Sparse Additive Interaction Learning

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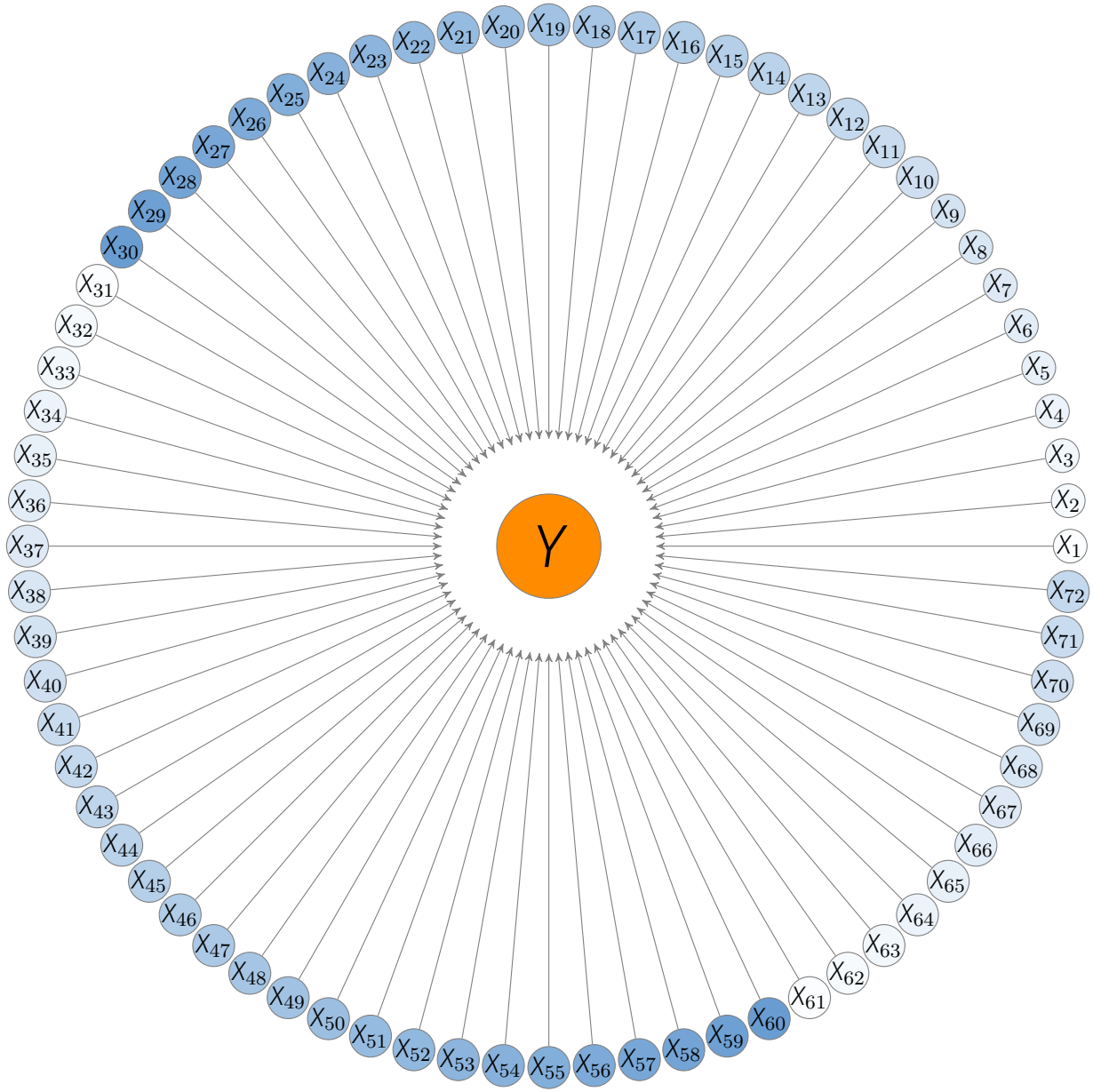
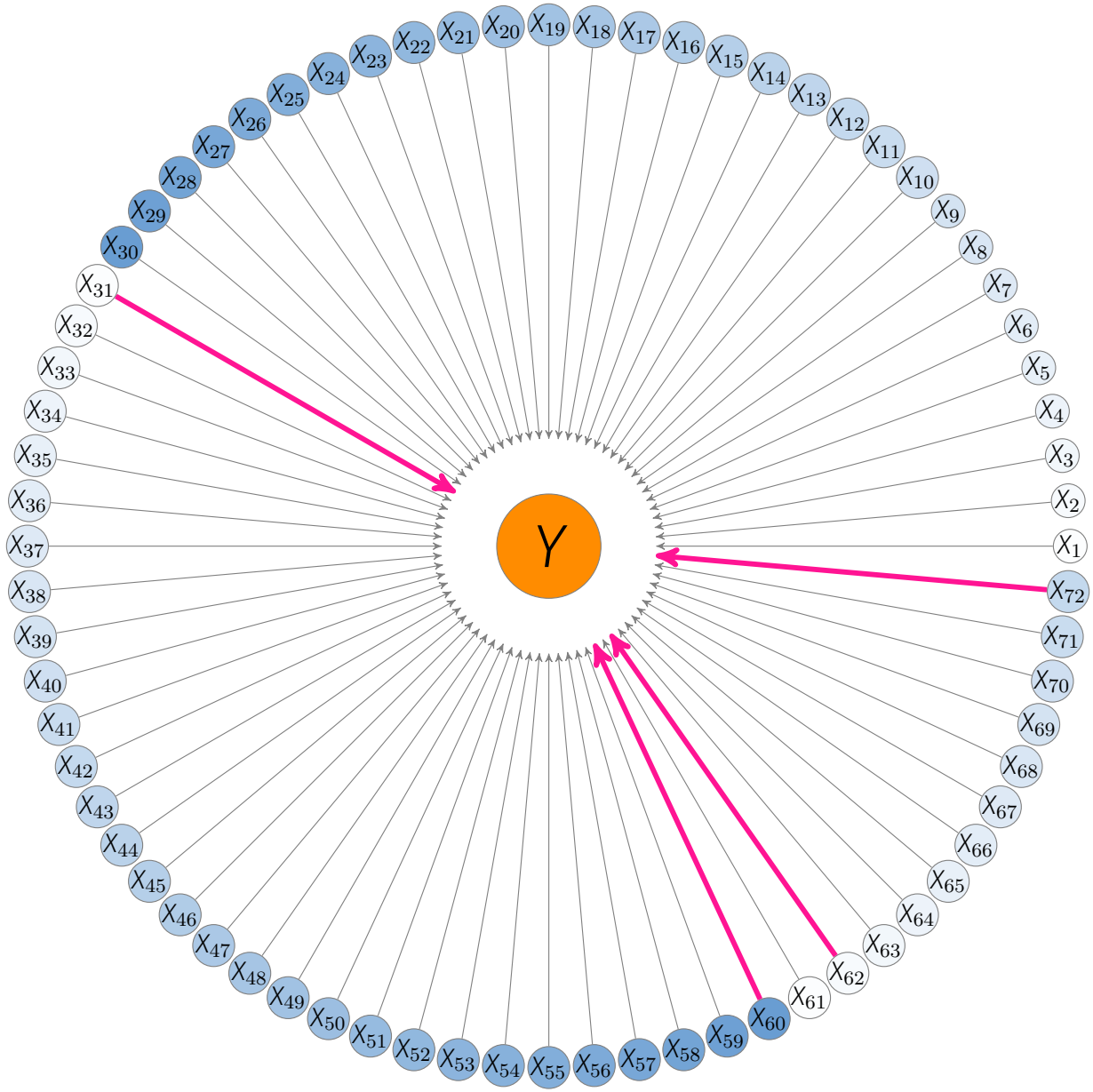
https://sahirbhatnagar.com/sail

Betting on Sparsity

Bet on Sparsity Principle



Bet on Sparsity Principle



Bet on Sparsity Principle

Use a procedure that does well in sparse problems, since no procedure does well in dense problems.[1]

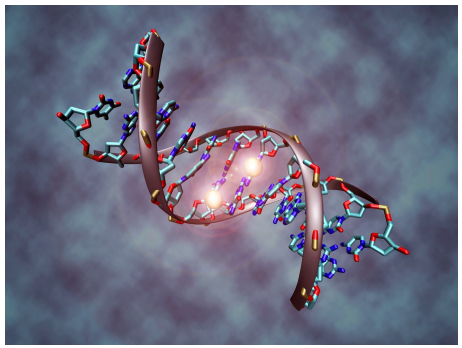
- We often don't have enough data to estimate so many parameters
- Even when we do, we might want to identify a **relatively small number of predictors** ($k < N$) that play an important role
- Faster computation, easier to understand, and stable predictions on new datasets.

[1] The elements of statistical learning. Springer series in statistics, 2001.

