

## 12 Specialist Mathematics Summarised Notes (Work in Progress)

2022

Functions

Composite Functions

Given  $f : x \mapsto f(x)$  and  $g : x \mapsto g(x)$ , the composite function of  $f$  and  $g$  is:

$$(f \circ g)(x) = f(g(x))$$
  
or  
$$f \circ g : x \mapsto f(g(x))$$

In general,  $(f \circ g)(x) \neq (g \circ f)(x)$ .  
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Inverse Functions

An inverse function returns the original value from the output of a function.  
 $f(x)$  has an inverse if it is one-to-one, if  $f(a) = f(b)$  only when  $a = b$ ,  $\therefore$  passes the horizontal line test.

- For  $f^{-1}(x)$ , the inverse of  $f(x)$
- Is a reflection of  $y = f(x)$  over  $y = x$ .
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
  - Domain of  $f^{-1}$  = Range of  $f$ .
  - Range of  $f^{-1}$  = Domain of  $f$ .
- .....

Self-Inverse Functions

An invertible function which is symmetrical about  $y = x$ .  
$$f^{-1}(x) = f(x)$$
  
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Reciprocal Functions

A function of the form  $f(x) = \frac{k}{x}$ , where  $k \neq 0$  is a constant.  
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Reciprocal of Other Functions

The reciprocal of a function  $f(x)$  is  $\frac{1}{f(x)}$ .  
Graphing  $y = \frac{1}{f(x)}$  from  $y = f(x)$ :

- Zero  $f(x) \rightarrow$  vertical asymp  $\frac{1}{f(x)}$
- Vertical asymp  $f(x) \rightarrow$  zero  $\frac{1}{f(x)}$
- Local max  $f(x) \rightarrow$  local min  $\frac{1}{f(x)}$
- Local min  $f(x) \rightarrow$  local max  $\frac{1}{f(x)}$
- When  $f(x) > 0$ ,  $\frac{1}{f(x)} > 0$
- When  $f(x) < 0$ ,  $\frac{1}{f(x)} < 0$
- When  $f(x) \rightarrow 0$ ,  $\frac{1}{f(x)} \rightarrow \pm\infty$
- When  $f(x) \rightarrow \pm\infty$ ,  $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:  
Points which do not move under a transformation occurring at  $y = \pm 1$ .  
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Rational Functions

Results from the division of one polynomial by another.  
Vertical asymptote occurs when denominator is zero.  
Horizontal asymptote ascertained from behaviour of graph as  $|x| \rightarrow \infty$ .

- If the degree of denominator > numerator, horizontal asymptote at  $y = 0$ .
  - If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
  - If the degree of denominator = numerator horizontal asymptote at  $y = \frac{a}{b}$  where  $a$  and  $b$  are the leading coefficients.
- .....

Absolute Value Functions

The absolute value or modulus  $|x|$  of a real number  $x$  is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If  $|x| = a$  where  $a > 0$ , then  $x = \pm a$ .  
If  $|x| = |b|$  then  $x = \pm b$ .  
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Graphs Involving the Absolute Value Function

Graphing  $y = f(|x|)$  from  $y = f(x)$ :

- Discard the graph for  $x < 0$
- Reflect the graph for  $x \geq 0$  in the  $y$ -axis
- Points on the  $y$ -axis are invariant

Graphing  $y = |f(x)|$  from  $y = f(x)$ :

- Keep the graph for  $f(x) \geq 0$
  - Reflect the graph for  $f(x) < 0$  in the  $x$ -axis
  - Points on the  $x$ -axis are invariant
- .....

Mathematical Induction

The Principle of Mathematical Induction

Suppose  $P_n$  is a proposition which is defines for every integer  $n \geq a$ ,  $a \in \mathbb{Z}$ . If  $P_a$  is true, and if  $P_{k+1}$  is true whenever  $P_k$  is true, then  $P_n$  is true for all  $n \geq a$ .  
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Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form  $ai$  where  $a \in \mathbb{R}$  and  $i = \sqrt{-1}$ .  
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Complex Numbers

Any number in the form  $a + bi$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

If  $z = a + bi$   
$$\Re(z) = a \qquad \Im(z) = b$$
  
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The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the  $x$ -axis is the real axis and the  $y$ -axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ represents } x + yi$$
  
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Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad \bar{z} = a - bi$$

In the complex plane,  $\bar{z}$  is the reflection of  $z$  in the real axis.  
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Modulus and Argument

The modulus of the complex number  $z = a + bi$  is the length of the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ , which is the real number:

|z| = \sqrt{a^2 + b^2}

The argument of  $z$ ,  $\arg z$  is the angle  $\theta$  between the positive real axis and  $\begin{pmatrix} a \\ b \end{pmatrix}$ . Real numbers have argument of 0 or  $\pi$ . Purely imaginary numbers have argument of  $\frac{\pi}{2}$  or  $-\frac{\pi}{2}$ .

Properties of Modulus:

- $|\bar{z}| = |z|$
- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \ z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$
- $|z^n| = |z|^n, \ n \in \mathbb{Z}^+$

Polar Form

cis \theta = \cos \theta + i \sin \theta

A complex number  $z$  has polar form

z = |z| cis \theta

where  $\theta = \arg z$ .  
The conjugate of  $z$  is:

\bar{z} = |z| cis -\theta

Properties of cis \theta:

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - k2\pi) = \text{cis } \theta, \ k \in \mathbb{Z}$

Euler’s Form

e^{i\theta} = \cos \theta + i \sin \theta

De Moivre’s Theorem

(|z| cis \theta)^n = |z|^n cis n\theta, \text{ for all } n \in \mathbb{Q}

Trigonometric Identities

Angle Relationships

\sin(-\theta) = -\sin \theta \qquad \cos(-\theta) = \cos \theta  
\sin(\pi - \theta) = \sin \theta \qquad \cos(\pi - \theta) = -\cos \theta  
\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta

Pythagorean Theorem

\sin^2 \theta + \cos^2 \theta = 1  
\tan^2 \theta + 1 = \sec^2 \theta  
1 + \cot^2 \theta = \csc^2 \theta

Double Angle Identities

\sin 2\theta = 2 \sin \theta \cos \theta  
\cos 2\theta = \cos^2 \theta - \sin^2 \theta  
= 1 - 2 \sin^2 \theta  
= 2 \cos^2 \theta - 1  
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}

Angle Sum and Difference

\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B  
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B  
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}

Sum to Product

\sin A \pm \sin B = 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right)  
\cos A + \cos B = 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right)  
\cos A - \cos B = -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)

Product to Sum

2 \sin A \cos B = \sin(A + B) + \sin(A - B)  
2 \sin A \sin B = \cos(A - B) - \cos(A + B)  
2 \cos A \cos B = \cos(A + B) + \cos(A - B)

Vectors

Vectors in Space

Any point  $P$  in space can be specified  $(x, y, z)$  corresponding to steps in the  $X$ ,  $Y$  and  $Z$  direction from the origin  $O$ .  
The position vector of  $P$  is

\vec{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{j} + z\vec{k}

where  $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , and  $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ , the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  is

|\vec{a}| = \sqrt{a\_1^2 + a\_2^2 + a\_3^2}

Operations with Vectors

If  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$  then:

-\vec{a} = \begin{pmatrix} -a\_1 \\ -a\_2 \\ -a\_3 \end{pmatrix} \qquad \vec{a} + \vec{b} = \begin{pmatrix} a\_1 + b\_1 \\ a\_2 + b\_2 \\ a\_3 + b\_3 \end{pmatrix}  
\vec{a} - \vec{b} = \begin{pmatrix} a\_1 - b\_1 \\ a\_2 - b\_2 \\ a\_3 - b\_3 \end{pmatrix} \qquad k\vec{a} = \begin{pmatrix} ka\_1 \\ ka\_2 \\ ka\_3 \end{pmatrix}

Vector Algebra

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $|k\vec{a}| = |k||\vec{a}|$
- If  $\vec{x} + \vec{a} = \vec{b}$  then  $\vec{x} = \vec{b} - \vec{a}$
- If  $k\vec{x} = \vec{a}$ ,  $k \neq 0$ , then  $\vec{x} = \frac{1}{k}\vec{a}$

Vector Between Two Points

If  $A(a_1, a_2, a_3)$  and  $B(b_1, b_2, b_3)$  then the position vector of  $B$  relative to  $A$  is

\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} b\_1 - a\_1 \\ b\_2 - a\_2 \\ b\_3 - a\_3 \end{pmatrix}

The distance from  $A$  to  $B$  is

|\vec{AB}| = \sqrt{(b\_1 - a\_1)^2 + (b\_2 - a\_2)^2 + (b\_3 - a\_3)^2}