

12 Mathematical Methods Summarised Notes

Last updated: March 2022

Differential Calculus

Derivatives

The derivative of $y = f(x)$ is:

dy/dx = f'(x) = lim_{h -> 0} (f(x+h) - f(x))/h

Simple Rules of Differentiation

d/dx [k] = 0
d/dx [x^n] = nx^{n-1}
d/dx [kf(x)] = k d/dx [f(x)] = kf'(x)
d/dx [u(x) +/- v(x)] = u'(x) +/- v'(x)

Chain Rule

If y = f(u(x)), then
dy/dx = dy/du * du/dx
or
d/dx [f(g(x))] = f'(g(x)) * g'(x)

Product Rule

d/dx [u(x)v(x)] = u'(x)v(x) + u(x)v'(x)

Quotient Rule

d/dx [u(x)/v(x)] = (u'(x)v(x) - u(x)v'(x))/[v(x)]^2

Exponential Functions

d/dx [e^x] = e^x
d/dx [e^{f(x)}] = e^{f(x)} * f'(x)

Natural Logarithm

ln x = log_e x

Natural Logarithm Laws:

ln a + ln b = ln ab
ln e = 1
ln a - ln b = ln (a/b)
ln e^x = x
ln (a^n) = n ln a
e^{ln x} = x

Graphing:

y = ln x

- asymptotic to the y-axis (x = 0)
- passes through (1, 0)
- domain: x > 0, range: y in R

General Function:

y = k ln (b(x - c))

- horizontal translation by c units
- vertical translation by k ln b units
- vertical dilation scale factor k

Logarithmic Functions

d/dx [ln x] = 1/x, x > 0
d/dx [ln f(x)] = f'(x)/f(x)

Trigonometric Functions

d/dx [sin x] = cos x
d/dx [cos x] = -sin x
d/dx [tan x] = 1/cos^2 x = sec^2 x
d/dx [sin [f(x)]] = cos [f(x)] f'(x)
d/dx [cos [f(x)]] = -sin [f(x)] f'(x)
d/dx [tan [f(x)]] = f'(x)/cos^2 [f(x)] = sec^2 [f(x)] f'(x)

Applications of Derivatives

Tangents

A line that touches a curve, matching the gradient of the curve at that point.
To find the tangent to a curve y = f(x) at x = a:

- Find dy/dx or f'(x)
- Find y-coordinate y = f(a) and gradient m = f'(a).
- Substitute values into y = mx + c to find c.

Or use general formula:

y = f'(a)(x - a) + f(a)

Normals

A line that intersects a curve, perpendicular to the tangent at that point.

For a normal to y = f(x) at x = a, the gradient m is the negative reciprocal of the tangent slope.

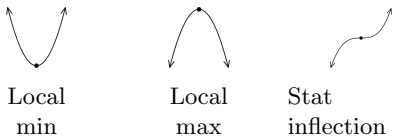
m = -1/f'(a)

Normal general formula:

y = -1/f'(a)(x - a) + f(a)

Stationary Points

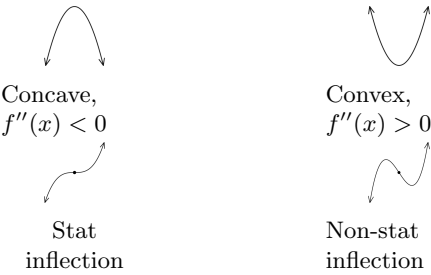
The stationary point of a function is where the tangent is horizontal, and so, f'(x) = 0.



Solve f'(x) = 0 for x.
Draw sign diagram of f'(x)
Classify stationary points.

Inflection Points

The inflection point on a curve is where the shape of the curve changes, when f''(x) = 0.



Solve f''(x) = 0 for x.
Classify inflection points:
If f'(x) = 0, stationary.
If f'(x) != 0, non-stationary.