`sail`: Strong Additive Interaction Learning

Motivation 1: Non-linear Interactions

$\sim$



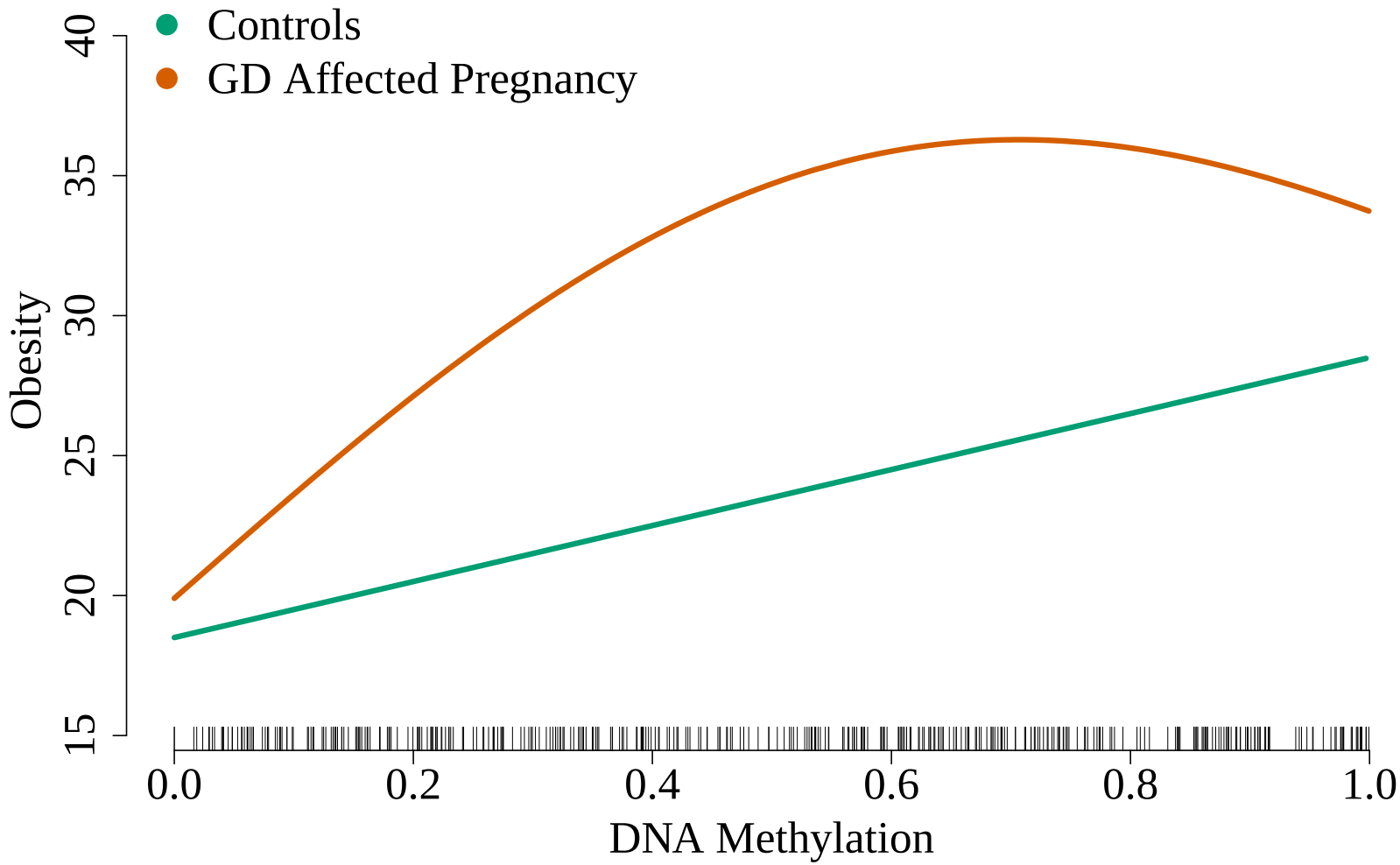
$\times$

Phenotype
Obesity measures

Large Data
Child's epigenome
$(p \approx 450\text{k})$

Environment
Gestational
Diabetes

Motivation 1: Non-linear Interactions



Motivation 2: Heredity Property

$$Y = \beta_0 \cdot \mathbf{1} + \underbrace{\sum_{j=1}^{p} \beta_j X_j + \beta_E X_E}_{\text{main effects}} + \underbrace{\sum_{j=1}^{p} \tau_j X_E X_j}_{\text{interactions}} + \varepsilon$$

[1] Chipman. Canadian Journal of Statistics (1996)

[2] McCullagh and Nelder. Generalized Linear Models (1983)

[3] Cox. International Statistical Review (1984)

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Strong Heredity[1]

$$\hat{\tau}_j \neq 0 \quad \Rightarrow \quad \hat{\beta}_j \neq 0 \quad \text{and} \quad \hat{\beta}_E \neq 0$$

- Heredity property is desired for the purposes of **interpretability**[2]
- Large main effects are more likely to lead to appreciable interactions[3]

[1] Chipman. Canadian Journal of Statistics (1996)
[2] McCullagh and Nelder. Generalized Linear Models (1983)
[3] Cox. International Statistical Review (1984)

Lasso interaction model

- $Y \rightarrow$ response
- $X_E \rightarrow$ environment
- $X_j \rightarrow$ predictors, $j = 1, \ldots, p$

$$Y = \beta_0 \cdot \mathbf{1} + \sum_{j=1}^{p} \beta_j X_j + \beta_E X_E + \sum_{j=1}^{p} \tau_j X_E X_j + \varepsilon$$

$$\underset{\boldsymbol{\Theta}:=(\beta_0,\boldsymbol{\beta},\boldsymbol{\tau})}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(\|\boldsymbol{\beta}\|_1 + \|\boldsymbol{\tau}\|_1)$$

Strong Heredity Interactions: Current State of the Art

Type	Model	Software
Linear	`CAP` (Zhao et al. 2009, *Ann. Stat*)	✗
	`SHIM` (Choi et al. 2009, *JASA*)	✗
	`hiernet` (Bien et al. 2013, *Ann. Stat*)	`hierNet(x, y)`
	`GRESH` (She and Jiang 2014, *JASA*)	✗
	`FAMILY` (Haris et al. 2014, *JCGS*)	`FAMILY(x, z, y)`
	`glinternet` (Lim and Hastie 2015, *JCGS*)	`glinternet(x, y)`
	`RAMP` (Hao et al. 2016, *JASA*)	`RAMP(x, y)`
	`LassoBacktracking` (Shah 2018, *JMLR*)	`LassoBT(x, y)`
Non-linear	`VANISH` (Radchenko and James 2010, *JASA*)	✗
	`sail` (Bhatnagar et al. 2019+, bioRxiv)	`sail(x, e, y, basis)`

Our Extension to Nonlinear Effects

Consider the basis expansion

$$f_j(X_j) = \sum_{\ell=1}^{m_j} \psi_{j\ell}(X_j)\beta_{j\ell}$$

$$f(X_1) = \underbrace{\begin{bmatrix} \psi_{11}(X_{11}) & \psi_{12}(X_{12}) & \cdots & \psi_{11}(X_{15}) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{11}(X_{i1}) & \psi_{12}(X_{i2}) & \cdots & \psi_{11}(X_{i5}) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{11}(X_{N1}) & \psi_{12}(X_{N2}) & \cdots & \psi_{11}(X_{N5}) \end{bmatrix}_{N\times 5}}_{\boldsymbol{\Psi_1}} \times \underbrace{\begin{bmatrix} \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \beta_{14} \\ \beta_{15} \end{bmatrix}_{5\times 1}}_{\theta_1}$$

`sail`: Additive Interactions

- $\boldsymbol{\theta}_j = (\beta_{j1}, \ldots, \beta_{jm_j}) \in \mathbb{R}^{m_j}$
- $\boldsymbol{\tau}_j = (\tau_{j1}, \ldots, \tau_{jm_j}) \in \mathbb{R}^{m_j}$
- $\boldsymbol{\Psi}_j \to n \times m_j$ matrix of evaluations of the $\psi_{j\ell}$
- In our implementation, we use cubic **`bsplines`** with 5 degrees of freedom

Model

$$Y = \beta_0 \cdot \mathbf{1} + \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j + \beta_E X_E + \sum_{j=1}^{p} (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\tau}_j + \varepsilon$$

`sail`: Strong Heredity

Reparametrization[1]

$$\boldsymbol{\tau}_j = \gamma_j \beta_E \boldsymbol{\theta}_j$$

Model

$$Y = \beta_0 \cdot \mathbf{1} + \sum_{j=1}^{p} \boldsymbol{\Psi}_j \boldsymbol{\theta}_j + \beta_E X_E + \sum_{j=1}^{p} \gamma_j \beta_E (X_E \circ \boldsymbol{\Psi}_j) \boldsymbol{\theta}_j + \varepsilon$$

Objective Function

$$\underset{\boldsymbol{\Theta} := (\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma})}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha) \left(w_E |\beta_E| + \sum_{j=1}^{p} w_j \|\boldsymbol{\theta}_j\|_2 \right) + \lambda\alpha \sum_{j=1}^{p} w_{jE} |\gamma_j|$$

[1] Choi et al. JASA (2010)

`sail`: Weak Heredity

Reparametrization

$$\boldsymbol{\tau}_j = \gamma_j(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j)$$

Model

$$Y = \beta_0 \cdot \mathbf{1} + \sum_{j=1}^{p} \boldsymbol{\Psi}_j \theta_j + \beta_E X_E + \sum_{j=1}^{p} \gamma_j (X_E \circ \boldsymbol{\Psi}_j)(\beta_E \cdot \mathbf{1}_{m_j} + \boldsymbol{\theta}_j) + \varepsilon$$

Objective Function

$$\underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

Algorithm

Block Relaxation (De Leeuw, 1994)

Algorithm 1: Block Relaxation Algorithm

Set the iteration counter $k \leftarrow 0$ and fix $\alpha \in (0, 1)$;

for *each* λ **do**

  repeat

$$\boldsymbol{\gamma}^{(k+1)} \leftarrow \underset{\boldsymbol{\gamma}}{\operatorname{argmin}} \quad Q_\lambda\left(\boldsymbol{\gamma}, \beta_E^{(k)}, \boldsymbol{\theta}^{(k)}\right)$$

$$\boldsymbol{\theta}^{(k+1)} \leftarrow \underset{\boldsymbol{\theta}}{\operatorname{argmin}} \quad Q_\lambda\left(\boldsymbol{\theta}, \beta_E^{(k)}, \boldsymbol{\gamma}^{(k+1)}\right)$$

$$\beta_E^{(k+1)} \leftarrow \underset{\boldsymbol{\beta}_E}{\operatorname{argmin}} \quad Q_\lambda\left(\boldsymbol{\theta}^{(k+1)}, \beta_E, \boldsymbol{\gamma}^{(k+1)}\right)$$

    $k \leftarrow k + 1$

  until *convergence criterion is satisfied*;

end

Implementation

Objective Function

$$\underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(Y; \boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

[1] https://github.com/sahirbhatnagar/sail

Implementation

Objective Function

$$\underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(Y; \boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

Lasso problem

$$\underset{\boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(Y; \boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

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Group Lasso problem

$$\underset{\beta_E,\boldsymbol{\theta}}{\operatorname{argmin}} \quad \mathcal{L}(Y;\boldsymbol{\Theta})+\lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j\|\theta_j\|_2\right)+\lambda\alpha\sum_{j=1}^{p} w_{jE}|\gamma_j|$$

[1] https://github.com/sahirbhatnagar/sail

Theory

Sparsity

Theorem 1

$$\widehat{\boldsymbol{\Theta}}_n = \underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

$$\mathcal{A}_1 = \left\{j : \theta_j \neq 0, \beta_j \neq 0\right\}$$
$$\mathcal{A}_2 = \left\{k : \gamma_k \neq 0\right\}, \qquad \mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

Under certain regularity conditions and the existence of a local minimizer $\widehat{\boldsymbol{\Theta}}_n$ that is $\sqrt{n}$-consistent

$$P\left(\widehat{\boldsymbol{\Theta}}_{\mathcal{A}^c} = 0\right) \to 1$$

Sparsity

Theorem 1

$$\widehat{\boldsymbol{\Theta}}_n = \underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

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Under certain regularity conditions and the existence of a local minimizer $\widehat{\boldsymbol{\Theta}}_n$ that is $\sqrt{n}$-consistent

$$P\left(\widehat{\boldsymbol{\Theta}}_{\mathcal{A}^c} = 0\right) \to 1$$

Theorem 1 shows that when the tuning parameters for the nonzero coefficients converge to 0 faster than $n^{-1/2}$ `sail` can consistently remove the noise terms with probability tending to 1.

