

Consumer Resource Model

May 23, 2022

Consumer Resource Models from literature

Moran, Tikhonov model

$$\begin{aligned}\frac{dN_u}{dt} &= N_u(t) \left[\sum_i h_{ui}(t) - \left(c + \sum_i \chi_i \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \right) \right] \\ h_{ui}(t) &= \sigma_{ui} b_i \frac{1}{1 + \frac{\sum_v N_v(t) \sigma_{vi}}{k_i}}\end{aligned}$$

Mehta model

$$\begin{aligned}\frac{dN_u}{dt} &= g_u N_u(t) \left[\sum_i \omega_i (1 - l_i) h_{ui}(t) - c_u \right] \\ \frac{dR_i}{dt} &= \rho_i - \tau_i^{-1} R_i(t) - \sum_u N_u(t) h_{ui}(t) + \sum_{u,j} l_j N_u(t) \sigma_{uj} h_{uj}(t) D_{ij} \frac{\omega_j}{\omega_i} \\ h_{ui}(t) &= \sigma_{ui} R_i(t)\end{aligned}$$

O'Dwyer model

$$\begin{aligned}\frac{dN_u}{dt} &= N_u(t) \left[\sum_i g h_{ui}(t) - c \right] \\ \frac{dR_i}{dt} &= \rho_i - \sum_u N_u(t) h_{ui}(t) \\ h_{ui}(t) &= b_{ui} R_i(t)\end{aligned}$$

Consensus Tikhonov Group Consumer Resource Model

Generalized, with cross-feeding

$$\begin{aligned}
 \frac{dN_u}{dt} &= N_u(t) \underbrace{\gamma_u \left(\underbrace{\sum_i \omega_i (1 - \lambda_{ui}) h_{ui}(t) \sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}} \\
 \frac{dR_i}{dt} &= \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui}}_{\text{consumption}} + \underbrace{\sum_{u,j} D_{ij} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{uj}} h_{uj}(t) \sigma_{uj}}_{\text{by-production}} \\
 h_{ui}(t) &= \beta_{ui} R_i(t) \\
 c_u &= \begin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\ \dots \end{cases}
 \end{aligned}$$

Simplification 1: No cross-feeding, fix energy/growth conversion factors

(Assume $\lambda_{ui} = 0 \ \forall \ u, i$; $\omega_i = 1 \ \forall \ i$; $\gamma_u = 1 \ \forall \ u$)

$$\begin{aligned}
 \frac{dN_u}{dt} &= N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t) \sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}} \\
 \frac{dR_i}{dt} &= \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui}}_{\text{consumption}} \\
 h_{ui}(t) &= \beta_{ui} R_i(t) \\
 c_u &= \begin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\ \dots \end{cases}
 \end{aligned}$$

Simplification 2: Fast resource equilibration

(Assume $\frac{dR_i}{dt} = 0$)

$$\begin{aligned}
 \frac{dN_u}{dt} &= N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t) \sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}} \\
 h_{ui}(t) &= \beta_{ui} \frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}} \\
 c_u &= \begin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\ \dots \end{cases}
 \end{aligned}$$

Derivation of $h_{ui}(t)$:

$$\begin{aligned}
\frac{dR_i}{dt} &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} && \text{as above} \\
0 &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} && \text{assume } \frac{dR_i}{dt} = 0 \text{ (resource dynamics are fast relative to abundances)} \\
\rho_i &= \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} \\
\rho_i &= \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} R_i(t) \sigma_{ui} && h_{ui}(t) = \beta_{ui} R_i(t) \text{ as above} \\
\rho_i &= R_i(t) \left(\frac{1}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} \sigma_{ui} \right) \\
R_i(t) &= \frac{\rho_i}{\frac{1}{\tau_i} + \sum_u \frac{\beta_{ui} N_u(t) \sigma_{ui}}{\kappa_{ui}}}
\end{aligned}$$

$$h_{ui}(t) = \beta_{ui} R_i(t) = \beta_{ui} \frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}}$$

Simplification 3: Fix resource time scale parameters, hold carrying capacities constant across types

(Assume $\rho_i = 1 \forall i$; $\tau_i = 1 \forall i$; $\beta_{ui} = 1 \forall u, i$; $\kappa_{ui} = \kappa_{vi} \forall u, v, i \rightarrow$ Moran, Tikhonov (2022) model)

$$\begin{aligned}
\frac{dN_u}{dt} &= N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t) \sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}} \\
h_{ui}(t) &= \frac{1}{1 + \frac{\sum_v N_v(t) \sigma_{vi}}{\kappa_i}} = \sigma_{ui} \frac{1}{1 + \frac{T_i(t)}{\kappa_i}} && \text{Let } T_i(t) = \sum_v \sigma_{vi} N_v(t) \\
c_u &= \begin{cases} \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\ \dots \end{cases}
\end{aligned}$$

Dimensional analysis for Tikhonov Group CRM model:

	Description	Units
$N_u(t)$	Abundance of type u at time t	individuals
$R_i(t)$	Quantity of resource i at time t	mass
$h_{ui}(t)$	Rate of uptake of resource i by type u	mass/time
σ_{ui}	Utilization of resource i by type u (expression of "trait" i)	unitless
β_{ui}	Timescale for uptake of resource i by type u	1/time
λ_{ui}	Fraction of energy from resource i converted into byproducts by type u	unitless
γ_u	Conversion of net energy uptake to growth of type u	1/energy
c_u	Minimal energy uptake to maintain type u	energy/time
ξ_u	Minimal baseline energy uptake to maintain type u	energy/time
χ_{ui}	Minimal marginal energy uptake to maintain utilization of resource i by type u	energy/time
J_{ij}	Minimal marginal energy uptake to maintain utilization of <i>both</i> resource i and j	energy/time
ρ_i	Influx rate of resource i	mass/time
τ_i	Timescale for dilution/decay of resource i	time
κ_{ui}	"Carrying capacity" of resource i for type u (resource depletion rate scaling factor)	individuals
ω_i	Energy content of resource i	energy/mass
D_{ij}	Fraction of byproducts of resource i that are converted to resource j	unitless

$$\begin{aligned}
 \frac{dN_u}{dt} &= N_u(t) \gamma_u \left(\underbrace{\sum_i \omega_i (1 - \lambda_{ui}) h_{ui}(t)}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right) \\
 &\quad \underbrace{\hspace{10em}}_{\text{growth rate}} \\
 &= \text{individuals} \frac{1}{\text{energy}} \left(\underbrace{\sum_i \frac{\text{energy}}{\text{mass}} (1 - \text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right) \\
 &\quad \underbrace{\hspace{10em}}_{\text{growth rate}} = \frac{\text{individuals}}{\text{time}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dR_i}{dt} &= \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}} \\
 &= \underbrace{\frac{\text{mass}}{\text{time}}}_{\text{influx}} - \underbrace{\frac{\text{mass}}{\text{time}}}_{\text{decay}} - \underbrace{\sum_u \frac{\text{individuals}}{\text{individuals}} \frac{\text{mass}}{\text{time}}}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\frac{\text{energy}}{\text{mass}}}{\frac{\text{energy}}{\text{mass}}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{by-production}} = \frac{\text{mass}}{\text{time}}
 \end{aligned}$$

$$\begin{aligned}
 h_{ui}(t) &= \sigma_{ui} \beta_{ui} R_i(t) \\
 &= (\text{unitless}) \frac{1}{\text{time}} \text{mass} = \frac{\text{mass}}{\text{time}} \\
 c_u &= \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i < j} J_{ij} \sigma_{ui} \sigma_{uj} \\
 &= \frac{\text{energy}}{\text{time}} + \sum_i \frac{\text{energy}}{\text{time}} (\text{unitless}) - \sum_{i < j} \frac{\text{energy}}{\text{time}} (\text{unitless}) (\text{unitless}) = \frac{\text{energy}}{\text{time}}
 \end{aligned}$$

ConsumerResourceSystem dynamics notes

Consensus Tikhonov Group Consumer Resource Model

$$h_{ui}(t) = \beta_{ui} R_i(t)$$

$$\begin{aligned} \frac{dN_u}{dt} &= N_u(t) \gamma_u \left(\sum_i \omega_i (1 - \lambda_{ui}) h_{ui}(t) \sigma_{ui} - c_u \right) \\ &= N_u(t) \gamma_u \left(\sum_i \omega_i (1 - \lambda_{ui}) \beta_{ui} \sigma_{ui} R_i(t) - c_u \right) \end{aligned}$$

$$\begin{aligned} \frac{dR_i}{dt} &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t) \sigma_{ui} + \sum_{u,j} D_{ij} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{uj}} h_{uj}(t) \sigma_{uj} \\ &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} R_i(t) \sigma_{ui} + \sum_{u,j} D_{ij} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{uj}} \beta_{uj} R_j(t) \sigma_{uj} \\ &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{\sigma_{ui} \beta_{ui}}{\kappa_{ui}} N_u(t) R_i(t) + \sum_{u,j} D_{ij} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{\sigma_{uj} \beta_{uj}}{\kappa_{uj}} N_u(t) R_j(t) \\ &= \rho_i - \frac{1}{\tau_i} R_i(t) - \sum_u \frac{\beta_{ui} \sigma_{ui}}{\kappa_{ui}} N_u(t) R_i(t) + \sum_{u,j} M_{ij} \lambda_{uj} \frac{\beta_{uj} \sigma_{uj}}{\kappa_{uj}} N_u(t) R_j(t) \quad (M_{ij} = D_{ij} \frac{\omega_j}{\omega_i}) \end{aligned}$$

Simplification 2: Fast resource equilibration

$$h_{ui}(t) = \beta_{ui} \frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}}$$

$$\begin{aligned} \frac{dN_u}{dt} &= N_u(t) \left(\sum_i h_{ui}(t) \sigma_{ui} - c_u \right) \\ &= N_u(t) \left(\sum_i \beta_{ui} \sigma_{ui} \frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} N_v(t) \sigma_{vi}}{\kappa_{vi}}} - c_u \right) \\ &= N_u(t) \left(\sum_i \beta_{ui} \sigma_{ui} \frac{1}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} \sigma_{vi}}{\kappa_{vi}} N_v(t)} - c_u \right) \end{aligned}$$