Multivariate Calculus

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Partial Derivatives

A partial derivative of a function of several variables is its derivative with respect to on of the variables in the function; in which case; the rest of the other variables become / get held as constants.

- $z = f(x,y)$ $\begin{cases} \dfrac{\partial z}{\partial x} \; {\uparrow}\, y\text{ const} \; ; \; \dfrac{\partial}{\partial x} f(x,y) \; ; \; f_x(x,y) \\ \dfrac{\partial z}{\partial y} \; {\uparrow}\, x\text{ const} \; ; \; \dfrac{\partial}{\partial y} f(x,y) \; ; \; f_y(x,y) \end{cases}$

Theorem Clairaut / Schwarz

If $f(x,y)$ is defined on a disk D that contains the point (a,b) & $f_{xy}(a,b)$ & $f_{yx}(a,b)$ are continuous; then $f_{xy}(a,b) = f_{yx}(a,b)$

ie the second partial derivatives of f_{xy} & f_{yx} will be the same; ∴ we only need to compute one of them.

Tangent Lines, Planes & Linear Approximations

- Equation of tangent plane:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

- Linear approximation of a function at point

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

if partial derivatives f_x & f_y exist near (a,b) & are continuous at (a,b); then f is differentiable at (a,b)

Gradient of function

- $$\nabla f(a,b) = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}$$

The gradient of a function at a point (a,b) is a vector that points in the direction of greatest rate of increase in that direction.
In 2-D it is the partial derivative of function f w.r to x & y at point (a,b)

Chain Rule: Multivariate

$z =$ function in terms of x & y ; where $\begin{cases} x = \text{function in terms of other variable ; eg: } t \\ y = \text{also another function in terms of another variable} \end{cases}$

To find the partial derivative of z w.r to another variable; apply tree branching.

eg: $z = x^2y + 3xy^4$ $\quad x = \sin 2t \quad y = \cos t$

$$\begin{tikzcd} & z \arrow[dl, "\frac{dz}{dx}"'] \arrow[dr, "\frac{dz}{dy}"] & \\ x \arrow[d, "\frac{dx}{dt}"'] & & y \arrow[d, "\frac{dy}{dt}"] \\ \boxed{t} & & \boxed{t} \end{tikzcd}$$

arrow represent 'in terms of' in which the given function is described

Multiply through end variable branches and add them up:

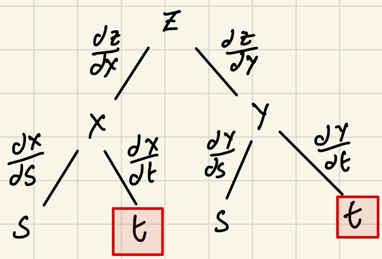
$$\left\{ \frac{dz}{\boxed{dt}} = \frac{dz}{dx}\cdot\frac{dx}{dt} + \frac{dz}{dy}\cdot\frac{dy}{dt} \right\}$$

example 2:

$z = 4x^3y^5 \qquad x = 3s^2t \qquad y = 2s^3t^2$

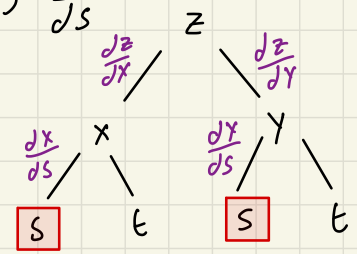
Here x & y are represented by 2 sub variables
∴ branch out for respective end partial derivatieve:

1) $\frac{dz}{dt}$



$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

2) $\frac{dz}{ds}$



$$\frac{dz}{ds} = \frac{dz}{dx} \cdot \frac{dx}{ds} + \frac{dz}{dy} \cdot \frac{dy}{ds}$$

Directional Derivative:

If a function f is differentiable w.r to x & y then f has a directional derivative in the direction of of a vector $\vec{v}$ given by:

$$D_u f = \nabla f \cdot \vec{u} \quad \text{where} \quad \vec{u} = \frac{1}{|\vec{v}|} \cdot \vec{v}$$

- if u given with angle made with +ve x-axis

  $\vec{u} = (\cos\alpha, \sin\alpha)$ or $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$

confirm it is unit vector

$$\vec{u} = \sqrt{\cos^2\alpha + \sin^2\alpha}$$

- if a vector $\langle a, b \rangle$ is given

$$\vec{u} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right)$$

Then compute $D_u f = \nabla f \cdot \vec{u}$

where $\nabla f = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}$

Absolute Minima & Maxima

1) Compute the critical point by equating $\nabla f = 0$
solve the equations to find different critical points

2) Compute f_{xx}, f_{yy} & f_{xy}

3) For each critical point (a,b) compute
$f_{xx}(a,b)$; $f_{yy}(a,b)$ & $f_{xy}(a,b)$

4) Compute the Hessian Matrix Determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

to obtain D.

5) Create a table noting the respective $D(a,b)$ for each critical point (a,b)

point:	(a_1, b_1)	(a_2, b_2)	(a_3, b_3) ....
D value:			

6) if $D > 0$ & $f_{xx} > 0$ $\rightarrow$ f has local min at that (a,b)

if $D > 0$ & $f_{xx} < 0$ $\rightarrow$ f has local max at point (a,b)

if $D < 0$ $\rightarrow$ Saddle point

if $D = 0$ $\rightarrow$ Test inconclusive

Two Types of Question:

Local Maxima / Minima

- **Find Critical Points** $\nabla f = 0$

Compute the gradient of the function (the vector of the partial derivatives). Set it equal to zero vector & solve for the variables

- **Classify the Critical Points**

Generate the Hessian Matrix (H)

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \text{ or } \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Compute determinant of H

$$D = \Delta = \det(H)$$

- **Interpret the results:**

if $D > 0$; $f_{xx} > 0$ local minimum

$D > 0$; $f_{xx} < 0$ local maximum

$D < 0$; saddle point

$D = 0$; test inconclusive

(evaluate overall function)

Absolute Maxima / Minima

- **Find Critical Points** $\nabla f = 0$

same as in local max/minima

- **Evaluate the function at CP's**

Substitute the coordinates of the critical points into the original function & evaluate the function value at these points

- **Evaluate the function on the boundaries**

The boundary region is typically composed of several lines or curves. For each boundary part:

Parameterize the boundary:

- if it is a vertical line ; you have a fixed x value eg: $x = 3$. for that line find the max & min y vals & evaluate $f(3, y_1)$ & $f(3, y_2)$

- if it is a line segment ; express the coordinates in terms of single param t ; parameterizing the line segment.

eg: line seg. b/w (1,1) & (2,3)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Think of t as time parameter going

from 0 to 1; where we start from $(1,1)$ to end at $(2,3)$.

generate the $\begin{pmatrix} x \\ y \end{pmatrix}$ equations & substitute them to the $f(x,y)$ function in the question to obtain new function $g(t)$.

Substitute t's range $[0,1]$ to obtain the max & min values from $g(t)$.

- If it is a circle with centre (h,k) and radius r

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + r\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$t: [0, 2\pi]$

eg: $(h,k):(0,0)$ & $r=1$
& $f(x,y) = x^2 + y^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}; \ t \in [0, 2\pi]$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \Rightarrow g(t)$$

$f(x,y) = x^2 + y^2$
$g(x,y) = \cos^2 t + \sin^2 t$

<u>when $t=0$</u>

$g(x,y) = 1 + 0 = 1$

<u>when $t=2\pi$</u>

$g(x,y) = \cos^2(2\pi) + \sin^2(2\pi) = 1$

In both cases we have 1; $\therefore$ max & min of $f(x,y)$ are 1

- If it is an ellipse with (h,k) centre; semimajor axis a & semiminor axis b; Apply the same logic as circle but parameterize the equation properly:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

- **Evaluate Function at Vertices**

If it is a region of a polygon; triangle or rectangle; you will have vertices; substitute those coordinates into the function to obtain function values

- **Find the Absolute Maxima & Minima:**

From the above values evaluated from the CP's; boundary lines & vertices; the largest value will be the maximum & smallest would be the minimum absolutes.

Tangent lines & planes

Tangent Lines & Planes

Plane passing through the point $P(x_0, y_0, z_0)$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$\div C$ $$\frac{A}{C}(x - x_0) + \frac{B}{C}(y - y_0) + (z - z_0) = 0$$

Taking $a = -\frac{A}{C}$ & $b = -\frac{B}{C}$

$$a(x - x_0) + b(y - y_0) = z - z_0$$

$a = f_x(x_0, y_0)$ & $b = f_y(x_0, y_0)$

$\therefore$ Equation of tangent plane to surface of $z = f(x, y)$ at point $P(x, y, z)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

1) Given $z = 2x^2 + y^2$ $\quad P(1, 1, 3)$

we have $z = f(x, y) = 2x^2 + y^2$

$$f_x(x,y) \overset{y\text{const}}{=} \frac{\partial}{\partial x}(2x^2 + y^2)$$

$$= 4x + 0 = 4x$$

$$f_y(x,y) \overset{x\text{ const}}{=} \frac{\partial}{\partial y}(2x^2 + y^2)$$

$$= 0 + 2y = 2y$$

$$f_x(x_0, y_0) = f_x(1,1) = 4$$

$$f_y(x_0, y_0) = f_y(1,1) = 2$$

Equation of tangent plane:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 3 = 4(x-1) + 2(y-1)$$

$$z = 4x - 4 + 2y - 2 + 3$$

$$z = 4x + 2y - 3$$

This is also called linearization of $f(x,y)$ at $(1,1)$

written as: $L(x,y)$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\boxed{L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

Task 3

Read section 14.4 *Tangent Planes and Linear Approximations*, but leave out *Differentials*. Solve the exercises 1-18.
For 7-10: Plot with Matlab. Calculate all partial derivatives yourself. No computer is needed for that.

1–6 Find an equation of the tangent plane to the given surface the specified point.

1. $z = 2x^2 + y^2 - 5y, \quad (1, 2, -4)$

2. $z = (x + 2)^2 - 2(y - 1)^2 - 5, \quad (2, 3, 3)$

3. $z = e^{x-y}, \quad (2, 2, 1)$

4. $z = x/y^2, \quad (-4, 2, -1)$

5. $z = x\sin(x + y), \quad (-1, 1, 0)$

6. $z = \ln(x - 2y), \quad (3, 1, 0)$

1) Equation of tangent plane

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$(x_0, y_0, z_0) = (1, 2, -4)$$

$$f(x, y) = 2x^2 + y^2 - 5y$$

$$f_x(x, y) \overset{\uparrow\, y \text{ const}}{=} 4x + 0 - 0 = 4x$$

$$f_y(x, y) \overset{\uparrow\, x \text{ const}}{=} 0 + 2y - 5 = 2y - 5$$

$$f_x(1, 2) = 4$$

$$f_y(1, 2) = 2(2) - 5 = 4 - 5 = 1$$

$$\therefore \quad z - (-4) = 4(x - 1) + 1(y - 2)$$

$$z + 4 = 4x - 4 + y - 2$$

$$z + 4 = 4x + y - 6$$

$$z = 4x + y - 10$$

2) $z = (x+2)^2 - 2(y-1)^2 - 5 \qquad (2, 3, 3)$

$$\frac{\partial z}{\partial x} \overset{\uparrow\, y \text{ const}}{=} 2(x+2) \cdot 1 - 0 - 0$$

$$= 2(x+2)$$

$$\frac{\partial z}{\partial y} \overset{\uparrow\, x \text{ const}}{=} 0 - 2 \cdot 2(y-1) \cdot 1 - 0$$

$$= -4(y-1)$$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2(x+2) \\ -4(y-1) \end{pmatrix}$$

$$\begin{pmatrix} f_x(2,3) \\ f_y(2,3) \end{pmatrix} = \begin{pmatrix} 2(2+3) \\ -4(3-1) \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$$

$$z - 3 = 10(x-2) + (-8)(y-3)$$

$$z - 3 = 10x - 20 - 8y + 24$$

$$z - 3 = 10x - 8y + 4$$

$$z = 10x - 8y + 7$$

3) $z = e^{x-y}$ $\quad (2, 2, 1)$

$$\begin{pmatrix} f_x \\ \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{d}{dx}\left(e^{x-y}\right) \text{ y const} \\ \\ \frac{d}{dy}\left(e^{x-y}\right) \text{ x const} \end{pmatrix} = \begin{pmatrix} e^{x-y} \cdot (1) \\ \\ e^{x-y} \cdot (-1) \end{pmatrix} = \begin{pmatrix} e^{x-y} \\ \\ -e^{x-y} \end{pmatrix}$$

$$\begin{pmatrix} f_x(2,2) \\ \\ f_y(2,2) \end{pmatrix} = \begin{pmatrix} e^{2-2} \\ \\ -e^{2-2} \end{pmatrix} = \begin{pmatrix} 1 \\ \\ -1 \end{pmatrix}$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 1 = 1(x-2) - 1(y-2)$$

$$z - 1 = x - 2 - y + 2$$

$$z = x - y + 1$$

4) $z = x/y^2 \quad (-4, 2, -1)$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{dz}{dx} \uparrow \text{y const} \\ \frac{dz}{dy} \uparrow \text{x const} \end{pmatrix} = \begin{pmatrix} \frac{1}{y^2} \\ x \cdot -2 \cdot y^{-3} \end{pmatrix} = \begin{pmatrix} 1/y^2 \\ -2\frac{x}{y^3} \end{pmatrix}$$

$$\begin{pmatrix} f_x(-4,2) \\ f_y(-4,2) \end{pmatrix} = \begin{pmatrix} 1/4 \\ -2\frac{(-4)}{8} \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z + 1 = \frac{1}{4}(x + 4) + 1(y - 2)$$

$$z = \frac{x}{4} + 1 + y - 2 - 1$$

$$z = \frac{x}{4} + y - 2 = \frac{x + 4y - 4}{4}$$

5) $z = x\sin(x+y)$
$(-1, 1, 0)$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} \frac{dz}{dx} \uparrow \text{y const} \\ \frac{dz}{dy} \uparrow \text{x const} \end{pmatrix} = \begin{pmatrix} x\cos(x+y) \cdot 1 + \sin(x+y) \cdot 1 \\ x\cos(x+y) \cdot 1 + \sin(x+y) \cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} x\cos(x+y) + \sin(x+y) \\ x\cos(x+y) \end{pmatrix}$$

$$\begin{pmatrix} f_x(-1,1) \\ f_y(-1,1) \end{pmatrix} = \begin{pmatrix} -1 \ \cos(0) + \sin(0) \\ -1 \ \cos 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

