CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair

DIS 4A

1 RSA Warm-Up

Note 7

Consider an RSA scheme with modulus $N = pq$, where p and q are distinct prime numbers larger than 3.

(a) What is wrong with using the exponent $e = 2$ in an RSA public key?

$\gcd(e, (p-1)(q-1)) = 1 \ldots (1)$

$(p-1)(q-1)$ is even.

p, q is prime, so they are odd

then can not satisfy (1)

(b) Recall that e must be relatively prime to $p-1$ and $q-1$. Find a condition on p and q such that $e = 3$ is a valid exponent

$\gcd(e, (p-1)(q-1)) = 1$

$(p-1)(q-1) \rightarrow$ every p, q larger than 3 is ok

(c) Now suppose that $p = 5$, $q = 17$, and $e = 3$. What is the public key?

$e=3$ is the public key

$N = 5 * 17 = 85$ and

(d) What is the private key?

$3x + 64y = 1$

$3p \equiv 1 \pmod{(5-1)(17-1)}$

$x = -21, y = 1$

$\pmod{64}$.

$-21 \equiv 43 \pmod{64}$

(e) Alice wants to send a message $x = 10$ to Bob. What is the encrypted message $E(x)$ she sends using the public key?

$10^3 \pmod{85} \equiv 100 * 10 \pmod{85}$

$\equiv 15 * 10 \equiv 150 \equiv 65 \pmod{85}$

17*(-2)+5*7=1

(f) Suppose Bob receives the message $y = 19$ from Alice. What equation would he use to decrypt the message? What is the decrypted message?

$y^{43} \pmod{85}$

$19^{43} \pmod{85}$? ← how to

$x \equiv 19^{43} \pmod{85}$

$x \equiv a \pmod 5$

$x \equiv b \pmod{17}$

$19^{43} \equiv 4^{43} = 4*(4^2)^{21} \equiv 4 \pmod 5$ $a=1$

$19^{43} \equiv 2^{43} = 2^3*(2^4)^{10} \equiv 2^3 = 8$

CRT $x = a p_2 p_2^{-1} + b p_1 p_1^{-1} \pmod{85}$

$p_2 p_2^{-1} \equiv 1 \pmod 5$ $p_1 p_1^{-1} \equiv 1 \pmod{17}$

$p_2 = 17$ $p_2^{-1} = -2 \equiv 3$

5 17

$x = 51a + 35b \pmod{85}$ $p_1^{-1} = 7$

2 RSA with Multiple Keys

Note 7

Members of a secret society know a secret word. They transmit this secret word x between each other many times, each time encrypting it with the RSA method. Eve, who is listening to all of their communications, notices that in all of the public keys they use, the exponent e is the same. Therefore the public keys used look like $(N_1, e), \ldots, (N_k, e)$ where no two N_i's are the same. Assume that the message is x such that $0 \le x < N_i$ for every i.

$= 51*4 + 35*8 \equiv 484 \equiv 39 \pmod{85}$

Further, in all of the subparts, you may assume that Eve knows the details of the modified RSA schemes (i.e. Eve knows the format of the N_i's, but not the specific values used to compute the N_i's).

3
85 | 331
255
76

(a) Suppose Eve sees the public keys $(p_1q_1, 7)$ and $(p_1q_2, 7)$ as well as the corresponding transmissions. Can Eve use this knowledge to break the encryption? If so, how? Assume that Eve cannot compute prime factors efficiently. Think of p_1, q_1, q_2 as massive 1024-bit numbers. Assume p_1, q_1, q_2 are all distinct and are valid primes for RSA to be carried out.

p_1q_1

p_1q_2

$\gcd(p_1q_1, p_1q_2) = p_1$

how? 因子分解至多一个素数, ↓共同的素数就是最大公约数

(b) The secret society has wised up to Eve and changed their choices of N, in addition to changing their word x. Now, Eve sees keys $(p_1q_1, 3)$, $(p_2q_2, 3)$, and $(p_3q_3, 3)$ along with their transmissions. Argue why Eve cannot break the encryption in the same way as above. Assume $p_1, p_2, p_3, q_1, q_2, q_3$ are all distinct and are valid primes for RSA to be carried out.

N_1, N_2, N_3, relative prime. $\uparrow p_1q_1 \quad p_2q_2 \quad p_3q_3$

(c) Let's say the secret x was not changed ($e = 3$), so they used the same public keys as before, but did not transmit different messages. How can Eve figure out x?

(N_1, e) $(x^e)^d \pmod N$ $y = x^3 \pmod{N_1} = a$ \ know

(N_2, e) ? $y = x^3 \pmod{N_2} = b$ (

$y = x^3 \pmod{N_3} = c$ $0 < x < N$.

y CRT $x^3 \pmod{N_1 * N_2 * N_3}$ → can get y

$0 < x^3 < N_1 * N_2 * N_3$ $\sqrt[3]{y}$ will be the result

3 RSA for Concert Tickets

Note 7

Alice wants to tell Bob her concert ticket number, m, which is an integer between 0 and 100 inclusive. She wants to tell Bob over an insecure channel that Eve can listen in on, but Alice does not want Eve to know her ticket number.

(a) Bob announces his public key $(N = pq, e)$, where N is large (512 bits). Alice encrypts her message using RSA. Eve sees the encrypted message, and figures out what Alice's ticket number is. How did she do it?

she can try from 0, ~100, calculate $x^e \pmod N$

compare it with the encrypted message.

(b) Alice decides to be a bit more elaborate. She picks a random number r that is 256 bits long, so that it is too hard to guess. She encrypts that and sends it to Bob, and also computes rm, encrypts that, and sends it to Bob. Eve is aware of what Alice did, but does not know the value of r. How can she figure out m? (You may assume that r is coprime to N.)

$r^e \pmod N = x$ $x * x^{-1} \equiv 1 \pmod N$

$(rm)^e \pmod N = y$ x^{-1}

$r^e m^e \pmod N = y$ $m^e \pmod N = x^{-1} * y$

$x * m^e \pmod N = y$ ↓ the same as (a)