Consumer Resource Model

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Consumer Resource Models from literature

Moran, Tikhonov model

$$egin{aligned} rac{dN_u}{dt} &= N_u(t) \left[\sum_i h_{ui}(t) - \left(c + \sum_i \chi_i \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj}
ight)
ight] \ h_{ui}(t) &= \sigma_{ui} b_i rac{1}{1 + rac{\sum_v N_v(t) \sigma_{vi}}{k_i}} \end{aligned}$$

Mehta model

$$egin{aligned} rac{dN_u}{dt} &= g_u N_u(t) \left[\sum_i \omega_i (1-l_i) h_{ui}(t) - c_u
ight] \ rac{dR_i}{dt} &=
ho_i - au_i^{-1} R_i(t) - \sum_u N_u(t) h_{ui}(t) + \sum_{u,j} l_j N_u(t) \sigma_{uj} h_{uj}(t) D_{ij} rac{\omega_j}{\omega_i} \ h_{ui}(t) &= \sigma_{ui} R_i(t) \end{aligned}$$

O'Dwyer model

$$egin{aligned} rac{dN_u}{dt} &= N_u(t) \left[\sum_i g h_{ui}(t) - c
ight] \ rac{dR_i}{dt} &=
ho_i - \sum_u N_u(t) h_{ui}(t) \ h_{ui}(t) &= b_{ui} R_i(t) \end{aligned}$$

Consensus Tikhonov Group Consumer Resource Model

Generalized, with cross-feeding

$$\frac{dN_u}{dt} = N_u(t) \, \gamma_u \left(\underbrace{\sum_i \omega_i (1 - \lambda_{ui}) h_{ui}(t) \sigma_{ui}}_{\text{energy uptake}} - \underbrace{\sum_{\text{energy costs}}^{C_u} }_{\text{energy uptake}} \right)$$

$$\frac{dR_i}{dt} = \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui}}_{\text{consumption}} + \underbrace{\sum_{u,j} D_{ij} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) \sigma_{ui}}_{\text{by-production}}$$

$$h_{ui}(t) = \beta_{ui} R_i(t)$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\ \dots \end{cases}$$

Simplification 1: No cross-feeding, fix energy/growth conversion factors

(Assume
$$\lambda_{ui} = 0 \ \forall \ u, i; \ \omega_i = 1 \ \forall \ i; \ \gamma_u = 1 \ \forall \ u)$$

$$egin{aligned} rac{dN_u}{dt} &= N_u(t) \left(\underbrace{\sum_i h_{ui}(t) \sigma_{ui}}_{ ext{energy uptake}} - \underbrace{c_u}_{ ext{energy costs}}
ight) \ rac{dR_i}{dt} &= \underbrace{
ho_i}_{ ext{influx}} - \underbrace{rac{R_i(t)}{ au_i}}_{ ext{decay}} - \underbrace{\sum_u rac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui}}_{ ext{consumption}}
ight. \ h_{ui}(t) &= eta_{ui} R_i(t) \ c_u &= \left\{ eta_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj}
ight. \end{aligned}$$

Simplification 2: Fast resource equilibration

(Assume
$$\frac{dR_i}{dt} = 0$$
)

$$egin{aligned} rac{dN_u}{dt} &= N_u(t) \left(\underbrace{\sum_i h_{ui}(t) \sigma_{ui}}_{ ext{energy uptake}} - \underbrace{c_u}_{ ext{energy costs}}
ight) \ & \underbrace{p_i}_{ ext{growth rate}} \ h_{ui}(t) &= eta_{ui} rac{
ho_i}{rac{1}{ au_i} + \sum_v rac{eta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}} \ c_u &= egin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \ ... \end{cases} \end{aligned}$$

Derivation of $h_{ui}(t)$:

$$\frac{dR_i}{dt} = \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} \qquad \text{as above}$$

$$0 = \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} \qquad \text{assume } \frac{dR_i}{dt} = 0 \text{ (resource dynamics are fast relative to abundances)}$$

$$\rho_i = \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui}$$

$$\rho_i = \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} R_i(t) \sigma_{ui}$$

$$\rho_i = R_i(t) \left(\frac{1}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} \sigma_{ui} \right)$$

$$R_i(t) = \frac{\rho_i}{\frac{1}{\tau_i} + \sum_u \frac{\beta_{ui} N_u(t) \sigma_{ui}}{\kappa_{ui}}}$$

$$h_{ui}(t) = eta_{ui} R_i(t) = eta_{ui} rac{
ho_i}{rac{1}{ au_i} + \sum_v rac{eta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}}$$