Asymptotic normality

Theorem 2

$$\widehat{\boldsymbol{\Theta}}_n = \underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

Under certain regularity conditions, the component $\widehat{\boldsymbol{\Theta}}_{\mathcal{A}}$ of the local minimizer $\widehat{\boldsymbol{\Theta}}_n$ satisfies

$$\sqrt{n}\left(\widehat{\boldsymbol{\Theta}}_{\mathcal{A}} - \boldsymbol{\Theta}_{\mathcal{A}}\right) \to_d \mathcal{N}\left(0, \mathbf{I}^{-1}\left(\boldsymbol{\Theta}_{\mathcal{A}}\right)\right)$$

Theorem 2 shows that the `sail` estimates for nonzero coefficients in the true model have the same asymptotic distribution as they would have if the zero coefficients were known in advance.

Asymptotic normality

Theorem 2

$$\widehat{\boldsymbol{\Theta}}_n = \underset{\beta_E, \boldsymbol{\theta}, \boldsymbol{\gamma}}{\operatorname{argmin}} \quad \mathcal{L}(\boldsymbol{\Theta}) + \lambda(1-\alpha)\left(w_E|\beta_E| + \sum_{j=1}^{p} w_j \|\theta_j\|_2\right) + \lambda\alpha \sum_{j=1}^{p} w_{jE}|\gamma_j|$$

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Theorem 2 shows that the `sail` estimates for nonzero coefficients in the true model have the same asymptotic distribution as they would have if the zero coefficients were known in advance.
Theorem 1 + 2 –> Oracle property (Fan and Li, 2001)

Simulations

Simulation Scenarios

1. Truth obeys strong hierarchy (right in our wheel house):

$$Y = \sum_{j=1}^{4} f_j(X_j) + \beta_E \cdot X_E + X_E \times (f_3(X_3) + f_4(X_4)) + \varepsilon$$

Simulation Scenarios

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2. Truth obeys weak hierarchy

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2. Truth obeys weak hierarchy
3. Truth only has interactions

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5. Truth only has main effects

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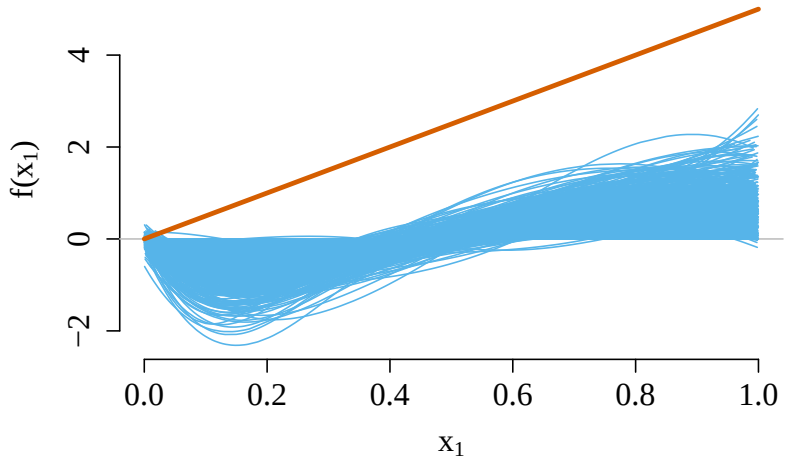
2. Truth obeys weak hierarchy
3. Truth only has interactions
4. Truth is linear
5. Truth only has main effects

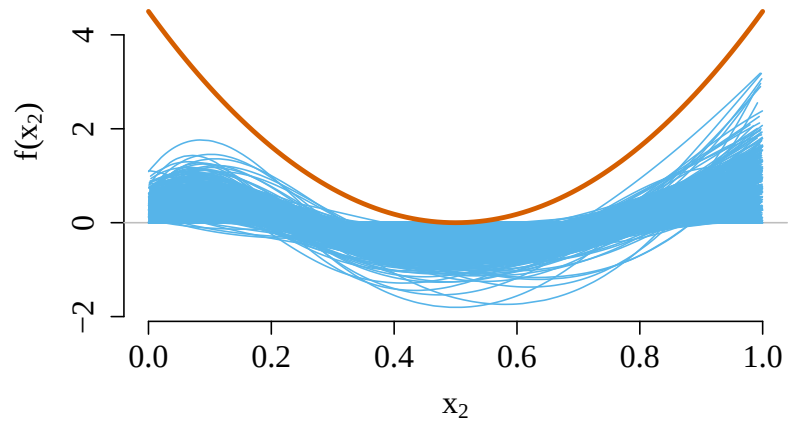
- $n_{train} = n_{tuning} = 200$, $n_{test} = 800$, $p = 1000$, $\beta_E = 1$, $SNR = 2$
- $X_j \sim$ `truncnorm(0,1)`, $j = 1, \ldots, 1000$,
  $E \sim$ `truncnorm(-1,1)`
- `sail` needs to estimate $1000 \times 5 \times 2 = 10$k parameters

Scenario 1: Main Effects for 500 Simulations

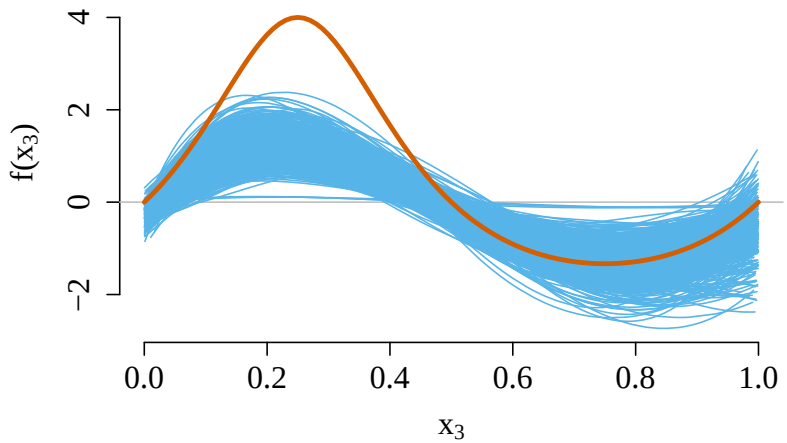
$f(x_1) = 5x_1$



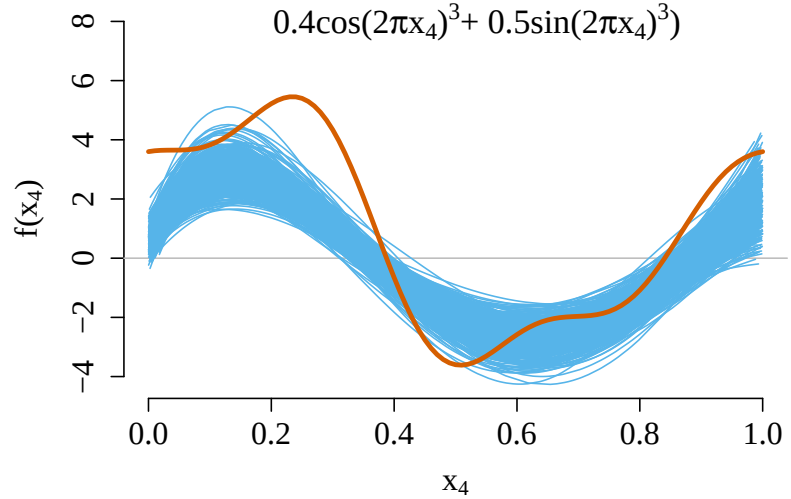
$f(x_2) = 4.5(2x_2 - 1)^2$



$f(x_3) = \frac{4\sin(2\pi x_3)}{2 - \sin(2\pi x_3)}$

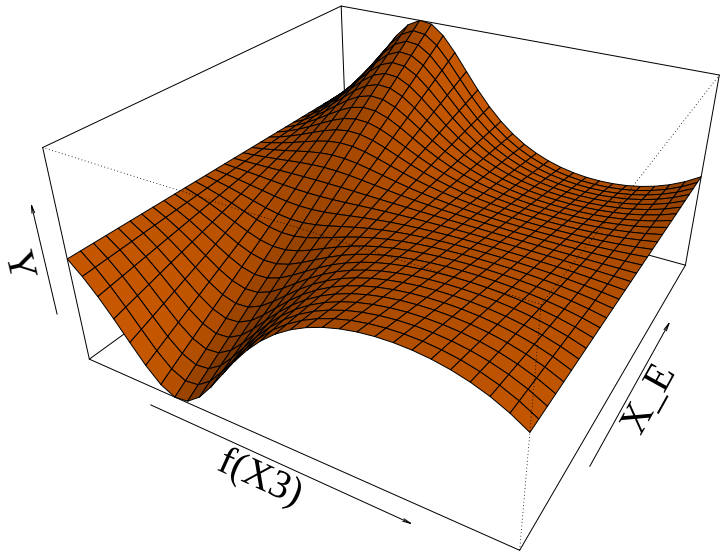


$f(x_4) = 6(0.1\sin(2\pi x_4) + 0.2\cos(2\pi x_4) + 0.3\sin(2\pi x_4)^2 + 0.4\cos(2\pi x_4)^3 + 0.5\sin(2\pi x_4)^3)$

