Integration

github.com/royceanton

(1) $\int \sin x \, dx = -\cos x + C$

(2) $\int \cos x \, dx = \sin x + C$

(3) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$

(4) $\int 1 \, dx = x + C$

(5) $\int \sec^2 x \, dx = \tan x + C$

(6) $\int \sec x \tan x \, dx = \sec x + C$

(7) $\int \operatorname{cosec}^2 x = -\cot x + C$

(8) $\int \operatorname{cosec} x \cot x = -\operatorname{cosec} x + C$

(9) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$

(10) $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1}(x) + C$

(11) $\int \frac{1}{1+x^2} dx = \tan^{-1}(x) + C$

(12) $\int \frac{1}{1+x^2} dx = -\cot^{-1}(x) + C$

(13) $\int \frac{1}{|x|\sqrt{x^2-1}} dx = \sec^{-1}(x) + C$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

sec sec tan

cosec -cosec cot

$\sin^{-1}(x)$ $\frac{1}{\sqrt{1-x^2}}$

$\tan^{-1}(x)$ $\frac{1}{1+x^2}$

$\sec^{-1}(x)$ $\frac{1}{|x|\sqrt{x^2-1}}$

(14) $\int \frac{1\, dx}{|x|\sqrt{x^2-1}} = -\operatorname{cosec}^{-1}(x) + C$

(15) $\int e^x \, dx = e^x + C$

(16) $\int \frac{1}{x} \, dx = \ln x + C$

$\frac{d}{dx} a^x = a^x \log a$

(17) $\int a^x \, dx = \frac{a^x}{\ln x} + C$

Example

1) $\int (6x^2 + 3x^4 - 9)\, dx$

$$= 6\frac{x^3}{3} + 3\frac{x^5}{5} - 9x + C$$

$$= 2x^3 + \frac{3x^5}{5} - 9x + C \quad ; \; C \in R$$

2) $\int (x+2)(4x^2 - 1)\, dx$

$$\int (4x^3 - x + 8x^2 - 2)\, dx$$

$$= 4\frac{x^4}{4} - \frac{x^2}{2} + \frac{8x^3}{3} - 2x + C$$

$$= x^4 - \frac{x^2}{2} + \frac{8x^3}{3} - 2x + C \quad ; \; C \in R$$

3) $\int \left(\sqrt{x} + 2\sqrt{x^3} + \frac{3}{x^2} - \frac{1}{x^3} \right) dx$

$$= 2\frac{x^{1/2+1}}{3} + 2\frac{x^{3/2+1}}{5} \cdot 2 + 3\frac{x^{-2+1}}{-1} - \frac{x^{-3+1}}{-2} + C$$

$$= 2\frac{x^{3/2}}{3} + 4\frac{x^{5/2}}{5} - 3x^{-1} + \frac{x^{-2}}{2} + C$$

4) $f(x) = (x+2)(5-x)$

$f(x) = 0$ gives zero crossing

$x+2 = 0$ $5-x = 0$

$x = -2$ $x = 5$

x	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	-8	0	6	10	12	12	10	6	0	-8

$f(-3) = (-1)(8) = -8$

$f(-2) = 0$

$f(-1) = (1)(6) = 6$

$f(0) = (2)(5) = 10$

$f(1) = 3(4) = 12$

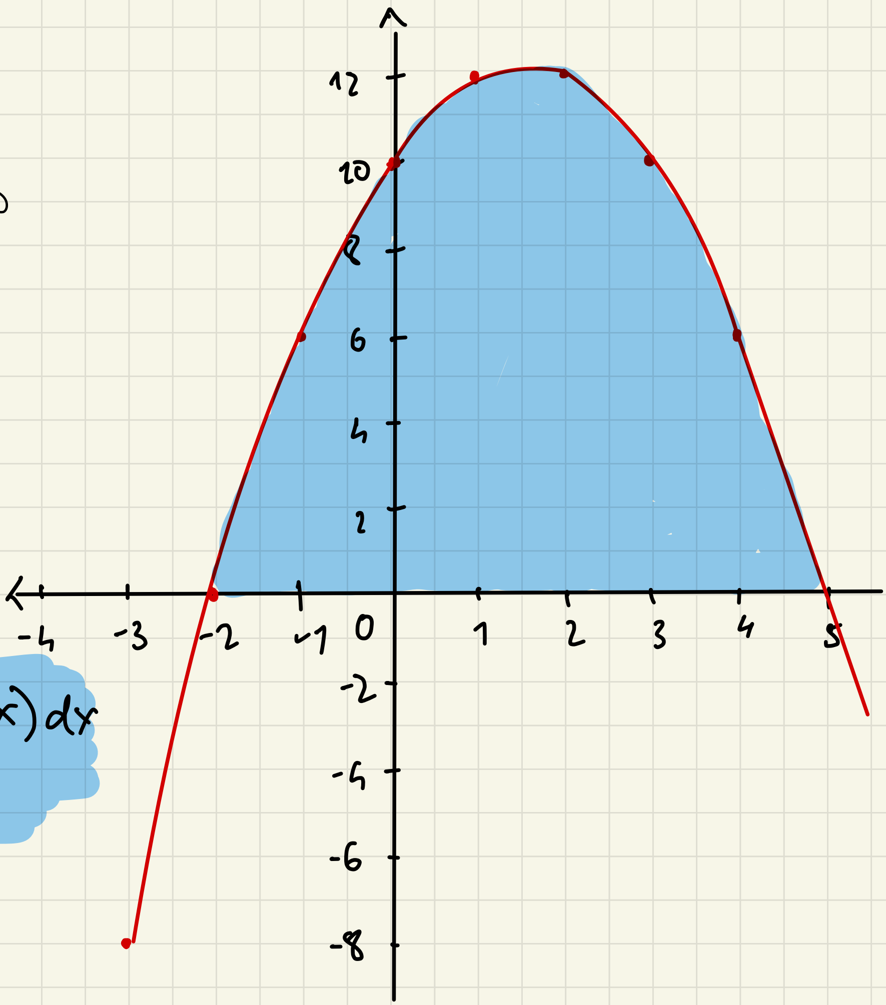
$f(2) = 4(3) = 12$

$f(3) = 5(2) = 10$

$f(4) = 6(1) = 6$

$f(5) = 0$

$f(6) = 8(-1) = -8$



$$\int_{-2}^{5} (x+2)(5-x)\,dx$$

$$\int_{-2}^{5} (x+2)(5-x)\,dx$$

$$\int_{-2}^{5} (5x - x^2 + 10 - 2x)\,dx$$

$$\int_{-2}^{5} (-x^2 + 3x + 10)\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{3x^2}{2} + 10x\right]_{-2}^{5}$$

$$= \left[-\frac{5^3}{3} + \frac{3(25)}{2} + 50\right] - \left[-\frac{(-2)^3}{3} + \frac{3(-2)^2}{2} + 10(-2)\right]$$

$$= \left[-\frac{125}{3} + \frac{75}{2} + 50\right] - \left[\frac{-8}{3} + \frac{12}{2} - 20\right]$$

$$= \left[-\frac{125}{3} + \frac{75}{2} + 50 + \frac{8}{3} - \frac{12}{2} + 20\right]$$

$$= -\frac{116}{3} + \frac{63}{2} + 70$$

$$= -35.33 + 31.5 + 70$$

$$= -3.83 + 70 = 66.17$$

$$\begin{array}{r} 35.3 \\ 3\overline{)116} \\ 9 \\ \hline 16 \\ 15 \\ \hline 10 \end{array} \qquad \begin{array}{r} 5 \\ 6\overline{)343} \\ 30 \\ \hline 43 \end{array}$$

I Reverse Power Rule

step 1 Step 2

$$\frac{d}{dx}\left(x^{n}\right)^{-1} = n x^{n-1}$$

step 1 step2

$$\int \frac{1}{n+1} x^{n+1} = \frac{1}{n+1} x^{n+1} + C$$

II Know the famous first steps:

Part 1 the famous first step.

① $\int \tan x \, dx$

Multiply & divide by $\sec x \, dx$

$$\int \tan x \, dx \Rightarrow \int \frac{1}{\sec x} \cdot \sec x \tan x \, dx$$

we have integral for this.
∴ apply usub

Take $\sec x = u$

$\sec x \tan x = du$

$\therefore \int \frac{1}{u}\, du = \ln|u| + C$

$= \ln|\sec x| + C$

② $\int \sec x \, dx$

Multiply & divide by $(\sec x + \tan x)$

$$\int \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x}\, dx$$

sec sec tan

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

Now let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

we have $\displaystyle\int \frac{1}{u} \cdot du = \ln|u| + C$

$= \ln\left| \sec x + \tan x \right| + C$

3) $\displaystyle\int \frac{1}{x^3 + x} \, dx$

Take x^3 common instead of x

$$\int \frac{1}{x^3\left(1+\frac{x}{x^3}\right)} dx = \int \frac{1}{x^3\left(1+\frac{1}{x^2}\right)} dx$$

Let $1 + \frac{1}{x^2} = u$

$1 + x^{-2} = u$

$0 + -2x^{-2-1} = \frac{du}{dx}$

$-2x^{-3} \, dx = du$

$\frac{1}{x^3} \, dx = -\frac{1}{2} du$

$\therefore$ we have $\displaystyle\int \frac{1}{u} \cdot \frac{-1}{2}\,du$

$$= \frac{-1}{2} \int \frac{1}{u}\,du$$

$$= -\frac{1}{2} \ln|u| + C$$

$$= \frac{-1}{2} \ln\left|1+\frac{1}{x^2}\right| + C$$

III

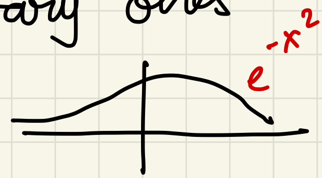
1. $\displaystyle\int \sin^3 x\,dx$ — Do not apply reverse chain rule
2. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$ — Do not write in 'log' terms
3. $\displaystyle\int e^{x^2}\,dx$ — Do not directly apply reverse chain rule
4. $\displaystyle\int \tan^{-1} x\,dx$ — Do not write $\frac{1}{1+x^2}$ thats deriv!
5. $\displaystyle\int x^2 \sin x\,dx$ — Do not apply reverse chain rule.

