Differential Equations:

- Ordinary Differential Equations
- Second Order Differential Equations

github.com/royceanton

Ordinary Differential Equations

In the exam question: first check the order of the eqn ie, if $y' \to 1^{st}$ Order
$y'' \to 2^{nd}$ Order

Check if you can separate the eqn by grouping the all 'x' variables along with 'dx' on one side and all 'y' variable along with 'dy' on the other side. If you can write: It is separable
or else : It is non separable

If separable

1) Group x & y on both sides
2) Integrate both sides to get an eqn in 'y'
3) That is one of your solution. Dont forget to add: where $c, k \in R$ depending on variable constant that you use while solving the integral
5) For the other soln plug $y = \gamma$ in to the question and solve to find a value for γ
6) write $y = \gamma$ is also a solution

If it is non separable

1) Take the constant part to 0 and solve for y. This is called variation of constant. Duh!! :>
2) when varying the constant & solving the integral you will encounter natural log on both sides. Raise the power on both sides by 'e'
$\therefore e^{\ln x}$ can be written as x
3) Solve & write one soln of y with const $k, c \in R$
4) Now that you have varied the constant, you need to find another soln by trial method; ie turn $k \to k(x)$ from step 3. Now you have a new function. $y = k(x)$....
5) Find y' of this new function Plug y & y' in the original question. Notice that some terms will cancel & you have $k'(x)$ remaining
6) Integrate $k'(x)$ to get $k(x)$ & plug it to the function from step 3. This is your other soln with $c, k \in R$
7) If IVP plug the question condition $y(..) = \dots$ into step 3 & solve the constant k or c & rewrite the eqn.

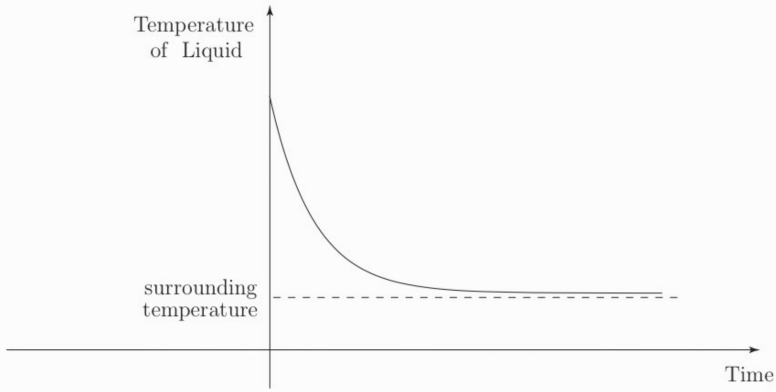


Figure 1

After an initially rapid decrease the temperature changes progressively less rapidly and eventually the curve appears to 'flatten out'.

Newton's law of cooling states that the rate of cooling of liquid is proportional to the difference between its temperature and the temperature of its environment (the ambient temperature). To convert this into mathematics, let t be the time elapsed (in seconds, s), θ the temperature of the liquid (°C),and θ_0 the temperature of the liquid at the start $(t = 0)$. The temperature of the surroundings is denoted by θ_s.

From the text above we can derive the following symbols:

$\theta \Rightarrow$ temp of liquid that changes with time t

$\theta_s \Rightarrow$ Outside temp

$t \Rightarrow$ time

$\theta_0 \Rightarrow$ initial temp.

Thus: a matematical model for the description would be:

$$\frac{d\theta}{dt} \propto \theta - \theta_s$$

$$= \quad \frac{d\theta}{dt} = -k(\theta - \theta_s)$$

$-k$ as θ decreases with time

Here $t \to$ independent variable

$t = 0$ would be the initial condition

θ = dependent variable

Here the Order of equation is 1 as it is $\frac{d}{dt}$ & not $\frac{d^2}{dt^2}$ ∴

and also the degree is 1 as it is $\left(\frac{d}{dt}\right)^1$

The first Order Differential equation has $\to$ One Independent variable $\to$ One or more independent variable

example:

(i) $y' = \cos x$

$\frac{dy}{dx} = \cos x$

degree: 1
Order : 1
dependent : y
independent : x

(2) $\frac{d^2y}{dx^2} + 9y = e^{-2x}$

degree : 1
order : 2
independent : x
dependent : y

(3) $y'y''' - \frac{3}{2}y'^2 = 0$

degree : 1
order : 3
independent : x
dependent : y

(4) $\frac{dy}{dt} + \frac{dx}{dt} = \sin t$

degree : 1
order : 1
independent : t
dependent : y, x

1 independent & one or more dependent
$\therefore$ Still 1st ODE.

⑤ $\frac{d^4y}{dx^4} + \sin(y''') = 0$

degree: not defined

order: 4

independent: x

dependent: y

To define the degree; the function must be polynomial.

⑥ $y''' + y^2 + e^{y'} = 0$

degree: not defined

order: 3

independent: x

dependent: y

> An equation that contain only the 1st derivative & may contain y and any given functions of x is called first ODE. Hence we can write them as:
>
> $$F(x, y, y') = 0 \quad \text{or} \quad y' = f(x, y)$$

Verify that $y = \frac{c}{x}$ is a solution of the ODE $xy' = -y$ for all $x \neq 0$ where c is constant

we are given y as a solution & we have to verify if the y' when substituted into the ODE's LHS produces the RHS of the ODE. Thus we can easily verify its a solution or not

$$y = \frac{c}{x}$$

$$y' = c \cdot \left(-\frac{1}{x^2}\right) = -\frac{c}{x^2}$$

we have $xy' = -y$

LHS:

$$\therefore \quad x\left(-\frac{c}{x^2}\right) = -\frac{c}{x}$$

RHS:

$$-y = -\frac{c}{x}$$

LHS = RHS $\quad \therefore \quad \frac{c}{x}$ is a solution of ODE $xy' = y$

Verify that $y = \sin x + C$ is a solution of the ODE $y' = \cos x$; where C is constant

Given $\quad y = \sin x + C$

$$y' = \cos x$$

plug into ODE

(LHS) $= \cos x = \cos x$ (RHS)

LHS = RHS $\quad y = \sin x + C$ is a solution of the ODE $y' = \cos x$

DE of the form $y' = kx$ is exponential growth model

DE of the form $y' = -kx$ is exponential decay model

General Solution

The general solution (primitive) of a DE is the solution which contains as many arbitrary constants (parameters) as the order of the DE.

Particular Solution

Obtained from the general solution by plugging in a particular value to the arbitrary constant. Hence the particular soln would be free from arbitrary constants.

1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.

2. Verify that $y = -t\cos t - t$ is a solution of the initial-value problem

$$t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$$

① We have $y = \frac{2}{3}e^x + e^{-2x} \rightarrow ①$

$y' + 2y = 2e^x \rightarrow ②$

$y' = \frac{2}{3}e^x - 2e^{-2x} \rightarrow ③$

③ & ① in ②'s LHS:

$$\frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x}$$

$$= \frac{2}{3}e^x + \frac{4}{3}e^x = \frac{6e^x}{3} = 2e^x = RHS$$

② $t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$

This is non separable.
∴ taking the constant to 0.

$$t\frac{dy}{dt} = y + 0$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{t}\,dt$$

$$e^{\ln y} = e^{\ln t + c} \quad ; \quad c \in R$$

$$y = e^{\ln t} \cdot e^{c}$$

$$y = t \cdot k \quad ; \quad k \in R$$

$y_t = t \cdot k(t) \to ①$ varying the constant
where $y_t = y$ trial

$y_t' = t \cdot k'(t) + k(t) \cdot 1 \longrightarrow ②$

① & ② in q

$$t\,\frac{dy}{dt} = y + t^2 \sin t$$

$$t^2 k'(t) + t\,k(t) = t\,k(t) + t^2 \sin t$$

$$t^2 k'(t) = t^2 \sin t$$

$$k'(t) = \sin t$$

$k(t) = \int \sin t = -\cos t + c \rightarrow$ y_t's $k(t)$

$$\therefore \quad y_t = (-\cos t + c)\,t$$

$\boxed{y = (c - \cos t)\,t}$ $\rightarrow$ is one solution from y_{trial}

This was done for practice. But we are already given one solution in the question. We can use that one & plug it to the question.

$$t\frac{dy}{dt} = y + t^2 \sin t \qquad y(\pi) = 0$$

& one soln: $y = -t\cos t - t \rightarrow (1)$

$$\therefore \quad \frac{dy}{dt} = -t(-\sin t) - \cos t \cdot 1 - 1$$

$$\frac{dy}{dt} = t\sin t - \cos t - 1 \rightarrow (2)$$

(1) & (2) in q's

$$t^2 \sin t - t\cos t - t = -t\cos t - t + t^2 \sin t$$

we can see LHS = RHS $\therefore$ it is indeed a solution.

Finally we are given $y(\pi) = 0$

we substitute $t = \pi$ in the solution

$\therefore$

3. (a) For what values of r does the function $y = e^{rx}$ satisfy the differential equation $2y'' + y' - y = 0$?

(b) If r_1 and r_2 are the values of r that you found in part (a), show that every member of the family of functions $y = ae^{r_1 x} + be^{r_2 x}$ is also a solution.

4. (a) For what values of k does the function $y = \cos kt$ satisfy the differential equation $4y'' = -25y$?

(b) For those values of k, verify that every member of the family of functions $y = A \sin kt + B \cos kt$ is also a solution.

5. Which of the following functions are solutions of the differential equation $y'' + y = \sin x$?

(a) $y = \sin x$ (b) $y = \cos x$

(c) $y = \frac{1}{2}x \sin x$ (d) $y = -\frac{1}{2}x \cos x$

3- a) we have $y = e^{rx}$

$y' = r e^{rx}$ $y'' = r \cdot r e^{rx} + e^{rx} \cdot 0$
$= r^2 e^{rx}$

$2y'' + y' - y = 0$

$2(r^2 e^{rx}) + re^{rx} - e^{rx} = 0$

$2r^2 e^{rx} + re^{rx} - e^{rx} = 0$

$e^{rx}(2r^2 + r - 1) = 0$

$e^{rx} = 0$ $r_{1,2} = \dfrac{-1 \pm \sqrt{1 - 4(-2)}}{4}$

$e^{rx} = e^0$

$rx = 0$ $r_{1,2} = \dfrac{-1 \pm \sqrt{9}}{4} = \dfrac{-1 \pm 3}{4} = \dfrac{-1+3}{4}$ or $\dfrac{-1-3}{4}$

$r = 0$ $r_{1,2} = \dfrac{2}{4} = \dfrac{1}{2}$ or -1

given: $y = ae^{r_1 x} + be^{r_2 x}$

we have $y = ae^{\frac{1}{2}x} + be^{-x} \longrightarrow (1)$

$$\therefore \quad y' = a\cdot\left(e^{\frac{1}{2}x}\cdot\frac{1}{2}\right) + e^{1/2x}\cdot 0 + b\left(e^{-x}\cdot(-1)\right) + e^{-x}\cdot 0$$

$$= a\cdot\frac{1}{2}e^{1/2x} - b\,e^{-x} \longrightarrow 2$$

$$y'' = \left[\frac{a}{2}\cdot\left(\frac{1}{2}\cdot e^{1/2x}\right) + e^{1/2x}\cdot 0\right] - \left[b\left(-1e^{-x}\right) + e^{-x}\cdot 0\right]$$

$$= \frac{a}{4}e^{1/2x} + b\,e^{-x} \longrightarrow 3$$

1, 2 & 3 in q's LHS:

$$2\left[\frac{a}{4}e^{1/2x} + be^{-x}\right] + \left[a\frac{1}{2}e^{1/2x} - b\,e^{-x}\right] - \left[ae^{1/2x} + be^{-x}\right]$$

$$\frac{a}{2}e^{1/2x} + 2be^{-x} + \frac{a}{2}e^{1/2x} - be^{-x} - ae^{1/2x} - be^{-x}$$

$$= ae^{1/2x} + 2be^{-x} - 2be^{-x} - ae^{1/2x} = 0 = \text{RHS}$$

$$\therefore \quad y = a_1e^{r_1x} + b\,e^{r_2x}$$

family of functions is a solution to the diff eqn

4)a) $4y'' = -25y$

$y = \cos kt$

$4y'' = -25y$ we have $y = \cos kt$

$\therefore \; y' = k \cdot (-\sin kt) = -k\sin kt$

$y'' = -k\cos kt - \sin kt(0)$

$= -k\cos kt$

$\therefore \; 4(-k\cos kt) = -25(\cos kt)$

$k\cos kt = \frac{25}{4}\cos kt$

$$\boxed{k = \frac{25}{4}}$$

$y = A\sin kt + B\cos kt$

$y' = A.k.\cos kt + 0 + B(-\sin kt.k) + 0$

$= A.k.\cos kt - B.k.\sin kt$

$y'' = Ak^2(-\sin kt) - Bk^2(\cos kt)$

$= -Ak^2\sin kt - Bk^2\cos kt$

when $k = \frac{25}{4} \Rightarrow$ $-A\frac{25^2}{4^2}\sin\frac{25}{4}t - B\frac{25^2}{4^2}\cos\frac{25}{4}t$

