- Improper Integrals
- Sequence & Series

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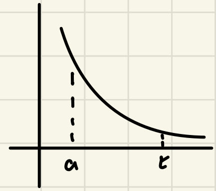
Improper Integrals

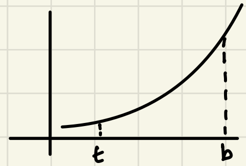
Improper Integrals

Type 1: Continuous Functions

- $\displaystyle\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx \quad , \quad t \geq a$



- $\displaystyle\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to -\infty} \int_t^{b} f(x)\,dx \quad , \quad t \leq b$



If limit exists in the above cases; they are called convergent, else divergent

- If both $\displaystyle\int_{-\infty}^{a} f(x)\,dx$ & $\displaystyle\int_a^{\infty} f(x)\,dx$ are convergent

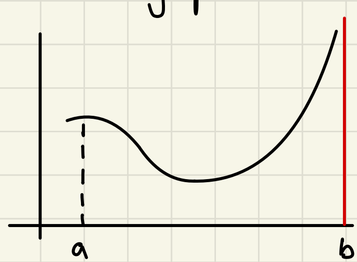
then we can write:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Type 2: For discontinuous Functions

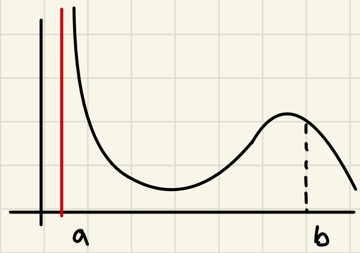
- if function discontinuous at b ie vertical asymptode

$$\int_a^{b} f(x)\,dx = \lim_{t\to b^-} \int_a^{t} f(x)\,dx$$



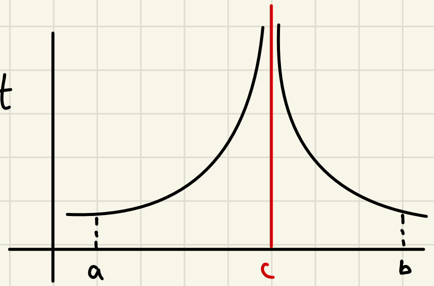
- if function is discontinuous at a,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$



- if function is discontinuous at c where $a < c < b$ & $\int_a^c f(x)\,dx$ & $\int_c^b f(x)\,dx$ are convergent

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Comparison Test for Improper Integrals

① Simplify the integral by taking highest power from the numerator & denominator if both has polynomials. Integrate that & test for convergence or divergence

By finding p-value for the p-series integral

$$\int_1^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges if } p \le 1 \\ \text{converges if } p > 1 \end{cases}$$

② Prove the convergence by comparison:

For convergence

$0 \le$ Improper Integral $\le$ Convergent Integral we derived

if known simplified integral converges everything below it converges.

For divergence

$$\text{Improper Integral} \geq \text{derived divergent Integral} \geq 0$$

if known simplified integral diverges everything above it diverges

given they both are +ve & above zero

- write both the integrals; solve the inequalities with " ∀ boundary start of integral "
- plug the initial start of boundary & see if the comparison sign holds; then the theory holds from the p-integral test.

③ Write conclusion

Examples:

1) $\int_1^\infty \frac{1}{x} dx$

$= \lim_{t \to \infty} \int_1^t \frac{1}{x} dx = \lim_{t \to \infty} \ln x \Big|_1^t$

$= \lim_{t \to \infty} [\ln t - \ln 1]$

$= \infty - 0 = \infty$ limit DNE divergent

2) $\int_1^\infty \frac{1 dx}{x^2} = \lim_{t \to \infty} \int_1^t \frac{1}{x^2} dx$

$= \lim_{t \to \infty} \frac{x^{-2+1}}{-2+1} \Big|_1^t = \lim_{t \to \infty} \frac{x^{-1}}{-1} \Big|_1^t$

← check on wolfram later not sure about answer

$$= \lim_{t \to \infty} -\frac{1}{x}\Big|_1^t$$

$$= \lim_{t \to \infty} \left[-\frac{1}{t} + \frac{1}{1}\right] = \quad 0 + 1 = 1$$

Limit exists
converges

3) $\int_{-\infty}^{0} x e^x \, dx$

$$= \lim_{t \to -\infty} \int_t^0 x e^x \, dx$$

$\int u v dv = uv - \int v du$

$u = x \qquad v dv = e^x dx$

$du = dx \qquad v = e^x$

$$\lim_{t \to -\infty} \int_t^0 x e^x dx = \lim_{t \to -\infty} x e^x - \int e^x dx \Big|_t^0$$

$$= \lim_{t \to -\infty} x e^x - e^x \Big|$$

$$= \lim_{t \to -\infty} \left[(0 - 1) - (te^t - e^t)\right]$$

$$= \lim_{t \to -\infty} \left[-1 - te^t + e^t\right]$$

$$= -1 + \infty - \infty = -1$$

Limit exists
convergent

4) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \tan^{-1}(x)\Big|_{t}^{0} + \lim_{t\to\infty} \tan^{-1}(x)\Big|_{0}^{t}$$

$$= 0 - \lim_{t\to-\infty} \tan^{-1}(t) + \lim_{t\to\infty} \tan^{-1}(t) + 0$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

Limit exits converges.

5) $$\int_{1}^{\infty} \frac{1}{x^p}\,dx$$

For what value of p; the function converges?

$$= \lim_{t\to\infty} \int_{1}^{t} x^{-p}\,dx$$

$$= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_{1}^{t}$$

$$= \lim_{t\to\infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right]$$

At this state if we apply limit it leads to ∞ in numerator but we can bring it down to denominator to see if that is the case

$$= \lim_{t \to \infty} \frac{1}{-p+1} \left[\frac{1}{t^{-(-p+1)}} - \frac{1}{1^{p+1}} \right]$$

$$= \lim_{t \to \infty} \frac{1}{-p+1} \left[\frac{1}{t^{1-p}} - 1 \right]$$

Now applying limit

$$= \frac{1}{-p+1} \left[\frac{1}{\infty} - 1 \right]$$

$$= \frac{1}{-p+1} \left[0 - 1 \right] = \frac{1}{1+p}$$

In the given interval

$$\int_1^\infty \frac{1}{x^p} dx = \frac{1}{1+p}$$

for $p > 1$ the integral converges.

6) $$\int_2^5 \frac{1}{\sqrt{x-2}} dx$$

This is type 2 integral ; discontinuous at $x = 2$

$$\therefore \lim_{x \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} dx$$

$u = x - 2$

$\frac{du}{dx} = 1$

$du = 1dx$

$$\lim_{x \to 2^+} \int_t^5 \frac{1}{\sqrt{u}} dx$$

$$\lim_{x \to 2^+} \frac{u^{-1/2+1}}{-\frac{1}{2}+1} \Big|_t^5$$

$$\lim_{x \to 2^+} 2\sqrt{u} \Big|_t^5$$

$$= \lim_{x \to 2^+} 2\sqrt{x-2} \Big|_t^5$$

$$= \lim_{x \to 2^+} \left[2\sqrt{5-2} - 2\sqrt{t-2} \right]$$

$$= 2\sqrt{3} - 0 = 2\sqrt{3}$$

7) $\int_0^{\pi/2} \sec x$

checking boundaries: $\int_0^{\pi/2} \frac{1}{\cos x}$

$$\text{Left} = \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

$$\text{Right} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{\infty} = \text{Not defined}$$

Discontinuous on the right end

$$\therefore \lim_{t \to \frac{\pi}{2}^-} \int_0^t \frac{1}{\cos x} dx$$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec x \, dx \times \frac{\sec x + \tan x}{\sec x + \tan x}$$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} \cdot$$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$\sec x + \tan x = u$$

$$\sec x \tan x + \sec^2 x = \frac{du}{dx}$$

$$du = (\sec x \tan x + \sec^2 x) \, dx$$

$$= \lim_{t \to \frac{\pi}{2}^-} \int_0^t \frac{1}{u} \, du$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln |\sec x + \tan x| \Big|_0^t$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left[\ln |\sec t + \tan t| - \ln |\sec 0 + \tan 0| \right]$$

$$= \lim_{t \to \frac{\pi}{2}^-} \ln |\sec t + \tan t| - \ln 1$$

$$= \quad \infty - 1 = \infty$$

8) $$\int_0^3 \frac{dx}{x-1}$$

= the function is discontinuous at $x = 1$ which is b/w the boundaries

$$= \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

Now with limits:

$$= \lim_{t \to 1^-} \int_0^t \frac{dx}{x-1} + \lim_{t \to 1^+} \int_t^3 \frac{dx}{x-1}$$

$x - 1 = u$

$du = dx$

$$\lim_{t \to 1^-} \ln|x-1| \Big|_0^t + \lim_{t \to 1^+} \ln|x-1| \Big|_t^3$$

$$= \lim_{t \to 1^-} \Big[\ln|t-1| - \ln|-1| \Big] + \lim_{t \to 1^+} \Big[\ln|3-1| - \ln|t-1| \Big]$$

$$= \lim_{t \to 1^-} \Big[\ln|t-1| \Big] + \lim_{t \to 1^+} \Big[\ln|2| - \ln|t-1| \Big]$$

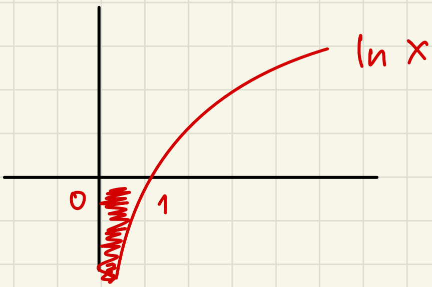
$$= -\infty + \big[\ln|2| - \infty \big]$$

$$= -\infty + -\infty = -\infty$$

in another way:

$=$ (1) divergent $\therefore$ (2) is divergent

9) $\int_0^1 \ln x \, dx$ we know



The boundary is b/w 0 & 1

$$= \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

sub problem

$$\int 1 \cdot \ln x \, dx =$$

$u = \ln x \quad v dv = 1 dx$

$du = \frac{1}{x} dx \quad v = x$

$$= x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - x$$

$$= \lim_{t \to 0^+} x \ln x - x \Big|_t^1$$

$$= \lim_{t \to 0^+} \left[\ln 1 - 1 - (t \ln t - t) \right]$$

$$= \lim_{t \to 0^+} \left[0 - 1 + t - t \ln t \right]$$

sub problem

$$\lim_{t \to 0^+} t \ln t = \lim_{t \to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t \to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}}$$

$$= \lim_{t \to 0^+} \frac{1}{t} \cdot \frac{t^2}{-1}$$

$$= \lim_{t \to 0^+} (-t)$$

$$= 0$$

$\therefore$ we have

$$= 0 - 1 + 0 - 0 = -1$$

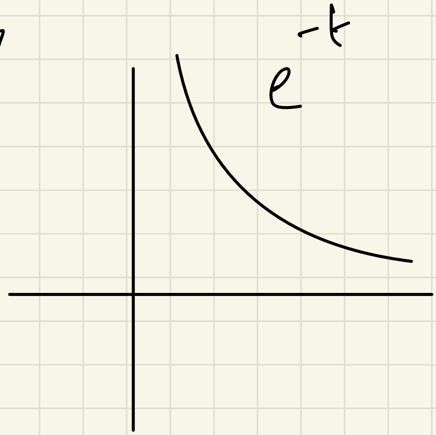
(10) $\displaystyle\int_0^{-\infty} e^{-x^2}\,dx$

$$= \int_0^1 e^{-x^2}\,dx + \int_0^{\infty} e^{-x^2}\,dx$$

$$= \int_0^1 e^{-x^2}\,dx + \lim_{t\to\infty} \int_0^t e^{-x^2}\,dx$$

$$= \lim_{t\to\infty} e^{-x}\Big|_0^t$$

$$= \lim_{t\to\infty} \left[e^{-t} - e^{0}\right]$$



$$= \lim_{t\to\infty} e^{-t} - 1$$

$$= e^{-\infty} \approx 1$$

Look up Limit Comparison Theorem shortcuts.

Black Pen Red Pen examples.

(1) $\int_e^\infty \frac{1}{x \ln x}\, dx$

$= \lim_{t\to\infty} \int_e^t \frac{1}{x\ln x}\, dx$

$=$ $\ln x = u$

$\frac{1}{x}\, dx = du$

$= \lim_{t\to\infty} \int_e^t \frac{1}{u}\, du = \lim_{t\to\infty} \ln|u| \Big|_e^t = \lim_{t\to\infty} \ln|\ln x| \Big|_e^t$

$= \lim_{t\to\infty} \left[\ln|\ln t| - \ln|\ln e| \right]$

$= \infty - \ln|1| = \infty - 0 = \infty$ diverges

(2) $\int_e^\infty \frac{1}{x(\ln x)^2}\, dx$

$= \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^2}\, dx$

$\ln x = u$

$\frac{1}{x} dx = du$

$= \lim_{t\to\infty} \int_{x=e}^t \frac{1}{u^2}\, du$

$= \lim_{t\to\infty} \frac{u^{-2+1}}{-1} \Big|_e^t$

$$= \lim_{t\to\infty} -\frac{1}{u}\Big|_e^t$$

$$= \lim_{t\to\infty} -\frac{1}{\ln x}\Big|_e^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln e}\right]$$

$$= -\frac{1}{\infty} + \frac{1}{1} = 0 + 1 = 1$$

③ $$\int_{-2}^{1} \frac{1}{x^2}\,dx$$

At 0, there is discontinuity ∴ it is Improper

$$\int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$\lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^2}\,dx + \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx$$

$$\int \frac{1}{x^2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = -\frac{1}{x} + C$$

$$\lim_{t\to 0^-} -\frac{1}{x}\Big|_{-2}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{1}$$

$$= \lim_{t \to 0^-} \left[\frac{-1}{t} - \frac{1}{2} \right] + \lim_{t \to 0^+} \left[-\frac{1}{1} + \frac{1}{t} \right]$$

$$= \frac{-1.1}{0^-} = -1(-\infty) = +\infty \qquad -1 + \frac{1}{+\infty} = +\infty$$

$$= \infty - \frac{1}{2} = +\infty$$

$$= +\infty + \infty = \infty$$

Limit DNE
diverges

(3) $\displaystyle\int_{-2}^{2} \frac{1}{1+x}\,dx$

Discontinuous at $x = -1$

$$\int_{-2}^{-1} \frac{1}{1+x}\,dx + \int_{-1}^{2} \frac{1}{1+x}\,dx$$

$$\lim_{t \to -1^-} \int_{-2}^{t} \frac{1}{1+x}\,dx + \lim_{t \to -1^+} \int_{t}^{2} \frac{1}{1+x}\,dx$$

$=$ $\quad 1+x = u$

$\quad 1dx = du$

$$\lim_{t \to -1^-} \ln|u| \Big|_{-2}^{t} + \lim_{t \to -1^+} \ln|u| \Big|_{t}^{2}$$

$$= \lim_{t \to -1^-} \ln|1+x| \Big|_{-2}^{t} + \lim_{t \to -1^+} \ln|1+x| \Big|_{t}^{2}$$

$$= \lim_{t\to -1^-} \ln|1+t| - \ln|1-2| + \ln|1+2| - \lim_{t\to -1^+} \ln|1+t|$$

$$= \ln 0 - \ln|-1| + \ln|3| - \ln 0$$

$$= \ln|3| - \ln|-1| = \ln 3 - \ln 1$$

④ $$\int_0^\infty \frac{dx}{x^2+4} = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+4}\,dx$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \therefore \frac{1}{2}\cdot\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{a^2+x^2}$$

$$= \lim_{t\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\cdot\left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\cdot\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$$

⑤ $$\int_3^\infty \frac{1}{(x-2)^{3/2}}\,dx$$

$$= \lim_{t\to\infty} \int_3^t \frac{1}{(x-2)^{3/2}}\,dx$$

let $x-2=u$

$dx=du$

$$\int = \frac{u^{-3/2+1}}{-3/2+1} = \frac{u^{-1/2}}{-1/2} = -\frac{2}{u^{1/2}} =$$

Don't forget to change u back in terms of x

$$= \lim_{t\to\infty} -\frac{2}{u^{1/2}}\Bigg|_3^t$$

$$= \lim_{t\to\infty} -\frac{2}{(x-2)^{1/2}}\Bigg|_3^t$$

$$= \lim_{t\to\infty} \frac{-2}{(t-2)^{1/2}} + \lim_{t\to\infty} \frac{2}{(3-2)^{1/2}}$$

$$= 0 + \frac{2}{1} = 2$$

6) $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2}\,dx$

$-x^2 = u$

$-2x\,dx = du$

$x\,dx = -\frac{1}{2}\,du$

Sub problem:

$$\int x e^{-x^2}dx = \int -\frac{1}{2}\cdot e^u du = -\frac{1}{2}\int e^u du$$

$$= -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

$$\lim_{t\to-\infty^+}\int_t^0 xe^{-x^2} + \lim_{t\to\infty^-}\int_0^t xe^{-x^2}$$

$$= \lim_{t\to-\infty^+} \left.\frac{-1}{2e^{x^2}}\right|_t^0 + \lim_{t\to\infty^-} \left.\frac{-1}{2e^{x^2}}\right|_0^t$$

$$= -\frac{1}{2e^0} + \lim_{t\to-\infty^+}\frac{1}{2e^{t^2}} - \lim_{t\to\infty^-}\frac{1}{2e^{t^2}} + \frac{1}{2e^0}$$

$$= -\frac{1}{2} + 0 - 0 + \frac{1}{2} = 0$$

(7) $\int x e^{-x^2} dx$

$=$ $-x^2 = u$

$-2x\,dx = du$

$x\,dx = -\frac{1}{2}du$

$$\int -\frac{1}{2}e^u du = -\frac{1}{2}e^u + C$$

$$= -\frac{1}{2}e^{-x^2} + C$$

(8) $\int_0^\infty \sin^2 x\, dx$

$$\lim_{t\to\infty}\int_0^t \sin^2 x\, dx$$

$$\int \sin^2 x = \frac{1}{2}\int 1 - \cos 2x$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x \right]$$

$$= \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$$\lim_{t\to\infty} \frac{x}{4} - \frac{1}{4}\sin 2x \Big|_0^t$$

$$= \lim_{t\to\infty} \frac{t}{4} - \frac{1}{4}\sin 2t - 0$$

$$= \infty$$

Limit DNE
diverges.

(9) $$\int_1^\infty \frac{\ln x}{x}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx$$

Let $\ln x = u$

$\frac{1}{x}dx = du$

$$= \lim_{t\to\infty} \int_1^t u\,du = \lim_{t\to\infty} \frac{u^2}{2}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} - 0$$

$$= \infty$$

Limit DNE
∴ Diverges.

(10) $\displaystyle\int_e^{\infty} \frac{1}{x\ln x}\,dx$

$\displaystyle\lim_{t\to\infty} \int_e^{t} \frac{1}{x\ln x}\,dx$

$$= \quad \begin{aligned} \ln x &= u \\ \tfrac{1}{x}\,dx &= du \end{aligned}$$

$\displaystyle\lim_{t\to\infty} \int_e^{t} \frac{du}{u} = \lim_{t\to\infty} \ln u \Big|_e^t$

$\displaystyle = \lim_{t\to\infty} \ln|\ln|x|| \Big|_e^t$

$\displaystyle = \lim_{t\to\infty} \ln|\ln|t| - \ln|\ln|e||$

$= \infty - \ln|1| = \infty - 0 = \infty$

(11) $\displaystyle\int_e^{\infty} \frac{1}{x(\ln x)^2}\,dx$

$\displaystyle = \int_e^{\infty} \frac{1}{u^2}\,du = \lim_{t\to\infty} -u^{-1} \Big|_e^t$

$\displaystyle = \lim_{t\to\infty} -\frac{1}{\ln x} \Big|_e^t$

$\displaystyle = \lim_{t\to\infty} -\frac{1}{\ln t} + \frac{1}{\ln e} = 0 + 1 = 1$

(12) $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx$

$=$ discontinuity at $x=0$

$\therefore \displaystyle\int_{-2}^{0} \frac{1}{x^4}dx + \int_{0}^{3} \frac{1}{x^4}dx$

$= \displaystyle\lim_{t\to 0} \int_{-2}^{t} \frac{1}{x^4}dx + \lim_{t\to 0} \int_{t}^{3} \frac{1}{x^4}dx$

$\displaystyle\lim_{t\to 0^-} -\frac{1}{3x^3}\Bigg|_{-2}^{t} + \lim_{t\to 0^+} -\frac{1}{3x^3}\Bigg|_{t}^{3}$

Dont forget 0^+ & 0^- case

$\displaystyle\lim_{t\to 0^-} -\frac{1}{3t^3} + \frac{1}{2(-2)^3} + \lim_{t\to 0} -\frac{1}{3(3)^3} + \lim_{t\to 0} \frac{1}{3t^3}$

$= -\dfrac{1}{16} - \dfrac{1}{81} = \dfrac{-81-16}{1296}$

$\displaystyle\underbrace{\lim_{t\to 0^-} -\frac{1}{3t^3}}_{-\frac{1}{3(0^-)} = \infty} + \frac{1}{3(-2)^3} + -\frac{1}{3(3)^3} - \underbrace{\lim_{t\to 0^+} -\frac{1}{3t^3}}_{+\lim_{t\to 0^+}\frac{1}{3t^3} = +\infty}$

$-(-\infty) + \dfrac{1}{3(-8)}$

$\underbrace{= \infty}_{\text{one side diverges}}$

Diverges as Limit DNE

(13) $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$

$x^3+1=u$

$3x^2\,dx = du$

$x\,dx = \frac{1}{3x}\,du$

$x^3+1=u$

$x^3 = u-1$

$x = \sqrt[3]{u-1}$

$x\,dx = \frac{1}{3(u-1)^{1/3}}\,du$

$\displaystyle\int_0^\infty \frac{1}{3}\cdot\frac{1}{(u-1)^{1/3}}\cdot\frac{1}{u}\,dx$

$\displaystyle = \frac{1}{3}\int_0^\infty \frac{1}{\quad}$

Use comparison theorem test !

Strategy:

1) Use comparison test + p-integral test
2) Trick: In order to apply p-integral test the series/Integral has to start from 1 to ∞ ie $\int_1^\infty \frac{1}{x^p}$ & not $\int_0^\infty \frac{1}{x^p}$ & has to be continuous ie denom $\neq 0$
3) Therefore split the integral & solve
4) For the remaining part, check whether it is defined in the remaining interval & if there exists any vertical asymptotes in the interval. If not it is a convergent integral.

Important

ie $\int_0^\infty \frac{x}{1+x^3} \approx \int_0^\infty \frac{x}{x^3} = \int_0^\infty \frac{1}{x^2}$

But p-series => $\int_1^\infty \frac{1}{x^p}$ $\quad p \leq 1$ diverges

$p > 1$ converges

∴ split original question as:

$$\underbrace{\int_0^1 \frac{x}{1+x^3}}_{} + \underbrace{\int_1^\infty \frac{x}{1+x^3}}_{}$$

This is defined
no vertical
asymptodes
on the
interval
∴ converges

$$\underbrace{\int_1^\infty \frac{1}{x^2}}_{}$$

converges

Hence the complete integral converges by Comparison Theorem.

(14) $\displaystyle\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

Type 1 continuous

$$\lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^3}\,dx$$

$=$ $2x+1 = u$

$2\,dx = du$

$dx = \frac{1}{2}\,du$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{2} \cdot \frac{1}{u^3}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\frac{u^{-3+1}}{-2}\right]_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{4}\left[\frac{1}{u^2}\right]_1^t$$

$$= \lim_{t\to\infty} -\frac{1}{4}\left[\frac{1}{t^2} - 1\right]$$

$$= -\frac{1}{4}\left[0 - 1\right] = \frac{1}{4}$$

limit exists
converges to $\frac{1}{4}$

(15) $$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$$

Integral starts from 1 to ∞
$\therefore$ p-series possibility with comparison theorem
But the function has to be continuous
Here denominator $= 0$ when $x = 1$ $\therefore$ split

when denom $= 0$ Type 2
when cont. & $\int_a^\infty$ & $x \geq a$ Type 1 & apply p-test

$$= \int_1^2 \frac{x+1}{\sqrt{x^4-x}} + \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}$$

$$= \lim_{t\to 1^+} \int_t^2 \frac{x+1}{\sqrt{x^4-x}} + \underbrace{\lim_{t\to\infty} \int_2^t \frac{x+1}{\sqrt{x^4-x}}}_{\text{Apply Comparison theorem}}$$

$=$

$$\lim_{t\to\infty} \int_2^t \frac{x}{x^{4/2}}$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x^1}$$

$p = 1$

$p \leq 1 \rightarrow$ diverges

$\therefore$ this function part diverges.

