Complex Numbers

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Natural numbers : 0, 1, 2 .... $\mathbb{N}$

Integers : ... -2, -1, 0, 1, 2 .... $\mathbb{Z}$

Rational nos: $\frac{3}{10}$, 2.111 $\mathbb{Q}$

Irrational nos: e, $\sqrt{2}$, π $\mathbb{R}$

Complex nos: $2+3i$, $7-\sqrt{3}i$ $\mathbb{C}$

$z = a + ib$

$i = \sqrt{-1}$

polar form:

$z = r(\cos\theta + i\sin\theta)$

cartesian form:

$z = a + ib$

exponential form

$e^{i\theta} = \cos\theta + i\sin\theta$ $\quad r = \sqrt{a^2+b^2}$

$z = re^{i\theta}$

Exponential Form:

We know

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\therefore e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 + \frac{i\theta}{1} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!}$$

$$= \underbrace{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right]}_{\cos\theta} + i\underbrace{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right]}_{\sin\theta}$$

$$=$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Hyperbolic & trig functions

$$e^{i\theta} = \cos\theta + i\sin\theta \rightarrow (1)$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \rightarrow (2)$$

(1)+(2)

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

in trigonometry:

$$\cos hx = \frac{e^{x} + e^{-x}}{2} \qquad \sin hx = \frac{e^{x} - e^{-x}}{2}$$

$\therefore$

$$\cos h(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$$

$$\sin h(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta$$

Root Finding

$$z = re^{i\theta}$$

n^{th} power $\rightarrow$ $z^n = r^n (e^{i\theta})^n$

n^{th} root $\rightarrow$ $z^{1/n} = r^{1/n} (e^{i\theta})^{1/n}$

n n-th roots :

$$w_0 = r^{1/n} \cdot e^{i\theta/n}$$

$$w_1 = r^{1/n} \cdot e^{i\frac{(\theta+2\pi)}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\right)}$$

$$w_2 = r^{1/n} e^{i\frac{(\theta+2\cdot 2\pi)}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + 2\cdot\frac{2\pi}{n}\right)}$$

$\vdots$

$$w_n = r^{\frac{1}{n}} e^{i\left(\frac{\theta + n\cdot 2\pi}{n}\right)} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + k\cdot\frac{2\pi}{n}\right)}$$

where Argument is $= \frac{\theta}{n}$

periodicity $= \frac{2\pi}{n}$

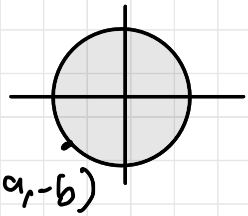
Solve $z^4 + 8\sqrt{3} + 8i = 0$

$$z^4 = -8\sqrt{3} - 8i$$

4th root of $z = -8\sqrt{3} - 8i$

$a = -8\sqrt{3}$

$b = -8$



3rd quadrant

$$r = \sqrt{a^2+b^2} = \sqrt{(-8\sqrt{3})^2 + 8^2}$$

$$= \sqrt{64(3) + 64}$$

$$= \sqrt{256} = 16$$

$\frac{1}{2} \cdot \frac{2}{\sqrt{3}}$

$$\tan\theta = \frac{-8}{-8\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \; ; \quad \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ = \frac{\pi}{6}$$

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

as we are in 3rd quadrant

polar form:

$$z = 16\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

4 n-th roots

$$w = r^{1/n} \cdot e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}\cdot k\right)} \qquad k = 0, 1, 2 \dots n-1$$

$$\text{Argument} = \frac{\theta}{n} = \frac{7\pi}{6} \cdot \frac{1}{4} = \frac{7\pi}{24} \; *$$

$$\text{periodicity} = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \; *$$

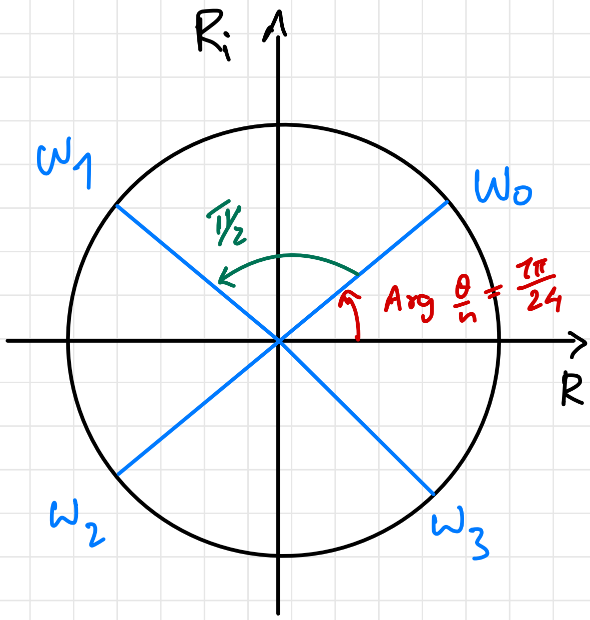
$$r^{1/n} = 16^{1/4} = 2^{4\cdot\frac{1}{4}} = 2$$

$$w_0 = 2e^{i\left(\frac{7\pi}{24} + 0\cdot\frac{2\pi}{4}\right)}$$

$$w_1 = 2e^{i\left(\frac{7\pi}{24} + 1\cdot\frac{2\pi}{4}\right)}$$

$$w_2 = 2e^{i\left(\frac{7\pi}{24} + 2\cdot\frac{2\pi}{4}\right)}$$

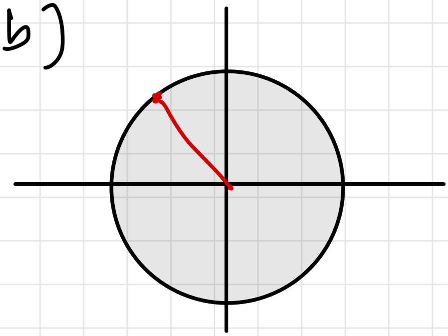
$$w_3 = 2e^{i\left(\frac{7\pi}{24} + 3\cdot\frac{2\pi}{4}\right)}$$