$=$

we have $4y'' = -25y$

LHS:

$$4\left(-A\frac{25^2}{4^2}\sin\frac{25}{4}t - B\frac{25^2}{4^2}\cos\frac{25}{4}t\right)$$

$$= -A\frac{25^2}{4}\sin kt - B\frac{25^2}{4}\cos 25\,kt$$

$$= -25\left(\frac{25}{4}A\sin kt - \frac{25}{4}B\cos kt\right)$$

$$= -25y = RHS$$

Hence it is also a family of solutions

5) $y'' + y = \sin x$

a) $y = \sin x \longrightarrow y' = \cos x \quad y'' = -\sin x$

LHS: $-\sin x + \cos x \neq RHS$ Hence not a solution

b) $y = \cos x \longrightarrow y' = -\sin x \quad y'' = -\cos x$

LHS : $-\cos x + \cos x \neq RHS$ Hence not a solution

c) $y = \frac{1}{2}x\sin x \longrightarrow y' = \frac{1}{2}x\cos x + \sin x \cdot \frac{1}{2}$

$y'' = \frac{1}{2}x(-\sin x) + \cos x \cdot \frac{1}{2} +$

$\sin x \cdot 0 + \frac{1}{2}\cos x$

$$y''+y = -\frac{1}{2}x\sin x + \frac{1}{2}\cos x + \frac{1}{2}\cos x + \frac{1}{2}x\sin x$$

$$= \cos x \neq RHS$$

Hence it is not a solution.

d) $y = -\frac{1}{2}x\cos x$

$$y' = -\frac{1}{2}x(-\sin x) + \cos x\left(-\frac{1}{2}\right)$$

$$= \frac{1}{2}x\sin x - \frac{1}{2}\cos x$$

$$y'' = \frac{1}{2}x\cos x + \sin x\cdot\frac{1}{2} - \left(\frac{1}{2}(-\sin x) + \cos x\cdot 0\right)$$

$$= \frac{1}{2}x\cos x + \frac{1}{2}\sin x + \frac{1}{2}\sin x$$

$$y''+y = -\frac{1}{2}x\cos x + \frac{1}{2}x\cos x + \underbrace{\frac{1}{2}\sin x + \frac{1}{2}\sin x}_{\sin x}$$

$$y''+y = \sin x = RHS$$

Hence it is actually a solution.

7. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?

(b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).

(c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?

(d) Find a solution of the initial-value problem

$$y' = -y^2 \qquad y(0) = 0.5$$

b) we have $y' = -y^2$

$$\int \frac{1}{y^2}\,dy = \int -1\,dx$$

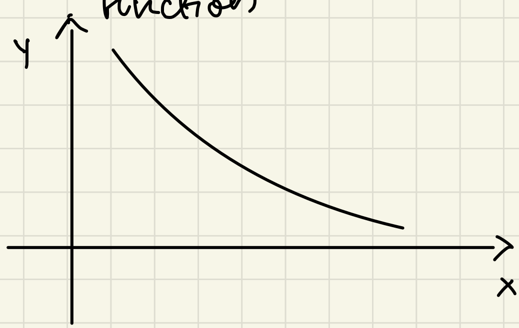
$$\frac{y^{-2+1}}{-2+1} = -x + C$$

$$\frac{y^{-1}}{-1} = -x + C$$

$$-\frac{1}{y} = -x + C$$

$$y = \frac{1}{x+C}$$

a) It is a decreasing function



c) e^x

d) given $y(0) = 0.5$

when $x = 0$ $y = 0.5$

$$\therefore \quad y = \frac{1}{x+C} \iff 0.5 = \frac{1}{0+C} \Rightarrow C = \frac{1}{0.5} = 2$$

$$y = \frac{1}{x+2} \quad \text{at } y(0) = 0.5$$

1^{st} ODE (separable)

1) $y' = y^2 \sin x$

$$\frac{dy}{dx} = y^2 \sin x$$

$$\frac{1}{y^2} dy = \sin x dx$$

$$= \int y^{-2} dy = \int \sin x \, dx$$

$$= \frac{y^{-2+1}}{-2+1} = -\cos x + C \qquad C \in R$$

$$\frac{1}{-y} = \cos x + 3c$$

$$\boxed{\begin{array}{l} y = \dfrac{1}{\cos x + 3c} = \dfrac{1}{\cos x + k} \quad \text{is a soln} \\ y = \gamma \quad \text{is another soln} \; ; \; \gamma \in R \end{array}}$$

I^{st} ODE (Non-separable)

2) $y' - \frac{1}{2x} y = 6x$

This is non separable

$\therefore$ varying the const to 0

$$y' - \frac{1}{2x} y = 0$$

$$\frac{dy}{dx} = \frac{1}{2x} \cdot y$$

$$\int \frac{1}{y} dy = \int \frac{1}{2x} dx$$

$$\ln y = \frac{1}{2} \ln x + C$$

$$e^{\ln y} = e^{\ln x^{1/2} + C} \quad ; C \in R$$

$$y = e^{\ln x^{1/2}} \cdot e^{C}$$

$$y = x^{1/2} \cdot K \quad ; \quad K \in R$$

$$\boxed{y_{trial} = k(x) \cdot x^{1/2}} \rightarrow (1)$$

$$y'_{trial} = k(x) \cdot \frac{1}{2} x^{-1/2} + x^{1/2} \cdot k'(x) \rightarrow (2)$$

(1) & (2) in q

$$y' - \frac{1}{2x} y = 6x \longrightarrow q$$

$$\underbrace{K(x)\frac{1}{2}x^{-1/2}} + x^{1/2}\cdot K'(x) - \underbrace{\frac{1}{2x}K(x)\cdot x^{1/2}} = 6x$$

$$x^{1/2}\cdot K'(x) = 6x$$

$$K'(x) = \frac{6x}{x^{1/2}}$$

$$K(x) = \int 6x^{1/2}$$

$$= 6\frac{x^{3/2}}{3/2} + C$$

$$\boxed{= 4x^{3/2} + C \quad , \ C \in \mathbb{R}}$$

∴ plugging $K(x)$ back into y_{trial}

we have:

$$\boxed{y = \left(4x^{3/2} + C\right)x^{1/2} \ ; \ C, K \in \mathbb{R}}$$

eg:) $y' = \dfrac{\cos x}{\sin 2y}$

$$\int \sin 2y \, dy = \int \cos x \, dx$$

$$\frac{1}{2}(-\cos 2y) = \sin x + c$$

$$-\frac{1}{2}\cos 2y = \sin x + C$$

$$\cos 2y = -2\sin x - 2c$$

$$\cos 2y = -2\sin x + k \qquad c, k \in R$$

$$y = \frac{\cos^{-1}(-2\sin x + k)}{2}$$

$$y = \frac{1}{2} \cdot \cos^{-1}(k - 2\sin x) \quad ; \text{ where } k = -2c$$

1. $\dfrac{dy}{dx} = \dfrac{e^{-x}}{y}$

$$\int y \, dy = \int e^{-x} dx$$

$$\frac{y^2}{2} = -e^{-x} + C \quad ; c \in R$$

$$y^2 = -2e^{-x} + 2c$$

$$y = \pm\sqrt{k - 2e^{-x}} \; ; \; k \in R$$

2. $y(0) = 1$

$$\int e^y dy = \int 3x^2 dx$$

$$e^y = \frac{3x^3}{3} + c \qquad c \in R$$

$$\ln e^y = \ln(x^3 + c)$$

given $y(0) = 1$

$\therefore \; c = e$

$$y = \ln(x^3 + e)$$

1) $$\frac{dy}{dx} = 3x^2y^2$$

$$\int \frac{1}{y^2}\,dy = \int 3x^2\,dx$$

$$-\frac{1}{y} = 3\frac{x^3}{3} + C \quad ; \; C \in R$$

$$y = \frac{1}{x^3 + C}$$

2) $$\frac{dy}{dx} = x\sqrt{y}$$

$$\frac{1}{\sqrt{y}}\,dy = x\,dx$$

$$\frac{y^{-1/2+1}}{-\frac{1}{2}+1} = \frac{x^2}{2} + C \quad ; \; C \in R$$

$$2\sqrt{y} = \frac{x^2 + 2C}{2}$$

$$y = \left(\frac{x^2 + K}{4}\right)^2 \quad ; \; K \in R$$

3) $$xyy' = x^2 + 1$$

$$yy' = x + \frac{1}{x}$$

$$\int y\,dy = \int \left(x + \frac{1}{x}\right)dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \ln x + C \quad ; \ C \in R$$

$$y^2 = x^2 + 2\ln x + 2C$$

$$y = \sqrt{x^2 + 2\ln x + K} \quad ; \ K \in R$$

4) $y' + x e^y = 0$

$$y' = -x e^y$$

$$\frac{1}{e^y} dy = -x \, dx$$

$$\int e^{-y} dy = \int -x \, dx$$

$$-e^{-y} = -\frac{x^2}{2} + C \quad ; \ C \in R$$

$$e^{-y} = \frac{x^2 + 2C}{2}$$

$$\ln e^{-y} = \ln\left(\frac{x^2 + 2C}{2}\right)$$

$$-y = \ln(x^2 + 2C) - \ln|2|$$

$$y = \ln|2| - \ln|x^2 + 2C|$$

5) $(e^y - 1) y' = 2 + \cos x$

$$\int (e^y - 1) \, dy = \int 2 + \cos x \, dx$$

$$e^y - y = 2x + \sin x + C$$

6) $\dfrac{du}{dt} = \dfrac{1+t^4}{ut^2+u^4t^2}$

$$\frac{du}{dt} = \frac{1+t^4}{t^2(u+u^4)}$$

$$\int (u+u^4)\,du = \int \left(\frac{1+t^4}{t^2}\right) dt$$

$$\frac{u^2}{2} + \frac{u^5}{5} = \frac{t^{-2+1}}{-2+1} + \frac{t^3}{3} + c \quad ; \ c \in R$$

$$\frac{5u^2+2u^5}{10} = \frac{t^{-1}}{-1} + \frac{t^3}{3} + c \quad ; \ c \in R$$

$$\frac{5u^2+2u^5}{10} = -\frac{1}{t} + \frac{t^3}{3} + c \quad ; \ c \in R$$

7) $\dfrac{d\theta}{dt} = \dfrac{t \sec\theta}{\theta e^{t^2}}$

$$\frac{\theta}{\sec\theta}\, d\theta = \frac{t}{e^{t^2}}\, dt$$

$$\int \theta \cos\theta \, d\theta = \int \frac{t}{e^{t^2}} dt$$

$$\int x \cos x \, dx = \int \frac{t}{e^{t^2}} dt$$

$t^2 = u$

$2t\,dt = du$

$t\,dt = \frac{1}{2} du$

LHS:

$u = x \quad vdv = \cos x\,dx$

$du = dx \quad v = \sin x$

$= x\sin x - \int \sin x \, dx$

$= x\sin x - (-\cos x) + C$

$= x\sin x + \cos x + C$

RHS:

$$\int \frac{1}{2} \cdot \frac{1}{e^u} du$$

$$= \frac{1}{2} \int e^{-u} du$$

$$= \frac{1}{2} \cdot -e^{-u} + C$$

$$= -\frac{1}{2} e^{t^2} + C$$

$\therefore$ we have

$$x \sin x + \cos x = -\frac{1}{2} e^{t^2} + C \; ; \; C \in R$$

$$\theta \sin\theta + \cos\theta = C - \frac{1}{2} e^{t^2} \; ; \; C \in R$$

⑧ $$\frac{dH}{dR} = \frac{RH^2\sqrt{1+R^2}}{\ln H}$$

$$\int \frac{\ln H}{H^2}\, dH = \int \left(R\sqrt{1+R^2}\right) dR$$

$\frac{1}{H} = u$ (LHS) $\qquad 1+R^2 = u$ (RHS)