Sketching Graphs of Derivatives

Sketching $y = f'(x)$ from $y = f(x)$:

- 1. Stat point $f(x) \rightarrow$ zero of $f'(x)$
- 2. If $f(x)$ increasing, $f'(x)$ positive
- 3. If $f(x)$ decreasing, $f'(x)$ negative
- 4. If $f(x)$ convex, $f'(x)$ increasing
- 5. If $f(x)$ concave, $f'(x)$ decreasing
- 6. Infl point $f(x) \rightarrow$ stat point $f'(x)$
- 7. Asymp $f(x) \rightarrow$ asymp $f'(x)$

Reverse to sketch $f(x)$ from $f'(x)$.

Sketching $y = f''(x)$ from $y = f(x)$:

- 1. Infl point $f(x) \rightarrow$ zero of $f''(x)$
- 2. If $f(x)$ convex, $f''(x)$ positive
- 3. If $f(x)$ concave, $f''(x)$ negative
- 4. Asymp $f(x) \rightarrow$ asymp $f''(x)$

Reverse to sketch $f'(x)$ from $f''(x)$.

Kinematics

Describing the motion of moving objects.

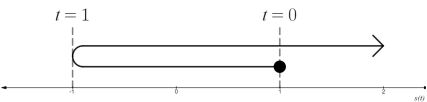
Displacement: $s(t)$

Velocity: $v(t) = s'(t)$

Acceleration: $a(t) = v'(t) = s''(t)$

Motion Diagrams:

- Start position
- Time and position of direction change
- End position



Speed increasing when $v(t)$ and $a(t)$ have same sign. Decreasing when different sign.

Optimisation

Optimisation is finding the max/min value of a function, often called the optimal solution.

- 1. Draw a diagram.
- 2. Construct a formula with value to be optimised as the subject.
- 3. Find the first derivative and its zeros.

- 4. Use a sign diagram to determine the nature of stationary points.
- 5. Identify the optimal solution.
- 6. Write the answer as a sentence.

Integral Calculus

Antidifferentiation

The reverse process of differentiation. $F(x)$ is a function where $F'(x) = f(x)$.

- The derivative of $F(x)$ is $f(x)$
- The antiderivative of $f(x)$ is $F(x)$

Indefinite Integrals

$$\int k \, dx = kx + c$$
$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c, \, n \neq -1$$
$$\int e^x \, dx = e^x + c$$
$$\int \frac{1}{x} \, dx = \ln |x| + c$$
$$\int \cos x \, dx = \sin x + c$$
$$\int \sin x \, dx = -\cos x + c$$

Integrating $f(ax + b)$

$$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c, \, n \neq -1$$
$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$
$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + c$$
$$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + c$$
$$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + c$$

Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

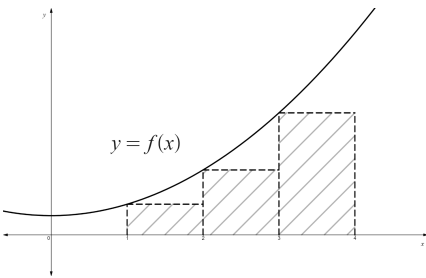
$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Properties of Definite Integrals

$$\int_a^a f(x) \, dx = 0$$
$$\int_a^b k \, dx = k(b-a)$$
$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$$
$$\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$$
$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$
$$\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

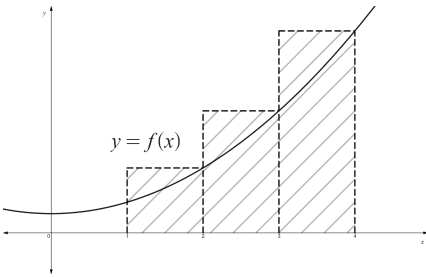
Underestimating and Overestimating

Underestimating:



$$A_L = 1 \times f(1) + 1 \times f(2) + 1 \times f(3)$$

Overestimating:



$$A_U = 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$$
$$\therefore A_L \leq A \leq A_U$$

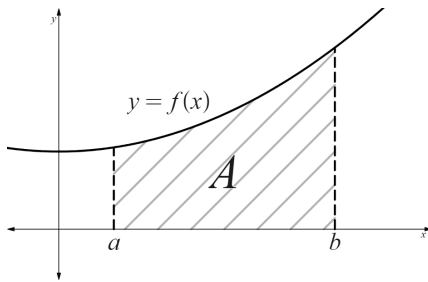
If graph is convex, underestimate is more accurate.

