Consumer Resource Model

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Consumer Resource Models from literature

Moran, Tikhonov model

$$\frac{dN_u}{dt} = N_u(t)\left[\sum_i h_{ui}(t) - \left(c + \sum_i \chi_i \sigma_{ui} - \sum_{i<j} J_{ij}\sigma_{ui}\sigma_{uj}\right)\right]$$
$$h_{ui}(t) = \sigma_{ui} b_i \frac{1}{1 + \frac{\sum_v N_v(t)\sigma_{vi}}{k_i}}$$

Mehta model

$$\frac{dN_u}{dt} = g_u N_u(t)\left[\sum_i \omega_i (1 - l_i) h_{ui}(t) - c_u\right]$$
$$\frac{dR_i}{dt} = \rho_i - \tau_i^{-1} R_i(t) - \sum_u N_u(t) h_{ui}(t) + \sum_{u,j} l_j N_u(t)\sigma_{uj} h_{uj}(t) D_{ij}\frac{\omega_j}{\omega_i}$$
$$h_{ui}(t) = \sigma_{ui} R_i(t)$$

O'Dwyer model

$$\frac{dN_u}{dt} = N_u(t)\left[\sum_i g h_{ui}(t) - c\right]$$
$$\frac{dR_i}{dt} = \rho_i - \sum_u N_u(t) h_{ui}(t)$$
$$h_{ui}(t) = b_{ui} R_i(t)$$

Consensus Tikhonov Group Consumer Resource Model

Generalized, with cross-feeding

$$\frac{dN_u}{dt} = N_u(t)\,\gamma_u \underbrace{\left(\underbrace{\sum_i \omega_i(1-\lambda_{ui})h_{ui}(t)\sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}}\right)}_{\text{growth rate}}$$

$$\frac{dR_i}{dt} = \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)\sigma_{ui}}_{\text{consumption}} + \underbrace{\sum_{u,j} D_{ij}\frac{\omega_j}{\omega_i}\lambda_{uj}\frac{N_u(t)}{\kappa_{ui}} h_{uj}(t)\sigma_{ui}}_{\text{by-production}}$$

$$h_{ui}(t) = \beta_{ui}R_i(t)$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j} J_{ij}\sigma_{ui}\sigma_{uj} \\ \ldots \end{cases}$$

Simplification 1: No cross-feeding, fix energy/growth conversion factors

(Assume $\lambda_{ui} = 0 \;\forall\; u, i$; $\omega_i = 1 \;\forall\; i$; $\gamma_u = 1 \;\forall\; u$)

$$\frac{dN_u}{dt} = N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t)\sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}}\right)}_{\text{growth rate}}$$

$$\frac{dR_i}{dt} = \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)\sigma_{ui}}_{\text{consumption}}$$

$$h_{ui}(t) = \beta_{ui}R_i(t)$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j} J_{ij}\sigma_{ui}\sigma_{uj} \\ \ldots \end{cases}$$

Simplification 2: Fast resource equilibration

(Assume $\frac{dR_i}{dt} = 0$)

$$\frac{dN_u}{dt} = N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t)\sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}}\right)}_{\text{growth rate}}$$

$$h_{ui}(t) = \beta_{ui}\frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi}N_v(t)\sigma_{vi}}{\kappa_{vi}}}$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j} J_{ij}\sigma_{ui}\sigma_{uj} \\ \ldots \end{cases}$$

Derivation of $h_{ui}(t)$:

$$\begin{aligned}
\frac{dR_i}{dt} &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)\sigma_{ui} && \text{as above} \\
0 &= \rho_i - \frac{R_i(t)}{\tau_i} - \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)\sigma_{ui} && \text{assume } \tfrac{dR_i}{dt} = 0 \text{ (resource dynamics are fast relative to abundances)} \\
\rho_i &= \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)\sigma_{ui} \\
\rho_i &= \frac{R_i(t)}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui} R_i(t)\sigma_{ui} && h_{ui}(t) = \beta_{ui} R_i(t) \text{ as above} \\
\rho_i &= R_i(t) \left(\frac{1}{\tau_i} + \sum_u \frac{N_u(t)}{\kappa_{ui}} \beta_{ui}\sigma_{ui} \right) \\
R_i(t) &= \frac{\rho_i}{\frac{1}{\tau_i} + \sum_u \frac{\beta_{ui} N_u(t)\sigma_{ui}}{\kappa_{ui}}}
\end{aligned}$$

$$h_{ui}(t) = \beta_{ui} R_i(t) = \beta_{ui} \frac{\rho_i}{\frac{1}{\tau_i} + \sum_v \frac{\beta_{vi} N_v(t)\sigma_{vi}}{\kappa_{vi}}}$$

