- Matrices
- Determinants
- Eigenvalues & Eigenvectors

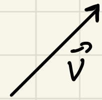
github.com/soyceanton

Matrices

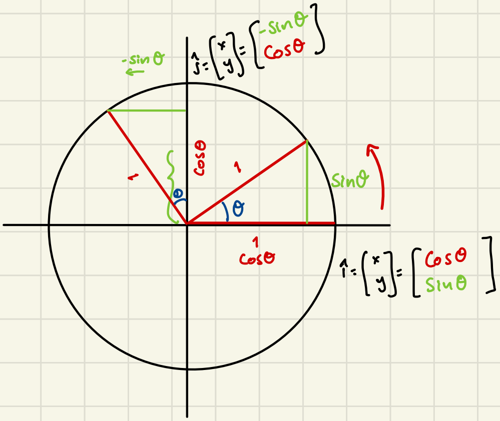
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$ $m \times n$ matrix

Vectors

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$



Linear Transformation:



For 2x2

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

diff. this to get

diff

For 3x3

Notice the Rotating axis remains unchanged.

$$R_x(\theta) = \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \end{array}$$

diff

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

diff

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

diff

1. Generate the rotation matrix with the angle depending on the x, y or z axes.
2. Multiply each vertices (if more than one given) with the rotation matrix to generate new position.

Multiplying Matrix by Matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{pmatrix} a(e)+b(g) & a(f)+b(h) \\ c(e)+d(g) & c(f)+d(h) \end{pmatrix}$$

$$\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Multiplying Matrix by Vector

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2} \cdot \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}$$

$$x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}_{2\times 1}$$

Rank of a matrix

The rank is essentialy how many of the rows are unique

So if any of the row or column is a multiple or is formed by adding or subtracting one of the already rank counted rows; then this new row does not count

eg:
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}$$

$\mathrm{II} \rightarrow \mathrm{I} \times 3$

$\therefore$ rank $= 1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{pmatrix}$$

$\underline{\mathrm{III}} \rightarrow \underline{\mathrm{I}} + \underline{\mathrm{II}}$

$\therefore$ rank $= 2$ as 3^{rd} does not count

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ -1 & -2 & -1 \end{pmatrix}$$

$\underline{\mathrm{III}} \rightarrow \underline{\mathrm{I}} - 2\,\underline{\mathrm{II}}$

$\therefore$ rank $= 2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Identity matrix

$\therefore$ rank $= 3$

Special cases:

1 x n matrices

1 row, n columns

eg: $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

m x 1 matrices

m rows, 1 columns

eg $\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$

1 x 1 matrices

1 row
1 column.

(-3)

identity matrix

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

I_n refers to identity matrix of size $n \times n$

Matrix Operations

Addition: can only add same size

$A: m \times n$ $B: m \times n$

Subtraction: Be careful to apply minus sign properly

Scalar Multiplication: Any matrix can be multiplied by a scalar

Matrix Transposition: A^T is defined by switching rows A with its columns

Diagonals: $n \times n$ square matrix transpose leaves them unmoved.

$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$$

Inverse of a Matrix

$A \cdot A^{-1} = I$

So how is it solved? Just like SLE

$A\vec{J} = B$

$a_1 x + a_2 y = b_1$

$a_3 x + a_4 y = b_2$

can be written as:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Similarly for $AA^{-1} = I$

→ Start with $(A \mid I)$ & do GE to obtain $(I \mid A^{-1})$

→ A^{-1} may or may not exist
doesnt exist if pivot on RHS.

if it is 2x2 matrix

$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$ $\Rightarrow$ ie $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

How to check if Matrix is solvable?

→ Check the number of pivots in the reduced form for $(A \mid b)$ & also of matrix A in reduced form

$$\underbrace{\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{(A \mid B)}$$

with the first three columns marked A.

if rank: (after reducing)

$A = (A \mid B)$ solvable

$A < (A \mid B)$ not solvable as pivot on RHS

Also introduce params based on SLE after reducing the system

$\#\text{params} = n_{\text{variables}} - \text{rank}_A$

if params $= 1$; introduce $t \in R$

if params $= 2$; introduce $t, s \in R$

$$\underbrace{\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \end{array}\right)}_{(A|B)}$$

A

- Reduced
- Rank A = Rank $(A|B)$
  solvable
- params:
  $n \to$ num of variables $= 3$
  $R_A \to$ Rank of matrix $A = 3$
  $\# \text{params} = n - R_A$
  $= 3 - 3 = 0$
  no parameters introduced

$$\underbrace{\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{(A|B)}$$

A

- Reduced
- Rank A = Rank $(A|B)$
  solvable
- params:
  $n = 3$
  $R_A = 2$
  $\# p = n - R_A$
  $= 3 - 2 = 1$
  $\therefore t \in R$
  apply back substitution

$$\underbrace{\left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right)}_{(A|B)}$$

A

- Reduced
- $R_A < R(A|B)$
- not solvable

Matrix Inverse & Transpose properties

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Symmetric & Skew Symmetric Matrix

if $A = A^T$; it is symmetric matrix

if $A = -A^T$ with diagonals 0
& elements above diagonals
with opposite sign compared
to elements below the diagonal
it is skew symmetric

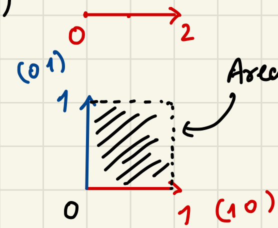
To generate a symmetric matrix, C
$C = A + A^T$ where A is $n \times n$ matrix

To generate a skew symmetric matrix, B
$B = A - A^T$ where A is $n \times n$ matrix

Determinants

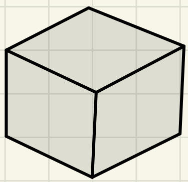
$n = 1$ eg: $|2|$ $\det(2)$

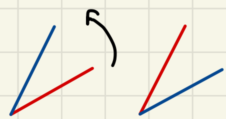
$n = 2$ eg: $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$



Area = 1 = determinant

$n = 3$ eg: $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$





$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$ $\det = 1$ $\qquad$ $\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$ $\det = -1$

Properties

① switching rows; multiply by -1 as:

② Take a common factor out from row

③ A row can be split out into its sum

Not possible in matrix in the same way ie:

$$\begin{vmatrix} 6 & 4 \\ 5 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} \quad \text{but} \quad \begin{pmatrix} 6 & 4 \\ 5 & 2 \end{pmatrix} \neq 2\begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix}$$

or

$$\det\begin{pmatrix} 6 & 4 \\ 5 & 2 \end{pmatrix} = 2\det\begin{pmatrix} 3 & 2 \\ 5 & 2 \end{pmatrix}$$

also

$$\begin{vmatrix} 6 & 4 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} \quad \text{but} \quad \begin{pmatrix} 6 & 4 \\ 5 & 2 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 \\ 5 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$$

$12 - 20$
$= -8$

$6 \quad + \quad 6 - 20$
$= -8$

(4) If Matrix A contains a rows of 0's then
det A = 0

$$A = \begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix} \qquad \det A = 0$$

(5) if Matrix A's any two rows are the same
det A = 0

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ a & b & c \end{pmatrix} \qquad \det A = 0$$

(6) Adding one row to another does not change the determinant.

(7) if you have an upper triangular matrix, A
the det A = product of diagonals

$$A = \begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & c & * \\ 0 & 0 & 0 & d \end{pmatrix} \qquad \det A = a \cdot b \cdot c \cdot d$$

(8) for matrix A; det A = 0 matrix is not invertible
if det A ≠ 0 matrix is invertable

(9) $\det(AB) = \det A \cdot \det B$

(10) $\det A = \det A^T$; determinant of A & A^T will be same.

Applications of Determinants

Find inverse of Matrix

$A^{-1} = \frac{1}{\det(A)} \cdot C^T$; where C is the cofactors of A

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Find $\det A = \begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d^- & e^+ & f \\ g^+ & h^- & i \end{vmatrix} \begin{matrix} + \\ - \\ + \end{matrix}$

$C_{11} = \begin{vmatrix} e & f \\ h & i \end{vmatrix}$ $C_{12} = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ .....

$C = \begin{pmatrix} +C_{11} & -C_{12} & +C_{13} \\ -C_{21} & +C_{22} & -C_{23} \\ +C_{31} & -C_{32} & +C_{33} \end{pmatrix}$

Apply transpose to that to get C^T

Solve SLE's with Cramers Rule

$A\vec{x} = \vec{b}$ if you have:
$$ax + by = c$$
$$dx + ey = f$$

then find $A = \det\begin{pmatrix} a & b \\ d & e \end{pmatrix}$

$A_x = \det\begin{pmatrix} c & b \\ f & e \end{pmatrix}$

$A_y = \det\begin{pmatrix} a & c \\ d & f \end{pmatrix}$

$x = \frac{A_x}{A}$ $y = \frac{A_y}{A}$

Eigen Values & Eigen Vectors

Given an $n \times n$ matrix A there exists an $n \times 1$ vector $\vec{v} \neq 0$ and a scalar λ such that:

$$\underbrace{A}_{\text{matrix } A}\underbrace{\vec{v}}_{\text{eigen vector}} = \underbrace{\lambda}_{\text{eigen value}} \vec{v}$$

important: $\vec{v} \neq 0$ but λ can be 0

$$A\vec{v} = \lambda \vec{v} = \lambda I \vec{v}$$

$$A\vec{v} - \lambda I \vec{v} = 0$$

$$(A - \lambda I)\vec{v} = 0$$

This is of the form $M\vec{x} = \vec{0}$

This system can have solutions other than $\vec{x} = 0$ if and only if $\det(A - \lambda I) = 0$

This is a property of linear algebra if $\det M = 0$; M is non invertible and M can have infinitely many soln's as it has non-trivial (ie meaningful) values. if it was $\det M \neq 0$ then we have a unique solution & in this case $(A - \lambda I)x = 0$; $\vec{x}$ is a zero vector; which is non-trivial (ie meaningless). Thus $\det(A - \lambda I) = 0$ makes sense as we are interested in the soln's other than zero vector which are our eigen vector's.

Additionally:

For any given vector; lest say $\vec{x}$ can be written as its linear combinations of eigen vectors

ie $\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 \rightarrow$ ①

$\therefore$ $A\vec{x}$ can be written as:

$$A(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3)$$

$$a_1 \underbrace{A\vec{v}_1}_{\lambda_1 \vec{v}_1} + a_2 \underbrace{A\vec{v}_2}_{\lambda_2 V_2} + a_3 \underbrace{A\vec{v}_3}_{\lambda_3 \vec{v}_3}$$

$$A\vec{x} = a_1 \lambda_1 \vec{v}_1 + a_2 \lambda_2 \vec{v}_2 + a_3 \lambda_3 \vec{v}_3$$

where scalars a_1, a_2 & a_3 can be found through gauss elimination.

Study Guide for Applied Mathematics
Matrices 1–4: Basic Operations

You can use these selected problems (used as homework problems in past runs of this course) to pace your study progress.

Prof Megill, Dr Camps and Ms Neh recommend that you regularly e-meet with your study group and discuss the material.

Task 1

Let $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, $x = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$,

$y = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and $z = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}$.

Compute all possible matrix products

a) MN, with $M, N \in \{A, B, C\}$, (two matrices),

b) Mv, with $M \in \{A, B, C\}$, $v \in \{x, y, z\}$, (matrix and vector),

c) MNP, with $M, N, P \in \{A, B, C\}$, (three matrices),

d) $v^T M$, with $M \in \{A, B, C\}$, $v \in \{x, y, z\}$, (vector and matrix),

e) $v^T M w$, with $M \in \{A, B, C\}$, $v, w \in \{x, y, z\}$, (vector-matrix-vector),

f) $v^T w$, with $v, w \in \{x, y, z\}$, (vector-vector),

g) vw^T, with $v, w \in \{x, y, z\}$, (vector-vector).

Which of these products does the same as the dot product for vectors?
Why are the matrix products vw and $v^T w^T$ not defined for any vectors?

Hint:
Part a) means: Choose a matrix A, B, or C for M, and also choose a matrix for N. Then decide if the product is defined. If so, compute it, if not, give the reason.

Check all possible combinations (for a): nine in total).

Continue in the same manner for b) up to g).

Task 2

Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

a) Compute AB, $(AB)^T$, $A^T B^T$, and $B^T A^T$. Memorize the formula

$$(AB)^T = B^T A^T.$$

b) Compute $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$. Memorize the formula

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Does this change of order look weird to you? Well, in the morning you first put your socks on, then your shoes. In the evening, you first pull of your shoes, then your socks. Note the same change of order when inverting the operations.

Task 3

a) What condition does a matrix A have to satisfy, so that

(i) both the products AA^T and $A^T A$ exist?

(ii) A^2 exists?

(iii) Check i) and ii) with the matrices $B = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix}$.

b) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Calculate A^{50}.

Task 4

With the matrices from Task 2 compute $(A+B)^2$, $A^2 + 2AB + B^2$, and $A^2 + AB + BA + B^2$. Note that the binomial formula does not work for matrix multiplication.

Task 5 (Rotation Matrices)

A counterclockwise rotation can be performed by a matrix multiplication with the rotation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad .$$

a) Draw the unit vector $\vec{\imath} = \binom{1}{0}$ and the result of the matrix product $R(30°)\vec{\imath}$ in one coordinate system. Do the same with the unit vector $\vec{\jmath}$.

b) What does $R(\alpha)$ look like for $\alpha = 0$?

c) Compute the matrix product

$$R(\beta)R(\gamma) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & cos\gamma \end{pmatrix}$$

and simplify the result by using the addition theorems for cosine and sine. Write the product in the form of $R(\alpha)$ for a suitable α. What is the geometrical meaning of the product?

d) Compute the determinant of $R(\alpha)$. How does the result depend on the angle α?

e) Compute the inverse matrix of $R(\alpha)$. Write the answer in the form

$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & cos\gamma \end{pmatrix} .$$

What is the value of γ? What is the geometrical meaning of this matrix?

Task 6

a) Find the 3×3 matrix Y that describes a rotation by $30°$ about the y-axis in 3-dimensional space.

b) Find the 3×3 matrix X that describes a rotation by $45°$ about the x-axis in 3-dimensional space.

c) Compute the products YX and XY. Draw sequences of pictures of cubes to visualize both rotations (imitate what we did for $90°$ rotations in the lecture: draw the cube in original position, then after having applied the first rotation, and in final position after the second rotation.)

Task 7

A furniture company produces tables, chairs and benches, and sells these separately or in sets. For a table they need 12 units wood and 3 units metal. For a bench it is 6 units wood, 2 units metal and 5 units fabric. A chair is made from 2 units wood, 1 unit metal and 2 units fabric. Two types of sets are composed: set A consists of a table and four chairs, and set B contains a table, three chairs and a bench.

a) Find the production matrix M that is used to calculate the necessary amounts of wood, metal and fabric to produce x tables, y chairs and z benches.

b) Find the production matrix P that is used to calculate the necessary amounts of wood, metal and fabric to produce m sets of type A and n sets of type B.

c) An order is for 40 sets of type A, 60 sets of type B and 10 (extra) benches. How many units of wood, metal and fabric are needed for this order?

d) If a unit of wood costs \$2, a unit of metal \$4, and a unit of fabric \$7, how much does the furniture company pay for materials to produce the order in task c)?

Note: Solve tasks c) and d) with appropriate matrix products.

Task 8

a) We have seen, that (in general) $AB \neq BA$. There are some exceptions, but they are rare. To see how rare, solve the following question:
Let $A = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$. Find all 2×2-matrices B, so that $AB = BA$.

b) Find a 2×2-matrix $C \neq 0$ with $C^2 = 0$. To make it more interesting: No element of C may be zero.

Task 9

Check if the following matrices are invertible. If they are, compute their inverses.

$$\text{(i) } A = \begin{pmatrix} 3 & -3 \\ 2 & -3 \end{pmatrix}, \quad \text{(ii) } B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix}, \quad \text{(iii) } C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 3 & 2 & 3 \\ 4 & 4 & 4 & 3 \end{pmatrix}$$

Task 10

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 1 & 2 & -2 \end{pmatrix}.$$

Compute
(i) A^{-1}, (ii) $(2A)^{-1}$, (iii) $(A^T)^{-1}$ (iv), $(A^T A)^{-1}$

Note: For this task, only one inverse has to be computed.

Task 11

Let $A \in \mathbb{R}^{n \times n}$ be a matrix.

a) Show that the matrix $B = A + A^T$ is symmetric.

b) Show that the matrix $C = A - A^T$ is skew–symmetric. That means: $C = -C^T$.

c) Show that A is the sum of a symmetric and a skew-symmetric matrix.

d) Use c) to write the matrix

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -3 & 3 \\ 5 & -1 & 2 \end{pmatrix}$$

as the sum of a symmetric and a skew–symmetric matrix.

Task 12

There is something special about the main diagonal of a skew-symmetric matrix. What is it?

Task 1

Let $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, $x = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$,

$y = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and $z = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}$.