Simplification 3: Fix resource time scale parameters, hold carrying capacities constant across types

(Assume $\rho_i=1\ \forall\ i;\ au_i=1\ \forall\ i;\ eta_{ui}=1\ \forall\ u,i;\ \kappa_{ui}=\kappa_{vi}\ \forall\ u,v,i\ o$ Moran, Tikhonov (2022) model)

$$\frac{dN_u}{dt} = N_u(t) \left(\underbrace{\sum_i h_{ui}(t)\sigma_{ui}}_{\text{energy costs}} - \underbrace{c_u}_{\text{energy costs}} \right)$$

$$\frac{1}{1 + \underbrace{\sum_v N_v(t)\sigma_{vi}}_{\kappa_i}} = \sigma_{ui} \frac{1}{1 + \frac{T_i(t)}{\kappa_i}}$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i < j} J_{ij}\sigma_{ui}\sigma_{uj} \\ \dots \end{cases}$$
Let $T_i(t) = \sum_v \sigma_{vi}N_v(t)$

Dimensional analysis for Tikhonov Group CRM model:

37 (1)	Description	Units
$egin{aligned} N_u(t)\ R_i(t) \end{aligned}$	Abundance of type u at time t Quantity of resource i at time t	individuals mass
		4. •
$h_{ui}(t)$	Rate of uptake of resource i by type u	mass/time
σ_{ui}	Utilization of resource i by type u (expression of "trait" i)	unitless
eta_{ui}	Timescale for uptake of resource i by type u	1/time
λ_{ui}	Fraction of energy from resource i converted into byproducts by type u	unitless
γ_u	Conversion of net energy uptake to growth of type u	1/energy
c_u	Minimal energy uptake to maintain type u	energy/time
ξ_u	Minimal baseline energy uptake to maintain type u	energy/time
χ_{ui}	Minimal marginal energy uptake to maintain utilization of resource i by type u	energy/time
J_{ij}	Minimal marginal energy uptake to maintain utilization of $both$ resource i and j	energy/time
$ ho_i$	Influx rate of resource i	mass/time
$ au_i$	Timescale for dilution/decay of resource i	time
κ_{ui}	"Carrying capacity" of resource i for type u (resource depletion rate scaling factor)	individuals
ω_i	Energy content of resource i	energy/mass
D_{ij}	Fraction of byproducts of resource i that are converted to resource j	unitless
at	$u(t) \gamma_u \left(\underbrace{\sum_i \omega_i (1 - \lambda_{ui}) h_{ui}(t)}_{ ext{energy uptake}} - \underbrace{c_u}_{ ext{energy costs}} \right)$	
	dividuals $\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy total energy costs}}{\text{time}}}_{\text{energy costs}} \right)$	$=rac{ ext{individual}}{ ext{time}}$
	$\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right)$	=
$= { m in} \epsilon$ $rac{dR_i}{dt} = rac{ ho}{2} \epsilon$	dividuals $\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) $ $\frac{v_{i}}{\sqrt{t_{i}}} - \underbrace{\frac{N_{u}(t)}{\kappa_{ui}} h_{ui}(t)}_{\text{decay}} + \underbrace{\sum_{u,j} \frac{\omega_{j}}{\omega_{i}} \lambda_{uj} \frac{N_{u}(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{kui}} \right)$	=
$= ext{inc}$ $rac{dR_i}{dt} = rac{ ho}{ ext{inf}}$ $= rac{ ho}{ ext{inf}}$	dividuals $\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) $ $= \underbrace{\frac{1}{\text{energy uptake}}}_{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) $ $= \underbrace{\frac{1}{\text{energy uptake}}}_{\text{energy uptake}} \left(\sum_{i} \frac{N_u(t)}{N_u(t)} h_{ui}(t) + \sum_{i} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij} \right) $ $= \underbrace{\frac{1}{\text{consumption}}}_{\text{consumption}} \underbrace{\frac{1}{\text{consumption}}}_{\text{by-production}} \underbrace{\frac{1}{\text{consumption}}}_{\text{by-production}} \underbrace{\frac{1}{\text{consumption}}}_{\text{energy uptake}} \underbrace{\frac{1}{consumpt$	= time
$= ext{inc}$ $rac{dR_i}{dt} = ho rac{ ho}{ ext{inf}}$ $= rac{ ext{m}}{ ext{ti}}$	dividuals $\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) $ $\frac{v_{i}}{v_{i}} - \underbrace{\frac{N_{u}(t)}{\kappa_{ui}} h_{ui}(t)}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\omega_{j}}{\omega_{i}} \lambda_{uj} \frac{N_{u}(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{individuals}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{decay}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{mass}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} $	= time