$2R\,dR = du$

$R\,dR = \frac{1}{2}\,du$

$\ln H\,dH = du$

$$\int \frac{1}{u^2}\,du = \int \frac{1}{2}\sqrt{u}\,du$$

$$\frac{u^{-2+1}}{-2+1} = \frac{1}{2}\,\frac{u^{1/2+1}}{1/2+1} + c$$

$$\frac{u^{-1}}{-1} = \frac{1}{2}\,\frac{u^{3/2}}{3/2} + c$$

$$-\frac{1}{u} + c = \frac{1}{3}\,u^{3/2} + c$$

$$-H + C = \frac{1}{3}\left(1+R^2\right)^{3/2} + C$$

$$-H = \frac{1}{3}\sqrt{\left(1+R^2\right)^3} + C$$

$$-3H = \left(1+R^2\right)^{3/2} + 3C \quad ; \; C, K \in \mathbb{R}$$

$$3H = -\left(1+R^2\right)^{3/2} + K \quad ; \; C, K \in \mathbb{R}$$

(9) $$\frac{dp}{dt} = t^2 p - p + t^2 - 1$$

$$\frac{dp}{dt} = p(t^2-1) + t^2 - 1$$

$$\frac{dp}{dt} = (t^2-1)\left[p + 1\right]$$

$$\frac{1}{p+1}\, dp = (t^2-1)\, dt$$

$$e^{\ln|p+1|} = e^{\frac{t^3}{3} - t + c} \quad ; \; c \in R$$

$$p+1 = e^{\frac{t^3-3t}{3} + c} \quad ; \; c \in R$$

$$p+1 = e^{\frac{t^3-3t}{3}} \cdot e^{c} \quad ; \; c \in R$$

$$p = e^{\frac{t^3-3t}{3}} \cdot k - 1 \quad ; \; c \in R$$

(10) $$\frac{dz}{dt} + e^{t+z} = 0$$

$$\frac{dz}{dt} = -e^{t+z} = -e^{t} \cdot e^{z}$$

$$\frac{1}{e^{z}}\, dz = -e^{t}\, dt$$

$$\int e^{-z}\,dz = \int -e^{t}\,dt$$

$$= -e^{-z} = -e^{t} + c \ ; \ c \in R$$

$$e^{-z} - e^{t} + c = 0$$

11. $$\frac{dy}{dx} = x e^{y} \ ; \ y(0) = 0$$

$$\int \frac{1}{e^{y}}\,dy = \int x\,dx$$

$$-e^{-y} = \frac{x^2}{2} + c$$

for $y(0)=0$; $x=0$ then $y=0$

$$-\frac{1}{e^{0}} = \frac{0^2}{2} + c$$

$$-1 = c$$

$$c = -1$$

$$-\frac{1}{e^{y}} = \frac{x^2}{2} - 1$$

$$-\frac{1}{e^{y}} = \frac{x^2 - 2}{2}$$

$$\frac{-2}{x^2-2} = e^{y}$$

$$\ln e^{y} = \ln\left|\frac{-2}{x^2-2}\right|$$

$$y = \ln|2| - \ln|x^2-2|$$

(12) $$\frac{dy}{dx} = \frac{x\sin x}{y}$$

$u = x \quad dv = \sin x$

$du = dx \quad v = -\cos x$

$$\int y\,dy = \int x\sin x\,dx$$

$$\frac{y^2}{2} = -x\cos x + \int \cos x\,dx + c$$

$$\frac{y^2}{2} = -x\cos x + \sin x + C$$

$$y^2 = 2(\sin x - x\cos x + c)$$

$y(0) = -1$

when $x = 0$

$y = -1$

$$\therefore\ 1 = 2(0 - 0 + c)$$

$$1 = 2C$$

$$c = \frac{1}{2}$$

$$\frac{y^2}{2} = -x\cos x + \sin x + \frac{1}{2}$$

$$y = \sqrt{2\sin x - 2x\cos x + 1}$$

(13) $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$ $u(0) = -5$

$\int 2u\,du = \int 2t + \sec^2 t \; dt$

$2\frac{u^2}{2} = 2\frac{t^2}{2} + \int \sec^2 t \, dt + c$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\therefore \int \sec^2 x = \tan x + C$

$u^2 = t^2 + \tan t + C$; $u(0) = -5$

$t = 0 ; u = -5$

$25 = 0 + \tan 0 + C$

$c = 25$

$\therefore \; u^2 = t^2 + \tan t + 25$

(14) $x + 3y^2 \sqrt{x^2 + 1} \frac{dy}{dx} = 0$

$3y^2 \sqrt{x^2 + 1} \, dy = -x \, dx$

$3y^2 dy = \frac{-x}{\sqrt{x^2 + 1}} dx$

$\int y^2 dy = -\frac{1}{3} \int \frac{x}{\sqrt{x^2+1}} dx$

$x^2 + 1 = u$

$2x\,dx = du$

$x\,dx = \frac{1}{2} du$

$$\frac{y^3}{3} = -\frac{1}{3}\int \frac{1}{2}\frac{1}{\sqrt{u}}\,du$$

$$y^3 = -\frac{1}{2}\frac{u^{-1/2+1}}{-1/2+1} + C$$

$$-2y^3 = 2u^{1/2} + C \qquad \text{given } y(0)=1$$

$$-2y^3 = 2(x^2+1)^{1/2} + C \qquad x=0;\ y=1$$

$$-2 = 2(0+1)^{1/2} + C$$

$$C = -1$$

$$\therefore\ -2y^3 = 2(x^2+1)^{1/2} - 1$$

$$y^3 = \frac{1}{2} - \sqrt{x^2+1}$$

? check later.

15 $x\ln x = y\left(1+\sqrt{3+y^2}\right)y'$; $y(1)=1$

$$\int x\ln x\,dx = \int\left(y + y\sqrt{3+y^2}\right)dy$$

$3+y^2 = u$

$2y\,dy = du$

$u = \ln x \qquad v\,dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{x^2}{2}$

$$\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx$$

$$\frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + C$$

$$\frac{y^2}{2} + \int \frac{1}{2}\sqrt{u}\,du$$

$$\frac{y^2}{2} + \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$\frac{x^2}{2}\ln x - \frac{x^2}{4} + C = \frac{y^2}{2} + \frac{1}{3}\cdot(3+y^2)^{3/2} + C$$

$$\frac{x^2}{2}\ln x - \frac{x^2}{4} = \frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3} + C$$

given $y(1) = 1$

$x = 1$; $y = 1$

$$\frac{1}{2}\cdot 0 - \frac{1}{4} = \frac{1}{2} + \frac{3^{3/2}}{3} + C$$

$$-\frac{1}{4} = \frac{1}{2} + \frac{9\cdot\sqrt{3}}{3} + C$$

$$C = -\frac{1}{4} - \frac{1}{2} - \frac{9\sqrt{3}}{3}$$

$$= \frac{-2-4}{8} - \frac{9\sqrt{3}}{3}$$

$$= \frac{-6(3) - 72\sqrt{3}}{24}$$

$$= \frac{-18 - 72\sqrt{3}}{24}$$

check later on Wolfram.

(16) $\frac{dP}{dt} = \sqrt{Pt}$, $P(1) = 2$

$$dP = \sqrt{P} \cdot \sqrt{t}\, dt$$

$$\int \frac{1}{\sqrt{P}}\, dP = \int \sqrt{t}\, dt$$

$$\frac{P^{-1/2+1}}{-\frac{1}{2}+1} = \frac{t^{3/2}}{3/2} + C$$

$$2P^{1/2} = \frac{2}{3} t^{3/2} + C \qquad P(1) = 2$$

$t = 1$; $P = 2$

$$2\sqrt{2} = \frac{2}{3} \cdot 1 + C$$

$$2\sqrt{2} - \frac{2}{3} = C$$

$$C = \frac{6\sqrt{2} - 2}{3}$$

$$2\sqrt{P} = \frac{2}{3}\sqrt{t^3} + \frac{6\sqrt{2} - 2}{3}$$

$$2\sqrt{P} = \frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{3}$$

$$P = \left(\frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{6}\right)^2$$

(17) $y' \tan x = a + y \qquad y(\pi/3) = a$

$0 < x < \pi/2$

$$\int \frac{1}{a+y}\,dy = \int \frac{1}{\tan x}\,dx$$

$x = \frac{\pi}{3}$; $y = a$

RHS: $\int \frac{1}{\tan x} \Rightarrow \int \frac{\sec^2 x\,dx}{\sec^2 x \tan x}$

$\sec x \tan x = u$

$\sec^2 x\,dx = du$

(11) $\frac{dL}{dt} = kL^2 \ln t \quad , \quad L(1) = -1$

$$\frac{1}{L^2}\, dL = k \ln t \, dt$$

$$\frac{L^{-2+1}}{-2+1} = k \int \ln t \, dt$$

$\ln t = u$

$\frac{1}{t}\, dt = du$

$e^{\ln t} = e^u$

$t = e^u$

$$= k \; \frac{1}{t}\, u \, du$$

$$= k \int \frac{u \, du}{e^u}$$

$$= k \int u \cdot e^{-u} du$$

$$\approx k \int x e^{-x} dx$$

$u = x \quad v dv = e^{-x} dx$

$du = dx \quad v = -e^{-x}$

$$= k \left[-u e^{-u} + \int e^u du \right]$$

$$= k \left[e^u - \frac{u}{e^u} \right]$$

$$= k \left[e^{\ln t} - \frac{\ln t}{e^{\ln t}} \right] = k \left[t - \frac{\ln(t)}{t} \right]$$

$$-\frac{1}{L} = k\left[t - \frac{\ln t}{t}\right] + C \quad ; \quad C \in R$$

given $L(1) = -1$

$$t = 1 \; ; \; L = -1$$

$$-\frac{1}{-1} = k\left[1 - \frac{0}{1}\right] + C$$

$$1 = k + C$$

$$\boxed{C = 1 - k}$$

$$-\frac{1}{L} = k\left[t - \frac{\ln t}{t}\right] + 1 - k$$

$$-\frac{1}{L} = k\left[\frac{t^2 - \ln t}{t}\right] + 1 - k$$

$$-\frac{1}{L} = \frac{kt^2 - k\ln t + t - kt}{t}$$

$$L = -\frac{t}{kt^2 - k\ln t + t - kt}$$

$$f'(x) = x f(x) - x \qquad f(0) = 2$$

$$y' = xy - x$$

$$y' = x(y-1)$$

$$\int \frac{1}{y-1}\, dy = \int x \, dx$$

$$\ln|y-1| = \frac{x^2}{2} + c$$

given $f(0) = 2$
$x = 0$

$$y' = -3x^2 y + 6x^2 \to 0$$

$$\frac{dy}{dx} = -3x^2 y$$

$$\int \frac{1}{y}\, dy = \int -3x^2\, dx$$

$$\ln|y| = -3\frac{x^3}{3} + C$$

$$e^{\ln|y|} = e^{-x^3 + C}$$

$$y = e^{-x^3} \cdot e^{C}$$

$$y = e^{-x^3} \cdot K$$

vary the k to constant

$$y = e^{-x^3} k(x)$$

$$y' = e^{-x^3} k'(x) + k(x) - e^{-x^3} \cdot 3x^2$$

plugging back.

$$e^{-x^3} k'(x) - 3x^2 e^{-x^3} k(x) + 3x^2\left(e^{-x^3} k(x)\right) = 6x^2$$

$$e^{-x^3} k'(x) = 6x^2$$

$$k'(x) = \frac{6x^2}{e^{-x^3}}$$

$$k'(x) = 6x^2 e^{x^3}$$

$$k(x) = \int 6x^2 e^{x^3} dx = 2\int 3x^2 e^{x^3} dx$$

$$x^3 = u$$
$$3x^2 dx = du$$

$$= 2\int e^u du$$
$$= 2e^u + C$$
$$= 2e^{x^3} + C$$

$$\boxed{\therefore \quad y = e^{-x^3}\left(2e^{x^3} + C\right)}$$

$$y = 2 + e^{-x^3} \cdot C$$

Now trying seperable

$$y' = -3x^2 y + 6x^2$$
$$y' = -3x^2(y+2)$$

$$\int \frac{1}{y+2}\,dy = \int -3x^2\,dx$$

$$\ln|y+2| = -3\frac{x^3}{3} + c$$

$$e^{\ln|y+2|} = e^{-x^3+c}$$

$$y+2 = e^{-x^3} \cdot e^{c}$$

$$y = e^{-x^3} \cdot K + 2$$

9.3 Exercises

1–10 Solve the differential equation.

1. $\dfrac{dy}{dx} = xy^2$

2. $\dfrac{dy}{dx} = xe^{-y}$

3. $xy^2y' = x + 1$

4. $(y^2 + xy^2)y' = 1$

5. $(y + \sin y)y' = x + x^3$

6. $\dfrac{dv}{ds} = \dfrac{s+1}{sv+s}$

7. $\dfrac{dy}{dt} = \dfrac{t}{ye^{y+t^2}}$

8. $\dfrac{dy}{d\theta} = \dfrac{e^y \sin^2\theta}{y \sec\theta}$

9. $\dfrac{dp}{dt} = t^2p - p + t^2 - 1$

10. $\dfrac{dz}{dt} + e^{t+z} = 0$

11–18 Find the solution of the differential equation that satisfies the given initial condition.