Compute all possible matrix products

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f) $v^T w$, with $v, w \in \{x, y, z\}$, (vector-vector),

g) $v w^T$, with $v, w \in \{x, y, z\}$, (vector-vector).

$$A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}_{2\times 3} \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}_{4\times 2} \quad C = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}_{2\times 2}$$

a) $M, N \in \{A, B, C\}$

$= \quad A_{2\times 3} \quad B_{4\times 2} \quad C_{2\times 2} \qquad x_{3\times 1} \quad y_{2\times 1} \quad z_{4\times 1}$

possible combos:

$A_{2\times 3}\, A_{2\times 3}$

$A_{2\times 3}\, B_{4\times 2}$

$A_{2\times 3}\, C_{2\times 2}$

$B_{4\times 2}\, A_{2\times 3}$

$B_{4\times 2}\, A_{2\times 3}$

$B_{4\times 2}\, C_{2\times 2}$

$C_{2\times 2}\, A_{2\times 3}$

$C_{2\times 2}\, C_{2\times 2}$

For the rest the columns of first doesnt match the rows of the 2nd

$B_{4\times2}\ B_{4\times2}$

$B_{4\times2}\ C_{2\times2}$

$C_{2\times2}\ A_{2\times3}$

$C_{2\times2}\ B_{4\times2}$

$C_{2\times2}\ C_{2\times2}$

One solution:

$B_{4\times2}\ A_{2\times3}$

a1) $$BA = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1(-2)+3(3) & 1(1)+3(1) & 1(4)+3(8) \\ 0(-2)+2(3) & 0(1)+2(1) & 0(4)+2(8) \\ 1(-2)-2(3) & 1(1)-2(1) & 1(4)-2(8) \\ 0(-2)+1(3) & 0(1)+1(1) & 0(4)+1(8) \end{pmatrix}$$

$$= \begin{pmatrix} -2+9 & 1+3 & 4+24 \\ 0+6 & 0+2 & 0+16 \\ -2-6 & 1-2 & 4-16 \\ 0+3 & 0+1 & 0+8 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 4 & 28 \\ 6 & 2 & 16 \\ -8 & -1 & -12 \\ 3 & 1 & 8 \end{pmatrix}$$

a2) $\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$ $\quad B_{2\times 2} \; C_{2\times 2}$

$$\begin{pmatrix} 2+12 & -1-9 \\ 0+8 & 0-6 \\ 2-8 & -1+6 \\ 0+4 & 0-3 \end{pmatrix} = \begin{pmatrix} 14 & -10 \\ 8 & -6 \\ -6 & -5 \\ 4 & -3 \end{pmatrix} \qquad C_{2\times 2} \; A_{2\times 3}$$

a3) $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}_{2\times 2} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}_{2\times 3}$

$$\begin{pmatrix} -4-3 & 2-1 & 8-8 \\ -8-9 & 4-3 & 16-24 \end{pmatrix} = \begin{pmatrix} -7 & 1 & 0 \\ -17 & 1 & 8 \end{pmatrix}$$

a4) $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$ $\quad C_{2\times 2} \; C_{2\times 2}$

$$= \begin{pmatrix} 4-4 & -2+3 \\ 8-12 & -4+9 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$$

b) Mv $\quad M\{A, B, C\}$

$v\{x, y, z\}$ ⇐ vector ∴ use matrix by vector mutiplication

$\underline{Mv}$

$A_{2\times3}\ x_{3\times1}$

$A_{2\times3}\ y_{2\times1}$

$A_{2\times3}\ z_{4\times1}$

$B_{4\times2}\ x_{3\times1}$

$B_{4\times2}\ y_{2\times1}$

$B_{4\times2}\ z_{4\times1}$

$C_{2\times2}\ x_{3\times1}$

$C_{2\times2}\ y_{2\times1}$

$C_{2\times2}\ z_{4\times1}$

$A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$ $\quad C = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$

$x = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}_{3\times1}$ $\quad y = \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{2\times1}$ $\quad z = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}_{4\times1}$

$x = \langle 3\ -1\ 1 \rangle$ $\quad y = \langle 2\ 2 \rangle$ $\quad z = \langle -4\ 1\ 0\ 8 \rangle$

solve

1) $A = \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$ $\quad x = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ $\qquad$ 2) $B = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$ $\quad y = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

3) $C = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$ $y = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

b1) $Ax = \begin{pmatrix} -2(3) + 1(-1) + 4(1) \\ 3(3) + 1(-1) + 8(1) \end{pmatrix}$

$= \begin{pmatrix} -6 - 1 + 4 \\ 9 - 1 + 8 \end{pmatrix} = \begin{pmatrix} -3 \\ 16 \end{pmatrix}$

b2) $By = \begin{pmatrix} 1(2) + 3(2) \\ 0(2) + 2(2) \\ 1(2) + -2(2) \\ 0(2) + 1(2) \end{pmatrix}$

$= \begin{pmatrix} 2 + 6 \\ 0 + 4 \\ 2 - 4 \\ 0 + 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ -2 \\ 2 \end{pmatrix}$

b3) $Cy = \begin{pmatrix} 2(2) + 1(2) \\ 4(2) - 3(2) \end{pmatrix} = \begin{pmatrix} 4 + 2 \\ 8 - 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

c) $MNP \in \{A, B, C\}$

M	N	
$A_{2\times3}$	$A_{2\times3}$	
$A_{2\times3}$	$B_{4\times2}$	
$A_{2\times3}$	$C_{2\times2}$	
$B_{4\times2}$	$A_{2\times3}$	$= BA_{4\times3}$
$B_{4\times2}$	$B_{4\times2}$	
$B_{4\times2}$	$C_{2\times2}$	$= BC_{4\times2}$
$C_{2\times2}$	$A_{2\times3}$	$= CA_{2\times3}$
$C_{2\times2}$	$B_{4\times2}$	
$C_{2\times2}$	$C_{2\times2}$	$= CC_{2\times2}$

MN	P	
$BC_{4\times2}$	$A_{2\times3}$	✓
$BC_{4\times2}$	$B_{4\times2}$	
$BC_{4\times2}$	$C_{2\times2}$	✓
$BA_{4\times3}$	$A_{2\times3}$	
$BA_{4\times3}$	$B_{4\times2}$	
$BA_{4\times3}$	$C_{2\times2}$	
$CA_{2\times3}$	$A_{2\times3}$	
$CA_{2\times3}$	$B_{4\times2}$	
$CA_{2\times3}$	$C_{2\times2}$	
$CC_{2\times2}$	$A_{2\times3}$	✓
$CC_{2\times2}$	$B_{4\times2}$	
$CC_{2\times2}$	$C_{2\times2}$	✓

Possible MNP combinations are:

$BCA_{4\times3}$

$BCC_{4\times2}$

$CCA_{2\times3}$

$CCC_{2\times2}$

Find:

c1) $BCA_{4\times3}$ $\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$

c2) $BCC_{4\times2}$ $\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$

c3) $CCA_{2\times3}$ $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$

c4) $CCC_{2\times2}$ $\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$

c1) $\begin{pmatrix} 1(2)+3(4) & 1(1)+3(-3) \\ 0(2)+2(4) & 0(1)+2(-3) \\ 1(2)-2(4) & 1(1)-2(-3) \\ 0(2)+1(4) & 0(1)+1(-3) \end{pmatrix}$

$BCA_{4\times3}$

$= \begin{pmatrix} 2+12 & 1-9 \\ 0+8 & 0-6 \\ 2-8 & 1+6 \\ 0+4 & 0-3 \end{pmatrix} = \begin{pmatrix} 14 & -8 \\ 8 & -6 \\ -6 & 7 \\ 4 & -3 \end{pmatrix} \Rightarrow BC$

$BC \cdot A = \begin{pmatrix} 14 & -8 \\ 8 & -6 \\ -6 & 7 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$

$$\begin{pmatrix} 14(-2)-8(3) & 14(1)-8(1) & 14(4)-8(8) \\ 8(-2)-6(3) & 8(1)-6(1) & 8(4)-6(8) \\ -6(-2)+7(3) & -6(1)+7(1) & -6(4)+7(8) \\ 4(-2)-3(3) & 4(1)-3(1) & 4(4)-3(8) \end{pmatrix}$$

$$= \begin{pmatrix} -28-24 & 14-8 & 56-64 \\ -16-18 & 8-6 & 32-48 \\ 12+21 & -6+7 & -24+56 \\ -12-9 & 4-3 & 16-24 \end{pmatrix} = \begin{pmatrix} -52 & -6 & -8 \\ -34 & 2 & -16 \\ 33 & 1 & 31 \\ -21 & 1 & -8 \end{pmatrix}$$

c2) $BCC_{4\times 2}$ $\begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$ $BCC_{4\times 2}$

$$\begin{pmatrix} 1(2)+3(4) & 1(1)+3(-3) \\ 0(2)+2(4) & 0(1)+2(-3) \\ 1(2)+-2(4) & 1(1)-2(-3) \\ 0(2)+1(4) & 0(1)+1(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 2+12 & 1-9 \\ 0+8 & 0-6 \\ 2-8 & 1+6 \\ 0+4 & 0-3 \end{pmatrix} = \begin{pmatrix} 14 & -8 \\ 8 & -6 \\ -6 & 7 \\ 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 14 & -8 \\ 8 & -6 \\ -6 & 7 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 14(2)-8(4) & 14(1)-8(-3) \\ 8(2)-6(4) & 8(1)-6(-3) \\ -6(2)+7(4) & -6(1)+7(-3) \\ 4(2)-3(4) & 4(1)-3(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 28-32 & 14+24 \\ 16-24 & 8+18 \\ -12+28 & -6-21 \\ 8-12 & 4+9 \end{pmatrix} = \begin{pmatrix} -4 & 38 \\ -8 & 26 \\ 16 & -27 \\ -4 & 13 \end{pmatrix}$$

c3) $$\underbrace{\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$$

$CCA_{2\times 3}$

$$\begin{pmatrix} 2(2)+1(4) & 2(1)+1(-3) \\ 4(2)-3(4) & 4(1)-3(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 4+8 & 2-3 \\ 8-12 & 4+9 \end{pmatrix} = \begin{pmatrix} 12 & -1 \\ -4 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 12 & -1 \\ -4 & 13 \end{pmatrix} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 12(-2)-1(3) & 12(1)-1(1) & 12(4)-1(8) \\ -4(-2)+13(3) & -4(1)+13(1) & -4(4)+13(8) \end{pmatrix}$$

$$= \begin{pmatrix} -24-3 & 12-1 & 48-8 \\ 8+39 & -4+13 & -16+104 \end{pmatrix} = \begin{pmatrix} -27 & 11 & 40 \\ 47 & 9 & 88 \end{pmatrix}$$

c4) $$CC = \begin{pmatrix} 12 & -1 \\ -4 & 13 \end{pmatrix} \cdot C = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

$CCC_{2\times 2}$

$$= \begin{pmatrix} 24-4 & 12+3 \\ -8+52 & -4-39 \end{pmatrix} = \begin{pmatrix} 20 & 15 \\ 44 & 43 \end{pmatrix}$$

d) $v^T M \qquad v \in \{x, y, z\}$

$M \in \{A \; B \; C\}$

$x^T = 1\times 3 \qquad y^T = 1\times 2 \qquad z^T = 1\times 4$

$A = 2\times 3 \qquad B = 4\times 2 \qquad C = 2\times 2$

$x^T_{1\times 3} \; A_{2\times 3} \qquad x^T_{1\times 3} \; B_{4\times 2} \qquad x^T_{1\times 3} \; C_{2\times 2}$

$y^T_{1\times 2} \; A_{2\times 3} \qquad y^T_{1\times 2} \; B_{4\times 2} \qquad y^T_{1\times 2} \; C_{2\times 2}$

$z^T_{1\times 4} \; A_{2\times 3} \qquad z^T_{1\times 4} \; B_{4\times 2} \qquad z^T_{1\times 4} \; C_{2\times 2}$

d1) $y^T A$ d2) $z^T B$ d3) $y^T C$

d1) $\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 1 & 4 \\ 3 & 1 & 8 \end{pmatrix}$

$y^T A$

$= \begin{pmatrix} -4+6 & 2+2 & 8+16 \end{pmatrix} = \begin{pmatrix} -2 & 4 & 24 \end{pmatrix}$

d2) $\underset{1\times 4}{\begin{pmatrix} -4 & 1 & 0 & 8 \end{pmatrix}} \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}_{4\times 2}$

$= \begin{pmatrix} -4+0+0+0 & -12+2+0+8 \end{pmatrix}$

$= \begin{pmatrix} -4 & -2 \end{pmatrix}$

$z^T B$

d3) $\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$ y^TC

$= \begin{pmatrix} 4+8 & -2-6 \end{pmatrix} = \begin{pmatrix} 12 & -8 \end{pmatrix}$

e) $v^TM \cdot w$ $w \in \{x, y\ z\}$

$\underline{v^TM}$ $\underline{w}$

$y^TA = 1\times 3$ $x = 3\times 1$

$z^TB = 1\times 2$ $y = 2\times 1$

$y^TC = 1\times 2$ $z = 4\times 1$

$y^TA \cdot x = 1\times 3 \cdot 3\times 1$

$y^TA \cdot y = 1\times 3 \cdot 2\times 1$

$y^TA \cdot z = 1\times 3 \cdot 4\times 1$

$z^TB \cdot x = 1\times 2 \cdot 3\times 1$

$z^TB \cdot y = 1\times 2\ \ 2\times 1$

$z^TB \cdot z = 1\times 2\ \ 4\times 1$

$y^T \cdot C \cdot x = 1\times 2\ \ 3\times 1$

$y^T \cdot C \cdot y = 1\times 2\ \ 2\times 1$

$y^TC \cdot z = 1\times 2\ \ 4\times 1$

e1) $y^TA \cdot x$

$= \begin{pmatrix} -2 & 4 & 24 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$= -6-4+24 = 14$

e2) $z^T \cdot B \cdot y$

$= \begin{pmatrix} -4 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$= -8-4 = -12$

e3) $y^TC \cdot z$

$= \begin{pmatrix} 12 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$= 24-16 = 8$

f) $v^T \cdot w$ $\quad v, w \; \{x, y, z\}$

	v^T	w
x^T	1×3	3×1
y^T	1×2	2×1
z^T	1×4	4×1

1) $x^T \cdot x = 1 \times 3 \cdot 3 \times 1 = 1 \times 1$

$x^T \cdot y = 1 \times 3 \quad 2 \times 1$

$x^T \cdot z = 1 \times 3 \quad 4 \times 1$

$y^T \cdot x = 1 \times 2 \quad 3 \times 1$

2) $y^T \cdot y = 1 \times 2 \quad 2 \times 1 = 1 \times 1$

$y^T \cdot z = 1 \times 2 \quad 4 \times 1$

$z^T \cdot x = 1 \times 4 \quad 3 \times 1$

$z^T \cdot y = 1 \times 4 \quad 2 \times 1$

3) $z^T \cdot z = 1 \times 4 \quad 4 \times 1 = 1 \times 1$

f1) $\begin{pmatrix} 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$= 9 + 1 + 1 = 11$

f2) $\begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$= 4 + 4 = 8$

f3) $\begin{pmatrix} -4 & 1 & 0 & 8 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}$

$= (16 + 1 + 0 + 64)$

$= 81$

9) $v w^T \qquad v, w \in \{x, y, z\}$

$$\begin{array}{ll} v & w^T \\ x = 3 \times 1 & x^T = 1 \times 3 \\ y = 2 \times 1 & y^T = 1 \times 2 \\ z = 4 \times 1 & z^T = 1 \times 4 \end{array}$$

$$\left.\begin{array}{l} x x^T = 3\times 1 \cdot 1\times 3 \\ x y^T = 3\times 1 \cdot 1\times 2 \\ x z^T = 3\times 1 \cdot 1\times 4 \\ y x^T = 2\times 1 \quad 1\times 3 \\ y y^T = 2\times 1 \quad 1\times 2 \\ y z^T = 2\times 1 \quad 1\times 4 \\ z x^T = 4\times 1 \quad 1\times 3 \\ z y^T = 4\times 1 \quad 1\times 2 \\ z z^T = 4\times 1 \quad 1\times 4 \end{array}\right\} \text{All combos are possible}$$

91) $x x^T = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}_{3\times 1} \begin{pmatrix} 3 & -1 & 1 \end{pmatrix}_{1\times 3} = \begin{pmatrix} 9 & -3 & 3 \\ -3 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}_{3\times 3}$

92) $x y^T = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}_{3\times 1} \begin{pmatrix} 2 & 2 \end{pmatrix}_{1\times 2} = \begin{pmatrix} 6 & 6 \\ -2 & -2 \\ 2 & 2 \end{pmatrix}_{3\times 2}$

g3) $xz^T = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}_{3\times 1} \begin{pmatrix} -4 & 1 & 0 & 8 \end{pmatrix}_{1\times 4} = \begin{pmatrix} -12 & 3 & 0 & 24 \\ 4 & -1 & 0 & -8 \\ -4 & 1 & 0 & 8 \end{pmatrix}_{3\times 4}$

g4) $yx^T = \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{2\times 1} \begin{pmatrix} 3 & -1 & 1 \end{pmatrix}_{1\times 3} = \begin{pmatrix} 6 & -2 & 2 \\ 6 & -2 & 2 \end{pmatrix}_{2\times 3}$

g5) $yy^T = \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{2\times 1} \begin{pmatrix} 2 & 2 \end{pmatrix}_{1\times 2} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}_{2\times 2}$

g6) $yz^T = \begin{pmatrix} 2 \\ 2 \end{pmatrix}_{2\times 1} \begin{pmatrix} -4 & 1 & 0 & 8 \end{pmatrix}_{1\times 4} = \begin{pmatrix} -8 & 1 & 0 & 16 \\ -8 & 1 & 0 & 16 \end{pmatrix}_{2\times 4}$

g7) $zx^T = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}_{4\times 1} \begin{pmatrix} 3 & -1 & 1 \end{pmatrix}_{1\times 3} = \begin{pmatrix} -12 & 4 & -4 \\ 3 & -1 & 1 \\ 0 & 0 & 0 \\ 24 & -8 & 8 \end{pmatrix}_{4\times 3}$

g8) $zy^T = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}_{4\times 1} \begin{pmatrix} 2 & 2 \end{pmatrix}_{1\times 2} = \begin{pmatrix} -8 & -8 \\ 2 & 2 \\ 0 & 0 \\ 16 & 16 \end{pmatrix}_{4\times 2}$

g9) $zz^T = \begin{pmatrix} -4 \\ 1 \\ 0 \\ 8 \end{pmatrix}_{4\times 1} \begin{pmatrix} -4 & 1 & 0 & 8 \end{pmatrix}_{1\times 4} = \begin{pmatrix} 16 & -4 & 0 & -32 \\ -4 & 1 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ -32 & 8 & 0 & 64 \end{pmatrix}_{4\times 4}$

Task 2

Let

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

a) Compute AB, $(AB)^T$, $A^T B^T$, and $B^T A^T$. Memorize the formula

$$(AB)^T = B^T A^T.$$

b) Compute $(AB)^{-1}$, $A^{-1}B^{-1}$, and $B^{-1}A^{-1}$. Memorize the formula

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Does this change of order look weird to you? Well, in the morning you first put your socks on, then your shoes. In the evening, you first pull of your shoes, then your socks. Note the same change of order when inverting the operations.

a)

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1(1) + 0(0) & 1(3) + 0(1) \\ 2(1) + 1(0) & 2(3) + 1(1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$$

$\therefore$

$$(AB)^T = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$(AB)^T = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

$$A^T B^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 0+2 \\ 0+3 & 0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 2+0 \\ 3+0 & 6+1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

b) $AB = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+0 & 3+0 \\ 2+0 & 6+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}$

$(AB)^{-1} = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right) \quad II - \frac{2}{1} I$

$= \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \quad I - \frac{3}{1} II$

$= \left(\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right)$

$(AB)^{-1} = \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$

$A^{-1} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \quad II - \frac{2}{1} I$

$= \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$B^{-1} = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \qquad \mathrm{I} - 3\,\mathrm{II}$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & 0-3 \\ 0-2 & 0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix}$$

Task 3

a) What condition does a matrix A have to satisfy, so that

(i) both the products AA^T and A^TA exist?

(ii) A^2 exists?

(iii) Check i) and ii) with the matrices $B = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix}$.

b) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Calculate A^{50}.

a) Matrix is solvable

AA^T eg: let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2}$ $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}_{2\times 2}$

$$AA^T = \begin{pmatrix} 1+4 & 3+8 \\ 3+8 & 9+16 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$$

Now $A^T \cdot A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 1+9 & 2+12 \\ 2+12 & 4+16 \end{pmatrix} = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

if $A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}_{4\times 1}$ $\quad A^T = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}_{1\times 4}$

$AA^T = 4\times 1 \; 1\times 4$
$= 4\times 4$

$A^T \cdot A = 1\times 4 \cdot 4\times 1$
$= 1\times 1$

$$AA^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1 + 4 + 9 + 16$$
$$= 30$$

given $B = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & -1 \end{pmatrix}_{2\times 3}$ $\quad C = \begin{pmatrix} -2 & 3 \\ 2 & 1 \end{pmatrix}_{2\times 2}$

$$B \cdot B^T = \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & -1 \end{pmatrix}_{2\times 3} \begin{pmatrix} 2 & -3 \\ 1 & 4 \\ -1 & -1 \end{pmatrix}_{3\times 2}$$

$$= \begin{pmatrix} 4+1+1 & -6+4+1 \\ -6+4+1 & 9+16+1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -1 \\ -1 & 26 \end{pmatrix}$$

$$B^T B = \begin{pmatrix} 2 & -3 \\ 1 & 4 \\ -1 & -1 \end{pmatrix}_{3\times 2} \begin{pmatrix} 2 & 1 & -1 \\ -3 & 4 & -1 \end{pmatrix}_{2\times 3}$$

$$= \begin{pmatrix} 4+9 & 2-12 & -2+3 \\ 2-12 & 1+16 & -1-4 \\ -2+3 & -1-4 & 1+1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -10 & 1 \\ -10 & 17 & -5 \\ 1 & -5 & 2 \end{pmatrix}$$

In all these cases AA^T & A^TA is possible as the rank of the matrix is full

Similarly for A^2 to exist; it must be a square matrix.

b) $A^{50} = (A^{25})^2 = ((A^5)^5)^2$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 2+2 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

$$A^2 \cdot A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 4+4 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

$$A^4 \cdot A^4 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 8+8 \\ 0+0 & 0+1 \end{pmatrix}$$

$$A^8 = \begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix}$$

$$A^8 \cdot A^8 = \begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 16 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 16+16 \\ 0+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 32 \\ 0 & 1 \end{pmatrix} = A^{16}$$

$$A^{16} \cdot A^{16} = \begin{pmatrix} 1 & 32 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 32 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 32+32 \\ 0+0 & 0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 64 \\ 0 & 1 \end{pmatrix} = A^{32}$$

$$A^{32} \cdot A^{16} = \begin{pmatrix} 1 & 64 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 32 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 64+32 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 96 \\ 0 & 1 \end{pmatrix}$$

$$A^{48} \cdot A^{2} = \begin{pmatrix} 1 & 96 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 98 \\ 0 & 1 \end{pmatrix}$$

correct ans $\begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}$

Let's say:

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ & $A = I + N$

where I is the identity matrix

$$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{N}$$

special case for N

Det $|N| = 0$ always.

