

Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Master Theorem

For solving recurrence relations asymptotically, it often helps to use the *Master Theorem*:

Master Theorem. If $T(n) = aT(n/b) + \mathcal{O}(n^d)$ for $a > 0$, $b > 1$, and $d \geq 0$, then

$$T(n) = \begin{cases} \mathcal{O}(n^d) & \text{if } d > \log_b a \\ \mathcal{O}(n^d \log n) & \text{if } d = \log_b a \\ \mathcal{O}(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Note: You can replace \mathcal{O} with Θ and you get an alternate (but still true) version of the Master Theorem that produces Θ bounds.

$d_{crit} = \log_b a$ is called the critical exponent. Notice that whichever of d_{crit} and d is greater determines the growth of $T(n)$, except in the case where they are perfectly balanced.

Solve the following recurrence relations and give a \mathcal{O} bound for each of them.

(a) (i) $T(n) = 3T(n/4) + 4n$

(ii) $T(n) = 45T(n/3) + 1n^3$

(i)

$a=3 \quad b=4, \quad d=1$

$\log_b a = \log_4 3 < 1 \quad \mathcal{O}(n)$

(ii) $a=45 \quad b=3 \quad d=3$

$\log_b a = \log_3 45 > d$

$\mathcal{O}(n^{\log_3 45})$

(b) $T(n) = 2T(\sqrt{n}) + 3$, and $T(2) = 3$.

Hint: Try repeatedly expanding the recurrence.

$T(n) = 2T(\sqrt{n}) + 3 \Rightarrow (2T(\sqrt[4]{n}) + 3) + 3$

$= 2^{\log_2 n} * 3 + 3 + 2 * 3 + 2^2 * 3 + \dots + 2^{\log_2 n} * 3$

$= 3 \log_2 n + 3 * \left[\frac{1 - 2^{1 + \log_2 n}}{1 - 2} \right]$

$2^k = \log n$

$k = \log(\log n)$

$n^{\frac{1}{2^k}} = 2$

$\frac{1}{2^k} = \log n$

(c) Consider the recurrence relation $T(n) = 2T(n/2) + n \log n$. We can't plug it directly into the Master Theorem, so solve it by adding the size of each layer.

Hint: split up the $\log(n/(2^i))$ terms into $\log n - \log(2^i)$, and use the formula for arithmetic series.

↓ layers then calculate it

2 Sorted Array

Given a sorted array A of n (possibly negative) distinct integers, you want to find out whether there is an index i for which $A[i] = i$. Devise a divide-and-conquer algorithm that runs in $O(\log n)$ time.

look at $\lfloor \frac{n}{2} \rfloor$ is = return true

if $<$ look at right half

index i bigger \geq look at left half

$$\lfloor \frac{n}{2} \rfloor \quad T(n) = T\left(\frac{n}{2}\right) + \underline{O(1)}$$

$$\log_2 1 = \underline{0} = \underline{n^0}$$

$$\underline{O(\log n)}$$

3 Quantiles

Let A be an array of length n . The boundaries for the k quantiles of A are $\{\underline{a^{(n/k)}}, a^{(2n/k)}, \dots, a^{((k-1)n/k)}\}$ where $a^{(\ell)}$ is the ℓ -th smallest element in A .

Devise an algorithm to compute the boundaries of the k quantiles in time $\mathcal{O}(n \log k)$. For convenience, you may assume that k is a power of 2.

Hint: Recall that $\text{QUICKSELECT}(A, \ell)$ gives $a^{(\ell)}$ in $\mathcal{O}(n)$ time.

$$a^{(\frac{\frac{n}{2}}{2})} = a^{(\frac{n}{4})}$$

$$\begin{array}{c} \swarrow \\ \frac{n}{2} \\ \downarrow \end{array}$$

$$\begin{array}{c} \swarrow \\ \frac{n}{2} \\ \downarrow \end{array} \rightarrow \underline{\mathcal{O}(n)}$$

complex - time analysis : $\frac{n}{2^i}$ $\underline{2^i = k \rightarrow \text{will stop}}$
 $\underline{i = \log k}$

4 Complex numbers review

A *complex number* is a number that can be written in the rectangular form $a + bi$ (i is the imaginary unit, with $i^2 = -1$). The following famous equation (*Euler's formula*) relates the polar form of complex numbers to the rectangular form:

$$re^{i\theta} = r(\cos \theta + i \sin \theta)$$

In polar form, $r \geq 0$ represents the distance of the complex number from 0, and θ represents its angle. The n roots of unity are the n complex numbers satisfying $\omega^n = 1$. They are given by

$$\omega_k = e^{2\pi i k/n}, \quad k = 0, 1, 2, \dots, n-1$$

- (a) Let $x = e^{2\pi i 3/10}$, $y = e^{2\pi i 5/10}$ which are two 10-th roots of unity. Compute the product $x \cdot y$. Is this a root of unity? Is it an 10-th root of unity?

What happens if $x = e^{2\pi i 6/10}$, $y = e^{2\pi i 7/10}$?

$$e^{2\pi i k/10}$$

$$x \cdot y = e^{2\pi i 8/10} \leftarrow \text{res}$$

$$x \cdot y = e^{2\pi i 13/10} = e^{2\pi i} \cdot e^{2\pi i 3/10} \rightarrow \text{also is root.}$$

- (b) Show that for any n -th root of unity ω , $\sum_{k=0}^{n-1} \omega^k = 0$, when $n > 1$.

Hint: Use the formula for the sum of a geometric series $\sum_{k=0}^{n-1} \alpha^k = \frac{\alpha^n - 1}{\alpha - 1}$. It works for complex numbers too!

$$\sum_{k=0}^{n-1} \omega^k = \frac{\omega^n - 1}{\omega - 1} = 0$$

- (c) (i) Find all ω such that $\omega^2 = -1$.

$$\omega = re^{i\theta}$$

$$r^2 e^{2i\theta} = e^{i\pi}$$

$$r=1$$

$$2\theta = \pi + 2\pi k$$

$$\theta = \frac{\pi}{2} + \pi k$$

$$\theta = \frac{\pi}{2}$$

- (ii) Find all ω such that $\omega^4 = -1$.

$$\theta = \pi + 2\pi k$$

$$\theta = \frac{3}{2}\pi$$

$$\theta = \frac{\pi}{4} + \frac{\pi}{2}k$$

$$k=1 \quad \frac{3}{4}\pi$$

$$k=0 \quad \frac{\pi}{4}$$

$$k=2 \quad \frac{5}{4}\pi$$

$$k=3 \quad \frac{7}{4}\pi$$

$$e^{2\pi i \cdot \frac{1}{8}} \quad \frac{3}{8}$$

$$\frac{5}{8} \quad \frac{7}{8}$$