12 Specialist Mathematics Summarised Notes (Unofficial) Work in Progress

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Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$
or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function.
$f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, $\therefore$ passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f.
- Range of f^{-1} = domain of f.

Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \dfrac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.
Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \to$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \to$ zero $\frac{1}{f(x)}$
- Local max $f(x) \to$ local min $\frac{1}{f(x)}$
- Local min $f(x) \to$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- When $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$

Invariant Points:
Points which do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \to \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y-axis
- Points on the y-axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x-axis
- Points on the x-axis are invariant

Trigonometric Identities

Angle Relationships

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

Pythagorean Theorem

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1 \\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta} \end{aligned}$$

Angle Sum and Difference

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Sum to Product

$$\sin A \pm \sin B = 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A + B}{2}\right)\sin\left(\frac{A - B}{2}\right)$$

Product to Sum

$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$
$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$
$$2\cos A\cos B = \cos(A + B) + \cos(A - B)$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $\quad z = a + bi$

$$\mathfrak{Re}(z) = a \qquad \mathfrak{Im}(z) = b$$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x-axis is the real axis and the y-axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{represents} \quad x + yi$$

Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad z^* = a - bi$$

In the complex plane, z^* is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\binom{a}{b}$, which is the real number:

$$|z| = \sqrt{a^2 + b^2}$$

The argument of z, $\arg(z)$ is the angle θ between the positive real axis and $\binom{a}{b}$.
Real numbers have an argument of 0 or π.
Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$
- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \ z_2 \neq 0$
- $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3| \ldots |z_n|$
- $|z^n| = |z|^n, \ n \in \mathbb{Z}^+$

Polar Form

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta$$

A complex number z has polar form

$$z = |z| \operatorname{cis}\theta$$

where $\theta = \arg(z)$.
The conjugate of z is:

$$z^* = |z| \operatorname{cis}(-\theta)$$

Properties of $\operatorname{cis}\theta$:

- $\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta + \phi)$
- $\frac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta - \phi)$
- $\operatorname{cis}(\theta - 2k\pi) = \operatorname{cis}\theta, \ k \in \mathbb{Z}$

De Moivre's Theorem

$$(|z| \operatorname{cis}\theta)^n = |z|^n \operatorname{cis} n\theta, \ \text{for all } n \in \mathbb{Q}$$

Roots of Complex Numbers

The n^{th} roots of the complex number c are the solutions of $z^n = c$.

The n^{th} Roots of Unity

The n^{th} roots of unity are the solutions of $z^n = 1$.

Distances in the Complex Plane

If $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2.

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
α is a zero of polynomial

$$P(x) \iff P(\alpha) = 0$$

The roots of a polynomial equation are the solutions to the equation.
α is a root (or solution) of

$$P(x) \iff P(\alpha) = 0$$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x-intercepts of the graph $y = P(x)$

Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$$

where $ex + f$ is the remainder.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n, then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X, Y and Z direction from the origin O. The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i} = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\mathbf{j} = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $\mathbf{k} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, the base unit vectors.

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The Magnitude of a Vector

The magnitude or length of the vector $\mathbf{a} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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Operations with Vectors

$$\text{If } \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ then:}$$

$$-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

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Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{c} - \mathbf{b}$
- If $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$

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Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B is $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

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Unit Vectors

The unit vector $\hat{\mathbf{v}}$, a vector of length 1 in the direction of $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

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Dot Product (Scalar Product)

If $\mathbf{a} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ and $\mathbf{b} = \left(\begin{smallmatrix} b_1 \\ b_2 \\ b_3 \end{smallmatrix}\right)$, then the scalar dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

<u>Properties</u>

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{c}+\mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
- $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$, $k \in \mathbb{R}$

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The Angle Between Two Vectors

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ cacn be found using

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

<u>Scalar Product Geometric Properties</u>

- For non-zero vectors $\mathbf{a}$ and $\mathbf{b}$:
  $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and $\mathbf{b}$ are perpendicular.
- $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \iff \mathbf{a}$ and $\mathbf{b}$ are non-zero parallel vectors.
- Given $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
  If θ is acute, $\cos\theta > 0$ and so $\mathbf{a} \cdot \mathbf{b} > 0$
  If θ is obtuse, $\cos\theta < 0$ and so $\mathbf{a} \cdot \mathbf{b} < 0$

Integration

Indefinite Integrals

$$\int k \;\mathrm{d}x = kx + c$$
$$\int x^n \;\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c,\; n \neq -1$$
$$\int e^x \;\mathrm{d}x = e^x + c$$
$$\int \frac{1}{x} \;\mathrm{d}x = \ln|x| + c$$
$$\int \cos x \;\mathrm{d}x = \sin x + c$$
$$\int \sin x \;\mathrm{d}x = -\cos x + c$$

Integrating $f(ax+b)$

$$\int (ax+b)^n \;\mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c,$$
$$n \neq -1$$
$$\int e^{ax+b} \;\mathrm{d}x = \frac{1}{a}e^{ax+b} + c$$
$$\int \frac{1}{ax+b} \;\mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$
$$\int \cos(ax+b) \;\mathrm{d}x$$
$$\frac{1}{a}\sin(ax+b) + c$$
$$\int \sin(ax+b) \;\mathrm{d}x$$
$$= -\frac{1}{a}\cos(ax+b) + c$$

Inverse Trigonomentric Functions

$$\int \frac{1}{\sqrt{1-x^2}} \;\mathrm{d}x = \arcsin x + c$$
$$\int -\frac{1}{\sqrt{1-x^2}} \;\mathrm{d}x = \arccos x + c$$
$$\int \frac{1}{1+x^2} \;\mathrm{d}x = \arctan x + c$$
$$\int \frac{1}{\sqrt{a^2-x^2}} \;\mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + c$$
$$\int -\frac{1}{\sqrt{a^2-x^2}} \;\mathrm{d}x = \arccos\left(\frac{x}{a}\right) + c$$
$$\int \frac{a}{a^2+x^2} \;\mathrm{d}x = \arctan\left(\frac{x}{a}\right) + c$$

Integrating $\sin^2 x$ and $\cos^2 x$

Use double angle identities when integrating.

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$$

Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$$\int_a^b f(x) \;\mathrm{d}x = F(b) - F(a)$$

Properties of Definite Integrals

$$\int_a^a f(x) \;\mathrm{d}x = 0$$
$$\int_a^b k \;\mathrm{d}x = k(b-a)$$
$$\int_a^b f(x) \;\mathrm{d}x = -\int_b^a f(x) \;\mathrm{d}x$$
$$\int_a^b kf(x) \;\mathrm{d}x = k\int_a^b f(x) \;\mathrm{d}x$$
$$\int_a^b f(x) \;\mathrm{d}x + \int_b^c f(x) \;\mathrm{d}x$$
$$= \int_a^c f(x) \;\mathrm{d}x$$
$$\int_a^b [f(x) \pm g(x)] \;\mathrm{d}x$$
$$= \int_a^b f(x) \;\mathrm{d}x \pm \int_a^b g(x) \;\mathrm{d}x$$

Integration by Substitution

$$\int f(u)\,\frac{\mathrm{d}u}{\mathrm{d}x} \;\mathrm{d}x = \int f(u) \;\mathrm{d}u$$

When solving definite integrals with bounds a and b, adjust as $u(a)$ and $u(b)$.

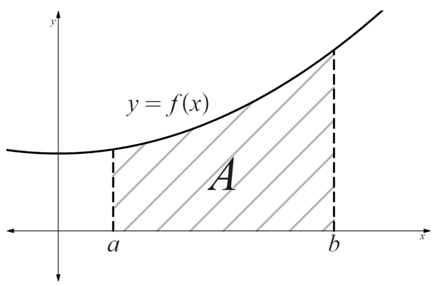
Integration by Parts

$$\int uv' \;\mathrm{d}x = uv - \int u'v \;\mathrm{d}x$$

Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x) \;\mathrm{d}x \qquad \text{or} \qquad A = \int_a^b y \;\mathrm{d}x$$



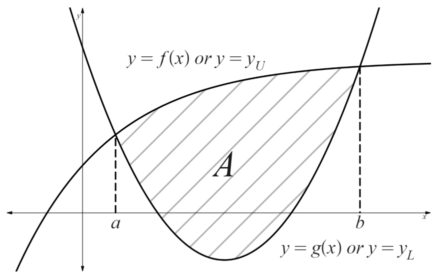
Area Between Two Curves

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)] \;\mathrm{d}x$$

or

$$A = \int_a^b [y_U - y_L] \;\mathrm{d}x$$



Solids of Revolution

When the region enclosed by $y = f(x)$, the x-axis, $x = a$ and $x = b$ is revolved through 2π about the x-axis, the volume is:

$$V = \pi\int_a^b y^2 \;\mathrm{d}x$$

When the region enclosed by $y = f(x)$, the y-axis, $y = f(a) = c$ and $y = f(b) = d$ is revolved through 2π about the y-axis, the volume is:

$$V = \pi\int_c^d x^2 \;\mathrm{d}y$$

Volumes for Two Defining Functions

Upper function: $f(x)$ or y_U
Lower function: $g(x)$ or y_L

$$A = \int_a^b \left([f(x)]^2 - [g(x)]^2\right) \;\mathrm{d}x$$

or

$$A = \int_a^b \left(y_U^2 - y_L^2\right) \;\mathrm{d}x$$