Now we have:

$$A = I + N$$

$$N^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\therefore A^{50} = (I + N)^{50}$$

$$= I^{50} + 50C_1 N + 50C_2 N^2 + \dots + N^{50}$$

we have $N^2 = 0$

$$\therefore A^{50} = I^{50} + 50N + 0$$

$$A^{50} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 50 \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}$$

Task 4

With the matrices from Task 2 compute $(A+B)^2$, $A^2+2AB+B^2$, and $A^2+AB+BA+B^2$. Note that the binomial formula does not work for matrix multiplication.

Task 5 (Rotation Matrices)

A counterclockwise rotation can be performed by a matrix multiplication with the rotation matrix

$$R(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix} .$$

$$A=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}_{2\times 2} \quad B=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$$

$$A^2=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+0 & 0+0\\ 2+2 & 0+1\end{pmatrix}=\begin{pmatrix}1 & 0\\ 4 & 1\end{pmatrix}$$

$$B^2=\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1+0 & 3+3\\ 0+0 & 0+1\end{pmatrix}=\begin{pmatrix}1 & 6\\ 0 & 1\end{pmatrix}$$

$$AB=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1+0 & 3+0\\ 2+0 & 6+1\end{pmatrix}$$

$$=\begin{pmatrix}1 & 3\\ 2 & 7\end{pmatrix}$$

$$2AB = \begin{pmatrix} 2 & 6 \\ 4 & 14 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+6 & 0+3 \\ 0+2 & 0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}$$

$$(A+B)^2 = \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right)^2$$

$$= \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix}^2$$

$$= \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 4+6 & 6+6 \\ 4+4 & 6+4 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 12 \\ 8 & 10 \end{pmatrix}$$

$$A^2 + 2AB + B^2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 4 & 14 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 12 \\ 8 & 16 \end{pmatrix}$$

$$A^2 + AB + BA + B^2 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 12 \\ 8 & 10 \end{pmatrix}$$

Task 5 (Rotation Matrices)

A counterclockwise rotation can be performed by a matrix multiplication with the rotation matrix

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad .$$

a) Draw the unit vector $\vec{i} = \binom{1}{0}$ and the result of the matrix product $R(30^\circ)\vec{i}$ in one coordinate system. Do the same with the unit vector $\vec{j}$.

b) What does $R(\alpha)$ look like for $\alpha = 0$?

c) Compute the matrix product

$$R(\beta)R(\gamma) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & cos\gamma \end{pmatrix}$$

and simplify the result by using the addition theorems for cosine and sine. Write the product in the form of $R(\alpha)$ for a suitable α. What is the geometrical meaning of the product?

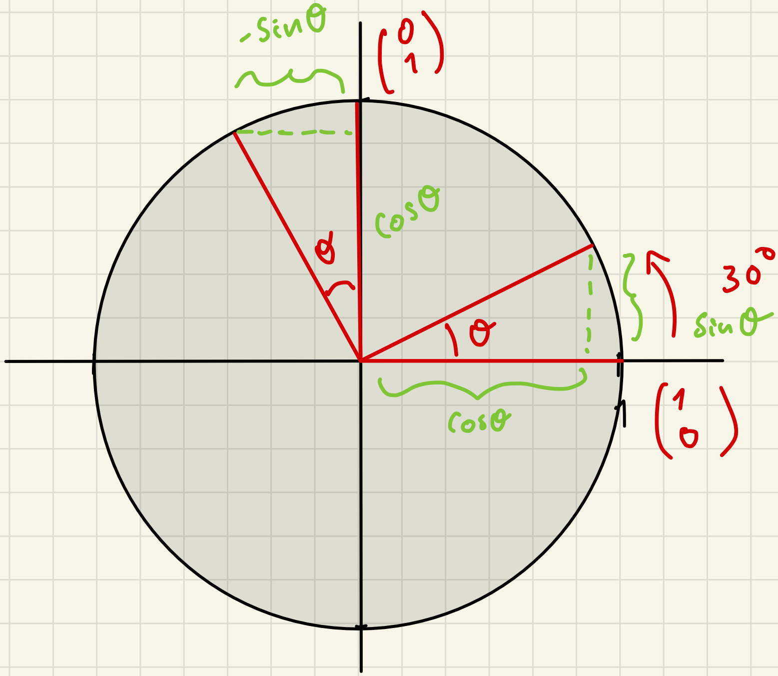
d) Compute the determinant of $R(\alpha)$. How does the result depend on the angle α?

e) Compute the inverse matrix of $R(\alpha)$. Write the answer in the form

$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & cos\gamma \end{pmatrix} \quad .$$

What is the value of γ? What is the geometrical meaning of this matrix?

$$R(\alpha) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$\langle 1, 0 \rangle$

$$\therefore R(30^\circ)\vec{i} = \begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 1\begin{pmatrix} \cos 30^\circ \\ \sin 30^\circ \end{pmatrix} + 0\begin{pmatrix} -\sin 30^\circ \\ \cos 30^\circ \end{pmatrix}$$

$$= \begin{pmatrix} \cos 30^\circ \\ \sin 30^\circ \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ \frac{1}{2} \end{pmatrix}$$

b) $R(0^\circ) = \begin{pmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $R(\beta) R(\gamma) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$

$= \begin{pmatrix} \cos\beta\cos\gamma - \sin\beta\sin\gamma & -\cos\beta\sin\gamma - \sin\beta\cos\gamma \\ \sin\beta\cos\gamma + \cos\beta\sin\gamma & -\sin\beta\sin\gamma + \cos\beta\cos\gamma \end{pmatrix}$

$= \begin{pmatrix} \cos(\beta+\gamma) & -\sin(\beta+\gamma) \\ \sin(\beta+\gamma) & \cos(\beta+\gamma) \end{pmatrix}$

Comparing to $R(\alpha)$ we have

$\cos\alpha = \cos(\beta+\gamma)$ $\qquad -\sin\alpha = -\sin(\beta+\gamma)$

$\sin\alpha = \sin(\beta+\gamma)$ $\qquad \cos\alpha = \cos(\beta+\gamma)$

$\alpha = \beta + \gamma$

This means that there is a rotation of the matrices in the plane by an angle α

whic is the sum of the othe two matrices angles.

d) $|R(\alpha)| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

Determinant can refer to the area regard less of the angle

$|R(\alpha)| = \cos^2\alpha + \sin^2\alpha = 1$

that means rotating a matrix does not change any area of any region in the plane.

ie apply rotation to any shape in the plane; the resulting shape will have original size and shape

Also det $|R(\alpha)|$ is not 0. It is non zero

$\therefore$ it's invertable. This means we can find a matrix $R(-\alpha)$ that undoes the rotation performed by $R(\alpha)$

$\therefore$ to get back to original orientation apply $R(-\alpha)$ to $R(\alpha)$

e) $\left(\begin{array}{cc|cc} \cos\gamma & -\sin\gamma & 1 & 0 \\ \sin\gamma & \cos\gamma & 0 & 1 \end{array}\right)$ $\begin{array}{l} \text{I} \to \text{I} \div \sin\gamma \\ \text{II} \to \text{II} \div \sin\gamma \end{array}$

$$\left(\begin{array}{cc|cc} \cot\gamma & -1 & \operatorname{cosec}\gamma & 0 \\ 1 & \cot\gamma & 0 & \operatorname{cosec}\gamma \end{array}\right) \text{II} - \frac{1}{\cot\gamma}\text{I}$$

$$= \left(\begin{array}{cc|cc} \cot\gamma & -1 & \operatorname{cosec}\gamma & 0 \\ 0 & \cot\gamma + \frac{1}{\cot\gamma} & -\frac{\operatorname{cosec}\gamma}{\cot\gamma} & \operatorname{cosec}\gamma \end{array}\right)$$

Too complicated ∴ using 2x2 A^{-1} rule

$$\begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{vmatrix}} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \left(\cos^2\gamma + \sin^2\gamma\right) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$$ } This is a rotation matrix with $\gamma = -\alpha$

So inversion of rotation matrix is another matrix which represents a rotation by the negative of original angle.

Task 6

a) Find the 3×3 matrix Y that describes a rotation by $30°$ about the y-axis in 3-dimensional space.

b) Find the 3×3 matrix X that describes a rotation by $45°$ about the x-axis in 3-dimensional space.

c) Compute the products YX and XY. Draw sequences of pictures of cubes to visualize both rotations (imitate what we did for $90°$ rotations in the lecture: draw the cube in original position, then after having applied the first rotation, and in final position after the second rotation.)

For 2×2 we have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$R(\theta)_x$
$R(\theta)_y$
$R(\theta)_z$

For 3×3

$\underline{x}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$\underline{y}$

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$\underline{z}$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) 3x3 Y matrix $= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

$$R(30^\circ)_y = \begin{pmatrix} \cos 30^\circ & 0 & \sin 30^\circ \\ 0 & 1 & 0 \\ -\sin 30^\circ & 0 & \cos 30^\circ \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix}$$

$$R(45^\circ)_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45 & -\sin 45 \\ 0 & \sin 45 & \cos 45 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

c) YX

$$= \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 + 0 + 0 & 0 + 0 + \frac{1}{2\sqrt{2}} & 0 + 0 + \frac{1}{2\sqrt{2}} \\ 0 & 0 + \frac{1}{\sqrt{2}} + 0 & 0 + \frac{1}{\sqrt{2}} + 0 \\ -\frac{1}{2} + 0 + 0 & 0 + 0 + \frac{\sqrt{3}}{2\sqrt{2}} & 0 + 0 + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

$$XY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 + 0 + 0 & 0 + 0 + 0 & \frac{1}{2} + 0 + 0 \\ 0 + 0 - \frac{1}{2\sqrt{2}} & 0 + \frac{1}{\sqrt{2}} + 0 & 0 + 0 + \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 + 0 - \frac{1}{2\sqrt{2}} & 0 + \frac{1}{\sqrt{2}} + 0 & 0 + 0 + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix}$$

original vertices with X Y   New coordinates

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + 0 \\ -1/2\sqrt{2} + 0 + 0 \\ -\frac{1}{2\sqrt{2}} + 0 + 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2\sqrt{2} \\ -1/2\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + 0 + 0 \\ 0 + 1/\sqrt{2} + 0 \\ 0 + 1/\sqrt{2} + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + 0 + 1/2 \\ 0 + 0 + \sqrt{3}/2 \\ 0 + 0 + \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + 0 \\ -1/2\sqrt{2} + 1/\sqrt{2} + 0 \\ -1/2\sqrt{2} + \frac{1}{\sqrt{2}} + 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ \frac{-\sqrt{2}+2\sqrt{2}}{4} \\ \frac{-\sqrt{2}+2\sqrt{2}}{4} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + 1/2 \\ -\frac{1}{2\sqrt{2}} + 0 + \sqrt{3}/2 \\ -1/2\sqrt{2} + 0 + \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}+1}{2} \\ \frac{-2+\sqrt{3}}{4\sqrt{2}} \\ \frac{-2+\sqrt{3}}{4\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + \frac{1}{2} \\ 1/\sqrt{2} + \sqrt{3}/2 \\ 1/\sqrt{2} + \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \frac{2+\sqrt{3}}{2\sqrt{2}} \\ \frac{2+\sqrt{3}}{2\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & 0 & \frac{1}{2} \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \\ -1/2\sqrt{2} & 1/\sqrt{2} & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + \frac{1}{2} \\ -1/2\sqrt{2} + \frac{1}{\sqrt{2}} + \sqrt{3}/2 \\ -\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}+1}{2} \\ \frac{4\sqrt{3}+2-2\sqrt{2}}{8} \\ \frac{4\sqrt{3}+4\sqrt{2}-2\sqrt{2}}{8} \end{pmatrix}$$

YX

$$\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + 0 \\ 0 + 0 + 0 \\ -\frac{1}{2} + 0 + 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + \frac{1}{2\sqrt{2}} + 0 \\ 0 + \frac{1}{\sqrt{2}} + 0 \\ 0 + \frac{\sqrt{3}}{2\sqrt{2}} + 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} \\ \sqrt{3}/2\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + 0 + \frac{1}{2\sqrt{2}} \\ 0 + 0 + \frac{1}{\sqrt{2}} \\ 0 + 0 + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} \\ \sqrt{3}/2\sqrt{2} \end{pmatrix}$$

$$
\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}
=
\begin{pmatrix} \sqrt{3}/2 + \frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} \\ -\frac{1}{2} + \sqrt{3}/2\sqrt{2} \end{pmatrix}
=
\begin{pmatrix} \frac{2\sqrt{6}+2}{4\sqrt{2}} \\ 1/\sqrt{2} \\ \frac{-2\sqrt{2}+2\sqrt{3}}{4\sqrt{2}} \end{pmatrix}
$$

$$
\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}
=
\begin{pmatrix} \sqrt{3}/2 + \frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} \\ -\frac{1}{2} + 0 + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
=
\begin{pmatrix} \frac{2\sqrt{6}+2}{4\sqrt{2}} \\ 1/\sqrt{2} \\ \frac{-2\sqrt{2}+2\sqrt{3}}{4\sqrt{2}} \end{pmatrix}
$$

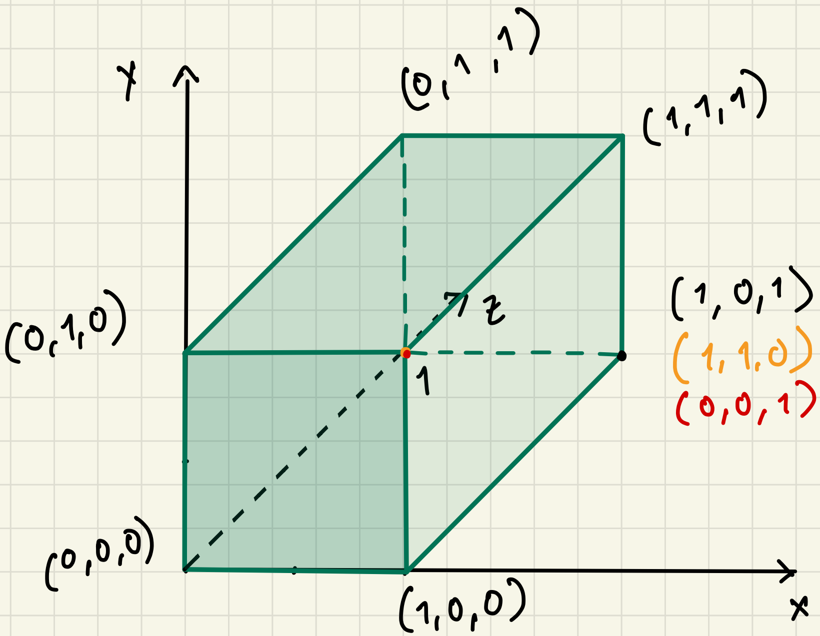
$$
\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
=
\begin{pmatrix} 0 + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ 0 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ 0 + \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
=
\begin{pmatrix} \frac{2}{2\sqrt{2}} \\ \frac{2}{\sqrt{2}} \\ \frac{2\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
$$

$$
\begin{pmatrix} \sqrt{3}/2 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
=
\begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} + 1/\sqrt{2} \\ -\frac{1}{2} + \frac{\sqrt{3}}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix}
=
\begin{pmatrix} \frac{4+\sqrt{3}}{4\sqrt{2}} \\ 2/\sqrt{2} \\ \frac{4\sqrt{3}-1}{4\sqrt{2}} \end{pmatrix}
$$

Rotation (sequential)

Cube vertices $(\pm 1, \pm 1, \pm 1)$ $\rightarrow$ X rotation $\rightarrow$ XY rotation

Cube vertices $(\pm 1, \pm 1, \pm 1)$ $\rightarrow$ Y rotation $\rightarrow$ YX rotation.



vertices vectors

(0, 0, 0)
(1, 0, 0)
(0, 1, 0)
(0, 0, 1)
(1, 1, 0)
(1, 0, 1)
(0, 1, 1)
(1, 1, 1)

vector rotation: (1^{st} rotation)

$$R_x(45^\circ) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

1. $R_x(45^\circ) \cdot \langle 0, 0, 0 \rangle$
2. $R_x(45^\circ) \cdot \langle 1, 0, 0 \rangle$
3. $R_x(45^\circ) \cdot \langle 0, 1, 0 \rangle$

4 $R_x(45^\circ) \cdot \langle 0, 0, 1 \rangle$

5 $R_x(45^\circ) \cdot \langle 1, 1, 0 \rangle$

6 $R_x(45^\circ) \cdot \langle 1, 0, 1 \rangle$

7 $R_x(45^\circ) \cdot \langle 0, 1, 1 \rangle$

8 $R_x(45^\circ) \cdot \langle 1, 1, 1 \rangle$

1 $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{3\times1} \text{(old)} = \begin{pmatrix} 0+0+0 \\ 0+0+0 \\ 0+0-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{(new)}$$

2 $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{3\times1} \text{(old)} \quad \begin{pmatrix} 1+0+0 \\ 0+0+0 \\ 0+0-0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{(new)}$$

3) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{3\times1} \text{(old)} = \begin{pmatrix} 0+0+0 \\ 0+1/\sqrt{2}+0 \\ 0+1/\sqrt{2}-0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}_{3\times1} \text{(new)}$$

4) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{3\times1} \text{(old)} = \begin{pmatrix} 0+0+0 \\ 0+0+1/\sqrt{2} \\ 0+0-1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}_{3\times1} \text{(new)}$$

5) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}_{3\times1} = \begin{pmatrix} 1+0+0 \\ 0+1/\sqrt{2}+1/\sqrt{2} \\ 0+1/\sqrt{2}-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \text{(new)}$$

6) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}_{3\times1} = \begin{pmatrix} 1+0+0 \\ 0+0+1/\sqrt{2} \\ 0+0-1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

7) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}_{3\times1} = \begin{pmatrix} 0+0+0 \\ 0+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} \\ 0+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 2/\sqrt{2} \\ 0 \end{pmatrix} \text{(new)}$$

8) $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}_{3\times3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}_{3\times1} = \begin{pmatrix} 1+0+0 \\ 0+1/\sqrt{2}+1/\sqrt{2} \\ 0+1/\sqrt{2}-1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix} \text{(new)}$$

2nd rotation

1. $R_y(30^\circ) \cdot \langle 0, 0, 0 \rangle$
2. $R_y(30^\circ) \cdot \langle 1, 0, 0 \rangle$
3. $R_y(30^\circ) \cdot \langle 0, 1/\sqrt{2}, 0 \rangle$
4. $R_x(30^\circ) \cdot \langle 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$
5. $R_x(30^\circ) \cdot \langle 1, \frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$
6. $R_x(30^\circ) \cdot \langle 1, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$
7. $R_x(30^\circ) \cdot \langle 0, 2/\sqrt{2}, 0 \rangle$
8. $R_x(30^\circ) \cdot \langle 1, \sqrt{2}, 0 \rangle$

1) $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

2) $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 0 \\ -1/2 \end{pmatrix}$$

3. $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

4. $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 + 0 - \frac{1}{2\sqrt{2}} \\ 0 + \frac{1}{\sqrt{2}} + 0 \\ 0 + 0 - \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2\sqrt{2}} \\ 1/\sqrt{2} \\ -\sqrt{3}/2\sqrt{2} \end{pmatrix}$$