Find cube roots of $-\frac{27\sqrt{2}}{2} + \frac{27\sqrt{2}}{2} i$

$a = -\frac{27\sqrt{2}}{2}$

$(-a, b)$



$b = \frac{27\sqrt{2}}{2}$

$r = \sqrt{a^2 + b^2}$

$= \sqrt{\left(-\frac{27\sqrt{2}}{2}\right)^2 + \left(\frac{27\sqrt{2}}{2}\right)^2}$

$= \sqrt{\frac{27^2 \cdot 2}{4} + \frac{27^2 \cdot 2}{4}}$

$= \sqrt{\frac{27^2 + 27^2}{2}} = \sqrt{\frac{2(27)^2}{2}} = 27$

$\tan\theta = \frac{b}{a} = \frac{27\sqrt{2}}{2} \div -\frac{27\sqrt{2}}{2} = -1$

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$

$= 45^\circ = -\frac{\pi}{4}$

We have θ on 2nd quadrant

$\therefore \pi - \frac{\pi}{4}$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad \frac{\pi}{2} + \frac{\pi}{4} = \frac{6\pi}{8} = \frac{3\pi}{4}$$

$$w_k = r^{\frac{1}{n}} e^{i\theta\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$$ where $k = 0, 1, 2 \ldots n-1$

Here $n = 3$

$\therefore$ we find w_0, w_1, w_2

Also $r^{\frac{1}{n}} = 27^{\frac{1}{3}} = 3^{3 \cdot \frac{1}{3}} = 3$

$$\text{Argument} = \frac{\theta}{n} = \frac{3\pi}{4} \cdot \frac{1}{3} = \frac{3\pi}{12} = \frac{\pi}{4}$$

$$\text{periodicity} = \frac{2\pi}{n} = \frac{2\pi}{3}$$

$$w_0 = 3e^{i\left(\frac{\pi}{4} + 0 \cdot \frac{2\pi}{3}\right)} = 3e^{i\left(\frac{\pi}{4}\right)}$$

$$w_1 = 3e^{i\left(\frac{\pi}{4} + 1 \cdot \frac{2\pi}{3}\right)} = 3e^{i\left(\frac{19\pi}{12}\right)}$$

$$w_2 = 3e^{i\left(\frac{\pi}{4} + 2 \cdot \frac{2\pi}{3}\right)} = 3e^{\left(i\frac{19\pi}{12}\right)}$$

$$\frac{\pi}{4} + 3 \cdot \frac{2\pi}{3}$$

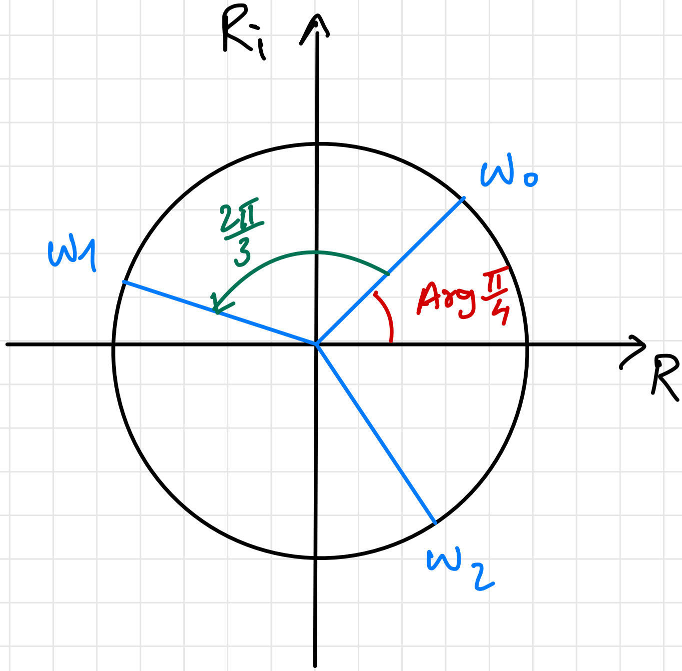
$$\frac{\pi}{4} + 2\pi \longrightarrow$$ comes back to starting point a full 360°

Plot:

we have

arg: $\frac{\pi}{4}$

periodicity: $\frac{2\pi}{3}$



$\frac{\pi}{4} = 45^\circ$

$\frac{2\pi}{3} = 2 \cdot \frac{180}{3} = 120^\circ$

45°

$45^\circ + 120^\circ = 165^\circ$

$165^\circ + 120^\circ = 285^\circ$

Use the following **complex numbers** to perform algebraic operations

$$z_{11} = 3 + 2i \qquad z_{12} = -2 + 3i \qquad z_{13} = 0 + 6i$$

$$z_{21} = 2 - 3i \qquad z_{22} = -1 - 4i \qquad z_{23} = 0 - 2i$$

$$z_{31} = 6 + 0i \qquad z_{32} = -8 + 0i \qquad z_{33} = 0 + 0i$$

Addition

a) $z_{11} + z_{22}$.

