11 Specialist Mathematics Semester 2 Summarised Notes

2021

Matrices

Notation

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Order: 2×3

Addition and Subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

Scalar Multiplication

$$k \times \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix}$$

Order:
$n \times m \ \leftarrow$ must match $\rightarrow \ m \times p$
Resultant matrix: $n \times p$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad AI = IA = A$$

Inverse of a Matrix

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$\det\{A\} = |A| = ad - bc$$

- If $\det\{A\} \neq 0$, A is invertible
- If $\det\{A\} = 0$, A is singular

$$AA^{-1} = A^{-1}A = I$$

Matrix Algebra

- $A + O = O + A = A$ where O is zero/null matrix
- $A + (-A) = (-A) + A = O$
- $A+B$ exists if both have same order
- $A + B = B + A$ {commutative}
- $(A + B) + C = A + (B + C)$ {associative}
- In general, $AB \neq BA$ {non-commutative}
- $(AB)C = A(BC)$ {associative}
- $AO = OA = O$
- AB may be O without $A = O$ or $B = O$
- $A(B+C) = AB + AC$ {distributive law}
- $AI = IA = A$
- A^n exists provided A is square and $n \in \mathbb{Z}^+$
- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = \frac{1}{k}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$

Simultaneous Linear Equations

$$\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$$
$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \end{bmatrix}$$
$$AX = B, \ X = A^{-1}B$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 4 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 17 \end{bmatrix}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$
$$\therefore x = 5, \ y = -2$$

Translations and Lines in 2D

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \text{ translated through } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$
$$A' = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

Line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Where λ is a parameter

Linear Transformations

Maps an object initial vector onto its image.
T is a linear transformation if:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(k\vec{u}) = kT(\vec{u})$
- $T(\vec{0}) = \vec{0}$
- $T(-\vec{u}) = -T(\vec{u})$
- $T(k_1\vec{u_1} + k_2\vec{u_2} + \cdots + k_r\vec{u_r})$ $= k_1T(\vec{u_1}) + k_2T(\vec{u_2}) + \cdots + k_rT(\vec{u_r})$

Geometric Linear Transformations

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$v' = Av, \ v = A^{-1}v'$$

To transform a function or relation, substitute in x and y from v.

Rotations About the Origin

For a rotation anticlockwise about $O(0, 0)$ through θ,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

with $\det\{A\} = 1$

Reflections

For a reflection in the mirror line $y = (\tan\alpha)x$,

$$A = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

with $\det\{A\} = -1$
If $m = \tan\alpha$ then:

$$\cos 2\alpha = \frac{1 - m^2}{1 + m^2} \qquad \sin 2\alpha = \frac{2m}{1 + m^2}$$
$$\tan 2\alpha = \frac{2m}{1 - m^2}$$

Dilatations

For a dilation with scale factor m parallel to the x-axis and k parallel to the y-axis:

$$A = \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix}$$

Compositions of Transformations

If v is transformed under T_A followed by T_B,

$$v' = BAv$$

The transformation is $T_B \circ T_A$

Real and Complex Numbers

Number Sets

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ natural numbers
$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ all integers
$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ positive integers
$\mathbb{Z}^- = \{-1, -2, -3, \dots\}$ negative integers
$\mathbb{Q}$ all rational numbers
$\mathbb{Q}'$ all irrational numbers
$\mathbb{R}$ all real numbers
$\mathbb{I}$ all imaginary numbers
$\mathbb{C}$ all complex numbers

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Real Numbers

Numbers that can be placed on a number line.

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Rational Numbers

A real number which can be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. Have a decimal expansion that either terminates or recurs.

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Irrational Numbers

A real number which cannot be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. For example, surds, $(\sqrt{a})$.

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Interval Notation

$$A = \{x \mid a \leq x \leq b,\ x \in \mathbb{R}\}$$

"The set of real numbers x such that x is between a and b, including a and b."

