Probability

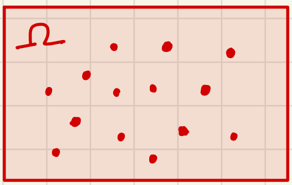
github.com/royceanton

- **Probabilistic Model:** A random phenomenon / random experiment

Involves 2 steps
↳ Describing possible outcomes
↳ Describing beliefs about the likelihood of outcomes

Sample space:

- List (Set) of all possible outcomes, Ω



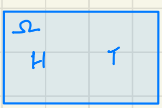
- A set must be **mutually exclusive**, ie if one of the outcomes (dots) in the set happened, the other outcomes must also not have occured at the end of the experiment.

- Must be **collectively exhaustive**: ie. together all the elements in the set exhaust all the possibilities. Meaning we will be able to pinpoint exactly one event from this set happened

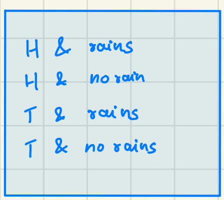
- The set must also be at the **right granularity**

- Why do we do this? So that when we put together a model, we decide how detailed our model has to be, which also inturn captures all the aspects that are of relevant interest to us

eg:



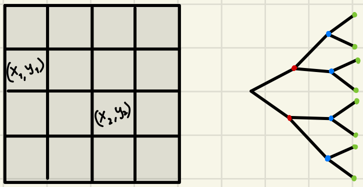
Sample space of heads & tails



Checking the weather & flipping a coin

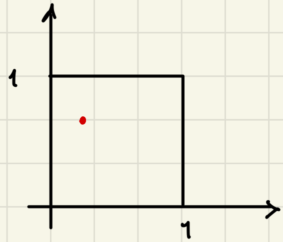
Sample spaces can be discrete, finite infinite, continuous & so on

Discrete Sample Space



out comes are finite & can be distinguished to be put into one of the boxes of branches

Continuous Sample Space

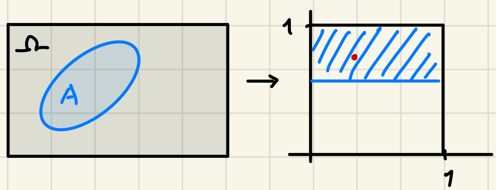


(x, y) such that $0 \leq x, y \leq 1$

- Throwing a dart on the unit square & noting the coordinates
- But here throwing a dart to the centre of square with infinite precision exactly would be zero.
  (individual points would have 0 probability on a continuous model)

Hence we create subsets in such continous models & assign probabilities to those subsets to get a probability than simply 0.

eg: Sub set A in a sample space:



This subset of a sample space is called an **event**

PROBABILITY AXIOMS:

- **Events:** A subset of sample space
  - → Probability is assigned to these events

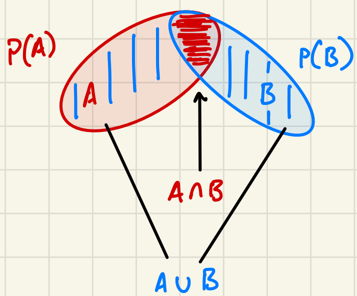
- **Axioms:**

$\boxed{\rightarrow \text{ Non negativity } \quad P(A) \geq 0}$ (a)

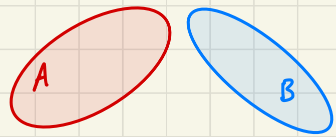
$\boxed{\rightarrow \text{ Normalization } \quad P(\Omega) = 1}$ (b)

→ (Finite) additivity:
During modelling of outcomes, there can be events which can overlapping (joint) or disjoint events:

→ Joint events



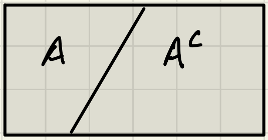
→ Disjoint events



if $P(A \cap B) = \phi$ (empty set)

$\boxed{P(A \cup B) = P(A) + P(B)}$ (c)

Now let us look at a sample space A & consider the compliment of its subset A^c



$A \cup A^c = \Omega$

$A \cap A^c = \phi$

Using axiom (b) we have:

$P(\Omega) = 1 \quad \therefore \quad P(A \cup A^c) = 1 \rightarrow$ 1

Using axiom (c) we have for disjoint events

$$P(A \cup B) = P(A) + P(B)$$

$$\therefore \; P(A \cup A^c) = P(A) + P(A^c) \longrightarrow ②$$

① in ② gives:

$$1 = P(A) + P(A^c)$$

$$\boxed{P(A) = 1 - P(A^c)}$$

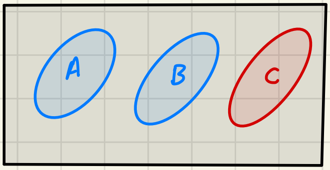
Hence when it comes to a complete set :

$$\underbrace{P(\Omega)}_{1} = 1 - \underbrace{P(\Omega^c)}_{\phi}$$

$$\therefore \; 1 = 1 - P(\Omega^c)$$

$$\boxed{\therefore \; P(\Omega^c) = 0 = \phi}$$

→ Now when it comes to disjoint events and additive property, if we have 3 disjoint events or more:



$$P(A \cup B \cup C)$$

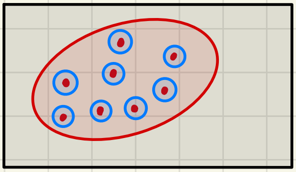
$$= P(A \cup B) + P(C)$$

$$= P(A) + P(B) + P(C)$$

$$\boxed{\begin{gathered}\therefore \text{ if } A_1, A_2, A_3 \ldots\ldots A_k \text{ disjoint events; then:} \\ P(A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_k) = \sum_{i=1}^{k} P(A_i)\end{gathered}}$$

→ Calculating Probability of a finite set:

Imagine we have sample space Ω filled with particular elements $\{S1\}, \{S2\}, \ldots \{S_k\}$ & these individual elements form together a larger finite set. If we have to calculate the probability of such a set:



$$P(\{s_1, s_2, s_3 \ldots s_k\}):$$
$$= P(\{s_1\} \cup \{s_2\} \cup \{s_3\} \ldots \cup \{s_k\})$$
$$= P(s_1) + P(s_2) + \ldots P(s_k)$$

EXERCISE

Let A, B, and C be disjoint subsets of the sample space. For each one of the following statements, determine whether it is true or false. *Note:* "False" means "not guaranteed to be true."

a) $\mathbf{P}(A) + \mathbf{P}(A^c) + \mathbf{P}(B) = \mathbf{P}(A \cup A^c \cup B)$

False   Answer: False

b) $\mathbf{P}(A) + \mathbf{P}(B) \leq 1$

True   Answer: True

c) $\mathbf{P}(A^c) + \mathbf{P}(B) \leq 1$

False   Answer: False

d) $\mathbf{P}(A \cup B \cup C) \geq \mathbf{P}(A \cup B)$

True   Answer: True

Solution:

a) False. For a counterexample, let $A = \emptyset$, $B = \Omega$, and $C = \emptyset$. In that case, the left-hand side of the equation equals 2, whereas the right-hand side equals 1.

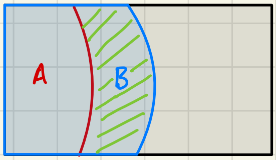
b) True. Since A and B are disjoint, we have $\mathbf{P}(A) + \mathbf{P}(B) = \mathbf{P}(A \cup B) \leq 1$.

c) False. For a counterexample, let $A = \emptyset$, $B = \Omega$, and $C = \emptyset$. In that case, $\mathbf{P}(A^c) + \mathbf{P}(B) = 2$.

d) True. Since A, B, and C are disjoint, we have
$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) \geq \mathbf{P}(A) + \mathbf{P}(B) = \mathbf{P}(A \cup B)$.

→ $\boxed{\text{if } A \subset B \text{ then } P(A) \leq P(B)}$

Proof :



Here, the set B can be expressed as a union of two pieces. One piece is set A itself & the other would be whatever is remaining in B after removing A from B which can be represented as a combination of $B \cap A^c$

$$\therefore \quad B = A + B \cap A^c$$

Hence $\quad P(B) = P(A) + \underline{P(B \cap A^c)}$

Now if we look at the RHS of the equation, we know probabilities are non negative. Therefore :

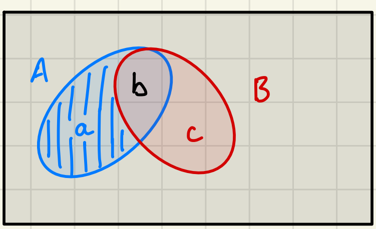
$$P(B \cap A^c) \geq P(A)$$

Hence, we can write :

$$P(B) = P(A) + P(B \cap A^c) \geq P(A)$$

$$\therefore \quad P(A) \leq P(B)$$

→ $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$



By indicating the individual probabilities inside the subsets each pieces as a, b & c, we can represent :

$$a = P(A \cap B^c)$$

$$b = P(A \cap B)$$

$$c = P(B \cap A^c)$$

Now looking at the standard equation in green box above; if we look at the LHS & RHS seperately, we can write them in terms of a, b & C to equal & check if they match.

