



**FACULTY** **institute**  
**OF MECHANICAL** **of automation**  
**ENGINEERING** **and computer science**

# Programming for robots and manipulators

## Lecture 5

1.

Introduction

2.

Differential Kinematics

3.

Dynamics

4.

Robotic Control

5.

Why do we need knowledge of Kinematics and Dynamics?



# Introduction

**Kinematics** describes the analytical relationship between the joint positions and the end-effector position and orientation.

**Differential Kinematics** describes the analytical relationship between the joint motion and the end-effector motion in terms of velocities.

**Robot dynamics** is concerned with the relationship between the forces acting on a robot mechanism and the accelerations they produce.

Typically, the robot mechanism is modelled as a rigid-body system, in which case robot dynamics is the application of rigid-body dynamics to robots.

The two main problems in robot dynamics are:

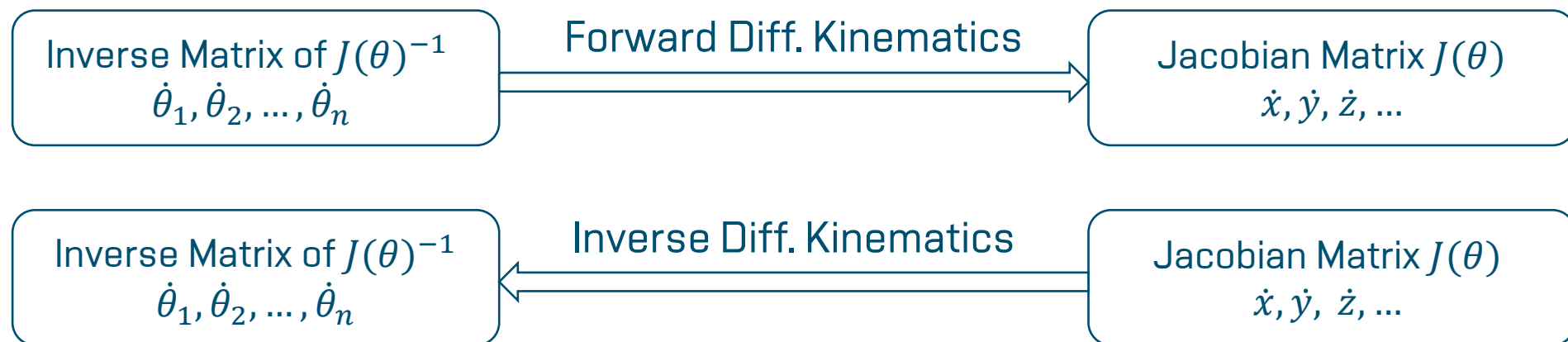
- A) Forward dynamics – Given the forces, work out the accelerations.
- B) Inverse dynamics – Given the accelerations, work out the forces.

# Differential Kinematics

# Differential Kinematics

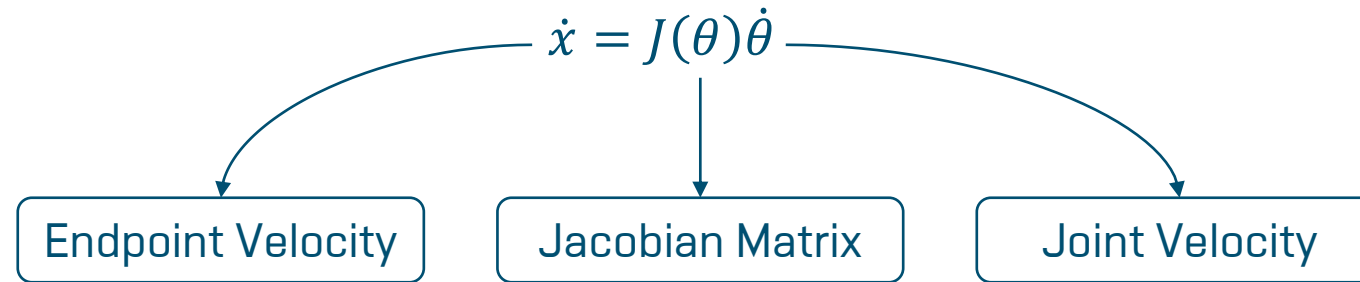
Differential kinematics relates the velocities of the manipulator components. These velocities may be the velocities at the joints of the manipulator or the velocities of one or more links in the manipulator kinematic chain.

The standard approach to differential kinematics is to relate joint and end-effector velocities through the **Jacobian matrix**, which allows the calculation of the end-effector velocities given the joint velocities (Forward differential kinematics) or, to determine the joint velocities in order to move the end-effector with a prescribed speed (Inverse differential kinematics).



# Differential Kinematics

**Jacobian** is a multidimensional form of the derivative, in robotics, Jacobian is a time-varying linear transformations, which determines the relationship between joint velocity (input) and end-effector velocity (output).



Most manipulator have values of  $\theta$  where the Jacobian becomes singular. If the Jacobian matrix is non singular, we can calculate "Inverse Jacobian matrix".

$$\dot{\theta} = J(\theta)^{-1}\dot{x}$$

The Jacobian is singular when its determinant is equal to 0  $\rightarrow \det(J) = 0$ .