$(3+2i) + (-1-4i)$

$= 2 - 2i$

b) $z_{21} + z_{12}$

$(2-3i) + (-2+3i)$

$= 0$

c) $z_{31} + z_{13}$

$(6+0i) + (0+6i)$

$= 6 + 6i$

d) $z_{32} + z_{33}$

$(-8+0i) + (0+0i)$

$= -8$

e) $z_{12} + z_{22}$

$= (-2+3i) + (-1-4i)$

f) $\bar{z}_{33} + z_{13}$

$= (0-0i) + (0+6i)$

$= 6i$

g) $z_{23} + z_{12}$

$(0 - 2i) + (-2 + 3i)$

$= -2 + i$

h) $z_{32} + z_{31}$

$(-8 + 0i) + (6 + 0i)$

$= -2 + 0i = -2$

i) $\overline{z_{23}} + z_{31}$

$(-8 - 0i) + (6 + 0i)$

$-2 + 0i = -2$

Subtraction

a) $z_{12} - z_{22}$

$= (-2 + 3i) - (-1 - 4i)$

$= (-2 + 1) + (3i + 4i)$

$= -1 + 7i$

b) $z_{21} - z_{13}$

$= (2 - 3i) - (0 + 6i)$

$= (2 - 0) + (-3i - 6i)$

$= 2 - 3i$

c) $z_{31} - z_{13}$

$= (6 + 0i) - (0 + 6i)$

$= (6 - 0) + (0i - 6i)$

$= 6 - 6i$

d) $z_{32} - z_{33}$

$(-8 + 0i) - (0 + 0i)$

$= -8$

e) $-2\overline{z_{11}} - z_{22}$

$-2(3-2i) - (-1-4i)$

$(-6+4i) - (-1-4i)$

$= (-6 - -1) + (4i - -4i)$

$= (-6+1) + (4i+4i)$

$= -5+8i$

f) $z_{33} - z_{13}$

$(0+0i) - (0+6i)$

$(0) + (0i - 6i)$

$0 - 6i = -6i$

g) $-2z_{23} - 3z_{12}$

$-2(0-2i) - 3(-2+3i)$

$= (0+4i) - (-6+9i)$

$= (0+6) + (4i - 9i)$

$= 6-5i$

h) $z_{32} - z_{31}$

$(-8+0i) - (6+0i)$

$= (-8-6) + (0i - 0i)$

$= -14$

i) $z_{11} - 3z_{31}$

$(3+2i) - 3(6+0i)$

$= (3+2i) - (18+0i)$

$= (3-18) + (2i - 0i)$

$= -15+2i$

Multiplication

a) $z_{11} \cdot z_{22}$

$(3+2i) \cdot (-1-4i)$

$= -3 - 12i - 2i - 8i^2$

$= -3 + 8 - 14i$

$= 5 - 14i$

b) $z_{21} \cdot z_{12}$

$(2-3i)(-2+3i)$

$= -4 + 6i + 6i - 9i^2$

$= -4 + 9 + 12i$

$= 5 + 12i$

c) $z_{31} \cdot z_{13}$

$= (6+0i) \cdot (0+6i)$

$= 36i$

d) $\overline{z_{32}} \cdot z_{23}$

$(-8-0i)(0-2i)$

$16i$

e) $z_{12} \cdot z_{22}$

$(-2+3i)(-1-4i)$

$= 2 + 8i - 3i - 12i^2$

$= 2 + 12 + 5i$

$= 14 + 5i$

f) $z_{33} \cdot z_{13}$

$(0+0i) \cdot (0+6i)$

$= 0$

g) $z_{23} \cdot z_{12}$

$= (0-2i) \cdot (-2+3i)$

$= 0 + 0 + 4i - 6i^2$

$= 6 + 4i$

h) $\overline{z_{21}} \cdot z_{21}$

$= (2+3i)(2-3i)$

$= 4 - 6i + 6i - 9i^2$

$= 4 - 9i^2 = 4 + 9 = 13$

i) $\bar{z}_{22} \cdot z_{22}$

$(-1+4i)(-1-4i)$

$= (-1)^2 + 4i - 4i - 16i^2$

$= 1 - 16i^2 = 1 + 16 = 17$

Division

a) $\overline{z_{11}} / z_{22} = \dfrac{3-2i}{-1-4i}$

$\dfrac{(3-2i)(-1+4i)}{(-1)^2 - (4i)^2}$

$= \dfrac{-3 + 12i + 2i - 8i^2}{1 - 16i^2}$

$= \dfrac{-3 + 14i + 8}{1 + 16}$

$= \dfrac{5 + 14i}{17}$

$= \dfrac{5}{17} + \dfrac{14}{17} i$

b) $z_{21} / z_{12} = \dfrac{2-3i}{-2+3i}$

$\dfrac{(2-3i)(-2-3i)}{(-2)^2 - (9i^2)}$

$= \dfrac{-4 - 6i + 6i + 9i^2}{4 + 9}$

$= \dfrac{-4-9}{4+9} = \dfrac{-13}{13} = -1$

c) $z_{31} / z_{13} = \dfrac{6+0i}{0+6i}$

$= \dfrac{(6+0i)(0-6i)}{0^2 - 36i^2}$

$= \dfrac{0 - 36i + 0 - 0}{36}$

$= -i$

d) $z_{32} / z_{21} = \dfrac{-8+0i}{2-3i}$

$\dfrac{(-8+0i)(2+3i)}{4 - 9i^2}$

$\dfrac{-16 - 24i + 0 + 0}{4 + 9}$

$= \dfrac{-16}{13} - \dfrac{24}{13} i$

e) $z_{12} / z_{22} = \frac{-2+3i}{-1-4i}$

$$\frac{(-2+3i)(-1+4i)}{(-1-4i)(-1+4i)}$$

$$= \frac{2-8i-3i+12i^2}{(-1)^2-16i^2}$$

$$= \frac{2-12-11i}{1+16}$$

$$= -\frac{10}{17} - \frac{11}{17}i$$

f) $z_{33} / z_{13} = \frac{0+0i}{6+0i}$

$$= \frac{0}{6} = 0$$

Mixed

a) $z_{11} \cdot \frac{z_{13}}{z_{12}}$

b) $\frac{z_{21}}{z_{12}} \cdot z_{22}$

$$z_{11} \cdot \frac{z_{13}}{z_{12}} = (3+2i)\,\frac{(0+6i)}{(-2+3i)}$$

$$i = \sqrt{-1}$$

