CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair **DIS 12B**

1 Interesting Gaussians

Note 21

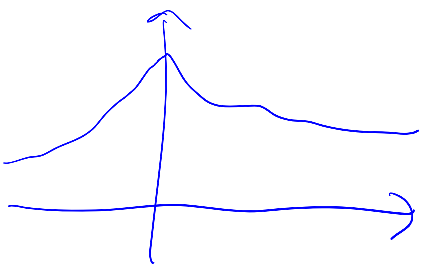
(a) If $X \sim N(0,\sigma_X^2)$ and $Y \sim N(0,\sigma_Y^2)$ are independent, then what is $\mathbb{E}\left[(X+Y)^k\right]$ for any *odd* $k \in \mathbb{N}$?

$E(X) = 0$ $E(Y) = 0$ $E(XY) = E(X) * E(Y) = 0.$

$Var(X) = \sigma_X^2$ $Var(Y) = \sigma_Y^2$ $(X+Y) = 0$

$= E(X^2)$ $= E(Y^2)$ $(X+Y)^3$

$X+Y \in N(0, \sigma_x^2 + \sigma_y^2)$



$E((X+Y)^k) = 0$

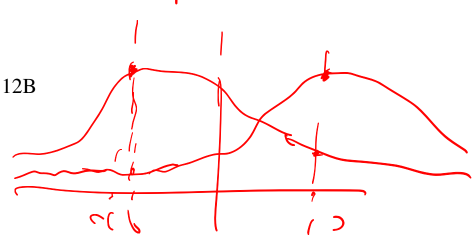
(b) Let $f_{\mu,\sigma}(x)$ be the density of a $N(\mu,\sigma^2)$ random variable, and let X be distributed according to $\alpha f_{\mu_1,\sigma_1}(x) + (1-\alpha) f_{\mu_2,\sigma_2}(x)$ for some $\alpha \in [0,1]$. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. Is X normally distributed?

$\alpha f_{\mu_1,\sigma_1}(x) + (1-\alpha) f_{\mu_2,\sigma_2}(x)$ Yes NO.

$\alpha\mu_1 + (1-\alpha)\mu_2 = E(X)$ ✓

$\cancel{\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 = Var(X)}$ $Var(X) = E(X^2) - (E(X))^2$

↗ must be indenpent $= \alpha(\sigma_1^2+\mu_1^2) + (1-\alpha)(\sigma_2^2+\mu_2^2) - (E(X))^2$



2 Binomial Concentration

Note 21

Here, we will prove that the binomial distribution is *concentrated* about its mean as the number of trials tends to ∞. Suppose we have i.i.d. trials, each with a probability of success $1/2$. Let S_n be the number of successes in the first n trials (n is a positive integer).

$S_n \sim \text{Binomial}(n, \frac{1}{2})$

(a) Compute the mean and variance of S_n.

$E(S_n) = E(\sum_{i=1}^{n} X_i) = n * \frac{1}{2} = \frac{n}{2}$

$Var(S_n) = E(S_n^2) - (E(S_n))^2$ $\quad E(S_n^2) = E((\sum_{i=1}^{n} X_i)^2)$

$= \sum_{i=1}^{n} E(X_i^2) + (n^2 - n) E(X_i X_j)$

$= \frac{n}{2} + \frac{n^2-n}{4} = \frac{n^2+n}{4}$

$Var(S_n) = \frac{n^2+n}{4} - \frac{n^2}{4} = \frac{n}{4}$

(b) How should we define Z_n in terms of S_n to ensure that Z_n has mean 0 and variance 1?

$Z_n = \frac{S_n - \frac{n}{2}}{\sqrt{\frac{n}{4}}} = \frac{S_n - \frac{n}{2}}{\frac{\sqrt{n}}{2}}$

(c) What is the distribution of Z_n as $n \to \infty$?

$n \to \infty$

The central limit theorem: $Z_n \sim N(0, 1)$

(d) Use the bound $\mathbb{P}[Z > z] \leq (\sqrt{2\pi}z)^{-1} e^{-z^2/2}$ when Z is a standard normal in order to approximately bound $\mathbb{P}[S_n/n > 1/2 + \delta]$, where $\delta > 0$.

$P[\frac{S_n}{n} > \frac{1}{2} + \delta] = P[\frac{S_n - \frac{n}{2}}{\frac{n}{2\sqrt{n}}} > 2\delta\sqrt{n}]$

$= P[\frac{S_n - \frac{n}{2}}{\frac{\sqrt{n}}{2}} > 2\delta\sqrt{n}] \approx P[Z > 2\delta\sqrt{n}]$

$\leq \frac{1}{\sqrt{2\pi} * 2\delta\sqrt{n}} * e^{-\frac{4\delta^2 n}{2}}$

$= \frac{1}{2^{\frac{3}{2}}\delta\sqrt{n\pi}} e^{-2\delta^2 n}$

3 Erasures, Bounds, and Probabilities

Note 21

Alice is sending 1000 bits to Bob. The probability that a bit gets erased is p, and the erasure of each bit is independent of the others.

Alice is using a scheme that can tolerate up to one-fifth ($\frac{1}{4}$) of the bits being erased. That is, as long as Bob receives at least 801 of the 1000 bits correctly, he can decode Alice's message.

In other words, Bob becomes unable to decode Alice's message only if 200 or more bits are erased. We call this a "communication breakdown", and we want the probability of a communication breakdown to be at most 10^{-6}.

(a) Use Chebyshev's inequality to upper bound p such that the probability of a communications breakdown is at most 10^{-6}.

$$P[|X-\mu| \geq c] \leq \frac{Var(X)}{c^2} = 10^{-6}$$

(对分布要求不高)

X : the number of bit be earsed

$P(X \geq 200)$

$$E(X) = E\left(\sum_{i=1}^{1000} X_i\right) = np$$

$X \leq 200$

$$Var(X) = np(1-p)$$

$P[X \geq \mu + c]$

$c = 200$

$\mu - x \geq c \qquad x \leq \mu - c$

$\mu - c = 200$

$c = np - 200$

$$\frac{1000p\, p(1-p)}{(1000p - 200)^2} \leq 10^{-6}$$

$p \leq 3.948 \times 10^{-5}$

更保守

Central Limit Theorom

(b) As the CLT would suggest, approximate the fraction of erasures by a Gaussian random variable (with suitable mean and variance). Use this to find an approximate bound for p such that the probability of a communications breakdown is at most 10^{-6}.

You may use that $\Phi^{-1}(1-10^{-6}) \approx 4.753$.

p ?

正 标准正态分布的 CDF (累积分布函数)

$p \leq 0.1468$

$\Rightarrow$ more tighter