# Differential Kinematics

The Jacobian,  $J$ , is defined as:

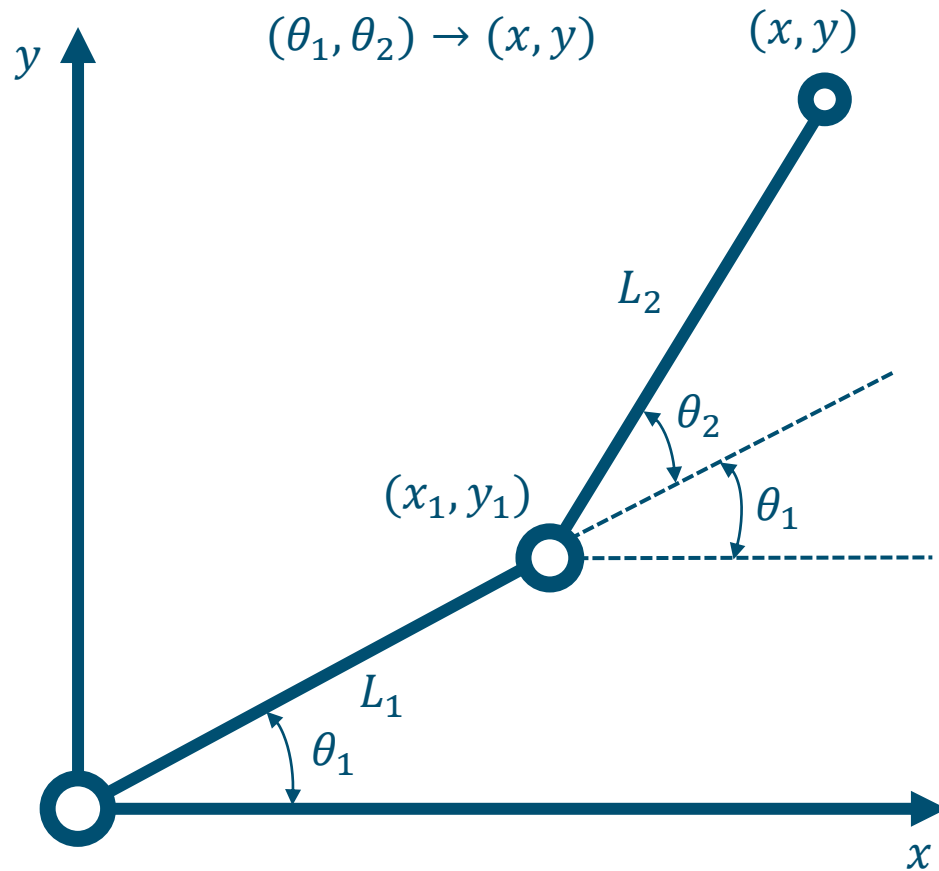
$$J(\theta) = \begin{bmatrix} \frac{\partial x_1}{\partial \theta_1} & \dots & \frac{\partial x_1}{\partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_m}{\partial \theta_1} & \dots & \frac{\partial x_m}{\partial \theta_n} \end{bmatrix}$$

where is the  $m \times n$  analytical Jacobian matrix.

Jacobians are useful for planning and executing smooth trajectories, determining singular configurations, executing coordinated anthropomorphic motion, deriving dynamic equations of motion, and transforming forces and torques from the end-effector to the manipulator joints.

Example

# Example: Differential Kinematics



## Forward Kinematics

$$\begin{aligned}x &= L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) \\y &= L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2)\end{aligned}$$

## Jacobian Matrix

$$\begin{bmatrix} p\dot{x} \\ p\dot{y} \end{bmatrix} = J(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

# Example: Differential Kinematics

## Jacobian Matrix

$$\begin{bmatrix} p\dot{x} \\ p\dot{y} \end{bmatrix} = J(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} =$$

$$= \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

## Singular Jacobian

$$\det \left( \begin{bmatrix} -L_1 \sin \theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \right) = 0$$

# Dynamics

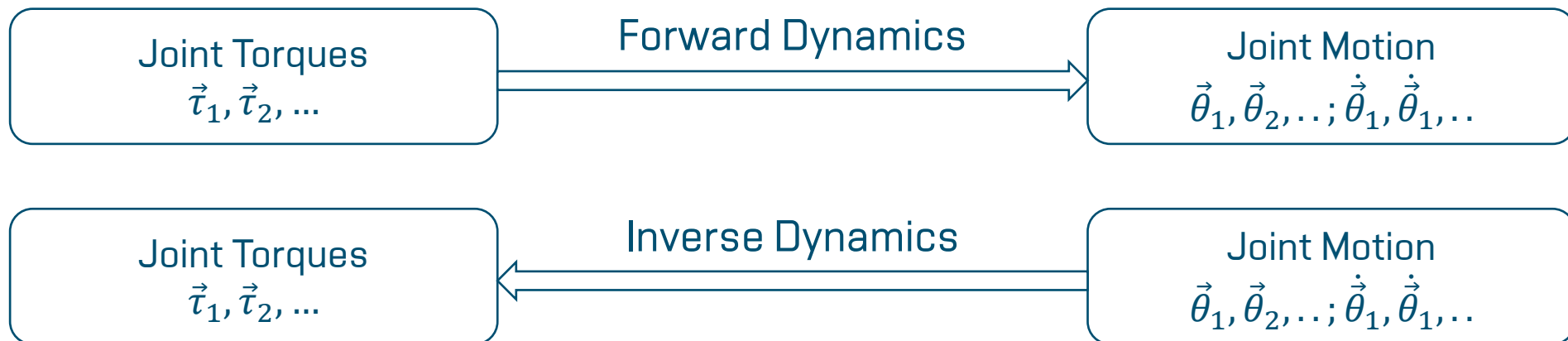
# Robotic Dynamics

The dynamics of a system describes how the controls  $\theta_t$  influence the change-of-state of the system:

$$x_{t+1} = f(x_t, \theta_t)$$

The notation  $x_t$  refers to the dynamic state of the system: e.g., joint positions and velocities  $x_t = (\theta_t, \dot{\theta}_t)$ . Function  $f$  is an arbitrary function, often smooth.

Describes the relationship between Forces ( $\vec{f}_n$ )/Torques ( $\vec{\tau}_n$ ):



# Robotic Dynamics

For many applications with fixed-based robots we need to find a multi-body dynamics formulated as:

$$M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta) = \tau + J_c(\theta)^T F_c$$

$M(\theta)$  → Generalized mass matrix (orthogonal).

$\theta, \dot{\theta}, \ddot{\theta}$  → Generalized position, velocity and acceleration vectors.

$b(\theta, \dot{\theta})$  → Coriolis and centrifugal terms.

