

12 Specialist Mathematics Summarised Notes  
(Unofficial)  
Work in Progress

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Functions

Composite Functions

Given  $f : x \mapsto f(x)$  and  $g : x \mapsto g(x)$ , the composite function of  $f$  and  $g$  is:

$$(f \circ g)(x) = f(g(x))$$
  
or  
$$f \circ g : x \mapsto f(g(x))$$

In general,  $(f \circ g)(x) \neq (g \circ f)(x)$ .

Inverse Functions

An inverse function returns the original value from the output of a function.  $f(x)$  has an inverse if it is injective (one-to-one), if  $f(a) = f(b)$  only when  $a = b$ ,  $\therefore$  passes the horizontal line test.

For  $f^{-1}(x)$ , the inverse of  $f(x)$ :

- Is a reflection of  $y = f(x)$  over  $y = x$ .
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of  $f^{-1}$  = range of  $f$ .
- Range of  $f^{-1}$  = domain of  $f$ .

Self-Inverse Functions

An invertible function which is symmetrical about  $y = x$ .

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form  $f(x) = \frac{k}{x}$ , where  $k \neq 0$  is a constant.

Reciprocal of Other Functions

The reciprocal of a function  $f(x)$  is  $\frac{1}{f(x)}$ . Graphing  $y = \frac{1}{f(x)}$  from  $y = f(x)$ :

- Zero  $f(x) \rightarrow$  vertical asymp  $\frac{1}{f(x)}$
- Vertical asymp  $f(x) \rightarrow$  zero  $\frac{1}{f(x)}$
- Local max  $f(x) \rightarrow$  local min  $\frac{1}{f(x)}$
- Local min  $f(x) \rightarrow$  local max  $\frac{1}{f(x)}$
- When  $f(x) > 0$ ,  $\frac{1}{f(x)} > 0$
- When  $f(x) < 0$ ,  $\frac{1}{f(x)} < 0$
- When  $f(x) \rightarrow 0$ ,  $\frac{1}{f(x)} \rightarrow \pm\infty$
- When  $f(x) \rightarrow \pm\infty$ ,  $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points which do not move under a transformation occurring at  $y = \pm 1$ .

Rational Functions

Results from the division of one polynomial by another. Vertical asymptote occurs when denominator is zero. Horizontal asymptote ascertained from behaviour of graph as  $|x| \rightarrow \infty$ .

- If the degree of denominator > numerator, horizontal asymptote at  $y = 0$ .
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at  $y = \frac{a}{b}$  where  $a$  and  $b$  are the leading coefficients.

Absolute Value Functions

The absolute value or modulus  $|x|$  of a real number  $x$  is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If  $|x| = a$  where  $a > 0$ , then  $x = \pm a$ . If  $|x| = |b|$  then  $x = \pm b$ .

Graphs Involving the Absolute Value Function

Graphing  $y = f(|x|)$  from  $y = f(x)$ :

- Discard the graph for  $x < 0$
- Reflect the graph for  $x \geq 0$  in the  $y$ -axis
- Points on the  $y$ -axis are invariant

Graphing  $y = |f(x)|$  from  $y = f(x)$ :

- Keep the graph for  $f(x) \geq 0$
- Reflect the graph for  $f(x) < 0$  in the  $x$ -axis
- Points on the  $x$ -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose  $P_n$  is a proposition which is defined for every integer  $n \geq a$ ,  $a \in \mathbb{Z}$ . If  $P_a$  is true, and if  $P_{k+1}$  is true whenever  $P_k$  is true, then  $P_n$  is true for all  $n \geq a$ .

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form  $ai$  where  $a \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

Complex Numbers

Any number in the form  $a + bi$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

If  $z = a + bi$   
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the  $x$ -axis is the real axis and the  $y$ -axis is the imaginary axis.

$\vec{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$  represents  $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$  is  $z^* = a - bi$

In the complex plane,  $z^*$  is the reflection of  $z$  in the real axis.

Modulus and Argument

The modulus of the complex number  $z = a + bi$  is the length of the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ , which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of  $z$ ,  $\arg(z)$  is the angle  $\theta$  between the positive real axis and  $\begin{pmatrix} a \\ b \end{pmatrix}$ . Real numbers have an argument of 0 or  $\pi$ . Purely imaginary numbers have argument of  $\frac{\pi}{2}$  or  $-\frac{\pi}{2}$ .