LHS:

$$P(A \cup B) = \underbrace{a + b + c}_{\text{denotes the set pieces it consists of}}$$

RHS:

$$P(A) + P(B) - P(A \cap B)$$

$P(A) \rightarrow a+b$

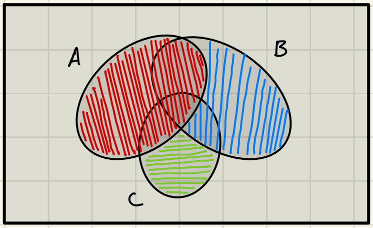
$P(B) \rightarrow b+c$

$P(A \cap B) \rightarrow b$

$$= a + b + b + c - b = a + b + c = LHS$$

$\rightarrow$

$$\boxed{P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)}$$



Using set theoretic relations, if we denote:

$A \longrightarrow$ set A, then we can express the remaing ones in terms of A, ie:

$B \longrightarrow$ set B & whatever is removed from A

$C \longrightarrow$ set C & whatever is removed from A & B which shows up in C

$$\therefore \; A \cup B \cup C$$

$A \rightarrow A$

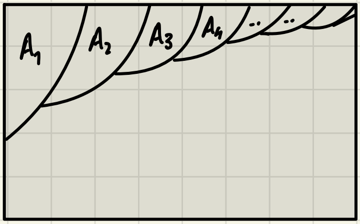
$B \rightarrow (B \cap A^c)$

$C \rightarrow (C \cap A^c \cap B^c)$

Hence $P(A \cup B \cup C) = P(A) + P(B \cap A^c) + P(C \cap A^c \cap B^c)$

→ Countable Additivity Axiom

if we have ==infinite sequence== of disjoint events



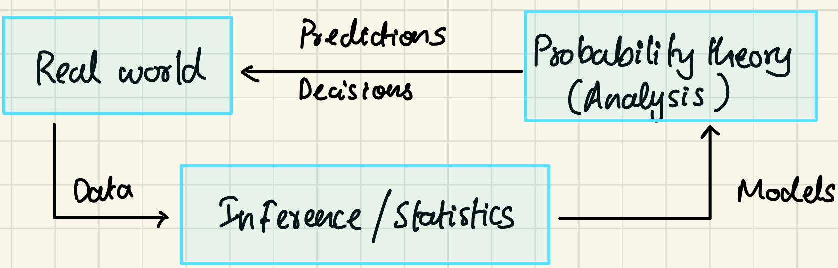
then :

$$P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A) + \dots$$

- It is important to notice that this additivity only holds for 'countable' sequence of events
- The elements of unit square (similarly, the real line, etc) is not countable (ie, its elements cannot be arranged in a sequence)

The role of Probability Theory

- A framework for analyzing phenomena with certain outcomes
  - → Rules for consistent reasoning
  - → Used for predictions & decisions



Exercises due May 24, 2023 13:59 CEST Past due

Exercise: More properties

2 points possible (graded)

Let A, B, and C be subsets of the sample space, not necessarily disjoint. For each one of the following statements, determine whether it is true or false. *Note:* “False" means “not guaranteed to be true."

a) $\mathbf{P}\big((A\cap B)\cup(C\cap A^c)\big)\leq \mathbf{P}(A\cup B\cup C)$

True Answer: True

b) $\mathbf{P}(A\cup B\cup C)=\mathbf{P}(A\cap C^c)+\mathbf{P}(C)+\mathbf{P}(B\cap A^c\cap C^c)$

True Answer: True

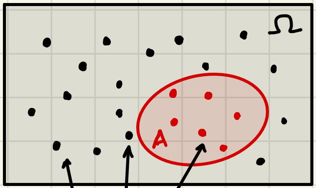
Solution:

a) True. This is because the set $(A\cap B)\cup(C\cap A^c)$ is a subset of $A\cup B\cup C$.

b) True. This is the same property shown in the last segment, with the three sets appearing in a different order.

Discrete Uniform Law

When we have a discrete uniform probability law, we can calculate probabilities by simply counting the number of elements of Ω & then by counting the number of elements of the set A.

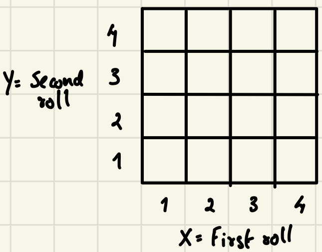


Prob = $\frac{1}{n}$ (whole sample space)

Assume Ω consists of n equally likely elements
Assume A consists of k elements

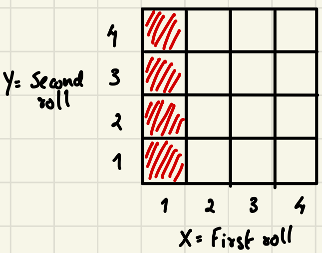
$\therefore \quad P(A) = k \cdot \frac{1}{n}$

Example: Two rolls of a Tetrahedral die (Discrete)



From the discrete events for two rolls of a tetrahedral die every possible outcome will have a probability of $\frac{1}{16}$

- $P(X=1)$



: Denotes the event X (First roll) where the first roll leads to a '1' & second roll Y can be any

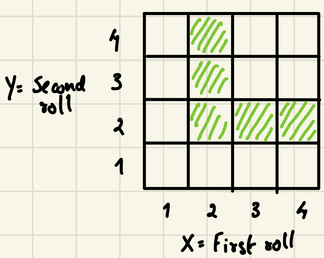
ie $\underbrace{(1,1),\ (1,2),\ (1,3),\ (1,4)}_{K}$

∴ $$P(X=1) = 4 \cdot \frac{1}{16} = \frac{4}{16}$$

- Let $Z = \min(X, Y)$

  ie if $X = 2$ & $Y = 3$; $Z = 2$

- $P(Z=2)$



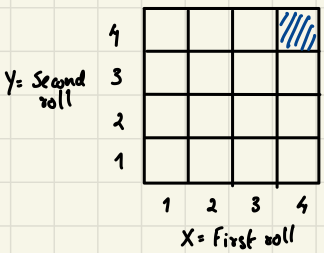
: Would be the event where if X rolls to a 2 Y can have 2, or even larger number. Similarly if Y rolls to a 2; it can also be possible on the previous X roll we may have gotten 2, or a larger number than 2.

So it can be: $\left.\begin{matrix} (2,2),\ (2,3),\ (2,4), \\ (3,2),\ (4,2) \end{matrix}\right\}$ $k = 5$

(2,2) not counted again

∴ $$P(Z=2) = 5 \cdot \frac{1}{16} = \frac{5}{16}$$

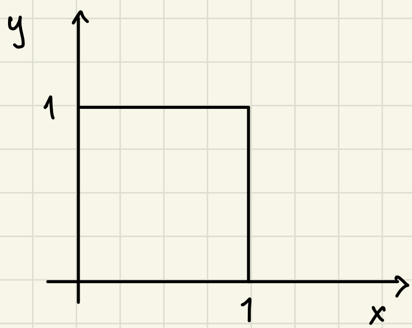
- $P(Z=4)$



: Would be the event where first roll gets a 4 & second roll gets a number larger than 4 but since this is a tetrahedral die the highest valid number on the die is 4 so essentially $Z = \min(X,Y) \Leftrightarrow (4,4)$ ie just one event

$$P(Z=4) = 1 \cdot \frac{1}{16}$$

Example: Continuous

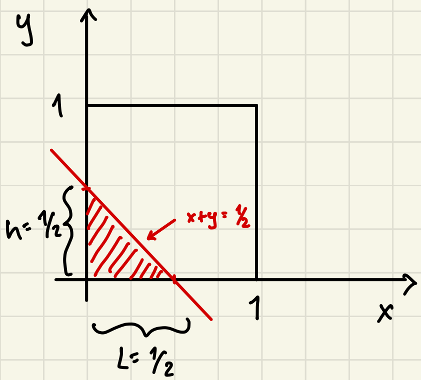


By uniform probability law, we know probability = Area

- (x,y) such that $0 \leq x, y \leq 1$

imagine (x,y) as a point on the unit square wall on to which a dart is thrown into. Since we cannot discretize as it a continous space, we will usually have a constraint kind of function to generate our subset; Thus finding the probability the point lies in the region.

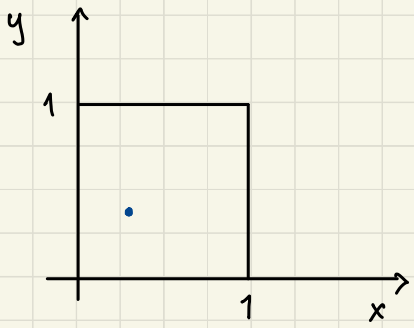
- Find $P(\{(x,y) \mid x+y \leq 1/2\})$



$\therefore P(\{(x,y) \mid x+y \leq 1/2\})$

$$= \frac{1}{2} \cdot l \cdot h = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

- Find $P(\{(0.5, 0.3)\})$:

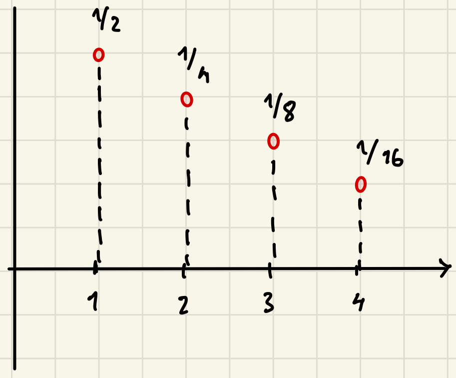


Here we are asked to find the probability that the dart falls into an exact point. In a continuous space this would be a point on the wall which we cannot define with the precise probability. Hence it would denote to calculating the area of the point => which is 0

$$P(\{(0.5, 0.3)\}) = 0$$

Discrete but infinite Sample Space

example ; think about tossing a coin untill we observe a head for the first time. So the sample space here can be any positive integer on the number line the first head can appear in the 1^{st}, 2^{nd} or 3^{rd} or so on.



