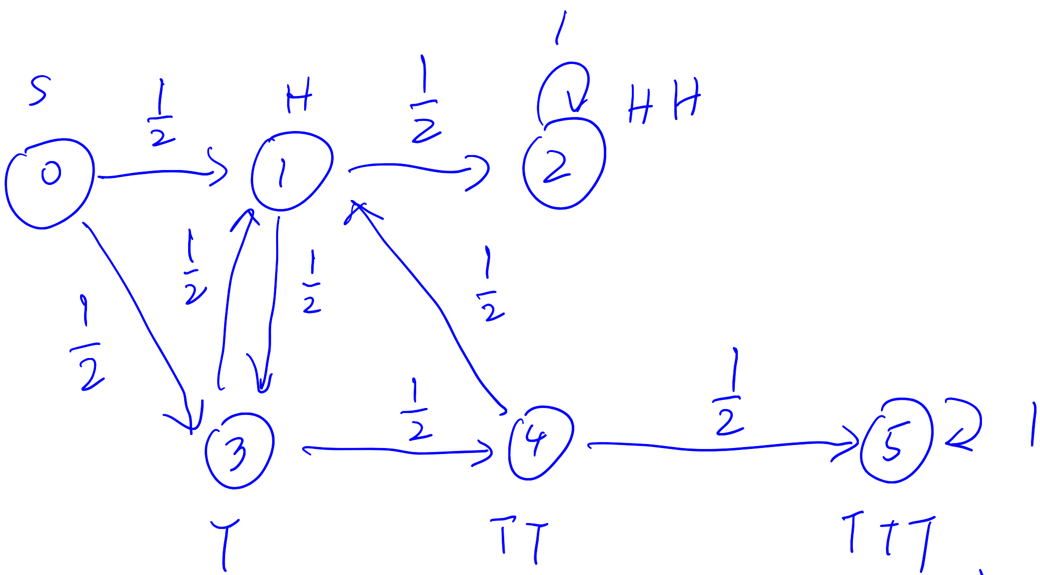
CS 70 Spring 2024 — Discrete Mathematics and Probability Theory — Seshia, Sinclair — HW 14

1 Rahil's Dilemma

Note 22

Youngmin and Rahil decided to play a game: A fair coin is flipped until either the last two flips were all heads - then Youngmin wins, or the last three flips were all tails - then Rahil wins. Compute the probability that Rahil wins.



$\alpha(0) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(3)$

$\alpha(1) = \frac{1}{2}\alpha(2) + \frac{1}{2}\alpha(3)$

$\alpha(2) = 0$

$\alpha(3) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(4)$

$\alpha(4) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(5)$

$\alpha(5) = 1$

$\alpha(0) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(3)$

$\alpha(1) = \frac{1}{2}\alpha(3)$

$\alpha(2) = 0$

$\alpha(3) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(4)$

$\alpha(4) = \frac{1}{2}\alpha(1) + \frac{1}{2}$

$2\alpha(1) = \frac{1}{2}\alpha(1) + \frac{1}{2}\alpha(4)$

$\frac{3}{2}\alpha(1) = 1\ \frac{1}{2}\alpha(4)$

$3\alpha(1) = \frac{1}{2}\alpha(1) + \frac{1}{2}$ $\qquad \alpha(1) = \frac{1}{5}$

$\alpha(0) = \frac{3}{10}$

$$\begin{cases} \alpha(1) = \frac{1}{5} \\ \alpha(2) = 0 \\ \alpha(3) = \frac{2}{5} \\ \alpha(4) = \frac{3}{5} \\ \alpha(5) = 1 \end{cases}$$

$\frac{3}{10}$

2 A Bit of Everything

Suppose that $X_0, X_1, \ldots$ is a Markov chain with finite state space $S = \{1, 2, \ldots, n\}$, where $n > 2$, and transition matrix P. Suppose further that

$$P(1,i) = \frac{1}{n} \quad \text{for all states } i \text{ and}$$
$$P(j, j-1) = 1 \quad \text{for all states } j \neq 1,$$

with $P(i,j) = 0$ everywhere else.

(a) Prove that this Markov chain is irreducible and aperiodic.

(b) Suppose you start at state 1. What is the distribution of T, where T is the number of transitions until you leave state 1 for the first time?

(c) Again starting from state 1, what is the expected number of transitions until you reach state n for the first time?

(d) Again starting from state 1, what is the probability you reach state n before you reach state 2?

