12 Specialist Mathematics Summarised Notes (Work in Progress)

2022

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or

$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function.
$f(x)$ has an inverse if it is one-to-one, if $f(a) = f(b)$ only when $a = b$, $\therefore$ passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = Range of f.
- Range of f^{-1} = Domain of f.

Self-Inverse Functions

An invertible function which is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$.

Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \to$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \to$ zero $\frac{1}{f(x)}$
- Local max $f(x) \to$ local min $\frac{1}{f(x)}$
- Local min $f(x) \to$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \to 0$, $\frac{1}{f(x)} \to \pm\infty$
- When $f(x) \to \pm\infty$, $\frac{1}{f(x)} \to 0$

Invariant Points:
Points which do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another.
Vertical asymptote occurs when denominator is zero.
Horizontal asymptote ascertained from behaviour of graph as $|x| \to \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slated asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$.
If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y-axis
- Points on the y-axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x-axis
- Points on the x-axis are invariant

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defines for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$

$$\mathfrak{Re}(z) = a \qquad \mathfrak{Im}(z) = b$$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x-axis is the real axis and the y-axis is the imaginary axis.

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{represents} \quad x + yi$$

Complex Conjugates

The complex conjugate of

$$z = a + bi \qquad \text{is} \qquad \bar{z} = a - bi$$

In the complex plane, $\bar{z}$ is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a+bi$ is the length of the vector $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$, which is the real number:

$$|z| = \sqrt{a^2+b^2}$$

The argument of z, $\arg z$ is the angle θ between the positive real axis and $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. Real numbers have argument of 0 or π. Purely imaginary numbers have arguement of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|\bar{z}| = |z|$
- $|\bar{z}| = z\bar{z}$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1||z_2||z_3|\dots|z_n|$
- $|z^n| = |z|^n,\ n \in \mathbb{Z}^+$

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Polar Form

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta$$

A complex number z has polar form

$$z = |z|\operatorname{cis}\theta$$

where $\theta = \arg z$.
The conjugate of z is:

$$\bar{z} = |z|\operatorname{cis}-\theta$$

Properties of $\operatorname{cis}\theta$:

- $\operatorname{cis}\theta \times \operatorname{cis}\phi = \operatorname{cis}(\theta+\phi)$
- $\frac{\operatorname{cis}\theta}{\operatorname{cis}\phi} = \operatorname{cis}(\theta-\phi)$
- $\operatorname{cis}(\theta - k2\pi) = \operatorname{cis}\theta,\ k \in \mathbb{Z}$

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Euler's Form

$$e^{i\theta} = \cos\theta + i\sin\theta$$

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De Moivre's Theorem

$$(|z|\operatorname{cis}\theta)^n = |z|^n\operatorname{cis}n\theta,\ \text{for all } n \in \mathbb{Q}$$

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin\theta & \cos(-\theta) &= \cos\theta \\ \sin(\pi-\theta) &= \sin\theta & \cos(\pi-\theta) &= -\cos\theta \\ \sin\left(\frac{\pi}{2}-\theta\right) &= \cos\theta & \cos\left(\frac{\pi}{2}-\theta\right) &= \sin\theta \end{aligned}$$

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Pythagorean Theorem

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

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Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1 \\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}$$

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Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B \\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \end{aligned}$$

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Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2\sin\left(\frac{A \pm B}{2}\right)\cos\left(\frac{A \mp B}{2}\right) \\ \cos A \pm \cos B &= 2\sin\left(\frac{A + B}{2}\right)\cos\left(\frac{A - B}{2}\right) \end{aligned}$$

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Product to Sum

$$\begin{aligned} 2\sin A\cos B &= \sin(A+B) + \sin(A-B) \\ 2\sin A\sin B &= \cos(A-B) - \cos(A+B) \\ 2\cos A\cos B &= \sin(A+B) + \cos(A-B) \end{aligned}$$

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X, Y and Z direction from the origin O. The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\vec{i} + y\vec{k} + z\vec{k}$$

where $\vec{i} = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\vec{j} = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $\vec{k} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$, the base unit vectors.

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The Magnitude of a Vector

The magnitude or length of the vector $\vec{a} = \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

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Operations with Vectors

$$\text{If } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ then:}$$

$$-\vec{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\vec{a} - \vec{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\vec{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

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Vector Algebra

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- $\vec{a} + \vec{0} = \vec{0} + \vec{a}$
- $\vec{a} + (\vec{-a}) = (\vec{-a}) + \vec{a} = \vec{0}$
- $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$
- $|k\vec{a}| = |k||\vec{a}|$
- If $\vec{x} + \vec{a} = \vec{b}$ then $\vec{x} = \vec{b} = \vec{a}$
- If $k\vec{x} = a$, $k \neq 0$, then $\vec{x} = \frac{1}{k}\vec{a}$

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Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B Is $|\overrightarrow{AB}| = \sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2}$