11. $\dfrac{dy}{dx} = \dfrac{x}{y}, \quad y(0) = -3$

12. $\dfrac{dy}{dx} = \dfrac{\ln x}{xy}, \quad y(1) = 2$

13. $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

14. $y' = \dfrac{xy \sin x}{y+1}, \quad y(0) = 1$

15. $x \ln x = y\left(1 + \sqrt{3 + y^2}\right)y', \quad y(1) = 1$

16. $\dfrac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

17. $y' \tan x = a + y, \quad y(\pi/3) = a, \quad 0 < x < \pi/2$

18. $\dfrac{dL}{dt} = kL^2 \ln t, \quad L(1) = -1$

9.3 1) $\frac{dy}{dx} = xy^2$

$y' = xy^2$ This is seperable

$\int \frac{1}{y^2} dy = \int x \, dx$

$\frac{y^{-2+1}}{-2+1} = \frac{x^2}{2} + C$

$-\frac{1}{y} = \frac{x^2}{2} + C$

$y = \frac{1}{-\left(\frac{x^2}{2} + C\right)} = \frac{1}{-\frac{x^2}{2} - C}$; $C \in R$

other solution $y = \gamma$; $y' = 0$; $\gamma \in R$

$\therefore 0 = x\gamma^2$; $\gamma = 0$

2) $\frac{dy}{dx} = xe^{-y}$

$\int e^y dy = \int x \, dx$

$e^y = \frac{x^2}{2} + C$

$\ln e^y = \ln\left(\frac{x^2}{2} + C\right)$

$y = \ln\left(\frac{x^2}{2} + C\right)$; $C \in R$

For the other solution:

$y = \gamma \quad ; \quad y' = 0 \quad$ where $\gamma \in \mathbb{R}$

3) $x y^2 y' = x + 1$

This is I^{st} order Seperable:

$$\int y^2 \, dy = \int \frac{x+1}{x} \, dx$$

$$\frac{y^3}{3} = \int \left(1 + \frac{1}{x}\right) dx$$

$$= x + \ln|x| + c$$

$$y = \sqrt{3x + 3\ln|x| + 3c} \quad ; \quad c \in \mathbb{R}$$

& for other solution ; $y = \gamma$; $\gamma \in \mathbb{R}$

$\therefore \ y' = 0$

4) $(y^2 + x y^2) y' = 1$

This is sepeable

$$y^2 (1 + x^2) y' = 1$$

$$\int y^2 \, dy = \int \frac{1}{1+x} \, dx$$ if it was $\int \frac{1}{1+x^2} dx$

$$\frac{y^3}{3} = \ln|1+x| \qquad = \tan^{-1}(x) + C$$

$$y = \sqrt[3]{3\ln|1+x| + 3c} \quad ; \quad c \in R \quad \& \; y = \gamma ; \gamma \in R \text{ soln.}$$

5) $(y + \sin y)\, y' = x + x^3$

$$\int (y + \sin y)\, dy = \int (x + x^3)\, dx$$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$$\frac{y^2}{2} - \cos y = \frac{x^2}{2} + \frac{x^4}{4} + c$$

$$y^2 - 2\cos y = x^2 + \frac{x^4}{2} + 2c \quad ; \quad c \in R$$

cannot be expressed in terms of elementary function. The soln is an implicit soln.

6) $\frac{dv}{ds} = \frac{s+1}{sv + s}$

$$(v+1)\, dv = \frac{s+1}{s}\, ds$$

$$\int (v+1)\, dv = \int \left(1 + \frac{1}{s}\right) ds$$

$$\frac{v^2}{2} + v = s + \ln|s| + c \quad ; \quad c \in R$$

$$v^2 + 2v = 2s + 2\ln|s| + 2c \; ; \; c \in R$$

& $v = \gamma$ is also another soln; $\gamma \in R$

7) $\dfrac{dy}{dt} = \dfrac{t}{ye^{y+t^2}}$

$$y e^{y} \cdot e^{t^2} dy = t\, dt$$

$$ye^y dy = \frac{t}{e^{t^2}} dt$$

$$\int y \cdot e^y \, dy = \int t \cdot \frac{1}{e^{t^2}} dt$$

$u = y \qquad dv = e^y dy$

$du = dy \qquad v = e^y$

$$ye^y - \int e^y dy$$

$$= ye^y - e^y + C$$

$t^2 = u$

$2t\, dt = du$

$t\, dt = \frac{1}{2} du$

$$\int \frac{1}{2} \cdot \frac{1}{e^u} du$$

$$= \frac{1}{2} \int e^{-u} du$$

$$= -\frac{1}{2} \cdot e^{-u} + C$$

$$= -\frac{1}{2} e^{-t^2} + C$$

$$\therefore \quad ye^y - e^y = -\frac{1}{2} e^{-t^2} + C \quad ; \quad C \in \mathbb{R}$$

$y = \gamma$ is another soln ; $\gamma \in \mathbb{R}$

8. $$\frac{dy}{d\theta} = \frac{e^y \sin^2\theta}{y \sec\theta}$$

$$\frac{y}{e^y}\,dy = \sin^2\theta \cos\theta \, d\theta$$

$$\int y \cdot e^{-y}\,dy = \int \sin^2\theta \cos\theta \, d\theta$$

$u = y \quad dv = e^{-y}dy$

$du = dy \quad v = -e^{-y}$

$\sin\theta = u$

$\cos\theta \, d\theta = du$

$-ye^{-y} - \int -e^{-y}dy$

$-ye^{-y} + e^{-y} + C$

$e^{-y}(1-y)$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \frac{\sin^3\theta}{3} + C$

$\therefore \quad e^{-y}(1-y) = \frac{\sin^3\theta}{3} + C$ is one solution

other solution is $y = \gamma$ & $c, \gamma \in R$

9) $$\frac{dp}{dt} = t^2p - p + t^2 - 1$$

$$\frac{dp}{dt} = p(t^2-1) + t^2 - 1$$

$$\frac{dp}{dt} = (t^2-1)\left[p+1\right]$$

$$\int \frac{1}{p+1} = \int (t^2-1)\, dt$$

$$e^{\ln|p+1|} = e^{\frac{t^3}{3} - t + c} \quad ; \quad c \in R$$

$$p+1 = e^{t^3/3 - t + c}$$

$$p = e^{t^3/3 - t} \cdot e^{c} \overset{-}{\cancel{+}} 1$$

$$p = e^{\frac{t^3-3t}{3}} \cdot K \overset{-}{\cancel{+}} 1 \quad ; \quad c, K \in R$$

is one solution

other solution is $p = \gamma$; $\gamma \in R$

10. $$\frac{dz}{dt} + e^{t+z} = 0$$

$$\frac{dz}{dt} = -e^{t+z} = -e^{t} \cdot e^{z}$$

$$\int \frac{1}{e^z}\, dz = \int -e^{t}\, dt$$

$$\int e^{-z}\, dz = -1 \int e^{t}\, dt$$

$$-e^{-z} = -e^{t} + C \quad ; \quad C \in R$$

$$e^{-z} = e^{t} + K \quad ; \quad K \in R$$

$$\ln e^{-z} = \ln(e^t + k)$$

$$-z = \ln(e^t + k)$$

$$z = -\ln(e^t + k) \quad ; \quad k \in R$$

other soln is $z = \gamma$; $\gamma \in R$

11) $\dfrac{dy}{dx} = \dfrac{x}{y}$; $y(0) = -3$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C \quad ; C \in R$$

$y(0) = -3$ $\quad x = 0 \quad y = -3$

$$\frac{9}{2} = 0 + C \quad ; \quad C = \frac{9}{2}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{9}{2}$$

$$y^2 = x^2 + 9$$

$$y = \pm\sqrt{x^2 + 9} \quad ; \quad y(0) = -3$$

since $y(x)$ is -ve we choose the negative square root

$$\therefore \quad y = -\sqrt{x^2 + 9} \quad ; \quad y(0) = -3$$

12) $\frac{dy}{dx} = \frac{\ln x}{xy}$, $y(1) = 2$

$\int y\,dy = \int \frac{\ln x}{x}\,dx$ $\qquad$ $\ln x = u$

$\frac{1}{x}\,dx = du$

$\frac{y^2}{2} = \int u\,du$

$\frac{y^2}{2} = \frac{u^2}{2} + c$

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + c$

$\frac{y^2}{2} = 2\frac{\ln x}{2} + c$ wrong!

$\boxed{\begin{array}{l} \ln a^n = n \log a \\ (\ln a)^n \neq n \log a \end{array}}$

~~Now $y(1) = 2$; $x = 1$ $y = 2$~~

~~$\frac{4}{2} = 2\frac{\ln 1}{2} + c$; $c \in R$~~

~~$c = 2$~~

~~$\therefore \frac{y^2}{2} = \ln x + 2$~~

~~$y^2 = 2\ln x + 4$~~

~~$y = \sqrt{2\ln x + 4}$~~

$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$

$\frac{y^2}{2} = \frac{\ln^2 x}{2} + C$

$x = 1 \qquad y = 2$

$C = 2$

$\therefore y = \sqrt{\ln^2 x + 4}$

13. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$; $u(0) = -5$

$\int 2u\,du = \int (2t + \sec^2 t)\,dt$; $u(0) = -5$

$2\frac{u^2}{2} = 2\frac{t^2}{2} + \tan t + c$; $c \in R$

$u^2 = t^2 + \tan t + C$

$t = 0 \; ; \quad u = -5$

$25 = 0 + 0 + C$

$C = 25$

$u^2 = t^2 + \tan t + 25$

$u = \pm\sqrt{t^2 + \tan t + 25}$

14. $y' = \dfrac{xy \sin x}{y+1} \quad , \quad y(0) = 1$

$f'(\sin x) \Rightarrow \cos x$

$f'(\cos x) \Rightarrow -\sin x$

$\int \dfrac{(y+1)}{y} \, dy = \int x \sin x \, dx$

$\int 1 + \dfrac{1}{y} \, dy = \int x \cdot \sin x \, dx$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x$

$-x\cos x + \int \cos x \, dx$

$y + \ln|y| = \sin x - x\cos x + C$

$y(0) = 1 \qquad x = 0 \; ; \; y = 1$

$1 + 0 = 0 - 0 + C$

$C = 1$

$\therefore \quad y + \ln|y| = \sin x - x\cos x + 1$

15. $x \ln x = y\left(1 + \sqrt{3+y^2}\right) y', \quad y(1) = 1$

$$\int \left(y + y\sqrt{3+y^2}\right) dy = \int x \ln x \, dx$$

$$\frac{y^2}{2} + \int y\sqrt{3+y^2}\, dy = \int x \ln x \, dx$$

$3+y^2 = u$
$2y\,dy = du$
$y\,dy = \frac{1}{2} du$

$u = \ln x \qquad v\,dv = x\,dx$
$du = \frac{1}{x} dx \qquad v = \frac{x^2}{2}$

$$\frac{y^2}{2} + \frac{1}{2} \cdot \frac{u^{1/2+1}}{3/2} + C = \frac{1}{2} x^2 \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \quad ; C \in R$$

$$\frac{y^2}{2} + \frac{\left(3+y^2\right)^{3/2}}{3} = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C_1 \quad ; C_1 \in R$$

given $y(1) = 1$

$$\frac{1}{2} + \frac{(4)^{3/2}}{3} = \frac{1}{2} \cdot 0 - \frac{1}{2} \cdot \frac{1}{2} + C_1$$

$$\frac{1}{2} + \frac{8}{3} = -\frac{1}{4} + C_1$$

$$C_1 = \frac{1}{2} + \frac{8}{3} + \frac{1}{4}$$

$$24 C_1 = 12 + 8 \cdot 8 + 6$$

$$C_1 = 82/24 = \frac{41}{12}$$

$$\frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + \frac{41}{12}$$

16. $\frac{dP}{dt} = \sqrt{Pt}$; $P(1) = 2$

$$\int \frac{1}{\sqrt{P}}\, dP = \int \sqrt{t}\, dt$$

$$\frac{P^{-1/2+1}}{-\frac{1}{2}+1} = \frac{t^{1/2+1}}{\frac{1}{2}+1} + C \quad ; \quad C \in R$$

$$2P^{1/2} = 2 \cdot \frac{t^{3/2}}{3} + C$$

$P(1)=2$ $t=1$; $P: 2$

$$2\sqrt{2} = \frac{2}{3} + C$$

$$C = 2\sqrt{2} - \frac{2}{3} = \frac{6\sqrt{2}-2}{3}$$

$$2\sqrt{P} = \frac{2\sqrt{t^3}}{3} + \frac{6\sqrt{2}-2}{3}$$

$$4P = \left(\frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{3}\right)^2$$

$$P = \frac{(2\sqrt{t^3} + 6\sqrt{2} - 2)^2}{36}$$

17. $y' \tan x = a + y$, $y(\pi/3) = a$; $0 < x < \pi/2$

$$\frac{dy}{dx} = \frac{a+y}{\tan x}$$

$$\int \frac{1}{a+y}\, dy = \int \frac{1}{\tan x}\, dx$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