Now proof

$$\frac{x+1}{\sqrt{x^4 - x}} \geq \frac{1}{x} \quad \forall\ x \geq 2$$

$$x(x+1) \geq \sqrt{x^4 - x}$$

$$6 \geq \sqrt{16-2} = \sqrt{14} \approx 3.9$$

Since one part is divergent we know the whole thing has to be divergent

(16) $$\int_0^1 \frac{1}{x^p}\,dx$$

when $p = 1$ & when $p \neq 1$

$$\lim_{t\to 0^+} \ln x \Big|_{0^+}^{1}$$

$$= \ln|1| - \ln|0^+|$$

$$= 0 - (-\infty) = +\infty$$

$$\lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Bigg|_t^1$$

$$\lim_{t\to 0^+} \frac{1}{1-p}\left[1^{-p+1} - t^{-p+1}\right]$$

$$\frac{1}{1-p}\left[1 - 0\right] = \frac{1}{1-p} \quad \forall\, p \neq 1 \ \& \ p > 0$$

&

$$\infty \quad \forall\, p \neq 1 \ \& \ p < 0$$

(17) $$\int_e^\infty \frac{1}{x(\ln x)^p}\,dx$$

$$= \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^p}\,dx$$

let $\ln x = u$

$\frac{1}{x}\,dx = du$

$$\lim_{t\to\infty} \int_e^t \frac{1}{u^p}\,du$$

when $p = 1$

$$\lim_{t\to\infty} \int_e^t \frac{1}{u}\,du$$

when $p \neq 1$

$$\lim_{t\to\infty} \int_e^t \frac{1}{u^p}$$

$$= \lim_{t\to\infty} \ln|u| \Big|_e^t$$

$$= \lim_{t\to\infty} \ln|\ln x| \Big|_e^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|1|$$

$$= \infty - 0$$

$$\underbrace{= \infty}$$

when $p = 1$
diverges

$$\lim_{t\to\infty} \frac{u^{-p+1}}{-p+1} \Big|_e^t$$

$$\lim_{t\to\infty} \frac{1}{-p+1} u^{-p+1} \Big|_e^t$$

$$\lim_{t\to\infty} \frac{1}{-p+1} (\ln x)^{-p+1} \Big|_e^t$$

$$\frac{1}{1+p}\left[\lim_{t\to\infty} (\ln t)^{-p+1} - (\ln(e))^{-p+1}\right]$$

$$\frac{1}{1+p}\left[\lim_{t\to\infty} \frac{1}{(\ln t)^{-(p+1)}} - 1^{-p+1}\right]$$

$$\frac{1}{1+p}\left[0 - 1\right]$$

$$\underbrace{= -\frac{1}{1+p} = \frac{1}{p-1}}$$

converges

$p > 1 \qquad p \neq 1$

Essentially p-test condition

$$\begin{cases} p \leq 1 & \text{diverges} \\ p > 1 & \text{converges.} \end{cases}$$

(18) Given $\int_1^\infty \frac{1}{e^x}dx$ converges. Which of the following can we conclude to be convergent as well, by using the Comparison theorem

a) $\int_1^\infty \frac{x}{e^x}dx$

Given convergent

$\therefore\ 0 \leq$ Improper Integral $\leq$ Convergent integral

$$\int_1^\infty \frac{x}{e^x}dx \overset{?}{\leq} \int_1^\infty \frac{1}{e^x}dx$$

$$\frac{x}{e^x} \overset{?}{\leq} \frac{1}{e^x} \qquad \forall\ x \geq 1$$

$$xe^x \overset{?}{\leq} e^x$$; Not satisfying

$\therefore$ This is not convergent

b) $\int_1^\infty \frac{1}{e^x+x}dx$

$$\frac{1}{e^x+x} \leq \frac{1}{e^x} \qquad \forall\ x \geq 1$$

$$e^x \leq e^x + x$$

$1 \leq x$ in other way $x \geq 1$ which is true!

c) $\int_1^\infty \frac{1}{e^x-1}\,dx$

$$\frac{1}{e^x-1} \le \frac{1}{e^x} \quad \forall\ x \ge 1$$

$$e^x \le e^x - 1$$

$$0 \le -1 \quad \text{Not satisfying}$$

(19) Does $\int_1^\infty \frac{x^2-1}{2x^5+3x+17}\,dx$ converge or diverge?

$\int_1^\infty$ satisfies p-series bounderies & denom when $x = 1 \neq 0$ $\therefore$ continuous

$\therefore$ By p-series of the highes powers from numerator & denominator

$$\int_1^\infty \frac{x^2}{x^5} = \int_1^\infty \frac{1}{x^3} \qquad p = 3$$

$p = 3$ & $p > 1$ $\therefore$ converges

if $p > 1$ converges

$p \le 1$ diverges

Now proof:

$$\frac{x^2-1}{2x^5+3x+17} \leq \frac{1}{x^3} \quad \forall \quad x \geq 1$$

$$\left. x^3(x^2-1) \leq 2x^5+3x+17 \right\} \text{ Always True}$$

when $x = 1$

$$0 \leq 2+3+17$$

$$0 \leq 22 \quad \text{satisfies.}$$

$\therefore$ By comparison theorem this function converges.

(20) Does $\int_2^\infty \frac{x^2}{\sqrt{x^5-1}}\,dx$ conv. or diverge?

$$\lim_{t\to\infty} \int_2^t \frac{x^2}{\sqrt{x^5-1}}$$

satisfies p-series test condition as $a > 1$ & function is continuous as denom $\neq 0$

$$\therefore \lim_{t\to\infty} \int_2^t \frac{x^2}{x^{5/2}}$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x^{5/2-2}}\,dx$$

$$= \lim_{t\to\infty} \int_2^t \frac{1}{x^{1/2}}\,dx$$

$p = 1/2$ $\quad p \leq 1 \quad \therefore$ diverges

Do not forget to prove divergence

proof $\lim_{t\to\infty} \int_2^t \frac{x^2}{\sqrt{x^5-1}} \geq \lim_{t\to\infty} \int_2^t \frac{1}{x^{1/2}} \quad \forall \; x \geq 2$

$$\frac{x^2}{\sqrt{x^5-1}} \geq \frac{1}{\sqrt{x}}$$

$$x^2\sqrt{x} \geq \sqrt{x^5-1}$$

$$x^4(x) \geq x^5-1$$

$$\left.x^5 \geq x^5-1\right\} \text{Always True}$$

conclusion:
The improper integral diverges as the integral is always larger than the divergent integral we have taken

(21) Does $\int_e^\infty \frac{1}{\ln x}\,dx$ conv or div?

This does not at the moment satisfy p series test rule where $\int_1^\infty$ we have $\int_e^\infty$

we ~~apply u sub~~

~~let $\ln x = u$~~

~~$\frac{1}{x}dx = du$~~

~~$1dx = xdu$~~

This is where the list could come to play.

The list $\ln x < x^n < a^x < x^x$

$$\frac{1}{\ln x} > \frac{1}{x^n} > \frac{1}{a^x} > \frac{1}{x^x}$$

if $n=1$, everything beyond.

diverges. | converges

$\therefore$ we take: $\int_e^\infty \frac{1}{x^1}$ & this diverges

Now we prove it

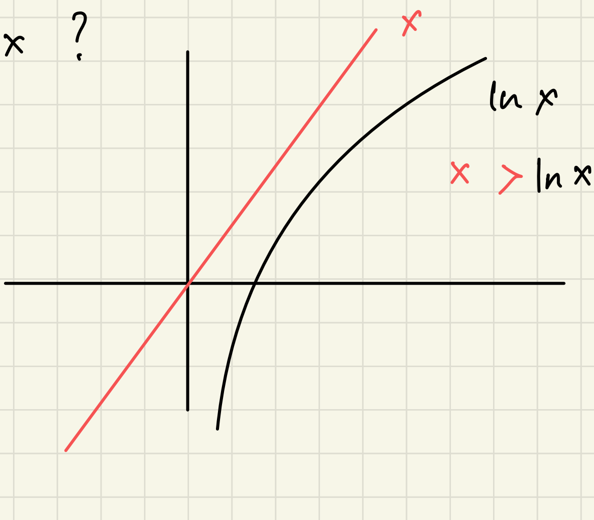
$$\int_e^\infty \frac{1}{\ln x} > \int_e^\infty \frac{1}{x} \quad \forall \; x \geq e$$

$$\frac{1}{\ln x} > \frac{1}{x}$$

$$x \overset{?}{>} \ln x$$

is $x > \ln x$?

Looking at graph:



(22) Does $\int_2^\infty \frac{1}{x^2-1}\,dx$ conv or div?

$$\lim_{t\to\infty} \int_2^t \frac{1}{x^2-1}\,dx$$

From the list:

$$\ln x < x^n < a^x < x^x$$

$$\frac{1}{\ln x} > \frac{1}{x^n} > \frac{1}{a^x} > \frac{1}{x^x}$$

$n=1$ ⟶ converges.

$$\lim_{t\to\infty} \int_2^t \frac{1}{x^2} = p=2$$

converges

∴ proof

$$0 \le \lim_{t\to\infty} \int_2^t \frac{1}{x^2-1} \le \lim_{t\to\infty} \int_2^t \frac{1}{x^2} \quad \forall\ x \ge 2$$

$$\frac{1}{x^2-1} \le \frac{1}{x^2}$$

$$x^2 \le x^2-1$$ Does not satisfy

we cannot draw any conclusion.

Now we apply normal Improper Integrals

$$\lim_{t\to\infty} \int_2^t \frac{1}{x^2-1}\,dx$$

$$\lim_{t\to\infty} \int_2^t \frac{1}{(x+1)(x-1)} = \lim_{t\to\infty} \int_2^t \frac{A}{x+1} + \lim_{t\to\infty} \int_2^t \frac{B}{(x-1)}$$

$1 = A(x-1) + B(x+1)$

when $x = 1$

$1 = 0 + B(2)$

$B = \frac{1}{2}$

when $x = -1$

$1 = -2A + 0$

$A = -\frac{1}{2}$

$$\therefore \lim_{t\to\infty} \int_2^t \frac{-1}{2(x+1)}\,dx + \lim_{t\to\infty} \int_2^t \frac{1}{2(x-1)}\,dx$$

Let $x+1 = u$

$1\,dx = du$

$$-\frac{1}{2} \lim_{t\to\infty} \int_2^t \frac{1}{u}\,du + \frac{1}{2}\lim_{t\to\infty} \int_2^t \frac{1}{u}\,du$$

$$-\frac{1}{2} \lim_{t\to\infty} \ln|x+1| \Big|_2^t + \frac{1}{2} \lim_{t\to\infty} \ln|x-1| \Big|_2^t$$

$$= -\frac{1}{2}\left[\lim_{t\to\infty} \ln|t+1| - \ln 3\right] + \frac{1}{2}\left[\lim_{t\to\infty} \ln|t-1| - \ln|3|\right]$$

$$= -\frac{1}{2}\left[\ln|\infty+1| - \ln 3\right] + \frac{1}{2}\left[\ln|\infty-1| - \ln|3|\right]$$

$$-\frac{1}{2}(\infty) + \frac{1}{2}\ln 3 + \frac{1}{2}(\infty) - \frac{1}{2}\ln 3$$

$= 0$

Improper Integral Remarks

- Plug in given boundaries → no zeros → f(x) continuous
  ↘ zeros → Type 2 Improper

- Check if any value b/w the boundaries leads to zeros → Split Improper Integral at that value into two

- If both boundaries 'inf'; split at any value & Apply limits

- when to use $\lim_{x \to a^+}$ or $\lim_{x \to a^-}$: when the point 'a' has a vertical asymptode : } Sometimes 'a' would be the boundary ∴ either a^+ or a^-. If a in b/w both are necessary

- Apply limits : → if value exists Improper Integral converges
  common encounters:
  $\ln|e| = 1$ $\ln 1 = 0$ → if value DNE Diverges

- Test for Convergence or Divergence using Comparison theorem

- To take highest power of numerator and denominator to apply p-series test the Integral must start from: $\int_1^\infty$; if not split it

- Also importantly; the integral must be Type 1; ie not Type 2 where denom goes to 0 when values plugged in

Solving structure:

1. Apply p-series
   if $p \leq 1$ diverges
   if $p > 1$ converges
2. Compare improper with the convergent function or divergent function
   $0 <$ Improper $\leq$ Convergent
   Improper $\geq$ Divergent > 0
3. Prove the inequality statement is true ∀ all x values starting from initial boundary

- Keep in mind of the list if the above method does not work

④ write conclusion

$$\ln x < x^n < a^x < x^x$$

$$\therefore \frac{1}{\ln x} > \frac{1}{x^n} > \frac{1}{a^x} > \frac{1}{x^x}$$

for $n=1$

diverges　　converges

- If the above still does not help & the integral is $\int_1^\infty$ bot not continous ie Type 2; split the integral $\int_1^2 + \int_2^\infty$ & apply p series to the other one
  solve the other one using Type 2
- Always need $\lim\limits_{x\to a^+}$ or $\lim\limits_{x\to a^-}$ when integral is Type 2 as there is discontinuity.

Sequence & Series

Test Summary

① TFD if $\lim_{n \to \infty} a_n \neq 0$ diverges lim neq zero div

② Geometric Series Test

$\sum_{n=1}^{\infty} ar^n$ if $|r| < 1$ converges Sum $= \frac{a}{1-r}$

③ The list: ∞ $\lim_{n \to \infty}$ $\ln n < n^p < a^n < n! < n^n$

$$\frac{1}{\ln n} > \frac{1}{n^p} > \frac{1}{a^n} > \frac{1}{n!} > \frac{1}{n^n}$$

can be used for deriving known convergent or divergent series.

④ AST lim zer ast con zerascon

$\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ if $\lim_{n \to \infty} a_n = 0$ converges

if it converges prove it by comparison:

$a_{n+1} < a_n$

⑤ $\sum_{n=1}^{\infty} a_n$ Integral Test $\lim_{t \to \infty} \int_1^t f(x)dx$ converges or diverges (∞) after solving

But to apply Integral Test the function has to be +ve, continuous and decreasing

⑥ LCT: $\sum_{n=1}^{\infty} a_n$: question series

$\sum_{n=1}^{\infty} b_n$: derived convergent or divergent based on list or highest num denom powers

Apply $\lim_{n\to\infty} \frac{a_n}{a_n} \neq \infty$ or 0 if converges or diverges with respect to the type of b_n (converges or diverges)

⑦ P- Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ if $p \leq 1$ diverges

$p > 1$ converges

⑧ Ratio Test

given $\sum_{n=1}^{\infty} a_n$

then $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda$ if $\lambda \begin{cases} \lambda < 1 & \text{converges} \\ \lambda > 1 & \text{diverges} \\ \lambda = 1 & \text{inconclusive} \end{cases}$

⑨ DCT $\sum_{n=1}^{\infty} a_n$ given

Use the list to derive a known convergent or divergent

Compare them $\rightarrow 0 < a_n \overset{?}{<}$ known convergent is True

then a_n converges

$\rightarrow a_n \overset{?}{>}$ known divergent > 0 is True

a_n diverges

Sequence : Just a succession of numbers

eg: $1, -1, 1, -1, 1$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4} \ldots$

denoted by $a_1, a_2, a_3 \ldots a_n$ leading to n^{th} term expression

Apply limit to n^{th} term & if it can be found then the sequence converges

ie $\lim_{n \to \infty} a_n$ exists

Arithmetic progression: observed when the next particular term is obtained by addition of a constant value (eg: -2, 3)

eg: $1, 4, 7, 10 \ldots$

$3, 1, -1, -3 \ldots$

general form: for n^{th} term $a + (n-1)d$

Geometric progression: observed when the next term is obtained by multiplying previous term by a constant factor

eg: $-1, -\frac{1}{3}, -\frac{1}{9}, -\frac{1}{27}$

$1, 2, 4, 8$

$a, ar, ar^2, ar^3 \ldots$

$r =$ common ratio

Sum of an Arithematic series

$A = a + \dots \quad a + (n-1)d$

$A = a + (n-1)d + \dots \dots \quad a$

Same ∴

(+)

$2A = 2a + (n-1)d$ ∴ for every n^{th} term

$2A = n\left[2a + (n-1)d\right]$

Sum of Arithematic series having n-terms $\Rightarrow$ $A = \frac{1}{2} n \left[2a + (n-1)d\right]$

Sum of a Geometric Series

$G = a + \cancel{ar} + \cancel{ar^2} + \cancel{\dots} \quad ar^{n-1}$ × by r

$rG = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots \cancel{ar^{n-1}} + ar^n$ (−)

$G(1-r) = a - ar^n$

$G(1-r) = a(1-r^n)$

$G = a\frac{(1-r^n)}{1-r}$ if $r \neq 1$

$G = n \cdot a$ if $r = 1$

Infinite Series

$$a_1 + a_2 + a_3 + \dots \doteq \sum_{k=1}^{\infty} a_k$$

To evaluate such series we can apply limit and find the partial sum

$$\underbrace{\sum_{k=1}^{\infty} a_k}_{\text{infinite series}} = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} a_k}_{\text{Finite Sum}}$$

Now check if limit exists for this function.

If it does, the series converges;
If limit does not exist, the series diverges.

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \Rightarrow \qquad S_n = \frac{2^{n+1} - 1}{2^n}$$

Test for divergence

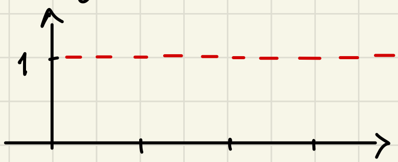
If a series is given & you want to test if it diverges; you can apply limit on the corresponding sequences' n-th term.

if limit is 0 ; the series may converge.

if limit $\neq 0$; the series diverges.

eg: $1 + 1 + 1 + 1 \ldots$

$a_n =$



lim $\neq 0$ $\therefore$ diverges

sequence is: a_n	Apply limit on the sequence	$\Rightarrow$	$\lim_{n\to\infty} a_n$	$\neq 0$ diverges
series is: $\sum_{k=1}^{\infty} a_k$				$= 0$ may converge

useless information

liman neq zero div

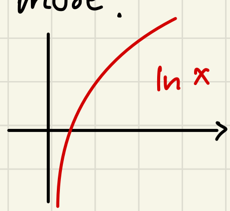
Famous Series to know

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series)

if you apply divergence test: $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$ might converge

test is unsatisfactory; we need to research more!

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x} dx = \lim_{n\to\infty} \ln n = +\infty$$



$\therefore$ the series diverges!

② Alternating Harmonic Series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$ => This series converges

if and only if

① $a_1 > a_2 > a_3 \dots$ terms decrease

② $\lim_{p \to \infty} a_p = 0$; partial summands derived to $a_n \to a_p$

③ Geometric Series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

or

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

} converges when $|r| < 1$

Ratio Test

if series has +ve terms then.

$$\lim_{p \to \infty} \frac{a_{p+1}}{a_p} = \lambda$$ if

- $\lambda > 1$; diverges
- $\lambda < 1$; converges
- $\lambda = 1$; inconclusive further test required.

Alternating Series Test

Usually applied on alternating series when the divergence test is inconclusive

ie An alternating series: $a_1 - a_2 + a_3 - a_4 \cdots\cdots$

is convergent if

- we observe the positive terms of the series to be decreasing; ie $a_1 > a_2 > a_3 \cdots\cdots$

&

- $\lim_{n \to \infty} a_n = 0$

Absolute Convergence Test

Suppose you have series $\sum_{n=1}^{\infty} a_n$ converging

But $\sum_{n=1}^{n} |a_n|$ is divergent, then the series is conditionally convergent. Happens mostly with alternating series. If both $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} |a_n|$ is converging then there exist absolute convergence

$\sum_{n=1}^{\infty} a_n$ converging & $\sum_{n=1}^{\infty} \lvert a_n \rvert$ diverging	partial convergence
$\sum_{n=1}^{\infty} a_n$ converging & $\sum_{n=1}^{\infty} \lvert a_n \rvert$ converging	absolute convergence

Binomial Series

Binomial series of the form $(1+x)^n$ is convergent if $|x| < 1$

multiply

$(a+b)^n$ terms	$nC_r = \frac{n!}{r!(n-r)!}$ terms	a terms	b terms	
① $nC_0 a^n b^0$	1	a^n	1	+
② $nC_1 a^{n-1} b^1$	n	a^{n-1}	b^1	+
③ $nC_2 a^{n-2} b^2$	$\frac{n(n-1)}{2!}$	a^{n-2}	b^2	+
④ $nC_3 a^{n-3} b^3$	$\frac{n(n-1)(n-2)}{3!}$	a^{n-3}	b^3	+
⑤ $nC_4 a^{n-4} b^4$	$\frac{n(n-1)(n-2)(n-3)}{4!}$	a^{n-4}	b^4	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$(1+x)^n$				
	1	1^n	1	
	n	1^{a-1}	x	
	$\frac{n(n-1)}{2!}$	1^{n-2}	x^2	
	$\frac{n(n-1)(n-2)}{3!}$	1^{n-3}	x^3	
	$\frac{n(n-1)(n-2)(n-3)}{4!}$	1^{n-4}	x^4	

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \dots \quad |x| < 1$$

↑

① Try to convert given series to this form to extract x & find $|x|$

② if $|x| < 1$; series is convergent

③ Find validity region by takin $|x| < 1$ to $-1 < x < 1$

Power Series

form: $\sum_{n=0}^{1} b_n x^n$

$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 \dots\dots$

Find the Radius of Convergence R

if $|x| < R$ series converges

$|x| > R$ series diverges

$x = \pm R$ may be convergent or divergent

R is obtained from the Ratio test

Q: eg: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

• Checking $\sum_{n=1}^{\infty} a_n$:

Ratio Test: $\lim_{n\to\infty} \frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}} = \lim_{n\to\infty} \frac{\cancel{(-1)^n}(-1)^2 \cdot n}{(n+1)\cancel{(-1)^n}(-1)}$

$= \lim_{n\to\infty} \frac{n}{(n+1)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

if $\lambda = 1$ Ratio Test is inconclusive

Alternating Test: $\lim_{n\to\infty} \frac{1}{n} = 0$ & $a_1 > a_2 > a_3 \dots$

$1 > \frac{1}{2} > \frac{1}{3} \dots$

$\therefore \lim_{n\to\infty} a_n$ converges

• Checking $\sum_{n=1}^{\infty} |a_n|$

div test $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n (-1)^1}{n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$

By divergence test this is inconclusive
as lim neqzerodiv we can say it diverges
if and only if the lim is not equal to zero

Here we have lim as 0

But this $\lim_{n \to \infty} \frac{1}{n}$ is

famous series ① $\therefore$ it diverges.