If graph is concave, overestimate is more accurate.

Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x) \, dx \quad \text{or} \quad A = \int_a^b y \, dx$$



Area Between Two Curves

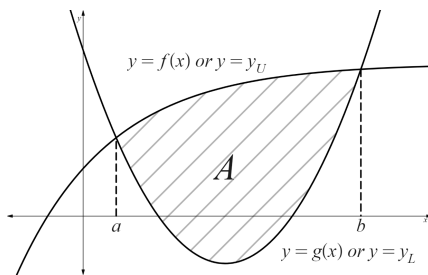
Upper function: $f(x)$ or y_U

Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)] \, dx$$

or

$$A = \int_a^b [y_U - y_L] \, dx$$



Kinematics

For a velocity time function $v(t)$ where $v(t) \geq 0$ on the interval $t_1 \leq t \leq t_2$

$$\text{distance travelled} = \int_{t_1}^{t_2} |v(t)| \, dt$$

Displacement function:

$$s(t) = \int_{t_1}^{t_2} v(t) \, dt$$

Statistics

Discrete Random Variables

Random variable X with distinct possible values $x_1, x_2, x_3, \dots, x_n$ and corresponding probabilities $\{p_1, p_2, p_3, \dots, p_n\}$

$$P(X = x_i) = p_i$$

$$0 \leq p_i \leq 1 \quad \forall i = 1, 2, 3, \dots, n$$

$$\sum_{i=1}^n p_i = p_1 + p_2 + p_3 + \dots + p_n = 1$$

Mean/Expected Value

The mean/expected value of discrete random variable X is:

$$\begin{aligned} E(X) = \mu &= \sum_{i=1}^n x_i p_i \\ &= x_1 p_1 + x_2 p_2 + \dots + x_n p_n \end{aligned}$$

Fair Games

If X represents the gain of a player from each game, the game is fair if $E(X) = 0$.

Variance and Standard Deviation

Variance: Average squared deviation from the mean.

$$\text{Var}(X) = \sigma^2 = \sum (x_i - \mu)^2 p_i$$

or

$$\text{Var}(X) = \sigma^2 = \sum x_i^2 p_i - \mu^2$$

Standard Deviation: Average deviations from the mean.

$$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

or

$$\sigma = \sqrt{\sum x_i^2 p_i - \mu^2}$$

Bernoulli Distribution

A Bernoulli random variable X can only take two values:

- $X = 1$ is "success"
- $X = 0$ is "failure"
- Only one trial is conducted

Mean: $E(X) = \mu = p$

Variance: $\text{Var}(X) = \sigma^2 = p(1 - p)$

Standard Deviation: $\sigma = \sqrt{p(1 - p)}$

Binomial random variable X is denoted:

$$X \sim B(n, p)$$

Mean: $E(X) = \mu = np$

Variance: $\text{Var}(X) = \sigma^2 = np(1 - p)$

Standard Deviation: $\sigma = \sqrt{np(1 - p)}$

Probability Density Functions

For a continuous random variable X on domain $a \leq x \leq b$ has probability density function $f(x)$ such that:

$$P(a \leq x \leq b) = \int_a^b f(x) \, dx$$

$$f(x) \geq 0 \quad \forall a \leq x \leq b$$

$$\int_a^b f(x) \, dx = 1$$

Mode: Value of x which maximises $f(x)$ on $a \leq x \leq b$.

Median: Value of m such that:

$$\int_a^m f(x) \, dx = \frac{1}{2}$$

Mean:

$$E(X) = \mu = \int_a^b x f(x) \, dx$$

Variance:

$$\text{Var}(X) = \sigma^2 = \int_a^b x^2 f(x) \, dx - \mu^2$$

or

$$\text{Var}(X) = \sigma^2 = \int_a^b (x - \mu)^2 f(x) \, dx$$

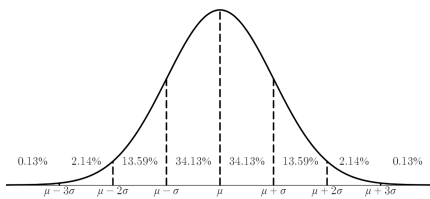
Standard Deviation:

$$\sigma = \sqrt{\int_a^b x^2 f(x) \, dx - \mu^2}$$

or

$$\sigma = \sqrt{\int_a^b (x - \mu)^2 f(x) \, dx}$$

Normal Distribution



f(x) = 1 / (σ√2π) * e^(-1/2 * ((x-μ)/σ)^2)

for -∞ < x < ∞

- Symmetric about the mean
- Bell shaped curve
- Asymptotic to the x-axis
- Empirical rule (“68-95-99” rule)
- Inflection point one σ from the mean
- More score distributed closer to the mean
- Peak at (μ, 1 / (σ√2π))

