

## 11 Specialist Mathematics Semester 2 Summarised Notes

2021

Matrices

Notation

[a b c; d e f]

Order: 2 x 3

Addition and Subtraction

[a b; c d] +/- [e f; g h] = [a +/- e b +/- f; c +/- g d +/- h]

Scalar Multiplication

k x [a b c; d e f] = [ka kb kc; kd ke kf]

Matrix Multiplication

[a b; c d] [w x; y z] = [aw + by ax + bz; cw + dy cx + dz]

Order:
n x m <- must match -> m x p
Resultant matrix: n x p

Identity Matrix

I = [1 0; 0 1] AI = IA = A

Inverse of a Matrix

A^-1 = 1/(ad - bc) [d -b; -c a]
det A = |A| = ad - bc

- If det A ≠ 0, A is invertible
- If det A = 0, A is singular

AA^-1 = A^-1A = I

Matrix Algebra

- A + O = O + A = A where O is zero/null matrix
- A + (-A) = (-A) + A = O
- A + B exists if both have same order
- A + B = B + A {commutative}
- (A + B) + C = A + (B + C) {associative}

- In general, AB ≠ BA {non-commutative}
- (AB)C = A(BC) {associative}
- AO = OA = O
- AB may be O without A = O or B = O
- A(B + C) = AB + AC {distributive law}
- AI = IA = A
- A^n exists provided A is square and n ∈ Z+
- AA^-1 = A^-1A = I
- (A^-1)^-1 = A
- (kA)^-1 = 1/k A^-1
- (AB)^-1 = B^-1A^-1

Simultaneous Linear Equations

{ 2x + 3y = 4; 5x + 4y = 17 }
[2 3; 5 4] [x; y] = [4; 17]
AX = B, X = A^-1B
[x; y] = -1/7 [4 -3; -5 2] [4; 17]
[x; y] = [5; -2]
∴ x = 5, y = -2

Translations and Lines in 2D

A = (a1; a2) translated through b = (b1; b2)
A' = (a1 + b1; a2 + b2)

Line:

(x; y) = (a1; a2) + λ (b1; b2)

Where λ is a parameter

Linear Transformations

Maps an object initial vector onto its image.
T is a linear transformation if:

- T(u + v) = T(u) + T(v)
- T(ku) = kT(u)
- T(0) = 0

- T(-u) = -T(u)
- T(k1u1 + k2u2 + ... + krur) = k1T(u1) + k2T(u2) + ... + krT(ur)

Geometric Linear Transformations

[x'; y'] = [a b; c d] [x; y]
v' = Av, v = A^-1v'

To transform a function or relation, substitute in x and y from v.

Rotations About the Origin

For a rotation anticlockwise about O(0,0) through θ,

A = [cos θ -sin θ; sin θ cos θ]

with det A = 1

Reflections

For a reflection in the mirror line y = (tan α)x,

A = [cos 2α sin 2α; sin 2α -cos 2α]

with det A = -1

If m = tan α then:

cos 2α = (1 - m^2)/(1 + m^2) sin 2α = 2m/(1 + m^2)
tan 2α = 2m/(1 - m^2)

Dilatations

For a dilation with scale factor m parallel to the x-axis and k parallel to the y-axis:

A = [m 0; 0 k]

Compositions of Transformations

If v is transformed under TA followed by TB,

v' = BAv

The transformation is TB ∘ TA

Real and Complex Numbers

Number Sets

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$  natural numbers  
 $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$  all integers  
 $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$  positive integers  
 $\mathbb{Z}^- = \{-1, -2, -3, \dots\}$  negative integers  
 $\mathbb{Q}$  all rational numbers  
 $\mathbb{Q}'$  all irrational numbers  
 $\mathbb{R}$  all real numbers  
 $\mathbb{I}$  all imaginary numbers  
 $\mathbb{C}$  all complex numbers

Real Numbers

Numbers that can be placed on a number line.

Rational Numbers

A real number which can be written in the form  $\frac{p}{q}$  where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . Have a decimal expansion which either terminates or recurs.

Irrational Numbers

A real number which cannot be written in the form  $\frac{p}{q}$  where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . For example, surds,  $(\sqrt{a})$ .

Interval Notation

$A = \{x \mid a \leq x \leq b, x \in \mathbb{R}\}$

“The set of real numbers  $x$  such that  $x$  is between  $a$  and  $b$ , including  $a$  and  $b$ .”

- An interval is a connected subset of the number line  $\mathbb{R}$ .
- An interval is closed if both of its endpoints are included.
- An interval is open if both of its endpoints are not included.