Part 2: Know the famous Ones.

Do not integrate by hand.

Know the non elementary ones

$\int e^{x^2}\,dx$, $\int e^{-x^2}$



$\int \frac{e^x}{x}\,dx$, $\int \frac{1}{\ln x}\,dx$

$\int \frac{\sin x}{x}\,dx$, $\int \frac{\cos x}{x}\,dx$

$\int \sin(x^2)\,dx$, $\int \cos(x^2)\,dx$ Fresnels integrals

$\int x^x\,dx$, $\int \sqrt{1+x^3}\,dx$

INTEGRATION BY PARTS

(reverse the product rule)

IBP: $\int u\,dV = uv - \int v\,du$

1) $\int x \cos(x^2)\,dx$ ✓ LIATE

$u = x$ $vdV = \cos(x^2)\,dx$

$du = dx$ $V =$

integrating this is not nice!
Therefore try u-sub for the problem.

Let $u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

we have $\int \cos(u)\,\frac{1}{2}du$

$= \int \frac{1}{2}\cos(u) = \frac{1}{2}\sin(x^2) + C$

2) $\int x \cos x \, dx$

$u = x \qquad vdV = \cos x \, dx$

$du = dx \qquad V = \sin x$

$\int uv = uv - \int v \, du$

$= x \sin x - \int \sin x \, dx$

$= x \sin x - (-\cos x) + c$

$= x \sin x + \cos x + C \; ; \; c \in R$

3) $\int x^3 \ln x \, dx$ $\qquad$ u choose LIATE

$u = \ln x \qquad vdv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^4}{4}$

$\int uv = \ln x . \frac{x^4}{4} + \int \frac{x^4}{4} . \frac{1}{x} dx$

$= \ln x \frac{x^4}{4} + \frac{1}{4} \int x^3 dx$

$= \ln x . \frac{x^4}{4} + \frac{1}{4} . \frac{x^4}{4}$

$= \frac{x^4}{4} \left(\ln x + \frac{1}{4} \right) + C \; ; \; C \in R$

④ $\int x^2 \sin x \, dx$ $\qquad$ u LIATE

$u = x^2 \qquad v dv = \sin x \, dx$

$du = 2x\,dx \qquad v = -\cos x$

$\int uv = -x^2 \cos x - \int -\cos x \, 2x \, dx$

$= -x^2 \cos x + \int \cos x \, 2x \, dx$

$u = 2x \qquad v dv = \cos x \, dx$

$du = 2\,dx \qquad v = \sin x$

$\int uv = 2x \sin x - 2\int \sin x \, dx$

$= 2x \sin x + 2 \cos x$

$\int \cos x \, 2x \, dx$

Back to original question

$= -x^2 \cos x + 2x \sin x + 2\cos x$

$= 2x \sin x + 2\cos x - x^2 \cos x + C \; ; \quad C \in \mathbb{R}$

DI Method

$$\int x^2 \sin x \, dx$$

		D	I
	+	x^2	$\sin x$
	−	$2x$	$-\cos x$
	+	2	$-\sin x$
1st stop →	−	0	$+\cos x$

$$(+x^2)(-\cos x) + (-2x)(-\sin x) + (+2)(+\cos x)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C \,;\, C \in \mathbb{R}$$

1st stop → 0 in the D column; then start to multiply the diagonals.

⑤ $\int x^3 \ln x \, dx$

	D	I
	$\ln x$	x^3
	$\frac{1}{x}$	$\frac{x^4}{4}$
This ↓	$-\frac{1}{x^2}$	$\frac{x^5}{5} \cdot \frac{1}{4}$

$-x^{-2}$

$-(-2)x^{-3}$

never stops as D never goes to 0

$\frac{2}{x^3}$

$\frac{1}{20} \cdot \frac{x^6}{6}$

2nd stop → whenever we can integrate the product of a row

In this case; stop at 1st DI iteration and write the traditional $\int uv$ form and find the answer

$$\ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx$$

$$\ln x \frac{x^4}{4} - \frac{1}{4} \int x^3 dx$$

$$\ln \frac{x^4}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + C \quad ; C \in \mathbb{R}$$

(26) $\int e^x \sin(2x) dx$ $\quad$ u choose LIATE

$u = \sin(2x)$ $\quad$ $v dv = e^x dx$

$du = 2 \cdot \cos(2x) dx$ $\quad$ $v = e^x$

$$\int uv = uv - \int v du$$

$$\sin(2x)e^x - \int e^x \cdot 2\cos(2x)\,dx$$

$$e^x \sin(2x) - 2\int e^x \cos(2x)\,dx \rightarrow ①$$

$$\int e^x \cos(2x)\,dx$$

$u = \cos(2x)$ $\quad$ $dv = e^x\,dx$

$du = -2\sin(2x)\,dx$ $\quad$ $v = e^x$

$$= e^x \cos(2x) - \int e^x (-2\sin(2x))\,dx$$

$$= e^x \cos(2x) + 2\underbrace{\int e^x \sin(2x)\,dx}_{\text{original question}} \rightarrow ②$$

Put ② in ① and rearrange LHS & RHS bringing question on one side

$$\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2\left[e^x\cos(2x) + 2\int e^x \sin(2x)\,dx\right]$$

$$\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x\cos(2x) - \underbrace{4\int e^x \sin(2x)\,dx}_{\text{Take this to LHS}}$$

$$5\int e^x \sin(2x)\,dx = e^x \sin(2x) - 2e^x \cos(2x)$$

$$\int e^x \sin(2x)\,dx = \frac{e^x \sin(2x) - 2e^x \cos(2x)}{5} + C \quad ; C \in \mathbb{R}$$

Method 1 ↗

Using DI Method

LHS DI

	D	I
+	$\sin(2x)$	e^x
−	$2\cos(2x)$	e^x
+	$-4\sin(2x)$	e^x
−	$-8\cos(2x)$	e^x
+		
−		

RHS DI

	D	I
+	e^x	$\sin(2x)$
−	e^x	$-\cos(2x)\,\frac{1}{2}$
+	e^x	$-\sin(2x)\cdot\frac{1}{4}$
−	e^x	

stop 3 → when the function part repeats.

LHS DI ← Method 2

$$\sin(2x)e^x - e^x 2\cos(2x) + \int -4\sin(2x)e^x$$

$$\int e^x \sin(2x)\,dx = \sin(2x)e^x - 2e^x\cos(2x) - 4\int \sin(2x)e^x + C$$

$$5\int e^x \sin(2x)\,dx = \sin(2x)e^x - 2e^x\cos(2x) + C$$

$$\int e^x \sin(2x)\,dx = \frac{\sin(2x)e^x - 2e^x\cos(2x)}{5} + C$$

RHS DI ← Method 3

$$-e^x\cos(2x)\frac{1}{2} + e^x\sin(2x)\frac{1}{4} + \int -\frac{1}{4}\sin(2x)e^x$$

$$\int \sin(2x)e^x = -e^x\cos(2x)\frac{1}{2} + e^x\sin(2x)\frac{1}{4} - \frac{1}{4}\int \sin(2x)e^x$$

x throughout by 4 LHS & RHS

$$4\int \sin(2x)e^x = -2e^x\cos(2x) + e^x\sin(2x) - \int \sin(2x)e^x$$

Bringing question part to LHS

$$5\int \sin(2x)e^x = -2e^x\cos(2x) + e^x\sin(2x)$$

$$\int \sin(2x)e^x = \frac{-2e^x\cos(2x) + e^x\sin(2x)}{5} + C$$

21) $\int \tan^{-1} x \; dx$

This can be written as:

$\int 1 \cdot \tan^{-1} x \; dx$ $\quad u \quad$ LIATE

	D	I	
+	$\tan^{-1} x$	1	
-	$\frac{1}{1+x^2}$	x	step 2 stop
+	⋮	$\frac{x^2}{2}$	
-		⋮	
+			
⋮			

$$\int \tan^{-1} x \, dx = x \tan^{-1} x + \int x \left(-\frac{1}{1+x^2} \right) dx$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx$$

$\int \frac{x}{1+x^2} dx$ $\quad$ Let $u = 1 + x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$$\int \frac{x}{1+x^2} dx \Rightarrow \int \frac{1}{2} du \cdot \frac{1}{u}$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + c$$

$$= \frac{1}{2} \ln |1+x^2| + C$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + C \; ; C \in R$$

