12 Specialist Mathematics Summarised Notes (Work in Progress)

2022

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$
or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$).

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Inverse Functions

An inverse function returns the original value from the output of a function.
$f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, $\therefore$ passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = Range of f.
- Range of f^{-1} = Domain of f.

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Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

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Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

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Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.

<u>Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:</u>

- Zero $f(x) \to$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \to$ zero $\frac{1}{f(x)}$
- Local max $f(x) \to$ local min $\frac{1}{f(x)}$
- Local min $f(x) \to$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- When $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$

<u>Invariant Points:</u>
Points which do not move under a transformation occurring at $y = \pm 1$.

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Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \to \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slated asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

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Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

<u>Properties:</u>

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.

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Graphs Involving the Absolute Value Function

<u>Graphing $y = f(|x|)$ from $y = f(x)$:</u>

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y-axis
- Points on the y-axis are invariant

<u>Graphing $y = |f(x)|$ from $y = f(x)$:</u>

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x-axis
- Points on the x-axis are invariant

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defines for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

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Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $\quad z = a + bi$
$$\mathfrak{Re}(z) = a \qquad \mathfrak{Im}(z) = b$$

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The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x-axis is the real axis and the y-axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{represents} \quad x + yi$$

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Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad \bar{z} = a - bi$$

In the complex plane, $\bar{z}$ is the reflection of z in the real axis.

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Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\binom{a}{b}$, which is the real number:

$$|z| = \sqrt{a^2 + b^2}$$

The argument of z, $\arg(z)$ is the angle θ between the positive real axis and $\binom{a}{b}$. Real numbers have argument of 0 or π. Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|\bar{z}| = |z|$
- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$
- $|z^n| = |z|^n,\ n \in \mathbb{Z}^+$

Polar Form

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta$$

A complex number z has polar form

$$z = |z| \operatorname{cis}\theta$$

where $\theta = \arg(z)$.
The conjugate of z is:

$$\bar{z} = |z| \operatorname{cis} -\theta$$

Properties of $\operatorname{cis}\theta$:

- $\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$
- $\frac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta - \phi)$
- $\operatorname{cis}(\theta - 2k\pi) = \operatorname{cis}\theta,\ k \in \mathbb{Z}$

Euler's Form

$$e^{i\theta} = \cos\theta + i\sin\theta$$

De Moivre's Theorem

$$(|z| \operatorname{cis}\theta)^n = |z|^n \operatorname{cis} n\theta, \text{ for all } n \in \mathbb{Q}$$

Trigonometric Identities

Angle Relationships

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Pythagorean Theorem

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1\end{aligned}$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Angle Sum and Difference

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Sum to Product

$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X, Y and Z direction from the origin O. The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{k} + z\vec{k}$$

where $\vec{i} = \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, $\vec{j} = \left(\begin{smallmatrix}0\\1\\0\end{smallmatrix}\right)$, and $\vec{k} = \left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$, the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector $\vec{a} = \left(\begin{smallmatrix}a_1\\a_2\\a_3\end{smallmatrix}\right)$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations with Vectors

$$\text{If } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ then:}$$

$$-\vec{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$
$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\vec{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

Vector Algebra

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{0} + \vec{a}$
- $\vec{a} + (\overrightarrow{-a}) = (\overrightarrow{-a}) + \vec{a} = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $|k\vec{a}| = |k||\vec{a}|$
- If $\vec{x} + \vec{a} = \vec{b}$ then $\vec{x} = \vec{b} = \vec{a}$
- If $k\vec{x} = a$, $k \neq 0$, then $\vec{x} = \frac{1}{k}\vec{a}$

Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B Is $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Integration

Integration by Parts

Suppose we have a function $f = u \times v$ where u and v are also functions. We know that to find the derivative of f, we utilise the product rule:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}(u \times v)}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Then, integrate both sides with respect to x:

$$\int \frac{\mathrm{d}(u \times v)}{\mathrm{d}x}\,\mathrm{d}x = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Simplifying, we get:

$$uv = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Rearranging:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$