CS 170 HW 5 (Optional)

Due **2020-02-04, at 10:00 pm**
You may submit your solutions if you wish them to be graded, but they will be worth no points

1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write none.

2 Arbitrage

Shortest-path algorithms can also be applied to currency trading. Suppose we have n currencies $C = \{c_1, c_2, \ldots, c_n\}$: e.g., dollars, Euros, bitcoins, dogecoins, etc. For any pair i, j of currencies, there is an exchange rate $r_{i,j}$: you can buy $r_{i,j}$ units of currency c_j at the price of one unit of currency c_i. Assume that $r_{i,i} = 1$ and $r_{i,j} \geq 0$ for all i, j.
The Foreign Exchange Market Organization (FEMO) has hired Oski, a CS170 alumnus, to make sure that it is not possible to generate a profit through a cycle of exchanges; that is, for any currency $i \in C$, it is not possible to start with one unit of currency i, perform a series of exchanges, and end with more than one unit of currency i. (That is called *arbitrage.*)
More precisely, arbitrage is possible when there is a sequence of currencies $c_{i_1}, \ldots, c_{i_k}$ such that $r_{i_1,i_2} \cdot r_{i_2,i_3} \cdot \cdots \cdot r_{i_{k-1},i_k} \cdot r_{i_k,i_1} > 1$. This means that by starting with one unit of currency c_{i_1} and then successively converting it to currencies $c_{i_2}, c_{i_3}, \ldots, c_{i_k}$ and finally back to c_{i_1}, you would end up with more than one unit of currency c_{i_1}. Such anomalies last only a fraction of a minute on the currency exchange, but they provide an opportunity for profit.
We say that a set of exchange rates is arbitrage-free when there is no such sequence, i.e. it is not possible to profit by a series of exchanges.

(a) Give an efficient algorithm for the following problem: given a set of exchange rates $r_{i,j}$ which is *arbitrage-free*, and two specific currencies s, t, find the most advantageous sequence of currency exchanges for converting currency s into currency t.

Hint: represent the currencies and rates by a graph whose edge weights are real numbers.

(b) Oski is fed up of manually checking exchange rates, and has asked you for help to write a computer program to do his job for him. Give an efficient algorithm for detecting the possibility of arbitrage. You may use the same graph representation as for part (a).

3 Bounded Bellman-Ford

Modify the Bellman-Ford algorithm to find the weight of the lowest-weight path from s to t with the restriction that the path must have at most k edges.

4 Money Changing.

Fix a set of positive integers called *denominations* $x_1, x_2, \ldots, x_n$ (think of them as the integers 1, 5, 10, and 25). The problem you want to solve for these denominations is the following: Given an integer A, express it as

$$A = \sum_{i=1}^{n} a_i x_i$$

for some nonnegative integers $a_1, \ldots, a_n \geq 0$.

1. Under which conditions on the denominations x_i are you able to do this for all integers $A > 0$?

2. Suppose that you want, given A, to find the nonnegative a_i's that satisfy $A = \sum_{i=1}^{n} a_i x_i$, and such that the sum of all a_i's is minimal —that is, you use the smallest possible number of coins. Define a *greedy algorithm* for this problem. (Your greedy algorithm may not necessarily solve the problem, i.e., it may fail on some inputs)

3. Show that the greedy algorithm finds the optimum a_i's in the case of the denominations 1, 5, 10, and 25, and for any amount A.

4. Give an example of a denomination where the greedy algorithm fails to find the optimum a_i's for some A. Do you know of an actual country where such a set of denominations exists?

5. How far from the optimum number of coins can the output of the greedy algorithm be, as a function of the denominations?