NPTEL MOOC,JAN-FEB 2015
Week 4, Module 7

DESIGN AND ANALYSIS OF ALGORITHMS

Spanning trees: Kruskal's algorithm

MADHAVAN MUKUND, CHENNAI MATHEMATICAL INSTITUTE
http://www.cmi.ac.in/~madhavan

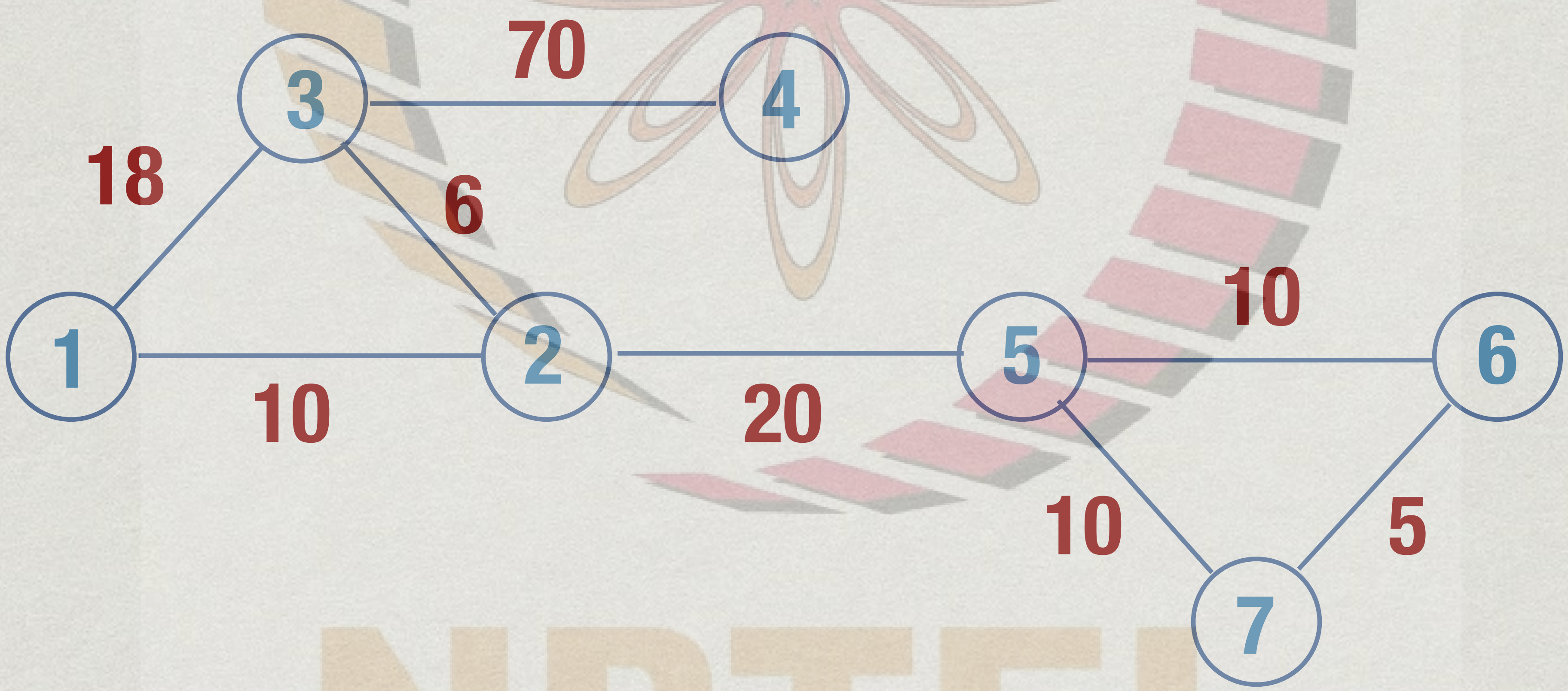
Spanning tree

- Weighted undirected graph, $G = (V,E,w)$
  - Assume G is connected
- Identify a spanning tree with minimum weight
  - Tree connecting all vertices in V
- **Strategy 2:**
  - Order edges in ascending order by weight
  - Keep adding edges to combine components

