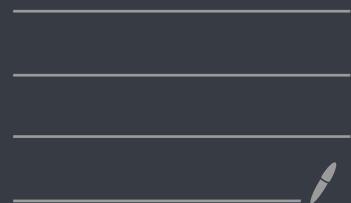


Differential Equations:

- Ordinary Differential Equations
 - Second Order Differential Equations
-



Ordinary Differential Equations

In the exam question: first check the order of the eqn ie, if $y' \rightarrow$ 1st Order
 $y'' \rightarrow$ 2nd Order

Check if you can separate the eqn by grouping the all 'x' variables along with 'dx' on one side and all 'y' variable along with 'dy' on the other side. If you can write: It is separable or else : It is non separable

If separable

- 1) Group x & y on both sides
- 2) Integrate both sides to get an eqn in 'y'
- 3) That is one of your solution. Don't forget to add: where C, K ∈ R depending on variable constant that you use while solving the integral
- 4) For the other soln plug $y=0$ in to the question and solve to find a value for x
- 5) write $y=0$ is also a solution

If it is non separable

- 1) Take the constant part to 0 and solve for y. This is called variation of constant. $Duh!! \rightarrow$
- 2) When varying the constant & solving the integral you will encounter natural log on both sides. Raise the power on both sides by 'e'
 $\therefore e^{\ln x}$ can be written as x
- 3) Solve & write one soln of y with const K, C ∈ R
- 4) Now that you have varied the constant, you need to find another soln by trial method; ie turn $K \rightarrow K(x)$ from step 3. Now you have a new function. $y = k(x) \dots$
- 5) Find y' of this new function
Plug y & y' in the original question. Notice that some terms will cancel & you have $k'(x)$ remaining
- 6) Integrate $k'(x)$ to get $K(x)$ & plug it to the function from step 3. This is your other soln with $C, K \in R$
- 7) If IVP plug the question condition $y(\cdot) = \dots$ into step 8 & solve the constant K or C & rewrite the eqn.

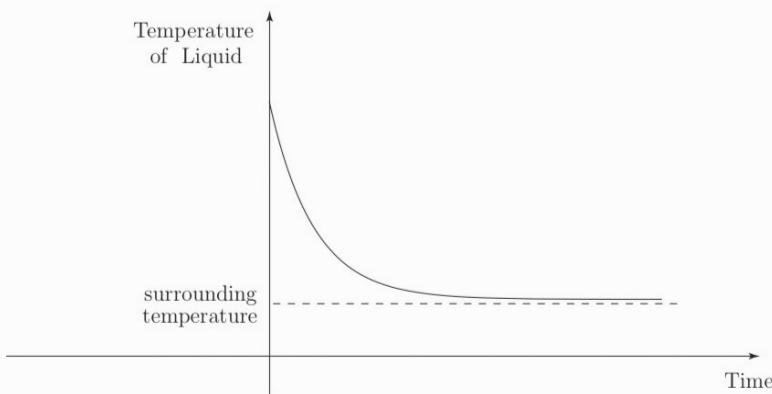


Figure 1

After an initially rapid decrease the temperature changes progressively less rapidly and eventually the curve appears to 'flatten out'.

Newton's law of cooling states that the rate of cooling of liquid is proportional to the difference between its temperature and the temperature of its environment (the ambient temperature). To convert this into mathematics, let t be the time elapsed (in seconds, s), θ the temperature of the liquid ($^{\circ}\text{C}$), and θ_0 the temperature of the liquid at the start ($t = 0$). The temperature of the surroundings is denoted by θ_s .

From the fact above we can derive the following symbols:

$\theta \rightarrow$ Temp of liquid that changes with time $\cdot t$

$\theta_s \rightarrow$ Outside temp

$t \rightarrow$ time

$\theta_0 \rightarrow$ initial temp.

thus: a mathematical model for the description would be:

$$\frac{d\theta}{dt} \propto \theta - \theta_s$$

$$= \frac{d\theta}{dt} = -k(\theta - \theta_s)$$

$-k$ as θ decreases with time

Here $t \rightarrow$ independent variable

$t=0$ would be the initial condition

$\theta =$ dependent variable

Here the Order of equations is 1

as it's $\frac{d}{dt}$ & not $\frac{d^2}{dt^2}$.

and also the degree is 1 as it is
 $\left(\frac{d}{dt}\right)^1$

The First Order Differential equation
has \rightarrow One Independent variable
 \rightarrow One or more independent variable

example:

① $y' = \cos x$ degree : 1

$$\frac{dy}{dx} = \cos x$$
 Order : 1

dependent : y

independent : x

② $\frac{d^2y}{dx^2} + 9y = e^{-2x}$ degree : 1
 order : 2

independent : x

dependent : y

③ $y'y''' - \frac{3}{2}y'^2 = 0$ degree : 1
 order : 3

independent : x

dependent : y

④ $\frac{dy}{dt} + \frac{dx}{dt} = \sin t$ degree : 1
 order : 1

independent : t

dependent : y, x

? independent & one or more dependent
 \therefore still 1st ODE.

$$\textcircled{5} \quad \frac{d^4y}{dx^4} + \sin(y''') = 0$$

degree: not defined
order: 4

To define the degree;
the function must be
polynomial.

independent: x
dependent: y

$$\textcircled{6} \quad y''' + y^2 + e^{y'} = 0$$

degree: not defined
order: 3
independent: x
dependent: y

An equation that contains only the 1st derivative & may contain y and any given functions of x is called first ODE. Hence we can write them as:

$$F(x, y, y') = 0 \quad \text{or} \quad y' = f(x, y)$$

Verify that $y = \frac{c}{x}$ is a solution of the ODE $xy' = -y$ for all $x \neq 0$ where c is constant

We are given y as a solution & we have to verify if the y' when substituted into the ODE's LHS produces the RHS of the ODE. Thus we can easily verify its a solution or not

$$y = \frac{c}{x}$$

$$y' = c \cdot \left(-\frac{1}{x^2}\right) = -\frac{c}{x^2}$$

$$\text{we have } xy' = -y$$

LHS:

$$\therefore x \left(-\frac{c}{x^2}\right) = -\frac{c}{x}$$

RHS:

$$-y = -\frac{c}{x}$$

$$\text{LHS} = \text{RHS} \quad \therefore \quad \frac{c}{x} \text{ is a solution of ODE}$$

$$xy' = y$$

Verify that $y = \sin x + C$ is a solution of
the ODE $y' = \cos x$; where C is constant

$$\text{Given } y = \sin x + C$$

$$y' = \cos x$$

plug into ODE

$$(\text{LHS}) = \cos x = \cos x \quad (\text{RHS})$$

$\text{LHS} = \text{RHS}$ $y = \sin x + C$ is a solution
of the ODE $y' = \cos x$

DE of the form $y' = kx$ is exponential growth model

DE of the form $y' = -kx$ is exponential decay model

General Solution

The general solution (primitive) of a DE is the solution which contains as many arbitrary constants (parameters) as the order of the DE.

Particular Solution

Obtained from the general solution by plugging in a particular value to the arbitrary constant. Hence the particular soln would be free from arbitrary constants.

- Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$.
- Verify that $y = -t \cos t - t$ is a solution of the initial-value problem

$$t \frac{dy}{dt} = y + t^2 \sin t \quad y(\pi) = 0$$

① we have $y = \frac{2}{3}e^x + e^{-2x} \rightarrow ①$

$$y' + 2y = 2e^x \rightarrow ②$$

$$y' = \frac{2}{3}e^x - 2e^{-2x} \rightarrow ③$$

③ & ① in ②'s LHS:

$$\begin{aligned} & \frac{2}{3}e^x - 2e^{-2x} + \frac{4}{3}e^x + 2e^{-2x} \\ &= \frac{2}{3}e^x + \frac{4}{3}e^x = \frac{6e^x}{3} = 2e^x = \text{RHS} \end{aligned}$$

② $t \frac{dy}{dt} = y + t^2 \sin t \quad y(\pi) = 0$

This is non separable.

\therefore taking the constant to 0.

$$t \frac{dy}{dt} = y + 0$$

$$\int \frac{1}{y} dy = \int \frac{1}{t} dt$$

$$e^{\ln y} = e^{\ln t + c}; c \in \mathbb{R}$$

$$y = e^{\ln t} \cdot e^c$$

$$y = t \cdot k; k \in \mathbb{R}$$

$$y_t = t \cdot k(t) \rightarrow \textcircled{1} \quad \text{varying the constant where } y_t = y_{\text{trial}}$$

$$y'_t = t \cdot k'(t) + k(t) \cdot 1 \rightarrow \textcircled{2}$$

\textcircled{1} & \textcircled{2} in q

$$t \frac{dy}{dt} = y + t^2 \sin t$$

$$t^2 k'(t) + t k(t) = t k(t) + t^2 \sin t$$

$$t^2 k'(t) = t^2 \sin t$$

$$k'(t) = \sin t$$

$$k(t) = \int \sin t dt = -\cos t + c \rightarrow y_t \text{ 's } k(t)$$

$$\therefore y_t = (-\cos t + c) t$$

$$y = (c - \cos t) t \rightarrow \text{is one solution from } y_{\text{trial}}$$

This was done for practice. But we are already given one solution in the question. We can use that

one & plug it to the question.

$$t \frac{dy}{dt} = y + t^2 \sin t \quad y(\pi) = 0$$

& one soln: $y = -t \cos t - t \rightarrow ①$

$$\therefore \frac{dy}{dt} = -t(-\sin t) - \cos t \cdot 1 - 1$$

$$\frac{dy}{dt} = t \sin t - \cos t - 1 \rightarrow ②$$

① & ② in q's

$$t^2 \sin t - t \cos t - t = -t \cos t - t + t^2 \sin t$$

we can see LHS = RHS \therefore it is indeed a solution.

Finally we are given $y(\pi) = 0$

we substitute $t = \pi$ in the solution

\therefore

$$3-a) \text{ we have } y = e^{\gamma x} \\ y' = \gamma e^{\gamma x} \quad y'' = \gamma \cdot \gamma e^{\gamma x} + e^{\gamma x} \cdot 0 \\ = \gamma^2 e^{\gamma x}$$

$$\begin{aligned}2y'' + y' - y &= 0 \\2(\sigma^2 e^{\sigma x}) + \sigma e^{\sigma x} - e^{\sigma x} &= 0 \\2\sigma^2 e^{\sigma x} + \sigma e^{\sigma x} - e^{\sigma x} &= 0 \\e^{\sigma x}(2\sigma^2 + \sigma - 1) &= 0\end{aligned}$$

$$e^{sx} = 0 \quad \delta_{1,2} = \frac{-1 \pm \sqrt{1 - 4(-2)}}{4}$$

$$\begin{array}{l} 8x=0 \\ y=0 \end{array} \quad y_{1,2} = -\frac{1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = \frac{-1+3}{4} \text{ or } \frac{-1-3}{4}$$

given: $y = a e^{x_1 x} + b e^{x_2 x}$

we have $y = a e^{\frac{1}{2}x} + b e^{-x}$ → ①

$$\therefore y' = a \cdot \left(e^{\frac{1}{2}x} \cdot \frac{1}{2} \right) + e^{\frac{1}{2}x} \cdot 0 + b \left(e^{-x} \cdot (-1) \right) + e^{-x} \cdot 0$$

$$= a \cdot \frac{1}{2} e^{\frac{1}{2}x} - b e^{-x} \quad \rightarrow \textcircled{2}$$

$$y'' = \left[\frac{a}{2} \cdot \left(\frac{1}{2} \cdot e^{\frac{1}{2}x} \right) + e^{\frac{1}{2}x} \cdot 0 \right] - \left[b (-1 e^{-x}) + e^{-x} \cdot 0 \right]$$

$$= \frac{a}{4} e^{\frac{1}{2}x} + b e^{-x} \quad \rightarrow \textcircled{3}$$

①, ② & ③ in q's LHS:

$$2 \left[\frac{a}{4} e^{\frac{1}{2}x} + b e^{-x} \right] + \left[a \frac{1}{2} e^{\frac{1}{2}x} - b e^{-x} \right] - \left[a e^{\frac{1}{2}x} + b e^{-x} \right]$$

$$\frac{a}{2} e^{\frac{1}{2}x} + 2 b e^{-x} + \frac{a}{2} e^{\frac{1}{2}x} - b e^{-x} - a e^{\frac{1}{2}x} - b e^{-x}$$

$$= a e^{\frac{1}{2}x} + 2 b e^{-x} - 2 b e^{-x} - a e^{\frac{1}{2}x} = 0 = \text{RHS}$$

$\therefore y = a_1 e^{\alpha_1 x} + b e^{\alpha_2 x}$ family of functions
is a solution to the
diff eqn

$$1) a) 4y'' = -25y$$

$$y = \cos kt$$

$$4y'' = -25y \quad \text{we have } y = \cos kt$$

$$\therefore y' = k \cdot (-\sin kt) = -k \sin kt$$

$$\begin{aligned} y'' &= -k \cos kt - \sin kt \cdot 0 \\ &= -k \cos kt \end{aligned}$$

$$\therefore 4(-k \cos kt) = -25(\cos kt)$$

$$k \cos kt = \frac{25}{4} \cos kt$$

$$k = \frac{25}{4}$$

$$y = A \sin kt + B \cos kt$$

$$y' = A \cdot k \cdot \cos kt + 0 + B (-\sin kt \cdot k) + 0$$

$$= A \cdot k \cdot \cos kt - B \cdot k \cdot \sin kt$$

$$\begin{aligned} y'' &= Ak^2(-\sin kt) - Bk^2(\cos kt) \\ &= -Ak^2 \sin kt - Bk^2 \cos kt \end{aligned}$$

$$\text{when } k = \frac{25}{4} \Rightarrow -A \frac{25^2}{4^2} \sin \frac{25}{4} t - B \frac{25^2}{4^2} \cos \frac{25}{4} t$$