$$i^2 = -1$$

$$(3+2i)\,\frac{6i\,(-2-3i)}{(-2+3i)\,(-2-3i)}$$

$$\frac{(3+2i)\,(-12i-18i^2)}{4+6i-6i-9i^2}$$

$$\frac{(3+2i)\,(18-12i)}{4+9}$$

$$\frac{54-36i+36i-24i^2}{13}$$

$$= \frac{54+24}{13} = \frac{78}{13} = 6$$

$$\cancel{(3+2i)\,\frac{(0+6i)\,(-2-3i)}{(-2+3i)\,(-2-3i)}}$$

$$\cancel{(3+2i)\,\frac{(0-18i^2)}{4-9i^2}}$$

$$\cancel{(3+2i)\,\frac{18}{4+9} = \frac{18}{13}\,(3+2i)}$$

$$\cancel{= \frac{54}{13} + \frac{36}{13}\,i}$$

c) $z_{31} + \dfrac{z_{21}}{z_{12} \cdot z_{23}}$

$(6+0i) + \dfrac{2-3i}{(-2+3i)(0-2i)}$

$= (6+0i) + \dfrac{2-3i}{0+4i+0-6i^2}$

$= (6+0i) + \dfrac{2-3i}{4i-6i^2}$

$= (6+0i) + \dfrac{2-3i}{4i+6}$

$= 6 + \dfrac{2-3i}{6+4i}$

$= 6 + \dfrac{(2-3i)(6-4i)}{36+16}$

$= 6 + \dfrac{(2)(6) + (2)(-4i) + (-3i)(6) + (-3i)(-4i)}{52}$

$= 6 + \dfrac{12 - 8i - 18i - 12}{52}$

$= 6 - \dfrac{26i}{52} = 6 - \dfrac{1}{2}i$

d) $\overline{z_{22}} \cdot z_{21} + \dfrac{z_{11}}{z_{23}}$

$$\underbrace{(-1+4i)(2-3i)}_{1} + \underbrace{\frac{3+2i}{0-2i}}_{2}$$

1

$$= (-1)(2) + (-1)(-3i) + (4i)(2) + (4i)(-3i)$$
$$= -2 + 3i + 8i - 12i^2$$
$$= -2 + 12 + 11i$$
$$= 10 + 11i$$

2 $\dfrac{(3+2i)(0+2i)}{(0-2i)(0+2i)}$

$$\frac{3(0) + 3(2i) + 2i(0) + 2i(2i)}{0^2 - (2i)^2}$$

$$= \frac{0 + 6i + 0 - 4}{4} = \frac{6i-4}{4} = \frac{-4+6i}{4} = -1 + \frac{6}{4}i$$

1 + 2 $\Rightarrow$ $(10 + 11i) + (-1 + \frac{6}{4}i)$

$$= 9 + \left(11 + \frac{6}{4}\right)i$$
$$= 9 + \frac{50}{4}i$$
$$= 9 + \frac{25}{2}i$$

(e) $\dfrac{z_{21}}{z_{12}} \cdot \overline{z_{12}}$

$$\frac{2-3i}{-2+3i} \cdot (-2-3i)$$

① ②

① $\dfrac{(2-3i)(-2-3i)}{(-2)^2-(3i)^2}$

$$= \frac{(2)(-2)+(2)(-3i)+(-3i)(-2)+(-3i)(-3i)}{4+9}$$

$$= \frac{-4-6i+6i-9}{4+9}$$

$$= \frac{-4-9}{4+9} = -1$$

①-② $= -1(-2-3i) = 2+3i$

(f) $Z_{22} \cdot \dfrac{Z_{24}}{Z_{13}} \left(\overline{Z_{22}} - Z_{31} \cdot \overline{Z_{22}} \right)$:49 :57

$$(-1-4i) \cdot \frac{(2-3i)}{(0+6i)} \left(-1-4i - (6+0i)(-1-4i) \right)$$

$$6(-1-4i) = -6-24i$$

$$(-1-4i) - (-6-24i)$$
$$= (-1+6) + (-4i-24i)$$
$$= 5-28i$$

$$\frac{2-3i}{(0+6i)} \frac{(0-6i)}{(0-6i)} = \frac{-12i+18i^2}{0-36i^2} = \frac{-12i-18}{36}$$

$$= -\frac{18}{36} - \frac{12}{36} i$$

$$= -2 - \frac{1}{3} i$$

$$(-1-4i)\left(-2-\frac{1}{3}i\right)(5-28i)$$

$$\left[(-1)(-2) + (-1)\left(-\frac{1}{3}i\right) + (-4i)(-2) + (-4i)\left(-\frac{1}{3}i\right)\right](5-28i)$$

$$= \left[2 + \frac{1}{3}i + 8i - \frac{4}{3}\right](5-28i)$$

$$= \left[2 - \frac{4}{3} + \left(\frac{1}{3} + 8\right)i\right](5-28i)$$

$$= \left(\frac{2}{3} + \frac{25}{3}i\right)(5-28i)$$

$$= \left(\frac{2}{3}\right)(5) + \left(\frac{2}{3}\right)(-28i) + \left(\frac{25}{3}i\right)(5) + \left(\frac{25}{3}i\right)(-28i)$$

$$= \frac{10}{3} - \frac{56}{3}i + \frac{125i}{3} + \frac{700}{3}$$

$$= \frac{710}{3} - \frac{61}{3}i$$

6. **Converting Form**
Convert these complex numbers into polar (cos + i.sin), exponential (e^i.theta) and phasor (r angle theta) forms

(a) z_{11} (c) z_{21} (e) z_{13} (g) z_{33}
(b) z_{22} (d) z_{12} (f) z_{23}

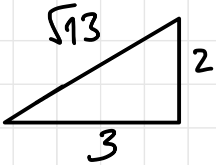
a) $Z_{11} = 3 + 2i$

$a = 3$

$b = 2$

$r = \sqrt{3^2 + 2^2}$
$= \sqrt{9+4} = \sqrt{13}$

$\tan\theta = \frac{2}{3}$



$\theta = \tan^{-1}\left(\frac{2}{3}\right)$ using calculator: $\theta = 33.69^{\circ}$
$= 33 \cdot \frac{\pi}{180}$