Normal way 10:05

$$\int \tan^{-1} x = \int 1 \cdot \tan^{-1}(x)$$ U LIATE

$u = \tan^{-1} x$ $\quad$ $vdv = 1dx$

$du = \frac{1}{1+x^2} dx$ $\quad$ $v = x$

$$\int uv = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

Let $u = 1+x^2$

$du = 2x \, dx$

$\frac{1}{2} du = x \, dx$

$$= x \tan^{-1} x - \int \frac{1}{u} \cdot \frac{1}{2} du$$

$= x\tan^{-1}x - \frac{1}{2}\ln|u| + C$

$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + C \; ; C \in R$

(27) $\int \frac{\ln x}{\sqrt{x}}\,dx$

$u = \sqrt{x}$ $\quad$ $u = \ln x$

$du = \frac{1}{2\sqrt{x}}\,dx$ $\quad$ $du = \frac{1}{x}\,dx$

Both would not be ideal ; rather apply $\int uv$

$\int \frac{1}{\sqrt{x}} \cdot \ln x \,dx$

	D	I
+	$\ln x$	$\frac{1}{\sqrt{x}}$
−	$\frac{1}{x}$	$2\sqrt{x}$
+		
−		

Step 2
stop as
integrable
products

$+(\ln x)(2\sqrt{x}) + \int\left(-\frac{1}{x}\right)(2\sqrt{x})\,dx$

$= 2\sqrt{x}\ln x - 2\int \frac{\sqrt{x}}{x}\,dx$

$= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$

$$= 2\sqrt{x}\ln x - 2\,\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

28) $\int x^2 e^{3x}\,dx$

u-sub not ideal
$\int uv$ better

u choose LIATE

	D	I
+	x^2	e^{3x}
−	$2x$	$e^{3x}\cdot\frac{1}{3}$
+	2	$e^{3x}\cdot\frac{1}{9}$
−	0	$e^{3x}\cdot\frac{1}{27}$

$\int x^2 e^{3x}\,dx$

$$= x^2\frac{e^{3x}}{3} - \frac{2}{9}x\,e^{3x} + \frac{2}{27}e^{3x}$$

$u = x^2 \qquad vdv = e^{3x}$

$du = 2x \qquad v = e^{3x}\cdot\frac{1}{3}$

$\int uv = uv - \int v\,du$

$$= \frac{x^2}{3}e^{3x} - \int \frac{2}{3}x e^{3x}$$

$\int x e^{3x}$

$= \quad u = x \qquad vdv = e^{3x}$

$du = dx \qquad v = e^{3x}\cdot\frac{1}{3}$

$$= \frac{x}{3}e^{3x} - \int \frac{e^{3x}}{3}dx$$

$$= \frac{x}{3}e^{3x} - \frac{1}{3}e^{3x}\cdot\frac{1}{3}$$

$$= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]$$

$$= \frac{x^2}{3}e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x}$$

29) $\int x \sec x \tan x \, dx$ U LIATE

$u = x$ $\quad$ $v dv = \sec x \tan x \, dx$

$du = dx$ $\quad$ $v = \sec x$

$\int uv = uv - \int v du$

sec sec tan

$= x \sec x - \int \sec x dx$

$\frac{d}{dx} \sec x = \sec x \tan x$

Now $\int \sec x$ is special integral

$\therefore$ Multiply & divide by $(\sec x + \tan x)$

$$= \int \sec x \, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

$\therefore$ Back to original problem

$$\int x \sec x \tan x \, dx$$

$$= x \sec x - \ln\left| \sec x + \tan x \right| + C \; ; C \in \mathbb{R}$$

30) $\int \sin^2 x \cos x \, dx$

$u = \sin^2 x$

$du = 2 \sin x \cdot \cos x$

(complicated)

$u = \sin x$

$du = \cos x \, dx$

(easier)

$$= \int u^2 \, dx = \frac{u^3}{3} + C$$

$$= \frac{\sin^3 x}{3} + C$$

31) $\int \sin^2 x \, dx$

=

	D	I
$+$	$\sin^2 x$	1
$-$	$2 \sin x \cos x$	x

$= x \sin^2 x - 2\int x \sin x \cos x \, dx$

Tricky

Apply trig identities!

$\sin^2 x = 1 - \cos^2 x$ ⇐ still tricky

$\sin 2\theta = 2 \sin\theta \cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$ * pick this one

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\therefore \int \sin^2 x = \int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{2}\int 1 - \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x \cdot \frac{1}{2} + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin x + C$$

Using Trig Identities

Remember these for applying integrals:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$$

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha-\beta) + \sin(\alpha+\beta)\right)$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$$

Strategies:

Case 1 : $\sin x \quad \cos x$

$$\int \underbrace{(\qquad)}_{\substack{\text{an expression} \\ \text{in terms of } \sin x \\ u = \sin x}} \underbrace{\cos x\, dx}_{du} \quad \text{or} \quad \int \underbrace{(\qquad)}_{\substack{\text{an expression} \\ \text{in terms of } \cos x \\ u = \cos x}} \underbrace{\sin x\, dx}_{du}$$

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$$\int \sin^3 x\, dx$$

$$\int \underbrace{\sin^2 x}_{\text{we now need this in } \cos x \text{ form}} \sin x\, dx$$

$$\int (1 - \cos^2 x) \sin x\, dx$$

Let $\cos x = u$

$-\sin x\, dx = du$

$\sin x\, dx = -du$

$$\int (1-u^2)(-1\,du)$$

$$\int (u^2-1)\,du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C$$

Case 2: tan x , sec x

$$\int (\qquad) \sec^2 x \, dx$$

an expression in terms of tan x

$u = \tan x$

$$\int (\qquad) \sec x \tan x \, dx$$

an expression in terms of sec x

also make use of $\tan^2 x + 1 = \sec^2 x$ & $\tan^2 x = \sec^2 x - 1$

(32) $\int \sec^4 x \, dx$

$\sec x \quad \sec x \tan x$

$\int \sec^2 x \sec^2 x \, dx$

$\tan^2 x + 1 = \sec^2 x$

$\int (\tan^2 x + 1) \underline{\underline{\sec^2 x \, dx}}$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$\therefore \int (u^2 + 1) \, du$

$= \frac{u^3}{3} + u + C$

$= \frac{\tan^3 x}{3} + \tan x + C$

(33) $\int \sec^4 x \tan x \, dx$

$: \int \sec^3 x \underline{\underline{\sec x \tan x \, dx}}$

Let $u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int u^3 \, du$$

$$= \frac{u^4}{4} + C = \frac{\sec^4 x}{4} + C$$

(34) $\int \tan^3 x \, dx$

we know

sec sec tan

$\frac{d}{dx} \sec x = \underline{\underline{\sec x \tan x}}$

$\frac{d}{dx} \tan x = \underline{\underline{\sec^2 x}}$

$$= \int (\sec^2 x - 1) \tan x \, dx$$

$$= \int (\sec^2 x \tan x - \tan x) \, dx$$

$$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

$$= \int \underbrace{\sec x \, \sec x \tan x \, dx}_{1} - \int \underbrace{\tan x \, dx}_{2}$$

This is the 1st special integral $\times$ & $\div$ by $\sec x$

(1) Let $u_1 = \sec x$

$du_1 = \sec x \tan x \, dx$

$$= \int u_1 \, du_1$$

(2) $\int \tan x \, dx = \int \frac{\tan x \sec x}{\sec x} \, dx$

Let $u_2 = \sec x$

$du_2 = \sec x \tan x$

$$= \int \frac{1}{u_2} du_2$$

1 + 2 $\Rightarrow$ $$\int u_1 du_1 + \int \frac{1}{u_2} du_2$$

$$= \frac{u_1^2}{2} + \ln|u_2| + C$$

$$= \frac{\sec^2 x}{2} + \ln|\sec x| + C$$

35 $$\int \sec^3 x \, dx$$

Don't do this :)

$$= \int \sec x \sec^2 x \, dx$$

$\tan^2 x + 1 = \sec^2 x$

$$= \int (\tan^2 x + 1) \sec x \, dx$$

sec sec tan

$$= \int \tan^2 x \sec x + \int \sec x \, dx$$

Do IBP when possible

$$= \int (\sec^2 x - 1) \sec x \, dx + \int \sec x \, dx$$

$$= \int (\sec^3 x - \sec x) dx + \int \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec \cancel{x} dx + \int \sec \cancel{x} dx$$

$$\int \sec^3 x \, dx$$

$$= \int \sec^2 x \sec x \, dx$$

sec sec tan

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\int \sec^2 x = \tan x + c$

$u = \sec x \qquad v\,dv = \sec^2 x \, dx$

$du = \sec x \tan x \qquad v = \tan x$

$$\int uv = \sec x \tan x - \int \tan x \sec x \tan x \, dx$$

1

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$\tan^2 x + 1 = \sec^2 x$

$\sec^2 x - 1 = \tan^2 x$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int (\sec^3 x - \sec x) dx$$

$$\int \sec^3 x = \sec x \tan x - \int \sec^3 x \, dx - \int \sec x \, dx$$

$$2 \int \sec^3 x = \sec x \tan x - \underbrace{\int \sec x \, dx}_{\text{special integral } 2}$$

special integral ②

$$\int \sec x \, dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

Let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

$$= \int \frac{1}{u} du = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x = \frac{\sec x \tan x - \ln|\sec x + \tan x|}{2} + C$$

Category - IV

Trig - Substitution.