$\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges

$\therefore$ it is partially convergent

Binomial series

$(1+x)^p$ form is convergent if $|x| < 1$

$$(1+x)^p = 1 + p\frac{x}{1!} + p(p-1)\frac{x^2}{2!} + p(p-1)(p-2)\frac{x^3}{3!} + \dots$$

$$(a+b)^n = nC_0 a^n b^0 + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \dots$$

$$\dots\dots nC_{n-1} a^0 b^n$$

$$nC_r = \frac{n!}{r!(n-r)!}$$

$$nC_0 = \frac{n!}{0!(n-0)!} = 1$$

$$nC_1 : \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$$nC_2 = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2!}$$

$$(a+b)^n = a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots b^n$$

eg: $(1-x)^{1/2}$ $\quad n = \frac{1}{2} \quad a = 1 \quad b = -x$

$n = 1/2$

Terms	$nC_r = \frac{n!}{r!(n-r)!}$	$a = 1$	$b = (-x)$
① $nC_0 a^n b^0$	$\frac{n!}{0!\,n!} = 1$	$1^{1/2} = 1$	$(-x)^0 = 1$
② $nC_1 a^{n-1} b^1$	$\frac{n!}{1!(n-1)!} = n$	$1^{1/2-1} = 1$	$(-x)^1 = -x$
③ $nC_2 a^{n-2} b^2$	$\frac{n(n-1)}{2!}$	$1^{1/2-2} = 1^{-3/2} = \frac{1}{1^{3/2}} = 1$	$(-x)^2 = x^2$

④ $nC_3\, a^{n-3} b^3$	$\dfrac{n(n-1)(n-2)}{3!}$	$1^{1/2-3} = 1$	$(-x)^3 = -x^3$
⑤ $nC_4\, a^{n-4} b^4$	$\dfrac{n(n-1)(n-2)(n-3)}{4!}$	$1^{1/2-4} = 1$	$(-x)^4 = x^4$

$$1 \cdot 1 \cdot 1 + \left(n \cdot 1(-x)\right) + \frac{n(n-1)}{2!} x^2 + \dots$$

$$1 - xn + \frac{n(n-1)}{2!} x^2 - \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

$n = \frac{1}{2}$

$$\therefore 1 - \frac{x}{2} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} x^2 - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} x^3 + \dots$$

$$\approx 1 - \frac{x}{2} + \left(-\frac{1}{4} \cdot \frac{1}{2!}\right) x^2 - \frac{1}{2} \cdot \frac{1}{4^2} \cdot \frac{1}{3!} x^3$$

$$\approx 1 - \frac{x}{2} - \frac{1}{8} x^2 - \frac{1}{96} x^3$$

$$\frac{1}{(2+x)} = \frac{1}{2\left(1+\frac{x}{2}\right)^{1}} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$

$n = -1 \qquad a = 1 \qquad b = \frac{x}{2}$

Terms:
$(a+b)^n$

		$nC_r = \frac{n!}{r!(n-r)!}$	a	b
1	$nC_0 a^n b^0$	1	1^{-1}	1
2	$nC_1 a^{n-1} b^1$	n	1^{-1-1}	$x/2$
3	$nC_2 a^{n-2} b^2$	$\frac{n(n-1)}{2!}$	1^{-1-2}	$(x/2)^2$
4	$nC_3 a^{n-3} b^3$	$\frac{n(n-1)(n-2)}{3!}$	1^{-1-3}	$(x/2)^3$

$$1 + n\frac{x}{2} + \frac{n(n-1)}{2!}\left(\frac{x}{2}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{2}\right)^3 + \dots$$

$n = -1$

$$1 - \frac{x}{2} + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3$$

$$= \left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3\right)\frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{x^2}{8} - \frac{x^3}{16}$$

validity of x

we have condition $|x| < 1$

$$\therefore \quad \left| \frac{x}{2} \right| < 1$$

$$-1 < \frac{x}{2} < 1$$

$$\times 2 \Rightarrow \quad -2 < x < 2$$

Binomial Series $(1+x)^n$ form validity $|x|<1$

Power Series $\sum_{n=0}^{\infty} b_n x^n$ form ROC needed.

radius of convergence

$|x|<R$ abs convergence

$|x|>R$ divergent

At $x=R$ & $x=-R$ series may converge or diverge.

eg: $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$

$= \frac{1}{1} x^0 + \frac{1}{2} x^1 + \frac{1}{3} x^2 + \frac{1}{4} x^3 + \cdots$

$= \frac{1}{n} x^{n-1} = \sum_{n=0}^{1} \frac{x^{n-1}}{n}$ or $\sum_{n=0}^{1} \frac{x^n}{n+1}$

Ratio test

$$\lim_{n \to \infty} \left| \frac{(n+1)^{th} \text{ term}}{n^{th} \text{ term}} \right| = \lambda$$

$(n+1)^{th}$ term $= \frac{x^{n+1-1}}{n+1}$ or $\frac{x^{n+1}}{n+2}$

n^{th} term $= \frac{x^{n-1}}{n}$ or $\frac{x^n}{n+1}$

$$\lim_{n \to \infty} \left| \frac{x^n}{n+1} \cdot \frac{n}{x^{n-1}} \right| \text{ or } \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{n x^n}{\frac{x^n}{x}(n+1)} \right| \text{ or } \lim_{n\to\infty} \left| \frac{x \cdot x^n (n+1)}{x^n (n+2)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{nx}{(n+1)} \right| \text{ or } \lim_{n\to\infty} \left| \frac{x(n+1)}{(n+2)} \right|$$

Apply L'Hôpitals

$$\lim_{n\to\infty} \left| \frac{x}{1} \right| \quad \text{or} \quad \lim_{n\to\infty} |x|$$

wehave $\lambda = |x|$

For Ratio test $\lambda = \begin{cases} <1 & \text{converge} \\ 1 & \text{neither converge or diverge} \\ >1 & \text{diverges.} \end{cases}$

$\therefore \quad |x| < 1$ converges

$|x| > 1$ diverges

when $|x| = 1$

$\therefore \quad x = \pm 1$

when $x = +1$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$ harmonic series & diverges

when $x = -1$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ Alternating harmonic series & converges.

$\therefore \quad 1 + \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{4} + \cdots$

convergent if $-1 \leq x < 1$

(2) $\quad 1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$

$$b = \frac{1\ x^0}{3^0} + \frac{1 \cdot x^1}{3} + \frac{1 \cdot x^2}{3^2} + \frac{1 \cdot x^3}{3^3} + \cdots$$

$$= \frac{x^0}{3^0} + \frac{x^1}{3^1} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$$

$$= \frac{x^n}{3^n} = \left(\frac{x}{3}\right)^n \Rightarrow \left(\frac{1}{3}\right)^n$$

Apply ratio test.

$$\lim_{n\to\infty} \left| \frac{\left(\frac{1}{3}\right)^{n+1}}{\left(\frac{1}{3}\right)^{n}} \right| = \left| \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n}}{\left(\frac{1}{3}\right)^{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^{n}}{3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{3} \right|$$

Contents 16

Sequences and Series

Learning outcomes

In this Workbook you will learn about sequences and series. You will learn about arithmetic and geometric series and also about infinite series. You will learn how to test the for the convergence of an infinite series. You will then learn about power series, in particular you will study the binomial series. Finally you will apply your knowledge of power series to the process of finding series expansions of functions of a single variable. You will be able to find the Maclaurin and Taylor series expansions of simple functions about a point of interest.

1. Introduction

Many of the series considered in Section 16.1 were examples of **finite series** in that they all involved the summation of a finite number of terms. When the number of terms in the series increases without bound we refer to the sum as an **infinite series**. Of particular concern with infinite series is whether they are convergent or divergent. For example, the infinite series

$$1 + 1 + 1 + 1 + \cdots$$

is clearly divergent because the sum of the first n terms increases without bound as more and more terms are taken. It is less clear as to whether the harmonic and alternating harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \qquad\qquad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converge or diverge. Indeed you may be surprised to find that the first is divergent and the second is convergent. What we shall do in this Section is to consider some simple convergence tests for infinite series. Although we all have an intuitive idea as to the meaning of convergence of an infinite series we must be more precise in our approach. We need a definition for convergence which we can apply rigorously.

First, using an obvious extension of the notation we have used for a finite sum of terms, we denote the infinite series:

$$a_1 + a_2 + a_3 + \cdots + a_p + \cdots \qquad \text{by the expression} \quad \sum_{p=1}^{\infty} a_p$$

where a_p is an expression for the p^{th} term in the series. So, as examples:

$$1 + 2 + 3 + \cdots = \sum_{p=1}^{\infty} p \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p$$

$$1^2 + 2^2 + 3^2 + \cdots = \sum_{p=1}^{\infty} p^2 \qquad \text{since the } p^{th} \text{ term is } a_p \equiv p^2$$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} \qquad \text{here} \qquad a_p \equiv \frac{(-1)^{p+1}}{p}$$

Consider the infinite series:

$$a_1 + a_2 + \cdots + a_p + \cdots = \sum_{p=1}^{\infty} a_p$$

We consider the **sequence of partial sums**, $S_1, S_2, \ldots,$ of this series where

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n \end{aligned}$$

That is, S_n is the sum of the first n terms of the infinite series. If the limit of the sequence $S_1,\ S_2, \ldots, S_n, \ldots$ can be found; that is

2. General tests for convergence

The techniques we have applied to analyse the harmonic and the alternating harmonic series are 'one-off':- they cannot be applied to infinite series in general. However, there are many tests that can be used to determine the convergence properties of infinite series. Of the large number available we shall only consider two such tests in detail.

The alternating series test

An alternating series is a special type of series in which the sign changes from one term to the next. They have the form

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ is a **positive** number)
Examples are:

(a) $1 - 1 + 1 - 1 + 1 \cdots$

(b) $\dfrac{1}{3} - \dfrac{2}{4} + \dfrac{3}{5} - \dfrac{4}{6} + \cdots$

(c) $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots.$

For series of this type there is a simple criterion for convergence:

Key Point 6

The Alternating Series Test

The alternating series

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

(in which each $a_i,\ i = 1, 2, 3, \ldots$ are **positive** numbers) is convergent if and only if

- the terms continually decrease:

$$a_1 > a_2 > a_3 > \ldots$$

- the terms decrease to zero:

$$a_p \to 0 \quad \text{as } p \text{ increases} \qquad (\text{mathematically } \lim_{p\to\infty} a_p = 0)$$

Task

Which of the following series are convergent?

(a) $\displaystyle\sum_{p=1}^{\infty}(-1)^p\frac{(2p-1)}{(2p+1)}$ (b) $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p^2}$

3rd $-1\cdot\frac{5}{7} = -\frac{5}{7}$

2nd term $p=2$ $(1)\left(\frac{3}{5}\right) = \frac{3}{5}$

(a) First, write out the series:

Your solution

1st term $p=1$ $(-1)\cdot\left(\frac{1}{3}\right) = \frac{-1}{3}$

Answer

$-\frac{1}{3}+\frac{3}{5}-\frac{5}{7}+\cdots$

Now examine the series for convergence:

Your solution

$\lim_{p\to\infty}\frac{2p-1}{2p+1} = \lim_{p\to\infty}\frac{1-\frac{1}{2p}}{1-\frac{1}{2p}} = 1$

limneqzerodiv diverges

lim not eq to zero Harmonic series

but w.r.t to diverges

test it neq0 ∴

Answer

$$\frac{(2p-1)}{(2p+1)} = \frac{(1-\frac{1}{2p})}{(1+\frac{1}{2p})} \to 1 \text{ as } p \text{ increases.}$$

Since the individual terms of the series do not converge to zero this is therefore a divergent series.

(b) Apply the procedure used in (a) to problem (b):

Your solution

1st: $\frac{1}{1} = 1$ 4th $= -\frac{1}{16}$

2nd $= -\frac{1}{4}$

3rd $= \frac{1}{9}$

$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}$ condition 1 ✓

$a_1 > a_2 > a_3 > a_4$

$\therefore \lim_{p\to\infty}\frac{1}{p^2} = \frac{1}{\infty} = 0$

limneqzerodiv but here limeqzeromay may converge but condition 1 is true ∴ condition 2 is also true and converges

Answer

This series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$ is an alternating series of the form $a_1-a_2+a_3-a_4+\cdots$ in which $a_p = \frac{1}{p^2}$. The a_p sequence is a decreasing sequence since $1 > \frac{1}{2^2} > \frac{1}{3^2} > \ldots$

Also $\lim_{p\to\infty}\frac{1}{p^2} = 0$. Hence the series is convergent by the alternating series test.

3. The ratio test

This test, which is one of the most useful and widely used convergence tests, applies only to series of **positive terms.**

Key Point 7

The Ratio Test

Let $\displaystyle\sum_{p=1}^{\infty} a_p$ be a series of **positive** terms such that, as p increases, the limit of $\dfrac{a_{p+1}}{a_p}$ equals a number λ. That is $\displaystyle\lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \lambda$.

It can be shown that:

- if $\lambda > 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ diverges
- if $\lambda < 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ converges
- if $\lambda = 1$, then $\displaystyle\sum_{p=1}^{\infty} a_p$ may converge or diverge.

That is, the test is inconclusive in this case.

$1 \cdot \frac{x}{x} + \frac{x \cdot x}{x} + \frac{x^2 \cdot x}{x} + \frac{x^3}{x} + x^{3-1} \ldots$

$\frac{x}{x} + \frac{x^2}{x} + x^{2-1}$

x^{n-1}

$\frac{1}{x^n}$

Example 1

Use the ratio test to examine the convergence of the series

(a) $1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \cdots$ (b) $1 + x + x^2 + x^3 + \cdots$

$1^{st} = \frac{1}{n!}$ $2^{nd} = \frac{1}{(n+1)!}$

$\lim_{n\to\infty} \frac{(n+1)!}{n!}$ $\lim_{n\to\infty} \frac{1}{n} = 0 < 1$ converges

$x = 0$

Solution

(a) The general term in this series is $\dfrac{1}{p!}$ i.e.

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{p=1}^{\infty} \frac{1}{p!} \qquad a_p = \frac{1}{p!} \qquad \therefore \qquad a_{p+1} = \frac{1}{(p+1)!}$$

and the ratio

$$\frac{a_{p+1}}{a_p} = \frac{p!}{(p+1)!} = \frac{p(p-1)\ldots(3)(2)(1)}{(p+1)p(p-1)\ldots(3)(2)(1)} = \frac{1}{(p+1)}$$

$$\therefore \qquad \lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \lim_{p\to\infty} \frac{1}{(p+1)} = 0$$

Since $0 < 1$ the series is convergent. In fact, it will be easily shown, using the techniques outlined in HELM 16.5, that

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \mathrm{e} - 1 \approx 1.718$$

(b) Here we must assume that $x > 0$ since we can only apply the ratio test to a series of positive terms.

Now

$$1 + x + x^2 + x^3 + \cdots \quad = \sum_{p=1}^{\infty} x^{p-1}$$

not arithmetic progression

$\therefore$ geometric C

$a \cdot x^{n-1}$

$a \cdot r^{n-1}$

so that

$$a_p = x^{p-1}\,, \quad a_{p+1} = x^p$$

$G = a + ar + ar^2 \ldots ar^{n-1}$

$Gr = ar + ar^2 + \ldots ar^n$

$G(1-r) = a - ar^n$

$G = \frac{a(1-r^n)}{1-r}$

and

$$\lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \lim_{p\to\infty} \frac{x^p}{x^{p-1}} = \lim_{p\to\infty} x = x$$

Thus, using the ratio test we deduce that (if x is a positive number) this series will only converge if $x < 1$.

We will see in Section 16.4 that

$$1 + x + x^2 + x^3 + \cdots \quad = \frac{1}{1-x} \qquad \text{provided} \quad 0 < x < 1.$$

$1 \cdot x^0 + 1 \cdot x^1 + 1 \cdot x^2 + \ldots$

$\therefore \sum_{n=1}^{\infty} x^{n-1}$

$\sum_{n=0}^{\infty} x^n$

$\frac{1^3}{x} + \frac{2^3}{x^2} + \frac{3^3}{x^3}$

$\frac{n^3}{(\ln 3)^n}$

Task

Use the ratio test to examine the convergence of the series:

$$\frac{1}{\ln 3} + \frac{8}{(\ln 3)^2} + \frac{27}{(\ln 3)^3} + \cdots$$

$a_n = \frac{n^3}{(\ln 3)^n}$

now $a_{n+1} = \frac{(n+1)^3}{(\ln 3)^{n+1}}$

First, find the general term of the series:

Your solution

$a_p =$

Answer

$$\frac{1}{\ln 3} + \frac{8}{(\ln 3)^2} + \cdots = \sum_{p=1}^{\infty} \frac{p^3}{(\ln 3)^p} \quad \text{so} \quad a_p = \frac{p^3}{(\ln 3)^p}$$

Now find a_{p+1}:

Your solution

$a_{p+1} =$

Answer

$$a_{p+1} = \frac{(p+1)^3}{(\ln 3)^{p+1}}$$

Finally, obtain $\lim_{p\to\infty} \frac{a_{p+1}}{a_p}$:

Your solution

$\frac{a_{p+1}}{a_p} =$ $\therefore$ $\lim_{p\to\infty} \frac{a_{p+1}}{a_p} =$

Answer

$$\frac{a_{p+1}}{a_p} = \left(\frac{p+1}{p}\right)^3 \frac{1}{(\ln 3)}. \text{ Now } \left(\frac{p+1}{p}\right)^3 = \left(1 + \frac{1}{p}\right)^3 \to 1 \text{ as } p \text{ increases}$$

$$\therefore \quad \lim_{p\to\infty} \frac{a_{p+1}}{a_p} = \frac{1}{(\ln 3)} < 1$$

Hence this is a convergent series.

$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

$= \lim_{n\to\infty} \frac{(n+1)^3}{(\ln 3)^{n+1}} \cdot \frac{(\ln 3)^n}{n^3}$

$= \lim_{n\to\infty} \frac{(n+1)^3}{(\ln 3)\, n^3}$

$= \lim_{n\to\infty} \frac{3(n+1)^2 \cdot 1}{\ln 3 \cdot 2n^2}$

$= \lim_{n\to\infty} \frac{3(n+1)^2}{2\ln 3 \cdot n^2}$

$= \lim_{n\to\infty} \frac{6(n+1) \cdot 1}{2\ln 3 \cdot 2n}$

$\lim_{n\to\infty} \frac{6}{2\ln 3 \cdot 2}$

$\lim_{n\to\infty} \frac{6}{4\ln 3}$

$\lim_{n\to\infty} \frac{3}{2\ln 3} = \frac{1.5}{\ln 3} = 1$

< 1 $\therefore$ converges

Note that in all of these Examples and Tasks we have decided upon the convergence or divergence of various series; we have not been able to use the tests to discover what actual number the convergent series converges to.

4. Absolute and conditional convergence

The ratio test applies to series of positive terms. Indeed this is true of many related tests for convergence. However, as we have seen, not all series are series of positive terms. To apply the ratio test such series must first be converted into series of positive terms. This is easily done. Consider two series $\sum_{p=1}^{\infty} a_p$ and $\sum_{p=1}^{\infty} |a_p|$. The latter series, obviously directly related to the first, is a series of positive terms.

Using imprecise language, it is harder for the second series to converge than it is for the first, since, in the first, some of the terms may be negative and cancel out part of the contribution from the positive terms. No such cancellations can take place in the second series since they are all positive terms. Thus it is plausible that if $\sum_{p=1}^{\infty} |a_p|$ converges so does $\sum_{p=1}^{\infty} a_p$. This leads to the following definitions.

Key Point 8

Conditional Convergence and Absolute Convergence

A convergent series $\sum_{p=1}^{\infty} a_p$ is said to be **conditionally convergent** if $\sum_{p=1}^{\infty} |a_p|$ is divergent.

A convergent series $\sum_{p=1}^{\infty} a_p$ is said to be **absolutely convergent** if $\sum_{p=1}^{\infty} |a_p|$ is convergent.

For example, the alternating harmonic series:

$$\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is **conditionally convergent** since the series of positive terms (the harmonic series):

$$\sum_{p=1}^{\infty} \left| \frac{(-1)^{p+1}}{p} \right| \equiv \sum_{p=1}^{\infty} \frac{1}{p} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is divergent.

Task Show that the series $-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}+\cdots$ is absolutely convergent.

First, find the general term of the series:

Your solution

$$-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}(\qquad) \qquad \therefore \qquad a_p \equiv$$

Answer

$$-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p)!} \qquad \therefore \qquad a_p \equiv \frac{(-1)^p}{(2p)!}$$

Write down an expression for the related series of positive terms:

Your solution

$$\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots \quad =\sum_{p=1}^{\infty}(\qquad\qquad) \qquad \therefore \qquad a_p =$$

Answer

$$\sum_{p=1}^{\infty}\frac{1}{(2p)!} \qquad \text{so} \qquad a_p=\frac{1}{(2p)!}$$

Now use the ratio test to examine the convergence of this series:

Your solution

p^{th} term $=$ $\qquad\qquad$ $(p+1)^{th}$ term $=$

Answer

p^{th} term $=\dfrac{1}{(2p)!}$ $\qquad\qquad$ $(p+1)^{th}$ term $=\dfrac{1}{(2(p+1))!}$

Find $\displaystyle\lim_{p\to\infty}\left[\frac{(p+1)^{th}\text{ term}}{p^{th}\text{ term}}\right]$:

Your solution

$$\lim_{p\to\infty}\left[\frac{(p+1)^{th}\text{ term}}{p^{th}\text{ term}}\right]=$$

Answer

$$\frac{(2p)!}{(2(p+1))!}=\frac{2p(2p-1)\ldots}{(2p+2)(2p+1)2p(2p-1)\ldots}=\frac{1}{(2p+2)(2p+1)}\to 0 \text{ as } p \text{ increases.}$$

So the series of positive terms is convergent by the ratio test. Hence $\displaystyle\sum_{p=1}^{\infty}\frac{(-1)^p}{(2p)!}$ is absolutely convergent.

Exercises

1. Which of the following alternating series are convergent?

(a) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^p \ln(3)}{p}$ (b) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p^2+1}$ (c) $\displaystyle\sum_{p=1}^{\infty} \frac{p\sin(2p+1)\frac{\pi}{2}}{(p+100)}$

2. Use the ratio test to examine the convergence of the series:

(a) $\displaystyle\sum_{p=1}^{\infty} \frac{e^4}{(2p+1)^{p+1}}$ (b) $\displaystyle\sum_{p=1}^{\infty} \frac{p^3}{p!}$ (c) $\displaystyle\sum_{p=1}^{\infty} \frac{1}{\sqrt{p}}$

(d) $\displaystyle\sum_{p=1}^{\infty} \frac{1}{(0.3)^p}$ (e) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{3^p}$

3. For what values of x are the following series absolutely convergent?

(a) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^p x^p}{p}$ (b) $\displaystyle\sum_{p=1}^{\infty} \frac{(-1)^p x^p}{p!}$

Answers

1. (a) convergent, (b) convergent, (c) divergent

2. (a) $\lambda = 0$ so convergent
   (b) $\lambda = 0$ so convergent
   (c) $\lambda = 1$ so test is inconclusive. However, since $\dfrac{1}{p^{1/2}} > \dfrac{1}{p}$ then the given series is divergent by comparison with the harmonic series.
   (d) $\lambda = 10/3$ so divergent, (e) Not a series of positive terms so the ratio test cannot be applied.

3. (a) The related series of positive terms is $\displaystyle\sum_{p=1}^{\infty} \frac{|x|^p}{p}$. For this series, using the ratio test we find $\lambda = |x|$ so the original series is absolutely convergent if $|x| < 1$.
   (b) The related series of positive terms is $\displaystyle\sum_{p=1}^{\infty} \frac{|x|^p}{p!}$. For this series, using the ratio test we find $\lambda = 0$ (irrespective of the value of x) so the original series is absolutely convergent for **all** values of x.

The Binomial Series 16.3

Introduction

In this Section we examine an important example of an infinite series, the **binomial** series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

We show that this series is only convergent if $|x| < 1$ and that in this case the series sums to the value $(1+x)^p$. As a special case of the binomial series we consider the situation when p is a positive integer n. In this case the infinite series reduces to a **finite** series and we obtain, by replacing x with $\frac{b}{a}$, the **binomial theorem**:

$$(b+a)^n = b^n + nb^{n-1}a + \frac{n(n-1)}{2!}b^{n-2}a^2 + \cdots + a^n.$$

Finally, we use the binomial series to obtain various polynomial expressions for $(1+x)^p$ when x is 'small'.

Prerequisites

Before starting this Section you should ...

- understand the factorial notation
- have knowledge of the ratio test for convergence of infinite series.
- understand the use of inequalities

Learning Outcomes

On completion you should be able to ...

- recognise and use the binomial series
- state and use the binomial theorem
- use the binomial series to obtain numerical approximations

1. The binomial series

A very important infinite series which occurs often in applications and in algebra has the form:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

in which p is a given number and x is a variable. By using the ratio test it can be shown that this series converges, irrespective of the value of p, as long as $|x| < 1$. In fact, as we shall see in Section 16.5 the given series converges to the value $(1+x)^p$ as long as $|x| < 1$.

Key Point 9

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \qquad |x| < 1$$

The binomial theorem can be obtained directly from the binomial series if p is chosen to be a **positive integer** (here we need not demand that $|x| < 1$ as the series is now finite and so is always convergent irrespective of the value of x). For example, with $p = 2$ we obtain

$$\begin{aligned} (1+x)^2 &= 1 + 2x + \frac{2(1)}{2}x^2 + 0 + 0 + \cdots \\ &= 1 + 2x + x^2 \quad \text{as is well known.} \end{aligned}$$

With $p = 3$ we get

$$\begin{aligned} (1+x)^3 &= 1 + 3x + \frac{3(2)}{2}x^2 + \frac{3(2)(1)}{3!}x^3 + 0 + 0 + \cdots \\ &= 1 + 3x + 3x^2 + x^3 \end{aligned}$$

Generally if $p = n$ (a positive integer) then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n$$

which is a form of the binomial theorem. If x is replaced by $\dfrac{b}{a}$ then

$$\left(1 + \frac{b}{a}\right)^n = 1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots + \left(\frac{b}{a}\right)^n$$

Now multiplying both sides by a^n we have the following Key Point:

Key Point 10

The Binomial Theorem

If n is a positive integer then the expansion of $(a+b)$ raised to the power n is given by:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + b^n$$

This is known as the **binomial** theorem.

Task

Use the binomial theorem to obtain (a) $(1+x)^7$ (b) $(a+b)^4$

(a) Here $n = 7$:

Your solution

$(1+x)^7 =$

Answer

$(1+x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

(b) Here $n = 4$:

Your solution

$(a+b)^4 =$

Answer

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Task

Given that x is so small that powers of x^3 and above may be ignored in comparison to lower order terms, find a quadratic approximation of $(1-x)^{\frac{1}{2}}$ and check for accuracy your approximation for $x = 0.1$.

First expand $(1-x)^{\frac{1}{2}}$ using the binomial series with $p = \dfrac{1}{2}$ and with x replaced by $(-x)$:

Your solution

$(1-x)^{\frac{1}{2}} =$

Answer

$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3 + \cdots$

Now obtain the quadratic approximation:

Your solution

$(1-x)^{\frac{1}{2}} \simeq$

Answer

$(1-x)^{\frac{1}{2}} \simeq 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$

Now check on the validity of the approximation by choosing $x = 0.1$:

Your solution

Answer

On the left-hand side we have

$(0.9)^{\frac{1}{2}} = 0.94868$ to 5 d.p. obtained by calculator

whereas, using the quadratic expansion:

$$(0.9)^{\frac{1}{2}} \approx 1 - \frac{1}{2}(0.1) - \frac{1}{8}(0.1)^2 = 1 - 0.05 - (0.00125) = 0.94875.$$

so the error is only 0.00007.

