

12 Specialist Mathematics Summarised Notes
(Unofficial)
Work in Progress

Last updated: 7 July 2022

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function. $f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f .
- Range of f^{-1} = domain of f .

Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$. Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points which do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another. Vertical asymptote occurs when denominator is zero. Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$. If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x -axis
- Points on the x -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a, a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$ is $z^* = a - bi$

In the complex plane, z^* is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of z , $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have an argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$\text{cis } \theta = \cos \theta + i \sin \theta$

A complex number z has polar form

$z = |z| \text{cis } \theta$

where $\theta = \arg(z)$.
The conjugate of z is:

$z^* = |z| \text{cis } (-\theta)$

Properties of cis θ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

De Moivre's Theorem

$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$

Roots of Complex Numbers

The n^{th} roots of the complex number c are the solutions of $z^n = c$.

The n^{th} Roots of Unity

The n^{th} roots of unity are the solutions of $z^n = 1$.

Distances in the Complex Plane

If $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2 .

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
 α is a zero of polynomial

$P(x) \iff P(\alpha) = 0$

The roots of a polynomial equation are the solutions to the equation.
 α is a root (or solution) of

$P(x) \iff P(\alpha) = 0$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x -intercepts of the graph $y = P(x)$

Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$

where $ex + f$ is the remainder.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n , then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X , Y and Z direction from the origin O .
The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,
the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations with Vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

$$\begin{aligned} -\mathbf{a} &= \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} & \mathbf{a} + \mathbf{b} &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ \mathbf{a} - \mathbf{b} &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} & k\mathbf{a} &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \end{aligned}$$

Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If $\mathbf{c} + \mathbf{a} = \mathbf{b}$ then $\mathbf{c} = \mathbf{b} - \mathbf{a}$
- If $k\mathbf{b} = \mathbf{a}$, $k \neq 0$, then $\mathbf{b} = \frac{1}{k}\mathbf{a}$

Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B is
 $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Unit Vector

For a vector \mathbf{u} , the unit vector would be:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

Dot Product (Scalar Product)

The algebraic definition of the dot product is defined as thus:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$$

The geometric definition is as follows:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Properties of the Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b})$, $\lambda \in \mathbb{R}$

Integration

Indefinite Integrals

$$\int k \, dx = kx + c$$
$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \, n \neq -1$$
$$\int e^x \, dx = e^x + c$$
$$\int \frac{1}{x} \, dx = \ln|x| + c$$
$$\int \cos x \, dx = \sin x + c$$
$$\int \sin x \, dx = -\cos x + c$$

.....

Integrating $f(ax + b)$

$$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c,$$
$$n \neq -1$$
$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$$
$$\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln|ax + b| + c$$
$$\int \cos(ax + b) \, dx$$
$$\frac{1}{a} \sin(ax + b) + c$$
$$\int \sin(ax + b) \, dx$$
$$= -\frac{1}{a} \cos(ax + b) + c$$

.....

Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$$
$$\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + c$$
$$\int \frac{1}{1+x^2} \, dx = \arctan x + c$$
$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + c$$
$$\int -\frac{1}{\sqrt{a^2-x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + c$$
$$\int \frac{a}{a^2-x^2} \, dx = \arctan\left(\frac{x}{a}\right) + c$$

.....

Integrating $\sin^2 x$ and $\cos^2 x$

Use double angle identities when integrating.

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

.....

Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

.....

Properties of Definite Integrals

$$\int_a^a f(x) \, dx = 0$$
$$\int_a^b k \, dx = k(b-a)$$
$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$$
$$\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$$
$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx$$
$$= \int_a^c f(x) \, dx$$

$$\int_a^b [f(x) \pm g(x)] \, dx$$
$$= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

.....

Integration by Substitution

$$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du$$

When solving definite integrals with bounds a and b , adjust as $u(a)$ and $u(b)$.

.....

Integration by Parts

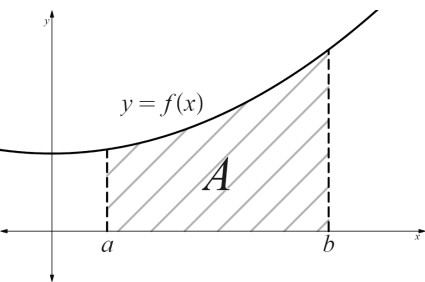
$$\int uv' \, dx = uv - \int u'v \, dx$$

.....

Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x) \, dx \quad \text{or} \quad A = \int_a^b y \, dx$$



.....

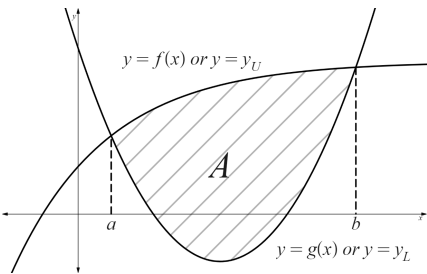
Area Between Two Curves

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)] \, dx$$

or

$$A = \int_a^b [y_U - y_L] \, dx$$



.....

Solids of Revolution

When the region enclosed by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is revolved through 2π about the x -axis, the volume is:

$$V = \pi \int_a^b y^2 \, dx$$

When the region enclosed by $y = f(x)$, the y -axis, $y = f(a) = c$ and $y = f(b) = d$ is revolved through 2π about the y -axis, the volume is:

$$V = \pi \int_c^d x^2 \, dx$$

.....

Volumes for Two Defining Functions

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$$

or

$$A = \int_a^b (y_U^2 - y_L^2) \, dx$$