

12 Specialist Mathematics Summarised Notes
(Unofficial)

Functions

Composite Functions

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the composite function of f and g is:

$$(f \circ g)(x) = f(g(x))$$

or
$$f \circ g : x \mapsto f(g(x))$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Inverse Functions

An inverse function returns the original value from the output of a function. $f(x)$ has an inverse if it is injective (one-to-one), if $f(a) = f(b)$ only when $a = b$, \therefore passes the horizontal line test.

For $f^{-1}(x)$, the inverse of $f(x)$:

- Is a reflection of $y = f(x)$ over $y = x$.
- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$
- Domain of f^{-1} = range of f .
- Range of f^{-1} = domain of f .

Self-Inverse Functions

An invertible function that is symmetrical about $y = x$.

$$f^{-1}(x) = f(x)$$

Reciprocal Functions

A function of the form $f(x) = \frac{k}{x}$, where $k \neq 0$ is a constant.

Reciprocal of Other Functions

The reciprocal of a function $f(x)$ is $\frac{1}{f(x)}$. Graphing $y = \frac{1}{f(x)}$ from $y = f(x)$:

- Zero $f(x) \rightarrow$ vertical asymp $\frac{1}{f(x)}$
- Vertical asymp $f(x) \rightarrow$ zero $\frac{1}{f(x)}$
- Local max $f(x) \rightarrow$ local min $\frac{1}{f(x)}$
- Local min $f(x) \rightarrow$ local max $\frac{1}{f(x)}$
- When $f(x) > 0$, $\frac{1}{f(x)} > 0$
- When $f(x) < 0$, $\frac{1}{f(x)} < 0$
- When $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
- When $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$

Invariant Points:

Points that do not move under a transformation occurring at $y = \pm 1$.

Rational Functions

Results from the division of one polynomial by another. Vertical asymptote occurs when denominator is zero. Horizontal asymptote ascertained from behaviour of graph as $|x| \rightarrow \infty$.

- If the degree of denominator > numerator, horizontal asymptote at $y = 0$.
- If the degree of denominator < numerator, function has slanted asymptote found through polynomial division.
- If the degree of denominator = numerator horizontal asymptote at $y = \frac{a}{b}$ where a and b are the leading coefficients.

Absolute Value Functions

The absolute value or modulus $|x|$ of a real number x is its distance from 0 on the number line.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Alternatively,

$$|x| = \sqrt{x^2}$$

Properties:

- $|x| \geq 0$
- $|x|^2 = x^2$
- $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
- $|-x| = |x|$
- $|xy| = |x||y|$
- $|x - y| = |y - x|$

If $|x| = a$ where $a > 0$, then $x = \pm a$. If $|x| = |b|$ then $x = \pm b$.

Graphs Involving the Absolute Value Function

Graphing $y = f(|x|)$ from $y = f(x)$:

- Discard the graph for $x < 0$
- Reflect the graph for $x \geq 0$ in the y -axis
- Points on the y -axis are invariant

Graphing $y = |f(x)|$ from $y = f(x)$:

- Keep the graph for $f(x) \geq 0$
- Reflect the graph for $f(x) < 0$ in the x -axis
- Points on the x -axis are invariant

Trigonometric Identities

Angle Relationships

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \sin(\pi - \theta) &= \sin \theta & \cos(\pi - \theta) &= -\cos \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \end{aligned}$$

Pythagorean Theorem

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ \cot^2 \theta + 1 &= \csc^2 \theta \end{aligned}$$

Double Angle Identities

$$\begin{aligned} \sin 2\theta &= 2 \sin \theta \cos \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 1 - 2 \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

Angle Sum and Difference

$$\begin{aligned} \sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \end{aligned}$$

Sum to Product

$$\begin{aligned} \sin A \pm \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cos\left(\frac{A \mp B}{2}\right) \\ \cos A + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cos\left(\frac{A - B}{2}\right) \\ \cos A - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Product to Sum

$$\begin{aligned} 2 \sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2 \sin A \sin B &= \cos(A - B) - \cos(A + B) \\ 2 \cos A \cos B &= \cos(A + B) + \cos(A - B) \end{aligned}$$

Mathematical Induction

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. If P_a is true, and if P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Complex Numbers

Imaginary Numbers

A number which cannot be placed on a real number line in the form ai where $a \in \mathbb{R}$ and $i = \sqrt{-1}$.

