

12 Specialist Mathematics Summarised Notes (Work in Progress)

2022

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or

$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.
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Inverse Functions

An inverse function returns the original value from the output of a function.
 $f(x)$ has an inverse if it is one-to-one, if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

- For $f^{-1}(x)$, the inverse of $f(x)$
- Is a reflection of $y = f(x)$ over $y = x$.
 - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
 - Domain of f^{-1} = Range of f .
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Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.
$$f^{-1}(x) = f(x)$$

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Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.
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Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.
Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:
Points which do not move under a transformation occurring at $y = \pm 1$.
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Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
 - If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
 - If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.
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Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.
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Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
 - Reflect the graph for $f(x) < 0$ in the x -axis
 - Points on the x -axis are invariant
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Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defines for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.
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Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.
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Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$

$$\Re(z) = a \qquad \Im(z) = b$$

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The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ represents } x + yi$$

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Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad \bar{z} = a - bi$$

In the complex plane, \bar{z} is the reflection of z in the real axis.
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Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

|z| = \sqrt{a^2 + b^2}

The argument of z , $\arg z$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|\bar{z}| = |z|$
- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

cis \theta = \cos \theta + i \sin \theta

A complex number z has polar form

z = |z| cis \theta

where $\theta = \arg z$.
The conjugate of z is:

\bar{z} = |z| cis -\theta

Properties of cis \theta:

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - k2\pi) = \text{cis } \theta, k \in \mathbb{Z}$

Euler’s Form

e^{i\theta} = \cos \theta + i \sin \theta

De Moivre’s Theorem

(|z| cis \theta)^n = |z|^n cis n\theta, for all n \in \mathbb{Q}

Pythagorean Theorem

\sin^2 \theta + \cos^2 \theta = 1
\tan^2 \theta + 1 = \sec^2 \theta
1 + \cot^2 \theta = \csc^2 \theta

Double Angle Identities

\sin 2\theta = 2 \sin \theta \cos \theta
\cos 2\theta = \cos^2 \theta - \sin^2 \theta
= 1 - 2 \sin^2 \theta
= 2 \cos^2 \theta - 1
\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}

Angle Sum and Difference

\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B
\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B
\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}

Sum to Product

\sin A \pm \sin B = 2 \sin \left(\frac{A \pm B}{2}\right) \cos \left(\frac{A \mp B}{2}\right)
\cos A \pm \cos B = 2 \sin \left(\frac{A + B}{2}\right) \cos \left(\frac{A - B}{2}\right)

Product to Sum

2 \sin A \cos B = \sin (A + B) + \sin (A - B)
2 \sin A \sin B = \cos (A - B) - \cos (A + B)
2 \cos A \cos B = \sin (A + B) + \cos (A - B)

Trigonometric Identities

Angle Relationships

\sin (-\theta) = -\sin \theta \qquad \cos (-\theta) = \cos \theta
\sin (\pi - \theta) = \sin \theta \qquad \cos (\pi - \theta) = -\cos \theta
\sin \left(\frac{\pi}{2} - \theta\right) = \cos \theta \qquad \cos \left(\frac{\pi}{2} - \theta\right) = \sin \theta