(36) $\int \sqrt{x^2-6x+9}\, dx$

$-3 \times -3 = 9$
$-3 + -3 = -6$

$$\int \sqrt{x^2-3x-3x+9}\, dx$$
$$= \int \sqrt{x(x-3)-3(x-3)}\, dx$$
$$= \int \sqrt{(x-3)^2}\, dx$$
$$= \int x-3\, dx$$
$$= \frac{x^2}{2} - 3x + C$$

(37) $\int \sqrt{x^2+9}$

You cannot complete the squares here & try to solve it; it's not useful as the complexity increases

Instead play around with $\sec\theta$ & $\tan\theta$ formulae:

$$\tan^2\theta + 1 = \sec^2\theta$$
$$\sec^2\theta - 1 = \tan^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

x by 9 on both sides $\Rightarrow$

$$9(\tan^2\theta + 1) = 9(\sec^2\theta)$$

$$9\tan^2\theta + 9 = 9\sec^2\theta$$

$$(3\tan\theta)^2 + 9 = 9\sec^2\theta \quad *$$

Let $x = 3\tan\theta$

Then differentiate x w.r.t θ

$$\frac{dx}{d\theta} = 3\sec^2\theta$$

$$dx = 3\sec^2\theta \, d\theta$$

re write question in terms of θ

$$\int \sqrt{x^2+9}\,dx$$

$$= \int \left(\underbrace{\sqrt{(3\tan\theta)^2 + 9}}_{\text{This would be the * expression.}}\right) \cdot \left(3\sec^2\theta \, d\theta\right)$$

$$= \int \sqrt{9\sec^2\theta}\; 3\sec^2\theta \, d\theta$$

$$= \int 3\sec\theta \cdot 3\sec^2\theta \, d\theta$$

$$= 9\int \sec^3\theta \, d\theta$$

Solving $\int \sec^3\theta$ as $\int \sec^3 x$

$$\int \sec^3 x = \int \sec^2 x \sec x \, dx$$

$u = \sec x \qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad v = \tan x$

$$\int uv = \sec x \tan x - \int \sec x \tan x \cdot \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$\int \sec^3 x = \sec x \tan x - \int \left[\sec x \left(\sec^2 x - 1\right)\right] dx$$

$$\int \sec^3 x = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$2\int \sec^3 x = \sec x \tan x + \int \sec x \, dx$$

$$2\int \sec^3 x = \sec x \tan x + \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) dx$

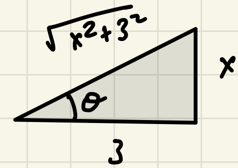
$$2\int \sec^3 x = \sec x \tan x + \int \frac{1}{u} du$$

$$\int \sec^3\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

$$\therefore 9\int \sec^3\theta = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta) + C_2$$

Now we have $x = 3\tan\theta$

ie $\tan\theta = \frac{x}{3}$



$\therefore \sec\theta = \frac{\sqrt{x^2+9}}{3}$

re writing the expression back in terms of 'x'

$$= \frac{9}{2}\frac{\sqrt{x^2+9}}{3}\cdot\frac{x}{3} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C_2$$

$$= \frac{x}{2}\sqrt{x^2+9} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C_2$$

You can also use log properties to further split it

$\ln\frac{A}{B} = \ln A - \ln B$

$\ln\frac{\sqrt{x^2+9}+x}{3} = \ln\sqrt{x^2+9}+x - \ln 3$

When you see the following:

You see	You let	You use
$\sqrt{x^2+a^2}$	$x = a\tan\theta$	$\tan^2\theta + 1 = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$

take $d\theta$ of it.

(38) $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$

form
$x^2 - a^2$ $\quad a\sec\theta$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$4(\sec^2\theta - 1) = 4\tan^2\theta$$

$$(2\sec\theta)^2 - 4 = 4\tan^2\theta$$

$$x = 2\sec\theta$$

$$dx = 2\sec\theta\tan\theta\, d\theta$$

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{2\sec\theta\,\sqrt{4\tan^2\theta}}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2\sec\theta\cdot 2\tan\theta}$$

$$= \int \frac{1}{2}\,d\theta = \frac{1}{2}\theta + C$$

we know $\quad x = 2\sec\theta$

$\theta = \sec^{-1}\left(\frac{x}{2}\right)$

$$\therefore \int \frac{1}{2}\,d\theta = \frac{1}{2}\sec^{-1}\left(\frac{x}{2}\right) + C$$

(39) $\int \sqrt{1-x^2}\,dx$

This is of the form $a^2 - x^2$

$\sin^2\theta + \cos^2\theta = 1$

$1 - \sin^2\theta = \cos^2\theta$

Let $x^2 = \sin^2\theta$

we have $\quad 1 - x^2 = \cos^2\theta$

$$1 - x^2 = \cos^2\theta$$

$$x^2 = \sin^2\theta$$

$$x = \sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta \, d\theta$$

rewriting question as $\Rightarrow \int \sqrt{1-x^2}\,dx = \int \sqrt{\cos^2\theta}\, \cos\theta \, d\theta$

$$= \int \cos^2\theta \, d\theta$$

$$= \int \frac{1}{2}\cos 2\theta + \frac{1}{2} \, d\theta$$

$$= \frac{1}{2}\int \cos 2\theta \, d\theta + \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\,\frac{\sin 2\theta}{2} + \frac{1}{2}\,\theta + C$$

$$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\sin^{-1}(x) + C$$

$$= \frac{1}{4}\, 2\sin\theta\cos\theta + \frac{1}{2}\sin^{-1}(x) + C$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

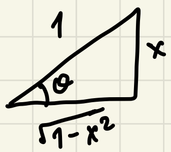
$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\therefore \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$



$$= \frac{1}{2}\, x \cdot \sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) + C$$

(40) $\int \frac{1}{(25+x^2)^{3/2}} dx$

This is of the form $a^2 + x^2$

$\therefore$ $\sin^2\theta + \cos^2\theta = 1$ is for $a^2 - x^2$

$\tan^2\theta + 1 = \sec^2\theta$ is for $x^2 + a^2$

$\therefore$ $1 + \tan^2\theta = \sec^2\theta$

$25(1 + \tan^2\theta) = 25\sec^2\theta$

$25 + 25\tan^2\theta = 25\sec^2\theta$

$25 + (5\tan\theta)^2 = 25\sec^2\theta$

$x = 5\tan\theta$

$dx = 5\sec^2\theta \, d\theta$

$$\int \frac{1}{(25+x^2)^{3/2}} dx = \int \frac{1}{(25\sec^2\theta)^{3/2}} \cdot 5\sec^2\theta \, d\theta$$

$$= \int \frac{5\sec^2\theta \, d\theta}{25^{3/2}(\sec^2\theta)^{3/2}}$$

$$= \int \frac{5}{5^{2 \cdot 3/2}} \cdot \frac{\sec^2\theta \, d\theta}{\sec^3\theta}$$

$$= \int \frac{5}{5^3} \cdot \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

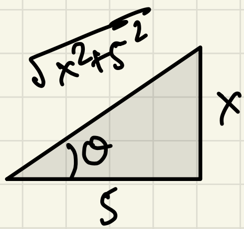
$$= \frac{1}{25} \int \frac{1}{\sec\theta}\, d\theta$$

$$= \frac{1}{25} \int \cos\theta \quad \cancel{+C}$$

$$= \frac{1}{25} \sin\theta + C$$

We have $x = 5\tan\theta$

$$\tan\theta = \frac{x}{5}$$



$$\sin\theta = \frac{x}{\sqrt{x^2+5^2}}$$

$$\therefore \int \frac{1}{(25+x^2)^{3/2}} = \frac{1}{25} \cdot \frac{x}{\sqrt{x^2+5^2}} + C$$

$$= \frac{1}{5} \cdot \frac{x}{\sqrt{x^2+5^2}} + C$$

(41) $\displaystyle\int \frac{x}{(25+x^2)^{3/2}}\,dx$

Let $u = 25 + x^2$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}\,du$

$$\int \frac{1}{2}\,du \cdot \frac{1}{u^{3/2}} = \frac{1}{2}\int u^{-3/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} + C$$

$$= \frac{1}{2}\,\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} + C$$

$$= -\frac{1}{\sqrt{u}} + C = -\frac{1}{\sqrt{25+x^2}} + C$$

(42) $\int \frac{1}{x^2+a^2}\,dx$

$\tan^2\theta + 1 = \sec^2\theta$

$a^2(\tan^2\theta + 1) = a^2\sec^2\theta$

$a^2\tan^2\theta + a^2 = a^2\sec^2\theta$

Let $x = a\tan\theta$

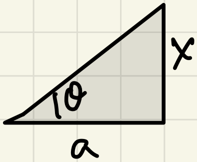
$dx = a\sec^2\theta\,d\theta$

x^2+a^2 can be written as $a^2\sec^2\theta$

$\int \frac{1}{a^2\sec^2\theta}\,a\sec^2\theta\,d\theta$

$= \int \frac{1}{a}\,d\theta = \frac{1}{a}\theta$

$\tan\theta = \frac{x}{a}$



$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

Partial Fractions

(43) $\displaystyle\int \frac{x^3}{x^2+9}\,dx$

Degree on top greater than degree on bottom.