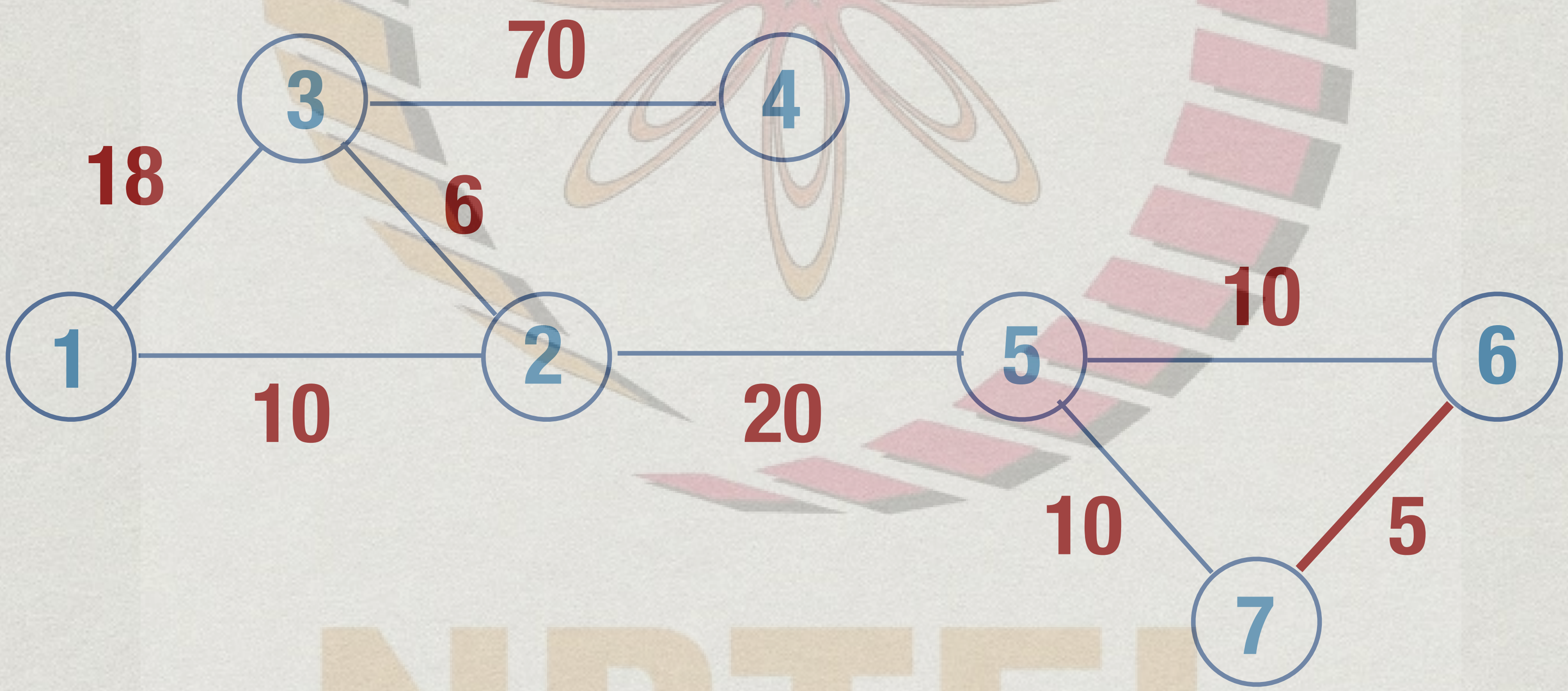
Kruskal's algorithm

```
algorithm Kruskal_V1

Let E = [e1,e2,…,em] be edges sorted by weight

TE = [] // List of edges added so far
i = 1   // Index of edge to try next

while TE.length() < n-1 //n-1 edges form a tree
  if adding E[i] to TE does not form a cycle
    TE.append(E[i])
    i = i+1
```