We know: eqn of tangent to plane

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z = -1(x+1) - 1(y-1)$$

$$z = -x - 1 - y + 1$$

$$\boxed{x + y + z = 0}$$

6) $z = \ln(x-2y)$ $\quad (3,1,0)$ $\qquad \frac{d}{dx}\ln x = \frac{1}{x}$

$$\begin{pmatrix} \frac{dz}{dx} \uparrow y \text{ const} \\ \frac{dz}{dy} \uparrow x \text{ const} \end{pmatrix} = \begin{pmatrix} \frac{1}{x-2y} \cdot 1 \\ \frac{1}{x-2y} \cdot (-2) \end{pmatrix} = \begin{pmatrix} \frac{1}{x-2y} \\ \frac{-2}{x-2y} \end{pmatrix}$$

$$\begin{pmatrix} f_x(3,1) \\ f_y(3,1) \end{pmatrix} = \begin{pmatrix} \frac{1}{3-2} \\ \frac{-2}{3-2} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z = (1)(x-3) + (-2)(y-1)$$

$$z = x-3-2y+2 = x-2y-1$$

$$\boxed{z = x-2y+1}$$

7–8 Graph the surface and the tangent plane at the given point. (Choose the domain and viewpoint so that you get a good view of both the surface and the tangent plane.) Then zoom in until the surface and the tangent plane become indistinguishable.

7. $z = x^2 + xy + 3y^2$, $(1, 1, 5)$

8. $z = \sqrt{9 + x^2y^2}$, $(2, 2, 5)$

(graphs after python)

7. $z = x^2 + xy + 3y^2$; $(1,1,5)$

Eqn of tangent line

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$(x_0, y_0, z_0) = (1, 1, 5)$$

$$f_x(x_0, y_0) = \frac{dz}{dx} \overset{\uparrow \text{ycoast}}{=} 2x + y + 0 = 2x + y$$

$$f_y(x_0, y_0) = \frac{dz}{dy} \overset{\uparrow \text{xcoust}}{=} 0 + x + 6y = x + 6y$$

$$f_x(x_0, y_0) = 2 + 1 = 3$$

$$f_y(x_0, y_0) = 1 + 30 = 31$$

$$z - 5 = 3(x - 1) + 31(y - 1)$$

$$z - 5 = 3x - 3 + 31y - 31$$

$$z - 5 = 3x + 3y - 34$$

$$3x + 3y - z = 29$$

8. $z = \sqrt{9 + x^2 y^2}$ $(2, 2, 5)$

$$f_x(x_0, y_0) = \frac{dz}{dx} = \frac{1}{2\sqrt{9 + x^2 y^2}} \cdot (y^2 \cdot 2x)$$

$$= \frac{x y^2}{\sqrt{9 + x^2 y^2}}$$

$$f_y(x_0, y_0) = \frac{dz}{dy} = \frac{1}{2\sqrt{9 + x^2y^2}} \left(x^2 \cdot 2y\right)$$

$$= \frac{x^2 y}{\sqrt{9 + x^2y^2}}$$

$$(x_0, y_0, z_0) = (2, 2, 5)$$

$$f_x(x_0, y_0) = \frac{2 \cdot 2^2}{\sqrt{9 + 2^2 \cdot 2^2}} = \frac{8}{\sqrt{9 + 16}} = \frac{8}{5}$$

$$f_y(x_0, y_0) = \frac{4 \cdot 2}{\sqrt{9 + 4 \cdot 4}} = \frac{8}{5}$$

$$z - 5 = \frac{8}{5}(x - 2) + \frac{8}{5}(y - 2)$$

$$5z - 25 = 8x - 16 + 8y - 16$$

$$32 - 25 = 8x + 8y - 5z$$

$$8x + 8y - 5z = 7$$

Then zoom in until the surface and the tangent plane become indistinguishable.

9. $f(x, y) = \dfrac{1 + \cos^2(x - y)}{1 + \cos^2(x + y)}, \quad \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

10. $f(x, y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right), \quad (1, 1, 3e^{-0.1})$

11–16 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = 1 + x \ln(xy - 5), \quad (2, 3)$

12. $f(x, y) = \sqrt{xy}, \quad (1, 4)$

13. $f(x, y) = x^2 e^y, \quad (1, 0)$

14. $f(x, y) = \dfrac{1 + y}{1 + x}, \quad (1, 3)$

15. $f(x, y) = 4 \arctan(xy), \quad (1, 1)$

16. $f(x, y) = y + \sin(x/y), \quad (0, 3)$

17–18 Verify the linear approximation at $(0, 0)$.

17. $e^x \cos(xy) \approx x + 1$

18. $\dfrac{y - 1}{x + 1} \approx x + y - 1$

9. $f(x,y) = \dfrac{1 + \cos^2(x-y)}{1 + \cos^2(x+y)} \quad \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

$(x_0, y_0, z_0) = \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{7}{4}\right)$

$$z - \frac{\pi}{4} = f_x(x_0, y_0)\left(x - \frac{\pi}{3}\right) + f_y(x_0, y_0)\left(x - \frac{\pi}{6}\right)$$

$$f_x = \frac{dz}{dx} \quad \uparrow y\text{const}$$

$$\frac{\left(1+\cos^2(x+y)\right)\cdot\left(0+2\cos(x-y)\left(-\sin(x-y)\cdot 1\right)\right) - \left(1+\cos^2(x+y)\right)\left(0+2\cos(x+y)\left(-\sin(x+y)\cdot 1\right)\right)}{\left(1+\cos^2(x+y)\right)^2}$$

$$\frac{\left(1+\cos^2(x+y)\right)\left(-2\sin(x-y)\cos(x-y)\right) - \left(1+\cos^2(x+y)\right)\left(-2\sin(x+y)\cos(x+y)\right)}{\left(1+\cos^2(x+y)\right)^2}$$

$$\frac{\pi}{3} + \frac{\pi}{6} = \frac{9\pi}{18} = \frac{\pi}{2} = 90^\circ$$

$$\frac{\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{18} = \frac{\pi}{6} = \frac{180}{6} = 60^\circ$$

$$f_x\left(\frac{\pi}{3}, \frac{\pi}{6}\right) =$$

$$(1+\cos^2 90^\circ)\left(-2\sin(60^\circ)\cos(60^\circ)\right) - (1+\cos^2 90^\circ)\left(-2\sin(90^\circ)\cos(90^\circ)\right)$$

$$\overline{\qquad\qquad\qquad\qquad}$$

$$\left(1+\cos^2 90^\circ\right)^2$$

$\cos 90^\circ = 0 \qquad \sin 60^\circ = \dfrac{\sqrt{3}}{2} \qquad \cos 60^\circ = \dfrac{1}{2}$

$$= (1+0)\left(-2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) - (1+0)\left(-2(1)\cdot(0)\right)$$

$$= 1\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$$

$$f_y = \frac{dz}{dy}\Big|^{x\text{ const}} = \frac{d}{dy}\left(\frac{1+\cos^2(x-y)}{1+\cos^2(x+y)}\right)^{x\text{ const}}$$

$$= \frac{\left[1+\cos^2(x+y)\right]\left[\frac{d}{dy}\left(1+\cos^2(x-y)\right)\right]^{x\text{ const}} - \left[1+\cos^2(x-y)\right]\left[\frac{d}{dy}\left(1+\cos^2(x+y)\right)\right]^{x\text{ const}}}{\left(1+\cos^2(x+y)\right)^2}$$

$$= \frac{\left[1+\cos^2(x+y)\right]\left[0+2\cos(x-y)\cdot\sin(x-y)\cdot(-1)\right] - \left[1+\cos^2(x-y)\right]\left[0+2\cos(x+y)\sin(x+y)\cdot(1)\right]}{\left(1+\cos^2(x+y)\right)^2}$$

$= \quad x+y = 90^\circ \qquad x-y = 60^\circ$

$$= \frac{\left[1+\cos^2(90^\circ)\right]\left[-2\cos(60^\circ)\cdot\sin(60^\circ)\right] - \left[1+\cos^2(60^\circ)\right]\left[2\cos(90^\circ)\sin(90^\circ)\right]}{\left(1+\cos^2(90^\circ)\right)^2}$$

$= \quad \cos 90 = 0 \qquad \cos 60^\circ = \frac{1}{2} \qquad \sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \sin 90^\circ = 1$

$$= \frac{\left[1+0\right]\left[-2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] - \left[1+\frac{1}{2}\right]\left[2\cdot 0\cdot 1\right]}{(1+0)^2}$$

$$= -\frac{\sqrt{3}}{2} - 0 = \frac{-\sqrt{3}}{2}$$

$$\therefore z - \frac{7}{4} = \frac{\sqrt{3}}{2}\left(x + \frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}\left(y + \frac{\sqrt{3}}{2}\right)$$

$$z = \frac{\sqrt{3}}{2}x + \frac{3}{4} - \frac{\sqrt{3}}{2}y + \frac{3}{4} + \frac{7}{4}$$

$$z = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y + \frac{13}{4}$$

10) $f(x,y) = e^{-xy/10}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)$; $\left(1, 1, 3e^{-0.1}\right)$

Tangent line eqn:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - 3e^{-0.1} = f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1)$$

$f_x(1, 1)$

$$\frac{dz}{dx} \overset{\uparrow \text{ y const}}{=}$$

$$= e^{-xy/10} \cdot \frac{d}{dx}\left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right) + \left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)\frac{d}{dx}\left(e^{-xy/10}\right)$$

$$= e^{-xy/10}\left(\frac{1}{2\sqrt{x}} + 0 + \frac{\sqrt{y}}{2\sqrt{x}}\right) + \left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)\left(-e^{-xy/10} \cdot\right)\left(\frac{-y}{10}\right)$$

$$= \left(e^{-xy/10}\right)\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{y}}{2\sqrt{x}}\right) + \left(\sqrt{x} + \sqrt{y} + \sqrt{xy}\right)\left(\frac{ye^{-xy/10}}{10}\right)$$

$=$ $x = 1$; $y = 1$

$$= e^{-1/10}\left(\frac{1}{2}+\frac{1}{2}\right)+(1+1+1)\left(\frac{e^{-1/10}}{10}\right)$$

$$= e^{-0.1} + 0.3\, e^{-0.1}$$

$f_y(1,1)$

$$= \frac{dz}{dY} \quad \uparrow y\text{const}$$

$$= \left(e^{-xy/10}\right)\frac{d}{dY}\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right) + \left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right)\frac{d}{dY}\left(e^{-xy/10}\right) \quad \uparrow x\text{const}$$

$$= \left(e^{-xy/10}\cdot\right)\left(0+\frac{1}{2\sqrt{y}}+\frac{\sqrt{x}}{2\sqrt{y}}\right)+\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right)\left(-e^{-xy/10}\cdot -\frac{x}{10}\right)$$

$$= \left(e^{-xy/10}\right)\left(\frac{1}{2\sqrt{y}}+\frac{\sqrt{x}}{2\sqrt{y}}\right)+\left(\sqrt{x}+\sqrt{y}+\sqrt{xy}\right)\left(\frac{x\,e^{-xy/10}}{10}\right)$$

$x = 1 \quad y = 1$

$$= e^{-0.1}\left(\frac{1}{2}+\frac{1}{2}\right)+(1+1+1)\left(\frac{e^{-1/10}}{10}\right)$$

$$= e^{-0.1} + 0.3\, e^{-0.1}$$

$\therefore$

$$z - 3e^{-0.1} = e^{-0.1} + 0.3e^{-0.1}(x-1) + e^{-0.1} + 0.3e^{-0.1}(y-1)$$

$$z - 3e^{-0.1} = e^{-0.1} + 0.3e^{-0.1}x - 0.3e^{-0.1} + e^{-0.1} + 0.3e^{-0.1}y - 0.3e^{-0.1}$$

Let $e^{-0.1} = A$

$$\underline{z - 3A} = \cancel{A} + 0.3Ax - \underline{0.3A} + \underline{A} + 0.3Ay - \cancel{A}$$

$$-3A - 0.7A = 0.3Ax + 0.3Ay - z$$

$$-3.7A = 0.3Ax + 0.3Ay - z$$

$$0.3e^{-0.1}x + 0.3e^{-0.1}y - z = -3.7e^{-0.1}$$

11–16 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = 1 + x\ln(xy - 5)$, $(2, 3)$