polar form

$z = \sqrt{13}\left(\cos\frac{33\pi}{180} + i\sin\frac{33\pi}{180}\right) =$

exponential

$z = \sqrt{13}\, e^{i\frac{33\pi}{180}}$

phasor

$z = \sqrt{13} \angle \frac{33\pi}{180}$

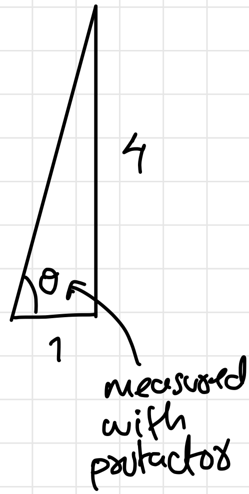
b) $z_{22} = -1 - 4i$

$r = \sqrt{1 + 16} = \sqrt{17}$

$\tan\theta = \frac{-4}{-1} = 4$

$\theta = \tan^{-1}\left(\frac{4}{1}\right) \approx 74^{\circ}$

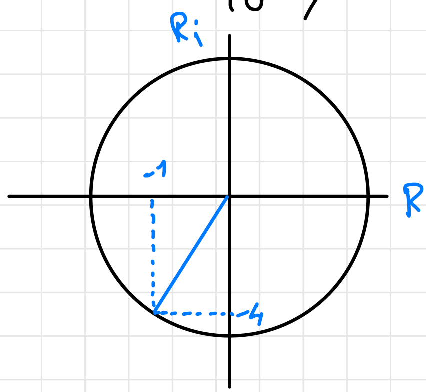
$= \frac{74\pi}{180} = \frac{37\pi}{90}$



polar $= \sqrt{17}\left(\cos\frac{37\pi}{90} + i\sin\frac{37\pi}{90}\right)$

expo: $\sqrt{17}\, e^{i\frac{37\pi}{90}}$

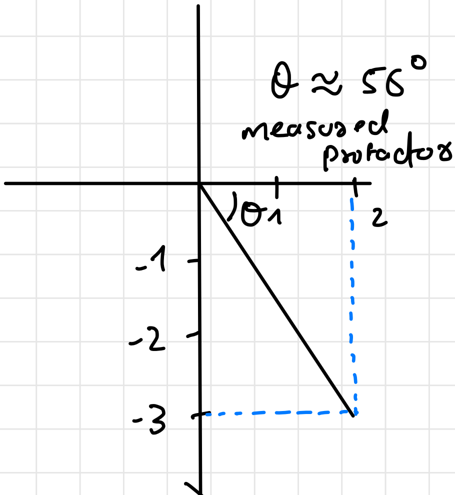
phasor: $\sqrt{17} \angle \frac{37\pi}{90}$



c) $z_{21} = 2 - 3i$

$r = \sqrt{4 + 9} = \sqrt{13}$

$\tan \theta = \frac{-3}{2}$



polar form

$z = \sqrt{13}\left(\cos \theta + i \sin \theta \right)$

exponential form

$z = \sqrt{13}\, e^{i\theta}$

phasor form

$z = \sqrt{13} \angle \theta$

where:

$\theta = \tan^{-1}\left(\frac{-3}{2} \right)$

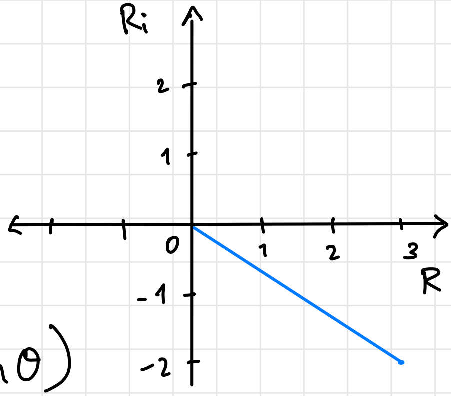
$\cos \theta = \frac{2}{\sqrt{13}}$ $\quad \sin \theta = \frac{-3}{\sqrt{13}}$

$\theta \approx -38^\circ = -38 \cdot \frac{\pi}{180}$

d) $z_{12} = -2 + 3i$

$r = \sqrt{4 + 9} = \sqrt{13}$

$\tan\theta = \frac{3}{-2}$



$z = \sqrt{13}\,(\cos\theta + i\sin\theta)$

exponential

$z = \sqrt{13}\,e^{i\theta}$

where:

$\theta = \tan^{-1}\left(-\frac{3}{2}\right)$

phasor

$\sin\theta = \frac{-2}{\sqrt{13}}$ $\cos\theta = \frac{3}{\sqrt{13}}$

$z = \sqrt{13}\angle\theta$

e) $z_{13} = 0 + 6i$

$r = \sqrt{0 + 36} = 6$

$\theta = \tan^{-1}\left(\frac{6}{0}\right) = \frac{\pi}{2}$

polar form

$= z = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

expo:

$z = 6e^{i\frac{\pi}{2}}$

phasor

$z = 6\angle\frac{\pi}{2}$

f) $z_{23} = 0 - 2i$

$r = \sqrt{0+4} = 2$

$\theta = \tan^{-1}\left(\frac{-2}{0}\right) = \frac{\pi}{2}$

polar $= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

expo: $e^{i\frac{\pi}{2}}$

phasor: $2\angle\frac{\pi}{2}$

g) $z_{33} = 0 + 0i$

$= 0$

7. **Operations using Polar Form**

Use answers from question above and perform the following operations using exponential form only

(a) $\overline{z_{11}} \cdot \frac{z_{13}}{z_{12}}$

(b) $\frac{z_{21}}{z_{12}} \cdot z_{22}$

(c) $\frac{z_{21}}{z_{12}} \cdot \overline{z_{12}}$

(d) $z_{22} \cdot \frac{z_{21}}{z_{13}} \left(\overline{z_{22}} - z_{13} \cdot \overline{z_{22}}\right)$

a) $\overline{z}_{11} \cdot z_{13}$

$z_{11} = 3 + 2i$

$\overline{z}_{11} = 3 - 2i$

$\theta = \tan^{-1}\left(-\frac{2}{3}\right)$

$= -\frac{33\pi}{180}$

$$= \sqrt{13}\, e^{i \frac{-33\pi}{180}}$$

$$z_{13} = 6 e^{i\frac{\pi}{2}}$$

$$\bar{z}_{11} \cdot z_{13} = \sqrt{13} e^{\frac{-33\pi}{180}} \cdot 6 e^{\frac{\pi}{2}}$$

