1 Continuous Intro

Note 21

(a) Is

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$\to F(x) = \int_{-\infty}^{x} f(z)\,dz$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1$

$f'(x) = \frac{dF(x)}{dx} = 2$

$\int_{-\infty}^{+\infty} f'(x)\,dx = \int_0^1 2\,dx = 2$

(b) Calculate the PDF $f_X(x)$, along with $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ if the CDF of X is

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ \frac{x}{\ell}, & 0 \le x \le \ell, \\ 1, & x \ge \ell \end{cases}$$

$Var(X) = E(X^2) - (E(X))^2$

$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 & x \le 0 \\ \frac{1}{l} & 0 \le x < l \\ 0 & x \ge l \end{cases}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$

$= \int_0^l x^2 * \frac{1}{l}\,dx$

$= \frac{x^3}{3l}\Big|_0^l = \frac{l^2}{3}$

$E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^l \frac{x}{l}\,dx = \frac{x^2}{2l}\Big|_0^l = \frac{l}{2}$

$Var(X) = \frac{l^2}{3} - \frac{l^2}{4} = \frac{l^2}{12}$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

What is their joint distribution? (Hint: for parts (c) and (d), we can use independence in much the same way that we did in discrete probability)

$f(x,y) = f_X(x) * f_Y(y)$

$= \begin{cases} 2x & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$E(x) = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$ $\quad E(y) = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

(d) Calculate $\mathbb{E}[XY]$ for the X and Y in part (c).

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy \cdot 2x\,dx\,dy = \int_0^1 y*\left(\frac{2}{3}x^3\Big|_0^1\right)dy$$

$$= \int_0^1 \frac{2}{3}y\,dy = \frac{1}{3}y^2\Big|_0^1 = \frac{1}{3}$$

$E(X)*E(Y)$

$= \frac{2}{3}*\frac{1}{2} = \frac{1}{3}$

2 Darts Again

Note 21

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim (but his throws may land outside of the dartboard); the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}[X < x]$.]

$P[Y]$

$$f_Y(y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x) =$

$$F_X(x) = \frac{\pi x^2}{\pi * 10^2} = \frac{x^2}{100} \quad (0 \leq x \leq 10)$$

$$f(x,y) = f_X(x) * f_Y(y)$$

$$f_X(x) = \frac{x}{50} \quad (0 \leq x < 10)$$

$$P[X<Y] = \cancel{\int_x^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy} = \int_0^{10} P[Y>X \mid X=x]\,f_X(x)\,dx$$

$$= \int_0^{10} P[Y>x]\,f_X(x)\,dx$$

$$-e^{-\frac{y}{2}}\Big|_x^{+\infty}$$

$$e^{-\frac{x}{2}}$$

$$= \int_x^{+\infty}\int_0^{10} \frac{1}{2}e^{-\frac{y}{2}} * \frac{x}{50}\,dx\,dy$$

simply

$$= \int_0^{10}\int_x^{+\infty} \frac{1}{2}e^{-\frac{y}{2}}\,dy * \frac{x}{50}\,dx = \int_0^{10} e^{-\frac{x}{2}}\,\frac{x}{50}\,dx$$

3 Lunch Meeting

Note 21

Alice and Bob agree to try to meet for lunch between 12 PM and 1 PM at their favorite sushi restaurant. Being extremely busy, they are unable to specify their arrival times exactly, and can say only that each of them will arrive (independently) at a time that is uniformly distributed within the hour. In order to avoid wasting precious time, if the other person is not there when they arrive they agree to wait exactly fifteen minutes before leaving.

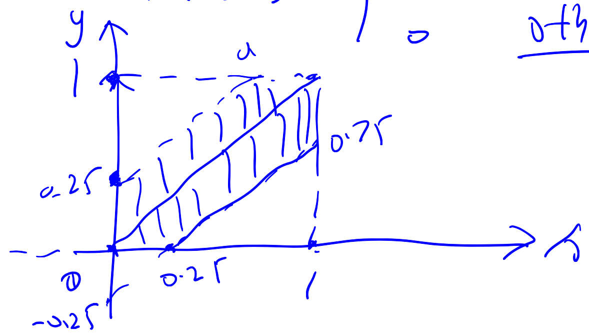
(a) Provide a sketch of the joint distribution of the arrival times of Alice and Bob. For which region of the graph will Alice and Bob actually meet?

0 ~ 1

$F_X(x) = \frac{x}{1} = x \Rightarrow x \in (0, 1)$

$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{others} \end{cases}$

$f_Y(Y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{others} \end{cases}$



A meat

$P[A] = P[|X - Y| \le 0.25]$

(b) Based on your sketch, what is the probability that they will actually meet for lunch?

$f_X(x, y) = f_X(x) * f_Y(y) = \begin{cases} 1 & 0 \le x \le 1 \quad 0 \le y \le 1 \\ 0 & 0 \end{cases}$

$x > y \quad x - y \le 0.25$

$x < y \quad y - x \le 0.25$

(b) $1 \times 1 - \frac{3}{4} * \frac{3}{4} * \frac{1}{2} \times 2$

$= 1 - \frac{9}{16} = \frac{7}{16}$