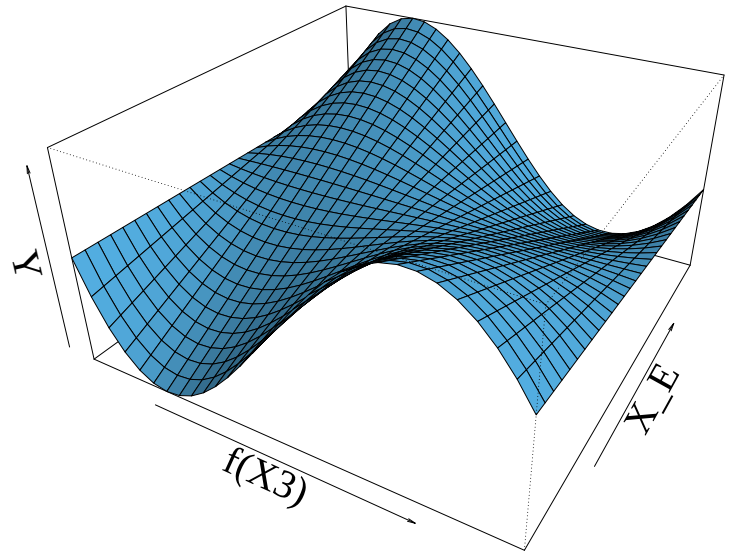


Scenario 1: Estimated Interaction Effects for $E \cdot f(X_3)$

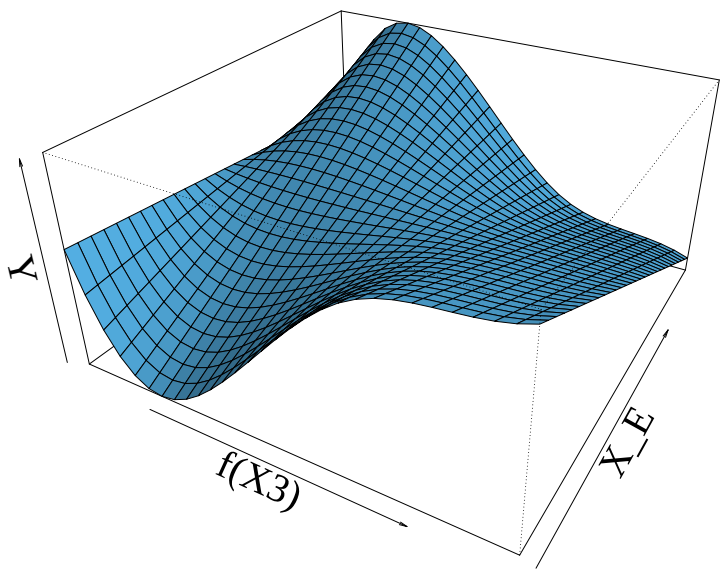
Truth



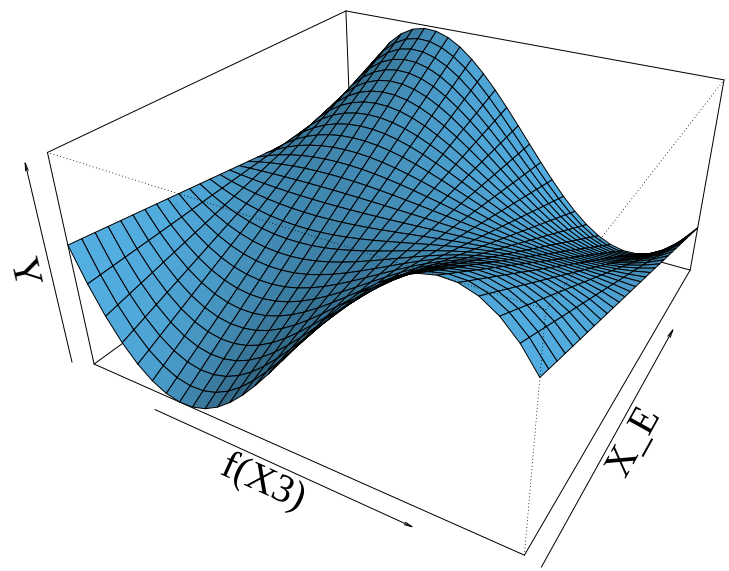
Estimated: 25th Percentile



Estimated: 50th Percentile

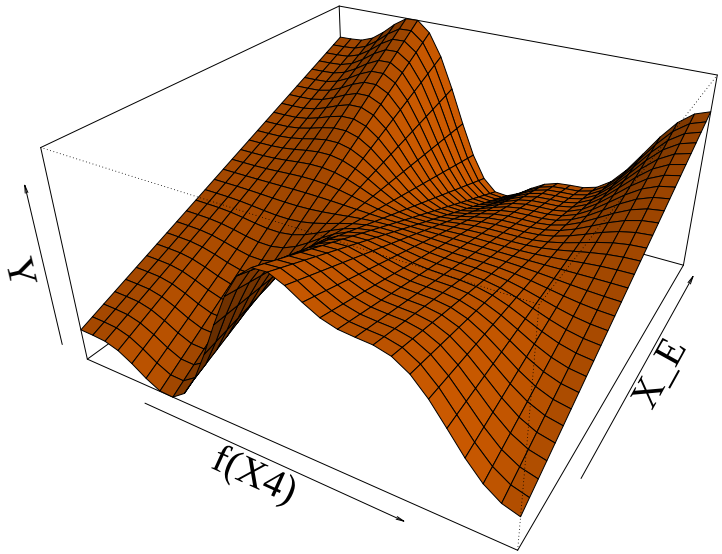


Estimated: 75th Percentile

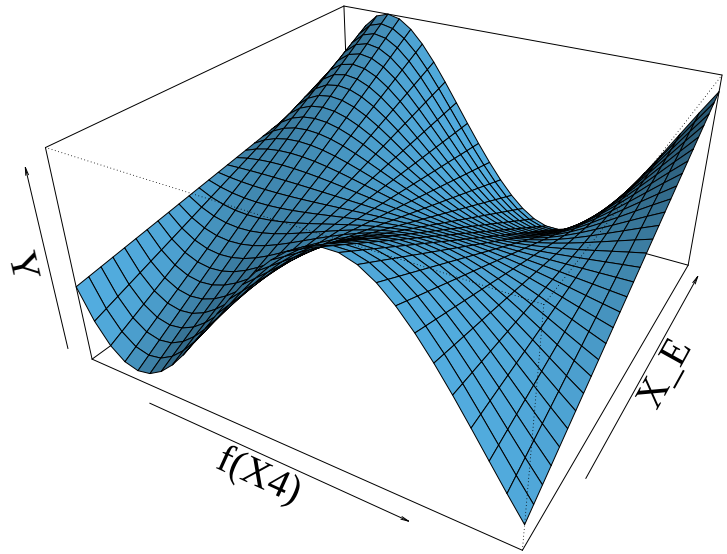


Scenario 1: Estimated Interaction Effects for $E \cdot f(X_4)$

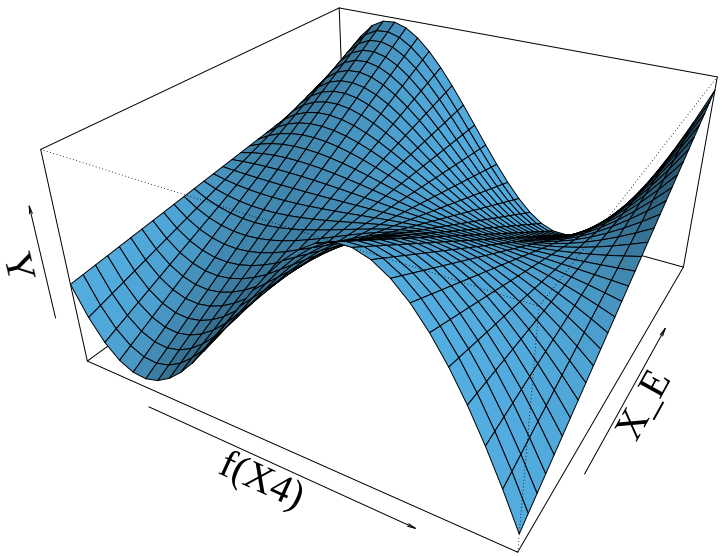
Truth



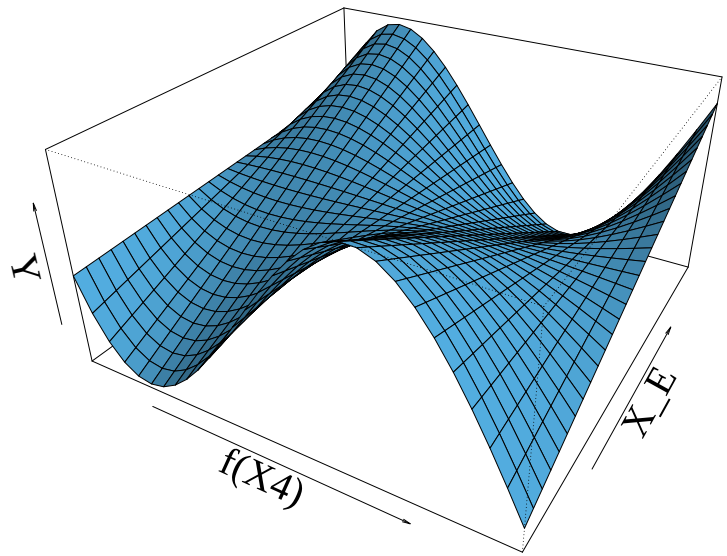
Estimated: 25th Percentile



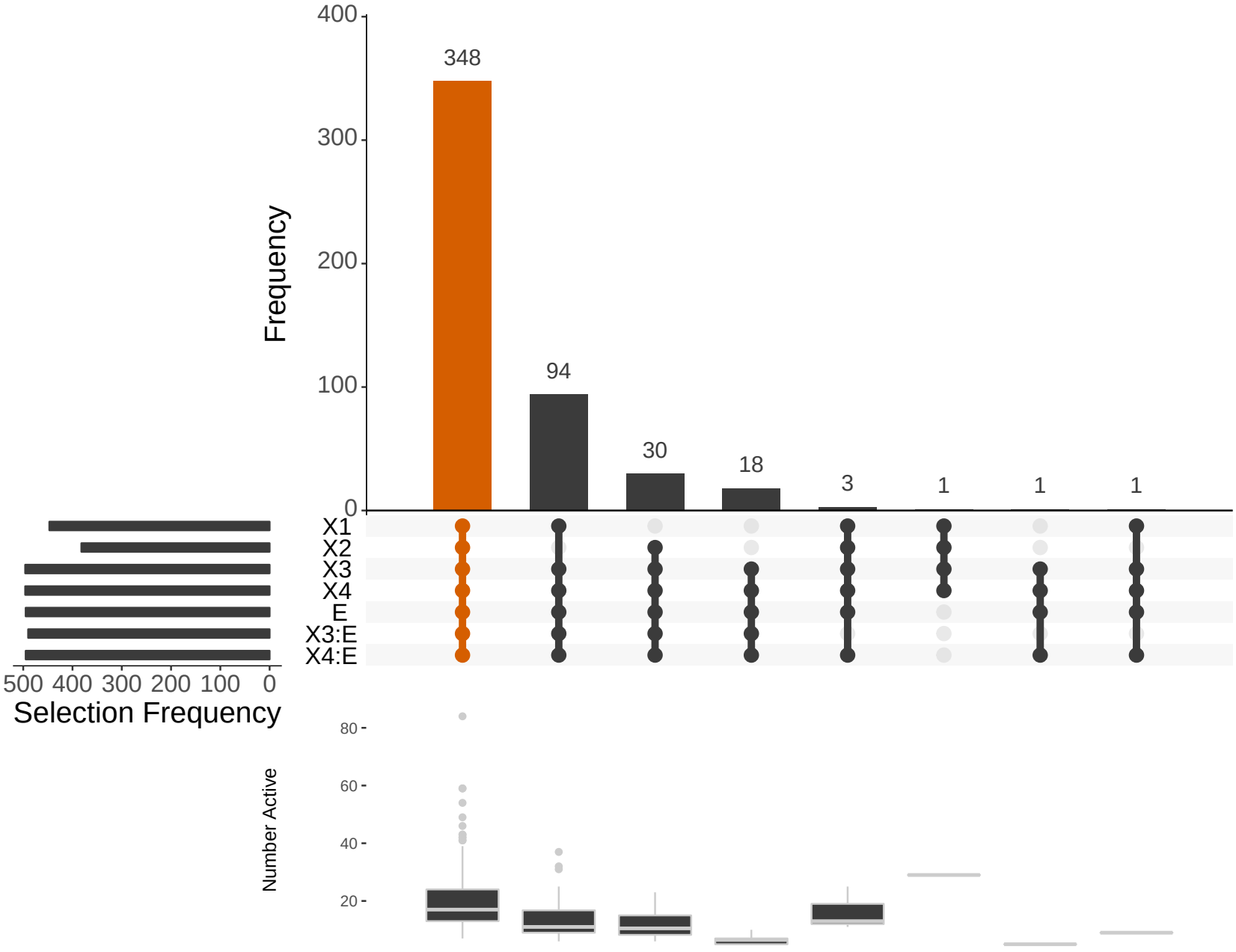
Estimated: 50th Percentile