$g(\theta)$  → Gravitational terms.

$\tau$  → External generalized forces.

$J_c(\theta)$  → Geometric Jacobian corresponding to the external forces.

$F_c$  → External Cartesian forces (e.g. from contacts).

There are typically two ways to derive the equation of motion for an open-chain robot:

## Lagrangian Method

Energy-based method.

Dynamic equations in closed form.

Often used for study of dynamic properties and analysis of control methods.

## Newton-Euler Method

Balance of forces/torques.

Dynamic equations in numeric/recursive form.

Often used for numerical solution of forward/inverse dynamics.

**Many other formal methods based on basic principles in mechanics are available for the derivation of the robot dynamic model:**

Principle of d'Alembert, of Hamilton, of virtual works, etc.



# Lagrangian Method

Lagrange Method is one of the analytical methods used to describe the motion of physical systems (closely tied to both the d'Alembert and Hamilton principles).

**The method is centered around three fundamental concepts:**

1. The definition of generalized coordinates  $\theta$  and generalized velocities  $\dot{\theta}$ , which may or may not encode the information regarding the constraints applicable to the system.
2. A scalar function called the Lagrangian function  $\mathcal{L}$ . For mechanical systems, it is exactly the difference between the total Kinetic energy  $T$  and the total Potential energy  $U$ , of the system at each instant:

$$\mathcal{L} = T - U$$

# Lagrangian Method

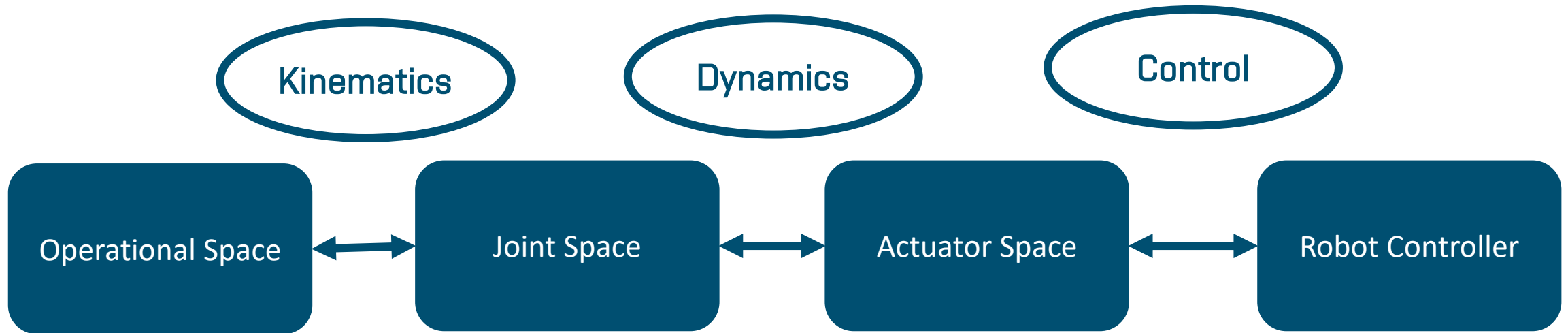
3. The so-called Euler-Lagrange equation, also known as the Euler-Lagrange of the second kind, which applies to the Lagrangian function  $\mathcal{L}$  and to the total external generalized forces  $\tau$ :

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \left( \frac{\partial \mathcal{L}}{\partial \theta} \right) = \tau$$

In the most general case, the Lagrangian is a function of the generalized coordinates and velocities  $\theta$  and  $\dot{\theta}$ , and it may also have an explicit dependence on time  $t$ , hence we redefine the aforementioned scalar energy functions as:

$$\begin{aligned} T &= T(t, \theta, \dot{\theta}) \\ U &= U(t, \theta) \\ \mathcal{L} &= \mathcal{L}(t, \theta, \dot{\theta}) \end{aligned}$$