$$= 6\sqrt{13}\; e^{\frac{-33\pi}{180} + \frac{\pi}{2}}$$

$$= 6\sqrt{13}\; e^{\frac{(-66+180)\pi}{360}}$$

$$= 6\sqrt{13}\, e^{\frac{14\pi}{360}}$$

b) $\dfrac{z_{21}}{z_{12}} \cdot z_{22}$

$$\frac{c}{d} \cdot b$$

$$\frac{\sqrt{13} e^{\frac{-38\pi}{180}}}{\sqrt{13} e^{\frac{-38\pi}{180}}} \cdot \sqrt{17}\, e^{i\frac{37\pi}{90}} = 17\, e^{i\frac{37\pi}{90}}$$

c) $\frac{z_{21}}{z_{12}} \cdot \bar{z}_{12}$

$$\frac{c}{d} \cdot \bar{d}$$

$$\frac{\sqrt{13}\, e^{i\frac{-38\pi}{180}}}{\sqrt{13}\, e^{i\frac{-38\pi}{180}}} \cdot \sqrt{13}\, e^{i\frac{38\pi}{180}} = \sqrt{13}\, e^{i\frac{38\pi}{180}}$$

d) $z_{22} \cdot \frac{z_{21}}{z_{13}} \left(\bar{z}_{22} - z_{13} \cdot \bar{z}_{22}\right)$

$$b \cdot \frac{c}{e} \left(\bar{b} - e \cdot \bar{b}\right)$$

$$\sqrt{17}\, e^{i\frac{37\pi}{40}} \cdot \frac{\sqrt{13} \cdot e^{i\frac{-56\pi}{180}}}{6\, e^{i\frac{\pi}{2}}} \left(\sqrt{13}\, e^{\frac{-37\pi i}{180}} - 6\, e^{i\frac{\pi}{2}} \cdot \sqrt{13}\, e^{\frac{-37\pi i}{180}}\right)$$

$$\frac{\sqrt{17} \cdot \sqrt{13}}{6} \cdot e^{i\left(\frac{37\pi}{40} - \frac{56\pi}{180} - \frac{\pi}{2}\right)} \cdot \left(\sqrt{13}\, e^{\frac{-37\pi i}{180}} - 6\sqrt{13}\, e^{i\left(\frac{\pi}{2} + \frac{-37\pi}{180}\right)}\right)$$

8. **Convert to Cartesian Form**
Convert the following complex numbers into Cartesian form. Hint: Draw Diagrams

(a) $2e^{i\pi/4}$ (b) $2\angle\pi/4$ (c) $-1.5e^{-i\pi/3}$

9. **Finding Roots**

(a) $\sqrt[4]{1}$
(b) $\sqrt[3]{i}$
(c) $\sqrt[5]{3+4i}$

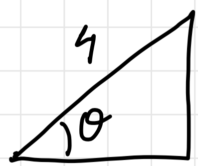
8) a) $z = 2e^{i\frac{\pi}{4}}$

$z = re^{i\theta} \Rightarrow$

$r = 2$

$\theta = \frac{\pi}{4} \times \frac{180}{\pi} = 45^\circ$

$\tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$



$\therefore \quad z = \sqrt{2} + \sqrt{2}i$

b) $2\angle\frac{\pi}{4}$

$r = 2 \qquad \theta = \frac{\pi}{4}$

$z = 2 + \sqrt{2}i$

c) $-1.5\, e^{-i\pi/3}$

$r = -1.5$

$\theta = -\frac{\pi}{3} = -\frac{\pi}{3} \times \frac{180}{\pi} = -60^\circ$

$\frac{\sqrt{3}}{2} \cdot \frac{2}{1}$

$\tan 60 = \frac{\sqrt{3}}{1}$

$z = (\sqrt{3} + 1i) \cdot (-1)$

$z = -\sqrt{3} - 1i$

9.1) Finding roots 82

$\sqrt[4]{1}$

$z = 1 + 0i$

$a = 1$

$b = 0$

$r = \sqrt{a^2 + b^2} = \sqrt{1} = 1$

$\tan\theta = \frac{b}{a}$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$n = 4 \qquad k = 0, 1, 2, 3$

$$w_k = r^{1/n} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$$

$$\text{Argument} = \frac{\theta}{n} = \frac{0}{4} = 0$$

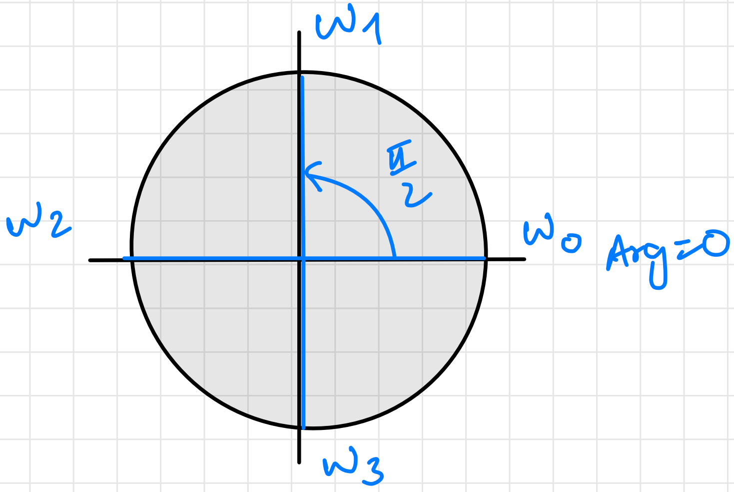
$$\text{periodicity} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$w_0 = 1^{\frac{1}{4}} e^{i(0)} = 1$$

$$w_1 = 1^{\frac{1}{4}} e^{i\left(0 + \frac{2\pi}{4}\cdot 1\right)} = 1 e^{i\pi/2}$$

$$w_2 = 1^{1/4} e^{i\left(0 + \frac{2\pi}{4}\cdot 2\right)} = 1 \cdot e^{i\pi}$$

$$w_3 = 1^{\frac{1}{4}} e^{i\left(0 + \frac{2\pi}{4}\cdot 3\right)} = 1 \cdot e^{i\frac{3\pi}{2}}$$



