1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

↳ not coprime to 100

↓ divide by 2, 5 (A: 2, B: 5)

$100 = 2 * 5 * 2 * 5$

divide by 2 : 49

divide by $2*5$: 9

divide by 5 : 19

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 49 + 19 - 9 = 59$

$99 - 59 = 40$

2 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

two grope A : n

B : n

2 all from A $\binom{n}{2}$

2 all from B $\binom{n}{2}$

1 from A and 1 from B. n^2

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

$\binom{n}{k} \rightarrow$ random select a people

if not inclue a : $\binom{n-1}{k}$

include a $\binom{n-1}{k-1}$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

first choose a leader : n

$\downarrow$ other people $(n-1)$ choose or not choose : 2^{n-1}

rHS : $n * 2^{n-1}$

LHS : choose k people ; from k people choose a leader

$k\binom{n}{k} \rightarrow \sum_{k=1}^{n} k\binom{n}{k}$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

RHS choose j people → $\binom{n}{j}$

→ other people choose or not choose 2^{n-j}

LHS: choose k people → k people choose j people

$\binom{n}{k}$ $\binom{k}{j}$

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j}$$

3 Farmer's Market

Note 10

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

$\binom{k+1}{1}$

(b) There are pumpkins, apples, oranges, and pears at the market.

$\binom{k+3}{3}$

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

n

$\binom{k+n-1}{n-1} - n$

4 The Count

Note 10

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

one : 10

two : 10 + 9 + 8 + . . . + 1

three :

7 → 10 → digit, 9 interval

9 + 7 = 16

$$\binom{16}{9}$$

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

9, 8, 7, 6, 5, 4, 3, 2, 1, 0

7 digit

$$\binom{10}{7}$$

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

11

first start of five consecutive 0's →

A: 1 0 0 0 0 0 1

redundant

5 5

$6 * 2^5$

$2^5 + 5 * 2^4 = 112$