1 Short Tree Proofs

Note 5

Let $G = (V,E)$ be an undirected graph with $|V| \geq 1$.

(a) Prove that every connected component in an acyclic graph is a tree.

tree is connected and acyclic.

(b) Suppose G has k connected components. Prove that if G is acyclic, then $|E| = |V| - k$.

V_1 . V_k is connected

$V_1 + .. V_k - k = |V| - k$

(c) Prove that a graph with $|V|$ edges contains a cycle.

use (b) $k \geq 1$, $|E| \leq |V| - 1$ is acyclic

have $|V|$ can not be acyclic.

2 Secret Sharing Practice

Consider the following secret sharing schemes and solve for asked variables.

(a) Suppose there is a bag of candy locked with a passcode between 0 and an integer n. Create a scheme for 5 trick-or-treaters such that they can only open the bag of candy if 3 of them agree to open it.

(b) Create a scheme for the following situation: There are 4 cats and 3 dogs in the neighborhood, and you want them to only be able to get the treats if the majority of the animals of each type are hungry. The treats are locked by a passcode between 0 and an integer n.

(a) mod 7 degree 2 $f(x) = ax^2 + bx + c \pmod 7$

$f(1), f(2), f(3), f(4), f(5)$ $f(0)$ is n.

(b) ~~2~~ 3 cats 1 2 dogs

3 dogs $g(x) = ax + b, \pmod 5$

4 cats $f(x) = ax^2 + bx + c \pmod 5$

$g(1)$ $g(2)$ $g(3)$. $g(0)$

$f(1)$ $f(2)$ $f(3)$ $f(4)$ $\rightarrow f(0)$

$f(0) + g(0)$ as n

$h(x) = a_2 x + b_2 \pmod 3$

$f(0) = h(1)$

$h(0)$ = passcode

$g(0) = h(2)$

3 Counting Subsets

Note 11

Consider the set S of all (possibly infinite) subsets of $\mathbb{N}$.

(a) Show that there is a bijection between S and $T = \{f : \mathbb{N} \rightarrow \{0,1\}\}$ (the set of all functions that map each natural number to 0 or 1).

(b) Prove or disprove: S is countable.

(c) Say that a function $f : \mathbb{N} \rightarrow \{0,1\}$ has *finite support* if it is non-zero on only a finite set of inputs. Let F denote the set of functions $f : \mathbb{N} \rightarrow \{0,1\}$ with finite support.

Prove that F is countably infinite.

(a) Let $X \subseteq \mathbb{N}$, Define $f(x) = 1$ if $x \in X$ and 0 otherwise.

$f(x) = a_x$

(b) uncountable $\rightarrow 0.a_0 a_1 a_2 \ldots a_x$

f can be binary encoding of real number $[0,1]$

$S \Leftrightarrow [0,1]$

(c) ? bijection F and $\mathbb{N}$

binary number ith position set to 1 if $f(i) = 1$.

finite length.

4 Strings

Note 10

How many different strings of length 5 only contain A, B, C? And how many such strings contain at least one of each characters?

$\cup$ 3^5

$5 \times 4 \times 3 \times 2^3$ × have redundant

$E(A)$, $E(B)$ $E(C)$

A not used

$E(A) = E(B) = E(C) = 2^5$

$E(A \cup B \cup C) = 3 * 2^5 - [E_A \cap E_B] * 3 + [E_A \cap E_B \cap E_C] = 0$

$= 3 * 2^5 - 3 = 93$

$3^5 - 93 = 150$

5 Mario's Coins

Note 14

Mario owns three identical-looking coins. One coin shows heads with probability $1/4$, another shows heads with probability $1/2$, and the last shows heads with probability $3/4$.

(a) Mario randomly picks a coin and flips it. He then picks one of the other two coins and flips it. Let X_1 and X_2 be the events of the 1st and 2nd flips showing heads, respectively. Are X_1 and X_2 independent? Please prove your answer.

~~Yes~~? NO

$$P(X_2 | X_1) = \frac{P(X_1 \cap X_2)}{P(X_1)}$$

$\boxed{\text{proof by calcucate}}$ ✓

$P(X_1) = P(X_2) = \frac{1}{3}\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right) = \frac{1}{2}$ $\quad P(X_1)\,P(X_2) = \frac{1}{4}$

$P(X_1 \cap X_2) = \frac{1}{3}\left(\frac{1}{4} * \frac{1}{2} + \frac{1}{4} * \frac{3}{4} + \frac{1}{2} \times \frac{3}{4}\right) = \frac{11}{48}$ ✗

(b) Mario randomly picks a single coin and flips it twice. Let Y_1 and Y_2 be the events of the 1st and 2nd flips showing heads, respectively. Are Y_1 and Y_2 independent? Please prove your answer.