5. $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} + 0 + \frac{1}{2\sqrt{2}} \\ 0 + \frac{2}{\sqrt{2}} + 0 \\ -\frac{1}{2} + 0 + \sqrt{3}/2\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{6}+2}{4\sqrt{2}} \\ 2/\sqrt{2} \\ \frac{-2\sqrt{2}+2\sqrt{3}}{4\sqrt{2}} \end{pmatrix}$$

6. $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 - \frac{1}{2\sqrt{2}} \\ 0 + \frac{1}{\sqrt{2}} + 0 \\ -\frac{1}{2} + 0 - \frac{\sqrt{3}}{2\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{6}-2}{4\sqrt{2}} \\ 1/\sqrt{2} \\ \frac{-2\sqrt{2}-2\sqrt{3}}{4\sqrt{2}} \end{pmatrix}$$

7. $$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 + 0 + 0 \\ 0 + \sqrt{2} + 0 \\ -\frac{1}{2} + 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \\ -\frac{1}{2} \end{pmatrix}$$

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$$\begin{pmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 + 0 + 0 \\ 0 + \sqrt{2} + 0 \\ -\frac{1}{2} + 0 + 0 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ \sqrt{2} \\ -\frac{1}{2} \end{pmatrix}$$

Sequential rotation

Original	X rotation	X Y rotation
O_v	$O_v \cdot R_x(45^\circ)$	$O_v \cdot R_x(45^\circ) \cdot R_y(30^\circ)$
original vertices	1^{st} rotation	2^{nd} rotation.
$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
$(1, 0, 0)$	$(1, 0, 0)$	$(\sqrt{3}/2, 0, -1/2)$
$(0, 1, 0)$	$(0, \frac{1}{\sqrt{2}}, 0)$	$(0, 1/\sqrt{2}, 0)$
$(0, 0, 1)$	$(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$(-\frac{1}{2\sqrt{2}}, 1/\sqrt{2}, -\sqrt{3}/2\sqrt{2})$
$(1, 1, 0)$	$(1, \sqrt{2}, 1/\sqrt{2})$	$(\frac{2\sqrt{6}+2}{4\sqrt{2}}, 2/\sqrt{2}, \frac{-2\sqrt{2}+2\sqrt{3}}{4\sqrt{2}})$
$(1, 0, 1)$	$(1, 1/\sqrt{2}, -1/\sqrt{2})$	$(\frac{2\sqrt{6}-2}{4\sqrt{2}}, 1/\sqrt{2}, \frac{-2\sqrt{2}-2\sqrt{3}}{4\sqrt{2}})$
$(0, 0, 1)$	$(0, \sqrt{2}, 0)$	$(0, \sqrt{2}, -\frac{1}{2})$
$(1, 1, 1)$	$(1, \sqrt{2}, 0)$	$(\sqrt{3}/2, \sqrt{2}, -\frac{1}{2})$

Task 7

A furniture company produces tables, chairs and benches, and sells these separately or in sets. For a table they need 12 units wood and 3 units metal. For a bench it is 6 units wood, 2 units metal and 5 units fabric. A chair is made from 2 units wood, 1 unit metal and 2 units fabric. Two types of sets are composed: set A consists of a table and four chairs, and set B contains a table, three chairs and a bench.

a) Find the production matrix M that is used to calculate the necessary amounts of wood, metal and fabric to produce x tables, y chairs and z benches.

b) Find the production matrix P that is used to calculate the necessary amounts of wood, metal and fabric to produce m sets of type A and n sets of type B.

c) An order is for 40 sets of type A, 60 sets of type B and 10 (extra) benches. How many units of wood, metal and fabric are needed for this order?

d) If a unit of wood costs \$2, a unit of metal \$4, and a unit of fabric \$7, how much does the furniture company pay for materials to produce the order in task c)?

Note: Solve tasks c) and d) with appropriate matrix products.

	tables	chairs	benches.
wood	12	2	6
metal	3	1	2
fabric	0	2	5

set A

1 tables 4 chairs 0 bench

set B

1 table 3 chairs 1 bench

a) production matrix M

$x_{tables} = \begin{pmatrix} 12 \\ 3 \\ 0 \end{pmatrix}$ $y_{chairs} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $z_{benches} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$

$$\begin{array}{cl} M = \begin{pmatrix} 12 & 2 & 6 \\ 3 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix} & \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array} \\ \begin{array}{ccc} \text{tables} & \text{chair} & \text{benches} \end{array} & \end{array}$$

Set A = 1 table
4 Chairs
0 benches.

$$\left(1\begin{pmatrix} 12 \\ 3 \\ 0 \end{pmatrix} \quad 4\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad 0\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \right)$$

$$= \begin{array}{cl} \begin{pmatrix} 12 & 8 & 0 \\ 3 & 4 & 0 \\ 0 & 8 & 0 \end{pmatrix} & \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array} \\ \begin{array}{ccc} \text{tables} & \text{chairs} & \text{benches} \end{array} & \end{array}$$

$$= \begin{array}{cl} \begin{pmatrix} 20 \\ 7 \\ 8 \end{pmatrix} & \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array} \\ \text{set A} & \end{array}$$

Set B = 1 table
3 Chairs
1 bench

$$= \left(1\begin{pmatrix} 12 \\ 3 \\ 0 \end{pmatrix} \quad 3\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 12 & 6 & 6 \\ 3 & 3 & 2 \\ 0 & 6 & 5 \end{pmatrix} \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array}$$

table chair bench.

$$= \begin{pmatrix} 12+6+6 \\ 3+3+2 \\ 0+6+5 \end{pmatrix} = \begin{array}{cl} \begin{pmatrix} 24 \\ 8 \\ 11 \end{pmatrix} & \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array} \end{array}$$

set B set B

c) $40(\text{Set } A) + 60(\text{Set } B) + 10 \begin{array}{cl} \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} & \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array} \end{array}$

benches

$$= 40\begin{pmatrix} 20 \\ 7 \\ 8 \end{pmatrix} + 60\begin{pmatrix} 24 \\ 8 \\ 11 \end{pmatrix} + 10\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabric} \end{array}$$

$$= \begin{pmatrix} 80 + 1440 + 60 \\ 28 + 480 + 20 \\ 32 + 66 + 50 \end{pmatrix} \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabic} \end{array}$$

$$\begin{array}{r} {}^{2}24 \\ 6 \\ \hline 144 \end{array}$$

$$= \begin{pmatrix} 1580 \\ 528 \\ 148 \end{pmatrix} \begin{array}{l} \text{wood} \\ \text{metal} \\ \text{fabic} \end{array}$$

d) wood \$2
metal \$4
fabric \$7

Total cost:

$$\Rightarrow \begin{pmatrix} 1580 \\ 528 \\ 148 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

$$= 1580(2) + 528(4) + 148(7)$$
$$= 3160 + 2112 + 1036 = \$6308$$

Task 8

a) We have seen, that (in general) $AB \neq BA$. There are some exceptions, but they are rare. To see how rare, solve the following question:
Let $A = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$. Find all 2×2-matrices B, so that $AB = BA$.

b) Find a 2×2-matrix $C \neq 0$ with $C^2 = 0$. To make it more interesting: No element of C may be zero.

a) $$\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2a + c & 2b + a \\ 4a - 3c & 4b - 3a \end{pmatrix} = \begin{pmatrix} 2a + 4b & a - 3b \\ 2c + 4d & c - 3d \end{pmatrix}$$

$2a + c = 2a + 4b$

$c = 4b$

$c = 0$

$2b + a = a - 3b$

$2b + 3b = 0$

$5b = 0$

$b = 0$

$4a - 3c = 2c + 4d$

$4a = 5c + 4d$

$4a = 4d$

$a = d$

$4b - 3a = c - 3d$

$-3a = c - 3d$

$a = d$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

let $a, d = 1$

$$\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2+0 & 0+1 \\ 4+0 & 0-3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2+0 & 1+0 \\ 0+4 & 0-3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

Let $a, d = 2$

$$\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4+0 & 0+2 \\ 8+0 & 0-6 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 8 & -6 \end{pmatrix}$$

$$= 2\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

Let $a, b = -1$

$$\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2+0 & 0-1 \\ -4+0 & 0+3 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -4 & 3 \end{pmatrix}$$

$$= -1\begin{pmatrix} 2 & 1 \\ 4 & -3 \end{pmatrix}$$

$$\therefore \quad B_{2\times 2} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{2\times 2} \text{ where } a, b \in R$$

b) Let $C_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ & $C \neq 0$ $C^2 = 0$

$$C^2_{2\times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= a^2 + bc = 0 \qquad ab + bd = 0$$

$$ac + cd = 0 \qquad cb + d^2 = 0$$

given no element of C may be 0

$\therefore$ let $a = 1$ & $c = -1$ (random choice)

$$= \quad 1 - b = 0 \qquad 1 + d = 0$$

$$b = 1 \qquad d = -1$$

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 - 1 & 1 - 1 \\ -1 + 1 & -1 + 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Task 9

Check if the following matrices are invertible. If they are, compute their inverses.

$$\text{(i) } A = \begin{pmatrix} 3 & -3 \\ 2 & -3 \end{pmatrix}, \quad \text{(ii) } B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix}, \quad \text{(iii) } C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 3 & 2 & 3 \\ 4 & 4 & 4 & 3 \end{pmatrix}$$

Condition for A^{-1} & solvability

$AA^{-1} = I$ also $|A| \neq 0$ then matrix invertable.

1) $\left(\begin{array}{cc|cc} 3 & -3 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right)$ $\underline{II} - \frac{2}{3} I$

$\underline{II} - \frac{2}{3} I$

$2 - \frac{2}{3}(3) = 0$

$-3 - \frac{2}{3}(-3) = \frac{-9+6}{3} = -1$

$0 - \frac{2}{3}(1) = -2/3$

$1 - \frac{2}{3}(0) = 1$

$\left(\begin{array}{cc|cc} 3 & -3 & 1 & 0 \\ 0 & -1 & -2/3 & 1 \end{array}\right)$ $I \div 3$

$\left(\begin{array}{cc|cc} 1 & -1 & 1/3 & 0 \\ 0 & -1 & -2/3 & 1 \end{array}\right)$ $\underline{I} - \underline{II}$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & -1 & -2/3 & 1 \end{array}\right) \quad \mathrm{II} \times -1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 2/3 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 2/3 & -1 \end{pmatrix}$$

b)

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\;2\;3}}^{A}} & \multicolumn{3}{c}{\overbrace{\hphantom{1\;0\;0}}^{I}} \\ 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & 4 & 7 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \mathrm{II} - \frac{2}{1}\mathrm{I} = \mathrm{II} - 2\mathrm{I} \\ \mathrm{III} - \mathrm{I} = \mathrm{III} - \mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 0 \end{array}\right) \quad \mathrm{III} - \frac{2}{-1}\mathrm{II} = \mathrm{III} + 2\,\mathrm{II}$$

$$\begin{array}{l} 0 + 2(0) \\ 2 + 2(-1) \\ 4 + 2(-2) \\ -1 + 2(-2) \\ 0 + 2(1) \\ 0 + 2(0) \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & -5 & 2 & 0 \end{array}\right)$$

$R(A) = 2 \qquad R(A \mid I) = 3$

$R(A) < R(A \mid I) \quad \therefore$ matrix is not solvable

pivot on RHS

Further confirming through det A

$$\left|\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 4 & 2 & 3 & 4 \\ 1 & 4 & 7 & 1 & 4 & 7 \end{array}\right.$$

$16 + 28 + 9 = 53$

$21 + 8 + 24 = 53$

$53 - 53 = 0$

$$= 14 + 8 + 24 - 16 - 28 - 6$$

$$14 - 16 + 8 - 28 + 24 - 6$$

$$= -2 - 20 + 18 = 2$$

$$1 \begin{vmatrix} 3 & 4 \\ 4 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= 21 - 16 - 2(14 - 4) + 3(8 - 3)$$

$$= 5 - 20 - 15 = 0$$

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3)

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 3 & 2 & 3 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \quad \div \frac{1}{2}$$

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 3 & 2 & 3 \\ 4 & 4 & 4 & 3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 2 & 3 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{matrix} \mathrm{III} - \frac{3}{1}\mathrm{I} \\ \mathrm{IV} - \frac{4}{1}\mathrm{I} \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \quad \times 2$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 0 & -3 & 2 & 0 \\ 0 & 2 & 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right) \quad \begin{matrix} \mathrm{III} - 3\,\mathrm{II} \\ \mathrm{IV} - \frac{2}{1}\mathrm{II} \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & -3 & -3 & -3 & 2 & 0 \\ 0 & 0 & -2 & -3 & -2 & -2 & 0 & 1 \end{array}\right) \quad \begin{matrix} \times -1 \\ \times -1 \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 3 & 3 & -2 & 0 \\ 0 & 0 & 2 & 3 & 2 & 2 & 0 & -1 \end{array}\right) \quad \mathrm{IV} - \frac{2}{5}\mathrm{III}$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 3 & 3 & -2 & 0 \\ 0 & 0 & 0 & \frac{9}{5} & \frac{4}{5} & \frac{4}{5} & \frac{4}{5} & -1 \end{array}\right) \quad \times 5$$

$$\left(\begin{array}{cccc|cccc} 1 & \frac{1}{2} & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 3 & 3 & -2 & 0 \\ 0 & 0 & 0 & 9 & 4 & 4 & 4 & -5 \end{array}\right) \quad \begin{matrix} \mathrm{I} - \frac{1}{9}\mathrm{IV} \\ \mathrm{II} - \frac{1}{9}\mathrm{IV} \\ \mathrm{III} - \frac{3}{9}\mathrm{IV} \end{matrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1/2 & 1 & 0 & -\frac{4}{9} & 1/18 & -\frac{4}{9} & \frac{5}{9} \\ 0 & 1 & 1 & 0 & 5/9 & -4/9 & -4/9 & 5/9 \\ 0 & 0 & 5 & 0 & -23/9 & -23/9 & -22/9 & 5/9 \\ 0 & 0 & 0 & 9 & 4 & 4 & 4 & -5 \end{array}\right)$$

$I - \frac{1}{5}\,\underline{III}$

$\underline{II} - \frac{1}{5}\,\underline{III}$

$$\left(\begin{array}{cccc|cccc} 1 & 1/2 & 0 & 0 & 1/15 & 17/30 & 2/45 & 4/9 \\ 0 & 1 & 0 & 0 & 16/15 & 1/15 & 2/45 & 4/9 \\ 0 & 0 & 5 & 0 & -23/9 & -23/9 & -22/9 & 5/9 \\ 0 & 0 & 0 & 9 & 4 & 4 & 4 & -5 \end{array}\right)$$

II

$$\frac{5}{9} + \frac{1}{5} \cdot \frac{23}{9} = \frac{225 + 207}{405} = \frac{432}{405} = \frac{144}{135} = \frac{48}{45} = \frac{16}{15}$$

$$-\frac{4}{9} + \frac{1}{5}\,\frac{23}{9} = \frac{-180 + 207}{405} = \frac{27}{405} = \frac{9}{135} = \frac{3}{45} = \frac{1}{15}$$

$$-\frac{4}{9} + \frac{1}{5}\,\frac{22}{9} = \frac{-180 + 198}{405} = \frac{18}{405} = \frac{6}{135} = \frac{2}{45} = \frac{2}{45}$$

$$\frac{5}{9} - \frac{1}{5} \cdot \frac{5}{9} = \frac{5}{9} - \frac{1}{9} = \frac{4}{9}$$

I

$$-\frac{4}{9} + \frac{1}{5}\,\frac{23}{9} = \frac{1}{15}$$

$$\frac{1}{18} + \frac{1}{5}\,\frac{23}{9} = \frac{45 + 414}{810} = \frac{459}{810} = \frac{153}{270} = \frac{51}{90} = \frac{17}{30}$$

$$-\frac{4}{9} + \frac{1}{5}\,\frac{22}{9} = \frac{2}{45}$$

$$\frac{5}{9} - \frac{1}{5}\,\frac{5}{9} = \frac{4}{9}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1/2 & 0 & 0 & 1/15 & 17/30 & 2/45 & 4/9 \\ 0 & 1 & 0 & 0 & 16/15 & 1/15 & 2/45 & 4/9 \\ 0 & 0 & 5 & 0 & -23/9 & -23/9 & -22/9 & 5/9 \\ 0 & 0 & 0 & 9 & 4 & 4 & 4 & -5 \end{array}\right) \begin{array}{l} \mathrm{I} - \frac{1}{2}\,\mathrm{II} \\ \\ \div 5 \\ \div 9 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -7/15 & 16/30 & 1/45 & 2/9 \\ 0 & 1 & 0 & 0 & 16/15 & 1/15 & 2/45 & 4/9 \\ 0 & 0 & 1 & 0 & -23/45 & -23/45 & -22/45 & 1/9 \\ 0 & 0 & 0 & 1 & 4/9 & 4/9 & 4/9 & -5/9 \end{array}\right)$$

$$\frac{1}{15} - \frac{1}{2}\cdot\frac{16}{15} = \frac{1-8}{15} = \frac{-7}{15}$$

$$\frac{17}{30} - \frac{1}{2}\cdot\frac{1}{15} = \frac{16}{30}$$

$$\frac{2}{45} - \frac{1}{2}\cdot\frac{2}{45} = 1/45$$

$$\frac{4}{9} - \frac{1}{2}\cdot\frac{4}{9}$$

Answer is wrong for row 1, 2 & 3
need to correct it later.

Task 10

Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 1 & 2 & -2 \end{pmatrix}.$$

Compute
(i) A^{-1}, (ii) $(2A)^{-1}$, (iii) $(A^T)^{-1}$ (iv), $(A^TA)^{-1}$

Note: For this task, only one inverse has to be computed.

$$A^{-1} = \left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & -1 \\ 3 & 1 & -3 \\ 1 & 2 & -2 \end{matrix}}_{A} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \end{array}\right) \begin{array}{l} \\ \mathrm{II} - 3\,\mathrm{I} \\ \mathrm{III} - \mathrm{I} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 2 & -1 & -1 & 0 & 1 \end{array}\right) \quad \text{III} - 2\,\text{II}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & 5 & -2 & 1 \end{array}\right) \quad \text{I} - \text{III}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & -1 & 5 & -2 & 1 \end{array}\right) \quad \times -1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & -1 \end{array}\right)$$

$$\underbrace{\begin{matrix} -4 & 2 & -1 \\ -3 & 1 & 0 \\ -5 & 2 & -1 \end{matrix}}_{A^{-1}}$$

② $2A^{-1} = 2^{-1}A^{-1} = \frac{1}{2}A^{-1}$

$$= 2A^{-1} = \begin{pmatrix} -2 & 1 & -1/2 \\ -3/2 & 1/2 & 0 \\ -5/2 & 1 & -1/2 \end{pmatrix}$$

③ $(A^T)^{-1} = (A^{-1})^T$

$$= \begin{pmatrix} -4 & -3 & -5 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$$

④ $(A^T A)^{-1} = (A^T)^{-1} \cdot A^{-1}$

$$= (A^{-1})^T \cdot A^{-1}$$

$$\begin{pmatrix} -4 & -3 & -5 \\ 2 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -4 & 2 & -1 \\ -3 & 1 & 0 \\ -5 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 16+9+25 & -8-3-10 & 4+0+5 \\ -8-3-10 & 4+1+4 & -2+0-2 \\ 4+0+5 & -2+0-2 & 1+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 50 & -21 & 9 \\ -21 & 9 & -4 \\ 9 & -4 & 2 \end{pmatrix}$$

Task 11

Let $A \in \mathbb{R}^{n \times n}$ be a matrix.

a) Show that the matrix $B = A + A^T$ is symmetric.

b) Show that the matrix $C = A - A^T$ is skew–symmetric. That means: $C = -C^T$.

c) Show that A is the sum of a symmetric and a skew-symmetric matrix.

d) Use c) to write the matrix

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -3 & 3 \\ 5 & -1 & 2 \end{pmatrix}$$

as the sum of a symmetric and a skew–symmetric matrix.