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$\text{cis } \theta = \cos \theta + i \sin \theta$

A complex number  $z$  has polar form

$z = |z| \text{cis } \theta$

where  $\theta = \arg(z)$ .  
The conjugate of  $z$  is:

$z^* = |z| \text{cis } (-\theta)$

Properties of cis  $\theta$ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

De Moivre's Theorem

$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$

Roots of Complex Numbers

The  $n^{\text{th}}$  roots of the complex number  $c$  are the solutions of  $z^n = c$ .

The  $n^{\text{th}}$  Roots of Unity

The  $n^{\text{th}}$  roots of unity are the solutions of  $z^n = 1$ .

Distances in the Complex Plane

If  $z_1 \equiv \vec{OP_1}$  and  $z_2 \equiv \vec{OP_2}$  then  $|z_1 - z_2|$  is the distance between points  $P_1$  and  $P_2$ .

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.  
 $\alpha$  is a zero of polynomial

$P(x) \iff P(\alpha) = 0$

The roots of a polynomial equation are the solutions to the equation.  
 $\alpha$  is a root (or solution) of

$P(x) \iff P(\alpha) = 0$

The roots of  $P(x) = 0$  are the zeros of  $P(x)$  and the  $x$ -intercepts of the graph  $y = P(x)$

Factors

$(x - \alpha)$  is a factor of the polynomial  $P(x) \iff$  there exists a polynomial  $Q(x)$  such that  $P(x) = (x - \alpha)Q(x)$ .

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If  $P(x)$  is divided by  $D(x) = ax + b$  until a quotient  $Q(x)$  and constant remainder  $R$  is obtained, then

$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$

Notice that  $P(x) = Q(x) \times (ax + b) + R$ .

Polynomial Division by Quadratics

If  $P(x)$  is divided by  $D(x) = ax^2 + bx + c$ , then

$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$

where  $ex + f$  is the remainder.

The Remainder Theorem

When a polynomial  $P(x)$  is divided by  $x - k$  until a constant remainder  $R$  is obtained, then  $R = P(k)$ .

The Factor Theorem

For any polynomial  $P(x)$ ,  $k$  is a zero of  $P(x) \iff (x - k)$  is a factor of  $P(x)$ .

The Fundamental Theorem of Algebra

If  $P(x)$  is a polynomial of degree  $n$ , then  $P(x)$  has  $n$  zeros, each in the form  $a + bi$  where  $a, b \in \mathbb{R}$ , some of which may be repeated.

Vectors

Vectors in Space

Any point  $P$  in space can be specified  $(x, y, z)$  corresponding to steps in the  $X$ ,  $Y$  and  $Z$  direction from the origin  $O$ .  
The position vector of  $P$  is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where  $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , and  $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,  
the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations with Vectors

If  $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$  then:

$$\begin{aligned} -\mathbf{a} &= \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} & \mathbf{a} + \mathbf{b} &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ \mathbf{a} - \mathbf{b} &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} & k\mathbf{a} &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \end{aligned}$$

Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If  $\mathbf{c} + \mathbf{a} = \mathbf{b}$  then  $\mathbf{c} = \mathbf{b} - \mathbf{a}$
- If  $k\mathbf{b} = \mathbf{a}$ ,  $k \neq 0$ , then  $\mathbf{b} = \frac{1}{k}\mathbf{a}$

Vector Between Two Points

If  $A(a_1, a_2, a_3)$  and  $B(b_1, b_2, b_3)$  then the position vector of  $B$  relative to  $A$  is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from  $A$  to  $B$  is  $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Unit Vector

For a vector  $\mathbf{u}$ , the unit vector would be:

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}$$

Dot Product (Scalar Product)

The algebraic definition of the dot product is defined as thus:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a_i b_i$$

The geometric definition is as follows:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Properties of the Dot Product

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $\lambda(\mathbf{a} \cdot \mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\lambda\mathbf{b})$ ,  $\lambda \in \mathbb{R}$