Based on the probability law we should be able to determine the probability of every event (which is a subset) of the sample space. But insted; notice the probability of events that contain a single element.

therefore for sample space : $\{1, 2, 3, \ldots\}$

we are given $P(n) = \frac{1}{2^n}$; $n = 1, 2, 3, \ldots$

if we evaluate the geometric series : $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1 - 1/2} = 1$

- $P(\text{outcome is even}) = P(\{2,4,6,\dots\})$

$$= P(\{2\} \cup \{4\} \cup \{6\} \cup \dots) = P(2) + P(4) + P(6)$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} \dots$$

$$= \frac{1/4}{1 - 1/4} = \frac{1}{3}$$

Axioms Summary Sheet:

- Non negativity $P(A) > 0$
- Normalization $P(\Omega) = 1$

- Disjoint events finite additivity: if $P(A \cap B) = \emptyset$
  $P(A \cup B) = P(A) + P(B)$

- $P(A) = 1 - P(A)'$ $\qquad P(A)' = P(A)^c =$ complement of A
- $P(\Omega)' = 0$
- For disjoint events $A_1\ A_2\ A_3 \dots A_k$: $P(A_1 \cup A_2 \cup A_3 \dots A_k) = P(A_1) + P(A_2) + \dots = \sum_{i=1}^{k} P(A_i)$
- For A subset of B; $A \subset B$ then $P(A) \leq P(B)$
- For joint events $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Multiple joint events $P(A \cup B \cup C) = P(A) + P(A' \cap B) + P(A' \cap B' \cap C)$
- For infinite sequence of disjoint events $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) \dots$

EXERCISE:

1. The probability of the symmetric difference of two events

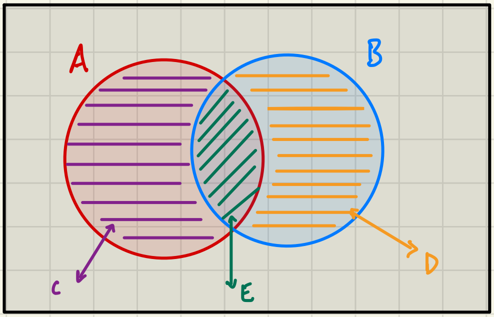
Bookmark this page

The probability of the symmetric difference of two events. Give a mathematical derivation of the formula

$$\mathbf{P}\Big((A\cap B^c)\cup(A^c\cap B)\Big)=\mathbf{P}(A)+\mathbf{P}(B)-2\cdot\mathbf{P}(A\cap B).$$

The event of interest, $\Big((A\cap B^c)\cup(A^c\cap B)\Big)$, is the event that exactly one of A and B occurs, and is called the **symmetric difference of $\mathbf{A}$ and $\mathbf{B}$**. Your derivation should be a sequence of steps, with each step justified by appealing to one of the probability axioms.

Teaching Assistant: Kuang Xu



Lets denote the subsets in sample space:

A → the whole red circular event

B → the blue circular event

C (≡) → the dashed lines creating a disjoint event inside A seperating it from $A \cap B$

Furthermore we can introduce notations to create disjoint events from these joint events →

D (≡) → the similar disjoint event from B

E (////) → the event from intersection $A \cap B$

Now that we have done that, we can rewrite the LHS of the question as:

$$P\big((A\cap B') \cup (A'\cap B)\big) = P(C\cup D)$$

$(A\cap B') \to C$, $(A'\cap B) \to D$

Now $P(C \cup D) = P(C) + P(D)$ is our LHS

we can also write $P(A)$ & $P(B)$ in terms of C & E

As $C \cap E = \emptyset$ $A = C \cup E$; $\therefore$ $P(A) = P(C \cup E)$

$$= P(C) + P(E)$$

$$P(C) = P(A) - P(E)$$

Similarly $D \cap E = \emptyset$ $B = D \cup E$; $\therefore$ $P(B) = P(D \cup E)$

$$= P(D) + P(E)$$

$$P(D) = P(B) - P(E)$$

Therefore we can rewrite LHS as

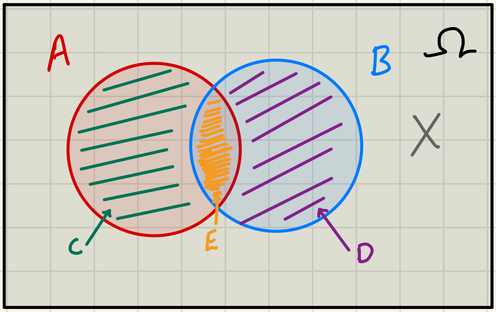
$$\begin{aligned} P(C \cup D) &= P(C) + P(D) \\ &= P(A) - P(E) + P(B) - P(E) \\ &= P(A) + P(B) - 2P(E) \end{aligned}$$

Now $P(E) = P(A \cap B)$

$\therefore$ $P(C \cup D) = P(A) + P(B) - 2P(A \cap B) = \text{RHS}$

Geniuses and chocolates. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Teaching Assistant: Katie Szeto



Let A be the set of geniuses

Let B be the set of chocolate lovers

$P(A) = 60\%$
$P(B) = 70\%$
$P(A \cap B) = 40\%$

& $P(\Omega) = 1$

we need to create disjoint events from these joint events to weed out the information we require

$P(A) = P(C) + P(E)$
$P(B) = P(D) + P(E)$

& $P(E) = P(A \cap B) = 0.4$

we need to find the probability that the randomly picked student is neither a genius nor a chocolate lover. (ie X)

First we find the individual disjoint events probabilities

$P(C) = P(A) - P(E)$
$= 0.6 - 0.4 = 0.2$

$P(D) = P(B) - P(E)$
$= 0.7 - 0.4 = 0.3$

$\therefore$ we know: $X + A + B = 1$

$= X + C + E + D = 1$

neither a chocolate lover or a genius; $X = 1 - (P(C) + P(E) + P(D))$

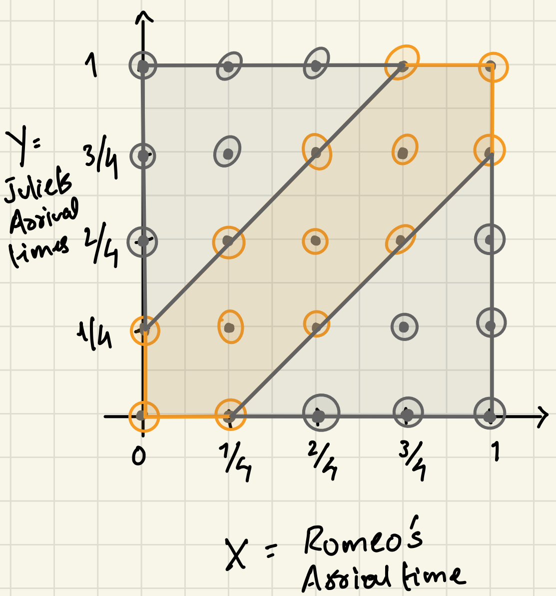
$= 1 - (0.2 + 0.4 + 0.3)$

$= 1 - 0.9$

$= 0.1$

Uniform probabilities on a square. Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being "equally likely," that is, according to a uniform probability law on the unit square. The first to arrive will wait for 15 minutes and will leave if the other has not arrived. What is the probability that they will meet?

Teaching Assistant: Jimmy Li



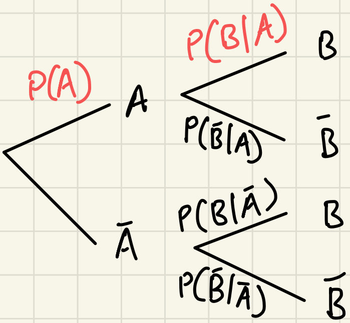
$P(\text{meet}) =$

$= 1 - \left(\frac{1}{2} \cdot l \cdot h\right) - \left(\frac{1}{2} \cdot l \cdot h\right)$

$= 1 - \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4}\right) - \left(\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}\right)$

Bayes Theorem

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B \mid A)}{P(B)}$$



$\rightarrow P(A \cap B) = P(A) \cdot P(B \mid A)$

Discrete Probability Distributions

- Permutations: A permutation of a set of objects places the object in order.

eg: Arranging set of numbers $\{1, 2, 3\} \rightarrow n! = 3! = 3 \cdot 2 \cdot 1 = 6$

$$nP_r = \frac{n!}{(n-r)!}$$

- Combinations : Takes no account of order

$$nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

- Expected Value of Discrete Random Variable

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i) = p_1 x_1 + p_2 x_2 + \dots p_n x_n$$

- Variance of Discrete Random Variable

$$V(X) = \sum_{i=1}^{n} p_i (x_i - \mu)^2$$

$$V(X) = E(X^2) - [E(X)]^2$$

where $E(X^2) = \sum_{i=1}^{n} x_i^2 P(X = x_i)$

- Standard deviation, σ :

$$\sigma = \sqrt{V(X)}$$

- Normal distribution → has mean μ & variance σ^2

$$X \sim N(\mu, \sigma^2) \quad ; \quad \text{std Norm dist } N \sim (\mu = 0, \sigma^2 = 1)$$

- The random variable Z from $N(\mu, \sigma^2)$

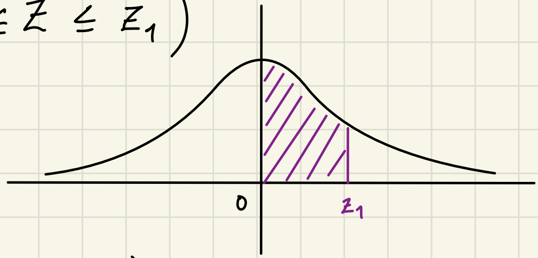
$$Z = \frac{X - \mu}{\sigma}$$

Table 1: **The Standard Normal Probability Integral**

$Z = \frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	4332	4345	4357	4370	4382	4394	4406	4418	4429	4441
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857
2.2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986

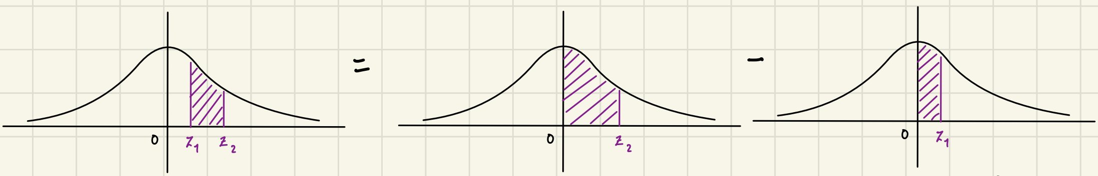
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

• $P(0 \le Z \le z_1)$



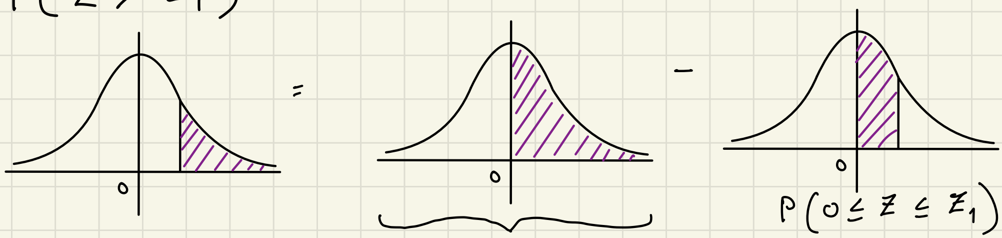
Fetch z_1 value directly from table when X is b/w $\mu = 0$ & z_1