Simplification 3: Fix resource time scale parameters, hold carrying capacities constant across types

(Assume $\rho_i = 1 \ \forall \ i$; $\tau_i = 1 \ \forall \ i$; $\beta_{ui} = 1 \ \forall \ u, i$; $\kappa_{ui} = \kappa_{vi} \ \forall \ u, v, i \rightarrow$ Moran, Tikhonov (2022) model)

$$\frac{dN_u}{dt} = N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t)\sigma_{ui}}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}}$$

$$h_{ui}(t) = \frac{1}{1 + \frac{\sum_v N_v(t)\sigma_{vi}}{\kappa_i}} = \sigma_{ui} \frac{1}{1 + \frac{T_i(t)}{\kappa_i}} \qquad \text{Let } T_i(t) = \textstyle\sum_v \sigma_{vi} N_v(t)$$

$$c_u = \begin{cases} \xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j} J_{ij}\sigma_{ui}\sigma_{uj} \\ \ldots \end{cases}$$

Dimensional analysis for Tikhonov Group CRM model:

	Description	**Units**
$N_u(t)$	Abundance of type u at time t	individuals
$R_i(t)$	Quantity of resource i at time t	mass
$h_{ui}(t)$	Rate of uptake of resource i by type u	mass/time
σ_{ui}	Utilization of resource i by type u (expression of "trait" i)	*unitless*
β_{ui}	Timescale for uptake of resource i by type u	1/time
λ_{ui}	Fraction of energy from resource i converted into byproducts by type u	*unitless*
γ_u	Conversion of net energy uptake to growth of type u	1/energy
c_u	Minimal energy uptake to maintain type u	energy/time
ξ_u	Minimal baseline energy uptake to maintain type u	energy/time
χ_{ui}	Minimal marginal energy uptake to maintain utilization of resource i by type u	energy/time
J_{ij}	Minimal marginal energy uptake to maintain utilization of *both* resource i and j	energy/time
ρ_i	Influx rate of resource i	mass/time
τ_i	Timescale for dilution/decay of resource i	time
κ_{ui}	"Carrying capacity" of resource i for type u (resource depletion rate scaling factor)	individuals
ω_i	Energy content of resource i	energy/mass
D_{ij}	Fraction of byproducts of resource i that are converted to resource j	*unitless*

$$\frac{dN_u}{dt} = N_u(t)\,\gamma_u \underbrace{\left(\underbrace{\sum_i \omega_i (1-\lambda_{ui}) h_{ui}(t)}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}}$$

$$= \text{individuals}\,\underbrace{\frac{1}{\text{energy}} \left(\underbrace{\sum_i \frac{\text{energy}}{\text{mass}} (1-\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{energy uptake}} - \underbrace{\frac{\text{energy}}{\text{time}}}_{\text{energy costs}} \right)}_{\text{growth rate}} \qquad = \frac{\text{individuals}}{\text{time}}$$

$$\frac{dR_i}{dt} = \underbrace{\rho_i}_{\text{influx}} - \underbrace{\frac{R_i(t)}{\tau_i}}_{\text{decay}} - \underbrace{\sum_u \frac{N_u(t)}{\kappa_{ui}} h_{ui}(t)}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\omega_j}{\omega_i} \lambda_{uj} \frac{N_u(t)}{\kappa_{ui}} h_{uj}(t) D_{ij}}_{\text{by-production}}$$

$$= \underbrace{\frac{\text{mass}}{\text{time}}}_{\text{influx}} - \underbrace{\frac{\text{mass}}{\text{time}}}_{\text{decay}} - \underbrace{\sum_u \frac{\text{individuals}}{\text{individuals}} \frac{\text{mass}}{\text{time}}}_{\text{consumption}} + \underbrace{\sum_{u,j} \frac{\frac{\text{energy}}{\text{mass}}}{\frac{\text{energy}}{\text{mass}}} (\text{unitless}) \frac{\text{individuals}}{\text{individuals}} (\text{unitless}) \frac{\text{mass}}{\text{time}}}_{\text{by-production}} \qquad = \frac{\text{mass}}{\text{time}}$$

$$h_{ui}(t) = \sigma_{ui} \beta_{ui} R_i(t)$$

$$= (\text{unitless}) \frac{1}{\text{time}} \text{mass} \qquad = \frac{\text{mass}}{\text{time}}$$

$$c_u = \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i<j} J_{ij} \sigma_{ui} \sigma_{uj}$$

$$= \frac{\text{energy}}{\text{time}} + \sum_i \frac{\text{energy}}{\text{time}} (\text{unitless}) - \sum_{i<j} \frac{\text{energy}}{\text{time}} (\text{unitless})(\text{unitless}) \qquad = \frac{\text{energy}}{\text{time}}$$