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

$A + A^T = \begin{pmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{pmatrix} = B$

Now $B^T = \begin{pmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{pmatrix}$

$B^T = B$ $\therefore$ it is a symmetric matrix

b) $A - A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

$= \begin{pmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{pmatrix} = C$

Now $C^T = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 0 & 2 \\ -4 & -2 & 0 \end{pmatrix}$

Here $C^T = -C$ $\therefore$ C is a skew symmetric matrix

Also the diagonals are 0

c) $B + C = A + A^T + A - A^T$

$= 2A = \text{scaled } A$

d) $M = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -3 & 3 \\ 5 & -1 & 2 \end{pmatrix}$

$M^T = \begin{pmatrix} 1 & 4 & 5 \\ 2 & -3 & -1 \\ -1 & 3 & 2 \end{pmatrix}$

$M + M^T = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -3 & 3 \\ 5 & -1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 5 \\ 2 & -3 & -1 \\ 1 & 3 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2 & 6 & 4 \\ 6 & -6 & 2 \\ 6 & 2 & 4 \end{pmatrix}$

$M - M^T = \begin{pmatrix} 0 & -2 & -6 \\ 2 & 0 & 4 \\ 4 & -4 & 0 \end{pmatrix}$

Sum: $\begin{pmatrix} 2 & 6 & 4 \\ 6 & -6 & 2 \\ 6 & 2 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -2 & -6 \\ 2 & 0 & 4 \\ 4 & -4 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 4 & -2 \\ 8 & -6 & 6 \\ 10 & -2 & 4 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1 & 2 & -1 \\ 4 & -3 & 3 \\ 5 & -1 & 2 \end{pmatrix}$$

$$= 2M$$

Task 12

There is something special about the main diagonal of a skew-symmetric matrix. What is it?

The diagonals are zero
The elements above the diagonal are of the opposite signs compared to the ones below the diagonal.

ie $A^T = -A$ $\qquad A_{(i,j)} = -A_{(j,i)}$

Study Guide for Applied Mathematics
Matrices 5-10: Determinants

You can use these selected problems (used as homework problems in past runs of this course) to pace your study progress.

Ms Neh, Dr Camps, and Prof Megill recommend that you regularly e-meet with your study group and discuss the material.

Task 0 (voluntary)

Read Chapter 31 *Tensors* in the Feynman Lectures on Physics, Volume II (s. link on Moodle, next to the lectures on determinants) to learn about some applications of matrices in optics, dynamics, elastostatics, and electromagnetics. Section 31-6 "The Tensor of Stress" should be particularly interesting to you.

Task 1

For the following example, determine geometrically that the splitting up into sums of rows works for determinants. Use the example

$$\det\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix},$$

and plot appropriate parallelograms.

Task 2

a) Compute the determinants of the following matrices:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 3 & 5 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 \\ 2 & 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$$

b) Compute the following determinants for the matrix A from a):

$$\det(2A), \quad \det(A^{-1}), \quad \det(A^2), \quad \det(A^T A A^T).$$

c) For what $x \in \mathbb{R}$ is the matrix C invertible?

Task 3

Determine whether the following matrices are invertible. If they are, compute the inverse.

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Also compute their determinants.

Task 4

a) Use Cramer's rule to solve the system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where b_1, b_2, b_3 are the last three digits of your student ID number. For example, if your student ID is 12345 then use 3, 4, 5.

b) Compute the inverse of the coefficient matrix (left-hand side) from task a) in two ways: On the one hand, use Gauss-Jordan; on the other hand, use the determinant formula

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T .$$

c) Use the inverse matrix found in b) to solve the system of linear equations in a) again.

Task 5

a) Solve the following SLE with Cramer's Rule:

$$\begin{aligned} 2x - y - 2z &= 5 \\ 4x + y + 2z &= 1 \\ 8x - y + z &= 5 \end{aligned}$$

b) For

$$M = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 5 & 1 \end{pmatrix}$$

find the inverse M^{-1}, using the determinant formula.

Task 6

a) Create 4×4 matrices with random single digit numbers (Write the numbers 0, 1, ..., 9 on small sheets of paper and draw them randomly from a box "with replacement", i.e. put them back in the box and reshuffle.). Avoid zero columns and identical columns.

   Alternatively: Write a Matlab (or C) programme that creates such a random matrix <u>and</u> checks for zero and identical columns.

b) Compute the determinant and the inverse of that matrix (Hint: Why are Laplace and Leibniz a bad idea here?). Show all steps and then check your results using Matlab (or C).

Task 7

For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix},$$

find the determinant $\det(A)$ using the Leibniz formula.

Task 8

We introduced the determinant of a 3×3 matrix as the volume of the box that has the three row vectors of the matrix as edges.

a) What does it mean for the determinant, if the three row vectors of the matrix lie in a plane?

b) Use the determinant of a (well-chosen) matrix to check if a point (x, y, z) and the points $(2, 1, 0)$ and $(1, 1, 0)$ lie in a plane.

c) Are the points $(2, 1, 0)$, $(1, 1, 0)$ and $(1, 0, -1)$ in a plane?

Task 9

a) Use the determinant formula to find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of a, b, c, and d.

b) Find the inverse of $B = \begin{pmatrix} 16 & 5 \\ 13 & 4 \end{pmatrix}$ with the formula from a). Check your result by multiplying B and B^{-1}.

Task 10

Find the determinant of

$$A = \begin{pmatrix} a & 2 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & b & 1 & -1 \end{pmatrix},$$

where a, b are real numbers. For what choices of a and b is the matrix invertible?

Task 11

Find a formula for the determinant of the matrix

$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

in terms of a, b, c.
Check the lectures for a matrix of that type, and verify with your formula that we got its determinant correctly.

Hint: The determinant is a product of three easy factors.

$$
\begin{array}{c}
\begin{array}{cccc} + & - & + & - \end{array} \\
\left|\begin{array}{cccc} 1 & 3 & 2 & 4 \\ -1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 \\ 5 & 0 & 1 & 2 \end{array}\right|
\end{array}
\begin{array}{c} - \\ + \\ - \\ + \end{array}
$$

$$
\underset{1}{-1\left|\begin{array}{ccc} 1 & 2 & 4 \\ -1 & 1 & 1 \\ 5 & 1 & 2 \end{array}\right|} - 3 \underset{2}{\left|\begin{array}{ccc} -1 & 1 & 1 \\ 2 & -1 & 0 \\ 5 & 1 & 2 \end{array}\right|} = -1(-9) - 3(5) = 9 - 15 = -6
$$

1

$$
\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 2 & 4 \\ -1 & 1 & 1 & -1 & 1 & 1 \\ 5 & 1 & 2 & 5 & 1 & 2 \end{array}
$$

$(2 + 10 - 4) - (1 - 4 + 20)$

$8 - 17 = -9$

$$
\begin{array}{ccc|ccc} -1 & 1 & 1 & -1 & 1 & 1 \\ 2 & -1 & 0 & 2 & -1 & 0 \\ 5 & 1 & 2 & 5 & 1 & 2 \end{array}
$$

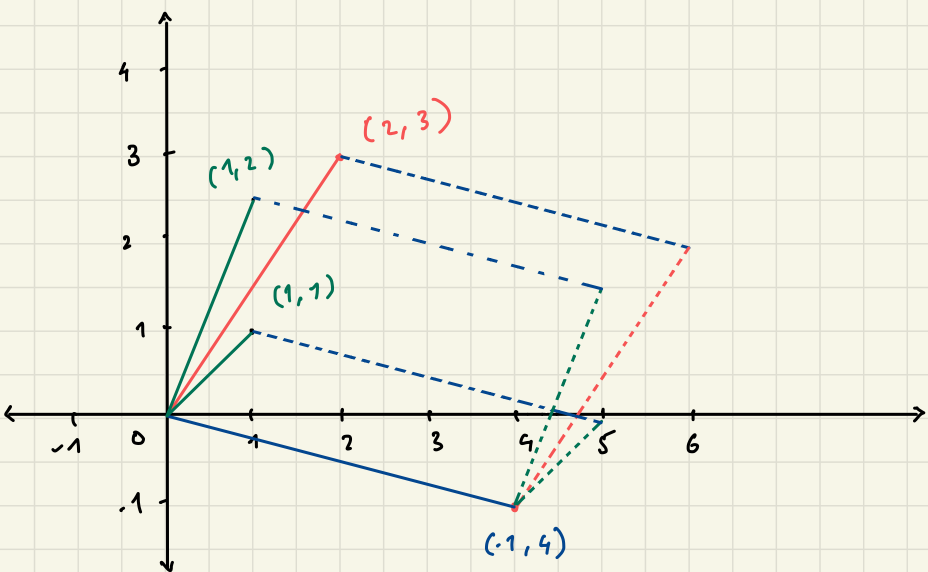
$(2 + 0 + 2) - (0 + 4 - 5) = 5$

Task 1

For the following example, determine geometrically that the splitting up into sums of rows works for determinants. Use the example

$$
\det\begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix},
$$

and plot appropriate parallelograms.



Area of smaller:

$4-(-1) = 5$

Area = bh

$4-(-2) = 6$

Total = $5+6 = 11$

b = 4

bh = 5

$h = \frac{5}{4}$

bh = 6

$h = \frac{6}{4}$

Area of larger:

$8-(-3) = 8+3 = 11$

Task 2

a) Compute the determinants of the following matrices:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 3 & 5 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 \\ 2 & 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$$

b) Compute the following determinants for the matrix A from a):

$$\det(2A), \quad \det(A^{-1}), \quad \det(A^2), \quad \det(A^T A A^T).$$

c) For what $x \in \mathbb{R}$ is the matrix C invertible?

7:59

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$

$$= \begin{array}{ccc|ccc} 2 & 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 0 & -1 & 2 & 0 & -1 & 2 \end{array}$$

$$= 4 + 0 + 0 - (-6 + 2 + 0)$$
$$= 4 - (-4) = 8$$

8:00

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 2 \end{pmatrix}$$
$$= 2 \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} + 0$$
$$= 2(2+3) - 1(2-0)$$
$$= 10 - 2 = 8$$

$$B = \begin{pmatrix} 0 & 2 & 3 & 5 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

8:04

$$-1 \begin{pmatrix} 1 & 3 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 5 \\ 2 & 0 & 1 & 3 \end{pmatrix} \quad \begin{array}{l} \underline{II} + \underline{I} \\ \\ \underline{IV} - \frac{2}{1}\underline{I} \end{array}$$

$$-1 \begin{pmatrix} 1 & 3 & 1 & 4 \\ 0 & 4 & 1 & 4 \\ 0 & 2 & 3 & 5 \\ 0 & -6 & -1 & -5 \end{pmatrix}$$

$$= (-1)(1) \begin{vmatrix} 4 & 1 & 4 \\ 2 & 3 & 5 \\ -6 & -1 & -5 \end{vmatrix}$$

$$\begin{array}{ccc|ccc} 4 & 1 & 4 & 4 & 1 & 4 \\ 2 & 3 & 5 & 2 & 3 & 5 \\ -6 & -1 & -5 & -6 & -1 & -5 \end{array}$$

8:11

$= (-60-30-8)-(-20-10-72)$

wrong!

$= -98-(-102) = 102-98 = 4$

8:11

$$B \quad \begin{array}{cccc} + & - & + & - \\ \end{array}$$
$$\begin{pmatrix} 0 & 2 & 3 & 5 \\ -1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 4 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

$$-(-1)\begin{vmatrix} 2 & 3 & 5 \\ 3 & 1 & 4 \\ 0 & 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 & 5 \\ 1 & 1 & 4 \\ 2 & 1 & 3 \end{vmatrix}$$

$$\begin{array}{ccc|ccc} 2 & 3 & 5 & 2 & 3 & 5 \\ 3 & 1 & 4 & 3 & 1 & 4 \\ 0 & 1 & 3 & 0 & 1 & 3 \end{array} \qquad \begin{array}{ccc|ccc} 0 & 3 & 5 & 0 & 3 & 5 \\ 1 & 1 & 4 & 1 & 1 & 4 \\ 2 & 1 & 3 & 2 & 1 & 3 \end{array}$$

$(6+0-15)-(8+27+0)$

$(-9) - 35$

$=$

$(0+24+5)-(0+3+10)$

$= 29-30$

$= -1$

$-(-1)(-2) \quad +1(-1)$

$1(-2) \quad +1(-1)$

$-2-1=-3$

8:17

wrong!

$$1\begin{vmatrix} 2 & 3 & 5 \\ 3 & 1 & 4 \\ 0 & 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 & 5 \\ 1 & 1 & 4 \\ 2 & 1 & 3 \end{vmatrix}$$

8:20

(1) (2)

(1) $= 2(3-4) - 3(9-0) + 5(3-0)$

$= -2 - 27 + 15$

$= -29 + 15 = -14$

(2) $0 - 3(3-8) + 5(1-2)$

$= -3(-5) + 5(-1)$

$= 15 - 5 = 10$

correct

$= -14 + 10 = -4$

c) $C = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$

$$x\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & x \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ x & 1 \end{vmatrix}$$

$= x(x^2-1) - (x-1) + 1(1-x)$

if $\det(C) = 0$ it is not invertible

$\therefore$ every other possibility will be:

all nos where x does not lead to 0 for RHS ie $\quad x(x^2-1)-(x-1)-(x-1)=0$

$$(x-1)(x(x+1)-1-1)=0$$

$$(x-1)(x^2+x-2)=0$$

$-1 \cdot 2 = -2$ $\quad (x-1)(x^2-x+2x-2)=0$

$-1+2=1$ $\quad (x-1)(x(x-1)+2(x-1))=0$

$$(x-1)(x-1)(x+2)=0$$

$$x=1 \qquad x=-2$$

C is invertable for $\{x \in R;\ \text{where } x \neq 1, 2\}$

Task 3

Determine whether the following matrices are invertible. If they are, compute the inverse.

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Also compute their determinants.

A: $\quad \det A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \quad \underline{II} - \frac{1}{-2} I$

$\underline{II} + \frac{1}{2} I$

$1 + \frac{1}{2}(-2) = 0$

$-2 + \frac{1}{2}(1) = \frac{-4+1}{2} = \frac{-3}{2}$

$$1 + \frac{1}{2}(0) = 1$$

$$0 + \frac{1}{2}(0) = 0$$

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$= -2 \begin{vmatrix} -3/2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{vmatrix}$$

1

$$-2 \left(-\frac{3}{2}(4-1) - 1(-2-0) \right)$$

$$= -2 \left(-\frac{3}{2}(3) + 2 \right)$$

$$= -2 \left(\frac{-9+4}{2} \right) = -\frac{5}{2}(-2) = 5$$

det $A \neq 0$ $\therefore$ inverse is possible

$$\left(\begin{matrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{matrix} \;\middle|\; \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right) \quad \text{II} - \frac{1}{-2}\,\text{I}$$

$\underline{II} + \frac{1}{2}\ \underline{I}$

$1 + \frac{1}{2}(-2) = 0$

$-2 + \frac{1}{2} \cdot 1 = -\frac{3}{2}$

$1 + \frac{1}{2} \cdot 0 = 1$

$0 + \frac{1}{2} \cdot 0 = 0$

$0 + \frac{1}{2} \cdot 1 = \frac{1}{2}$

$1 + 1/2 \cdot 0 = 1$

$0 + 1/2 \cdot 0 = 0$

$0 + 1/2 \cdot 0 = 0$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \underline{III} - \frac{1}{-3/2}\ \underline{II}$$

$\underline{III} + \frac{2}{3}\ \underline{II}$

$0 + \frac{2}{3}\ 0 = 0$

$1 + \frac{2}{3}\ -\frac{3}{2} = 0$

$-2 + \frac{2}{3} \cdot 1 = -\frac{4}{3}$

$1 + \frac{2}{3} \cdot 0 = 1$

$$0 + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

$$0 + \frac{2}{3} \cdot 1 = \frac{2}{3}$$

$$1 + \frac{2}{3} \cdot 0 = 1$$

$$0 + \frac{2}{3} \cdot 0 = 0$$

$$\begin{array}{c} \left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & -4/3 & 1 & 1/3 & 2/3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \times 2 \\ \times 3 \\ \\ \end{array} \end{array}$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -4 & 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ \underline{IV} - \frac{1}{-4}\, \underline{III} \end{array}$$

$$\underline{IV} + \frac{1}{4}\, \underline{III}$$

$$0 + \frac{1}{4} \cdot 0 = 0 \qquad 0 + \frac{1}{4}\, 1 = 1/4$$

$$0 + \frac{1}{4} \cdot 0 = 0 \qquad 0 + \frac{1}{4}\, 2 = 1/2 \quad \times 4 = 2$$

$$1 + \frac{1}{4} \cdot -4 = 0 \qquad 0 + \frac{1}{4}\, 3 = 3/4$$

$$-2 + \frac{1}{4} \cdot 3 = \frac{-8+3}{4} = \frac{-5}{4} \qquad 1 + \frac{1}{4}\, 0 = 1$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -4 & 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -5 & 1 & 2 & 3 & 4 \end{array}\right) \quad \underline{III} - \frac{3}{-5}\underline{IV}$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -20 & 0 & 8 & 16 & 24 & 32 \\ 0 & 0 & 0 & -5 & 1 & 2 & 3 & 4 \end{array}\right) \quad \underline{II} - \frac{2}{-20}\underline{III}$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -120 & 0 & 0 & 16 & 72 & 48 & 64 \\ 0 & 0 & -20 & 0 & 8 & 16 & 24 & 32 \\ 0 & 0 & 0 & -5 & 1 & 2 & 3 & 4 \end{array}\right) \quad \underline{I} - \frac{1}{-120}\underline{II}$$

$$\left(\begin{array}{cccc|cccc} -2 & 0 & 0 & 0 & 136/120 & \frac{72}{120} & \frac{48}{120} & \frac{64}{120} \\ 0 & -120 & 0 & 0 & 16 & 72 & 48 & 64 \\ 0 & 0 & -20 & 0 & 8 & 16 & 24 & 32 \\ 0 & 0 & 0 & -5 & 1 & 2 & 3 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{272}{120} & -\frac{144}{120} & -\frac{96}{120} & -\frac{128}{120} \\ 0 & 1 & 0 & 0 & -16/120 & -72/120 & -\frac{48}{120} & -64/120 \\ 0 & 0 & 1 & 0 & -8/20 & -16/20 & -24/20 & -32/20 \end{array}\right.$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & -1/5 & -2/5 & -3/5 & -4/5 \end{array}\right)$$

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(2)

$$B = \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\det B = -2 \begin{vmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\left[-2(-2-1) - 1(1-0)\right] - 1\left[1(-2-1) - 1(0-0)\right]$$

$$= -2\left[6-1\right] - 1\left[-3-0\right]$$

$$= -10 + 3 = -7 \neq 0 \therefore B^{-1} \text{ is possible}$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad II - \frac{1}{-2} I$$

$$II + \frac{1}{2} I$$

$$1 + \frac{1}{2} -2 = 0 \qquad 0 + \frac{1}{2} 1 = 1/2$$

$$-2 + \frac{1}{2} 1 = \frac{-3}{2} \qquad 1 + \frac{1}{2} 0 = 1$$

$\Bigg\} \times 2$

$1 + \frac{1}{2} 0 = 1 \qquad 0 + \frac{1}{2} 0 = 0$

$0 + \frac{1}{2} 0 = 0 \qquad 0 + \frac{1}{2} 0 = 0$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \text{III} - \frac{1}{-3}\text{II}$$

