

Due: Saturday, 3/23, 4:00 PM
Grace period until Saturday, 3/23, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Symmetric Marbles

Note 14

A bag contains 4 red marbles and 4 blue marbles. Rachel and Brooke play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Rachel wins if there are more red than blue marbles, and Brooke wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

- Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?
- What is the probability that Rachel wins the game?
- Given that Rachel wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Rachel wins if the third marble is red, and Brooke wins if the third marble is blue. All other rules stay the same.

- What is the probability that the third marble is red?
- Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k .
- Given that the third marble is red, what is the probability that Rachel wins the game?

2 Man Speaks Truth

Note 14

Consider a man who speaks the truth with probability $\frac{3}{4}$.

1. (a) No

$$P(A_1) = \frac{1}{2}$$

$$P(A_2) = \frac{1}{2}$$

$$P(A_1 \cap A_2) = \frac{\binom{4}{2}}{\binom{8}{2}} = \frac{\frac{4 \times 3}{2 \times 1}}{\frac{2 \times 7}{2 \times 1}} = \frac{3}{14} \neq P(A_1) * P(A_2)$$

(b) 3 r 1b

$$\frac{4r}{\binom{8}{4}} = \frac{\binom{3}{4} * \binom{1}{4}}{\binom{8}{4}} + \frac{\binom{4}{4}}{\binom{8}{4}}$$

$$\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}$$

$$= \frac{4 \times 4 + 1}{70} = \frac{17}{70}$$

(c) $P(E|R) = \frac{P(\bar{E}) * P(R|\bar{E})}{P(R)} = \frac{\frac{1}{70}}{\frac{17}{70}} = \frac{1}{17}$

(d) $P(F) = \frac{8}{12} = \frac{2}{3}$

(e) $P(F|G) = \frac{k}{4}$

(f) $P(\bar{I}|F) = \frac{P(I \cap F)}{P(F)} = \frac{P(I \cap F \cap G_0) + P(I \cap F \cap G_1)}{\frac{2}{3}}$

$$P(G_k) * P(F|G_k) * P(I|F \cap G_k) \quad k=2, 3, 4$$

$$P(G_k) = \frac{\binom{k}{8} \binom{4-k}{4}}{\binom{4}{12}}$$

$$\frac{\binom{8}{2} \binom{4}{2}}{\binom{12}{4}} = \frac{\frac{8 \times 7}{2 \times 1} \times \frac{4 \times 3}{2 \times 1}}{\frac{8 \times 7 \times 6 \times 5}{(4 \times 3 \times 2 \times 1) \times 10 \times 9}} = \frac{28 \times 6}{55 \times 9} = \frac{56}{165}$$

$$\frac{\binom{8}{3} \binom{4}{1}}{\binom{12}{4}} = \frac{\frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{4 \times 3 \times 2}{3 \times 2 \times 1}}{11 \times 5 \times 9} = \frac{56 \times 4}{11 \times 5 \times 9}$$

$$\frac{\binom{8}{4} \binom{4}{0}}{\binom{12}{4}} = \frac{\frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}}{11 \times 5 \times 9} = \frac{14}{99}$$

to complicated
to lose.

all other three is blue.

$$\frac{456}{165} \times \frac{2}{4} + \frac{56 \times 4}{11 \times 5 \times 9} \times \frac{3}{4} + \frac{14}{99}$$

$$\frac{\frac{4}{123} \times \frac{3}{11} \times \frac{8}{10} \times \frac{2}{9}}{\frac{2}{3} - \frac{4}{165}}$$

$$\frac{84}{165} + \frac{14}{99} = \frac{161}{165} = \frac{4}{165}$$

$$1 - \frac{4}{165} = \frac{161}{165}$$

the same

- (a) Suppose the man flips a biased coin that comes up heads $1/3$ of the time, and reports that it is heads.
- What is the probability that the coin actually landed on heads?
 - Unconvinced, you ask him if he just lied to you, to which he replies “no”. What is the probability now that the coin actually landed on heads?
 - Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.
- (b) Suppose the man rolls a fair 6-sided die. When you ask him if the die came up with a 6, he answers “yes”.
- What is the probability that the die actually came up with a 6?
 - Skeptical, you also ask him whether the die came up with a 1, to which he replies “yes”. What is the probability now that the die actually came up with a 6?
 - Did the probability go up, go down, or stay the same with this new information? Explain in words why this should be the case.

