

12 Specialist Mathematics Summarised Notes  
(Work in Progress)

2022

Functions

Composite Functions

Given  $f : x \mapsto f(x)$  and  $g : x \mapsto g(x)$ , the composite function of  $f$  and  $g$  is:

$$(f \circ g)(x) = f(g(x))$$
  
or  
$$f \circ g : x \mapsto f(g(x))$$

In general,  $(f \circ g)(x) \neq (g \circ f)(x)$ .

Inverse Functions

An inverse function returns the original value from the output of a function.  $f(x)$  has an inverse if it is injective (one-to-one), if  $f(a) = f(b)$  only when  $a = b$ ,  $\therefore$  passes the horizontal line test.

For  $f^{-1}(x)$ , the inverse of  $f(x)$

- Is a reflection of  $y = f(x)$  over  $y = x$ .
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of  $f^{-1}$  = Range of  $f$ .
- Range of  $f^{-1}$  = Domain of  $f$ .

Self-Inverse Functions

An invertible function which is symmetrical about  $y = x$ .

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form  $f(x) = \frac{k}{x}$ , where  $k \neq 0$  is a constant.

Reciprocal of Other Functions

The reciprocal of a function  $f(x)$  is  $\frac{1}{f(x)}$ .

Graphing  $y = \frac{1}{f(x)}$  from  $y = f(x)$ :

- Zero  $f(x) \rightarrow$  vertical asymp  $\frac{1}{f(x)}$
- Vertical asymp  $f(x) \rightarrow$  zero  $\frac{1}{f(x)}$
- Local max  $f(x) \rightarrow$  local min  $\frac{1}{f(x)}$
- Local min  $f(x) \rightarrow$  local max  $\frac{1}{f(x)}$
- When  $f(x) > 0$ ,  $\frac{1}{f(x)} > 0$
- When  $f(x) < 0$ ,  $\frac{1}{f(x)} < 0$
- When  $f(x) \rightarrow 0$ ,  $\frac{1}{f(x)} \rightarrow \pm\infty$
- When  $f(x) \rightarrow \pm\infty$ ,  $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points which do not move under a transformation occurring at  $y = \pm 1$ .

Rational Functions

Results from the division of one polynomial by another.

Vertical asymptote occurs when denominator is zero.

Horizontal asymptote ascertained from behaviour of graph as  $|x| \rightarrow \infty$ .

- If the degree of denominator > numerator, horizontal asymptote at  $y = 0$ .
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at  $y = \frac{a}{b}$  where  $a$  and  $b$  are the leading coefficients.

Absolute Value Functions

The absolute value or modulus  $|x|$  of a real number  $x$  is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If  $|x| = a$  where  $a > 0$ , then  $x = \pm a$ .

If  $|x| = |b|$  then  $x = \pm b$ .

Graphs Involving the Absolute Value Function

Graphing  $y = f(|x|)$  from  $y = f(x)$ :

- Discard the graph for  $x < 0$
- Reflect the graph for  $x \geq 0$  in the  $y$ -axis
- Points on the  $y$ -axis are invariant

Graphing  $y = |f(x)|$  from  $y = f(x)$ :

- Keep the graph for  $f(x) \geq 0$
- Reflect the graph for  $f(x) < 0$  in the  $x$ -axis
- Points on the  $x$ -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose  $P_n$  is a proposition which is defines for every integer  $n \geq a$ ,  $a \in \mathbb{Z}$ . If  $P_a$  is true, and if  $P_{k+1}$  is true whenever  $P_k$  is true, then  $P_n$  is true for all  $n \geq a$ .

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form  $ai$  where  $a \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

Complex Numbers

Any number in the form  $a + bi$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

If  $z = a + bi$   
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the  $x$ -axis is the real axis and the  $y$ -axis is the imaginary axis.

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$  represents  $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$  is  $\bar{z} = a - bi$

In the complex plane,  $\bar{z}$  is the reflection of  $z$  in the real axis.