# Robotic Control

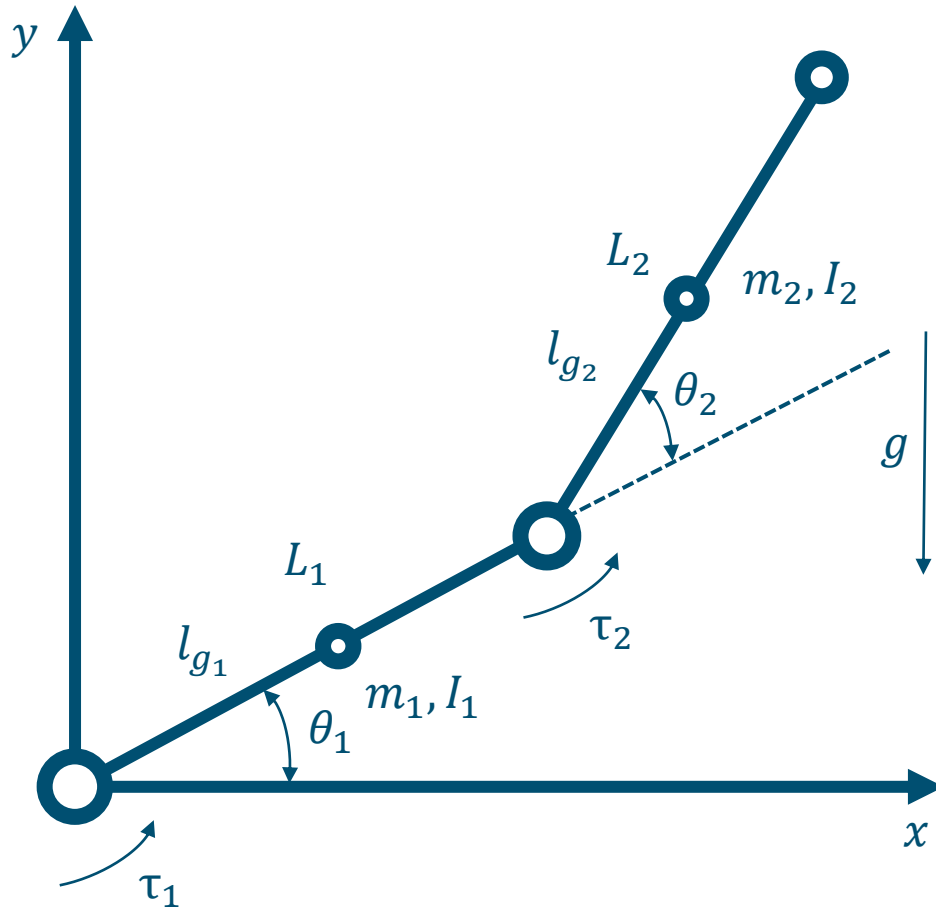


Example

# Example: Differential Kinematics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$

Lagrangian Method



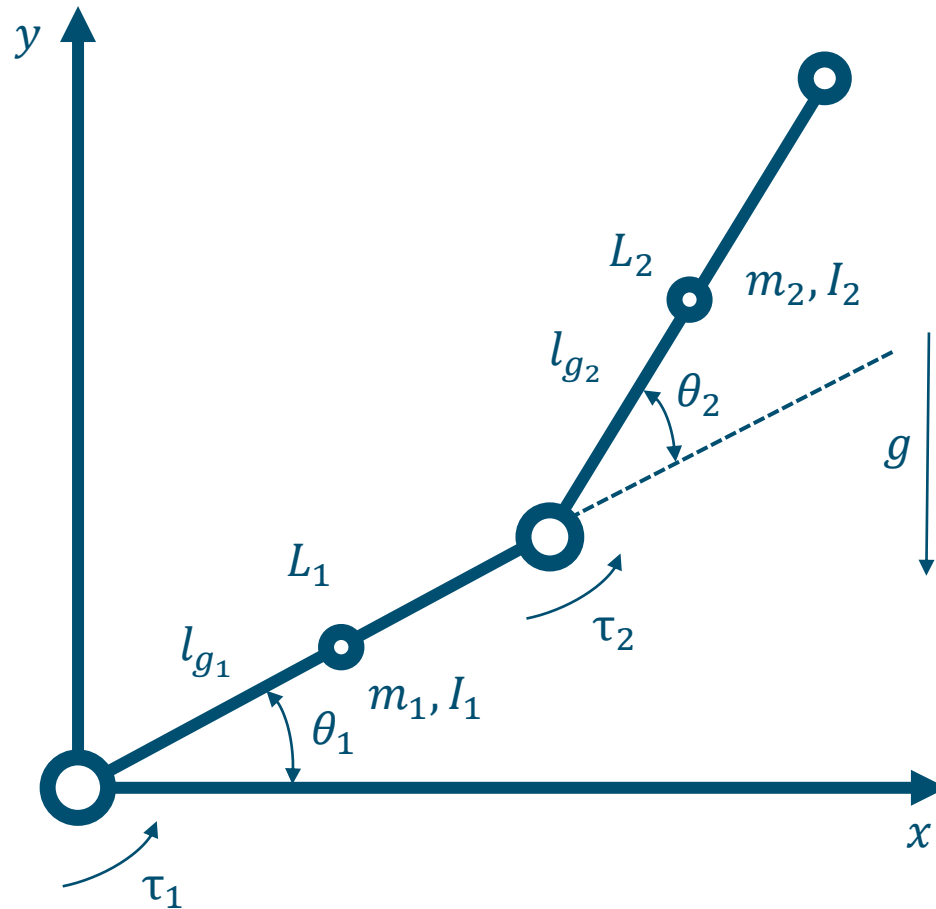
$$\tau_i = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \right) - \left( \frac{\partial \mathcal{L}}{\partial \theta_i} \right)$$

$$\mathcal{L} = T - U$$

$$\mathcal{L}(\theta, \dot{\theta}) = T(\theta, \dot{\theta}) - U(\theta)$$

# Example: Differential Kinematics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$



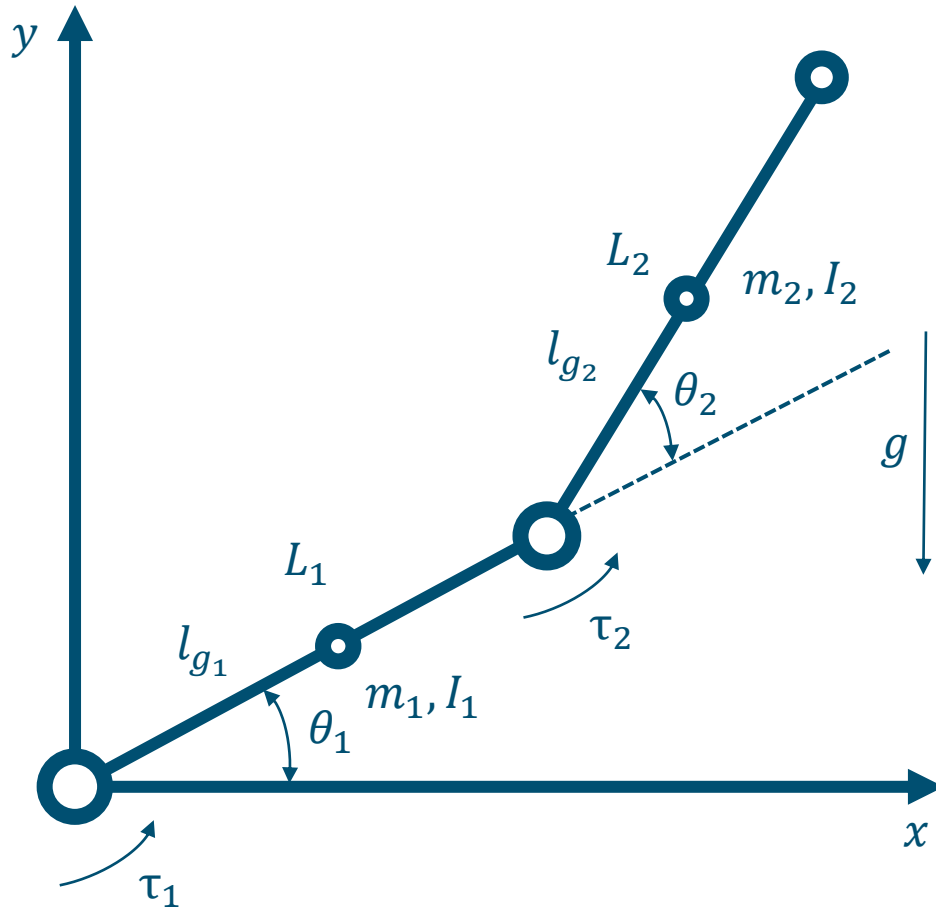
$$\begin{aligned} x_{m_1} &= l_{g_1} \cos(\theta_1) & \dot{x}_{m_1} &= -l_{g_1} \dot{\theta}_1 \sin(\theta_1) \\ y_{m_1} &= l_{g_1} \sin(\theta_1) & \dot{y}_{m_1} &= l_{g_1} \dot{\theta}_1 \cos(\theta_1) \end{aligned}$$