12. $f(x, y) = \sqrt{xy}$, $(1, 4)$

13. $f(x, y) = x^2e^y$, $(1, 0)$

14. $f(x, y) = \dfrac{1 + y}{1 + x}$, $(1, 3)$

15. $f(x, y) = 4\arctan(xy)$, $(1, 1)$

16. $f(x, y) = y + \sin(x/y)$, $(0, 3)$

11. $f(x,y) = 1 + x\ln(xy-5)$, $(2,3)$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$f(x_0,y_0) = 1 + 2\ln(6-5)$

$= 1 + 2\ln|1| = 1 + 2\cdot 0 = 1$

$(x-x_0) = (x-2)$

$(y-y_0) = (y-3)$

$f_x(x_0,y_0) = \frac{dz}{dx}\Big\uparrow^{y\,\text{const}} = 0 + x\frac{1}{xy-5}\cdot y + \ln(xy-5)\cdot 1$

$= \frac{xy}{xy-5} + \ln(xy-5)$

$f_x(2,3) = \frac{6}{6-5} + \ln(6-5) = \frac{6}{1} + 0 = 6$

$f_y(x_0,y_0) = \frac{dz}{dy}\Big\uparrow^{x\,\text{const}} = 0 + x\frac{1}{xy-5}\cdot x + 0$

$= \frac{x^2}{xy-5}$

$f_y(2,3) = \frac{4}{6-5} = 4$

$$L(x,y) = 1 + 6(x-2) + 4(y-3)$$

$$= 1 + 6x - 12 + 4y - 12$$

$$= 1 - 24 + 6x + 4y$$

$$\boxed{L(x,y) = 6x + 4y - 23}$$

As to answer why the function is differentiable at the given point is because:
if the partial derivative is continuous & exists near (a,b) then f is differentiable at (a,b)

12. $f(x,y) = \sqrt{xy}$; $(1,4)$

$$L(x,y) = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$$

$$f(x_0,y_0) = \sqrt{1 \cdot 4} = \sqrt{4} = 2$$

$$f_x(x,y) = \frac{dz}{dx} \overset{y\,\text{const}}{=} \sqrt{y} \cdot \frac{1}{2\sqrt{x}} \qquad f_y(x,y) \overset{x\,\text{const}}{=} \frac{dz}{dy} = \sqrt{x} \cdot \frac{1}{2\sqrt{y}}$$

$$f_x(1,4) = \sqrt{4} \cdot \frac{1}{1} = 2 \qquad f_y(1,4) = \sqrt{1} \cdot \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$L(x,y) = 2 + 2(x-1) + \frac{1}{4}(y-4)$$

$$= 2 + 2x - 2 + \frac{y}{4} - 1$$

$$\boxed{L(x,y) = 2x + \frac{y}{4} - 1 = \frac{8x + y - 4}{4}}$$

13- $f(x,y) = x^2 e^y \quad (1,0)$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(1,0) = 1 e^0 = 1$$

$$f_x(1,0) \overset{\uparrow y\,const}{=} x^2 \cdot 0 + e^y \cdot 2x = 2xe^y$$

$$= 2 \cdot 1 \cdot e^0 = 2$$

$$f_y(1,0) \overset{\uparrow x\,const}{=} x^2 \cdot e^y + e^y \cdot 0 = x^2 e^y$$

$$= 1 \cdot 1 = 1$$

$$L(x,y) = 1 + 2(x - 1) + 1(y - 0)$$

$$= 1 + 2x - 2 + y$$

$$\boxed{L(x,y) = 2x + y - 1}$$

14- $f(x,y) = \dfrac{1+y}{1+x} \; ; \quad (1, 3)$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(x_0, y_0) = \frac{1+3}{1+1} = \frac{4}{2} = 2$$

$$f_x(x,y) = \frac{d}{dx}\left(\frac{1+y}{1+x}\right)$$

$$= \frac{1+x \cdot (0) - (1+y) \cdot 1}{(1+x)^2} = -\frac{1+y}{(1+x)^2}$$

$$f_x(1,3) = \frac{-1+3}{(1+1)^2} = \frac{2}{2^2} = \frac{2}{4} = \frac{1}{2}$$

$$f_y(x,y) = \frac{d}{dy}\left(\frac{1+y}{1+x}\right)$$

$$= \frac{(1+x)(1) - (1+y)\cdot 0}{(1+x)^2} = \frac{1+x}{(1+x)^2}$$

$$f_y(1,3) = \frac{1+1}{(1+1)^2} = \frac{1}{1} = 1$$

$$\boxed{L(x,y) = 2 + \frac{1}{2}(x-1) + 1(y-3)}$$

15. $f(x,y) = 4\tan^{-1}(xy)$; $(1,1)$

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f_x(x,y) = \frac{d}{dx}\left(4\tan^{-1}(xy)\right)^{\uparrow y \text{ const}}$$

$$= 4\left[\frac{1}{1+(xy)^2}\cdot y\right] = \frac{4y}{1+(xy)^2}$$

$$f_y(x,y) = \frac{d}{dy}\left(4\tan^{-1}(xy)\right)^{\uparrow x \text{ const}}$$

$$= 4\left[\frac{1}{1+(xy)^2}\cdot x\right] = \frac{4x}{1+(xy)^2}$$

$$f_x(1,1) = \frac{4}{1+1} = \frac{4}{2} = 2$$

$$f_y(1,1) = \frac{4}{1+1} = \frac{4}{2} = 2$$

$$f(1,1) = 4\tan^{-1}(1) = 4 \cdot \frac{\pi}{4} = \pi$$

$$\boxed{L(x,y) = \pi + 2(x-1) + 2(y-1)}$$

16. $f(x,y) = y + \sin(x/y)$; $(0,3)$

$$f(0,3) = 3 + \sin(0) = 3$$

$$f_x(0,3) \overset{\uparrow y\,\text{const}}{=} 0 + \cos\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right)$$

$$= \frac{1}{3}\cos 0 = \frac{1}{3}$$

$$f_y(0,3) \overset{\uparrow x\,\text{const.}}{=} 1 + \cos\left(\frac{x}{y}\right)\cdot x \cdot\left(-\frac{1}{y^2}\right)$$

$$= 1 - \cos\left(\frac{x}{y}\right)\cdot\frac{x}{y^2}$$

$$f_y(0,3) = 1 - \cos(0)\cdot 0 = 1$$

$$L(x,y) = 3 + \frac{1}{3}(x-0) + 1(y-1)$$

$$= 3 + \frac{1}{3}x + y - 1 = \frac{1}{3}x + y + 2$$

$$\boxed{L(x,y) = \frac{1}{3}x + y + 2}$$

17–18 Verify the linear approximation at (0, 0).

17. $e^x \cos(xy) \approx x + 1$ **18.** $\dfrac{y-1}{x+1} \approx x + y - 1$

17. $e^x \cos xy \approx x+1$; $(0,0)$

$f(0,0) = e^0 \cos 0 = 1$

$f_x(x,y) = \dfrac{d}{dx}\underbrace{(e^x \cos xy)}_{\text{y const}} = e^x \cdot -\sin(xy)\cdot y + \cos xy \cdot e^x$

$= -ye^x \sin(xy) + e^x \cos xy$

$f_x(0,0) = 0 + 1\cdot \cos 0 = 1$

$f_y(x,y) = \dfrac{d}{dy}\underbrace{(e^x \cos xy)}_{\text{x const}} = e^x \cdot \sin(xy)\cdot x + \cos(xy)\cdot 0$

$= xe^x \cdot \sin(xy)$

$f_y(0,0) = 0$

$L(x,y) = 1 + 1(x-0) + 0(y-0)$

$\approx 1 + x$

18. $\dfrac{y-1}{x+1} \approx x+y-1$; $(0,0)$

$f(0,0) = \dfrac{0-1}{0+1} = -1$

$f_x(0,0) = \dfrac{d}{dx}\underbrace{\left(\dfrac{y-1}{x+1}\right)}_{\text{y const}} = \dfrac{(x+1)\cdot 0 - (y-1)(1)}{(x+1)^2}$

$= \dfrac{1+y}{(x+1)^2} = \dfrac{1}{1} = 1$

$$f_y(0,0) = \frac{d}{dy}\left(\frac{y-1}{x+1}\right)^{\uparrow x \text{ const}} = \frac{(x+1)(1) - (y-1)\cdot(0)}{(x+1)^2}$$

$$= \frac{x+1}{(x+1)^2} = \frac{1}{x+1} = 1$$

$$\therefore \quad L(x,y) = -1 + 1(x-0) + 1(y-0)$$

$$\approx x + y - 1 \quad \Rightarrow \text{RHS approximation}$$

Directional derivatives & Gradient vector

4–6 Find the directional derivative of f at the given point in the direction indicated by the angle θ.

4. $f(x, y) = xy^3 - x^2$, $(1, 2)$, $\theta = \pi/3$

$$D_u f = \nabla f \cdot \vec{u}$$

here $\theta = \frac{\pi}{3}$ $\therefore$ $(\cos\alpha, \sin\alpha) = \vec{u}$

$$\vec{u} = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$$

$$\nabla f = \begin{pmatrix} \frac{d}{dx}(xy^3 - x^2) \uparrow y\,\text{const} \\ \frac{d}{dy}(xy^3 - x^2) \uparrow x\,\text{const} \end{pmatrix}$$

$$= \begin{pmatrix} y^3 - 2x \\ 3xy^2 - 0 \end{pmatrix}$$

$$D_u f = \begin{pmatrix} y^3 - 2x \\ 3xy^2 \end{pmatrix} \begin{pmatrix} \cos\pi/3 \\ \sin\pi/3 \end{pmatrix} = \nabla f \cdot \vec{u}$$

$$D_u f(1,2) = \begin{pmatrix} 8 - 2 \\ 3 \cdot 1 \cdot 4 \end{pmatrix} \begin{pmatrix} \cos\pi/3 \\ \sin\pi/3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 12 \end{pmatrix} \begin{pmatrix} \cos \pi/3 \\ \sin \pi/3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 12 \end{pmatrix} \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$= 6 \cdot \frac{1}{2} + 12 \cdot \frac{\sqrt{3}}{2}$$

$$\boxed{D_u f(1,2) = 3 + 6\sqrt{3}}$$

5. $f(x, y) = y\cos(xy)$, $(0, 1)$, $\theta = \pi/4$

6. $f(x, y) = \sqrt{2x + 3y}$, $(3, 1)$, $\theta = -\pi/6$

5. $f(x,y) = y\cos(xy)$ $(0,1)$; $\theta = \pi/4$

$$D_u f(x,y) = \vec{u} \cdot \nabla f$$

$$\vec{u} = \begin{pmatrix} \cos \pi/4 \\ \sin \pi/4 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \frac{d}{dx}(y\cos(xy)) & \uparrow \text{ y const} \\ \frac{d}{dy}(y\cos(xy)) & \uparrow \text{ x const} \end{pmatrix}$$

$$= \begin{pmatrix} y(-\sin(xy)\cdot y) + 0 \\ y(-\sin(xy)\cdot x) + \cos(xy)\cdot 1) \end{pmatrix}$$

$$= \begin{pmatrix} -y^2 \sin(xy) \\ -xy\sin(xy) + \cos(xy) \end{pmatrix}$$

$$\nabla f(0,1) = \begin{pmatrix} -1\cdot 0 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D_u f(0,1) = \nabla f \cdot \vec{u}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\boxed{D_u f(0,1) = 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}}$$

6. $f(x,y) = \sqrt{2x+3y}$, $(3,1)$; $\theta = -\pi/6$

$$D_u f(3,1) = \nabla f(3,1) \cdot \vec{u}$$

$$\vec{u} = \left\langle \cos\left(-\frac{\pi}{6}\right), \sin\left(-\pi/6\right) \right\rangle$$

$$= \left\langle \cos\frac{\pi}{6}, -\sin \pi/6 \right\rangle$$

$$= \langle \cos 30^\circ , -\sin 30^\circ \rangle$$

$$= \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle$$

$$\vec{u} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

y const

x const

$$\nabla f(x,y) = \begin{pmatrix} \frac{d}{dx}\left(\sqrt{2x+3y}\right) \\ \frac{d}{dY}\left(\sqrt{2x+3y}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2x+3y}} \cdot 2 \\ \frac{1}{2\sqrt{2x+3y}} \cdot 3 \end{pmatrix}$$

$$D_u f(x,y) = \begin{pmatrix} \frac{1}{\sqrt{2x+3y}} \\ \frac{3}{2\sqrt{2x+3y}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{3}/2\sqrt{2x+3y} \\ -3/4\sqrt{2x+3y} \end{pmatrix}$$

$$D_a f(3,1) = \frac{\sqrt{3}}{2\sqrt{6+3}} - \frac{3}{4\sqrt{6+3}}$$

$$= \frac{\sqrt{3}}{2 \cdot 3} - \frac{3}{4 \cdot 3}$$

$$\boxed{D_u f(3,1) = \frac{\sqrt{3}}{2}\left(\frac{1}{3} - \frac{\sqrt{3}}{2 \cdot 3}\right)}$$

11–17 Find the directional derivative of the function at the given point in the direction of the vector **v**.