What we have done in this last Task is to replace (or approximate) the function $(1-x)^{\frac{1}{2}}$ by the simpler (polynomial) function $1-\dfrac{1}{2}x-\dfrac{1}{8}x^2$ which is reasonable provided x is very small. This approximation is well illustrated geometrically by drawing the curves $y = (1-x)^{\frac{1}{2}}$ and $y = 1 - \dfrac{1}{2}x - \dfrac{1}{8}x^2$. The two curves coincide when x is 'small'. See Figure 2:

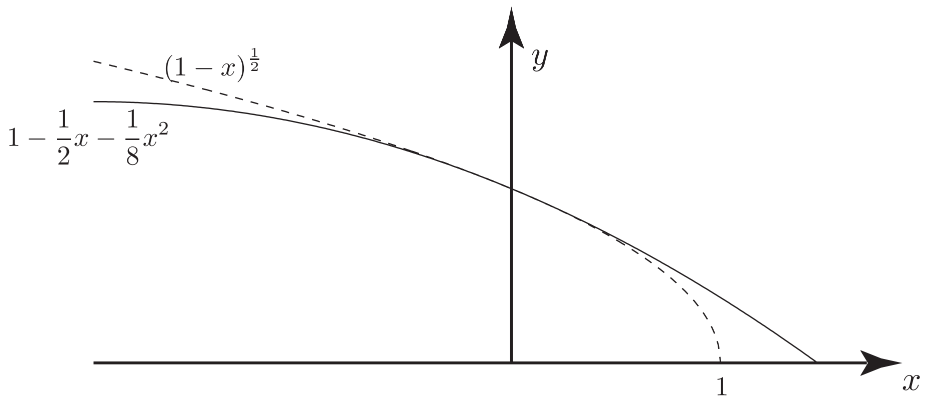


Figure 2

Task

Obtain a cubic approximation of $\dfrac{1}{(2+x)}$. Check your approximation for accuracy using appropriate values of x.

First write the term $\dfrac{1}{(2+x)}$ in a form suitable for the binomial series (refer to Key Point 9):

Your solution $\dfrac{1}{(2+x)} =$

Answer $\dfrac{1}{2+x} = \dfrac{1}{2\left(1+\dfrac{x}{2}\right)} = \dfrac{1}{2}\left(1+\dfrac{x}{2}\right)^{-1}$

Now expand using the binomial series with $p = -1$ and $\dfrac{x}{2}$ instead of x, to include terms up to x^3:

Your solution $\dfrac{1}{2}\left(1+\dfrac{x}{2}\right)^{-1} =$

Answer

$$\begin{aligned}\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} &= \frac{1}{2}\left\{1+(-1)\frac{x}{2}+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3\right\}\\ &= \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}\end{aligned}$$

State the range of x for which the binomial series of $\left(1+\dfrac{x}{2}\right)^{-1}$ is valid:

Your solution   The series is valid if

Answer   valid as long as $\left\lvert\dfrac{x}{2}\right\rvert < 1$ i.e. $\lvert x\rvert < 2$ or $-2 < x < 2$

Choose $x = 0.1$ to check the accuracy of your approximation:

Your solution

$$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} =$$

$$\frac{1}{2}-\frac{0.1}{4}+\frac{0.01}{8}-\frac{0.001}{16} =$$

Answer

$\frac{1}{2}\left(1+\frac{0.1}{2}\right)^{-1} = 0.47619$ to 5 d.p.

$\frac{1}{2}-\frac{0.1}{4}+\frac{0.01}{8}-\frac{0.001}{16} = 0.4761875.$

Figure 3 below illustrates the close correspondence (when x is 'small') between the curves $y = \frac{1}{2}(1+\frac{x}{2})^{-1}$ and $y = \frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\frac{x^3}{16}$.

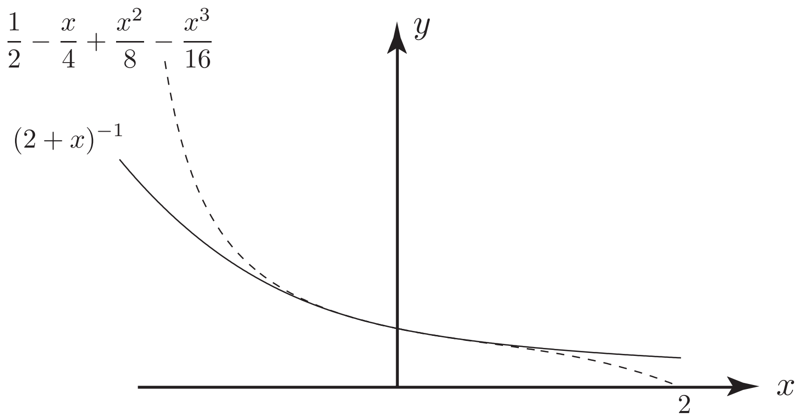


Figure 3

Exercises

1. Determine the expansion of each of the following

   (a) $(a+b)^3$, (b) $(1-x)^5$, (c) $(1+x^2)^{-1}$, (d) $(1-x)^{1/3}$.

2. Obtain a cubic approximation (valid if x is small) of the function $(1+2x)^{3/2}$.

Answers

1. (a) $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

   (b) $(1-x)^5 = 1-5x+10x^2-10x^3+5x^4-x^5$

   (c) $(1+x^2)^{-1} = 1-x^2+x^4-x^6+\cdots$

   (d) $(1-x)^{1/3} = 1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3+\cdots$

2. $(1+2x)^{3/2} = 1+3x+\frac{3}{2}x^2-\frac{1}{2}x^3+\cdots$

Power Series

16.4

Introduction

In this Section we consider power series. These are examples of infinite series where each term contains a variable, x, raised to a positive integer power. We use the ratio test to obtain the **radius of convergence** R, of the power series and state the important result that the series is absolutely convergent if $|x| < R$, divergent if $|x| > R$ and may or may not be convergent if $x = \pm R$. Finally, we extend the work to apply to general power series when the variable x is replaced by $(x - x_0)$.

Prerequisites

Before starting this Section you should ...

- have knowledge of infinite series and of the ratio test
- have knowledge of inequalities and of the factorial notation.

Learning Outcomes

On completion you should be able to ...

- explain what a power series is
- obtain the radius of convergence for a power series
- explain what a general power series is

1. Power series

A power series is simply a sum of terms each of which contains a variable raised to a non-negative integer power. To illustrate:

$$x - x^3 + x^5 - x^7 + \cdots$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

are examples of power series. In HELM 16.3 we encountered an important example of a power series, the binomial series:

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

which, as we have already noted, represents the function $(1+x)^p$ as long as the variable x satisfies $|x| < 1$.

A power series has the general form

$$b_0 + b_1x + b_2x^2 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \cdots$ are constants. Note that, in the summation notation, we have chosen to start the series at $p = 0$. This is to ensure that the power series can include a constant term b_0 since $x^0 = 1$.

The convergence, or otherwise, of a power series, clearly depends upon the value of x chosen. For example, the power series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $x = -1$ (for then it is the alternating harmonic series) and divergent if $x = +1$ (for then it is the harmonic series).

2. The radius of convergence

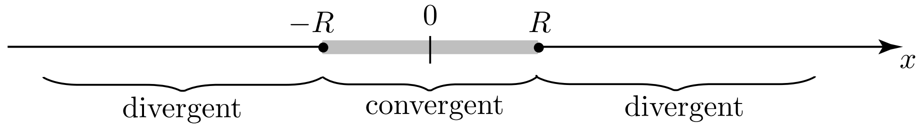
The most important statement one can make about a power series is that there exists a number, R, called the radius of convergence, such that if $|x| < R$ the power series is absolutely convergent and if $|x| > R$ the power series is divergent. At the two points $x = -R$ and $x = R$ the power series may be convergent or divergent.

Key Point 11

Convergence of Power Series

For a power series $\sum_{p=0}^{\infty} b_p x^p$ with radius of convergence R then

- the series converges absolutely if $|x| < R$
- the series diverges if $|x| > R$
- the series may be convergent or divergent at $x = \pm R$



For any particular power series $\sum_{p=0}^{\infty} b_p x^p$ the value of R can be obtained using the ratio test. We know, from the ratio test that $\sum_{p=0}^{\infty} b_p x^p$ is absolutely convergent if

$$\lim_{p\to\infty} \frac{|b_{p+1}x^{p+1}|}{|b_p x^p|} = \lim_{p\to\infty}\left|\frac{b_{p+1}}{b_p}\right| |x| < 1 \quad \text{implying} \quad |x| < \lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right| \quad \text{and so} \quad R = \lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|.$$

Example 2

(a) Find the radius of convergence of the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

(b) Investigate what happens at the end-points $x = -1$, $x = +1$ of the region of absolute convergence.

Solution

(a) Here $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots \quad = \sum_{p=0}^{\infty} \frac{x^p}{p+1}$

so

$$b_p = \frac{1}{p+1} \qquad \therefore \qquad b_{p+1} = \frac{1}{p+2}$$

In this case,

$$R = \lim_{p \to \infty} \left| \frac{p+2}{p+1} \right| = 1$$

so the given series is absolutely convergent if $|x| < 1$ and is divergent if $|x| > 1$.

(b) At $x = +1$ the series is $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ which is divergent (the harmonic series). However, at $x = -1$ the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ which is convergent (the alternating harmonic series).

Finally, therefore, the series

$$1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots$$

is convergent if $-1 \leq x < 1$.

Task

Find the range of values of x for which the following power series converges:

$$1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \cdots$$

First find the coefficient of x^p:

Your solution

$b_p =$

Answer

$b_p = \frac{1}{3^p}$

Now find R, the radius of convergence:

Your solution

$$R = \lim_{p \to \infty} \left| \frac{b_p}{b_{p+1}} \right| =$$

Answer

$$R = \lim_{p \to \infty} \left| \frac{b_p}{b_{p+1}} \right| = \lim_{p \to \infty} \left| \frac{3^{p+1}}{3^p} \right| = \lim_{p \to \infty} (3) = 3.$$

When $x = \pm 3$ the series is clearly divergent. Hence the series is convergent only if $-3 < x < 3$.

3. Properties of power series

Let P_1 and P_2 represent two power series with radii of convergence R_1 and R_2 respectively. We can combine P_1 and P_2 together by addition and multiplication. We find the following properties:

Key Point 12

If P_1 and P_2 are power series with respective radii of convergence R_1 and R_2 then the sum (P_1+P_2) and the product (P_1P_2) are each power series with the radius of convergence being the **smaller** of R_1 and R_2.

Power series can also be differentiated and integrated on a term by term basis:

Key Point 13

If P_1 is a power series with radius of convergence R_1 then

$$\frac{d}{dx}(P_1) \quad \text{and} \quad \int (P_1)\,dx$$

are each power series with radius of convergence R_1

Example 3

Using the known result that $\quad (1+x)^p = 1 + px + \dfrac{p(p-1)}{2!}x^2 + \cdots \qquad |x| < 1,$

choose $p = \frac{1}{2}$ and by differentiating obtain the power series expression for $(1+x)^{-\frac{1}{2}}$.

Solution

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots$$

Differentiating both sides: $\quad \dfrac{1}{2}(1+x)^{-\frac{1}{2}} = \dfrac{1}{2} + \dfrac{1}{2}\left(-\dfrac{1}{2}\right)x + \dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

Multiplying through by 2: $\quad (1+x)^{-\frac{1}{2}} = 1 - \dfrac{1}{2}x + \dfrac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 + \cdots$

This result can, of course, be obtained directly from the expansion for $(1+x)^p$ with $p = -\frac{1}{2}$.

Task

Using the known result that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1,$$

(a) Find an expression for $\ln(1+x)$

(b) Use the expression to obtain an approximation to $\ln(1.1)$

(a) Integrate both sides of $\dfrac{1}{1+x} = 1 - x + x^2 - \cdots$ and so deduce an expression for $\ln(1+x)$:

Your solution

$$\int \frac{dx}{1+x} =$$

$$\int (1 - x + x^2 - \cdots)\, dx =$$

Answer

$\displaystyle\int \frac{dx}{1+x} = \ln(1+x) + c$ where c is a constant of integration,

$\displaystyle\int (1 - x + x^2 - \cdots)\, dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k$ where k is a constant of integration.

So we conclude $\qquad \displaystyle\ln(1+x) + c = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + k \qquad \text{if} \quad |x| < 1$

Choosing $x = 0$ shows that $c = k$ so they cancel from this equation.

(b) Now choose $x = 0.1$ to approximate $\ln(1 + 0.1)$ using terms up to cubic:

Your solution

$$\ln(1.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \cdots \simeq$$

Answer

$\ln(1.1) \simeq 0.0953$ which is easily checked by calculator.

4. General power series

A general power series has the form

$$b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots = \sum_{p=0}^{\infty} b_p(x - x_0)^p$$

Exactly the same considerations apply to this general power series as apply to the 'special' series $\sum_{p=0}^{\infty} b_p x^p$ except that the variable x is replaced by $(x - x_0)$. The radius of convergence of the general series is obtained in the same way:

$$R = \lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right|$$

and the interval of convergence is now shifted to have centre at $x = x_0$ (see Figure 4 below). The series is absolutely convergent if $|x - x_0| < R$, diverges if $|x - x_0| > R$ and may or may not converge if $|x - x_0| = R$.

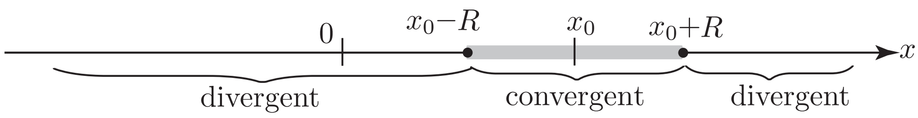


Figure 4

Task

Find the radius of convergence of the general power series

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots$$

First find an expression for the general term:

Your solution

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots = \sum_{p=0}^{\infty}$$

Answer

$\sum_{p=0}^{\infty}(x - 1)^p(-1)^p$ so $b_p = (-1)^p$

Now obtain the radius of convergence:

Your solution

$\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right| =$ $\therefore$ $R =$

Answer

$\lim_{p\to\infty}\left|\frac{b_p}{b_{p+1}}\right| = \lim_{p\to\infty}\left|\frac{(-1)^p}{(-1)^{p+1}}\right| = 1.$

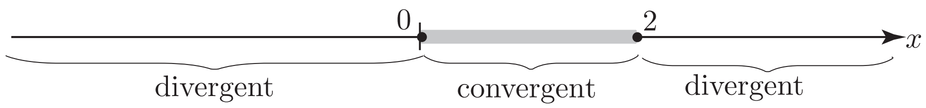
Hence $R = 1$, so the series is absolutely convergent if $|x - 1| < 1$.

Finally, decide on the convergence at $|x-1| = 1$ (i.e. at $x - 1 = -1$ and $x - 1 = 1$ i.e. $x = 0$ and $x = 2$):

Your solution

Answer

At $x = 0$ the series is $1 + 1 + 1 + \cdots$ which diverges and at $x = 2$ the series is $1 - 1 + 1 - 1 \cdots$ which also diverges. Thus the given series only converges if $|x - 1| < 1$ i.e. $0 < x < 2$.



Exercises

1. From the result $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ , \quad |x| < 1$

   (a) Find an expression for $\ln(1 - x)$

   (b) Use this expression to obtain an approximation to $\ln(0.9)$ to 4 d.p.

2. Find the radius of convergence of the general power series $1-(x+2)+(x+2)^2-(x+2)^3+\ldots$

3. Find the range of values of x for which the power series $1 + \dfrac{x}{4} + \dfrac{x^2}{4^2} + \dfrac{x^3}{4^3} + \ldots$ converges.

4. By differentiating the series for $(1 + x)^{1/3}$ find the power series for $(1 + x)^{-2/3}$ and state its radius of convergence.

5. (a) Find the radius of convergence of the series $1 + \dfrac{x}{3} + \dfrac{x^2}{4} + \dfrac{x^3}{5} + \ldots$

   (b) Investigate what happens at the points $x = -1$ and $x = +1$

Answers

1. $\ln(1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots \qquad \ln(0.9) \approx -0.1054$ (4 d.p.)

2. $R = 1$. Series converges if $-3 < x < -1$. If $x = -1$ series diverges. If $x = -3$ series diverges.

3. Series converges if $-4 < x < 4$.

4. $(1 + x)^{-2/3} = 1 - \dfrac{2}{3}x + \dfrac{5}{3}x^2 + \ldots$ valid for $|x| < 1$.

5. (a) $R = 1$. (b) At $x = +1$ series diverges. At $x = -1$ series converges.

Maclaurin and Taylor Series

16.5

Introduction

In this Section we examine how functions may be expressed in terms of power series. This is an extremely useful way of expressing a function since (as we shall see) we can then replace 'complicated' functions in terms of 'simple' polynomials. The only requirement (of any significance) is that the 'complicated' function should be *smooth*; this means that at a point of interest, it must be possible to differentiate the function as often as we please.

Prerequisites

Before starting this Section you should ...

- have knowledge of power series and of the ratio test
- be able to differentiate simple functions
- be familiar with the rules for combining power series

Learning Outcomes

On completion you should be able to ...

- find the Maclaurin and Taylor series expansions of given functions
- find Maclaurin expansions of functions by combining known power series together
- find Maclaurin expansions by using differentiation and integration

1. Maclaurin and Taylor series

As we shall see, many functions can be represented by power series. In fact we have already seen in earlier Sections examples of such a representation:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots \qquad |x| < 1 \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad -1 < x \leq 1 \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x \end{aligned}$$

The first two examples show that, as long as we constrain x to lie within the domain $|x| < 1$ (or, equivalently, $-1 < x < 1$), then in the first case $\dfrac{1}{1-x}$ has the **same numerical value** as $1+x+x^2+\cdots$ and in the second case $\ln(1+x)$ has the same numerical value as $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$. In the third example we see that e^x has the same numerical value as $1 + x + \dfrac{x^2}{2!} + \cdots$ but in this case there is no restriction to be placed on the value of x since this power series converges for all values of x. Figure 5 shows this situation geometrically. As more and more terms are used from the series $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} \cdots$ the curve representing e^x is a better and better approximation. In (a) we show the linear approximation to e^x. In (b) and (c) we show, respectively, the quadratic and cubic approximations.

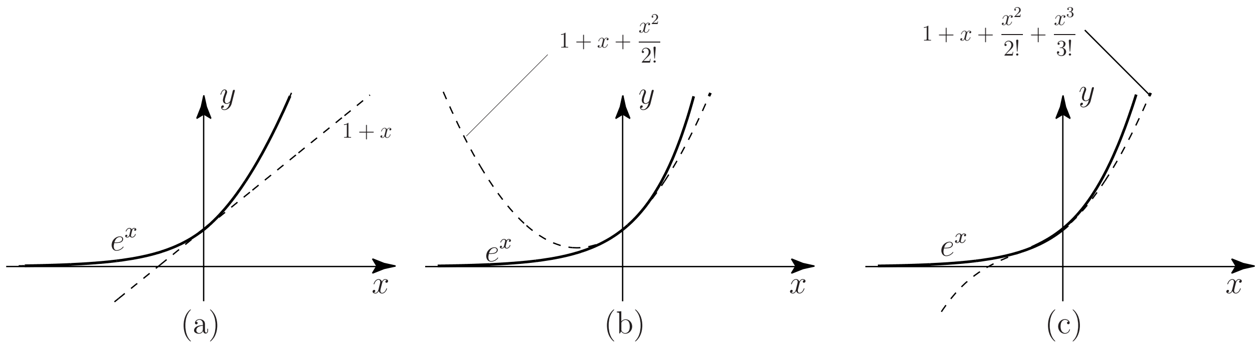


Figure 5: Linear, quadratic and cubic approximations to e^x

These power series representations are extremely important, from many points of view. Numerically, we can simply replace the function $\dfrac{1}{1-x}$ by the quadratic expression $1 + x + x^2$ as long as x is so small that powers of x greater than or equal to 3 can be ignored in comparison to quadratic terms. This approach can be used to approximate more complicated functions in terms of simpler polynomials. Our aim now is to see how these power series expansions are obtained.

2. The Maclaurin series

Consider a function $f(x)$ which can be differentiated at $x = 0$ as often as we please. For example $e^x, \cos x, \sin x$ would fit into this category but $|x|$ would not.
Let us assume that $f(x)$ can be represented by a power series in x:

$$f(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots = \sum_{p=0}^{\infty} b_p x^p$$

where $b_0, b_1, b_2, \ldots$ are constants to be determined.

If we substitute $x = 0$ then, clearly $\quad f(0) = b_0$

The other constants can be determined by further differentiating and, on each differentiation, substituting $x = 0$. For example, differentiating once:

$$f'(x) = 0 + b_1 + 2b_2 x + 3b_3 x^2 + 4b_4 x^3 + \cdots$$

so, putting $x = 0$, we have $f'(0) = b_1$.
Continuing to differentiate:

$$f''(x) = 0 + 2b_2 + 3(2)b_3 x + 4(3)b_4 x^2 + \cdots$$

so

$$f''(0) = 2b_2 \qquad \text{or} \qquad b_2 = \frac{1}{2} f''(0)$$

Further:

$$f'''(x) = 3(2)b_3 + 4(3)(2)b_4 x + \cdots \qquad \text{so} \qquad f'''(0) = 3(2)b_3 \qquad \text{implying} \qquad b_3 = \frac{1}{3(2)} f'''(0)$$

Continuing in this way we easily find that (remembering that $0! = 1$)

$$b_n = \frac{1}{n!} f^{(n)}(0) \qquad n = 0, 1, 2, \ldots$$

where $f^{(n)}(0)$ means the value of the n^{th} derivative at $x = 0$ and $f^{(0)}(0)$ means $f(0)$.
Bringing all these results together we have:

Key Point 14

Maclaurin Series

If $f(x)$ can be differentiated as often as required:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots = \sum_{p=0}^{\infty} \frac{x^p}{p!} f^{(p)}(0)$$

This is called the **Maclaurin expansion** of $f(x)$.

Example 4

Find the Maclaurin expansion of $\cos x$.

Solution

Here $f(x) = \cos x$ and, differentiating a number of times:

$$f(x) = \cos x, \qquad f'(x) = -\sin x, \qquad f''(x) = -\cos x, \qquad f'''(x) = \sin x \quad \text{etc.}$$

Evaluating each of these at $x = 0$:

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0 \text{ etc.}$$

Substituting into $f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \cdots$, gives:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The reader should confirm (by finding the radius of convergence) that this series is convergent for **all** values of x. The geometrical approximation to $\cos x$ by the first few terms of its Maclaurin series are shown in Figure 6.

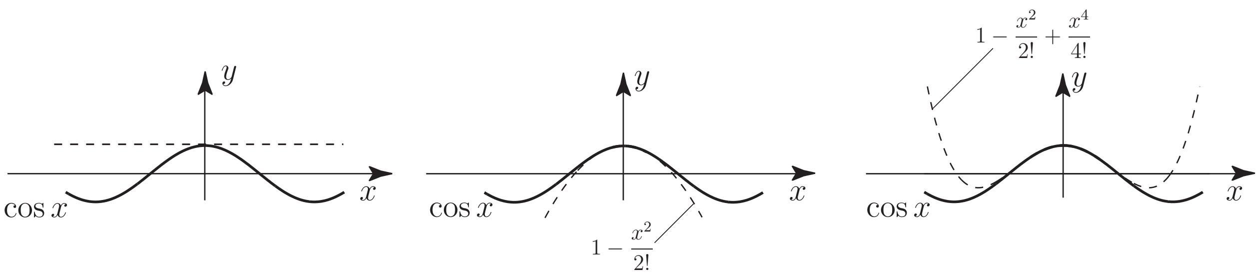


Figure 6: Linear, quadratic and cubic approximations to $\cos x$

Task

Find the Maclaurin expansion of $\ln(1 + x)$.

(Note that we **cannot** find a Maclaurin expansion of the function $\ln x$ since $\ln x$ does not exist at $x = 0$ and so cannot be differentiated at $x = 0$.)

Find the first four derivatives of $f(x) = \ln(1 + x)$:

Your solution

$f'(x) =$ $\qquad f''(x) =$ $\qquad f'''(x) =$ $\qquad f''''(x) =$

Answer

$$f'(x) = \frac{1}{1+x}, \qquad f''(x) = \frac{-1}{(1+x)^2}, \qquad f'''(x) = \frac{2}{(1+x)^3},$$

generally: $\quad f^{(n)}(x) = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n}$

Now obtain $f(0),\ f'(0),\ f''(0),\ f'''(0)$:

Your solution

$f(0) = \qquad f'(0) = \qquad f''(0) = \qquad f'''(0) =$

Answer

$f(0) = 0 \qquad f'(0) = 1, \qquad f''(0) = -1, \qquad f'''(0) = 2,$

generally: $\quad f^{(n)}(0) = (-1)^{n+1}(n-1)!$

Hence, obtain the Maclaurin expansion of $\ln(1+x)$:

Your solution

$\ln(1+x) =$

Answer

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \ldots + \dfrac{(-1)^{n+1}}{n}x^n + \cdots$ (This was obtained in Section 16.4, page 37.)

Now obtain the radius of convergence and consider the situation at the boundary values:

Your solution

Radius of convergence $\qquad R =$

Answer

$R = 1$. Also at $x = 1$ the series is convergent (alternating harmonic series) and at $x = -1$ the series is divergent. Hence this Maclaurin expansion is only valid if $-1 < x \leq 1$.

