Differentiation

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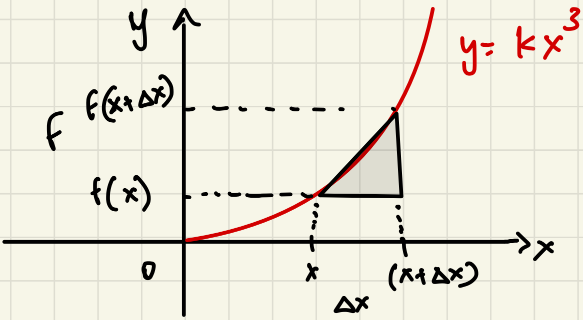
Notes

Differentiation can be defined as the rate of change of a function with respect to its independent variable's

For example if we have the function

$$F = kx^3$$



The slope would be the derivative

if $f(x) = kx^3$

$$f(x+\Delta) = k(x+\Delta x)^3$$

$$\text{slope} = \frac{\Delta F}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\therefore$ here it would imply $\Rightarrow$ $$\frac{\Delta F}{\Delta x} = \frac{k(x+\Delta x)^3 - kx^3}{\Delta x}$$

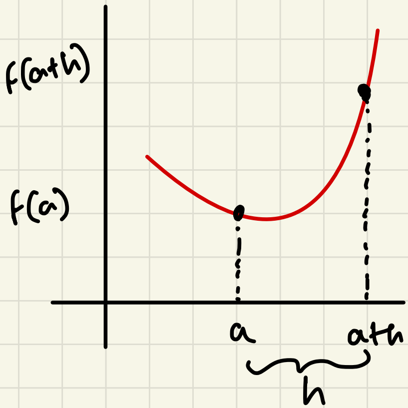
Now we improve the accuracy by shrinking Δx thus Δx approaches 0 as we minimize the distance to approximate that means we apply limit to the slope as $\Delta x \to 0$

$$\therefore \text{slope} = \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} \frac{k(x+\Delta x)^3 - x}{\Delta x}$$

So in essence, procedure

$$\text{slope} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This would actually mean, graphically:



if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

then the function is differentiable at a

Equation of a line with a point (x_0, y_0) we calculate the slope as:

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y - y_1}{x - x_1}$$

if a is point on the x axis ie $x_1 = a$ then y_1 would be $f(a)$ & slope m is $f'(a)$

$$\frac{df(a)}{dx} = f'(a) = \frac{y - f(a)}{x - a}$$

$$y = \underbrace{f'(a)(x-a)}_{\text{instantaneous rate of change at } x=a} + f(a)$$

1) Why derivative of constant is zero?

Lets have point $f(x) = c$

From what we know the slope would be defined as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Here it would mean

$$f'(x) = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 0 = 0$$

2) $$\frac{d}{dx} x^n = n x^{n-1}$$

3) $$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$$

4) $$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

5) $$\frac{d}{dx}(\underset{\substack{\downarrow \\ u(x)}}{u} \cdot \underset{\substack{\downarrow \\ v(x)}}{v}) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

6) $\frac{d}{dx}\left(\frac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v}$

7) $\frac{d}{dx}\sin x = \cos x$

8) $\frac{d}{dx}\cos x = -\sin x$

9) $\frac{d}{dx}e^x = e^x$

10) $\frac{d}{dx}\ln x = \frac{1}{x}$ $\{\ln x = \log_e x\}$

11) $\frac{d}{dx}(\tan x) = \sec^2 x = 1 + \tan^2 x$

12) Chain Rule: evaluate inner most paranthesis first then move out.

ie $h(x) = f(g(x))$

then $h'(x) = g'(x) \cdot f'(g(x))$

$= f'(g(x)) \cdot g'(x)$

13) In implicit differention; take y as a fraction of x & apply chain rule to get something. $\frac{dy}{dx}$ form. check example

14) $\frac{d}{dx} kx = k$

15) $\frac{d}{dx} kx^n = k \cdot nx^{n-1}$

16) $\frac{d}{dx} e^{kx} = k \cdot e^{kx}$

17) $\frac{d}{dx} \ln kx = k \cdot \frac{1}{kx} = \frac{1}{x}$

18) $\frac{d}{dx} \sin kx = k \cdot \cos kx$

19) $\frac{d}{dx} \sin(kx+c) = k \cdot \cos(kx+c)$

20) $\frac{d}{dx} \cos kx = k \cdot (-\sin(kx))$

21) $\cos(kx+c) = k(-\sin(kx+c))$

22) $\tan kx = k \cdot \sec^2(kx) = k(1+\tan^2(kx))$

23) $\tan(kx+c) = k \cdot \sec^2(kx+c)$

24) $\sec kx = k \sec(kx)\tan(kx) = k(\sec^2(kx))$

25) $\sin^{-1}(kx) = k \cdot \frac{1}{\sqrt{1-(kx)^2}}$

26) $\frac{1}{k}\tan^{-1}\left(\frac{x}{k}\right) = \frac{1}{k} \cdot \frac{1}{k} \cdot \frac{1}{1+(x/k)^2} + \tan^{-1}\left(\frac{x}{k}\right) \cdot 0$

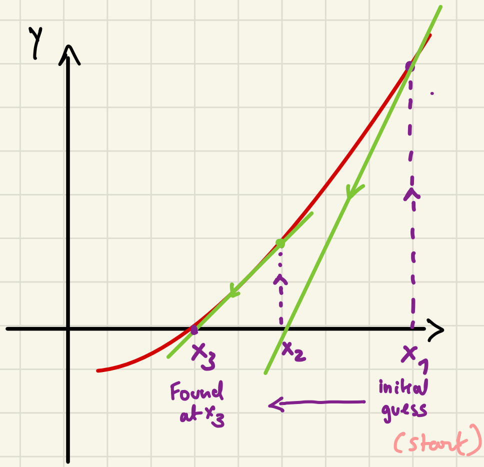
$$= \frac{1}{k^2} \frac{1 \cdot k^2}{k^2+x^2} = \frac{1}{k^2+x^2}$$

Newtons Method

We can use this approach to approximate a function that crosses the zero level on the x axis. Interesting approach for machines to learn

How this is done:

Take an inital point on the axis & extend it to the point it touches the function



Draw a tangent from the point to the zero level (ie crossing the x axis)

From there repeat until the datapoint of the function and tangent line becomes the same.
it could take from iteration x_1, x_2, x_3.... maybe you find it at x_n depending on your initial guess of x_1

Numerically:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

The next closest point is equal to the initial guess minus the function value with initial guess divided by the derivative of the function value

why minus; because we are going back to match the data point.

When Newtons method initial guess or approach is too long divide the function by a fictional $(x-a)$. Keep note of power of function and solve the result fictional quadratic eqn.

Derivatives : Trigs

$\sin x = \cos x$

$\cos x = -\sin x$

$\tan x = \sec^2 x = 1 + \tan^2 x$

$\cot x = -\operatorname{cosec}^2 x$

$\sec x = \sec x \tan x$

$\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

$\sinh x = \cosh x$

$\cosh x = \sinh x$

$\tanh x = \sec^2 hx$

$\coth x = \operatorname{cosech}^2 x$

$\operatorname{sech} x = -\operatorname{sech} x \tanh x$

$\operatorname{cosec} h x = -\operatorname{cosec} x \cot x$

$$\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$$

$$\frac{d}{dx} \cos u(x) = -\sin u(x) \frac{du(x)}{dx}$$

$$\frac{d}{dx} \cot u(x) = -\operatorname{cosec}^2 u(x) \frac{du(x)}{dx}$$

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx} \operatorname{cosec}^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Examples

1) $f(x) = \sin(x^2+1)$

$$f'(x) = (2x+0)\cos(x^2+1)$$
$$= 2x\cos(x^2+1)$$

2) $f(x) = (x^3-1)^4$

$$f'(x) = (3x^2-0)\cdot 4(x^3-1)^3$$
$$= 4(3x^2)(x^3-1)^3$$
$$= 12x^2(x^3-1)^3$$

$$\begin{array}{ccccccccc}
0 & & & & 1 & & & & \\
1 & & & 1 & & 1 & & & \\
2 & & 1 & & 2 & & 1 & & \\
3 & 1 & & 3 & & 3 & & 1 & \\
 & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 &
\end{array}$$

$$(a+(-b))^3 = {}^nC_r\, a\, b$$
$$3C_0 a^3 b^0 + 3C_1 a^2 b^1 + 3C_2 a^1 b^2 + 3C_3 a^0 b^3$$
$$a^3 + 3a^2(-b) + 3a(-b)^2 + 1(-b)^3$$
$$= a^3 - 3a^2 b + 3ab^2 - b^3$$

$$= 12x^2\left((x^3)^3 - 3(x^3)^2(1) + 3(x^3)(1)^2 - 1\right)$$
$$= 12x^2(x^9 - 3x^6 + 3x^3 - 1)$$
$$= 12x^{11} - 36x^8 + 36x^5 - 12x^2$$

Implicit Differentiation

$x^2 + y = 1 + y^3$ Find $\frac{dy}{dx}$

$$\frac{d(x^2)}{dx} + \frac{dy}{dx} = \frac{d(1)}{dx} + \frac{d(y^3)}{dx}$$

$$2x^{2-1} + \frac{dy}{dx} = 0 + \frac{d(y^3)}{dx} \to y(x)$$

Take y as function of x

$$2x + \frac{dy}{dx} = \frac{d}{dx}\left(y^3(x)\right)$$

and diff it by applying chain rule

For RHS apply chain rule

internal $\frac{d}{dx} y(x) = \frac{dy}{dx}$

external $\frac{d}{dx} y^3 = 3y^2$

chain rule gives $3y^2 \frac{dy}{dx}$

$$\therefore \quad 2x + \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

$$2x = (3y^2 - 1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 - 1}$$

Should Newtons method be applied always?

Find the zero crossing for $x^3 + 1 = 6x$

Newtons method:

next guess = initial guess − $\dfrac{\text{function val w/ initial guess}}{\text{function deriv val w/ initial guess}}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We have $x^3 + 1 = 6x$

$x^3 - 6x + 1 = 0$

$f(x) = x^3 - 6x + 1$

$f'(x) = 3x^2 - 6$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}

Here newtons method would be painful!

Solve the function by dividing & simplifying.