11. $f(x, y) = e^x \sin y, \quad (0, \pi/3), \quad \mathbf{v} = \langle -6, 8 \rangle$

12. $f(x, y) = \dfrac{x}{x^2 + y^2}, \quad (1, 2), \quad \mathbf{v} = \langle 3, 5 \rangle$

13. $g(s, t) = s\sqrt{t}, \quad (2, 4), \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j}$

14. $g(u, v) = u^2 e^{-v}, \quad (3, 0), \quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

15. $f(x, y, z) = x^2 y + y^2 z, \quad (1, 2, 3), \quad \mathbf{v} = \langle 2, -1, 2 \rangle$

16. $f(x, y, z) = xy^2 \tan^{-1} z, \quad (2, 1, 1), \quad \mathbf{v} = \langle 1, 1, 1 \rangle$

17. $h(r, s, t) = \ln(3r + 6s + 9t), \quad (1, 1, 1),$
$\mathbf{v} = 4\mathbf{i} + 12\mathbf{j} + 6\mathbf{k}$

11. $f(x,y) = e^x \sin y \quad (0, \pi/3) \quad v = \langle -6, 8 \rangle$

$$\nabla f = \begin{pmatrix} \frac{dz}{dx} \\ \frac{dz}{dy} \end{pmatrix} \qquad u = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right)$$

$$\vec{u} = \left(\frac{-6}{\sqrt{36+64}}, \frac{8}{\sqrt{36+64}} \right)$$

$$\nabla f = \begin{pmatrix} \frac{d}{dx} e^x \sin y \uparrow y\,\text{const} \\ \frac{d}{dy} e^x \sin y \uparrow x\,\text{const} \end{pmatrix} \qquad \vec{u} = \left(\frac{-6}{10}, \frac{8}{10}\right) = \begin{pmatrix} -\frac{6}{10} \\ \frac{8}{10} \end{pmatrix}$$

$$= \begin{pmatrix} e^x \cdot 0 + \sin y \cdot e^x \\ e^x \cdot \cos y + \sin y \cdot 0 \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} e^x \sin y \\ e^x \cos y \end{pmatrix}$$

$$\nabla f\left(0, \frac{\pi}{3}\right) = \begin{pmatrix} e^0 \sin\frac{\pi}{3} \\ e^0 \cos \pi/3 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$D_u f = \nabla f \cdot \vec{u} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} -6/10 \\ 8/10 \end{pmatrix}$$

$$\boxed{D_u f\left(0, \frac{\pi}{3}\right) = \frac{-6\sqrt{3}}{20} + \frac{8}{20} = \frac{8 - 6\sqrt{3}}{20}}$$

12. $f(x,y) = \dfrac{x}{x^2+y^2}$; $(1,2)$, $v = \langle 3,5 \rangle$

$$D_u f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

$$\vec{u} = \begin{pmatrix} \dfrac{3}{\sqrt{3^2+5^2}} \\[2ex] \dfrac{5}{\sqrt{3^2+5^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{3}{\sqrt{34}} \\[2ex] \dfrac{5}{\sqrt{34}} \end{pmatrix}$$

$$\nabla f(x,y) = \begin{pmatrix} \dfrac{d}{dx}\left(\dfrac{x}{x^2+y^2}\right) \uparrow \text{y const} \\[2ex] \dfrac{d}{dy}\left(\dfrac{x}{x^2+y^2}\right) \uparrow \text{x const} \end{pmatrix} = \begin{pmatrix} \dfrac{(x^2+y^2)\,1 - x(2x+0)}{(x^2+y^2)^2} \\[2ex] \dfrac{x(0+2y) - (x^2+y^2)\cdot 0}{(x^2+y^2)^2} \end{pmatrix}$$

$$\nabla f(1,2) = \begin{pmatrix} \dfrac{5-2}{25} \\[2ex] \dfrac{4-0}{25} \end{pmatrix} = \begin{pmatrix} 3/25 \\[1ex] 4/25 \end{pmatrix}$$

$$D_u f(1,2) = \nabla f \cdot \vec{u}$$

$$= \begin{pmatrix} 3/25 \\[1ex] 4/25 \end{pmatrix} \cdot \begin{pmatrix} \dfrac{3}{\sqrt{34}} \\[2ex] \dfrac{5}{\sqrt{34}} \end{pmatrix}$$

$$\boxed{D_u f(1,2) = \frac{9}{25\sqrt{34}} + \frac{20}{25\sqrt{34}} = \frac{29}{25\sqrt{34}}}$$

13. $g(s,t) = s\sqrt{t}$; $(2,4)$ $v = 2i - \hat{j}$

$$v = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \vec{u} = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$$

t const

$$\nabla g = \begin{pmatrix} \frac{d(s\sqrt{t})}{ds} \\ \frac{d(s\sqrt{t})}{dt} \end{pmatrix} = \begin{pmatrix} \sqrt{t} \\ s \cdot \frac{1}{2\sqrt{t}} \end{pmatrix}$$

s const

$$\nabla g(2,4) = \begin{pmatrix} \sqrt{4} \\ \frac{2}{2\cdot\sqrt{4}} \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}$$

$$D_u g(2,4) = \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$\boxed{D_u g(2,4) = \frac{4}{\sqrt{5}} - \frac{1}{2\sqrt{5}}}$$

14. $g(u,v) = u^2 e^{-v}$; $(3,0)$; $v = 3\hat{\imath} + 4\hat{\jmath}$

$$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\nabla f = \begin{pmatrix} \frac{d(u^2 e^{-v})}{du} \uparrow \text{v const} \\ \frac{d(u^2 e^{-v})}{dv} \uparrow \text{u const} \end{pmatrix} = \begin{pmatrix} u^2 \cdot 0 + e^{-v} \cdot 2u \\ u^2 \cdot (-e^{-v}) + e^{-v} \cdot 0 \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} 2ue^{-v} \\ -u^2 e^{-v} \end{pmatrix}$$

$$\nabla f(3,0) = \begin{pmatrix} 2 \cdot 3 \cdot e^{-0} \\ -9 \cdot e^{0} \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

$$D_u g(3,0) = \begin{pmatrix} 6 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$= \frac{18}{5} - \frac{36}{5} = \frac{-18}{5}$$

15. $f(x,y,z) = x^2 y + y^2 z$; $(1,2,3)$; $v = \langle 2, -1, 2 \rangle$

$$\nabla f = \begin{pmatrix} \frac{dz}{dx} & \uparrow \text{ y \& z const} \\ \frac{dz}{dy} & \uparrow \text{ x \& z const} \\ \frac{dz}{dz} & \uparrow \text{ x \& y const} \end{pmatrix} = \begin{pmatrix} y \cdot 2x + 0 \\ x^2 + z \cdot 2y \\ 0 + y^2 \end{pmatrix} = \begin{pmatrix} 2xy \\ x^2 + 2zy \\ y^2 \end{pmatrix}$$

$$\nabla f(1,2,3) = \begin{pmatrix} 4 \\ 1 + 2 \cdot 3 \cdot 2 \\ 2^2 \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \\ 4 \end{pmatrix}$$

$$\vec{u} = \left(\frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}} \right)$$

$$= \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right)$$

$$D_u f(1,2,3) = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$= \frac{8}{3} - \frac{1}{3} + \frac{6}{3} = \frac{13}{3}$$

16. $f(x,y,z) = xy^2 \tan^{-1} z \; ; \; (2,1,1) \quad v = \langle 1,1,1 \rangle$

$$\vec{u} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} \frac{d}{dx}\left(xy^2 \tan^{-1} z\right)^{\uparrow yz\ \text{const}} \\ \frac{d}{dy}\left(xy^2 \tan^{-1} z\right)^{\uparrow xz\ \text{const}} \\ \frac{d}{dz}\left(xy^2 \tan^{-1}(z)\right)^{\uparrow xy\ \text{const}} \end{pmatrix} = \begin{pmatrix} y^2 \tan^{-1}(z) \\ x \tan^{-1}(z) \cdot 2y \\ xy \cdot \frac{1}{1+z^2} \end{pmatrix}$$

$$\nabla f(2,1,1) = \begin{pmatrix} 1 \cdot \frac{\pi}{4} \\ 2\frac{\pi}{4} \cdot 2 \\ 2 \cdot \frac{1}{1+1} \end{pmatrix} = \begin{pmatrix} \pi/4 \\ \pi \\ 1 \end{pmatrix}$$

$$D_u f = \nabla f \cdot \vec{u} = \begin{pmatrix} \pi/4 \\ \pi \\ 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\boxed{D_v f = \frac{\pi}{4\sqrt{3}} + \frac{\pi}{\sqrt{3}} + \frac{1}{\sqrt{3}}}$$

17. $h(r,s,t) = \ln(3r + 6s + 9t)$; $(1,1,1)$ $v = 4i + 12\hat{j} + 6k$

$$\vec{u} = \left(\frac{4}{\sqrt{4^2 + 12^2 + 6^2}}, \frac{12}{\sqrt{16 + 144 + 36}}, \frac{6}{\sqrt{196}} \right)$$

$$= \left(\frac{4}{8\sqrt{3}}, \frac{12}{8\sqrt{3}}, \frac{6}{8\sqrt{3}} \right)$$

$$\nabla f = \begin{pmatrix} \frac{dz}{dr} \uparrow s,t \text{ const} \\ \frac{dz}{ds} \uparrow r,t \text{ const} \\ \frac{dz}{dt} \uparrow r,s \text{ const} \end{pmatrix} = \begin{pmatrix} \frac{1}{3r + 6s + 9t} \cdot 3 \\ \frac{1}{3r + 6s + 9t} \cdot 6 \\ \frac{1}{3r + 6s + 9t} \cdot 9 \end{pmatrix}$$

$$\nabla f(1,1,1) = \left(\frac{1}{3 + 6 + 9} \cdot 3 \right.$$

$$\begin{pmatrix} \frac{1}{18} \cdot 6 \\ \frac{1}{18} \cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{18} \\ \frac{6}{18} \\ \frac{9}{18} \end{pmatrix}$$

$$D_u f(1,1,1) = \nabla f(1,1,1) \cdot \vec{u}$$

$$= \begin{pmatrix} \frac{3}{18} \\ \frac{6}{18} \\ \frac{9}{18} \end{pmatrix} \cdot \begin{pmatrix} \frac{4}{8\sqrt{3}} \\ \frac{12}{8\sqrt{3}} \\ \frac{6}{8\sqrt{3}} \end{pmatrix}$$

$$\boxed{D_u f(1,1,1) = \frac{12}{18 \cdot 8\sqrt{3}} + \frac{6 \cdot 12}{18 \cdot 8\sqrt{3}} + \frac{9 \cdot 6}{18 \cdot 8\sqrt{3}}}$$

Maximum & Minimum values: Multivariate

Two Types of Question:

Local Maxima / Minima

- **Find Critical Points** $\nabla f = 0$

Compute the gradient of the function (the vector of the partial derivatives). Set it equal to zero vector & solve for the variables

- **Classify the Critical Points**

Generate the Hessian Matrix (H)

$$H = \begin{bmatrix} \frac{d^2 f}{dx^2} & \frac{d^2 f}{dx\,dy} \\ \frac{d^2 f}{dy\,dx} & \frac{d^2 f}{dy^2} \end{bmatrix} \text{ or } \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Compute determinant of H

$$D = \Delta = \det(H)$$

- **Interpret the results:**

if $D > 0$; $f_{xx} > 0$ local minimum

$D > 0$; $f_{xx} < 0$ local maximum

$D < 0$; saddle point

$D = 0$; test inconclusive

(evaluate overall function)

Absolute Maxima / Minima

- **Find Critical Points** $\nabla f = 0$

same as in local max/minima

- **Evaluate the function at CP's**

Substitute the coordinates of the critical points into the original function & evaluate the function value at these points

- **Evaluate the function on the boundaries**

The boundary region is typically composed of several lines or curves. For each boundary part:

Parameterize the boundary:

- if it is a vertical line; you have a fixed x value eg: $x = 3$. for that line find the max & min y vals & evaluate $f(3, y_1)$ & $f(3, y_2)$

- if it is a line segment; express the coordinates in terms of single param t; parameterizing the line segment.

eg: line seg. b/w $(1,1)$ & $(2,3)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

Think of t as time parameter going

from 0 to 1; where we start from (1, 1) to end at (2, 3).

generate the $\begin{pmatrix} x \\ y \end{pmatrix}$ equations & substitute them to the $f(x, y)$ function in the question to obtain new function $g(t)$.

Substitute t's range [0, 1] to obtain the max & min values from $g(t)$.

- If it is a circle with centre (h, k) and radius r

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + r \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

$t : [0, 2\pi]$

eg: $(h, k) : (0, 0)$ & $r = 1$
& $f(x, y) = x^2 + y^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}; \quad t \in [0, 2\pi]$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \Rightarrow g(t)$$

$f(x, y) = x^2 + y^2$

$g(x, y) = \cos^2 t + \sin^2 t$

<u>when $t = 0$</u>

$g(x, y) = 1 + 0 = 1$

<u>when $t = 2\pi$</u>

$g(x, y) = \cos^2(2\pi) + \sin^2(2\pi) = 1$

In both cases we have 1; $\therefore$ max & min of $f(x, y)$ are 1

- If it is an ellipse with (h, k) centre; semimajor axis a & semiminor axis b; Apply the same logic as circle but parameterize the equation properly:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

- **Evaluate Function at Vertices**

If it is a region of a polygon; triangle or rectangle; you will have vertices; substitute those coordinates into the function to obtain function values

- **Find the Absolute Maxima & Minima:**

From the above values evaluated from the CP's; boundary lines & vertices; the largest value will be the maximum & smallest would be the minimum absolutes.