Both did not work

$$\ln|a+y| = \int \sec x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$\ln|a+y| = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$$

recall the trick
Multiply & divide
by $(\sec x + \tan x)$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

$\sec x + \tan x = u$

$\sec x \tan x + \sec^2 x = du$

$$= \int \frac{1}{u}\, du$$

$$e^{\ln|a+y|} = e^{\ln|\sec x + \tan x| + c} \quad ; \quad c \in \mathbb{R}$$

$$a + y = (\sec x + \tan x) \cdot e^c$$

$y\left(\frac{\pi}{3}\right) = a$ ie $x = \frac{\pi}{3}$ $y = a$

$$2a = \left(\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right) \cdot e^c$$

$$2a = \left(\frac{1}{\cos 60^\circ} + \tan 60^\circ\right) \cdot e^c$$

$$\frac{\sqrt{3}}{2} \div \frac{1}{2}$$

$$2a = (2 + \sqrt{3})\, e^c$$

$$\frac{2a}{2+\sqrt{3}} = e^c \quad \therefore \quad c = \ln\left|\frac{2a}{2+\sqrt{3}}\right|$$

$$\therefore \quad y = (\sec x + \tan x) \cdot e^{\ln\left|\frac{2a}{2+\sqrt{3}}\right|} - a$$

$$y = \frac{(\sec x + \tan x) \cdot 2a}{2+\sqrt{3}} - a$$

18. $\frac{dL}{dt} = KL^2 \ln t \qquad L(1) = -1$

$$\frac{1}{KL^2} dL = \ln t \, dt$$

$$= \frac{1}{K} \cdot \int \frac{1}{L^2} dL = \int \ln t \, dt$$

LHS:

$$\frac{1}{K} \cdot \frac{L^{-2+1}}{-2+1} + C$$

$$= \frac{1}{K} \cdot \frac{L^{-1}}{-1} + C$$

$$= -\frac{1}{K} \frac{1}{L} + C$$

$$= -\frac{1}{KL} + C_1; \quad C_1 \in R$$

RHS:

$u = \ln t \qquad v\,dv = 1\,dt$

$du = \frac{1}{t} dt \qquad v = t$

$$t \ln t - \int 1\,dt$$

$$= t \ln t - t + C_2 ; \quad C_2 \in R$$

$$-\frac{1}{KL} = t \ln t - t + C_3 ; \quad C_3 \in R$$

given $L(1) = -1$ ie $t = 1$; $L = -1$

$$-\frac{1}{K(-1)} = 0 - 1 + C_3$$

$$\frac{1}{K} = -1 + c_3$$

$$c_3 = 1 + \frac{1}{K}$$

$\therefore$ we have:

$$-\frac{1}{KL} = t\ln t - t + 1 + \frac{1}{K}$$

$$-\frac{1}{KL} - \frac{1}{K} = t\ln t - t$$

$$-\frac{1}{K}\left(\frac{1}{L} + 1\right) = t(\ln t - 1)$$

$$\frac{1}{L} + 1 = -kt(\ln t - 1)$$

$$\frac{1}{L} = -kt(\ln t - 1) - 1$$

$$L = \frac{-1}{kt(\ln t - 1) + 1}$$

Math Café

Applied Mathematics Mock Exam

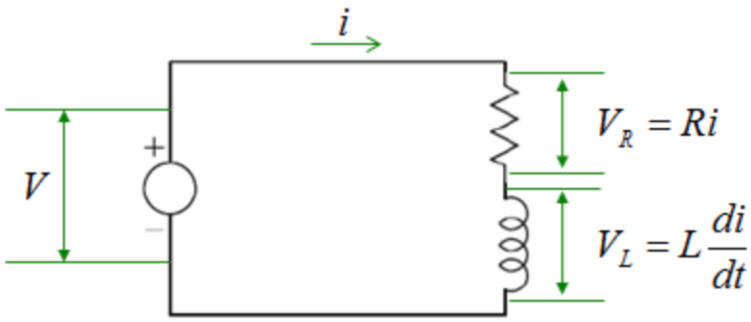
Why did the differential equation go to therapy? Because it had trouble separating variables.

1. Ordinary Differential equations

1. $\frac{dP}{dt} = t^2P - P + t^2 - 1$

2. $\frac{dy}{dx} = \frac{xysin(x)}{y+1} \qquad y(0) = 1$

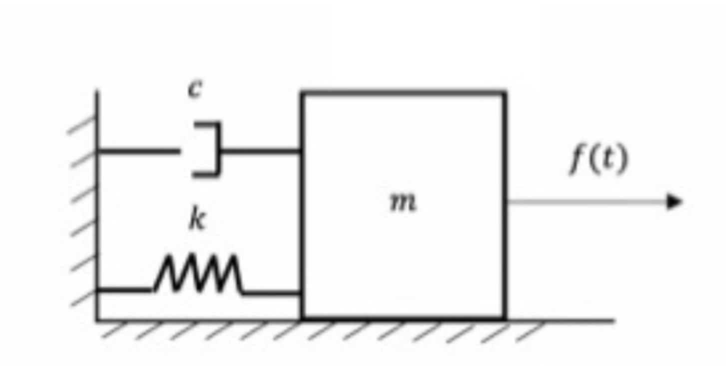
3. $t\ln(t)\frac{dr}{dt} = te^t - r$



4.

Find the function I(t) with respect to time given this circuit has a Resistance R = 12 Ω and Inductance of L = 4H. This circuit is connected to a DC power supply of 60V. Find the Function **I(t)** if I(0) = 0 and find the Maximum current in the circuit.

Hints: $V_{DC} = V_R + V_L$



5.

If for the spring mass damper shown below $f(t) = 100\sin(2t)$, find the equations that describes its path. Given that m = 2kg, k = 40N/m and c = 10Ns/m. x(0) = 1, x'(0) = 0.

Hints: $ma = f(t) - c.v - k.x \quad v = \frac{dx}{dt} \quad a = \frac{d^2x}{dt^2}$

1) $\frac{dP}{dt} = t^2P - P + t^2 - 1$ (seperable)

$\frac{dP}{dt} = P(t^2 - 1) + (t^2 - 1)$

$dP = (t^2 - 1)[P + 1]\,dt$

$\int \frac{1}{P+1}\,dP = \int (t^2 + 1)\,dt$

$e^{\ln|P+1|} = e^{\frac{t^3}{3} + t + c}$

$P + 1 = e^{t^3/3 + t + c}$

$P = e^{t^3/3 + t + c} - 1 \quad ; \quad c \in R$

& $y = \gamma \quad ; \quad \gamma \in R$

is another solution

2) $\frac{dy}{dx} = \frac{xy\sin(x)}{y+1} \qquad y(0) = 1$ (seperable)

$y+1\,dy = xy\sin(x)\,dx \qquad y(0) = 1$

$\int \frac{y+1}{y}\,dy = \int x\sin(x)\,dx$

$\int \left(1 + \frac{1}{y}\right)dy = \int x\sin(x)\,dx$

$u = x \qquad vdv = \sin x\,dx$

$du = dx \qquad v = -\cos x$

$y + \ln|y| = -x\cos x + \int \cos x\,dx$

$y + \ln|y| = \sin x - x\cos x + C \quad ; \quad C \in R$

given $y(0) = 1$

ie $x = 0 \quad ; \quad y = 1$

$1 + \ln|1| = \sin 0 - 0 + C$

$$1 + 0 = 0 - 0 + C$$
$$C = 1$$

$$\therefore \quad y + \ln|y| = \sin x - x\cos x + 1$$

3) $$t\ln|t|\frac{dr}{dt} = te^t - r$$

(Non separable)

$\therefore$ varying constant to 0

$$t\ln|t|\frac{dr}{dt} = -r$$

$$\int \frac{1}{r}\,dr = \int -\frac{1}{t\ln|t|}\,dt \qquad \begin{aligned} \ln|t| &= u \\ \frac{1}{t}\,dt &= du \end{aligned}$$

$$\ln|r| = -1\int \frac{du}{u}$$

$$\ln|r| = -\ln|u| + c_1 \quad ; \quad c_1 \in R$$

$$e^{\ln|r|} = e^{-\ln|\ln t| + c_1}$$

$$r = e^{\ln|\ln t|^{-1} + c_1}$$

$$r = \frac{1}{\ln t} \cdot e^{c_1}$$

$$r = \frac{1}{\ln t} \cdot K \quad ; \quad K \in R$$

Varying the constant:

$$r_{trial} = \frac{1}{\ln t} \cdot k(t) = \frac{K(t)}{\ln(t)}$$

$$\gamma'_{total} = \frac{\ln(t)\, k'(t) - k(t)\frac{1}{t}}{(\ln(t))^2}$$

Substituting them back in question:

$$t \ln|t| \cdot \frac{\left[\ln(t)\, k'(t) - k(t)\cdot\frac{1}{t}\right]}{(\ln(t))^2} = te^t - \frac{k(t)}{\ln|t|}$$

$$\frac{t}{\ln|t|}\left[\ln|t|\, k'(t) - k(t)\cdot\frac{1}{t}\right] = te^t - \frac{k(t)}{\ln|t|}$$

$$t\, k'(t) - \cancel{\frac{k(t)}{\ln|t|}} = te^t - \cancel{\frac{k(t)}{\ln|t|}}$$

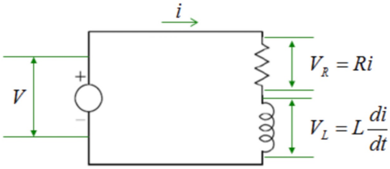
$$t\, k'(t) = te^t$$

$$k'(t) = e^t$$

$$k(t) = e^t + c_2 \quad ; \quad c_2 \in R$$

$$\therefore \quad \gamma = \frac{e^t + c_2}{\ln t} \quad ; \quad c_2 \in R.$$

4)



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Hints: $V_{DC} = V_R + V_L$

$R = 12\,\Omega \qquad L = 4\,H \qquad V = 6\,V \qquad$ Find $I(t)$; $I(0) = 0$

$$V_{DC} = V_R + V_L$$

$$V_{DC} = Ri + L\frac{di}{dt}$$

$$V_{DC} = 12i + 4\frac{di}{dt}$$

$$6 = 12i + 4\frac{di}{dt} \iff$$

is similar to the diff eqn of form

$$12y + 4\frac{dy}{dt} = 6$$

where $y = i$

(Non separable)

$$\therefore \quad 12y + 4\frac{dy}{dt} = 6$$

$$\therefore \quad 12y + 4\frac{dy}{dt} = 0$$

$$4\frac{dy}{dt} = -12y$$

$$\int \frac{1}{y}\,dy = \int -\frac{12}{4}\,dt$$

$$e^{\ln y} = e^{-3t + c_1} \quad ; \quad c_1 \in \mathbb{R}$$

$$y = e^{-3t} \cdot e^{c_1}$$

$$y = e^{-3t} \cdot K \quad ; \quad K \in \mathbb{R}$$

varying the constant:

$$y_{trial} = e^{-3t} \cdot k(t)$$

$$y'_{trial} = e^{-3t} \cdot k'(t) + k(t) \cdot e^{-3t} \cdot -3$$

$$= e^{-3t} \cdot k'(t) - 3e^{-3t} k(t)$$

$$12(e^{-3t} k(t)) + 4\left(e^{-3t} k'(t) - 3e^{-3t} k(t)\right) = 6$$

$$12e^{-3t} k(t) + 4e^{-3t} k'(t) - 12e^{-3t} k(t) = 6$$

$$4e^{-3t} k'(t) = 6$$

$$k'(t) = \frac{6}{4} \cdot \frac{1}{e^{-3t}}$$

$$k'(t) = \frac{3}{2} e^{3t}$$

$$k(t) = \frac{3}{2} \cdot \frac{1}{3} e^{3t} = \frac{1}{2} e^{3t} + c_2$$

$$\therefore \quad y = e^{-3t} \cdot \left(\frac{1}{2} e^{3t} + c_2\right) \quad ; \quad c_2 \in R$$

$$y = \frac{1}{2} \cdot 1 + e^{-3t} c_2$$

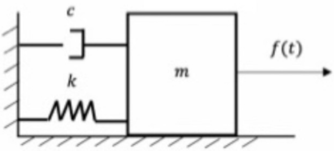
$$y = \frac{1}{2} + c_2 e^{-3t} \quad ; \quad i(0) = 0$$

$$I = \frac{1}{2} + c_2 e^{-3t}$$

$$0 = \frac{1}{2} + c_2 e^{0} \quad \therefore \quad c_2 = -\frac{1}{2}$$

$\therefore$ when $i(0) = 0$ $\quad I = \frac{1}{2} - \frac{1}{2} e^{-3t}$

5)



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Hints: $ma = f(t) - c.v - k.x \quad v = \frac{dx}{dt} \quad a = \frac{d^2x}{dt^2}$

$f(t) = 100 \sin(2t)$

$m = 2\,kg \qquad k = 40 \frac{N}{m} \qquad c = 10 \frac{Ns}{m}$

$x(0) = 1 \; ; \; x'(0) = 0$

$ma = f(t) - c.v - k.x \qquad v = \frac{dx}{dt} \qquad a = \frac{d^2x}{dt^2}$

$$2\frac{d^2x}{dt^2} = 100 \sin(2t) - 10\frac{dx}{dt} - 40x$$

$$= 2x'' = 100 \sin(2t) - 10x' - 40x$$

is similar to $\quad 2y'' = 100 \sin(2t) - 10y' - 40y$

where $y = x$; differential eqn.

$$2y'' + 10y' + 40y = 100 \sin(2t)$$

This is 2^{nd} order Diff eqn:

Solving charateristic eqn:

$$2r^2 + 10r + 40 = 0$$

$$r^2 + 5r + 20 = 0$$

$$\frac{-5 \pm \sqrt{25-80}}{2} = 0$$

$$r_{1,2} = \frac{-5 \pm \sqrt{-55}}{2} = \frac{-5}{2} \pm \frac{\sqrt{55}\,i}{2} \quad \Big\} \text{ complex roots}$$

$$y_h = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right)$$

$$= e^{-\frac{5}{2}x}\left(C_1 \cos \frac{\sqrt{55}}{2}x + C_2 \sin \frac{\sqrt{55}}{2}x\right);\ C_1, C_2 \in R$$

6. Consider a tank that is being filled with water at a constant rate of 3 liters per minute. The tank has a drainage pipe that allows water to flow out of the tank at a rate proportional to the square root of the water level in the tank.

