12 Mathematical Methods Summarised Notes

Last updated: March 2022

Differential Calculus

Derivatives

The derivative of $y = f(x)$ is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Simple Rules of Differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}[c] = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[cu(x)] = cu'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[u(x) \pm v(x)] = u'(x) \pm v'(x)$$

Chain Rule

If $y = f(u(x))$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x)$$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Exponential Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{f(x)}\right] = e^{f(x)} \times f'(x)$$

Natural Logarithm

$$\ln x = \log_e x$$

Natural Logarithm Laws:

$\ln a + \ln b = \ln ab$	$\ln e = 1$
$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$	$\ln e^x = x$
$\ln(a^n) = n \ln a$	$e^{\ln x} = x$

Graphing:

$$y = \ln x$$

- asymptotic to the y-axis ($x = 0$)
- passes through $(1, 0)$
- domain: $x > 0$, range: $y \in \mathbb{R}$

General Function:

$$y = k \ln(b(x - c))$$

- horizontal translation by c units
- vertical translation by $k \ln b$ units
- vertical dilation scale factor k

Logarithmic Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x},\ x > 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln f(x)] = \frac{f'(x)}{f(x)}$$

Trigonometric Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \frac{1}{cos^2 x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin[f(x)]] = \cos[f(x)]f'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos[f(x)]] = -\sin[f(x)]f'(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}[\tan[f(x)]] = \frac{f'(x)}{cos^2[f(x)]}$$

Applications of Derivatives

Tangents

A line that touches a curve, matching the gradient of the curve at that point.
To find the tangent to a curve $y = f(x)$ at $x = a$:

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$
2. Find y-coordinate $y = f(a)$ and gradient $m = f'(a)$.
3. Substitute values into $y = mx + c$ to find c.

Or use general formula:

$$y = f'(a)(x - a) + f(a)$$

Normals

A line that intersects a curve, perpendicular to the tangent at that point.
For a normal to $y = f(x)$ at $x = a$, the gradient m is the negative reciprocal of the tangent slope.

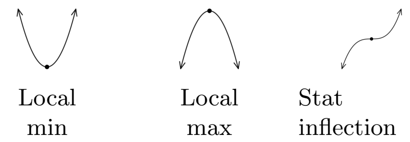
$$m = -\frac{1}{f'(a)}$$

Normal general formula:

$$y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Stationary Points

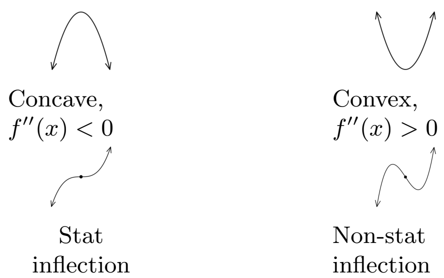
The stationary point of a function is where the tangent is horizontal, and so, $f'(x) = 0$.



Solve $f'(x) = 0$ for x.
Draw sign diagram of $f'(x)$
Classify stationary points.

Inflection Points

The inflection point on a curve is where the shape of the curve changes, when $f''(x) = 0$.



Solve $f''(x) = 0$ for x.
Classify inflection points:
If $f'(x) = 0$, stationary.
If $f'(x) \neq 0$, non-stationary.

Sketching Graphs of Derivatives

Sketching $y = f'(x)$ from $y = f(x)$:

1. Stat point $f(x) \to$ zero of $f'(x)$
2. If $f(x)$ increasing, $f'(x)$ positive
3. If $f(x)$ decreasing, $f'(x)$ negative
4. If $f(x)$ convex, $f'(x)$ increasing
5. If $f(x)$ concave, $f'(x)$ decreasing
6. Infl point $f(x) \to$ stat point $f'(x)$
7. Asymp $f(x) \to$ asymp $f'(x)$

Reverse to sketch $f(x)$ from $f'(x)$.

Sketching $y = f''(x)$ from $y = f(x)$:

1. Infl point $f(x) \to$ zero of $f''(x)$
2. If $f(x)$ convex, $f''(x)$ positive
3. If $f(x)$ concave, $f''(x)$ negative
4. Asymp $f(x) \to$ asymp $f''(x)$

Reverse to sketch $f'(x)$ from $f''(x)$.

....