=

$$\text{we have } 4y'' = -25y$$

LHS:

$$\begin{aligned} & \left(-A \frac{25}{4^2} \sin \frac{25}{4} t - B \frac{25}{4^2} \cos \frac{25}{4} t \right) \\ &= -A \frac{25}{4}^2 \sin kt - B \frac{25}{4}^2 \cos 25kt \\ &= -25 \left(\frac{25A}{4} \sin kt - \frac{25B}{4} \cos kt \right) \\ &= -25y = \text{RHS} \end{aligned}$$

Hence it is also a family of solutions

5) $y'' + y = \sin x$

a) $y = \sin x \rightarrow y' = \cos x \quad y'' = -\sin x$

LHS: $-\sin x + \cos x \neq \text{RHS}$ Hence not a solution

b) $y = \cos x \rightarrow y' = -\sin x \quad y'' = -\cos x$

LHS: $-\cos x + \cos x \neq \text{RHS}$ Hence not a solution

c) $y = \frac{1}{2} \times \sin x \rightarrow y' = \frac{1}{2} \times \cos x + \sin x \cdot \frac{1}{2}$

$$\begin{aligned} y'' &= \frac{1}{2} \times (-\sin x) + \cos x \cdot \frac{1}{2} + \\ &\quad \sin x \cdot 0 + \frac{1}{2} \cos x \end{aligned}$$

$$\begin{aligned}
 y'' + y &= -\frac{1}{2}x \sin x + \frac{1}{2} \cos x + \frac{1}{2} \cos x + \frac{1}{2}x \sin x \\
 &= \cos x \neq \text{RHS} \quad \text{Hence it is not a solution.}
 \end{aligned}$$

d) $y = -\frac{1}{2}x \cos x$

$$\begin{aligned}
 y' &= -\frac{1}{2}x(-\sin x) + \cos x \left(-\frac{1}{2}\right) \\
 &= \frac{1}{2}x \sin x - \frac{1}{2} \cos x
 \end{aligned}$$

$$\begin{aligned}
 y'' &= \frac{1}{2}x \cos x + \sin x \cdot \frac{1}{2} - \left(\frac{1}{2}(-\sin x) + \cos x \cdot 0 \right) \\
 &= \frac{1}{2}x \cos x + \frac{1}{2} \sin x + \frac{1}{2} \sin x
 \end{aligned}$$

$$y'' + y = -\frac{1}{2}x \cos x + \frac{1}{2}x \cos x + \frac{1}{2} \sin x + \frac{1}{2} \sin x$$

$\underbrace{\hspace{10em}}$
Sin x

$$y'' + y = \sin x = \text{RHS} \quad \text{Hence it is actually a solution.}$$

7. (a) What can you say about a solution of the equation $y' = -y^2$ just by looking at the differential equation?
 (b) Verify that all members of the family $y = 1/(x + C)$ are solutions of the equation in part (a).
 (c) Can you think of a solution of the differential equation $y' = -y^2$ that is not a member of the family in part (b)?
 (d) Find a solution of the initial-value problem

$$y' = -y^2 \quad y(0) = 0.5$$

b) we have $y' = -y^2$

$$\int \frac{1}{y^2} dy = \int -1 dx$$

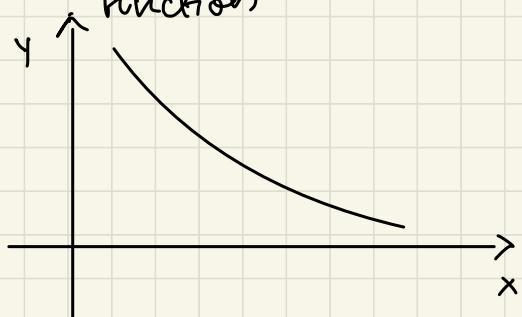
$$\frac{y^{-2+1}}{-2+1} = -x + C$$

$$\frac{y^{-1}}{-1} = -x + C$$

$$-\frac{1}{y} = -x + C$$

$$y = \frac{1}{x+C}$$

a) It is a decreasing function



c) e^x

d) given $y(0) = 0.5$

when $x=0 \quad y = 0.5$

$$\therefore y = \frac{1}{x+C} \Leftrightarrow 0.5 = \frac{1}{0+C} \Rightarrow C = \frac{1}{0.5} = 2$$

$$y = \frac{1}{x+2} \quad \text{at } y(0) = 0.5$$

1st ODE (separable)

$$-1) \quad y' = y^2 \sin x$$

$$\frac{dy}{dx} = y^2 \sin x$$

$$\frac{1}{y^2} dy = \sin x dx$$

$$= \int y^{-2} dy = \int \sin x dx$$

$$= \frac{y^{-2+1}}{-2+1} = -\cos x + C \quad C \in \mathbb{R}$$

$$\frac{1}{-y} = \cos x + 3C$$

$$y = \frac{1}{\cos x + 3C} = \frac{1}{\cos x + k} \quad \text{is a soln}$$

$y = \gamma$ is another soln ; $\gamma \in \mathbb{R}$

I^{sr} ODE (Non-separable)

2) $y' - \frac{1}{2x}y = 6x$

This is non separable

∴ varying the const to 0

$$y' - \frac{1}{2x}y = 0$$

$$\frac{dy}{dx} = \frac{1}{2x}y$$

$$\int \frac{1}{y} dy = \int \frac{1}{2x} dx$$

$$\ln y = \frac{1}{2} \ln x + C$$

$$e^{\ln y} = e^{\ln x^{1/2} + C}; C \in \mathbb{R}$$

$$y = e^{\ln x^{1/2}} \cdot e^C$$

$$y = x^{1/2} \cdot K; K \in \mathbb{R}$$

$$y_{\text{trial}} = k(x) \cdot x^{1/2} \rightarrow ①$$

$$y'_{\text{trial}} = k(x) \cdot \frac{1}{2} x^{1/2} + x^{1/2} \cdot k'(x) \rightarrow ②$$

① & ② in q

$$y' - \frac{1}{2x}y = 6x \rightarrow \textcircled{q}$$

$$k(x) \underbrace{\frac{1}{2}x^{1/2}}_{\text{---}} + x^{1/2} \cdot k'(x) - \underbrace{\frac{1}{2}k(x) \cdot x^{1/2}}_{\text{---}} = 6x$$

$$x^{1/2} \cdot k'(x) = 6x$$

$$k'(x) = \frac{6x}{x^{1/2}}$$

$$\begin{aligned} k(x) &= \int 6x^{1/2} \\ &= 6 \frac{x^{3/2}}{3/2} + C \end{aligned}$$

$$= 4x^{3/2} + C, C \in \mathbb{R}$$

\therefore plugging $k(x)$ back into y_{trial}

we have:

$$y = (4x^{3/2} + C)x^{1/2}; C, k \in \mathbb{R}$$

$$\text{eg: } y' = \frac{\cos x}{\sin 2y}$$

$$\int \sin 2y \, dy = \int \cos x \, dx$$

$$\frac{1}{2} (-\cos 2y) = \sin x + C$$

$$-\frac{1}{2} \cos 2y = \sin x + C$$

$$\cos 2y = -2 \sin x - 2C$$

$$\cos 2y = -2 \sin x + k \quad c, k \in R$$

$$y = \frac{\cos^{-1}(-2 \sin x + k)}{2}$$

$$y = \frac{1}{2} \cos^{-1}(k - 2 \sin x); \text{ where } k = -2c$$

$$1. \frac{dy}{dx} = \frac{e^{-x}}{y}$$

$$\int y \, dy = \int e^{-x} \, dx$$

$$\frac{y^2}{2} = -e^{-x} + C; \quad c \in R$$

$$y^2 = -2e^{-x} + 2c$$
$$y = \pm \sqrt{k - 2e^{-x}}, \quad k \in \mathbb{R}$$

2. $y(0) = 1$

$$\int e^y dy = \int 3x^2 dx$$

$$e^y = 3 \frac{x^3}{3} + c \quad c \in \mathbb{R}$$

$$\ln e^y = \ln(x^3 + c)$$

given $y(0) = 1$

$$\therefore c = e$$

$$y = \ln(x^3 + e)$$

$$1) \frac{dy}{dx} = 3x^2y^2$$

$$\int \frac{1}{y^2} dy = \int 3x^2 dx$$

$$-\frac{1}{y} = 3 \frac{x^3}{3} + C \quad ; \quad C \in \mathbb{R}$$

$$y = \frac{1}{x^3 + C}$$

$$2) \frac{dy}{dx} = x\sqrt{y}$$

$$\frac{1}{\sqrt{y}} dy = x dx$$

$$\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{x^2}{2} + C \quad ; \quad C \in \mathbb{R}$$

$$2\sqrt{y} = \frac{x^2 + 2C}{2}$$

$$y = \left(\frac{x^2 + K}{4} \right)^2 \quad ; \quad K \in \mathbb{R}$$

$$3) xy y' = x^2 + 1$$

$$yy' = x + \frac{1}{x}$$

$$\int y dy = \int \left(x + \frac{1}{x}\right) dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \ln x + C ; C \in \mathbb{R}$$

$$y^2 = x^2 + 2 \ln x + 2C$$

$$y = \sqrt{x^2 + 2 \ln x + k} ; k \in \mathbb{R}$$

$$4) y' + x e^y = 0$$

$$y' = -x e^y$$

$$\frac{1}{e^y} dy = -x dx$$

$$\int e^{-y} dy = \int -x dx$$

$$-e^{-y} = -\frac{x^2}{2} + C ; C \in \mathbb{R}$$

$$e^{-y} = \frac{x^2 + 2C}{2}$$

$$\ln e^{-y} = \ln \left(\frac{x^2 + 2C}{2} \right)$$

$$-y = \ln(x^2 + 2C) - \ln(2)$$

$$y = \ln(2) - \ln(x^2 + 2C)$$

$$5) (e^y - 1) y' = 2 + \cos x$$

$$\int (e^y - 1) dy = \int 2 + \cos x dx$$

$$e^y - y = 2x + \sin x + C$$

$$6) \frac{du}{dt} = \frac{1+t^4}{ut^2+u^4t^2}$$

$$\frac{du}{dt} = \frac{1+t^4}{t^2(u+u^4)}$$

$$\int (u+u^4)du = \int \left(\frac{1+t^4}{t^2} \right) dt$$

$$\frac{u^2}{2} + \frac{u^5}{5} = \frac{t^{-2+1}}{-2+1} + \frac{t^3}{3} + C ; C \in R$$

$$\frac{5u^2+2u^5}{10} = \frac{t^{-1}}{-1} + \frac{t^3}{3} + C ; C \in R$$

$$\frac{5u^2+2u^5}{10} = -\frac{1}{t} + \frac{t^3}{3} + C ; C \in R$$

$$7) \frac{d\theta}{dt} = \frac{t \sec \theta}{\theta e^{t^2}}$$

$$\frac{\theta}{\sec \theta} d\theta = \frac{t}{e^{t^2}} dt$$

$$\int \theta \cos \theta d\theta = \int \frac{t}{e^{t^2}} dt$$

$$\int x \cos x dx = \int \frac{t}{e^{t^2}} dt$$

LHS:

$$u = x \quad v dv = \cos x dx \\ du = dx \quad v = \sin x$$

$$t^2 = u \\ 2t dt = du$$

$$tdt = \frac{1}{2} du$$

RHS:

$$\begin{aligned} & \int \frac{1}{2} \cdot \frac{1}{e^u} du \\ &= \frac{1}{2} \int e^{-u} du \\ &= \frac{1}{2} \cdot e^{-u} + C \\ &= -\frac{1}{2} e^{t^2} + C \end{aligned}$$

\therefore we have

$$x \sin x + \cos x = -\frac{1}{2} e^{t^2} + C ; C \in \mathbb{R}$$

$$\theta \sin \theta + \cos \theta = C - \frac{1}{2} e^{t^2} ; C \in \mathbb{R}$$

$$(8) \quad \frac{dH}{dR} = \frac{RH^2 \sqrt{1+R^2}}{\ln H}$$

$$\int \frac{\ln H}{H^2} dH = \int (R \sqrt{1+R^2}) dR$$

$$\frac{1}{H} = u \quad (\text{LHS}) \quad 1+R^2 = u \quad (\text{RHS})$$

$$2R dR = du$$

$$R dR = \frac{1}{2} du$$

$$\ln H dH = du$$

$$\int \frac{1}{u^2} du = \int \frac{1}{2} \sqrt{u} du$$

$$\frac{u^{-2+1}}{-2+1} = \frac{1}{2} \frac{u^{1/2+1}}{1/2+1} + C$$

$$\frac{u^{-1}}{-1} = \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$-\frac{1}{u} + C = \frac{1}{3} u^{3/2} + C$$

$$-H + C = \frac{1}{3} (1+R^2)^{3/2} + C$$

$$-H = \frac{1}{3} \sqrt{(1+R^2)^3} + C$$

$$-3H = (1+R^2)^{3/2} + 3C ; C, k \in R$$

$$3H = - (1+R^2)^{3/2} + K ; C, K \in R$$

$$⑨ \frac{dp}{dt} = t^2 p - p + t^2 - 1$$

$$\frac{dp}{dt} = p(t^2 - 1) + t^2 - 1$$

$$\frac{dp}{dt} = (t^2 - 1)[p + 1]$$

$$\frac{1}{p+1} \frac{dp}{dt} = (t^2 - 1) dt$$

$$e^{\ln |p+1|} = e^{\frac{t^3 - t}{3} + c}; c \in \mathbb{R}$$

$$p+1 = e^{\frac{t^3 - 3t}{3} + c}; c \in \mathbb{R}$$

$$p+1 = e^{\frac{t^3 - 3t}{3}} \cdot e^c; c \in \mathbb{R}$$

$$p = e^{\frac{t^3 - 3t}{3}} \cdot k - 1; c \in \mathbb{R}$$

$$⑩ \frac{dz}{dt} + e^{t+z} = 0$$

$$\frac{dz}{dt} = -e^{t+z} = -e^t \cdot e^z$$

$$\frac{1}{e^z} dz = -e^t dt$$

$$\int e^{-z} dz = \int -e^t dt$$

$$= -e^{-z} = -e^t + c ; c \in \mathbb{R}$$

$$e^{-z} - e^t + c = 0$$

(11.)