Let h(t) represent the height of water in the tank at time t, measured in meters.

a) Write a differential equation that represents the rate of change of water level in the tank.

b) If the tank is initially empty (h(0) = 0), and the drainage constant is k = 0.2 liters per minute per square root meter, find the particular solution of the differential equation that satisfies this initial condition.

c) Determine the maximum height that the water level in the tank will reach.

Given that the tank is being filled with water at constant rate of 3 l/min

Given $h(t)$ represents the height of the water

$\therefore$ $h(t)$ depends on the area of the tank, A

$\therefore$ if $V(t)$ is the volume of the water filled inside the tank; it can be represented as:

$$V(t) = A \cdot h(t)$$

$\therefore$ rate of change of vol & height would be:

$$\frac{dV}{dt} = A\frac{dh}{dt} \quad \rightarrow (1)$$

Given outflow is prop to sqrt(h)
also given inflow is 3 l/min
∴ rate of change of vol can also be represented in its terms as:

$$\frac{dv}{dt} = 3 - k\sqrt{h(t)} \rightarrow (2)$$

(1) in (2)

$$A \frac{dh}{dt} = 3 - k\sqrt{h(t)}$$

$$\frac{dh}{dt} = \frac{3 - k\sqrt{h(t)}}{A}$$

Assuming $A = 1m^2$ / since not given

$$\frac{dh}{dt} = 3 - k\sqrt{h(t)} \longrightarrow$$

1st ODE
non-seperable

given $h(0) = 0$ & $k = 0.2$

ie $t = 0$; $h = 0$

$$\frac{dh}{dt} = 3 - k\sqrt{h}$$

$$\int \frac{1}{3 - k\sqrt{h}} dh = \int dt$$

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Applied Mathematics Mock Exam

water level → h
water flow out ∝ $\sqrt{h}$

$h(t) = 3 \quad - k\sqrt{h}$

$h(t) = 3 - k\sqrt{h}$

$3\ell/min$

6. Consider a tank that is being filled with water at a constant rate of 3 liters per minute. The tank has a drainage pipe that allows water to flow out of the tank at a rate proportional to the square root of the water level in the tank.

Let h(t) represent the height of water in the tank at time t, measured in meters.

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c) Determine the maximum height that the water level in the tank will reach.

7. Consider a pendulum of length (L) with a small amplitude. The motion of the pendulum is governed by the second-order differential equation:

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right) * sin(\theta) = 0$$

where θ represents the angular displacement of the pendulum from the vertical equilibrium position, t represents time, and g is the acceleration due to gravity.

a) Rewrite the differential equation in terms of the linear displacement (x) of the pendulum bob.

b) If the length of the pendulum is 1.5 meters and the acceleration due to gravity is 10 m/s^2 determine the general solution of the differential equation.

c) If the initial conditions are θ(0) = π/4 (45 degrees) and dθ/dt(0) = 0, find the particular solution that satisfies these initial conditions.

Hint: since θ << 1 rad, sin(θ) ≅ θ **(Bonus points (+100): Guess whose favorite approximation is this?)**

8. $\ddot{y} + 2\dot{y} + 50y = 12\cos(5t) + \sin(5t)$ → just to Relax

9. Consider a mechanical drive system consisting of a motor, a load, and a rotational shaft. The motion of the drive system is governed by the following second-order differential equation:

$$J\frac{d^2\theta}{dt^2} = T_m - T_L$$

where **θ** represents the angular displacement of the shaft, **t** represents time, **J** is the moment of inertia of the system, $\mathbf{T_m}$ is the torque produced by the motor, and $\mathbf{T_L}$ is the external load torque acting on the shaft.

a) If the moment of inertia of the system is $J = 0.5\ kg\cdot m^2$, and the load torque is given by $T_L = k\theta$, where $k = 2Nm/rad$, determine the general solution of the differential equation.

b) Suppose the motor produces a constant torque of $T_m = 10$ Nm. Find the particular solution of the differential equation that satisfies this condition.

How do you make a multivariate function laugh? Give it a domain that's full of complex twists and turns!

2. Equations with multiple variables

10. Find and sketch the domain of $\arcsin(x^2 + y^2 - 2)$

11. Find the approximate value of $\sqrt{(3.02^2 + 1.97^2 + 5.99^2)}$. Hint: First find 3 points close to the values then apply a tangent Plane approximation.

12. Given the function $f(x, y) = \frac{x}{x^2+y^2}$. Find the direction derivate at (1, 2) in the direction of

V = (3,5)'

13. $z = x^{-2}y^6 - 4x, x = u^2v,\ y = v - 3u$. Given the information find $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$

14. $Z = e^r\cos(\theta),\ r = st,\ \theta = \sqrt{s^2 + t^2}\ find\ \frac{dZ}{dt}$

15. Find the Partial Derivates of the following

a) $u = x^{y/z}$ b) $p = \sqrt{t^4 + u^2 cosv}$ c) $Z = \ln(x + \sqrt{x^2 + y^2})$ and find $F_x(3,4)$

16. Find the local properties of $f(x, y) = e^x cosy$

17. Find the Global Properties of $f(x, y) = 4x + 6y - x^2 - y^2 \quad D\!:\{\,(x, y) \mid 0 \le x \le 3, 0 \le y \le 2\}$

18. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, V = IR, to find how the current I is changing at the moment when R = 400 Ω ,I = 0.08 A, $\frac{dI}{dt}$ = -0.01 V/s, and $\frac{dR}{dt}$ = 0.03 Ω /s.

19. The temperature at a point (x, y) is T(x, y), measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimetres. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

Why did the math book want to wish the student good luck before the exam?

Because it knew that a little "study" and "commitment" could solve any "problem"!

All the best in your Assessment, and remember to Practice hard!

Q : SoSe 2022

c. Solve ODE: $t \ln t \frac{dr}{dt} + r = t e^{t}$

This is non separable

$\therefore \quad t \ln t \frac{dr}{dt} = -r$

$$\frac{1}{r} dr = -1 \int \frac{1}{t \ln t} dt$$

$u = \ln t$

$du = \frac{1}{t} dt$

$$\ln r = -1 \ln|\ln t| + c \; ; \; c \in R$$

$$e^{\ln r} = e^{\ln|\ln t|^{-1} + c}$$

$$r = (\ln|t|)^{-1} \cdot e^{c}$$

$$r = \frac{1}{\ln t} \cdot e^{c}$$

$$r = \frac{1}{\ln t} k \quad ; \quad k \in R$$

variation of param:

$$r_p = \frac{1}{\ln t} k(t) = \frac{k(t)}{\ln t}$$

$$r_p' = \frac{\ln t \; k'(t) - k(t) \cdot \frac{1}{t}}{(\ln t)^2}$$

$$= \frac{k'(t)}{\ln t} - \frac{k(t)}{t \cdot (\ln t)^2}$$

$$t \ln t \left[\frac{k'(t)}{\ln t} - \frac{k(t)}{t(\ln t)^2} \right] + \frac{k(t)}{\ln t} = t e^t$$

$$t k'(t) - \cancel{\frac{k(t)}{\ln t}} + \cancel{\frac{k(t)}{\ln t}} = t e^t$$

$$t k'(t) = t e^t$$

$$k'(t) = e^t$$

$$k(t) = e^t + c_2 \quad ; \quad c_2 \in R$$

$$\therefore \quad y_p = \frac{e^t + c_2}{\ln t} \quad ; \quad c_2 \in R$$

is another solution.

Second Order Differential Equations

Approaching Exam Questions:

Solve the Homogenous Linear ODE with constant coefficients:

Homogenous (RHS : 0)

$$ay'' + by' + cy = 0 \qquad a, b, c \in R \text{ constants} \qquad a \neq 0$$

characteristic eqn:

$$ar^2 + br + c = 0$$

- if $r_1 \neq r_2$

  $$y_h = c_1 e^{r_1 x} + c_2 e^{r_2 x} \; ; \; c_{1,2} \in R$$

- if $r_1 = r_2 = r$

  $$y_h = c_1 e^{rx} + c_2 x e^{rx} \; ; \; c_1, c_2 \in R$$

- if $r_{1,2} = \alpha \pm i\beta$; extract α & β values

  $$y_h = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \; ; \; c_1, c_2 \in R$$

Solve the non-homogenous linear ODE with constant coefficients:

$$ay'' + by' + cy = G(x)$$

Take RHS as 0 & turn it into Homogenous eqn & solve the characteristic eqn for y_h

Now for y_p : use method of undetermined coefficents
if $G(x)$ is of the form, take a y_{trial} as follows:

Form	y_{trial}
1	A

$5x+7$	$Ax+B$
$4x^2-8$	Ax^2+Bx+C
e^{3x}	Ae^{3x}
$\sin 4x$	$A\cos 4x + B\sin 4x$
$\cos kx$	$A\cos kx + B\sin kx$
e^{kx}	Ae^{kx}
$x^2 e^{5x}$	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin 4x$	$e^{3x}(A\cos 4x + B\sin 4x)$
$5x^2 \sin 4x$	$(Ax^2+Bx+C)\cos x + (Dx^2+Ex+F)\sin x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x + (Cx+D)e^{3x}\sin 4x$

Basic gist of coming up with y_p:

The method of undetermined coefficients is a bit more nuanced than just breaking up $G(x)$ into $G_1(x) \cdot G_2(x)$ or $G_1(x) + G_2(x)$

When we apply method of undetermined coefficients, we are essentially making a guess on how the particular solution would look like based on the RHS of the given differential equation.

The guess is based on the form of the 'function' on the RHS, but; we ignore the multiplicative constants when making the initial guess

eg: $25 \sin(2x)$ $\Rightarrow$ function is $\sin 2x$ $\therefore$ ignoring the constant we have $y_p = A \sin 2x + B \cos 2x$
the method of undetermined coefficients is designed to account for that ignored constant while solving.

$(4x-7)e^{5x}$ $\Rightarrow$ $4x-7$ becomes $Ax+B$ & $e^{5x} = Ce^{5x}$ & is also a function but when we match the form of the questions differential equation A & B will account for the case where we may have multiplied it prior. $\therefore$ $y_p = (Ax+B)e^{5x}$

$5x^2 \sin(4x)$: when dealing with functions like this; this is product of a polynomial & trig function we guess the particular solution to have the form of the same degree of polynomial.

we know for $5x^2 \rightarrow Ax^2 + Bx + C$
& we know $\sin(4x) \rightarrow A \cos 4x + B \sin 4x$

So if we multiply them; we have to generate y_p as:

$$y_p = (Ax^2 + Bx + C)\cos 4x + (Dx^2 + Ex + G)\sin 4x$$

as A & B are not the same! Hence we introduce new set of variable with its polynomial form.

Second ODE (Notes)

Two types:

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = G(x) \leftarrow \text{Inhomogenous}$$

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \leftarrow \text{Homogenous}$$

If $y_1(x)$ & $y_2(x)$ are both solutions of a homogenous 2nd order linear ODE, then for every choice of constants $c_1, c_2 \in R$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution of same ODE.

