CS 61B
Spring 2018

More Asymptotic Analysis

Discussion 8: March 6, 2018

Here is a review of some formulas that you will find useful when doing asymptotic analysis.

- $\sum_{i=1}^{N} i = 1 + 2 + 3 + 4 + \cdots + N = \frac{N(N+1)}{2} = \mathbf{\frac{N^2+N}{2}}$
- $\sum_{i=0}^{N-1} 2^i = 1 + 2 + 4 + 8 + \cdots + 2^{N-1} = 2 \cdot 2^{N-1} - 1 = \mathbf{2^N - 1}$

Intuition

For the following recursive functions, give the worst case and best case running time in the appropriate $O(\cdot)$, $\Omega(\cdot)$, or $\Theta(\cdot)$ notation.

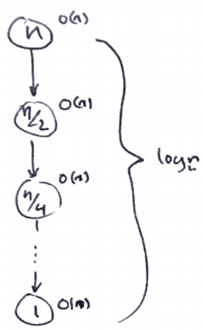
The meta-strat on this problem is to explore a rigourous framework to analyze running time for recursive procedures. Specifically, one can derive the running time by drawing the recursive tree and accounting for three pieces of information.

- The height of the tree.
- The branching factor of each node.
- The amount of work each node contributes relative to its input size.

1.1 Give the running time in terms of N.

```
public void andslam(int N) {
    if (N > 0) {
        for (int i = 0; i < N; i += 1) {
            System.out.println("datboi.jpg");
        }
        andslam(N / 2);
    }
}
```

`andslam(N)` runs in time $\Theta(N)$ worst and best case. One potentially tricky portion is that the $\sum_{i=0}^{\log n} 2^{-i}$ is at most 2 because the geometric sum as it goes to infinity is bounded by 2! (2 *exclamation mark* not "2 factorial")



[1] Height of tree

→ how many times can you divide n by 2 until you get n = 1. Let h be height.

$$\frac{n}{2^h} = 1 \to n = 2^h \to h = \log_2 n$$

[2] Branching Factor

→ Note each time andslam is called, it makes 1 recursive call on n/2.

→ # nodes per layer = 1.

[3] Amount of work each node does.

→ linear relative to input size. So O(n).

Now running time ~~can be~~ of entire recursive procedure can be calcula by summing over entire recursive tree.

$$\text{running time} = \underset{\text{sum over}}{\#\text{ layers}} \cdot \left(\frac{\#\text{ nodes}}{\#\text{ layer}}\right) \cdot \left(\frac{\text{amount work}}{1\text{ node}}\right)$$

$$= \underbrace{\sum_{i=0}^{\log n}}_{\text{layers}} \overbrace{(1)}^{\#\text{nodes/layer}} \cdot \underbrace{\left(\frac{n}{2^i}\right)}_{\text{work / node}}$$

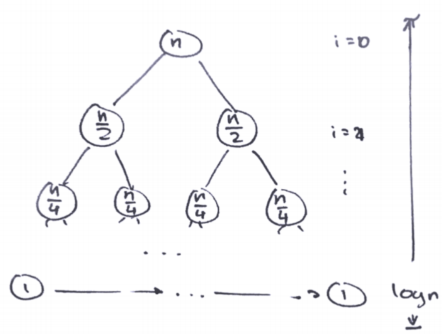
$$= \sum_{i=0}^{\log n} \frac{n}{2^i} = n \sum_{i=0}^{\log n} \frac{1}{2^i} \leq 2n \in \Theta(n)$$

1.2 Give the running time for `andwelcome(arr, 0, N)` where N is the length of the input array `arr`.

```java
public static void andwelcome(int[] arr, int low, int high) {
    System.out.print("[ ");
    for (int i = low; i < high; i += 1) {
         System.out.print("loyal  ");
    }
    System.out.println("]");
    if (high - low > 0) {
        double coin = Math.random();
        if (coin > 0.5) {
            andwelcome(arr, low, low + (high - low) / 2);
        } else {
            andwelcome(arr, low, low + (high - low) / 2);
            andwelcome(arr, low + (high - low) / 2, high);
        }
    }
}
```

`andwelcome(arr, 0, N)` runs in time $\Theta(N \log N)$ worst case and $\Theta(N)$ best case. The recurrence relation is different for each case. In the worst case you always flip the wrong side of the coin resulting in a branching factor of 2. Because there is a branching factor of 2, there are 2^i nodes in the i-th layer. Meanwhile, the work you do per node is linear with respect to the size of the input. Hence in the i-th layer, the work done is about $\frac{n}{2^i}$. In the best case you always flip the right side of the coin giving a branching factor of 1. The analysis is then the same as the previous problem!

