

12 Specialist Mathematics Summarised Notes
(Unofficial)
Work in Progress

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Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function. $f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f .
- Range of f^{-1} = domain of f .

Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$. Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points which do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another. Vertical asymptote occurs when denominator is zero. Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$. If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x -axis
- Points on the x -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$\vec{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$ is $z^* = a - bi$

In the complex plane, z^* is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of z , $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have an argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$\text{cis } \theta = \cos \theta + i \sin \theta$

A complex number z has polar form

$z = |z| \text{cis } \theta$

where $\theta = \arg(z)$.
The conjugate of z is:

$z^* = |z| \text{cis } (-\theta)$

Properties of cis θ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

De Moivre's Theorem

$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$

Roots of Complex Numbers

The n^{th} roots of the complex number c are the solutions of $z^n = c$.

The n^{th} Roots of Unity

The n^{th} roots of unity are the solutions of $z^n = 1$.

Distances in the Complex Plane

If $z_1 \equiv \vec{OP_1}$ and $z_2 \equiv \vec{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2 .

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
 α is a zero of polynomial

$P(x) \iff P(\alpha) = 0$

The roots of a polynomial equation are the solutions to the equation.
 α is a root (or solution) of

$P(x) \iff P(\alpha) = 0$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x -intercepts of the graph $y = P(x)$

Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$

where $ex + f$ is the remainder.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n , then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X , Y and Z direction from the origin O .
The position vector of P is

OP→ = (x y z) = xi + yj + zk

where i = (1 0 0), j = (0 1 0), and k = (0 0 1), the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector a = (a1 a2 a3) is

|a| = sqrt(a1^2 + a2^2 + a3^2)

Operations with Vectors

If a = (a1 a2 a3) and b = (b1 b2 b3) then:

-a = (-a1 -a2 -a3) a+b = (a1+b1 a2+b2 a3+b3) a-b = (a1-b1 a2-b2 a3-b3) ka = (ka1 ka2 ka3)

Vector Algebra

- a + b = b + a
- (a + b) + c = a + (b + c)
- a + 0 = 0 + a = a
- a + (-a) = (-a) + a = 0
- k(a + b) = ka + kb
- |ka| = |k||a|
- If a + b = c, then a = c - b
- If ka = b, k ≠ 0, then a = 1/k b

Vector Between Two Points

If A(a1, a2, a3) and B(b1, b2, b3) then the position vector of B relative to A is

AB→ = OB→ - OA→ = (b1-a1 b2-a2 b3-a3)

The distance from A to B is |AB→| = sqrt((b1-a1)^2 + (b2-a2)^2 + (b3-a3)^2)

Parallelism

a = kb ⇔ a and b are non-zero parallel vectors.

Collinear Points

A, B and C are collinear if AB→ = kBC→.

Unit Vectors

The unit vector v-hat, a vector of length 1 in the direction of v is

v-hat = 1/|v| v

Dot Product (Scalar Product)

If a = (a1 a2 a3) and b = (b1 b2 b3), then the scalar dot product is defined as

a · b = a1b1 + a2b2 + a3b3

Properties

- a · b = b · a
- a · a = |a|^2
- a · (b + c) = a · b + a · c
- (a+b) · (c+d) = a · c + a · d + b · c + b · d
- k(a · b) = (ka) · b = a · (kb), k ∈ R

The Angle Between Two Vectors

The angle θ between two vectors a and b can be found using

cos θ = (a · b) / (|a||b|)

Scalar Product Geometric Properties

- For non-zero vectors a and b: a · b = 0 ⇔ a and b are perpendicular.
- |a · b| = |a||b| ⇔ a and b are non-zero parallel vectors.
- Given a · b = |a||b| cos θ If θ is acute, cos θ > 0 and so a · b > 0 If θ is obtuse, cos θ < 0 and so a · b < 0

Cross Product (Vector Product)

The vector cross product of a = (a1 a2 a3) and b = (b1 b2 b3) is

a × b = (a2b3 - a3b2 a3b1 - a1b3 a1b2 - a2b1)

Alternatively,

a × b = (i j k a1 a2 a3 b1 b2 b3) = (a2b3 - a3b2)i - (a1b3 - a3b1)j + (a1b2 - a2b1)k

Properties

- a × b is a vector perpendicular to both a and b.
- a × a = 0 for all a
- a × b = -b × a, ∴ a × b and b × a hav equal length but opposite direction.
- a · (b × c) is called the scalar triple prodcut.
- a × (b × c) = a × b + a × c
- (a + b) × (c + d) = a × c + a × d + b × c + b × d

Direction of a × b

|a × b| = |a||b| sin θ

where θ is the angle between a and b.