$[a, b]$  is  $\{x \mid a \leq x \leq b, x \in \mathbb{R}\}$   
 $[a, b)$  is  $\{x \mid a \leq x < b, x \in \mathbb{R}\}$   
 $(a, b]$  is  $\{x \mid a < x \leq b, x \in \mathbb{R}\}$   
 $(a, b)$  is  $\{x \mid a < x < b, x \in \mathbb{R}\}$

Set Notation

$\in$  element of                       $\notin$  not element  
 $\subset$  subset                             $\not\subset$  not subset  
 $\subseteq$  equal subset                     $\emptyset$  empty set  
 $\cup$  union                             $\cap$  intersection

Imaginary Numbers

A number which cannot be placed on a number line in the form  $ai$  where  $a \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

Complex Numbers

Any number in the form  $a + bi$  where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ .

If  $z = a + bi$   
 $\Re(z) = a$                        $\Im(z) = b$

Conjugates (Surds)

Rationalising the denominator.

$\frac{a + \sqrt{b}}{c + \sqrt{d}} = \frac{a + \sqrt{b}}{c + \sqrt{d}} \times \frac{c - \sqrt{d}}{c - \sqrt{d}}$

Complex Conjugates

$z = a + bi$                       and                       $z^* = a - bi$

are complex conjugates. To write a fraction of two complex numbers with a real denominator:

$\frac{z}{w} = \frac{z}{w} \times \frac{w^*}{w^*} = \frac{zw^*}{ww^*}$   
 $\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$

Properties:

- $(z^*)^* = z$
- $(z_1 + z_2)^* = z_1^* + z_2^*$
- $(z_1 z_2)^* = z_1^* \times z_2^*$
- $\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$
- $(z^n)^* = (z^*)^n, n \in \mathbb{Z}^+$
- $z + z^*$  and  $zz^*$  are real

Complex Solutions of Quadratics

If a real quadratic equation has  $\Delta < 0$ , and root  $c + di$ , the other root is its conjugate  $c - di$ .  
Then the equation is:

$a(x^2 - 2cx + (c^2 + d^2)) = 0, a \neq 0$

For the roots of  $ax^2 + bx + c = 0$ :

sum =  $-\frac{b}{a}$                       product =  $\frac{c}{a}$

Complex Numbers as 2D Vectors

Complex numbers can be represented on an Argand diagram where the  $x$ -axis is the real axis and the  $y$ -axis is the imaginary axis.

$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$  represents  $x + yi$

Modulus

The modulus of the complex number  $z = a + bi$  is the length of the vector  $\begin{pmatrix} a \\ b \end{pmatrix}$ , which is the real number:

$|z| = \sqrt{a^2 + b^2}$

Properties:

- $|z^*| = |z|$
- $|z^*| = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left|\frac{z_1}{z_2}\right|^* = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Coordinate Geometry

If  $z_1 \equiv \overrightarrow{OP_1}$  and  $z_2 \equiv \overrightarrow{OP_2}$

Distance between  $P_1$  and  $P_2$  is:

$|z_1 - z_2|$

If  $M$  is the midpoint between  $P_1$  and  $P_2$ ,

$\overrightarrow{OM} \equiv \frac{z_1 + z_2}{2}$

Sequences and Series

Arithmetic Sequences

Sequence where each term differs from the previous by a fixed term, the common difference.

First term:  $t_1$

Common difference:  $d$

General Formula:

$$t_n = t_1 + (n - 1)d$$

.....

Geometric Sequences

Sequence where each is the product of the previous term and a fixed common ratio.

First term:  $t_1$

Common ratio:  $r$

General Formula:

$$t_n = t_1 r^{n-1}$$

.....

Series

A series is the sum of the terms in a sequence.

Sigma Notation:

$$\sum_{k=1}^n t_k = t_1 + t_2 + t_3 + \cdots + t_n$$

Properties:

$$\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$$

If  $c$  is a constant,

$$\sum_{k=1}^n ca_k = c \sum_{k=1}^n a_k$$

$$\sum_{k=1}^n c = cn$$

.....

Arithmetic Series

The sum of the terms of an arithmetic sequence.

$$S_n = \sum_{k=1}^n (t_1 + (k - 1)d)$$

$$S_n = \frac{n}{2}(t_1 + t_n)$$

or

$$S_n = \frac{n}{2}(2t_1 + (n - 1)d)$$

.....

Geometric Series

The sum of the terms of a geometric sequence.

$$S_n = \sum_{k=1}^n t_1 r^{k-1}$$

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

or

$$S_n = \frac{t_1(1 - r^n)}{1 - r}$$

$$r \neq 1$$

Sum of Infinite Geometric Series:

If  $|r| < 1$ , an infinite series of the form

$$t_1 + t_1 r + t_1 r^2 + \cdots = \sum_{k=1}^{\infty} t_1 r^{k-1}$$

will converge to the sum

$$S = \frac{t_1}{1 - r}$$

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