* worst case *



height = logn

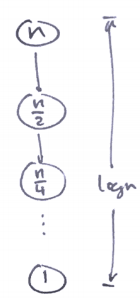
$\frac{\text{nodes}}{\text{layer}} = 2^i$ for layer i

$\frac{\text{work}}{\text{node}} = \frac{n}{2^i}$

$$\sum_{i=0}^{\log n} 2^i \left(\frac{n}{2^i}\right) = \sum_{i=0}^{\log n} n = n \sum_{i=0}^{\log n} 1$$

$$= n \log n \in \Theta(n \log n)$$

* best case *



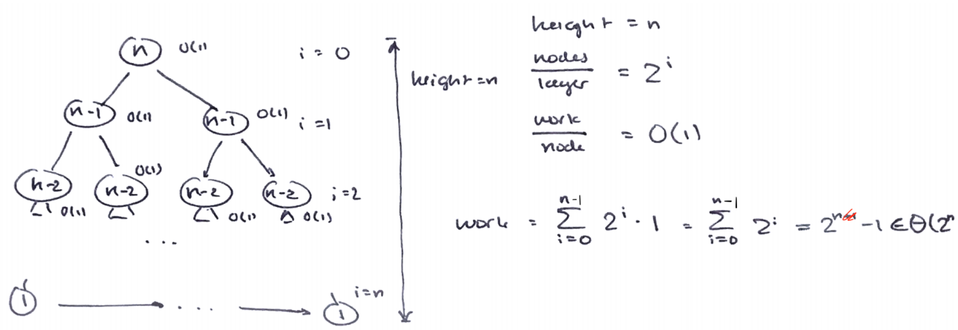
Same analysis ~~for~~ as question 2.a

work = $\Theta(n)$.

1.3 Give the running time in terms of N.

```java
public int tothe(int N) {
    if (N <= 1) {
        return N;
    }
    return tothe(N - 1) + tothe(N - 1);
}
```

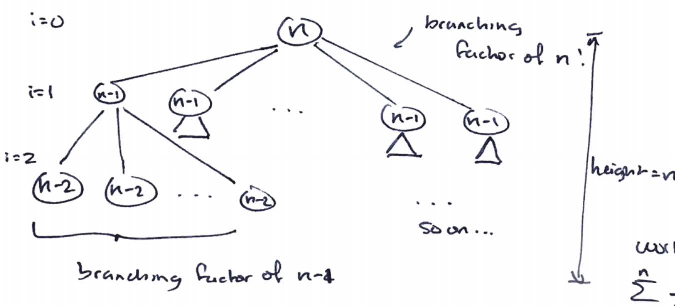
For `tothe(N)` the worst and best case are $\Theta(2^N)$. Notice that at the i-th layer, there are 2^i nodes. Each node does constant amount of work so with the fact that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$, we can derive the following.



1.4 Give the running time in terms of N.

```
public static void spacejam(int N) {
    if (N <= 1) {
        return;
    }
    for (int i = 0; i < N; i += 1) {
        spacejam(N - 1);
    }
}
```

For `spacejam(N)` the worst and best case is $O(N \cdot N!)$. Now for the i-th layer, the number of nodes is $n \cdot (n-1) \cdot \ldots \cdot (n-i)$ since the branching factor starts at n and decrements by 1 each layer. Actually calculating the sum is a bit tricky because there is a pesky $(n-i)!$ term in the denominator. We can upper bound the sum by just removing the denominator, but in the strictest sense we would now have a big-O bound instead of big-Θ.



height = n

$$\frac{\text{nodes}}{\text{layer}} = \frac{n!}{(n-i)!}$$

$$\frac{\text{work}}{\text{node}} = O(n)$$

work ...

$$\sum_{i=0}^{n} \frac{n!}{(n-i)!}(n-i) = \sum_{i=0}^{n} \frac{n!}{(n-i-1)!}$$

$$\le \sum_{i=0}^{n} n! = n \cdot n!$$

$$\in O(n \cdot n!)$$

EXTRA PRACTICE!

Hey you watchu gon do

2.1 For each example below, there are two algorithms solving the same problem. Given the asymptotic runtimes for each, is one of the algorithms **guaranteed** to be faster? If so, which? And if neither is always faster, explain why.

