1 Probabilistic Bounds

Note 17

A random variable X has variance $\mathrm{Var}(X) = 9$ and expectation $\mathbb{E}[X] = 2$. Furthermore, the value of X is never greater than 10. Given this information, provide either a proof or a counterexample for the following statements.

(a) $\mathbb{E}[X^2] = 13$.

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = E[X^2] - 4$

$E[X^2] = 9 + 4 = 13$

(b) $\mathbb{P}[X = 2] > 0$.

$P[X=2] = 0$

$P[X=-3] = \frac{1}{3}$ $P[X=4] = \frac{1}{4}$

$P[X=0] = \frac{1}{4}$ $P[X=6] = \frac{1}{6}$

(c) $\mathbb{P}[X \geq 2] = \mathbb{P}[X \leq 2]$.

$P[X \geq 2] = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$

$P[X \leq 2] = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

(d) $\mathbb{P}[X \leq 1] \leq 8/9$.

? $Y = 10 - X$

$P[10 - X \geq a] = P(Y \geq a) \leq \frac{E(Y)}{a} = \frac{8}{a}$

$\rightarrow a = 9 \rightarrow P[X \leq 1] \leq \frac{8}{9}$

(e) $\mathbb{P}[X \geq 6] \leq 9/16$.

$P[|X-2| \geq c] \leq \frac{9}{c^2}$

$c = 4$

$P[X \geq 6 \cup X \leq -2] \leq \frac{9}{16}$

$\downarrow$

$P[X \geq 6) \leq \frac{9}{16}$

2 Vegas

Note 17

On the planet Vegas, everyone carries a coin. Many people are honest and carry a fair coin (heads on one side and tails on the other), but a fraction p of them cheat and carry a trick coin with heads on both sides. You want to estimate p with the following experiment: you pick a random sample of n people and ask each one to flip their coin. Assume that each person is independently likely to carry a fair or a trick coin.

(a) Let X be the proportion of coin flips which are heads. Find $\mathbb{E}[X]$.

$* \frac{1}{n}$

$$E[X] = E[\sum_{i=1}^{n} X_i] = \frac{(1+p)}{2} * n * \frac{1}{n} = \frac{p+1}{2}$$

$X_i \to i$ people are tail

$$P[X_i = 1] = (1-p) \times \frac{1}{2} + p = \frac{1+p}{2}$$

(b) Given the results of your experiment, how should you estimate p? (*Hint:* Construct an unbiased estimator for p using part (a). Recall that $\hat{p}$ is an unbiased estimator if $\mathbb{E}[\hat{p}] = p$.)

$$2E[X] - 1 = p$$

$$p = E(2X - 1)$$

$$\hat{p} = 2X - 1$$

(c) How many people do you need to ask to be 95% sure that your answer is off by at most 0.05?

$$? \to P[|\hat{p} - p| \leq 0.05] \geq 0.95$$

$$P[|\hat{p} - p| > 0.05] \leq 0.05$$

$$\leq P[|\hat{p} - p| \geq 0.05] \leq \frac{Var(\hat{p})}{0.05^2}$$

$$Var(\hat{p}) \leq 0.05^3$$

$$Var(\hat{p}) = Var(2X - 1) = 4 Var(X) = \frac{4}{n^2} Var(\sum_{i=1}^{n} X_i) = \frac{4}{n} Var(X_1)$$

$$Var(X_1) = p(1-p) \leq \frac{1}{4}$$

$$\leq \frac{4}{n} \times \frac{1}{4} < \frac{1}{n}$$

$$n \geq \frac{1}{0.05^3} = 8000$$

3 Working with the Law of Large Numbers

Note 17

(a) A fair coin is tossed multiple times and you win a prize if there are more than 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

⇓ 10 tosses

more tosses, more close to 50%

(b) A fair coin is tossed multiple times and you win a prize if there are more than 40% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

⇓

(c) A fair coin is tossed multiple times and you win a prize if there are between 40% and 60% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.

(d) A fair coin is tossed multiple times and you win a prize if there are exactly 50% heads. Which number of tosses would you prefer: 10 tosses or 100 tosses? Explain.