The geometrical closeness of the polynomial terms with the function $\ln(1+x)$ for $-1 < x \leq 1$ is displayed in Figure 7:

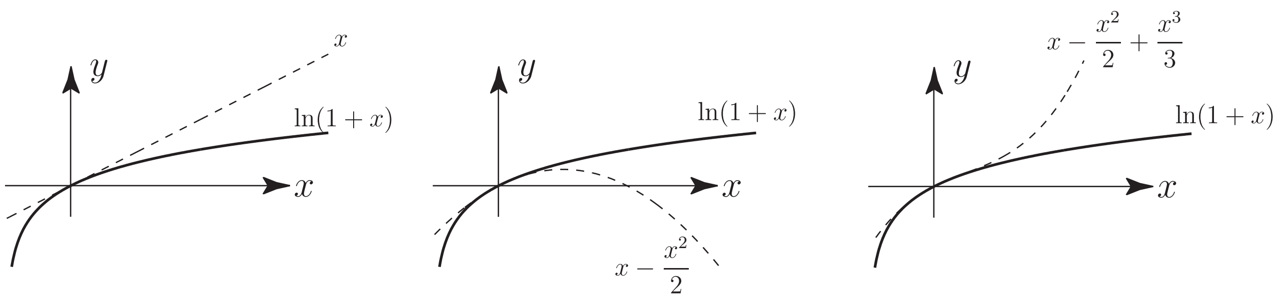


Figure 7: Linear, quadratic and cubic approximations to $\ln(1+x)$

Note that when $x = 1$ $\quad \ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} \cdots$ so the alternating harmonic series converges to $\ln 2 \simeq 0.693$, as stated in Section 16.2, page 17.

The Maclaurin expansion of a product of two functions: $f(x)g(x)$ is obtained by multiplying together the Maclaurin expansions of $f(x)$ and of $g(x)$ and collecting like terms together. The product series will have a radius of convergence equal to the **smaller** of the two separate radii of convergence.

Example 5

Find the Maclaurin expansion of $e^x \ln(1 + x)$.

Solution

Here, instead of finding the derivatives of $f(x) = e^x \ln(1+x)$, we can more simply multiply together the Maclaurin expansions for e^x and $\ln(1 + x)$ which we already know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{all } x$$

and

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad -1 < x \leq 1$$

The resulting power series will only be convergent if $-1 < x \leq 1$. Multiplying:

$$\begin{aligned}
e^x \ln(1 + x) &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
&\quad + x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \cdots \\
&\quad + \frac{x^3}{2} - \frac{x^4}{4} \cdots \\
&\quad + \frac{x^4}{6} \cdots \\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots \qquad -1 < x \leq 1
\end{aligned}$$

(You must take care not to miss relevant terms when carrying through the multiplication.)

Task

Find the Maclaurin expansion of $\cos^2 x$ up to powers of x^4. Hence write down the expansion of $\sin^2 x$ to powers of x^6.

First, write down the expansion of $\cos x$:

Your solution

$\cos x =$

Answer

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Now, by multiplication, find the expansion of $\cos^2 x$:

Your solution

$\cos^2 x =$

Answer

$$\cos^2 x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)$$

$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots) + (-\frac{x^2}{2!} + \frac{x^4}{4} \cdots) + (\frac{x^4}{4!} \cdots) + \cdots = 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} \cdots$$

Now obtain the expansion of $\sin^2 x$ using a suitable trigonometric identity:

Your solution

$\sin^2 x =$

Answer

$$\sin^2 x = 1 - \cos^2 x = 1 - \left(1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \cdots\right) = x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + \cdots$$

As an alternative approach the reader could obtain the power series expansion for $\cos^2 x$ by using the trigonometric identity $\cos^2 x \equiv \frac{1}{2}(1 + \cos 2x)$.

Example 6

Find the Maclaurin expansion of $\tanh x$ up to powers of x^5.

Solution

The first two derivatives of $f(x) = \tanh x$ are

$$f'(x) = \text{sech}^2 x \qquad f''(x) = -2\text{sech}^2 x \tanh x \qquad f'''(x) = 4\text{sech}^2 x \tanh^2 x - 2\text{sech}^4 x \quad \cdots$$

giving $\qquad f(0) = 0, \quad f'(0) - 1, \quad f''(0) = 0 \quad f'''(0) = -2 \quad \cdots$

This leads directly to the Maclaurin expansion as $\qquad \tanh x = 1 - \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 \quad \cdots$

Example 7

The relationship between the wavelength, L, the wave period, T, and the water depth, d, for a surface wave in water is given by: $\qquad L = \dfrac{gT^2}{2\pi}\tanh\left(\dfrac{2\pi d}{L}\right)$

In a particular case the wave period was 10 s and the water depth was 6.1 m. Taking the acceleration due to gravity, g, as 9.81 m s^{-2} determine the wave length.

[Hint: Use the series expansion for $\tanh x$ developed in Example 6.]

Solution

Substituting for the wave period, water depth and g we get

$$L = \frac{9.81 \times 10^2}{2\pi}\tanh\left(\frac{2\pi \times 6.1}{L}\right) = \frac{490.5}{\pi}\tanh\left(\frac{12.2\pi}{L}\right)$$

The series expansion of $\tanh x$ is given by $\qquad \tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \cdots$

Using the series expansion of $\tanh x$ we can approximate the equation as

$$L = \frac{490.5}{\pi}\left\{\left(\frac{12.2\pi}{L}\right) - \frac{1}{3}\left(\frac{12.2\pi}{L}\right)^3 + \cdots\right\}$$

Multiplying through by πL^3 the equation becomes

$$\pi L^4 = 490.5 \times 12.2\pi L^2 - \frac{490.5}{3} \times (12.2\pi)^3$$

This equation can be rewritten as $\qquad L^4 - 5984.1L^2 + 2930198 = 0$

Solving this as a quadratic in L^2 we get $L = 74$ m.

Using Newton-Raphson iteration this can be further refined to give a wave length of 73.9 m.

3. Differentiation of Maclaurin series

We have already noted that, by the binomial series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

Thus, with x replaced by $-x$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

We have previously obtained the Maclaurin expansion of $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x \leq 1$$

Now, we differentiate both sides with respect to x:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

This result matches that found from the binomial series and demonstrates that the Maclaurin expansion of a function $f(x)$ may be differentiated term by term to give a series which will be the Maclaurin expansion of $\dfrac{df}{dx}$.

As we noted in Section 16.4 the derived series will have the **same** radius of convergence as the original series.

Find the Maclaurin expansion of $(1-x)^{-3}$ and state its radius of convergence.

First write down the expansion of $(1-x)^{-1}$:

Your solution

$$\frac{1}{1-x}$$

Answer

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad |x| < 1$$

Now, by differentiation, obtain the expansion of $\dfrac{1}{(1-x)^2}$:

Your solution

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) =$$

Answer

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(1 + x + x^2 + \cdots\right) = 1 + 2x + 3x^2 + 4x^3$$

Differentiate again to obtain the expansion of $(1-x)^{-3}$:

Your solution

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}[\qquad\qquad]$$
$$=$$

Answer

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{2}[2+6x+12x^2+20x^3+\cdots] = 1+3x+6x^2+10x^3+\cdots$$

Finally state its radius of convergence:

Your solution

Answer

The final series: $1+3x+6x^2+10x^3+\cdots$ has radius of convergence $R=1$ since the original series has this radius of convergence. This can also be found directly using the formula $R = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ and using the fact that the coefficient of the n^{th} term is $b_n = \frac{1}{2}n(n+1)$.

4. The Taylor series

The **Taylor series** is a generalisation of the Maclaurin series being a power series developed in powers of $(x-x_0)$ rather than in powers of x. Thus

Key Point 15

Taylor Series

If the function $f(x)$ can be differentiated as often as required at $x = x_0$ then:

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$$

This is called the Taylor series of $f(x)$ about the point x_0.

The reader will see that the Maclaurin expansion is the Taylor expansion obtained if x_0 is chosen to be zero.

Task

Obtain the Taylor series expansion of $\dfrac{1}{1-x}$ about $x = 2$. (That is, find a power series in powers of $(x-2)$.)

First, obtain the first three derivatives and the n^{th} derivative of $f(x) = \dfrac{1}{1-x}$:

Your solution
$f'(x) =$ $f''(x) =$ $f'''(x) =$ $f^{(n)}(x) =$

Answer

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f''(x) = \frac{2}{(1-x)^3}, \qquad f'''(x) = \frac{6}{(1-x)^4}, \qquad \cdots \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Now evaluate these derivatives at $x_0 = 2$:

Your solution
$f'(2) =$ $f''(2) =$ $f'''(2) =$ $f^{(n)}(2) =$

Answer

$f'(2) = 1, \qquad f''(2) = -2, \qquad f'''(2) = 6, \qquad f^{(n)}(2) = (-1)^{n+1}n!$

Hence, write down the Taylor expansion of $f(x) = \dfrac{1}{1-x}$ about $x = 2$:

Your solution
$\dfrac{1}{1-x} =$

Answer

$$\frac{1}{1-x} = -1 + (x-2) - (x-2)^2 + (x-2)^3 + \cdots + (-1)^{n+1}(x-2)^n + \cdots$$

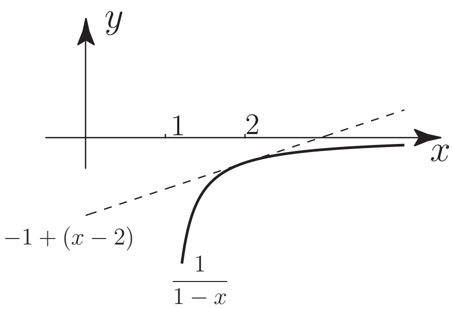
Exercises

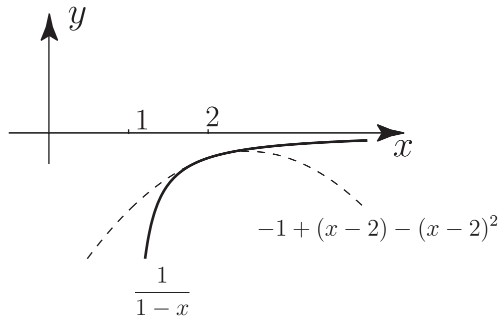
1. Show that the series obtained in the last Task is convergent if $|x-2|<1$.

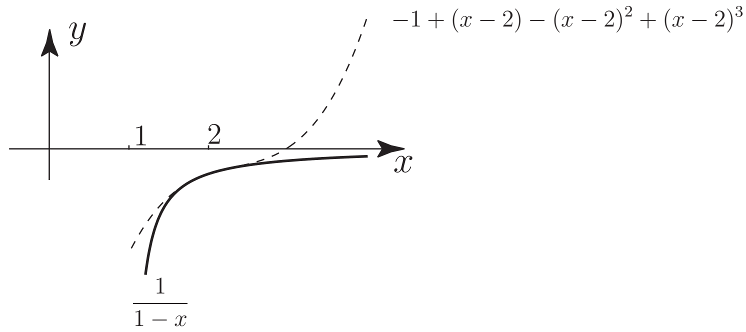
2. Sketch the linear, quadratic and cubic approximations to $\dfrac{1}{1-x}$ obtained from the series in the last task and compare to $\dfrac{1}{1-x}$.

Answer

2. In the following diagrams some of the terms from the Taylor series are plotted to compare with $\dfrac{1}{(1-x)}$.







Black pen Red Pen

(1) $\sum_{n=1}^{\infty} 2^{-3n} \cdot 7^{n}$

$$= \sum_{n=1}^{\infty} \frac{7^n}{2^{3n}} = \sum_{n=1}^{\infty} \frac{7^n}{8^n} = \sum_{n=1}^{\infty} 1 \cdot \left(\frac{7}{8}\right)^n$$

This can be taken as geometric series of the form $\sum_{m=0}^{\infty} ar^{m}$ when $m = n-1$

$\therefore |r| = \frac{7}{8}$ & Sum $= \frac{a}{1-r} = \frac{7/8}{1-7/8}$

$= \frac{7/8}{1/8} = 7$

Also by:

$\therefore$ Ratio Test

$$\lambda = \lim_{n\to\infty} \frac{(7/8)^{n+1}}{(7/8)^n}$$

$$\lambda = \lim_{n\to\infty} \frac{(7/8)^n (7/8)}{(7/8)^n}$$

$$= \lim_{n\to\infty} 7/8 = 7/8$$

$\lambda = 7/8$ & $\lambda < 1$ $\therefore$ converges

③ $\sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n}$

There is limit in ratio Test But with absolute values-sign(.

Ratio test.

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(-2)^n n!} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{(-2)^n}(-2)\cancel{(n+1)}\cancel{(n)!}}{(n+1)^n\cancel{(n+1)}} \cdot \frac{n^n}{\cancel{(-2)^n}\cancel{n!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-2)(n^n)}{(n+1)^n} \right|$$

$$\lim_{n\to\infty} |-2| \left| \frac{(n)^n}{(n+1)^n} \right|$$

$$\lim_{n\to\infty} |-2| \frac{n(n)^{n-1}}{n(n+1)^{n-1}}$$

$$\lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n$$

This is where you use the secrect weapon The Fact

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^{bn} = e^{ab}$$

$$2\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n}$$

$= 2 \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{-n}$ is of the form : $\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$

here $a = 1$ $b = -1$

$\therefore = 2 e^{-1} = \left|\frac{2}{e}\right|$

& $\left|\frac{2}{e}\right| < 1$ $\therefore$ converges.

④ $\sum_{n=1}^{\infty} \frac{\sin(2n)}{n + 3^n}$

remember $|\sin \theta| \leq 1$

$\therefore \sum_{n=1}^{\infty} \frac{\sin(2n)}{n+3^n} = \sum_{n=1}^{\infty} \left|\frac{1}{n+3^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{3^n}$

$= \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n}_{\text{geometric series}}$

$|r| = \left|\frac{1}{3}\right| < 1$

$\therefore$ converges

Therefore the original series converges by Direct Comparison Test

|abs| series converges $\therefore$ absolute convergence

Series Formulae Sheet & Tips

- T.F.D

For $\sum_{n=1}^{\infty} a_n$ $\lim_{n\to\infty} a_n \neq 0$ diverges

limneqzerodiv

- AST

For $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

① prove $a_1 > a_2$

② $\lim_{n\to\infty} a_n = 0$

limzerastcon

key tricks to remember for solving series

- if you see $\sin\theta$ in question remember: $|\sin\theta| \leq 1$

- transform a series

$$\sum_{n=2}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - x^0 - x^1 = \frac{a}{1-r} - 1 - x$$

- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

- Remember the secrect fact:

$$\left.\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}\ \right\}$$

Then this automatically becomes a test for divergence

⑥ $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$

By Integral Test

$\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx$ Type 1 Improper Integral

$\displaystyle\lim_{t\to\infty} \int_2^{t} \frac{\ln x\,dx}{x^2}$

$u = \ln x$ $\quad vdv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = \frac{x^{-2+1}}{-2+1} = -\frac{1}{x}$

$\displaystyle\lim_{t\to\infty} \left.\frac{-\ln x - 1}{x}\right|_2^t$

$\displaystyle = -\frac{\ln x}{x} - \int -\frac{1}{x^2}\,dx$

$\displaystyle = -\frac{\ln x}{x} + \frac{x^{-2+1}}{-2+1} + c$

$\displaystyle = -\frac{\ln x}{x} - \frac{1}{x} = \frac{-\ln x - 1}{x} + c$

$\displaystyle\left(\lim_{t\to\infty} \frac{-\ln t - 1}{t}\right) - \left(\frac{-\ln 2 - 2}{2}\right)$

$\displaystyle\left(\frac{-\infty}{\infty}\right) + \left(\frac{\ln 2 + 2}{2}\right)$

$\displaystyle -1 + \frac{\ln 2 + 2}{2}$

$\displaystyle \frac{-2 + \ln 2 + 2}{2} = \frac{\ln 2}{2}$

limit exist

∴ converges

or

use comparison

$$\ln(n) < \sqrt{n}$$

$$\therefore \sum_{n=2}^{\infty} \ln(n) < \sum_{n=2}^{\infty} \sqrt{n}$$

$\div$ by n^2

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} < \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2}$$

$$\sum_{n=2}^{\infty} \frac{n^{1/2}}{n^2}$$

$$= \sum_{n=2}^{\infty} (n)^{\frac{1}{2}-2} = \sum_{n=2}^{\infty} n^{-3/2}$$

$$= \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

$p = \frac{3}{2}$ $\quad p > 1$

$\therefore$ converges.

(7) $$\sum_{n=1}^{\infty} \frac{3n+1}{\sqrt{n^5+4n^2+12}}$$

This series is continuous & from $\int_1^{\infty}$ & Type 1

$$\therefore \lim_{t\to\infty} \int_1^t \frac{3n+1}{\sqrt{n^5+4n^2+12}} \approx \lim_{t\to\infty} \int_1^t \frac{n}{n^{5/2}}$$

$$= \lim_{t \to \infty} \int_1^t \frac{1}{n^{5/2 - 1}}$$

$$= \lim_{t \to \infty} \int_1^t \frac{1}{n^{3/2}}$$

$p = \frac{3}{2} > 1 \therefore$ converges

(8) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$

writing it in terms of x

$$\sum_{n=3}^{\infty} \frac{1}{x \ln(x)} = \int_3^{\infty} \frac{1}{x \ln(x)}$$

Type 1 Integral continuous.

$$\lim_{t \to \infty} \int_3^t \frac{1}{x \ln(x)} dx$$

let $\ln x = u$

$\frac{1}{x} dx = du$

$$\lim_{t\to\infty} \int_3^t \frac{1}{u}\,du$$

$$= \lim_{t\to\infty} \ln|u| \Big|_3^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 3|$$

$$= \infty$$

Limit DNE
∴ diverges

(9) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

Secret Fact: $\sum_{n=1}^{\infty} \left(1 + \frac{a}{n}\right)^{bn} = e^{ab}$

we have $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n = \sum_{n=1}^{\infty} \left(1 + \frac{-1}{n}\right)^{1n}$

$$\therefore \underbrace{\lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^{1n}}_{\text{when you use the fact it automatically becomes test for divergence limneqzerodiv}} = e^{-1} = \frac{1}{e} \neq 0$$

∴ diverges

(10) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$

we know $|\sin\theta| \le 1$

but we cannot use it

rewrit n as x

$$\therefore \int_1^{\infty} 1 \cdot \sin^2\left(\frac{1}{x}\right)$$

$$= \lim_{t\to\infty} \int_1^t 1 \cdot \sin^2\left(\frac{1}{x}\right) dx$$

$u = \sin^2\left(\frac{1}{x}\right)$ $\quad$ $vdv = 1dx$

$du = \cos^2\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right)$ $\quad$ $v = x$

$$\int \sin^2\left(\frac{1}{x}\right) dx = x\sin^2\left(\frac{1}{x}\right) + \int \cos^2\left(\frac{1}{x}\right) \cdot x + C$$

$$\int \cos^2\left(\frac{1}{x}\right) \cdot x \, dx$$

$u = \cos^2\left(\frac{1}{x}\right)$ $\quad$ $vdv = x\,dx$

$du = \sin^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)dx$ $\quad$ $v = \frac{x^2}{2}$

$$\int \sin^2\left(\frac{1}{x}\right) dx = x\sin^2\left(\frac{1}{x}\right) + \cos^2\left(\frac{1}{x}\right) \cdot \frac{x^2}{2} + \frac{1}{2}\int \sin^2\left(\frac{1}{x}\right)$$

$$\frac{1}{2}\int \sin^2\left(\frac{1}{x}\right) dx = x\sin^2\left(\frac{1}{x}\right) + \frac{x^2}{2}\cos^2\left(\frac{1}{x}\right)$$

$$\int \sin^2\left(\frac{1}{x}\right) dx = 2x\sin^2\left(\frac{1}{x}\right) + x^2\cos^2\left(\frac{1}{x}\right)$$

Now back to limit

$$\lim_{t \to \infty} \left[2x \sin^2\left(\frac{1}{x}\right) + x^2 \cos^2\left(\frac{1}{x}\right) \right]_1^t$$

$$= \lim_{t \to \infty} \left[2t \sin^2\left(\frac{1}{t}\right) + t^2 \cos^2\left(\frac{1}{t}\right) \right] -$$

$$\left[2 \sin^2(1) + \cos^2(1) \right]$$

$$\left[2 \left[1 - \cos^2(1) + \cos^2(1) \right] \right]$$

$$\left[2 \left[1 - 0 \right] \right]$$

$$\left[2 \right]$$

$$\infty \sin^2(0) + \infty \cos^2(0) - 2$$

$$= \infty$$

Limit DNE $\therefore$ diverges

$$\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$$

$$a_n = \sin^2\left(\frac{1}{n}\right)$$

let $b_n = \frac{1}{n} \Rightarrow$ diverges

$$\therefore \lim_{n \to \infty} \frac{\sin^2\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\cos^2\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \cos^2\left(\frac{1}{n}\right)$$

$$= \cos^2(0) = 1 > 0 \quad \therefore \text{ converges}$$

According to Limit Comparison Test

for Σa_n & Σb_n ; if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

where $c \neq 0$ or ∞ & $c > 0$

(11) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Ratio Test $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lambda$

$\lambda < 1$ converges

$\lambda > 1$ diverges

$\lambda = 1$ inconclusive

$$a_{n+1} = \left|\frac{(2(n+1))!}{((n+1)!)^2}\right|$$

$$a_n = \left|\frac{(2n)!}{(n!)^2}\right|$$

$$\lim_{n\to\infty} \left|\frac{(2n+2)!}{((n+1)(n)!)^2} \cdot \frac{n!\,.\,n!}{(2n)!}\right|$$

$$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}}{((n+1)\cancel{(n)!})^{\cancel{2}}}\cdot\frac{\cancel{n!}\,\cancel{n!}}{\cancel{(2n)!}}\right|$$

$$\lim_{n\to\infty}\left|\frac{2(n+1)(2n+1)}{(n+1)^2}\right|$$

$$\lim_{n\to\infty}\left|\frac{2(2n+1)}{(n+1)}\right|$$

$$\lim_{n\to\infty}\left|\frac{4n\left(1+\frac{1}{2n}\right)}{n\left(1+\frac{1}{n}\right)}\right| = \frac{4\left(1+\frac{1}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)} = \frac{4}{1} = 4$$

$4 > 1 \therefore$ diverges by Ratio Test

(12) $\frac{3}{5} - \frac{1}{5} + \frac{1}{15} - \frac{1}{45} + \frac{1}{135} - \dots$

ratio: $\frac{1}{5}\cdot\frac{5}{3} = \frac{1}{3}$

$$\sum_{n=0}^{\infty} ar^n = (-1)^n\frac{3}{5}\cdot\left(\frac{1}{3}\right)^n$$

$|r| = \frac{1}{3} < 1$

$\therefore$ converges by Geometric Series Test

This above looks complicated hence we transform:

$$\sum_{n=1}^{\infty}\frac{3}{5}\left(-\frac{1}{3}\right)^{n-1}$$

$|r| = \left|-\frac{1}{3}\right| = \frac{1}{3} < 1$

$\therefore$ converges

$$\text{sum} = \frac{3/5}{1-\left(-\frac{1}{3}\right)} = \frac{3/5}{1+\frac{1}{3}} = \frac{3/5}{4/3} = \frac{3}{5}\cdot\frac{3}{4} = \frac{9}{20}$$

(12) $$\frac{\tan^{-1}(1)}{2} + \frac{\tan^{-1}(2)}{5} + \frac{\tan^{-1}(3)}{10} + \frac{\tan^{-1}(4)}{17} + \cdots$$

$$\sum_{n=1}^{\infty} \tan^{-1}(n) \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots\right)$$

$$\underbrace{\frac{1}{1\cdot1+1} + \frac{1}{2\,2+1} + \frac{1}{3\cdot3+1} + \cdots}_{\frac{1}{n^2+1}}$$

$$= \sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2+1}$$

in terms of x : $\displaystyle\int_1^{\infty} \frac{\tan^{-1}(x)}{1+x^2}\,dx$

$$u = \tan^{-1}(x)$$

$$du = \frac{1}{1+x^2}\,dx$$

$$\therefore \lim_{t\to\infty} \int_1^t u\,du$$

$$\lim_{t\to\infty} \left.\frac{u^2}{2}\right|_1^t$$

$$\frac{1}{2} \lim_{t\to\infty} \left.\left(\tan^{-1}(x)\right)^2\right|_1^t$$

$$= \frac{1}{2}\left[\lim_{t\to\infty} \left(\tan^{-1}(t)\right)^2 - \left(\tan^{-1}(1)\right)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\pi}{16}\right]$$

$$= \frac{16\pi - 4\pi}{64 \cdot 2} = \frac{12\pi}{64 \cdot 2} = \frac{6\pi}{64}$$

$$= \frac{3\pi}{32}$$

Limit exists

$\therefore$ the Improper Integral converges

(13) $\frac{1}{4} - \frac{1}{7} + \frac{1}{10} - \frac{1}{13} + \frac{1}{16} \dots$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \frac{1}{13} + \frac{1}{16}\right]$$

