CS 70 Spring 2024	Discrete Mathematics and Probability Theory Seshia, Sinclair	HW 11

Due: Saturday, 4/6, 4:00 PM
Grace period until Saturday, 4/6, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Fishy Computations

Note 19

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2024?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

(d) Denote $X \sim \text{Pois}(\lambda)$. Prove that

$$\mathbb{E}[Xf(X)] = \lambda\,\mathbb{E}[f(X+1)]$$

for any function f.

2 Such High Expectations

Note 19

Suppose X and Y are independently drawn from a Geometric distribution with parameter p.

(a) Compute $\mathbb{E}[\min(X,Y)]$.

(b) Compute $\mathbb{E}[\max(X,Y)]$.

1. $P[X=i] = \frac{\lambda^i}{i!} e^{-\lambda}$

(a) $\sim$ Poisson (20)

$$P[X=7] = \frac{20^7}{7!} e^{-20} \approx 0.0005235$$

(b) $\sim$ Possion (2)

$$P[X=0 \cup X=1] = e^{-2} + \frac{2}{1!} e^{-2} = 3e^{-2} = 0.406$$

(c) $5.7 * 2 = 11.4$ $\sim$ Possion(11.4) $\frac{11.4^i}{i!} e^{-11.4}$

$$1 - P[X=0 \cup X=1 \cup X=2]$$

$$= 1 - e^{-11.4} - 11.4 e^{-11.4} - \frac{11.4^2}{2!} e^{-11.4} \approx 0.999134$$

(d) $E[Xf(X)] = \sum_{i=0}^{\infty} P[X=i]\, i f(i) = \sum_{i=1}^{\infty} P[X=i] * f(i)$

$$\sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!} e^{-\lambda} f(i)$$

$$E[f(X+1)] = \sum_{i=0}^{\infty} P[X=i] * f(i+1)$$

$$= \sum_{i=1}^{\infty} P[X=i-1] * f(i) = \sum_{i=1}^{\infty} \frac{\lambda^{(i-1)}}{(i-1)!} e^{-\lambda} f(i)$$

$$E[Xf(X)] = \lambda E[f(X+1)]$$

2. $E(X) = \frac{1}{p}$ $\quad Var(X) = \frac{1-p}{p^2}$

$P[X=i] = (1-p)^{i-1} p$

$(1-p)^{2i-2} p^2$

$A = \min(X, Y)$

$E[\min(X, Y)]$

$P[A=i] = P[X=i \cap y>i] + P[x>i \cap y=i] + P[x=i \cap y=i]$

$= (1-p)^{i-1} p * P(y>i) + (1-p)^{2i-2} p^2$ Too complicated

$E(A) = \sum_{i=1}^{\infty} 2i (1-p)^{i-1} p * \underbrace{P(y>i)}_{(1-p)^i} + \sum_{i=1}^{\infty} i p^2 (1-p)^{2i-2}$.

$\sum_{i=1}^{\infty} 2i (1-p)^{2i-1} p + i p^2 (1-p)^{2i-2}$

$(1-p)^{i-1}$

$i p (1-p)^{2i-1} [2 + p(1-p)^{-1}]$

$\frac{p+2-2p}{1-p}$

$p[2+p(1-p)] \sum_{i=1}^{\infty} i (1-p)^{2i-1}$

$\frac{2-p}{1-p}$

tail probolity

$P(A \geq t) = P(X \geq t \cap y \geq t)$

$= P(X \geq t) * P(y \geq t) = (1-p)^{t-1} * (1-p)^{t-1}$

$= (1-p)^{2t-2} = ((1-p)^2) \sim$ Geometric$(1-(1-p)^2)$

$E(A) = \frac{1}{1-(1-p)^2}$ tail sum formula

(b) $\max(X, Y) = B$

$$P(B \geq i) = P(X \geq i \cup Y \geq i)$$

$(1-p)^{i-1} + (1-p)^{i-1}$

$$= P(X \geq i) + P(Y \geq i) - P(X \geq i \cap Y \geq i)$$

$$= 2(1-p)^{i-1} - \left((1-p)^{i-1}\right)^2$$

$$= (1-p)^{i-1}\left(2 - (1-p)^{i-1}\right)$$

$$E(B) = \sum_{i=1}^{\infty} P(B \geq i) \leftarrow \text{Tail sum formual}$$

$$= \sum_{i=1}^{\infty} 2(1-p)^{i-1} - \sum_{i=1}^{\infty} \left((1-p)^2\right)^{i-1}$$

$$\Rightarrow \sum_{i=0}^{\infty} x^i = \frac{1-x^i}{1-x}$$

$$= 2 * \frac{1}{p} - \frac{1}{1-(1-p)^2}$$

$i \to \infty$, $x < 1$

$$\to \frac{1}{1-x}$$

3 Diversify Your Hand

Note 15
Note 16

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $\mathbb{E}[X]$. (Hint: Consider indicator variables X_i representing whether i appears in the hand.)

(b) Calculate $\text{Var}(X)$.

4 Swaps and Cycles

Note 15

We'll say that a permutation $\pi = (\pi(1), \dots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \dots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *k-cycle* if there exist $i_1, \dots, i_k \in \{1, \dots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \dots, \pi(i_k) = i_1$. Compute the expectation of the number of k-cycles.

5 Double-Check Your Intuition Again

Note 16

(a) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

(i) What is $\text{cov}(X+Y, X-Y)$?

(ii) Prove that $X+Y$ and $X-Y$ are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

(b) If X is a random variable and $\text{Var}(X) = 0$, then must X be a constant?