$\Rightarrow \dfrac{x^3 + 0x^2 - 6x + 1}{x - a}$ that returns 0.

where a can be any number

$$
\begin{array}{r l}
 & x^2 + ax + (a^2-6) \\
\cline{2-2}
x - a \,\big) & x^3 + 0x^2 - 6x + 1 \\
 & x^2 - ax^2 \\
\cline{2-2}
 & \quad ax^2 - 6x \\
 & \quad ax^2 - a^2x \\
\cline{2-2}
 & \qquad (a^2-6)x + 1 \\
 & \qquad (a^2-6)x - (a^3-6a) \\
\cline{2-2}
 & \qquad\qquad 1 + a^3 - 6a \Rightarrow 0
\end{array}
$$

would be 0 by definition

$$\therefore \quad x^2 + ax + (a^2-6) = 0$$

Now apply quadratic eqn:

$$x_{1,2} = \frac{-a \pm \sqrt{a^2 - 4(a^2-6)}}{2} \qquad a \in \mathbb{R}$$

100 Derivatives

1) $\frac{d}{dx}(ax^2+bx+c)$

$= a\cdot 2x + b = 2ax+b$

2) $\frac{d}{dx}\left(\frac{\sin x}{1+\cos x}\right)$

$$\frac{(1+\cos x)\frac{d}{dx}(\sin x) \cancel{+} - \sin x \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$\frac{(1+\cos x)\cos x \cancel{+} - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$\frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x}{(1+\cos x)^2}$$

3) $\frac{d}{dx}\left(\frac{1+\cos}{\sin x}\right)$

$$= \frac{\sin x \frac{d}{dx}(1+\cos x) - (1+\cos x)\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x\,(-\sin x) \;-\; (1+\cos x)\cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x \;-\; \cos x \;-\; \cos^2 x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x \;-\; \cos x}{\sin^2 x}$$

$$= \frac{-1 - \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x} - \frac{\cos}{\sin^2 x}$$

$$= -\operatorname{cosec}^2 x \;-\; \cos x \operatorname{cosec}^2 x$$

4) $\frac{d}{dx}\left(\sqrt{3x+1}\right)$

$$= \frac{d}{dx}(3x+1) = 3$$

$$\frac{d}{dx}\left(\sqrt{3x+1}\right) = 3 \cdot \frac{1}{2\sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}$$

5) $\frac{d}{dx}\left(\sin^3(x) + \sin(x^3)\right)$

$$= \frac{d}{dx}\left(\sin^3(x)\right) + \frac{d}{dx}\left(\sin(x^3)\right)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}\left(\sin^3(x)\right) = \cos x \cdot 3\sin^2 x$$

$$= 3\cos x \sin^2 x$$

$$\frac{d}{dx} x^3 = 3x^2$$

$$\frac{d}{dx}(\sin x^3) = 3x^2 \cdot \cos x^3$$

$$= 3\cos x \sin^2 x + 3x^2 \cos x^3$$

6) $\frac{d}{dx}\left(\frac{1}{x^4}\right)$

$$= \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}$$

7) $\frac{d}{dx}\left((1+\cot x)^3\right)$

sec — sec tan

cosec — -cosec cot

$$\frac{d}{dx}(1+\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(1+\cot x)^3 = -\operatorname{cosec}^2 x \cdot 3(1+\cot x)^2$$

$$= -3\operatorname{cosec}^2 x(1+\cot x)^2$$

8) $\frac{d}{dx}\left(x^2(2x^3+1)^{10}\right)$

$$\frac{d}{dx}(2x^3+1) = 6x^2$$

$$\frac{d}{dx}(2x^3+1)^{10} = 6x^2 \cdot 10(2x^3+1)^9$$

$$= 60x^2(2x^3+1)^9$$

$$\frac{d}{dx}\left(x^2 \cdot (2x^3+1)^9\right)$$

$$= x^2 \frac{d}{dx}(2x^3+1)^9 + (2x^3+1)^{10} \frac{d}{dx}(x^2)$$

$$= x^2\, 60x^2 (2x^3+1)^9 + (2x^3+1]^{10} (2x)$$

$$= 60x^4 (2x^3+1)^9 + (2x^3+1)^{10} (2x)$$

$$= (2x^3+1)^9 \left[60x^4 + (2x^3+1)(2x)\right]$$

$$= (2x^3+1)^9 \left[60x^4 + 4x^4 + 2x\right]$$

$$= (2x^3+1)^9 (64x^4 + 2x)$$

$$= 2x(32x^3+1)(2x^3+1)^9$$

9) $$\frac{d}{dx}\left[\frac{x}{(x^2+1)^2}\right]$$

$$\frac{(x^2+1)^2 \frac{d}{dx}(x) - x \frac{d}{dx}(x^2+1)^2}{\left((x^2+1)^2\right)^2}$$

$$\frac{d}{dx}(x^2+1) = 2x$$

$$\frac{d}{dx}(x^2+1)^2 = 2x \cdot 2(x^2+1)$$

$$= 4x(x^2+1) = \cancel{4x^3+4x}$$

$$\frac{(x^2+1)^2 - 4x^2(x^2+1)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)\left[x^2+1 - 4x^2\right]}{(x^2+1)^4} = \frac{-3x^2+1}{(x^2+1)^3}$$

10. $$\frac{d}{dx}\left[\frac{20}{1+5e^{-2x}}\right]$$

$$= \frac{(1+5e^{-2x})\frac{d}{dx}(20) - 20\frac{d}{dx}(1+5e^{-2x})}{(1+5e^{-2x})^2}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{-2x}) = -2e^{-2x}$$

$$\frac{d}{dx}(5e^{-2x}) = 5(-2e^{-2x}) = -10e^{-2x}$$

$$\frac{d}{dx}(1+5e^{-2x}) = 0 + (-10e^{-2x})$$

$$= -10e^{-2x}$$

$$= \frac{0 - 20(-10e^{-2x})}{(1+5e^{-2x})^2}$$

$$= \frac{200e^{-2x}}{(1+5e^{-2x})^2}$$

11) $\frac{d}{dx}\left(\sqrt{e^x} + e^{\sqrt{x}}\right)$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\sqrt{e^x}) = e^x \cdot \frac{1}{2\sqrt{e^x}}$

$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx} e^{\sqrt{x}} = \frac{1}{2\sqrt{x}} \cdot e^{\sqrt{x}}$

$= \frac{e^x}{2\sqrt{e^x}} + \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

$= \frac{\sqrt{e^x}}{2} + \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

12) $\frac{d}{dx}\left(\sec^3(2x)\right)$

sec sec tan

$\frac{d}{dx}(2x) = 2$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\sec 2x) = 2\sec 2x \tan 2x$

$$\frac{d}{dx}\left(\sec^3(2x)\right) = 2\sec 2x \tan 2x \cdot 3\sec^2(2x)$$

$$= 6\sec^2(2x)\sec 2x \tan 2x$$

$$= 6\sec^3(2x)\tan 2x$$

13) $\frac{d}{dx}\left(\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln(\sec x + \tan x)\right)$

$$\frac{1}{2}\frac{d}{dx}(\sec x \tan x) + \frac{1}{2}\frac{d}{dx}(\ln(\sec x + \tan x))$$

(1) (2)

(1) $\frac{1}{2}\left[\sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x)\right]$

$$\frac{1}{2}\left[\sec x \sec^2 x + \tan x \sec x \tan x\right]$$

sec sec tan

$$= \frac{1}{2}\left[\sec^3 x + \sec x \tan^2 x\right]$$

$$= \frac{\sec^3 x + \sec x \tan^2 x}{2}$$

$$= \frac{1}{2}\sec x\left(\sec^2 x + \tan^2 x\right)$$

② $\frac{d}{dx}(\sec x + \tan x)$

$= \sec x \tan x + \sec^2 x$

$\frac{d}{dx} \ln(\sec x + \tan x)$

$= (\sec x \tan x + \sec^2 x) \cdot \frac{1}{\sec x + \tan x}$

$= \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x}$

$\frac{1}{2}\left[\frac{d}{dx}(\ln(\sec x + \tan x))\right]$

$= \frac{1}{2} \sec x$

① + ② ⇒ $\frac{1}{2} \sec x \left[\sec^2 x + \tan^2 x + 1\right]$

$= \frac{1}{2} \sec x \left[\sec^2 x + \sec^2 x\right]$

$= \frac{1}{2} \sec x \left[2 \sec^2 x\right]$

$= \sec^3 x$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec$

14) $\frac{d}{dx}\left[\frac{xe^x}{1+e^x}\right]$

$$= \frac{(1+e^x)\frac{d}{dx}(xe^x) - xe^x\frac{d}{dx}(1+e^x)}{(1+e^x)^2}$$

$$\frac{d}{dx}(xe^x) = xe^x + e^x$$

$$\frac{d}{dx}(1+e^x) = e^x$$

$$= \frac{(1+e^x)\ e^x(x+1) - xe^x \cdot e^x}{(1+e^x)^2}$$

$$= \frac{e^x\left[(1+e^x)(x+1) - xe^x\right]}{(1+e^x)^2}$$

$$= \frac{e^x\left[x+1+\cancel{e^x x}+e^x - \cancel{xe^x}\right]}{(1+e^x)^2}$$

$$= \frac{xe^x+e^x+e^{2x}}{(1+e^x)^2}$$

15) $\frac{d}{dx}\left(e^{4x}\cos\left(\frac{x}{2}\right)\right)$

$$= e^{4x}\frac{d}{dx}\cos\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(e^{4x}\right)$$

$$\frac{d}{dx}\left(e^{4x}\right) = 4\cdot e^{4x}$$

$$\frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\frac{d}{dx}\left(\cos\frac{x}{2}\right) = \frac{1}{2}\cdot -\sin\frac{x}{2} = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$$

$$= e^{4x}\cdot -\frac{1}{2}\sin\frac{x}{2} + \cos\frac{x}{2}\cdot 4e^{4x}$$

$$= 4e^{4x}\cos\frac{x}{2} - \frac{e^{4x}}{2}\sin\frac{x}{2}$$

16) $$\frac{d}{dx}\left[\frac{1}{\sqrt[4]{x^3-2}}\right]$$

$$= \frac{d}{dx}\left[(x^3-2)^{-1/4}\right]$$

$$\frac{d}{dx}(x^3-2) = 3x^2$$

$$\frac{d}{dx}(x^3-2)^{-1/4} = 3x^2 \cdot \frac{-1}{4}(x^3-2)^{-1/4-1}$$

$$= -\frac{3}{4}x^2(x^3-2)^{-5/4}$$

$$= \frac{-3x^2}{4\sqrt[4]{(x^3-2)^5}}$$

17) $$\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{x^2-1}\right)\right)$$

$$\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\tan^{-1}(x) = \frac{1}{1+x^2}$$

$$\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sqrt{x^2-1} = \frac{1}{2\sqrt{x^2-1}} \cdot 2x = \frac{x}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\sqrt{x^2-1}\right)\right) = \frac{x}{\sqrt{x^2-1}} \cdot \frac{1}{1+\left(\sqrt{x^2-1}\right)^2}$$

$$= \frac{x}{\left(\sqrt{x^2-1}\right)\left(1+x^2-1\right)}$$

$$= \frac{x}{x^2\sqrt{x^2-1}}$$

$$= \frac{1}{x\sqrt{x^2-1}}$$

(18) $\frac{d}{dx}\left(\frac{\ln x}{x^3}\right)$

$$\frac{x^3 \frac{d}{dx}\ln x - \ln x \frac{d}{dx}x^3}{(x^3)^2}$$

$$= \frac{x^3 \frac{1}{x} - \ln x \; 3x^2}{(x^3)^2}$$

$$= \frac{x^2 - 3x^2 \ln x}{(x^3)^2}$$

$$= \frac{x^2(1 - 3\ln x)}{x^6}$$

$$= \frac{1 - 3\ln x}{x^4}$$

19. $\frac{d}{dx}\left(x^{x}\right)$

$$x^{x} = y$$

$$\ln x^{x} = \ln y$$

$$x \ln x = \ln y$$

$$e^{x \ln x} = e^{\ln y}$$

$$\cancel{e^{x} e^{\ln x} = y}$$ ← got wrong here not possible as $e^{x} e^{\ln x} = e^{x + \ln x}$

$$\cancel{e^{x} \cdot x = y}$$

$$\cancel{\frac{d}{dx}\left(x e^{x}\right)} = \cancel{x \frac{d}{dx} e^{x} + e^{x} \frac{d}{dx}(x)}$$

$$= x e^{x} + e^{x}$$

$$e^{x \ln x} = e^{\ln y}$$

$$e^{x \ln x} = y$$

$$\frac{d}{dx}\left(e^{x \ln x}\right)$$

We have inner $\frac{d}{dx}(x\ln x) = x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x)$