(a) Algorithm 1: $\Theta(N)$, Algorithm 2: $\Theta(N^2)$

Algorithm 1: $\Theta(N)$ - Θ gives tightest bounds therefore the slowest algorithm 1 could run is relative to N while the fastest algorithm 2 could run is relative to N^2.

(b) Algorithm 1: $\Omega(N)$, Algorithm 2: $\Omega(N^2)$

Neither, $\Omega(N)$ means that algorithm 1's running time is lower bounded by N, but does not provide an upper bound. Hence the bound on algorithm 1 could not be tight and it could also be in $\Omega(N^2)$ or lower bounded by N^2.

(c) Algorithm 1: $O(N)$, Algorithm 2: $O(N^2)$

Neither, same reasoning for part (b) but now with upper bounds. $O(N^2)$ could also be in $O(1)$.

(d) Algorithm 1: $\Theta(N^2)$, Algorithm 2: $O(\log N)$

Algorithm 2: $O(\log N)$ - Algorithm 2 cannot run SLOWER than $O(\log N)$ while Algorithm 1 is constrained on to run FASTEST and SLOWEST by $\Theta(N^2)$.

(e) Algorithm 1: $O(N \log N)$, Algorithm 2: $\Omega(N \log N)$

Neither, Algorithm 1 CAN be faster, but it is not guaranteed - it is guaranteed to be "as fast as or faster" than Algorithm 2.

Would your answers above change if we did not assume that N was very large (for example, if there was a maximum value for N, or if N was constant)?

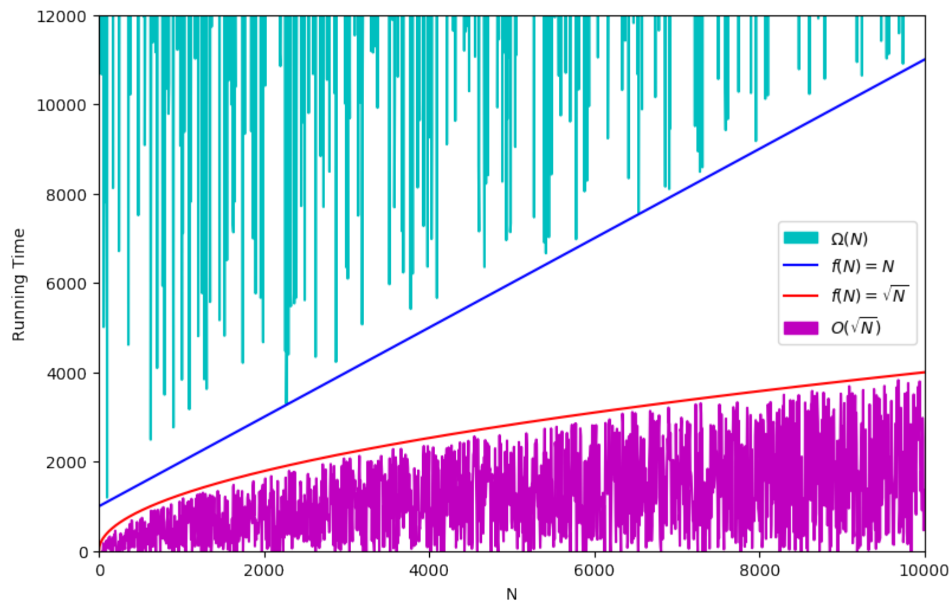
Depends, because for fixed N, constants and lower order terms may dominate the function we are trying to bound. For example N^2 is asymptotically larger than $10000N$, yet when N is less than 10000, $10000N$ is larger than N^2. This highlights the power in using big-O because these lower order terms don't affect the running time as much as our input size grows very large!

However, part of the definition of $O(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ is the limit to infinity ($\lim_{N\to\infty}$) so, when working with asymptotic notation, we must always assume large inputs.

Asymptotic Notation

3.1 Draw the running time graph of an algorithm that is $O(\sqrt{N})$ in the best case and $\Omega(N)$ in the worst case. Assume that the algorithm is also trivially $\Omega(1)$ in the best case and $O(\infty)$ in the worst case.

Below we have the graph an example solution. Assume that the cyan region extends arbitrarily towards infinity. Ignoring constants, the algorithm running time takes at most $\sqrt{N}$ time (and trivially at least constant time) in the worst case. The algorithm also takes at least N time (and trivially at most infinite time) in the worst case.



Extra: Following is a question from last week, now that you have properly learned about $O(\cdot)$, $\Omega(\cdot)$, or $\Theta(\cdot)$.

