CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair **DIS 1A**

1 Natural Induction on Inequality

Note 3

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

Base step: $n=0$ $(1+x)^n = 1 \geq 1+nx = 0$

Induction hypothesis: $\forall k \in N, k \geq 0$, $(1+x)^k \geq 1+kx$

Inductive step: $n=k+1$, $(1+x)^{k+1} = (1+x)\underline{(1+x)^k}$ target: $1+(k+1)x$

$\geq (1+x)(1+kx)$

$= \underline{1+kx+x} + kx^2 = 1+(k+1)x + kx^2 \geq 1+(k+1)x$.

proof. done.

2 Make It Stronger

Note 3

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2$ target: $3^{2^{n+1}}$.

$= \underline{3 * 3^{2^{n+1}}}$ can not get the target

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

Base case: $n=1$, $a_1 = 1 \leq 3^{(2^n)} \underline{= 9}$.

Induction hypothesis: $a_n \leq 3^{(2^n-1)}$

Inductive step: $a_{n+1} = 3a_n^2 \leq 3 * (3^{(2^n-1)})^2 = 3 * 3^{(2^{n+1}-2)}$

$= \frac{1}{3} \times 3^{2^{n+1}} \leq 3^{2^{n+1}}$ proof done

(c) Why does the hypothesis in part (b) imply the overall claim?

The induction in part (b) is more strength

3 Binary Numbers

Note 3

Prove that every positive integer n can be written in binary. In other words, prove that for any positive integer n, we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

for some $k \in \mathbb{N}$ and $c_i \in \{0,1\}$ for all $i \le k$.

Base case: $n=1$ $C_0 = 1$, $k=0$

Induction hypothesis: $\forall n \geq 1$, the proposition is true

Induction step: $n+1 = C_k * 2^k + C_{k-1} \cdot 2^{k-1} + \ldots\ldots + C_0 \cdot 2^0 + 1$

if $C_0 = 0$, then $C_0' = 1$, will be the answer.

else, find the minimum i, let $C_i = 0$, the $\forall j (j<i) \to C_j = 1$

if there is one, let $C_i' = 1$, $\forall j$, $C_j' = 0$, $k' = k$

else $C_0 \sim C_k$ is all one, let $k' = k+1$, $C_{k+1}' = 1$, $C_k' \ldots C_0' = 0$

$(n+1)$ is divided by 2.

$(n+1)/2 = C_k * 2^k + \ldots + C_0 * 2^0$

$n+1 = C_k * 2^{k+1} + \ldots + C_0 * 2^1 + 0 \cdot 2^0$

$(n+1)$ not divided by 2.

n is odd

$n = C_k * 2^k + \ldots + 0 * 2^0$

$n+1 = C_k * 2^k + \ldots + 1 * 2^0$.

4 Fibonacci for Home

Note 3

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

$F_1 = 1$

$F_2 = 1$

$F_3 = 2$

$F_4 = 3$

$F_5 = 5$

$F_6 = 8$

prove that for n

$\forall n \geq 1$ if $n \% 3 == 0$, F_n is even else F_n is odd.

Base case: $F_1 = 1$ $F_2 = 1$ $F_3 = 2$.

Induction hypothesis: $\to F_{3k}$ is even for all $k \geq 1$.

Inductive step: $\boxed{n+1}$

$F_{3(k+1)} = F_{3k+3} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k}$

$F_{3k} = 2g$

$F_{3k+3} = 2(F_{3k+1} + g)$ is even

if $(n+1) \% 3 == 0$, the $n \% 3 == 2$, $(n-1) \% 3 == 1$.

$F_{n+1} = F_n + F_{n-1} = \text{odd} + \text{odd} = \text{even}$

if $(n+1) \% 3 == 1$ $n \% 3 == 0$ $(n-1) \% 3 == 2$

$F_{n+1} = \underset{\text{even}}{F_n} + \underset{\text{odd}}{F_{n-1}} = \text{even}.$

if $(n+1) \% 3 == 2$ $n \% 3 == 1$ $(n-1) \% 3 == 0$.

$\underset{\text{odd}}{F_{n+1}} = \underset{\text{odd}}{F_n} + \underset{\text{even}}{F_{n-1}}$

prove done.