CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair
HW 01

Due: Saturday, 1/27, 4:00 PM
Grace period until Saturday, 1/27, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Logical Equivalence?

Note 1 Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \forall x\, P(x) \vee \forall x\, Q(x)$

(c) $\exists x\ (P(x) \vee Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \vee \exists x\, Q(x)$

(d) $\exists x\ (P(x) \wedge Q(x)) \stackrel{?}{\equiv} \exists x\, P(x) \wedge \exists x\, Q(x)$

2 Prove or Disprove

Note 2 For each of the following, either prove the statement, or disprove by finding a counterexample.

(a) $(\forall n \in \mathbb{N})$ if n is odd then $n^2 + 4n$ is odd.

(b) $(\forall a, b \in \mathbb{R})$ if $a + b \leq 15$ then $a \leq 11$ or $b \leq 4$.

(c) $(\forall r \in \mathbb{R})$ if r^2 is irrational, then r is irrational.

(d) $(\forall n \in \mathbb{Z}^+)$ $5n^3 > n!$. (Note: $\mathbb{Z}^+$ is the set of positive integers)

(e) The product of a non-zero rational number and an irrational number is irrational.

1. Logical Equivalence

(a) True

$\forall x (P(x) \wedge Q(x)) \overset{\Longrightarrow}{\Longleftarrow} \forall x P(x) \wedge \forall x Q(x)$

(b) False, $P(x) = \begin{cases} \text{true} & x = \\ \text{false} & x \neq 1 \end{cases}$ $\quad Q(x) = \begin{cases} \text{False} & x = 1 \\ \text{true} & x \neq 1 \end{cases}$

then $\forall x (P(x) \vee Q(x))$ is true, while $\forall x P(x) \vee \forall x Q(x)$ is False

(c) True $\quad \exists x (P(x) \vee Q(x)) \Rightarrow \exists x P(x) \vee \exists x Q(x)$

$\Leftarrow$

(d) False $\quad \exists x (P(x) \wedge Q(x)) \Rightarrow \exists x P(x) \wedge \exists x Q(x)$

$\not\Leftarrow$ the same example as (b)

2. proof or Disprove

(a) True let $n = 2k+1$ ($k \geq 0$, $k \in N$)

direct proof

$(2k+1)^2 + 4(2k+1) = 4k^2 + 4k + 1 + 8k + 4 = 4k^2 + 12k + 5 = 2(2k^2 + 6k + 2) + 1$

is odd

(b) True, proof by contraposition

$a+b \leq 15 \Rightarrow a \leq 11 \vee b \leq 4 \Longleftrightarrow \underbrace{a > 11 \wedge b > 4 \Rightarrow a+b > 15}_{\uparrow \text{ this is true}}$

then (b) is true

(c) True, proof by contraposition

equal to if r is rational, then r^2 is rational.

$r = \frac{b}{a} \Rightarrow r^2 = \frac{b^2}{a^2}$

(d) False $\quad n = 7, \quad 5n^3 = 1715 < n! = 5040$

(e) True, proof by contradiction.

Hypothesis: there exist a none zero rational number $\frac{a}{b}$ ($a, b \in Z$ $b \neq 0, a \neq 0$) and an irrational number c that $\frac{a}{b} * c$ is rational

exist ($d, e \in Z$, $e \neq 0$) that $\frac{a}{b} * c = \frac{d}{e}$

$a \neq 0$, so $c = \frac{d}{e} \times \frac{b}{a} = \frac{bd}{ae}$

$bd \in Z$, $ae \in Z$ then c is rational

contradiction here, proof done.

3 Twin Primes

Note 2

(a) Let $p > 3$ be a prime. Prove that p is of the form $3k+1$ or $3k-1$ for some integer k.

(b) *Twin primes* are pairs of prime numbers p and q that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

4 Airport

Note 3

Suppose that there are $2n+1$ airports, where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

5 A Coin Game

Note 3

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of n coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be $3 \cdot 2 = 6$). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of n coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be $\frac{n(n-1)}{2}$. (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

6 Grid Induction

Note 3

Pacman is walking on an infinite 2D grid. He starts at some location $(i, j) \in \mathbb{N}^2$ in the first quadrant, and is constrained to stay in the first quadrant (say, by walls along the x and y axes).

Every second he does one of the following (if possible):

(i) Walk one step down, to $(i, j-1)$.

