CS 70 Spring 2024 — Discrete Mathematics and Probability Theory — Seshia, Sinclair — DIS 2B

1 Short Answers

Note 5

In each part below, provide the number/equation and brief justification.

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

$v+f=e+2$ $\quad f=e+2-v=5+2=7$

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

$3*2^2=12$ edges $\quad$ 8 $\quad$ 7

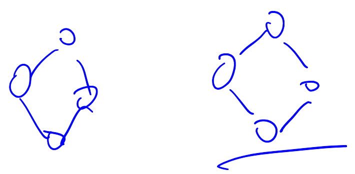
$12-7=5$

(c) The Euler's formula $v-e+f=2$ requires the planar graph to be connected. What is the analogous formula for planar graphs wth k connected components?

$v-e+f=k+1$

$\cancel{v-e+f=3k-1}$

$v-e+f+k-1=2k$



$v_1-e_1+f_1=2$

$\vdots$

$v_k-e_k+f_k=2$

$v-e+f_1+\dots+f_k=2k$

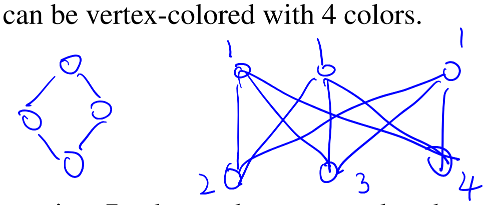
$\cancel{v-e+f-k+1=2k}$

2 Always, Sometimes, or Never

Note 5

In each part below, you are given some information about a graph G. Using only the information in the current part, say whether G will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

(a) G can be vertex-colored with 4 colors.



either

(b) G requires 7 colors to be vertex-colored.

→ planar can be color with 4.

non-planar

(c) $e \leq 3v - 6$, where e is the number of edges of G and v is the number of vertices of G.

either → $K_{3,3}$ → $e=9$, $V=6$ $e \leq 3v-6$ non-planar.

 → $e=3$ $V=3$ planar.

(d) G is connected, and each vertex in G has degree at most 2.

planar

non-planar have $K_{5,5}$ or $K_{3,3}$.

the degree of them is greater than 2.

(e) Each vertex in G has degree at most 2.

planar

not induction on degree

3 Graph Coloring

Note 5

Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

$k=1$ true.

induction on vertex numbers

hypothesis: $1 \leq k \leq n$ is true.

$k=n+1$, remove vertice at degree $n+1$ and all its' edge one by one until there is no degree $n+1$ nodes; the removed vertix must not be adjacent; the remail graph can be colored with at most $n+1$ color.

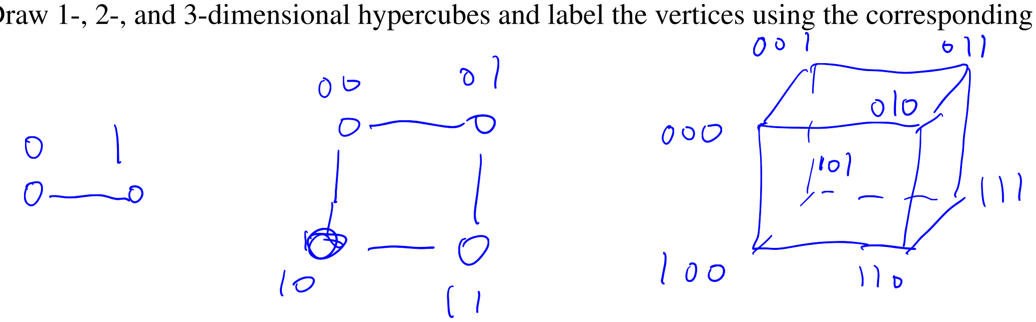
add removed vertices back, at most $n+2$ color

4 Hypercubes

Note 5

The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$ (recall that $\{0,1\}^n$ denotes the set of all n-bit strings). There is an edge between two vertices x and y if and only if x and y differ in exactly one bit position.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.



(b) Show that the edges of an n-dimensional hypercube can be colored using n colors so that no pair of edges sharing a common vertex have the same color.

all degree is n

Induction

$0\,a_1 \ldots a_{n-1} \to n-1$ color

$\updownarrow \to 1$ color

$1\,a_1 \ldots a_{n-1}$

$\downarrow$ n color

(c) Show that for any $n \geq 1$, the n-dimensional hypercube is bipartite.

$0\,a_1 \ldots a_{n-1} \to$ one part

$1\,a_1 \ldots a_{n-1} \to$ one part

will be bipartite.