(e) Compute the stationary distribution of this Markov chain.

(a) from 1 can go every vertex

from a vertex x other than 1, go to $x-1$ until one, then go to n, $\to$

$p(1,1) = \frac{1}{n} \to$ aperiodic

(b) Geometric $\left(\frac{n-1}{n}\right)$

(c) $\beta(n) = 0$

$\beta(1) = 1 + \frac{1}{n}(\beta(1) + \ldots + \beta(n))$

$\beta(2) = 1 + \beta(1) = 1 + \beta(1)$

$\beta(3) = 1 + \beta(2) = 2 + \beta(1)$

$\beta(n-1) = n-2 + \beta(1)$

$\frac{n-1}{n}\beta(1) = 1 + \frac{1}{n}(1 + 2 + \ldots + n-2 + (n-2)\beta(1))$

$= 1 + \frac{1}{n}\left(\frac{(n-2)(n-1)}{2} + (n-2)\beta(1)\right)$

$\frac{1}{n}\beta(1) = 1 + \frac{1}{n}\left(\frac{(n-2)(n-1)}{2}\right)$

$\beta(1) = n + \frac{(n-2)(n-1)}{2}$

(d) $\alpha(n) = 1$

$\alpha(2) = 0$

$$\alpha(1) = \frac{1}{n}\left(\alpha(1) + \overset{0}{\alpha(2)} + \dots + \overset{1}{\alpha(n)}\right)$$

$\alpha(2) = 0$ $\qquad$ $n\alpha(1) = \alpha(1) + 1$

$\alpha(3) = \alpha(2)$ $\qquad$ $\alpha(1) = \frac{1}{n-1}$

$\vdots$

$\alpha(n-1) = \alpha(n-2)$

(e) $D = n + n - 1 = 2n-1$ $\qquad$ not Random walk

$\cancel{\pi(1) = \frac{n}{2n-1}}$ $\qquad$ $\pi(2 \dots n) = \frac{1}{2n-1}$

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$$\pi(i) = \frac{1}{n}\pi(1) + \pi(i+1) \quad (i \neq n)$$

$$\pi(n) = \frac{1}{n}\pi(1)$$

$$\pi(n-1) = \frac{1}{n}\pi(1) + \pi(n)$$

$$\pi(n-2) = \frac{1}{n}\pi(1) + \frac{1}{n}\pi(1) + \pi(n)$$

$$\pi(i) = \frac{n-i}{n}\pi(1) + \pi(n) = \frac{n-i+1}{n}\pi(1)$$

$$\pi(1)\left[\frac{n}{n} + \frac{n-1}{n} + \dots + \frac{1}{n}\right] = 1$$

$$\pi(1) * \frac{\frac{n(n+1)}{2}}{n} \qquad \pi(1) * \frac{n+1}{2} = 1 \qquad \pi(1) = \frac{2}{n+1}$$

$$\pi = \frac{2}{n(n+1)}\left(n, n-1, \dots\dots, 1\right)$$

3 Playing Blackjack

Note 22

Suppose you start with \$1, and at each turn, you win \$1 with probability p, or lose \$1 with probability $1-p$. You will continually play games of Blackjack until you either lose all your money, or you have a total of n dollars.

(a) Formulate this problem as a Markov chain.

(b) Let $\alpha(i)$ denote the probability that you end the game with n dollars, given that you started with i dollars.

Notice that for $0 < i < n$, we can write $\alpha(i+1) - \alpha(i) = k(\alpha(i) - \alpha(i-1))$. Find k.

(c) Using part (b), find $\alpha(i)$, where $0 \le i \le n$. (You will need to split into two cases: $p = \frac{1}{2}$ or $p \neq \frac{1}{2}$.)

Hint: Try to apply part (b) iteratively, and look at a telescoping sum to write $\alpha(i)$ in terms of $\alpha(1)$. The formula for the sum of a finite geometric series may be helpful when looking at the case where $p \neq \frac{1}{2}$:

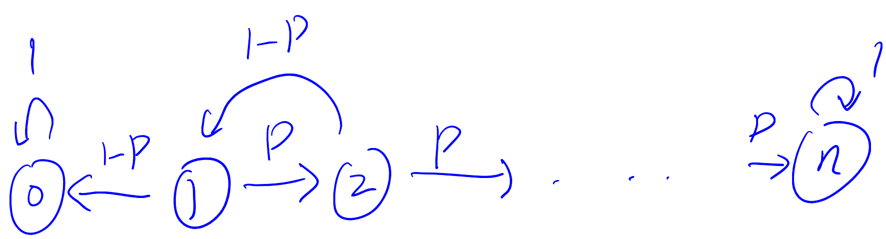
$$\sum_{k=0}^{m} a^k = \frac{1-a^{m+1}}{1-a}.$$

Lastly, it may help to use the value of $\alpha(n)$ to find $\alpha(1)$ for the last few steps of the calculation.