Modulus and Argument

The modulus of the complex number  $z = a + bi$  is the length of the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ , which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of  $z$ ,  $\arg(z)$  is the angle  $\theta$  between the positive real axis and  $\begin{pmatrix} a \\ b \end{pmatrix}$ . Real numbers have argument of 0 or  $\pi$ . Purely imaginary numbers have argument of  $\frac{\pi}{2}$  or  $-\frac{\pi}{2}$ .

Properties of Modulus:

$|\bar{z}| = |z|$

- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$\text{cis } \theta = \cos \theta + i \sin \theta$

A complex number  $z$  has polar form

$z = |z| \text{cis } \theta$

where  $\theta = \arg(z)$ .  
The conjugate of  $z$  is:

$\bar{z} = |z| \text{cis } (-\theta)$

Properties of cis  $\theta$ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

Euler's Form

$e^{i\theta} = \cos \theta + i \sin \theta$

De Moivre's Theorem

$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.

$\alpha$  is a zero of polynomial

$P(x) \iff P(\alpha) = 0$

The roots of a polynomial equation are the solutions to the equation.

$\alpha$  is a root (or solution) of

$P(x) \iff P(\alpha) = 0$

The roots of  $P(x) = 0$  are the zeros of  $P(x)$  and the  $x$ -intercepts of the graph  $y = P(x)$

Factors

$(x - \alpha)$  is a factor of the polynomial  $P(x) \iff$  there exists a polynomial  $Q(x)$  such that  $P(x) = (x - \alpha)Q(x)$ .

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If  $P(x)$  is divided by  $D(x) = ax + b$  until a quotient  $Q(x)$  and constant remainder  $R$  is obtained, then

$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$

Notice that  $P(x) = Q(x) \times (ax + b) + R$ .

Polynomial Division by Quadratics

If  $P(x)$  is divided by  $D(x) = ax^2 + bx + c$ , then

$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$

where  $ex + f$  is the remainder.

The Remainder Theorem

When a polynomial  $P(x)$  is divided by  $x - k$  until a constant remainder  $R$  is obtained, then  $R = P(k)$ .

The Factor Theorem

For any polynomial  $P(x)$ ,  $k$  is a zero of  $P(x) \iff (x - k)$  is a factor of  $P(x)$ .

The Fundamental Theorem of Algebra

If  $P(x)$  is a polynomial of degree  $n$ , then  $P(x)$  has  $n$  zeros, each in the form  $a + bi$  where  $a, b \in \mathbb{R}$ , some of which may be repeated.

Sum and Product of Roots

For the polynomial equation

$\sum_{r=0}^n a_r x^r = 0, a_n \neq 0$

Sum of roots:  $\frac{-a_{n-1}}{a_n}$

Product of roots:  $\frac{(-1)^n a_0}{a_n}$

Vectors

Vectors in Space

Any point  $P$  in space can be specified  $(x, y, z)$  corresponding to steps in the  $X$ ,  $Y$  and  $Z$  direction from the origin  $O$ .  
The position vector of  $P$  is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{j} + z\vec{k}$$

where  $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , and  $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ,  
the base unit vectors.  
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The Magnitude of a Vector

The magnitude or length of the vector  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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Operations with Vectors

If  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$  then:

$$\begin{aligned} -\vec{a} &= \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} & \vec{a} + \vec{b} &= \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ \vec{a} - \vec{b} &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} & k\vec{a} &= \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix} \end{aligned}$$

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Vector Algebra

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{0} + \vec{a}$
- $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $|k\vec{a}| = |k||\vec{a}|$
- If  $\vec{x} + \vec{a} = \vec{b}$  then  $\vec{x} = \vec{b} - \vec{a}$
- If  $k\vec{x} = \vec{a}$ ,  $k \neq 0$ , then  $\vec{x} = \frac{1}{k}\vec{a}$

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Vector Between Two Points

If  $A(a_1, a_2, a_3)$  and  $B(b_1, b_2, b_3)$  then the position vector of  $B$  relative to  $A$  is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from  $A$  to  $B$  is

$$|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$