$$= x\frac{1}{x} + \ln x \cdot 1$$

$$= 1 + \ln x$$

$$\frac{d}{dx}\left(e^{x\ln x}\right) = (1+\ln x)\, e^{x\ln x}$$

$$= (1+\ln x)\cdot x^x$$

(20) Find $\frac{dy}{dx}$ for $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3 + y^3 - 6xy\right) = 0$$

$$= 3x^2 + 3y^2\frac{dy}{dx} - 6\frac{d(xy)}{dx} = 0$$

$$\frac{d(xy)}{dx} = x\frac{dy}{dx} + y\frac{dx}{dx}$$

$$= \frac{x\,dy}{dx} + y$$

$$= 3x^2 + 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} + 6y = 0$$

$$= (3y^2 - 6x)\frac{dy}{dx} = -3x^2 - 6y$$

$$\frac{dy}{dx} = \frac{-3x^2 - 6y}{3y^2 - 6x}$$

$$\frac{dy}{dx} = \frac{-x^2 - 2y}{y^2 - 2x}$$

$$\frac{dy}{dx} = \frac{x^2 - 2y}{-1(y^2 - 2x)} = \frac{x^2 - 2y}{2x - y^2}$$

(21) Find $\frac{dy}{dx}$ for $y\sin y = x\sin x$

$$y\sin y - x\sin x = 0$$

$$\frac{d}{dx}(y\sin y) = y\frac{d}{dx}(\sin y) + \sin y\frac{dy}{dx}$$

$$= y\cos y \cdot \frac{dy}{dx} + \sin y\frac{dy}{dx}$$

$$\frac{d}{dx}(x\sin x) = x\frac{d}{dx}(\sin x) + \sin x\frac{dx}{dx}$$

$$= x\cos x + \sin x$$

$$= y\cos y\frac{dy}{dx} + \sin y\frac{dy}{dx} - x\cos x - \sin x = 0$$

$$\frac{dy}{dx} = \frac{x\cos x + \sin x}{y\cos y + \sin y}$$

$$\frac{dy}{dx} = \frac{\sin x + x\cos x}{\sin y + y\cos y}$$

(22) Find $\frac{dy}{dx}$ for $\ln\left(\frac{x}{y}\right) = e^{xy^3}$

$$\underline{\underline{\frac{d}{dx}\left(\ln\left(\frac{x}{y}\right)\right)}}$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}$$

something to be done here?

$$= \frac{y - x\frac{dy}{dx}}{y^2}$$

$$= \frac{1}{y^2}\left(y - x\frac{dy}{dx}\right) = \left(\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(\ln\left(\frac{x}{y}\right)\right) = \left[\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right]\frac{1}{\frac{x}{y}}$$

$$= \frac{y}{x}\left[\frac{1}{y} - \frac{x}{y^2}\frac{dy}{dx}\right]$$

$$= \left[\frac{1}{x} - \frac{1}{y}\frac{dy}{dx}\right]$$

$$\frac{d}{dx} e^{xy^3}$$

inner: $\frac{d}{dx}(xy^3) = x\,\frac{d(y^3)}{dx} + y^3\,\frac{d(x)}{dx}$

$$= x\,3y^2\frac{dy}{dx} + y^3$$

$$= 3xy^2\frac{dy}{dx} + y^3$$

$$\frac{d}{dx} e^{xy^3} = \left(3xy^2\frac{dy}{dx} + y^3\right) e^{xy^3}$$

$\frac{dy}{dx}$?

$$= 3xy^2 e^{xy^3}\frac{dy}{dx} + y^3 e^{xy^3}$$

LHS = RHS

$$\frac{1}{x} - \frac{1}{y}\frac{dy}{dx} = 3xy^2 e^{xy^3}\frac{dy}{dx} + y^3 e^{xy^3}$$

$$\frac{1}{x} - y^3 e^{xy^3} = 3xy^2 e^{xy^3}\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx}$$

$$\frac{1 - xy^3 e^{xy^3}}{x} = \frac{(3xy^3 e^{xy^3} + 1)}{y}\frac{dy}{dx}$$

$$\frac{y - xy^4 e^{xy^3}}{3x^2y^3e^{xy^3} + x} = \frac{dy}{dx}$$

easier way:

Notice LHS is of form $\ln \frac{A}{B}$

$$\frac{d}{dx}\left(\ln\left(\frac{x}{y}\right)\right) = \frac{d}{dx}\left(\ln(x) - \ln(y)\right)$$

$$= \frac{d}{dx}(\ln x) - \frac{d}{dx}(\ln y)$$

$$= \frac{1}{x} - \frac{1}{y} \cdot \frac{dy}{dx}$$

(23) Find $\frac{dy}{dx}$ for $x = \sec y$ 10:30

$\sin^{-1}$ $\frac{1}{\sqrt{1-x^2}}$

$\tan^{-1}$ $\frac{1}{1+x^2}$

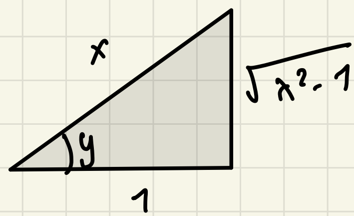
$\sec^{-1}$ $\frac{1}{x\sqrt{x^2-1}}$

$$x = \sec y$$

$$\sec^{-1}(x) = y$$

$$\frac{1}{x\sqrt{x^2-1}} = \frac{dy}{dx}$$

or



$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

$$= \cos y \cdot \frac{\cos y}{\sin y}$$

$$= \frac{1}{x^2} \cdot \frac{1}{\sqrt{x^2-1}} \cdot x = \frac{1}{x\sqrt{x^2-1}}$$

(24) Find $\frac{dy}{dx}$ for $(x-y)^2 = \sin(x) + \sin(y)$

LHS = inner = $1 - \frac{dy}{dx}$

$$\frac{d}{dx}(x-y)^2 = \left(1-\frac{dy}{dx}\right) 2(x-y)$$

$$= \left(1-\frac{dy}{dx}\right)(2x-2y)$$

$$= 2x - 2y - 2x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$= 2x - 2y + \frac{dy}{dx}(2y-2x)$$

RHS

$$\frac{d}{dx}(\sin x) + \frac{d}{dx}(\sin y)$$

$$= \cos x + \cos y \frac{dy}{dx}$$

LHS = RHS

$$= \frac{dy}{dx}(2y-2x) - \cos y\frac{dy}{dx} = \cos x + 2y - 2x$$

$$\frac{dy}{dx} = \frac{\cos x + 2y - 2x}{2y - 2x - \cos y}$$

$$= \frac{2y - 2x + \cos x}{2y - 2x - \cos y} \begin{matrix} \times(-1) \\ \times(-1) \end{matrix}$$

$$= \frac{2x - 2y - \cos x}{2x - 2y + \cos y}$$

(25) Find $\frac{dy}{dx}$ for $x^y = y^x$

Apply ln on both sides

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$\frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y)$$

$$y\frac{1}{x} + \ln x \frac{dy}{dx} = x\frac{1}{y}\frac{dy}{dx} + \ln y \cdot 1$$

$$\frac{y}{x} + \ln x \frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \ln y$$

$$\left(\ln x - \frac{x}{y}\right)\frac{dy}{dx} = \ln y - \frac{y}{x}$$

$$\left(\frac{y\ln x - x}{y}\right)\frac{dy}{dx} = \frac{x\ln y - y}{x}$$

$$\frac{dy}{dx} = \frac{xy\ln y - y^2}{xy\ln x - x^2}$$

(26) Q26.) Find $\frac{dy}{dx}$ for $\tan^{-1}(x^2y) = x + y^3$

(A) $\frac{2xy+1-x^4y^2}{3y^2+3x^4y^2-2xy}$ (B) $\frac{2xy-1-x^4y^2}{3y^2+3x^4y^4-x^2}$ (C) $\frac{3xy-1+x^4y^2}{2y^2-3x^4y^4+x^2}$

$$\tan^{-1}(x^2y) = x + y^3$$

$$x^2y = \tan(x+y^3)$$

$$\frac{d}{dx}(x^2 y) = \frac{d}{dx}\left(\tan(x+y^3)\right)$$

$$\left.x^2 \cdot \frac{dy}{dx} + y\,2x \quad\right\} \text{LHS}$$

RHS: inner $\frac{d}{dx}(x+y^3) = 1 + 3y^2\frac{dy}{dx}$

$$\frac{d}{dx}\left(\tan(x+y^3)\right) = \left(1+3y^2\frac{dy}{dx}\right)\left(\sec^2(x+y^3)\right)$$

$$= \sec^2(x+y^3) + 3\sec^2(x+y^3)\,y^2\,\frac{dy}{dx}$$

LHS = RHS

$$\left(x^2 - 3y^2\sec^2(x+y^3)\right)\frac{dy}{dx} = \sec^2(x+y^3) - 2xy$$

$$\frac{dy}{dx} = \frac{\sec^2(x+y^3) - 2xy}{x^2 - 3y^2\sec^2(x+y^3)}$$

$$\tan^{-1}(x^2y) = x + y^3$$

$$\left.\frac{1}{1+(x^2y)^2}\cdot\left(x^2\frac{dy}{dx} + y2x\right)\right\} LHS$$

RHS: $1 + 3y^2\frac{dy}{dx}$

LHS = RHS:

$$\frac{x^2}{1+x^4y^2}\frac{dy}{dx} + \frac{2xy}{1+x^4y^2} = 1 + 3y^2\frac{dy}{dx}$$

$$\left(\frac{x^2}{1+x^4y^2} - 3y^2\right)\frac{dy}{dx} = 1 - \frac{2xy}{1+x^4y^2}$$

$$\left(\frac{x^2 - 3y^2(1+x^4y^2)}{1+x^4y^2}\right)\frac{dy}{dx} = \frac{1+x^4y^2 - 2xy}{1+x^4y^2}$$

$$\frac{dy}{dx} = \frac{1+x^4y^2-2xy}{x^2-3y^2-3x^4y^4}$$

$$\frac{dy}{dx} = \frac{2xy - x^4y^2 - 1}{3y^2 + 3x^4y^4 - x^2}$$

27 $\frac{dy}{dx}$ for $\frac{x^2}{x^2 - y^2} = 3y$

:30

$$\frac{(x^2-y^2)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(x^2-y^2)}{(x^2-y^2)^2} = \frac{d}{dx}(3y)$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^2-y^2) = 2x - 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(3y) = \frac{3dy}{dx}$$

$$\frac{(x^2-y^2)(2x) - x^2\left(2x - 2y\frac{dy}{dx}\right)}{(x^2-y^2)^2} = \frac{3dy}{dx}$$

$$2x^3 - 2xy^2 - 2x^3 + 2x^2y\frac{dy}{dx} = 3(x^2-y^2)^2\frac{dy}{dx}$$

$$2x^2y\frac{dy}{dx} - 3(x^2-y^2)^2\frac{dy}{dx} = +2xy^2$$

$$\frac{dy}{dx} = \frac{2xy^2}{2x^2y - 3(x^2-y^2)^2}$$

$$(x^2-y^2)^2 = (x^2)^2 + (y^2)^2 - 2x^2y^2$$

$$\frac{dy}{dx} = \frac{2xy^2}{2x^2y - 3x^4 - 3y^4 + 6x^2y^2}$$

or

$$x^2 = 3y(x^2-y^2)$$

$$x^2 = 3x^2y - 3y^3$$

$$2x = 3\left(x^2\frac{dy}{dx} + y2x\right) - 3\cdot 3y^2\frac{dy}{dx}$$

$$2x = 3x^2\frac{dy}{dx} + 6xy - 9y^2\frac{dy}{dx}$$

$$2x - 6xy = (3x^2 - 9y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - 6xy}{3x^2 - 9y^2}$$