(d) As $n \to \infty$, what happens to the probability of ending the game with n dollars, given that you start with i dollars, with the following values of p?

(i) $p > \frac{1}{2}$
(ii) $p = \frac{1}{2}$
(iii) $p < \frac{1}{2}$

(a)



$\alpha(n) = 1$

(b) $\alpha(i+1) = (1-p)\,\underline{\alpha(i)} + p\,\alpha(i+2)$

$\alpha(i) = (1-p)\,\alpha(i-1) + p\,\alpha(i+1)$

$\alpha(2) = (1-p)\alpha_1 + p\,\alpha_3$

$\alpha(1) = (1-p)\,\alpha(0) + \underline{p\,\alpha(2)}$

$\alpha(0) = 0$

start at $\underline{i}$

$$\alpha(i+1)-\alpha(i) = (1-p)\left[\alpha(i)-\alpha(i-1)\right] + p(\alpha(i+2)-\alpha(i+1))$$

$$\alpha(i) = (1-p)\alpha(i-1) + p\,\alpha(i+1)$$

rewrite it

$$\alpha(i)p + (1-p)\alpha(i)$$

$$(1-p)(\alpha(i)-\alpha(i-1)) = p(\alpha(i+1)-\alpha(i))$$

$$\alpha(i+1)-\alpha(i) = \frac{1-p}{p}(\alpha(i)-\alpha(i-1))$$

$$k = \frac{1-p}{p}$$

$$i-k=0$$

(C.) $\alpha(0)=0$ $\qquad$ $\alpha(n)=1$

$$\alpha(i+1)-\alpha(i) = \frac{1-p}{p} * \frac{1-p}{p}(\alpha(i-1)-\alpha(i-2))$$

$$= \left(\frac{1-p}{p}\right)^{i}(\alpha(1)-\alpha(0))$$

$$\alpha(i)-\alpha(i-1) = \left(\frac{1-p}{p}\right)^{i-1}(\alpha(1)-\alpha(0))$$

$$\vdots$$

$$\alpha(1)-\alpha(0) = (\alpha(1)-\alpha(0))$$

$$\alpha(i) = \alpha(1) * \left[1 + \left(\frac{1-p}{p}\right) + \ldots \left(\frac{1-p}{p}\right)^{i-1}\right]$$

$p=\frac{1}{2}$ $\qquad$ $\alpha(i) = i\alpha(1)$ $\qquad$ $p \neq \frac{1}{2}$

$$\frac{1-\left(\frac{1-p}{p}\right)^{i}}{1-\frac{1-p}{p}}\,\alpha(1)$$

$\alpha(1) = (1-p)\alpha(0) + p\,\alpha(2)$

$\alpha(n) = n\,\alpha(1)$ $\quad p = \frac{1}{2}$

$\alpha(1) = \frac{1}{n}$ $\quad \alpha(i) = \frac{i}{n}$

$p \neq \frac{1}{2}$

$$\frac{1-\left(\frac{1-p}{p}\right)^n}{1-\frac{1-p}{p}}\,\alpha(1) = 1$$

$$\alpha(1) = \frac{1-\frac{1-p}{p}}{1-\left(\frac{1-p}{p}\right)^n}$$

very good

$$\alpha(i) = \frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^n}$$

(3) $p = \frac{1}{2}$ $\quad \alpha(i) = \frac{i}{n}$ $\quad \lim_{n\to\infty} \frac{i}{n} = 0$.

$p > \frac{1}{2}$ $\quad \frac{1-p}{p} = \left(\frac{1}{p}-1\right)^n$ $\quad \alpha(i) = 1-\left(\frac{1-p}{p}\right)^i$

$\frac{1}{p} \leq 2$ $\quad \frac{1}{p}-1 \leq 1$

$p < \frac{1}{2}$ $\quad \alpha(i) \to 0$ $\quad \alpha$

$\frac{1}{p} > 2$