Handout

i)

① a) $f: R \rightarrow R$

if $f(x_1) = f(x_2)$, then function is injective

$f(x_1) = \frac{1}{x_1}$ $\quad f(x_2) = \frac{1}{x_2}$

$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$

$x_1 = x_2$

$x \in \mathbb{R}$ has a unique image in codomain

$\therefore f$ is one-to-one (surjective)

② ii) there is a unique pre-image of y

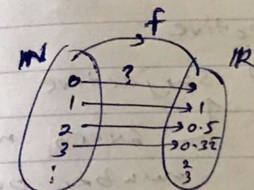
$\therefore f$ is onto

b) $f: N \rightarrow R$

if $f(x_1) = f(x_2)$,

$\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2 \quad (x_1, x_2 \in \mathbb{N})$

$\therefore f$ is one-to-one



f, one-to-one not onto

b) $f: N \rightarrow R$

If $f(x_1) = f(x_2)$

$\frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2 \quad (x_1, x_2 \in \mathbb{N})$

$\therefore$ one-to-one

a ~~if~~ every real number belong to codomain may not have a pre-image in N.

$\therefore f$ is not onto

(2) i) $f: N \longrightarrow N \quad f(x) = x^2$

if $f(x_1) = f(x_2)$,

$$x_1^2 = x_2^2$$

$$x_1 = x_2 \quad (x_1, x_2 \in \mathbb{N})$$

$$\left[\text{if } x_1, x_2 \in \mathbb{R} \; ; \; x_1 \neq x_2\right]$$

∴ injective.

∴ not surjective

every natural number belongs to the codomain doesn't have a pre-image in the domain //

ex: $y = 3$

$y = 5$ likewise

ii) $f: Z \longrightarrow Z \quad f(x) = x^2$

if $f(x_1) = f(x_2)$,

$$x_1^2 = x_2^2$$

$$\pm x_1 = \pm x_2 \quad (x_1, x_2 \in \mathbb{Z})$$

$$\cancel{\therefore} \; x_1 = \pm x_2$$

$$\cancel{x_1 = x_2}$$

∴ not injective //

~~every~~ also not surjective //

~~eg: y = -1 & y = 1~~

~~y = 2 & y = 2~~

ex: if $f(x) = 3$, 3 has no pre-image in the domain //

iii) $f: R \longrightarrow R \quad f(x) = x^2$

if, $f(x_1) = f(x_2)$

$$x_1^2 = x_2^2$$

$$\pm x_1 = \pm x_2 \quad (x_1, x_2 \in \mathbb{R})$$

$$\therefore \; x_1 = \pm x_2$$

∴ not injective //

also, not surjective. //

ex: when $f(x) = (-2)$

iv) $f: N \longrightarrow N \quad f(x) = x^3$

if $f(x_1) = f(x_2)$,

$$x_1^3 = x_2^3$$

$$x_1 = x_2 \quad (x_1, x_2 \in \mathbb{N})$$

∴ injective

~~also~~ not surjective

v) $f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = x^3$

if $f(x_1) = f(x_2)$,

$n_1^3 = n_2^3$ & $(n_1; n_2 \in \mathbb{Z})$

$n_1 = n_2$

$\therefore$ injective //

but not surjective //

ex: ~~if f(x)~~

if $f(x) = 3$

$x = \sqrt{3}$

but $\sqrt{3} \notin \mathbb{Z}$

4) $f: R \to R$ $f(x) = |x|$

$f(x_1) = f(x_2)$

$|x_1| = |x_2|$

$\pm x_1 = \pm x_2$

$\therefore x_1 = \pm x_2$

$\therefore$ not injective //

~~No Negative~~ all negative values in the codomain has no pre-image in the domain //

$\therefore$ f is not onto

Hence, f is neither one-to-one nor onto. //

5) $f: R \to R$

$$f(x) = \begin{cases} 1, & \text{if } x>0 \\ 0, & \text{if } x=0 \\ -1, & \text{if } x<0 \end{cases}$$

if $x = 2$; $f(x) = 1$

$x = 1$; $f(x) = 1$

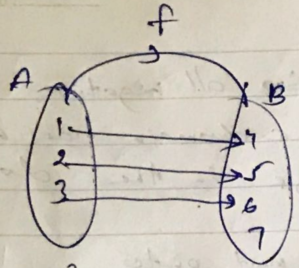
both 1 and 2 for x has the same image in the codomain 1.