(28) Find $\frac{dy}{dx}$ for $e^{\frac{x}{y}} = x + y^2$

:15

$$\frac{d}{dx}\left(e^{\frac{x}{y}}\right) = \frac{d}{dx}\left(x + y^2\right)$$

$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{y\,1 - x\frac{dy}{dx}}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2}$$

$$\frac{d}{dx}\left(e^{\frac{x}{y}}\right) = \frac{\left(y - x\frac{dy}{dx}\right)e^{\frac{x}{y}}}{y^2}$$

$$= \frac{e^{x/y}}{y} - \frac{x}{y^2}\frac{dy}{dx}e^{x/y}$$

$$\frac{d}{dx}(x+y^2) = 1 + 2y\frac{dy}{dx}$$

$$\frac{e^{x/y}}{y} - 1 = \left(2y + \frac{xe^{x/y}}{y^2}\right)\frac{dy}{dx}$$

$$\frac{e^{x/y} - y}{y} = \frac{2y^3 + xe^{x/y}}{y^2}\frac{dy}{dx}$$

$$\frac{(e^{x/y} - y)\,y}{2y^3 + xe^{x/y}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{ye^{x/y} - y^2}{xe^{x/y} + 2y^3}$$

(29) Find $\frac{dy}{dx}$ for $(x^2+y^2-1)^3 = y$

$$\Downarrow$$

$$x^2+y^2-1 = \sqrt[3]{y}$$

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

$$\frac{d}{dx}(1) = 0$$

$$\frac{d}{dx}(x^2+y^2-1) = 2x + 2y\frac{dy}{dx} + 0$$

$$\frac{d}{dx}(y^{1/3}) = \frac{1}{3}y^{\frac{1}{3}-1} \cdot \frac{dy}{dx}$$

$$= \frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx}$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt[3]{y^2}}\frac{dy}{dx} = \frac{1}{3\sqrt[3]{y^2}}\frac{dy}{dx}$$

$$2x + 2y\frac{dy}{dx} = \frac{1}{3\sqrt[3]{y^2}}\frac{dy}{dx}$$

$$2x = \left(\frac{1}{3\sqrt[3]{y^2}} - 2y\right)\frac{dy}{dx}$$

$$2x = \frac{1 - 6y\sqrt[3]{y^2}}{3y^{2/3}} \frac{dy}{dx}$$

$$\frac{6xy^{2/3}}{1-6y \cdot y^{2/3}} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6xy^{2/3}}{1-6yy^{2/3}}$$

we have $y = (x^2+y^2-1)^3$

$$\therefore \quad \frac{dy}{dx} = \frac{6x(x^2+y^2-1)^{3\cdot 2/3}}{1-6y(x^2+y^2-1)^{3\cdot 2/3}}$$

$$\frac{dy}{dx} = \frac{6x(x^2+y^2-1)^2}{1-6y(x^2+y^2-1)^2}$$

(30) Find $\frac{d^2y}{dx^2}$ for $9x^2 + y^2 = 9$

$$\frac{d}{dx}(9x^2 + y^2) = \frac{d}{dx}(9)$$

$$9 \cdot 2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -18x$$

$$\frac{dy}{dx} = \frac{-9x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y\frac{d}{dx}(-9x) - -9x\frac{dy}{dx}}{y^2}$$

$$= \frac{-9y + 9x\frac{dy}{dx}}{y^2}$$

we already have value for $\frac{dy}{dx} = \frac{-9x}{y}$

$$= \frac{-9y + 9x\left(\frac{-9x}{y}\right)}{y^2}$$

$$= \frac{-9y^2 - 81x^2}{y^3}$$

in question $9x^2 + y^2 = 9$

$$y^2 = 9 - 9x^2$$

$$= \frac{-9(9 - 9x^2) - 81x^2}{y^3}$$

$$= \frac{-81 + 81x^2 - 81x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-81}{y^3}$$

(31) $\frac{d^2}{dx}\left(\frac{1}{9}\sec(3x)\right)$

$\frac{d}{dy}\left(\frac{1}{9}\sec(3x)\right)$

$\frac{d}{dx}(3x)=3$

$\frac{d}{dx}(\sec(3x))=3\sec 3x\tan 3x$

$\frac{d}{dx}\left(\frac{1}{9}\sec(3x)\right)=\frac{3}{9}\sec 3x\tan 3x$

$\frac{d}{dx}\left(\frac{1}{3}\sec 3x\tan 3x\right)$

$\frac{1}{3}\left[\sec 3x\frac{d}{dx}(\tan 3x)+\tan 3x\frac{d}{dx}(\sec 3x)\right]$

$\frac{d}{dx}(\tan 3x)=3.\sec^2 3x$

$$\frac{d}{dx}(\sec 3x) = 3 \cdot \sec 3x \tan 3x$$

$$\frac{1}{3}\left[\sec 3x \; 3\sec^2 3x + \tan 3x \cdot 3\sec 3x \tan 3x\right]$$

$$= \sec^3 3x + \sec 3x \tan^2 3x$$

(32) $$\frac{d^2}{dx^2}\left[\frac{x+1}{\sqrt{x}}\right]$$

$$\frac{d}{dx}\left[\sqrt{x} + \frac{1}{\sqrt{x}}\right]$$

$$= \frac{1}{2\sqrt{x}} + \left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right)$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$

$$\frac{d}{dx}\left[\frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-3/2}\right]$$

$$= \frac{1}{2}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) - \frac{1}{2}\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right)$$

$$= -\frac{1}{4}x^{-\frac{3}{2}} + \frac{3}{4}x^{-5/2}$$

$$= \frac{3}{4\sqrt{x^5}} - \frac{1}{4\sqrt{x^3}}$$

$$= \frac{12\sqrt{x^3} - 4\sqrt{x^5}}{16\sqrt{x^8}}$$

$$= \frac{12x\sqrt{x} - 4x^2\sqrt{x}}{16x^4}$$

$$= \frac{4x(3\sqrt{x} - x\sqrt{x})}{16x^4}$$

$$= \frac{3\sqrt{x} - x\sqrt{x}}{4x^3} = \frac{3-x}{4x^2x^{\frac{1}{2}}} = \frac{3-x}{4x^{5/2}}$$

(33) $\frac{d^2}{dx^2}\left(\sin^{-1}(x^2)\right)$

$\frac{d}{dx}(x^2) = 2x$

$\frac{d}{dx}\left(\sin^{-1}(x^2)\right) = 2x \cdot \frac{1}{\sqrt{1-(x^2)^2}}$

$= \frac{2x}{\sqrt{1-x^4}}$

$\frac{d}{dx}\left(\frac{2x}{\sqrt{1-x^4}}\right)$

$$\frac{\sqrt{1-x^4} \cdot \frac{d}{dx}(2x) - 2x \frac{d}{dx}\left(\sqrt{1-x^4}\right)}{\left(\sqrt{1-x^4}\right)^2}$$

$\frac{d}{dx}(2x) = 2$

$\frac{d}{dx}\left(\sqrt{1-x^4}\right) = -4x^3 \cdot \frac{1}{2\sqrt{1-x^4}} = \frac{-2x^3}{\sqrt{1-x^4}}$

$$= \frac{2\sqrt{1-x^4} - 2x \cdot \frac{-2x^3}{\sqrt{1-x^4}}}{1-x^4}$$

$$= \frac{2(1-x^4) + 4x^4}{(1-x^4)\sqrt{1-x^4}} = \frac{2-2x^4+4x^4}{(1-x^4)^{3/2}}$$

$$= \frac{2+2x^4}{(1-x^4)^{3/2}}$$

(34) $\frac{d^2}{dx}\left(\frac{1}{1+\cos x}\right)$ 7:29

$$= \frac{1+\cos x \cdot 0 - 1 \frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$= -\frac{(0-\sin x)}{(1+\cos x)^2} = \frac{\sin x}{(1+\cos x)^2}$$

$$\frac{d}{dx}\left(\frac{\sin x}{(1+\cos x)^2}\right)$$

$$= \frac{(1+\cos x)^2 \cos x - \sin x \frac{d}{dx}(1+\cos x)^2}{(1+\cos x)^4}$$

$$\frac{d}{dx}(1+\cos x)^2 = -\sin x \cdot 2(1+\cos x)$$
$$= -2\sin x - 2\sin x \cos x$$

$$(1+\cos x)^2 = 1+\cos^2 x + 2\cos x$$

$$\frac{(1+\cos x)^2 \cos x - 2\sin^2 x(1+\cos x)}{(1+\cos x)^4}$$

$$= \frac{(1+\cos x)\cos x - 2\sin^2 x}{(1+\cos x)^3}$$

$$= \frac{\cos x + \cos^2 x - 2\sin^2 x}{(1+\cos x)^3}$$

35 $\frac{d^2}{dx^2}\left(x \tan^{-1}(x)\right)$ 7:40

$$= x\,\frac{1}{1+x^2} + \tan^{-1}(x)\cdot 1$$

$$= \frac{d}{dx}\left(\frac{x}{1+x^2} + \tan^{-1}(x)\right)$$

$$= \frac{d}{dx}\left(\frac{x}{1+x^2}\right) + \frac{d}{dx}\tan^{-1}(x)$$

$$= \frac{(1+x^2) - x(2x)}{(1+x^2)^2} + \frac{1}{(1+x^2)}$$

$$= \frac{(1+x^2)^2 - 2x^2(1+x^2) + (1+x^2)^2}{(1+x^2)^3}$$

$$= \frac{(1+x^2)\left[1+x^2 - 2x^2 + 1+x^2\right]}{(1+x^2)^3}$$

$$= \frac{2}{(1+x^2)^2}$$

(36) $\frac{d^2}{dx^2}\left(x^4 \ln(x)\right)$ 8:07

$$= x^4 \frac{1}{x} + \ln(x) \cdot 4x^3$$

$$= x^3 + 4x^3 \ln(x)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}\left(4x^3 \ln(x)\right)$$

$$= 3x^2 + 4\left[x^3 \cdot \frac{1}{x} + \ln x \cdot 3x^2\right]$$

$$= 3x^2 + 4x^2 + 12x^2 \ln x$$

$$= 7x^2 + 12x^2 \ln x$$

37 $\dfrac{d^2}{dx^2}\left(e^{-x^2}\right)$

$$= \frac{d}{dx}(-x^2) = -2x$$

$$\frac{d}{dx}\left(e^{-x^2}\right) = -2x \cdot e^{-x^2}$$

$$\frac{d}{dx}\left(-2xe^{-x^2}\right)$$

$$= -2\left[x\frac{d}{dx}e^{-x^2} + e^{-x^2} \cdot 1\right]$$

$$= -2\left[x\left(-2xe^{-x^2}\right) + e^{-x^2}\right]$$

$$= -2\left[-2x^2e^{-x^2} + e^{-x^2}\right]$$

$$= 4x^2e^{-x^2} - 2e^{-x^2}$$

$$= e^{-x^2}\left(4x^2-2\right)$$

(38) $\frac{d}{dx}\left(\cos(\ln x)\right)$

$$= \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}\cos(\ln x) = -\frac{1}{x}\sin(\ln x)$$

$$\frac{d}{dx}\left(-\frac{\sin(\ln x)}{x}\right)$$

$$= -1\left[\frac{x\,\frac{1}{x}\cdot\cos(\ln x) - \sin(\ln x)\cdot 1}{x^2}\right]$$

$$= -1\left[\frac{\cos(\ln x) - \sin(\ln x)}{x^2}\right]$$

$$= \frac{\sin(\ln x) - \cos(\ln x)}{x^2}$$

(39) $\frac{d^2}{dx}\left(\ln(\cos x)\right)$

$\frac{d}{dx}((\cos x)) = -\sin x$

$\frac{d}{dx}\ln(\cos x) = -\sin x \cdot \frac{1}{\cos x} = \frac{-\sin x}{\cos x}$