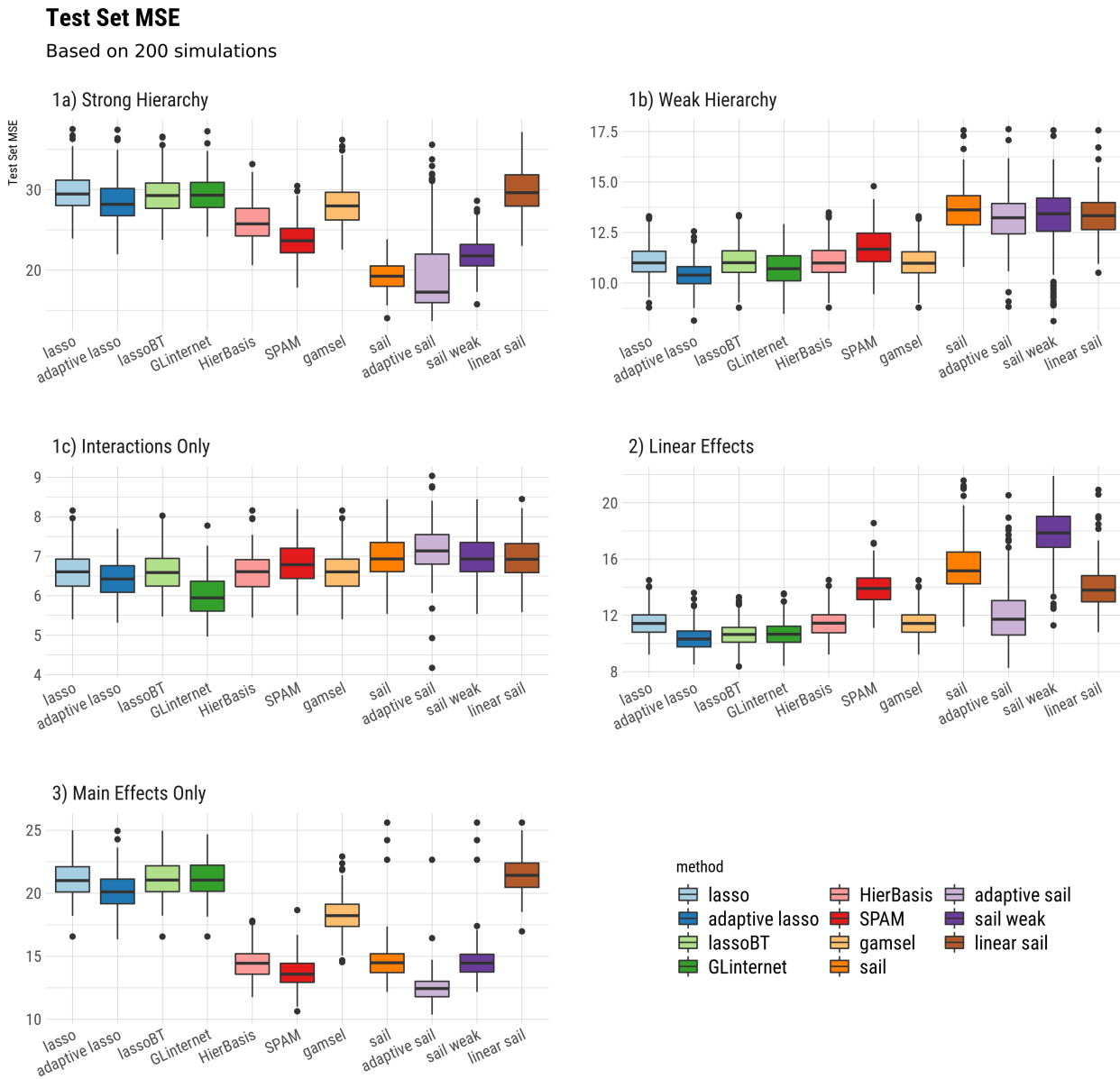
Estimated: 75th Percentile



Right in Our Wheel House Simulation Results



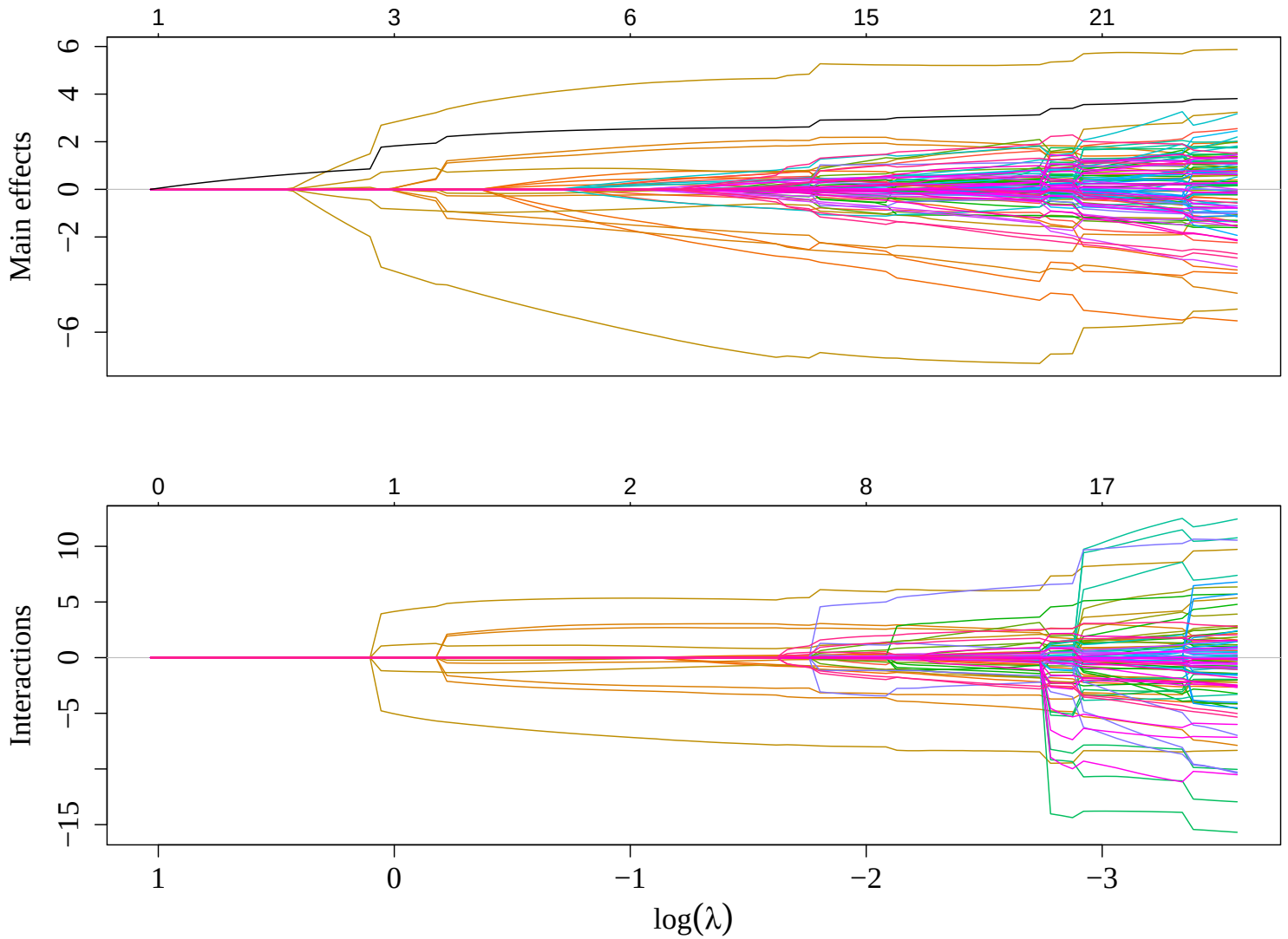
All Scenarios MSE



`sail` R package

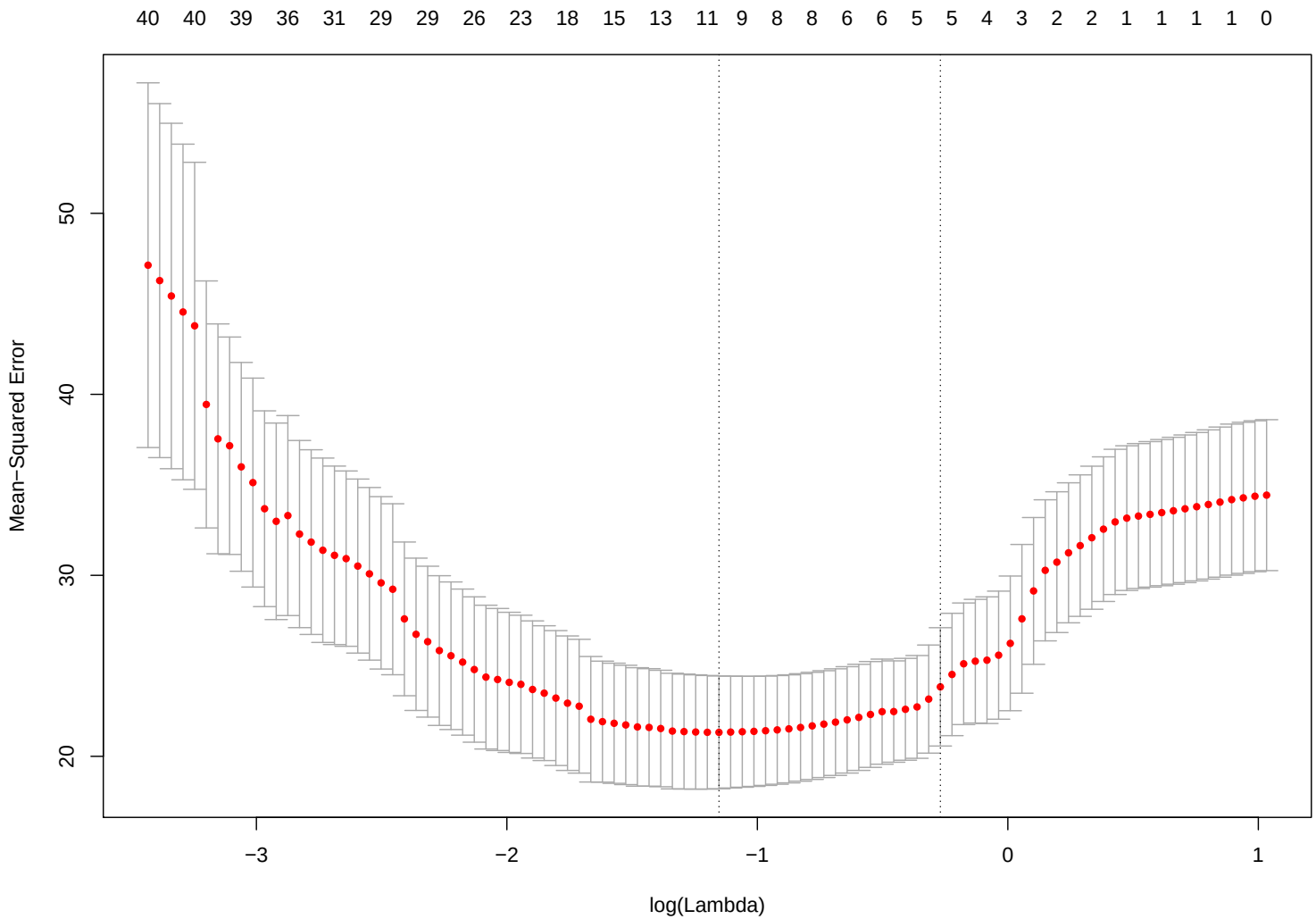
`sail` R package: Solution Path results

```
f.basis <- function(x) splines::bs(x, degree = 5)
fit <- sail(x, y, e, basis = f.basis)
plot(fit)
```



`sail` R package: Cross-validation results

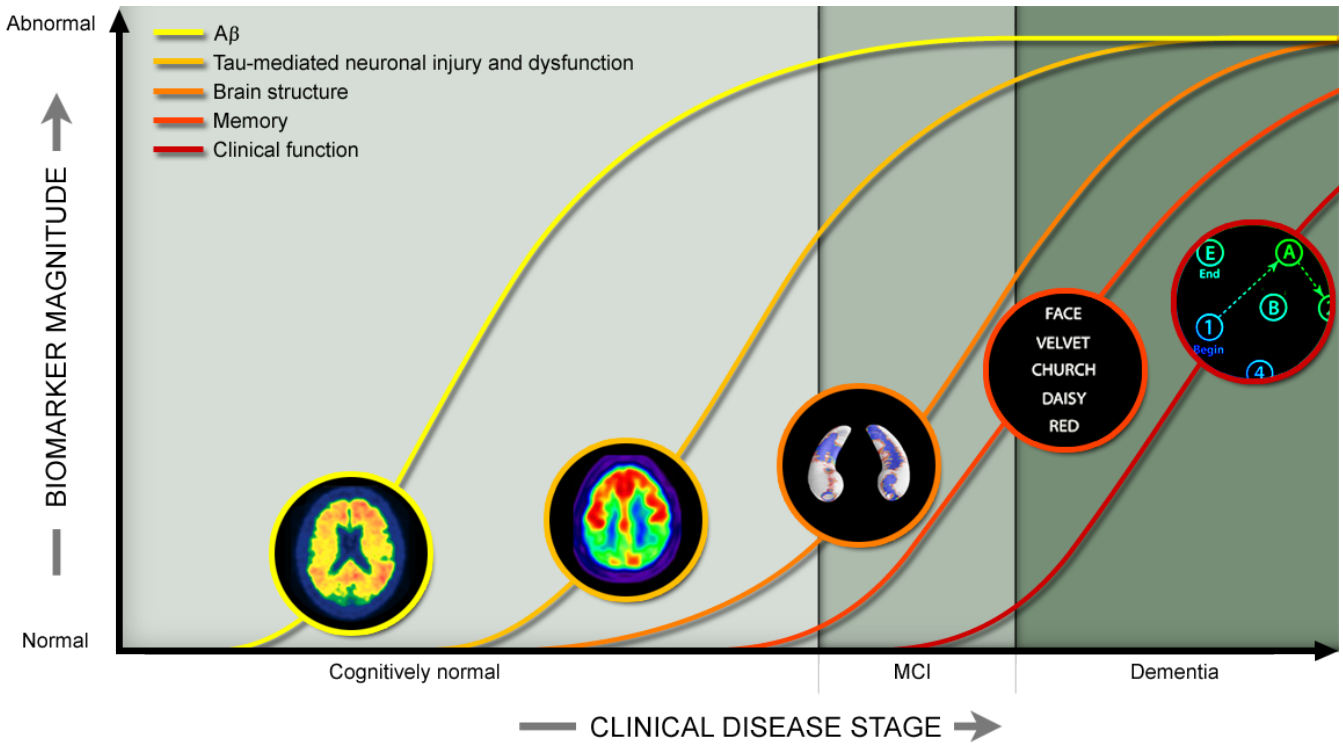
```
sail::plot(cvfit)
```



Real Data Application

Alzheimer's Disease Neuroimaging Initiative (ADNI)

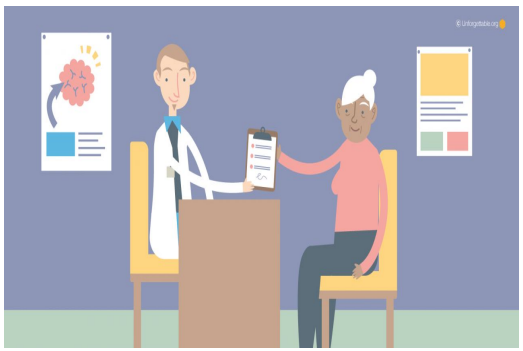