$$\begin{aligned} x_{m_2} &= L_1 \cos(\theta_1) + l_{g_2} \cos(\theta_1 + \theta_2) \\ y_{m_2} &= L_1 \sin(\theta_1) + l_{g_2} \sin(\theta_1 + \theta_2) \end{aligned}$$

$$\begin{aligned} \dot{x}_{m_2} &= -L_1 \dot{\theta}_1 \sin(\theta_1) - l_{g_2} (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \\ \dot{y}_{m_2} &= L_1 \dot{\theta}_1 \cos(\theta_1) + l_{g_2} (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{aligned}$$

# Example: Differential Kinematics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$



## Kinetic energy

$$T_1 = \frac{1}{2}m_1(\dot{x}_{m_1}^2 + \dot{y}_{m_1}^2) + \frac{1}{2}I_1\dot{\theta}_1^2$$

$$T_2 = \frac{1}{2}m_2(\dot{x}_{m_2}^2 + \dot{y}_{m_2}^2) + \frac{1}{2}I_2(\dot{\theta}_1 + \dot{\theta}_2)^2$$

## Potential energy

$$U_1 = m_1 l_{g1} g \sin(\theta_1)$$

$$U_2 = m_2 g (L_1 \sin(\theta_1) + l_{g2} \sin(\theta_1 + \theta_2))$$



# Example: Differential Kinematics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$

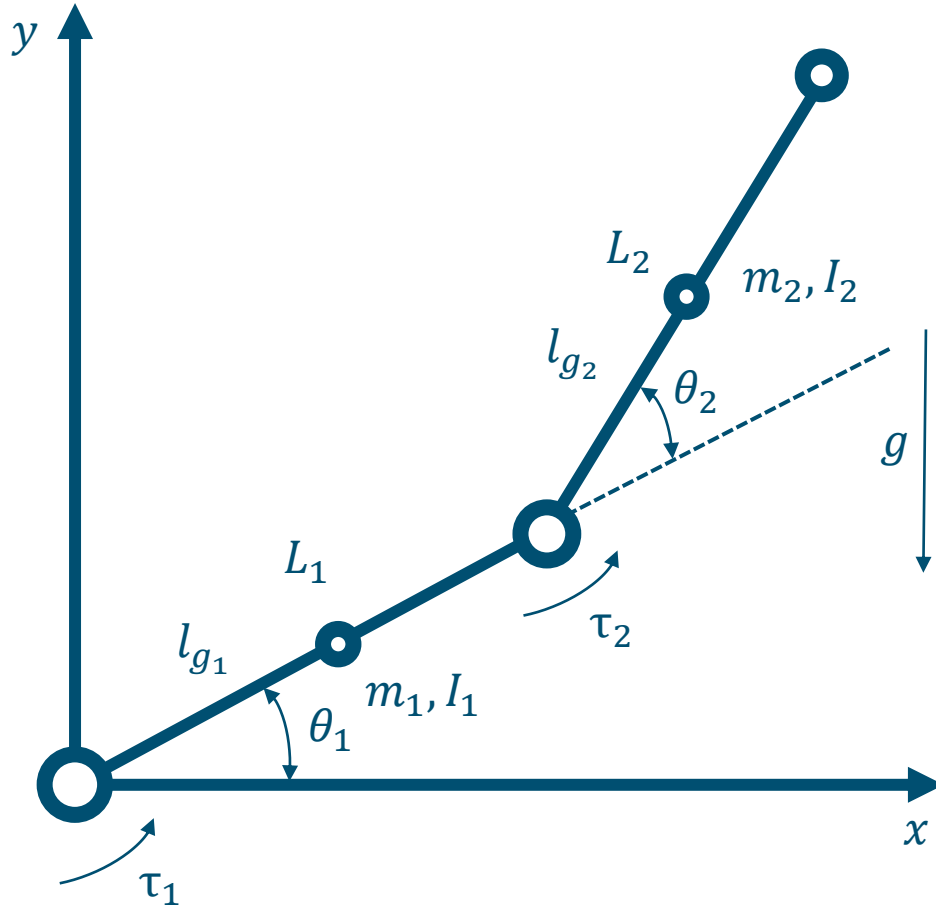
Torque equation

$$\mathcal{L} = T - U$$

$$\tau_i = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \right) - \left( \frac{\partial \mathcal{L}}{\partial \theta_i} \right)$$

$$\mathcal{L} = (T_1 + T_2) - (U_1 + U_2)$$

$$\tau = [\tau_1; \tau_2]^T, \theta = [\theta_1; \theta_2]^T$$



# Example: Differential Kinematics

Torque equations can then be split up into the form of the general equation. The different parameters can be extracted from this, by isolating the angular acceleration, velocities and positions.