Complex Numbers

Any number in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

If $z = a + bi$
 $\Re(z) = a \quad \Im(z) = b$

The Complex Plane

Complex numbers can be plotted on the complex plane or Argand plane as a vector where the x -axis is the real axis and the y -axis is the imaginary axis.

$\vec{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$

Complex Conjugates

The complex conjugate of

$z = a + bi$ is $z^* = a - bi$

In the complex plane, z^* is the reflection of z in the real axis.

Modulus and Argument

The modulus of the complex number $z = a + bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number:

$|z| = \sqrt{a^2 + b^2}$

The argument of z , $\arg(z)$ is the angle θ between the positive real axis and $\begin{pmatrix} a \\ b \end{pmatrix}$. Real numbers have an argument of 0 or π . Purely imaginary numbers have argument of $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Properties of Modulus:

- $|z^*| = |z|$

- $|z^*|^2 = zz^*$
- $|z_1 z_2| = |z_1| |z_2|$
- $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$
- $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- $|z^n| = |z|^n, n \in \mathbb{Z}^+$

Polar Form

$$\text{cis } \theta = \cos \theta + i \sin \theta$$

A complex number z has the polar form

$$z = |z| \text{cis } \theta$$

where $\theta = \arg(z)$.
The conjugate of z is:

$$z^* = |z| \text{cis } (-\theta)$$

Properties of cis θ :

- $\text{cis } \theta \times \text{cis } \phi = \text{cis } (\theta + \phi)$
- $\frac{\text{cis } \theta}{\text{cis } \phi} = \text{cis } (\theta - \phi)$
- $\text{cis } (\theta - 2k\pi) = \text{cis } \theta, k \in \mathbb{Z}$

De Moivre's Theorem

$$(|z| \text{cis } \theta)^n = |z|^n \text{cis } n\theta, \text{ for all } n \in \mathbb{Q}$$

Roots of Complex Numbers

The n^{th} roots of the complex number c are the solutions of $z^n = c$.

The n^{th} Roots of Unity

The n^{th} roots of unity are the solutions of $z^n = 1$.

Distances in the Complex Plane

If $z_1 \equiv \vec{OP_1}$ and $z_2 \equiv \vec{OP_2}$ then $|z_1 - z_2|$ is the distance between points P_1 and P_2 .

Real Polynomials

Zeros and Roots

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero.
 α is a zero of polynomial

$$P(x) \iff P(\alpha) = 0$$

The roots of a polynomial equation are the solutions to the equation.
 α is a root (or solution) of

$$P(x) \iff P(\alpha) = 0$$

The roots of $P(x) = 0$ are the zeros of $P(x)$ and the x -intercepts of the graph $y = P(x)$

Factors

$(x - \alpha)$ is a factor of the polynomial $P(x) \iff$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Polynomial Equality

Two polynomials are equal if and only if they have the same degree (order) and corresponding terms have equal coefficients.

Polynomial Division by Linears

If $P(x)$ is divided by $D(x) = ax + b$ until a quotient $Q(x)$ and constant remainder R is obtained, then

$$\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$$

Notice that $P(x) = Q(x) \times (ax + b) + R$.

Polynomial Division by Quadratics

If $P(x)$ is divided by $D(x) = ax^2 + bx + c$, then

$$\frac{P(x)}{ax^2 + bx + c} = Q(x) + \frac{ex + f}{ax^2 + bx + c}$$

where $ex + f$ is the remainder.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x - k$ until a constant remainder R is obtained, then $R = P(k)$.

The Factor Theorem

For any polynomial $P(x)$, k is a zero of $P(x) \iff (x - k)$ is a factor of $P(x)$.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree n , then $P(x)$ has n zeros, each in the form $a + bi$ where $a, b \in \mathbb{R}$, some of which may be repeated.

Vectors

Vectors in Space

Any point P in space can be specified (x, y, z) corresponding to steps in the X , Y and Z direction from the origin O .
The position vector of P is

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,
the base unit vectors.

The Magnitude of a Vector

The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Operations with Vectors

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

$$-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix} \qquad \mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix} \qquad k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$$

Vector Algebra

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
- $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$
- $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$
- $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$
- $|k\mathbf{a}| = |k||\mathbf{a}|$
- If $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then $\mathbf{a} = \mathbf{c} - \mathbf{b}$
- If $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$

Vector Between Two Points

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ then the position vector of B relative to A is

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

The distance from A to B is
$$|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

Parallelism

$\mathbf{a} = k\mathbf{b} \iff \mathbf{a}$ and \mathbf{b} are non-zero parallel vectors.