- Alzheimer's is an irreversible neurodegenerative disease that results in a loss of mental function due to the **deterioration of brain tissue**.
- The overall goal of ADNI is to **validate biomarkers** for use in Alzheimer's disease clinical treatment trials



Interaction between Aβ Protein and APOE gene

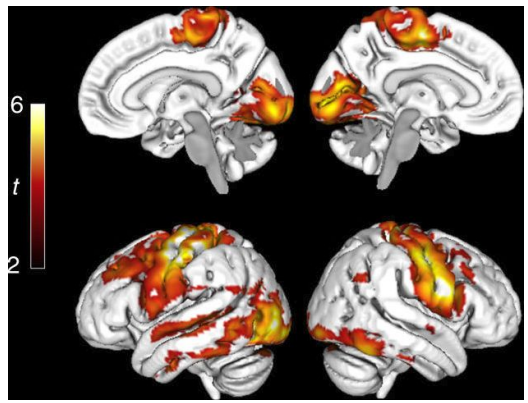
- $\mathbf{E}$: APOE4 allele increases the risk for Alzheimer's and lowers the age of onset
- $\mathbf{X}$: PET amyloid imaging to assess Aβ protein load in 96 brain regions
- $\mathbf{Y}$: General cognitive decline measured by mini-mental state examination
- $3 \times 96 \times 2 + 1 = 577$ parameters to estimate