ConsumerResourceSystem dynamics
Fast resource equilibration

$$h_{ui}(t) = \sigma_{ui}\beta_{ui}\frac{1}{1+\frac{\sum_v N_v(t)\sigma_{vi}}{\kappa_{ui}}} = \sigma_{ui}\beta_{ui}\frac{\kappa_{ui}}{\kappa_{ui}+\sum_v N_v(t)\sigma_{vi}}$$

$$\begin{aligned}
\frac{dN_u}{dt} &= N_u(t)\gamma_u\left(\sum_i \omega_i(1-\lambda_{ui})h_{ui}(t) - \left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)\right)\\
&= N_u(t)\gamma_u\left(\sum_i \omega_i(1-\lambda_{ui})\sigma_{ui}\beta_{ui}\frac{\kappa_{ui}}{\kappa_{ui}+\sum_v N_v(t)\sigma_{vi}} - \left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)\right)\\
&= N_u(t)\gamma_u\left(\sum_i \underbrace{\omega_i(1-\lambda_{ui})\sigma_{ui}\beta_{ui}}_{\text{energy uptake coeffs}} \times \frac{\kappa_{ui}}{\kappa_{ui}+\underbrace{\sum_v N_v(t)\times\sigma_{vi}}_{\text{resource demand}}} - \underbrace{\left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)}_{\text{energy costs}}\right)
\end{aligned}$$

Linear resource consumption

$$h_{ui}(t) = \sigma_{ui}\beta_{ui}R_i(t)$$

$$\begin{aligned}
\frac{dN_u}{dt} &= N_u(t)\gamma_u\left(\sum_i \omega_i(1-\lambda_{ui})h_{ui}(t) - \left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)\right)\\
&= N_u(t)\gamma_u\left(\sum_i \omega_i(1-\lambda_{ui})\sigma_{ui}\beta_{ui}R_i(t) - \left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)\right)\\
&= N_u(t)\gamma_u\left(\sum_i \underbrace{\omega_i(1-\lambda_{ui})\sigma_{ui}\beta_{ui}}_{\text{energy uptake coeffs}} \times R_i(t) - \underbrace{\left(\xi_u + \sum_i \chi_{ui}\sigma_{ui} - \sum_{i<j}\mathbf{J}_{ij}\sigma_{ui}\sigma_{uj}\right)}_{\text{energy costs}}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{dR_i}{dt} &= \rho_i - \tau_i^{-1}R_i(t) - \sum_u N_u(t)h_{ui}(t) + \sum_{u,j}\mathbf{D}_{ij}\frac{\omega_j}{\omega_i}N_u(t)\lambda_{uj}h_{uj}(t)\\
&= \rho_i - \tau_i^{-1}R_i(t) - \sum_u N_u(t)\sigma_{ui}\beta_{ui}R_i(t) + \sum_{u,j}\mathbf{D}_{ij}\frac{\omega_j}{\omega_i}N_u(t)\lambda_{uj}\sigma_{uj}\beta_{uj}R_i(t)
\end{aligned}$$

u : strain index

i : resource index

$$\frac{dN_u}{dt} = N_u(t) \underbrace{\left(\underbrace{\sum_i h_{ui}(t)}_{\text{energy uptake}} - \underbrace{c_u}_{\text{energy costs}} \right)}_{\text{growth rate}}$$

$$h_{ui}(t) = \begin{cases} \sigma_{ui} \left[\beta_{ui} R_i(t) \right] & \text{Linear} \\ \sigma_{ui} \left[\beta_{ui} \frac{R_i(t)^{\eta_{ui}}}{R_i(t)^{\eta_{ui}} + \kappa_{ui}^{\eta_{ui}}} \right] & \text{Monod} \\ \sigma_{ui} \left[\beta_{ui} \frac{1}{1 + \frac{\sum_v N_v(t) \sigma_{vi}}{\kappa_i}} \right] & \text{Fast resource equilibration} \end{cases}$$

$$c_u = \begin{cases} \xi_u \sim \mathcal{N}(\mu, \sigma) \\ \sum_i \chi_i \sigma_{ui} \\ \xi_u + \sum_i \chi_{ui} \sigma_{ui} - \sum_{i,j} J_{ij} \sigma_{ui} \sigma_{uj} \end{cases}$$