$$

$$= \frac{(x-2)\left[\cancel{8x^2} - \cancel{32x} - 40 - \cancel{8x^2} + \cancel{32x} + 18\right]}{(x+1)^2(x-5)^2}$$

$$= \frac{(x-2)(-22)}{(x+1)^2(x-5)^2} = \frac{-22(x-2)}{(x+1)^2(x-5)^2}$$

critical nos: $\begin{cases} x = 2 \\ \text{undefined at } x=-1 \;\; x=5 \end{cases}$

intervals $(-\infty, -1)$:54

$$f''(x) = \frac{(x+1)^2(x-5)^2 \frac{d}{dx}(-22(x-2)) - -22(x-2)\frac{d}{dx}((x+1)^2(x-5)^2)}{}$$

$$(x+1)^2 \quad (x-5)^2$$

$$= \frac{d}{dx}\left((x+1)^2(x-5)^2\right)$$

$$= (x+1)^2 \cdot 2(x-5) + (x-5)^2 \cdot 1 \cdot 2(x+1)$$

$$= 2(x-5)(x+1)^2 + 2(x+1)(x-5)^2$$

$$= 2(x-5)(x+1)\left[x+1 + x-5\right]$$

$$= 2(x-5)(x+1)(2x-4)$$

$$= 4(x-5)(x+1)(x-1)$$

$$f''(x) = \frac{(x+1)^2(x-5)^2(-22x) - 22(x+2)\,4(x-5)(x+1)(x-1)}{(x+1)^4(x-5)^4}$$

$$= \frac{22(x-5)(x+1)\left[(x+1)(x-5)(-x) - (x+2)4(x-1)\right]}{(x+1)^4(x-5)^4}$$

$$= \frac{22(x-5)(x+1)\left[(x^2-5x+x-5)(-x) - 4(x^2-x+2x-2)\right]}{(x+1)^4(x-5)^4}$$

$$= \frac{22(x-5)(x+1)\left[(x^2-4x-5)(-x) - (4x^2+4x-8)\right]}{(x+1)^4(x-5)^4}$$

$$= \frac{22(x-5)(x+1)\left[-x^3 + \cancel{4x^2} + 5x - \cancel{4x^2} - 4x + 8\right]}{(x+1)^4(x-5)^4}$$

$$= \frac{22(x-5)(x+1)\left[-x^3 + x + 8\right]}{}$$

$$y = \frac{-22\,(x-2)}{(x+1)^2(x-5)^2}$$

$$\ln y = \ln\left|-22(x-2)\right| - \ln\left|(x+1)^2(x-5)^2\right|$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\ln|-22| + \ln|x-2|\right] - \left[\ln\left|(x+1)^2\right| + \ln\left|(x-5)^2\right|\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \left[0 + 1 \cdot \frac{1}{x-2}\right] - \left[\frac{2(x+1)}{x+1} + \frac{2(x-5)}{(x-5)}\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-2} - \left[2 + 2\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-2} - 4$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1 - 4(x-2)}{x-2}$$

$$\frac{dy}{dx} = \frac{y\,(1 - 4x + 8)}{x-2} = \frac{y\,(9-4x)}{x-2}$$

$$\frac{dy}{dx} = \frac{-22(x-2)\,(9-4x)}{(x+1)^2\,(x-5)^2\,(x-2)}$$

$$\frac{dy}{dx} = \frac{-22\,(9-4x)}{(x+1)^2(x-5)^2}$$

$$x^2 - 4x + 7$$

$$2ax = -4x$$

$$2a = -4$$

$$a = -2$$

$$a^2 = 4$$

$$x^2 - 4x + 4 + 7 - 4$$

$$x^2 - 2x - 2x + 4 + 3$$

$$x(x+2) - 2(x+2) + 3$$

$$(x+2)(x-2) + 3$$

$$(x+2)^2 + 3$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\ln|-22| + \ln(x-2|\right] - \left[\ln|(x+1)^2| + \ln|(x-5)^2|\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \left[\ln|-22| + \ln|x-2|\right] - \left[2\ln|x+1| + 2\ln|x-5|\right]$$

$$\frac{1}{y}dy = \left[0 + \frac{1}{x-2}\right] - \left[2\ \frac{1}{x+1} + \frac{2}{x-5}\right]$$

$$\frac{1}{y}dy = \frac{1}{x-2} - \frac{2}{x+1} - \frac{2}{x-5}$$

$$\frac{1}{y}dy = \frac{x+1-2(x-2)}{(x-2)(x+1)} - \frac{2}{(x-5)}$$

$$\frac{1}{y}\,dy = \frac{x+1-2x+2}{(x-2)(x+1)} - \frac{2}{(x-5)}$$

$$\frac{1}{y}\,dy = \frac{(3-x)(x-5) - 2(x-2)(x+1)}{(x-2)(x+1)(x-5)}$$

$$\frac{1}{y}\,dy = \frac{3x-15-x^2+5x - 2\left[x^2+x-2x-2\right]}{(x-2)(x+1)(x-5)}$$

$$\frac{1}{y}\,dy = \frac{3x-15-x^2+5x-2x^2+2x+4}{(x-2)(x+1)(x-5)}$$

$$\frac{1}{y}\,dy = \frac{10x-11-3x^2}{(x-2)(x+1)(x-5)}$$

$$dy = \frac{-22(x-2)}{(x+1)^2(x-5)^2}\,\frac{(10x-11-3x^2)}{(x-2)(x+1)(x-5)}$$

$$= \frac{-22\,(-3x^2+10x-11)}{(x+1)^3(x-5)^3}$$

$$\lim_{x \to -1} f(x) = \frac{4x^2 - 16x - 9}{x^2 - 4x - 5}$$

$$\lim_{x \to -1} \frac{4x^2 - 18x + 2x - 9}{x^2 - 5x + x - 5}$$

$$\lim_{x \to -1} \frac{2x(2x-9) + 1(2x-9)}{x(x-5) + 1(x-5)}$$

$$\lim_{x \to -1} \frac{(2x-9)(2x+1)}{(x-5)(x+1)} = (-2-9)(-2+1)$$

$$= (-11)(-1)$$

$$= 11$$

$$\lim_{x \to -1^+} \frac{11}{(x-5)(x+1)} \qquad \frac{+ve}{-ve} = -\infty$$

$$\underbrace{-0.9-5}_{-ve} \times \underbrace{-0.9+1}_{+ve} = -ve$$

$$\lim_{x \to -1^-} \frac{11}{(x-5)(x+1)} = \frac{+ve}{+ve} = +\infty$$

$$\underset{-ve}{-2-5} \times \underset{-ve}{-2+1} =$$