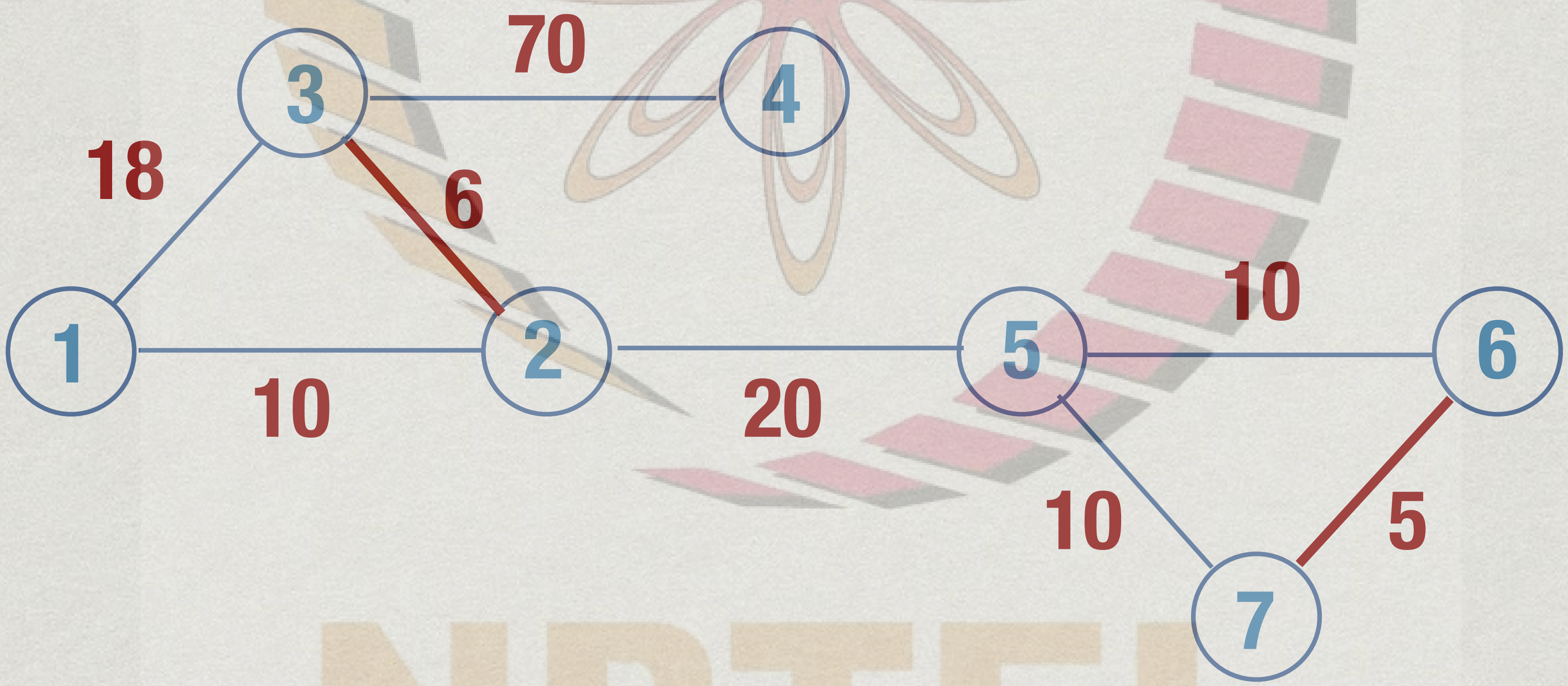
Kruskal's algorithm



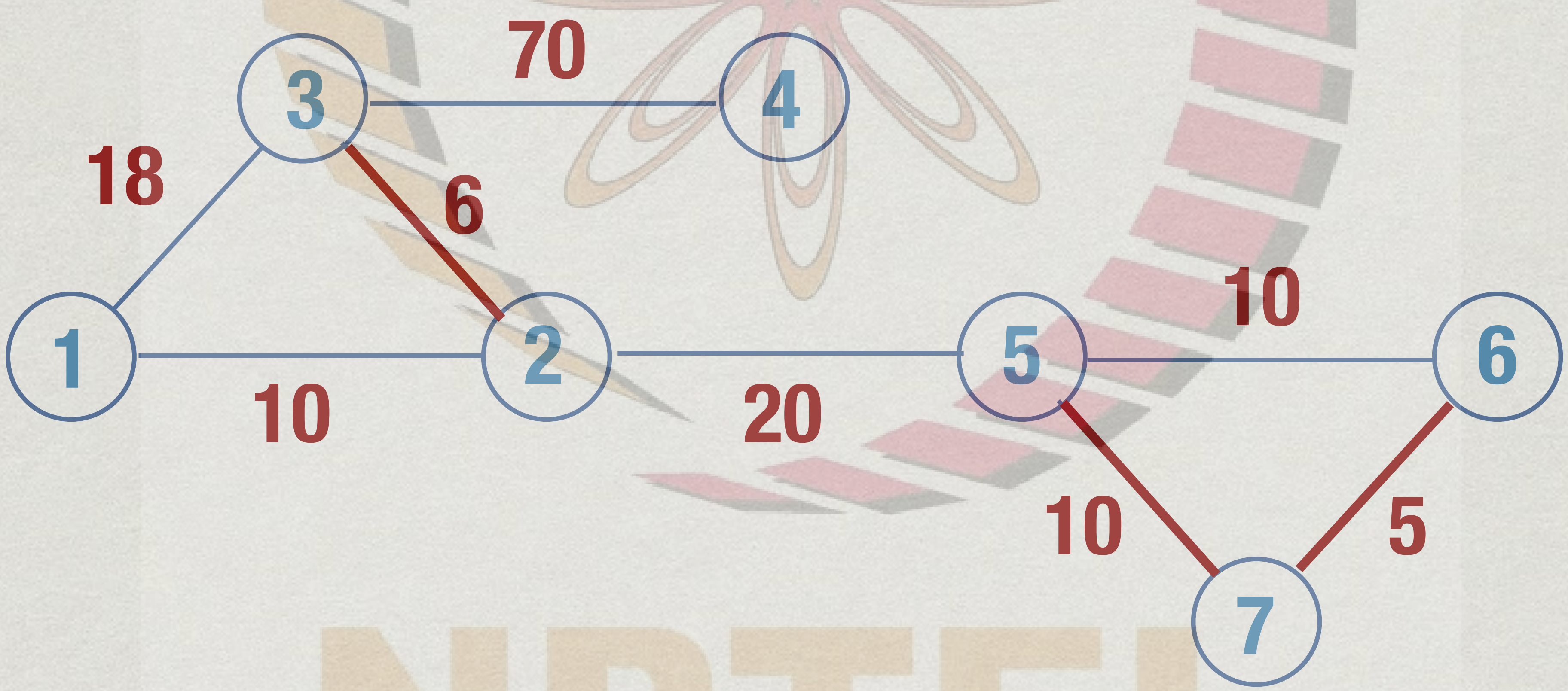
Kruskal's algorithm



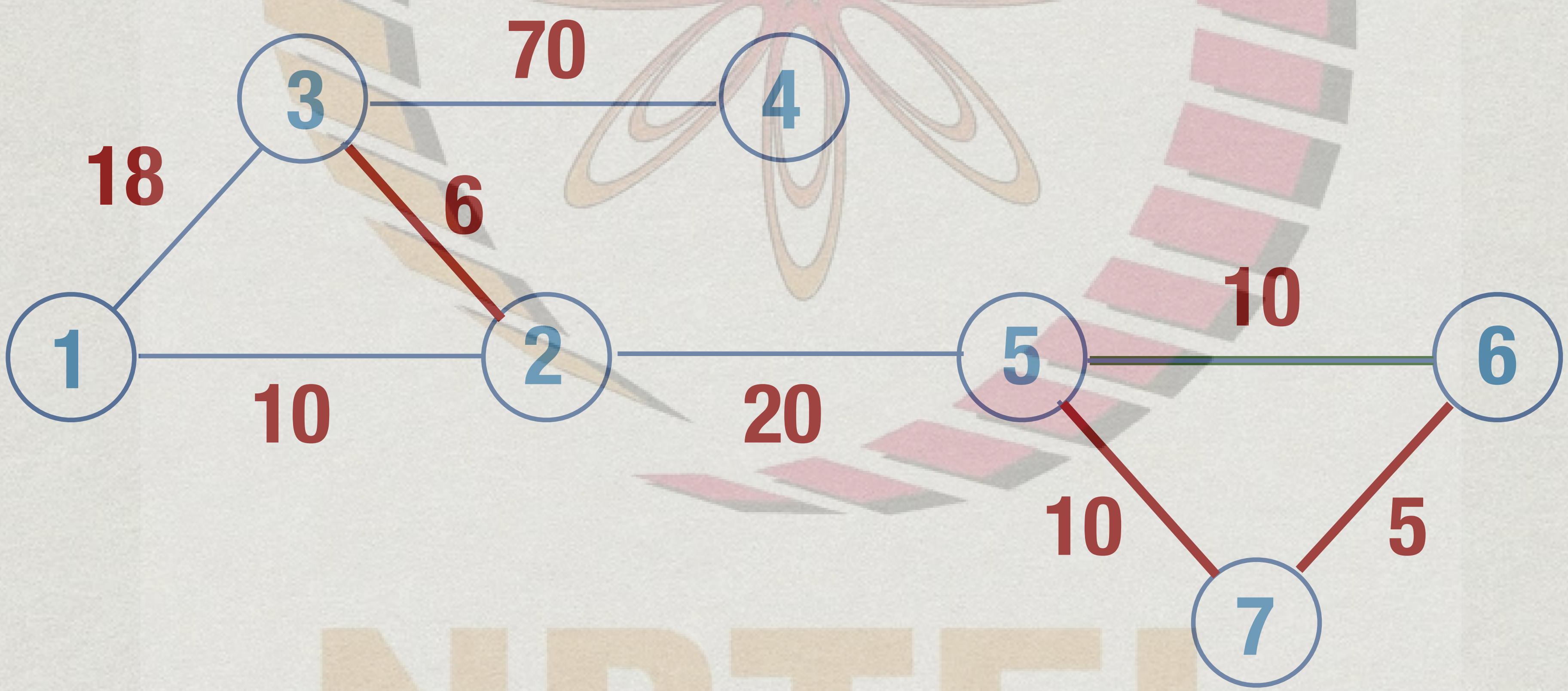
Kruskal's algorithm



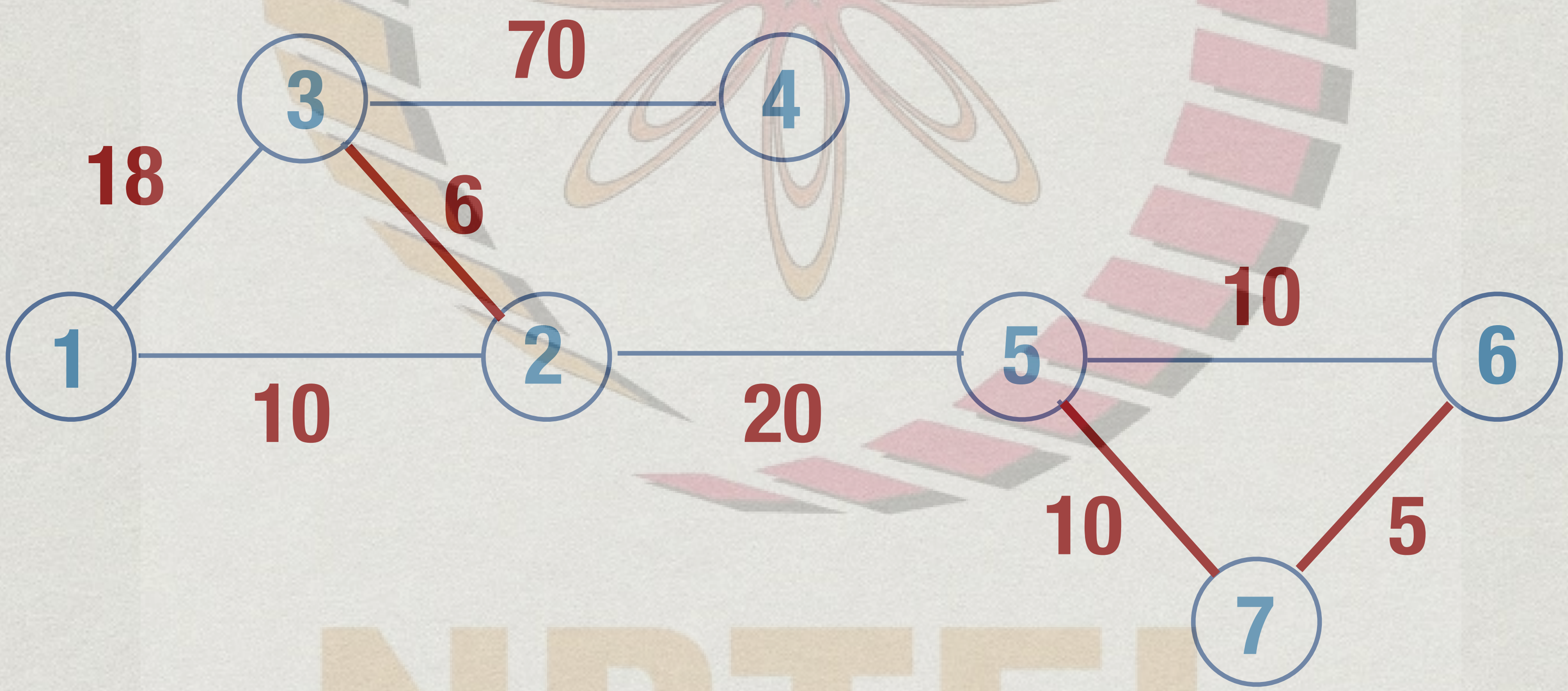
Kruskal's algorithm



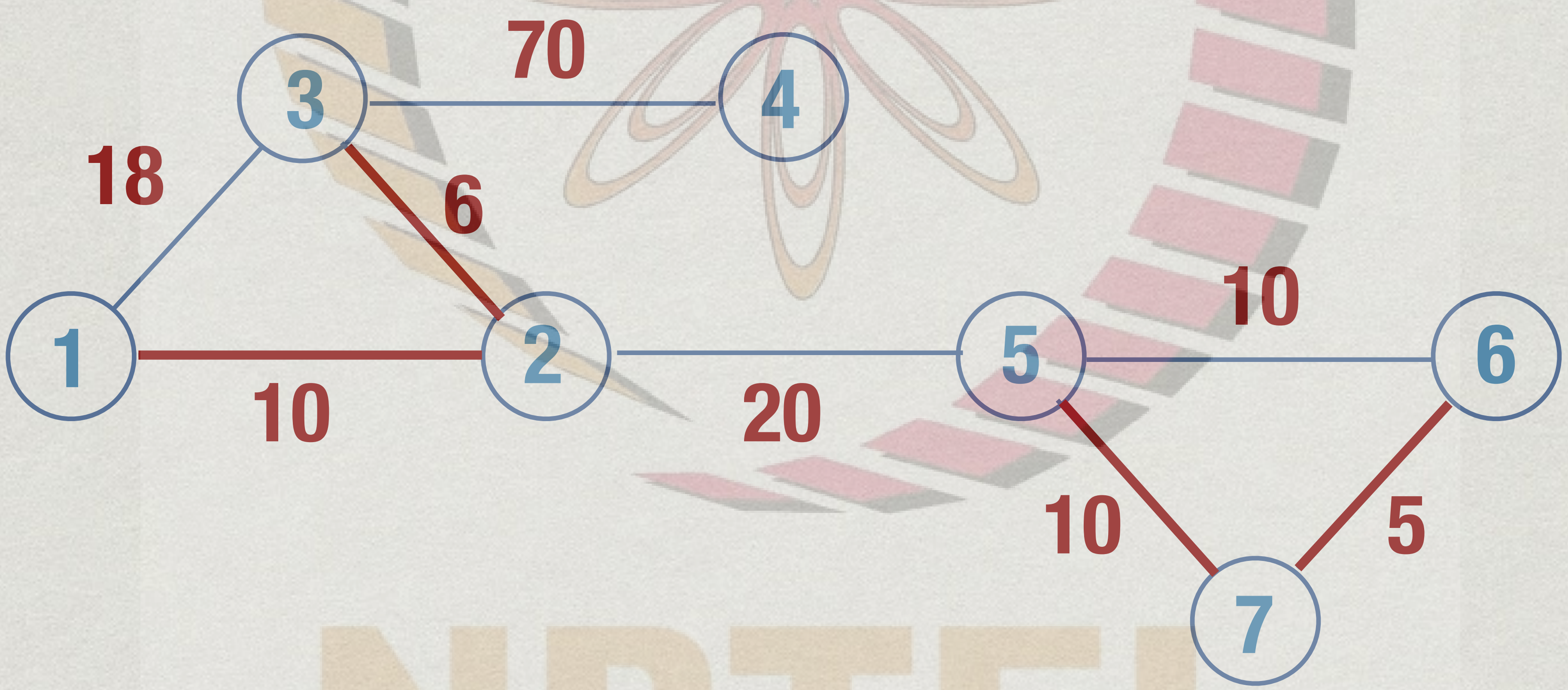
Kruskal's algorithm



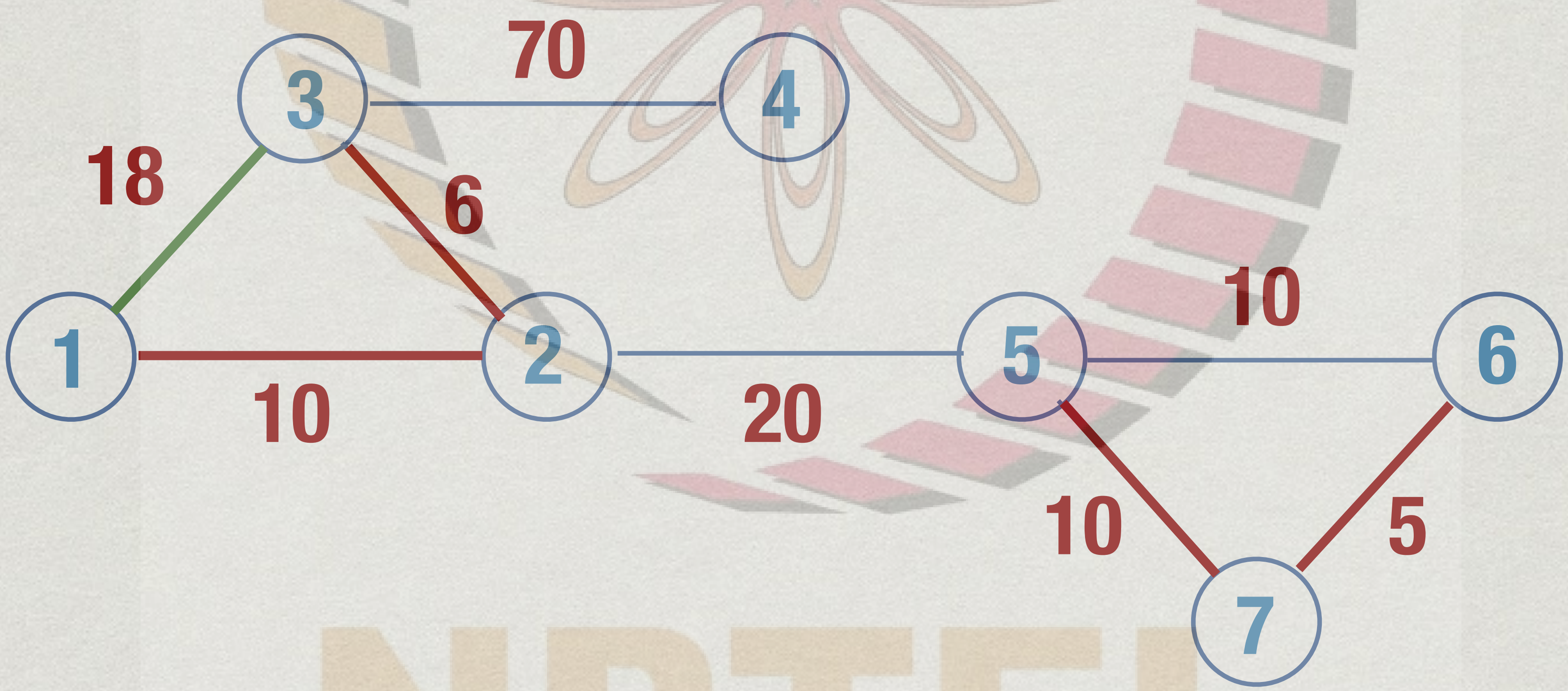
Kruskal's algorithm



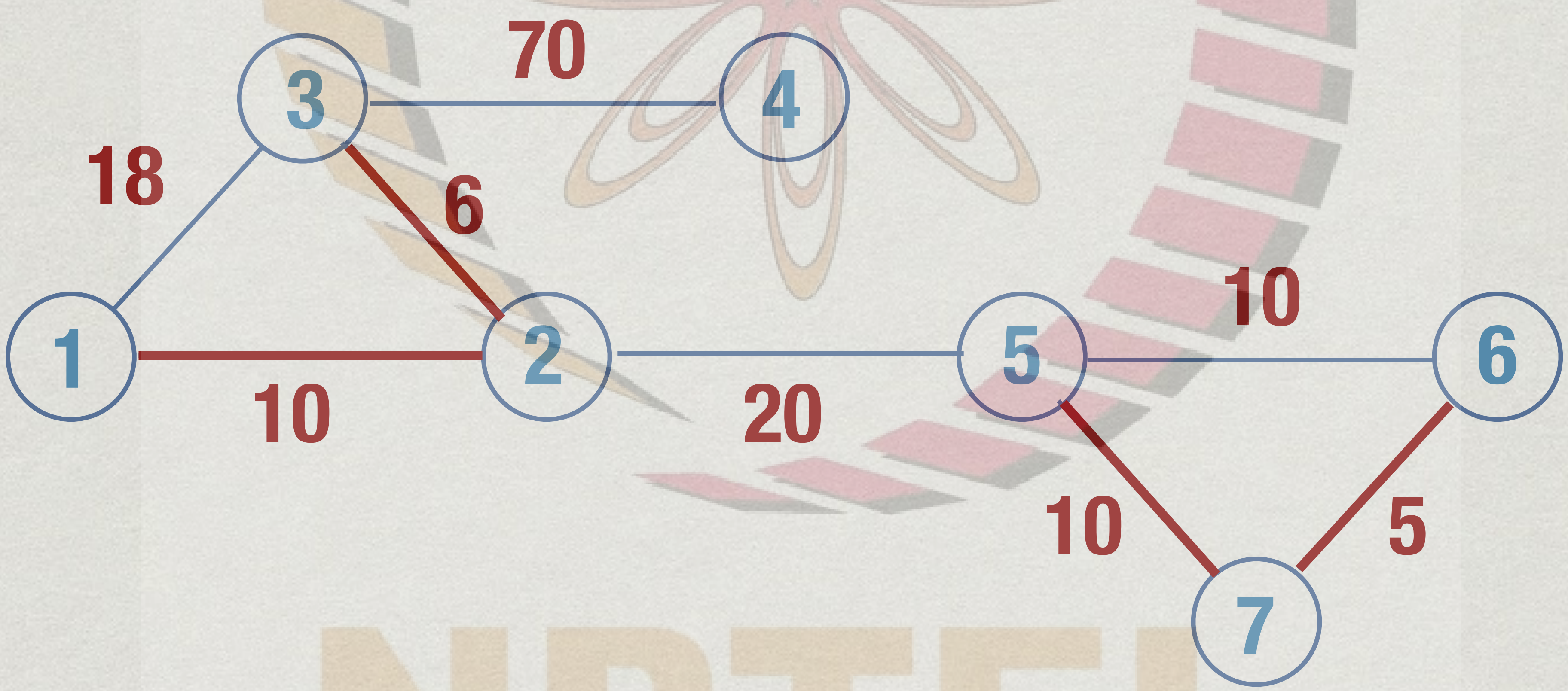
Kruskal's algorithm



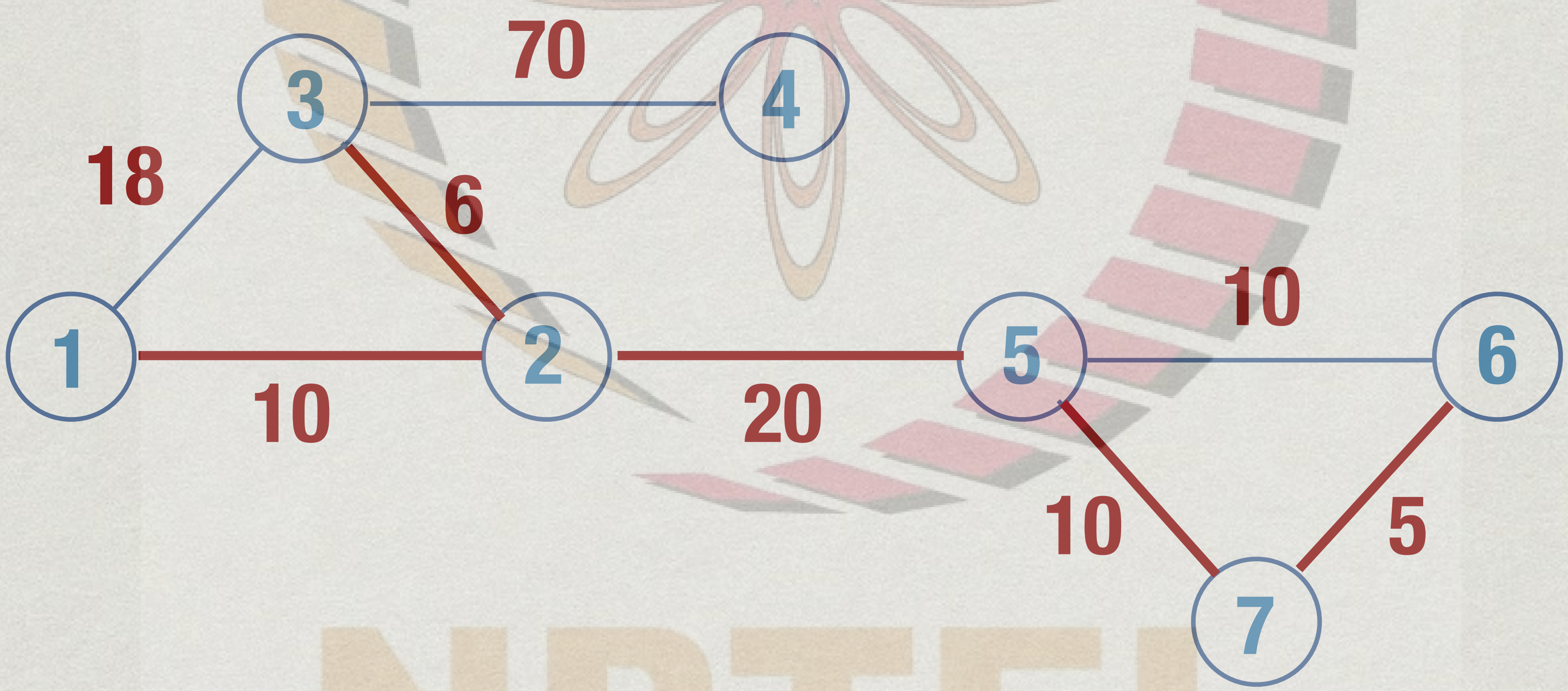
Kruskal's algorithm



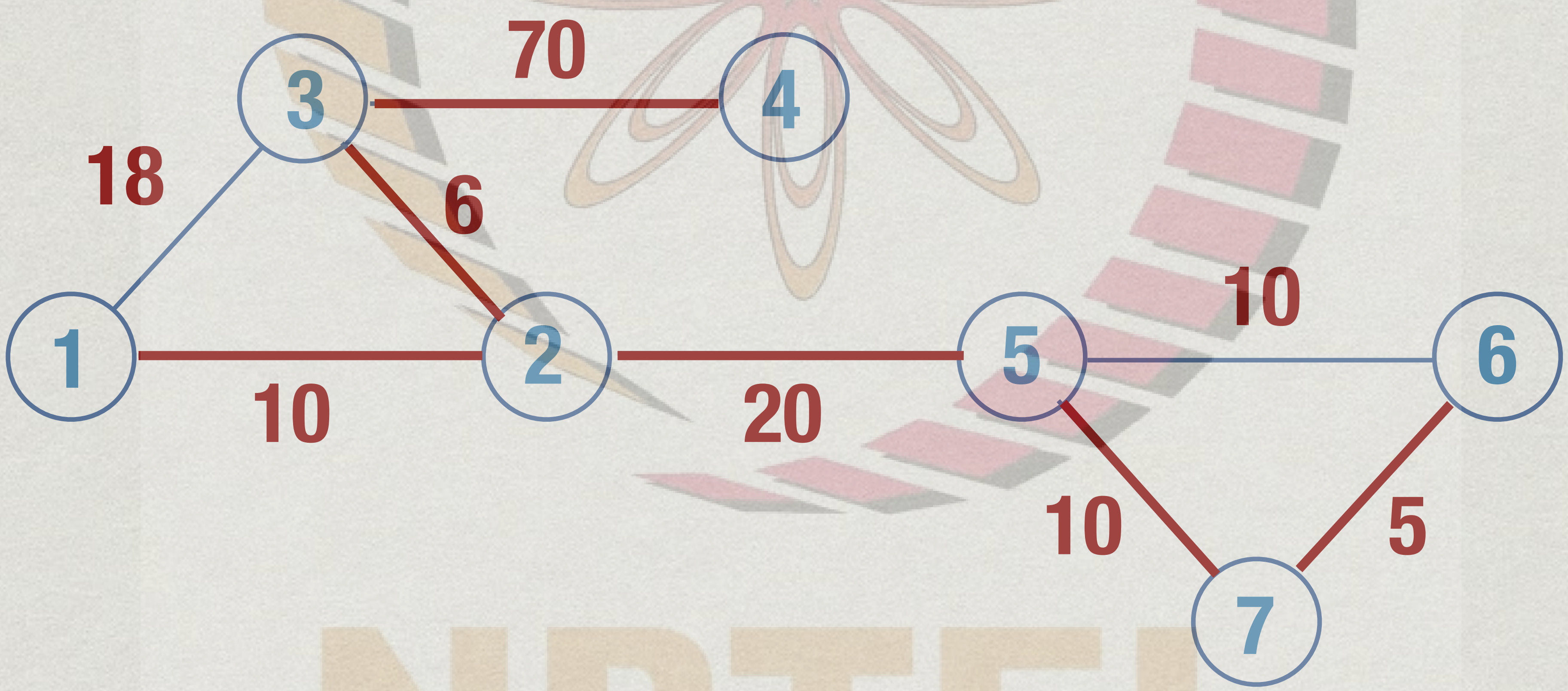