$$\frac{dy}{dx} = x e^y ; y(0) = 0$$

$$\int \frac{1}{e^y} dy = \int x dx$$

$$-e^{-y} = \frac{x^2}{2} + c$$

for $y(0) = 0$; $x=0$ then $y=0$

$$-\frac{1}{e^0} = \frac{0^2}{2} + c$$

$$-1 = c$$

$$c = -1$$

$$-\frac{1}{e^y} = \frac{x^2}{2} - 1$$

$$-\frac{1}{e^y} = \frac{x^2 - 2}{2}$$

$$\frac{-2}{x^2 - 2} = e^y$$

$$\ln e^y = \ln \left| \frac{-2}{x^2 - 2} \right|$$

$$y = \ln |2| - \ln |x^2 - 2|$$

(12)

$$\frac{dy}{dx} = \frac{x \sin x}{y}$$

$$\int y dy = \int x \sin x dx$$

$$u = x \quad v du = \sin x \\ du = dx \quad v = -\cos x$$

$$\frac{y^2}{2} = -x \cos x + \int \cos x dx + C$$

$$\frac{y^2}{2} = -x \cos x + \sin x + C$$

$$y^2 = 2(\sin x - x \cos x + C)$$

$$y(0) = -1 \\ \text{when } x=0 \\ y = -1$$

$$\therefore 1 = 2(0 - 0 + C)$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

$$\frac{y^2}{2} = -x \cos x + \sin x + \frac{1}{2}$$

$$y = \sqrt{2 \sin x - 2x \cos x + 1}$$

$$(13) \quad \frac{du}{dt} = \frac{2t + \sec^2 t}{2u} \quad u(0) = -5$$

$$\int 2u du = \int 2t + \sec^2 t dt$$

$$2 \frac{u^2}{2} = 2 \frac{t^2}{2} + \int \sec^2 t dt + C$$

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

$$\therefore \int \sec^2 x = \tan x + C$$

$$u^2 = t^2 + \tan t + C ; \quad u(0) = -5$$

$t=0; u=-5$

$$25 = 0 + \tan 0 + C$$

$$C = 25$$

$$\therefore u^2 = t^2 + \tan t + 25$$

$$(14) \quad x + 3y^2 \sqrt{x^2 + 1} \frac{dy}{dx} = 0$$

$$3y^2 \sqrt{x^2 + 1} dy = -x dx$$

$$3y^2 dy = \frac{-x}{\sqrt{x^2 + 1}} dx$$

$$\int y^2 dy = -\frac{1}{3} \int \frac{x}{\sqrt{x^2 + 1}} dx$$

$$\begin{aligned} x^2 + 1 &= u \\ 2x dx &= du \\ x dx &= \frac{1}{2} du \end{aligned}$$

$$\frac{y^3}{3} = -\frac{1}{3} \int \frac{1}{2} \frac{1}{\sqrt{u}} du$$

$$y^3 = -\frac{1}{2} \frac{u^{-1/2+1}}{-1/2+1} + C$$

$$-2y^3 = 2u^{1/2} + C \quad \text{given } y(0) = 1$$

$$-2y^3 = 2(x^2+1)^{1/2} + C \quad x=0; y=1$$

$$-2 = 2(0+1)^{1/2} + C$$

$$C = -1$$

$$\therefore -2y^3 = 2(x^2+1)^{1/2} - 1$$

$$y^3 = \frac{1}{2} - \sqrt{x^2+1}$$

? check later.

(15)

$$x \ln x = y \left(1 + \sqrt{3+y^2} \right) y' ; \quad y(1) = 1$$

$$\int x \ln x dx = \int \left(y + y \sqrt{3+y^2} \right) dy$$

$$3+y^2 = u \\ 2y dy = du$$

$$u = \ln x \quad v dv = x dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$$

$$\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$\frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C$$

$$\frac{y^2}{2} + \int \frac{1}{2} \frac{u}{\sqrt{u}} du$$

$$\frac{y^2}{2} + \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$\frac{x^2}{2} \ln x - \frac{x^2}{4} + C = \frac{y^2}{2} + \frac{1}{3} \cdot (3+y^2)^{3/2} + C$$

$$\frac{x^2}{2} \ln x - \frac{x^2}{4} = \frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3} + C$$

given $y(1) = 1$

$$x=1 ; y=1$$

$$\frac{1}{2} \cdot 0 - \frac{1}{4} = \frac{1}{2} + \frac{3^{3/2}}{3} + C$$

$$-\frac{1}{4} = \frac{1}{2} + \frac{9\sqrt{3}}{3} + C$$

$$C = -\frac{1}{4} - \frac{1}{2} - \frac{9\sqrt{3}}{3}$$

$$= -\frac{2-4}{8} - \frac{9\sqrt{3}}{3}$$

$$= -\frac{6(3)}{24} - \frac{72\sqrt{3}}{24}$$

$$= -\frac{18 - 72\sqrt{3}}{24}$$

check
 (at-6)
 on W_o(fram)

$$⑯ \quad \frac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$$

$$dP = \sqrt{P} \cdot \sqrt{t} dt$$

$$\int \frac{1}{\sqrt{P}} dP = \int \sqrt{t} dt$$

$$\frac{P^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$2P^{\frac{1}{2}} = \frac{2}{3} t^{\frac{3}{2}} + C \quad P(1) = 2$$

$$t=1; P=2$$

$$2\sqrt{2} = \frac{2}{3} \cdot 1 + C$$

$$2\sqrt{2} - \frac{2}{3} = C$$

$$C = \frac{6\sqrt{2} - 2}{3}$$

$$2\sqrt{P} = \frac{2}{3} \sqrt{t^3} + \frac{6\sqrt{2} - 2}{3}$$

$$2\sqrt{P} = \frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{3}$$

$$P = \left(\frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{6} \right)^2$$

$$(17) \quad y' \tan x = a + y \quad y(\pi/3) = a$$

$$\int \frac{1}{a+y} dy = \int \frac{1}{\tan x} dx$$

$$0 < x < \pi/2$$

$$x = \frac{\pi}{3}, y = a$$

$$\text{RHS: } \int \frac{1}{\tan x} \Rightarrow \int \frac{\sec^2 x dx}{\sec^2 x \tan x}$$

$$\sec x \tan x = u$$

$$\sec^2 x dx = du$$

$$(11) \quad \frac{dL}{dt} = k L^2 \ln t, \quad L(1) = -1$$

$$\frac{1}{L^2} dL = k \ln t dt$$

$$\frac{L^{-2+1}}{-2+1} = K \int \ln t dt$$

$$= K \frac{1}{t} u du$$

$$= K \int \frac{u du}{e^u}$$

$$= K \int u \cdot e^{-u} du$$

$$\approx K \int x e^{-x} dx$$

$$\ln t = u$$

$$\frac{1}{t} dt = du$$

$$e^{\ln t} = e^u$$

$$t = e^u$$

$$u = x \quad v dv = e^{-x} dx$$

$$du = dx \quad v = -e^{-x}$$

$$= K \left[-u e^{-u} + \int e^u du \right]$$

$$= K \left[e^u - \frac{u}{e^u} \right]$$

$$= K \left[e^{\ln t} - \frac{\ln t}{e^{\ln t}} \right] = K \left[t - \frac{\ln(t)}{t} \right]$$

$$-\frac{1}{L} = k \left[t - \frac{\ln t}{t} \right] + C ; \quad C \in R$$

given $L(1) = -1$

$$t=1 ; L=-1$$

$$-\frac{1}{-1} = k \left[1 - \frac{0}{1} \right] + C$$

$$1 = k + C$$

$$C = 1 - k$$

$$-\frac{1}{L} = k \left[t - \frac{\ln t}{t} \right] + 1 - k$$

$$-\frac{1}{L} = k \left[\frac{t^2 - \ln t}{t} \right] + 1 - k$$

$$-\frac{1}{L} = \frac{k t^2 - k \ln t + t - kt}{t}$$

$$L = -\frac{t}{kt^2 - k \ln t + t - kt}$$

$$f'(x) = x f(x) - x \quad f(0) = 2$$

$$y' = x y - x$$

$$y' = x(y-1)$$

$$\int \frac{1}{y-1} dy = \int x dx$$

$$\ln|y-1| = \frac{x^2}{2} + C$$

given $f(0) = 2$
 $x=0$

$$y' = -3x^2y + \cancel{6x^2}$$

$$\frac{dy}{dx} = -3x^2y$$

$$\int \frac{1}{y} dy = \int -3x^2 dx$$

$$\ln|y| = -3 \frac{x^3}{3} + C$$

$$e^{\ln|y|} = e^{-x^3} + C$$

$$y = e^{-x^3} \cdot e^C$$

$$y = e^{-x^3} \cdot K$$

vary the K to constant

$$y = e^{-x^3} k(x)$$

$$y' = e^{-x^3} k'(x) + k(x) - e^{-x^3} \cdot 3x^2$$

plugging back.

$$e^{-x^3} k'(x) - 3x^2 e^{-x^3} k(x) + 3x^2 (e^{-x^3} k(x)) = 6x^2$$

$$e^{-x^3} k'(x) = 6x^2$$

$$k'(x) = \frac{6x^2}{e^{-x^3}}$$

$$k'(x) = 6x^2 e^{x^3}$$

$$k(x) = \int 6x^2 e^{x^3} dx = 2 \int 3x^2 e^{x^3} dx$$

$$\begin{aligned} x^3 &= u \\ 3x^2 dx &= du \end{aligned}$$

$$= 2 \int e^u du$$

$$= 2 e^u + C$$

$$= 2 e^{x^3} + C$$

$$\therefore y = e^{-x^3} (2 e^{x^3} + C)$$

$$y = 2 + e^{-x^3} \cdot C$$

Now trying separable

$$y' = -3x^2 y + 6x^2$$

$$y' = -3x^2 (y + 2)$$

$$\int \frac{1}{y+2} dy = \int -3x^2 dx$$

$$\ln|y+2| = -3 \frac{x^3}{3} + C$$

$$e^{\ln|y+2|} = e^{-x^3} \cdot e^C$$

$$y+2 = e^{-x^3} \cdot e^C$$

$$y = e^{-x^3} \cdot k + 2$$

9.3 Exercises

1–10 Solve the differential equation.

1. $\frac{dy}{dx} = xy^2$

2. $\frac{dy}{dx} = xe^{-y}$

3. $xy^2y' = x + 1$

4. $(y^2 + xy^2)y' = 1$

5. $(y + \sin y)y' = x + x^3$

6. $\frac{dv}{ds} = \frac{s+1}{sv+s}$

7. $\frac{dy}{dt} = \frac{t}{ye^{y+t^2}}$

8. $\frac{dy}{d\theta} = \frac{e^y \sin^2 \theta}{y \sec \theta}$

9. $\frac{dp}{dt} = t^2 p - p + t^2 - 1$

10. $\frac{dz}{dt} + e^{iz} = 0$

11–18 Find the solution of the differential equation that satisfies the given initial condition.