$\displaystyle\int \frac{\text{polynomial}}{\text{polynomial}}\,dx$ ← The form we want for these kind of problems

We like:

1) $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

2) $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3) $\displaystyle\int \frac{x}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b| + C$

Case 1: degree (top) $\geq$ deg (bottom)

Do polynomial division

$$\int \frac{x^3}{x^2+9} \qquad x^3 > x^2$$

$$\therefore \quad x^2+9 \, \overline{\smash{)}\, x^3} \quad \text{quotient } x$$

$$\begin{array}{r} x^3 \\ \underline{x^3 + 9x} \\ -9x \end{array}$$

← stop when degree less than denom.

$$\int \frac{x^3 \, dx}{x^2+9} = \int \left(x - \frac{9x}{x^2+9} \right) dx$$

$$= \int x \, dx - \int \frac{9x}{x^2+9} \, dx$$

Let $x^2+9 = u$

$2x \, dx = du$

$x \, dx = \frac{1}{2} du$

$$= \frac{x^2}{2} - \frac{9}{2} \ln |x^2+9| + C$$

Case 2: Two linear factors; that are distinct

(44) $\int \frac{8x-17}{x^2-5x+4}\,dx$

① > ②

Deg Num > Deg Denom

do not do long division.

x^2-5x+4 $\quad -1 \times -4 = 4$

$-1 + -4 = -5$

$x^2-x-4x+4 = x(x-1)-4(x-1)$

$= (x-1)(x-4)$

$= \frac{8x-17}{(x-1)(x-4)} = \frac{A}{(x-1)} + \frac{B}{(x-4)}$

x throughout by LHS denom. $= \quad 8x-17 = A(x-4) + B(x-1)$

Now when $x=+4$; $A \to 0$

$= \quad 8(4)-17 = 0 + B(4-1)$

$32-17 = 3B$

$15 = 3B$

$\boxed{B=5}$

Now when $x=1$; $B \to 0$

$8-17 = A(1-4)$

$-9 = -3A \longrightarrow \boxed{A=3}$

$$\therefore \int \frac{8x-17}{(x-1)(x-4)} = \int \frac{3}{x-1}\,dx + \int \frac{5}{x-4}\,dx$$

$$= 3\ln|x-1| + 5\ln|x-4| + C$$

Case 3 : Quadratic factors distinct (irreducible quadratic factors) like x^2+4

x^2+4 ↑ cannot reduce

x^2-5x+4 ↑ reducable

(45) $$\int \frac{4x^2-9x+2}{(x+3)(x^2+4)}\,dx$$

↑ Notice irreducable quadratic factors

Also deg Num not greater than deg Denom

$\therefore$ no long divison required.

$$\frac{4x^2-9x+2}{(x+3)(x^2+4)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+4)}$$

↑ Degree 1 $\therefore$ top $\rightarrow x^0$

↑ Degree 2 $\therefore$ top has 1 degree less $\rightarrow x^1$

$$4x^2 - 9x + 2 = A(x^2+4) + (Bx+C)(x+3)$$

when $x = -3$

$$4(9) - 9(-3) + 2 = A(9+4) + 0$$

$$36 + 27 + 2 = 13A$$

$$65 = 13A$$

$$\boxed{A = 5}$$

The same method does not work with B with A in the eqn.
∴ open up the brackets

$$4x^2 - 9x + 2 = 5(x^2+4) + (Bx+C)(x+3)$$

$$= 5x^2 + 20 + Bx^2 + 3Bx + Cx + 3C$$

$$= 5x^2 + Bx^2 + 3Bx + Cx + 3C + 20$$

$$4x^2 - 9x + 2 = (5+B)x^2 + (3B+C)x + 3C + 20$$

Now relate groups

$$4 = 5 + B$$

$$\boxed{B = -1}$$

$$-9 = 3B + C$$

$$-9 = -3 + C$$

$$\boxed{C = -6}$$

$$2 = 3C + 20$$

$$2 = 3(-6) + 20$$

$$2 = -18 + 20$$

$$2 = 2$$

c is confirmed
∴ rest is correct!

$$\therefore \int \frac{4x^2-9x+2}{(x+3)(x^2+4)}\,dx = \int \frac{5}{x+3}\,dx + \int \frac{-x-6}{x^2+4}\,dx$$

$$= \int \frac{5}{x+3}\,dx - \int \frac{x+6}{x^2+4}\,dx$$

$$= \int \frac{5}{x+3}\,dx - \int \frac{x}{x^2+4}\,dx - \underbrace{\int \frac{6}{x^2+4}\,dx}_{\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C}$$

$$= 5\ln|x+3| - \frac{1}{2}\ln|x^2+4| - \frac{6}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= 5\ln|x+3| - \frac{1}{2}\ln|x^2+4| + 3\tan^{-1}\left(\frac{x}{2}\right) + C$$

(46) $\int \frac{1}{x^2+6x+13}$

= cannot reduce denom quadratically
$\therefore$ completing the squares

$$\left.\begin{aligned} 2ax &= 6x \\ 2a &= 6 \\ a &= 3 \end{aligned}\right\}$$

completing through squares of a
ie $a^2 = 9$

$x^2+6x+13+9-9$

$x^2+6x+9+13-9$

$\underline{3} \times \underline{3} = 9$

$\underline{3} + \underline{3} = 6$

$x^2+3x+3x+9+4$

$x(x+3)+3(x+3)+4$

$= (x+3)^2+4$ $\leftarrow$ New Denominator

$$\int \frac{1\,dx}{(x+3)^2+4}$$

Let $x+3 = u$

$dx = du$

$\int \frac{du}{u^2+4}$ we have of the form x^2+a^2 $\therefore$ trig substitution

$\tan^2\theta + 1 = \sec^2\theta$

~~$u = \tan\theta$~~

~~$du = \sec^2\theta \, d\theta$~~

$4\tan^2\theta + 4 = 4\sec^2\theta$

$u = 2\tan\theta$

$du = 2\sec^2\theta \, d\theta$

$u^2+4 = 4\sec^2\theta$

$$\therefore \int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta} = \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$$

Case 4: Repeating Squares

pay attention to $(\quad)^2$, $(\quad)^3$ ...

$$x^2 = x \cdot x$$

$$(x^2+5)^3 = (x^2+5)(x^2+5)(x^2+5)$$

(47) $$\int \frac{2x-5}{x^3+x^2}\,dx$$

$$= \int \frac{2x-5}{x^2(x+1)} = \int \frac{2x-5}{x \cdot x \cdot (x+1)}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\frac{2x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$2x-5 = Ax(x+1) + B(x+1) + Cx^2$$

when $x = -1$

$$2(-1)-5 = 0 + 0 + C(-1)^2$$

$$-7 = C \qquad \boxed{C=-7}$$

Now grouping for rest.

$$2x - 5 = Ax^2 + Ax + Bx + B + Cx^2$$

$$0x^2 + 2x - 5 = Ax^2 + Cx^2 + Ax + Bx + A + B$$

$$(A+B)x = 2x$$

$$7 + B = 2$$

$$\boxed{B = -5}$$

$$(A+C)x^2 = 0x^2$$

$$A + C = 0$$

$$A - 7 = 0$$

$$\boxed{A = 7}$$

$$\int \frac{2x-5}{x^3+x^2} = \int \frac{7}{x} dx - \int \frac{5}{x^2} dx - \int \frac{7}{x+1} dx$$

$$= 7\ln|x| - 5\frac{x^{-1}}{-1} - 7\ln|x+1| + C$$

$$= 7\ln|x| + \frac{5}{x} - 7\ln|x+1| + C$$

Note:

$$\frac{2x-5}{x^3(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{(x+1)} + \frac{E}{(x+1)^2}$$

$$\frac{2x-5}{(x+1)(x^2+9)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

increase power up.

(48) $\displaystyle\int \frac{2x^2+8x+5}{x^2+4x+13}\,dx$

Denominator irreducable factor

↳ use completing through squares

$2ax = 4x$

$a = 2$

completing square $a^2 = 4$

$\therefore \quad x^2+4x+13+4-4$

$x^2+4x+4+13-4$

$x^2+2x+2x+4+9$

$x(x+2)+2(x+2)+9$

$(x+2)^2+9 \leftarrow$ New denominator

when numerator & denominator degrees are same ; do long division.

$$\begin{array}{r|l} & \quad 2 \\ \hline x^2+4x+13 & 2x^2+8x+5 \\ & 2x^2+8x+26 \\ \hline & \qquad\quad -21 \end{array}$$

$$\begin{array}{r|l} & 6 \\ \hline 4 & 21 \\ & \underline{24} \\ & 1 \end{array}$$