Kruskal's algorithm



Kruskal's algorithm



Kruskal's algorithm



Correctness

- Kruskal's algorithm is also a greedy algorithm
- We fix in advance that edges will be added in ascending order of weight
- Why does this achieve a global optimum?

Minimum separator lemma

- Let V be partitioned into two non-empty sets U and W = V - U
- Let e = (u,w) be minimum cost edge with u in U and w in W
  - Assume all edges have different weights
- Then every minimum cost spanning tree must include e

Correctness of Kruskal's algorithm …

- Unlike Prim's algorithm, at intermediate stages TE is not a tree
- Edges in TE partition vertices into connected components
  - Initially, each vertex is a separate component
  - Adding e = (u,v) merges components of u and v
    - If u and v are already in same component, e forms a cycle, hence discarded

Correctness of Kruskal's algorithm …

- Suppose $e_j = (u,v)$ with u and v in disjoint components
  - Let U = Component(u), W = V - Component(u)
  - No smaller weight edge in $[e_1,e_2,\ldots,e_{j-1}]$ connects U and W
  - By minimum separator lemma, e_j must be in the minimum cost spanning tree

Kruskal's algorithm revisited

- To check if e = (u,v) forms a cycle, keep track of components
- Initially, Component[i] = i for each vertex i
- e = (u,v) can be added if Component[u] is different from Component[v]
  - Merge the two components

Kruskal's algorithm, refined

```
algorithm Kruskal

Let E = [e1,e2,…,em] be edges sorted by weight

for j in 1 to n       //Initially, each vertex is isolated
  Component[j] = j    //Component names are 1..n

TE = []               //List of edges added so far
i = 1                 //Index of edge to try next

while TE.length() < n-1  //n-1 edges form a tree
  Let E[i] = (u,v)
  if Component[u] != Component[v] //E[i] does not form cycle
    TE.append(E[i])
    for j in 1 to n //Merge Component[v] into Component[u]
      if Component[j] == Component[v]
        Component[j] = Component[u]
```

Complexity

- Initially, sort edges, O(m log m)
  - m is at most n^2, so this is also O(m log n)
- Outer loop runs upto m times
  - In each iteration, we examine one edge
  - If we add the edge, we have to merge components
    - O(n) scan to update components
    - This is done once for each tree edge—O(n) times
- Overall $O(n^2)$

Bottleneck

- Naive strategy for labelling and merging components is inefficient
- Components form a partition of the vertex set V
- Union-find data structure implements the following operations efficiently
  - find(v) — find the component containing v
  - union(u,v) — merge the components of u and v
- This will bring down the complexity to O(m log n)