b) $\sqrt[3]{i}$ $\quad n=3$

$z = 0 + 1i$

$a = 0 \qquad r = \sqrt{0^2+1^2} = 1$

$b = 1 \qquad \tan\theta = \frac{1}{0}$

$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

$n = 3 \;;\; k = 0, 1, 2$

$w_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$

$\cancel{90}\ 45$

$\cancel{180}$

$\overline{\cancel{8}\,4\,2}$

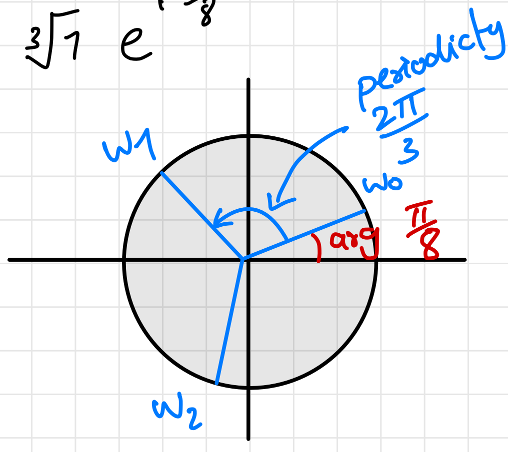
Argument : $\frac{\theta}{n} = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8}$

periodicity : $\frac{2\pi}{n} = \frac{2\pi}{3}$ or 120°

$w_0 = 1^{\frac{1}{3}} e^{i\left(\frac{\pi}{8} + \frac{\pi}{2}\cdot 0\right)} = \sqrt[3]{1}\, e^{i\frac{\pi}{8}}$

$w_1 = 1^{1/3} e^{i\left(\frac{\pi}{8} + \frac{\pi}{2}\cdot 1\right)} = \sqrt[3]{1}\, e^{i\frac{10\pi}{16}}$

$w_2 = 1^{1/3} e^{i\left(\frac{\pi}{8} + \frac{\pi}{2}\cdot 2\right)} = \sqrt[3]{1}\, e^{i\frac{9\pi}{8}}$

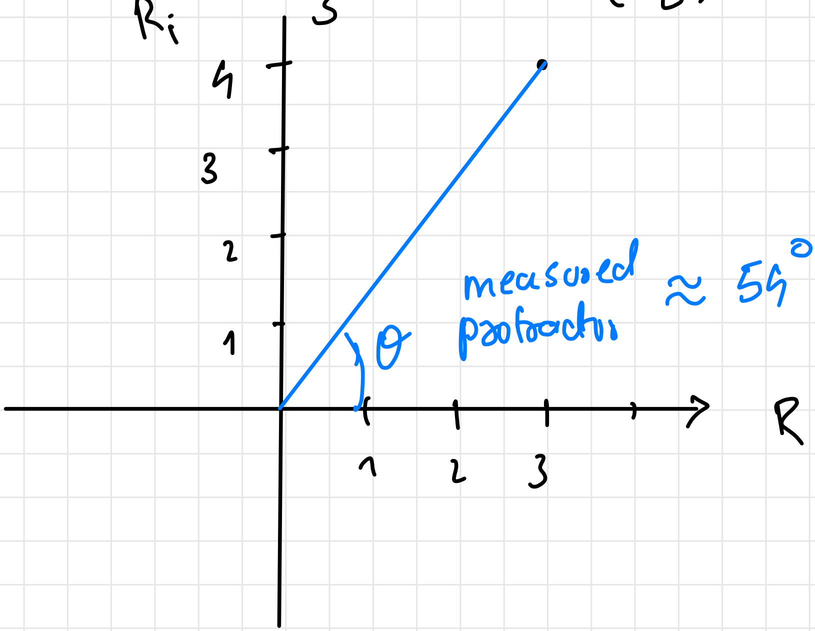


c) $\sqrt[5]{3+4i}$

$n = 5$

$a = 3$ $\qquad r = \sqrt{9+16} = 5$

$b = 4$ $\qquad \tan\theta = \frac{4}{3} \qquad \theta = \tan^{-1}\left(\frac{4}{3}\right)$



$$\theta = 54^\circ = \frac{54\pi}{180} = \frac{9\pi}{30} = \frac{3\pi}{10}$$

$r = 5 \qquad \theta = \frac{3\pi}{10} \qquad n = 5$

$\therefore K = 0, 1, 2, 3, 4$

$$w_k = r^{1/n} \cdot e^{i\left(\frac{\theta}{n} + \frac{2\pi}{n}k\right)}$$

$$\text{Argument} = \frac{\theta}{n} = \frac{3\pi}{10} \cdot \frac{1}{5} = \frac{3\pi}{50}$$

$$\text{periodicity} = \frac{2\pi}{n} = \frac{2\pi}{5}$$

$$\frac{3\pi}{50} = \frac{3 \cdot 18\cancel{0}}{5\cancel{0}} = \frac{54}{50} = \approx 1.0\ldots^{\circ}$$

$$\frac{2\pi}{5} = 2 \cdot \frac{180}{5} = 2 \cdot 36 = 72^{\circ}$$

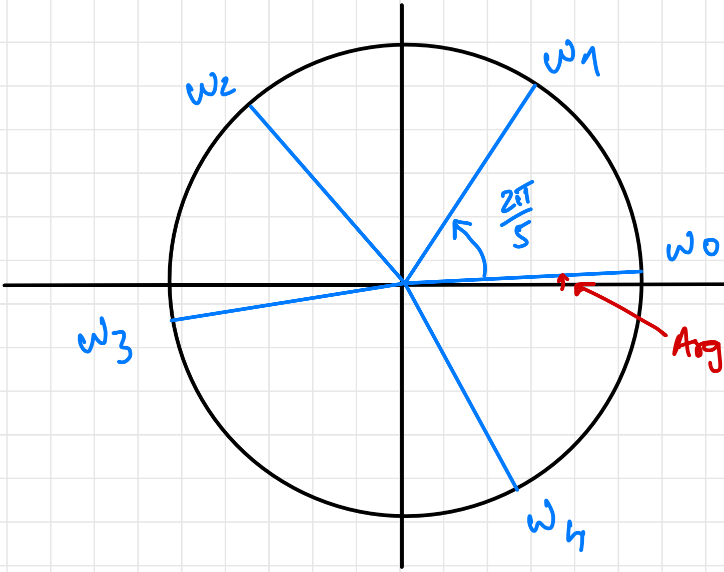
$$w_0 = \sqrt[5]{5}\, e^{i\left(\frac{3\pi}{50} + \frac{2\pi}{5} \cdot 0\right)} = \sqrt[5]{5}\, e^{i\frac{3\pi}{50}}$$