$6+\frac{1}{4}$

$$\int \frac{2x^2+8x+5}{x^2+4x+13} = \int \left(2 - \frac{21}{x^2+4x+13}\right) dx$$

$$= \int 2\,dx - \int \frac{21}{x^2+4x+13}\,dx$$

denom: $x^2+4x+13$

completing through squares

$2ax = 4x$

$a = 2$

$a^2 = 4$

$x^2+4x+4-4+13$

$(x+2)^2+9$ ← New denominator

$$= \int 2\,dx - \int \frac{21\,dx}{(x+2)^2+9}$$

$$= 2x - 21 \cdot \frac{1}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + C$$

$$= 2x - \frac{21}{3} \tan^{-1}\left(\frac{x+2}{3}\right) + C$$

(49) $$\int \frac{6x^2+31x+45}{x^3+6x^2+9x}\,dx$$

$$= \int \frac{6x^2+31x+45}{x(x^2+6x+9)}$$

x^2+6x+9 $\qquad$ $\underline{3} \times \underline{3} = 3$

$x^2+3x+3x+9$ $\qquad$ $\underline{3} + \underline{3} = 6$

$x(x+3)\ 3(x+3)$

$(x+3)^2 \leftarrow$ New form of denominator

$$= \int \frac{6x^2+31x+45}{x\,(x+3)^2}$$

$$= \frac{A}{x} + \frac{B}{(x+3)} + \frac{C}{(x+3)^2}$$

Mult by

$$x(x+3)^2 = A(x+3)^2 + Bx(x+3) + Cx$$

$$6x^2 + 37x + 45 = A(x+3)^2 + Bx(x+3) + Cx$$

when $x = -3$

$$6(-3)^2 + 31(-3) + 45 = 0 + 0 + C(-3)$$

$$54 - 93 + 45 = -3C$$

$$99 - 93 = -3C$$

$$6 = -3C$$

$$\boxed{C = -2}$$

$$6x^2 + 37x + 45 = A(x^2 + 9 + 6x) + Bx^2 + 3Bx + Cx$$

$$= Ax^2 + 9A + 6Ax + Bx^2 + 3B + Cx$$

$$= Ax^2 + Bx^2 + 6Ax + 3Bx + Cx + 9A$$

$$(A+B)x^2 + (6A + 3B + C)x + 9A$$

$9A = 45$ $\quad$ $A + B = 6$

$\boxed{A = 5}$ $\quad$ $B = 6 - 5 = 1$

$\boxed{B = 1}$

$$= \int \frac{6x^2 + 31x + 45}{x(x^2+6x+9)} = \int \frac{5\,dx}{x} + \int \frac{1\,dx}{(x+3)} + \int \frac{-2\,dx}{(x+3)^2}$$

① ② ③

① $\int \frac{5}{x} dx = 5 \ln|x| + C_1$

② $\int \frac{1}{x+3} dx = \ln|x+3| + C_2$

③ $\int \frac{-2}{(x+3)^2} dx =$ Let $u = x+3$

$du = dx$

$$-2 \int \frac{1}{(x+3)^2} dx = -2 \int \frac{1}{u^2} dx$$

$$= -2 \frac{u^{-2+1}}{-2+1} + C_3$$

$$= -2 \frac{u^{-1}}{-1} + C_3$$

$$= \frac{-2}{u} + C_3$$

$$= -\frac{2}{x+3} + C_3$$

① + ② + ③

$$= \int \frac{6x^2+31x+45}{x(x^2+6x+9)} = 5\ln|x| + \ln|x+3| - \frac{2}{x+3} + C_4$$

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(50) $\int \frac{1}{x^2-a^2}\,dx$

$=$ This is ideal for trig sub.

$=$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

$$(a\sec\theta)^2 - a^2 = a^2\tan^2\theta \quad *$$

$$x = a\sec\theta$$

$$dx = a\sec\theta\tan\theta\,d\theta \quad *$$

Dont do this just do $(x+a)(x-a)$

$$\int \frac{1}{x^2-a^2} = \int \frac{1}{(x+a)(x-a)}$$

$$= \frac{A}{x+a} + \frac{B}{x-a}$$

$$1 = A(x-a) + B(x+a)$$

when $x = a$

$$1 = 0 + B(2a)$$

$$\boxed{B = \frac{1}{2a}}$$

when $x = -a$

$$1 = A(-2a) + 0$$

$$\boxed{A = -\frac{1}{2a}}$$

$$= \int \frac{1}{x^2 - a^2} = \int \frac{-1}{2a(x+a)}\,dx + \int \frac{1}{2a(x-a)}\,dx$$

$$= -\frac{1}{2a}\int \frac{1\,dx}{x+a} + \frac{1}{2a}\int \frac{1}{x-a}\,dx$$

$$= -\frac{1}{2a}\ln|x+a| + \frac{1}{2a}\ln|x-a| + C$$

$$= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Exam solutions

b. [3 pts] Find antiderivatives to solve the following integrals

$$F(x) = \int 3x^2 dx \qquad F(x) = \int 3\cos(3x+2)\, dx \qquad F(x) = \int \frac{4}{x}\, dx$$

c. [6 pts] Evaluate the following definite integrals

$$\int_{-6}^{-10} \frac{1}{x+2}\, dx \qquad \int_{-2}^{2} \frac{1}{x^2+4}\, dx$$

d. [12 pts] Use the substitution rule to evaluate the following integrals:

$$\int \frac{e^{1/x^2}}{x^3}\, dx \qquad \int (\cos x - \sin x)^2\, dx$$

e. [6 pts] Use the integration by parts method to perform the following integral:

$$\int x^6 \ln x\, dx$$

f. [6 pts] Perform the following integral using the method of partial fraction decomposition

$$\int \frac{x+2}{x^2+x}\, dx$$

b.1) $F(x) = \int 3x^2 dx$

$= 3\int x^2 dx$

$= 3\frac{x^3}{3} + C \; ; \; C \in R$

$= x^3 + C \; ; \; C \in R$

b.2) $\int 3\cos(3x+2)\,dx$

$= 3\int \cos(3x+2)\,dx$

$\frac{d}{dx}\sin x = \cos x$

$= 3\sin(3x+2)\cdot\frac{1}{3} + C$

$= \sin(3x+2) + C \; ; \; C \in R$

b.3) $\int \frac{4}{x}\,dx$

$\frac{d}{dx}\ln x = \frac{1}{x}$

$= 4\int \frac{1}{x}\,dx$

$= 4\ln|x| + C$

c.1) $\int_{-6}^{-10} \frac{1}{x+2}\,dx$

$x+2 = u$

$1\,dx = du$

LL: $x+2 = -6+2 = -4$

UL: $x+2 = -10+2 = -8$

$$\int_{-8}^{-4} \frac{1}{u}\,du$$

$$= \ln|u| \Big|_{-8}^{-4}$$

$$= \ln|-4| - \ln|-8|$$

$$= \ln 4 - \ln 8$$

$$= 2 - 3 = -1$$

$\log_2 4 = x$

$2^x = 4$

$2^x = 2^2$

$x = 2$

$\log_2 8 = x$

$2^x = 8$

$2^x = 2^3$

$x = 3$

C. 2) $$\int_{-2}^{2} \frac{1}{x^2+4}\,dx$$

This is of the form x^2+a^2 in denominator, $\therefore$ trig substitution

$$\tan^2\theta + 1 = \sec^2\theta$$

$$4\tan^2\theta + 4 = 4\sec^2\theta$$

$$(2\tan\theta)^2 + 4 = 4\sec^2\theta \quad *$$

$$x = 2\tan\theta$$

$$dx = 2\sec^2\theta\, d\theta \quad *$$

$$\int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta}$$

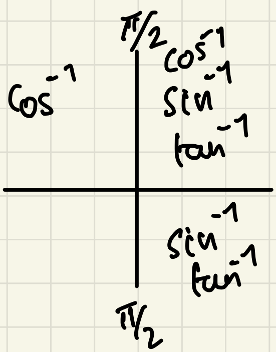
$$\int \frac{1}{4}\, d\theta = \frac{1}{4}\theta + C$$

$$x = 2\tan\theta$$
$$\theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\int \frac{1}{4}\, d\theta = \frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\int_{-2}^{2} \frac{1}{x^2+4}\, dx = \frac{1}{4}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= \frac{1}{4}\left\{\tan^{-1}(1) - \tan^{-1}(-1)\right]$$



$$\tan^{-1}(1) = \theta$$
$$\tan\theta = 1$$
$$\theta = 45^\circ = \frac{\pi}{4}$$

$$\tan^{-1}(-1) = \theta$$
$$\tan\theta = -1$$
$$\theta = -45^\circ$$
$$= -\frac{\pi}{4}$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - \frac{-\pi}{4}\right]$$

$$= \frac{1}{4}\left[\frac{2\pi}{4}\right] = \frac{1}{4}\;\frac{\pi}{2} = \frac{\pi}{8}$$

Homework contd.

Schaum: IP. teil 1

(96) $\int (4x^3 + 3x^2 + 2x + 5)\,dx$

$$= 4\frac{x^4}{4} + 3\frac{x^3}{3} + 2\frac{x^2}{2} + 5x + C$$

$$= x^4 + x^3 + x^2 + 5x + C$$

(97) $\int (3 - 2x - x^4)\,dx$

$$= 3x - \frac{2x^2}{2} - \frac{x^5}{5} + C$$

$$= 3x - x^2 - \frac{x^5}{5} + C$$

(98) $\int (2 - 3x + x^3)\,dx$

$$= 2x - \frac{3x^2}{2} + \frac{x^4}{4} + C$$

(99) $\int (x^2 - 1)^2 dx$

$= \int x^4 + 1 - 2x^2$

$= \frac{x^5}{5} + x - \frac{2x^3}{3} + C$

(100) $\int \left(\sqrt{x} - \frac{1}{2}x + \frac{2}{\sqrt{x}} \right) dx$

$= \frac{x^{\frac{1}{2}+1}}{3/2} - \frac{1}{2}\frac{x^2}{2} + 2\frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}}$