Integration

Indefinite Integrals

$\int k \, dx = kx + c$   
 $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \, n \neq -1$   
 $\int e^x \, dx = e^x + c$   
 $\int \frac{1}{x} \, dx = \ln|x| + c$   
 $\int \cos x \, dx = \sin x + c$   
 $\int \sin x \, dx = -\cos x + c$

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Integrating  $f(ax + b)$

$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c,$   
 $n \neq -1$   
 $\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$   
 $\int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln|ax + b| + c$   
 $\int \cos(ax + b) \, dx$   
 $\frac{1}{a} \sin(ax + b) + c$   
 $\int \sin(ax + b) \, dx$   
 $= -\frac{1}{a} \cos(ax + b) + c$

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Inverse Trigonometric Functions

$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + c$   
 $\int -\frac{1}{\sqrt{1-x^2}} \, dx = \arccos x + c$   
 $\int \frac{1}{1+x^2} \, dx = \arctan x + c$   
 $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + c$   
 $\int -\frac{1}{\sqrt{a^2-x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + c$   
 $\int \frac{a}{a^2+x^2} \, dx = \arctan\left(\frac{x}{a}\right) + c$

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Integrating  $\sin^2 x$  and  $\cos^2 x$

Use double angle identities when integrating.

$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$   
 $\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$

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Definite Integrals

If  $F(x)$  is the antiderivative of  $f(x)$  where  $f(x)$  is continuous over  $a \leq x \leq b$ , the definite integral is:

$\int_a^b f(x) \, dx = F(b) - F(a)$

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Properties of Definite Integrals

$\int_a^a f(x) \, dx = 0$   
 $\int_a^b k \, dx = k(b-a)$   
 $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$   
 $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$   
 $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx$   
 $= \int_a^c f(x) \, dx$

$\int_a^b [f(x) \pm g(x)] \, dx$   
 $= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

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Integration by Substitution

$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du$

When solving definite integrals with bounds  $a$  and  $b$ , adjust as  $u(a)$  and  $u(b)$ .

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Integration by Parts

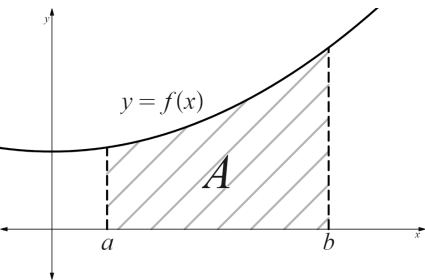
$\int uv' \, dx = uv - \int u'v \, dx$

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Area Under a Curve

If  $f(x)$  is positive and continuous for  $a \leq x \leq b$ , the area bound by  $y = f(x)$ , the  $x$ -axis,  $x = a$  and  $x = b$  is:

$A = \int_a^b f(x) \, dx$       or       $A = \int_a^b y \, dx$



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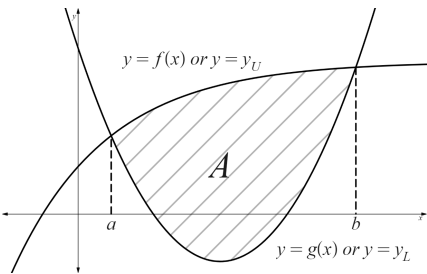
Area Between Two Curves

Upper function:  $f(x)$  or  $y_U$   
Lower function:  $g(x)$  or  $y_L$

$A = \int_a^b [f(x) - g(x)] \, dx$

or

$A = \int_a^b [y_U - y_L] \, dx$



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Solids of Revolution

When the region enclosed by  $y = f(x)$ , the  $x$ -axis,  $x = a$  and  $x = b$  is revolved through  $2\pi$  about the  $x$ -axis, the volume is:

$V = \pi \int_a^b y^2 \, dx$

When the region enclosed by  $y = f(x)$ , the  $y$ -axis,  $y = f(a) = c$  and  $y = f(b) = d$  is revolved through  $2\pi$  about the  $y$ -axis, the volume is:

$V = \pi \int_c^d x^2 \, dx$

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Volumes for Two Defining Functions

Upper function:  $f(x)$  or  $y_U$   
Lower function:  $g(x)$  or  $y_L$

$A = \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$

or

$A = \int_a^b (y_U^2 - y_L^2) \, dx$