1) Find the absolute max & min:

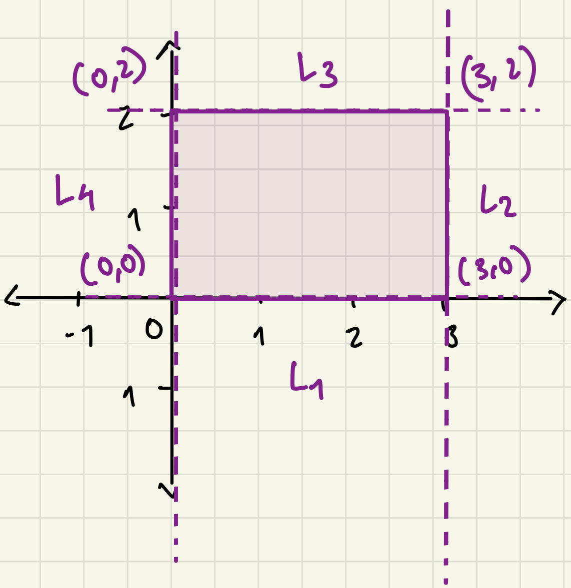
$$f(x,y) = x^2 - 2xy + 2y$$

$$D = \{ (x,y) \mid 0 \leq x \leq 3 \; ; \; 0 \leq y \leq 2 \}$$

step summary: (make a plot first)

1) evaluate CP's & its function vals
2) evaluate Boundaries function vals
3) evaluate each vertices function vals

Plot:



- Generated vertices:
  (0,2) (0,0) (3,2) (3,0)

- Lines $L_1 \to L_4$:

  L_1: $y = 0$; $x_1 = 0$ $x_2 = 3$

  L_2: $x = 3$; $y_1 = 0$ $y_2 = 2$

  L_3: $y = 2$; $x_1 = 0$ $x_2 = 3$

  L_4: $x = 0$; $y_1 = 0$ $y_2 = 2$

generated points to evaluate

L_1: (0,0), (3,0)

L_2: (3,0), (3,2)

L_3: (0,2), (3,2)

L_4: (0,0), (0,2)

Finding CP's:

$$\nabla f = \begin{pmatrix} f_x \uparrow \text{y const} \\ f_y \uparrow \text{x const} \end{pmatrix} = 0$$

$$= \begin{pmatrix} 2x - 2y \\ -2x + 2 \end{pmatrix} = 0$$

$$= \begin{pmatrix} 2x - 2y \\ 2 - 2x \end{pmatrix} = 0$$

$$2 - 2x = 0 \qquad \therefore \qquad 2 - 2y = 0$$

$$x = 1 \qquad\qquad\qquad 2 = 2y$$

$$y = 1$$

generated CP: $(1, 1)$

Function value chart:

All the common Data points	$f(x,y) = x^2 - 2xy + 2y$
$(0,0)$	$0 - 0 + 0 = 0$ ← absolute minimum
$(3,0)$	$9 - 0 + 0 = 9$ ← absolute maximum
$(3,2)$	$9 - 12 + 4 = 1$
$(0,2)$	$0 - 0 + 4 = 4$
$(1,1)$	$1 - 2 + 2 = 1$

Task 5

Read section 14.7 *Maximum and Minimum Values*. Solve the exercises 29-36 (7th ed.) or 31-38 (8th. ed.).

To practice finding just the local min/max, there are examples in the exercises 5-18.

31–38 Find the absolute maximum and minimum values of f on the set D.

31. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

32. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$

33. $f(x, y) = x^2 + y^2 + x^2y + 4$,
$D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

34. $f(x, y) = x^2 + xy + y^2 - 6y$,
$D = \{(x, y) \mid -3 \leqslant x \leqslant 3, 0 \leqslant y \leqslant 5\}$

35. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$,
$D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$

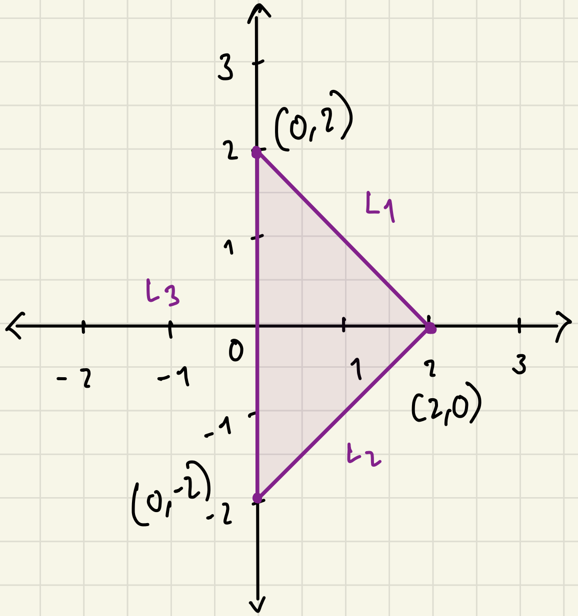
36. $f(x, y) = xy^2$, $D = \{(x, y) \mid x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\}$

37. $f(x, y) = 2x^3 + y^4$, $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$

38. $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$

31) $f(x,y) = x^2 + y^2 - 2x$ $D \to$ closed triangular region

vertices: $(2,0)$ $(0,2)$ $(0,-2)$



Solving for critical points:

$$\nabla f = 0 \Rightarrow \begin{pmatrix} \frac{d}{dx}(x^2+y^2-2x) \overset{\uparrow}{\text{y const}} \\ \frac{d}{dy}(x^2+y^2-2x) \overset{\uparrow}{\text{x const}} \end{pmatrix} = 0$$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x + 0 - 2 \\ 0 + 2y - 0 \end{pmatrix} = 0$$

$2x - 2 = 0$

$x = 1$

$2y = 0$

$y = 0$

we have one critical point : CP: $(x, y) = (1, 0)$

$f(1,0) = 1 + 0 - 2 = -1$

Evaluating function values at boundaries

L1: b/w $(0,2)$ & $(2,0)$

L2: b/w $(0,-2)$ & $(2,0)$

L3 : $x = 0$; $y_1 = 2$ & $y_2 = -2$

parameterising L_1 & L_2

L_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2-0 \\ 0-2 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \begin{matrix} = 2t \\ = 2-2t \end{matrix}$

$f(x,y) = x^2 + y^2 - 2x$

$f(2t, 2-2t) = (2t)^2 + (2-2t)^2 - 2(2t)$

when $t = 0$

$g(t) = 0 + 4 - 0$
$= 4$

when $t = 1$

$4 + 0 - 4 = 0$

L_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2-0 \\ 0+2 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ -2+2t \end{pmatrix}$$

$$f(2t, 2t-2) = 4t^2 + (2t-2)^2 - 4t$$

$$g(t) = 4t^2 + 4t^2 + 4 - 8t - 4t$$

$$= 8t^2 - 12t + 4$$

$t \in [0, 1]$

when $t = 0$ (min)

$g(t) = 0 - 0 + 4 = 4$

$= 4$

when $t = 1$ (max)

$g(t) = 8 - 12 + 4$

$= 0$

L_3: $x = 0$ $y_1 = 2$, $y_2 = -2$ $\therefore$ point: $(0, 2)$, $(0, -2)$

$$f(0, 2) = 0 + 4 - 0 = 4$$

$$f(0, -2) = 0 + 4 - 0 = 4$$

Evaluating function value at vertices:

$f(0, 2) = 4$

$f(0, -2) = 4$

$f(2, 0) = 4 + 0 - 4 = 0$

Finding D

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

we have $\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 2x-2 \\ 2y \end{pmatrix}$

$f_{xx} = \frac{d}{dx}(2x-2)^{\uparrow y\,const} = 2$

$f_{xy} = \frac{d}{dx}(2x-2)^{\uparrow x\,const} = 0$

$f_{yy} = \frac{d}{dy}(2y)^{\uparrow x\,const} = 2$

$f_{yx} = \frac{d}{dx}(2y)^{\uparrow y\,const} = 0$

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 - 0 = 4$$

we have $D > 0$ & $f_{xx} > 0$
$\therefore$ we have a local minimum at $(1,0)$

Function value chart:

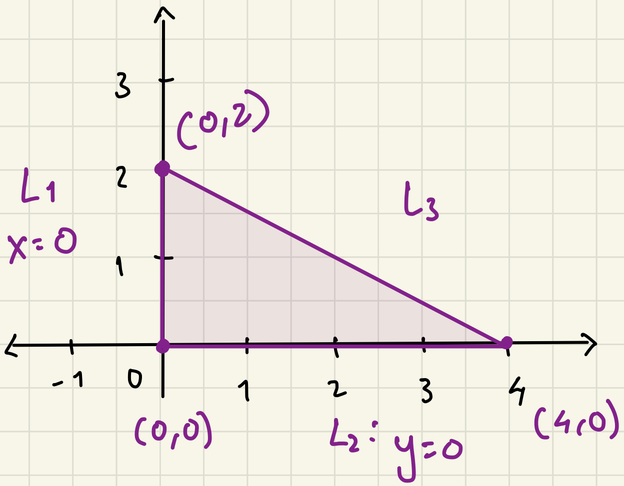
Data points:	$f(x,y)$ values		
$f(0,2)$	4	L_1 max	4
$f(0,-2)$	4	L_1 min	0
$f(2,0)$	0	L_2 max	0
$f(1,0)$	-1	L_2 min	4
		L_3	

↑ absolute minimum

abs. maximum on the boundary.

32. $f(x,y) = x + y - xy$ $\quad (0,0), (0,2), (4,0)$

Plot



Function values at vertices:

Data point	$f(x,y) = x+y-xy$
$(0,0)$	$0+0-0=0$
$(4,0)$	$4+0-0=4$
$(0,2)$	$0+2-0=2$

Function value at Boundaries

L_1: $\quad x=0 \quad y_{max}=2 \quad y_{min}=0$

L_2: $\quad y=0 \quad x_{min}=0 \quad x_{max}=4$

L_3: $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 4-0 \\ 0-2 \end{pmatrix}$

$$x = 4t \qquad \forall\, t \in [0,1]$$
$$y = 2 - 2t$$

$$f(x, y) = x + y - xy$$

$$g(t) = f(4t, 2-2t) = 4t + 2 - 2t - (4t)(2-2t)$$

when $t = 0$; $g(t) = 0 + 2 - 0 - 0 = 2$

$t = 1$; $g(t) = 4 + 2 - 2 - (4)(0)$
$= 4$

	Data points	$f(x,y) = x + y - xy$
L_1:	$(0, 2)$	$0 + 2 - 0 = 2$
	$(0, 0)$	$0 + 0 - 0 = 0$
L_2:	$(0, 0)$	$0 + 0 - 0 = 0$
	$(4, 0)$	$4 + 0 - 0 = 4$

Line segments	values	
	max	min
L_1	2	0
L_2	0	4
L_3	4	2

Critical Point & it function value

$$\nabla f = 0 \quad \text{ie} \quad \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$$

$$f(x,y) = x + y - xy$$

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} 1 + 0 - y \\ 0 + 1 - x \end{pmatrix} = 0$$

↑ y const

↑ x const

$$= \quad 1 - y = 0 \quad \& \quad 1 - x = 0$$

$$y = 1 \qquad\qquad x = 1$$

we have one critical point : $CP = (1, 1)$

Data Point	Function Value
$f(1,1)$	$1 + 1 - 1 = 1$

therefore we have: absolute max at $f(4,0)$

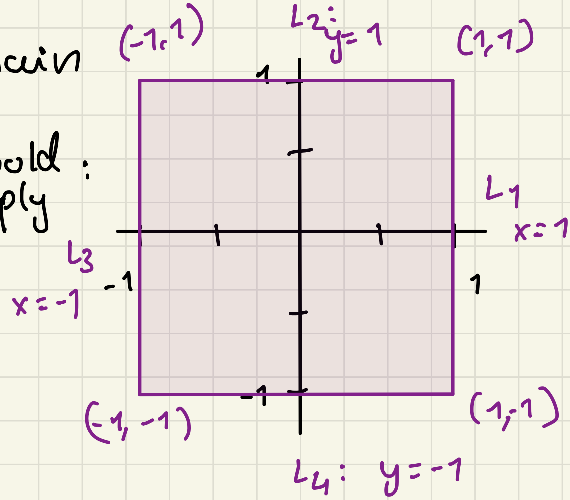
absolute min at $f(0,0)$

33. $f(x,y) = x^2 + y^2 + x^2y + 4$

$D = \{ (x,y) \mid |x| \leq 1, \ |y| \leq 1 \}$

For the given domain

$\left.\begin{array}{l} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{array}\right\}$ would imply:



Function Value at vertices:

Datapoint	$f(x,y)$: $x^2+y^2+x^2y+4$
$(1,1)$	$1+1+1+4 = 7$
$(1,-1)$	$1+1-1+4 = 5$
$(-1,-1)$	$1+1-1+4 = 5$
$(-1,1)$	$1+1+1+4 = 7$

Function Value at line segments

Data points:

L_1: $x = 1$ $\quad y_{11} = 1 \quad y_{12} = -1 \quad \therefore (1,1)$ & $(1,-1)$

L_2: $y = 1$ $\quad x_{21} = -1 \quad x_{22} = 1 \quad \therefore (-1,1)$ & $(1,1)$

$L_3: \ x = -1 \qquad y_{31} = -1 \qquad y_{32} = 1 \quad \therefore (-1,-1) \ \& \ (-1,1)$

$L_4: \ y = -1 \qquad x_{41} = -1 \qquad x_{42} = 1 \quad \therefore (-1,-1) \ \& \ (1,-1)$

Function values for these repeated datapoints are already computed.

