Formulae

FRL General

- Number of choices -
  - $A = \{1, 2, 3, 4, 5\}$
  - Let's form a sub-set of A with no constraints. Here, for each of the 5 elements, we may or may not include them in the subset. So, we have 2 choices per element. Total number of choices:
    > 2^5 for 5 elements
    > 2^n for n elements
  - If we had 3 choices per element, Total number of choices would be:
    > 3^5 for 5 elements
    > 3^n for n elements

FRL Reflexive Relations

- Total number of relations: n^2
- Total number of Diagonal elements: n
- Total number of Non-Diagonal elements: Total number of elements in AxA - Total number of diagonal elements = $n^2 - n$
- Total number of reflexive relations: $2^{Non-Diagonal}$ = 2^{n^2-n}
- Total number of non-reflexive relations: $2^{n^2} - 2^{n^2-n}$
- Smallest possible size: n
- Largest possible size: n^2

FRL Irreflexive Relations

- Total number of Non-Diagonal elements: Total number of elements in AxA - Total number of diagonal elements = $n^2 - n$

- All non-reflexive relations are not irreflexive relations.
- Total number of irreflexive relations: 2^{n^2-n}
- Smallest possible size: 0 ($\{\}$)
- Largest possible size: $n^2 - n$

FRL Comparison of relations

> $A = \{1, 2, 3\}$, $AxA = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

Parameter	Reflexive	Irreflexive	Symmetric	Anti-symmetric	Asymı
Cardinality of smallest relation	n	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
Cardinality of largest relation	n^2	$n^2 - n$	n^2	$(n^2 + n)/2$	$(n^2 -$
Number of relations	2^{n^2-n}	2^{n^2-n}	$2^n.2^{(n^2-n)/2)}$	$2^n.3^{(n^2-n)/2}$	$3^{(n^2-n}$

Set Theory

Set

A well-defined unordered collection of distinct elements.

- Unordered: Set {1,2,3,4} is same as {2,3,4,1}.
- Distinct: Set {1,2,3,4} is same as {1,1,2,2,3,4}.

Null/Empty set

A set with no elements, denoted by Φ or {}.

- Cardinality of an empty set is 0.
- {Φ} is not an empty set.
- Φ is present in every set.

Subset

If every element of Set A also exists in Set B, denoted by $\subseteq$.

- **Example**: If A = {1,2,3,4}, A’ = {1,2,3,4,5}, then A $\subseteq$ A’.
- Every set is a subset of itself.
- Trivial subset: A set which contains all elements of A, is if A={1,2,3,4} and A’={1,2,3,4}.
- Proper subset: A set which is not a trivial subset of A, ie the subset can’t be same / can’t have the same length as A.
- If A$\subseteq$B & B$\subseteq$A, then A=B. #Note

Cardinality

Number of elements in a set.

Power Set

If ‘A’ is a finite set, then it is the set of all subsets of ‘A’.

- **Example**: If $A = \{1, 2, 3\},\ P(A) = \{\{\Phi\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

- If cardinality of a set is n, the number of elements in the power set is $2^n = 8$, ie $|P(a)| = |2^n|$
- Write all the sets starting with cardinality 0, then 1, then 2 … and so on.

Cartesian product

Product of all elements of 1 set, with all elements of the other set.

- **Example**: If A={a,b} and B={1,2,3}, then:
  - $AxB = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$ | If an element is (m,n), then m ∈ A & n ∈ B.
  - $BxA = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$
- Total number of products, assuming Cardinality(A)=m & Cardinality(B)=n, is mxn.
- First multiply element 1 of A with element 1 of B, then element 1 of A with element 2 of B … and so on.
- The order needs to be maintained, as is apparent in the example.
- AxB is not same as BxA, unless A=B.

Relations, also check Cartesian Product

If A and B are two sets and AxB is their cartesian product, then any subset of AxB can form a relation from A to B.

- **Example**:
  - $\{(a,1),(a,2)\}$ | Valid
  - $\{(b,1),(b,2),(a,1)\}$ | Valid
  - $\{(1,a),(2,a)\}$ | Not Valid
- Total number of relations: 2^{mxn}.