$= \frac{2x^{3/2}}{3} - \frac{x^2}{4} + 4x^{1/2} + C$

(101) $\int (a+x)^3 dx$

$= \int a^3 + x^3 + 3a^2x + 3ax^2 \, dx$

$= a^3x + \frac{x^4}{4} + 3a^2\frac{x^2}{2} + 3a\frac{x^3}{3} + C$

or $\frac{a+x^4}{4} + C$

(102) $\int (x-2)^{3/2}\,dx$

$= \frac{(x-2)^{3/2+1}}{\frac{3}{2}+1} + C$

$= \frac{2}{5}(x-2)^{5/2} \cdot \frac{1}{\frac{d}{dx}(x-2)} + C$

$= \frac{2}{5}(x-2)^{5/2}$

(103) $\int \frac{1}{x^3}\,dx = \frac{x^{-2}}{-2} + C$

$= -\frac{1}{2x^2} + C$

(104) $\int \frac{dx}{(x-1)^3} = \int (x-1)^{-3}\,dx$

$$= \frac{(x-1)^{-3+1}}{-3+1} \cdot \frac{1}{\frac{d}{dx}(x-1)} + C$$

$$= \frac{(x-1)^{-2}}{-2} + C$$

$$= -\frac{1}{2(x-1)^2} + C$$

105 $$\int \frac{dx}{\sqrt{x+3}} = \int (x+3)^{-\frac{1}{2}} dx$$

$$= \frac{(x+3)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cdot \frac{1}{1} + C$$

$$= 2(x+3)^{\frac{1}{2}} + C$$

(106) $\int \sqrt{3x-1}\,dx$

$$\int (3x-1)^{\frac{1}{2}}\,dx$$

$$= \frac{(3x-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \cdot \frac{1}{\frac{d}{dx}(3x-1)} + C$$

$$= 2\frac{(3x-1)^{3/2}}{3} \cdot \frac{1}{3} + C$$

$$= \frac{2}{9}(3x-1)^{3/2} + C$$

(107) $\int \sqrt{2-3x}\,dx$

$$= 2\frac{(2-3x)^{3/2}}{3} \cdot \frac{1}{-3} + C$$

$$= -\frac{2}{9}(2-3x)^{3/2} + C$$

(108) $\int (2x^2+3)^{1/3} x\,dx$

$$2x^2+3 = u$$
$$4x\,dx = du$$

$$x\,dx = \frac{1}{4}\,du$$

$$\int u^{1/3} \cdot \frac{1}{4}\,du = \frac{1}{4}\,\frac{3u^{\frac{1}{3}+1}}{4} + C$$

$$= \frac{3}{16}\,u^{\frac{4}{3}} + C$$

$$= \frac{3}{16}\left(2x^2+3\right)^{\frac{4}{3}} + C$$

109 $\int (x-1)^2 x\,dx$

$$\int (x^2+1-2x)\,x\,dx$$

$$= \int x^3 + x - 2x^2\,dx$$

$$= \frac{x^4}{4} + \frac{x^2}{2} - 2\frac{x^3}{3} + C$$

110 $\int (x^2-1)\,x\,dx$

$$x^2-1 = u$$
$$2x\,dx = du$$
$$x\,dx = \frac{1}{2}\,du$$

$$\int u \frac{1}{2} du = \frac{1}{2} \frac{u^2}{2} + C$$

$$= \frac{1}{4} (x^2 - 1)^2 + C$$

(111) $\int \sqrt{1+y^4}\, y^3 dy$

$$1 + y^4 = u$$

$$4y^3 dy = du$$

$$y^3 dy = \frac{1}{4} du$$

$$= \int \sqrt{u} \cdot \frac{1}{4} du = \frac{1}{4} \frac{2u^{\frac{1}{2}+1}}{3} + C$$

$$= \frac{1}{6} (1+y^4)^{3/2} + C$$

(112) $\int (x^3 + 3) x^2 dx$

$$x^3 + 3 = u$$

$$3x^2 dx = du$$

$$x^2 dx = \frac{1}{3} du$$

$$\int u \cdot \frac{1}{3} du = \frac{1}{3} \frac{u^2}{2} + C$$

$$= \frac{1}{6} (x^3+3)^2 + C$$

(113) $\int (4-x^2)^2 x^2 \, dx$

$$\int (16 + x^4 - 8x^2) x^2 \, dx$$

$$\int (16x^2 + x^6 - 8x^4) \, dx$$

$$= 16 \frac{x^3}{3} + \frac{x^7}{7} - 8 \frac{x^5}{5} + C$$

(114) $\int \frac{dy}{(2-y)^3} = \int (2-y)^{-3}$

$$= \frac{(2-y)^{-3+1}}{-3+1} \cdot \frac{1}{\frac{d}{dy}(2-y)} + C$$

$$= \frac{(2-y)^{-2}}{-2} \cdot \frac{1}{-1} + C = \frac{1}{2} \cdot \frac{1}{(2-y)^2} + C$$

(115) $\displaystyle\int \frac{x\,dx}{(x^2+4)^3}$

$x^2+4=u$

$2x\,dx=du$

$x\,dx=\frac{1}{2}\,du$

$\displaystyle\int \frac{1}{2}\,\frac{1\,du}{u^3} = \frac{1}{2}\,\frac{u^{-2}}{-2}+C$

$= -\frac{1}{4}\cdot\frac{1}{(x^2+4)^2}+C$

(116) $\displaystyle\int (1-x^3)^2\,dx$

$\displaystyle\int (1+x^6-2x^3)\,dx$

$= x+\frac{x^7}{7}-2\frac{x^4}{4}+C$

$= x-\frac{x^4}{2}+\frac{x^7}{7}+C$

(117) $\int (1-x^3)^2 x \, dx$

$$1 - x^3 = u \quad \Rightarrow \quad x = \sqrt[3]{1-u}$$

$$-3x^2 dx = du$$

$$x \, dx = -\frac{1}{3x} du \quad \Rightarrow \quad -\frac{1}{3} \frac{1}{\sqrt[3]{1-u}} du$$

Makes things complicated.

$\therefore$ expanding & simplifying

$$\int (1 + x^6 - 2x^3) x \, dx$$

$$\int x + x^7 - 2x^4 \, dx$$

$$= \frac{x^2}{2} + \frac{x^8}{8} - \frac{2x^5}{5} + C$$

(118) $\int (1-x^3)^2 x^2 \, dx$

$$1 - x^3 = u$$

$$-3x^2 dx = du$$

$$x^2 dx = -\frac{1}{3} du$$

$$= \int u^2 \cdot \frac{-1}{3}\, du$$

$$= -\frac{1}{3} \frac{u^3}{3} + C$$

$$= -\frac{1}{9}(1-x^3)^3 + C$$

$$= -\frac{(1-x^3)^3}{9} + C$$

(119) $\int (x^2 - x)^4 (2x-1)\, dx$

$$x^2 - x = u$$

$$2x - 1\, dx = du$$

$$\int u^4\, du = \frac{u^5}{5} + C$$

$$= \frac{(x^2-x)^5}{5} + C$$

(120) $\int \frac{3t\,dt}{\sqrt[3]{t^2+3}} = 3 \int \frac{t\,dt}{\sqrt[3]{t^2+3}}$

$$t^2 + 3 = u$$

$$2t\,dt = du$$

$$t\,dt = \frac{1}{2}du$$

$$= 3\int \frac{1}{2}\,\frac{1}{\sqrt[3]{u}}\,du$$

$$= \frac{3}{2}\int u^{-\frac{1}{3}}\,du$$

$$= \frac{3}{2}\,\frac{3u^{-\frac{1}{3}+1}}{2} + C$$

$$= \frac{9}{4}\,u^{\frac{2}{3}} + C$$

$$= \frac{9}{4}\left(t^2+3\right)^{\frac{2}{3}} + C$$

(121) $$\int \frac{(x+1)\,dx}{\sqrt{x^2+2x-4}}$$

$$x^2+2x-4 = u$$

$$2x+2\,dx = du$$

$$x+1 = \frac{1}{2}\,du$$

$$\int \frac{1}{2} \frac{1}{\sqrt{u}} \, du$$

$$\frac{1}{2} \, 2 \, \frac{u^{-\frac{1}{2}+1}}{1} + C = \sqrt{u} + C$$

$$= \sqrt{x^2 + 2x - 4} + C$$

(122) $$\int \frac{dx}{(a+bx)^{\frac{1}{3}}}$$

$$= \frac{(a+bx)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \cdot \frac{1}{\frac{d}{dx}(a+bx)} + C$$

$$= \frac{3(a+bx)^{\frac{2}{3}}}{2} \cdot \frac{1}{b} + C$$

$$= \frac{3}{2b}(a+bx)^{\frac{2}{3}} + C$$

Schaum : 3

(169) $\int e^{\tan 2x} \sec^2 2x \, dx$

$=$ $\tan 2x = u$

$2 \cdot \sec^2 2x \, dx = du$

$\sec^2 2x \, dx = \frac{1}{2} du$

$= \int e^u \cdot \frac{1}{2} du = \frac{1}{2} e^u + c$

$= \frac{1}{2} e^{\tan 2x} + C$

170 $\int e^{2 \sin 3x} \cos 3x \, dx$

$\sin 3x = u$

$3 \cos 3x \, dx = du$

$\cos 3x \, dx = \frac{1}{3} du$

$\int e^{2u} \frac{1}{3} u \, du$

$$= \frac{1}{3} e^{2u} \cdot \frac{1}{2} + C$$

$$= \frac{1}{6} e^{2\sin 3x} + C$$

171. $\int \frac{dx}{\sqrt{5-x^2}}$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$5\cos^2\theta = 5 - 5\sin^2\theta$$

$$5\cos^2\theta = 5 - (\sqrt{5}\sin\theta)^2$$

use θ not dx to avoid confusion!