Function value at Critical Point

$$\nabla f = 0 \ ; \quad \begin{pmatrix} f_x \ \uparrow y\text{ const} \\ f_y \ \uparrow x\text{ const} \end{pmatrix} = 0$$

$$f_x = 2x + y \cdot 2x = 0$$
$$f_y = 2y + x^2 = 0$$

$$2x(y+1) = 0$$

$$2x = 0 \qquad y = -1$$
$$x = 0$$

also

→ Mistake: $x(2y+2) = 0$

$x = 0 \quad 2y = -2 \qquad\qquad x_1 = 0 \quad y_1 = -2 \quad y_2 = -1$

$y = -1$

$$2y + x^2 = 0$$

when $x_1 = 0$: $\quad 2y = 0 \ ; \quad y_1 = 0 \quad \therefore CP_1 : (0,0)$

This is not $\quad y_1 = -2$: $\quad x^2 = 4 \ ; \quad x_{1,2} = \pm 2 \quad CP_2 : (4, 2)$

$CP_3 : (4, -3)$

a CP

$y_2 = -1 \qquad x^2 = 2 \; ; \quad x = \pm\sqrt{2} \quad CP_4 : (\sqrt{2}, -1)$

$CP_5 : (-\sqrt{2}, -1)$

Critical Data Points $\qquad f(x,y) = x^2 + y^2 + x^2 y + 4$

$(0,0) \qquad 0+0+0+4 = 4$

abs $\left\{ \begin{array}{l} \cancel{(4,2)\ (\text{max})} \qquad \cancel{16+4+32+4 = 56} \\ \cancel{(4,-3)\ (\text{min})} \qquad \cancel{16+9+16\cdot(-3)+4 = 29-48 = -19} \end{array} \right.$ Nota CP!

$(\sqrt{2}, -1) \qquad 2+1-2+4 = 5$

$(-\sqrt{2}, -1) \qquad 2+1-2+4 = 5$

Incase local max & min needed

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$= \quad f_{xx} = 2 + 2y$$

$$f_{xy} = 0 + 2x$$

$$f_{yx} = 2x$$

$$f_{yy} = 2$$

$$D = \begin{vmatrix} 2+2y & 2x \\ 2x & 2 \end{vmatrix} = (4+4y) - 4x^2$$

Data point	$D = (4+4y) - 4x^2$	$f_{xx} = 2+2y$
~~$(4, 2)$~~ Not a CP.		
~~$(4, -3)$~~		
$(0, 0)$	$(4+0) - 0 = \cancel{0}\ 4$	$2+0=2$
$(\sqrt{2}, -1)$	$(4-4) - 4(2) = -8$	$2-2=0$
$(-\sqrt{2}, -1)$	$0 - 4(2) = -8$	$2-2=0$

$D > 0$; f_{xx} +ve $\to$ local min

$D > 0$; f_{xx} +ve $\to$ local max

$D < 0$; saddle point

$D = 0$; inconclusive

$\therefore$ we have local minimum at $(0,0)$
the other two are saddle points.

34. $f(x,y) = x^2 + xy + y^2 - 6y$

$D = \{(x,y) \mid -3 \le x \le 3,\ 0 \le y \le 5\}$

Evaluating function Value at Critical Points

$\nabla f = 0$ ie $\begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$

$\uparrow y$ cons, $\uparrow x$ cons

$$= \begin{pmatrix} 2x + y \\ x + 2y - 6 \end{pmatrix} = 0$$

$$2x + y = 0 \overset{\times 2}{\Rightarrow} 4x + 2y = 0$$

$$x + 2y = 6 \qquad x + 2y = 6 \ (-)$$

$$3x = -6$$

$$x = -2$$

when $x = -2$; $-2 + 2y = 6$

$2y = 8$

$y = 4$

one critical point : $CP_1: (-2, 4)$

$f(x,y) = x^2 + xy + y^2 - 6y$

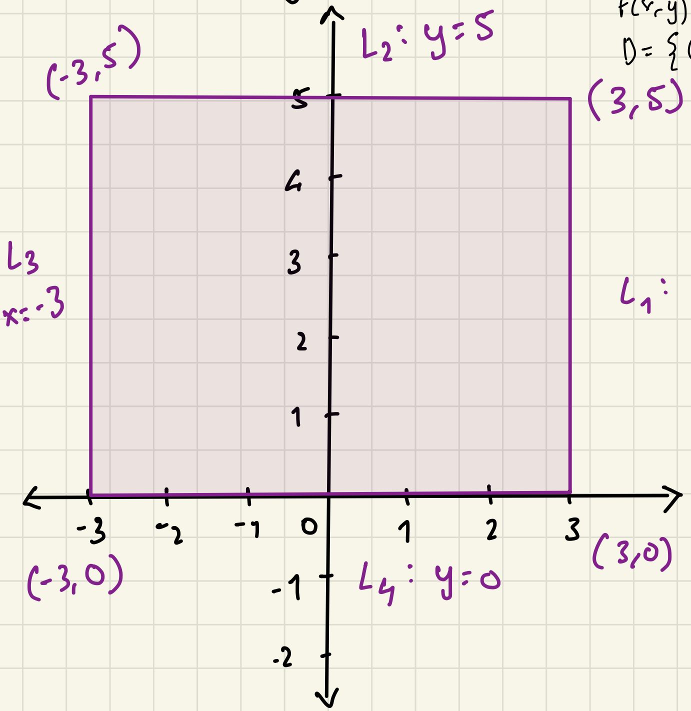
$f(-2,4) = 4 - 8 + 16 - 24$

$= 20 - 32 = -12$

Evaluating function at vertices

$f(x,y) = x^2 + xy + y^2 - 6y$

$D = \{(x,y) \mid -3 \le x \le 3,\ 0 \le y \le 5\}$



Datapoints	$f(x,y) = x^2 + xy + y^2 - 6y$
(3, 0)	$9 + 0 + 0 - 0 = 9$
(3, 5)	$9 + 15 + 25 - 30 = 19$
(-3, 5)	$9 - 15 + 25 - 30 = -11$
(-3, 0)	$9 - 0 + 0 - 0 = 9$

Function value at Boundaries	Data points to evaluate
L_1: $x = 3$	$(3, 0)$, $(3, 5)$
L_2: $y = 5$	$(-3, 5)$, $(3, 5)$

$L_3: x = -3 \qquad (-3, 5), (-3, 0)$

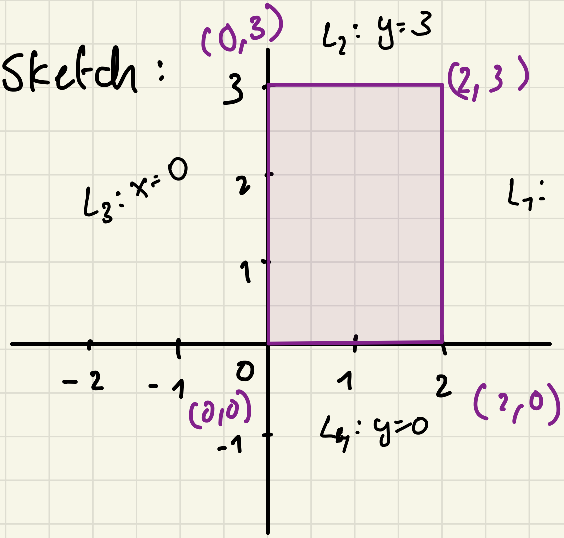
$L_4: y = 0 \qquad (-3, 0), (3, 0)$

Data point's function values has already been computed therefore from this we can estimate that we have a function's absolute max at $f(3,5)$ &
absolute min at $f(-2,4)$ which is also the critical point

35. $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1$

$D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3\}$

Sketch:



Evaluating function value at Critical Point:

$\nabla f = 0 \qquad \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$

$$= \begin{pmatrix} 2x + 0 - 2 \\ 4y - 4 \end{pmatrix} = 0$$

$$2x = 2 \qquad\qquad 4y = 4$$
$$x = 1 \qquad\qquad y = 1$$

We have one critical point ; CP_1: $(1, 1)$

Function value at CP: $f(1,1) = 1 + 2 - 2 - 4 + 1$
$= 2 - 4 = -2$

Evaluating Function at Boundaries & Vertices:

	Datapoints	
$L_1: x = 2$	$(2, 3)$	$(2, 0)$
$L_2: y = 3$	$(0, 3)$	$(2, 3)$
$L_3: x = 0$	$(0, 3)$	$(0, 0)$
$L_4: y = 0$	$(0, 0)$	$(2, 0)$

Data points	$f(x,y) = x^2 + 2y^2 - 2x - 4y + 1$
$(2, 3)$	$4 + 18 - 4 - 12 + 1 = 7$
$(2, 0)$	$4 + 0 - 4 - 0 + 1 = 1$
$(0, 3)$	$0 + 18 - 0 - 12 + 1 = 7$
$(0, 0)$	$0 + 0 - 0 - 0 + 1 = 1$

Absolute max at $f(2, 3)$ & $f(0, 3)$

Absolute min at CP_1 : $f(1, 1)$

36. $f(x,y) = xy^2 \quad D = \{(x,y) \mid x \geq 0, y \geq 0, x^2+y^2 \leq 3\}$

we have $x^2+y^2 \leq 3$
This boundary represents a circle
with radius $\pm\sqrt{3}$ $\therefore$ parametric boundary eqn is

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pm \sqrt{3} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} ; \quad t \in [0, 2\pi]$$

$$f(x,y) = f(\pm\sqrt{3}\cos(t), \pm\sqrt{3}\sin(t))$$

when $t=0$: $f(\pm\sqrt{3}\cos(0), \pm\sqrt{3}\sin(0))$
$= f(\pm\sqrt{3}, 0)$
$= \sqrt{3} \cdot 0^2 = 0$

when $t=2\pi$: $f(\pm\sqrt{3}\cos(2\pi), \pm\sqrt{3}\sin(2\pi))$
$= f(\pm\sqrt{3}, 0)$
$= \sqrt{3} \cdot 0^2 = 0$

Critical Point of the function:

$$\nabla f = 0 \qquad \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$$

(f_x: ↑ y const; f_y: ↑ x const)

$$= \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = 0$$

(2 complex roots) $\quad y^2 = 0$, $y = 0$ $\qquad y \cdot 2x = 0$, $x = 0$ or $y = 0$

we have one critical point at $(0,0)$

$f(0,0) = 0$

Computing local Min & Max

$f_{xx} = 0 \qquad f_{xy} = 2y$

$f_{yx} = 2 \qquad f_{yy} = 0$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2y \\ 2 & 0 \end{vmatrix} = 0 - 4y = 4y = 0$$

$D = 0$; inconclusive.

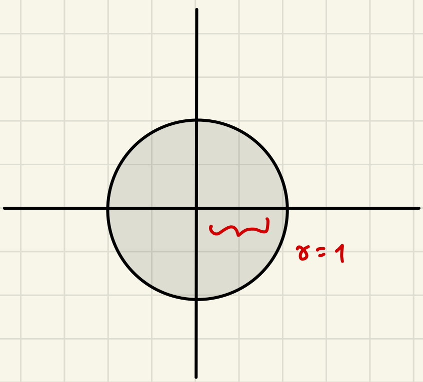
37. $f(x,y) = 2x^3 + y^4 \qquad D = \{(x,y) \mid x^2 + y^2 \leq 1\}$

Evaluating Critical Point:

$$\nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$$

↑ y const

↑ x const



$$= \begin{pmatrix} 6x^2 + 0 \\ 0 + 4y^3 \end{pmatrix} = 0$$

$x^2 = 0 \qquad 4y^3 = 0$

$x = 0 \qquad y^3 = 0$ (3 complex roots)

$y = 0$

CP_1: $(0,0) \qquad f(0,0) = 0 + 0 = 0$

Evaluating at Boundaries

This is an equation of a circle with radius 1 from the centre

parameterizing it; we have:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$\therefore (x, y) = (\cos t, \sin t) \quad ; \quad t \in [0, 2\pi]$

when $t = 0$

$$f(\cos t, \sin t) = 2\cos^3 t + \sin^4 t$$
$$= 2 \cdot 1 + 0 = 2$$

when $t = 2\pi$ (same...)

$$f(\cos t, \sin t) = 2\cos^3 t + \sin^4 t$$
$$= 2 + 0 = 2$$

We have Absolute Maximum on the boundaries and absolute minimum at the critical point

Abs max: $f(1, 0) \Leftarrow (\cos 0, \sin 0)$

Abs min: $f(0, 0)$

38. $f(x,y) = x^3 - 3x - y^3 + 12y$ $\quad D \to$ quadrilateral

vertices: $(-2,3)$, $(2,3)$, $(2,2)$ & $(-2,-2)$

Evaluating Critical point:

$f(x,y) = x^3 - 3x - y^3 + 12y$

$$\nabla f = 0 \Rightarrow \begin{pmatrix} f_x \\ f_y \end{pmatrix} = 0$$

$$= \begin{pmatrix} 3x^2 - 3 \\ -3y^2 + 12 \end{pmatrix} = 0$$

$3x^2 - 3 = 0$

$x^2 = 1$

(2 complex roots)

$x = \pm 1$

$-3y^2 + 12 = 0$

$3y^2 = 12$

$y^2 = 4$

$y = \pm 2$

CP_1: $(1, 2, f(1,2))$

CP_2: $(1, -2, f(1,-2))$

CP_3: $(-1, 2, f(-1,2))$

CP_4: $(-1, -2, f(-1,-2))$

Evaluating function values & classifying:

$f_{xx} = 6x$ $\quad f_{xy} = 0$

$f_{yx} = 0$ $\quad f_{yy} = -6y$

$$D = \begin{vmatrix} 6x & 0 \\ 0 & -6y \end{vmatrix} = -36xy \quad \& \quad f_{xx} = 6x$$

if $D > 0$ & $f_{xx} > 0$; local min

$D > 0$ & $f_{xx} < 0$; local max

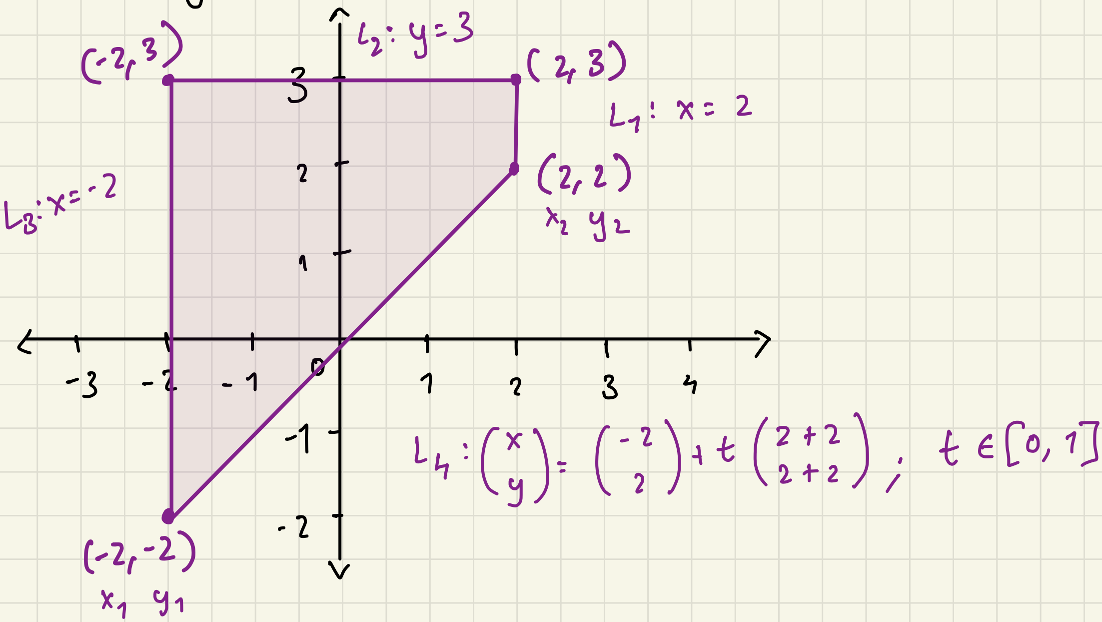
$D < 0$ saddle point

$D = 0$ inconclusive

Critical Data point	$f_{xx} = 6x$	$D = -36xy$	
1, 2	6	−72	saddle
1, −2	6	72	local min
−1, 2	−6	72	local max
−1, −2	−6	72	local max

Critical Data point	$f(x,y) = x^3 - 3x - y^3 + 12y$
1, 2	$1 - 3 - 8 + 24 = 14$
1, −2	$1 - 3 + 8 - 24 = -18$
−1, 2	$-1 + 3 - 8 + 24 = 18$
−1, −2	$-1 + 3 - 8 - 24 = -30$

Evaluating function values at Vertices & Boundaries



vertices: $(-2, 3)$, $(2, 3)$, $(2, 2)$ & $(-2, -2)$

Datapoints

For L_1: $x = 2$; $(2, 2)$, $(2, 3)$

For L_2: $y = 3$; $(-2, 3)$, $(2, 3)$

For L_3: $x = -2$; $(-2, 3)$, $(-2, -2)$

For L_4: $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 + 4t \\ 2 + 4t \end{pmatrix}$$

$$(x, y) = (-2 + 4t, \; 2 + 4t)$$

$t = 0$; $(x, y) = (-2, 2)$

$t = 1$; $(x, y) = (-2+4, \; 2+4) = (2, 6)$

$\underline{t=0}$

$f(-2+4t,\ 2+4t)$

$f(x,y) = x^3 - 3x - y^3 + 12y$

$= (4t-2)^3 - 3(4t-2) - (2+4t)^3 + 12(2+4t)$

when $t=0 \Rightarrow (-2)^3 - 3(-2) - (2)^3 + 12(2)$

$= -8 + 6 - 8 + 24$

$= 30 - 16 = 14$

when $t=1 \Rightarrow (4-2)^3 - 3(4-2) - (2+4)^3 + 12(2+4)$

$= 8 - 6 - 216 + 72$

$= 80 - 222$

$= 60 - 202 = -142$

For the other data points:

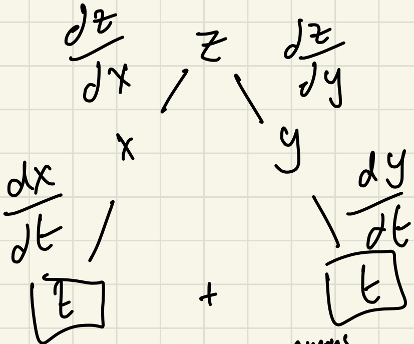
Datapoints	$f(x,y) = x^3 - 3x - y^3 + 12y$
$(2,2)$	$8-6-8+24 = 18$
$(2,3)$	$8-6-27+36 = 11$
$(-2,3)$	$-8+6-27+36 = 7$
$(-2,-2)$	$-8+6-8-24 = -34$

Chain rule

1) $z = x^2 y + 3xy^4$

$x = \sin 2t \qquad y = \cos t$

Find a) $\dfrac{dz}{dt}$ when $t = 0$



$$\frac{dz}{dt} = \overset{\uparrow y\,\text{const}}{\frac{dz}{dx}} \cdot \frac{dz}{dt} + \overset{\uparrow x\,\text{const}}{\frac{dz}{dy}} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^4) \cdot (\cos 2t \cdot 2) + (x^2 + 12xy^3)(-\sin t)$$

$$= 2\cos 2t\,(2xy + 3y^4) - \sin t\,(x^2 + 12xy^3)$$

$$\frac{dz}{dt} = 4\cos 2t\, xy + 6\cos 2t\, y^4 - x^2 \sin t - 12 \sin t\, xy^3$$

when $t = 0$;

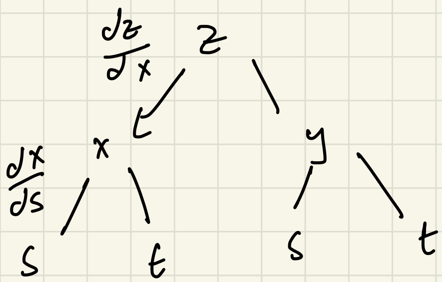
$$\frac{dz}{dt} = 4xy + 6y^4 - 0 - 0 = 4xy + 6y^4$$

we also know $x = \sin 2t$ & $y = \cos t$

$x = 0$ & $y = 1$

$\therefore \frac{dz}{dt} = 4 \cdot 0 + 6 = 6$

3) $z = e^x \sin y \qquad x = st^2 \qquad y = s^2 t$



Find $\frac{dz}{ds}$ & $\frac{dz}{dt}$

1) $\frac{dz}{ds} = \frac{dz}{dx} \cdot \frac{dx}{ds} + \frac{dz}{dy} \cdot \frac{dy}{ds}$

$= \sin y \, e^x \cdot t^2 + e^x \cdot \cos y \cdot t \cdot 2s$

$= e^x t^2 \sin y + 2st \, e^x \cos y$

Don't forget to write : where $x = st^2 \qquad y = s^2 t$

2) $\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$

$= \sin y \cdot e^x \cdot 2st + e^x \cos y \cdot s^2$

$= 2st \, e^x \sin y + s^2 e^x \cos y$

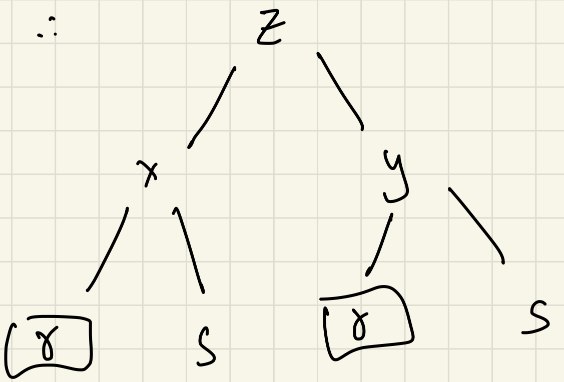
Don't forget to write : where $x = st^2 \qquad y = s^2 t$

3) Given $z = f(x, y)$; $x = r^2 + s^2$ & $y = 2rs$

Find $\dfrac{dz}{dr}$ & $\dfrac{d^2 z}{dr^2}$

we have no explicit z function ; we just know it is comprised of x & y

$\therefore$



But we know what r & s is comprised of.

$\therefore$ $\dfrac{dz}{dr}$ can be constructed

$$\frac{dz}{dr} = \frac{dz}{dx} \cdot \frac{dx}{dr} + \frac{dz}{dy} \cdot \frac{dy}{dr}$$

$$= \frac{dz}{dx} \cdot (2r) + \frac{dz}{dy} \cdot (2s)$$

$$\frac{d}{dr}\left(\frac{dz}{dr}\right) = \frac{d}{dr}\left(\frac{dz}{dx}(2r) + \frac{dz}{dy}(2s)\right)$$

u . v

$$= \frac{d}{dr}\left(\frac{dz}{dx} \cdot 2r\right) + \frac{d}{dr}\left(\frac{dz}{dy} \cdot 2s\right)$$

$$= \frac{dz}{dx}\frac{d(2r)}{dr} + 2r\frac{d}{dr}\left(\frac{dz}{dx}\right) +$$

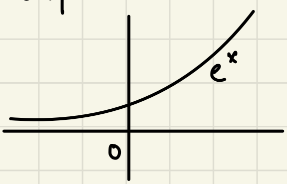
$$\frac{dz}{dy}\frac{d(2s)}{dr} + 2s\frac{d}{dr}\left(\frac{dz}{dy}\right)$$

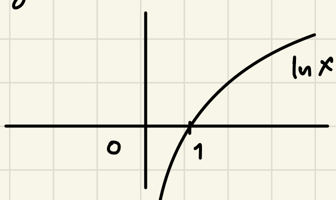
$$= \frac{dz}{dx}.2 + 2r\frac{d}{dr}\left(\frac{dz}{dx}\right) +$$

$$0 + 2s \cdot \frac{d}{dr}\left(\frac{dz}{dy}\right)$$

Sketching Domains:

- Focus on denominator as it should not be equal to 0. so equate for case denom of $f(x,y) > 0$
- Polynomial functions: Domain $\mathbb{R}^n$; where n is the number of variables (all real nos)
- Root functions: The quantity under the root must be greater than or equal to zero
  For $f(x) = \sqrt{g(x)}$; $D = \{ x \in R : g(x) > 0 \}$
- For odd roots (cubic for example) does not have any domain restrictions. $D = \{ x \in R \}$
- Exponential & Logarithmic Functions:





all real nos

$D = \{ x \in R \}$

the argument of ln (the qty inside) must be greater than 0

$D = \{ x \in R ; g(x) > 0 \}$

- Trig Functions

sin, cos & its reciprocals $\rightarrow$ all real nos $D = \{ \mathbb{R} \}$

tan $\rightarrow$ $D = \{ x \in R : x \neq (n + \frac{1}{2})\pi \ \forall n \in Z \}$

cot $\rightarrow$ $D = \{ x \in R : x \neq n\pi \ \forall n \in Z \}$

$\sin^{-1}$ & $\cos^{-1}$ $\rightarrow$ $D = -1 \leq x \leq 1$

$\tan^{-1}(x)$ $\rightarrow$ $D = \{\mathbb{R}\}$

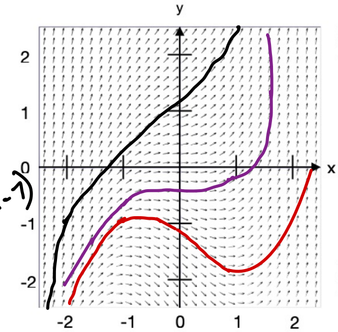
3. Differential Equations [35 pts]

a. [2 pts] Indicate which of the following statements are correct (+) or incorrect (-)

+	-	
		A second order ODE involves functions of two variables.
		In the explicit form of an ODE, the highest derivative depends on one variable only.
		The RHS of a "separable ODE" can be broken into a sum of independent terms.
		The sum of the homogeneous solutions of an ODE is also a solution of the ODE.

b. [3 pts] The direction field below is the solution to the 1st order ODE $y' = \frac{y}{2} + x^2$.