$\therefore$ f is not one-to-one

except 1, 0, −1 in codomain, other values & in the real number set have no pre-image in the domain //

$\therefore$ f is ~~not~~ neither

(6)



$\therefore$ f is one to one // but not onto

(7) i) $f: R \to R$ $\quad f(x) = 3 - 4x$

~~if $f(x_1) = f(x_2)$~~

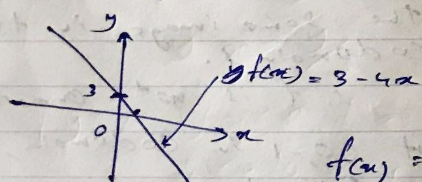
a) $f(x_1) = f(x_2)$

$3 - 4x_1 = 3 - 4x_2$

$x_1 = x_2$

$\therefore$ injective

b)



$f(x) = y = 3 - 4x$

$\therefore x = \dfrac{3-y}{4}$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ onto

f is

$\therefore$ bijective //

ii) $f: R \to R$ $\quad f(x) = 1 + x^2$

$f(x_1) = f(x_2)$

$1 + x_1^2 = 1 + x_2^2$

$x_1^2 = x_2^2$

$\pm x_1 = \pm x_2$

$x_1 = \pm x_2$

$x_1 = x_2$ or $x_1 = -x_2$

$\therefore$ not injective

$f(x) = 1 + x^2 = y$

$y - 1 = x^2$

$x = \sqrt{y-1}$

$\forall y \in \mathbb{R}, \nexists x \in \mathbb{R} : f(x) = y$

ex: $y = -1$; $x = \sqrt{-2} \notin \mathbb{R}$

$\therefore$ not onto (surjective)

$\therefore$ f is niether injective nor surjective

$\therefore$ not bijective //

(11) $f: R \to R \quad f(x) = x^4$

- not one-to-one (many to one) $\begin{pmatrix} -1 \\ +1 \end{pmatrix} \to 1$
- not onto

(d) f is neither one-one nor onto

(12) $f: R \to R \quad f(x) = 3x$

$f(x_1) = f(x_2)$

$3x_1 = 3x_2$

$x_1 = x_2$

$\therefore$ One-One (injective)

$f(x) = 3x = y$

$x = y/3$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ onto (surjective)

(a) bijective

(5) $f: R \to R \quad f(x) = x^3$

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$x_1 = x_2$

$\therefore$ f is injective

Additional

$f(x) = x^3 = y$

$x = y^{1/3}$

$x = \sqrt[3]{y}$

~~$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$~~

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ f is ~~onto~~ surjective too

f is bijective

⑥ $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$f = \{(1,4), (2,5), (3,6)\}$

$f(1) = 4$

$f(2) = 5$

$f(3) = 6$

4 ~~is~~ if $4 = f(x), \nexists x \in A$

Additional (surjective check) There is no x value in the set A as $f(x)$ gives 4.

$\therefore f$ is not onto

$f(x_1) = f(x_2) \quad (x_1, x_2 \in A)$

$x_1 \neq x_2$

$\Rightarrow f(x_1) \neq f(x_2)$

$\therefore$ not one-one

Inverse Function

If it is bijection, then it has an inverse fuction.

⑦ $f: R \longrightarrow R \quad f(x) = 4x+3$

$f(x_1) = f(x_2)$

$4x_1 + 3 = 4x_2 + 3$

$x_1 = x_2$

$\therefore f$ is injective

$f(x) = 4x + 3 = y$

$\Rightarrow x = \frac{y-3}{4}$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore f$ is surjective

$\therefore f$ is bijective

$\therefore f$ is invertible

$f^{-1}(y) = \frac{y-3}{4}$

⑤ i) not injective, surjective $\}$ ∴ not bijective

∴ have no inverse function

ii) not a injective ∴ have no inverse function.

iii) ~~on~~ injective, surjective $\}$ ∴ invertible

(ii) ~~$f^{-1}: \{a, b, c\} \rightarrow \{1, 2, 3\}$~~

~~$f: \{1, 2, 3\} \rightarrow \{a, b, c\}$~~

$f(1) = a$ ~~$f(x) \rightarrow$~~

$f(2) = b$

$f(3) = c$

(ii) $f: \{1, 2, 3\} \longrightarrow \{a, b, c\}$

if $X = \{1, 2, 3\}$

$Y = \{a, b, c\}$, $f: X \longrightarrow Y$

$\therefore f^{-1}: Y \rightarrow X$

$\therefore f^{-1}(a) = 1$

$f^{-1}(b) = 2$

$f^{-1}(c) = 3$

$\therefore f^{-1}: \{a, b, c\} \longrightarrow \{1, 2, 3\}$

Likewise, $(f^{-1})^{-1}: \{1, 2, 3\} \longrightarrow \{a, b, c\}$

but $f: \{1, 2, 3\} \longrightarrow \{a, b, c\}$

$\therefore (f^{-1})^{-1} = f$

⑧ $f: \mathbb{R}^+ \rightarrow [4, \infty)$ $f(x) = x^2 + 4$

$f(x_1) = f(x_2)$

$x_1^2 + 4 = x_2^2 + 4$

$\pm x_1 = \pm x_2$

$x_1 = x_2$ $(x_1, x_2 \in \mathbb{R}^+)$

∴ f is injective

$f(x) = x^2+4 = y$

~~y~~ $x = \sqrt{y-4}$

$y \geq 4$

Hence, ~~$\forall y$~~

~~$y \geq 4$~~

$\forall y \in \mathbb{R} \geq 4, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ ~~and~~ f is surjective

$\therefore f$ is bijective

$\therefore f$ in invertible //

$f^{-1}(y) = \sqrt{y-4}$ //

(c) i) $F^{-1} = \{(3,a), (2,b), (1,c)\}$ //

ii) inveribtible //

Exercise 6

$f : \mathbb{R} \longrightarrow \mathbb{R} \qquad f(x) = x+1$

$f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1$

$x_1 = x_2$

$\therefore f$ is injective

$f(x) = x+1 = y$

$x = y-1$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore f$ is surjective

$\therefore f$ is bijective

$\therefore f$ is invertible.

~~$f^{-1}(y) = y-1$ // or $f^{-1}(x)$~~

$f^{-1}(x) = x-1$ //

Tutorial 6

(01) a)

f_1	f_2	f_3
$f_1(a) = u$	$f_2(a) = u$	$f_3(a) = v$
$f_1(b) = v$	$f_2(b) = u$	$f_3(b) = v$

~~$f_1(a) = v$~~

f_4

$f_4(a) = v$

$f_4(b) = u$

4 functions

b) f_1

$f_1(a) = u$

$f_1(b) = u$

$f_1(c) = u$

1 function

c)

$f_1(a) = u$	$f_2(a) = u$	$f_3(a) = u$
$f_1(b) = v$	$f_2(b) = v$	$f_3(b) = u$
$f_1(c) = u$	$f_2(c) = v$	$f_3(c) = u$
$f_4(a) = v$	$f_5(a) = v$	$f_6(a) = v$
$f_4(b) = v$	$f_5(b) = u$	$f_6(b) = u$
$f_4(c) = v$	$f_6(c) = u$	$f_6(c) = v$

$f_7(a) = u$	$f_8(a) = v$
$f_7(b) = u$	$f_8(b) = v$
$f_8(c) = v$	$f_8(c) = u$

(02) $f(x) = 2x \qquad g(x) = \dfrac{2x^3 + 2x}{x^2+1}$

$$f(x) = \frac{2x(x^2+1)}{(x^2+1)}$$

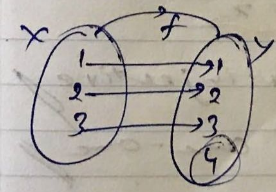
$$f(x) = \frac{2x^3 + 2x}{x^2+1} = g(x)$$

$\therefore f(x) = g(x) \quad (x \in \mathbb{R})$

$\therefore f = g$ & $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

(04) a) $f: X \to Y$

$f: \{1, 2, 3\} \to \{1, 2, 3, 4\}$



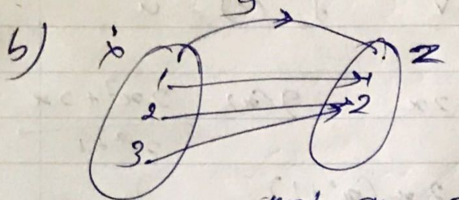
One-one

not onto

~~$f(x) = 4$~~

~~There is no x value exists~~

b)



not one-to-one but onto

(05) $f: R \longrightarrow R \quad f(x) = \frac{x+1}{x}$

$f(x_1) = f(x_2)$

$\frac{x_1+1}{x_1} = \frac{x_2+1}{x_2}$

$1 + \frac{1}{x_1} = \frac{1}{x_2} + 1$

$x_1 = x_2$

$\therefore$ f is injective

one-one

(06) $f: [0,2] \longrightarrow [-1,2] \quad f(x) = \sqrt{x} - 1$

$f(x) = \sqrt{x} - 1 = y$

$\Rightarrow x = (y+1)^2$

$\forall y \in [0,2], \exists x \in [-1,2] : f(x) = y,$

$\therefore$ f is onto fn

(07) $f: R \longrightarrow R$

$f(x) = |x|$

$f(x_1) = f(x_2)$

$|x_1| = |x_2|$

$\pm x_1 = \pm x_2$

$x_1 = +x_2$ or $x_1 = -x_2$

$\therefore$ not one to one

$\therefore$ not invertible

08) a) $f: R \longrightarrow R$ $\quad f(x) = -3x + 4$

$f(x_1) = f(x_2)$

$-3x_1 + 4 = -3x_2 + 4$

$x_1 = x_2$

$\therefore$ ~~one to one~~ injective //

$f(x) = -3x + 4 = y$

$\Rightarrow y = -3x + 4$

$x = \cancel{y}\ \dfrac{4-y}{3}$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ ~~onto~~ surjective //