• $P(z_1 \le Z \le z_2)$



$P(0 \le Z \le z_2) - P(0 \le Z \le z_1)$

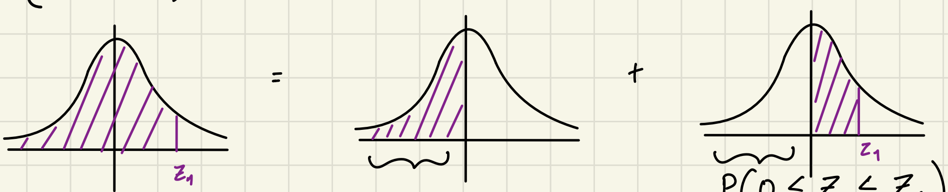
• $P(Z > z_1)$



area = 0.5 (shaded)

$P(0 \le Z \le z_1)$

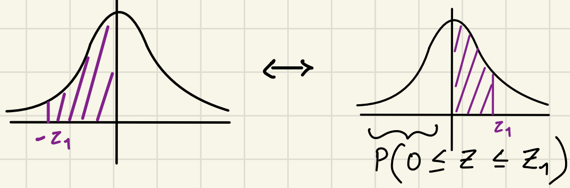
• $P(Z < z_1)$



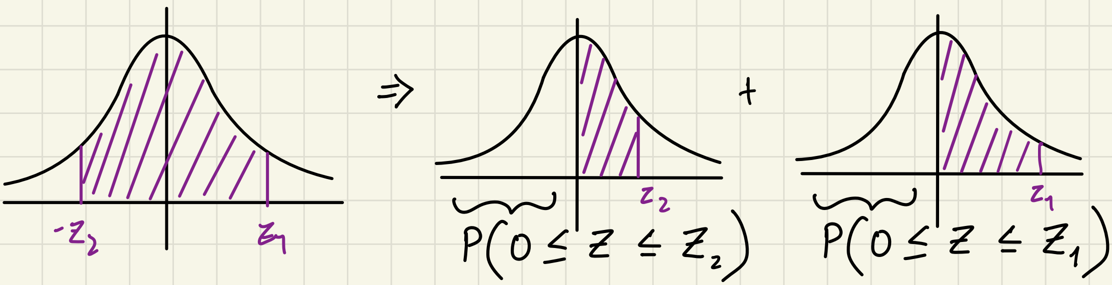
0.5

$P(0 \le Z \le z_1)$

- $P(-z_1 < Z < 0) \iff$ symmetry applies $\therefore \Rightarrow P(0 \le Z \le z_1)$

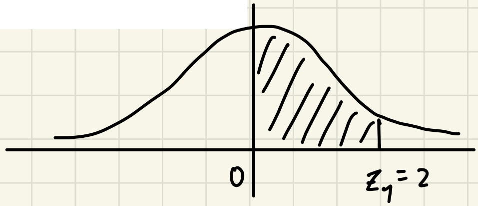


- $P(-z_2 < Z < z_1)$



What are the probabilities that Z takes values between

(a) 0 and 2 (b) 0 and 2.3 (c) 0 and 2.33 (d) 0 and 2.333?



a) $\mu = 0$ range $z_1 = 2$

From table

$P(0 \leq Z \leq 2) = 0.4772$

b) $P(0 \leq Z_1 \leq 2.3) = 0.4893$

c) $P(0 \leq Z_1 \leq 2.33) = 0.4901$

d) $P(0 \leq Z_1 \leq 2.333)$

Since we do not have 2.333 on the table; we can apply linear interpolation. b/w 2.33 & 2.34

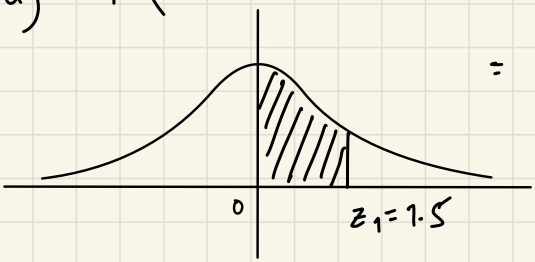
$2.33 = 0.4901$
$2.34 = 0.4904$
$2.333 = 0.4901 + 0.3\,(0.4904 - 0.4901)$

Find the following probabilities.

(a) $\mathrm{P}(0 < Z < 1.5)$	(b) $\mathrm{P}(Z > 1.8)$
(c) $\mathrm{P}(1.5 < Z < 1.8)$	(d) $\mathrm{P}(Z < 1.8)$
(e) $\mathrm{P}(-1.5 < Z < 0)$	(f) $\mathrm{P}(Z < -1.5)$
(g) $\mathrm{P}(-1.8 < Z < -1.5)$	(h) $\mathrm{P}(-1.5 < Z < 1.8)$

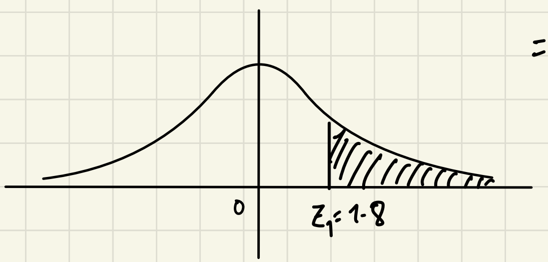
(A simple sketch of the standard normal curve will help.)

a) $P(0 < Z < 1.5)$



$= 0.4332$

b) $P(Z > 1.8)$

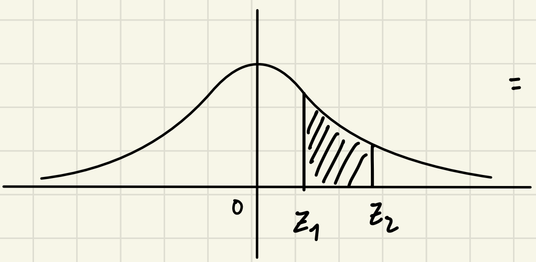


$= 0.5 - P(0 \leq Z \leq Z_1)$

$= 0.5 - 0.4641$

$= 0.0359$

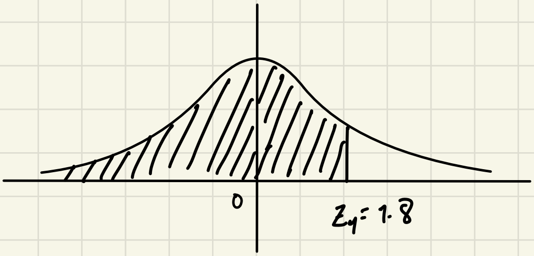
c) $P(1.5 < Z < 1.8)$



$= P(0 \leq Z \leq Z_2) - P(0 \leq Z \leq Z_1)$

$0.4641 - 0.4332$

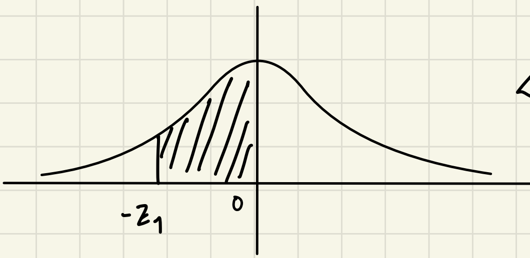
d) $P(Z < 1.8)$



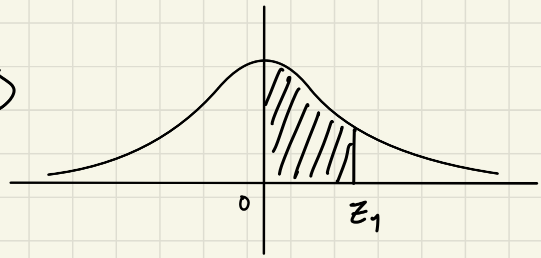
$= 0.5 + P(0 \le Z \le Z_1)$

$= 0.5 + 0.4641$

e) $P(-1.5 < Z < 0)$



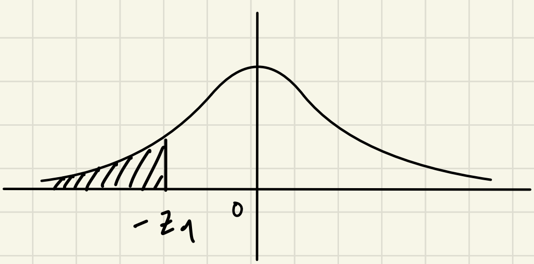
$\Leftrightarrow$



$P(0 < Z \le Z_1)$

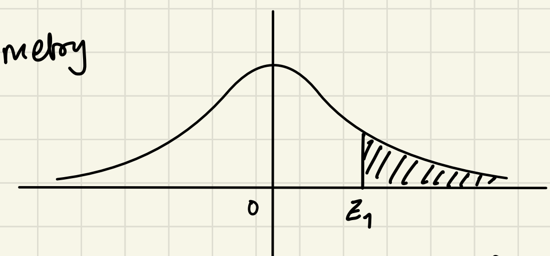
$= P(0 < Z < 1.5) = 0.4332$

f) $P(Z < -1.5)$



Symmetry

$\Leftrightarrow$



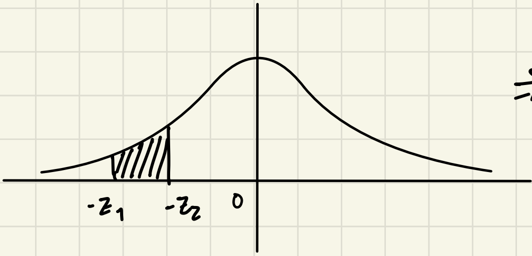
$= 0.5 - P(0 \le Z \le Z_1)$

$= 0.5 - 0.4332$

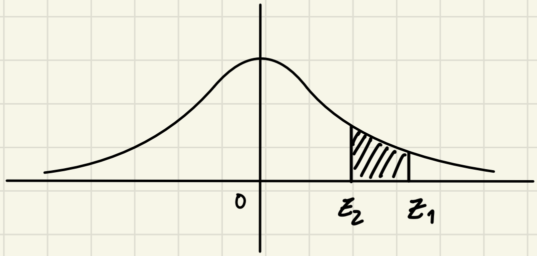
$= 0.0668$

g) $P(-1.8 < Z < -1.5)$

$\quad\; z_1 \qquad\qquad z_2$



$\Rightarrow$



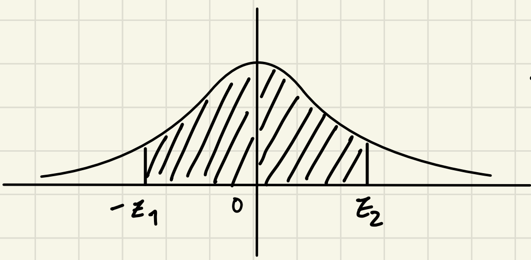
$P(0 \leq Z \leq z_1) - P(0 \leq Z \leq z_2)$