Normally distributed variable X is denoted:

X ~ N(μ, σ^2)

Standard Normal Distribution (Z-Distribution)

A normal distribution X can be transformed into a normal distribution.

Z ~ N(0, 1^2)

A z-score is used to compare a data value x across different data sets.

z = (x - μ) / σ

Sampling and Confidence Intervals

Sampling Distributions

For the sum of n independent observations of a random variable X

S_n = X_1 + X_2 + X_3 + ... + X_n

The distribution of S_n is a sampling distribution.

Mean: μ_{S_n} = nμ

Standard Deviation: σ_{S_n} = σ√n

Distributions of Sample Means

The mean of n independent observations of a random variable X.

X̄_n = (X_1 + X_2 + X_3 + ... + X_n) / n

Mean: μ_{X̄_n} = μ

Standard Deviation: σ_{X̄_n} = σ / √n

Central Limit Theorem

Suppose X is a random variable which is not necessarily normally distributed. X has population mean μ and standard deviation σ.

For Sufficient large n (generally n ≥ 30), X̄_n is approximately normally distributed.

Mean: μ_{X̄_n} = μ

Standard Deviation: σ_{X̄_n} = σ / √n

Confidence Intervals for Means

A confidence interval for a population mean is an interval in which we are a certain percentage confident the population mean will lie.

95% Confidence Interval:

x̄ - 1.96 * (σ / √n) ≤ μ ≤ x̄ + 1.96 * (σ / √n)

General Confidence Interval:

x̄ - z_{a/2} * (σ / √n) ≤ μ ≤ x̄ + z_{a/2} * (σ / √n)

Common Confidence Percentages:

- 90%: z_{a/2} = 1.64
- 95%: z_{a/2} = 1.96
- 98%: z_{a/2} = 2.33
- 99%: z_{a/2} = 2.58

To interpret a confidence interval in words, use the following template:

“We are <percentage>% confident that the mean of <quantity of interest> lies within <lower limit> <units> and <upper limit> <units>.”

Determine Sample Size

n = ((2 * 1.96σ) / w)^2

Where w is the width of the confidence interval.

Always round n up to an integer.

Test a Claim About μ

- If μ_0 lies outside confidence interval, reject μ = μ_0.
- If μ_0 lies within confidence interval, cannot reject μ = μ_0.

Sample Proportions

A sample of size n is taken with X successes to find proportion p̂. p̂ is an estimate of p.

p̂ = X / n

Mean: μ_{p̂} = p

Standard Deviation: σ_{p̂} = √(p(1-p) / n)

Generally, the distribution of p̂ is approximately normal if np ≥ 5 and n(1-p) ≥ 5.

Confidence Intervals for Proportions

A confidence interval for the population mean p where p̂ is the sample mean and n is sample size.

95% Confidence Interval:

p̂ - 1.96 * √(p̂(1-p̂) / n) ≤ p ≤ p̂ + 1.96 * √(p̂(1-p̂) / n)

General Confidence Interval:

p̂ - z_{a/2} * √(p̂(1-p̂) / n) ≤ p ≤ p̂ + z_{a/2} * √(p̂(1-p̂) / n)

To interpret a confidence interval in words, use the following template:

“We are <percentage>% confident that the proportion of <quantity of interest> lies within <lower limit>% and <upper limit>%.”

Choosing Sample Size

n = ((2 * 1.96) / w)^2 * p̂(1-p̂)

Where w is the width of the confidence interval.

Always round n up to an integer. If p̂ is unknown, assume p̂ = p*, p* = 0.5 for worst case scenario.