11. $\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -3$

12. $\frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2$

13. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}, \quad u(0) = -5$

14. $y' = \frac{xy \sin x}{y+1}, \quad y(0) = 1$

15. $x \ln x = y(1 + \sqrt{3 + y^2})y', \quad y(1) = 1$

16. $\frac{dP}{dt} = \sqrt{Pt}, \quad P(1) = 2$

17. $y' \tan x = a + y, \quad y(\pi/3) = a, \quad 0 < x < \pi/2$

18. $\frac{dL}{dt} = kL^2 \ln t, \quad L(1) = -1$

$$9.3) \quad \frac{dy}{dx} = xy^2$$

$$y' = xy^2$$

This is separable

$$\int \frac{1}{y^2} dy = \int x dx$$

$$\frac{y^{-2+1}}{-2+1} = \frac{x^2}{2} + C$$

$$-\frac{1}{y} = \frac{x^2}{2} + C$$

$$y = \frac{1}{-\left(\frac{x^2}{2} + C\right)} = -\frac{1}{\frac{x^2}{2} + C} ; \quad C \in R$$

$$\text{Other solution } y = \gamma ; \quad y = 0 ; \quad \gamma \in R$$

$$\therefore 0 = x\gamma^2 ; \quad \gamma = 0$$

$$2) \quad \frac{dy}{dx} = x e^{-y}$$

$$\int e^y dy = \int x dx$$

$$e^y = \frac{x^2}{2} + C$$

$$\ln e^y = \ln \left(\frac{x^2}{2} + C \right)$$

$$y = \ln \left(\frac{x^2}{2} + C \right) ; \quad C \in R$$

For the other solution:

$$y = \gamma ; \quad y' = 0 \quad \text{where} \quad \gamma \in \mathbb{R}$$

3) $xy^2 y' = x + 1$

This is 1st order Separable:

$$\int y^2 dy = \int \frac{x+1}{x} dx$$
$$\frac{y^3}{3} = \int \left(1 + \frac{1}{x}\right) dx$$
$$= x + \ln|x| + C$$

$$y = \sqrt[3]{3x + 3\ln|x| + 3C} ; \quad C \in \mathbb{R}$$

& for other solution; $y = \gamma ; \quad \gamma \in \mathbb{R}$
 $\therefore y' = 0$

4) $(y^2 + xy^2)y' = 1$

This is separable

$$y^2(1+x^2)y' = 1$$

$$\int y^2 dy = \int \frac{1}{1+x^2} dx \quad \text{if it was}$$

$$\int \frac{1}{1+x^2} dx$$

$$\frac{y^3}{3} = \ln|1+x| = \tan^{-1}(x) + C$$

$$y = 3\sqrt{3\ln|1+x| + 3C}; \quad C \in R \quad \text{& } y = \delta; \quad \delta \in R$$

5) $(y + \sin y) y' = x + x^3$

$$\int (y + \sin y) dy = \int (x + x^3) dx$$

$$\frac{y^2}{2} - \cos y = \frac{x^2}{2} + \frac{x^4}{4} + C$$

$$y^2 - 2\cos y = x^2 + \frac{x^4}{2} + 2C; \quad C \in R$$

cannot be expressed in terms of elementary function. The soln is an implicit soln.

6) $\frac{dv}{ds} = \frac{s+1}{sv+s}$

$$(v+1)dv = \frac{s+1}{s} ds$$

$$\int (v+1)dv = \int \left(1 + \frac{1}{s}\right) ds$$

$$\frac{v^2}{2} + v = s + \ln|s| + C; \quad C \in R$$

$$v^2 + 2v = 2s + 2\ln|s| + 2C; \quad C \in R$$

& $v=1$ is also another soln; $\gamma \in R$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$7) \frac{dy}{dt} = \frac{t}{ye^{y+t^2}}$$

$$ye^y \cdot e^{t^2} dy = t dt$$

$$ye^y dy = \frac{t}{e^{t^2}} dt$$

$$\int ye^y dy = \int t \cdot \frac{1}{e^{t^2}} dt$$

$$\begin{aligned}
 u &= y & v dv &= e^y dy & t^2 &= u \\
 du &= dy & v &= e^y & 2t dt &= du \\
 &&&& t dt &= \frac{1}{2} du \\
 ye^y - \int e^y dy &&&&& \\
 = ye^y - e^y + C && \int \frac{1}{2} \cdot \frac{1}{e^u} du &&& \\
 &&= \frac{1}{2} \int e^{-u} du &&& \\
 &&= -\frac{1}{2} \cdot e^{-u} + C &&& \\
 &&= -\frac{1}{2} e^{-t^2} + C &&&
 \end{aligned}$$

$$\therefore ye^y - e^y = -\frac{1}{2} e^{-t^2} + C \quad ; \quad C \in \mathbb{R}$$

$y = r$ is another soln ; $r \in \mathbb{R}$

$$8. \frac{dy}{d\theta} = \frac{e^y \sin^2 \theta}{y \sec \theta}$$

$$\frac{y}{e^y} dy = \sin^2 \theta \cos \theta d\theta$$

$$\int y \cdot e^{-y} dy = \int \sin^2 \theta \cos \theta d\theta$$

$$u = y \quad v dv = e^{-y} dy \\ du = dy \quad v = -e^{-y}$$

$$\sin \theta = u \\ \cos \theta d\theta = du$$

$$-ye^{-y} - \int e^{-y} dy$$

$$= \int u^2 du$$

$$-ye^{-y} + e^{-y} + C$$

$$= \frac{u^3}{3} + C$$

$$e^{-y}(1-y)$$

$$= \frac{\sin^3 \theta}{3} + C$$

$$\therefore e^{-y}(1-y) = \frac{\sin^3 \theta}{3} + C \quad \text{is one solution}$$

other solution is $y = \gamma$ & $C, \gamma \in R$

$$9) \frac{dp}{dt} = t^2 p - pt + t^2 - 1$$

$$\frac{dp}{dt} = p(t^2 - 1) + t^2 - 1$$

$$\frac{dp}{dt} = (t^2 - 1) [p + 1]$$

$$\int \frac{1}{p+1} dt = \int (t^2 - 1) dt$$

$$e^{\ln(p+1)} = e^{\frac{t^3}{3} - t + C}; C \in \mathbb{R}$$

$$p+1 = e^{\frac{t^3}{3} - t + C}$$

$$p = e^{\frac{t^3}{3} - t} \cdot e^{C - 1}$$

$$p = e^{\frac{t^3 - 3t}{3}} \cdot K \quad ; \quad C, K \in \mathbb{R}$$

is one solution

other solution is $p = \gamma$; $\gamma \in \mathbb{R}$

$$10. \quad \frac{dz}{dt} + e^{t+z} = 0$$

$$\frac{dz}{dt} = -e^{t+z} = -e^t \cdot e^z$$

$$\int \frac{1}{e^z} dz = \int -e^t dt$$

$$\int e^{-z} dz = -1 \int e^t dt$$

$$-e^{-z} = -e^t + C; C \in \mathbb{R}$$

$$e^{-z} = e^t + K; K \in \mathbb{R}$$

$$\ln e^{-z} = \ln(e^t + k)$$

$$-z = \ln(e^t + k)$$

$$z = -\ln(e^t + k) ; \quad k \in R$$

other soln is $z = \gamma ; \quad \gamma \in R$

11) $\frac{dy}{dx} = \frac{x}{y} ; \quad y(0) = -3$

$$\int y dy = \int x dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C ; \quad C \in R$$

$$y(0) = -3 \quad x=0 \quad y=-3$$

$$\frac{9}{2} = 0 + C ; \quad C = \frac{9}{2}$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{9}{2}$$

$$y^2 = x^2 + 9 \\ y = \pm \sqrt{x^2 + 9} ; \quad y(0) = -3$$

since $y(x)$ is -ve we choose the negative square root

$$\therefore y = -\sqrt{x^2 + 9} ; \quad y(0) = -3$$

$$12) \frac{dy}{dx} = \frac{\ln x}{xy}, \quad y(1) = 2$$

$$\int y dy = \int \frac{\ln x}{x} dx$$

$$\ln x = u \\ \frac{1}{x} dx = du$$

$$\frac{y^2}{2} = \int u du$$

$$\frac{y^2}{2} = \frac{u^2}{2} + C$$

$$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$$

$$\cancel{\frac{y^2}{2} = 2 \frac{\ln x}{2} + C}$$

worry!

$$\ln a^n = n \log a$$

$$(\ln a)^n \neq n \log a$$

$$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + C$$

$$\frac{y^2}{2} = \frac{(\ln^2 x)}{2} + C$$

$$x=1 \quad y=2$$

$$C=2$$

$$\therefore y = \sqrt{\ln^2 x + 4}$$

Now $y(1) = 2$; $x=1 \quad y=2$

$$\frac{1}{2} = 2 \frac{\ln 1}{2} + C \quad C \in R$$

$$C=2$$

$$\therefore \frac{y^2}{2} = \ln x + 2$$

$$y^2 = 2 \ln x + 4$$

$$y = \sqrt{2 \ln x + 4}$$

$$13. \quad \frac{du}{dt} = \frac{2t + \sec^2 t}{2u}; \quad u(0) = -5$$

$$\int 2u du = \int (2t + \sec^2 t) dt; \quad u(0) = -5$$

$$2 \frac{u^2}{2} = 2 \frac{t^2}{2} + \tan t + C; \quad C \in R$$

$$u^2 = t^2 + \tan t + C$$

$$t=0; u=-5$$

$$25 = 0 + 0 + C$$

$$C = 25$$

$$u^2 = t^2 + \tan t + 25$$

$$u = \pm \sqrt{t^2 + \tan t + 25}$$

14. $y' = \frac{xg \sin x}{y+1}, y(0)=1$

$$\int \left(\frac{y+1}{y}\right) dy = \int x \sin x dx$$

$$\int 1 + \frac{1}{y} dy = \int x \cdot \sin x dx$$

$$\begin{aligned} u &= x & v dv &= \sin x dx \\ du &= dx & v &= -\cos x \end{aligned}$$

$$-x \cos x + \int \cos x dx$$

$$y + \ln|y| = \sin x - x \cos x + C$$

$$y(0)=1 \quad x=0; y=1$$

$$1 + 0 = 0 - 0 + C$$

$$C = 1$$

$$\therefore y + \ln|y| = \sin x - x \cos x + 1$$

$$\begin{aligned} f'(\sin x) &\Rightarrow \cos x \\ f'(\cos x) &\Rightarrow -\sin x \end{aligned}$$

$$15. \quad x \ln x = y(1 + \sqrt{3+y^2})y', \quad y(1) = 1$$

$$\int (y + y\sqrt{3+y^2}) dy = \int x \ln x dx$$

$$\frac{y^2}{2} + \int y\sqrt{3+y^2} dy = \int x \ln x dx$$

$$3+y^2 = u$$

$$2y dy = du$$

$$y dy = \frac{1}{2} du$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$vdv = x \frac{dx}{2} \quad v = \frac{x^2}{2}$$

$$\frac{y^2}{2} + \frac{1}{2} \cdot \frac{u^{3/2+1}}{3/2} + C = \frac{1}{2} x^2 \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx ; C \in R$$

$$\frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3} = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C_1 ; C_1 \in R$$

$$\text{given } y(1) = 1$$

$$\frac{1}{2} + \frac{(4)^{3/2}}{3} = \frac{1}{2} \cdot 0 - \frac{1}{2} \cdot \frac{1}{2} + C_1$$

$$\frac{1}{2} + \frac{8}{3} = -\frac{1}{4} + C_1$$

$$C_1 = \frac{1}{2} + \frac{8}{3} + \frac{1}{4}$$

$$24C_1 = 12 + 8 \cdot 8 + 6$$

$$C_1 = \frac{82}{24} = \frac{41}{12}$$

$$\frac{y^2}{2} + \frac{(3+y^2)^{3/2}}{3} = \frac{x^2}{2} \ln x - \frac{x^2}{4} + \frac{49}{12}$$

$$16. \quad \frac{dp}{dt} = \sqrt{pt} ; \quad p(1) = 2$$

$$\int \frac{1}{\sqrt{p}} dp = \int \sqrt{t} dt$$

$$\frac{p^{-1/2+1}}{-\frac{1}{2}+1} = \frac{t^{1/2+1}}{\frac{1}{2}+1} + C ; \quad C \in R$$

$$2p^{1/2} = 2 \cdot \frac{t^{3/2}}{3} + C$$

$$p(1) = 2 \quad t=1 ; \quad p=2$$

$$2\sqrt{2} = \frac{2}{3} + C$$

$$C = 2\sqrt{2} - \frac{2}{3} = \frac{6\sqrt{2}-2}{3}$$

$$2\sqrt{p} = \frac{2\sqrt{t^3}}{3} + \frac{6\sqrt{2}-2}{3}$$

$$4p = \left(\frac{2\sqrt{t^3} + 6\sqrt{2} - 2}{3} \right)^2$$

$$p = \frac{(2\sqrt{t^3} + 6\sqrt{2} - 2)^2}{36}$$

$$17. \quad y' \tan x = a+y, \quad y(\pi/3) = a, \quad 0 < x < \pi/2$$

$$\frac{dy}{dx} = \frac{a+y}{\tan x}$$

$$\int \frac{1}{a+y} dy = \int \frac{1}{\tan x} dx$$

$$\ln|a+y| = \int \sec x \, dx$$

$$\ln|a+y| = \int \frac{\sec x (\sec x + \tan x)}{(\sec x + \tan x)} dx$$

$$= \int \frac{(\sec^2 x + \sec x \tan x)}{\sec x + \tan x} dx$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

