12 Specialist Mathematics Summarised Notes (Unofficial)
Work in Progress

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Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:
$$(f \circ g)(x) = f(g(x))$$
$$\text{or}$$
$$f \circ g : x \mapsto f(g(x))$$
In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

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Inverse Functions

An inverse function returns the original value from the output of a function.
$f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, $\therefore$ passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f.
- Range of f^{-1} = domain of f.

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Self-Inverse Functions

An invertible function that is symmetrical about $y = x$.
$$f^{-1}(x) = f(x)$$

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Reciprocal Functions

A function of the form $f(x) = \dfrac{k}{x}$, where $k \neq 0$ is a constant.

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Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.
Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \to$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \to$ zero $\frac{1}{f(x)}$
- Local max $f(x) \to$ local min $\frac{1}{f(x)}$
- Local min $f(x) \to$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- When $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$

Invariant Points:
Points that do not move under a transformation occurring at $y = \pm 1$.

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Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \to \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

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Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$
Alternatively,
$$|x| = \sqrt{x^2}$$
Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.

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Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y-axis
- Points on the y-axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x-axis
- Points on the x-axis are invariant

Trigonometric Identities

Angle Relationships

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

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Pythagorean Theorem

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

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Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

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Angle Sum and Difference

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

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Sum to Product

$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

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Product to Sum

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$
$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$
$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

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Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $\quad z = a + bi$
$$\mathfrak{Re}(z) = a \qquad \mathfrak{Im}(z) = b$$

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The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x-axis is the real axis and the y-axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{represents} \quad x + yi$$

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Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad z^* = a - bi$$

In the complex plane, z^* is the reflection of z in the real axis.

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Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

$$|z| = \sqrt{a^2 + b^2}$$

The argument of z, $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have an argument of 0 or π.
Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$

- $|z_1 z_2| = |z_1||z_2|$

- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \ z_2 \neq 0$

- $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$

- $|z^n| = |z|^n, \ n \in \mathbb{Z}^+$

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Polar Form

$$\operatorname{cis} \theta = \cos \theta + i \sin \theta$$

A complex number z has the polar form

$$z = |z| \operatorname{cis} \theta$$

where $\theta = \arg(z)$.
The conjugate of z is:

$$z^* = |z| \operatorname{cis} (-\theta)$$

Properties of $\operatorname{cis} \theta$:

- $\operatorname{cis} \theta \times \operatorname{cis} \phi = \operatorname{cis} (\theta + \phi)$

- $\frac{\operatorname{cis} \theta}{\operatorname{cis} \phi} = \operatorname{cis} (\theta - \phi)$

- $\operatorname{cis} (\theta - 2k\pi) = \operatorname{cis} \theta, \ k \in \mathbb{Z}$

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De Moivre's Theorem

$$(|z| \operatorname{cis} \theta)^n = |z|^n \operatorname{cis} n\theta, \text{ for all } n \in \mathbb{Q}$$

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Roots of Complex Numbers

The nth roots of the complex number c are the solutions of $z^n = c$.

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The nth Roots of Unity

The nth roots of unity are the solutions of $z^n = 1$.

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Distances in the Complex Plane

If $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2.

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
α is a zero of polynomial

$$P(x) \iff P(\alpha) = 0$$

The roots of a polynomial equation are the solutions to the equation.
α is a root (or solution) of

$$P(x) \iff P(\alpha) = 0$$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x-intercepts of the graph $y = P(x)$

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Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

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Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

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Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

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Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$$

where $ex + f$ is the remainder.

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The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

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The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

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The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n, then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X, Y and Z direction from the origin O. The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the base unit vectors.

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The Magnitude of a Vector

The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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Operations with Vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

$$-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

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Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{c} - \mathbf{b}$
- If $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$

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Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B is

$$|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

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Parallelism

$\mathbf{a} = k\mathbf{b} \iff \mathbf{a}$ and $\mathbf{b}$ are non-zero parallel vectors.

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Collinear Points

A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$.