- An interval is a connected subset of the number line $\mathbb{R}$.
- An interval is closed if both of its endpoints are included.
- An interval is open if both of its endpoints are not included.

$$\begin{array}{rcl} [a, b] & \text{is} & \{x \mid a \leq x \leq b,\ x \in \mathbb{R}\} \\ [a, b) & \text{is} & \{x \mid a \leq x < b,\ x \in \mathbb{R}\} \\ (a, b] & \text{is} & \{x \mid a < x \leq b,\ x \in \mathbb{R}\} \\ (a, b) & \text{is} & \{x \mid a < x < b,\ x \in \mathbb{R}\} \end{array}$$

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Set Notation

$\in$	element of	$\notin$	not element
$\subset$	subset	$\not\subset$	not subset
$\subseteq$	equal subset	$\varnothing$	empty set
$\cup$	union	$\cap$	intersection

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Imaginary Numbers

A number which cannot be placed on a number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

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Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $\quad z = a + bi$

$$\mathfrak{Re}(z) = a \qquad \mathfrak{Im}(z) = b$$

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Conjugates (Surds)

Rationalising the denominator.

$$\frac{a + \sqrt{b}}{c + \sqrt{d}} = \frac{a + \sqrt{b}}{c + \sqrt{d}} \times \frac{c - \sqrt{d}}{c - \sqrt{d}}$$

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Complex Conjugates

$$z = a + bi \qquad \text{and} \qquad z^* = a - bi$$

are complex conjugates. To write a fraction of two complex numbers with a real denominator:

$$\frac{z}{w} = \frac{z}{w} \times \frac{w^*}{w^*} = \frac{zw^*}{ww^*}$$
$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

Properties:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$
- $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$
- $(z^n)^* = (z^*)^n,\ n \in \mathbb{Z}^+$
- $z + z^*$ and zz^* are real

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Complex Solutions of Quadratics

If a real quadratic equation has $\Delta < 0$, and root $c + di$, the other root is its conjugate $c - di$.
Then the equation is:

$$a\left(x^2 - 2cx + \left(c^2 + d^2\right)\right) = 0,\ a \neq 0$$

For the roots of $ax^2 + bx + c = 0$:

$$\text{sum} = -\frac{b}{a} \qquad \text{product} = \frac{c}{a}$$

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Complex Numbers as 2D Vectors

Complex numbers can be represented on an Argand diagram where the x-axis is the real axis and the y-axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{represents} \quad x + yi$$

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Modulus

The modulus of the complex number $z = a + bi$ is the length of the vector $\binom{a}{b}$, which is the real number:

$$|z| = \sqrt{a^2 + b^2}$$

Properties:

- $|z^*| = |z|$
- $|z^*| = zz^*$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right|^* = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$
- $|z^n| = |z|^n,\ n \in \mathbb{Z}^+$

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Coordinate Geometry

$$\text{If} \quad z_1 \equiv \overrightarrow{OP_1} \quad \text{and} \quad z_2 \equiv \overrightarrow{OP_2}$$

Distance between P_1 and P_2 is:

$$|z_1 - z_2|$$

If M is the midpoint between P_1 and P_2,

$$\overrightarrow{OM} \equiv \frac{z_1 + z_2}{2}$$

Sequences and Series

Arithmetic Sequences

Sequence where each term differs from the previous by a fixed term, the common difference.
First term: t_1
Common difference: d
General Formula:

$$t_n = t_1 + (n-1)d$$

....

Geometric Sequences

Sequence where each is the product of the previous term and a fixed common ratio.
First term: t_1
Common ratio: r
General Formula:

$$t_n = t_1 r^{n-1}$$

....

Series

A series is the sum of the terms in a sequence.
Sigma Notation:

$$\sum_{k=1}^{n} t_k = t_1 + t_2 + t_3 + \cdots + t_n$$

Properties:

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

If c is a constant,

$$\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} c = cn$$

....

Arithmetic Series

The sum of the terms of an arithmetic sequence.

$$S_n = \sum_{k=1}^{n} (t_1 + (k-1)d)$$

$$S_n = \frac{n}{2}(t_1 + t_n)$$

or

$$S_n = \frac{n}{2}(2t_1 + (n-1)d)$$

....

Geometric Series

The sum of the terms of a geometric sequence.

$$S_n = \sum_{k=1}^{n} t_1 r^{k-1}$$

$$S_n = \frac{t_1(r^n - 1)}{r-1}$$

or

$$S_n = \frac{t_1(1-r^n)}{1-r}$$

$r \neq 1$

Sum of Infinite Geometric Series:
If $|r| < 1$, an infinite series of the form

$$t_1 + t_1 r + t_1 r^2 + \cdots = \sum_{k=1}^{\infty} t_1 r^{k-1}$$

will converge to the sum

$$S = \frac{t_1}{1-r}$$