Both did not work

$$u = \sec x$$

$$du = \sec x \tan x \, dx$$

recall the trick

Multiply & divide
by $(\sec x + \tan x)$

$$\sec x + \tan x = u$$

$$\sec x \tan x + \sec^2 x = du$$

$$= \int \frac{1}{u} du$$

$$e^{\ln|a+y|} = e^{\ln|\sec x + \tan x|} + C \quad ; \quad C \in \mathbb{R}$$

$$a+y = (\sec x + \tan x) \cdot e^C$$

$$y\left(\frac{\pi}{3}\right) = a \quad \text{i.e.} \quad x = \frac{\pi}{3} \quad y = a$$

$$2a = \left(\sec \frac{\pi}{3} + \tan \frac{\pi}{3}\right) \cdot e^C$$

$$2a = \left(\frac{1}{\cos 60^\circ} + \tan 60^\circ\right) \cdot e^C$$

$$\frac{\sqrt{3}}{2} \div \frac{1}{2}$$

$$2a = (2 + \sqrt{3}) e^C$$

$$\frac{2a}{2+\sqrt{3}} = e^C$$

∴

$$C = \ln \left| \frac{2a}{2+\sqrt{3}} \right|$$

$$\therefore y = (\sec x + \tan x) \cdot e^{\ln \left| \frac{2x}{2\sqrt{3}} \right|} - a$$

$$y = \frac{(\sec x + \tan x) \cdot 2x}{2 + \sqrt{3}} - a$$

$$18. \frac{dL}{dt} = KL^2 \ln t \quad L(1) = -1$$

$$\frac{1}{KL^2} dL = \ln t dt$$

$$= \frac{1}{K} \cdot \int \frac{1}{L^2} dL = \int \ln t dt$$

LHS:

$$\begin{aligned} & \frac{1}{K} \cdot \frac{L^{-2+1}}{-2+1} + C \\ &= \frac{1}{K} \cdot \frac{L^{-1}}{-1} + C \\ &= -\frac{1}{K} \frac{1}{L} + C \\ &= -\frac{1}{KL} + C_1; \quad C_1 \in R \end{aligned}$$

RHS:

$$\begin{aligned} u &= \ln t & v du &= 1 dt \\ du &= \frac{1}{t} dt & v &= t \\ t \ln t - \int 1 dt & \\ &= t \ln t - t + C_2; \quad C_2 \in R \end{aligned}$$

$$-\frac{1}{KL} = t \ln t - t + C_3; \quad C_3 \in R$$

given $L(1) = -1$ ie $t = 1$; $L = -1$

$$-\frac{1}{K(-1)} = 0 - 1 + C_3$$

$$\frac{1}{K} = -1 + C_3$$

$$C_3 = 1 + \frac{1}{K}$$

\therefore we have:

$$-\frac{1}{KL} = t \ln t - t + 1 + \frac{1}{K}$$

$$-\frac{1}{KL} - \frac{1}{K} = t \ln t - t$$

$$-\frac{1}{K} \left(\frac{1}{L} + 1 \right) = t (\ln t - 1)$$

$$\frac{1}{L} + 1 = -kt (\ln t - 1)$$

$$\frac{1}{L} = -kt (\ln t - 1) - 1$$

$$L = \frac{-1}{kt (\ln t - 1) + 1}$$

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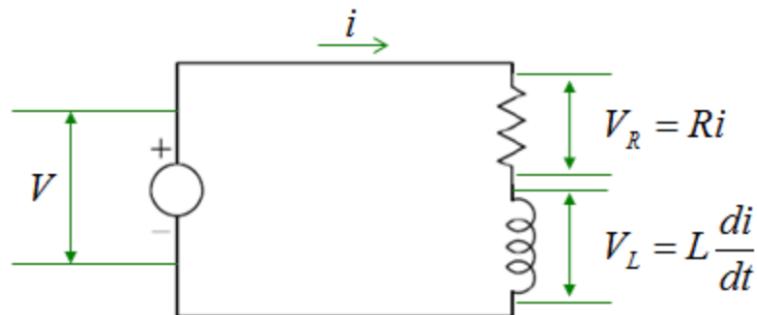
Why did the differential equation go to therapy? Because it had trouble separating variables.

1. Ordinary Differential equations

1. $\frac{dP}{dt} = t^2 P - P + t^2 - 1$

2. $\frac{dy}{dx} = \frac{x y \sin(x)}{y+1} \quad y(0) = 1$

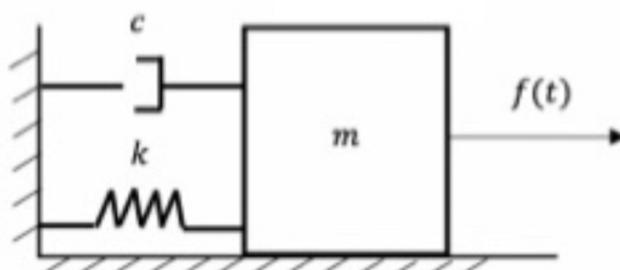
3. $t \ln(t) \frac{dr}{dt} = t e^t - r$



4.

Find the function $I(t)$ with respect to time given this circuit has a Resistance $R = 12 \Omega$ and Inductance of $L = 4H$. This circuit is connected to a DC power supply of 60V. Find the Function $I(t)$ if $I(0) = 0$ and find the Maximum current in the circuit.

Hints: $V_{DC} = V_R + V_L$



5.

If for the spring mass damper shown below $f(t) = 100 \sin(2t)$, find the equations that describes its path. Given that $m = 2\text{kg}$, $k = 40\text{N/m}$ and $c = 10\text{Ns/m}$. $x(0) = 1$, $x'(0) = 0$.

Hints: $ma = f(t) - c \cdot v - k \cdot x$ $v = \frac{dx}{dt}$ $a = \frac{d^2x}{dt^2}$

$$1) \frac{dP}{dt} = t^2 P - P + t^2 - 1 \quad (\text{separable})$$

$$\frac{dP}{dt} = P(t^2 - 1) + (t^2 - 1)$$

$$dP = (t^2 - 1)[P + 1] dt$$

$$\int \frac{1}{P+1} dP = \int (t^2 + 1) dt$$

$$e^{\ln|P+1|} = e^{t^3/3 + t + C}$$

$$P+1 = e^{t^3/3 + t + C}$$

$$P = e^{t^3/3 + t + C} - 1 ; \quad C \in \mathbb{R}$$

↳ $y = e^{t^3/3 + t + C}$; $C \in \mathbb{R}$
is another solution

$$2) \frac{dy}{dx} = \frac{xy \sin(x)}{y+1} \quad y(0) = 1 \quad (\text{separable})$$

$$y+1 \ dy = xy \sin(x) dx \quad y(0) = 1$$

$$\int \frac{y+1}{y} dy = \int x \sin(x) dx$$

$$\int \left(1 + \frac{1}{y}\right) dy = \int x \sin(x) dx \quad u = x \quad v \sin(x) dx \\ du = dx \quad v = -\cos(x)$$

$$y + \ln|y| = -x \cos x + \int \cos x dx$$

$$y + \ln|y| = \sin x - x \cos x + C \quad ; \quad C \in \mathbb{R}$$

given $y(0) = 1$
ie $x=0 ; y=1$

$$1 + \ln|1| = \sin 0 - 0 + C$$

$$1 + 0 = 0 - 0 + C$$

$$C = 1$$

$$\therefore y + |\ln|y|| = \sin x - x \cos x + 1$$

3) $t \ln|t| \frac{dy}{dt} = te^t - r$

(Non separable)

\therefore varying constant to 0

$$t \ln|t| \frac{dy}{dt} = -r$$

$$\int \frac{1}{0} dy = \int -\frac{1}{t \ln|t|} dt$$

$$\ln|r| = -1 \int \frac{du}{u}$$

$$\ln|t| = u$$

$$\frac{1}{t} dt = du$$

$$\ln|r| = -\ln|u| + C_1 ; C_1 \in R$$

$$e^{\ln|r|} = e^{-\ln|\ln t| + C_1}$$

$$r = e^{\ln|\ln t| + C_1}$$

$$r = \frac{1}{\ln t} \cdot e^{C_1}$$

$$r = \frac{1}{\ln t} \cdot k ; k \in R$$

Varying the constant:

$$r_{\text{trial}} = \frac{1}{\ln t} \cdot k(t) = \frac{k(t)}{\ln(t)}$$

$$\gamma'_{\text{total}} = \frac{\ln(t) k'(t) - k(t) \frac{1}{t}}{(\ln(t))^2}$$

Substituting them back in question:

$$\frac{t \ln|t| \cdot \left[\ln(t) k'(t) - k(t) \cdot \frac{1}{t} \right]}{(\ln(t))^2} = t e^t - \frac{k(t)}{\ln|t|}$$

$$\frac{t}{\ln|t|} \left[\ln|t| k'(t) - k(t) \cdot \frac{1}{t} \right] = t e^t - \frac{k(t)}{\ln|t|}$$

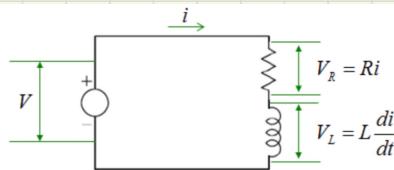
$$t k'(t) - \frac{k(t)}{\ln|t|} = t e^t - \frac{k(t)}{\ln|t|}$$

$$t k'(t) = t e^t$$

$$k'(t) = e^t$$

$$k(t) = e^t + c_2 \quad ; \quad c_2 \in \mathbb{R}$$

$$\therefore \gamma = \frac{e^t + c_2}{\ln t} \quad ; \quad c_2 \in \mathbb{R}.$$



4.

Find the function $I(t)$ with respect to time given this circuit has a Resistance $R = 12 \Omega$ and Inductance of $L = 4H$. This circuit is connected to a DC power supply of 60V. Find the Function $I(t)$ if $I(0) = 0$ and find the Maximum current in the circuit.

Hints: $V_{DC} = V_R + V_L$

$$R = 12 \Omega \quad L = 4 H \quad V = 6 V \quad \text{Find } I(t); \quad I(0) = 0$$

$$V_{DC} = V_R + V_L$$

$$V_{DC} = R i + L \frac{di}{dt}$$

$$V_{DC} = 12i + 4 \frac{di}{dt}$$

$$6 = 12i + 4 \frac{di}{dt}$$

\Leftrightarrow is similar to the diff eqn of form
 $12y + 4 \frac{dy}{dt} = 6$

where $y = i$

$$\therefore 12y + 4 \frac{dy}{dt} = 6 \quad (\text{Non separable})$$

$$\therefore 12y + 4 \frac{dy}{dt} = 0$$

$$4 \frac{dy}{dt} = -12y$$

$$\int \frac{1}{y} dy = \int -\frac{12}{4} dt$$

$$e^{\ln y} = e^{-3t + C_1}; \quad C_1 \in \mathbb{R}$$

$$y = e^{-3t} \cdot e^{C_1}$$

$$y = e^{-3t} \cdot k; \quad k \in \mathbb{R}$$

varying the constant:

$$y_{\text{trial}} = e^{-3t} \cdot k(t)$$

$$\begin{aligned}y'_{\text{trial}} &= e^{-3t} \cdot k'(t) + k(t) \cdot e^{-3t} \cdot -3 \\&= e^{-3t} \cdot k'(t) - 3e^{-3t} k(t)\end{aligned}$$

$$12(e^{-3t} k(t)) + 4\left(e^{-3t} k'(t) - 3e^{-3t} k(t)\right) = 6$$

$$12e^{-3t} k(t) + 4e^{-3t} k'(t) - 12e^{-3t} k(t) = 6$$

$$4e^{-3t} k'(t) = 6$$

$$k'(t) = \frac{6}{4} \cdot \frac{1}{e^{-3t}}$$

$$k'(t) = \frac{3}{2} e^{3t}$$

$$k(t) = \frac{3}{2} \cdot \frac{1}{3} e^{3t} = \frac{1}{2} e^{3t} + c_2$$

$$\therefore y = e^{-3t} \cdot \left(\frac{1}{2} e^{3t} + c_2 \right) ; c_2 \in \mathbb{R}$$

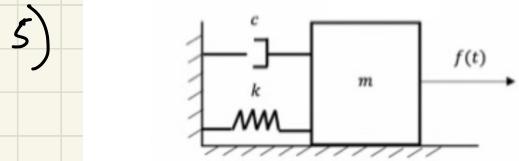
$$y = \frac{1}{2} \cdot 1 + e^{-3t} c_2$$

$$y = \frac{1}{2} + c_2 e^{-3t} ; i(0) = 0$$

$$I = \frac{1}{2} + c_2 e^{-3t}$$

$$0 = \frac{1}{2} + c_2 e^0 \quad \therefore c_2 = -\frac{1}{2}$$

$$\therefore \text{when } i(0) = 0 \quad I = \frac{1}{2} - \frac{1}{2} e^{-3t}$$



5.

If for the spring mass damper shown below $f(t) = 100 \sin(2t)$, find the equations that describes its path. Given that $m = 2\text{kg}$, $k = 40\text{N/m}$ and $c = 10\text{Ns/m}$. $x(0) = 1$, $x'(0) = 0$.

Hints: $ma = f(t) - c \cdot v - k \cdot x$ $v = \frac{dx}{dt}$ $a = \frac{d^2x}{dt^2}$

$$f(t) = 100 \sin(2t)$$

$$m = 2\text{kg} \quad k = 40 \frac{\text{N}}{\text{m}} \quad c = 10 \frac{\text{Ns}}{\text{m}}$$

$$x(0) = 1 \quad ; \quad x'(0) = 0$$

$$ma = f(t) - c \cdot v - k \cdot x \quad v = \frac{dx}{dt} \quad a = \frac{d^2x}{dt^2}$$

$$2 \frac{d^2x}{dt^2} = 100 \sin(2t) - 10 \frac{dx}{dt} - 40x$$

$$= 2x'' = 100 \sin(2t) - 10x' - 40x$$

is similar to $2y'' = 100 \sin(2t) - 10y' - 40y$
where $y = x$; differential eqn.

$$2y'' + 10y' + 40y = 100 \sin(2t)$$

This is 2nd order Diff eqn:

Solving characteristic eqn:

$$2\gamma^2 + 10\gamma + 40 = 0$$

$$\gamma^2 + 5\gamma + 20 = 0$$

$$\frac{-5 \pm \sqrt{25-80}}{2} = 0$$

$$\gamma_{1,2} = \frac{-5 \pm \sqrt{55}}{2} = \frac{-5}{2} \pm \frac{\sqrt{55}}{2} i \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{complex roots}$$

$$y_h = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

$$= e^{-\frac{5}{2}x} \left(C_1 \cos \frac{\sqrt{55}}{2}x + C_2 \sin \frac{\sqrt{55}}{2}x \right); C_1, C_2 \in \mathbb{R}$$

6. Consider a tank that is being filled with water at a constant rate of 3 liters per minute. The tank has a drainage pipe that allows water to flow out of the tank at a rate proportional to the square root of the water level in the tank.