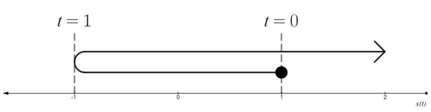
Kinematics

Describing the motion of moving objects.

Displacement: $s(t)$
Velocity: $v(t) = s'(t)$
Acceleration: $a(t) = v'(t) = s''(t)$

Motion Diagrams:

- Start position
- Time and position of direction change
- End position



Speed increasing when $v(t)$ and $a(t)$ have same sign. Decreasing when different sign.

....

Optimisation

Optimisation is finding the max/min value of a function, often called the optimal solution.

1. Draw a diagram.
2. Construct a formula with value to be optimised as the subject.
3. Find the first derivative and its zeros.
4. Use a sign diagram to determine the nature of stationary points.
5. Identify the optimal solution.
6. Write the answer as a sentence.

Integral Calculus

Antidifferentiation

The reverse process of differentiation. $F(x)$ is a function where $F'(x) = f(x)$.

- The derivative of $F(x)$ is $f(x)$
- The antiderivative of $f(x)$ is $F(x)$

....

Indefinite Integrals

$$\int k \, \mathrm{d}x = kx + c$$

$$\int x^n \, \mathrm{d}x = \frac{1}{n+1}x^{n+1} + c, \; n \neq -1$$

$$\int e^x \, \mathrm{d}x = e^x + c$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + c$$

$$\int \cos x \, \mathrm{d}x = \sin x + c$$

$$\int \sin x \, \mathrm{d}x = -\cos x + c$$

....

Integrating $f(ax+b)$

$$\int (ax+b)^n \, \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \quad n \neq -1$$

$$\int e^{ax+b} \, \mathrm{d}x = \frac{1}{a}e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

$$\int \cos(ax+b) \, \mathrm{d}x$$

$$\frac{1}{a}\sin(ax+b) + c$$

$$\int \sin(ax+b) \, \mathrm{d}x$$

$$= -\frac{1}{a}\cos(ax+b) + c$$

....

Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

....

Properties of Definite Integrals

$$\int_a^a f(x) \, \mathrm{d}x = 0$$

$$\int_a^b k \, \mathrm{d}x = k(b-a)$$

$$\int_a^b f(x) \, \mathrm{d}x = -\int_b^a f(x) \, \mathrm{d}x$$

$$\int_a^b kf(x) \, \mathrm{d}x = k\int_a^b f(x) \, \mathrm{d}x$$

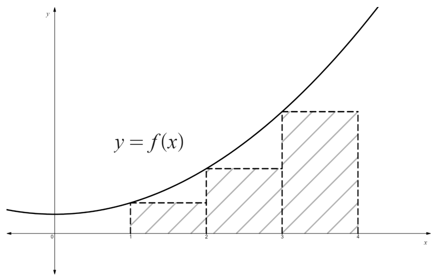
$$\int_a^b f(x) \, \mathrm{d}x + \int_b^c f(x) \, \mathrm{d}x = \int_a^c f(x) \, \mathrm{d}x$$

$$\int_a^b [f(x) \pm g(x)] \, \mathrm{d}x = \int_a^b f(x) \, \mathrm{d}x \pm \int_a^b g(x) \, \mathrm{d}x$$

....

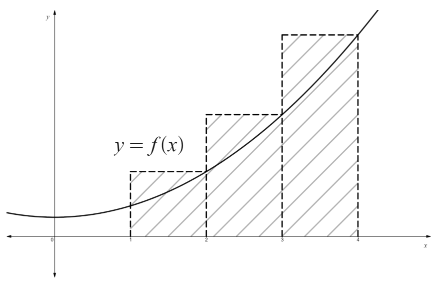
Underestimating and Overestimating

Underestimating:



$$A_L = 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$$

Overestimating:



$$A_U = 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$$

$$\therefore A_L \leq A \leq A_U$$

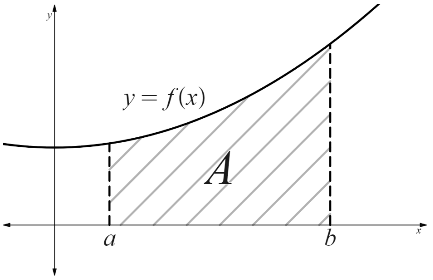
If graph is convex, underestimate is more accurate.
If graph is concave, overestimate is more accurate.

....

Area Under a Curve

If $f(x)$ is positive and continuous for $a \le x \le b$, the area bound by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x)\ \mathrm{d}x \qquad \text{or} \qquad A = \int_a^b y\ \mathrm{d}x$$



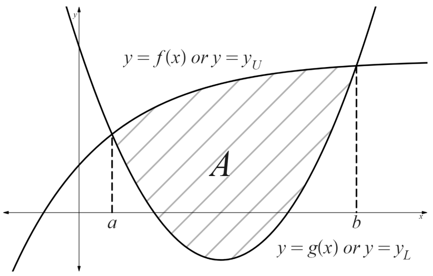
Area Between Two Curves

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)]\ \mathrm{d}x$$

or

$$A = \int_a^b [y_U - y_L]\ \mathrm{d}x$$



Kinematics

For a velocity time function $v(t)$ where $v(t) \ge 0$ on the interval $t_1 \le t \le t_2$

$$\text{distance travelled} = \int_{t_1}^{t_2} |v(t)|\,\mathrm{d}t$$

Displacement function:

$$s(t) = \int_{t_1}^{t_2} v(t)\,\mathrm{d}t$$

Statistics

Discrete Random Variables

Random variable X with distinct possible values $x_1, x_2, x_3, \dots, x_n$ and corresponding probabilities $\{p_1, p_2, p_3, \dots, p_n\}$

$$P(X = x_i) = p_i$$

$$0 \le p_i \le 1\ \forall\ i = 1, 2, 3, \dots, n$$

$$\sum_{i=1}^{n} p_i = p_1 + p_2 + p_3 + \cdots + p_n = 1$$

Mean/Expected Value

The mean/expected value of discrete random variable X is:

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

Fair Games

If X represents the gain of a player from each game, the game is fair if $E(X) = 0$.

Variance and Standard Deviation

Variance: Average squared deviation from the mean.

$$\mathrm{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i$$

or

$$\mathrm{Var}(X) = \sigma^2 = \sum x_i^2 p_i - \mu^2$$

Standard Deviation: Average deviations from the mean.

$$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

or

$$\sigma = \sqrt{\sum x_i^2 p_i - \mu^2}$$

Bernoulli Distribution

A Bernoulli random variable X can only take two values:

- $X = 1$ is "success"
- $X = 0$ is "failure"
- Only one trial is conducted

Mean: $E(X) = \mu = p$
Variance: $\mathrm{Var}(X) = \sigma^2 = p(1-p)$
Standard Deviation: $\sigma = \sqrt{p(1-p)}$

Binomial Distribution

If there are n independent trials with probability p of success, the probability there are k successes is:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $k = 1, 2, 3, \dots, n$

Binomial random variable X is denoted:

$$X \sim B(n, p)$$

Mean: $E(X) = \mu = np$
Variance: $\mathrm{Var}(X) = \sigma^2 = np(1-p)$
Standard Deviation: $\sigma = \sqrt{np(1-p)}$

Probability Density Functions

For a continuous random variable X on domain $a \le x \le b$ has probability density function $f(x)$ such that:

$$P(a \le x \le b) = \int_a^b f(x)\ \mathrm{d}x$$

$$f(x) \ge 0\ \forall\ a \le X \le b$$

$$\int_a^b f(x)\ \mathrm{d}x = 1$$

Mode: Value of x which maximises $f(x)$ on $a \le x \le b$.
Median: Value of m such that:

$$\int_a^m f(x)\ \mathrm{d}x = \frac{1}{2}$$

Mean:

$$E(X) = \mu = \int_a^b x f(x)\ \mathrm{d}x$$

Variance:

$$\mathrm{Var}(X) = \sigma^2 = \int_a^b x^2 f(x)\ \mathrm{d}x - \mu^2$$

or

$$\mathrm{Var}(X) = \sigma^2 = \int_a^b (x-\mu)^2 f(x)\ \mathrm{d}x$$

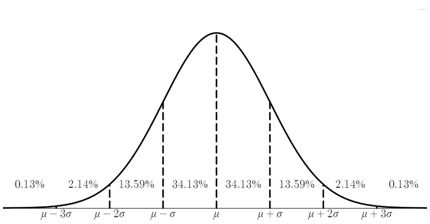
Standard Deviation:

$$\sigma = \sqrt{\int_a^b x^2 f(x)\ \mathrm{d}x - \mu^2}$$

or

$$\sigma = \sqrt{\int_a^b (x-\mu)^2 f(x)\ \mathrm{d}x}$$

Normal Distribution



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$
$$\text{for } -\infty < x < \infty$$

- Symmetric about the mean
- Bell shaped curve
- Asymptotic to the x-axis
- Empirical rule ("68-95-99" rule)
- Inflection point one σ from the mean
- More score distributed closer to the mean
- Peak at $(\mu, \frac{1}{\sigma\sqrt{2\pi}})$

Normally distributed variable X is denoted:

$$X \sim N(\mu, \sigma^2)$$

....