$= P(0 \leq Z \leq 1.8) - P(0 \leq Z \leq 1.5)$

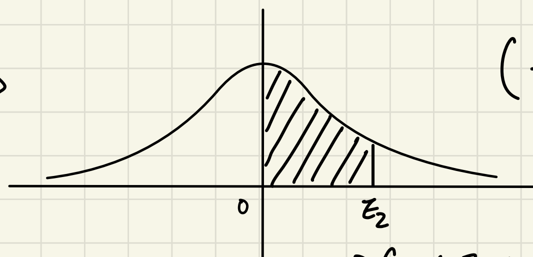
$= 0.4641 - 0.4332$

$= 0.0309$

h) $P(-1.5 < Z < 1.8)$

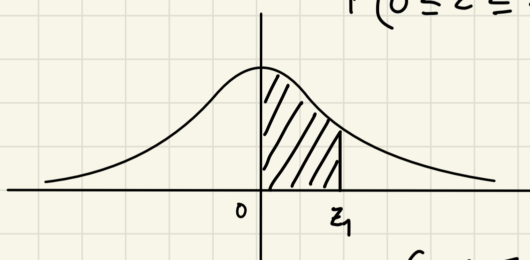


$\Rightarrow$



(+)

$P(0 \leq Z \leq z_2)$



$P(0 \leq Z \leq z_1)$

$= 0.4641 + 0.4332$

$= 0.8973$

Exercises

1. A machine is operated by two workers. There are sixteen workers available. How many possible teams of two workers are there?

2. A factory has 52 machines. Two of these have been given an experimental modification. In the first week after this modification, problems are reported with thirteen of the machines. What is the probability that both of the modified machines are among the thirteen with problems assuming that all machines are equally likely to give problems,?

3. A factory has 52 machines. Four of these have been given an experimental modification. In the first week after this modification, problems are reported with thirteen of the machines. What is the probability that exactly two of the modified machines are among the thirteen with problems assuming that all machines are equally likely to give problems?

4. A random number generator produces sequences of independent digits, each of which is as likely to be any digit from 0 to 9 as any other. If X denotes any single digit, find $\mathrm{E}(X)$.

5. A hand-held calculator has a clock cycle time of 100 nanoseconds; these are positions numbered $0, 1, \ldots, 99$. Assume a flag is set during a particular cycle at a random position. Thus, if X is the position number at which the flag is set.

$$\mathrm{P}(X = k) = \frac{1}{100} \qquad k = 0, 1, 2, \ldots, 99.$$

Evaluate the average position number $\mathrm{E}(X)$, and σ, the standard deviation.

(Hint: The sum of the first k integers is $k(k+1)/2$ and the sum of their squares is: $k(k+1)(2k+1)/6$.)

6. Concentric circles of radii 1 cm and 3 cm are drawn on a circular target radius 5 cm. A darts player receives 10, 5 or 3 points for hitting the target inside the smaller circle, middle annular region and outer annular region respectively. The player has only a 50-50 chance of hitting the target at all but if he does hit it he is just as likely to hit any one point on it as any other. If X = 'number of points scored on a single throw of a dart' calculate the expected value of X.

1 $16C_2 = \frac{16!}{2!\,14!} = \frac{16 \cdot 15 \cdot 14!}{2 \cdot 1 \cdot 14!} = \frac{16 \cdot 15}{2 \cdot 1} = 8 \cdot 15 = 120$

2 $52C_{13} = \binom{52}{13} = \frac{52!}{13!\,(52-13)!}$

But there is only $2C_2$ ways to pick

$$\therefore \binom{52-2}{13-2} = \binom{52}{11} = 52C_{11}$$

$$\therefore \text{Req prob} : \frac{\binom{2}{2}\binom{50}{11}}{\binom{52}{13}}$$

3) $$\frac{\binom{4}{2}\binom{52-4}{13-2}}{\binom{52}{13}}$$

4) $$X = \sum_{n=0}^{9} n \quad \Rightarrow 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

$$\therefore P(X = x) = \frac{1}{10}$$

$$E(x) = \frac{1}{10} \cdot 0 + \frac{1}{10} \cdot 1 + \frac{1}{10} \cdot 2 \dots + \frac{1}{10} \cdot 9$$

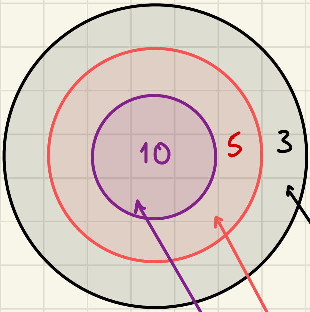
5) $$P(X = k) = \frac{1}{100} \qquad k = 0, 1, 2 \dots 99$$

$$E(x) = \frac{1}{100} \frac{n(n+1)}{2} \quad ; \; n = k$$

$$E(x^2) = \frac{1}{100} \frac{n(n+1)(2n+1)}{6}$$

6)

$X =$ no of points scored on a single draw



$(9\pi - \pi) = 8\pi$

$(25\pi - 9\pi) = 16\pi$

$(\pi) = \pi$

$$P(X=0) = 0.5 = \frac{1}{2}$$

$$\therefore P(X=3) = \frac{16\pi}{25\pi} \cdot \frac{1}{2} = \frac{8}{50}$$

$$P(X=5) = \frac{8\pi}{25\pi} \cdot \frac{1}{2} = \frac{8}{50}$$

1) Bayes Theorem → Example 1

A home pregnancy test was given to women. The pregnancy was verified through blood tests. The following table shows the home pregnancy results:

	positive	negative
pregnant	70	4
not pregnant	5	14

a) How many women are pregnant & tested positive ?

b) How many are not pregnant or tested negative ?

Let : P → event of being pregnant
T → event of being testest positive

		T positive	$\bar{T}$ negative	
P	pregnant	70	4	74
$\bar{P}$	not pregnant	5	14	19
		75	18	93

$P(P) = 74/93$

$P(\bar{P}) = 19/93$

$P(T) = 75/93$

$P(\bar{T}) = 18/93$

we need a) $P(P \cap T)$ b) $P(\bar{P} \cup \bar{T})$

$$P(P \cap T) = \frac{70}{93}$$

$$P(\bar{P} \cap \bar{T}) = P(\bar{P}) + P(\bar{T}) - P(\bar{P} \cap \bar{T})$$

$$= \frac{19}{93} + \frac{18}{93} - \frac{14}{93} = \frac{23}{93}$$

c) what is the probability of a woman testing -ve given that she is pregnant

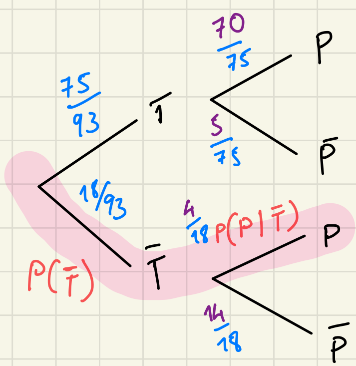
$$P(\bar{T} \mid P) = \frac{4}{74}$$

(Answer directly from table
→ Filter out 'given' (P's) row or column
→ Filter out whats needed ($\bar{T}$s) row or column)

Same Answer found using Bayes Theorem:

we start branch with T & $\bar{T}$

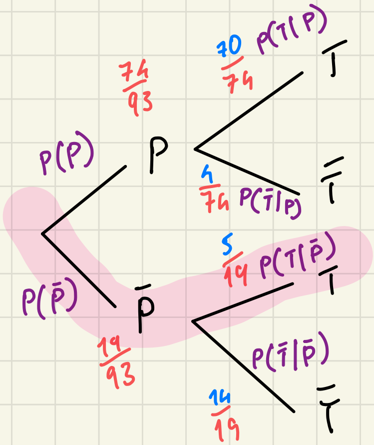
$$P(\bar{T} \mid P) = \frac{P(\bar{T} \cap P)}{P(P)} = \frac{P(\bar{T}) \cdot P(P \mid \bar{T})}{P(P)}$$



$$P(\bar{T} \cap P) = P(\bar{T}) \cdot P(P \mid \bar{T})$$
$$\frac{18}{93} \cdot \frac{4}{18}$$

$$P(\bar{T} \mid P) = \frac{\frac{18}{93} \cdot \frac{4}{18}}{\frac{74}{93}} = \frac{4}{93} \cdot \frac{93}{74} = \frac{4}{74}$$

d) $P(\text{not pregnant} \mid \text{positive}) = P(\bar{P} \mid T) = \dfrac{P(\bar{P} \cap T)}{P(T)}$