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Unit Vectors

The unit vector $\hat{\mathbf{v}}$, a vector of length 1 in the direction of $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

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Dot Product (Scalar Product)

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the scalar dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
- $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}), k \in \mathbb{R}$

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The Angle Between Two Vectors

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be found using

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Scalar Product Geometric Properties

- For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$: $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \iff \mathbf{a}$ and $\mathbf{b}$ are non-zero parallel vectors.
- Given $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
  If θ is acute, $\cos\theta > 0$ and so $\mathbf{a} \cdot \mathbf{b} > 0$
  If θ is obtuse, $\cos\theta < 0$ and so $\mathbf{a} \cdot \mathbf{b} < 0$

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Cross Product (Vector Product)

The vector cross product of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Alternatively,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Properties

- $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all $\mathbf{a}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, $\therefore$ $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have equal length but opposite direction.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the scalar triple product.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}$

Direction of $\mathbf{a} \times \mathbf{b}$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

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Area

If a triangle has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units^2.
If a parallelogram has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $|\mathbf{a} \times \mathbf{b}|$ units^2.

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Lines in 2 and 3 Dimensions

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ is the vector equation of the line.

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The Shortest Distance From a Point to a Line

A point P is closest to a point R on a line in direction $\mathbf{b}$ when $\overrightarrow{PR}$ is perpendicular to $\mathbf{b}$.

$$\overrightarrow{PR} \cdot \mathbf{b} = 0$$

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Relationships Between Lines

Line Classification in 2 Dimensions

- Intersecting: Unique solution

- Parallel: No solutions

- Coincident: Infinite Solutions

Line Classification in 3 Dimensions

- Lines are coplanar if they lie in the same plane. In this case, they may be intersecting, parallel or coincident.

- Otherwise, they are skew.

Shortest Distance Between Skew Lines

For two skew lines with vector equations $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{b_1}$ and $\mathbf{r_2} = \mathbf{a_2} + \mu\mathbf{b_2}$, the shortest distance d between them is

$$d = \frac{|(\mathbf{a_1} - \mathbf{a_2}) \cdot (\mathbf{b_1} \times \mathbf{b_2})|}{|\mathbf{b_1} \times \mathbf{b_2}|}$$

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Planes

A plane is a flat surface that extends forever and has zero thickness.

The Vector Equation of a Plane

$$\mathbf{r} = \mathbf{a} + s\mathbf{v} + t\mathbf{w}$$

- $\mathbf{r}$ is the position vector of any point on the plane.

- $\mathbf{a}$ is the position vector of a known point on the plane.

- $\mathbf{v}$ and $\mathbf{w}$ are any two non-parallel vectors that are parallel to the plane.

- $s, t \in \mathbb{R}$ are two independent parameters.

The Normal of a Plane

A vector is normal to a plane if it is perpendicular to all vectors which are parallel to the plane. If $\mathbf{n}$ is a normal to a plane such as $\mathbf{n} = \mathbf{v} \times \mathbf{w}$, an equivalent vector equation of a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) \qquad \text{or} \qquad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

The Cartesian Equation of a Plane

If a plane has normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and passes through $P(X, Y, Z)$ then it has equation

$$ax + by + cz = aX = bY = cZ = d$$

where d is a constant.

Distance Between a Point and a Plane

The distance between a point $P(x_1, y_1, z_1)$ and the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + Bx_1 + Cx_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Angles in Space

The Angle Between a Line and a Plane

The acute angle ϕ between a line with direction vector $\mathbf{a}$ and a plane with normal vector $\mathbf{n}$ is

$$\phi = \sin^{-1}\left(\frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}||\mathbf{b}|}\right)$$

The Angle Between Two Planes

If two planes have normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ and θ is the acute angle between them then

$$\theta = \cos^{-1}\left(\frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{|\mathbf{n_1}||\mathbf{n_2}|}\right)$$

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Row Reduction

Linear systems of equations can be solved using augmented matrices. A general 3×3 system has the form:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

Using row operations it is reduced to the echelon form.

$$\left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

$\therefore hz = i \Rightarrow z = \frac{i}{h}$, $ey + fz = g$ and $ax + by + cz = d$

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Intersecting Planes

Two planes in space could have the following arrangements:

- Intersecting

- Parallel

- Coincident

Three planes in space could have the following arrangements:

- All coincident

- Two coincident and one intersecting

- Two coincident and one parallel

- Two parallel and one intersecting

- All parallel

- All meet at one point

- All meet in a common line

- The line of intersection of any two planes is parallel to the third plane.