(ii) Walk one step left, to $(i-1, j)$.

For example, if he is at $(5,0)$, his only option is to walk left to $(4,0)$; if Pacman is instead at $(3,2)$, he could walk either to $(2,2)$ or $(3,1)$.

Prove by induction that no matter how he walks, he will always reach $(0,0)$ in finite time.

(*Hint*: Try starting Pacman at a few small points like $(2,1)$ and looking all the different paths he could take to reach $(0,0)$. Do you notice a pattern in the number of steps he takes? Try to use this to strengthen the inductive hypothesis.)

3. Twin Primes

(a) direct proof.

If p is a prime the $p\%3 = 1$ or $p\%3 = 2$

if $p\%3 = 1$, the $p = 3k+1$ $k = \lfloor \frac{p}{3} \rfloor$

if $p\%3 = 2$ the $p = 3k-1$ $k = \lceil \frac{p}{3} \rceil$

(b) 3, 5 and 5, 7 is all twin primes,

so 5 is the prime number that takes part in two different twin prime pairs

Now, need to proof 5 is the only number

use contradiction

Hypothesis there exisit a prime $p > 5$, which takes part in two twins.

then $p-2$, p, $p+2$ is all prime

if p is prime, then $p = 3k+1$ or $p = 3k-1$ according to (a)

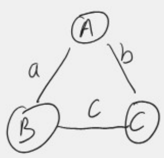
if $p = 3k+1$, then $p+2 = 3k+3 = 3(k+1)$, which is not a prime. contradiction

is $p = 3k-1$, then $p-2 = 3k-3 = 3(k-1)$, which is not a prime, contradiction.

so hypothesis is wrong, 5 is the only one

4. air port

Base case: $n=1$, $2n+1 = 3$



→ let $d = \min(a, b, c)$

then airport not in the edge of d has no air planes destined to it.

beaces the airport at edge d will choose to fly to each other.

Induction hypothesis: for all $n \geq 1$, the proposition is true

Inductive step: for $n+1$, there will be $2n+3$ airport

find the two airport A, B that have the minimum distance,

this two airport will fly to each other,

for the other $2n+1$ airport, if none of thom fly to A, or B,

according to hypothesis, there will be an airport has no planes destined to it

if at least one of them fly to A or B, then the $2n+1$ airport will only have less than $2n+1$ airport fly to them, so there will be an airport has no planes destined to it

→ remove A, B from airport then add back

5. A Coin Game

Proof by Induction choose. $n=1$ as the base case, will be easier

Base case: If $n=2$, only one choice, $1*1=1=\frac{2\times(2-1)}{2}$

Induction hypotheise. $\forall n \geq 2$, total score will be $\frac{n(n-1)}{2}$

for $n+1$, assume first split it as k, $n+1-k$.

$n \geq k \geq 1$ if $k=1$, or $k=n$

then split into 1 and n, the total score of stack n is $\frac{n(n-1)}{2}$

so $n+\frac{n(n-1)}{2}=\frac{n^2+n}{2}=\frac{(n+1)n}{2}$;

if $1<k<n$, then $1<n+1-k<n$

according to the hypothesis,

$$\text{total score} = \frac{k(k-1)}{2}+\frac{(n+1-k)(n+1-k-1)}{2}+k*((n+1)-k)$$

$$=\frac{k^2-k+(n-k)(n+1-k)+2k(n+1-k)}{2}$$

$$=\frac{k^2-k+(n+1-k)(n+k)}{2}=\frac{k^2-k+n^2+nk+n+k-nk-k^2}{2}$$

$$=\frac{n^2+n}{2}=\frac{(n+1)*n}{2}$$ proof done

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7 (Optional) Calculus Review

In the probability section of this course, you will be expected to compute derivatives, integrals, and double integrals. This question contains a couple examples of the kinds of calculus you will encounter.

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t}\,\mathrm{d}t\,.$$

(b) Compute the values of $x \in (-2,2)$ that correspond to local maxima and minima of the function

$$f(x) = \int_0^{x^2} t\cos\left(\sqrt{t}\right)\mathrm{d}t\,.$$

Classify which x correspond to local maxima and which to local minima.

(c) Compute the double integral

$$\iint_R 2x + y\,\mathrm{d}A\,,$$

where R is the region bounded by the lines $x = 1$, $y = 0$, and $y = x$.