

12 Specialist Mathematics Summarised Notes
(Unofficial)

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function. $f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f .
- Range of f^{-1} = domain of f .

Self-Inverse Functions

An invertible function that is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$. Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points that do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another. Vertical asymptote occurs when denominator is zero. Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$. If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x -axis
- Points on the x -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a, a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$\vec{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$ is $z^* = a - bi$

In the complex plane, z^* is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of z , $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have an argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$\text{cis } \theta = \cos \theta + i \sin \theta$

A complex number z has the polar form

$z = |z| \text{cis } \theta$

where $\theta = \arg(z)$.
The conjugate of z is:

$z^* = |z| \text{cis } (-\theta)$

Properties of cis θ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

De Moivre's Theorem

$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$

Roots of Complex Numbers

The n^{th} roots of the complex number c are the solutions of $z^n = c$.

The n^{th} Roots of Unity

The n^{th} roots of unity are the solutions of $z^n = 1$.

Distances in the Complex Plane

If $z_1 \equiv \vec{OP_1}$ and $z_2 \equiv \vec{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2 .

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
 α is a zero of polynomial

$P(x) \iff P(\alpha) = 0$

The roots of a polynomial equation are the solutions to the equation.
 α is a root (or solution) of

$P(x) \iff P(\alpha) = 0$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x -intercepts of the graph $y = P(x)$

Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$

where $ex + f$ is the remainder.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n , then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X , Y and Z direction from the origin O .
The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations with Vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

$$-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$
$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{c} - \mathbf{b}$
- If $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$

Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B is
$$|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

Parallelism

$\mathbf{a} = k\mathbf{b} \iff \mathbf{a}$ and \mathbf{b} are non-zero parallel vectors.

Collinear Points

A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$.

Unit Vectors

The unit vector $\hat{\mathbf{v}}$, a vector of length 1 in the direction of \mathbf{v} is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

Dot Product (Scalar Product)

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the scalar dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
- $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$, $k \in \mathbb{R}$

The Angle Between Two Vectors

The angle θ between two vectors \mathbf{a} and \mathbf{b} can be found using

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Scalar Product Geometric Properties

- For non-zero vectors \mathbf{a} and \mathbf{b} :
 $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and \mathbf{b} are perpendicular.
- $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \iff \mathbf{a}$ and \mathbf{b} are non-zero parallel vectors.
- Given $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
If θ is acute, $\cos \theta > 0$ and so $\mathbf{a} \cdot \mathbf{b} > 0$
If θ is obtuse, $\cos \theta < 0$ and so $\mathbf{a} \cdot \mathbf{b} < 0$

Cross Product (Vector Product)

The vector cross product of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Alternatively,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Properties

- $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both \mathbf{a} and \mathbf{b} .
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all \mathbf{a} .
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, $\therefore \mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have equal length but opposite direction.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the scalar triple product.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}$

Direction of $\mathbf{a} \times \mathbf{b}$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} .

Area

If a triangle has defining vectors \mathbf{a} and \mathbf{b} then its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units².
If a parallelogram has defining vectors \mathbf{a} and \mathbf{b} then its area is $|\mathbf{a} \times \mathbf{b}|$ units².

Lines in 2 and 3 Dimensions

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ is the vector equation of the line.

The Shortest Distance From a Point to a Line

A point P is closest to a point R on a line in direction \mathbf{b} when \overrightarrow{PR} is perpendicular to \mathbf{b} .

$$\overrightarrow{PR} \cdot \mathbf{b} = 0$$

Relationships Between Lines

Line Classification in 2 Dimensions

- Intersecting: Unique solution
- Parallel: No solutions
- Coincident: Infinite Solutions

Line Classification in 3 Dimensions

- Lines are coplanar if they lie in the same plane. In this case, they may be intersecting, parallel or coincident.
- Otherwise, they are skew.

Shortest Distance Between Skew Lines

For two skew lines with vector equations $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$, the shortest distance d between them is

d = (a1 - a2) · (b1 × b2) / |b1 × b2|

Planes

A plane is a flat surface that extends forever and has zero thickness. The Vector Equation of a Plane

r = a + sv + tw

- r is the position vector of any point on the plane.
- a is the position vector of a known point on the plane.
- v and w are any two non-parallel vectors that are parallel to the plane.
- s, t ∈ ℝ are two independent parameters.

The Normal of a Plane

A vector is normal to a plane if it is perpendicular to all vectors which are parallel to the plane. If n is a normal to a plane such as n = v × w, an equivalent vector equation of a plane is

n · (r - a) or r · n = a · n

The Cartesian Equation of a Plane

If a plane has normal vector n = (a, b, c) and passes through P(X, Y, Z) then it has equation

ax + by + cz = aX + bY + cZ = d

where d is a constant.

Distance Between a Point and a Plane

The distance between a point P(x1, y1, z1) and the plane Ax + By + Cz + D = 0 is

d = |Ax1 + Bx1 + Cx1 + D| / sqrt(A^2 + B^2 + C^2)

Angles in Space

The Angle Between a Line and a Plane The acute angle φ between a line with direction vector a and a plane with normal vector n is

φ = sin^-1 (|n · b| / (|n||b|))

The Angle Between Two Planes

If two planes have normal vectors n1 and n2 and θ is the acute angle between them then

θ = cos^-1 (|n1 · n2| / (|n1||n2|))

Row Reduction

Linear systems of equations can be solved using augmented matrices. A general 3 × 3 system has the form:

{ a1x + b1y + c1z = d1, a2x + b2y + c2z = d2, a3x + b3y + c3z = d3

In augmented matrix form, the system is:

[a1 b1 c1 | d1, a2 b2 c2 | d2, a3 b3 c3 | d3]

Using row operations it is reduced to the echelon form.

[a b c | d, 0 e f | g, 0 0 h | i]

∴ hz = i ⇒ z = i/h, ey + fz = g and ax + by + cz = d

Intersecting Planes

Two planes in space could have the following arrangements:

- Intersecting
- Parallel
- Coincident

Three planes in space could have the following arrangements:

- All coincident
- Two coincident and one intersecting
- Two coincident and one parallel
- Two parallel and one intersecting
- All parallel
- All meet at one point
- All meet in a common line
- The line of intersection of any two planes is parallel to the third plane.

Integration

Indefinite Integrals

∫ k dx = kx + c, ∫ x^n dx = 1/(n+1) x^(n+1) + c, n ≠ -1, ∫ e^x dx = e^x + c, ∫ 1/x dx = ln|x| + c, ∫ cos x dx = sin x + c, ∫ sin x dx = -cos x + c

Integrating f(ax + b)

∫ (ax + b)^n dx = (ax + b)^(n+1) / (a(n+1)) + c, n ≠ -1, ∫ e^(ax+b) dx = 1/a e^(ax+b) + c, ∫ 1/(ax + b) dx = 1/a ln|ax + b| + c, ∫ cos(ax + b) dx = 1/a sin(ax + b) + c, ∫ sin(ax + b) dx = -1/a cos(ax + b) + c

Inverse Trigonometric Functions

∫ 1/sqrt(1-x^2) dx = arcsin x + c, ∫ -1/sqrt(1-x^2) dx = arccos x + c, ∫ 1/(1+x^2) dx = arctan x + c, ∫ 1/sqrt(a^2-x^2) dx = arcsin(x/a) + c, ∫ -1/sqrt(a^2-x^2) dx = arccos(x/a) + c, ∫ a/(a^2+x^2) dx = arctan(x/a) + c

Integrating $\sin^2 x$ and $\cos^2 x$

Use double-angle identities when integrating.

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x$$

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Definite Integrals

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous over $a \leq x \leq b$, the definite integral is:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

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Properties of Definite Integrals

$$\int_a^a f(x) \, dx = 0$$

$$\int_a^b k \, dx = k(b - a)$$

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$$

$$\int_a^b k f(x) \, dx = k \int_a^b f(x) \, dx$$

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx$$

$$= \int_a^c f(x) \, dx$$

$$\int_a^b [f(x) \pm g(x)] \, dx$$

$$= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

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Integration by Substitution

$$\int f(u) \frac{du}{dx} \, dx = \int f(u) \, du$$

When solving definite integrals with bounds a and b , adjust as $u(a)$ and $u(b)$.

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Integration by Parts

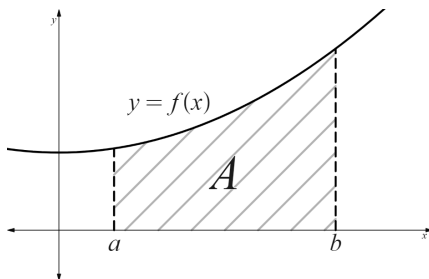
$$\int uv' \, dx = uv - \int u'v \, dx$$