$= ext{inc}$ $rac{dR_i}{dt} = ho rac{ ho}{ ext{inf}}$ $= rac{ ext{m}}{ ext{ti}}$	dividuals $\frac{1}{\text{energy uptake}} \left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) $ $\frac{v_{i}}{\sqrt{t_{i}}} - \underbrace{\frac{N_{u}(t)}{\kappa_{ui}} h_{ui}(t)}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\omega_{j}}{\omega_{i}} \lambda_{uj} \frac{N_{u}(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{time}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{decay}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{time}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{by-production}}$	= time
$= ext{in} $ $rac{dR_i}{dt} = rac{ ho}{ ext{inf}}$ $= rac{ ext{m}}{ ext{in}}$ int int	$\frac{1}{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}}\right)}_{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}}\right)}_{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy costs}} - \underbrace{\sum_{i} \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{mass}}}_{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{energy}}}_{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{energy}$	$=rac{ ext{mas}}{ ext{time}}$
$= inc$ $\frac{dR_i}{dt} = \rho$ $= \frac{m}{\text{tir}}$ inf $i(t) = \sigma_u$ $= (u$	dividuals $\frac{1}{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \frac{\text{energy}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy costs}} - \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\left(\sum_{i} \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} \right)}_{\text{by-production}}$ $\underbrace{\frac{\text{ass}}{\text{mass}}}_{\text{time}} - \underbrace{\frac{\text{individuals}}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{individuals}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{by-production}}$ $\underbrace{\frac{\text{individuals}}{\kappa_{ui}} (t)}_{\text{initless}} + \underbrace{\frac{1}{\kappa_{ui}} \frac{\text{individuals}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{$	$= \frac{1}{\text{time}}$
$= inc$ $\frac{dR_i}{dt} = \rho$ $= \frac{m}{\text{tir}}$ inf $i(t) = \sigma_u$ $= (u$	dividuals $\frac{1}{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \frac{\text{energy}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy costs}} - \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\left(\sum_{i} \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} \right)}_{\text{by-production}}$ $\underbrace{\frac{\text{ass}}{\text{mass}}}_{\text{time}} - \underbrace{\frac{\text{individuals}}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{individuals}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{by-production}}$ $\underbrace{\frac{\text{individuals}}{\kappa_{ui}} (t)}_{\text{initless}} + \underbrace{\frac{1}{\kappa_{ui}} \frac{\text{individuals}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} (\text{unitless})}_{\text{time}} + \underbrace{\frac{\text{individuals}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{ui}} \frac{\text{energy}}{\kappa_{$	$= rac{- ext{time}}{ ext{time}}$
$= ext{in} c$ $rac{dR_i}{dt} = \int_{ ext{inf}} rac{d}{dt} = \int_{ ext{inf}} rac{d}{dt} = c_u$ $= (u)$ $c_u = \xi_u$	$\frac{1}{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}}\right)}_{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}}\right)}_{\text{energy uptake}} \underbrace{\left(\sum_{i} \frac{\text{energy uptake}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy costs}} - \underbrace{\sum_{i} \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) + \sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} + \underbrace{\sum_{u,j} \frac{\text{energy}}{\text{mass}}}_{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{mass}}}_{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{energy}}}_{\text{individuals}} \cdot \underbrace{\frac{\text{energy}}{\text{energy}$	$= \frac{ ext{mas}}{ ext{time}}$

