Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Planting Trees

This problem will guide you through the process of writing a dynamic programming algorithm.
You have a garden and want to plant some apple trees in your garden, so that they produce as many apples as possible. There are n adjacent spots numbered 1 to n in your garden where you can place a tree. Based on the quality of the soil in each spot, you know that if you plant a tree in the ith spot, it will produce exactly x_i apples. However, each tree needs space to grow, so if you place a tree in the ith spot, you can't place a tree in spots $i-1$ or $i+1$. What is the maximum number of apples you can produce in your garden?

(a) Give an example of an input for which:

- Starting from either the first or second spot and then picking every other spot (e.g. either planting the trees in spots $1, 3, 5\ldots$ or in spots $2, 4, 6\ldots$) does not produce an optimal solution.
- The following algorithm does not produce an optimal solution: While it is possible to plant another tree, plant a tree in the spot where we are allowed to plant a tree with the largest x_i value.

(b) To solve this problem, we'll thinking about solving the following, more general problem: "What is the maximum number of apples that can be produced using only spots 1 to i?". Let $f(i)$ denote the answer to this question for any i. Define $f(0) = 0$, as when we have no spots, we can't plant any trees. What is $f(1)$? What is $f(2)$?

(c) Suppose you know that the best way to plant trees using only spots 1 to i does not place a tree in spot i. In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$. (Hint: What spots are we left with? What is the best way to plant trees in these spots?)

(d) Suppose you know that the best way to plant trees using only spots 1 to i places a tree in spot i. In this case, express $f(i)$ in terms of x_i and $f(j)$ for $j < i$.

(e) Describe a linear-time algorithm to compute the maximum number of apples you can produce. (Hint: Compute $f(i)$ for every i. You should be able to combine your results from the previous two parts to perform each computation in $O(1)$ time).

2 String Shuffling

Let x, y, and z be strings. We want to know if z can be obtained only from x and y by interleaving the characters from x and y such that the characters in x appear in order and the characters in y appear in order. For example, if $x =$ **efficient** and $y =$ **ALGORITHM**, then it is true for $z =$ **effALGiORciIenTHMt**, but false for $z =$ **efficientALGORITHMS** (extra characters), $z =$ **effALGORITHMicien** (missing the final t), and $z =$ **effOALGRicieITHMnt** (out of order). How can we answer this query efficiently? Your answer must be able to efficiently deal with strings with lots of overlap, such as $x =$ **aaaaaaaaaab** and $y =$ **aaaaaaaac**.

1. Design an efficient algorithm to solve the above problem and state its runtime.

2. Consider an iterative implementation of our DP algorithm in part (a). Naively if we want to keep track of every solved sub-problem, this requires $O(|x||y|)$ space (double check to see if you understand why this is the case). How can we reduce the amount of space our algorithm uses?

3 Complementary Connected Components

Given a graph G on vertex set $\{1, \ldots, n\}$ and m edges, give an $\widetilde{O}(m+n)$ time algorithm to output the number of connected components in the *complement* of G.

4 Greedy Cards

Ning and Evan are playing a game, where there are n cards in a line. The cards are all face-up (so they can both see all cards in the line) and numbered 2–9. Ning and Evan take turns. Whoever's turn it is can take one card from either the right end or the left end of the line. The goal for each player is to maximize the sum of the cards they've collected.

(a) Ning decides to use a greedy strategy: "on my turn, I will take the larger of the two cards available to me". Show a small counterexample ($n \leq 5$) where Ning will lose if he plays this greedy strategy, assuming Ning goes first and Evan plays optimally, but he could have won if he had played optimally.

(b) Evan decides to use dynamic programming to find an algorithm to maximize his score, assuming he is playing against Ning and Ning is using the greedy strategy from part (a). Help Evan develop the dynamic programming solution by providing an algorithm with its runtime and space complexity.