Reflexive Relation

- Formulae
- A Relation 'R' on a set 'A' is said to be Reflexive if $(x, x) \epsilon R \forall x \epsilon A$
- Every element of a set is related to itself.
- Points to remember:
  i. All elements must be present.
  ii. All of them must be related to themselves.
  iii. After putting in all valid elements, we can put extra elements.
  iv. The difference between a diagonal relation is that a reflexive relation may also contain extra elements.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
  - Possible Reflexive relations:
    - $\{(1,1), (2,2), (3,3)\}$: *Smallest*
    - $\{(1,1), (2,2), (3,3), (2,1)\}$: *We may also put extra elements as long as the original condition has been satisfied.*
    - $\{(1,1), (2,2), (3,3), (2,1), (2,3)\}$
    - $\{(1,1), (2,2), (3,3), (2,1), (2,3), (3,1)\}$
    - $\{(1,1), (2,2), (3,3), (2,1), (2,3), (3,1), (3,2)\}$
    - $\{(1,1), (2,2), (3,3), (2,1), (2,3), (3,1), (3,2), (3,3)\}$: *Largest*
- Example 1: Check for reflexive relation: {(x, y), $x - y$ is an integer}
  - $(2, 3)$ cannot be in an answer, since it doesn't satisfy point 2.
  - $(1, 1)$, $(2, 2)$, $(3, 3)$, etc. will satisfy the equation, as $1 - 1 = 0$, which is an integer. So **it is a reflexive relation**.
- Example 2: Check for reflexive relation: {(x, y), $x - y$ is an odd number}
  - $(1, 1)$, $(2, 2)$, $(3, 3)$, etc. will not satisfy the equation, as $1 - 1 = 0$, which is not an odd number. So **it is not a reflexive relation**.

Irreflexive Relation

- Formulae
- A Relation 'R' on a set 'A' is said to be Irreflexive if $(x, y) \notin R \; \forall \; (x, y) \epsilon A$
- No element of the set should be related to itself.
- Points to remember:
  i. We exclude all diagonal elements, and include the non-diagonal elements.
  ii. **A not reflexive relation is not the same as an Irreflexive relation**. Example:

    - $A = \{1,2,3\}$, $R = \{(1,1),(2,2)\}$
    - Here, R is not a reflexive relation since it does not include (3,3). But, R is not in Irreflexive relation either, because it includes (1,1) & (2,2). So, it is just not a reflexive relation.
- Example 0: $A = \{1,2,3\}$
    - $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
    - $R = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$

Symmetric Relation

- A Relation 'R' is symmetric, if for every $(x,y) \in R,\ (y,x) \in R\,\forall\,(x,y)\epsilon A$.
- If (x,y) is present in a Relation 'R' on a set 'A', then (y,x) must also be present in the set.
- We first check if (x,y) is present. We check for (y,x) only if (x,y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1,2,3\}$
    - $AxA = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$
    - $\{(1,2),(2,1),(1,3),(3,1),(1,1)\}$: for $(1,1)$, it's Symmetric pair will also be $(1,1)$, which is a duplicate.
    - $\{(1,2),(1,3),(1,1)\}$: This is NOT a symmetric relation, because $(3,1)$ is not present for $(1,3)$.
- Example 1: Check: 'perpendicular-to' on a set of all lines.

  > Solution: If a line $L1 \perp L2$, then obviously $L2 \perp L1$. So, **this is a symmetric relation**.

- Example 2: Check: 'brother of' of all humans on Earth.

  > Solution: If X is a brother of Y, Y may be a sister of X. So, **it's not symmetric**.

- Example 3: Check: 'complement of', on a set of numbers.

  > Solution: If `X` is a complement of `Y`, then `Y` is also a complement of `X`. So, **the relation is symmetric**.

Anti-symmetric Relation

- A Relation 'R' is anti-symmetric, if for every $(x,y) \in R,\ (y,x) \in R\,\forall\,(x,y)\epsilon A, only\ if\ (x = y)$.

