CS 70 Discrete Mathematics and Probability Theory
Spring 2024 Seshia, Sinclair

DIS 9B

1 Linearity

Note 15

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

(a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A, you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?

$E(C) = E(A) + E(B)$

$= 10 \times \frac{1}{3} \times 3 + 20 * \frac{1}{5} \times 4$

$= 10 + 16 = 26$

(b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence "book" appears? (*Hint*: Consider where the sequence "book" can appear in the string.)

10^6 $\quad (\frac{1}{26})^4$ $\quad$ position i is book

X_i

$E(X) = (10^6 - 3)\left(\frac{1}{26}\right)^4$

2 Head Count II

Note 19

Consider a coin with $\mathbb{P}[\text{Heads}] = 3/4$. Suppose you flip the coin until you see heads for the first time, and define X to be the number of times you flipped the coin.

(a) What is $\mathbb{P}[X = k]$, for some $k \geq 1$?

$$P[X=k] = \left(1-\frac{3}{4}\right)^{k-1} * \frac{3}{4}$$
$$= \left(\frac{1}{4}\right)^{k-1} * \frac{3}{4}$$

(b) Name the distribution of X and what its parameters are.

$$X \sim \text{Geometric}\left(\frac{3}{4}\right)$$

(c) What is $\mathbb{P}[X > k]$, for some $k \geq 0$?

$$P[X>k] = \left(\frac{1}{4}\right)^{k}$$

(d) What is $\mathbb{P}[X < k]$, for some $k \geq 1$?

$$P[X<k] = 1 - \left(\frac{1}{4}\right)^{k-1}$$

(e) What is $\mathbb{P}[X > k \mid X > m]$, for some $k \geq m \geq 0$? How does this relate to $\mathbb{P}[X > k - m]$?

$$P[X>k \mid X>m] = \frac{P(X>k)}{P(X>m)} = \frac{\left(\frac{1}{4}\right)^{k}}{\left(\frac{1}{4}\right)^{m}} = \left(\frac{1}{4}\right)^{k-m} = P[X>k-m]$$

(f) Suppose $X \sim \text{Geometric}(p)$ and $Y \sim \text{Geometric}(q)$ are independent. Find the distribution of $\min(X,Y)$ and justify your answer.

$X = z$ $\quad y > z \quad (1-p)^{z-1} * p * (1-q)^{z}$

$x > z$ $\quad y = z \quad (1-p)^{z} * q * (1-q)^{z-1}$

$x = z$ $\quad y = z \quad (1-p)^{z-1} * p * (1-q)^{z-1} * q$

$p+q-pq$

$$(1-p)^{z-1} * (1-q)^{z-1} * [p(1-q) + q(1-p) + pq]$$
$$= 1-(1-p)(1-q) \quad - \text{Geometric}[1-(1-p)(1-q)]$$

3 Shuttles and Taxis at Airport

Note 19

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson distribution. The shuttles arrive at a rate $\lambda_1 = 1/20$ (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate $\lambda_2 = 1/10$ (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

$P(E=i) = \frac{\lambda^i}{i!} e^{-\lambda}$ $\quad E(\lambda) = \lambda$

(a) What is the distribution of the following:

(i) The number of taxis that arrive between times 00:00 and 00:20? $\rightarrow$ Binomial$(20, \frac{1}{20})$

(ii) The number of shuttles that arrive between times 00:00 and 00:20? $\rightarrow$ Binomial$(20, \frac{1}{10})$

(iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?

Binomial$(20, \frac{3}{20})$

(b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?

$\binom{20}{1}\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^{19} * \binom{20}{3}\left(\frac{1}{10}\right)^{3}\left(\frac{9}{10}\right)^{17}$

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C

(c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi? B

$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{P(B) * P(C \mid B)}{P(C)} = \frac{\binom{20}{1}\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19} * \binom{20}{0}\left(\frac{19}{20}\right)^{20}}{\binom{20}{1}\left(\frac{3}{20}\right)\left(\frac{17}{20}\right)^{19}}$

$= \frac{2}{3} \times \frac{\left(\frac{9}{10}\right)^{19} \times \left(\frac{19}{20}\right)^{20}}{\left(\frac{17}{20}\right)^{19}}$

(d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

$1 - \left(\frac{17}{20}\right)^{10}$

(a) $P[T=t] = \underline{\text{poisson}(2)} = \frac{2^t}{t!} e^{-2}$

$P(S=s) = \text{poisson}(1) = \frac{1^s}{s!} e^{-1}$

$P(N=n) = \text{poisson}(3) = \frac{3^n}{n!} e^{-3}$

(b)

(c) $\frac{2}{3}$

$\underline{(d)}$ $\underline{10} * \left(\frac{1}{20} + \frac{1}{10}\right) = \underline{\frac{3}{2}}$

$\underline{\text{poisson}(1.5)}$

$P[X=0] = \frac{(1.5)^0}{0!} e^{-1.5}$