$\frac{1}{4} \vert n=1$	$\frac{1}{7} \vert n=2$	$\frac{1}{10} \vert n=3$	$\frac{1}{13} \vert n=4$
$1n+3=4$	$1n+5=7$	$1n+9=10$	$1n+9=13$
$2n+2=4$	$2n+3=7$	$2n+4=10$	$2n+5=13$
$3n+1=4$	$3n+1=7$	$3n+1=10$	$3n+1=13$
$4n-0=4$	$4n-1=7$	$4n-2=3$	$4n-3=13$

selection

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{3n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n+1}$$

} is an alternating series.

where $a_1 > a_2 > a_3 > \dots$

& $\lim_{n\to\infty} \frac{1}{3n+1} = 0$ } This usually does not help in divergence test but helps in alternating series test.

lim zero as t

& $a_{n+1} < a_n$ prove

(14) $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots$

5 7 9

$\sum_{n=1}^{\infty}$

n=1	n=2	n=3	n=4	n=5
$n+2=3$	$n+6=8$	$n+12=15$	$n+20=24$	$n+30=35$
$2n+1=3$	$2n+4=8$	$2n+9=15$	$2n+16=24$	$2n+25=35$

This can be useful

as: $2n+n^2 \Rightarrow n(2+n)$

$$= \sum_{n=1}^{\infty} \frac{1}{n(2+n)}$$

Using integral Test: (replacing n with x as Brains CNN is used to x)

$$= \int_1^{\infty} \frac{1}{x(2+x)}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x(2+x)}\,dx$$

$$= \frac{1}{x(2+x)} = \frac{A}{x} + \frac{B}{2+x}$$

$$1 = A(2+x) + B(x)$$

$x = -2 \Rightarrow \quad 1 = -2B \qquad B = -\frac{1}{2}$

$x = 0 \Rightarrow \quad 1 = 2A \qquad A = \frac{1}{2}$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{2x}\,dx + \lim_{t\to\infty} \int_1^t \frac{-1}{2(2+x)}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2} \ln|x| \Big|_1^t - \frac{1}{2} \lim_{t\to\infty} \ln|2+x| \Big|_1^t$$

$$\frac{1}{2}\left[\lim_{t\to\infty} \ln t - \ln 1\right] - \frac{1}{2}\left[\lim_{t\to\infty} \ln|2+t| - \ln|3|\right]$$

$$= \frac{1}{2}\left[\infty - 0\right] - \frac{1}{2}\left[\infty - \ln 3\right]$$

$$= \infty - \infty = 0$$

This actually is a Telescoping series

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{2+n}\right)$$

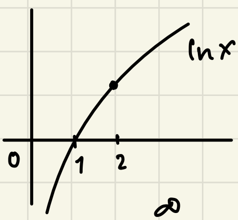
writing out first few terms

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} \dots - \frac{1}{2+n-1} - \frac{1}{2+n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{1+n} - \frac{1}{2+n}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{1}{2}\left(\frac{2+1}{2}\right) = \frac{3}{4}$$

(15) $1 + \frac{1}{2^{\ln 2}} + \frac{1}{3^{\ln 2}} + \frac{1}{4^{\ln 2}} + \frac{1}{5^{\ln 2}}$



$1^{\ln 2} = 1$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^{\ln 2}} = \text{p-series test}$$

$\ln 2 \approx 0.69$

$\ln 2 < 1 \quad \therefore$ diverges.

by p-series test

Bonus:

1) Conv or div: $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$

$$\int_1^{\infty} \cos\left(\frac{1}{n}\right) = \lim_{t\to\infty} \int_1^{t} \cos\left(\frac{1}{n}\right) dn$$

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

By p-series test

we have $p = 1 \leq 1 \therefore$ diverges

2) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ By integral test

$\Rightarrow \displaystyle\int_{2}^{\infty} \frac{1}{\ln x}\,dx$ is complicated to solve

But we know the list:

$\ln x < x^n < a^x < x^x$

$\dfrac{1}{\ln x} > \dfrac{1}{x^n} > \dfrac{1}{a^x} > \dfrac{1}{x^x}$

for $n = 1$ & below — above

$\longleftarrow$ diverges — converges $\longrightarrow$

3) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n^n)} = \sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$

$\displaystyle\int_{1}^{\infty} \frac{1}{x \ln(x)}\,dx$

$\ln x = u$

$\frac{1}{x}\,dx = du$

$$\lim_{t\to\infty} \int_1^t \frac{1}{u}\, du = \lim_{t\to\infty} \ln|\ln x| \Big|_1^t$$

$$= \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 1|$$

$$= \infty + \infty = \infty$$

4) $\displaystyle\sum_{n=1619}^{\infty} \frac{1}{(\ln n)^{\ln n}} = \cancel{\sum_{n=1619}^{\infty} \frac{1}{\ln n \cdot \ln n}} = \cancel{\sum_{n=1619}^{\infty} \frac{1}{(\ln n)^2}}$

$\displaystyle\cancel{= \sum_{n=1619}^{\infty} \frac{1}{2 \ln n}}$

we can rewrite it as:

$$= \left(e^{\ln n}\right)^{\ln n} = e^{\ln n \ln n}$$

$$= n^{\ln n}$$

$\therefore$ we have $\displaystyle\sum_{n=1619}^{\infty} \frac{1}{n^{\ln n}}$

$$= \sum_{n=1619}^{\infty} \frac{1}{n^{\ln n}} \overset{?}{<} \sum_{n=1619}^{\infty} \frac{1}{n^2}$$

$$n^2 \overset{?}{<} n^{\ln n} \quad \forall\ n \geq 1619$$

$2 < \ln n$ True

$\therefore$ converges.

According to the list

$$\frac{1}{n^p} > \frac{1}{n^n}$$

for $p > 1$ converges

$\therefore$ we prove our question summation function is less than $\frac{1}{n^2}$ to prove convergence

5) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\tan^{-1} n}$ This is an alternating series

$$\lim_{n\to\infty} \frac{(1)^{\infty}}{\tan^{-1} n} = \frac{(1)^{\infty}}{\pi/2} = \cancel{\frac{2}{\pi}} \quad \text{DNE} \neq 0 \ \therefore \text{Diverges}$$

limit not equal to zero diverges

But this is alternating series test.

$\therefore$ lim zero ast con. & the test is inconclusive

b) $\sum_{n=1}^{\infty} \frac{2^n}{3^n + n^3}$

By Ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lambda$ $\begin{cases} \lambda < 1 & \text{converges} \\ \lambda > 1 & \text{diverges} \\ \lambda = 1 & \text{inconclusive} \end{cases}$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1} + (n+1)^3} \cdot \frac{3^n + n^3}{2^n}$$

$$\lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2}{3^n \cdot 3 + (n^3 + 1 + 3n^2 + 3n)} \cdot \frac{3^n + n^3}{\cancel{2^n}}$$

$$\lim_{n\to\infty} \frac{2(3^n + n^3)}{3^n \cdot 3 + (n+1)^3} = \lim_{n\to\infty}$$

Instead of this find the dominant part of num & denom which makes things much easier

$\ln n < n^p < a^n < n^n$

$\frac{1}{\ln x} > \frac{1}{n^p} > \frac{1}{a^n} > \frac{1}{n^n}$

$\therefore$ we have as $\lim_{n\to\infty} \frac{2^n}{3^n}$

as $n \to \infty$ $n^3 < 3^n$

won't be large enough as in the enominator

$\therefore$ now $\lim_{n\to\infty} \frac{2^n}{3^n}$ is geometric series

$\therefore$ $\lim_{n\to\infty} 1 \cdot \left(\frac{2}{3}\right)^n$ & $|r| = \frac{2}{3} < 1$

$\therefore$ converges

By Direct Comparison Test.

⑦ $\sum_{n=1}^{\infty} \frac{3^n}{2^n + n^2}$

list

As $n \to \infty$ $\ln n < n^p < a^n < n! < n^n$

$\frac{1}{\ln n} > \frac{1}{n^p} > \frac{1}{a^n} > \frac{1}{n!} > \frac{1}{n^n}$

$=$ As $n \to \infty$

$\sum_{n=1}^{\infty} \underbrace{1 \cdot \left(\frac{3}{2}\right)^n}_{\text{geometric series}}$ as $n^p < a^n$ in denominator

with $|r| = 1.5 > 1$ diverges

what you got here is your known divergent
Now you have to prove the original question diverges
through Limit Comparison Test

Important thing you have to check the limit
as LCT states

$\sum_{n=1}^{\infty} a_n$: question limit is +ve & > 0

$\sum_{n=1}^{\infty} b_n$ is your derived divergent/convergent limit

Now $\lim_{n \to \infty} \frac{a_n}{b_n}$ is $\neq 0$ or ∞ ; is convergent or divergent depending on the type of b_n

8. $$\sum_{n=1}^{\infty} \frac{n\sin^2 n}{n^3+2}$$

$$< \lim_{n\to\infty} \frac{n\sin^2 n}{n^3+2} = \lim_{n\to\infty} \frac{n\,2\sin n\cos n}{3n^2}$$

$$= \lim_{n\to\infty} \frac{\sin 2n}{3n} = \lim_{n\to\infty} \frac{\cos 2n\cdot 2}{3}$$

$$= \frac{2}{3}\lim_{n\to\infty} \cos 2n$$

what to do

9. $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

This is an alternating series

$\therefore$ we attempt AST

when series converges if

$a_1 > a_2 \ldots$ & $\lim_{n\to 0} a_n = 0$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\infty} = 0$$

$$\frac{1}{\sqrt{n+1}} \overset{?}{>} \frac{1}{\sqrt{n+1+1}}$$

$$\sqrt{n+2} > \sqrt{n+1}$$

$n+2 > n+1$ always True

$\therefore$ converges

(10) $\frac{1}{2} - \frac{1}{3} + \frac{2}{9} - \frac{4}{27}$

$r = -\frac{1}{3} \times \frac{2}{1} = -\frac{2}{3}$

$\sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{2}{3}\right)^n$

$= n=2 \quad \frac{1}{2} \cdot \frac{+4}{9} = \frac{2}{9}$

$n=3 \quad \frac{1}{2} \cdot \left(\frac{-8}{27}\right) = \frac{-4}{27}$

$\therefore$ we have a geometric series

$\sum_{n=0}^{\infty} \frac{1}{2}\left(-\frac{2}{3}\right)^n$

$|r| = \frac{2}{3} < 1 \quad \therefore$ converges

$\text{Sum} = \frac{a}{1-r} = \frac{\frac{1}{2}}{1 - -\frac{2}{3}} = \frac{\frac{1}{2}}{\frac{5}{3}} = \frac{1}{2} \cdot \frac{3}{5}$

$= \frac{3}{15}$

(11) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} - \lim_{n\to\infty} \frac{1}{n}$

$= \infty - \infty$

? diverges or 0 ?

We know for separate:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}}_{p = \frac{1}{2}} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{1}}}_{p=1}$$

$p < 1$

$\therefore$ diverges

$p \leq 1$

diverges

$\therefore$ the whole series diverges.

(12) $$\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$$

$$= \cancel{\lim_{t\to\infty} \int_3^t \frac{1}{x^2 \ln x}\, dx}$$

difficult
∴ attempting
ratio test

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)^2 \ln(n+1)} \cdot \frac{n^2 \ln n}{1} \right|$$: Ratio Test

$$= \lim_{n\to\infty} \left| \frac{n^2 \ln n}{(n+1)^2 \ln(n+1)} \right|$$

= Applying L'Hôpitals

$$= \lim_{n\to\infty} \left| \frac{n^2 \cdot \frac{1}{n} + \ln n \cdot 2n}{(n+1)^2 \cdot \frac{1}{n+1} \cdot 1 + \ln(n+1)\, 2(n+1) \cdot 1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n + n \cdot 2 \ln n}{(n+1) + 2(n+1) \ln(n+1)} \right|$$

Applying L'Hôpitals:

$$= \lim_{n\to\infty} \left| \frac{1 + 2\left[n \cdot \frac{1}{n} + \ln n\right]}{1 + 2\left[(n+1)\frac{1}{(n+1)} + \ln(n+1) \cdot 1\right]} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+2(1+\ln n)}{1+2(1+\ln(n+1))} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+2+2\ln n}{1+2+2\ln(n+1)} \right|$$

Applying L'Hôpitals

$$= \lim_{n\to\infty} \left| \frac{2\cdot\frac{1}{n}}{2\cdot\frac{1}{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{\frac{1}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 1$$

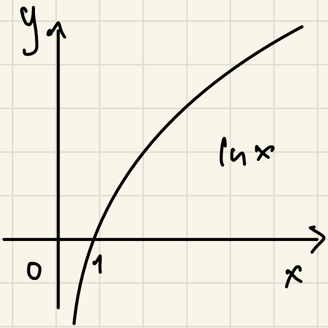
is inconclusive

Ratio Test also useless

Important! Check later!

Direct Comparison Test

$$\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n} \overset{?}{\le} \sum_{n=3}^{\infty} \underbrace{\frac{1}{n^2}}_{\text{The convergent we select to check}} \quad \forall n \ge 3$$



$$n^2 \leq n^2 \ln n$$

$$1 \leq \ln n \qquad \forall\, n \geq 3$$

is True

$\therefore$ Converges

13 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\, e^{\sqrt{n}}}$

$$\cancel{e^{\sqrt{x}} = u}$$

$$\cancel{e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\, dx = du}$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x}\, e^{\sqrt{x}}}\, dx$$

when we consider in x form:
take

$$\sqrt{x} = u$$

$$\frac{1}{2\sqrt{x}}\, dx = du$$

$$\frac{1}{\sqrt{x}}\, dx = 2du$$

$$= \lim_{t\to\infty} \int_1^t \frac{2}{e^u}\, du$$

$$= 2 \lim_{t\to\infty} -e^{-u} \Big|_1^t$$

$$= 2 \left[\lim_{t\to\infty} -\frac{1}{e^t} + \lim_{t\to\infty} \frac{1}{e} \right]$$

$$= 2\left[0 + \frac{1}{e}\right] = \frac{2}{e}$$ converges

(14) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{n^2}}$

According to the list

$\ln x \ll x^n < a^x < x^x$

$\frac{1}{\ln x} > \frac{1}{x^n} > \frac{1}{a^x} > \frac{1}{x^x}$

Too complicated

for n=1
& below

diverges

above n=1
converges.

Trying root test

$\lim\limits_{n\to\infty} \sqrt[n]{\frac{n^n}{3^{n^2}}}$

$= \lim\limits_{n\to\infty} \frac{n^{n/n}}{3^{n^2/n}} = \lim\limits_{n\to\infty} \frac{n^1}{3^n}$

~~Now acc to list:~~ This is still limit

$= \lim\limits_{n\to\infty} \frac{1}{3^n \ln 3} = \frac{1}{\infty} = 0$

(15) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

By ratio test

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n \cdot \cancel{(n+1)}}{((n+1)\cancel{(n)!})^{\cancel{2}}} \cdot \frac{\cancel{(n!)^2}}{n^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{(n+1)\, n^n} \right|$$

power series ?

Forgot chapter learn it!

(16) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

$$\int_1^{\infty} n \sin\left(\frac{1}{n}\right) = \lim_{t\to\infty} \int_1^{t} x \sin\left(\frac{1}{x}\right) dx$$

$$u = \sin\left(\frac{1}{x}\right) \qquad vdv = x\,dx$$

$$du = \cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)dx \qquad v = \frac{x^2}{2}$$

$$= \frac{x^2}{2}\sin\left(\frac{1}{x}\right) - \int \frac{x^2}{2}\cdot\cos\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right) dx$$

$$= \frac{x^2}{2}\sin\left(\frac{1}{x}\right) + \frac{1}{2}\int \cos\underbrace{\left(\frac{1}{x}\right)}_{\text{always confuses (look it up)}} dx$$

Power series

Best Friend

- BF: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $|x| < r$, $R = 1$
  $I = (-1, 1)$

same like geometric series

- BF diff on both sides

Also BF integrable
but $\frac{1}{1+x} = \ln x$
$\therefore \int \sum_{n=0}^{\infty} (-1x)^n$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

chain rule inside diff.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$; $R = 1$; $I = (-1, 1)$

R is always same

I is not always same $\therefore$ apply lim on both ends

$-\cancel{x}.$ is 1 as if we put 0 ; $n = 0$ $\therefore 0 \cdot x^{0-1} = 0$? . . .

Power Series Trick Sheet for Solving:

① Best Friend: when you see; transform & write as

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1 \qquad IOC: (-1,1)$$

becomes R; R=1

② Differentiate LHS & RHS of ① to get:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \qquad |x|<1 \qquad IOC: (-1,1)$$

But always check $\lim \to \infty$ on both ends

③ Also $\displaystyle\int \frac{1}{1+x} = \ln|1+x| + C = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1--x}$

④ Also $\displaystyle\int \frac{1}{1+x^2} = \tan^{-1}(x) + C = \int \sum_{n=0}^{\infty} (-1)^n (x^2)^n$

③ & ④ are some examples of manipulating & rewriting the functions to adjust to B.F.

⑤ $\displaystyle\frac{1}{(x+a)(x-b)} = \frac{A}{x--a} + \frac{B}{x-b}$; apply partial Fraction

⑥ Refer question ⑧ to solve cases when $a \neq 0$

⑦ Do not forget to apply completing through squares whenever needed

$2ax = num \cdot x$

square, $a^2 = \left(\frac{num}{2}\right)^2$

$$f'(\tan^{-1}(x))$$

$$\frac{1}{1+x^2} = \frac{1}{1-x} = \sum_{x=1}^{\infty} (x)^n$$

$$\frac{1}{1- -x^2}$$

$$\sum$$

use the square to solve the unsovable quadratic eqn.

⑧ Taylor series power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$c_n = \frac{f^n(a)}{n!}$$

for these series $|x-a| < R$

To find radius of convergence apply: $\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$ if lim is ∞ then IOC: $(-\infty, \infty)$

⑨ Remember famous power series to save time and rewrite them when needed

- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

(10) For the power series of type:

$$(1+x)^{r} = \sum_{n=0}^{\infty} \binom{r}{n} x^{n}$$

where, r choose n : denoted as $\binom{r}{n}$

is $$\frac{r(r-1)(r-2)\ldots(r-n+1)}{n!}$$

IOC : given r is a real number
converges if $-1 < r \leq 1$

① $\dfrac{x}{1-4x}$ at $a=0$

$$\frac{x}{1-4x} = x \cdot \frac{1}{1-4x}$$

$$= x \cdot \sum_{n=0}^{\infty} (4x)^n \quad \therefore \ |4x| < 1$$

$$= x \sum_{n=0}^{\infty} 4^n x^n \qquad 4|x| < 1$$

$$|x| < \frac{1}{4} \qquad R = \frac{1}{4}$$

$$= \sum_{n=0}^{\infty} 4^n \cdot x^{n+1} \qquad -\frac{1}{4} < x < \frac{1}{4}$$

$$\boxed{\sum_{n=0}^{\infty} 4^n x^{n+1} \qquad I = \left(-\frac{1}{4}, \frac{1}{4}\right) \qquad R = \frac{1}{4}}$$

② $\dfrac{x^4}{9+x^2}$ at $a=0$

we have $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n \qquad |R| = 1 \quad I = (-1, 1)$

$$\therefore \ x^4 \cdot \sum_{n=0}^{\infty} \frac{1}{9\left(1+\frac{1}{9}x^2\right)} = \frac{x^4}{9} \sum_{n=0}^{\infty} \frac{1}{1+\left(\frac{x}{3}\right)^2} \neq \frac{1}{1-x} \text{ form}$$

$$\therefore \ \frac{x^4}{9} \sum_{n=0}^{\infty} \frac{1}{1 - -\frac{x^2}{9}}$$

$\therefore$ we have $\left|-\frac{1}{9}x^2\right| < 1$

$\frac{1}{9}|x^2| < 1$

$|x^2| < 9$

$|x| < 3 \quad \therefore R = 3$

$-3 < x < 3$

IOC: $(-3, 3)$

since $a = 0$

Now back to the series

$$\frac{x^4}{9} \sum_{n=0}^{\infty} 1 \cdot \left(\frac{-x^2}{9}\right)^n$$

$$= \frac{x^4}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{9^n}$$

$$\boxed{= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{9^{n+1}} \Bigg\} \text{ is the power series.} \quad \text{IOC} = (-3,3) \quad a=0}$$

③ $\dfrac{1+2x}{1-x}$ at $a=0$

we have BF $= \dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$

$$\therefore \frac{1+2x}{1-x} = \frac{1}{1-x} + \frac{2x}{1-x}$$

$$= \sum_{n=0}^{\infty} \frac{1}{1-x} + 2x \sum_{n=0}^{\infty} \frac{1}{1-x}$$

$$= \sum_{n=0}^{\infty} x^n + 2x \sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty} x^{n+1}$$

$|x| < 1$

$\therefore R = 1$

$IOC = (-1, 1)$

To trans form $n+1$ to 1 we subtract 1 essentially making series start from $n=1$

$$= \sum_{n=0}^{\infty} x^n + 2\sum_{n=1}^{\infty+1} x^{n+1-1}$$

$$= \sum_{n=0}^{\infty} x^n + 2\sum_{n=1}^{\infty} x^n$$

$=$ writing out first term to make series start at $n=1$

$$= x^0 + \sum_{n=1}^{\infty} x^n + 2\sum_{n=1}^{\infty} x^n$$

$$\boxed{= 1 + 3\sum_{n=1}^{\infty} x^n \qquad IOC = (-1, 1) \qquad R = 1}$$

(4) $\dfrac{1}{x^2 - 5x - 6}$ at $a = 0$

$1 \times -6 = -6$

$1 + -6 = -5$

$$= \frac{1}{x^2 + 1x - 6x - 6}$$

$$= \frac{1}{x(1+x) - 6(x+1)} = \frac{1}{(1+x)(x-6)}$$

$$= \frac{1}{1 - -x} \cdot \frac{1}{-6 + x}$$

Do not split like this instead use partial fractions!

we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad |x| < 1 \quad (-1, 1)$

This does not work

wrong approach

$$\therefore \frac{1}{1 - -1x} \cdot \frac{1}{-6\left(1 + \frac{-1}{6}x\right)}$$

$$= \sum_{n=0}^{\infty} (-1x)^n \cdot \frac{1}{-6} \sum_{n=0}^{\infty} \left(\frac{-1}{6}x\right)^n$$

$$= \sum_{n=0}^{\infty} x^n \cdot \frac{1}{-6} \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{6^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^n \cdot x^n}{-6^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{-6^{n+1}}$$

$|-x| < 1$

$x < 1$

$-1 < x < 1$

also

$\left|-\frac{1}{6}x\right| < 1$

$\frac{1}{6}x < 1$

$|x| < 6$

$-6 < x < 6$

multiplication not possible must split denom using partial fractions

$$\frac{1}{x^2 - 5x - 6} = \frac{1}{(1+x)(x-6)} = \frac{1}{(1+x)(-6+x)}$$

$$\frac{1}{(1+x)(-6+x)} = \frac{A}{(1+x)} + \frac{B}{(-6+x)}$$

$$1 = A(-6+x) + B(1+x)$$

when $x = 6$

$1 = 0 + 7B$

$B = \frac{1}{7}$

when $x = -1$

$$1 = A(-7) + 0$$

$$A = -\frac{1}{7}$$

$$\frac{1}{(1+x)(-6+x)} = -\frac{1}{7(1+x)} + \frac{1}{7(-6+x)}$$

$$= -\frac{1}{7} \frac{1}{1 - -x} + \frac{1}{7} \frac{1}{-6\left(1 - \frac{1}{6}x\right)}$$

$$= -\frac{1}{7} \frac{1}{1 - -x} + \frac{1}{-42} \frac{1}{1 - \frac{1}{6}x}$$

$$= -\frac{1}{7} \sum_{n=0}^{\infty} (-x)^n + \frac{1}{-42} \sum_{n=0}^{\infty} \left(\frac{1}{6}x\right)^n$$

$$= -\frac{1}{7} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{1} - \frac{1}{42} \sum_{n=0}^{\infty} \frac{x^n}{6^n}$$

$$= \sum_{n=0}^{\infty} \left(-\frac{(-1)^n}{7} - \frac{1}{42 \cdot 6^n} \right) x^n$$

Now for Radius of Convergence we have

$$|-x| < 1 \quad \text{and} \quad \left|\frac{1}{6}x\right| < 1$$

$$|x| < 1 \qquad \frac{1}{6}|x| < 1$$

greater than

$$\therefore \quad R = 1$$

Interval of convergence

$$|x| < 1$$

$$I = -1 < x < 1$$

$$\boxed{\sum_{n=0}^{\infty}\left(-\frac{(-1)^n}{7} - \frac{1}{42 \cdot 6^n}\right)x^n \qquad R = 1 \qquad I = -1 < x < 1}$$

(5) $\left(\dfrac{1}{1-x}\right)^2$ at $a=0$

$= \dfrac{1}{(1-x)^2}$ is same as BF differentiated ie

We know $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \frac{d}{dx}\left((1-x)^{-1}\right) = \sum_{n=1}^{\infty} n x^{n-1}$$

$$= -1(1-x)^{-2} \cdot (-1) = \sum_{n=1}^{\infty} n x^{n-1}$$

$|x| < 1$

$$= \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad ; \quad \therefore R = 1$$