$$w_1 = \sqrt[5]{5}\, e^{i\left(\frac{3\pi}{50} + \frac{2\pi}{5} \cdot 1\right)} = \sqrt[5]{5}\, e^{i\frac{23\pi}{50}}$$

$$w_2 = \sqrt[5]{5}\, e^{i\left(\frac{3\pi}{50} + \frac{2\pi}{5} \cdot 2\right)} = \sqrt[5]{5}\, e^{i\frac{43\pi}{50}}$$

$$w_3 = \sqrt[5]{5}\, e^{i\left(\frac{3\pi}{50} + \frac{2\pi}{5} \cdot 3\right)} = \sqrt[5]{5}\, e^{i\frac{63\pi}{50}}$$

$$w_4 = \sqrt[5]{5}\, e^{i\left(\frac{3\pi}{50} + \frac{2\pi}{5}\cdot 4\right)}$$



$$\begin{array}{r} 73 \\ 72 \\ \hline 145 \\ 72 \\ \hline 217 \\ 72 \\ \hline 289 \end{array}$$

Find the roots of the polynomial

a) x^3+3x^2+3x+1

when $x=0 \Rightarrow \quad 0+0+0+1 \neq 0$

when $x=-1 \Rightarrow \quad -1+3-3+1=0$

$x=-1$ is a root

$$
\begin{array}{r|l}
 & x^2+2x+1 \\
\hline
x+1 & x^3+3x^2+3x+1 \\
 & \underline{x^3+x^2} \\
 & \quad 2x^2+3x \\
 & \quad \underline{2x^2+2x} \\
 & \qquad x+1 \\
 & \qquad \underline{x+1} \\
 & \qquad\quad 0
\end{array}
$$

$(x+1)(x^2+2x+1)=0$

$(x+1)(x^2+x+x+1)=0$

$(x+1)(x(x+1)(x+1))=0$

$(x+1)^3=0$ 3 roots are $x_{1,2,3}=1$

Find the roots of the polynomials:

b) $x^5 + x^3 + x^2 + 1 = 0$

$$x^5 + 0x^4 + x^3 + x^2 + 0x + 1 = 0$$

LHS: RHS:

$x = 0 \Rightarrow 0 + 0 + 0 + 1 \neq 0$

$x = 1 \Rightarrow 1 + 1 + 1 + 1 \neq 0$

$x = -1 \Rightarrow -1 + -1 + 1 + 1 = 0$

$x = -1$ is a factor

$\therefore$ dividing by $x + 1$ to simplify.

$$\begin{array}{r|l} & x^4 - x^3 + 2x^2 - x + 1 \\ \hline x+1 & x^5 + 0x^4 + x^3 + x^2 + 0x + 1 \\ & x^5 + x^4 \\ \hline & -x^4 + x^3 \\ & -x^4 - x^3 \\ \hline & 2x^3 + x^2 \\ & 2x^3 + 2x^2 \\ \hline & -x^2 + 0x \\ & -x^2 - 1x \\ \hline & x + 1 \end{array}$$

$$\frac{x+1}{0}$$

Now we have

$$(x+1)\underbrace{(x^4 - x^3 + 2x^2 - x + 1)}_{\Downarrow} = 0$$

	LHS	RHS
$x = 1$	$1 - 1 + 2 - 1 + 1$	0

$x - 1 = 0$ is a factor

$$
\begin{array}{r|l}
 & x^3 + 0x^2 + 2x + 1 \\
\hline
x - 1 & x^4 - x^3 + 2x^2 - x + 1 \\
 & x^4 - x^3 \\
\hline
 & 0x^3 + 2x^2 \\
 & 0x^3 - 0x^2 \\
\hline
 & 2x^2 - x \\
 & 2x^2 - 2x \\
\hline
 & 1x + 1 \\
 & 1x - 1 \\
\hline
 & 2
\end{array}
$$

$$x^5 + x^3 + x^2 + 1 = 0$$

$$x^3(x^2+1) + 1(x^2+1) = 0$$

$$(x^2+1)(x^3+1) = 0$$

$$x^2+1=0 \qquad x^3+1=0$$

$$x = \sqrt{-1}$$

$$x = i$$

is a factor

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$$

$$a^3+b^3 = (a+b)^3 - 3ab(a+b)$$

$$x^3+1^3 = (x+1)^3 - 3x(x+1)$$

$$= (x+1)\left[(x+1)^2 - 3x\right]$$

$$x+1 = 0$$

$x = -1$ is also a factor

$$(x+1)^2 - 3x = 0$$

$$(x+1)^2 = 3x$$

$$x^2 + 2x + 1 = 3x$$

$$x^2 + 1 = x$$

$$x^2 - x + 1 = 0$$

$$(x^2+1)(x+1)(x^2-x+1) = 0$$

$$\frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

roots are:

$x_{1,2} = \pm i$

$x_3 = -1$

$x_{4,5} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

c) $2x^3 - 3x^2 + 5$

$2x^3 - 3x^2 + 0x + 5 = 0$

$x = -1$ is a root

$$
\begin{array}{r|l}
 & 2x^2 - 5x + 5 \\
\hline
x+1 & 2x^3 - 3x^2 + 0x + 5 \\
 & \underline{2x^3 + 2x^2} \\
 & \quad -5x^2 + 0x \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad 5x + 5 \\
 & \qquad \underline{5x + 5} \\
 & \qquad\quad 0
\end{array}
$$

$(x+1)(2x^2 - 5x + 5) = 0$

$$\frac{-(-5) \pm \sqrt{25 - 40}}{4}$$

$$= \frac{5 \pm \sqrt{-15}}{4}$$

$$= \frac{5}{4} \pm \frac{\sqrt{15}}{4} i$$

roots are $x_1 = -1$

$$x_{2,3} = \frac{5}{4} \pm \frac{\sqrt{15}}{4} i$$