CS 70	Discrete Mathematics and Probability Theory	
Spring 2024	Seshia, Sinclair	HW 12

Due: Saturday, 4/13, 4:00 PM
Grace period until Saturday, 4/13, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Tellers

Note 17

Imagine that X is the number of customers that enter a bank at a given hour. To simplify everything, in order to serve n customers you need at least n tellers. One less teller and you won't finish serving all of the customers by the end of the hour. You are the manager of the bank and you need to decide how many tellers there should be in your bank so that you finish serving all of the customers in time. You need to be sure that you finish in time with probability at least 95%.

(a) Assume that from historical data you have found out that $\mathbb{E}[X] = 5$. How many tellers should you have?

(b) Now assume that you have also found out that $\text{Var}(X) = 5$. Now how many tellers do you need?

2 Polling Numbers

Note 17

Suppose the whole population of California has Democrats, Republicans, and no other parties. You choose N people independently and uniformly at random from the Californian population, and for each person, you record whether they are a Democrat or a Republican. We want to estimate the true percentage of Democrats among the polled Californians to within 1% with 95% confidence. According to Chebyshev's inequality, what is the minimum number of people you need to poll?

3 Tightness of Inequalities

Note 17

(a) Show by example that Markov's inequality is tight; that is, show that given some fixed $k > 0$, there exists a discrete non-negative random variable X such that $\mathbb{P}[X \geq k] = \mathbb{E}[X]/k$.

1. (a) $P[X \geq n] \leq \frac{E(X)}{n} = \frac{5}{n} \leq \frac{5}{100}$

$\Downarrow$

$n \geq 100 \rightarrow$ 99 is enough

(b) Y failed to serving all of the customers in time.

$Y = X \geq n \Rightarrow X-5 \geq n-5$

$\leq P(|X-5| \geq n-5) \leq \frac{5}{(n-5)^2} \leq \frac{5}{100}$

$(n-5)^2 \geq 100 \qquad n \geq 15 \rightarrow$ 14 is enough.

2. $\mu \quad \sigma^2 \qquad S_n = \sum_{i=1}^{N} X_i \qquad X_i \rightarrow$ support democrats. $p \Rightarrow \sigma^2$

$\hat{p} = \frac{S_n}{n} \qquad E(\hat{p}) = \frac{1}{n}\sum_{i=1}^{n} X_i = p \qquad \sigma^2 = p(1-p)$,

$\sigma(\hat{p}) = \frac{1}{n^2}\sum_{i=1}^{N} \sigma(X_i) = \frac{\sigma^2}{n} \qquad p = \frac{1}{2}$

$\sigma^2 = \frac{1}{4}$

$P[(\hat{p} - p) \geq 1\%] \leq \frac{\frac{\sigma^2}{n}}{(\frac{1}{100})^2} = \frac{10000\sigma^2}{n} \leq \frac{10000 \times \frac{1}{4}}{n} = \frac{2500}{n}$

$\leq 5\%$

$n \geq \frac{2500}{5\%} = 50000$

3. (a) get the result through the proof.

$X=0$ $\quad 1-p$ $\qquad E(X)=kp$

$X=k$ $\quad p$ $\qquad p[X\geq k]=p=\frac{E(X)}{k}$

(b) $P[|(X-\mu)|\geq c]\leq\frac{Var(X)}{c^2}$ $\rightarrow$ proof use (a).

same as (a)

$Y=(X-E(X))^2$ $\rightarrow$ only can be 0, c^2

$X-E(X)$ $\rightarrow$ let $E(X)=0$

+good.

$$\begin{cases} -a & \frac{1}{2k^2} \\ a & \frac{1}{2k^2} \\ 0 & 1-\frac{1}{k^2} \end{cases}$$

$\sigma=a$ $\qquad \sigma=\frac{a}{k}$ $\qquad \sigma^2=\frac{a^2}{k^2}$

$E(X)=0$

$P[|X|\geq a]=\frac{1}{k^2}$

(b) Show by example that Chebyshev's inequality is tight; that is, show that given some fixed $k \geq 1$, there exists a random variable X such that $\mathbb{P}[|X - \mathbb{E}[X]| \geq k\sigma] = 1/k^2$, where $\sigma^2 = \text{Var}(X)$.

4 Max of Uniforms

Note 21

Let $X_1, \ldots X_n$ be independent Uniform$(0,1)$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $\text{Var}(X)$?

5 Darts with Friends

Note 21

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

(i) What's the cumulative distribution function of X?

(ii) What's the cumulative distribution function of Y?

(iii) What's the probability density function of X?

(iv) What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max(X, Y)$?

$\downarrow$

not think about. x, y

jult the resulte

4. (a) $cdf(X_i) = x_i \rightarrow \underline{(0,1)}$

$pdf(x_i) = 1 \rightarrow (0,1)$

$X \leq x \Rightarrow X_1 \leq x \cap X_2 \leq x \cap \cdots \cap X_n \leq x$

$cdf(x) = x^n \quad \underline{(0,1)}$

$n \cdot x^n$

$\frac{n}{n+1} x^{n+1}$

(b) $pdf(x) = n \cdot x^{n-1} \quad \underline{(0,1)}$

(c) $E(x) = \int_0^1 x \, pdf(x) = \int_0^1 x \cdot n \cdot x^{n-1} dx = \frac{n}{n+1} x^{n+1} \Big|_0^1 = \frac{n}{n+1}$

(d) $E(x^2) = \int_0^1 x^2 pdf(x) = \int_0^1 n \cdot x^{n+1} dx = \frac{n}{n+2} x^{n+2} \Big|_0^1 = \frac{n}{n+2}$

$Var(x) = E(x^2) - [E(x)]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$

5. (a) $cdf(x) = \frac{\pi x^2}{\pi * 1^2} = x^2 \quad (0 < x < 1) \quad \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$

$cdf(y) = \frac{\pi * y^2}{\pi * 2^2} = \frac{y^2}{4} \quad (0 < y < 2) \quad \begin{cases} 0 & x < 0 \\ 1 & x \geq 2 \end{cases}$

$pdf(x) = 2x \quad (0 < x < 1)$

$pdf(y) = \frac{y}{2} \quad (0 < y < 2)$

$pdf(x, y) = \underline{xy}$

(b) $\underline{P(x \leq y)} = \int_0^1 \int_x^2 xy \, dy \, dx$

$= \int_0^1 x * \frac{1}{2} y^2 \Big|_x^2 \, dx$

$= \int_0^1 (2x - \frac{x^3}{2}) dx = (x^2 - \frac{x^4}{8}) \Big|_0^1 = \frac{7}{8}$

$P(x \geq y) = 1 - P(x \leq y) = \frac{1}{8}$

(c) $\underline{\max(x, y)} \leq u$

$\rightarrow (x \leq u \cap y \leq u)$

$P(U \leq u) = \underline{P(x \leq u) \, P(y \leq u)}$

$= u^2 \times \frac{u^2}{4} = \frac{u^4}{4}$ $\quad \underline{u \in [0, 1]}$

$u \in \underline{[1, 2]}$ $\quad P[x \leq u] = 1$

$P(U \leq u) = 1 \times \frac{u^2}{4} = \frac{u^2}{4}$