Now for IOC

check both ends

when $x = -1$ & $x = 1$

$$\sum_{n=1}^{\infty} n(-1)^{n-1}$$

$$\lim_{n \to \infty} n(-1)^{n-1}$$

DNE

$\therefore$ diverges

$$\sum_{n=1}^{\infty} n(1)^{n-1}$$

not equal to zero

by T.F.D

$\therefore$ Diverges

$$\therefore \quad I = (-1, 1)$$

$$\boxed{= \sum_{n=1}^{\infty} n x^{n-1} \; ; \quad R = 1 \; ; \; I = (-1, 1)}$$

6) $\ln(1+x)$ at $a = 0$

we know $\displaystyle\int \frac{1}{1+x} = \ln(1+x)$

also we know $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad |x| < 1 \quad R = 1$

IOC $(-1, 1)$

$$\therefore \quad \frac{1}{1+x} = \frac{1}{1 - -x}$$

$R = -1$

$|-1x| < 1$

$|x| < 1$

$$= \sum_{n=0}^{\infty} (-1x)^n \qquad -1 < x < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

For this ends we have to apply limits to the integrated function we get it $\int \frac{1}{1+x}$

$$\int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} \overset{\swarrow \text{const.}}{(-1)^n} x^n$$

u . v

$$\ln(1+x) = C + \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}_{\text{apply end lims here}}$$

when $x = -1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1}$$

Apply L'Hôpitals

$$= \lim_{n\to\infty} \frac{(\ln 1) \cdot 2^{2n+1}}{1} = 0$$

T.F.D is inconclusive

But if we rewrite

$$\sum_{n=0}^{\infty} \frac{\overset{\swarrow \text{even} \therefore 1}{(-1)^{2n}} (-1)}{n+1}$$

$$= (-1) \sum_{n=0}^{\infty} \frac{1}{n+1}$$

$x = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

by AST

$\lim_{n\to\infty} a_n = 0$

& decreases

$a_n > a_{n+1}$

if they both checks out it converges

$\therefore$ here $\lim_{n\to\infty} \frac{1}{n+1} = 0$

$$\frac{1}{n+1} \overset{?}{>} \frac{1}{n+2} \quad \forall n \geq 0$$

dominating power in denom & num

$\approx (-1) \sum_{n=0}^{\infty} \frac{1}{n}$

as $\lim n \to \infty$

$\therefore$ This is pseries with $p \leq 1$

$\therefore$ diverges

$\therefore$ open bracket

$n+2 \overset{?}{>} n+1$

True

$\therefore$ This end converges

$\therefore$ Square bracket

$IOC = (-1, 1]$

Now to find constant c

let $x = 0$

$\therefore \ln(0+1) = c + \sum_{n=0}^{\infty} (-1)^n \frac{0^{n+1}}{n+1}$

$0 = c + 0$

$c = 0$

$\ln(x+1)$ at $a = 0$

$\therefore$ is

$$\boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad R = -1 \qquad IOC = (-1, 1]}$$

⑦ $\tan^{-1} x$ at $a=0$

We know $\int \frac{1}{1+x^2} = \tan^{-1} x$

$\therefore \int \frac{1}{1 - -x^2}$ $|-x| < 1$

$|x| < 1$

$$\int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\tan^{-1}(x) = c + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \cdot (-1)^n$$ you forgot!

when $x=0$

$0 = c + 0 \quad \therefore \quad c=0$

$R = 1$

For IOC we apply limits on both ends

$|x| < 1$

$-1 < x < 1$

when $x = -1$

$$\lim_{n\to\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \; ; \; x = -1$$

$$\lim_{n\to\infty} (-1)^n \frac{(-1)^{2n}(-1)}{2n+1}$$

~~As $n \to \infty$~~

$$\cancel{\approx -1 \lim_{n\to\infty} \frac{1}{2n}}$$

$$\cancel{= -\frac{1}{2} \lim_{n\to\infty} \frac{1}{n}}$$

~~$p = 1 \; ; \; p \leq 1$~~

$\therefore$ diverges

when $x = 1$

$$\lim_{n\to\infty} \frac{x^{2n+1}}{2n+1}(-1)^n \; ; \; x = 1$$

$$\lim_{n\to\infty} \frac{1^{2n+1}}{2n+1}(-1)^n$$

$$\lim_{n\to\infty} \frac{1}{2n+1}(-1)^n$$

as $n \to \infty$

$$\cancel{\frac{1}{2} \lim_{n\to\infty} \frac{1}{n}}$$

~~diverges~~

$$\lim_{n\to\infty} \frac{1}{2n+1} = \frac{1}{\infty} = 0$$

$$\frac{1}{2n+1} \overset{?}{>} \frac{1}{2n+1+1}$$

$$2n+2 \overset{?}{>} 2n+1$$

True

converges

$\therefore \quad IOC = [\cancel{(}-1, 1)$

why?

as you forgot $(-1)^n$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n} (-1)}{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1}$$

Prove this By AST

as: $\lim_{n\to\infty} a_n = 0$ & $a_n > a_{n+1}$

$$\lim_{n\to\infty} \frac{1}{2n+1} = \frac{1}{\infty} = 0$$

$$\frac{1}{2n+1} \overset{?}{>} \frac{1}{2n+1+1}$$

$$2n+2 \overset{?}{>} 2n+1$$

True

$\therefore$ Converges by AST

Both converges

$\therefore$ IOC $= [-1, 1]$

(8) $\dfrac{1}{1-x}$ at $a=3$

we do not have $a=0$ anymore
$\therefore$ rewrite x as $x-a$ form of power series $= c_n(x-a)^n$

& then add or subtract what ever is necessary to make the other half 1
& also make sure to include '$-$' minus sign

Step (1) $\dfrac{1}{x-3}$

Step (2) $\dfrac{1}{-(x-3)}$

step (3) to make it add to 1 $\dfrac{1}{-2-(x-3)}$

Step ④ Take whatever added as factor: outside

$$\frac{1}{2\left(1-\frac{x-3}{-2}\right)}$$

your $|R|<1$

$$\frac{1}{1-x} \overset{a=3}{=} \frac{1}{-2-x-3}$$

$$= \frac{1}{-2\left(1-\frac{1}{-2}(x-3)\right)}$$

$$|R|<1$$

$$= \left|\frac{x-3}{-2}\right|<1$$

$$\frac{|x-3|}{2}<1$$

$$|x-3|<2 \quad \longleftarrow \text{ This is the } R$$

$$-2<x-3<2$$

+3 on whole equality relation

$$-2+3<x<2+3$$

$$\underbrace{1<x<5}$$

Now we do not have to check on ends. lims as we technically manipulated on R $\therefore$ open brackets

$$= \frac{1}{-2\left(1 - -\frac{x-3}{2}\right)} = \frac{1}{-2} \sum_{n=0}^{\infty} \left(\frac{(x-3)}{-2}\right)^n$$

$$= \frac{1}{-2} \sum_{n=0}^{\infty} \frac{(x-3)^n}{(-2)^n}$$

$\therefore$ $\frac{1}{1-x}$ at $a=3$ is:

$$\frac{1}{2} \sum_{n=0}^{\infty} \frac{(x-3)^n}{(-2)^n} \qquad R = 2 \; ; \; IOC: (1, 5)$$

(9) $\frac{1}{x^2}$ at $a = -2$

$$= \frac{1}{1-x} \xRightarrow{a=-2} \frac{1}{\left(3 - (x - -2)\right)^2}$$

$$= \frac{1}{\left(3 - (x+2)\right)^2}$$

Do not do this!

$$= \frac{1}{(1-x)^2}$$

and we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$|R| < 1$
$R = 1$ IOC $(-1, 1)$

we have $\frac{1}{x^2}$

we know $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

we also know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $|R| < 1$; $R = 1$
IOC $(-1, 1)$

at $a=0$

we have $a = -2$

$$\therefore \frac{1}{-2 + (x - -2)} = \frac{1}{x}$$

$$\frac{1}{-2 + x + 2} = \frac{1}{x}$$

$$\frac{1}{-2\left(1 + \frac{x+2}{-2}\right)} = \frac{1}{x}$$

$$\frac{1}{-2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{(2)^n} = \frac{1}{x}$$

$$\frac{d}{dx}\left(-\frac{1}{2} \sum_{n=0}^{\infty} \frac{(x+2)^n}{(2)^n} \right) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\sum_{n=0}^{\infty} \frac{-1}{2^{n+1}} \cdot n(x+2)^{n-1} = -\frac{1}{x^2}$$

$$= \sum_{n=0}^{\infty} \frac{n(x+2)^{n-1}}{2^{n+1}} = \frac{1}{x^2}$$

The power series form

Now for radius of convergence:

$$\left| \frac{x+2}{-2} \right| < 1$$

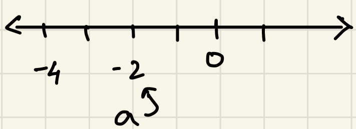
$$\frac{1}{2} |x+2| < 1$$

$$|x+2| < 2 \leftarrow R$$

$$-2 < x+2 < 2$$

$$-2-2 < x < 2-2$$

$$-4 < x < 0 \leftarrow IOC$$



(10) $\dfrac{1}{x^2+6x+10}$ at $a=-3$

roots: $\dfrac{-6 \pm \sqrt{36-40}}{2} = \dfrac{-6 \pm \sqrt{-4}}{2} = \dfrac{-6 \pm 4i}{2} = -3 \pm 2i$

$$\frac{1}{(-3+2i)(-3-2i)}$$

completing the squares:

$$-2ax = 6x$$
$$a = -3$$
$$a^2 = 9$$

$$\therefore \quad x^2+6x+10+9-9$$

$$x^2+6x+9+10-9$$

$\underline{3} \times \underline{3} = 9$

$\underline{3} + \underline{3} = 6$

$$x^2+3x+3x+9+1$$
$$= x(x+3)+3(x+3)+1$$
$$= (x+3)(x+3)+1$$

rewrite question as: $\dfrac{1}{1+(x+3)^2}$

we know: $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad |x|<1$

$R=1$

$IOC = (-1, 1)$

we want : $\frac{1}{1+x^2} = \frac{1}{1--x^2} = \sum_{n=0}^{\infty} (-x^2)^n$

$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$\therefore$ x+3 : $\frac{1}{1+(x+3)^2} = \sum_{n=0}^{\infty} (-1)^n (x+3)^{2n}$

$\hookrightarrow \frac{1}{1--(x+3)^2}$

$|-1(x+3)^2| < 1$

$|(x+3)^2| < 1$

$|x+3| < 1$

$\therefore R = 1$

$-1 < x+3 < 1$

$-1-3 < x+3-3 < 1-3$

$-4 < x < -2$ is the IOC

IOC: $(-4,-2)$

(11) e^x at $a=0$

You use Taylor series (create Table)

n	$f^{(n)}(x)$	$\frac{f^{(n)}(a=0)}{n!}$
0	e^x	$\frac{e^0}{0!} = \frac{1}{1}$

$f(x) = C_n(x-a)^n$

$C_n = \frac{f^{(n)}(a)}{n!}$

1	e^x	$\frac{e^0}{1!} = \frac{1}{1}$
2	e^x	$\frac{e^0}{2!} = \frac{1}{2!}$
3	e^x	$\frac{e^0}{3!} = \frac{1}{3!}$
	$\vdots$	
n	e^x	$\left.\frac{1}{n!}\right\} C_n$

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} (x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Radius of Convergence:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{1}{n!} \cdot \frac{(n+1)!}{1} \right| = \lim_{n\to\infty} |(n+1)| = \infty$$

$\therefore$ IOC $= (-\infty, \infty)$

(12) $\sin x$ at $a = 0$

n	$f^{(n)}(x)$	$\frac{f^{(n)}(a=0)}{n!}$	$(x-a)^n$
0	$\sin x$	$\frac{0}{0!} = 0$	$(x)^0$
1	$\cos x$	$\frac{1}{1!} = \frac{1}{1!}$	x^1
2	$-\sin x$	$\frac{0}{2!} = 0$	x^2
3	$-\cos x$	$-\frac{1}{3!} = -\frac{1}{3!}$	x^3
4	$\sin x$	$\frac{0}{4!} = 0$	x^4
5	$\cos x$	$\frac{1}{5!} = \frac{1}{5!}$	x^5

∴ we have power series formula

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a=0)}{n!} (x-a)^n$$

$$\sin x = 0 \cdot 1 + \frac{1}{1!} \cdot x^1 + 0 \cdot x^2 - \frac{1}{3!} x^3 + 0 \cdot x^4 + \frac{1}{5!} x^5 \ldots$$

$$= \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \dots -$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} \cdot (x-0)^{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Radius of Convergence

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

$$(2(n+1)+2)! = (2n+2+2)! = (2n+4)!$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n}{(2n+1)!} \cdot \frac{\cancel{(2n+2)!}}{(-1)^{n+1}} \right| \Leftarrow \text{wrong!}$$

$$= \lim_{n\to\infty} \left| \frac{2n+2}{-1} \right| = \lim_{n\to\infty} \left| 2n+2 \right| = \infty$$

$$\therefore\ R = \infty$$

$$IOC = (-\infty, \infty)$$

(13) $\cos x$ at $a = 0$

n	$f^{(n)}(x)$	$c_n = \dfrac{f^{(n)}(a=0)}{n!}$	$(x-a)^n$	$c_n(x-a)^n$
0	$\cos x$	$\cos 0 = \dfrac{1}{0!} = 1$	$x^0 = 1$	$1\,x^0$
1	$-\sin x$	$\dfrac{0}{1!} = 0$	x^1	0
2	$-\cos x$	$-\dfrac{1}{2!} = \dfrac{-1}{2!}$	x^2	$-\dfrac{1}{2!}x^2$
3	$\sin x$	$\dfrac{0}{3!} = 0$	x^3	0
4	$\cos x$	$\dfrac{1}{4!} = \dfrac{1}{4!}$	x^4	$+\dfrac{1}{4!}x^4$
5	$-\sin x$	$\dfrac{0}{5!} = 0$	x^5	0
6	$-\cos x$	$-\dfrac{1}{6!} = \dfrac{1}{6!}$	x^6	$-\dfrac{1}{6!}x^6$
7	$-\sin x$	$\dfrac{0}{7!} = 0$	x^7	0

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n)!}(x-a)^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$R = \infty$
IOC $= (-\infty, \infty)$

(14) e^{3x} at $x = 2$

we know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

we have e^{3x} $\therefore$ we manipulate e^x to e^{3x}

e^x at $x = 2 \Rightarrow e^{x-2}$

step① $e^{3x} = e^{(x-a)}$

step② we have $a = 2$ $\therefore$ $e^{3x} \stackrel{?}{=} e^{(x-2)}$

step③ make them equal $e^{3x} \stackrel{?}{=} e^3 \cdot e^{(x-2)}$

$e^{3x} \stackrel{?}{=} e^{3x-6}$

$$e^{3x} \stackrel{?}{=} e^{6} \cdot e^{3x-6}$$

$$e^{3x} = e^{3x}$$ yes! ∴ we take

$$e^{3x} = e^{6} \cdot \underbrace{e^{3x-6}}$$

This part can have the e^{x} power series expansion

$$= e^{6} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (3x-6)^{n}$$

$$= e^{6} \sum_{n=0}^{\infty} \frac{1}{n!} 3^{n}(x-2)^{n}$$

$$= \sum_{n=0}^{\infty} e^{6} \cdot \frac{3^{n}}{n!} (x-2)^{n}$$

Now for radius of Convergence

$$\lim_{n\to\infty} \left| \frac{3^{n}}{n!} \cdot \frac{(n+1)!}{3^{n+1}} \right|$$

R: $$\lim_{n\to\infty} \left| \frac{n+1}{3} \right| = \infty$$

IOC : $(-\infty, \infty)$

(15) $\sin x$ at $a = \frac{\pi}{2}$

we know $\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} x^{2n+1}$

when $a = 0$

we have $\sin x$ at $a = \frac{\pi}{2}$

$\sin x \overset{?}{=} \sin\left(x - \frac{\pi}{2}\right)$

at $x = 0$

$\sin\left(x - \frac{\pi}{2}\right)$

Mirror

S	$+\sin(90+\theta) = \cos\theta$ $\cos(90+\theta) = -\sin\theta$ $\tan(90+\theta) = -\cot\theta$ $+\sin(180-\theta) = \sin\theta$ $\cos(180-\theta) = \cos\theta$ $\tan(180-\theta) = \tan\theta$	$\sin(90-\theta) = \cos\theta$ $\cos(90-\theta) = \sin\theta$ ← we pick this $\tan(90-\theta) = \cot\theta$ $\sin(360+\theta) = \sin\theta$ $\cos(360+\theta) = \cos\theta$ $\tan(360+\theta) = \tan\theta$	A
T	$\sin(180+\theta) = -\sin\theta$ $\cos(180+\theta) = -\cos\theta$ $+\tan(180+\theta) = \tan\theta$ $\sin(270-\theta) = -\cos\theta$ $\cos(270-\theta) = -\sin\theta$ $+\tan(270-\theta) = \cot\theta$	$\sin(360-\theta) = -\sin\theta$ $+\cos(360-\theta) = +\cos\theta$ $\tan(360-\theta) = -\tan\theta$ $\sin(270+\theta) = -\cos\theta$ $+\cos(270+\theta) = +\sin\theta$ $\tan(270+\theta) = -\cot\theta$	C

So we have $\sin x$ at $x=0$ equivalent to $\Rightarrow$ $\cos(90-\theta) = \sin x$

$\therefore \cos\left(\frac{\pi}{2} - x\right)$

$=$ we have

$x-a = \left(x - \frac{\pi}{2}\right)$

$= -\left(\frac{\pi}{2} - x\right)$

we know $\cos(-\theta) = \cos\theta$

$\therefore$

$\therefore \cos\left(-\left(\frac{\pi}{2} - x\right)\right)$

$= \cos\left(\frac{\pi}{2} - x\right)$

we know $\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n)!} x^{2n}$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} \left(\frac{\pi}{2} - x\right)^{2n}$$

$R = \infty$ IOC: $(-\infty, \infty)$

(ii) $\sin x$ at $a = -\pi$

We know $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)} x^{2n+1}$

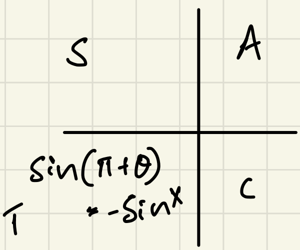
when $a = 0$

$\therefore$ when $a = -\pi$

we have $x - a = x - -\pi = \pi + x$

$$\sin(\pi + x) = -\sin(x+\pi)$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} (x+\pi)^{2n+1}$$



$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x+\pi)^{2n+1}}{(2n+1)!}$$

$R = \infty$ IOC: $(-\infty, \infty)$

(17) $\sin^2 x$ at $a=0$

we know $\sin^2 2x = \dfrac{1-\cos 2x}{2}$

$\therefore \quad \dfrac{1}{2} - \dfrac{1}{2}\underbrace{\cos 2x}_{\text{power series applied to it}}$

$$= \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}(2x)^{2n}$$

$R=\infty$ $\quad$ $IOC=(-\infty,\infty)$

$$R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

$$= \frac{1}{2} - \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2(2n)!}(2x)^{2n}$$

(18) $\cos x$ at $a=\frac{\pi}{4}$

we know $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}$ when $a=0$

we have $a=\frac{\pi}{4}$

$\therefore \quad x - a = x - \frac{\pi}{4}$

wrong its $\cos\left(\frac{\pi}{4}\right)$ not $\cos\frac{\pi}{2}$

n	$f^{(n)}(x)$	C_n $f^{(n)}(a)/n!$	$(x-a)^n$	$C_n(x-a)^n$
0	$\cos x$	$1/0!$	$\left(x-\frac{\pi}{4}\right)^0$	$\frac{1}{0!}\left(x-\frac{\pi}{4}\right)^0$
1	$-\sin x$	$0/1!$	$\left(x-\frac{\pi}{4}\right)^1$	0
2	$-\cos x$	$-1/2!$	$\left(x-\frac{\pi}{4}\right)^2$	$-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2$
3	$\sin x$	$0/3!$	$\left(x-\frac{\pi}{4}\right)^3$	0
4	$\cos x$	$1/4!$	$\left(x-\frac{\pi}{4}\right)^4$	$\frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4$
5	$-\sin x$	$-0/5!$	$\left(x-\frac{\pi}{4}\right)^5$	0

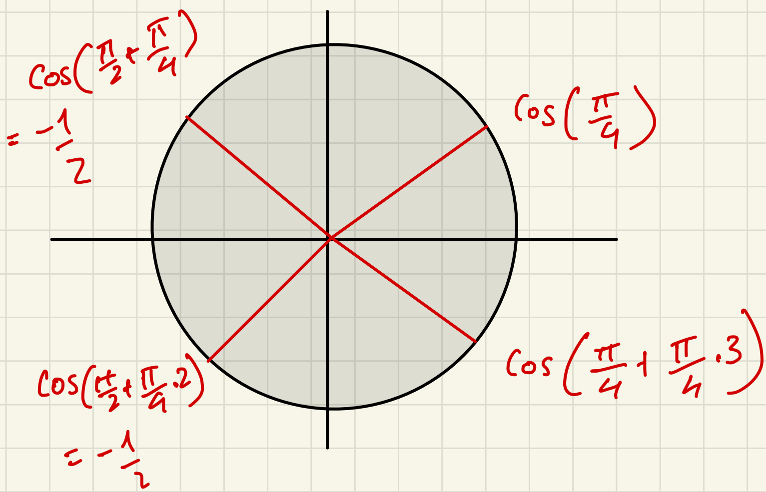
$\cos x$ at $a = \frac{\pi}{4}$

$$= \frac{1}{0!}\cdot 1 - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{4}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{4}\right)^6$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n}\left(x-\frac{\pi}{4}\right)^{2n}$$

$\cos x$ at $a = \frac{\pi}{4}$ $\quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ $\quad \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

n	$f^{(n)}(x)$	$c_n = f^{(n)}(a = \pi/4)/n!$	$(x-a)^n$
0	$\cos x$	$1/\sqrt{2} \cdot 0!$	$\left(x - \frac{\pi}{4}\right)^0$
1	$-\sin x$	$-1/\sqrt{2} \cdot 1!$	$\left(x - \frac{\pi}{4}\right)^1$
2	$-\cos x$	$-\frac{1}{\sqrt{2}} \cdot 2!$	$\left(x - \frac{\pi}{4}\right)^2$
3	$\sin x$	$\frac{1}{\sqrt{2}} \cdot 3!$	$\left(x - \frac{\pi}{4}\right)^3$
4	$\cos x$	$\frac{1}{\sqrt{2}} \cdot 4!$	$\left(x - \frac{\pi}{4}\right)^4$
5	$-\sin x$	$\frac{1}{\sqrt{2}} \cdot 5!$	$\left(x - \frac{\pi}{4}\right)^5$



$$= \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi}{4} + \frac{\pi}{2} n\right)}{n!} \left(x - \frac{\pi}{4}\right)^n$$

$R = \infty$ $IOC = (-\infty, \infty)$

(19) $\sinh x$ at $a = 0$

we know $\sinh x = \dfrac{e^x - e^{-x}}{2}$

& we know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$$\therefore \cancel{\sinh x = \frac{1}{2}\left(e^x - \frac{1}{e^x}\right)}$$ ← Do not write this

$$\cancel{= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n - \sum_{n=0}^{\infty} \frac{n!}{x^n}\right)}$$

$$\cancel{= \sum_{n=0}^{\infty} \left(\frac{x^n}{2 \cdot n!} - \frac{n!}{2 x^n}\right)}$$

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n - \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n\right)$$

$$= \frac{1}{2}\left(\left(\frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots\right) - \left(\frac{1}{0!}(-x)^0 + \frac{1}{1!}(-x)^1 + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \dots\right)\right)$$

$$= \frac{1}{2}\left(\left(1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots\right) - \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \dots\right)\right)$$

$$= \frac{1}{2}\left(\cancel{1} + \frac{x}{1!} + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} \dots - \cancel{1} + \frac{x}{1!} - \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} - \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} - \dots\right)$$

$$= \frac{1}{2}\left(\frac{2x}{1!} + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \dots\right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot x^{2n+1}$$

20. $\cosh x$ at $a = 0$

$$\frac{e^x + e^{-x}}{2} = \cosh x$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n + \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right)$$

$$= \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \frac{2x^6}{6!} + \cdots\right)$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$$

