1 Inclusion and Exclusion

Note 10

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

Solution: It is sufficient to count the opposite: what is the total number of positive integers strictly less than 100 and *not* coprime to 100?

If a number is not coprime to 100, this means that the number is either a multiple of 2 or a multiple of 5. In this case, we have:

- 49 multiples of 2
- 19 multiples of 5
- 9 multiples of both 2 and 5

By inclusion-exclusion, the total number of positive integers not coprime to 100 is $49+19-9=59$, and there are 99 positive integers strictly less than 100.

As such, in total there are $99-59=40$ different positive integers strictly less than 100 that are coprime to 100.

2 CS70: The Musical

Note 10

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

(b) Edward would now like to select a crew out of n people. Use this to provide a combinatorial argument that proves the following identity: (this is called Pascal's Identity)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}.$$

Solution:

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

(a) Both directors from the group of experienced directors,

(b) Both directors from the group of inexperienced directors, or

(c) One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

3 Farmer's Market

Note 10

Suppose you want k items from the farmer's market. Count how many ways you can do this, assuming:

(a) There are pumpkins and apples at the market.

(b) There are pumpkins, apples, oranges, and pears at the market.

(c) There are n kinds of fruits at the market, and you want to end up with at least two different types of fruit.

Solution:

This is a classic "balls and bins" (also known as "stars and bars") problem.

(a) $k+1$. We can have 0 pumpkins and k apples, or 1 pumpkin and $k-1$ apples, etc. all the way to k pumpkins and 0 apples. We can equivalently think about this as k balls and 2 bins, or k stars and 1 bar, giving us $\binom{k+1}{1} = \binom{k+1}{k}$.

(b) $\binom{k+3}{3}$. We have k balls and 4 bins, or k stars and 3 bars.

(c) There are $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ ways to choose k fruits of n types with no additional restrictions (i.e. k balls and n bins, or k stars and $n-1$ bars). n of these combinations, however, contain only one variety of fruit, so we subtract them for a total of $\binom{n+k-1}{n-1} - n = \binom{n+k-1}{k} - n$.

4 The Count

Note 10

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

Solution:

(a) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

(b) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

(c) This problem is a bit trickier to approach, since there is a strong possibility of overcounting - it is not sufficient to just choose 5 consecutive positions to be 00000, and let the rest of the positions be arbitrary values.

One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins (we look at the leftmost zero of a run of at least 5 zeros). It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits.

If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1 = 4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of valid passwords is $2^5 + 5 \cdot 2^4 = 112$. Note that, since there are only 10 digits, there can only be one occurence of the "100000" pattern.