Mini–Mental State Examination



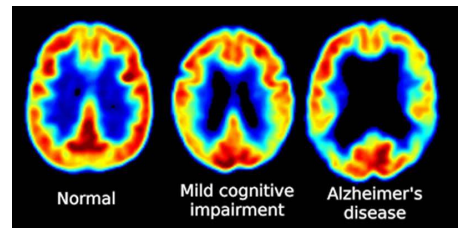
$\sim$

Amyloid Beta Protein



$\times$

Cognitive Status



$$Y_{343 \times 1} \qquad X_{343 \times 96} \qquad E_{343 \times 1}$$

MSE vs. number of active variables

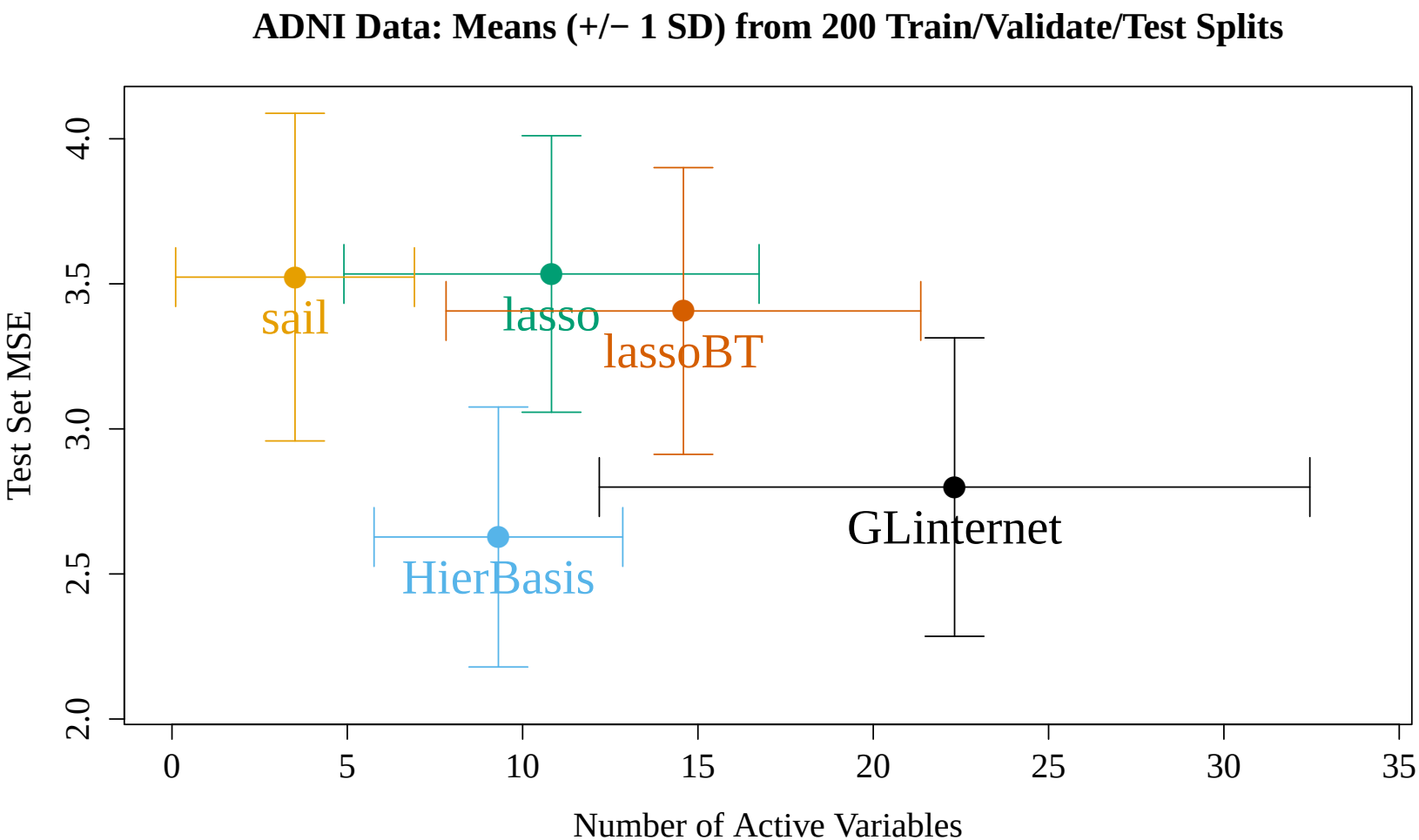
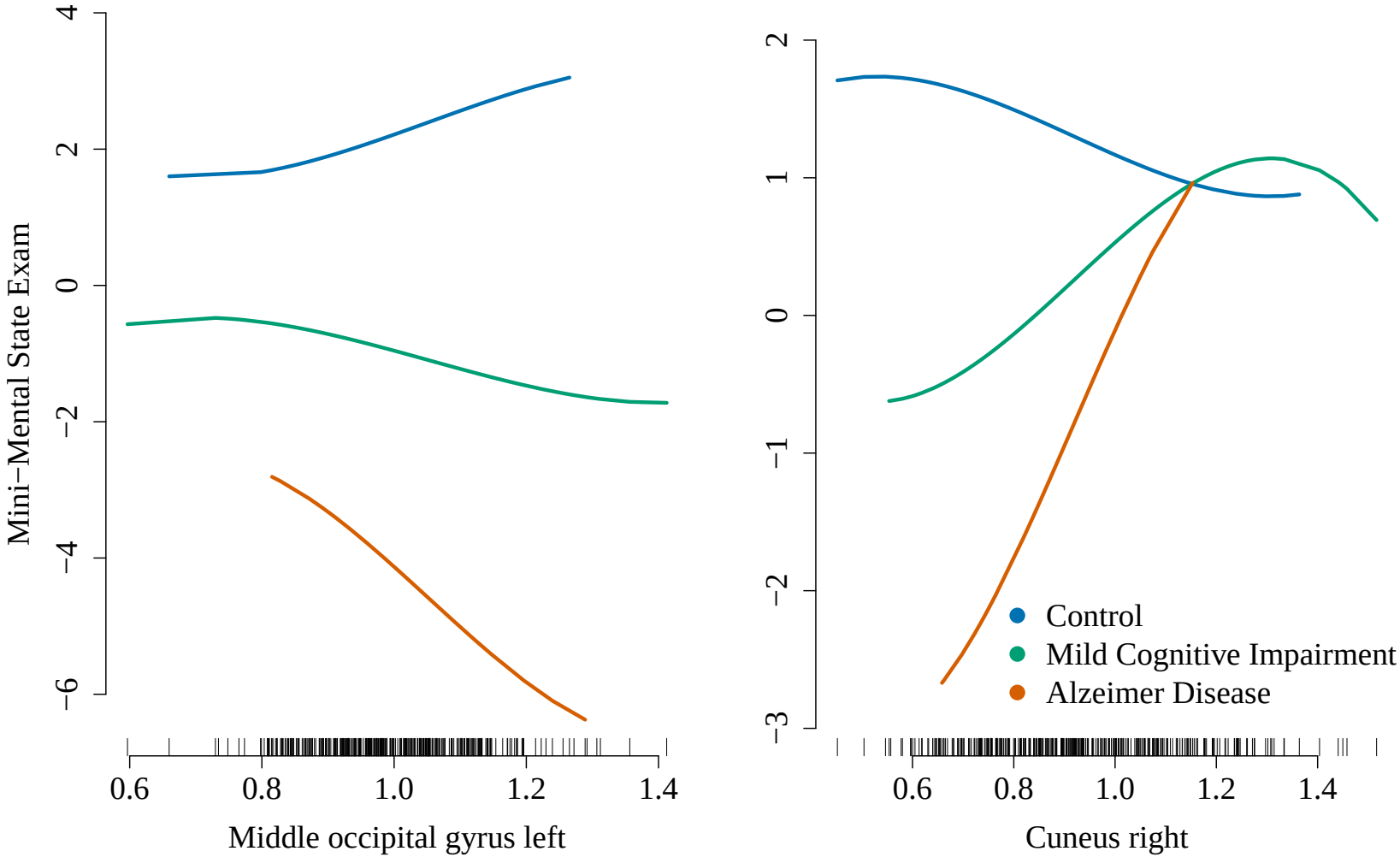


Fig.: Mean test set MSE vs. mean number of active variables (± 1 SD) for ADNI data based on 200 train/validation/test splits.

sail: Interactions



Discussion

Strengths and Limitations

Strengths

- Non-linear environment interactions with strong heredity property in $p >> N$
- `sail` allows for flexible modeling of input variables

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- Non-linear environment interactions with strong heredity property in $p >> N$
- `sail` allows for flexible modeling of input variables

Limitations

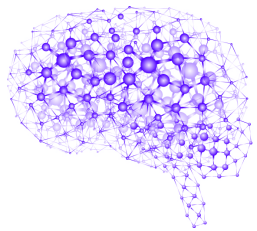
- `sail` can currently only handle $E \cdot f(X)$ or $f(E) \cdot X$
- Does not allow for $f(X_1, E)$ or $f(X_1, X_2)$
- Memory footprint is an issue

Future Directions

- Implement ADMM algorithm for scalability. Distributed computing (GPU)
- Binary Outcomes
- bi-level selection:

$$f(X_1) = \underbrace{\begin{bmatrix} X_{11} & \psi_{11}(X_{11}) & \psi_{12}(X_{12}) & \cdots & \psi_{11}(X_{15}) \\ & \vdots & \vdots & \cdots & \vdots \\ & \vdots & \vdots & \cdots & \vdots \\ X_{i1} & \psi_{11}(X_{i1}) & \psi_{12}(X_{i2}) & \cdots & \psi_{11}(X_{i5}) \\ & \vdots & \vdots & \cdots & \vdots \\ & \vdots & \vdots & \cdots & \vdots \\ X_{N1} & \psi_{11}(X_{N1}) & \psi_{12}(X_{N2}) & \cdots & \psi_{11}(X_{N5}) \end{bmatrix}_{N\times 5}}_{\boldsymbol{\Psi_1}} \times \underbrace{\begin{bmatrix} \beta_{\text{linear}} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \beta_{14} \\ \beta_{15} \end{bmatrix}_{6\times 1}}_{\theta_1}$$

Acknowledgements



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Jewish General Hospital
Lady Davis Institute for Medical Research

McGill

References

- Radchenko, P., & James, G. M. (2010). Variable selection using adaptive nonlinear interaction structures in high dimensions. Journal of the American Statistical Association, 105(492), 1541-1553.
- Choi, N. H., Li, W., & Zhu, J. (2010). Variable selection with the strong heredity constraint and its oracle property. Journal of the American Statistical Association, 105(489), 354-364.
- Chipman, H. (1996). Bayesian variable selection with related predictors. Canadian Journal of Statistics, 24(1), 17-36.
- Friedman, J., Hastie, T., & Tibshirani, R. (2010). Regularization paths for generalized linear models via coordinate descent. Journal of statistical software, 33(1)
- Yang, Y., & Zou, H. (2015). A fast unified algorithm for solving group-lasso penalize learning problems. Statistics and Computing, 25(6), 1129-1141
- De Leeuw, J. (1994). Block-relaxation algorithms in statistics. In Information systems and data analysis (pp. 308-324). Springer Berlin Heidelberg.