Collinear Points

A, B and C are collinear if $\overrightarrow{AB} = k\overrightarrow{BC}$.

Unit Vectors

The unit vector $\hat{\mathbf{v}}$, a vector of length 1 in the direction of \mathbf{v} is

$$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

Dot Product (Scalar Product)

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then the scalar dot product is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{c} + \mathbf{d}) = \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{d} + \mathbf{b} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{d}$
- $k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b})$, $k \in \mathbb{R}$

The Angle Between Two Vectors

The angle θ between two vectors \mathbf{a} and \mathbf{b} can be found using

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Scalar Product Geometric Properties

- For non-zero vectors \mathbf{a} and \mathbf{b} :
 $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a}$ and \mathbf{b} are perpendicular.
- $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \iff \mathbf{a}$ and \mathbf{b} are non-zero parallel vectors.
- Given $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$
If θ is acute, $\cos \theta > 0$ and so $\mathbf{a} \cdot \mathbf{b} > 0$
If θ is obtuse, $\cos \theta < 0$ and so $\mathbf{a} \cdot \mathbf{b} < 0$

Cross Product (Vector Product)

The vector cross product of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Alternatively,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

Properties

- $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both \mathbf{a} and \mathbf{b} .
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all \mathbf{a} .
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, $\therefore \mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have equal length but opposite direction.
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the scalar triple product.
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
- $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{d}$

Direction of $\mathbf{a} \times \mathbf{b}$

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} .

Area

If a triangle has defining vectors \mathbf{a} and \mathbf{b} then its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units².
If a parallelogram has defining vectors \mathbf{a} and \mathbf{b} then its area is $|\mathbf{a} \times \mathbf{b}|$ units².

Lines in 2 and 3 Dimensions

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$ is the vector equation of the line.

The Shortest Distance From a Point to a Line

A point P is closest to a point R on a line in direction \mathbf{b} when \overrightarrow{PR} is perpendicular to \mathbf{b} .

$$\overrightarrow{PR} \cdot \mathbf{b} = 0$$

Relationships Between Lines

Line Classification in 2 Dimensions

- Intersecting: Unique solution
- Parallel: No solutions
- Coincident: Infinite Solutions

Line Classification in 3 Dimensions

- Lines are coplanar if they lie in the same plane. In this case, they may be intersecting, parallel or coincident.
- Otherwise, they are skew.

Shortest Distance Between Skew Lines

For two skew lines with vector equations $\mathbf{r}_1 = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu \mathbf{b}_2$, the shortest distance d between them is

d = (a1 - a2) · (b1 × b2) / |b1 × b2|

Planes

A plane is a flat surface that extends forever and has zero thickness. The Vector Equation of a Plane

r = a + sv + tw

- r is the position vector of any point on the plane.
- a is the position vector of a known point on the plane.
- v and w are any two non-parallel vectors that are parallel to the plane.
- s, t ∈ ℝ are two independent parameters.

The Normal of a Plane

A vector is normal to a plane if it is perpendicular to all vectors which are parallel to the plane. If n is a normal to a plane such as n = v × w, an equivalent vector equation of a plane is

n · (r - a) or r · n = a · n

The Cartesian Equation of a Plane

If a plane has normal vector n = (a, b, c) and passes through P(X, Y, Z) then it has equation

ax + by + cz = aX = bY = cZ = d

where d is a constant.

Distance Between a Point and a Plane

The distance between a point P(x1, y1, z1) and the plane Ax + By + Cz + D = 0 is

d = |Ax1 + Bx1 + Cx1 + D| / sqrt(A^2 + B^2 + C^2)

Angles in Space

The Angle Between a Line and a Plane The acute angle φ between a line with direction vector a and a plane with normal vector n is

phi = sin^-1 (|n · b| / (|n| |b|))

The Angle Between Two Planes

If two planes have normal vectors n1 and n2 and θ is the acute angle between them then

theta = cos^-1 (|n1 · n2| / (|n1| |n2|))

Row Reduction

Linear systems of equations can be solved using augmented matrices. A general 3 × 3 system has the form:

{ a1x + b1y + c1z = d1, a2x + b2y + c2z = d2, a3x + b3y + c3z = d3 }

In augmented matrix form, the system is:

[a1 b1 c1 | d1, a2 b2 c2 | d2, a3 b3 c3 | d3]

Using row operations it is reduced to the echelon form.