$\text{III} + \frac{1}{3}\text{II}$

$0 + \frac{1}{3} 0 = 0 \qquad 0 + \frac{1}{3} \cdot 1 = \frac{1}{3}$

$1 + \frac{1}{3} -3 = 0 \qquad 0 + \frac{1}{3} \cdot 2 = \frac{2}{3}$

$-2 + \frac{1}{3} 2 = \frac{-4}{3} \qquad 1 + \frac{1}{3} \cdot 0 = 1$

$1 + \frac{1}{3} 0 = 1 \qquad 0 + \frac{1}{3} \cdot 0 = 0$

$\times 3$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -4 & 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \text{IV} - \frac{1}{-4}\text{III}$$

$\text{IV} + \frac{1}{4}\text{III}$

$0 + \frac{1}{4} 0 = 0 \qquad 0 + \frac{1}{4} 1 = \frac{1}{4}$

$0 + \frac{1}{4} 0 = 0 \qquad 0 + \frac{1}{4} 2 = \frac{2}{4}$

$\times 4$

$$1+\frac{1}{4}\cdot(-4)=0 \qquad 0+\frac{1}{4}\,3=\frac{3}{4}$$

$$1+\frac{1}{4}\,3=\frac{7}{4} \qquad 1+\frac{1}{4}\,0=1$$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -4 & 3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 7 & 1 & 2 & 3 & 4 \end{array}\right) \underline{III}-\frac{3}{7}\underline{IV}$$

$\underline{III}-\frac{3}{7}\underline{IV}$

$$0-\frac{3}{7}\,0=0 \qquad 1-\frac{3}{7}\,1=\frac{4}{7}$$

$$0-\frac{3}{7}\,0=0 \qquad 2-\frac{3}{7}\cdot 2=\frac{8}{7}$$

$$-4-\frac{3}{7}\cdot 0=-4 \qquad 3-\frac{3}{7}\cdot 3=\frac{12}{7}$$

$$3-\frac{3}{7}\cdot 7=0 \qquad 0-\frac{3}{7}\cdot 4=-\frac{12}{7}$$

$\times 7$

$$\left(\begin{array}{cccc|cccc} -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & -28 & 0 & 4 & 8 & 12 & -12 \\ 0 & 0 & 0 & 7 & 1 & 2 & 3 & 4 \end{array}\right) \underline{II}-\frac{2}{-28}\underline{III}$$

$$\mathrm{II} + \frac{2}{28}\,\mathrm{III}$$

$$
\begin{aligned}
0 + \frac{2}{28}\,0 &= 0 \\
-3 + \frac{2}{28}\,0 &= -3 \\
2 + \frac{2}{28}\,{-28} &= 0 \\
0 + \frac{2}{28}\,0 &= 0
\end{aligned}
\qquad
\left.
\begin{aligned}
1 + \frac{2}{28} \cdot 4 &= \frac{36}{28} \\
2 + \frac{2}{28} \cdot 8 &= \frac{72}{28} \\
0 + \frac{2}{28} \cdot 12 &= \frac{24}{28} \\
0 + \frac{2}{28} \cdot -12 &= -\frac{24}{28}
\end{aligned}
\right\} \times 28
$$

$$
\left(
\begin{array}{cccc|cccc}
-2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -84 & 0 & 0 & 36 & 72 & 24 & -24 \\
0 & 0 & -28 & 0 & 4 & 8 & 12 & -12 \\
0 & 0 & 0 & 7 & 1 & 2 & 3 & 4
\end{array}
\right)
\begin{array}{l}
\mathrm{I} - \frac{1}{-84}\,\mathrm{II} \\
\div -84 \\
\div -28 \\
\div 7
\end{array}
$$

$$\mathrm{I} + \frac{1}{84}\,\mathrm{II}$$

$$
\begin{aligned}
-2 + \frac{1}{84} \cdot 0 &= -2 \\
1 + \frac{1}{84} \cdot -84 &= 0 \\
0 + \frac{1}{84} \cdot 0 &= 0 \\
0 + \frac{1}{84} \cdot 0 &= 0
\end{aligned}
\qquad
\left.
\begin{aligned}
1 + \frac{1}{84} \cdot 36 &= \frac{120}{84} \\
0 + \frac{1}{84} \cdot 72 &= \frac{72}{84} \\
0 + \frac{1}{84} \cdot 24 &= \frac{24}{84} \\
0 + \frac{1}{84} \cdot -24 &= -\frac{24}{84}
\end{aligned}
\right\} \div -2
$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{60}{84} & -\frac{36}{84} & -\frac{12}{84} & \frac{12}{84} \\ 0 & 1 & 0 & 0 & -36/84 & -72/84 & -24/84 & 24/84 \\ 0 & 0 & 1 & 0 & -4/28 & -8/28 & -12/28 & 12/28 \\ 0 & 0 & 0 & 1 & 1/7 & 2/7 & 3/7 & 4/7 \end{array}\right)$$

Task 4

a) Use Cramer's rule to solve the system of linear equations

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

where b_1, b_2, b_3 are the last three digits of your student ID number. For example, if your student ID is 12345 then use 3, 4, 5.

b) Compute the inverse of the coefficient matrix (left-hand side) from task a) in two ways: On the one hand, use Gauss-Jordan; on the other hand, use the determinant formula

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T .$$

c) Use the inverse matrix found in b) to solve the system of linear equations in a) again.

a) $b_1 = 5$ $\quad x = \frac{D_x}{D}$ $\quad y = \frac{D_y}{D}$ $\quad z = \frac{D_z}{D}$

$b_2 = 6$

$b_3 = 2$

$$\therefore \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}$$

3×3 $\quad 3 \times 1$

3×1

$$D = \det(A)$$
$$= 1(4-2) + 1(4-1) + 1(2-1)$$
$$= 2 + 3 + 1 = 6$$

$$D_x = \begin{vmatrix} 5 & 1 & 1 \\ 6 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 5(4-2) - 6(4-1) + 2(2-1)$$
$$= 10 - 18 + 2 = -6$$

$$D_y = \begin{vmatrix} 1 & 5 & 1 \\ -1 & 6 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 1(24-4) + 1(20-2) + 1(10-6)$$
$$= 20 + 18 + 4$$
$$= 42$$

$$D_z = \begin{vmatrix} 1 & 1 & 5 \\ -1 & 1 & 6 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-6) + 1(2-5) + 1(6-5)$$
$$= -4 - 3 + 1 = -6$$

$$x = \frac{-6}{6} = -1 \qquad y = \frac{42}{6} = 7 \qquad z = \frac{-6}{6} = -1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ -1 \end{pmatrix}$$

Check:

$$x + y + z = -1 + 7 - 1 = 5$$
$$-x + y - 2z = 1 + 7 - 2 = 6$$
$$x + y + 4z = -1 + 7 - 4 = 2$$

$$\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix} \text{RHS}$$

A^{-1} is possible as $\det A \neq 0$

- now check $R(A) < R(A|B)$ no soln
$R(A) = R(A|B)$ solution exist.
$\#\text{params} =$ variable num $- R(A)$

} other gaussian SLE solution check recall.

b) Gauss Jordan: Method 1 for A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} II + 1\,I \\ III - 1\,I \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} I - \frac{1}{3}\,III \\ II - III \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 4/3 & 0 & -1/3 \\ 0 & 2 & 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \quad I - \frac{1}{2}\,II$$

$I - \frac{1}{3}\,III$

$1 - \frac{1}{3} \cdot 0 = 1 \qquad 1 - \frac{1}{3} \cdot -1 = 1 + \frac{1}{3} = \frac{4}{3}$

$1 - \frac{1}{3} \cdot 0 = 1 \qquad 0 - \frac{1}{3} \cdot 0 = 0$

$1 - \frac{1}{3} \cdot 3 = 0 \qquad 0 - \frac{1}{3} \cdot 1 = -\frac{1}{3}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/2 & -1/6 \\ 0 & 2 & 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right)$$

$$\frac{4}{3} - \frac{1}{2}\cdot 2 = \frac{4}{3} - 1 = \frac{1}{3}$$

$$0 - \frac{1}{2}\cdot 1 = -\frac{1}{2}$$

$$-\frac{1}{3} + \frac{1}{2}\cdot 1 = -\frac{1}{3} + \frac{1}{2} = \frac{-2+3}{6} = \frac{1}{6}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/2 & 1/6 \\ 0 & 1 & 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right)$$

$$\underbrace{\begin{pmatrix} 1/3 & -1/2 & 1/6 \\ 1 & 1/2 & -1/2 \\ -1/3 & 0 & 1/3 \end{pmatrix}}_{A^{-1}}$$

Correct on 1st attempt.

Method 2: Determinant application formula:

We have $\det A = -6$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$C_{11} = 4 - 2 = 2$ $\quad C_{12} = -4 - 2 = -6$ $\quad C_{13} = -1 - 1 = -2$

$C_{21} = 4 - 1 = 3$ $\quad C_{22} = 4 - 1 = 3$ $\quad C_{23} = 1 - 1 = 0$

$C_{31} = 2 - 1 = 1$ $\quad C_{32} = 2 + 1 = 3$ $\quad C_{33} = 1 + 1 = 2$

$$A^{-1} = \frac{1}{-6}\begin{pmatrix} +2 & - -6 & + -2 \\ -3 & +3 & -0 \\ +1 & -3 & +2 \end{pmatrix}^T$$

Take care of cofactors position signs!

$$= \frac{1}{-6} \begin{pmatrix} 2 & -3 & 1 \\ +6 & 3 & -3 \\ -2 & 0 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1/3 & -1/2 & -1/6 \\ 1 & -1/2 & -1/2 \\ 1/3 & 0 & -1/3 \end{pmatrix}$$

wrong

Method 3: Expansion matrix

we have $\det A = -\frac{1}{6}$

for cofactors:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & -1 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & -1 & 1 \end{array}$$

$\downarrow$ Column wisse

$\longrightarrow$ write row wise

$C_{11} = 4-2 = 3$ $C_{12} = 4-1 = 3$ $C_{13} = 2-1 = 1$

$C_{22} = 2+4 = 6$ $C_{22} = 4-1 = 3$ $C_{23} = -1-2 = -3$

$C_{33} = -1-1 = -2$ $C_{32} = 1-1 = 0$ $C_{33} = 1+1 = 2$

$$A^{-1} = \frac{1}{-6} \begin{pmatrix} 3 & 3 & 1 \\ 6 & 3 & -3 \\ -2 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 & -1/3 & -1/6 \\ -1 & -1/2 & 1/2 \\ 1/3 & 0 & -1/3 \end{pmatrix}$$

wrong.

Task 5

a) Solve the following SLE with Cramer's Rule:

$$\begin{aligned} 2x - y - 2z &= 5 \\ 4x + y + 2z &= 1 \\ 8x - y + z &= 5 \end{aligned}$$

b) For

$$M = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 2 & 1 \\ -1 & 5 & 1 \end{pmatrix}$$

find the inverse M^{-1}, using the determinant formula.

$$\underbrace{\begin{pmatrix} 2 & -1 & -2 \\ 4 & 1 & 2 \\ 8 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 5 \end{pmatrix}$$

$$A = \det(A) = 2(1+2) + 1(4-16) - 2(-4-8)$$

$$= 2(3) + (-12) - 2(-12)$$

$$= 6 - 12 + 24 = 12 + 6 = 18$$

$$A_x = \begin{pmatrix} 5 & -1 & -2 \\ 1 & 1 & 2 \\ 5 & -1 & 1 \end{pmatrix}$$

$$\det(A_x) = 5(1+2) + 1(1-10) - 2(-1-5)$$
$$= 15 - 9 + 12$$
$$= 27 - 9 = 18$$

$$A_y = \begin{pmatrix} 2 & 5 & -2 \\ 4 & 1 & 2 \\ 8 & 5 & 1 \end{pmatrix}$$

$$2(1-10) - 4(5+10) + 8(10+2)$$
$$2(-9) - 4(15) + 8(12)$$
$$= -18 - 60 + 96 = 18$$

$$\det(A_y) = 2(1-10) - 5(4-16) - 2(\overset{20}{\cancel{40}} - 8)$$
$$= -18 + 60 - 24$$
$$= -42 + 60 = 18$$

$$A_z = \begin{pmatrix} 2 & -1 & 5 \\ 4 & 1 & 1 \\ 8 & -1 & 5 \end{pmatrix}$$
$$= 2(5+1) + 1(20-8) + 5(-4-8)$$
$$= 12 + 12 - 60 = -36$$

$$x = \frac{\det A_x}{\det A} = \frac{18}{18} = 1 \qquad y = \frac{\det A_y}{\det A} = \frac{18}{18} = 1 \qquad z = \frac{\det A_z}{\det A} = \frac{-36}{18} = -2$$

check $\begin{pmatrix} 2 & -1 & -2 \\ 4 & 1 & 2 \\ 8 & -1 & 1 \end{pmatrix}_{3\times3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{3\times1}$

$$= \begin{aligned} 2x - y - 2z &= 2 - 1 + 4 = 5 \\ 4x + y + 2z &= 4 + 1 - 4 = 1 \\ 8x - y + 1z &= 8 - 1 - 2 = 5 \end{aligned}$$

Task 7

For

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix},$$

find the determinant $\det(A)$ using the Leibniz formula.

$\det A = \begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 & 2 \end{array}$: Sarrus

$1 + 12 + 18 - 4 - 6 - 9$

$= 31 - 19 = 12$

$$1\begin{vmatrix}1 & 2\\2 & 1\end{vmatrix} - 3\begin{vmatrix}2 & 3\\2 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix}$$

$$= 1(1-4) - 3(2-6) + 3(4-3)$$

$$= -3 + 12 + 3 = \cancel{10}\ 12$$

$$1\begin{vmatrix}1 & 2\\2 & 1\end{vmatrix} - 2\begin{vmatrix}3 & 2\\3 & 1\end{vmatrix} + 3\begin{vmatrix}3 & 1\\3 & 2\end{vmatrix}$$

$$(1-4) - 2(3-6) + 3(6-3)$$

$$= -3 + 6 + 9 = 12$$

Task 8

We introduced the determinant of a 3×3 matrix as the volume of the box that has the three row vectors of the matrix as edges.

a) What does it mean for the determinant, if the three row vectors of the matrix lie in a plane?

b) Use the determinant of a (well-chosen) matrix to check if a point (x, y, z) and the points $(2, 1, 0)$ and $(1, 1, 0)$ lie in a plane.

c) Are the points $(2, 1, 0)$, $(1, 1, 0)$ and $(1, 0, -1)$ in a plane?

a) The determinant will be zero as one of the row vector is a linearly dependent on the other and it does not spam to form 3-D.

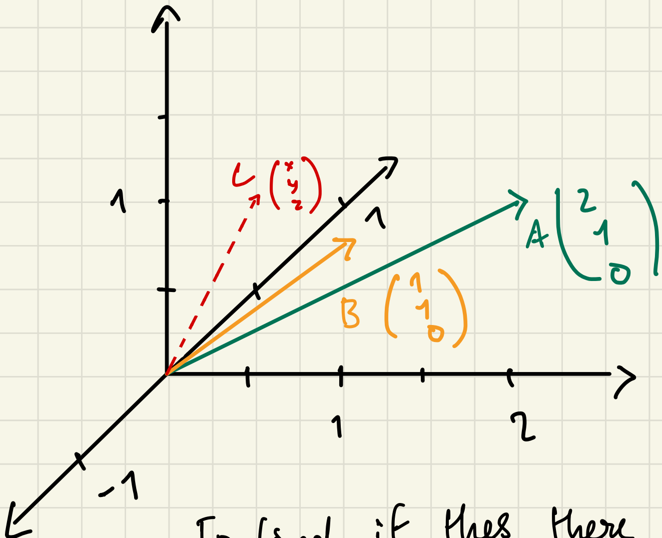
There fore volume will be zero

b)

Let A be $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Let B be $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Let C be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$



To find if thes there vectors lie on a plane we can form vectors combination AB, AC that can form a component together & then find its determinant

$$AB = B - A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$AC = C - A = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x-2 \\ y-1 \\ z \end{pmatrix}$$

For points to lie on the plane :

$$AB \times AC = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 0 \\ x-2 & y-1 & z \end{vmatrix} = 0$$

$$= 0\hat{i} - \hat{j}(-z) + \hat{k}(-y+1-0) = 0$$

$$\begin{pmatrix} 0 \\ 1z \\ -y+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x = 0 \qquad z = 0 \qquad y = 1$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a point on the plane.

c)
$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 2(-1) - 1(-1) + 0$$

$= -2 + 1 = -1 \neq 0 \quad \therefore$ they are not in the plane

Task 9

a) Use the determinant formula to find the inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of a, b, c, and d.

b) Find the inverse of $B = \begin{pmatrix} 16 & 5 \\ 13 & 4 \end{pmatrix}$ with the formula from a). Check your result by multiplying B and B^{-1}.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11}^{+} & C_{12}^{-} \\ C_{21}^{-} & C_{22}^{+} \end{pmatrix}^T$$

$$= \det A = ad - bc$$

$$C_{11} = d \qquad C_{12} = d \qquad C_{12} = -c$$

$$C_{21} = -b \qquad C_{22} = a$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

b) $B = \begin{pmatrix} 16 & 5 \\ 13 & 9 \end{pmatrix}$

$\det B = (64 - 65) = -1$

$B^{-1} = \frac{1}{-1} \begin{pmatrix} 4 & -5 \\ -13 & 16 \end{pmatrix} = \begin{pmatrix} -4 & 5 \\ 13 & -16 \end{pmatrix}$

Task 10

Find the determinant of

$$A = \begin{pmatrix} a & 2 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & b & 1 & -1 \end{pmatrix},$$

where a, b are real numbers. For what choices of a and b is the matrix invertible?

A^{-1} is possible when $\det A \neq 0$

$\therefore$ to find $a, b \in \mathbb{R}$ we find all possibilities of a, b that makes $\det A = 0$ and exclude them from the selection.

$$A = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{pmatrix} a & 2 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 0 & b & 1 & -1 \end{pmatrix} \end{array}$$

Finding det using 2^{nd} column

$$2\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} + b\begin{vmatrix} a & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$$2\left[1(-2-1) - 1(-1-2)\right] +$$

$$b\left[a(1-4) - 1(3-2) + 1(2-1)\right] = 0$$

$$2\left[1(-3) - 1(-3)\right] = 2\left[-3+3\right] = 0$$

$$\therefore \; b\left[-3a - 1 + 1\right] = 0$$

$$b = 0 \qquad -3a = 0$$

$$a = 0$$

A^{-1} is possible for all $a, b \in R; a, b \neq 0$

Task 11

Find a formula for the determinant of the matrix

$$A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

in terms of a, b, c.
Check the lectures for a matrix of that type, and verify with your formula that we got its determinant correctly.

Hint: The determinant is a product of three easy factors.

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} = A$$

$$\det(A) = 1\left(bc^2 - b^2c\right) - a\left(c^2 - b^2\right) + a^2(c-b)$$

$$= bc^2 - b^2c - ac^2 - ab^2 + a^2c - a^2b$$

Let $a = 1 \quad b = 2 \quad c = 3$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$\det A = 1(6) - 1(5) + 1(1) = 2$$

Eigen Values solving:

① $|A - \lambda I| = 0$

Solve det to get $\lambda_1, \lambda_2 \dots$

② Construct new vectors in the cases of each λ's

$A - \lambda_1 \qquad A - \lambda_2 \quad \dots$

③ Apply gaussian elimination to these $A - \lambda$ vectors to extract x, y, z if they exist or else use params based on num of pivots

④ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \vec{v}$ is your eigen vector

if num of pivots is less introduce params $t, s \in R$ or $t \in R$ depending on num of params required

$\therefore \vec{v}$ will become $\vec{v} = t \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Task 1

Compute the eigenvalues and all eigenvectors.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

Note that these matrices are not symmetric. In that case, eigenvalues can be complex and/or there may be "not enough" eigenvectors.