$R = -\infty$

$IOC = (-\infty, \infty)$

or you can also $\frac{d}{dx}(\sinh x)$ & differentiate the power series

(21) $\tanh x$ at $a=0$ $\Rightarrow$ $\dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{2}{e^x + e^x}$

$$= \frac{e^x - e^{-x}}{e^x + e^x}$$

Too complicated

instead use $\tanh x = \ln\left(\dfrac{1+x}{1-x}\right)$

$$\tanh x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$$

we know $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$=$ & $\dfrac{1}{1--x} = \sum_{n=0}^{\infty} (-x)^n$

$$\int \frac{1}{1+x} = \int \sum_{n=0}^{\infty} (-x)^n \qquad \int \frac{1}{1-x} = \int \sum_{n=0}^{\infty} x^n$$

$$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \qquad - \sum_{n=0}^{\infty} \frac{(-1)^n (-x)^{n+1}}{n+1} + C$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^n \cdot (-1)(x)^{n+1}}{n+1} + C$$

when $n=0$

$$= \frac{1}{2}\left(\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 + \dots\right)\right)$$

$$= \frac{1}{2}\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots\right)$$

$$= \frac{1}{2}\left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \dots\right)$$

$$= \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$R = 1$

$IOC = (-1, 1)$

$\ln(x)$ at $a=2$

we know $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $|R|<1$

$R=1$

IOC: $(-1, 1)$

n	$f^{(n)}(x)$	$f^{(n)}(a=2)/n!$	$(x-a)^n$
0	$\ln x$	$\ln 2 / 0!$	$(x-2)^0$
1	$\frac{1}{x}$	$\frac{1}{2} \cdot \frac{1}{1!}$	$(x-2)^1$
2	$-\frac{1}{x^2}$	$-\frac{1}{2^2} \cdot \frac{1}{2!}$	$(x-2)^2$
3	$-\frac{(-2)}{x^3}$	$\frac{2}{2^3} \cdot \frac{1}{3!}$	$(x-2)^3$
4	$\frac{2(-3)}{x^4}$	$-\frac{6}{2^4} \cdot \frac{1}{4!}$	$(x-2)^4$
5	$\frac{-6(-4)}{x^5}$	$\frac{24}{2^5} \cdot \frac{1}{5!}$	$(x-2)^5$
			$\vdots$

$$\ln|2| + \sum_{n=1}^{\infty} \frac{(-1)^n}{1} \cdot \frac{n!}{2^n} \cdot \frac{1}{n!} \cdot (x-2)^n$$

$$1 + 2 + 6 + 24 +$$

$$1 + 1.2 + 1.2.3 + 1.2.3.4$$

$n=1$ $\quad$ $n=2$ $\quad$ $n=3$ $\quad$ $n=4$

$1!$ $\quad$ $2!$ $\quad$ $3!$ $\quad$ $4!$ $\quad \ldots \ldots$ $\quad n!$

$$\ln|2| + \sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{2^n}$$

$$R = \lim_{n\to\infty} \left| \frac{1}{2^n} \cdot \frac{2^{n+1}}{1} \right|$$

$$= \lim_{n\to\infty} |2| = 2$$

IOC: $\quad |x-2| < 2$

$$-2 < x-2 < 2$$

$$-2+2 < x-2+2 < 2+2$$

$$0 < x < 4$$

checking lim at both ends:

when $x=0$

$$\lim_{n\to\infty} (-1)^n \frac{(x-2)^n}{2^n}$$

= This is an Alt. series:

$\therefore$ if $\lim_{n\to\infty} a_n = 0$ &

if $a_n > a_{n+1}$

is True series converges.

$$\lim_{n\to\infty} \frac{(x-2)^n}{2^n}$$

At $x=0$

$$\lim_{n\to\infty} \frac{(-2)^n}{2^n} \neq 0$$

diverges.

when $x=4$

$$\lim_{n\to\infty} (-1)^n \frac{(x-2)^n}{2^n}$$

when $x=4$

$$\lim_{n\to\infty} \frac{2^n}{2^n} = 1$$

(23) $2x^3 - 5x^2 + 1$; at $a = 1$

We know $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad |R| < 1$

$R = 1$

$IOC: (-1, 1)$

n	$f^{(n)}(x)$	$f^{(n)}(a=1)/n!$	$(x-a)^n$
0	$2x^3 - 5x^2 + 1$	$-2/0!$	$(x-1)^0 = 1$
1	$6x^2 - 10x$	$-4/1!$	$(x-1)^1$
2	$12x - 10$	$2/2!$	$(x-1)^2$
3	12	$12/3!$	$(x-1)^3$
4	0	0	$(x-1)^4$

$$-\frac{2}{0!} \cdot 1 - \frac{4}{1!} \cdot (x-1) + \frac{2}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3 + 0$$

$$= -2 - 4(x-1) + (x-1)^2 + 2(x-1)^3$$

$=$ $\quad R = \infty$ $\quad I: (-\infty, \infty)$

$\uparrow$ as it is polynomial

(26) $(1+x)^n$; where r is any real number at $a=0$

n	$f^{(n)}(x)$	$f^{(n)}(a=0)/n!$	$(x-a)^n$
0	$(1+x)^n$	$1/0!$	$x^0=1$
1	$n(1+x)^{n-1}$	$1/1!$	$x^1=x^1$
2	$n(n-1)(1+x)^{n-2}$	$2.1(2)^0/2!$	$x^2=x^2$
3	$n(n-1)(n-2)(1+x)^{n-3}$	$3.2.1.3^0/3!$	$x^3=x^3$
4	$n(n-1)(n-2)(n-3)(1+x)^{n-4}$	$4.3.2.1.4^0/4!$	x^4

In this column replace n with r

$\therefore \quad (1+x)^n$

$$= 1 + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 + \frac{5!}{5!}x^5 \dots$$

$$= 1x^0 + x^1 + x^2 + x^3 + x^4 + x^5 + \dots$$

$$= \sum_{n=0}^{\infty} x^n$$

But in terms of r:

$$\sum_{n=0}^{\infty} \frac{r(r-1)(r-2)(r-3)(1+x)^{r-4} \cdots (r-n+1)}{n!} \cdot x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\binom{r}{n}}_{r \text{ choose } n} x^n$$

$$R = \lim_{n\to\infty} \left| \binom{r}{n} \cdot \frac{1}{\binom{r}{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{r(r-1)(r-2)\ldots(r-n+1)}{n!} \cdot \frac{(n+1)!}{r(r-1)\ldots(r-n+1-1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{r-n} \right| = 1$$

$R = 1$

IOC depends on r : $-1, 1$

converges if $-1 < x \leq 1$ can have ( ) open bracket
or [ ] closed.

New formula

$$(1+x)^r = \binom{r}{n} x^n$$

$$= \frac{r(r-1)\cdots(r-n+1)}{n!} x^n$$

(25) $\sqrt{4+x}$ at $a=0$

n	$f^{(n)}(x)$	$f^{(n)}(a=0)/n!$	$(x-a)^n$
0	$\sqrt{4+x}$	$\frac{\sqrt{4}}{0!} = 2/0! = 2$	$x^0 = 1$
1	$\frac{1}{2\sqrt{4+x}}$	$\frac{1}{4} \cdot \frac{1}{1!} = \frac{1}{4}$	$x^1 = x^1$
2	$-\frac{1}{4(4+x)^{3/2}}$	$-\frac{1}{4(4)^{3/2}} \cdot \frac{1}{2!} = \frac{-1}{32} \cdot \frac{1}{2!}$	$x^2 = x^2$

$$2 + \frac{1}{4} x - \frac{1}{64} x^2$$

Do not use this instead

vs $(1+x)^r$ formula

$$f''\left(\frac{1}{2\sqrt{4+x}}\right)$$

$$= f''\left(\frac{1}{2}(4+x)^{-1/2}\right)$$

$$= \frac{1}{2}\cdot\frac{-1}{2}(4+x)^{-1/2-1}$$

$$= -\frac{1}{4}(4+x)^{-3/2}$$

$$= -\frac{1}{4}\frac{1}{(4+x)^{3/2}}$$

usin $(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n$

we have $\sqrt{4+x}$

$$= (4+x)^{1/2} = 4^{1/2}\left(1+\frac{x}{4}\right)^{1/2}$$

$$= 2\left(1+\frac{x}{4}\right)^{1/2} \leftarrow r$$

$r = \frac{1}{2}$

$$\therefore \quad 2\sum_{n=0}^{\infty} \binom{1/2}{n} \left(\frac{x}{4}\right)^n$$

$$= 2\ \frac{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} \left(\frac{x}{4}\right)^n$$

$$= 2\binom{1/2}{n}\frac{x^n}{2^{2n}} = \binom{1/2}{n}\frac{x^n}{2^{2n-1}}$$

$$\left|\frac{x}{4}\right| < 1$$

$$|x| < 4 \quad \in R$$

$$R = 4$$

$$IOC = \quad -4, 4$$

when $r \geq 0$ we include

$$\therefore \ IOC = \ [-4, 4]$$

(26) $\sin^{-1} x$ at $a = 0$

We know $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$\therefore \int \frac{1}{\sqrt{1-x^2}} = \sin^{-1}(x) + C$

we take:

$\therefore \frac{1}{\sqrt{1+(-x)^2}} = \left(1+(-x^2)\right)^{-1/2}$

is of the form

$(1+x)^{\gamma}$

where $x = -x^2$

$\gamma = -\frac{1}{2}$

$\therefore \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$

Now we Integrate:

$\int \frac{1}{\sqrt{1-x^2}} = \int \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n$

$$\sin^{-1}(x) = C + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= C + \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$$

let $x = 0$

$\sin^{-1}(0) = C + 0$

$C = 0$

$$\therefore \quad \sin^{-1} x = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$R = 1$ $\qquad I = -1, 1$ $\qquad$ where ( ) or [ ] depends on

(26-2) $\quad x^{0.2}$ at $a = 26$.

step① $\quad x^{0.2} \overset{?}{=} \quad x - 26$

$x - a$ form

step2 $\quad x^{0.2} \overset{?}{=} 26 + x - 26$

step 3 $x^{0.2} = (26 + x - 26)^{0.2}$

$$= \quad = 26^{0.2}\left(1 + \frac{x-26}{26}\right)^{0.2}$$

$$r = 0.2 \qquad x = \frac{x-26}{26}$$

$$= \sum_{n=0}^{\infty} \binom{0.2}{n} \left(\frac{x-26}{26}\right)^{n}$$

$$\left|\frac{x-26}{26}\right| < 1$$

$$\underbrace{|x-26|}_{|x|} < 26$$

$$\therefore \; R = 26$$

Now for IOC

$$-26 < x - 26 < 26$$

$$-26 + 26 < x - 26 + 26 < 26 + 26$$

$$0 < x < 52$$

IOC = $[0, 52]$ as r is +ve

2. Series

6. Find the Sum of these series

a) $\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$ b) $\sum_{n=1}^{\infty} (\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right))$ c) $\sum_{n=0}^{\infty} \frac{1}{2^n}$

7. Check if the following series converges

a) $\sum_{n=1}^{\infty} \arctan(n)$ b) $\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$

8. Use the integral test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{1}{\mathrm{n}^2+2\mathrm{n}+2}$ b) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ c) $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$

9. Use the comparison test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{5}{2\mathrm{n}^2+4\mathrm{n}+3}$ b) $\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\mathrm{n}}}{n+\cos^2 n}$

10. Use the Ratio test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2^n\, n^3}$ b) $\sum_{n=1}^{\infty} \cos\left(\mathrm{n} * \frac{\frac{\pi}{3}}{n!}\right)$ c) $\sum_{n=0}^{\infty} \frac{(-1)^n n^3}{3^n}$

Math Café

Applied Mathematics Mock Exam

1. Matrices and Determinants

1. You are developing a robot arm. The robot's arm is currently pointing towards the point (6, 3, -2), now you give instructions to rotate the arm with the Matrix $M_1 = \begin{pmatrix} 2 & -2 & 2 \\ 5 & 0 & 1 \\ -3 & 3 & -5 \end{pmatrix}$ and then by the Matrix $M_2 = \begin{pmatrix} 7 & 2 & 4 \\ 1 & -3 & 8 \\ 1 & 2 & 3 \end{pmatrix}$. Now towards which points does the arm point towards?

2. Find the determinant of $A = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 3 & 2 & -1 & 1 \end{pmatrix}$

3. You are conducting a thermodynamics experiment with pressure, volume and temperature as the affecting factors. After conducting the experiment 3 times you get the following equations:

$$\begin{aligned} 2P + V - T &= 1 \\ 3P + 2V + 2T &= 13 \\ 4P - 2V + 3T &= 9 \end{aligned}$$

Can you find the pressure, volume and temperature values used for this experiment?

4. Derive the characteristic equation, then find all the eigen values and one of the eigenvectors of the following matrix

$$C = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

5. From a matrix A, one element was lost it and it was replaced by a variable t. The matrix is then

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ t & 3 & 1 \end{pmatrix}$$

a) Compute det(A)
b) For which value of t is A invertible?
c) For the case that A is invertible, what is det(A^{-1})?
d) For t = 1, compute A^{-1}

Math Café

Applied Mathematics Mock Exam

2. Series

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a) $\sum_{n=1}^{\infty} \arctan(n)$ b) $\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$

8. Use the integral test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{1}{\mathrm{n}^2+2\mathrm{n}+2}$ b) $\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ c) $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$

9. Use the comparison test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{5}{2\mathrm{n}^2+4\mathrm{n}+3}$ b) $\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\mathrm{n}}}{n+\cos^2 n}$

10. Use the Ratio test to prove that the following series converge or diverge

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3^n}{2^n\, n^3}$ b) $\sum_{n=1}^{\infty} \cos\left(\mathrm{n} * \frac{\frac{\pi}{3}}{n!}\right)$ c) $\sum_{n=0}^{\infty} \frac{(-1)^n n^3}{3^n}$

6. a) $$\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$$

$$= \sum_{n=2}^{\infty} \left(\frac{A}{n} + \frac{B}{n+2} \right)$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + B(n)$$

when $n = -2$

$$1 = 0 - 2B$$

$$B = -\frac{1}{2}$$

when $n = 0$

$$A = \frac{1}{2}$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{n(n+2)} = \sum_{n=2}^{\infty} \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right)$$

= writing out terms:

$$= \frac{1}{2(2)} + \frac{1}{2(3)} + \frac{1}{2(4)} - \left(\frac{1}{2(4)} + \frac{1}{2(5)} + \frac{1}{2(6)} + \frac{1}{2(7)} \dots \right)$$

= It is Telescoping and cancels eachother

$\therefore$ what is left is:

$$= \frac{1}{2(2)} + \frac{1}{2(3)} = \frac{1}{4} + \frac{1}{6}$$

5) $\displaystyle\sum_{n=1}^{\infty} \left(\sin\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n+1}\right) \right)$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) - \left(\sum_{n=1}^{\infty} \sin\left(\frac{1}{n+1}\right) \right)$$

$$= \left(\sin\frac{1}{1} + \sin\left(\frac{1}{2}\right) + \sin\left(\frac{1}{3}\right) \cdots \right) - \left(\sin\left(\frac{1}{2}\right) + \sin\left(\frac{1}{3}\right) + \sin\left(\frac{1}{4}\right) \cdots \right)$$

Telescoping series. What is left is

$$= \sin 1$$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$= \sum_{n=0}^{\infty} \frac{1^n}{2^n} = \sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{2}\right)^n$$

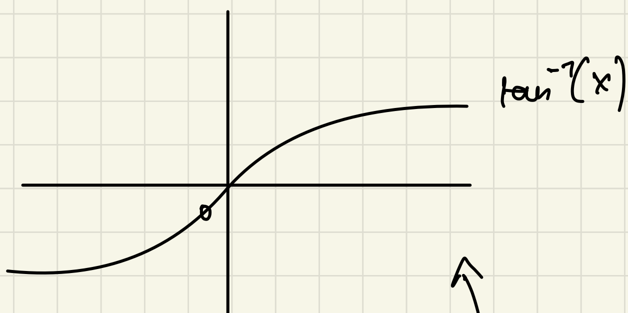
$$a = 1 \qquad r = \frac{1}{2}$$

Geometric series; sum $= \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}}$

$$= \frac{1}{\frac{1}{2}} = 2$$

7. a) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1}(x)$

$$= \sum_{n=1}^{\infty} \tan^{-1}(x)$$



$$\int_1^{\infty} \tan^{-1}(x) = \lim_{t \to \infty} \int_1^{t} \tan^{-1}(x)\, dx$$

diverges as n^{th} term goes to ∞ as we approach $\pi/2$

b) $\sum_{n=0}^{\infty} \frac{(-1)^{n-3} \sqrt{n}}{n+4}$

$= \sum_{n=0}^{\infty} \frac{(-1)^{3} \cdot (-1)^{n} \sqrt{n}}{n+4}$

$= \sum_{n=0}^{\infty} (-1)^{n} \cdot \frac{-\sqrt{n}}{n+4}$

this is an alternating series.

$\therefore$ we need to check if $\lim_{n \to \infty} a_n = 0$

& $a_n > a_{n+1}$; ie terms are decreasing

$= \lim_{n \to \infty} \frac{-\sqrt{n}}{n+4} = \lim_{n \to \infty} \frac{-\frac{1}{2\sqrt{n}}}{1+4}$

$= \lim_{n \to \infty} -\frac{5}{2} \cdot \frac{1}{\sqrt{n}} = \frac{1}{\infty} = 0$

$\frac{-\sqrt{n}}{n+4} > \frac{-\sqrt{n+1}}{n+4+1}$

$(n+5)\sqrt{n} \overset{?}{>} (n+4)\sqrt{n+1}$

if we put $n = 2$

$$(7)\sqrt{2} > (6)\sqrt{3}$$

is True

$\therefore$ The series converges

8. $$\sum_{n=1}^{\infty} \frac{1}{n^2 + 2n + 2}$$

$$= \int_1^{\infty} \frac{1}{x^2 + 2x + 2} dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^2 + 2x + 2} dx$$

Completing squares:

$$2ax = 2x$$

$$a = 1$$

$$a^2 = 1$$

$$x^2 + 2x + 2 + 1 - 1 = x^2 + 2x + 3 - 1$$

$$x^2 + 2x + 1 + 1 = (x^2 + x + x + 1) + 1$$

$$= (x(x+1) + (x+1)) + 1$$

$$(x+1)^2 + 1$$

$$\lim_{t \to \infty} \int_1^{t} \frac{1}{(x+1)^2 + 1} dx$$

$$x+1 = u$$

$$1\,dx = du$$

$$\lim_{t\to\infty} \int_1^t \frac{1}{u^2+1}\,du$$

$$= \lim_{t\to\infty} \tan^{-1}(u) \Big]_1^t$$

$$= \lim_{t\to\infty} \tan^{-1}(x+1) \Big]_1^t$$

$$= \lim_{t\to\infty} \left[\tan^{-1}(t+1) - \tan^{-1}(1)\right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}(1)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{4\pi - 2\pi}{8} = \frac{2\pi}{8} = \frac{\pi}{4}$$

is defined & converges.

8.b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

$$= \int_1^{\infty} \frac{n}{n^4+1} =$$

rewriting in x terms

$$\lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx$$

$$= \quad x^2 = u$$

$$2x\,dx = du$$

$$x\,dx = \frac{1}{2}\,du$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{2} \cdot \frac{du}{u^2+1}$$

<u>As $t\to\infty$</u> $= \displaystyle\frac{1}{2} \tan^{-1}(u) \Big]_1^t$

$$= \frac{1}{2} \tan^{-1}(x^2) \Big]_1^t$$

$$= \frac{1}{2}\left[\tan^{-1}(t^2) - \tan^{-1}(1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

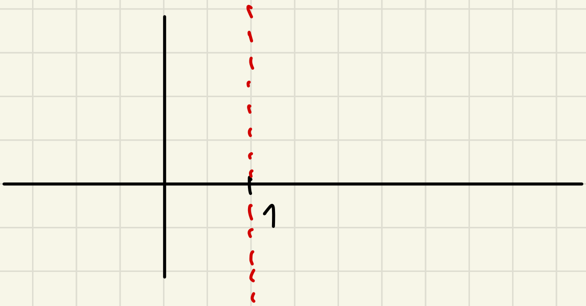
$$= \frac{1}{2}\,\frac{2\pi}{8} = \frac{\pi}{8}$$

8. c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln(n)}$

$\ln(1) = 0$

$=$ we have discontinuity at $n = 1$

$\therefore$



we have at $n = 1$ discontinuity.

$$\therefore \quad \lim_{t \to 1^+} \int_t^{\infty} \frac{1}{x \ln(x)} \, dx$$

$=$ let $\ln x = u$

$\frac{1}{x} dx = du$

$$\lim_{t \to 1^+} \int_1^{\infty} \frac{du}{u}$$

$$= \lim_{t \to 1^+} \ln|u| \Big]_1^{\infty}$$

$$= \lim_{t \to 1^+} \ln|\ln x| \Big]_1^\infty$$

$$= \ln|\ln \infty| - \ln|\ln 1|$$

$$= \infty - -\infty = \infty$$ diverges

9 a) $$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

By the largest power in num & denom.

As $n \to \infty$

$$\approx \sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

we have p-series with $p = 2$

for p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$

if $p \leq 1$ diverges

$\therefore$ this series converges.

Now don't forget to compare this with know convergent & prove it converges

$$= \sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3} \overset{?}{\leq} \frac{5}{2n^2} \quad \forall\, n \geq 1$$

$$= \sum_{n=1}^{\infty} \quad 10n^2 < 5(2n^2 + 4n + 3)$$

True! $\therefore$ converges.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2 n}$

$\dfrac{1}{n^p}$ $p=2$

$0 < \ln x$

DCT

$a_n \leq$ known convergent

$\uparrow$

a_n also converges

if $a_n \geq$ known divergent

$\uparrow$

a_n also diverges.

LCT

if $\displaystyle\lim_{n\to 0} \frac{a_n}{b_n} = L > 0$ $L \neq 0$ $L \neq \infty$

either both a_n & b_n converges

or a_n & b_n diverges.

$$\cancel{= \sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2 n} \overset{?}{>} \sum_{n=1}^{\infty} \frac{1}{n} \quad \forall\, n \geq 1}$$

$$\cancel{n e^{-n} \overset{?}{>} n+\cos^2 n}$$

$$\cancel{\frac{n}{e^n} \overset{?}{>} n+\cos^2 n.}$$

This is difficult to say

$\therefore$ Testing with a known convergent.

$$\frac{e^{-n}}{n+\cos^2 n} \overset{?}{<} \frac{1}{n^2}$$

$$n^2 e^{-n} \overset{?}{<} n + \cos^2 n$$

$$\frac{n^2}{e^n} \overset{?}{<} n + \cos^2 n$$

Also difficult to say.

Apply LCT

$$\lim_{n\to\infty} \left| \frac{\frac{e^{-n}}{n+\cos^2 n}}{\frac{1}{n^2}} \right|$$

$$\lim_{n\to\infty} \left| \frac{e^{-n}}{n+\cos^2 n} \cdot \frac{n^2}{1} \cdot \right|$$

$$\lim_{n\to\infty} \left| \frac{n^2}{e^n(n+\cos^2 n)} \right|$$

$$\lim_{n\to\infty} \frac{2n}{e^n \cdot 1 + n \cdot e^n + -e^n \sin^2 n + \cos^2 n \cdot e^n}$$

$$\lim_{n\to\infty} \left| \frac{2n}{e^n + ne^n - e^n \sin^2 n + e^n \cos^2 n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2}{e^n + ne^n + e^n - (e^n \cos^2 n + \sin^2 n \, e^n) + (-e^n \sin^2 n + \cos^2 n \, e^n)} \right|$$

$$\lim_{n\to\infty} \left| \frac{2}{e^n + ne^n + e^n - e^n \cos^2 n - \sin^2 n \, e^n - e^n \sin^2 n + \cos^2 n \, e^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2}{2e^n + ne^n - 2 \sin^2 n \, e^n} \right|$$

n $\quad f^{(n)}(x)$ $\quad f^{(n)}(a=-1)/n!$ $\quad (x-a)^n$

0 $\quad \frac{\ln(1-x)}{x}$ $\quad \frac{\ln(0)}{0} = -\infty/0!$ $\quad x+$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right), x=0$$

$$\ln(1-x) \quad = \sum_{n=0}^{\infty} n x^{n-1} \qquad \frac{1}{x}$$

$$= \sum_{n=0}^{\infty} \frac{n x^{n-1}}{x}$$

$$|x-a| < 1$$

$$\sum_{n=0}^{\infty} \frac{n^3}{n^4+1}$$

$$(a^4+b^4)$$

$$= \lim_{t\to\infty} \int_0^t \frac{x^3}{x^4+1}\,dx$$

$$x^3 = u$$

$$3x^2\,dx = du$$

$$x^2\,dx = \frac{1}{3}du$$

$$x^2 = u$$

$$=$$

$$2x\,dx = du$$

$$x\,dx = \frac{1}{2}du$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$x = \tan\theta$$

$$dx = \sec^2\theta\,d\theta$$

$$\int \frac{1}{(\tan\theta)^4+1}(\tan^3\theta)\sec^2\theta$$

$$\lim_{n \to \infty} \frac{e^{5n}}{3 - e^{2n}}$$

As $n \to \infty$

$$= \frac{e^{5n} \cdot 5}{0 - e^{2n} \cdot 2}$$

$$= \frac{e^{5n} \cdot 5}{e^{2n} \cdot (-2)}$$

$$= e^{3n} \cdot \frac{5}{-2}$$ diverges.

$$= \infty$$