- If (x, y) is present in a Relation 'R' on a set 'A', then (y, x) should only also be present in the relations if $x = y$. Otherwise, it must not be present.
- We first check if (x, y) is present. We check for (y, x) only if (x, y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
  - $\{(2, 1), (3, 1), (1, 1)\}$: TRUE, for $(2, 1)$, $(1, 2)$ must not be present since $(2 \neq 1)$. For $(1, 1)$, it's Symmetric pair is $(1, 1)$, which is fine since $(1 = 1)$.
  - $\{(1, 2), (2, 1), (1, 3), (1, 1)\}$: FALSE, because $(2, 1)$ is present.

Asymmetric Relation

- A Relation 'R' is asymmetric, if for every $(x, y) \in R,\ (y, x) \notin R \, \forall \, (x, y) \epsilon A$.
- If (x, y) is present in a Relation 'R' on a set 'A', then (y, x) should only also be present in the relationsif $x = y$. Otherwise, it must not be present.
- We first check if (x, y) is present. We check for (y, x) only if (x, y) is present. It is not necessary for all possible relations to be present.
- Example 0: $A = \{1, 2, 3\}$
  - $AxA = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
  - $\{(2, 1), (3, 1), (2, 3)\}$: TRUE, for all (x, y), (y, x) is not present in the relation.
  - $\{(2, 2), (3, 1), (2, 3), (1, 1)\}$: FALSE, because $(1, 1)$ is present, and it's duplicate ie $(1, 1)$ should not be present.
- Every Asymmetric relation is anti-symmetric. An Asymmetric relation is more restrictive than an anti-symmetric relation, since an asymmetric relation does not allow (y, x) (for every (x, y)) to be present even if $(x = y)$.
- Every Anti-symmetric relations is not asymmetric. An anti-symmetric relation allows (y, x) to be present if $(x = y)$.

Transitive Relation

- A Relation 'R' is Transitive, if for every $(x, y) \in R \ \& \ (y, z) \in R$, $(x, z) \in R$ ∀ $(x, y) \epsilon A$.
- If (x, y) & (y, z) is present in a Relation 'R' on a set 'A', then (x, z) should also be present in the relation.

- If any or x,y or z is not present in the relation, then the relation is transitive by default.
- Example 0: $A = \{1, 2, 3\}$
  - $\{(1, 1), (2, 2)\}$: TRUE, x,y & z are not present together.
  - ${Φ}: TRUE, none of the elements are present.
  - $\{(1, 2), (2, 3)\}$: FALSE, since (x, z) ie $(1, 3)$ is not present.
    > x = 1, y = 2, z = 3

  - $\{(1, 1), (1, 2), (2, 1)\}$: TRUE, since in both cases the conditions are matching.
    > x = 1, y = 1, z = 2 z = 1, y = 2, z = 1

Equivalence Relation

- A Relation 'R' is equivalence, if it is `reflexive` , `symmetric` or `transitive` .
- Example 0: $A = \{1, 2, 3\}$
  - $\{(1, 1), (2, 2), (3, 3)\}$: TRUE
  - $\{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2)\}$: TRUE
  - $\{(1, 1), (2, 2), (3, 3), (3, 2), (1, 3)\}$: FALSE, because $(2, 3)$ is not present for $(3, 2)$, so it's not symmetric.
  - $\{\ \}$: FALSE, since $(1, 1), (2, 2), (3, 3)$ is not present, so it is not a reflexive relation.

Partially Ordered Set

- A Relation 'R' is said to be a partial-ordering relation if R is `reflexive` , `anti-symmetric` as well as `transitive` .
- Partially odered set: A set 'A' with Partial-Ordering relation 'R' defined on 'A' is called POSET, defined by [A;R]
- Example 0: $A = \{1, 2, 3\}$
  - $\{(1, 1), (2, 2), (3, 3)\}$: TRUE
  - $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$: TRUE
  - [A,$\leq$] is a POSET: TRUE,
    > $(1 \leq 1)$ is true, so relation is reflexive.
    > $(1 \leq 1)$ is true, and for $(1.5 \leq 2.5)$, $(2.5 \leq 1, 5)$ is not true. So, relation

> is anti-symmetric.
> $(1 \leq 2.5)$, $(2.5 \leq 3)$, then $(1 \leq 3)$ is also true. So, the relation is transitive.