Theorem 2:

If y_1 & y_2 are linearly independent solutions of a homogenous 2nd order linear ODE, then the general soln (= all solutions) is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \;; \quad c_1, c_2 \in R$$

Keyword: linearly independent
ie not multiples of each other

x^3 & $8x^3$ are not linearly independent
e^{ax} & xe^{ax} are linearly independent.

$$ay'' + by' + cy = 0$$ (Homogeneous second ode)

soln. type: $y(x) = e^{rx}$

$$y'(x) = r e^{rx}$$

$$y''(x) = r^2 e^{rx}$$

$\therefore$ we have $ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

$$e^{rx}(ar^2 + br + c) = 0 \longrightarrow 7$$

we have $ar^2 + br + c = 0 \leftarrow$ characteristic equation.

solve the ODE

<u>case 1: $r_1 \neq r_2$</u>

then $y_1(x) = e^{r_1 x}$

$$y_2(x) = e^{r_2 x}$$

$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \; ; \; c_1, c_2 \in R \text{ when } r_1 \neq r_2}$$

case 2: $r_1 = r_2 = r$

1^{st} soln: $y_1(x) = e^{rx}$

$$y_2(x) = x e^{rx} \longrightarrow 1$$

$$\therefore \quad y_2'(x) = x \cdot r e^{rx} + e^{rx} \longrightarrow 2$$

$$y''(x) = r\left(x r e^{rx} + e^{rx}\right) + r e^{rx}$$

$$= x r^2 e^{rx} + r e^{rx} + r e^{rx}$$

$$= x r^2 e^{rx} + 2 r e^{rx} \longrightarrow 3$$

$$a y'' + b y' + c y = 0$$

$$a\left(x r^2 e^{rx} + 2 r e^{rx}\right) + b\left(x r e^{rx} + e^{rx}\right) + c x e^{rx} = 0$$

$$a x r^2 e^{rx} + 2 a r e^{rx} + b x r e^{rx} + b e^{rx} + c x e^{rx} = 0$$

$$a x r^2 + 2 a r + b x r + b + c x = 0$$

$$a x r^2 + b x r + c x + 2 a r + b = 0$$

$$x\underbrace{(a r^2 + b r + c)}_{7} + 2 a r + b = 0$$

$$\therefore \quad x(0) + 2 a r + b = 0$$

$$2 a r = -b$$

$$r = -\frac{b}{2a}$$

-! solution of ODE:

$$y(x) = C_1 e^{rx} + C_2 x e^{rx}$$ $r_1 = r_2 = r$

when roots are complex : $r_1, r_2 = \alpha + i\beta$

solution $y(x) = e^{\alpha x}\left(C_1 \cos \beta x + i \sin \beta x\right)$; $C_1, C_2 \in R$

(1) $y'' + 10y' + 25y = 0$

characteristic eqn:

$$r^2 + 10r + 25 = 0$$

$$\underline{5} \times \underline{5} = 25$$

$$\underline{5} + \underline{5} = 10$$

$$r^2 + 5r + 5r + 25 = 0$$

$$r(r+5) + 5(r+5) = 0$$

$$(r+5)^2 = 0$$

$$r_1, r_2 = -5$$

roots r_1, r_2 are the same

$\therefore$ Solution: $y = C_1 e^{-5x} + C_2 A e^{-5x}$

(2) $y''' + 3y'' + 3y' + y = 0$

$$r^3 + 3r^2 + 3r + 1 = 0$$

when $r = -1$

$$-1 + 3 - 3 + 1 = 0$$

$\therefore$ $r = -1$ is one factor

$$
\begin{array}{r l}
 & r^2 + 2r + 1 \\
r+1 & \overline{\big) \; r^3 + 3r^2 + 3r + 1} \\
 & \underline{r^3 + r^2} \\
 & \quad 2r^2 + 3r \\
 & \quad \underline{2r^2 + 2r} \\
 & \qquad 1r + 1 \\
 & \qquad \underline{1r + 1} \\
 & \qquad \quad 0
\end{array}
$$

$$\therefore \quad (r+1)\underbrace{(r^2 + 2r + 1)}_{\text{solving this}} = 0$$

$$\underline{1} \times \underline{1} = 1$$

$$\underline{1} + \underline{1} = 2$$

$$(r+1)(r^2 + r + r + 1) = 0$$

$$(r+1)(r(r+1) + (r+1)) = 0$$

$$(r+1)(r+1)(r+1) = 0$$

$$r_{1,2,3} = -1 \quad \leftarrow \text{roots are same}$$

$\therefore$ solution:

$$y = C_1 e^{-x} + C_2 A e^{-x} + C_3 A^2 e^{-x}$$

③

$$2y'' - y' - 3y = 25\sin(2x)$$

characteristic eqn:

$$2r^2 - r - 3 = 0$$

$$\frac{-(-1) \pm \sqrt{1-4(-6)}}{2(2)} = \frac{+1 \pm \sqrt{25}}{4} = \frac{+1 \pm 5}{4} =$$

$$= \frac{+1+5}{4} \text{ or } \frac{+1-5}{4}$$

$$= \frac{3}{2} \text{ or } -1$$

$\underline{2} \times \underline{-3} = -6$

$\underline{2} + \underline{-3} = -1$

$$2r^2 + 2r - 3r - 3 = 0$$

$$2r(r+1) - 3(r+1) = 0$$

$$(r+1)(2r-3) = 0$$

$$r = -1 \quad r = 3/2$$

$r_1 \neq r_2$

$$\therefore y_h = c_1 e^{-1x} + c_2 e^{3/2\,x}$$

For test function: we have: $G(x) = 25\sin(2x)$

$= G_1(x) \cdot G_2(x)$

ignored & rewritten.

$$\therefore \quad y_p = A(B\cos 2x + C\sin 2x)$$

proper coefficient $=$ $A\cos 2x + B\sin 2x$

$$y_p' = A \cdot -\sin 2x \cdot 2 + B\cos 2x \cdot 2$$

$$= -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -2A\cos 2x \cdot 2 + 2B(-\sin 2x) \cdot 2$$

$$= -4A\cos 2x - 4B\sin 2x$$

we have:

$$2y'' - y' - 3y = 25\sin(2x)$$

$$2(-4A\cos 2x - 4B\sin 2x)$$
$$-(-2A\sin 2x + 2B\cos 2x)$$
$$-3(A\cos 2x + B\sin 2x) = 25\sin(2x)$$

LHS:

$$-8A\cos 2x - 8B\sin 2x$$
$$+2A\sin 2x - 2B\cos 2x$$
$$-3A\cos 2x - 3B\sin 2x = 25\sin(2x)$$

$$(-8A - 2B - 3A)\cos 2x +$$
$$(-8B + 2A - 3B)\sin 2x = 25\sin(2x) + 0\cos(2x)$$

$$-11B - 2A = 0 \qquad (-)$$
$$-11B + 2A = 25$$

∴ we have:

$$-4A = -25$$
$$A = 25/4$$

$$-11B = 2A$$
$$-11B = \frac{50}{4}$$
$$B = -\frac{50}{44}$$

$$\therefore\ y_p = A\cos 2x + B\sin 2x$$

$$= \frac{25}{4}\cos 2x - \frac{50}{44}\sin 2x$$

$$y(x) = y_h + y_p$$

$$= C_1 e^{-x} + C_2 e^{3/2 x} + \frac{25}{4}\cos 2x - \frac{50}{44}\sin 2x$$

17.2 Exercises

1–10 Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' - 2y' - 3y = \cos 2x$
2. $y'' - y = x^3 - x$
3. $y'' + 9y = e^{-2x}$
4. $y'' + 2y' + 5y = 1 + e^x$
5. $y'' - 4y' + 5y = e^{-x}$
6. $y'' - 4y' + 4y = x - \sin x$
7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$
8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$
9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$
10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

11–12 Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $y'' + 3y' + 2y = \cos x$
12. $y'' + 4y = e^{-x}$

13–18 Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' - y' - 2y = xe^x \cos x$
14. $y'' + 4y = \cos 4x + \cos 2x$
15. $y'' - 3y' + 2y = e^x + \sin x$
16. $y'' + 3y' - 4y = (x^3 + x)e^x$
17. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$
18. $y'' + 4y = e^{3x} + x \sin 2x$

1) $y'' - 2y' - 3y = \cos 2x$

$$y = y_h + y_p$$

Characteristic eqn:

$$r^2 - 2r - 3 = 0$$

$$\left.\begin{aligned} \underline{1} \times \underline{-3} &= -3 \\ \underline{1} + \underline{-3} &= -2 \end{aligned}\right\} \Rightarrow \begin{aligned} r^2 + r - 3r - 3 &= 0 \\ r(r+1) - 3(r+1) &= 0 \\ (r-3)\ (r+1) &= 0 \\ r = 3 \qquad r &= -1 \end{aligned}$$

$$r_1 \neq r_2 \quad \boxed{\therefore \ y_h = c_1 e^{3x} + c_2 e^{-x}}$$

$$G(x) = \cos 2x$$

$$\therefore \quad y_p = A \cos 2x + B \sin 2x$$

$$y_p' = -2A \sin 2x + 2B \cos 2x$$

$$y_p'' = -4A \cos 2x - 4B \sin 2x$$

$\therefore$ we have

$$\left(-4A \cos 2x - 4B \sin 2x\right) - 2\left(-2A \sin 2x + 2B \cos 2x\right) - 3\left(A \cos 2x + B \sin 2x\right) = \cos 2x$$

$$-\underline{4A \cos 2x} - 4B \sin 2x + 4A \sin 2x - 4\underline{B \cos 2x} - 3\underline{A \cos 2x} - 3B \sin 2x = \cos 2x$$

$$(-4A-4B-3A)\cos 2x$$
$$+(-4B+4A-3B)\sin 2x = \cos 2x + 0\sin 2x$$

$$\left.\begin{array}{l} -4A-4B-3A = 1 \\ -4B+4A-3B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} -7A-4B = 1 \rightarrow (1) \\ 4A-7B = 0 \rightarrow (2) \end{array}$$

$$\left.\begin{array}{l} (1) \times 4 \\ (2) \times -7 \end{array}\right\} \Rightarrow \begin{array}{l} -28A-16B = 4 \\ -28A+49B = 0 \quad (-) \end{array}$$

$$-65B = 4$$
$$B = -\frac{4}{65}$$

$$4A = 7B$$
$$A = \frac{7}{4}\cdot\left(\frac{-4}{65}\right)$$
$$= -\frac{7}{65}$$

$$\boxed{\therefore\ y_p = -\frac{7}{65}\cos 2x - \frac{4}{65}\sin 2x}$$

$$\boxed{y = C_1 e^{3x} + C_2 e^{-x} + -\frac{7}{65}\cos 2x - \frac{4}{65}\sin 2x}$$

Do not forget! $\rightarrow C_{1,2} \in R$

2) $y'' - y = x^3 - x$

$y'' - 0y' - y = x^3 - x$

$r^2 - 0r - 1 = 0$

$r^2 - 1 = 0$

$r^2 = 1$

$r_{1,2} = \pm 1 \quad \therefore \quad y_h = c_1 e^{x} + c_2 e^{-x} \qquad c_{1,2} \in R$

$G(x) = G_1(x) + G_2(x)$

$= x^3 - x$ → ignored as D takes care of it

$y_p = Ax^3 + Bx^2 + Cx + D$

$y_p' = 3Ax^2 + 2Bx + C$

$y_p'' = 6Ax + 2B$

$\therefore \quad 6Ax + 2B - (Ax^3 + Bx^2 + Cx + D) = x^3 - x$

$6Ax + 2B - Ax^3 - Bx^2 - Cx - D = x^3 + 0x^2 - x + 0$

$-Ax^3 - Bx^2 + 6Ax - Cx + 2B - D = x^3 + 0x^2 - x + 0$

$-A = 1 \qquad -Bx^2 = 0x^2 \qquad (6A - C)x = -x \qquad 2B - D = 0$

$A = -1 \qquad B = 0 \qquad 6A - C = -1 \qquad D = 0$

$-6 - C = -1$

$-6 + 1 = C$

$C = -5$

$$\therefore \quad y_p = Ax^3 + Bx^2 + Cx + D$$
$$= -x^3 + 0x^2 - 5x + 0$$
$$= -x^3 - 5x$$

$$\boxed{y = y_H + y_p = c_1 e^{x} + c_2 e^{-x} - x^3 - 5x \; ; \; c_{1,2} \in R}$$