[a b c | d, 0 e f | g, 0 0 h | i]

∴ hz = i ⇒ z = i/h, ey + fz = g and ax + by + cz = d

Intersecting Planes

Two planes in space could have the following arrangements:

- Intersecting
- Parallel
- Coincident

Three planes in space could have the following arrangements:

- All coincident
- Two coincident and one intersecting
- Two coincident and one parallel
- Two parallel and one intersecting
- All parallel
- All meet at one point
- All meet in a common line
- The line of intersection of any two planes is parallel to the third plane.

Integration

Indefinite Integrals

∫ k dx = kx + c, ∫ x^n dx = 1/(n+1) x^(n+1) + c, n ≠ -1, ∫ e^x dx = e^x + c, ∫ 1/x dx = ln|x| + c, ∫ cos x dx = sin x + c, ∫ sin x dx = -cos x + c

Integrating f(ax + b)

∫ (ax + b)^n dx = (ax + b)^(n+1) / (a(n+1)) + c, n ≠ -1, ∫ e^(ax+b) dx = 1/a e^(ax+b) + c, ∫ 1/(ax + b) dx = 1/a ln|ax + b| + c, ∫ cos(ax + b) dx = 1/a sin(ax + b) + c, ∫ sin(ax + b) dx = -1/a cos(ax + b) + c

Inverse Trigonometric Functions

∫ 1/sqrt(1-x^2) dx = arcsin x + c, ∫ -1/sqrt(1-x^2) dx = arccos x + c, ∫ 1/(1+x^2) dx = arctan x + c, ∫ 1/sqrt(a^2-x^2) dx = arcsin(x/a) + c, ∫ -1/sqrt(a^2-x^2) dx = arccos(x/a) + c, ∫ a/(a^2+x^2) dx = arctan(x/a) + c

Integrating sin² x and cos² x

Use double-angle identities when integrating.

sin² x = 1/2 - 1/2 cos 2x
cos² x = 1/2 + 1/2 cos 2x

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Definite Integrals

If F(x) is the antiderivative of f(x) where f(x) is continuous over a ≤ x ≤ b, the definite integral is:

∫_a^b f(x) dx = F(b) - F(a)

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Properties of Definite Integrals

∫_a^a f(x) dx = 0
∫_a^b k dx = k(b - a)
∫_a^b f(x) dx = - ∫_b^a f(x) dx
∫_a^b kf(x) dx = k ∫_a^b f(x) dx
∫_a^b f(x) dx + ∫_b^c f(x) dx = ∫_a^c f(x) dx
∫_a^b [f(x) ± g(x)] dx = ∫_a^b f(x) dx ± ∫_a^b g(x) dx

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Integration by Substitution

∫ f(u) du/dx dx = ∫ f(u) du

When solving definite integrals with bounds a and b, adjust as u(a) and u(b).

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Integration by Parts

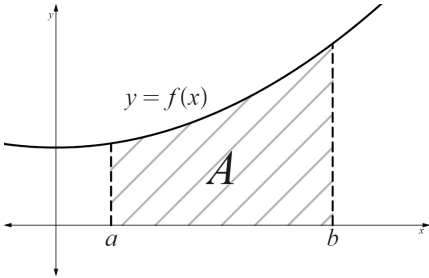
∫ uv' dx = uv - ∫ u'v dx

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Area Under a Curve

If f(x) is positive and continuous for a ≤ x ≤ b, the area bound by y = f(x), the x-axis, x = a and x = b is:

A = ∫_a^b f(x) dx or A = ∫_a^b y dx



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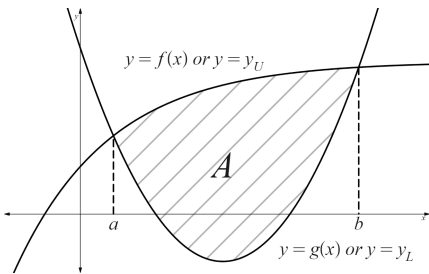
Area Between Two Curves

Upper function: f(x) or y_U
Lower function: g(x) or y_L

A = ∫_a^b [f(x) - g(x)] dx

or

A = ∫_a^b [y_U - y_L] dx



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Solids of Revolution

When the region enclosed by y = f(x), the x-axis, x = a and x = b is revolved through 2π about the x-axis, the volume is:

V = π ∫_a^b y² dx

When the region enclosed by y = f(x), the y-axis, y = f(a) = c and y = f(b) = d is revolved through 2π about the y-axis, the volume is:

V = π ∫_c^d x² dy

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Volumes for Two Defining Functions

Upper function: f(x) or y_U
Lower function: g(x) or y_L

V = ∫_a^b ([f(x)]² - [g(x)]²) dx

or

V = ∫_a^b (y_U² - y_L²) dx

Rates of Change and Differential Equations

Implicit Differentiation

To find dy/dx for an implicit relationship between x and y. A useful property is

d/dx [x^n] = nx^{n-1} and

d/dx [y^n] = ny^{n-1} dy/dx

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Related Rates

Two variables x and y may be related to each other and defined using an equation.

Eg: x² + y² = 5²

The equation can be differentiated in respect to a third parameter t resulting in a related rates equation.

Eg: 2x dx/dt + 2y dy/dt = 0

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Differential Equations

A differential equation is an equation that describes the relationship between a function and its derivatives.

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Differential Equations of the Form dy/dx = f(x)

Differential equations of the form dy/dx = f(x) can be solved using integration.

dy/dx = f(x)
dy = f(x) dx
∫ dy = ∫ f(x) dx
∴ y = ∫ f(x) dx

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Seperable Differential Equations

Differential equations of the form dy/dx = f(x)g(y) can be solved as follows:

dy/dx = f(x)g(y)
1/g(y) dy = f(x) dx
∫ 1/g(y) dy = ∫ f(x) dx

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Slope Fields

For a differential equation $\frac{dy}{dx} = f(x, y)$, any point (x, y) on the cartesian plane will have a gradient which can be graphically represented using a slope field.

Logistic Growth

Logistic growth is defined by the differential equation

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{A}\right)$$

The differential equation is solved with partial fractions using the identity:

$$\frac{A}{P(A - P)} = \frac{1}{P} + \frac{1}{A - P}$$

Vector Calculus

Parametric Equations

Cartesian equation can be expressed in terms of a parameter such as t , defining x and y independently such that $x = x(t)$ and $y = y(t)$.

Pairs of Uniformly Varying Quantities

An object initially at (x_0, y_0) and moving with velocity vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ has parametric equations

$$x(t) = x_0 + at, \quad y(t) = y_0 + bt, \quad t \geq 0$$

The speed of the object is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

Pairs of Non-Uniformly Varying Quantities

For the moving object $P(x(t), y(t))$, the velocity vector is:

$$\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

and the speed is:

$$\text{speed} = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

The gradient of the velocity vector at any point is:

$$\frac{y'(t)}{x'(t)} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dx} \frac{dt}{dx} = \frac{dy}{dx}$$

The velocity vector is tangential to the curve, therefore the direction of motion is the gradient.

Bézier Curves

Given the starting point (x_0, y_0) and the finishing point (x_3, y_3) , and control points (x_1, y_1) and (x_2, y_2) , the Bézier curve has parametric equations:

$$\begin{aligned} x(t) &= a_x t^3 + b_x t^2 + c_x t + d_x \\ y(t) &= a_y t^3 + b_y t^2 + c_y t + d_y, \\ 0 \leq t &\leq 1 \end{aligned}$$

where

$$\begin{cases} a_x = x_3 - 3x_2 + 3x_1 - x_0 \\ b_x = 3x_2 - 6x_1 + 3x_0 \\ c_x = 3x_1 - 3x_0 \\ d_x = x_0 \end{cases}$$

and

$$\begin{cases} a_y = y_3 - 3y_2 + 3y_1 - y_0 \\ b_y = 3y_2 - 6y_1 + 3y_0 \\ c_y = 3y_1 - 3y_0 \\ d_y = y_0 \end{cases}$$

Trigonometric Parameterisation

For an object P moving with parametric equations

$$x(t) = R \cos \omega t, \quad y(t) = R \sin \omega t, \quad t \geq 0$$

where $R > 0, \omega > 0$:

- R controls the radius of the path.
- ω controls the rate of revolutions. The duration of a revolution is $\frac{2\pi}{\omega}$.
- The speed of P is $R\omega$.

Arc Lengths of Parametric Curves

The length of the arc traced out by $P(x(t), y(t))$ from time $t = a$ to $t = b$ is:

$$\begin{aligned} \int_a^b |\mathbf{v}| \, dt &= \int_a^b \sqrt{\mathbf{v} \cdot \mathbf{v}} \, dt \\ &= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \end{aligned}$$