Totally Ordered Set

- A POSET {A,R} is called a totally ordered set if every pair of elements in set `A` are comparable, ie, if:
  > $aRb \; or \; bRa \; \forall \; (a, b) \in A$
- The set has to mandatorily be a Partially Ordered Set. In other words, every TOS is a POSET.
- Example 0: $(A, \div) \in R \; \forall (A \in I)$, I=Integers: FALSE.
  > A = {1,2,3,4,5,6,7,8…∞}
  > Reflexive: $(1, 1), (2, 2), (3, 3) \in R$
  > Anti-symmetric: $(4, 2) \in R$ but $(2, 4) \notin R$
  > Transitive: If $(8, 4) \in R \& (4, 2) \in R$, then $(8, 2) \in R$.
  > TOS: $(2, 3), (4, 8)$, etc. are not comparable when the domain consists only of positive integers.
- Example 1: $(A, \leq) \in R \; \forall (A \in R)$, I=Real numbers: TRUE.
- Example 2: $(A, \in) \in R \; \forall A \in \{\emptyset\}, \{\{a\}, \{b\}, \{a, b\}\}$: FALSE.
  > $\{a\}$ is not comparable with $\{b\}$.

Comparison of all relations

See comparison of relations in section `formulae` .

- Explanation:
  > $A = \{1, 2, 3\}$
  > $AxA = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (2, 2)\}$
  - Reflexive:
    - Smallest: $\{(1, 1), (2, 2), (3, 3)\}$ | Cardinality: n
    - Largest: $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ | Cardinality: n^2

        - Number of relations: For $\{(1,1),(2,2),(3,3)\}$ ie `n` elements, we don't have a choice. For the rest $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ ie (n^2-n) elements, we have 2 choices per pair. So, the total number of choices: $2^{n^2}-2^n=2^{n^2-n}$
    - Irreflexive:
        - Smallest: $\{\}$ | Cardinality: $\varnothing$
        - Largest: $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ | Cardinality: n^2-n
        - Number of relations: 2^{n^2-n}
    - Symmetric:
        - Smallest: $\{\}$ | Cardinality: $\varnothing$
        - Largest: $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ | Cardinality: n^2
        - Number of relations: For $\{(1,1),(2,2),(3,3)\}$ ie `n` elements, we have 2 choices per pair. For the rest $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ ie (n^2-n) elements, if (1,2) is present, (2,1) also needs to be present. So, we have 2 choices per respective double-pair. Total number of choices: $2^n.2^{(n^2-n)/2}$
    - Anti-symmetric:
        - Smallest: $\{\}$ | Cardinality: $\varnothing$
        - Largest: $\{(1,1),(2,2),(3,3),(1,2),(1,3),(2,3)\}$ OR $\{(1,1),(2,2),(3,3),(2,1),(3,1),(3,2)\}$ | Cardinality: $n+[(n^2-n)/2]=(2n+n^2-n)/2=(n^2+n)/2$
        - Number of relations: For $\{(1,1),(2,2),(3,3)\}$ ie `n` elements, we have 2 choices per pair. For the rest $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ ie (n^2-n) elements, if (1,2) is present, (2,1) must not be present and vice versa. So, we can either take (1,2) or (2,1) exclusively, or we can exclude both. We have 3 choices per double-pair. Total number of choices: $2^n.3^{(n^2-n)/2}$
    - Asymmetric:
        - Smallest: $\{\}$ | Cardinality: $\varnothing$
        - Largest: $\{(1,2),(1,3),(2,3)\}$ OR $\{(2,1),(3,1),(3,2)\}$ | Cardinality: $(n^2-n)/2$

    - Number of relations: For $\{(1,1),(2,2),(3,3)\}$ ie `n` elements, we cannot take them. For the rest $\{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$ ie (n^2-n) elements, if (1,2) is present, (2,1) must not be present and vice versa. So, we can either take (1,2) or (2,1) exclusively, or we can exclude both. We have 3 choices per double-pair. Total number of choices: $3^{(n^2-n)/2}$
  - Transitive:
    - Smallest: $\{\}$ | Cardinality: $\varnothing$
    - Largest: $\{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(2,2)\}$ | Cardinality: n^2