3) $y'' + 9y = e^{-2x}$

$$y'' + 0y + 9y = e^{-2x}$$

$$r^2 + 0r + 9 = 0$$
$$r^2 = -9$$
$$r = \sqrt{-9} = 0 \pm 3i$$

$$r = \alpha \pm \beta i \iff 0 \pm 3i$$
$$\alpha = 0 \qquad \beta = 3$$

$$y_h = e^{0x}\left(c_1 \cos 3x + c_2 \sin 3x\right)$$
$$y_h = c_1 \cos 3x + c_2 \sin 3x \quad ; \quad c_1, c_2 \in R$$

$$G(x) = e^{-2x}$$
$$y_p = Ae^{-2x}$$
$$y_p' = A \cdot e^{-2x} \cdot (-2)$$
$$= -2A e^{-2x}$$

$$y_p'' = -4Ae^{-2x}$$

$$-4Ae^{-2x} + 9Ae^{-2x} = e^{-2x}$$

$$5Ae^{-2x} = e^{-2x}$$

$$5A = 1$$

$$A = \frac{1}{5}$$

$$\therefore \; y_p = \frac{1}{5}e^{-2x}$$

$$\boxed{y = y_h + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}e^{-2x} \qquad c_1, c_2 \in \mathbb{R}}$$

4) $y'' + 2y' + 5y = 1 + e^x$

$$r^2 + 2r + 5 = 0$$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = -\frac{2}{2} \pm \frac{4i}{2} = -1 \pm 2i$$

$(\alpha \pm \beta i$ form$)$

$$y_h = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) \; ; \; c_1, c_2 \in \mathbb{R}$$

$$G(x) = 1 + e^x$$

$$\therefore \; y_p = A + Be^x$$

$$y_p' = Be^x$$
$$y_p'' = Be^x$$

$$\underline{Be^x} + 2\underline{Be^x} + 5A + \underline{5Be^x} = 1 + e^x$$

$$8Be^x + 5A = e^x + 1$$

$$8B = 1 \qquad 5A = 1$$
$$B = \frac{1}{8} \qquad A = \frac{1}{5}$$

$$y_p = \frac{1}{5} + \frac{1}{8}e^x$$

$$y = y_h + y_p$$
$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8}e^x + \frac{1}{5} \quad ; \quad C_1, C_2 \in \mathbb{R}$$

5) $y'' - 4y' + 5y = e^{-x}$

$$r^2 - 4r + 5 = 0$$
$$r_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2 \cdot 1} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm \frac{2}{2}i = 2 \pm i$$
($\alpha \pm \beta i$ form)

$$\therefore \quad y_h = e^{2x}(C_1 \cos x + C_2 \sin x) \quad ; \quad C_1, C_2 \in \mathbb{R}$$

For y_p: $\quad G(x) = e^{-x}$

$$y_p = Ae^{-x}$$

$$y_p' = -Ae^{-x}$$

$$y_p'' = Ae^{-x}$$

$$Ae^{-x} + 4Ae^{-x} + 5Ae^{-x} = e^{-x}$$

$$10Ae^{-x} = e^{-x}$$

$$A = \frac{1}{10}$$

$$\therefore \quad y_p = \frac{1}{10}e^{-x}$$

$$\boxed{\begin{aligned} y &= y_p + y_h \\ &= e^{2x}\left(C_1\cos x + C_2\sin x\right) + \frac{1}{10}e^{-x} \quad ; C_1, C_2 \in \mathbb{R} \end{aligned}}$$

SoSe 2022 Q 17:38

$\ddot{\psi} + 2\dot{\psi} + 50\psi = 12\cos 5t + \sin 5t$

replacing ψ as y to solve the question & $t = x$

$y'' + 2y' + 50y = 12\cos 5x + \sin 5x$

$r^2 + 2r + 50 = 0$

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 200}}{2}$$

$$= \frac{-2}{2} \pm \frac{\sqrt{-196}}{2} = -1 \pm 7i$$

$\alpha \pm \beta i$ form

$y_h = e^{-x}\left(C_1 \cos 7x + C_2 \sin 7x\right)$; $C_1, C_2 \in R$

$G(x) = 12\cos 5x + \sin 5x$

$= G_1(x) + G_2(x)$

$y_{p_1}: G_1(x) \Rightarrow A\cos 5x + B\sin 5x$

$y_{p_2}: G_2(x) = C\cos 5x + D\sin 5x$ } Not necessary as $G_1(x)$ guess already represent our complete RHS

$y_{p_1}' = -5A\sin 5x + 5B\cos 5x$

$y_{p_1}'' = -25A\cos 5x - 25B\sin 5x$

$$y_{p_2}' = -5C\sin 5x + 5D\cos 5x \quad ✗$$

$$y_{p_2}'' = -25C\cos 5x - 25D\sin 5x \quad ✗$$

not needed to compute!

$G_y(x)$ in $y'' + 2y' + 50y$

$$\begin{aligned} = & -25A\cos 5x - 25B\sin 5x \\ & + 2(-5A\sin 5x + 5B\cos 5x) \\ & + 50(A\cos 5x + B\sin 5x) \\ & = 12\cos 5x + \sin 5x \end{aligned}$$

$$\begin{aligned} = & (-25A + 10B + 50A)\cos 5x + \\ & (-25B - 10A + 50B)\sin 5x \\ & = 12\cos 5x + \sin 5x \end{aligned}$$

$$\begin{aligned} &(25A + 10B = 12) \times 10 \quad (+) \\ &(-10A + 25B = 1) \times 25 \end{aligned}$$

$$\begin{aligned} 250A + 100B &= 120 \quad (+) \\ -250A + 625B &= 25 \end{aligned}$$

$$725B = 145$$

$$B = \frac{145}{725} = \frac{1}{5}$$

$$\begin{array}{r} {}^{2}1{}^{2}45 \\ 5 \\ \hline 725 \end{array}$$

$$10A = 25B - 1$$

$$A = \frac{25 \cdot 1 - 5}{5 \cdot 10}$$

$$= \frac{25 - 5}{50} = \frac{20}{50} = \frac{2}{5}$$

$$\therefore \quad y_p = \frac{2}{5} \cos 5x + \frac{1}{5} \sin 5x$$

$$\boxed{\begin{aligned} y &= y_h + y_p \\ &= e^{-x}(C_1 \cos 7x + C_2 \sin 7x) + \frac{2}{5} \cos 5x + \frac{1}{5} \sin 5x \\ & \qquad \text{where } C_1, C_2 \in R \end{aligned}}$$

Q: Spring 2023:

d. $\ddot{y} + 3\dot{y} = x e^{x} + \sin 3x$

$$y'' + 3y' + 0y = xe^{x} + \sin 3x$$

$$r^2 + 3r + 0 = 0$$

$$r^2 + 3r = 0$$

$$r(r + 3) = 0$$

18:08

$r_1 = 0 \qquad r_2 = -3 \qquad r_1 \neq r_2$

$$\boxed{\therefore \quad y_h = C_1 e^{0x} + C_2 e^{-3x}} \qquad C_1, C_2 \in R.$$

$$G(x) = G_1(x) + G_2(x)$$

$$= \underbrace{(Ax+B)e^x}_{y_{p_1}} + \underbrace{(D\cos 3x + E\sin 3x)}_{y_{p_2}}$$

$$y_{p_1} = (Ax+B)e^x$$
$$= Axe^x + Be^x$$
$$y_{p_1}' = A[xe^x + e^x] + Be^x$$
$$= Axe^x + Ae^x + Be^x$$
$$y_{p_1}'' = A[xe^x + e^x] + Ae^x + Be^x$$
$$= Axe^x + 2Ae^x + Be^x$$

$G_1(x)$:

$$y'' + 3y' = xe^x$$

$$A\underline{xe^x} + 2Ae^x + Be^x + 3A\underline{xe^x} + 3Be^x = xe^x$$

$$4Axe^x + (2A+4B)e^x = xe^x + 0e^x$$

$$4A = 1 \qquad\qquad 2A + 4B = 0$$
$$A = \frac{1}{4} \qquad\qquad 2\cdot\frac{1}{4} + 4B = 0$$
$$4B = -\frac{1}{2}$$
$$B = -\frac{1}{4}$$

$$y_{p_1} = \left(\frac{1}{4}x - \frac{1}{4}\right)e^x$$

$G_2(x)$:

$$y_{p_2} = D\cos 3x + E\sin 3x$$

$$y_{p_2}' = -3D\sin 3x + 3E\cos 3x$$

$$y_{p_2}'' = -9D\cos 3x - 9E\sin 3x$$

$$y'' + 3y' = \sin 3x$$

$$-9D\underline{\cos 3x} - 9E\sin 3x + D\underline{\cos 3x} + E\sin 3x = \sin 3x$$

$$-8D\cos 3x + (-8E)\sin 3x = \sin 3x + 0\cdot\cos 3x$$

$$-8E = 1 \qquad\qquad -8D = 0$$

$$E = -\frac{1}{8} \qquad\qquad D = 0$$

$$y_{p_2} = 0 + \frac{1}{8}\sin 3x$$

$$y = y_h + y_{p_1} + y_{p_2}$$

$$\boxed{y = C_1 + C_2e^{-3x} + \frac{1}{4}xe^x - \frac{1}{4}e^x + \frac{1}{8}\sin 3x \qquad C_1, C_2 \in \mathbb{R}}$$

skipping 6 as repeated pattern

$7.\ y'' + y = e^x + x^3 \quad ; \quad y(0) = 2 \quad , \quad y'(0) = 0$

$r^2 + r + 0 = 0$ ← wrong!

wrong! do not confuse : $\left.\begin{aligned} y'' &= r^2 \\ y' &= r \\ \& \ y &= 1 \end{aligned}\right\}$

$r(r+1) = 0$

$r_1 = 0 \quad r_2 = -1 \quad ; \quad r_1 \neq r_2$

$\therefore \quad y_h = C_1 e^{0x} + C_2 e^{-x}$

$\therefore$ it's

$r^2 + 1 = 0$

$r = \pm i$

$y_h =$ complex form!

$$G(x) = e^x + x^3$$

$$= \underbrace{Ae^x}_{y_{p_1}} + \underbrace{Bx^3 + Cx^2 + Dx + E}_{y_{p_2}}$$

$$y_{p_1} = Ae^x$$

$$y'_{p_1} = Ae^x$$

$$y''_{p_1} = Ae^x$$

$$y_{p_2} = Bx^3 + Cx^2 + Dx + E$$

$$y'_{p_2} = 3Bx^2 + 2Cx + D$$

$$y''_{p_2} = 6Bx + 2C$$

$$y'' + y' = e^x + x^3$$

$\underline{G_1(x)}$:

$$Ae^x + Ae^x = e^x$$
$$2Ae^x = e^x$$
$$A = \frac{1}{2}$$

$$y_{p_1} = \frac{1}{2}e^x$$

$G_2(x)$:

$$6Bx + 2C + 3Bx^2 + 2Cx + D = x^3 + 0x^2 + 0x + 0$$

$$0x^3 + 3Bx^2 + (6B + 2C)x + 2C + D = x^3$$

$$3B = 0 \qquad 6B + 2C = 1 \qquad 2C + D = 0$$
$$B = 0 \qquad 2C = 1 \qquad 1 + D = 0$$
$$C = \frac{1}{2} \qquad D = -1$$

$$\therefore\ y_{p_2} = 0x^3 + \frac{1}{2}x^2 - x$$

& since E does not contribute to higher order terms; E=0

Redoing y_n as initial steps were incorrect:

$$r^2 + 1 = 0$$
$$r = \pm i$$
$$\alpha \pm i\beta \text{ form} \qquad \alpha = 0 \qquad \beta = 1$$

$$\therefore\ y_n = e^{0x}(C_1 \cos x + C_2 \sin x)$$

$$y = y_h + y_{p_1} + y_{p_2}$$

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x + \frac{1}{2} x^2 - x \rightarrow \text{*}$$

$C_1, C_2 \in \mathbb{R}$

given $y(0) = 2$ & $y'(0) = 0$

$y(0) = 2 \qquad x = 0 \quad y = 2$

$$2 = C_1 \cos 0 + C_2 \sin 0 + \frac{1}{2} + 0 - 0$$

$$2 = C_1 \cdot 1 + 0 + \frac{1}{2}$$

$$C_1 = 2 - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}$$

Applying derivative to: *

$$y' = -C_1 \sin x + C_2 \cos x + \frac{1}{2} e^x + \frac{1}{2} \cdot 2x - 1$$

$$y' = -C_1 \sin x + C_2 \cos x + \frac{e^x}{2} + x - 1$$

given $y'(0) = 0$

$x = 0 \quad y' = 0$

$$0 = 0 + C_2 + \frac{1}{2} + 0 - 1$$

$$C_2 + \frac{1}{2} = 1 \qquad C_2 = -\frac{1}{2}$$

$\therefore$

$$\boxed{y = \frac{3}{2}\cos x - \frac{1}{2}\sin x + \frac{1}{2}e^{x} + \frac{1}{2}x^{2} - x}$$