$\frac{d}{dx}\left(\frac{-\sin x}{\cos x}\right) = \frac{d}{dx}(-\tan x)$

$= -\sec^2 x$

(40) $\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1}x\right)$

$= \frac{d}{dx}\left(\sqrt{1-x^2}\right) + \frac{d}{dx}\left(x\sin^{-1}x\right)$

$= -2x \cdot \frac{1}{2\sqrt{1-x^2}} + x\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x$

$$= \frac{-x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x$$

$$= \sin^{-1} x$$

(41) $\frac{d}{dx}\left(x\sqrt{4-x^2}\right)$

$$= x \frac{d}{dx}\sqrt{4-x^2} + \sqrt{4-x^2} \cdot 1$$

$$= x \cdot \frac{-2x}{2\sqrt{4-x^2}} + \sqrt{4-x^2}$$

$$= \frac{-2x^2 + 2(4-x^2)}{2\sqrt{4-x^2}}$$

$$= \frac{-2x^2 + 8 - 2x^2}{2\sqrt{4-x^2}} = \frac{8-4x^2}{2\sqrt{4-x^2}}$$

$$= \frac{4-2x^2}{\sqrt{4-x^2}}$$

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$$\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x}\right)$$

$$\frac{x\,\frac{d}{dx}\left(\sqrt{x^2-1}\right) - \sqrt{x^2-1}\,\frac{d}{dx}(x)}{x^2}$$

$$\frac{d}{dx}\left(\sqrt{x^2-1}\right) = 2x \cdot \frac{1}{2\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(x) = 1$$

$$\left(\frac{x^2}{\sqrt{x^2-1}} - \sqrt{x^2-1}\right) \cdot \frac{1}{x^2}$$

$$\frac{x^2 - x^2 - 1}{x^2\sqrt{x^2-1}} = \frac{-1}{x^2\sqrt{x^2-1}}$$

(43) $\frac{d}{dx}\left(\frac{x}{\sqrt{x^2-1}}\right)$

$$\frac{\sqrt{x^2-1}\,(1) - x\,\frac{x}{\sqrt{x^2-1}}}{x^2-1}$$

$$= \frac{x^2-1-x^2}{(x^2-1)^{3/2}} = \frac{-1}{\sqrt{(x^2-1)^3}}$$

(44) $\frac{d}{dx}\left(\cos(\sin^{-1}x)\right)$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \cdot -\sin(\sin^{-1}x)$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

45 $\frac{d}{dx}\left(\ln\left(x^2+3x+5\right)\right)$

$$\frac{d}{dx}\left(x^2+3x+5\right) = 2x+3$$

$$\frac{d}{dx}\ln\left(x^2+3x+5\right) = 2x+3 \cdot \frac{1}{x^2+3x+5}$$

46 $\frac{d}{dx}\left(\left(\tan^{-1}(4x)\right)^2\right)$

$$\frac{d}{dx}(4x) = 4$$

$$\frac{d}{dx}\tan^{-1}(4x) = 4 \cdot \frac{1}{1+(4x)^2}$$

$$\frac{d}{dx}\left(\tan^{-1}(4x)\right)^2 = \frac{4}{1+(4x)^2} \cdot 2\tan^{-1}(4x)$$

$$= \frac{8\tan^{-1}(x)}{1+(4x)^2}$$

(47) $\frac{d}{dx}\left(\sqrt[3]{x^2}\right)$

$$= \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}\, x^{\frac{2}{3}-1} = \frac{2}{3}\, x^{-\frac{1}{3}}$$

$$= \frac{2}{3}\,\frac{1}{\sqrt[3]{x}} = \frac{2}{3\sqrt[3]{x}}$$

(48) $\frac{d}{dx}\left(\sin\left(\sqrt{x}\ln x\right)\right)$

$$\frac{d}{dx}\left(\sqrt{x}\ln x\right) = \sqrt{x}\,\frac{1}{x} + \ln x\,\frac{1}{2\sqrt{x}}$$

$$= \frac{\sqrt{x}}{x} + \frac{\ln x}{2\sqrt{x}}$$

$$= \frac{2x + x\ln x}{2x\sqrt{x}} = \frac{2 + \ln x}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\sin\left(\sqrt{x}\ln x\right)\right) = \frac{2+\ln x}{2\sqrt{x}}\,\cos\left(\sqrt{x}\ln x\right)$$

(49) $\frac{d}{dx}(\operatorname{cosec}(x^2))$

$\sec \to \sec \tan$

$\operatorname{cosec} \to -\operatorname{cosec} \cot$

$\frac{d}{dx}(x^2) = 2x$

$\frac{d}{dx}(\operatorname{cosec}(x^2)) = -2x \operatorname{cosec}(x^2)\cot(x^2)$

(50) $\frac{d}{dx}\left(\frac{x^2-1}{\ln x}\right)$

$$\frac{\ln x\,(2x) - (x^2-1)\frac{1}{x}}{(\ln x)^2}$$

$$= \frac{x\ln x\,(2x) - x^2 + 1}{x(\ln x)^2} = \frac{2x^2\ln x - x^2 + 1}{x(\ln x)^2}$$

(51) $\frac{d}{dx}(10^x)$

$\frac{d}{dx}a^x = a^x \log a$

$= 10^x \ln 10$

(52) $$\frac{d}{dx}\left(\sqrt[3]{x+(\ln x)^2}\right)$$

$$\frac{d}{dx}(\ln x)^2 = \frac{1}{x}\cdot 2\ln x = \frac{2\ln x}{x}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}\left(\sqrt[3]{x+(\ln x)^2}\right)$$

$$= \left(1+\frac{2\ln x}{x}\right)\frac{1}{3}\left(x+(\ln x)^2\right)^{\frac{1}{3}-1}$$

$$= \left(1+\frac{2\ln x}{x}\right)\frac{1}{3}\left(x+(\ln x)^2\right)^{-\frac{2}{3}}$$

$$= \left(1+\frac{2\ln x}{x}\right)\frac{1}{3}\frac{1}{\sqrt[3]{\left(x+(\ln x)^2\right)^2}}$$

$$= \frac{1+2\ln x}{3x\sqrt{\left(x+(\ln x)^2\right)^2}}$$

(53) $\frac{d}{dx}\left(x^{3/4} - 2x^{\frac{1}{4}}\right)$ 5:42

$$\frac{d}{dx}\left(x^{\frac{3}{4}}\right) - \frac{d}{dx}\left(2x^{\frac{1}{4}}\right)$$

$$= \frac{3}{4}x^{\frac{3}{4}-1} - 2\cdot\frac{1}{4}x^{\frac{1}{4}-1}$$

$$= \frac{3}{4}x^{-\frac{1}{4}} - \frac{2}{4}x^{-\frac{3}{4}}$$

$$= \frac{3}{4x^{1/4}} - \frac{2}{4x^{3/4}}$$

$$= \frac{3}{4x^{1/4}} - \frac{1}{2x^{3/4}}$$

$$= \frac{6x^{3/4} - 4x^{1/4}}{8x^{4/4}} = \frac{2x^{1/4}\left(3x^{2/4} - 2\right)}{8x^{4/4}}$$

$$= \frac{3x^{1/2} - 2}{4x^{3/4}} = \frac{3\sqrt{x} - 2}{4\sqrt[4]{x}}$$

(59) $\frac{d}{dx}\left(\log_2\left(x\sqrt{1+x^2}\right)\right)$

$\log_2 x = \frac{\ln x}{\ln 2}$

$\ln x = \log_e x$

$$\frac{d}{dx}\left(\frac{\ln\left(x\sqrt{1+x^2}\right)}{\ln 2}\right)$$

$$\frac{\ln 2 \frac{d}{dx}\ln\left(x\sqrt{1+x^2}\right) - \ln x\sqrt{1+x^2}\,\frac{d}{dx}(\ln 2)}{(\ln 2)^2}$$

inner:

$$\frac{d}{dx}\left(x\sqrt{1+x^2}\right)$$

$$= x\frac{d}{dx}\left(\sqrt{1+x^2}\right) + \sqrt{1+x^2}\cdot\frac{dx}{dx}$$

$$= x\cdot 2x\cdot\frac{1}{2\sqrt{1+x^2}} + \sqrt{1+x^2}$$

$$= \frac{x^2}{\sqrt{1+x^2}} + \sqrt{1+x^2} = \frac{x^2+1+x^2}{\sqrt{1+x^2}}$$

$$= \frac{2x^2+1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\ln\left(x\sqrt{1+x^2}\right)\right) = \frac{2x^2+1}{\sqrt{x^2+1}} \cdot \frac{1}{x\sqrt{1+x^2}}$$

$$= \frac{2x^2+1}{x(x^2+1)}$$

$$= \ln 2 \cdot \frac{(2x^2+1)}{x(x^2+1)} \cdot \frac{1}{(\ln 2)^2}$$

$$= \frac{2x^2+1}{x(x^2+1)\ln 2}$$

55 $\frac{d}{dx}\left(\frac{x-1}{x^2-x+1}\right)$

$$\frac{(x^2-x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x^2-x+1)}{(x^2-x+1)^2}$$

$$= \frac{x^2-x+1-[(x-1)(2x-1)]}{(x^2-x+1)^2}$$

$$(x-1)(2x-1) = 2x^2-x-2x+1$$
$$= 2x^2-3x+1$$

$$= x^2-x+1-2x^2+3x-1$$
$$= -x^2+2x = 2x-x^2$$

$$= \frac{2x-x^2}{(x^2-x+1)^2}$$

(56) $\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right)$

$$= \frac{1}{3}\frac{d}{dx}(\cos^3 x) - \frac{d}{dx}(\cos x)$$

$$= \frac{1}{3}(-\sin x) \cdot 3\cos^2 x - (-\sin x)$$

$$= -\sin x \cos^2 x + \sin x$$

$$= \sin x - \sin x \cos^2 x$$

$$= \sin x \,(1 - \cos^2 x)$$

$$= \sin x \cdot \sin^2 x$$

$$= \sin^3 x$$

(57) $\dfrac{d}{dx}\left(e^{x\cos x}\right)$

inner: $\dfrac{d}{dx}\left(x\cos x\right) = x(-\sin x) + \cos x\,(1)$

$$= \cos x - x\sin x$$

$$\frac{d}{dx}\left(e^{x\cos x}\right) = (\cos x - x\sin x)\cdot e^{x\cos x}$$

(58) $\dfrac{d}{dx}\left((x - \sqrt{x})(x + \sqrt{x})\right)$

$$= \frac{d}{dx}\left(x^2 - (\sqrt{x})^2\right)$$

$$= \frac{d}{dx}\left(x^2 - x\right) = 2x - 1$$

(59) $\frac{d}{dx}\left(\cot^{-1}\left(\frac{1}{x}\right)\right)$

cosec -cosec cot

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2} \qquad \cot^{-1}(x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}(x)) = \frac{1}{x\sqrt{x^2-1}} \qquad \operatorname{cosec}^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$$

inner : $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$$\frac{d}{dx}\left(\cot^{-1}\left(\frac{1}{x}\right)\right) = -\frac{1}{x^2} \cdot \frac{-1}{1+\left(\frac{1}{x}\right)^2}$$

$$= \frac{1}{x^2\left(1+\frac{1}{x^2}\right)}$$

$$= \frac{1}{x^2+1}$$

60 $\frac{d}{dx}\left(x\tan^{-1}x - \ln\left(\sqrt{x^2+1}\right)\right)$

$= \frac{d}{dx}\left(x\tan^{-1}x\right) - \frac{d}{dx}\left(\ln\sqrt{x^2+1}\right)$