Integration

Indefinite Integrals

$$\int k \ \mathrm{d}x = kx + c$$

$$\int x^n \ \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c, \ n \neq -1$$

$$\int e^x \ \mathrm{d}x = e^x + c$$

$$\int \frac{1}{x} \ \mathrm{d}x = \ln|x| + c$$

$$\int \cos x \ \mathrm{d}x = \sin x + c$$

$$\int \sin x \ \mathrm{d}x = -\cos x + c$$

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Integrating $f(ax + b)$

$$\int (ax+b)^n \ \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c,$$
$$n \neq -1$$

$$\int e^{ax+b} \ \mathrm{d}x = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \ \mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos(ax+b) \ \mathrm{d}x$$
$$\frac{1}{a}\sin(ax+b) + c$$

$$\int \sin(ax+b) \ \mathrm{d}x$$
$$= -\frac{1}{a}\cos(ax+b) + c$$

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Inverse Trigonometric Functions

$$\int \frac{1}{\sqrt{1-x^2}} \ \mathrm{d}x = \arcsin x + c$$

$$\int -\frac{1}{\sqrt{1-x^2}} \ \mathrm{d}x = \arccos x + c$$

$$\int \frac{1}{1+x^2} \ \mathrm{d}x = \arctan x + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \ \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int -\frac{1}{\sqrt{a^2-x^2}} \ \mathrm{d}x = \arccos\left(\frac{x}{a}\right) + c$$

$$\int \frac{a}{a^2+x^2} \ \mathrm{d}x = \arctan\left(\frac{x}{a}\right) + c$$

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Integrating $\sin^2 x$ and $\cos^2 x$

Use double-angle identities when integrating.

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

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Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$$\int_a^b f(x) \ \mathrm{d}x = F(b) - F(a)$$

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Properties of Definite Integrals

$$\int_a^a f(x) \ \mathrm{d}x = 0$$

$$\int_a^b k \ \mathrm{d}x = k(b-a)$$

$$\int_a^b f(x) \ \mathrm{d}x = -\int_b^a f(x) \ \mathrm{d}x$$

$$\int_a^b kf(x) \ \mathrm{d}x = k\int_a^b f(x) \ \mathrm{d}x$$

$$\int_a^b f(x) \ \mathrm{d}x + \int_b^c f(x) \ \mathrm{d}x$$
$$= \int_a^c f(x) \ \mathrm{d}x$$

$$\int_a^b [f(x) \pm g(x)] \ \mathrm{d}x$$
$$= \int_a^b f(x) \ \mathrm{d}x \pm \int_a^b g(x) \ \mathrm{d}x$$

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Integration by Substitution

$$\int f(u) \ \frac{\mathrm{d}u}{\mathrm{d}x} \ \mathrm{d}x = \int f(u) \ \mathrm{d}u$$

When solving definite integrals with bounds a and b, adjust as $u(a)$ and $u(b)$.

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Integration by Parts

$$\int uv' \ \mathrm{d}x = uv - \int u'v \ \mathrm{d}x$$

....

Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x) \ \mathrm{d}x \qquad \text{or} \qquad A = \int_a^b y \ \mathrm{d}x$$

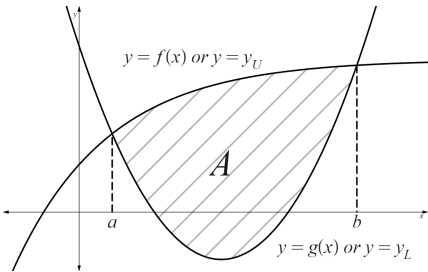
....
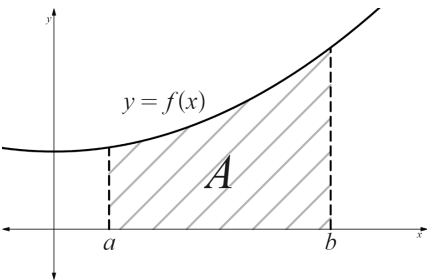
Area Between Two Curves

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)] \ \mathrm{d}x$$

or

$$A = \int_a^b [y_U - y_L] \ \mathrm{d}x$$

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Solids of Revolution

When the region enclosed by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is revolved through 2π about the x-axis, the volume is:

$$V = \pi \int_a^b y^2 \ \mathrm{d}x$$

When the region enclosed by $y = f(x)$, the y-axis, $y = f(a) = c$ and $y = f(b) = d$ is revolved through 2π about the y-axis, the volume is:

$$V = \pi \int_c^d x^2 \ \mathrm{d}y$$

....

Volumes for Two Defining Functions

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b \left([f(x)]^2 - [g(x)]^2\right) \ \mathrm{d}x$$

or

$$A = \int_a^b \left(y_U^2 - y_L^2\right) \ \mathrm{d}x$$