

12 Specialist Mathematics Summarised Notes
(Work in Progress)

2022

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or

$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.
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Inverse Functions

An inverse function returns the original value from the output of a function. $f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

- For $f^{-1}(x)$, the inverse of $f(x)$
- Is a reflection of $y = f(x)$ over $y = x$.
 - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
 - Domain of f^{-1} = Range of f .
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Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

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Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.
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Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.
Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:
Points which do not move under a transformation occurring at $y = \pm 1$.
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Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
 - If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
 - If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.
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Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.
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Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
 - Reflect the graph for $f(x) < 0$ in the x -axis
 - Points on the x -axis are invariant
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Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defines for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.
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Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$

$$\Re(z) = a \qquad \Im(z) = b$$

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The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ represents } x + yi$$

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Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad \bar{z} = a - bi$$

In the complex plane, \bar{z} is the reflection of z in the real axis.
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Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

|z| = \sqrt{a^2 + b^2}

The argument of z , $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|\bar{z}| = |z|$
- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, \quad n \in \mathbb{Z}^+$

Polar Form

cis \theta = \cos \theta + i \sin \theta

A complex number z has polar form

z = |z| cis \theta

where $\theta = \arg(z)$.

The conjugate of z is:

\bar{z} = |z| cis -\theta

Properties of cis \theta:

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, \quad k \in \mathbb{Z}$

Euler's Form

e^{i\theta} = \cos \theta + i \sin \theta

De Moivre's Theorem

(|z| cis \theta)^n = |z|^n cis n\theta, \text{ for all } n \in \mathbb{Q}

Trigonometric Identities

Angle Relationships

\sin(-\theta) = -\sin \theta \qquad \cos(-\theta) = \cos \theta

\sin(\pi - \theta) = \sin \theta \qquad \cos(\pi - \theta) = -\cos \theta

\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta

Pythagorean Theorem

\sin^2 \theta + \cos^2 \theta = 1

\tan^2 \theta + 1 = \sec^2 \theta

\cot^2 \theta + 1 = \csc^2 \theta

Double Angle Identities

\sin 2\theta = 2 \sin \theta \cos \theta

\cos 2\theta = \cos^2 \theta - \sin^2 \theta

\qquad\qquad = 1 - 2 \sin^2 \theta

\qquad\qquad = 2 \cos^2 \theta - 1

\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}

Angle Sum and Difference

\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B

\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B

\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}

Sum to Product

\sin A \pm \sin B = 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right)

\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right)

\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X , Y and Z direction from the origin O . The position vector of P is

\vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{j} + z\vec{k}

where $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}

Operations with Vectors

If $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

-\vec{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}

\vec{a} - \vec{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\vec{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}

Vector Algebra

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $|k\vec{a}| = |k| |\vec{a}|$
- If $\vec{x} + \vec{a} = \vec{b}$ then $\vec{x} = \vec{b} - \vec{a}$
- If $k\vec{x} = \vec{a}$, $k \neq 0$, then $\vec{x} = \frac{1}{k} \vec{a}$

Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}

The distance from A to B is $|\vec{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Integration

Integration by Parts

Suppose we have a function $f = u \times v$ where u and v are also functions. We know that to find the derivative of f , we utilise the product rule:

\frac{df}{dx} = \frac{d(u \times v)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}

Then, integrate both sides with respect to x :

\int \frac{d(u \times v)}{dx} dx = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx

Simplifying, we get:

uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx

Rearranging:

\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx