Formulae

FRL Permutation Combination

- (Combination) If we can choose `r` elements from `n` elements: $c_r^n = n! \div (n-r)!r!$
  - $c_n^n = 1$
  - If $c_r^n = c_S^n$, then $r = s$, or $n = r + s$
- (Permutation) If we can choose & arrange `r` elements from `n` elements: $n! \div (n-r)!r! * r! = n! \div (n-r)! = p_r^n$
  - Permutation: Arrangement
  - $p_r^n > c_r^n$
  - The number of ways `n` things can be arranged, if `p` things are of 1st kind, `q` things are of 2nd kind, `r` things are of 3rd kind, is $n! \div (p!q!r!)$

FRL General

- Factorials:
  - $3! = 6$
  - $4! = 24$

  > To get 6 (previous result), we divide 24 (current result) by 4 (current input), ie $24 \div 4 = 6$

- Number of choices -
  - $A = \{1, 2, 3, 4, 5\}$
  - Let's form a sub-set of A with no constraints. Here, for each of the 5 elements, we may or may not include them in the subset. So, we have 2 choices per element. Total number of choices:

    > 2^5 for 5 elements
    > 2^n for n elements

  - If we had 3 choices per element, Total number of choices would be:

    > 3^5 for 5 elements
    > 3^n for n elements

FRL Reflexive Relations

- Total number of relations: n^2
- Total number of Diagonal elements: n
- Total number of Non-Diagonal elements: Total number of elements in AxA - Total number of diagonal elements = $n^2 - n$
- Total number of reflexive relations: $2^{Non-Diagonal}$ = 2^{n^2-n}
- Total number of non-reflexive relations: $2^{n^2} - 2^{n^2-n}$
- Smallest possible size: n
- Largest possible size: n^2

FRL Irreflexive Relations

- Total number of Non-Diagonal elements: Total number of elements in AxA - Total number of diagonal elements = $n^2 - n$
- All non-reflexive relations are not irreflexive relations.
- Total number of irreflexive relations: 2^{n^2-n}
- Smallest possible size: 0 ($\{\}$)
- Largest possible size: $n^2 - n$

FRL Comparison of relations

> $A = \{1, 2, 3\}, AxA = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$

Parameter	Reflexive	Irreflexive	Symmetric	Anti-symmetric	Asymmetric	Tra
Cardinality of smallest relation	n	∅	∅	∅	∅	∅
Cardinality of largest relation	n^2	$n^2 - n$	n^2	$(n^2 + n)/2$	$(n^2 - n)/2$	n^2
Number of relations	2^{n^2-n}	2^{n^2-n}	$2^n.2^{(n^2-n)/2)}$	$2^n.3^{(n^2-n)/2}$	$3^{(n^2-n)/2}$	N/A

FRL Number Types

- **N** (Natural numbers) = {1,2,3,4,5 … ∞}
- **Z** (Integers) = {-∞, … +∞}
    - Z^+ = {1,2,3 … ∞}
    - Z^- = {-∞ … -3,-2,-1}
- **Q** (Rational numbers) = Numbers which fit into the following criteria:
    i. Can be written as a fraction, where the denominator is not zero.
    Example: $2.5/0$ is not a rational number
    ii. Has a fixed number of digits after decimel, ie terminating decimels.
    Example: $5/2 = 2.5$
    iii. Has a repeating pattern of numbers after the decimel.
    Example: $1/3 = 0.333... = 0.\bar{3}$ is a rational number
    - **Q*** = {All of Q, excluding 0}
- **I** (Irrational numbers) = Numbers which cannot be written as a fraction, ie don't have a fixed number of digits after decimel, like π, $\sqrt{2}$, etc.
- **R** (Real numbers) = Natural numbers, Integers, Rational numbers, Irrational numbers, etc. ie all types of numbers
- **C** = All complex numbers ie $(a + b_i)$

FRL Group Theory

- **Algebriac Structure**: $a * b = c$, c has to exist in the domain.
- **Semi-group**: $(a * b) * c = a * (b * c)$
- **Monoid**: $(e * a) = (a * e) = a,$ `e` = identity element
- **Group**: $(a * b) = (b * a) = e,$ `e` = identity element
- **Albenian Group**: $(a * b) = (b * a)$, does not have to be equal to `e`

FRL Functions

> $A = \{1, 2, 3, 4...(x\ elements)\}, B = \{a, b, c, d, ...(y\ elements)\}$

- AxB=$x * y$ elements
- Number of relations possible: $2^x + 2^y = 2^{xy}$
- Number of choices per element of `A` : y
- Number of functions possible: y^x
- Number of relations which are not functions: $2^{xy} - y^x$

> $A = \{1, 2, 3, 4...(x\ elements)\}$

- AxB=$x^x = x^2$ elements

- Number of relations possible: 2^{x^2}
- Number of choices per element of `A` : x
- Number of functions possible: x^x
- Number of relations which are not functions: $2^{x^2} - x^x$

FRL One-To-One Functions

> $A = \{1, 2, 3, 4 ... (x\ elements)\}, B = \{a, b, c, d, ... (y\ elements)\}$

- Number of elements: $x \leq y$
- In set `A` , number of choices for:
    - 1st element: y
    - 2nd element: $(y - 1)$
    - 3rd element: $(y - 2)$

      ...
    - x^{th} element: $y - (x - 1)$
- Number of functions possible: $y * (y - 1) * (y - 2) * ... * (y - (x - 1)) = P^y_x$
  > P: Permutation

FRL Onto Functions

> $A = \{1, 2, 3, 4 ... (x\ elements)\}, B = \{a, b, c, d, ... (y\ elements)\}$

- Number of elements: $x \geq y$
- If a function is one-to-one, it may not be onto.
- If a function is onto, it may not be one-to-one.
- Number of functions possible: $y * (y - 1) * (y - 2) * ... * (y - (x - 1)) = P^y_x$
  > P: Permutation

FRL Bijective Functions

> $A = \{1, 2, 3, 4 ... (x\ elements)\}, B = \{a, b, c, d, ... (y\ elements)\}$

- Number of elements: $x = y = n$
- Number of functions possible: $n + (n - 1) * (n - 2) * ... * 1 = n!$
  > !: Factorial

Set Theory

Set:

A well-defined unordered collection of distinct elements.

- Unordered: Set {1,2,3,4} is same as {2,3,4,1}.
- Distinct: Set {1,2,3,4} is same as {1,1,2,2,3,4}.

Null/Empty set

A set with no elements, denoted by Φ or {}.

- Cardinality of an empty set is 0.
- {Φ} is not an empty set.
- Φ is present in every set.

Subset

If every element of Set A also exists in Set B, denoted by ⊆.

- **Example**: If A = {1,2,3,4}, A' = {1,2,3,4,5}, then A ⊆ A'.
- Every set is a subset of itself.
- Trivial subset: A set which contains all elements of A, is if A={1,2,3,4} and A'={1,2,3,4}.
- Proper subset: A set which is not a trivial subset of A, ie the subset can't be same / can't have the same length as A.
- If A⊆B & B⊆A, then A=B. #Note

Cardinality

Number of elements in a set.

Power Set

If ‘A’ is a finite set, then it is the set of all subsets of ‘A’.

- **Example**: If $A = \{1, 2, 3\},\ P(A) = \{\{\Phi\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.
- If cardinality of a set is n, the number of elements in the power set is $2^n = 8$, ie $|P(a)| = |2^n|$
- Write all the sets starting with cardinality 0, then 1, then 2 … and so on.

Cartesian product

Product of all elements of 1 set, with all elements of the other set.

- **Example**: If A={a,b} and B={1,2,3}, then:
  - $AxB = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$ | If an element is (m,n), then m∈A & n∈B.
  - $BxA = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$
- Total number of products, assuming Cardinality(A)=m & Cardinality(B)=n, is mxn.
- First multiply element 1 of A with element 1 of B, then element 1 of A with element 2 of B … and so on.
- The order needs to be maintained, as is apparent in the example.
- AxB is not same as BxA, unless A=B.

Relations

> also check Cartesian Product

If A and B are two sets and AxB is their cartesian product, then any subset of AxB can form a relation from A to B.

- **Example**:
  - $\{(a, 1), (a, 2)\}$ | Valid
  - $\{(b, 1), (b, 2), (a, 1)\}$ | Valid
  - $\{(1, a), (2, a)\}$ | Not Valid
- Total number of relations: 2^{mxn}.

Reflexive Relation

- Formulae

- A Relation 'R' on a set 'A' is said to be Reflexive if $(x, x) \epsilon R \forall x \epsilon A$
- Every element of a set is related to itself.
- Points to remember:
    i. All elements must be present.
    ii. All of them must be related to themselves.
    iii. After putting in all valid elements, we can put extra elements.
    iv. The difference between a diagonal relation is that a reflexive relation may also contain extra elements.
- Example 0: $A = \{1, 2, 3\}$
    - $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
    - Possible Reflexive relations:
        - $\{(1,1),(2,2),(3,3)\}$: *Smallest*
        - $\{(1,1),(2,2),(3,3),(2,1)\}$: *We may also put extra elements as long as the original condition has been satisfied.*
        - $\{(1,1),(2,2),(3,3),(2,1),(2,3)\}$
        - $\{(1,1),(2,2),(3,3),(2,1),(2,3),(3,1)\}$
        - $\{(1,1),(2,2),(3,3),(2,1),(2,3),(3,1),(3,2)\}$
        - $\{(1,1),(2,2),(3,3),(2,1),(2,3),(3,1),(3,2),(3,3)\}$: *Largest*
- Example 1: Check for reflexive relation: {(x, y), $x - y$ is an integer}
    - $(2, 3)$ cannot be in an answer, since it doesn't satisfy point 2.
    - $(1,1),(2,2),(3,3)$, etc. will satisfy the equation, as $1 - 1 = 0$, which is an integer. So **it is a reflexive relation**.
- Example 2: Check for reflexive relation: {(x, y), $x - y$ is an odd number}
    - $(1,1),(2,2),(3,3)$, etc. will not satisfy the equation, as $1 - 1 = 0$, which is not an odd number. So **it is not a reflexive relation**.

Irreflexive Relation

- Formulae
- A Relation 'R' on a set 'A' is said to be Irreflexive if $(x, y) \notin R \; \forall \; (x, y) \epsilon A$
- No element of the set should be related to itself.
- Points to remember:
    i. We exclude all diagonal elements, and include the non-diagonal elements.
    ii. **A not reflexive relation is not the same as an Irreflexive relation**. Example:
        - $A = \{1, 2, 3\}$, $R = \{(1,1),(2,2)\}$
        - Here, R is not a reflexive relation since it does not include (3,3). But, R is not in Irreflexive relation either, because it includes (1,1) & (2,2). So, it is just not a reflexive relation.
- Example 0: $A = \{1, 2, 3\}$

- $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
- $R = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

Symmetric Relation

- A Relation 'R' is symmetric, if for every $(x, y) \in R,\ (y, x) \in R\, \forall\, (x, y) \epsilon A$.
- If (x, y) is present in a Relation 'R' on a set 'A', then (y, x) must also be present in the set.
- We first check if (x, y) is present. We check for (y, x) only if (x, y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
  - $\{(1,2),(2,1),(1,3),(3,1),(1,1)\}$: for $(1,1)$, it's Symmetric pair will also be $(1,1)$, which is a duplicate.
  - $\{(1,2),(1,3),(1,1)\}$: This is NOT a symmetric relation, because $(3,1)$ is not present for $(1,3)$.
- Example 1: Check: 'perpendicular-to' on a set of all lines.

  > Solution: If a line $L1 \perp L2$, then obviously $L2 \perp L1$. So, **this is a symmetric relation**.

- Example 2: Check: 'brother of' of all humans on Earth.

  > Solution: If X is a brother of Y, Y may be a sister of X. So, **it's not symmetric**.

- Example 3: Check: 'complement of', on a set of numbers.

  > Solution: If `X` is a complement of `Y`, then `Y` is also a complement of `X`. So, **the relation is symmetric**.

Anti-symmetric Relation

- A Relation 'R' is anti-symmetric, if for every $(x, y) \in R,\ (y, x) \in R\, \forall\, (x, y) \epsilon A, only\ if\ (x = y)$.
- If (x, y) is present in a Relation 'R' on a set 'A', then (y, x) should only also be present in the relations if $x = y$. Otherwise, it must not be present.
- We first check if (x, y) is present. We check for (y, x) only if (x, y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
  - $\{(2,1),(3,1),(1,1)\}$: TRUE, for $(2,1)$, $(1,2)$ must not be present since $(2 \neq 1)$. For $(1,1)$, it's Symmetric pair is $(1,1)$, which is fine since $(1 = 1)$.
  - $\{(1,2),(2,1),(1,3),(1,1)\}$: FALSE, because $(2,1)$ is present.

Asymmetric Relation

- A Relation ‘R’ is asymmetric, if for every $(x, y) \in R,\ (y, x) \notin R \, \forall \, (x, y) \epsilon A$.
- If (x, y) is present in a Relation ‘R’ on a set ‘A’, then (y, x) should only also be present in the relationsif $x = y$. Otherwise, it must not be present.
- We first check if (x, y) is present. We check for (y, x) only if (x, y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
  - $\{(2, 1), (3, 1), (2, 3)\}$: TRUE, for all (x, y), (y, x) is not present in the relation.
  - $\{(2, 2), (3, 1), (2, 3), (1, 1)\}$: FALSE, because $(1, 1)$ is present, and it’s duplicate ie $(1, 1)$ should not be present.
- Every Asymmetric relation is anti-symmetric. An Asymmetric relation is more restrictive than an anti-symmetric relation, since an asymmetric relation does not allow (y, x) (for every (x, y)) to be present even if $(x = y)$.
- Every Anti-symmetric relations is not asymmetric. An anti-symmetric relation allows (y, x) to be present if $(x = y)$.

Transitive Relation

- A Relation ‘R’ is Transitive, if for every $(x, y) \in R \,\&\, (y, z) \in R$, $(x, z) \in R$ ∀ $(x, y) \epsilon A$.
- If (x, y) & (y, z) is present in a Relation ‘R’ on a set ‘A’, then (x, z) should also be present in the relation.
- If any or x,y or z is not present in the relation, then the relation is transitive by default.
- Example 0: $A = \{1, 2, 3\}$
  - $\{(1, 1), (2, 2)\}$: TRUE, x,y & z are not present together.
  - ${Φ}: TRUE, none of the elements are present.
  - $\{(1, 2), (2, 3)\}$: FALSE, since (x, z) ie $(1, 3)$ is not present.

    > x = 1, y = 2, z = 3

  - $\{(1, 1), (1, 2), (2, 1)\}$: TRUE, since in both cases the conditions are matching.

    > x = 1, y = 1, z = 2 z = 1, y = 2, z = 1

Equivalence Relation

- A Relation ‘R’ is equivalence, if it is `reflexive`, `symmetric` or `transitive`.
- Example 0: $A = \{1, 2, 3\}$
  - $\{(1, 1), (2, 2), (3, 3)\}$: TRUE
  - $\{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2)\}$: TRUE

- $\{(1,1),(2,2),(3,3),(3,2),(1,3)\}$: FALSE, because $(2,3)$ is not present for $(3,2)$, so it's not symmetric.
- $\{\ \}$: FALSE, since $(1,1),(2,2),(3,3)$ is not present, so it is not a reflexive relation.

Partially Ordered Set

- A Relation 'R' is said to be a partial-ordering relation if R is `reflexive`, `anti-symmetric` as well as `transitive`.
- Partially odered set: A set 'A' with Partial-Ordering relation 'R' defined on 'A' is called POSET, defined by [A;R]
- Example 0: $A = \{1,2,3\}$
  - $\{(1,1),(2,2),(3,3)\}$: TRUE
  - $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$: TRUE
  - [A,$\leq$] is a POSET: TRUE,

    > $(1 \leq 1)$ is true, so relation is reflexive.
    > $(1 \leq 1)$ is true, and for $(1.5 \leq 2.5)$, $(2.5 \leq 1,5)$ is not true. So, relation is anti-symmetric.
    > $(1 \leq 2.5)$, $(2.5 \leq 3)$, then $(1 \leq 3)$ is also true. So, the relation is transitive.

Totally Ordered Set

- A POSET {A,R} is called a totally ordered set if every pair of elements in set `A` are comparable, ie, if:

  > $aRb\ or\ bRa\ \forall\ (a,b) \in A$

- The set has to mandatorily be a Partially Ordered Set. In other words, every TOS is a POSET.
- Example 0: $(A, \div) \in R\ \forall (A \in I)$, I=Integers: FALSE.

  > A = {1,2,3,4,5,6,7,8…∞}
  > Reflexive: $(1,1),(2,2),(3,3) \in R$
  > Anti-symmetric: $(4,2) \in R$ but $(2,4) \notin R$
  > Transitive: If $(8,4) \in R\ \&\ (4,2) \in R$, then $(8,2) \in R$.
  > TOS: $(2,3),(4,8)$, etc. are not comparable when the domain consists only of positive integers.

- Example 1: $(A, \leq) \in R\ \forall (A \in R)$, I=Real numbers: TRUE.
- Example 2: $(A, \in) \in R\ \forall A \in \{\{\emptyset\},\{a\},\{b\},\{a,b\}\}$: FALSE.

  > $\{a\}$ is not comparable with $\{b\}$.

Comparison of all relations

See comparison of relations in section `formulae` .

- Explanation:
  > $A = \{1, 2, 3\}$
  > $AxA = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (2, 2)\}$

  - Reflexive:
    - Smallest: $\{(1, 1), (2, 2), (3, 3)\}$ | Cardinality: n
    - Largest: $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ | Cardinality: n^2
    - Number of relations: For $\{(1, 1), (2, 2), (3, 3)\}$ ie `n` elements, we don't have a choice. For the rest $\{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ ie $(n^2 - n)$ elements, we have 2 choices per pair. So, the total number of choices: $2^{n^2} - 2^n = 2^{n^2-n}$
  - Irreflexive:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ | Cardinality: $n^2 - n$
    - Number of relations: 2^{n^2-n}
  - Symmetric:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ | Cardinality: n^2
    - Number of relations: For $\{(1, 1), (2, 2), (3, 3)\}$ ie `n` elements, we have 2 choices per pair. For the rest $\{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ ie $(n^2 - n)$ elements, if (1,2) is present, (2,1) also needs to be present. So, we have 2 choices per respective double-pair. Total number of choices: $2^n.2^{(n^2-n)/2}$
  - Anti-symmetric:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 3)\}$ OR $\{(1, 1), (2, 2), (3, 3), (2, 1), (3, 1), (3, 2)\}$ | Cardinality: $n + [(n^2 - n)/2] = (2n + n^2 - n)/2 = (n^2 + n)/2$
    - Number of relations: For $\{(1, 1), (2, 2), (3, 3)\}$ ie `n` elements, we have 2 choices per pair. For the rest $\{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ ie $(n^2 - n)$ elements, if (1,2) is present, (2,1) must not be present and vice versa. So, we can either take (1,2) or (2,1) exclusively, or we can exclude both. We have 3 choices per double-pair. Total number of choices: $2^n.3^{(n^2-n)/2}$
  - Asymmetric:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1, 2), (1, 3), (2, 3)\}$ OR $\{(2, 1), (3, 1), (3, 2)\}$ | Cardinality: $(n^2 - n)/2$

    - Number of relations: For $\{(1,1),(2,2),(3,3)\}$ ie `n` elements, we cannot take them. For the rest $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ ie (n^2-n) elements, if (1,2) is present, (2,1) must not be present and vice versa. So, we can either take (1,2) or (2,1) exclusively, or we can exclude both. We have 3 choices per double-pair. Total number of choices: $3^{(n^2-n)/2}$
  - Transitive:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(2,2)\}$ | Cardinality: n^2

Group Theory

Algebriac Structure >> Semi-group >> Monoid >> Group >> Abelian Group

Algebriac Structure

- A non-empty set S is called Algebriac Structure with respect to binary operation `*` if $(a*b)=c\,\forall\,\{(a,b,c)\in S\}$
- `c` needs to exist in the domain.
- `*` is closure operation on `S`
- Prerequisites: none
- Example 0:
  > N = Natural numbers
  - (N,+): TRUE
  - (N,-): FALSE
    > $3-5=-2$, which is an integer
  - (N,.): TRUE
    > We won't have 0 unless we multiply any number by 0, which won't happen in this case
  - (N,/): FALSE
    > $1/2=0.5$, which is not a rational number
- Example 1:
  > Z = Integers
  - (Z,+): TRUE
  - (Z,-): TRUE

    - (Z,.): TRUE
    - (Z,/): FALSE
        > $5/2 = 2.5$, which is a rational number

- Example 2:
    > R = Real numbers

    - (R,+): TRUE
    - (R,-): TRUE
    - (R,.): TRUE
    - (R,/): TRUE
- Example 3:
    > Q: Rational numbers

    - (Q,+): TRUE
    - (Q,-): TRUE
    - (Q,.): TRUE
    - (Q,/): FALSE
        > $2/0 = \infty$, where the denominator is zero

- Example 4:
    > Q*: Rational numbers, excluding 0

    - (Q*,+): TRUE
    - (Q*,-): TRUE
    - (Q*,.): TRUE
    - (Q*,/): TRUE
        > $5/0$ can't be done because 0 doesn't exist in the range of Q*

Semi-group

- An Algebriac Structure (S,*) is called Semi-group if it follows associative property, ie $(a * b) * c = a * (b * c) \forall \{(a, b, c) \in S\}$
- Prerequisites: `Algebriac Structure`
- Example 0:
    > N = Natural numbers

    - (N,+): TRUE | TRUE
    - (N,-): FALSE | null
    - (N,.): TRUE | TRUE

- (N,/): FALSE | null

- Example 1:

  > Z = Integers

  - (Z,+): TRUE | TRUE
  - (Z,-): TRUE | FALSE

    > $2-(3-5) \neq (2-3)-5$

  - (Z,.): TRUE | TRUE
  - (Z,/): FALSE | null

- Example 2:

  > R = Real numbers

  - (R,+): TRUE | TRUE
  - (R,-): TRUE | FALSE

    > $2-(3-5) \neq (2-3)-5$

  - (R,.): TRUE | TRUE
  - (R,/): TRUE | FALSE

    > $8/(4/2) \neq (8/4)/2$

- Example 3:

  > Q: Rational numbers

  - (Q,+): TRUE | TRUE
  - (Q,-): TRUE | FALSE

    > $3.5-(2.5-1.5) \neq (3.5-2.5)-1.5$

  - (Q,.): TRUE | TRUE
  - (Q,/): FALSE | null

- Example 4:

  > Q*: Rational numbers, excluding 0

  - (Q*,+): TRUE | TRUE
  - (Q*,-): TRUE | FALSE

    > $3.5-(2.5-1.5) \neq (3.5-2.5)-1.5$

  - (Q*,.): TRUE | TRUE
  - (Q*,/): TRUE | FALSE

    > $10/(5/2.5) \neq (10/5)/2.5$

- Example 5:

- $\{2^n$ | n is an integer w.r.t multiplication}
- $2^5.2^{10}.2^8 = 8388608$, which is an integer.
- $2^5.(2^{10}.2^8) = (2^5.2^{10}).2^8$, so TRUE

Monoid

- A Semi-group (S,*) is called Monoid if there exists one element $(e \in S)$, such that $(e * a) = (a * e) = a \forall \{a \in S\}$
- `e` is the identity element, $(a * e = a)$. Our goal is to find the identity element. If it exists, then the resultant group is a Monoid.

  > Example: 10+ `0` =10, a+ `e` =a

- Prerequisites: `Algebriac Structure` , `Semi-group`
- Example 0 (`Algebriac Structure` | `Semi-group` | `Monoid`):

  > N = Natural numbers

  - (N,+): TRUE | TRUE | FALSE
  - (N,-): FALSE | null | null
  - (N,.): TRUE | TRUE | TRUE

    > 2* `1` =2

  - (N,/): FALSE | null | null
- Example 1:

  > Z = Integers

  - (Z,+): TRUE | TRUE | TRUE
  - (Z,-): TRUE | FALSE | null
  - (Z,.): TRUE | TRUE | TRUE
  - (Z,/): FALSE | null | null
- Example 2:

  > R = Real numbers

  - (R,+): TRUE | TRUE | FALSE
  - (R,-): TRUE | FALSE | null
  - (R,.): TRUE | TRUE | TRUE
  - (R,/): TRUE | FALSE | null
- Example 3:

  > Q: Rational numbers

  - (Q,+): TRUE | TRUE | FALSE

> $4/2 + 0/0 = 4/2$, but `0/0` is not allowed.

  - (Q,-): TRUE | FALSE | null
  - (Q,.): TRUE | TRUE | FALSE
    > $0 * b = 0$, b cannot be found.
  - (Q,/): FALSE | null | null
- Example 4:
  > Q*: Rational numbers, excluding 0

  - (Q*,+): TRUE | TRUE | FALSE
  - (Q*,-): TRUE | FALSE | null
  - (Q*,.): TRUE | TRUE | TRUE
  - (Q*,/): TRUE | FALSE | null
- Example 5:
  - (2^n, `n` = Integer wrt multiplication): TRUE | TRUE | TRUE
    > $2^n.2^0 = 2^n$

Group

- A Monoid (S,*) is called Group if for each element $(a \in S)$, there exists an element $(b \in S)$, such that $(a * b) = (b * a) = e$
- `e` is the identity element.
- Prerequisites: `Algebriac Structure`, `Semi-group`, `Monoid`
- Example 0 (`Algebriac Structure` | `Semi-group` | `Monoid` | `Group`):
  > N = Natural numbers

  - (N,+): TRUE | TRUE | FALSE | null
  - (N,-): FALSE | null | null | null
  - (N,.): TRUE | TRUE | TRUE | FALSE
  - (N,/): FALSE | null | null | null
- Example 1:
  > Z = Integers

  - (Z,+): TRUE | TRUE | TRUE | TRUE
  - (Z,-): TRUE | FALSE | null | null
  - (Z,.): TRUE | TRUE | TRUE | FALSE
  - (Z,/): FALSE | null | null | null
- Example 2:

> R = Real numbers

  - (R,+): TRUE | TRUE | FALSE | null
  - (R,-): TRUE | FALSE | null | null
  - (R,.): TRUE | TRUE | TRUE | FALSE

    > For 0, it is invalid since we can't find `b` in $a.b = e$

  - (R,/): TRUE | FALSE | null | null
- Example 3:

  > Q: Rational numbers

  - (Q,+): TRUE | TRUE | FALSE | null
  - (Q,-): TRUE | FALSE | null | null
  - (Q,.): TRUE | TRUE | FALSE | null
  - (Q,/): FALSE | null | null | null
- Example 4:

  > Q*: Rational numbers, excluding 0

  - (Q*,+): TRUE | TRUE | FALSE | null
  - (Q*,-): TRUE | FALSE | null | null
  - (Q*,.): TRUE | TRUE | TRUE | TRUE
  - (Q*,/): TRUE | FALSE | null | null

Abelian Group

- A Group (G,*) is called an Abelian Group if $(a * b) = (b * a) \, \forall \, \{(a, b) \in G\}$
- Prerequisites: `Algebriac Structure`, `Semi-group`, `Monoid`, `Group`
- Example 0 (`Algebiac Structure` | `Semi-group` | `Monoid` | `Group` | `Abelian Group`):

  > N = Natural numbers

  - (N,+): TRUE | TRUE | FALSE | null | null
  - (N,-): FALSE | null | null | null | null
  - (N,.): TRUE | TRUE | TRUE | FALSE | null
  - (N,/): FALSE | null | null | null | null
- Example 1:

  > Z = Integers

  - (Z,+): TRUE | TRUE | TRUE | TRUE | TRUE
  - (Z,-): TRUE | FALSE | null | null | null
  - (Z,.): TRUE | TRUE | TRUE | FALSE | null

- (Z,/): FALSE | null | null | null | null
- Example 2:
  > R = Real numbers

  - (R,+): TRUE | TRUE | FALSE | null | null
  - (R,-): TRUE | FALSE | null | null | null
  - (R,.): TRUE | TRUE | TRUE | FALSE | null
  - (R,/): TRUE | FALSE | null | null | null
- Example 3:
  > Q: Rational numbers

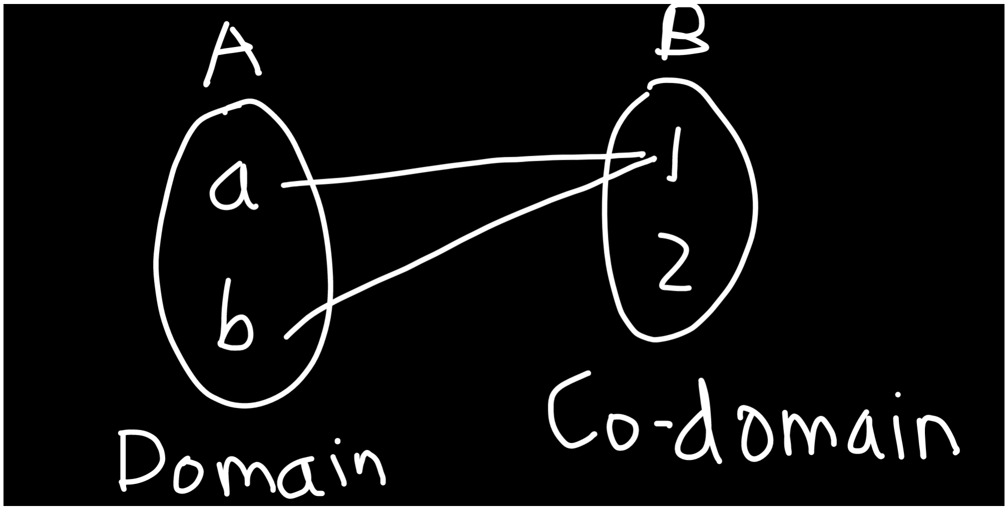
  - (Q,+): TRUE | TRUE | FALSE | null | null
  - (Q,-): TRUE | FALSE | null | null | null
  - (Q,.): TRUE | TRUE | FALSE | null | null
  - (Q,/): FALSE | null | null | null | null
- Example 4:
  > Q*: Rational numbers, excluding 0

  - (Q*,+): TRUE | TRUE | FALSE | null | null
  - (Q*,-): TRUE | FALSE | null | null | null
  - (Q*,.): TRUE | TRUE | TRUE | TRUE | TRUE
  - (Q*,/): TRUE | FALSE | null | null | null

Functions

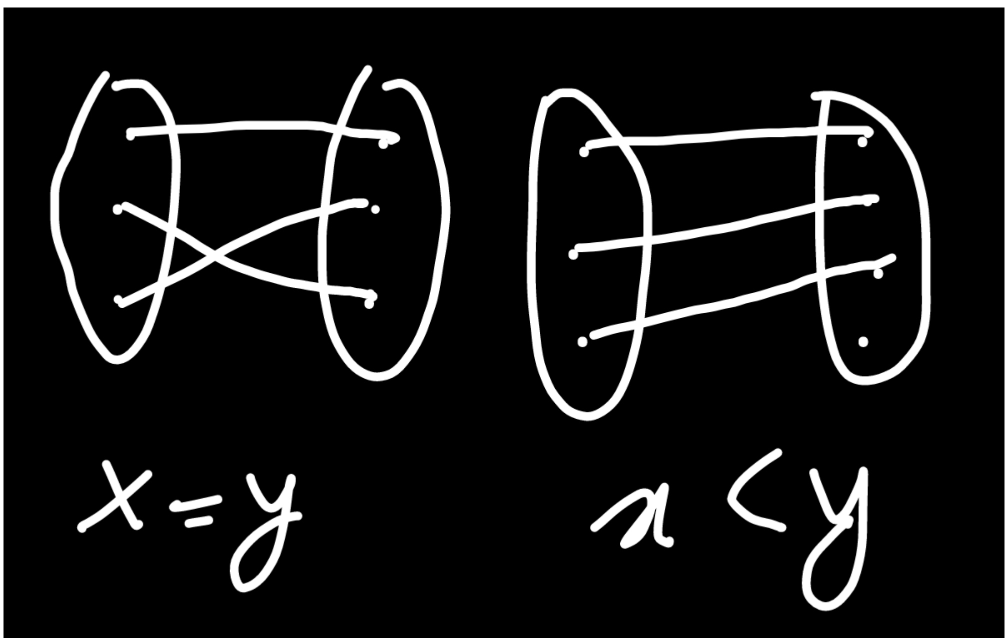
> also check Relations

- Formulae
- A Relation `f` from a set `A` to a set `B` is called a function if, for each element of `A` , we have assigned an unique (only one) element of `B` .
- Check `Example 0` for `Domain` , `Co-domain` , `Range` .
- Example 0, $A = \{a, b\}$, $B = \{1, 2\}$, $AxB = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$:
  - $(a, 1), (b, 2)$=TRUE
  - $(a, 2), (b, 1)$=TRUE
  - $(a, 1), (b, 1)$=TRUE
  - $(a, 1), (a, 2), (b, 2)$=FALSE, `a` cannot be mapped to both `1` & `2` at the same time.
  - **Domain**: Set $A = \{a, b\}$
  - **Co-domain**: Set $B = \{1, 2\}$
  - **Range**: Number of participants in Set `B` .



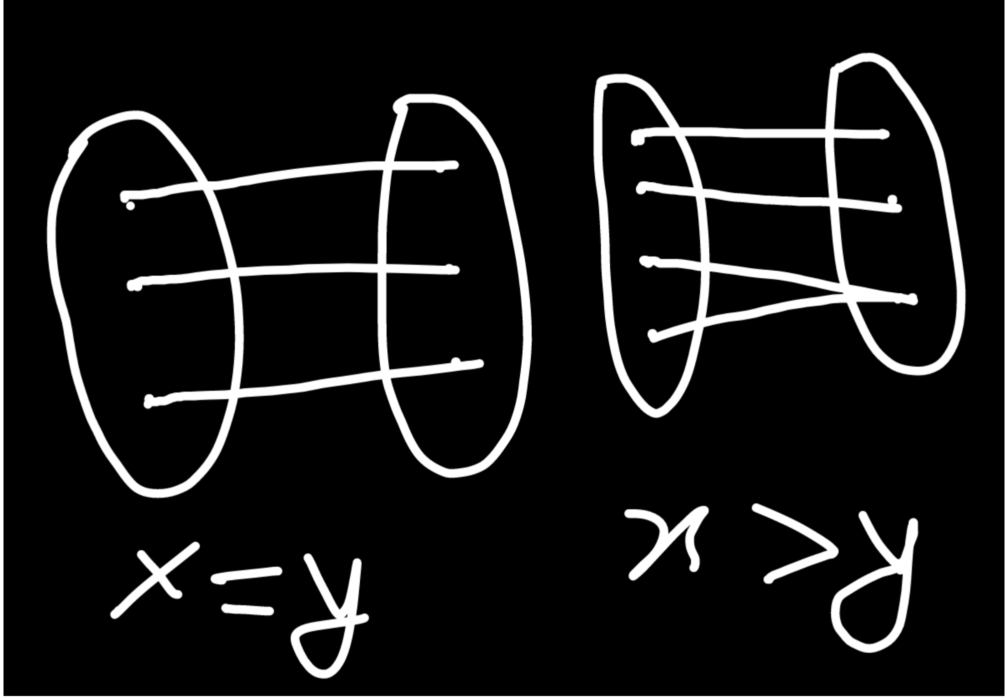
One-To-One Function (Injective)

- Formulae
- A function `f` from set `A` to set `B` is one-to-one if each element in `A` is mapped to only 1 element in `B`. All elements in `A` should be mapped.



Onto Function (Surjective)

- Formulae
- A function `f` from set `A` to set `B` is onto if each element in `B` is mapped to atleast 1 element of `A`.
- Every element in `A` must also participate in the function, since it is a function in the first place.



Bijective Function

- Formulae
- A function `f` from set `A` to set `B` is bijective if `f` is both one-to-one and onto.
- Number of elements in A should be equal to number of elements in B.

Probability & Statistics

- Prerequisite: Permutation & Combination
- **Random experiment**: The experiment for which the outcome is uncertain.
- **Sample space**: The set of all possible outcomes of a Random Experiment is called Sample space. Denoted by `S`.
  - Example 0: Flipping a coin, $S = \{head, tail\}$.
  - Example 1: Rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$
- **Event**: Any subset of Sample space is called an Event. Usually denoted by `E`.
  - Example 0: `head` appearing when a coin is flipped.
- **Probability of an Event**: Probability of Event `A` is given by:
  - $P(A) = Number\ of\ elements\ in\ `S`\ that\ favour\ occurence\ of\ `A` \div Total\ number\ of\ elements\ in\ `S`$
  - Example 0:
    - Rolling a dice; $S = \{1, 2, 3, 4, 5, 6\}; n(S) = 6$
    - A: Getting even number, $A = \{2, 4, 6\}; n(A) = 3$
    - $P(Getting\ an\ even\ number) = 3/6 = 1/2$

Axioms of Probability:

> Axiom: Statements that are correct by default, but have no proof as such.
> Theorem: Statements that are correct, and can also be proved.

1. Probability of an event always lies between `0` & `1`. $0 \leq P(A) \leq 1$
2. If S is the Sample space of a Random Experiment, then all outcomes of the Experiment will definitely be from S. $P(S) = 1$
3. If there are `n` elements in a Sample space, out of which `m` elements favour the occurence of `A`, then:
   - Odds in favour of `A` = $m/(n-m)$ = Number of ways `A` can occur / Number of ways `A` cannot occur.
   - Odds against `A` = $(n-m)/m$ = Number of ways `A` cannot occur / Number of ways `A` can occur.
4. If `A` is any event within Sample space `S`, and $n(A)$ is the number of Events that favour occurence of `A`, then the Probability of `A` to not happen: $P(A^c) = (n(S) - n(A))/n(S) = (1 - n(A))/n(S) = 1 - P(A)$.

Types of Events

- **Mutually Exhaustive**: If Events `A` & `B` together form the Sample space `S`, $A \cup B = S$
  - Example 0: Experiment: Rolling a dice.
    - `A`: Getting an even number; $A = \{2, 4, 6\}$
    - `B`: Getting an odd number; $B = \{1, 3, 5\}$
    - $A \cup B = \{1, 2, 3, 4, 5, 6\} = S$, so Mutually Exhaustive
    - $A \cap B = \ = \phi$, so Mutually Exclusive
- **Mutually Exclusive**: If Events `A` & `B` don't intersect each other, $A \cap B = \phi$
  - Example 0: Experiment: Rolling a dice.
    - `A`: Getting a prime number; $A = \{2, 3, 5\}$
    - `B`: Getting a composite number; $B = \{4, 6\}$
    - $A \cap B = \ = \phi$, so Mutually Exclusive
    - $A \cup B = \{2, 3, 4, 5, 6\} \neq S$, so not Mutually Exhaustive
- **Impossible**: If there is no feasable outcome of the event `A` in Sample space `S`. Denoted by `Φ`.
  - Example: Probability of Getting a number >6 when Rolling a dice.
    - $S = \{1, 2, 3, 4, 5, 6\}$
    - $A = \{\phi\}$
    - $P(A) = n(A)/n(S) = 0/6 = 0$
- **Independent**: If they are from different sample spaces (random experiments).

- $P(A \cap B) = P(A).P(B)$
- Example 0: Calculate the Probability that a leap year selected at random will contain 53 Sundays.
  - 52 weeks = $52 * 7 = 364$ days will mandatorily have a Sunday per week.
  - For the rest of the 2 days, Sunday can be 1 of the 2 days.
  - So, $S = \{(S, M), (M, T), (T, W), (W, T), (T, F), (F, S), (S, S), \}$
  - if `A` is the event, then we have 2 probable outcomes for `A`.
  - $P(A) = n(A)/n(S) = 2/7$
- Example 1: A bag contains 40 tickets numbered $1 - 40$, out of which 4 tickets are drawn at random and arranged in ascending order. What is the probability that the 3rd ticket is 25?
  - Out of 40 tickets, 4 can be drawn in c_4^{40} ways.
  - Combination: _ _ 25 _ .
  - For the 1st 2 positions, we have c_2^{24} choices.
  - For 25, we have c_1^1 choices.
  - For the last position, we have c_1^{15} choices.
  - Total number of choices: $c_2^{24} * c_1^1 * c_1^{15}$
  - Probability: $(c_2^{24} * c_1^1 * c_1^{15})/c_4^{40} = 0.0453$
- Example 2: A 5 digit number is formed by the digits 0,1,2,3,4 without repitition. The probability that the number formed is divisible by 4 is _.
  - Total number of 5 digits: _ _ _ _ _ = $4 * 4 * 3 * 2 * 1 = 96$ (we cannot have 0 in the 1st position).
  - Numbers divisible by 4 (last 2 digits have to be divisible by 4):
    - _ _ _ 0 4: $3 * 2 * 1 = 6$ choices
    - _ _ _ 1 2: $2 * 2 * 1 = 4$ choices
    - _ _ _ 2 0: $3 * 2 * 1 = 6$ choices
    - _ _ _ 2 4: $2 * 2 * 1 = 4$ choices
    - _ _ _ 3 2: $2 * 2 * 1 = 4$ choices
    - _ _ _ 4 0: $3 * 2 * 1 = 6$ choices
    - Total number of choices: $6 + 4 + 6 + 4 + 4 + 6 = 30$
  - Probability: $30/96 = 0.3125$
- Example 3: A bag contains 8 blue and 6 white balls. The probability of drawing 2 balls of same colour is _.
  - Total number of choices: c_2^{14}
  - Number of favourable choices: $c_2^8 + c_2^6$ (+ because they're alternative choices)
  - Probability: $(c_2^8 + c_2^6) \div c_2^{14} = 86/182 = 0.4725$
- Example 4: A fair dice is rolled twice. Find the probability that an odd number will follow an even number.
  - They are Independent events.

- Probability of getting an odd number, $P(A) = 3/6$
- Probability of getting an even number, $P(B) = 3/6$
- Total probability: $3/6 + 3/6 = 1/2 + 1/2 = 1/4 = 0.25$
- **Another method**: $S = \{(E, E), (E, O), (O, E), (O, O)\}$
- $n(S) = 4$, $N(A) = 1$
- $P(A) = 1/4$

Additional Theorem of Probability

- If E_1 & E_2 are two events within a sample space S, then: $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
- If E_1, E_2 & E_3 are three events within a sample space S, then: $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_3 \cap E_1) - P(E_1 \cap E_2 \cap E_3)$
- If Events E_1 & E_2 are Mutually Exclusive, then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
- If Events E_1 & E_2 are Independent of each other, then $P(E_1 \cap E_2) = P(E_1).P(E_2)$, and $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1).P(E_2)$
  - $P(E_1) + P(E_2)\{1 - P(E_1)\}$
  - $P(E_1) + P(E_2).(P(\bar{E}_1))$
- If E_1 & E_2 are events within a Sample space S, then $P(\bar{E}_1 \cap \bar{E}_2) = 1 - P(A \cup B)$
- Example 5: In view of Covid-19 pandemic, the probability that the state government cancels SSC exams is 0.6. The probability that it cancels Class 12 exams is 0.4. The probability that it cancels either SSC or Class 12 exams is 0.7. The probability that it cancels both of them is ? Assume that both events are dependent on each other.
  - E_1 = SSC Exams are cancelled.
  - E_2 = Class 12 Exams are cancelled.
  - $P(E_1 \cup E_2) = 0.7$, $P(E_1) = 0.6$, $P(E_2) = 0.4$ // ∪: Both events
  - $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
  - $0.7 = 0.6 + 0.4 - P(E_1 \cap E_2)$
  - $P(E_1 \cap E_2) = 0.3$ // ∩: Either of the events

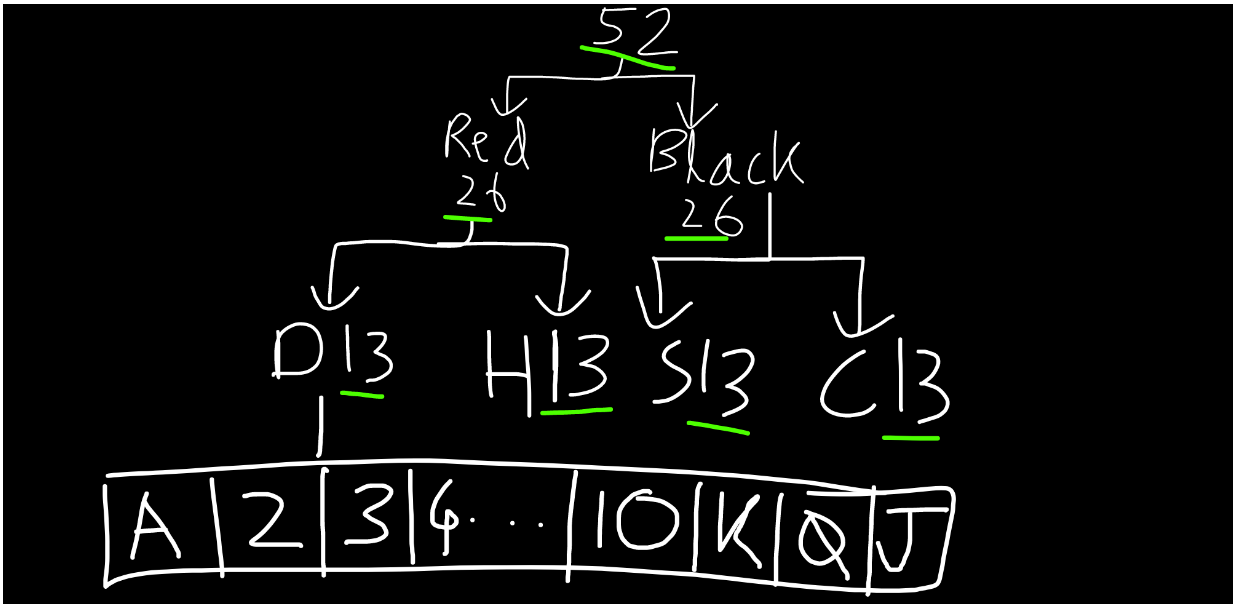
Conditional Probability

- The Probability of happening of `B` given that event `A` has already occured, is said to be the Conditional probability of `B`, denoted by $P(B|A)$.
- $P(B|A) = n(A \cap B) \div n(A) = (n(A \cap B) \div n(S))/n(A) \div n(S) = P(B \cap A)/P(A)$