$\therefore$ bijective //

b) $f: R \longrightarrow R$ $\quad f(x) = -3x^2 + 7$

$f(x_1) = f(x_2)$

$-3x_1^2 + 7 = -3x_2^2 + 7$

$x_1^2 = x_2^2$

$\pm x_1 = \pm x_2$

$x_1 = +x_2$ or $x_1 = -x_2$

$\therefore$ f is not injective //

$\therefore$ f is not bijective //

c) $f(x) = \dfrac{(x+1)}{(x+2)}$

$f(x_1) = f(x_2)$

$\dfrac{x_1 + 1}{x_1 + 2} = \dfrac{x_2 + 1}{x_2 + 2}$

$(x_1 + 1)(x_2 + 2) = (x_2 + 1)(x_1 + 2)$

$\cancel{x_1 x_2} + 2x_1 + \cancel{x_2} + \cancel{2} = \cancel{x_2 x_1} + 2x_2 + \cancel{x_1} + \cancel{2}$

$x_1 = x_2$

$\therefore$ f is injective //

$$f(x) = \frac{(x+1)}{(x+2)} = y$$

$$\frac{x+1}{x+2} = y$$

$$\frac{(x+2)-1}{(x+2)} = y$$

$$1 - \frac{1}{x+2} = y$$

$$\frac{1}{x+2} = 1-y$$

$$x = \frac{1-2(1-y)}{(1-y)}$$

$$x = \frac{2y-1}{1-y}$$

when $y=1$, $x \to \infty$

but $\infty \notin \mathbb{R}$

$\forall y \in \mathbb{R}, \nexists x \in \mathbb{R} : f(x) = y$

$\therefore$ f is ~~...~~ not surjective

$\therefore$ f is not bijective //

d) $f(x) = x^5 + 1$

$$f(x_1) = f(x_2)$$

$$x_1^5 + 1 = x_2^5 + 1$$

$$x_1 = x_2$$

$\therefore$ f is injective

$$f(x) = x^5 + 1 = y$$

$$\Rightarrow x^5 = y - 1$$

$$x = (y-1)^{1/5}$$

$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : f(x) = y$

$\therefore$ f is surjective

$\therefore$ f is bijective //

(09) a) one-to-one / onto
b) not one-to-one / not onto
c) not one-to-one / not onto

(10) $f: \mathbb{Z} \longrightarrow \mathbb{Z}$

a) $f(n) = n-1$

$f(n_1) = f(n_2)$

$n_1 - 1 = n_2 - 1$

$n_1 = n_2$ $\therefore$ one-to-one

$f(n) = n-1 = y$

$n = y+1$

$\forall y \in \mathbb{Z}, \exists n \in \mathbb{Z} : f(n) = y$

$\therefore$ onto

b) $f(n) = n^2+1$

$f(n_1) = f(n_2)$

$n_1^2 + 1 = n_2^2 + 1$

$\pm n_1 = \pm n_2$

$n_1 = +n_2$ or $n_1 = -n_2$

$\therefore$ not one-to-one

c) $f(n) = n^3$

$f(n_1) = f(n_2)$

$n_1^3 = n_2^3$

$n_1 = n_2$

$\therefore$ one-to-one //

$f(n) = n^3 = y$

$n = (y)^{\frac{1}{3}}$

$\forall y \in \mathbb{Z}, \nexists n \in \mathbb{Z} : f(n) = y$

$\therefore$ not onto //

d) $f(n) = \frac{n}{2}$ ← this integer division

$f(1) = 1$

$f(2) = 1$

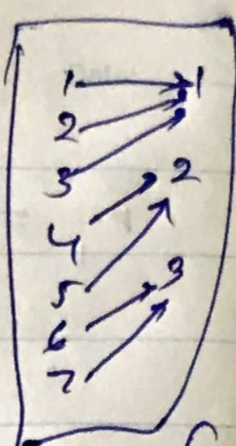
$\therefore$ not one-to-one

~~$f(n_1) = f(n_2)$~~

~~$\frac{n_1}{2} = \frac{n_2}{2}$~~

~~$n_1 = n_2$~~

~~$\therefore$ one-to-one~~



$f(n) = \frac{n}{2} = y$

$n = 2y$

$\forall y \in \mathbb{Z}, \exists n \in \mathbb{Z} : f(n) = y$

$\therefore$ onto

b) <u>missed part</u>

$f(n) = n^2 + 1 = y$

$n^2 = y - 1$

$n = \sqrt{y-1}$

$\forall y \in \mathbb{Z}, \nexists n \in \mathbb{Z} : f(n) = y$

$\therefore$ not onto