~~Yes~~ NO

$P(Y_1) = P(Y_2) = P(X_1) = \frac{1}{2}$ $\quad$ 事件的独立性

$P(Y_1) * P(Y_2) = \frac{1}{4}$

$P(Y_1 \cap Y_2) = \frac{1}{3} \times \left(\frac{1}{4}\right)^2 + \frac{1}{3} * \left(\frac{1}{2}\right)^2 + \frac{1}{3} * \left(\frac{3}{4}\right)^2 = \frac{14}{48} = \frac{7}{24} \neq \frac{1}{4}$

(c) Mario arranges his three coins in a row. He flips the coin on the left, which shows heads. He then flips the coin in the middle, which shows heads. Finally, he flips the coin on the right. What is the probability that it also shows heads?

$\cancel{\frac{1}{3} \times \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right)} = \cancel{\frac{1}{2}}$?

条件概率

$P(X_1 \cap X_2) = \frac{11}{48}$

$P(X_1 \cap X_2 \cap X_3) = \frac{1}{6}\left(6 \times \frac{1}{4} \times \frac{1}{2} \times \frac{3}{4}\right) = \frac{3}{32}$

$\frac{\frac{3}{32}}{\frac{11}{48}} = \frac{3}{32} \times \frac{48}{11} = \frac{3}{2} \times \frac{3}{11} = \frac{9}{22}$

6 Sum of Poisson Variables

Note 19

Assume that you were given two independent Poisson random variables X_1, X_2. Assume that the first has mean λ_1 and the second has mean λ_2. Prove that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

Hint: Recall the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$P(X_1 = i) = \frac{\lambda_1^i}{i!} e^{-\lambda_1}$$

$$P(X_1 + X_2 = k) = \sum_{j=0}^{k} P[X_1 = j, X_2 = k-j]$$

$$= \sum_{j=0}^{k} \frac{\lambda_1^j}{j!} e^{-\lambda_1} * \frac{\lambda_2^{k-j}}{(k-j)!} e^{-\lambda_2}$$

$$= e^{-(\lambda_1+\lambda_2)} \sum_{j=0}^{k} \frac{\lambda_1^j \lambda_2^{k-j}}{j! * (k-j)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{j=0}^{k} \frac{k!}{\underbrace{j!(k-j)!}_{\binom{k}{j}}} \lambda_1^j \lambda_2^{k-j} = \frac{(\lambda_1+\lambda_2)^k}{k!} e^{-(\lambda_1+\lambda_2)}$$

$$\sim \text{Poisson}(\lambda_1+\lambda_2)$$

7 Balls in Bins

Note 15

You are throwing k balls into n bins. Let X_i be the number of balls thrown into bin i.

(a) What is $\mathbb{E}[X_i]$?

$$X_i \sim \text{Bin}(k, \tfrac{1}{n})$$

$$E(X_i) = \frac{k}{n}$$

(b) What is the expected number of empty bins?

$$Y_i = \begin{cases} 1 & \text{bin } i \text{ is empty} \\ 0 & \text{others} \end{cases}$$

$$P(Y_i = 1) = \left(\frac{n-1}{n}\right)^k$$

$$E(Y_i) = \left(\frac{n-1}{n}\right)^k$$

$$E(Y) = n * \left(\frac{n-1}{n}\right)^k$$

(c) Define a collision to occur when a ball lands in a nonempty bin (if there are n balls in a bin, count that as $n-1$ collisions). What is the expected number of collisions?

??? like hash table

empty bins.

occupated bins)

$X_i \sim Bin(k, \frac{1}{n})$

$x_i = 0, 1, \ldots k$

$= \cancel{1 \ldots k-1}$

$n \sum_{i=2}^{k} (i-1) \left(\frac{1}{n}\right)^i \left(\frac{n-1}{n}\right)^{k-i}$

?

$E[\text{collisions}] = k - E[\text{occupide}]$

$= k - [n - E(\text{empty})]$

$= k - n + n * \left(\frac{n-1}{n}\right)^k$

8 Inequality Practice

Note 17

(a) X is a random variable such that $X \geq -5$ and $\mathbb{E}[X] = -3$. Find an upper bound for the probability of X being greater than or equal to -1.