Standard Normal Distribution (Z-Distribution)

A normal distribution X can be transformed into a normal distribution.

$$Z \sim N(0, 1^2)$$

A z-score is used to compare a data value x across different data sets.

$$z = \frac{x-\mu}{\sigma}$$

Sampling and Confidence Intervals

Sampling Distributions

For the sum of n independent observations of a random variable X

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

The distribution of S_n is a sampling distribution.

Mean: $\mu_{S_n} = n\mu$

Standard Deviation: $\sigma_{S_n} = \sigma\sqrt{n}$

....

Distributions of Sample Means

The mean of n independent observations of a random variable X.

$$\bar{X}_n = \frac{X_1 + X_2 + X_3 + \cdots + X_n}{n}$$

Mean: $\mu_{\bar{X}_n} = \mu$

Standard Deviation: $\sigma_{\bar{X}_n} = \dfrac{\sigma}{\sqrt{n}}$

....

Central Limit Theorem

Suppose X is a random variable which is not necessarily normally distributed. X has population mean μ and standard deviation σ.

For Sufficient large n (generally $n \geq 30$), $\bar{X}_n$ is approximately normally distributed.

Mean: $\mu_{\bar{X}_n} = \mu$

Standard Deviation: $\sigma_{\bar{X}_n} = \dfrac{\sigma}{\sqrt{n}}$

....

Confidence Intervals for Means

A confidence interval for a population mean is an interval in which we are a certain percentage confident the population mean will lie.

<u>95% Confidence Interval:</u>

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}$$

<u>General Confidence Interval:</u>

$$\bar{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

Common Confidence Percentages:

90%:	$z_{\frac{\alpha}{2}} = 1.64$
95%:	$z_{\frac{\alpha}{2}} = 1.96$
98%:	$z_{\frac{\alpha}{2}} = 2.33$
99%:	$z_{\frac{\alpha}{2}} = 2.58$

To interpret a confidence interval in words, use the following template:

"We are <percentage>% confident that the mean of <quantity of interest> lies within <lower limit> <units> and <upper limit> <units>."

....

Determine Sample Size

$$n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2$$

Where w is the width of the confidence interval.
Always round n up to an integer.

....

Test a Claim About μ

- If μ_0 lies outside confidence interval, reject $\mu = \mu_0$.
- If μ_0 lies within confidence interval, cannot reject $\mu = \mu_0$.

....

Sample Proportions

A sample of size n is taken with X successes to find proportion $\hat{p}$.
$\hat{p}$ is an estimate of p.

$$\hat{p} = \frac{X}{n}$$

Mean: $\mu_{\hat{p}} = p$

Standard Deviation: $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$

Generally, the distribution of $\hat{p}$ is approximately normal if $np \geq 5$ and $n(1-p) \geq 5$.

....

Confidence Intervals for Proportions

A confidence interval for the population mean p where $\hat{p}$ is the sample mean and n is sample size.

<u>95% Confidence Interval:</u>

$$\hat{p} - 1.96\sqrt{\frac{p(1-p)}{n}} \leq p \leq \hat{p} + 1.96\sqrt{\frac{p(1-p)}{n}}$$

<u>General Confidence Interval:</u>

$$\hat{p} - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

To interpret a confidence interval in words, use the following template:

"We are <percentage>% confident that the proportion of <quantity of interest> lies within <lower limit>% and <upper limit>%."

....

Choosing Sample Size

$$n = \left(\frac{2 \times 1.96}{w}\right)^2 \hat{p}(1-\hat{p})$$

Where w is the width of the confidence interval.
Always round n up to an integer.
If $\hat{p}$ is unknown, assume $\hat{p} = p^*$, $p^* = 0.5$ for worst case scenario.