ConsumerResourceSystem dynamics Fast resource equilibration

$$\begin{split} h_{ui}(t) &= \sigma_{ui}\beta_{ui} \frac{1}{1 + \frac{\sum_{v}N_{v}(t)\sigma_{vi}}{\kappa_{ui}}} = \sigma_{ui}\beta_{ui} \frac{\kappa_{ui}}{\kappa_{ui} + \sum_{v}N_{v}(t)\sigma_{vi}} \\ \frac{dN_{u}}{dt} &= N_{u}(t)\gamma_{u} \left(\sum_{i} \omega_{i}(1 - \lambda_{ui})h_{ui}(t) - \left(\xi_{u} + \sum_{i} \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj} \right) \right) \\ &= N_{u}(t)\gamma_{u} \left(\sum_{i} \omega_{i}(1 - \lambda_{ui})\sigma_{ui}\beta_{ui} \frac{\kappa_{ui}}{\kappa_{ui} + \sum_{v} N_{v}(t)\sigma_{vi}} - \left(\xi_{u} + \sum_{i} \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj} \right) \right) \\ &= N_{u}(t)\gamma_{u} \left(\sum_{i} \underbrace{\omega_{i}(1 - \lambda_{ui})\sigma_{ui}\beta_{ui}}_{\text{energy uptake coeffs}} \times \frac{\kappa_{ui}}{\kappa_{ui} + \sum_{v} N_{v}(t) \times \sigma_{vi}} - \underbrace{\left(\xi_{u} + \sum_{i} \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj} \right)}_{\text{energy costs}} \right) \end{split}$$

Linear resource consumption

$$h_{ui}(t) = \sigma_{ui}\beta_{ui}R_i(t)$$

$$\begin{split} \frac{dN_u}{dt} &= N_u(t)\gamma_u \Bigg(\sum_i \omega_i (1-\lambda_{ui}) h_{ui}(t) - \Big(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\Big) \Bigg) \\ &= N_u(t)\gamma_u \Bigg(\sum_i \omega_i (1-\lambda_{ui})\sigma_{ui}\beta_{ui}R_i(t) - \Big(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\Big) \Bigg) \\ &= N_u(t)\gamma_u \Bigg(\sum_i \underbrace{\omega_i (1-\lambda_{ui})\sigma_{ui}\beta_{ui}}_{\text{energy uptake coeffs}} \times R_i(t) - \underbrace{\Big(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i < j} \mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\Big)}_{\text{energy coeffs}} \Bigg) \end{split}$$

$$\begin{aligned} \frac{dR_i}{dt} &= \rho_i - \tau_i^{-1} R_i(t) - \sum_u N_u(t) h_{ui}(t) + \sum_{u,j} \mathbf{D}_{ij} \frac{\omega_j}{\omega_i} N_u(t) \lambda_{uj} h_{uj}(t) \\ &= \rho_i - \tau_i^{-1} R_i(t) - \sum_u N_u(t) \sigma_{ui} \beta_{ui} R_i(t) + \sum_{u,j} \mathbf{D}_{ij} \frac{\omega_j}{\omega_i} N_u(t) \lambda_{uj} \sigma_{uj} \beta_{uj} R_i(t) \end{aligned}$$

u: strain index

i: resource index

$$rac{dN_u}{dt} = N_u(t) \left(\underbrace{\sum_i h_{ui}(t) - \underbrace{c_u}_{ ext{energy costs}}}_{ ext{energy uptake}} \right)$$

$$h_{ui}(t) = egin{cases} \sigma_{ui} \left[eta_{ui} R_i(t)
ight] & ext{Linear} \ \sigma_{ui} \left[eta_{ui} rac{R_i(t)^{\eta_{ui}}}{R_i(t)^{\eta_{ui}} + \kappa^{\eta_{ui}}}
ight] & ext{Monod} \ \sigma_{ui} \left[eta_{ui} rac{1}{1 + rac{\sum_{v} N_v(t) \sigma_{vi}}{\kappa_i}}
ight] & ext{Fast resource equilibration} \end{cases}$$

$$c_u = egin{cases} \xi_u \sim \mathcal{N}(\mu, \sigma) \ \sum_i \chi_i \sigma_{ui} \ \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i,j} J_{ij} \sigma_{ui} \sigma_{uj} \end{cases}$$