$|A - \lambda I| = 0$ for eigen values

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & \cancel{0}\ 1 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$= \quad (1-\lambda)^4 = 0$$

$$\lambda = 1$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$= (a+b)^3$$
$$= a^3 + 3a^2b^1 + 3ab^2 + b^3$$

$$(a+b)^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

<u>when $\lambda = 1$</u>

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cancel{0}\,1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad II : II - I$$

$$\begin{array}{ccccc} w & x & y & z & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

System is reduced solvable.

4 variables 3 pivots

we choose $w = t$; $t \in \mathbb{R}$

$\therefore$

$z = 0$

$y = 0$

$x = 0$

$w = t$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is the eigenvector.

Back substitution practice

experiment:

① $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$z = t$

$y + t = 0$

$y = -t$

$x = s$

x is free variable

$0x = 0$

$\therefore x = s$

$\#\text{params} = n_{\text{variables}} - n_{\text{pivots}}$

$= 3 - 1 = 2 \quad \therefore s, t \in \mathbb{R}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix}$$

② $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\#_p \rightarrow$ No. of params

$n_v \rightarrow$ No. of variables

$n_p \rightarrow$ No of pivots

$\#_p = 3 - 2 = 1 \quad \therefore \; t \in R$

$z = t$

$1x + 2y + 3t = 0$

$x = -3t - 2y$

$y = -3t - x$

$z = t$

we already know z = 1 ∴ introducing z = t does not make sense

or

$y = t$

$z = 0$

$x + 2t + 3z = 0$

$x = -2t$

$x = -2t$

$y = t$

$z = 0$

this approach makes sense

Introduce param for variable that does not have explicit solution.

③

$$
\begin{array}{c}
\begin{array}{ccccc} x & y & z & w & \end{array} \\
\left(\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

$\#_p = 4 - 3 = 1 \quad \therefore \; t \in R$

$w = 0$

$x = 0$

let $y = t$

$t + 2z = 0$

$w = 0$

$x = 0$ } known explicit solns

$z = t$

$1y + 2t = 0$

$$z = -\frac{t}{2}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ t \\ -t/2 \\ 0 \end{pmatrix}$$

$$y = -2t$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ t \\ 0 \end{pmatrix}$$

(4)

$$\begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\#_p = 4 - 2 = 2 \qquad t, s \in R$$

Let $w = t$

$$z + 2w = 0$$
$$z + 2t = 0$$
$$z = -2t$$

Let $y = s$

$$1x + 2y + 3z = 0$$
$$x + 2s + 3(-2t) = 0$$
$$x + 2s - 6t = 0$$
$$x = 6t - 2s$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 6t - 2s \\ s \\ -2t \\ t \end{pmatrix}$$

(5)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\#_p: 3-2=1 \therefore t \in R$

explicit is $x=0$

$\therefore$ let $z=t$

$y+2z=0$

$y=-2t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ t \end{pmatrix}$$

(6)
$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$\#_p = 4-3=1 \quad t \in R$

explicit soln: $w=0$

$\therefore z+2w=0$

$z=0$

let $y=t$

$\therefore 1x+2t=0$

$x=-2t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \\ 0 \end{pmatrix}$$

(2) $B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

$|B - \lambda \underline{I}| = 0$ eigen value:

$$\left| \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{vmatrix} = 0$$

$$(\cos\alpha - \lambda)^2 + \sin^2\alpha = 0$$

$$\cos^2\alpha + 2\lambda\cos\alpha + \lambda^2 + \sin^2\alpha = 0$$

$$2\lambda\cos\alpha + \lambda^2 + 1 = 0$$

$$\lambda^2 + 2\cos\alpha\,\lambda + 1 = 0$$

$$\lambda = \frac{-2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{-2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2}$$

$$= -\cos\alpha \pm 2\sin\alpha\, i$$

$$= -\cos\alpha \pm i\, 2\sin\alpha$$

$$\lambda_1 = -\cos\alpha + i\, 2\sin\alpha$$

$$\lambda_2 = -\cos\alpha + i\, 2\sin\alpha.$$

$$(B - \lambda_1) =$$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} - \begin{pmatrix} -\cos\alpha + i\, 2\sin\alpha & 0 \\ 0 & -\cos\alpha + i\, 2\sin\alpha \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos\alpha + i\, 2\sin\alpha & -\sin\alpha \\ \sin\alpha & 2\cos\alpha + i\, 2\sin\alpha \end{pmatrix}$$

= Now apply gauss elimination

$$\left(\begin{array}{cc|c} 2\cos\alpha + i\, 2\sin\alpha & -\sin\alpha + 0i & 0 \\ \sin\alpha + 0i & 2\cos\alpha + i\, 2\sin\alpha & 0 \end{array}\right)$$

$$\mathrm{II} : \mathrm{II} - \frac{\sin\alpha}{2\cos\alpha + i\, 2\sin\alpha}\, \mathrm{I}$$

$$2\cos\alpha + i2\sin\alpha - \frac{\sin\alpha}{2\cos\alpha + i2\sin\alpha}(-\sin\alpha)$$

$$\frac{(2\cos\alpha + i2\sin\alpha)^2 + \sin^2\alpha}{2\cos\alpha + i2\sin\alpha}$$

$$\left(\begin{array}{cc|c} 2\cos\alpha + i2\sin\alpha & -\sin\alpha & 0 \\ 0 & \dfrac{(2\cos\alpha + i2\sin\alpha)^2 + \sin^2\alpha}{2\cos\alpha + i2\sin\alpha} & 0 \end{array}\right)$$

$$\vec{v} = \begin{pmatrix} \dfrac{\sin\alpha(2\cos\alpha + i2\sin\alpha)^2 + \sin^3\alpha}{2\cos\alpha + i2\sin\alpha} \\ (2\cos\alpha + i2\sin\alpha)^2 + \sin^2\alpha \end{pmatrix}$$

This is complex & the question is a Rotation Matrix. The eigen values & eigen vectors are not real unless α is a multiple of π

Rotation is circular & not linear (which is what eigen vals & vectors represent)

Task 2

Compute the eigenvalues and eigenvectors of the matrices

$$A = \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Task 3

A 3×3-matrix A has the eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$ and $\lambda_3 = 4$ with corresponding eigenvectors $\vec{v}_1 = (1,1,0)^T$, $\vec{v}_2 = (-2,1,3)^T$ and $\vec{v}_3 = (-3,2,2)^T$ respectively.
For $\vec{x} = (5,1,2)^T$, find the result of the product $A\vec{x}$.

TASK 3:

Given $\lambda_1 = 2 \quad \lambda_2 = -1 \quad \lambda_3 = 4$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} \quad \& \quad \vec{x} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

we know $A\vec{x} = \lambda \vec{x}$

& $\vec{x}$ can be written as linear combination of eigen vectors & scalar summed

ie $\vec{x} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 \rightarrow (1)$

$\therefore \quad A\vec{x} = a_1 A\vec{v}_1 + a_2 A\vec{v}_2 + a_3 A\vec{v}_3$

$= a_1 \lambda \vec{v}_1 + a_2 \lambda \vec{v}_2 + a_3 \lambda \vec{v}_3 \rightarrow (2)$

we can find a_1 a_2 & a_3 by solving (1)'s gaussian SLE & then find $A\vec{x}$ using (2)

$$\begin{array}{ccc|c} a & b & c & \vec{v} \\ \hline 1 & -2 & -3 & 5 \\ 1 & 1 & 2 & 1 \\ 0 & 3 & 2 & 2 \end{array} \qquad \mathrm{II}: \mathrm{II} - \mathrm{I}$$

$$\begin{array}{ccc|c} 1 & -2 & -3 & 5 \\ 0 & 3 & 5 & -4 \\ 0 & 3 & 2 & 2 \end{array} \qquad \mathrm{III}: \mathrm{III} - \mathrm{II}$$

$$\begin{array}{ccc|c} 1 & -2 & -3 & 5 \\ 0 & 3 & 5 & -4 \\ 0 & 0 & -3 & 6 \end{array}$$

System is Reduced ; Solvable
with atmost one pivot in each column
unique soln.

$$-3a_3 = 6$$

$$a_3 = -2$$

$$3a_2 + 5a_3 = -4$$
$$3a_2 = 10 - 4 = 6$$

$$a_2 = 2$$

$$a_1 - 2(2) - 3(-2) = 5$$
$$a_1 - 4 + 6 = 5$$
$$a_1 + 2 = 5$$
$$a_1 = 3$$

$a_1 = 3$ $\qquad a_2 = 2$ $\qquad a_3 = -2$

$$\therefore A\vec{x} = 3\cdot(2)\cdot\begin{pmatrix}1\\1\\0\end{pmatrix} + 2\cdot(-1)\begin{pmatrix}-2\\1\\3\end{pmatrix} - 2\cdot(4)\begin{pmatrix}-3\\5\\2\end{pmatrix}$$

$$= \begin{pmatrix}6\\6\\0\end{pmatrix} + \begin{pmatrix}4\\-2\\-6\end{pmatrix} + \begin{pmatrix}24\\-40\\-16\end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix}34\\-36\\-22\end{pmatrix}$$

TASK 2

A) $\begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix}$ $\quad |A - \lambda I| = 0$

$$\left| \begin{pmatrix} 5 & 1 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} 5-\lambda & 1 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$= (5-\lambda)(2-\lambda) - 4 = 0$$

$$= 10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$= \lambda^2 - 7\lambda + 6 = 0$$

$1 \times 6 = 6$

$-1 - 6 = -7$

$$= \lambda^2 - \lambda - 6\lambda + 6 = 0$$

$$= \lambda(\lambda - 1) - 6(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1 \qquad \lambda = 6$$

when $\lambda = 1$

$$= \underbrace{\left| \begin{matrix} 4 & 1 \\ 4 & 1 \end{matrix} \right|}_{R_A = 1} \begin{matrix} 0 \\ 0 \end{matrix} \Bigg|$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -4x \end{pmatrix} = x\begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$4x + 1y = 0$

$4x + 1y = 0$

$y = -4x$

$=$

Rank $= 1$

when $\lambda = 6$

$$\left|\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right| \quad II - \frac{4}{-1} I$$

$II + 4I$

$4 + 4(-1) = 0$

$-4 + 4(1) = 0$

$$\left|\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right|$$

System is Reduced, Solvable

$y = t \qquad t \in R$

$-1x + t = 0$

$x = -t$

$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(B) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $|A - \lambda I| = 0$

$$\left| \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$II: II - \frac{1}{2-\lambda} I$

$$= \quad II - \frac{1}{2-\lambda} I$$

$$1 - \frac{1}{2-\lambda}(2-\lambda) = 0$$

$$2-\lambda - \frac{1}{2-\lambda}(2-\lambda) = 1-\lambda$$

$$1 - \frac{1}{2-\lambda}(2-\lambda) = \lambda + 1$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & \lambda+1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\text{III} : \text{III} - \frac{1}{1-\lambda}\,\text{II}$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & \lambda+1 \\ 0 & 0 & \frac{(2-\lambda)(1-\lambda)+(1+\lambda)}{1-\lambda} \end{vmatrix} = 0$$

$$2-\lambda - \frac{1}{1-\lambda}(\lambda+1)$$

$$\frac{(2-\lambda)(1-\lambda)-(\lambda+1)}{1-\lambda}$$

$$= \frac{(2-\lambda)(1-\lambda)+(1+\lambda)}{1-\lambda}$$

$$= (2-\lambda)(1-\lambda)\left[\frac{(2-\lambda)(1-\lambda)+(1+\lambda)}{(1-\lambda)}\right] = 0$$

$$= (2-\lambda)^2 + (2-\lambda)(1+\lambda) = 0$$

$$= (2-\lambda)\big((2-\lambda)+1+\lambda\big) = 0$$

$$(2-\lambda)(3) = 0$$

$$6 - 3\lambda = 0$$
$$\lambda = 2$$

one eigen value ; $\lambda = 2$

When $\lambda = 2$; Now for eigen vectors:

$$\left|\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right|$$

$$\left|\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right| \quad \underline{III} : \underline{III} - \underline{II}$$

This is a Rank 1 matrix as 2nd row is identical to 3rd

$$\left|\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right|$$

# of params needed : num of variables − num of pivots

$= 3 - 2 = 1$

$\therefore$ introduce $t \in \mathbb{R}$

Known explicit: $y = 0 \quad \therefore$ let $z = t$

$x + z = 0 \qquad x = -t$

eigen vector $\vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix}$:

(c)

$$\det \begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\overline{\underline{IV}} : \underline{IV} - \frac{1}{1-\lambda} \underline{III}$$

$\underline{II}: \quad \underline{II} - \frac{1}{-\lambda} \underline{I}$

$\underline{II} + \frac{1}{\lambda} \underline{I}$ $\qquad\qquad$ $\underline{IV} - \frac{1}{1-\lambda} \underline{III}$

$1 + \frac{1}{\lambda}(-\lambda) = 0$ $\qquad\qquad$ 0

$-\lambda + \frac{1}{\lambda}(1) = \frac{-\lambda^2+1}{\lambda}$ $\qquad\qquad$ 0

$0 + \frac{1}{\lambda}(1) = \frac{1}{\lambda}$ $\qquad\qquad$ 0

$1 + \frac{1}{\lambda}(0) = 1$ $\qquad\qquad$ $(1-\lambda) - \frac{1}{1-\lambda} \cdot (0) = (1-\lambda)$

$$\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 0 & \frac{1-\lambda^2}{\lambda} & \frac{1}{\lambda} & 1 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(-\lambda) \frac{(1-\lambda^2)}{\lambda} \cdot (1-\lambda)^2 = 0$$

$$(1-\lambda^2)(1-\lambda)^2 = 0$$

$$(1+\lambda)(1-\lambda)(1-\lambda)^2 = 0$$

$$(1+\lambda)(1-\lambda)^3 = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = 1$$

when $\lambda_1 = -1$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \underline{II}: \underline{II} - \underline{I} \\ \\ \underline{IV}: \underline{IV} - \frac{1}{2}\,\overline{\underline{III}} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right)$$

Task 4

If a matrix A has an eigenvalue 7 with eigenvector $\vec{v} = (4, -11, 7)^T$, find an eigenvalue and eigenvector of the matrix $B = A - 4I_3$.

Task 5

Let

$$A = \begin{pmatrix} 8 & 9 & 2 \\ 1 & 0 & -2 \\ -8 & -2 & 7 \end{pmatrix}.$$

a) Find the eigenvalues of A.

b) To see that Gauss is not a good idea for eigenvalues: Take matrix A and use Gauss transformations to get it into reduced form (= upper triangular form). Find the eigenvalues of this upper triangular matrix to see that they are different from the ones of A.
So please: Always take $A - \lambda I_n$ of the given matrix A. Never mess with it!

c) Gauss is rarely an option after taking $A - \lambda I_n$, either. While it is correct to use it for finding the determinant, it is not helpful in the computation.
Try some Gauss transformations in the process of finding $\det(A - \lambda I_n)$. The lesson here is: They do not help much.

Task 4

$$\left.\lambda = 7 \qquad \vec{v} = \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix} \right\} \text{ matrix } A$$

$B = A - 4I_3$ Find B

We have $A\vec{v} = \lambda \vec{v}$

So : $(B = A - 4I_3)$ × both sides by $\vec{v}$

$$B\vec{v} = A\vec{v} - 4\vec{v}I_3$$

$$= \lambda \vec{v} - 4 \vec{v} I_3$$

$$= 7 \vec{v} - 4 \vec{v} I_3$$

$$\vec{v} I_3 = \underbrace{\begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{3\times 3}$$

is not possible. but since we multiplied the whole equation by $\vec{v}$ on both sides we write $\vec{v} I_3$ as $I_3 \vec{v}$

$$B\vec{v} = A\vec{v} - 4 I_3 \vec{v}$$

$$I_3 \vec{v} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{3\times 3} \underbrace{\begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix}}_{3\times 1}$$

$$= \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix} = \vec{v} \text{ itself}$$

$$\therefore \quad B\vec{v} = 7\vec{v} - 4\vec{v} = 3\vec{v}$$

we have $B\vec{v} = 3\vec{v}$

eigen value of $B = 3 = \lambda_1$

eigen vector of $B = \vec{v} = \begin{pmatrix} 4 \\ -11 \\ 7 \end{pmatrix}$

Task 5

$$A = \begin{pmatrix} 8 & 9 & 2 \\ 1 & 0 & -2 \\ -8 & -2 & 7 \end{pmatrix}$$

$$\det |A - \lambda I| = 0$$

$$\det \begin{pmatrix} 8-\lambda & 9 & 2 \\ 1 & -\lambda & -2 \\ 8 & -2 & 7-\lambda \end{pmatrix} = 0 \qquad \text{II}: \text{II} - \frac{1}{8-\lambda}\,\text{I}$$

$$\text{III}: \text{III} - \frac{8}{8-\lambda}\,\text{I}$$

$$\begin{pmatrix} 8-\lambda & 9 & 2 \\ 0 & \frac{(\lambda+1)(\lambda-9)}{8-\lambda} & \frac{2\lambda-18}{8-\lambda} \\ 0 & \frac{2\lambda-88}{8-\lambda} & \frac{7\lambda-72}{8-\lambda} \end{pmatrix}$$

$$-\lambda - \frac{1}{8-\lambda}\,9 = \frac{-\lambda(8-\lambda) - 9}{8-\lambda} = \frac{\lambda^2 - 8\lambda - 9}{8-\lambda}$$

$$= \frac{\lambda^2 + \lambda - 9\lambda - 9}{8-\lambda}$$

$\frac{1}{1}\lambda \frac{-9}{1} = -9$

$\frac{1}{1} + \frac{-9}{1} = -8$

$$= \frac{\lambda(\lambda+1) - 9(\lambda+1)}{(8-\lambda)}$$

$$= \frac{(\lambda+1)(\lambda-9)}{(8-\lambda)}$$

$$-2 - \frac{1}{8-\lambda} \, 2 = \frac{-2(8-\lambda) - 2}{8-\lambda}$$

$$= \frac{-16 + 2\lambda - 2}{8-\lambda} = \frac{2\lambda - 18}{8-\lambda}$$

$$\text{III} - \frac{8}{8-\lambda} \, \text{I}$$

$$-2 - \frac{8}{8-\lambda} \, 9 = \frac{-2(8-\lambda) - 72}{8-\lambda} = \frac{2\lambda - 88}{8-\lambda}$$

$$-7 - \frac{8}{8-\lambda} \, 2 = \frac{-7(8-\lambda) - 16}{8-\lambda} = \frac{7\lambda - 72}{8-\lambda}$$

$$\frac{-16 + 2\lambda - 72}{8-\lambda} = \frac{2\lambda - 88}{8-\lambda}$$

$$\frac{-56 + 7\lambda - 16}{8-\lambda} = \frac{7\lambda - 72}{8-\lambda}$$

$$\begin{pmatrix} 8-\lambda & 9 & 2 \\ 0 & \frac{(\lambda+1)(\lambda-9)}{8-\lambda} & \frac{2\lambda-18}{8-\lambda} \\ 0 & \frac{2\lambda-88}{8-\lambda} & \frac{7\lambda-72}{8-\lambda} \end{pmatrix}$$

$$\frac{1}{(8-\lambda)^2} \begin{pmatrix} 8-\lambda & 9 & 2 \\ 0 & (\lambda+1)(\lambda-9) & 2\lambda-18 \\ 0 & 2\lambda-88 & 7\lambda-72 \end{pmatrix}$$

$$\text{III}: \quad \text{III} - \frac{2\lambda-88}{(\lambda+1)(\lambda-9)} \, \text{II}$$

$$7\lambda-72 - \frac{2\lambda-88}{(\lambda+1)(\lambda-9)} (2\lambda-18)$$

This is too time consuming; use Sarrus' rule / normal way when 3x3

$$\det \begin{pmatrix} 8-\lambda & 9 & 2 \\ 1 & -\lambda & -2 \\ 8 & -2 & 7-\lambda \end{pmatrix} = 0$$

$$(8-\lambda)\begin{vmatrix} -\lambda & 2 \\ -2 & 7-\lambda \end{vmatrix} - 9\begin{vmatrix} 1 & -2 \\ 8 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 1 & -\lambda \\ 8 & -2 \end{vmatrix}$$

$$(8-\lambda)\left[-\lambda(7-\lambda)+4\right] - 9\left[7-\lambda+16\right] + 2\left[-2+8\lambda\right]$$

$$(8-\lambda)\left[\lambda^2-7\lambda+4\right] - 9\left[23-\lambda\right] + 2\left[8\lambda-2\right] = 0$$

$$(8-\lambda)\left[\lambda^2-7\lambda+4\right] - 207 + 9\lambda + 16\lambda - 4 = 0$$

$$8\lambda^2 - \underline{56\lambda} + 32 - \lambda^3 + 7\lambda^2 - \underline{4\lambda} - 207 + \underline{9\lambda} + \underline{16\lambda} - 4 = 0$$

$$-\lambda^3 + 15\lambda^2 - 60\lambda + 25\lambda + 28 - 207 = 0$$

$$-\lambda^3 + 15\lambda^2 - 35\lambda - 181 = 0$$

when $\quad 15\lambda^2 - 35\lambda - \lambda^3 = 181$

$\lambda = 3 \qquad 15(9) - 35(3) - 27 \stackrel{?}{=} 181$

$$\lambda\left(15\lambda - 35 \quad -\lambda^2\right) = 181$$

$\lambda_1 = 181 \qquad -\lambda^2 + 15\lambda - 35 = 181$

$$-\lambda^2 + 15\lambda - 206 = 0$$

Task 6

For the matrix

$$A = \begin{pmatrix} 3 & -10 & -10 \\ 0 & 3 & 0 \\ 0 & -5 & -2 \end{pmatrix},$$

find the eigenvalues and eigenvectors of

$$\text{(i) } A^{-1} \quad \text{(ii) } A^2 - 2A \quad \text{(iii) } A^4 - 7A^2 + 4I_3$$

Hint: The answer for each eigenvalue and eigenvector is one line.