Let $h(t)$ represent the height of water in the tank at time t , measured in meters.

- Write a differential equation that represents the rate of change of water level in the tank.
- If the tank is initially empty ($h(0) = 0$), and the drainage constant is $k = 0.2$ liters per minute per square root meter, find the particular solution of the differential equation that satisfies this initial condition.
- Determine the maximum height that the water level in the tank will reach.

Given that the tank is being filled with water at constant rate of 3 l/min

Given $h(t)$ represents the height of the water
 $\therefore h(t)$ depends on the area of the tank, A

\therefore if $V(t)$ is the volume of the water filled inside the tank; it can be represented as:

$$V(t) = A \cdot h(t)$$

\therefore rate of change of vol & height would be:

$$\frac{dV}{dt} = A \frac{dh}{dt} \rightarrow (1)$$

Given outflow is prop to \sqrt{h}
 also given inflow is 3 l/min
 \therefore rate of change of vol can also be represented in its terms as:

$$\frac{dV}{dt} = 3 - K\sqrt{h(t)} \rightarrow ②$$

(1) in (2)

$$A \frac{dh}{dt} = 3 - K\sqrt{h(t)}$$

$$\frac{dh}{dt} = \frac{3 - K\sqrt{h(t)}}{A}$$

Assuming $A: 1 m^2$ / since not given

$$\frac{dh}{dt} = 3 - K\sqrt{h(t)} \rightarrow$$

1st ODE
non-separable

$$\text{given } h(0) = 0 \quad \& \quad K = 0.2 \\ \text{i.e. } t=0; h=0$$

$$\frac{dh}{dt} = 3 - K\sqrt{h}$$

$$\int \frac{1}{3 - K\sqrt{h}} dh = \int dt$$

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water level $\rightarrow h$
 water flow out $\propto \sqrt{h}$

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$$h(t) = 8 - k\sqrt{h} \quad 3l/min$$

$$h(t) = 3 - k\sqrt{h}$$

6. Consider a tank that is being filled with water at a constant rate of 3 liters per minute. The tank has a drainage pipe that allows water to flow out of the tank at a rate proportional to the square root of the water level in the tank.

Let $h(t)$ represent the height of water in the tank at time t , measured in meters.

- a) Write a differential equation that represents the rate of change of water level in the tank.
- b) If the tank is initially empty ($h(0) = 0$), and the drainage constant is $k = 0.2$ liters per minute per square root meter, find the particular solution of the differential equation that satisfies this initial condition.
- c) Determine the maximum height that the water level in the tank will reach.

7. Consider a pendulum of length (L) with a small amplitude. The motion of the pendulum is governed by the second-order differential equation:

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{L}\right) * \sin(\theta) = 0$$

where θ represents the angular displacement of the pendulum from the vertical equilibrium position, t represents time, and g is the acceleration due to gravity.

- a) Rewrite the differential equation in terms of the linear displacement (x) of the pendulum bob.
- b) If the length of the pendulum is 1.5 meters and the acceleration due to gravity is 10 m/s^2 determine the general solution of the differential equation.
- c) If the initial conditions are $\theta(0) = \pi/4$ (45 degrees) and $d\theta/dt(0) = 0$, find the particular solution that satisfies these initial conditions.

Hint: since $\theta \ll 1 \text{ rad}$, $\sin(\theta) \approx \theta$ **(Bonus points (+100): Guess whose favorite approximation is this?)**

8. $\ddot{y} + 2\dot{y} + 50y = 12 \cos(5t) + \sin(5t) \rightarrow \text{just to Relax}$

9. Consider a mechanical drive system consisting of a motor, a load, and a rotational shaft. The motion of the drive system is governed by the following second-order differential equation:

$$J \frac{d^2\theta}{dt^2} = T_m - T_L$$

where θ represents the angular displacement of the shaft, t represents time, J is the moment of inertia of the system, T_m is the torque produced by the motor, and T_L is the external load torque acting on the shaft.

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- a) If the moment of inertia of the system is $J = 0.5 \text{ kg}\cdot\text{m}^2$, and the load torque is given by $T_L = k\theta$, where $k = 2\text{Nm/rad}$, determine the general solution of the differential equation.
- b) Suppose the motor produces a constant torque of $T_m = 10 \text{ Nm}$. Find the particular solution of the differential equation that satisfies this condition.

How do you make a multivariate function laugh? Give it a domain that's full of complex twists and turns!

2. Equations with multiple variables

10. Find and sketch the domain of $\arcsin(x^2 + y^2 - 2)$

11. Find the approximate value of $\sqrt{(3.02^2 + 1.97^2 + 5.99^2)}$. Hint: First find 3 points close to the values then apply a tangent Plane approximation.

12. Given the function $f(x, y) = \frac{x}{x^2+y^2}$. Find the direction derivate at $(1, 2)$ in the direction of

$$V = (3,5)'$$

13. $z = x^{-2}y^6 - 4x$, $x = u^2v$, $y = v - 3u$. Given the information find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

14. $Z = e^r \cos(\theta)$, $r = st$, $\theta = \sqrt{s^2 + t^2}$ find $\frac{dZ}{dt}$

15. Find the Partial Derivates of the following

a) $u = x^{y/z}$ b) $p = \sqrt{t^4 + u^2 \cos v}$ c) $Z = \ln(x + \sqrt{x^2 + y^2})$ and find $F_x(3,4)$

16. Find the local properties of $f(x, y) = e^x \cos y$

17. Find the Global Properties of $f(x, y) = 4x + 6y - x^2 - y^2$ $D: \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 2\}$

18. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400 \Omega$, $I = 0.08 \text{ A}$, $\frac{dI}{dt} = -0.01 \text{ V/s}$, and $\frac{dR}{dt} = 0.03 \Omega/\text{s}$.

19. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimetres. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

Why did the math book want to wish the student good luck before the exam?

Because it knew that a little "study" and "commitment" could solve any "problem"!

All the best in your Assessment, and remember to Practice hard!

Q: SoSe 2022

c. Solve ODE: $t \ln t \frac{dx}{dt} + x = t e^t$

This is non separable

$$\therefore t \ln t \frac{dx}{dt} = -x$$

$$\frac{1}{x} dx = -\frac{1}{t \ln t} dt$$

$u = \ln t$
 $du = \frac{1}{t} dt$

$$\ln x = -1 \ln |\ln t| + C, C \in \mathbb{R}$$

$$e^{\ln x} = e^{\ln |\ln t|^{-1} + C}$$

$$x = (\ln |t|)^{-1} \cdot e^C$$

$$x = \frac{1}{\ln t} \cdot e^C$$

$$x = \frac{1}{\ln t} k ; k \in \mathbb{R}$$

variation of param:

$$x_p = \frac{1}{\ln t} k(t) = \frac{k(t)}{\ln t}$$

$$x'_p = \frac{\ln t \cdot k'(t) - k(t) \cdot \frac{1}{t}}{(\ln t)^2}$$

$$= \frac{k'(t)}{\ln t} - \frac{k(t)}{t \cdot (\ln t)^2}$$

$$t \ln t \left[\frac{k'(t)}{\ln t} - \frac{k(t)}{t(\ln t)^2} \right] + \frac{k(t)}{\ln t} = te^t$$

$$tk'(t) - \frac{k(t)}{\ln t} + \frac{k(t)}{\ln t} = te^t$$

$$tk'(t) = te^t$$

$$k'(t) = e^t$$

$$k(t) = e^t + c_2 ; \quad c_2 \in \mathbb{R}$$

$$\therefore y_p = \frac{e^t + c_2}{\ln t} ; \quad c_2 \in \mathbb{R}$$

is another solution.

Second Order Differential Equations

Approaching Exam Questions:

Solve the Homogeneous Linear ODE with constant coefficients:

$$\left\{ \begin{array}{l} ay'' + by' + cy = 0 \\ \text{Homogeneous (RHS: 0)} \\ a, b, c \in \mathbb{R} \\ \text{constants} \\ \text{characteristic eqn:} \\ ax^2 + bx + c = 0 \end{array} \right.$$

- if $\gamma_1 \neq \gamma_2$
 $y_h = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x}; C_1, C_2 \in \mathbb{R}$
- if $\gamma_1 = \gamma_2 = \gamma$
 $y_h = C_1 e^{\gamma x} + C_2 x e^{\gamma x}; C_1, C_2 \in \mathbb{R}$
- if $\gamma_{1,2} = \alpha \pm i\beta$; extract α & β values
 $y_h = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x); C_1, C_2 \in \mathbb{R}$

Solve the non-homogeneous linear ODE with constant coefficients:

$$ay'' + by' + cy = G(x)$$

Take RHS as 0 & turn it into Homogeneous eqn & solve the characteristic eqn for y_h

Now for y_p : use method of undetermined coefficients
 if $G(x)$ is of the form; take a guess as follows:

Form	y_{trial}
1	A

$$5x + 7$$

$$5x^2 - 8$$

$$e^{3x}$$

$$\sin 4x$$

$$\cos kx$$

$$e^{kx}$$

$$x^2 e^{5x}$$

$$e^{3x} \sin 4x$$

$$5x^2 \sin 4x$$

$$x e^{3x} \cos 4x$$

$$Ax + B$$

$$Ax^2 + Bx + C$$

$$Ae^{3x}$$

$$A \cos 4x + B \sin 4x$$

$$A \cos kx + B \sin kx$$

$$Ae^{kx}$$

$$(Ax^2 + Bx + C) e^{5x}$$

$$e^{3x} (A \cos 4x + B \sin 4x)$$

$$(Ax^2 + Bx + C) \cos x + (Dx^2 + Ex + F) \sin x$$

$$(Ax + B) e^{3x} \cos 4x + (Cx + D) e^{3x} \sin 4x$$

Basic gist of coming up with Y_p :

The method of undetermined coefficients is a bit more nuanced than just breaking up $G(x)$ into $G_1(x), G_2(x)$ or $G_1(x) + G_2(x)$

When we apply method of undetermined coefficients, we are essentially making a guess on how the particular solution would look like based on the RHS of the given differential equation.

The guess is based on the form of the 'function' on the RHS, but; we ignore the multiplicative constants when making the initial guess

e.g: $25 \sin(2x) \Rightarrow$ function is $\sin 2x \therefore$ ignoring the constant we have $y_p = A \sin 2x + B \cos 2x$
the method of undetermined coefficients is designed to account for that ignored constant while solving.

$(4x-7)e^{5x} \Rightarrow 4x-7$ becomes $Ax+B$ & $e^{5x} = Ce^{5x}$ & is also a function but when we match the form of the questions differential equations A & B will account for the case where we may have multiplied it prior. $\therefore y_p = (Ax+B)e^{5x}$

$5x^2 \sin(4x)$: When dealing with functions like this; this is product of a polynomial & trig function we guess the particular solution to have the form of the same degree of polynomial.

We know for $5x^2 \rightarrow Ax^2 + Bx + C$

& we know $\sin(4x) \rightarrow A \cos 4x + B \sin 4x$

So if we multiply them; we have to generate y_p as:

$$y_p = (Ax^2 + Bx + C) \cos 4x + (Dx^2 + Ex + F) \sin 4x$$

as A & B are not the same! Hence we introduce new set of variable with its polynomial form.

Second ODE (Notes)

Two types:

$$P(x)y'' + Q(x)y' + R(x)y = G(x) \leftarrow \text{Inhomogeneous}$$

$$P(x)y'' + Q(x)y' + R(x)y = 0 \leftarrow \text{Homogeneous}$$

If $y_1(x)$ & $y_2(x)$ are both solutions of a homogeneous 2nd order linear ODE; Then for every choice of constants $c_1, c_2 \in \mathbb{R}$

$y(x) = c_1 y_1(x) + c_2 y_2(x)$
is also a solution of same ODE.