Area

If a triangle has defining vectors a and b then its area is 1/2 |a × b| units^2. If a parallelogram has defining vectors a and b then its area is |a × b| units^2.

Lines in 2 and 3 Dimensions

r = a + λb, λ ∈ R is the vector equation of the line.

The Shortest Distance From a Point to a Line

A point P is closest to a point R on a line in direction b when PR→ is perpendicular to b.

PR→ · b = 0

Relationships Between Lines

Line Classification in 2 Dimensions

- Intersecting: Unique solution
- Parallel: No solutions
- Coincident: Infinite Solutions

Line Classification in 3 Dimensions

- Lines are coplanar if they lie in the same plane. In this case they may be intersecting, parallel or coincident.
- Otherwise, they are skew.

Shortest Distance Between Skew Lines

For two skew lines with vector equations $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$, the shortest distance d between them is

$$d = \frac{|(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Integration

Indefinite Integrals

$\int k \, dx = kx + c$
 $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \, n \neq -1$
 $\int e^x \, dx = e^x + c$
 $\int \frac{1}{x} \, dx = \ln|x| + c$
 $\int \cos x \, dx = \sin x + c$
 $\int \sin x \, dx = -\cos x + c$

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Integrating $f(ax + b)$

$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c,$
 $n \neq -1$
 $\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$
 $\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln|ax + b| + c$
 $\int \cos(ax + b) \, dx$
 $\frac{1}{a} \sin(ax + b) + c$
 $\int \sin(ax + b) \, dx$
 $= -\frac{1}{a} \cos(ax + b) + c$

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Inverse Trigonometric Functions

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$
 $\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + c$
 $\int \frac{1}{1+x^2} \, dx = \arctan x + c$
 $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + c$
 $\int -\frac{1}{\sqrt{a^2-x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + c$
 $\int \frac{a}{a^2+x^2} \, dx = \arctan\left(\frac{x}{a}\right) + c$

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Integrating $\sin^2 x$ and $\cos^2 x$

Use double angle identities when integrating.

$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$
 $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$

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Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$\int_a^b f(x) \, dx = F(b) - F(a)$

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Properties of Definite Integrals

$\int_a^a f(x) \, dx = 0$
 $\int_a^b k \, dx = k(b-a)$
 $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$
 $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$
 $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx$
 $= \int_a^c f(x) \, dx$

$\int_a^b [f(x) \pm g(x)] \, dx$
 $= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

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Integration by Substitution

$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du$

When solving definite integrals with bounds a and b , adjust as $u(a)$ and $u(b)$.

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Integration by Parts

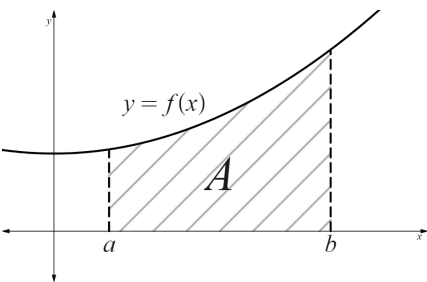
$\int uv' \, dx = uv - \int u'v \, dx$

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Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is:

$A = \int_a^b f(x) \, dx$ or $A = \int_a^b y \, dx$



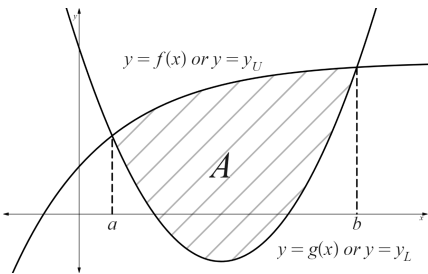
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Area Between Two Curves

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$A = \int_a^b [f(x) - g(x)] \, dx$
or

$A = \int_a^b [y_U - y_L] \, dx$



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Solids of Revolution

When the region enclosed by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is revolved through 2π about the x -axis, the volume is:

$V = \pi \int_a^b y^2 \, dx$

When the region enclosed by $y = f(x)$, the y -axis, $y = f(a) = c$ and $y = f(b) = d$ is revolved through 2π about the y -axis, the volume is:

$V = \pi \int_c^d x^2 \, dy$

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Volumes for Two Defining Functions

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$A = \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$
or

$A = \int_a^b (y_U^2 - y_L^2) \, dx$