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Area Under a Curve

If $f(x)$ is positive and continuous for $a \leq x \leq b$, the area bound by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is:

$$A = \int_a^b f(x) \, dx \quad \text{or} \quad A = \int_a^b y \, dx$$



Area Between Two Curves

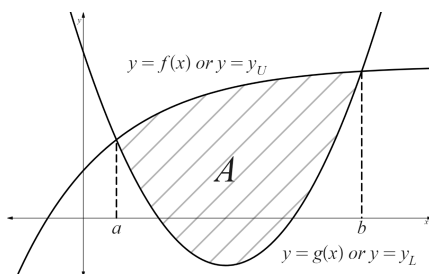
Upper function: $f(x)$ or y_U

Lower function: $g(x)$ or y_L

$$A = \int_a^b [f(x) - g(x)] \, dx$$

or

$$A = \int_a^b [y_U - y_L] \, dx$$



Solids of Revolution

When the region enclosed by $y = f(x)$, the x -axis, $x = a$ and $x = b$ is revolved through 2π about the x -axis, the volume is:

$$V = \pi \int_a^b y^2 \, dx$$

When the region enclosed by $y = f(x)$, the y -axis, $y = f(a) = c$ and $y = f(b) = d$ is revolved through 2π about the y -axis, the volume is:

$$V = \pi \int_c^d x^2 \, dy$$

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Volumes for Two Defining Functions

Upper function: $f(x)$ or y_U

Lower function: $g(x)$ or y_L

$$V = \int_a^b ([f(x)]^2 - [g(x)]^2) \, dx$$

or

$$V = \int_a^b (y_U^2 - y_L^2) \, dx$$

Rates of Change and Differential Equations

Implicit Differentiation

To find $\frac{dy}{dx}$ for an implicit relationship between x and y . A useful property is

$$\frac{d}{dx} [x^n] = nx^{n-1}$$

and

$$\frac{d}{dx} [y^n] = ny^{n-1} \frac{dy}{dx}$$

.....

Related Rates

Two variables x and y may be related to each other and defined using an equation.

$$\text{Eg: } x^2 + y^2 = 5^2$$

The equation can be differentiated in respect to a third parameter t resulting in a related rates equation.

$$\text{Eg: } 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

.....

Differential Equations

A differential equation is an equation that describes the relationship between a function and its derivatives.

.....

Differential Equations of the Form $\frac{dy}{dx} = f(x)$

Differential equations of the form $\frac{dy}{dx} = f(x)$ can be solved using integration.

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x) \, dx$$

$$\int dy = \int f(x) \, dx$$

$$\therefore y = \int f(x) \, dx$$

.....

Seperable Differential Equations

Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ can be solved as follows:

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{1}{g(y)} \, dy = f(x) \, dx$$

$$\int \frac{1}{g(y)} \, dy = \int f(x) \, dx$$

.....

Slope Fields

For a differential equation $\frac{dy}{dx} = f(x, y)$, any point (x, y) on the cartesian plane will have a gradient which can be graphically represented using a slope field.

Logistic Growth

Logistic growth is defined by the differential equation

dP/dt = kP(1 - P/A)

The differential equation is solved with partial fractions using the identity:

A/(P(A-P)) = 1/P + 1/(A-P)

Vector Calculus

Parametric Equations

Cartesian equation can be expressed in terms of a parameter such as t, defining x and y independently such that x = x(t) and y = y(t).

Pairs of Uniformly Varying Quantities

An object initially at (x0, y0) and moving with velocity vector v = (a, b) has parametric equations

x(t) = x0 + at, y(t) = y0 + bt, t >= 0

The speed of the object is

|v| = sqrt(a^2 + b^2)

Pairs of Non-Uniformly Varying Quantities

For the moving object P(x(t), y(t)), the velocity vector is:

v = (x'(t), y'(t))

and the speed is:

speed = |v| = sqrt([x'(t)]^2 + [y'(t)]^2)

The gradient of the velocity vector at any point is:

y'(t)/x'(t) = (dy/dt)/(dx/dt) = dy/dx

The velocity vector is tangential to the curve, therefore the direction of motion is the gradient.

Bézier Curves

Given the starting point (x0, y0) and the finishing point (x3, y3), and control points (x1, y1) and (x2, y2), the Bézier curve has parametric equations:

x(t) = axt^3 + bxt^2 + cxt + dx

y(t) = ayt^3 + byt^2 + cyt + dy

0 <= t <= 1

where

{ ax = x3 - 3x2 + 3x1 - x0, by = 3x2 - 6x1 + 3x0, cx = 3x1 - 3x0, dx = x0 }

and

{ ay = y3 - 3y2 + 3y1 - y0, by = 3y2 - 6y1 + 3y0, cy = 3y1 - 3y0, dy = y0 }

Trigonometric Parameterisation

For an object P moving with parametric equations

x(t) = R cos wt, y(t) = R sin wt, t >= 0

where R > 0, w > 0:

- R controls the radius of the path.
- w controls the rate of revolutions. The duration of a revolution is 2pi/w.
- The speed of P is R w.

Arc Lengths of Parametric Curves

The length of the arc traced out by P(x(t), y(t)) from time t = a to t = b is:

int_a^b |v| dt = int_a^b sqrt(v.v) dt = int_a^b sqrt([x'(t)]^2 + [y'(t)]^2) dt