1 $\qquad$ 2

1 $\quad x\frac{1}{1+x^2} + \tan^{-1}x \cdot 1$

$= \frac{x + (1+x^2)\tan^{-1}x}{1+x^2}$

2 $\quad f'\left(\ln\left(\sqrt{x^2+1}\right)\right) = 2x \cdot \frac{1}{2\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}}$

$= \frac{2x}{2(x^2+1)} = \frac{x}{1+x^2}$

1 - 2 $\Rightarrow \frac{\cancel{x} + (1+x^2)\tan^{-1}(x) - \cancel{x}}{1+x^2}$

$= \tan^{-1}(x)$

(G1) $\frac{d}{dx}\left(\frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} \right]$

$$= \frac{1}{2}\left[\underbrace{\frac{d}{dx}\left(x\sqrt{1-x^2}\right)}_{(1)} + \underbrace{\frac{d}{dx}\left(\sin^{-1}x\right)}_{(2)} \right]$$

$$\frac{d}{dx}\sqrt{1-x^2} = -2x \cdot \frac{1}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(x\sqrt{1-x^2}\right) = x \cdot \frac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2} \cdot 1$$

$$= \frac{-x^2 + 1 - x^2}{\sqrt{1-x^2}} = -2x^2 + 1$$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$(1) + (2) = \frac{1}{2}\left[\frac{-2x^2+1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right]$$

$$= \frac{1}{2}\,\frac{-2x^2+2}{\sqrt{1-x^2}} = \frac{1-x^2}{\sqrt{1-x^2}} = \sqrt{1-x^2}$$

62) $$\frac{d}{dx}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$$

$$= \frac{(\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x) - (\sin x - \cos x)\frac{d}{dx}(\sin x + \cos x)}{(\sin x + \cos x)^2}$$

$$\frac{d}{dx}(\sin x - \cos x) = \cos x + \sin x$$

$$\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$$

$$= \frac{(\sin x + \cos x)(\sin x + \cos x) - -1(\cos x - \sin x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{(\sin x + \cos x)^2 + (\cos x - \sin x)^2}{(\sin x + \cos x)^2}$$

$$= \frac{\sin^2 x + \cos^2 x + 2\sin x\cos x + \cos^2 x + \sin^2 x - 2\sin x\cos x}{(\sin x + \cos x)^2}$$

$$= \frac{2}{(\sin x + \cos x)^2}$$

(63) $$\frac{d}{dx}\left(4x^2(2x^3 - 5x^2)\right)$$

$$= \frac{d}{dx}\left(8x^5 - 20x^4\right)$$

$$= 40x^4 - 80x^3$$

(64) $$\frac{d}{dx}\left(\sqrt{x}\,(4 - x^2)\right)$$

$$= \sqrt{x}\,\frac{d}{dx}(4 - x^2) + (4 - x^2)\cdot\frac{1}{2\sqrt{x}}$$

$$= \sqrt{x}\,(-2x) + \frac{4 - x^2}{2\sqrt{x}}$$

$$= \frac{-2x\sqrt{x}\,(2\sqrt{x}) + 4 - x^2}{2\sqrt{x}}$$

$$= \frac{-4x^2 + 4 - x^2}{2\sqrt{x}} = \frac{4 - 5x^2}{2\sqrt{x}}$$

(65) $$\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right)$$

Inner: $$\frac{d}{dx}\left(\frac{1+x}{1-x}\right) =$$

$$\frac{(1-x)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$$

$$= \frac{1-x \ + 1+x}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$\frac{d}{dx}\left(\sqrt{\frac{1+x}{1-x}}\right) = \frac{2}{(1-x)^2} \cdot \frac{1}{2\sqrt{\frac{1+x}{1-x}}}$$

$$= \frac{1}{(1-x)^2 \cdot \frac{(1+x)^{1/2}}{(1-x)^{1/2}}} = \frac{1}{(1-x)^{3/2}(1+x)^{1/2}}$$

66) $\frac{d}{dx}\left(\sin(\sin x)\right)$

$$= \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\sin(\sin x)) = \cos x \cos(\sin x)$$

67) $\frac{d}{dx}\left(\frac{1+e^{2x}}{1-e^{2x}}\right)$

$$= \frac{1-e^{2x}\frac{d}{dx}(1+e^{2x}) - (1+e^{2x})\frac{d}{dx}(1-e^{2x})}{(1-e^{2x})^2}$$

$$\frac{d}{dx}(1+e^{2x}) = 2 \cdot e^{2x}$$

$$\frac{d}{dx}(1-e^{2x}) = -2e^{2x}$$

$$= \frac{(1-e^{2x})(2e^{2x}) - (1+e^{2x})(-2e^{2x})}{(1-e^{2x})^2}$$

$$= \frac{2e^{2x} - 2e^{4x} + 2e^{2x} + 2e^{4x}}{(1-e^{2x})^2}$$

$$= \frac{4e^{2x}}{(1-e^{2x})^2}$$

(68) $\frac{d}{dx}\left(\frac{x}{1+\ln x}\right)$

$$\frac{1+\ln x \quad 1 - x \cdot \frac{1}{x}}{(1+\ln x)^2}$$

$$= \frac{1+\ln x - 1}{(1+\ln x)^2} = \frac{\ln x}{(1+\ln x)^2}$$

(69) $\frac{d}{dx}\left(x^{\frac{x}{\ln x}}\right)$

Trick:
convert
$x \Rightarrow e^{\ln x}$

$x = e^{\ln x}$

$\therefore \frac{d}{dx}\left(e^{\ln x \cdot \frac{x}{\ln x}}\right)$

$x = e^{\ln x}$

$x^{\frac{1}{\ln x}} = e$

$= \frac{d}{dx} e^{x} = e^{x}$

$= \left(x^{\frac{1}{\ln x}}\right)^{x}$

$= x^{\frac{x}{\ln x}}$

70. $\frac{d}{dx}\left[\ln\left[\sqrt{\frac{x^2-1}{x^2+1}}\right]\right]$

chain rule inner to outer.

innermost

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{x^2+1\,(2x) - (x^2-1)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^3+2x \quad - 2x(x^2-1)}{(x^2+1)^2}$$

$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

$$\frac{d}{dx}\sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{2\sqrt{\frac{x^2-1}{x^2+1}}} \cdot \frac{4x}{(x^2+1)^2}$$

$$= \sqrt{\frac{x^2+1}{x^2-1}} \cdot \frac{2x}{(x^2+1)^2}$$

$$\frac{d}{dx}\ln\sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{\sqrt{\frac{x^2-1}{x^2+1}}} \cdot \sqrt{\frac{x^2+1}{x^2-1}} \cdot \frac{2x}{(x^2+1)^2}$$

$$= \sqrt{\frac{x^2+1}{x^2-1}} \cdot \sqrt{\frac{x^2+1}{x^2-1}} \cdot \frac{2x}{(x^2+1)^2}$$

$$= \frac{x^2+1}{x^2-1} \quad \frac{2x}{(x^2+1)^2}$$

$$= \frac{2x}{(x^2-1)(x^2+1)} = \frac{2x}{(x^2)^2-1}$$

$$= \frac{2x}{x^4-1}$$

71 $\frac{d}{dx}\left(\tan^{-1}(2x+3)\right)$ inner to outer

$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$ $\qquad \frac{d}{dx}(2x+3) = 2$

$$\frac{d}{dx} \tan^{-1}(2x+3) = 2 \cdot \frac{1}{1+(2x+3)^2}$$

$$= \frac{2}{1+(2x)^2+12x+9} = \frac{2}{4x^2+12x+10}$$

72. $\frac{d}{dx}\left(\cot^4(2x)\right)$

$\frac{d}{dx}(2x) = 2$ & $\frac{d}{dx}\cot x =$

$= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$

$= \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$

$= -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$

$$\frac{d}{dx}\cot^4(2x) = 2 \cdot -\operatorname{cosec}^2 2x \cdot 4\cot^3(2x)$$

$$= -8\,\frac{1}{\sin^2 x} \cdot \frac{\cos^3 2x}{\sin^3 2x}$$

$$= -8\cot^3(2x)\operatorname{cosec}^2 2x$$

73. $\frac{d}{dx}\left(\frac{x^2}{1+\frac{1}{x}}\right)$

$$= \frac{d}{dx}\left(\frac{x^2 \cdot x}{x+1}\right) = \frac{d}{dx}\left(\frac{x^3}{x+1}\right)$$

$$= \frac{(x+1)3x^2 - x^3(1)}{(x+1)^2}$$

$$= \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \frac{2x^3 + 3x^2}{(x+1)^2}$$

74. $$\frac{d}{dx}\left(e^{\frac{x}{1+x^2}}\right)$$

$$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x)^2}$$

$$= \frac{1+x^2-2x^2}{(1+x)^2} = \frac{1-x^2}{(1+x)^2}$$

$$= \frac{(1+x)(1-x)}{(1+x)^2}$$

$$= \frac{1-x}{1+x}$$

$$\frac{d}{dx}\left(e^{\frac{x}{1+x^2}}\right) = \frac{1-x}{1+x}\, e^{\frac{x}{1+x^2}}$$

75 $\frac{d}{dx}\left(\left(\sin^{-1}(x)\right)^3\right)$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right)^3 = \frac{1}{\sqrt{1-x^2}} \cdot 3\left(\sin^{-1}(x)\right)^2$$

$$= \frac{3\left(\sin^{-1}(x)\right)^2}{\sqrt{1-x^2}}$$

76. $\frac{d}{dx}\left(\frac{1}{2}\sec^2(x) - \ln(\sec x)\right)$

$$\left(\frac{1}{2}\frac{d}{dx}(\sec^2 x)\right) - \left(\frac{d}{dx}\ln(\sec x)\right)$$

$$\frac{1}{2}\frac{d}{dx}(\sec^2 x) = \frac{1}{2}\cdot(\sec x \tan x)\cdot(2\sec x)$$

$$\frac{d}{dx}\ln(\sec x) = \sec x \tan x \cdot \frac{1}{\sec x}$$

$$= \sec^2 x \tan x - \frac{\sec x \tan x}{\sec x}$$

$$= \frac{\sec^3 x \tan x - \sec x \tan x}{\sec x}$$

$$= \sec^2 x \tan x - \tan x$$

$$= (\sec^2 x - 1) \tan x$$

$$= \tan^2 x \tan x$$

$$= \tan^3 x$$

77 $\frac{d}{dx}\left(\ln\left(\ln\left(\ln x\right)\right)\right)$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln(\ln x) = \frac{1}{x} \cdot \frac{1}{\ln(x)}$$

$$\frac{d}{dx} \ln(\ln(\ln x)) = \frac{1}{x} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{\ln(\ln(x))}$$

$$= \frac{1}{x \ln(x) \ln(\ln(x))}$$

78. $\frac{d}{dx}\left(\pi^3\right) = 0$

79. $\frac{d}{dx}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)$

$$\frac{d}{dx}\sqrt{1+x^2} = 2x \cdot \frac{1}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\left(x+\sqrt{1+x^2}\right) = 1 + \frac{x}{\sqrt{1+x^2}}$$