- $P(B \cap A) = P(B|A).P(A)$ and $P(A \cap B) = P(A).P(A|B)$
- So, **Multiplication Theorem of Probability**: $P(A).P(A|B) = P(B|A).P(A)$
- If `A` & `B` are Independent Events, $P(B|A) = P(A).P(B) \div P(A) = P(B)$

Pack or Deck of Cards

- Total pack: (52), divided into Red (26) and Black (26) cards.
- Red Pack contains Diamonds (13), Hearts (13).
- Black Pack contains Spades (13), Clubs (13).
- Each suit contains:
  - Ace
  - 2,3,4,5,6,7,8,9,10
  - Jack, King, Queen (Face Cards)



Rolling a pair of dice

- The probability that the sum of outcomes to be `k` when a pair of dice is rolled is:
  - $P(k) = (k-1)/36; 2 \leq k \leq 7$
  - $P(k) = (13-k)/36; 8 \leq k \leq 12$
- Example 5: From a pack of regular playing cards, two cards are drawn one after the other without replacement. What is the probability that both cards will be kings?
  - $P(E) = (c_1^4 \div c_1^{52}) * (c_1^3 \div c_1^{51}) = 1/221 = 0.00452$
- Example 6: In a YouTube live class, the probability that a student likes the class is 0.2, and the probability that the student dislikes the class is 0.002. Assuming that a student can either like or dislike the class, or choose not to share his feedback, the probability that a student doesn't share feedback is ____.
  - $P(A \cup B)^c = 1 - P(A \cup B)$

- $P(A \cup B)^c = 1 - P(A) - P(B)$ // $P(A \cup B) = P(A) + P(B)$
- $1 - P(A) - P(B) = 1 - 0.2 - 0.02 = 0.798$

- Example 7: The probability of getting sum equal to `8` when a dice is rolled twice is $5/36 = 0.138$.
- Example 8: Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn at random. If it is known that the number of the card is more than 3, the probability that it is an even number is _____.
  - A: Number is even, B: Number is more than 3
  - $P(A|B) = P(A \cap B)/P(B) = 4/10 \div 7/10 = 4/7 = 0.571$
- Example 9: A dice is thrown twice and the sum of numbers appearing is observed to be `6`. The Conditional probability that `4` has appeared atleast once is _____.
  - A: 4 has appeared atleast once, B: Sum is 6
  - $S = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
  - $P(A|B) = P(A \cap B)/P(B) = 2/5 = 0.4$
- Example 10: A number is drawn at random from 100 numbers $00 - 99$. Let `x` denote the sum of digits, and `y` denote the product of digits of the numbers. The value of P(x=9|y=0) is?
  - $P(y=0): 19, $(00, 01, 02, 03...09, 10, 20, 30...90)$
  - $P(x = 9|y = 0) = 2/19 = 0.1052$
- Example 11: A & B are 2 players rolling a pair of dice on the condition that one who gets the sum 9 first, wins the game. The probability that `A` wins the game, if `B` starts the game is?
  - $P(sum = 9) = p = 4/36 = 1/9$, $S = \{(6,3),(5,4),(4,5),(3,6)\}$
  - $P(sum \neq 9) = q = 8/9$
  - For `A` to win, $P(A) = (q.p) + (q.q.q.p) + (q.q.q.q.q.p) + ...$

    > If `A` wins in 1st attempt, it means `B` started, but didn't get $sum = 6$. Then `A` rolled the dice and got $sum = 6$.
    > If `A` wins in 2nd attempt, it means `B` started, didn't get $sum = 6$, then `A` rolled the dice and didn't get $sum = 6$ either, then `B` rolled the dice, didn't get it either, but then `A` rolled the dice and got $sum = 6$.

  - $P(A) = (q.p) + (q.q.q.p) + (q.q.q.q.q.p) + ... = pq(1 + q^2 + q^4 + q^6) = pq(1 \div 1 - q^2)$ // Geometric Progression
  - $P(A) = pq(1/(1 - q^2)) = 1/9 * 8/9 \div 1 - (8/9)^2 = 0.4705$

Theorem of Total Probability

- If E_1, E_2, E_3 ... E_n, are Mutually Exclusive $(E_i \cap E_j = \emptyset \forall i \neq j)$ and collectively exclusive in a Sample Space S $(E_1 \cup E_2 \cup E_3 ... \cup E_n = S)$ and if `A` is any event withing Sample Space `S`, then $P(A)$ is:

- $A = (A \cap S)$
- $A = A \cap (E_1 \cup E_2 \cup E_3 ... \cup E_n)$
- $A = A \cap (E_1) \cup A \cap (E_2) \cup A \cap (E_3) \cup A \cap (E_4) ... \cup A \cap (E_n)$

  > A∩(E_1), A∩(E_2), etc are also Mutually exclusive

- $P(A) = P(A \cap (E_1) \cup A \cap (E_2) \cup A \cap (E_3) \cup A \cap (E_4) ... \cup A \cap (E_n))$
- $P(A) = P(A \cap (E_1)) + P(A \cap (E_2)) + P(A \cap (E_3)) + ... + P(A \cap (E_n)$
- $P(A) = P(A|E_1).P(E_1) + P(A|E_2).P(E_2) + P(A|E_3).P(E_3) + ... + P(A|E_n).P(E_n)$
- $P(A) = \sum(i = 1 - n)P(E_i).P(A|E_i)$ (i)
- $P(A \cap E_i) = P(E_i).P(A|E_i) = P(A).P(E_i|A)$
- $P(A \cap E_i) = P(E_i).P(A|E_i)/P(A) = P(E_i|A)$
- $P(E_i|A) = P(E_i).P(A|E_i)/\sum(i = 1 - n)P(E_i).P(A|E_i)$ Substituting (i), **Bayes' Theorem**

* In an exam, for a multiple choice question, if it's assumed that the answer is correct, then:
  - P(A)=P(I know the answer).P(Answer is correct|I know the answer) + P(I guessed the answer).P(Answer is correct|I guessed the answer)

Permutation & Combination

* Prerequisite: Formulae
* **Permutation**
  - Example 0: Find the number of Permutations of all the letters of the word 'KAMALUDDIN'.
    - `A` occurs twice
    - `D` occurs twice
    - The rest 8 occur once.
    - $p_r^n = 10! \div (2! * 2! * 1! * 1! * 1! * 1! * 1! * 1! * 1! * 1!) = 907200$
* **Combination**
  - Example 0: If $c_4^n = c_5^n$, then $n = ?$
    - If $c_4^n = c_5^n$, then either $r = s$ (impossible) or $n = r + s = 4 + 5 = 9$ (possible)