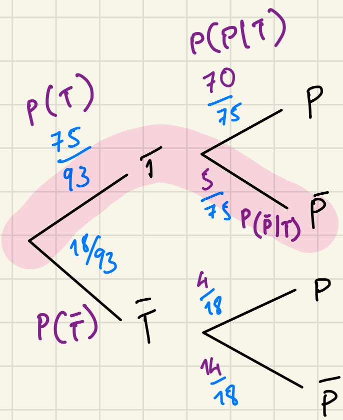
$$= \frac{P(\bar{P}) \cdot P(T \mid \bar{P})}{P(T)}$$

$$= \frac{\frac{19}{93} \cdot \frac{5}{19}}{\frac{75}{93}}$$

$$= \frac{19}{93} \cdot \frac{5}{19} \cdot \frac{93}{75} = \frac{5}{75}$$

e) $P(\text{positive test} / \text{not pregnant}) \Rightarrow P(T \mid \bar{P})$

$$= \frac{P(T \cap \bar{P})}{P(\bar{P})} = \frac{P(T)\, P(\bar{P} \mid T)}{P(\bar{P})}$$



$$= \frac{\frac{75}{93} \cdot \frac{5}{75}}{\frac{19}{93}}$$

$$= \frac{75}{93} \cdot \frac{5}{75} \cdot \frac{93}{19} = \frac{5}{19}$$

2) Example 2

A certain virus affects one in every 400 people. A test used to detect the virus in a person is positive 85% of the time. 5% of the time the person does not have the virus when tested +ve (false +ve)

a) Find the probability that the person has the virus given they tested positive

b) Do not have the virus given they tested negative

c) Do not have the virus given they tested positive

Approach :

We can create the following events:

$V \rightarrow$ The event the person has virus
$\bar{V} \rightarrow$ the event the person does not have the virus
$+ve \rightarrow$ The event the person is tested +ve
$-ve \rightarrow$ The event the person is tested -ve

we need $P(V \mid +ve) = P(V \cap +ve) = \dfrac{P(V) \cdot P(+ve \mid V)}{P(V)}$

$$P(\bar{V} \mid -ve) = P(\bar{V} \cap -ve) = \frac{P(\bar{V}) \cdot P(-ve \mid \bar{V})}{P(\bar{V})}$$

$$P(\bar{V} \mid +ve) = P(\bar{V} \cap +ve) = \frac{P(\bar{V}) \cdot P(+ve \mid \bar{V})}{P(\bar{V})}$$

~~we are given~~ $\cancel{P(V) = 85\%}$ $\therefore$ $\cancel{P(\bar{V}) = 15\%}$

we are told 1 in 400 has virus.

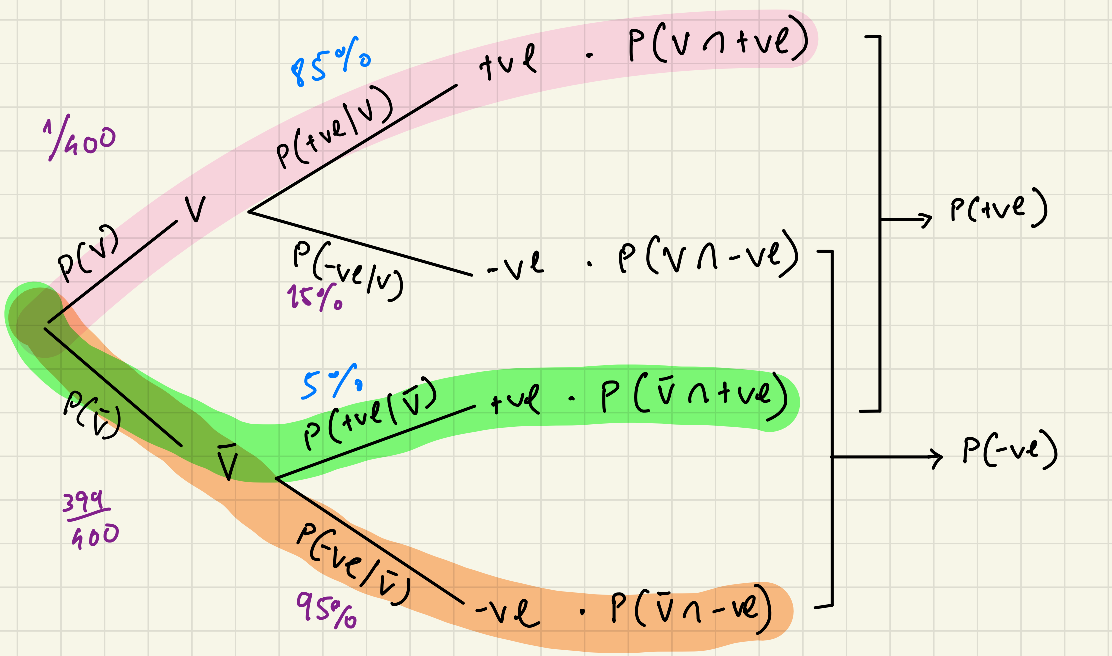
$\therefore P(V) = 1/400$ $\quad \therefore P(\bar{V}) = \frac{399}{400}$

$= 0.25\%$ (in % terms) $\quad = (100 - 0.25)\%$

$= 99.75\%$

also given $P(+ve \mid V) = 85\%$ &

$P(+ve \mid \bar{V}) = 5\%$



Additionally to solve the problem

we need $P(+ve) \Rightarrow P(+ve) = P(V \cap +ve) + P(\bar{V} \cap +ve)$

$$= \frac{1}{400} \cdot \frac{85}{400} + \frac{399}{400} \cdot \frac{5}{100}$$

&

$P(-ve) \Rightarrow P(-ve) = P(V \cap -ve) + P(\bar{V} \cap -ve)$

$$= \frac{1}{400} \cdot 15\% + \frac{399}{400} \cdot 95\%$$

$$\boxed{P(V \mid +ve)} = P(V \cap +ve) = \frac{P(V) \cdot P(+ve \mid V)}{P(+ve)} = \frac{\frac{1}{400} \cdot \frac{85}{100}}{\frac{1}{400} \cdot \frac{85}{100} + \frac{399}{400} \cdot \frac{5}{100}}$$

$$\boxed{P(\bar{V} \mid -ve)} = P(\bar{V} \cap -ve) = \frac{P(\bar{V}) \cdot P(-ve \mid \bar{V})}{P(-ve)} = \frac{\frac{399}{400} \cdot 95\%}{\frac{1}{400} \cdot 15\% + \frac{399}{400} \cdot 95\%}$$

$$\boxed{P(\bar{V} \mid +ve)} = P(\bar{V} \cap +ve) = \frac{P(\bar{V}) \cdot P(+ve \mid \bar{V})}{P(+ve)} = \frac{\frac{399}{400} \cdot 5\%}{\frac{1}{400} \cdot 85\% + \frac{399}{400} \cdot 5\%}$$

3) Example 3

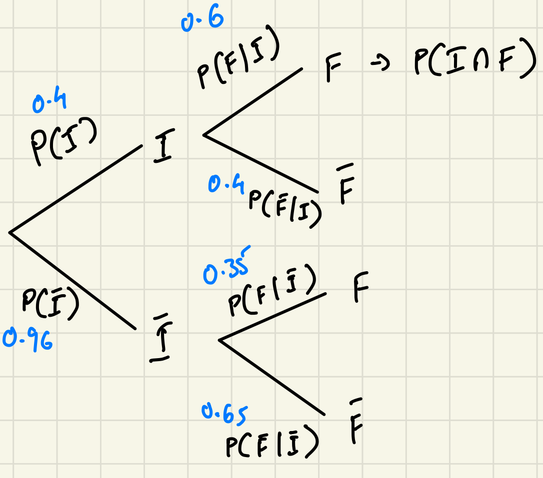
According to publicly traded companies 60% of the companies that increased their share price by more than 5% in the last 3 years replaced their CEO during the period. At the same time only 35% of the companies that did not increase their share price in the period replaced their CEO's. Knowing that the stock prices grow by more than 5% is 4%. Find the probability that the shares of the companies that fire their CEO's will increase by more than 5%.

we need to find the: Probability (Shares of company increase by more than 5% | fire their CEO's)

Let $F \to$ be the event the company fires the CEO
$\bar{F} \to$ the event the company does not fire

$I \to$ the event the share price increase by 5%
$\bar{I} \to$ the event the share price does not increase by 5%

We need $P(I|F) = \dfrac{P(I) \cdot P(F|I)}{P(F)}$



given $60\% \to I \to F$

$35\% \to \bar{I} \to F$

$I \to 4\%$

$\therefore \bar{I} \to 96\%$

$$P(F) = 0.4 \cdot 0.6 + 0.96 \cdot 0.35$$

$$P(\bar{F}) = 0.4 \cdot 0.4 + 0.96 \cdot 0.65$$

$$P(I|F) = \frac{P(I) \cdot P(F|I)}{P(F)} = \frac{0.4 \cdot 0.6}{P(F)}$$

$$= \frac{0.4 \cdot 0.6}{0.4 \cdot 0.6 + 0.96 \cdot 0.35}$$

4) Example 4

For the given contigency table, Find :

	T	$\bar{T}$
R	15	135
$\bar{R}$	45	410

a) $P(R \cap T)$
b) $P(T|R)$
c) $P(R \cup T)$

Approach

	T	$\bar{T}$	
R	15	135	150
$\bar{R}$	45	410	455
	60	545	605

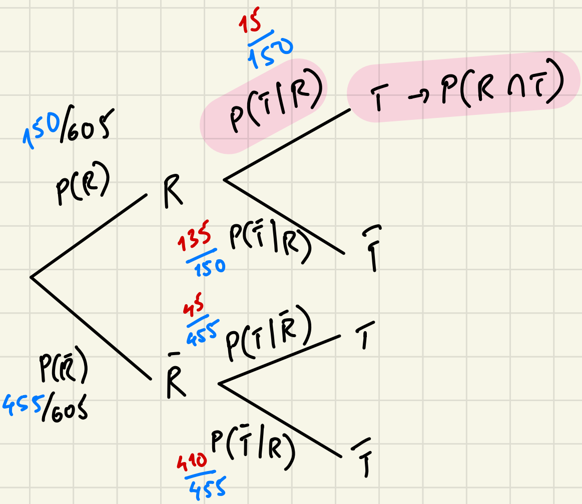
$P(R) = 150/605$
$P(\bar{R}) = 455/605$
$P(T) = 60/605$
$P(\bar{T}) = 545/605$