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$

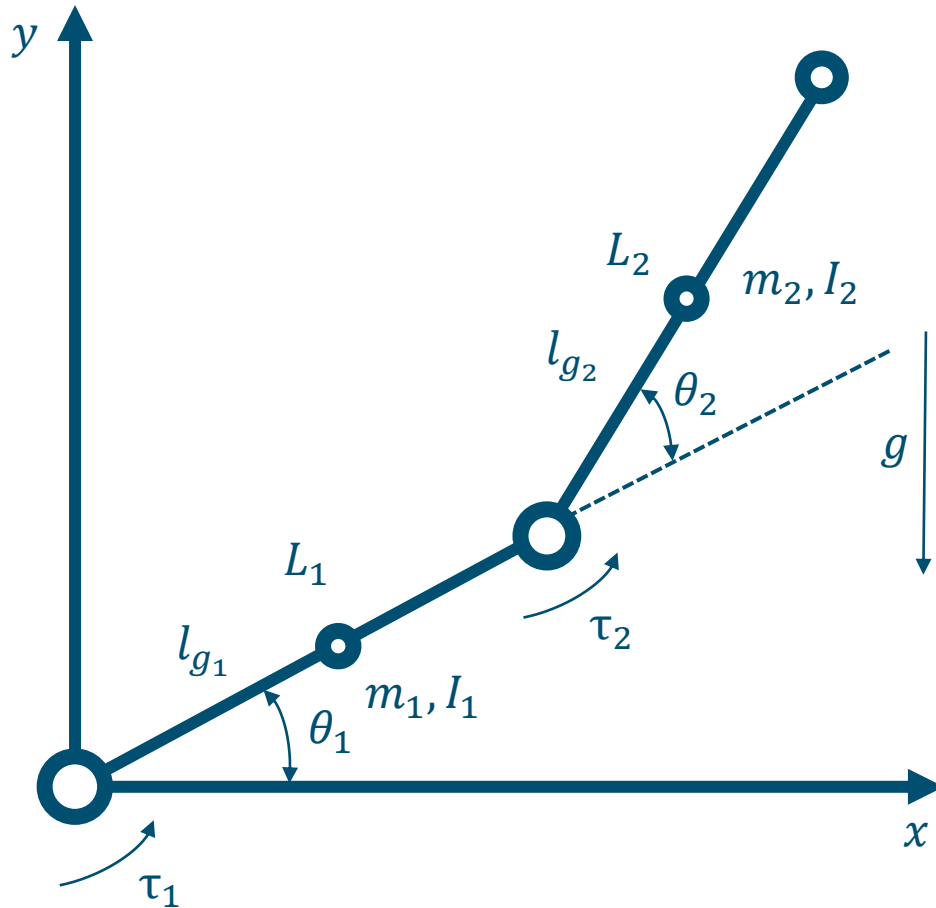
$$M(\theta) = \begin{bmatrix} I_1 + I_2 + m_1 l_{g_1}^2 + m_2 (L_1^2 + l_{g_2}^2 + 2L_1 l_{g_2} + 2L_1 l_{g_2} \cos(\theta_2)) & I_2 + m_2 (l_{g_2}^2 + L_1 l_{g_2} \cos(\theta_2)) \\ I_2 + m_2 (l_{g_2}^2 + L_1 l_{g_2} \cos(\theta_2)) & I_2 + m_2 l_{g_2}^2 \end{bmatrix}$$

$$b(\theta, \dot{\theta}) = \begin{bmatrix} -m_2 L_1 l_{g_2} \dot{\theta}_2 (2\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_2) \\ m_2 L_1 l_{g_2} \dot{\theta}_1^2 \sin(\theta_2) \end{bmatrix}$$

$$g(\theta) = \begin{bmatrix} m_1 g l_{g_1} \cos(\theta_1) + m_2 g (L_1 \cos(\theta_1) + l_{g_2} \cos(\theta_1 + \theta_2)) \\ m_2 g l_{g_2} \cos(\theta_1 + \theta_2) \end{bmatrix}$$

# Example: Differential Kinematics

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$



Motion equation

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}) + g(\theta)$$

$$\ddot{\theta} = M(\theta)^{-1}(-b(\theta, \dot{\theta}) - g(\theta) + \tau)$$

Ordinary Differential Equations (ODE)