$$\frac{d}{dx}\ln\left(x+\sqrt{1+x^2}\right) = \left(1+\frac{x}{\sqrt{1+x^2}}\right)\left(\frac{1}{x+\sqrt{1+x^2}}\right)$$

$$= \frac{\left(\sqrt{1+x^2}+x\right)}{\sqrt{1+x^2}} \cdot \frac{1}{\left(x+\sqrt{1+x^2}\right)}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{x+\sqrt{1+x^2}} + \frac{x}{x\sqrt{1+x^2}+1+x^2}$$

$$= \frac{x\sqrt{1+x^2}+1+x^2+x^2+x\sqrt{1+x^2}}{\left(x+\sqrt{1+x^2}\right)\left(x\sqrt{1+x^2}+1+x^2\right)}$$

$$= \frac{1+2x^2+2x\sqrt{1+x^2}}{}$$

80) $\dfrac{d}{dx}\left(\sinh^{-1} x\right)$

$$\sinh^{-1} x = y$$
$$x = \sinh y$$
$$x = \frac{e^y - e^{-y}}{2}$$
$$2x = e^y - e^{-y}$$
$$2x = e^y - \frac{1}{e^y}$$
$$2xe^y = e^{2y} - 1$$
$$e^{2y} - 2xe^y - 1 = 0$$
$$e^y_{1,2} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$
$$e^y_{1,2} = \frac{2x \pm 2\sqrt{(x^2+1)}}{2}$$
$$e^y_{1,2} = \cancel{1}x \pm \sqrt{x^2+1}$$
$$e^y = \cancel{1}x + \sqrt{x^2+1}$$
$$\ln e^y = \ln\left(\cancel{1}x + \sqrt{x^2+1}\right)$$
$$y = \ln\left(\cancel{1}x + \sqrt{x^2+1}\right)$$
$$\sinh^{-1} x = \ln\left(\cancel{1}x + \sqrt{x^2+1}\right)$$

$$\frac{d}{dx} \ln\left(\cancel{1}x + \sqrt{x^2+1}\right)$$

inner to outer

$$= \frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$
$$= \frac{x}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\left(\cancel{1}x + \sqrt{x^2+1}\right)$$
$$= \frac{d}{dx}(\cancel{1}x) + \frac{d}{dx}\left(\sqrt{x^2+1}\right)$$
$$= 1\cancel{0} + \frac{x}{\sqrt{x^2+1}}$$
$$= \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}$$

$$\frac{d}{dx} \ln\left(x + \sqrt{x^2+1}\right)$$
$$= \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \cdot \frac{1}{x + \sqrt{x^2+1}}$$
$$\boxed{= \frac{1}{\sqrt{x^2+1}}}$$

81) $\frac{d}{dx}(e^x \sinh x)$ $\qquad \sinh x = \frac{e^x - e^{-x}}{2}$

$$e^x \frac{d}{d} \sinh x + \sinh x \frac{d}{dx} e^x$$

$$e^x \cdot \frac{1}{2} \frac{d}{dx}(e^x - e^{-x}) + \left(\frac{e^x - e^{-x}}{2}\right) \cdot e^x$$

$$= \frac{e^x}{2} \cdot (e^x - -1e^{-x}) + \frac{e^{2x} - e^{-x+x}}{2}$$

$$= \frac{e^x}{2}(e^x + e^{-x}) + \frac{e^{2x} - 1}{2}$$

$$= \frac{e^{2x} + e^0}{2} + \frac{e^{2x} - 1}{2}$$

$$= \frac{e^{2x} + e^{2x} + 1 - 1}{2}$$

$$= \frac{2e^{2x}}{2} = e^{2x}$$

82) $\frac{d}{dx}(\operatorname{sech}(\frac{1}{x}))$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\operatorname{sech}\left(\frac{1}{x}\right)\right) = -\frac{1}{x^2} \cdot -\operatorname{sech}\left(\frac{1}{x}\right)\tanh\left(\frac{1}{x}\right)$$

$$= \frac{\operatorname{sech} x \tanh x}{x^2}$$

83) $\frac{d}{dx}(\cosh(\ln(x)))$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}((\cosh(\ln x)) = \sinh(\ln x) \cdot \frac{1}{x}$$

$$= \frac{e^{\ln x} - e^{-\ln x}}{2} \cdot \frac{1}{x}$$

$$= \frac{x - \frac{1}{x}}{2} \cdot \frac{1}{x}$$

$$= \frac{x^2 - 1}{x} \cdot \frac{1}{2} \cdot \frac{1}{x}$$

$$= \frac{x^2 - 1}{2x^2}$$

84. $\frac{d}{dx}\left(\ln(\cosh x)\right)$

$$\frac{d}{dx} \cosh x = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{d}{dx} \ln(x) = \frac{1}{x}$$

$$\frac{d}{dx} \ln(\cosh x) = \sinh x \cdot \frac{1}{\cosh x}$$

$$= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{2} \cdot \frac{2}{e^x + e^{-x}}$$

$$= \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

85. $\frac{d}{dx}\left(\frac{\sinh x}{1+\cosh x}\right)$

$$\frac{1+\cosh x\,\frac{d}{dx}(\sinh x) - \sinh x\,\frac{d}{dx}(1+\cosh x)}{(1+\cosh x)^2}$$

$\frac{d}{dx}\sinh x = \cosh x$

$\frac{d}{dx}(1+\cosh x) = 0 + \sinh x = \sinh x$

$$\frac{(1+\cosh x)(\cosh x) - \sinh x(\sinh x)}{(1+\cosh x)^2}$$

$$= \frac{\cosh x + \underbrace{\cosh^2 x - \sinh^2 x}_{=1}}{(1+\cosh x)^2}$$

$$= \frac{\cosh x + 1}{(1+\cosh x)^2} = \frac{1}{1+\cosh x}$$

86) $\frac{d}{dx}\left(\tanh^{-1}(\cos x)\right)$

$\sin^2 x + \cos^2 x = 1$ $\qquad$ $\cosh^2 x + \sinh^2 x = 1$

$\sec^2 x - \tan^2 x = 1$ $\qquad$ $\tanh^2 x + \operatorname{sech}^2 x = 1$

$\operatorname{cosec}^2 x - \cot^2 x = 1$ $\qquad$ $\operatorname{cosech}^2 x - \coth^2 x = 1$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tanh^{-1}(x) =$

$\tanh^{-1}(x) = y$

$x = \tanh(y)$

$x = \frac{\sinh y}{\cosh y}$

$x = \frac{e^{y} - e^{-y}}{2} \cdot \frac{2}{e^{y} + e^{-y}}$

$x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$

$xe^{y} + xe^{-y} = e^{y} - e^{-y}$

$$xe^{-y} + e^{-y} = e^{y} - xe^{y}$$

$$e^{-y}(x+1) = e^{y}(1-x)$$

$$x+1 = e^{2y}(1-x)$$

$$e^{2y} = \frac{x+1}{1-x}$$

$$\ln e^{2y} = \ln \frac{x+1}{1-x}$$

$$2y = \ln \frac{x+1}{1-x}$$

$$y = \frac{1}{2}\left(\ln \frac{x+1}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\left(\ln \frac{x+1}{1-x}\right)$$

$$\frac{d}{dx}\left(\tanh^{-1}(x)\right)$$

$$= \frac{1}{2} \frac{d}{dx}\left(\ln \frac{1+x}{1-x}\right)$$

$$= \frac{d}{dx}\left(\frac{1+x}{1-x}\right) = \frac{1-x(1) - (1+x)(-1)}{(1-x)^2}$$

$$= \frac{1-x+1+x}{(1-x)^2} = \frac{\cancel{x}\,2}{(1-x)^2}$$

$$= \frac{1}{2} \cdot \frac{1}{\frac{1+x}{1-x}} \cdot \frac{\cancel{1}\;2}{(1-x)^2}$$

$$= \frac{1}{2} \, \frac{1-x}{1+x} \cdot \frac{\cancel{1}\;2}{(1-x)^2} = \frac{\cancel{1}\;2}{2(1-x^2)}$$

$$\therefore \tanh^{-1}(\cos x)$$

$$= \frac{\cancel{1}\;2}{2(1-\cos^2 x)} \cdot -\sin x$$

$$= \frac{-\sin x \cdot 2}{2(1-\cos^2 x)} = -\frac{\sin x \cdot 2}{2\sin^2 x}$$

$$= -\frac{\operatorname{cosec} x \cdot 2}{2}$$

$$= -\operatorname{cosec} x$$

87) $\frac{d}{dx}\left(x \tanh^{-1} x + \ln\sqrt{1-x^2} \right)$

$$\frac{d}{dx}\left(x \tanh^{-1} x\right) + \frac{d}{dx}\left(\ln\sqrt{1-x^2}\right)$$

1 2

1

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$$

$$\frac{d}{dx}\left(x \tanh^{-1}(x)\right) = x\,\frac{1}{1-x^2} + \tanh^{-1}(x)\cdot 1$$

$$= \frac{x + (1-x^2)\tanh^{-1}(x)}{1-x^2}$$

2

$$\frac{d}{dx}\left(\ln\sqrt{1-x^2}\right)$$

$$\frac{d}{dx}(1-x^2) = 0 - 2x = -2x$$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$$

$$\frac{d}{dx}\ln\sqrt{1-x^2} = \frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$$

$$= \frac{1}{2(1-x^2)}\cdot(-2x) = \frac{-x}{1-x^2}$$

① + ②

$$\frac{x}{\cancel{1-x^2}} + \frac{\cancel{(1-x^2)} \tanh^{-1}(x)}{\cancel{1-x^2}} - \frac{x}{\cancel{1-x^2}}$$

$$= \tanh^{-1}(x)$$

88) $\frac{d}{dx}\left(\sinh^{-1}(\tan x)\right)$

sec	sec	tan
cosec	-cosec	cot

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sinh^{-1}(x)) = \frac{1}{\cosh(x)} = \frac{1}{\sqrt{1+x^2}}$$

$\cosh^2 x - \sinh^2 x = 1$
(logic trick!)

$$\frac{d}{dx}\left(\sinh^{-1}(\tan x)\right) = \sec^2 x \cdot \frac{1}{\sqrt{1+\tan^2 x}}$$

$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$

$$= \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \sec x$$

89) $\frac{d}{dx}\left(\sin^{-1}(\tan h x)\right)$

$\sec \to \sec \Leftarrow \tan$

$\sin = \frac{1}{\sqrt{1-x^2}}$

$\tan = \frac{1}{1+x}$

$\sec = \frac{1}{|x|\sqrt{x^2-1}}$

$= \frac{d}{dx}(\tan hx) = \operatorname{sech}^2 x$

$= \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

$= \frac{d}{dx}\sin^{-1}(\tan hx) = \operatorname{sech}^2 x \cdot \frac{1}{\sqrt{1-\tanh^2 x}}$

$\cosh^2 x - \sinh^2 x = 1$

$\div$ by $\cosh^2 x$ $\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$

$= \operatorname{sech}^2 x \cdot \frac{1}{\sqrt{\operatorname{sech}^2 x}}$

$= \operatorname{sech} x$

90) $\frac{d}{dx}\left(\frac{\tan h^{-1} x}{1-x^2}\right)$

$$\frac{(1-x^2)\frac{d}{dx}(\tan h^{-1}(x)) - \tan h^{-1} x \frac{d}{dx}(1-x^2)}{(1-x^2)^2}$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{\tanh(x)} = \frac{1}{\operatorname{sech}^2 x} = \frac{1}{1-x^2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(1-x^2) = -2x$$

$$\frac{(1-x^2)\cdot\frac{1}{1-x^2} - \tanh^{-1}(x)\cdot(-2x)}{(1-x^2)^2}$$

$$= \frac{1 + 2x\tanh^{-1}(x)}{(1-x^2)^2}$$

Use the definition of derivatives to solve the following:

Q1) $\frac{d}{dx}(x^3)$

$$\lim_{\cancel{\Delta}h \to 0} \frac{f(x+h) - f(x)}{\cancel{\Delta}h}$$

$$= \lim_{\cancel{\Delta}h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{\cancel{\Delta}h \to 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h}$$

$$= \lim_{\cancel{\Delta}h \to 0} \frac{h^3 + 3x^2h + 3xh^2}{h}$$

$$= \lim_{\cancel{\Delta}h \to 0} h^2 + 3x^2 + 3xh$$

$$= 3x^2$$

92) $\frac{d}{dx}\left(\sqrt{3x+1}\right)$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3x+3h+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3x+3h+1} + \sqrt{3x+1}}{\sqrt{3x+3h+1} + \sqrt{3x+1}}$$

$$= \lim_{h\to 0} \frac{3x+3h+1 - (3x+1)}{h\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{3x} + 3\cancel{h} \cancel{+1} - \cancel{3x} \cancel{-1}}{\cancel{h}\left(\sqrt{3x+3h+1} + \sqrt{3x+1}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h+1} + \sqrt{3x+1}}$$

$$= \frac{3}{\sqrt{3x+1} + \sqrt{3x+1}} = \frac{3}{2\sqrt{3x+1}}$$

93) $\frac{d}{dx}\left[\frac{1}{2x+5}\right]$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\frac{1}{2(x+h)+5}-\frac{1}{2x+5}}{h}$$

$$\lim_{h\to 0}\frac{2x+5-[2x+2h+5]}{(2x+5)[2x+2h+5]}\cdot\frac{1}{h}$$

$$\lim_{h\to 0}\frac{\cancel{2x}+\cancel{5}-\cancel{2x}-2h-\cancel{5}}{(2x+5)(h)(2x+2h+5)}$$

$$\lim_{h\to 0}\frac{-2h}{(2x+5)h(2x+2h+5)}$$

$$=\lim_{h\to 0}\frac{-2}{(2x+5)(2x+2h+5)}=\frac{-2}{(2x+5)(2x+5)}$$

$$=\frac{-2}{(2x+5)^2}$$

94) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)$

$$= \lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h^{\cancel{2}}}$$

$$\lim_{h\to 0} \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \cdot \frac{1}{h^{\cancel{2}}}$$

$$\lim_{h\to 0} \frac{x^2 - (x+h)^2}{x^2 h^{\cancel{2}} (x+h)^2}$$

$$= \lim_{h\to 0} \frac{x^2 - [x^2 + h^2 + 2xh]}{x^2 h^{\cancel{2}} (x+h)^2}$$

$$= \lim_{h\to 0} \frac{x^2 - x^2 - h^2 - 2xh}{x^2 h^{\cancel{2}} (x+h)^2}$$

$$= \lim_{h\to 0} \frac{-h - 2x}{x^2 \cancel{h}\,[x^2 + h^2 + 2xh]}$$

$$= \lim_{h\to 0} \frac{-2x - h}{x^4 \cancel{h} + x^2 h^{\cancel{3}\,2} + 2x^3 h^{\cancel{2}}}$$

$$= \cancel{\lim_{h\to 0}} \frac{-2x}{x^4} = \frac{-2}{x^3}$$

95) $\frac{d}{dx}(\sin x)$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\sin x(\cos h - 1)}{h}}_{①} + \underbrace{\frac{\cos x \sin h}{h}}_{②}$$

Final answer is $\cos x$

$\therefore$ ① goes to 0 when limit applied

& ② $\frac{\sin h}{h}$ limit is 1

$$\therefore = 0 + \lim_{h \to 0} \cos x \frac{\sin h}{h}$$

$$= 0 + \cos x \cdot 1$$

$$= \cos x$$

96) $\dfrac{d}{dx}(\sec x)$

$$\lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$\lim_{h \to 0} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \cdot \frac{1}{h}$$

$$\lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \cdot \cos x \, \cos(x+h)}$$

$$\cancel{\lim_{h \to 0} \frac{\cos x}{h \cdot \cos x \cdot \cos(x+h)} - \lim_{h \to 0} \frac{\cos(x+h)}{h \cos x \, \cos(x+h)}}$$

$$\cancel{= \lim_{h \to 0} \frac{1}{h \cos(x+h)} - \lim_{h \to 0} \frac{1}{h \cos x}}$$

$$\cancel{= \lim \frac{1}{h(\cos x \cosh - \sin x \sin h)}}$$

$$\lim_{h \to 0} \frac{1}{h \cos x \cos(x+h)} \cdot \left[\cos x - \cos(x+h) \right]$$

$$\cos x - [\cos x \cos h - \sin x \sin h]$$

$$\cos x - \cos x \cos h + \sin x \sin h$$

$$\lim_{h \to 0} \frac{1}{\cos x \cos(x+h)} \cdot \frac{[\cos x (1-\cos h) + \sin x \sin h]}{h}$$

$$= \frac{1}{\cos x \cos x} \cdot \sin x = \sec x \tan x$$

97. $\frac{d}{dx}(\sin^{-1} x)$

$$= \lim_{h \to 0} \frac{\overbrace{\sin^{-1}(x+h)}^{a} - \overbrace{\sin^{-1}(x)}^{b}}{h}$$

A complex tool to remember but not frequently used:

$$\sin^{-1}(\sin(a-b)) = \sin^{-1}(\sin a \cos b - \cos a \sin b)$$

$$\underbrace{a}_{\sin^{-1}(x+h)} - \underbrace{b}_{\sin^{-1}(x)} = \sin^{-1}\left(\sin a \sqrt{1-\sin^2 b} - \sqrt{1-\sin^2 a}\, \sin b\right)$$

$\sin^2 b + \cos^2 b = 1$

$\sin^2 a + \cos^2 a = 1$

$$\left.\begin{aligned} &\Rightarrow \sin(\sin^{-1}(x+h)) \sqrt{1-\sin^2(x)} \\ &\Rightarrow x+h \sqrt{1-(x)^2} \end{aligned}\right\} \quad (1)$$

Similarly

$$\Rightarrow \sqrt{1-\sin^2 \sin^{-1}(x+h)} \cdot \sin \sin^{-1} x$$

$$= \sqrt{1-(\sin \sin^{-1}(x+h))^2} \cdot \sin \cdot \sin^{-1}(x)$$

$$= \sqrt{1-(x+h)^2} \cdot (x)$$

$$= x\sqrt{1-(x+h)^2}$$

$$\therefore a-b = \sin^{-1}\left((x+h)\left(\sqrt{1-x^2}\right) - x\sqrt{1-(x+h)^2}\right)$$

we can remove $\sin^{-1}(\quad)$ based on the $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\lim_{\theta \to 0} \sin\theta = \lim_{\theta \to 0} \theta$$

$$\lim_{\theta \to 0} \theta = \lim_{\theta \to 0} \sin^{-1}(\theta)$$

$$\lim_{h \to 0} \frac{(x+h)\left(\sqrt{1-x^2}\right) - x\sqrt{1-(x+h)^2}}{h}$$

multiply by conjugate

divide

$$(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}$$

$$\lim_{h \to 0} \frac{(x+h)^2 \left(\sqrt{1-x^2}\right)^2 - \left[x^2 \left(1-(x+h)^2\right]\right.}{h(x+h)\left(\sqrt{1-x^2}\right)}$$

$$\lim_{h \to 0} \frac{(x+h)^2 (1-x^2) - x^2 + x^2 (x+h)^2}{h\left[(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}\right]}$$

$$\lim_{h \to 0} \frac{(x+h)^2 - \cancel{x^2 (x+h)^2} - x^2 + \cancel{x^2 (x+h)^2}}{h\left[(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}\right]}$$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h\left[(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}\right]}$$

$$\lim_{h \to 0} \frac{x^2 + h^2 + 2xh - x^2}{h\left[(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}\right]}$$

$$\lim_{h \to 0} \frac{h(h+2x)}{h\left[(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}\right]}$$

$$\lim_{h \to 0} \frac{h+2x}{(x+h)\left(\sqrt{1-x^2}\right) + x\sqrt{1-(x+h)^2}}$$

$$= \frac{2x}{x\sqrt{1-x^2} + x\sqrt{1-x^2}} = \frac{2x}{2x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

98- $\frac{d}{dx} \tan^{-1} x$

$$\qquad a \quad - \quad b$$

$$= \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h}$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

$$a - b = \tan^{-1}\left(\frac{\tan a - \tan b}{1 + \tan a \tan b}\right)$$

$$a = \tan^{-1}(x+h)$$
$$b = \tan^{-1}(x)$$

$$a - b = \tan^{-1}\left(\frac{\tan \tan^{-1}(x+h) - \tan \tan^{-1}(x)}{1 + \tan \tan^{-1}(x+h) \tan \tan^{-1}(x)}\right)$$

$$= \tan^{-1}\left(\frac{x+h - x}{1 + (x+h) \cdot x}\right)$$

Based on $\lim_{x \to 0} \frac{\tan\theta}{\theta} = 0$; $\tan^{-1}(\ )$ disappears.

$$\therefore \lim_{h \to 0} \frac{\frac{h}{1+x^2+h}}{h}$$

$$= \lim_{h \to 0} \frac{h}{1+x^2+h} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{1}{1+x^2+h} = \frac{1}{1+x^2}$$

99. $\frac{d}{dx}\left(f(x) \cdot g(x)\right)$

$$\lim_{x \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

Trick: add & subtract $f(x+h) \cdot g(x)$ on numerator

$$\lim_{x \to 0} \frac{f(x+h)\,g(x+h) + f(x+h)\,g(x) - f(x+h)\,g(x) - f(x) \cdot g(x)}{h}$$

$$\lim_{x \to 0} \frac{f(x+h)[g(x+h)+g(x)] - g(x)[f(x+h)+f(x)]}{h}$$

This would not group perfectly ∴ trying opposit sign grouping

$$\lim_{x \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$\lim_{x \to 0} \frac{f(x+h)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)]}{h}$$

$$\lim_{x \to 0} f(x+h)\frac{[g(x+h)-g(x)]}{h} + g(x)\frac{[f(x+h)-f(x)]}{h}$$

$$f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

$$= f(x)g'(x) + g(x)f'(x)$$

100) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$

$$\lim_{x \to 0} \left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] \cdot \frac{1}{h}$$

$$\lim_{x \to 0} \frac{f(x+h)\,g(x) - f(x)\,(g(x+h)}{g(x+h)\,g(x)\,h}$$

similar to last probleme add & subtract $f(x)\,g(x)$ this time.

$$\lim_{x \to 0} \frac{f(x+h)\,g(x) - f(x)g(x) - f(x)\,g(x+h) - f(x)g(x)}{g(x+h)\,g(x) \cdot h}$$

$$\lim_{x \to 0} \frac{1}{g(x+h)\,g(x)} \left[g(x)\,\frac{f(x+h) - f(x)}{h} - f(x)\,\frac{g(x+h) - g(x)}{h}\right]$$

$$= \frac{1}{(g(x))^2}\left[g(x)\,\frac{d\,f(x)}{dx} - f(x)\,\frac{d}{dx}g(x)\right]$$

$$= \frac{g(x)\,f'(x) - f(x)\,g'(x)}{(g(x))^2}$$