$P(X \geq c) \leq \frac{E(X)}{c}$ → only works for non-negative random variables

$\cancel{P(X \geq -1) \leq \frac{-3}{-1} = 3}$

new variable: $\tilde{X} = X + 5$

$E(\tilde{X}) = -3 + 5 = 2$

$P(\tilde{X} \geq 4) = P(X \geq 1) \leq \frac{E(\tilde{X})}{4} = \frac{2}{4} = \frac{1}{2}$

(b) Y is a random variable such that $Y \leq 10$ and $\mathbb{E}[Y] = 1$. Find an upper bound for the probability of Y being less than or equal to -1.

$-Y \geq -10$ $\quad -Y + 10 \geq 0$ $\quad \tilde{Y} = -Y + 10$

$Y \leq -1$ $\quad -Y + 10 \geq 11$ $\quad E(\tilde{Y}) = 9$

$P(\tilde{Y} \geq 11) \leq \frac{9}{11}$

$1 \sim 6 \quad X_i$

(c) You roll a die 100 times. Let Z be the sum of the numbers that appear on the die throughout the 100 rolls. Compute $\mathrm{Var}(Z)$. Then use Chebyshev's inequality to bound the probability of the sum Z being greater than 400 or less than 300.

$$P(|X-\mu| \geq c) \leq \frac{\mathrm{Var}(X)}{c^2}$$

$25 + 36 = 61$

$$E(X_i) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

$$E(X_i^2) = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$$

$10 \quad 20 \quad 30 \quad 61 \quad 91$

$$\sigma(X_i) = E(X_i^2) - (E(X_i))^2 = \frac{91}{6} - \frac{49}{4} = \frac{70}{24} = \frac{35}{12}$$

$$E(Z) = 100 \times \frac{7}{2} = 350$$

$$\sigma(Z) = 100 \times \frac{35}{12}$$

$$P(|X - 350| \geq 50) \leq \frac{100 \times \frac{35}{12}}{50^2} = \frac{2 \times 35}{12 \times 50} = \frac{7}{60}$$

9 Exponential Distributions: Lightbulbs

Note 21

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

$$f(x) = \lambda e^{-\lambda x} \qquad E(\lambda) = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{50}$$

$$\int_0^{30} \frac{1}{50} e^{-\frac{1}{50}x} = -e^{-\frac{1}{50}x}\Big|_0^{30} = 1 - e^{-\frac{3}{5}}$$

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

$$\int_{30}^{+\infty} \frac{1}{50} e^{-\frac{1}{50}x} = -e^{-\frac{1}{50}x}\Big|_{30}^{+\infty} = e^{-\frac{3}{5}}$$

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

No memory $\quad e^{-\frac{3}{5}}$

$P(X \geq 60 \mid X \geq 30) =$

10 Continuous Probability Continued

Note 17

For the following questions, please briefly justify your answers or show your work.

(a) Assume $\text{Bob}_1, \text{Bob}_2, \ldots, \text{Bob}_k$ each hold a fair coin whose two sides show numbers instead of heads and tails, with the numbers on Bob_i's coin being i and $-i$. Each Bob tosses their coin n times and sums up the numbers he sees; let's call this number X_i. For large n, what is the distribution of $(X_1 + \cdots + X_k)/\sqrt{n}$ approximately equal to?

(b) If $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables of mean μ and variance σ^2, what is $\lim_{n\to\infty} \mathbb{P}\left[\sum_{k=1}^{n} \frac{X_k - \mu}{\sigma n^{\alpha}} \in [-1,1]\right]$ for $\alpha \in [0,1]$ (your answer may depend on α and Φ, the CDF of a $N(0,1)$ variable)? Y

(a) $E(X_i) = 0 \qquad \sigma^2(X_i) = n i^2$

$E(X_1 + \cdots + X_k) = 0 \qquad \sigma^2(X_1 + \cdots + X_k) = kni^2 \qquad \sigma = i\sqrt{k} \cdot \sqrt{n}$

$\frac{X - \mu}{i \cdot \sqrt{k} \cdot \sqrt{n}} \sim N(0,1) \qquad \frac{X-\mu}{\sqrt{n}} \sim N(0, ki^2) \qquad \left|\frac{Y}{\sigma\sqrt{n}}\right| \leq \frac{1}{n^{\frac{1}{2}-\alpha}} = n^{\alpha - \frac{1}{2}}$

(b) $\frac{Y}{\sqrt{n}\,\sigma} \sim N(0,1) \qquad -1 \leq \frac{Y}{\sigma n^{\alpha}} \leq 1$

$n \to \infty \qquad \Phi(n^{\alpha-\frac{1}{2}}) - \Phi(-n^{\alpha-\frac{1}{2}})$