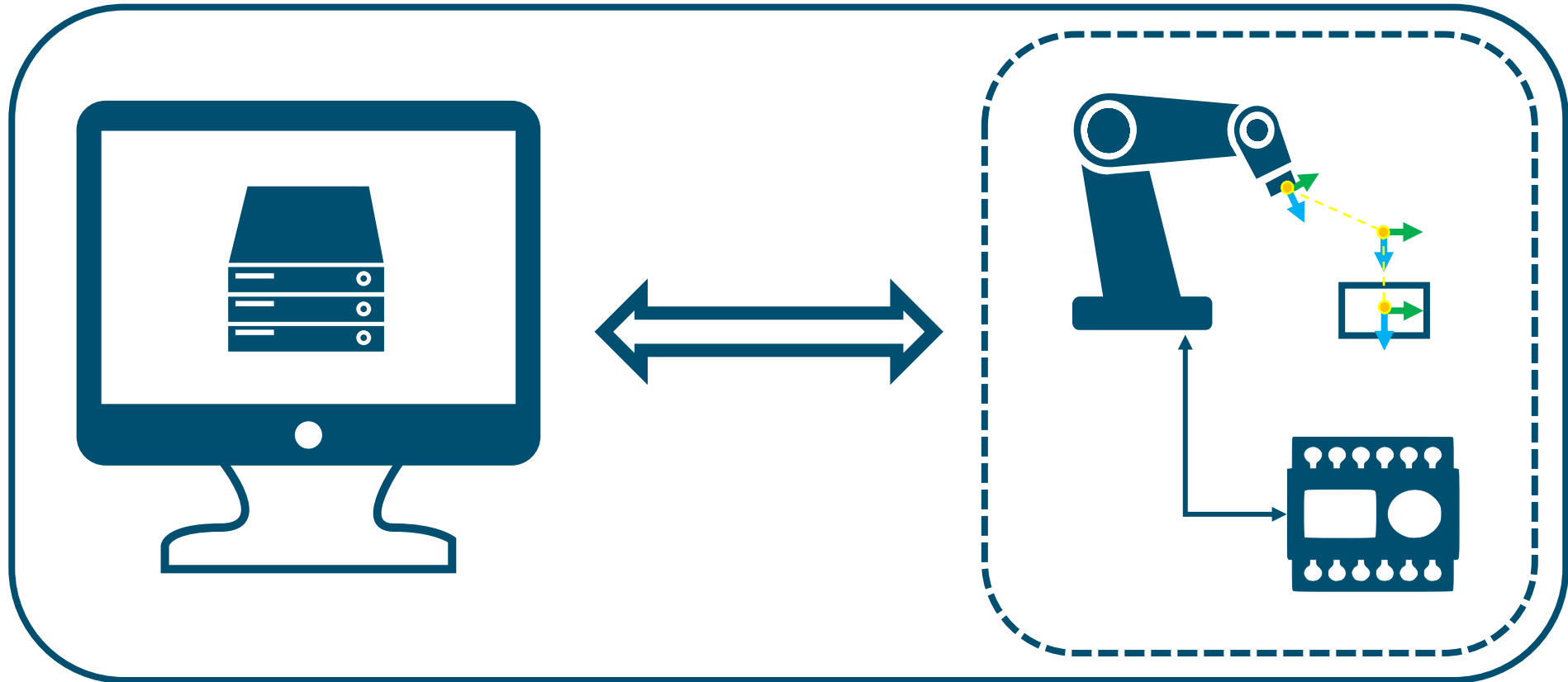
$$\frac{d}{dt} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ M(\theta)^{-1}(-b(\theta, \dot{\theta}) - g(\theta) + \tau) \end{bmatrix}$$

Why do we need knowledge of Kinematics  
and Dynamics?

# Design and Implementation of a Robotic Application

Budget: \$\$\$\$

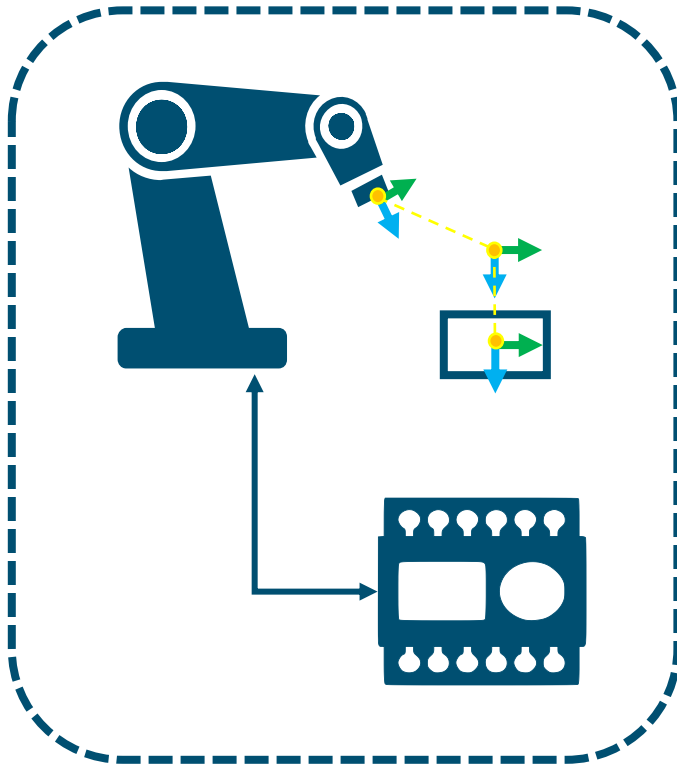
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# Design and Implementation of a Robotic Application

Budget: \$\$\$\$

Saved: ?



Robotic Construction

Controller

Additional Hardware

?

# Design and Implementation of a Robotic Application

Budget: \$\$\$\$\$

Saved: ?