3 Cliques in Random Graphs

Note 13

Note 14

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v , we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

- What is the size of the sample space?
- A k -clique in a graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example, a 3-clique is a triangle. Let E_S be the event that a set S forms a clique. What is the probability of E_S for a particular set S of k vertices?
- Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.
- Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e)).
- Prove that the probability that the graph contains a k -clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.
Hint: Use the union bound.

4 Combined Head Count

Note 19

Suppose you flip a fair coin twice.

- What is the sample space Ω generated from these flips?

2(a) C: tell the truth

A: say the coin is head up

B: coin is head up

(i)

$$P(B) = \frac{1}{3}$$

$$P(A|B) = \frac{3}{4}$$

$$P(A|\bar{B}) = \frac{1}{4}$$

$$\begin{aligned} P(A) &= P(B) * P(A|B) \\ &\quad + P(\bar{B}) * P(A|\bar{B}) \end{aligned}$$

$$P(B|A) = \frac{P(B) * P(A|B)}{P(A)}$$

$$\begin{aligned} &= \frac{\frac{1}{3} \times \frac{3}{4}}{\frac{5}{12}} + \frac{2}{3} \times \frac{1}{4} \\ &= \frac{5}{12} \end{aligned}$$

$$= \frac{\frac{1}{3} \times \frac{3}{4}}{\frac{5}{12}} = \frac{1}{4} \times \frac{12}{5} = \frac{3}{5}$$

(ii) C: he says that he doesn't lie to you.

$$A \cap B$$

$$P(C|A \cap B) = \frac{3}{4}$$

$$P(B|(A \cap C)) = \frac{P(A \cap B \cap C)}{P(A \cap C)} = \frac{9}{11}$$

$$\downarrow P(A \cap C) = P(A \cap C \cap B) + P(A \cap C \cap \bar{B}) = \frac{9}{48} + \frac{2}{3} \times \frac{1}{4} \times \frac{1}{4} = \frac{11}{48}$$

$$P(A \cap B \cap C) = P(B) * P(A|B) * P(C|A \cap B)$$

$$= \frac{1}{3} \times \frac{3}{4} \times \frac{3}{4} = \frac{9}{48}$$

positive,

(iii) go up, when you confirm, you get more information, because

he has more probability to tell the truth

(b) (i) A: 6 B: say if s t

$$P(A) = \frac{1}{6} \quad P(B|A) = \frac{3}{4} \quad P(B|\bar{A}) = \frac{1}{4}$$

$$P(B) = P(A) * P(B|A) + P(\bar{A}) * P(B|\bar{A})$$

$$= \frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4} = \frac{8}{24} = \frac{1}{3}$$

$$P(A|B) = \frac{P(A) * P(B|A)}{P(B)} = \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{3}} = \frac{9}{24} = \frac{3}{8}$$

(ii) C: 1 D: replay yes.

$$P(C|D) = \frac{1}{24}$$

$$P(D) = P(B) = \frac{1}{3}$$

$$P(A|(B \cap D)) = \frac{P(A \cap B \cap D)}{P(B \cap D)} = \frac{3}{10}$$

$$P(A \cap B \cap D) = P(A) * P(B|A) * P(D|(A \cap B)) \\ = \frac{1}{6} * \frac{3}{4} * \frac{1}{4}$$

$$\underline{P(B \cap D)} = P(A \cap B \cap D) + P(\bar{A} \cap B \cap D) \\ = \frac{3}{6 \times 4 \times 4} + \frac{5}{6} \times \frac{1}{4} \times \left[\frac{1}{5} \times \frac{3}{4} + \frac{4}{5} \times \frac{1}{4} \right]$$

$$= \frac{3}{6 \times 4 \times 4} + \frac{5}{6 \times 4} \times \frac{7}{4 \times 5} = \frac{10}{6 \times 4 \times 4}$$

down the man is impossible to tell truth in both situation, then one of the statement must be lie

first is true is lower than $\frac{1}{2}$.