11 Three Tails

Note 22

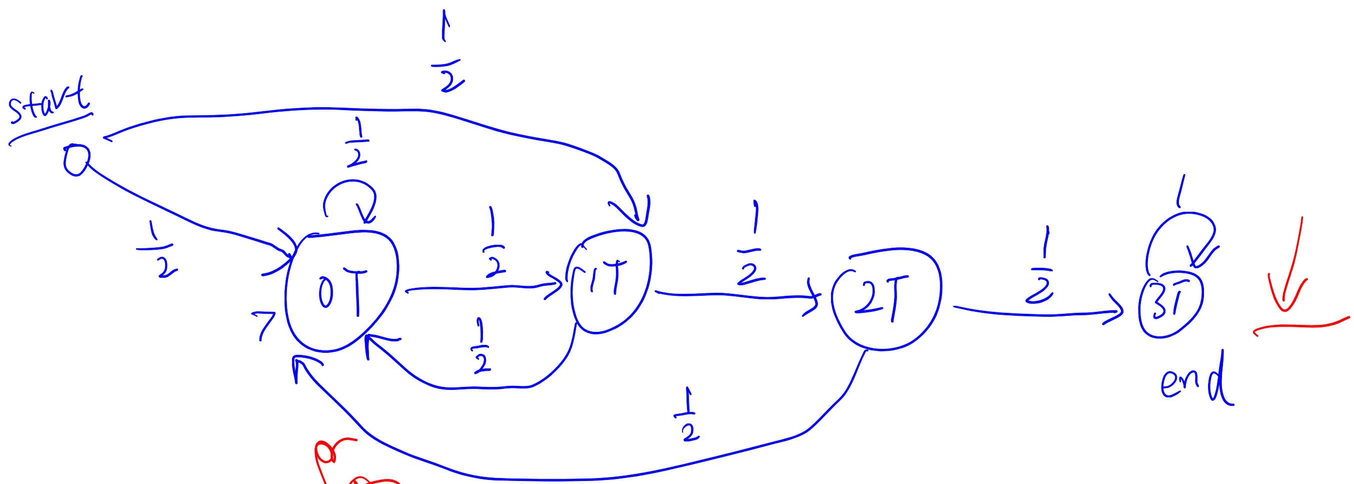
You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?

$\alpha > \frac{1}{2} \quad = 1$

$\alpha = \frac{1}{2} \quad \Phi(1) - \Phi(-1)$

$\alpha < \frac{1}{2} \quad 0$



$$\beta(s) = \cancel{0} \quad \text{or} \quad \tfrac{1}{2}\beta(0) + \tfrac{1}{2}\beta(1)$$

$$\beta(0) = 1 + \tfrac{1}{2}\beta(s) + \tfrac{1}{2}\beta(0) + \tfrac{1}{2}\beta(1) + \tfrac{1}{2}\beta(2)$$

$$\beta(1) = 1 + \tfrac{1}{2}\beta(s) + \tfrac{1}{2}\beta(0) \quad \times$$

$$\beta(2) = 1 + \tfrac{1}{2}\beta(1)$$

$$\beta(3) = 1 + \tfrac{1}{2}\beta(2)$$

$$\begin{cases} 2\beta(s) = \beta(0) + \beta(1) \\ \beta(0) = 2 + \beta(s) + \beta(1) + \beta(2) \\ 2\beta(1) = 2 + \beta(s) + \beta(0) \\ 2\beta(2) = 2 + \beta(1) \quad \times \\ 2\beta(3) = 2 + \beta(2) \end{cases}$$

$$\beta(1) = \tfrac{7}{4}\beta(0)$$

$$\beta(2) = \tfrac{1}{2}\beta(0)$$

Look the out, not in

memoryless

$\beta(t)$ ∝ 'pass to 0T's time

$$\beta(s) = \tfrac{1}{2}\beta(0) + \tfrac{1}{2}\beta(1)$$

$$\beta(0) = 1 + \tfrac{1}{2}\beta(0) + \tfrac{1}{2}\beta(1)$$

$$\beta(1) = \tfrac{1}{2}\beta(2) + \tfrac{1}{2}\beta(0)$$

$$\beta(2) = \tfrac{1}{2}\beta(0) + \tfrac{1}{2}\beta(3)$$

$$\beta(3) = 0$$

$$\begin{cases} \beta(s) = 7 \\ \beta(0) = 8 \\ \beta(1) = 6 \\ \beta(2) = 4 \\ \beta(3) = 0 \end{cases}$$

not steps

like hit