$$x = \sqrt{5}\sin\theta$$

$$dx = \sqrt{5}\cos\theta \, d\theta$$

$$\int \frac{\sqrt{5}\cos\theta \, d\theta}{\sqrt{5\cos^2\theta}}$$

$$= \int 1 \, d\theta = \theta + C$$

we have $\frac{x}{\sqrt{5}} = \sin\theta$

$\therefore \theta = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$

$\therefore \int \frac{dx}{\sqrt{5-x^2}} = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$

172. $\int \frac{dx}{5+x^2}$

$\tan^2\theta + 1 = \sec^2\theta$

$5\tan^2\theta + 5 = 5\sec^2\theta$

$(\sqrt{5}\tan\theta)^2 + 5 = 5\sec^2\theta$

$x = \sqrt{5}\tan\theta$

$dx = \sqrt{5}\sec^2\theta\, d\theta$

$\int \frac{\sqrt{5}\sec^2\theta\, d\theta}{5\sec^2\theta}$

$= \frac{1}{\sqrt{5}}\theta + C$

now $\theta = \tan^{-1}\left(\frac{x}{\sqrt{5}}\right)$

$$\therefore \quad \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

173. $\displaystyle\int \frac{dx}{x\sqrt{x^2-5}}$

$$\tan^2 \cancel{x}\theta + 1 = \sec^2 \cancel{x}\theta$$

$$\sec^2 \cancel{x}\theta - 1 = \tan^2 \cancel{x}\theta$$

$$5\sec^2 \cancel{x}\theta - 5 = 5\tan^2 \cancel{x}\theta$$

$$(\sqrt{5}\sec \cancel{x}\theta)^2 - 5 = 5\tan^2 \cancel{x}\theta$$

$$x = \sqrt{5}\sec\theta$$

$$dx = \sqrt{5}\sec\theta\tan\theta\, d\theta$$

$$\int \frac{\sqrt{5}\sec\theta\tan\theta\, d\theta}{\sqrt{5}\sec\theta\sqrt{5\tan^2\theta}}$$

$$= \int \frac{\sqrt{5}\sec\theta\tan\theta\, d\theta}{5\sec\theta\tan\theta}$$

$$= \frac{1}{\sqrt{5}}\theta + C = \frac{1}{\sqrt{5}}\sec^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

174. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$

$e^x = u$

$e^x dx = dx$

$\int \frac{du}{\sqrt{1-u^2}}$

$= \sin^{-1}|u| + C$

$= \sin^{-1}|e^x| + C$

175. $\int \frac{e^{2x}}{1+e^{4x}}\,dx$

$e^{2x} = u$

$2e^{2x}dx = du$

$e^{2x}dx = \frac{1}{2}du$

$= \int \frac{1}{2} \frac{du}{1+u^2}$

$= \frac{1}{2} \tan^{-1}|u| + C$

$= \frac{1}{2} \tan^{-1}|e^{2x}| + C$

176. $\int \frac{dx}{\sqrt{4-9x^2}}$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$4\sin^2 x = 4 - 4\cos^2 x$

$$= \int \frac{dx}{\sqrt{4} \cdot \sqrt{1 - \frac{9}{4}x^2}}$$

$$= \int \frac{dx}{2\sqrt{1 - \frac{9}{4}x^2}} = \int \frac{dx}{2\sqrt{1 - \left(\frac{3}{2}x\right)^2}}$$

$$= \frac{1}{2} \sin^{-1}\left(\frac{3x}{2}\right) + C$$

177. $$\int \frac{dx}{9x^2 + 4} = \int \frac{dx}{9\left(x^2 + \frac{4}{9}\right)}$$

$$= \int \frac{dx}{9\left(x^2 + \left(\frac{2}{3}\right)^2\right)}$$

$$= \frac{1}{9} \int \frac{1}{x^2 + \left(\frac{2}{3}\right)^2} = \int \frac{1}{x^2 + \frac{4}{9}}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\frac{4}{9}\tan^2\theta + \frac{4}{9} = \frac{4}{9}\sec^2\theta$$

$$\left(\frac{2}{3}\tan\theta\right)^2 + \frac{4}{9} = \frac{4}{9}\sec^2\theta$$

$$x = \frac{2}{3}\tan\theta$$

$$dx = \frac{2}{3} \cdot \sec^2\theta \, d\theta \quad *$$

$$= \frac{1}{9}\int \frac{1}{\frac{4}{9}\sec^2\theta} \cdot \frac{2}{3}\sec^2\theta \, d\theta$$

$$= \frac{1}{\cancel{9}} \cdot \frac{\cancel{9}}{\cancel{4}_{\,2}} \cdot \frac{\cancel{2}}{3} \theta + C$$

$$= \frac{1}{6}\theta + C = \frac{1}{6}\tan^{-1}\left(\frac{2}{3}\right) + C$$

178. $\displaystyle\int \frac{\sin 8x \, dx}{9 + \sin^2 4x}$

$= \sin 8x = \sin 2 \cdot 4x = 2\sin 4x \cos 4x$

$= \sin^2 4x = u$

$2 \cdot \cos 4x \cdot 2\sin 4x = du$

$= 2\cos 4x \cdot \sin 4x = \frac{1}{2}du$

$$\int \frac{2\sin 4x \cos 4x \, dx}{9 + \sin^2 4x} = \frac{1}{2}\int \frac{du}{9 + u}$$

$$= \frac{1}{2} \int \frac{1 \, du}{3^2 + (\sqrt{u})^2} = \frac{1}{2} \int \frac{1 \, du}{(\sqrt{u})^2 + 9}$$

$$= \quad \tan^2\theta + 1 = \sec^2\theta$$

$$9\tan^2\theta + 9 = 9\sec^2\theta$$

$$(3\tan\theta)^2 + 9 = 9\sec^2\theta$$

$$u = 3\tan\theta$$

$$du = 3\sec^2\theta \, d\theta$$

$$= \frac{1}{2} \cdot \int \frac{3\sec^2\theta \, d\theta}{9\sec^2\theta} = \frac{1}{3}\theta + C$$

$$= \frac{1}{2} \cdot \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \tan^{-1}\left(\frac{\sin^2 4x}{3}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{\sin^2 4x}{3}\right) + C$$

179. $$\int \frac{\sec^2 x \, dx}{\sqrt{1 - 4\tan^2 x}}$$

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$$\tan x = u$$
$$\sec^2 x \, dx = du$$

$$\int \frac{du}{\sqrt{1 - 4u^2}} = \int \frac{1 \, du}{\sqrt{1 - (2u)^2}}$$

$$2u = t$$
$$2du = dt$$
$$du = \frac{1}{2} dt$$

$$= \int \frac{1}{2} \frac{dt}{\sqrt{1 - t^2}} = \frac{1}{2} \sin^{-1}(t) + C$$

$$= \frac{1}{2} \sin^{-1}(2u) + C$$

$$= \frac{1}{2} \sin^{-1}(2\tan x) + C$$

180. $\displaystyle\int \frac{dx}{x\sqrt{4-9\ln^2 x}}$

$\ln x = u$

$\frac{1}{x}\,dx = du$

$$= \int \frac{du}{\sqrt{4-9u^2}} = \int \frac{du}{\sqrt{4\left(1-\frac{9}{4}u^2\right)}}$$

$$= \int \frac{du}{2\sqrt{1-\left(\frac{3}{2}u\right)^2}} = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3}{2}u\right)^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{3u}{2}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{3}{2}\ln x\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\ln x^{3/2}\right) + C$$

181. $\displaystyle\int \frac{2x^4 - x^2}{2x^2+1}\,dx$

numerator power greater than denominator:

$$\begin{array}{r|l} & x^2 + 1 \\ \hline 2x^2+1 & 2x^4 - x^2 \\ & 2x^2 + x^2 \\ \hline & -2x^2 \\ & 2x^2 + 1 \\ \hline & -1 \end{array}$$

$$= \int \left(x^2 + 1 - \frac{1}{2x^2+1} \right) dx$$

$$= \frac{x^3}{3} + x - \int \frac{1}{2x^2+1} dx$$

$$= \int \frac{1}{2x^2+1} = \int \frac{1}{2\left(x^2 + \frac{1}{2}\right)}$$

$$= \frac{1}{2} \int \frac{1}{\left(x^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)} = \int \frac{1}{x^2 + \frac{1}{2}}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\frac{1}{2}\tan^2\theta + \frac{1}{2} = \frac{1}{2}\sec^2\theta$$

$$\left(\frac{1}{\sqrt{2}}\tan\theta\right)^2 + \frac{1}{2} = \frac{1}{2}\sec^2\theta$$

$$x = \frac{1}{\sqrt{2}}\tan\theta$$

$$dx = \frac{1}{\sqrt{2}}\sec^2\theta$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{2}} \cdot \frac{\sec^2\theta}{\frac{1}{2}\sec^2\theta}\,d\theta$$

$$= \frac{1}{\sqrt{2}}\theta + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(x\sqrt{2}) + C$$

$\therefore$ complete integral:

$$\frac{x^3}{3} + x - \frac{1}{\sqrt{2}}\tan^{-1}(x\sqrt{2}) + C$$

182. $\int \frac{\cos 2x}{\sin^2 2x + 8}$

$u = \sin 2x$

$du = \cos 2x \, dx$

$\int \frac{du}{u^2 + 8}$