1 Tellers

Note 17

Imagine that X is the number of customers that enter a bank at a given hour. To simplify everything, in order to serve n customers you need at least n tellers. One less teller and you won't finish serving all of the customers by the end of the hour. You are the manager of the bank and you need to decide how many tellers there should be in your bank so that you finish serving all of the customers in time. You need to be sure that you finish in time with probability at least 95%.

(a) Assume that from historical data you have found out that $\mathbb{E}[X] = 5$. How many tellers should you have?

(b) Now assume that you have also found out that $\text{Var}(X) = 5$. Now how many tellers do you need?

Solution:

(a) Suppose we have t tellers, meaning that we can serve t customers, i.e., we fail to finish on time if more than t customers show up. So we want to choose t so that

$$\mathbb{P}[X \geq t] \leq 0.05.$$

Using Markov's inequality, we have $\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$. Therefore, we want to choose t so that $\frac{\mathbb{E}[X]}{t} = 0.05$. Since $\mathbb{E}[X] = 5$, this requires $t = 100$. Therefore, 99 tellers are needed in this case.

(b) Now that we have access to the variance as well, we can apply Chebyshev's inequality. Note that Markov's inequality is still correct, but Chebyshev's inequality gives us a tigher bound here. As in part (a), aiming for the probability of finishing in time to be least 95% is equivalent to aiming to limit the probability of not finishing (or in other words, taking more time to finish t customers) to 5%. So, we want

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 0.05$$

Using Chebyshev's inequality, we know $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\text{Var}(X)}{t^2}$. Plugging in $\mathbb{E}[X] = 5$ and $\text{Var}(X) = 5$, we get $\mathbb{P}[|X - 5| \geq t] \leq \frac{5}{t^2}$. Since we want to limit $\mathbb{P}[|X - 5| \geq t] \leq 0.05$, we get $\frac{5}{t^2} = 0.05$. Thus $t^2 = 100$ and $t = 10$. Now plugging $t = 10$:

$$\mathbb{P}[|X - 5| \geq 10] = \mathbb{P}[X \geq 5 + 10] = \mathbb{P}[X \geq 15] \leq 0.05$$

as wanted. Thus, 14 tellers are needed this time.

2 Polling Numbers

Note 17

Suppose the whole population of California has Democrats, Republicans, and no other parties. You choose N people independently and uniformly at random from the Californian population, and for each person, you record whether they are a Democrat or a Republican. We want to estimate the true percentage of Democrats among the polled Californians to within 1% with 95% confidence. According to Chebyshev's inequality, what is the minimum number of people you need to poll?

Solution: Let $S = \frac{1}{N}\sum_{i=1}^{N} X_i$, where $X_i = 1$ if the i-th person polled is a Democrat and 0 if they are a Republican. We need to be off by more than 1% no more than 5% of the time, which is represented by $\mathbb{P}[|S - \mathbb{E}[S]| \geq 0.01] \leq 0.05$. To do this, we can apply Chebyshev's inequality, which says $\mathbb{P}[|S - \mathbb{E}[S]| \geq 0.01] \leq \frac{\text{Var}(S)}{0.01^2}$. Thus, we would like $\frac{\text{Var}(S)}{0.01^2} \leq 0.05$. We have that $\text{Var}(S) = \frac{1}{N^2}\sum_{i=1}^{N} \text{Var}(X_i)$. Since X_i are all i.i.d. Bernoulli random variables, their variance is at most $\frac{1}{4}$. We have the following: $\frac{\text{Var}(S)}{0.01^2} \leq \frac{\frac{N}{4}}{N^2 \cdot 0.01^2} \leq 0.05$, which we solve to obtain $N \geq 50000$.

3 Tightness of Inequalities

Note 17

(a) Show by example that Markov's inequality is tight; that is, show that given some fixed $k > 0$, there exists a discrete non-negative random variable X such that $\mathbb{P}[X \geq k] = \mathbb{E}[X]/k$.

(b) Show by example that Chebyshev's inequality is tight; that is, show that given some fixed $k \geq 1$, there exists a random variable X such that $\mathbb{P}[|X - \mathbb{E}[X]| \geq k\sigma] = 1/k^2$, where $\sigma^2 = \text{Var}(X)$.

Solution:

(a) In the proof of Markov's Inequality ($\mathbb{P}[X \geq \alpha] \leq \frac{\mathbb{E}[X]}{\alpha}$), the first time we lose equality is at this step:

$$\boxed{\mathbb{E}[X] = \sum_a (a \cdot \mathbb{P}[X = a]) \geq \sum_{a \geq \alpha} (a \cdot \mathbb{P}[X = a])}$$

We get an inequality because we drop all $a \cdot \mathbb{P}[X = a]$ terms where $a < \alpha$. Thus, we can only maintain equality if all of these dropped terms were actually 0. This would mean either $a = 0$ or $\mathbb{P}[X = a] = 0$ for $a > 0$, which means X can put probability on 0, but should put no probability on any other value $< \alpha$.

The next time we lose equality in the proof is the step following the one above:

$$\boxed{\sum_{a \geq \alpha} (a \cdot \mathbb{P}[X = a]) \geq \alpha \cdot \sum_{a \geq \alpha} \mathbb{P}[X = a]}$$

We get an inequality because we treat all $a \geq \alpha$ in the summation as just α, so we can pull out the α term. The only way for us to maintain equality is if we never have to substitute α for some larger a. This tells us that X should not put probability on any value $> \alpha$.

Both of these facts drive the intuition behind our example: that X can only take values 0 and α.