From the extracted info from Contigency table; we have:

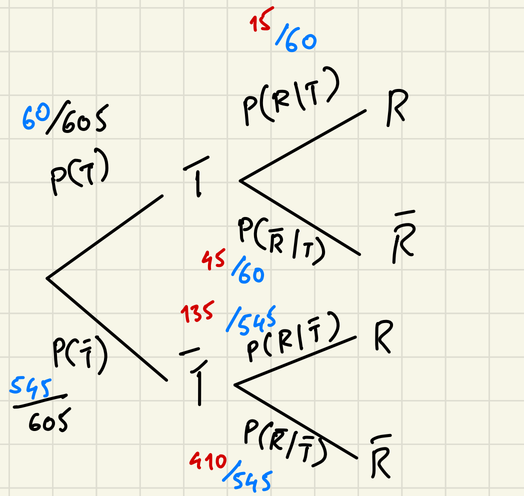
$$P(R \cap T) = P(R)\, P(T|R)$$

$$P(T|R) = \frac{P(T \cap R)}{P(R)} = \frac{P(T) \cdot P(R|T)}{P(R)} \quad \text{or directly from Tree: 1}$$

Tree: 1



Tree: 2



a) $P(R \cap T) = \frac{150}{605} \cdot \frac{15}{150} = \frac{15}{605}$

b) $P(T|R) = \frac{15}{150}$

b) $P(T|R) = \frac{P(T \cap R)}{P(R)} = \frac{P(T) \cdot P(R|T)}{P(R)}$

$$= \frac{\frac{60}{605} \cdot \frac{15}{60}}{\frac{150}{605}}$$

$$= \frac{60}{605} \cdot \frac{15}{60} \cdot \frac{605}{150}$$

$$= \frac{15}{150} \quad (\text{same as tree: 1})$$

c) $P(R \cup T) = P(R) + P(T) - P(R \cap T)$

$$= \frac{150}{605} + \frac{60}{605} - \frac{15}{605} = \frac{195}{605}$$

Example 5

Suppose the travel time from your apartment to the university is normally distributed with a mean 40 min and std dev. 7 min. If you want to be 95 % certain that you will not be late for the math exam at 13:00 what is the latest time you should leave home.

$$Z = \frac{x - \mu}{\sigma}$$

given $\mu = 40$ min
$\sigma = 7$ min

& $Z_1 = 0.95 \rightarrow$

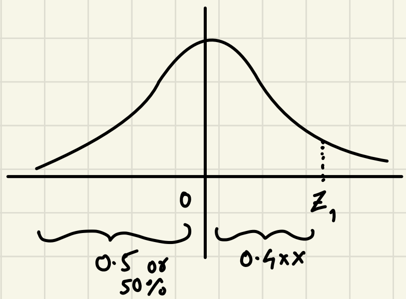


Table 1: The Standard Normal Probability Integral

$Z = \frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	4332	4345	4357	4370	4382	4394	4406	4418	4429	4441
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857
2.2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

we need to reverse estimate which Z_1 value would come approximately to 0.45 or 45% probability/certainity from the table & use the Z-score for it

Here it is 1.65

$\therefore Z = 1.65$

$$\therefore 1.65 = \frac{x - 40}{7}$$

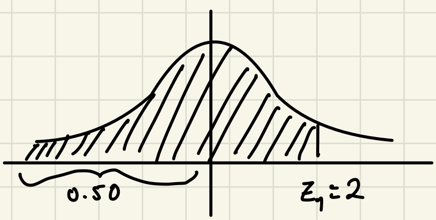
$$1.65 \cdot 7 + 40 = X$$
$$10.55 + 40 = X$$
$$X = 50.55 \text{ min}$$

$\therefore$ To reach the exam at 13:00 with 95% certainity, the student has to leave the apartment 50.55 min before the exam.

Example 6

The mean and std. dev of an IQ test is 100 & 15 respectively. You have an IQ of 130. What is your z-score? Is it usual or unusual?

given $\left.\begin{aligned} \mu &= 100 \\ \sigma &= 15 \\ X &= 130 \end{aligned}\right\} \Rightarrow \quad z = \dfrac{130-100}{15} = \dfrac{30}{15} = 2$



$z_1 = 2$'s probability is $= 0.4772$

which means $0.5 + 0.4772$
$= 0.9772$

meaning this IQ score is an outlier or simply unusual.

Table 1: The Standard Normal Probability Integral

$Z = \frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
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.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
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2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

Example 7

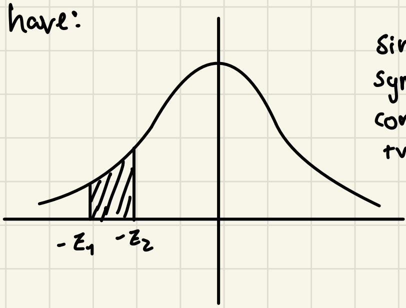
Find the probability that the student gets a test value b/w $74 < X < 78$. Assuming that the math exam is normally distributed with a mean of 82 and std. dev of 4.

given $\mu = 82$
$\sigma = 4$
$X_1 = 74 \quad X_2 = 78$

$$Z_1 = \frac{74 - 82}{4} \qquad \& \qquad Z_2 = \frac{78 - 82}{4}$$

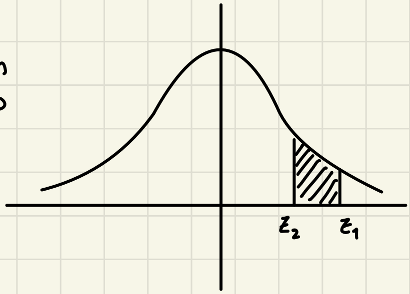
$$Z_1 = \frac{-8}{4} = -2 \qquad Z_2 = \frac{-4}{4} = -1$$

we have:



Normal dist. curve

since $N \sim (\mu, \sigma^2)$ is symmetrical; we can convert z-scores to +ve table



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.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
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.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
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2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

we need $P(Z_2 < Z < Z_1)$

$$= P(0 < Z < Z_1) - P(0 < Z < Z_2)$$
$$= 0.4772 - 0.3413$$
$$= 0.1359$$

So Se 2022

] a. Indicate which of the following statements are correct (+) or incorrect (-)

\+ -

- ☐ ☒ A probabilistic model requires at least three outcomes
- ☐ ☒ An elementary event occurs only in a deterministic model of a phenomenon
- ☒ ☐ Conditional probabilities are needed in all but the first level of a probability tree
- ☒ ☐ A random variable describes outcomes which lie in an interval in the set of real

bers

b) Given variable X follow $N(121, 12^2)$ calculate

1) $P(100 \leq X \leq 125)$

2) $P(X \leq 96)$

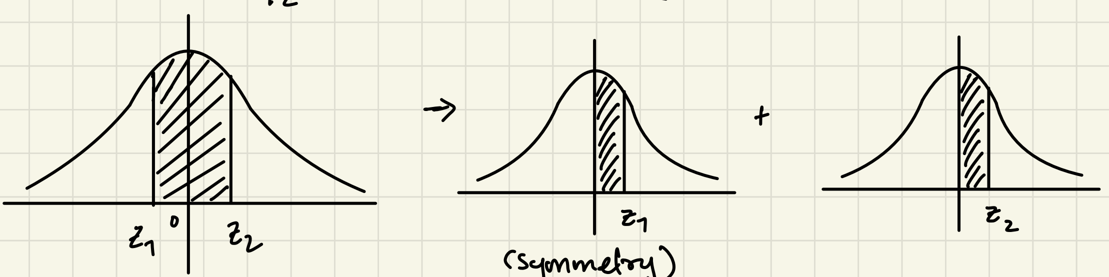
1) given $\mu = 121$

$\sigma = 12$

$x_1 = 100 \qquad x_2 = 125$

$$z_1 = \frac{100 - 121}{12} \qquad z_2 = \frac{125 - 121}{12}$$

$$z_1 = -\frac{21}{12} \approx 1.7 \qquad z_2 = \frac{4}{12} = \frac{1}{3} \approx 0.33$$



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.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
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.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
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1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
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1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
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2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

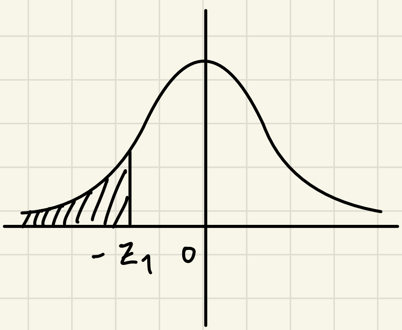
$$P(0 \le Z \le Z_1) + P(0 \le Z \le Z_2)$$

$$= 0.4554 + 0.1293$$

$$= 0.5847$$

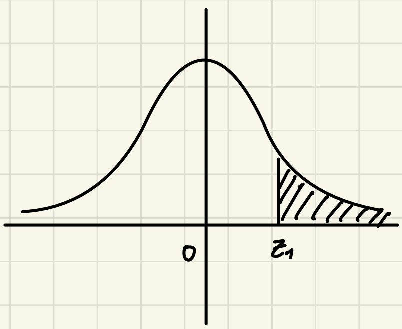
2) $P(x \le 96)$

$$Z = \frac{96 - 121}{12} = \frac{-25}{12} = -2.08$$



same as

$\Longleftrightarrow$



$0.5 - P(Z < -Z_1)$

$0.5 - P(z < -2.08)$

$0.5 - P(Z > Z_1)$

$0.5 - P(Z > 2.08)$

$= 0.5 - 0.4808$

$= 0.0188$

c)

[5] c. An airbag production company runs three looms H, J, and K to make the woven fabric disks used in their products. Loom H produces 30% of the disks, loom J makes 35%, and loom K produces the rest. The looms have been in the company for a while, and their reliability is well known. Of the cloth disks produced by loom H, 6% are defective, while from loom J, 3% are unusable. The newest machine, K, only messes up 2% of the time. A quality control engineer selects a random cloth disk for inspection and finds it to be unusable. Use Bayes' Theorem to determine the probability that it came from (i) loom H, (ii) loom J, or (iii) loom K.