Theorem 2:

If y_1 & y_2 are linearly independent solutions of a homogeneous 2nd order linear ODE; Then the general soln (= all solutions) is given by:

$$y(x) = c_1 y_1(x) + c_2 y_2(x); \quad c_1, c_2 \in \mathbb{R}$$

Keyword: linearly independent
ie not multiples of each other

x^3 & $8x^3$ are not linearly independent
 e^{ax} & $x e^{ax}$ are linearly independent.

$$ay'' + by' + cy = 0 \quad (\text{homogeneous second order})$$

soln . type: $y(x) = e^{\gamma x}$

$$y'(x) = \gamma e^{\gamma x}$$

$$y''(x) = \gamma^2 e^{\gamma x}$$

\therefore we have $a\gamma^2 e^{\gamma x} + b\gamma e^{\gamma x} + c e^{\gamma x} = 0$

$$e^{\gamma x}(a\gamma^2 + b\gamma + c) = 0 \rightarrow \textcircled{7}$$

we have $a\gamma^2 + b\gamma + c = 0 \leftarrow$ characteristic equation.

solve the ODE

case 1: $\gamma_1 \neq \gamma_2$

then $y_1(x) = e^{\gamma_1 x}$

$$y_2(x) = e^{\gamma_2 x}$$

$$y(x) = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x}; \quad C_1, C_2 \in \mathbb{R}$$

when $\gamma_1 \neq \gamma_2$

case 2: $\gamma_1 = \gamma_2 = \gamma$

1st soln: $y_1(x) = e^{\gamma x}$

$$y_2(x) = x e^{\gamma x} \longrightarrow (1)$$

$$\therefore y_2'(x) = x \cdot \gamma e^{\gamma x} + e^{\gamma x} \longrightarrow (2)$$

$$y''(x) = \gamma (x \gamma e^{\gamma x} + e^{\gamma x}) + \gamma e^{\gamma x}$$

$$= x \gamma^2 e^{\gamma x} + \gamma e^{\gamma x} + \gamma e^{\gamma x}$$

$$= x \gamma^2 e^{\gamma x} + 2\gamma e^{\gamma x} \longrightarrow (3)$$

$$ay'' + by' + cy = 0$$

$$a(x \gamma^2 e^{\gamma x} + 2\gamma e^{\gamma x}) + b(x \gamma e^{\gamma x} + e^{\gamma x}) + c x e^{\gamma x} = 0$$

$$ax\gamma^2 e^{\gamma x} + 2ax\gamma e^{\gamma x} + bx\gamma e^{\gamma x} + be^{\gamma x} + cx e^{\gamma x} = 0$$

$$ax\gamma^2 + 2ax\gamma + bx\gamma + b + cx = 0$$

$$ax\gamma^2 + bx\gamma + cx + 2ax\gamma + b = 0$$

$$x(ax^2 + bx + c) + 2ax\gamma + b = 0$$

(7)

$$\therefore x(0) + 2ax\gamma + b = 0$$

$$2ax\gamma = -b$$

$$\gamma = -\frac{b}{2ax}$$

-! Solution of ODE:

$$y(x) = C_1 e^{\gamma x} + C_2 x e^{\gamma x} \quad \gamma_1 = \gamma_2 = \gamma$$

when roots are complex: $\gamma_1, \gamma_2 = \alpha + i\beta$

$$\text{solution } y(x) = e^{\alpha x} (C_1 \cos \beta x + i \sin \beta x) ; \quad C_1, C_2 \in \mathbb{R}$$

$$\textcircled{1} \quad y'' + 10y' + 25y = 0$$

characteristic eqn:

$$\lambda^2 + 10\lambda + 25 = 0$$

$$\underline{\lambda} \times \underline{\lambda} = 25$$

$$\underline{\lambda} + \underline{\lambda} = 10$$

$$\begin{aligned}\lambda^2 + 5\lambda + 5\lambda + 25 &= 0 \\ \lambda(\lambda + 5) + 5(\lambda + 5) &= 0\end{aligned}$$

$$(\lambda + 5)^2 = 0$$

$$\lambda_1, \lambda_2 = -5$$

roots λ_1, λ_2 are the same

$$\therefore \text{solution: } y = C_1 e^{-5x} + C_2 x e^{-5x}$$

$$\textcircled{2} \quad y''' + 3y'' + 3y' + y = 0$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

when $\lambda = -1$

$$-1 + 3 - 3 + 1 = 0$$

$\therefore \lambda = -1$ is one factor

$$\begin{array}{r}
 x^2 + 2x + 1 \\
 \hline
 x+1 \left| \begin{array}{r} x^3 + 3x^2 + 3x + 1 \\ x^3 + x^2 \\ \hline 2x^2 + 3x \\ 2x^2 + 2x \\ \hline x + 1 \\ x + 1 \\ \hline 0 \end{array} \right.
 \end{array}$$

$$\therefore (x+1) \underbrace{(x^2 + 2x + 1)}_{\text{solving this}} = 0$$

$$\begin{array}{r}
 \frac{1}{1} \times \frac{1}{1} = 1 \\
 \frac{1}{1} + \frac{1}{1} = 2
 \end{array}$$

$$(x+1)(x^2 + x + 1) = 0$$

$$(x+1)(x(x+1) + (x+1)) = 0$$

$$(x+1)(x+1)(x+1) = 0$$

$x_1, x_2, x_3 = -1$ ← roots are same

∴ solution:

$$y = C_1 e^{-x} + C_2 A e^{-x} + C_3 A^2 e^{-x}$$

(3)

$$2y'' - y' - 3y = 25 \sin(2x)$$

characteristic eqn:

$$2\gamma^2 - \gamma - 3 = 0$$

$$\frac{-(-1) \pm \sqrt{1 - 4(-6)}}{2(2)} = \frac{+1 \pm \sqrt{25}}{4} = \frac{+1 \pm 5}{4} =$$

$$= \frac{+1+5}{4} \text{ or } \frac{+1-5}{4}$$

$$= \frac{3}{2} \text{ or } -1$$

$$\begin{aligned} 2\gamma^2 + 2\gamma - 3\gamma - 3 &= 0 \\ 2\gamma(\gamma + 1) - 3(\gamma + 1) &= 0 \\ (\gamma + 1)(2\gamma - 3) &= 0 \\ \gamma = -1 &\quad \gamma = \frac{3}{2} \end{aligned}$$

$$\gamma_1 \neq \gamma_2$$

$$\therefore y_h = C_1 e^{-1x} + C_2 e^{3/2 x}$$

For test function: we have: $G(x) = 25 \sin(2x)$
ignored & result from. $= G_1(x) \cdot G_2(x)$

$$\therefore y_p = A(B \cos 2x + C \sin 2x)$$

$$\text{proper coefficient} = A \cos 2x + B \sin 2x$$

$$\begin{aligned} y'_p &= A \cdot -\sin 2x \cdot 2 + B \cos 2x \cdot 2 \\ &= -2A \sin 2x + 2B \cos 2x \end{aligned}$$

$$y''_p = -2A \cos 2x \cdot 2 + 2B (-\sin 2x) \cdot 2$$

$$= -4A \cos 2x - 4B \sin 2x$$

we have:

$$2y'' - y' - 3y = 25 \sin(2x)$$

$$2(-4A \cos 2x - 4B \sin 2x)$$

$$- (-2A \sin 2x + 2B \cos 2x)$$

$$- 3(A \cos 2x + B \sin 2x) = 25 \sin(2x)$$

LHS:

$$\underline{-8A \cos 2x} - 8B \sin 2x$$

$$+ 2A \sin 2x - \underline{2B \cos 2x}$$

$$\underline{-3A \cos 2x} - 3B \sin 2x = 25 \sin(2x)$$

$$(-8A - 2B - 3A) \cos 2x +$$

$$(-8B + 2A - 3B) \sin 2x = 25 \sin(2x) + 0 \cos(2x)$$

$$-11B - 2A = 0 \quad (-)$$

$$-11B + 2A = 25$$

∴ we have:

$$-4A = -25$$

$$A = 25/4$$

$$-11B = 2A$$

$$-11B = \frac{50}{4}$$

$$B = -\frac{50}{44}$$

$$\begin{aligned}\therefore y_p &= A \cos 2x + B \sin 2x \\ &= \frac{25}{4} \cos 2x - \frac{50}{44} \sin 2x \\ y(x) &= y_h + y_p \\ &= C_1 e^{-x} + C_2 e^{3/2x} + \frac{25}{4} \cos 2x - \frac{50}{44} \sin 2x\end{aligned}$$

17.2 Exercises

1–10 Solve the differential equation or initial-value problem using the method of undetermined coefficients.

1. $y'' - 2y' - 3y = \cos 2x$

2. $y'' - y = x^3 - x$

3. $y'' + 9y = e^{-2x}$

4. $y'' + 2y' + 5y = 1 + e^x$

5. $y'' - 4y' + 5y = e^{-x}$

6. $y'' - 4y' + 4y = x - \sin x$

7. $y'' + y = e^x + x^3, \quad y(0) = 2, \quad y'(0) = 0$

8. $y'' - 4y = e^x \cos x, \quad y(0) = 1, \quad y'(0) = 2$

9. $y'' - y' = xe^x, \quad y(0) = 2, \quad y'(0) = 1$

10. $y'' + y' - 2y = x + \sin 2x, \quad y(0) = 1, \quad y'(0) = 0$

 **11–12** Graph the particular solution and several other solutions. What characteristics do these solutions have in common?

11. $y'' + 3y' + 2y = \cos x$

12. $y'' + 4y = e^{-x}$

13–18 Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

13. $y'' - y' - 2y = xe^x \cos x$

14. $y'' + 4y = \cos 4x + \cos 2x$

15. $y'' - 3y' + 2y = e^x + \sin x$

16. $y'' + 3y' - 4y = (x^3 + x)e^x$

17. $y'' + 2y' + 10y = x^2 e^{-x} \cos 3x$

18. $y'' + 4y = e^{3x} + x \sin 2x$

$$i) \quad y'' - 2y' - 3y = \cos 2x$$

$$y = y_h + y_p$$

Characteristic eqn:

$$\gamma^2 - 2\gamma - 3 = 0$$

$$\begin{array}{l} \underline{\gamma^2 - 3} = -3 \\ \underline{1 + \gamma} = -2 \end{array} \quad \left. \begin{array}{l} \gamma^2 - 2\gamma - 3 = 0 \\ \gamma(\gamma+1) - 3(\gamma+1) = 0 \\ (\gamma-3)(\gamma+1) = 0 \end{array} \right\} \quad \begin{array}{l} \gamma = 3 \\ \gamma = -1 \end{array}$$

$$\boxed{\gamma_1 \neq \gamma_2 \quad \therefore \quad y_h = C_1 e^{3x} + C_2 e^{-x}}$$

$$G(x) = \cos 2x$$

$$\therefore y_p = A \cos 2x + B \sin 2x$$

$$y_p' = -2A \sin 2x + 2B \cos 2x$$

$$y_p'' = -4A \cos 2x - 4B \sin 2x$$

\therefore we have

$$\begin{aligned} & (-4A \cos 2x - 4B \sin 2x) \\ & - 2(-2A \sin 2x + 2B \cos 2x) \\ & - 3(A \cos 2x + B \sin 2x) = \cos 2x \end{aligned}$$

$$\underline{-4A \cos 2x - 4B \sin 2x}$$

$$+ 4A \sin 2x - \underline{4B \cos 2x}$$

$$\underline{-3A \cos 2x - 3B \sin 2x} = \cos 2x$$

$$(-4A - 4B - 3A) \cos 2x$$

$$+ (-4B + 4A - 3B) \sin 2x = \cos 2x + 0 \sin 2x$$

$$\begin{aligned} -4A - 4B - 3A &= 1 \\ -4B + 4A - 3B &= 0 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \begin{aligned} -7A - 4B &= 1 \rightarrow ① \\ 4A - 7B &= 0 \rightarrow ② \end{aligned}$$

$$\begin{array}{rcl} ① \times 4 & \Rightarrow & -28A - 16B = 4 \\ ② \times -7 & \Rightarrow & -28A + 49B = 0 \end{array} \quad \underline{\quad} \quad \begin{aligned} -65B &= 4 \\ B &= -\frac{4}{65} \end{aligned}$$

$$\begin{aligned} 4A &= 7B \\ A &= \frac{7}{4} \cdot \left(-\frac{4}{65} \right) \\ &= -\frac{7}{65} \end{aligned}$$

$$\therefore y_p = -\frac{7}{65} \cos 2x - \frac{4}{65} \sin 2x$$

$$y = C_1 e^{3x} + C_2 e^{-x} + -\frac{7}{65} \cos 2x - \frac{4}{65} \sin 2x$$

Do not forget! $\rightarrow C_1, 2 \in \mathbb{R}$

$$2) y'' - y = x^3 - x$$

$$y'' - 0y' - y = x^3 - x$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda^2 = 1$$

$$\lambda_{1,2} = \pm 1$$

$$\therefore y_n = c_1 e^x + c_2 e^{-x} \quad c_{1,2} \in \mathbb{R}$$

$$G(x) = G_1(x) + G_2(x)$$

$$= x^3 - x \quad \text{→ ignored as } D \text{ takes care of it}$$

$$y_p = Ax^3 + Bx^2 + Cx + D$$

$$y'_p = 3Ax^2 + 2Bx + C$$

$$y''_p = 6Ax + 2B$$

$$\therefore 6Ax + 2B - (Ax^3 + Bx^2 + Cx + D) = x^3 - x$$

$$6Ax + 2B - Ax^3 - Bx^2 - Cx - D = x^3 + 0x^2 - x + 0$$

$$-Ax^3 - Bx^2 + 6Ax - Cx + 2B - D = x^3 + 0x^2 - x + 0$$

$$-A = 1 \quad -Bx^2 = 0x^2 \quad (6A - C)x = -x \quad 2B - D = 0$$

$$A = -1 \quad B = 0$$

$$6A - C = -1$$

$$-6 - C = -1$$

$$-6 + 1 = C$$

$$C = -5$$

$$\begin{aligned} \therefore y_p &= Ax^3 + Bx^2 + Cx + D \\ &= -x^3 + 0x^2 - 5x + 0 \\ &= -x^3 - 5x \end{aligned}$$