Let X be the random variable which is 0 with probability $1-p$ and k with probability p, where $k>0$. Then, $\mathbb{E}[X]=kp$, and Markov's inequality says

$$\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}[X]}{k} = \frac{kp}{k} = p,$$

which is tight.

(b) The proof of Chebyshev's Inequality ($\mathbb{P}[|X-\mathbb{E}[X]| \geq \alpha] \leq \frac{\mathrm{Var}(X)}{\alpha^2}$) comes from an application of Markov's Inequality to the variable $Y = (X-\mathbb{E}[X])^2$ being $\geq \alpha^2$. The only ways we can lose equality in the proof of Chebyshev's is if we lose equality in the application of Markov! Therefore, we need the variable Y to satisfy the conditions from Part (a) that ensure the application of Markov will be tight. To recap, we would need Y to only take values 0 and α^2. Thus, $(X-\mathbb{E}[X])$ can take on the values $\{-\alpha, 0, \alpha\}$.

Let

$$\boxed{X = \begin{cases} -a & \text{with probability } k^{-2}/2 \\ a & \text{with probability } k^{-2}/2 \\ 0 & \text{with probability } 1-k^{-2} \end{cases}}$$

for $a>0$. Note that $\mathbb{E}[X]=0$ and $\mathrm{Var}(X)=a^2k^{-2}$, so $k\sigma = a$, so Chebyshev's inequality gives

$$\boxed{\mathbb{P}(|X-\mathbb{E}[X]| \geq k\sigma) = \mathbb{P}(|X| \geq a) \leq \frac{1}{k^2},}$$

which is tight.

4 Max of Uniforms

Note 21

Let $X_1, \ldots X_n$ be independent Uniform$(0,1)$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $\mathrm{Var}(X)$?

Solution:

(a) $\mathbb{P}[X \leq x] = x^n$ since in order for $\max(X_1, \ldots X_n) < x$, we must have $X_i < x$ for all i. Since they are independent, we can multiply together the probabilities of each of them being less than x, which is x itself, as their distributions are uniform.

(b) The pdf is the derivative of the cdf, so we have $f_X(x) = nx^{n-1}$

(c) To find the expectation, we integrate $x f_X(x)$ over all values of x:

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^1 x f_X(x) \\
&= \int_0^1 n x^n dx \\
&= \frac{n}{n+1}
\end{aligned}$$

(d) First, we calculate $\mathbb{E}[X^2]$, then apply the formula $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2 f_X(x) = \int_0^1 n x^{n+1} dx = \frac{n}{n+2} \\
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2}
\end{aligned}$$

5 Darts with Friends

Note 21

Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

(i) What's the cumulative distribution function of X?

(ii) What's the cumulative distribution function of Y?

(iii) What's the probability density function of X?

(iv) What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max(X,Y)$?

Solution:

(a) (i) To get the cumulative distribution function of X, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X \le x] = \frac{\pi x^2}{\pi} = x^2, \qquad x \in [0,1].$$

(ii) Using the same approach as the previous part:

$$\mathbb{P}[Y \le y] = \frac{\pi y^2}{\pi \cdot 4} = \frac{y^2}{4}, \qquad y \in [0,2].$$

(iii) We'll take the derivative of the CDF to get the following:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}[X \le x] = 2x, \qquad x \in [0,1].$$

(iv) Using the same approach as the previous part:

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{P}[Y \le y] = \frac{y}{2}, \qquad y \in [0,2].$$

(b) We'll condition on Alex's outcome and then integrate over all the possibilities to get the marginal $\mathbb{P}[X \le Y]$ as following:

$$\begin{aligned}\mathbb{P}[X \le Y] &= \int_0^2 \mathbb{P}[X \le Y \mid Y = y] f_Y(y)\,\mathrm{d}y = \int_0^1 y^2 \times \frac{y}{2}\,\mathrm{d}y + \int_1^2 1 \times \frac{y}{2}\,\mathrm{d}y \\ &= \frac{1}{8} + \frac{3}{4} = \frac{7}{8}.\end{aligned}$$

Note the range within which $\mathbb{P}[X \le Y] = 1$. This allowed us to separate the integral to simplify our solution. Using this, we can get $\mathbb{P}[Y \le X]$ by the following:

$$\mathbb{P}[Y \le X] = 1 - \mathbb{P}[X \le Y] = \frac{1}{8}$$

A similar approach to the integral above could be used to verify this result:

$$\mathbb{P}[Y \le X] = \int_0^1 \mathbb{P}[Y \le X \mid X = x] f_X(x)\,\mathrm{d}x = \int_0^1 \frac{x^2}{4} 2x\,\mathrm{d}x = \frac{1}{2}\int_0^1 x^3\,\mathrm{d}x = \frac{1}{8}.$$

(c) Getting the CDF of U relies on the insight that for the maximum of two random variables to be smaller than a value, they both need to be smaller than that value. Using this we can get the following result for $u \in [0,1]$:

$$\mathbb{P}[U \le u] = \mathbb{P}[X \le u]\mathbb{P}[Y \le u] = \left(u^2\right)\left(\frac{u^2}{4}\right) = \frac{u^4}{4}.$$

For $u \in [1,2]$ we have $\mathbb{P}[X \le u] = 1$; this makes

$$\mathbb{P}[U \le u] = \mathbb{P}[Y \le u] = \frac{u^2}{4}.$$

For $u > 2$ we have $\mathbb{P}[U \le u] = 1$ since CDFs of both X and Y are 1 in this range.