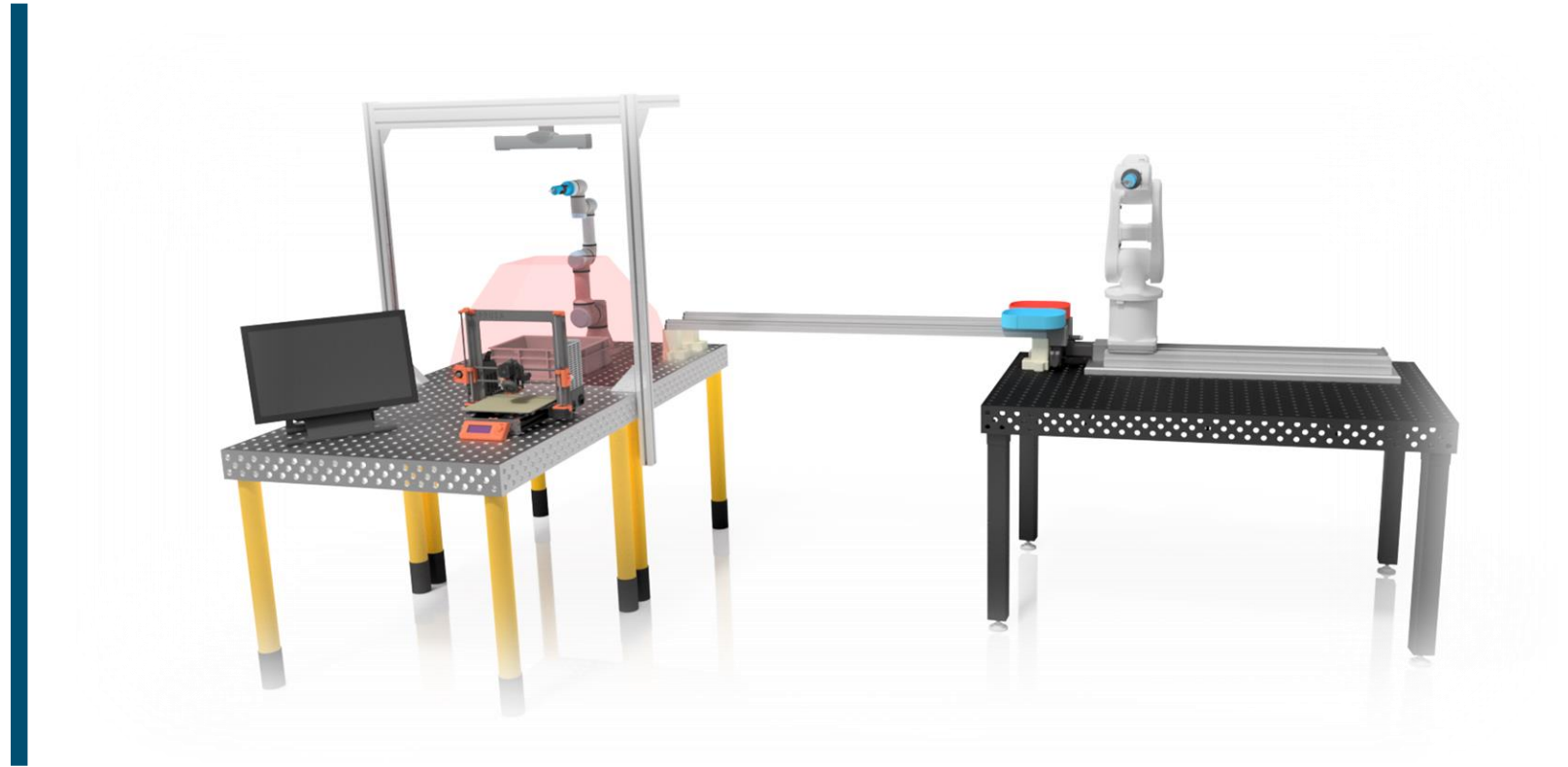
Simulation Software

Control Software

Additional Libraries

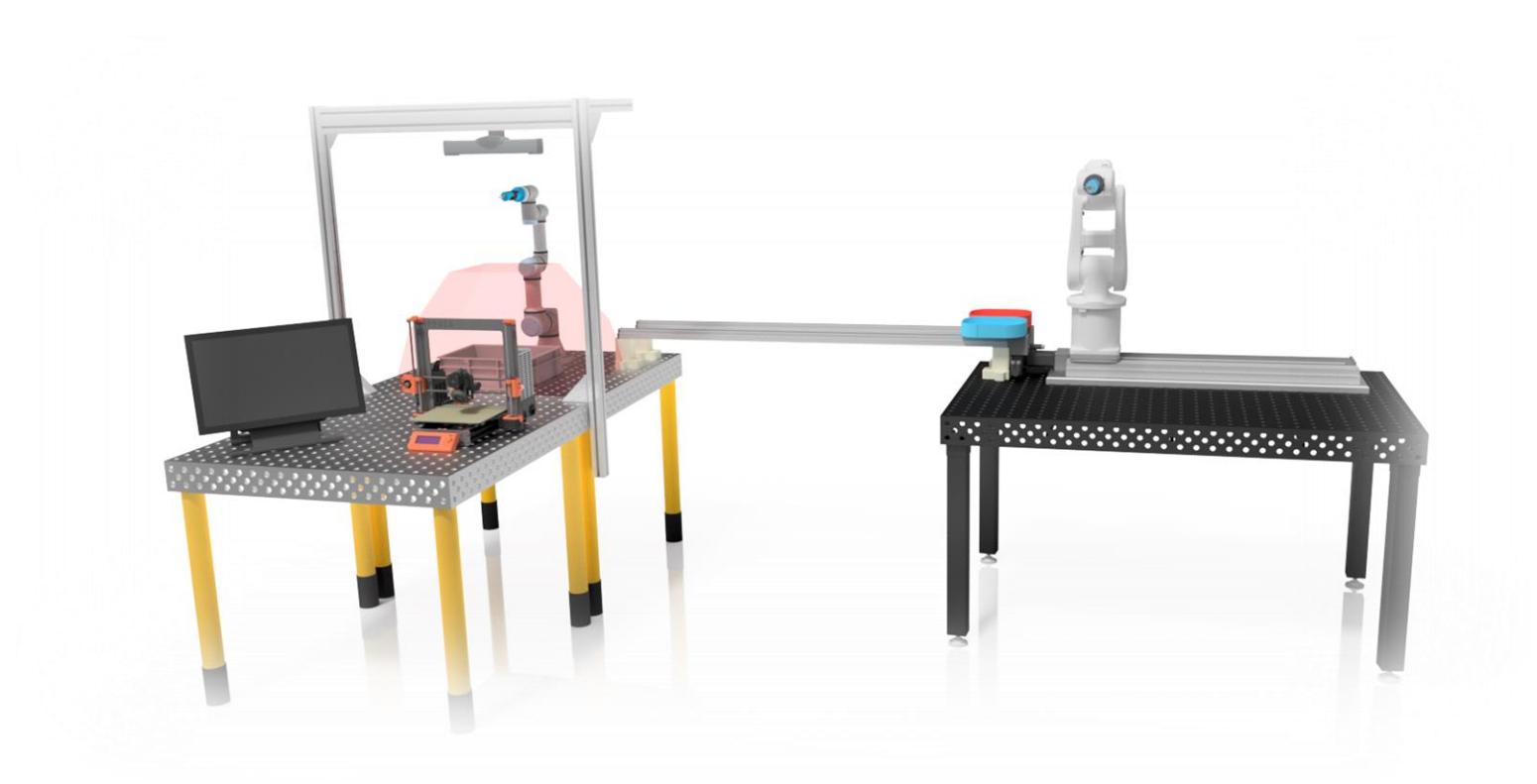
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Thank You!





# Questions?





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