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Session Info

```
R version 3.6.0 (2019-04-26)
Platform: x86_64-pc-linux-gnu (64-bit)
Running under: Pop!_OS 18.10

Matrix products: default
BLAS:   /usr/lib/x86_64-linux-gnu/blas/libblas.so.3.8.0
LAPACK: /usr/lib/x86_64-linux-gnu/lapack/liblapack.so.3.8.0

attached base packages:
[1] stats     graphics  grDevices utils     datasets  methods   base

other attached packages:
[1] xtable_1.8-4      rpart.plot_3.0.6  rpart_4.1-15      data.table_1
[5] ISLR_1.2          ggplot2_3.1.0     knitr_1.23

loaded via a namespace (and not attached):
 [1] Rcpp_1.0.1          magrittr_1.5        tidyselect_0.2.5
 [4] munsell_0.5.0       colorspace_1.4-0    R6_2.4.0
 [7] rlang_0.3.4         highr_0.8           stringr_1.4.0
[10] plyr_1.8.4          dplyr_0.8.0.1       tools_3.6.0
[13] grid_3.6.0          gtable_0.2.0        xfun_0.7
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[25] glue_1.3.1          evaluate_0.14       stringi_1.4.3
[28] compiler_3.6.0      pillar_1.4.0        scales_1.0.0
```
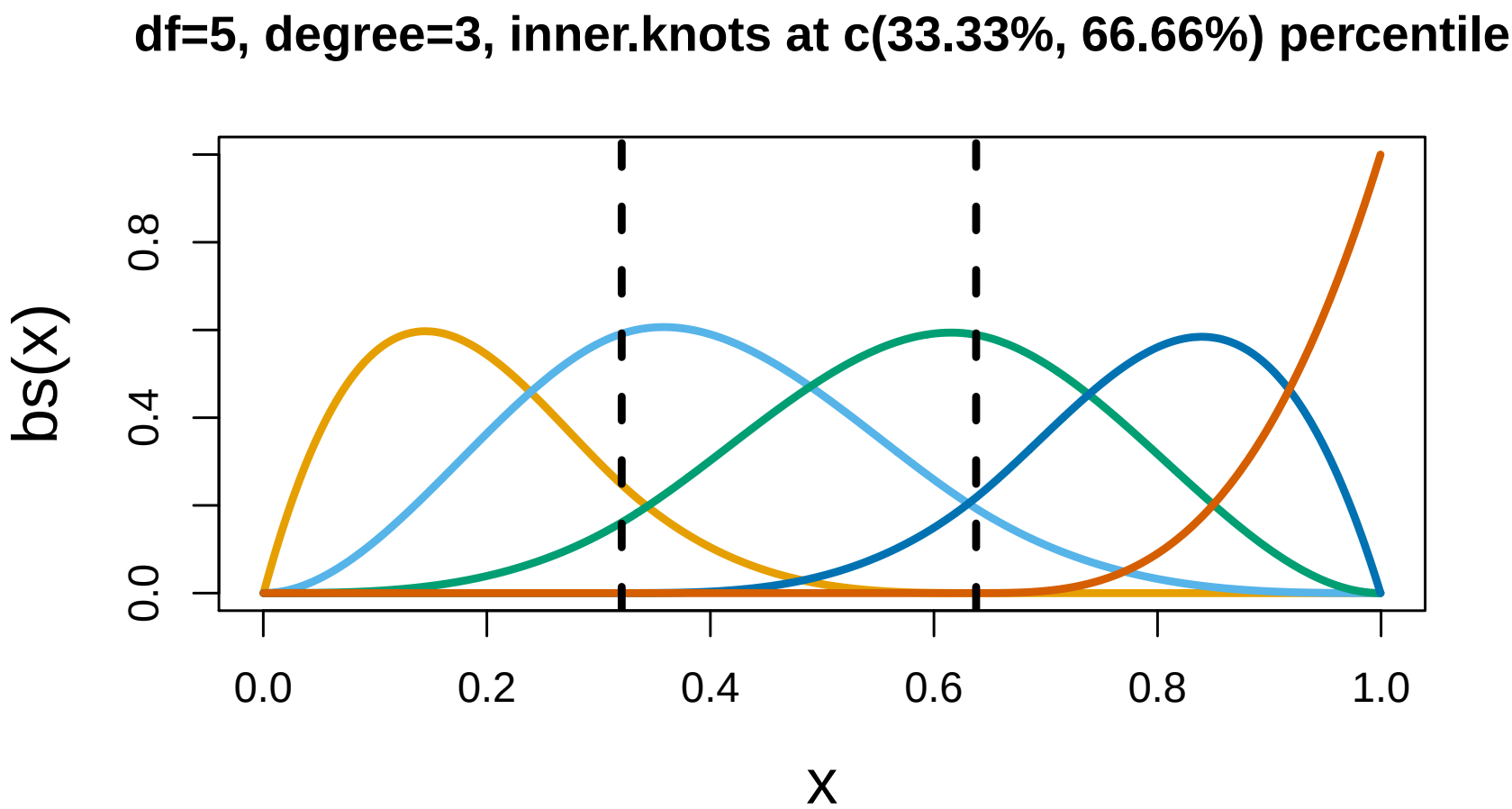
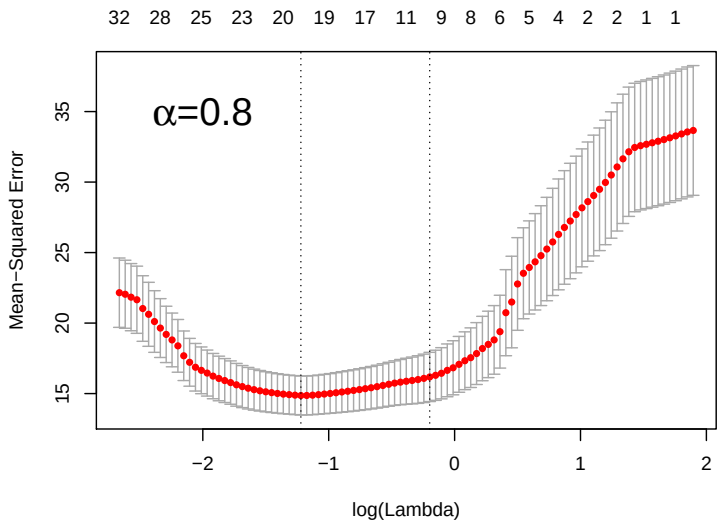
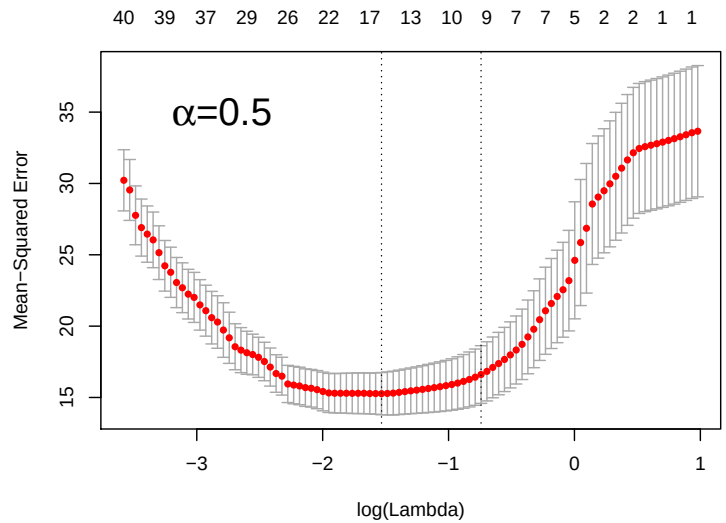

B-Spline Expansion

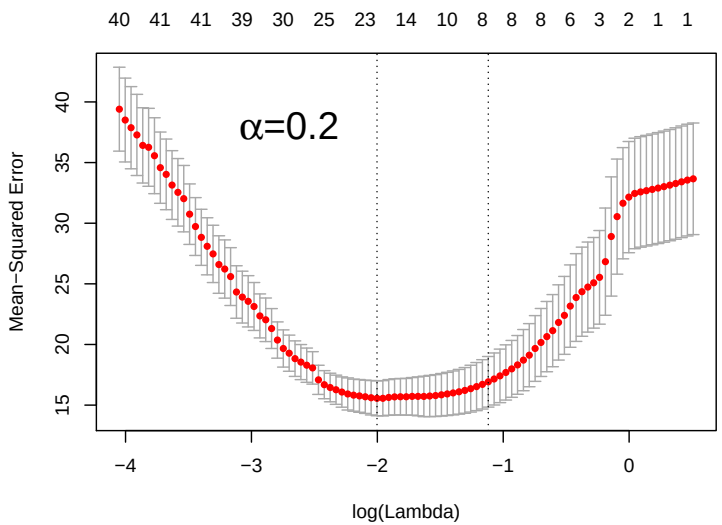
```r
x <- truncnorm::rtruncnorm(1000, a = 0, b = 1)
B <- splines::bs(x, df = 5, degree=3, intercept = FALSE)
```

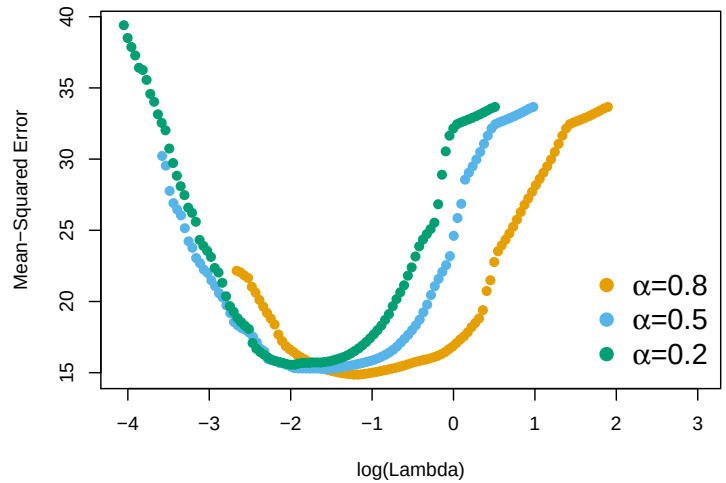


`sail` A Note on the Second Tuning Parameter results









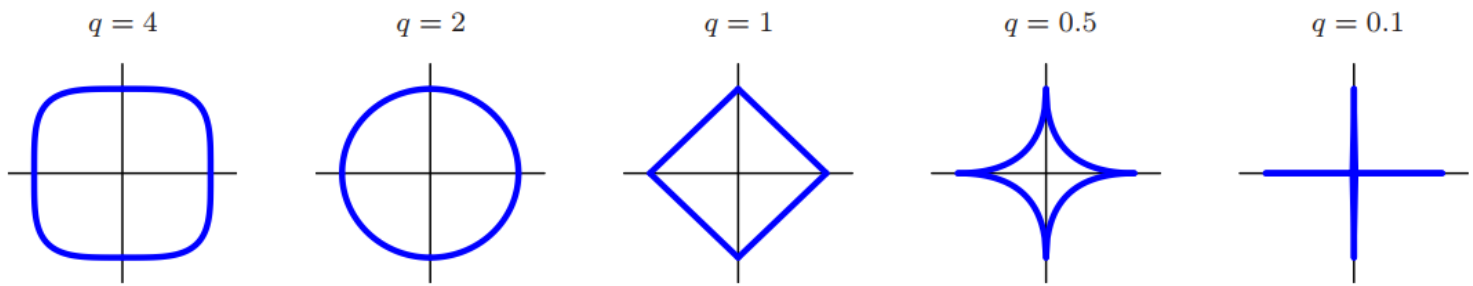
Appendix

Why the L1 norm ?

- For a fixed real number $q \geq 0$ consider the criterion

$$\widetilde{\boldsymbol{\beta}} = \arg\min_{\boldsymbol{\beta}} \left\{ \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} x_{ij}\beta_j \right)^2 + \lambda \sum_{j=1}^{p} |\beta_j|^q \right\}$$

- Why do we use the ℓ_1 norm? Why not use the $q = 2$ (Ridge) or any ℓ_q norm?



- $q = 1$ is the smallest value that yields a sparse solution and yields a **convex** problem $\rightarrow$ scalable to high-dimensional data
- For $q < 1$ the constrained region is **nonconvex**

Linear Effects Simulation - Comparison