↓ before is $\frac{3}{4}$

3.
(a) $\binom{n}{2}$ pair $2^{\binom{n}{2}}$ situation

(b) $P(E_S) \approx \left(\frac{1}{2}\right)^{\binom{R}{2}}$ (c) $|V_1 \cap V_2| < 2$

$|V_1 \cap V_2| \geq 2$ E_{V_1} is true
 a, b V_a, b is connected \rightarrow will influence E_{V_2}

(d) $\binom{n}{k} = \frac{n \times (n-1) \times (n-k+1)}{k \times (k-1) \dots \times 1} = \left(\frac{n}{k}\right) \times \left(\frac{n-1}{k-1}\right) \times \dots \times (n-k+1) \leq n^k$

(e) $P(E_k \cup E_{k+1} \dots \cup E_n) \leq P(E_k) + P(E_{k+1}) + \dots + P(E_n)$

$= \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}} + \binom{n}{k+1} \left(\frac{1}{2}\right)^{\binom{k+1}{2}} + \dots + \binom{n}{n} \left(\frac{1}{2}\right)^{\binom{n}{2}}$

?

$$P\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}$$

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} \leq \frac{n^k}{2^{\frac{k(k-1)}{2}}} \leq \frac{n^k}{\left(2^{\frac{k-1}{2}}\right)^k}$$

$$\leq \frac{n^k}{\left(2^{\log_2 n^2}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n} \quad k-1 \geq 4 \log_2 n$$

4.(a) $\{(T, T), (T, H), (H, T), (H, H)\}$

$$(b) P(X=x) = \binom{x}{2} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{2-x}$$

Binomial (2, $\frac{1}{2}$)

$$(c) P(Y=1) = \frac{1}{4} \rightarrow \text{Bernoulli } \left(\frac{1}{4}\right)$$

$$(Y=0) = \frac{3}{4}$$

$$(d) P(Y=0 | X=0, 2) = 1 \quad P(Y=0 | X=1) = \frac{1}{2}$$

$$P(Y=1 | X=0, 2) = 0 \quad P(Y=1 | X=1) = \frac{1}{2}$$

$$(e) P(Z=0) = \frac{1}{4} \quad P(Z=1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(Z=2) = \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{2}{4} = \frac{1}{2}$$

- (b) Define a random variable X to be the number of heads. What is the distribution of X ?
- (c) Define a random variable Y to be 1 if you get a heads followed by a tails and 0 otherwise. What is the distribution of Y ?
- (d) Compute the conditional probabilities $\mathbb{P}[Y = i | X = j]$ for all i, j .
- (e) Define a third random variable $Z = X + Y$. Use the conditional probabilities you computed in part (d) to find the distribution of Z .

5 Max/Min Dice Rolls

Note 15

Yining rolls three fair six-sided dice.

- (a) Let X denote the maximum of the three values rolled. What is the distribution of X (that is, $\mathbb{P}[X = x]$ for $x = 1, 2, 3, 4, 5, 6$)? You can leave your final answer in terms of "x". [Hint: Try to first compute $\mathbb{P}[X \leq x]$ for $x = 1, 2, 3, 4, 5, 6$]. If you want to check your answer, you can solve this problem using counting and make sure it matches with the formula you derived.
- (b) Let Y denote the minimum of the three values rolled. What is the distribution of Y ?

$$(a) P[X \leq 6] = 1$$

$$P[X \leq 5] = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \left(\frac{5}{6}\right)^3 \rightarrow P[X \leq x] = \left(\frac{x}{6}\right)^3$$

$$P[X \leq 4] = \left(\frac{4}{6}\right)^3 \quad P[x=1] = P[x \leq 1] = \left(\frac{1}{6}\right)^3$$

$$P[x=2] = P[x \leq 2] - P[x \leq 1]$$

$$P[X \leq 3] = \left(\frac{3}{6}\right)^3 \quad P[x=x] = P[x \leq x] - P[x \leq x-1]$$

$$= \left(\frac{x}{6}\right)^3 - \left(\frac{x-1}{6}\right)^3$$

$$(b) P[Y \geq y] = \left(\frac{7-y}{6}\right)^3 \rightarrow P[Y < y] = 1 - \left(\frac{7-y}{6}\right)^3$$

$$P[Y=y] = P[Y < y+1] - P[Y < y]$$

$$= \left(\frac{7-y}{6}\right)^3 - \left(\frac{6-y}{6}\right)^3$$