$$y = y_h + y_p = c_1 e^x + c_2 e^{-x} - x^3 - 5x ; c_1, c_2 \in \mathbb{R}$$

$$3) y'' + 9y = e^{-2x}$$

$$y'' + 0y + 9y = e^{-2x}$$

$$\gamma^2 + 0\gamma + 9 = 0$$

$$\begin{aligned} \delta^2 &= -9 \\ \delta &= \sqrt{-9} = 0 \pm 3i \end{aligned}$$

$$\begin{aligned} \delta &= \alpha \pm \beta i \Leftrightarrow 0 \pm 3i \\ \alpha &= 0 \quad \beta = 3 \end{aligned}$$

$$y_h = e^{0x} \left(c_1 \cos 3x + c_2 \sin 3x \right)$$

$$y_h = c_1 \cos 3x + c_2 \sin 3x ; c_1, c_2 \in \mathbb{R}$$

$$G(x) = e^{-2x}$$

$$y_p = A e^{-2x}$$

$$\begin{aligned} y'_p &= A \cdot e^{-2x} \cdot (-2) \\ &= -2A e^{-2x} \end{aligned}$$

$$y''_p = -4Ae^{-2x}$$

$$-4Ae^{-2x} + 9Ae^{-2x} = e^{-2x}$$

$$5Ae^{-2x} = e^{-2x}$$

$$5A = 1$$

$$A = \frac{1}{5}$$

$$\therefore y_p = \frac{1}{5}e^{-2x}$$

$$y = y_h + y_p = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}e^{-2x} \quad c_1, c_2 \in \mathbb{R}$$

$$4) y'' + 2y' + 5y = 1 + e^x$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = -\frac{2}{2} \pm \frac{4i}{2} = -1 \pm 2i \quad (\alpha \pm \beta i \text{ form})$$

$$y_h = e^{-x} (c_1 \cos 2x + c_2 \sin 2x); \quad c_1, c_2 \in \mathbb{R}$$

$$G(x) = 1 + e^x$$

$$\therefore y_p = A + Be^x$$

$$y_p = Be^x$$

$$y''_p = Be^x$$

$$\underline{Be^x} + \underline{2Be^x} + \underline{5A} + \underline{5Be^x} = 1 + e^x$$

$$8Be^x + 5A = e^x + 1$$

$$8B = 1 \quad 5A = 1$$

$$B = \frac{1}{8} \quad A = \frac{1}{5}$$

$$y_p = \frac{1}{5} + \frac{1}{8}e^x$$

$$y = y_h + y_p$$

$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8}e^x + \frac{1}{5}; \quad C_1, C_2 \in \mathbb{R}$$

$$5) \quad y'' - 4y' + 5y = e^{-x}$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 20}}{2 \cdot 1} = \frac{4 \pm \sqrt{-4}}{2} = \frac{2 \pm \frac{2i}{2}}{2} = \frac{2 \pm i}{2} \quad (\alpha \pm \beta i \text{ from })$$

$$\therefore y_h = e^{2x}(C_1 \cos x + C_2 \sin x); \quad C_1, C_2 \in \mathbb{R}$$

$$\text{For } y_p : \quad G(x) = e^{-x}$$
$$y_p = A e^{-x}$$
$$y'_p = -A e^{-x}$$
$$y''_p = A e^{-x}$$

$$A e^{-x} + 4 A e^{-x} + 5 A e^{-x} = e^{-x}$$

$$10 A e^{-x} = e^{-x}$$

$$A = \frac{1}{10}$$

$$\therefore y_p = \frac{1}{10} e^{-x}$$

$$\begin{aligned} y &= y_p + y_h \\ &= e^{2x} (C_1 \cos x + C_2 \sin x) + \frac{1}{10} e^{-x}; \quad C_1, C_2 \in \mathbb{R} \end{aligned}$$

SoSe 2022 Q

17:38

$$\ddot{\psi} + 2\dot{\psi} + 50\psi = 12 \cos 5t + \sin 5t$$

replacing ψ as y to solve the question & $t = x$

$$y'' + 2y' + 50y = 12 \cos 5x + \sin 5x$$

$$\lambda^2 + 2\lambda + 50 = 0$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4 - 200}}{2}$$
$$= \frac{-2}{2} \pm \frac{\sqrt{-196}}{2} = -1 \pm 7i$$

$\alpha \pm \beta i$ form

$$y_h = e^{-x} (C_1 \cos 7x + C_2 \sin 7x) ; C_1, C_2 \in \mathbb{R}$$

$$G_1(x) = 12 \cos 5x + \sin 5x$$

$$= G_{11}(x) + G_{12}(x)$$

$$y_{P_1}: G_{11}(x) \Rightarrow A \cos 5x + B \sin 5x$$

$$y_{P_2}: G_{12}(x) = C \cos 5x + D \sin 5x$$

Not necessary
as $G_{11}(x)$ guess
already represent
our complete RHS

$$y'_{P_1} = -5A \sin 5x + 5B \cos 5x$$

$$y''_{P_1} = -25A \cos 5x - 25B \sin 5x$$

$$y'_{P_2} = -5C \sin 5x + 5D \cos 5x$$

X not
needed
to compute!

$$y''_{P_2} = -25C \cos 5x - 25D \sin 5x \quad X$$

$$G_1(x) \text{ in } y'' + 2y' + 50y$$

$$\begin{aligned} &= -25A \cos 5x - 25B \sin 5x \\ &\quad + 2(-5A \sin 5x + 5B \cos 5x) \\ &\quad + 50(A \cos 5x + B \sin 5x) \\ &= 12 \cos 5x + 5 \sin 5x \end{aligned}$$

$$\begin{aligned} &= (-25A + 10B + 50A) \cos 5x + \\ &\quad (-25B - 10A + 50B) \sin 5x \\ &= 12 \cos 5x + 5 \sin 5x \end{aligned}$$

$$(25A + 10B = 12) \times 10 \quad (+)$$

$$(-10A + 25B = 1) \times 25$$

$$250A + 100B = 120 \quad (+)$$

$$-250A + 625B = 25$$

$$725B = 145$$

$$B = \frac{145}{725} = \frac{1}{5}$$

$$\begin{array}{r} 2 \\ 145 \\ \hline 5 \\ 725 \end{array}$$

$$10A = 25B - 1$$

$$A = \frac{25 \cdot 1 - 5}{5 \cdot 10}$$

$$= \frac{25 - 5}{50} = \frac{20}{50} = \frac{2}{5}$$

$$\therefore y_p = \frac{2}{5} \cos 5x + \frac{1}{5} \sin 5x$$

$$y = y_h + y_p$$

$$= e^{-x} (C_1 \cos 7x + C_2 \sin 7x) + \frac{2}{5} \cos 5x + \frac{1}{5} \sin 5x$$

where $C_1, C_2 \in R$

Q: Spring 2023:

$$d. \quad \ddot{y} + 3\dot{y} = x e^x + \sin 3x$$

$$y'' + 3y' + 0y = xe^x + \sin 3x$$

$$\lambda^2 + 3\lambda + 0 = 0$$

$$\lambda^2 + 3\lambda = 0$$

$$\lambda(\lambda + 3) = 0$$

$$\lambda_1 = 0 \quad \lambda_2 = -3$$

$$\lambda_1 \neq \lambda_2$$

18:08

$$\therefore y_h = C_1 e^{0x} + C_2 e^{-3x} \quad C_1, C_2 \in R$$

$$G(x) = G_1(x) + G_2(x)$$

$$= \underbrace{(Ax+B)e^x}_{y_{P_1}} + \underbrace{(D\cos 3x + E\sin 3x)}_{y_{P_2}}$$

$$y_{P_1} = (Ax+B)e^x$$

$$= Axe^x + Be^x$$

$$y'_{P_1} = A\{xe^x + e^x\} + Be^x$$

$$= Axe^x + Ae^x + Be^x$$

$$y''_{P_1} = A[xe^x + e^x] + Ae^x + Be^x$$

$$= Axe^x + 2Ae^x + Be^x$$

$G_1(x)$:

$$y'' + 3y' = xe^x$$

$$\underline{Axe^x + 2Ae^x + Be^x} + \underline{3Axe^x + 3Be^x} = xe^x$$

$$4Axe^x + (2A+4B)e^x = xe^x + 0e^x$$

$$4A = 1$$

$$A = \frac{1}{4}$$

$$2A + 4B = 0$$

$$2 \cdot \frac{1}{4} + 4B = 0$$

$$4B = -\frac{1}{2}$$

$$B = -\frac{1}{8}$$

$$y_{P_1} = \left(\frac{1}{4}x - \frac{1}{4} \right) e^x$$

$G_2(x)$:

$$y_{P_2} = D \cos 3x + E \sin 3x$$

$$y'_{P_2} = -3D \sin 3x + 3E \cos 3x$$

$$y''_{P_2} = -9D \cos 3x - 9E \sin 3x$$

$$y'' + 3y' = \sin 3x$$

$$\underline{-9D \cos 3x} - 9E \sin 3x + \underline{D \cos 3x + E \sin 3x} = \sin 3x$$

$$-8D \cos 3x + (-8E) \sin 3x = \sin 3x + 0 \cdot \cos 3x$$

$$\begin{aligned} -8E &= 1 \\ E &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned} -8D &= 0 \\ D &= 0 \end{aligned}$$

$$y_{P_2} = 0 + \frac{1}{8} \sin 3x$$

$$y = y_h + y_{P_1} + y_{P_2}$$

$$y = C_1 + C_2 e^{-3x} + \frac{1}{4} x e^x - \frac{1}{8} e^x + \frac{1}{8} \sin 3x$$

$C_1, C_2 \in \mathbb{R}$

skipping G as repeated pattern

$$7 \cdot y'' + y = e^x + x^3 ; \quad y(0) = 2 , \quad y'(0) = 0$$

~~$\lambda^2 + \lambda + 0 = 0$~~
 ~~$\lambda(\lambda + 1) = 0$~~

$$\lambda_1 = 0 \quad \lambda_2 = -1 ; \quad \lambda_1 \neq \lambda_2$$

$$\therefore y_h = C_1 e^{0x} + C_2 e^{-x}$$

$$g(x) = e^x + x^3$$

$$= Ae^x + Bx^3 + Cx^2 + Dx + E$$

$\underbrace{\hspace{1cm}}_{y_{P_1}}$ $\underbrace{\hspace{3cm}}_{y_{P_2}}$

$$y_{P_1} = Ae^x$$

$$y'_{P_1} = Ae^x$$

$$y''_{P_1} = Ae^x$$

$$y_{P_2} = Bx^3 + Cx^2 + Dx + E$$

$$y'_{P_2} = 3Bx^2 + 2Cx + D$$

$$y''_{P_2} = 6Bx + 2C$$

$$y'' + y' = e^x + x^3$$

wrong!
do not confuse:

$$\left. \begin{aligned} y'' &= \gamma^2 \\ y' &= \gamma \\ y &= 1 \end{aligned} \right\}$$

\therefore it is

$$\gamma^2 + 1 = 0$$

$$\gamma = \pm i$$

y_h = complex form?

$$\underline{G_1(x)}: \quad Ae^x + Ae^x = e^x$$

$$2Ae^x = e^x$$

$$A = \frac{1}{2}$$

$$y_{P_1} = \frac{1}{2}e^x$$

$$G_2(x): \quad 6Bx + 2C + 3Bx^2 + 2Cx + D = x^3 + 0x^2 + 0x + 0$$

$$0x^3 + 3Bx^2 + (6B + 2C)x + 2C + D = x^3$$

$$3B = 0$$

$$B = 0$$

$$6B + 2C = 1$$

$$2C = 1$$

$$C = \frac{1}{2}$$

$$2C + D = 0$$

$$1 + D = 0$$

$$D = -1$$

$$\therefore y_{P_2} = 0x^3 + \frac{1}{2}x^2 - x$$

& since E does not contribute to higher order terms; E = 0

Redoing y_n as initial steps were incorrect:

$$\gamma^2 + 1 = 0$$

$$\delta = \pm i$$

$$\alpha \pm i\beta \text{ form} \quad \alpha = 0 \quad \beta = 1$$

$$\therefore y_n = e^{0x} (C_1 \cos x + C_2 \sin x)$$

$$y = y_h + y_{p_1} + y_{p_2}$$

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^x + \frac{1}{2} x^2 - x \quad \rightarrow (*)$$

$C_1, C_2 \in \mathbb{R}$

given $y(0) = 2$ & $y'(0) = 0$

$$y(0) = 2 \quad x = 0 \quad y = 2$$

$$2 = C_1 \cos 0 + C_2 \sin 0 + \frac{1}{2} + 0 - 0$$

$$2 = C_1 \cdot 1 + 0 + \frac{1}{2}$$

$$C_1 = 2 - \frac{1}{2} = \frac{4-1}{2} = \frac{3}{2}$$

Applying derivative to: $\textcircled{*}$

$$y' = -C_1 \sin x + C_2 \cos x + \frac{1}{2} e^x + \frac{1}{2} \cdot 2x - 1$$

$$y' = -C_1 \sin x + C_2 \cos x + \frac{e^x}{2} + x - 1$$

given $y'(0) = 0$

$$x = 0 \quad y' = 0$$

$$0 = 0 + C_2 + \frac{1}{2} + 0 - 1$$

$$C_2 + \frac{1}{2} = 1 \quad C_2 = -\frac{1}{2}$$

∴

$$y = \frac{3}{2} \cos x - \frac{1}{2} \sin x + \frac{1}{2} e^x + \frac{1}{2} x^2 - x$$

