CS 70 Spring 2024 — Discrete Mathematics and Probability Theory — Seshia, Sinclair — HW 08

Due: Saturday, 3/16, 4:00 PM
Grace period until Saturday, 3/16, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Probability Warm-Up

Note 13

(a) Suppose that we have a bucket of 30 green balls and 70 orange balls. If we pick 15 balls uniformly out of the bucket, what is the probability of getting exactly k green balls (assuming $0 \leq k \leq 15$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?

(b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

2 Five Up

Note 13

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

(a) What is the size of the sample space, $|\Omega|$?

(b) How many elements of Ω have exactly three heads?

(c) How many elements of Ω have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

1. (a) $\left(\frac{3}{10}\right)^{k} * \left(\frac{7}{10}\right)^{15-k} \times C_{15}^{k}$

$$\frac{\binom{30}{k}\binom{70}{15-k}}{\binom{100}{15}}$$

(b) the same ~~as (a)~~

(c) $1-\frac{C_6^5 \times 5!}{6^5} = 1-\frac{6\times 5!}{6\times 6^4} = 1-\frac{5\times 4\times 3\times 2}{6\times 6\times 6\times 6}$

$= 1-\frac{5}{54} = \frac{49}{54}$

2. (a) $5 \to 2^5 = 32$

(b) $C_5^3 = \frac{5!}{3! * 2!} = \frac{5\times 4\times 3\times 2}{3\times 2\times 2} = 10$

(c) $C_5^3 + C_5^4 + C_5^5 = 10+5+1 = 16$

(d) $\left(\frac{1}{2}\right)^5$ (e) $1-\left(\frac{1}{2}\right)^5 = \frac{31}{32}$

$32 + 78 + 18 = 56$

(f) $\frac{1}{2}$

(g) $\left(\frac{2}{3}\right)^3 \times \left(\frac{1}{3}\right)^2 = \frac{8}{243}$ (h) $1-\left(\frac{1}{3}\right)^5 = \frac{242}{243}$

(h) $\frac{2}{3}$? 3H 2T 4H 1T 5H $\binom{5}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + \binom{5}{4}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)$

$\frac{2^3 + \cancel{2^4} + 2^5}{243} = \cancel{\frac{56}{243}} + \binom{5}{5}\left(\frac{2}{3}\right)^5$

(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?

(e) What is the probability of observing at least one head?

(f) What is the probability you will observe more heads than tails?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p = \frac{2}{3}$.

(g) What is the probability of observing the outcome HHHTT? What about HHHHT?

(h) What about the probability of at least one head?

(i) What is the probability you will observe more heads than tails?

3 Aces

Note 13
Note 14

Consider a standard 52-card deck of cards:

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

4 Independent Complements

Note 14

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

3.

(a) $4 + \cancel{13}^{26} - \cancel{1}^{2} = \cancel{16}\ 28$ $\qquad \frac{\cancel{16}\,28}{52} = \frac{\cancel{4}\,7}{13}$

(b) $4 + 13 - 1 - 1 = 15$ $\qquad \frac{15}{52}$

(c) $\frac{C_{51}^{4}}{C_{52}^{5}}$ $\qquad$ (e) $1 - \frac{C_{48}^{5}}{C_{52}^{5}}$ $\qquad 4 + 13 - 1 = 16$

(d) $\frac{C_4^2 * C_{\cancel{80}\,48}^{3}}{C_{52}^{5}}$ $\qquad$ (f) $1 - \frac{C_{36}^{5}}{C_{52}^{5}}$

De morgan's law

4. (a) prove

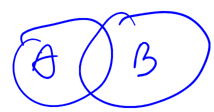
$P(A \cap B) = P(A) * P(B)$

$P(\bar{B} \mid \bar{A}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A) * P(B)$



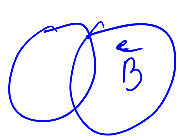
$P(A \cup B) = P(A) + P(B) - P(A) * P(B)$

$P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - P(A) * P(B) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$

$P(\bar{A} \cap \bar{B}) = P(\bar{A}) + P(\bar{B}) - 1 + P(A) * P(B)$

$= P(\bar{A}) * P(\bar{B})$ $\qquad (1 - P(\bar{A}))(1 - P(\bar{B}))$

$1 - P(\bar{B}) - P(\bar{A}) + P(\bar{A}) * P(\bar{B})$



(b) $P(A \cap \bar{B}) = P(A - A \cap B)$

$= P(A) - P(A) * P(B) = P(A)\, P(\bar{B})$ ✓

(c) $P(A \cap \bar{A}) = \underline{0} \neq \underline{P(A) * P(\bar{A})}$ <u>NO</u>

(d) Yes, <u>$A = B = \Omega$</u>

<u>or</u> $P(A) = P(B) = 0$

$\downarrow$

$P(A) = P(B) = 1$

5 Faulty Lightbulbs

Note 13
Note 14

Box 1 contains 1000 lightbulbs of which 10% are defective. Box 2 contains 2000 lightbulbs of which 5% are defective.

A C B

(a) Suppose a box is given to you at random and you randomly select a lightbulb from the box. If that lightbulb is defective, what is the probability you chose Box 1?

(b) Suppose now that a box is given to you at random and you randomly select two light- bulbs from the box. If both lightbulbs are defective, what is the probability that you chose from Box 1?

(a)

$P(C|A) = 10\%$ $\quad P(C|B) = 5\%$ $\quad P(C) = P(C \cap \sigma)$

$P(A) = \frac{1}{2}$ $\quad P(B) = \frac{1}{2}$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A) * P(C|A)}{P(C \cap A) + P(C \cap B)}$$

$$= \frac{\frac{1}{2} \times \frac{10}{100}}{P(A) * P(C|A) + P(B) * P(C|B)} = \frac{\frac{5}{100}}{\frac{1}{2} \times \frac{10}{100} + \frac{1}{2} \times \frac{5}{100}}$$

$$= \frac{\frac{10}{100}}{\frac{15}{100}} = \frac{10}{15} = \frac{2}{3}$$

(b) D

$$P(D|A) = \frac{100}{1000} \times \frac{99}{999}$$

$$P(D|B) = \frac{100}{2000} \times \frac{99}{1999}$$

$$P(A|D) = \frac{P(A) * P(D|A)}{P(D)}$$

$$= \frac{P(A) * P(D|A)}{P(A) * P(D|A) + P(B) * P(D*A)}$$

$$= \frac{\frac{100}{1000} \times \frac{99}{999}}{\frac{100}{1000} \times \frac{99}{999} + \frac{100}{2000} \times \frac{99}{1999}}$$

$$= \frac{\frac{1}{999}}{\frac{1}{999} + \frac{1}{2} \times \frac{1}{1999}}$$

$$\begin{array}{r} 1999 \\ {}^{1\,1\,1\,2} \\ \hline 3998 \end{array}$$

$$= \frac{\frac{1}{999}}{\frac{1}{999} + \frac{1}{3998}} = \frac{1}{1 + \frac{999}{3998}}$$

$$\begin{array}{r} 3998 \\ 999 \\ \hline \end{array}$$

$$= \frac{3998}{4997}$$