(-2,-1)



Sketch the graphs of the solutions to the following initial value problems

(A) $y' = \frac{y}{2} + x^2, (x_0, y_0) = (-2, -1)$

(B) $y' = \frac{y}{2} + x^2, (x_0, y_0) = (-2, -2)$

(C) $y' = \frac{y}{2} + x^2, (x_0, y_0) = (-1, -1)$

Do your sketching on the graph in the answer booklet. Label your sketches.

c. [12 pts] Solve the following 1st order ODE

$$xy' + 2y = x^2 - x + 1$$

d. [18 pts] Find the general solution of the following 2nd order ODE

$$\ddot{y} - 3\dot{y} = xe^x + \sin 3x$$

b. given $y' = \frac{y}{2} + x^2$

$$\frac{dy}{dx} = \frac{y}{2} + x^2$$

$$(x_0, y_0) = \quad \frac{-1}{2} + 4 = \frac{8-1}{2} = \frac{7}{2} = 3.5$$

C. $x y' + 2y = x^2 - x + 1$

$\div$ by $x \Rightarrow \quad y' + \frac{2}{x} y = x - 1 + \frac{1}{x}$

This is non separable form

$\therefore \quad y' = x - 1 + \frac{1}{x} - \frac{2}{x} y$

$y' = (x-1) + \frac{1}{x}(1-2)y$

$y' = (x-1) - \frac{1}{x} y$

$y' + \frac{1}{x} y = x - 1$

$y' + \frac{1}{x} y = 0$

$y' = -\frac{1}{x} y$

$\frac{dy}{dx} = -\frac{1}{x} y$

$\int \frac{1}{y} \, dy = \int -\frac{1}{x} \, dx$

$e^{\ln y} = e^{-\ln x + c} \quad ; \quad c \in R$

$y = x^{-1} \cdot e^{c}$

$y = \frac{e^{c}}{x} \Rightarrow \quad y = \frac{k}{x} \quad ; \quad k \in R$

$$y_t = \frac{k(x)}{x}$$

$$y_t' = \frac{x\,k'(x) - k(x)\cdot 1}{x^2}$$

$$= \frac{x\,k'(x) - k(x)}{x^2}$$

$$\frac{x\,k'(x)}{x^2} - \frac{k(x)}{x^2} + \frac{1}{x}\cdot\frac{k(x)}{x^2} = x - 1$$

$$\frac{k'(x)}{x} - \cancel{\frac{k(x)}{x^2}} + \cancel{\frac{k(x)}{x^2}} = x - 1$$

$$k'(x) = x^2 - x$$

$$k(x) = \frac{x^3}{3} - \frac{x^2}{2} + C_1 \; ; \; C_1 \in R$$

$$\therefore \; y = \frac{1}{x}\left(\frac{x^3}{3} - \frac{x^2}{2} + C_1\right) \; ; \; C_1 \in R$$

is one solution of the ODE.

d. $y'' - 3y' = xe^x + \sin 3x$

characteristic eqn:

$r^2 - 3r = 0$

$r(r-3) = 0$

$r_1 = 0 \quad r_2 = 3 \qquad r_1 \neq r_2$

$y_h = C_1 e^{0x} + C_2 e^{3x} \; ; \; C_1, C_2 \in R$

$$G(x) = xe^x + \sin 3x$$
$$= G_1(x) + G_2(x)$$
$$\quad y_{p_1} \qquad y_{p_2}$$

$\underline{\underline{G_1(x)}}$

$$y_{p_1} = (Ax + B)e^x = Axe^x + Be^x$$

$$y'_{p_1} = Axe^x + e^x \cdot A + Be^x$$

$$y''_{p_2} = Axe^x + e^x \cdot A + e^x \cdot A + Be^x$$
$$= Axe^x + 2Ae^x + Be^x$$

$$Axe^x + 2Ae^x + Be^x - 3Axe^x - 3Ae^x - 3Be^x = xe^x$$

$$-2Axe^x + (-Ae^x - 2Be^x) = xe^x$$
$$-2Axe^x + (-A - 2B)e^x = xe^x + 0e^x$$

$$-2A = 1 \qquad\qquad -A - 2B = 0$$
$$A = -\frac{1}{2} \qquad\qquad \frac{1}{2} = 2B$$
$$B = \frac{1}{4}$$

$$\therefore y_{p_1} = (Ax + B)e^x = \left(\frac{1}{4} - \frac{1}{2}x\right)e^x$$

$\underline{\underline{G_2(x)}}$:

$$y_{p_2} = C_1 \cos 3x + C_2 \sin 3x$$
$$y'_{p_2} = -3C_1 \sin 3x + 3C_2 \cos 3x$$

$$y''_{p_2} = -9C_1 \cos 3x - 9C_2 \sin 3x$$

$$y'' - 3y' = \sin 3x$$

$$-9C_1 \cos 3x - 9C_2 \sin 3x - 3(-3C_1 \sin 3x + 3C_2 \cos 3x) = \sin 3x$$

$$-9C_1 \cos 3x - 9C_2 \sin 3x + 9C_1 \sin 3x - 9C_2 \cos 3x = \sin 3x + 0 \cos 3x$$

$$(-9C_1 - 9C_2) \cos 3x + (9C_1 - 9C_2) \sin 3x = 0 \cos 3x + \sin 3x$$

$$-9C_1 - 9C_2 = 0 \quad (+)$$
$$9C_1 - 9C_2 = 1$$

$$0 - 18C_2 = 1$$
$$C_2 = -\frac{1}{18}$$

$$-9C_1 = 9C_2 = 9 \cdot -\frac{1}{18} = -\frac{1}{2}$$

$$-9C_1 = -\frac{1}{2}$$

$$C_1 = \frac{1}{18}$$

$$\therefore y_{p_2} = \frac{1}{18} \cos 3x - \frac{1}{18} \sin 3x$$

$$y = y_H + y_{p_1} + y_{p_2}$$

$$y = c_1 + c_2 e^{3x} + \left(\frac{1}{4} - \frac{1}{2}x\right)e^x + \frac{1}{18}\cos 3x - \frac{1}{18}\sin 3x \, ;$$

$$c_1, c_2 \in R$$

4. Functions of Several Variables [35 pts]

a. [2 pts] Indicate which of the following statements are correct (+) or incorrect (-)

	+	-	
✓	☐	☒	The range of a multivariate function is the set of all pairs (x, y) for which $f(x, y)$ is defined.
✗	☒	☐	In practice a multivariate function's domain is always set purely by mathematical conditions.
✓	☒	☐	The relationship $f_{xy}(a, b) = f_{yx}(a, b)$ is always true for real-valued functions.
✗	☐	☒	The local maximum of a closed set may lie on its boundary.

b. [5 pts] Given the function $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$, indicate the correct pair of partial derivatives.

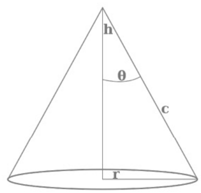
- ☐ $z_x(x, y) = -\dfrac{2x}{y^3} - \dfrac{1}{x^3}$
- ☒ $z_x(x, y) = \dfrac{1}{y^2} + \dfrac{2y}{x^3}$
- ☐ $z_x(x, y) = \dfrac{1 - y}{y^2 - 2x}$
- ☐ $z_x(x, y) = \dfrac{x^2}{2y^2} + \dfrac{y}{x}$
- ☐ $z_x(x, y) =$ none of the above

- ☐ $z_y(x, y) = \dfrac{x - 1}{2y - x^2}$
- ☐ $z_y(x, y) = -\dfrac{x}{y} - \dfrac{y^2}{2x^2}$
- ☐ $z_y(x, y) = \dfrac{1}{y^2} + \dfrac{2y}{x^3}$
- ☒ $z_y(x, y) = -\dfrac{2x}{y^3} - \dfrac{1}{x^3}$ ✗
- ☐ $z_y(x, y) =$ none of the above ✓

solved correctly-

c. [15 pts] Find the equation of the tangent plane to $z = x^2y^4 - \dfrac{12x}{y}$ at $(x, y) = (-1,2)$

d. [13 pts] A cone-shaped engine mount is being stretched along its long axis. The height is given by $h(t)$ and the radius is given by $r(t)$. The initial height at time $t = 0$ of a right circular cone is $h(0) = 15$cm and is increasing at $\dot{h}(t) = 0.2$cm/min. The initial radius of the base at time $t = 0$ is given by $r(0) = 10$cm and is decreasing at $\dot{r}(t) = -0.3$cm/min. The volume of a cone is given by $V(r, h) = \dfrac{1}{3}\pi r^2 h$ and the dimensions are functions of time: $r = r(t)$, $h = h(t)$. What are the functional forms of $r(t), h(t)$ and how fast is the volume of the cone changing? (hint: use the chain rule)



b. $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ Find z_x & z_y

① $\dfrac{dz}{dx}$ ↑ y const ② $\dfrac{dz}{dy}$ ↑ x const

① $\dfrac{dz}{dx} = \dfrac{d}{dx}\left(\dfrac{x}{y^2}\right) - \dfrac{d}{dx}\left(\dfrac{y}{x^2}\right)$

$= \dfrac{1}{y^2} - \dfrac{d}{dx}\left(x^{-2} y\right)$

$$= \frac{1}{y^2} - y \cdot -2 \cdot x^{-3}$$

$$= \frac{1}{y^2} + \frac{2y}{x^3}$$ ✓

② $$\frac{dz}{dy} = \frac{d}{dy}\left(xy^{-2}\right) - \frac{d}{dy}\left(\frac{1}{x^2} y\right)$$

$$= x \cdot -2y^{-3} - \frac{1}{x^2}$$

$$= -\frac{2x}{y^3} - \frac{1}{x^2}$$

Accidently selected wrong box
But solved correctly

c. Find eqn of tangent plane:

$z = x^2y^4 - \frac{12x}{y}$ at $(x, y) = (-1, 2)$

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is the eqn of tangent of plane. where $(x_0, y_0) = (-1, 2)$

$$f(x,y) = x^2y^4 - \frac{12x}{y}$$

$$f(x_0, y_0) = 1 \cdot 16 + \frac{12}{2} = 16 + 6 = 22$$

$$f_x(x,y) \overset{\uparrow\, y\text{ const}}{=} y^4 \cdot 2x - \frac{12}{y}$$

$$f_x(x_0,y_0) = 16 \cdot 2(-1) - \frac{12}{2}$$

$$= -32 - 6 = -38$$

$$f_y(x,y) \overset{\uparrow\, x\text{ const}}{=} x^2 \cdot 4y^3 - 12x \cdot \left(-\frac{1}{y^2}\right)$$

$$= 4x^2y^3 + \frac{12x}{y^2}$$

$$f_y(-1,2) = 4 \cdot 1 \cdot 8 + \frac{12 \cdot (-1)}{4}$$

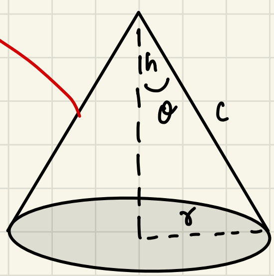
$$= 32 - 3 = 29$$

$$\boxed{L(x,y) = 22 - 38(x+1) + 29(y-2)}$$

opening brackets also good thing to do.

$$z - z_0 = -38(x+1) + 29(y-2)$$

d.



stretched along axis

height $h(t)$

radius $r(t)$

$t=0$; $h(0) = 15$ cm

$$\dot{h}(t) = \frac{dh}{dt} = 0.2 \ \frac{\text{cm}}{\text{min}}$$

$t=0$; $r(0) = 10$ cm

height & radius changes linearly

$\therefore h(t) = h(0) + h'(0)t$
$= 15 + 0.2t \to ①$

$r(t) = r(0) + r'(0)t$
$= 10 - 0.3t \to ②$

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} + \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$$\& \frac{dV}{dt} = \frac{2}{3}\pi . ② . ① . \frac{dr}{dt} + \frac{1}{3}\pi . ② . 0.2$$

Answer!

$$\dot{r}(t) = \frac{dr}{dt} = -0.3 \frac{cm}{min}$$

$$V(r,h) = \frac{1}{3}\pi r^2 h;$$

$r = r(t)$ & $h = h(t)$

we need $\frac{dV}{dt}$.

$$V_{(r,h)} = \frac{1}{3}\pi . r^2(t)\, h(t)$$

$$V = \frac{1}{3}\pi r^2 h$$

$$r = \sqrt{\frac{3V}{\pi h}} \qquad h = \frac{3V}{\pi r^2}$$

$$r(t) = \sqrt{\frac{3}{\pi} \cdot \frac{V(t)}{h(t)}} \qquad h(t) = \frac{3V(t)}{\pi (r(t))^2}$$

$$\frac{dV}{dt} = \frac{1}{3}\pi . \frac{d}{dt}(r(t) . h(t))$$

$$= \frac{1}{3}\pi . \left(r(t) . \frac{dh(t)}{dt} + h(t) . \frac{dr(t)}{dt} \right)$$

$$= \frac{1}{3}\pi . \left(10\,cm . 0.2 \frac{cm}{min} + 15\,cm . \left(-0.3 \frac{cm}{min}\right) \right)$$

$$= \frac{1}{3}\pi\left(2\,\frac{cm^2}{min} - 4.5\,\frac{cm^2}{min} \right)$$

$$\frac{dV}{dt} = \frac{2.5\,\pi}{3}\,\frac{cm^2}{min}$$