$AA^{-1} = I$

$$\therefore \left(\begin{array}{ccc|ccc} 3 & -10 & -10 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & -5 & -2 & 0 & 0 & 1 \end{array}\right) \quad \underline{\text{III}} : \underline{\text{III}} - \frac{-5}{3}\,\underline{\text{II}}$$

$\underline{\text{III}} + \frac{5}{3}\,\underline{\text{II}}$

$0 + \frac{5}{3}\,0 = 0$

$-5 + \frac{5}{3}\,3 = 0$

$-2 + 5/3\;0 = -2$

$0 + 5/3\;0 = 0$

$0 + 5/3\;1 = 5/3$

$1 + 5/3\;0 = 1$

$$\left(\begin{array}{ccc|ccc} 3 & -10 & -10 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 5/3 & 1 \end{array}\right) \quad \text{I: } \text{I} - \frac{-10}{-2}\,\text{III}$$

$$\text{I} - \frac{10}{2}\,\text{III}$$

$$3 - \frac{10}{2}\,0 = 3$$

$$-10 - \frac{10}{2}\,0 = -10$$

$$-10 - \frac{10}{2} \cdot -2 = 0$$

$$1 - \frac{10}{2} \cdot 0 = 1$$

$$0 - \frac{10}{2} \cdot \frac{5}{3} = -25/3$$

$$0 - \frac{10}{2} \cdot 1 = -5$$

$$\left(\begin{array}{ccc|ccc} 3 & -10 & 0 & 1 & -25/3 & -5 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 5/3 & 1 \end{array}\right)$$

$$\text{I} - \frac{-10}{3}\,\text{II}$$

$$\text{I} + \frac{10}{3}\,\text{II}$$

$$3 + 10/3\;0 = 3$$

$$-10 + \tfrac{10}{3}\, 3 = 0$$

$$0 + \tfrac{10}{3}\, 0 = 0$$

$$1 + \tfrac{10}{3}\, 0 = 1$$

$$-\tfrac{25}{3} + \tfrac{10}{3}\, 1 = -\tfrac{15}{3} = -5$$

$$-5 + \tfrac{10}{3}\, 0 = -5$$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & -5 & -5 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & \tfrac{5}{3} & 1 \end{array}\right) \begin{array}{l} \div 3 \\ \div 3 \\ \div -2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{3} & -\tfrac{5}{3} & -\tfrac{5}{3} \\ 0 & 1 & 0 & 0 & \tfrac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & -\tfrac{5}{6} & -\tfrac{1}{2} \end{array}\right)$$

$$\underbrace{\begin{pmatrix} \tfrac{1}{3} & -\tfrac{5}{3} & -\tfrac{5}{3} \\ 0 & \tfrac{1}{3} & 0 \\ 0 & -\tfrac{5}{6} & -\tfrac{1}{2} \end{pmatrix}}_{A^{-1}}$$

$$\begin{pmatrix} 3 & -10 & -10 \\ 0 & 3 & 0 \\ 0 & -5 & -2 \end{pmatrix} \begin{pmatrix} 3 & -10 & -10 \\ 0 & 3 & 0 \\ 0 & -5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 9+0+0 & -30-30+50 & -30+0+20 \\ 0+0+0 & 0+9+0 & 0+0+0 \\ 0+0+0 & 0-15+10 & 0+0+4 \end{pmatrix}$$

$$\begin{pmatrix} 9 & -10 & -10 \\ 0 & 9 & 0 \\ 0 & -5 & 4 \end{pmatrix} = A^2$$

$$A^4 = (N+I)^4$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 8 & -10 & -10 \\ 0 & & \\ & & \end{pmatrix}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$$(a+b)^3$$
$$= a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4a^1b^3 + b^4$$

possible but complicated.
when you can
simply do $A^2 \cdot A^2$

$$\begin{pmatrix} 9 & -10 & -10 \\ 0 & 9 & 0 \\ 0 & -5 & 4 \end{pmatrix} \begin{pmatrix} 9 & -10 & -10 \\ 0 & 9 & 0 \\ 0 & -5 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 81+0+0 & -90-90+50 & -90+0-40 \\ 0+0+0 & 0+81+0 & 0+0+0 \\ 0+0+0 & 0-45-20 & 0+0+16 \end{pmatrix}$$

$$\begin{pmatrix} 81 & -130 & -130 \\ 0 & 81 & 0 \\ 0 & -65 & 16 \end{pmatrix} = A^4$$

(1) $\det |A - \lambda I| = 0$

using A^{-1}

eigen val & eigen vectors of A^{-1}

$$\det \begin{pmatrix} 1/3-\lambda & -5/3 & -5/3 \\ 0 & 1/3-\lambda & 0 \\ 0 & -5/6 & -1/2-\lambda \end{pmatrix} = 0$$

$\underline{III}: \underline{III} - \dfrac{-5/6}{\frac{1}{3}-\lambda} \underline{II}$

$$\det \begin{pmatrix} 1/3-\lambda & -5/3 & -5/3 \\ 0 & 1/3-\lambda & 0 \\ 0 & 0 & -\frac{1}{2}-\lambda \end{pmatrix}$$

$$\left(\frac{1}{3}-\lambda\right)^2\left(-\frac{1}{2}-\lambda\right) = 0$$

$$\frac{(1-3\lambda)^2}{9}\,\frac{(-1-2\lambda)}{2} = 0$$

$$(1-3\lambda)^2(1+2\lambda) = 0$$

$$\cancel{(1+9\lambda^2-6\lambda)(1+2\lambda) = 0}$$

$$1-3\lambda = 0 \qquad 2\lambda = -1$$

$$3\lambda = 1 \qquad \lambda = -\frac{1}{2}$$

$$\lambda = \frac{1}{3}$$

when $\lambda = \frac{1}{3}$

$$\begin{pmatrix} 0 & -5/3 & -5/3 \\ 0 & 0 & 0 \\ 0 & -5/6 & -\frac{1}{2}-\frac{1}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5/3 & -5/3 \\ 0 & 0 & 0 \\ 0 & -5/6 & -5/6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -5 & -5 \\ 0 & 0 & 0 \\ 0 & -5 & -5 \end{pmatrix}$$

When $\lambda = -\frac{1}{2}$

$$\begin{pmatrix} \frac{1}{3}+\frac{1}{2} & -\frac{5}{3} & -5/3 \\ 0 & \frac{1}{3}+\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 5/6 & -5/3 & -5/3 \\ 0 & 5/6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -10 & -10 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Rank 1 matrix
as row 3 is identical
to row 1

$$\left(\begin{array}{ccc|c} 0 & -5 & -5 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3 variable
let $z = t$

$-5y - 5z = 0$
$-5y - 5t = 0$
$-5y = 5t$
$y = -t$

let $x = s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix}$$

$t, s \in R$

$$\left(\begin{array}{ccc|c} 5 & -10 & -10 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\#_{params} = 3 - 2 = 1$
$\therefore t \in R$

explicit soln

$5y = 0$
$y = 0$

let $z = t$

$5x - 10y - 10z = 0$
$5x + 0 - 10t = 0$
$5x = 10t$
$x = 2t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{v}_2$; eigen vector
when $\lambda = -\frac{1}{2}$

(2) $A^2 - 2A$

$$= \begin{pmatrix} 9 & -10 & -10 \\ 0 & 9 & 0 \\ 0 & -5 & 4 \end{pmatrix} - 2 \begin{pmatrix} 3 & -10 & -10 \\ 0 & 3 & 0 \\ 0 & -5 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9-6 & -10+20 & -10+20 \\ 0-0 & 9-6 & 0-0 \\ 0-0 & -5+10 & 4+4 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 10 & 10 \\ 0 & 3 & 0 \\ 0 & 5 & 8 \end{pmatrix}$$

$$\det\left((A^2 - 2A) - \lambda I \right) = 0$$

$$= \begin{pmatrix} 3-\lambda & 10 & 10 \\ 0 & 3-\lambda & 0 \\ 0 & 5 & 8-\lambda \end{pmatrix} \qquad III : III - \frac{5}{3-\lambda} II$$

$$\det\begin{pmatrix} 3-\lambda & 10 & 10 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 8-\lambda \end{pmatrix} = 0$$

$$= (3-\lambda)^2 (8-\lambda) = 0$$

$3-\lambda = 0$ $\qquad$ $8-\lambda = 0$

$\lambda_1 = 3$ $\qquad$ $\lambda_2 = 8$

$\lambda_1 = 3$

$$\begin{pmatrix} 0 & 10 & 10 \\ 0 & 0 & 0 \\ 0 & 5 & 5 \end{pmatrix}$$

Rank 1

SLE

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2t \\ 0 \\ t \end{pmatrix}$$

$\lambda_2 = 8$

$$\begin{pmatrix} -5 & 10 & 10 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Reduced Solvable

explicit soln:

$-5y = 0$

$y = 0$

let $z = t$ $\quad t \in \mathbb{R}$

$-5x + 10y + 10z = 0$

$-5x + 0 + 10t = 0$

$5x = 10t$

$x = 2t$

3) $A^4 - 7A^2 + 4I_3$

$$\begin{pmatrix} 81 & -130 & -130 \\ 0 & 81 & 0 \\ 0 & -65 & 16 \end{pmatrix} - 7\begin{pmatrix} 9 & -10 & -10 \\ 0 & 9 & 0 \\ 0 & -5 & 4 \end{pmatrix} + 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 81 & -130 & -130 \\ 0 & 81 & 0 \\ 0 & -65 & 16 \end{pmatrix} - \begin{pmatrix} 63 & -70 & -70 \\ 0 & 63 & 0 \\ 0 & -35 & 28 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 81-63+4 & -130+70+0 & -130+70+0 \\ 0-0+0 & 81-63+4 & 0-0+0 \\ 0+0+0 & -65+35+0 & 16-28+4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 22 & -50 & -50 \\ 0 & 22 & 0 \\ 0 & 30 & -8 \end{pmatrix}}_{=}$$

$$\det\left(\underbrace{A^4 - 7A^2 + 4I_3} - \lambda I\right) = 0$$

$$\det\begin{pmatrix} 22-\lambda & -50 & -50 \\ 0 & 22-\lambda & 0 \\ 0 & 30 & -8-\lambda \end{pmatrix} = 0$$

$$\underline{III} : \underline{III} - \frac{30}{22-\lambda}\,\underline{II}$$

$$\det\begin{pmatrix} 22-\lambda & -50 & -50 \\ 0 & 22-\lambda & 0 \\ 0 & 0 & -8-\lambda \end{pmatrix} = 0$$

$$(22-\lambda)^2(-8-\lambda) = 0$$

$\lambda_2 = -8$ $\qquad$ $\lambda_1 = 22$

When $\lambda_1 = 22$

$$\begin{pmatrix} 0 & -50 & -50 \\ 0 & 0 & 0 \\ 0 & 0 & -30 \end{pmatrix}$$

$$-30z = 0$$
$$z = 0$$
$$\therefore -50y + 0 = 0$$
$$y = 0$$

when $\lambda_2 = -8$

$$\begin{pmatrix} 30 & -50 & -50 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Reduced SLE

$$30y = 0$$
$$y = 0$$

Let $z = t$; $t \in \mathbb{R}$

$$30x - 50t = 0$$

Let $x = t$; $t \in \mathbb{R}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix}$$

eigen vector for $\lambda_1 = 22$

$x = \frac{50}{30} t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/3\, t \\ 0 \\ t \end{pmatrix}$$

Task 7

Find the eigenvalues of a triangular matrix. Try the matrices

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} a & e & h & l \\ 0 & b & f & k \\ 0 & 0 & c & g \\ 0 & 0 & 0 & d \end{pmatrix}.$$

$\det(A - \lambda I) = 0$

$$\begin{pmatrix} a-\lambda & c \\ 0 & b-\lambda \end{pmatrix} = 0$$

$(a-\lambda)(b-\lambda) = 0$

$a = \lambda_1 \quad b = \lambda_2$

when $\lambda_1 = a$

$$\begin{pmatrix} 0 & c \\ 0 & b-a \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 0 & c & 0 \\ 0 & b-a & 0 \end{array}\right)$$

$(b-a)z = 0$

$z = 0$

problem as eigenvector $\neq 0$ cannot be 0

When $\lambda_2 = b$

$$\begin{pmatrix} a-b & c \\ 0 & 0 \end{pmatrix} = 0$$

$c = 0$

$a - b = 0$

$a = b$

$c=0$

For a 2×2 matrix with: $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$

$a=b$; $c=0$ $\in R$

will have its diagonals elements as its eigen values

$$B=\begin{pmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{pmatrix}$$

$$\det\begin{pmatrix} a-\lambda & d & f \\ 0 & b-\lambda & e \\ 0 & 0 & c-\lambda \end{pmatrix}=0$$

$$(a-\lambda)(b-\lambda)(c-\lambda)=0$$

$\lambda_1=a$ $\lambda_2=b$ $\lambda_3=c$

When $\lambda_1=a$

$$\begin{pmatrix} 0 & d & f \\ 0 & b-a & e \\ 0 & 0 & c-a \end{pmatrix} \Rightarrow$$

$c-a=0$

$c=a$

$e=0$

$b-a=0$

$b=a$

To check later:

Is there anything special about a pivot free column?

Task 8

Find the eigenvalues and eigenvectors of the matrices

$$A = \begin{pmatrix} 6 & 7 & 8 & 9 \\ 0 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Are the matrices invertible?

Task 9

For the matrix

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix},$$

find all values for a, b, and c, so that the matrix B has eigenvalues $-3, 0$ and 3.

$$A: \det \begin{pmatrix} 6-\lambda & 7 & 8 & 9 \\ 0 & 3-\lambda & 4 & 5 \\ 0 & 0 & 1-\lambda & 2 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = 0$$

$$(6-\lambda)(3-\lambda)(1-\lambda)(-\lambda) = 0$$

$\lambda_1 = 6 \quad \lambda_2 = 3 \quad \lambda_3 = 1 \quad \lambda_4 = 0$

Eigen val can be zero
aber Eigen vector $\neq 0$

when $\lambda_1 = 6$

$$\begin{pmatrix} 0 & 7 & 8 & 9 \\ 0 & -3 & 4 & 5 \\ 0 & 0 & -5 & 2 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 0 & \boxed{7} & 8 & 9 & 0 \\ 0 & \boxed{-3} & 4 & 5 & 0 \\ 0 & 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right)$$

$\underline{II}: \underline{II} - \frac{-3}{7} \, I$

$\underline{II} + \frac{3}{7} \, I$

$0 + \frac{3}{7} \cdot 0 = 0$

$-3 + \frac{3}{7} \cdot 7 = 0$

$4 + \frac{3}{7} \cdot 8 = \frac{28 + 24}{7} = \frac{52}{7}$

$5 + \frac{3}{7} \cdot 9 = \frac{35 + 27}{7} = \frac{62}{7}$

$$\left(\begin{array}{cccc|c} 0 & 7 & 8 & 9 & 0 \\ 0 & 0 & 52 & 62 & 0 \\ 0 & 0 & -5 & 2 & 0 \\ 0 & 6 & 0 & 6 & 0 \end{array}\right)$$

$\underline{III}: \underline{III} - \frac{-5}{52}\underline{II}$

$\underline{III} + \frac{5}{52}\underline{II}$

$$2 + \frac{5}{52} \cdot 62 = \frac{104 + 310}{52} = \frac{414}{52}$$

$$\left(\begin{array}{cccc|c} 0 & 7 & 8 & 9 & 0 \\ 0 & 0 & 52 & 62 & 0 \\ 0 & 0 & 0 & 414 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right)$$

$\underline{IV}: \underline{IV} - \frac{6}{414}\underline{III}$

$$\begin{array}{cccc|c} 0 & 7 & 8 & 9 & 0 \\ 0 & 0 & 52 & 62 & 0 \\ 0 & 0 & 0 & 414 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x & y & z & w & \end{array}$$

$w = 0$

$52z + 62w = 0$

$z = 0$

$\therefore \; y = 0$

let $x = t$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix A as itsel is not invertible

Matrix is invertible if $\det A \neq 0$

Task 9:

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$$

$\lambda_1 = -3$

$\lambda_2 = 0$

$\lambda_3 = 3$

we know $\det |A - \lambda I| = 0$

$\therefore \quad \det |A - \lambda_1 I| = 0 \;\; \& \;\; \det |A - \lambda_2 I| = 0$

$\therefore$ when $\lambda_1 = -3$

$B - \lambda_1 I$

$$\det \begin{pmatrix} -\lambda_1 & 1 & 0 \\ 0 & -\lambda_1 & 1 \\ a & b & c-\lambda_1 \end{pmatrix} \qquad \lambda_1 = -3$$

$$\det \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ a & b & c+3 \end{pmatrix}$$

$$3 \begin{vmatrix} 3 & 1 \\ b & c+3 \end{vmatrix} - 0 \begin{vmatrix} & \\ & \end{vmatrix} + a \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 0$$

$$3(3c+9+b) + a(1-0) = 0$$

$$9c + 27 + 3b + a = 0$$

$$a + 3b + 9c + 27 = 0$$

$$a + 3b + 9c = -27 \longrightarrow 1$$

$$b + 3c = -9$$

when $\lambda_2 = 0$

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix} = 0$$

$$a \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0$$

$$a(1-0) = 0$$

$$a = 0 \longrightarrow (2)$$

<u>when $\lambda_3 = 3$</u>

$$\det \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ a & b & c-3 \end{pmatrix} = 0$$

$$-3 \begin{vmatrix} -3 & 1 \\ b & c-3 \end{vmatrix} - 0 \begin{vmatrix} & \\ & \end{vmatrix} + a \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} = 0$$

$$-3\left[-3c+9-b\right] + a\left[1-0\right] = 0$$

$$9c - 27 + 3b + a = 0$$

$$a + 3b + 9c = 27 \longrightarrow (3)$$

(2) in (1) & 3

$$b + 3c = -9$$

$$b + 3c = 9$$

inconsistent system ?