(c) If X is a random variable and c is a constant, then is $\text{Var}(cX) = c\,\text{Var}(X)$?

(d) If A and B are random variables with nonzero standard deviations and $\text{Corr}(A, B) = 0$, then are A and B independent?

(e) If X and Y are not necessarily independent random variables, but $\text{Corr}(X, Y) = 0$, and X and Y have nonzero standard deviations, then is $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$?

The two subparts below are **optional** and will not be graded but are recommended for practice.

(f) If X and Y are random variables then is $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$?

(g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\text{Corr}(\max(X,Y), \min(X,Y)) = \text{Corr}(X,Y)?$$

3. (a) $E[X] = E\left[\sum_{i=1}^{13} X_i\right] = \sum_{i=1}^{13} E[X_i]$

$$= 13 \times \left[1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right] = 4.43$$

(b) $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = E\left[\left(\sum_{i=1}^{13} X_i\right)^2\right] = E\left(\underbrace{\sum_{i=1}^{13} X_i^2}_{?} + [13^2 - 13]\, \underline{X_i X_j}\right)$$

$= E(X)$

$\underline{X_i X_j} = 1 \rightarrow \underline{X_i = 1 \cap X_j = 1}$

$$P(X_i X_j = 0) = \underbrace{P(X_i = 0) + P(X_j = 0)} - P(X_i = 0 \cap X_j = 0)$$

$$= 2\left[\frac{\binom{48}{5}}{\binom{52}{5}}\right] - \frac{\binom{44}{5}}{\binom{52}{5}}$$

$P(X_i X_j = 1) = 1 - P(X_i X_j = 0) = \underline{0,1002}$

$E(X^2) = E[X] + [13^2 - 13] * 0.1002 = 20.0629$

$Var(X) = 0.3932$

4 (a) X_i = position i have a swap

$n=2 \to E(X_n) = \frac{1}{2}$

$P(i, j \text{ are swapped}) = \frac{(n-2)!}{n!}$

$\downarrow I_{ij}$

$= \frac{1}{n(n-1)}$

$n=3$?

$\underbrace{\quad}_{n-1}$ n

$$E(N) = E\left(\sum_{(i,j) \subset (1 \dots n)} I_{ij}\right) = \binom{n}{2} \frac{1}{n(n-1)} = \frac{1}{2}$$

(b) $X_k \to k$ cycle

$i=i-1$

i_j $\frac{1}{2!}$

$X_2 \to$ specify the k

?

$$E(k) = \sum_{i=1}^{\infty} P(k \geq i) = \sum_{i=1}^{\infty} = \frac{1}{n} + \sum_{i=2}^{n} \frac{1}{1*(n-1)\dots(n-i+2)}$$

$$P(k \geq 2) = (n-1)\frac{1}{n(n-1)} = \frac{1}{n}$$

X 无必要是连续的。

$\pi(i_1) = i_2 \quad \pi(i_2) = i_3 \quad \dots \quad \pi(i_k) = i_1$

$$P(k \geq 3) = (n-2) * \frac{(n-3)!}{n!}$$

k position contains $(i_1, \dots i_k)$.

$$P(k \geq i) = (n-i+1) * \frac{(n-i)!}{n!}$$

$\binom{n}{k}$ subset $(i_1, \dots i_2, \dots i_k)$

$\downarrow$ cycle ordering.

$$= \frac{1}{n*(n-1)\dots(n-i+2)}$$

$(i_1, i_2 \dots i_k)$ $(i_2 \dots i_k, i_1)$ are the same,

have k redundant

for a selected $(i_1 \ldots i_2 \ldots i_k)$ is a total ordering

$$P(I_{(i_1 \ldots i_k)}) = \frac{(n-k)! * (k-1)!}{n!} = \frac{(n-k)! * k!}{n! \quad k}$$

$$E(N) = \frac{\cancel{n!}}{\cancel{(n-k)!}\ \cancel{k!}} \times \frac{\cancel{(n-k)!} * \cancel{k!}}{\cancel{n!} * k} = \frac{1}{k}$$

select a subset from the total set and analysis

5 (a) $Cov(X+Y,\ X-Y) =$

$$\boxed{Cov(X, Y) = E(XY) - E[X]E[Y]}$$

(i)
$$E(X) = E(Y) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$E(X+Y) = \frac{21}{3} \qquad E(X-Y) = 0$$

$$E((X+Y)(X-Y)) = E(X^2 - Y^2) = 0$$

$$Cov(X+Y,\ X-Y) = 0$$

(ii) $A : X+Y$ $\quad B : X-Y$

$$P(X+Y=2 \mid X-Y=0)$$

$$P(X+Y=2) = \frac{1}{36}$$

$$= \frac{P((X+Y=2) \cap (X-Y=0))}{P(X-Y=0)}$$

$$P(X-Y=0) = \frac{6}{36} = \frac{1}{6}$$

$$= \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6} \neq \frac{1}{36}$$

(b) Yes $\quad Var(X) = E(X^2) - (E(X))^2 = 0$

$E(X^2) = (E(X))^2$

$Var(X) = E[(X-\mu)^2] = 0$

X must be μ

(c) $Var(cX) = E(c^2X^2) - (E(cX))^2$

NO $\quad = c^2[E(X^2) - (E(X))^2] = c^2 \, Var(X)$

(d) NO, see (a)

(e) Yes $\quad Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

(f). Yes

(g) NO $\quad min(X, Y) \quad max(X, Y)$ may not be indepent.

$X, Y \in (0, 1)$