H	J	K	
30%	35%	35%	disks
6%	3%	2%	defects.

we need $P(H \mid \text{unusable})$

$P(J \mid \text{unusuable})$

$P(K \mid \text{unusuable})$

Let $U \to$ event of usable

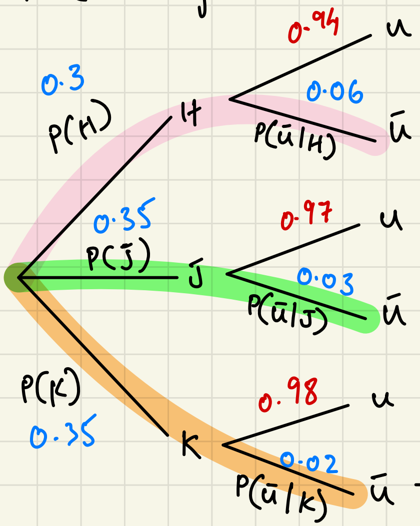
$\bar{U} \to$ event of unusable

$$P(H \mid \bar{U}) = \frac{P(H \cap \bar{U})}{P(\bar{U})} = \frac{P(H) \cdot P(\bar{U} \mid H)}{P(\bar{U})}$$

$$P(J \mid \bar{U}) = \frac{P(J \cap \bar{U})}{P(\bar{U})} = \frac{P(J)\ P(\bar{U} \mid J)}{P(\bar{U})}$$

$$P(K \mid \bar{U}) = \frac{P(K \cap \bar{U})}{P(\bar{U})} = \frac{P(K)\ P(\bar{U} \mid K)}{P(\bar{U})}$$

Tree diagram:



$\longrightarrow P(H \cap \bar{u}) = 0.3 \cdot 0.06$

$\longrightarrow P(J \cap \bar{u}) = 0.35 \cdot 0.03$

$\longrightarrow P(K \cap \bar{u}) = 0.35 \cdot 0.02$

Now we need $P(u)$ & $P(\bar{u})$

$$P(u) = 0.3 \cdot 0.94 + 0.35 \cdot 0.97 + 0.35 \cdot 0.98$$

$$P(\bar{u}) = 0.3 \cdot 0.06 + 0.35 \cdot 0.03 + 0.35 \cdot 0.02$$

$$P(H \mid \bar{u}) = \frac{0.3 \cdot 0.06}{0.3 \cdot 0.06 + 0.35 \cdot 0.03 + 0.35 \cdot 0.02}$$

$$P(J \mid \bar{u}) = \frac{0.35 \cdot 0.03}{0.3 \cdot 0.06 + 0.35 \cdot 0.03 + 0.35 \cdot 0.02}$$

$$P(K \mid \bar{u}) = \frac{0.35 \cdot 0.02}{0.3 \cdot 0.06 + 0.35 \cdot 0.03 + 0.35 \cdot 0.02}$$

Rhine-Waal University of Applied Sciences

[7] d. The manufacturing process for wire-wound resistors produces components with a normally distributed resistance. Resistors of type A have a mean resistance of 100Ω and a variance of $9\Omega^2$. Type B resistors have a mean resistance of 300Ω and a variance of $16\Omega^2$. If we form a series connection containing one of each type, what is the probability that the total resistance exceeds 388Ω?

A

$\mu_A = 100\,\Omega$

var, $\sigma_A^2 = 9\,\Omega^2$

std dev: $\sigma_A = 3\,\Omega$

B

$\mu_B = 300\,\Omega$

$\sigma_B^2 = 16\,\Omega^2$

$\sigma_B = 4\,\Omega$

given: we form a series connection

$\therefore$ we add up everything.

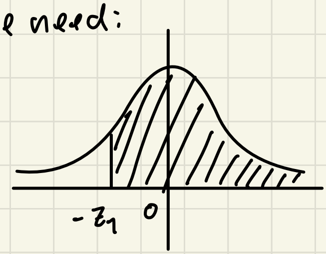
total $\begin{cases} \mu_T = \mu_A + \mu_B = 400\,\Omega \\ \sigma_T^2 = \sigma_A^2 + \sigma_B^2 = 9\,\Omega + 16\,\Omega = 25\,\Omega^2 \end{cases}$

$\sigma_T = 5\,\Omega$

we need $P(X \geq 388)$

$$z_1 = \frac{388 - 400}{5} = \frac{-12}{5} = -2.4$$

we need:



by symmetry it is same as:

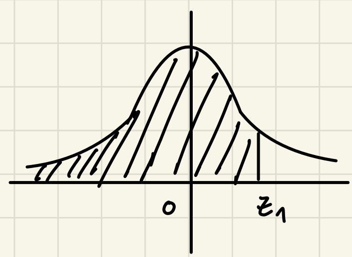


Table 1: The Standard Normal Probability Integral

$Z=\frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	4332	4345	4357	4370	4382	4394	4406	4418	4429	4441
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857
2.2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986

	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

& that is: $0.5 + P(0 \leq Z \leq z_1)$

$= 0.5 + 0.4918$

$= 0.9918$

e. In a factory in Augsburg, Germany, MAN makes sleeves to line the cylinders of ships' engines. The blanks for the sleeves are cast in a special iron alloy, then fixed into an enormous lathe which machines them to within a specified tolerance, cutting as much as 3mm of solid iron off both the inside and outside of the part in one pass. The target internal diameter of the sleeves is 45cm, but due to the violence of the cutting process, the actual inner diameter is normally distributed with a mean of 45cm and a standard deviation of 0.25mm. To ensure a tight seal with the piston ring, a diameter in the range of 44.95 to 45.05cm is required. What proportion of the sleeves are suitable for installation?

$\mu = 45$ cm

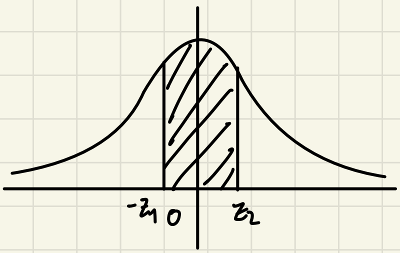
$\sigma = 0.25$ mm

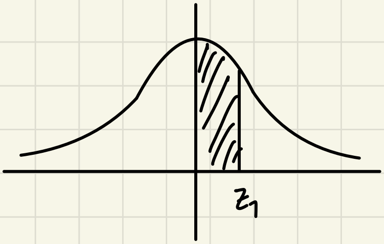
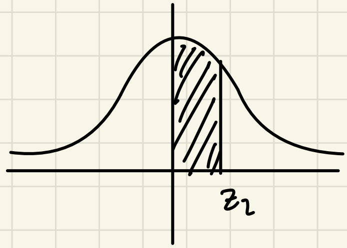
$P(\underset{x_1}{44.95} \leq X \leq \underset{x_2}{45.05})$ cm

$Z = \dfrac{X - 45}{0.25}$ $\therefore$ $z_1 = \dfrac{44.95 - 45}{0.25}$ $z_2 = \dfrac{45.05 - 45}{0.25}$

$z_1 = \dfrac{-0.05}{0.25} = -\dfrac{5}{25} = -0.2$

$z_2 = \dfrac{0.05}{0.25} = 0.2$



$=$  $+$ 

$$P(0 \le Z \le z_1) \quad + \quad P(0 \le Z \le z_2)$$

$$= P(0 \le Z \le 0.2) \quad + \quad P(0 \le Z \le 0.2)$$

Table 1: The Standard Normal Probability Integral

$Z=\frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
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2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

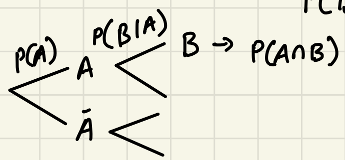
$$= 0.0793 + 0.0793$$

$$= 0.1586$$

Summary

Formula Sheet

- **Disjoint events:** $P(A \cup B) = P(A) + P(B)$
- **Joint events:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Bayseian theorem:** $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A) \cdot P(B|A)}{P(A)}$



- $nC_r = \dfrac{n!}{r!(n-r)!} = \binom{n}{r}$

For discrecte Random Variable

- Expectedvalue $E(x) = \sum_{i}^{n} x_i \, P(x = x_i)$
- Variance $V(x) = \sum_{i}^{n} P_i \, (x_i - \mu)^2$

$$V(x) = E(x^2) - (E(x))^2$$

x	$P(X = x)$
~	~
~	~
~	~

- Standard deviation, $\sigma = \sqrt{V(x)}$

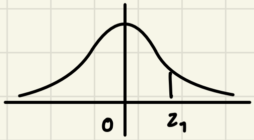
In Normal Distribution

$X \sim N(\mu, \sigma^2)$ std Norm. dist $N \sim (0, 1)$

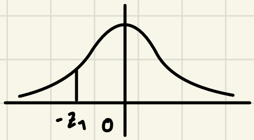
- Random variable Z from $N(\mu, \sigma^2)$

$$Z = \frac{X - \mu}{\sigma}$$

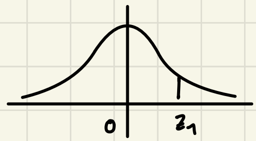
Use z-score table to calc probabilities.



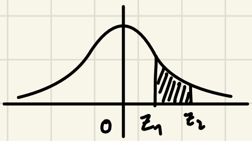
$P(0 \le Z \le Z_1) \rightarrow$ Take z_1 value from table



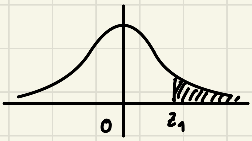
same as



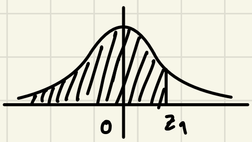
due to symmetry



$P(z_1 \le Z \le Z_2)$
$= P(0 \le Z \le Z_2) - P(0 \le Z \le Z_1)$



$P(Z > Z_1) = 0.5 - P(0 \le Z \le Z_1)$



$P(Z < Z_1) = 0.5 + P(0 \le Z \le Z_1)$