3.2 Are the statements in the right column true or false? If false, correct the asymptotic notation ($\Omega(\cdot)$, $\Theta(\cdot)$, $O(\cdot)$). Be sure to give the tightest bound. $\Omega(\cdot)$ is the opposite of $O(\cdot)$, i.e. $f(n) \in \Omega(g(n)) \iff g(n) \in O(f(n))$.

$f(n) = 20501$	$g(n) = 1$	$f(n) \in O(g(n))$
$f(n) = n^2 + n$	$g(n) = 0.000001n^3$	$f(n) \in \Omega(g(n))$
$f(n) = 2^{2n} + 1000$	$g(n) = 4^n + n^{100}$	$f(n) \in O(g(n))$
$f(n) = \log(n^{100})$	$g(n) = n \log n$	$f(n) \in \Theta(g(n))$
$f(n) = n \log n + 3^n + n$	$g(n) = n^2 + n + \log n$	$f(n) \in \Omega(g(n))$
$f(n) = n \log n + n^2$	$g(n) = \log n + n^2$	$f(n) \in \Theta(g(n))$
$f(n) = n \log n$	$g(n) = (\log n)^2$	$f(n) \in O(g(n))$

- True, although $\Theta(\cdot)$ is a better bound.
- False, $O(\cdot)$. Even though n^3 is strictly worse than n^2, n^2 is still in $O(n^3)$ because n^2 is always as good as or better than n^3 and can never be worse.
- True, although $\Theta(\cdot)$ is a better bound.
- False, $O(\cdot)$.
- True.
- True.
- False, $\Omega(\cdot)$.

Fall 2015 *Extra*

4.1 If you have time, try to answer this challenge question. For each answer true or false. If true, explain why and if false provide a counterexample.

(a) If $f(n) \in O(n^2)$ and $g(n) \in O(n)$ are positive-valued functions (that is for all n, $f(n), g(n) > 0$), then $\frac{f(n)}{g(n)} \in O(n)$.

Nope this does not hold in general! Consider if $f(n) = n^2$ and $g(n) = \frac{1}{n}$. Readily we have $f(n), g(n) \in O(n)$ but when divided they give us:

$$\frac{f(n)}{g(n)} = \frac{n^2}{n^{-1}} = n^3 \notin O(n)$$

(b) If $f(n) \in \Theta(n^2)$ and $g(n) \in \Theta(n)$ are positive-valued functions, then $\frac{f(n)}{g(n)} \in \Theta(n)$.

This does hold in general! We can think about this in two cases:

- First we ask, when can the ratio $\frac{f(n)}{g(n)}$ be larger than n. As $f(n)$ is tightly bounded (by Θ) by n^2, this is only true when $g(n)$ is asymptotically *smaller* than n because we are dividing n^2 (this is what happened in part a). However, $g(n)$ is tightly bounded, and thus lower bounded by n, this cannot happen.

- Next we ask, when can the ratio be smaller than n. Again as $f(n)$ is tightly bounded by n^2, this can only happen when $g(n)$ is asymptotically *bigger* than n as again we are dividing. But since $g(n)$ is tightly bounded, and thus upper bounded by n, this too cannot happen.

So what we note here is that $\frac{f(n)}{g(n)}$ is upper and lower bounded by n hence it is in $\Theta(n)$. We can also give a rigorous proof from definition of part b using the definitions provided in class.

Theorem 1. *If $f(n) \in \Theta(n^2)$ and $g(n) \in \Theta(n)$ are positive-valued functions, then $\frac{f(n)}{g(n)} \in \Theta(n)$.*

Proof. Given that $f \in \Theta(n^2)$ is positive, by definition there exists $k_0, k_0' > 0$ such that for all $n > N$, the following holds.

$$k_0 n^2 \leq f(n) \leq k_0' n^2$$

Similarly, $g \in \Theta(n)$ implies there exists $k_1, k_1' > 0$ such that

$$k_1 n \leq g(n) \leq k_1' n$$

Now consider $\frac{f(n)}{g(n)}$.

$$\frac{f(n)}{g(n)} \leq \frac{k_0' n^2}{k_1 n} = \frac{k_0' n}{k_1} \in O(n) \qquad \frac{f(n)}{g(n)} \geq \frac{k_0 n^2}{k_1' n} = \frac{k_0 n}{k_1'} \in \Omega(n)$$

As $\frac{f(n)}{g(n)}$ is in $O(n)$ and $\Omega(n)$ then it is in $\Theta(n)$. $\square$