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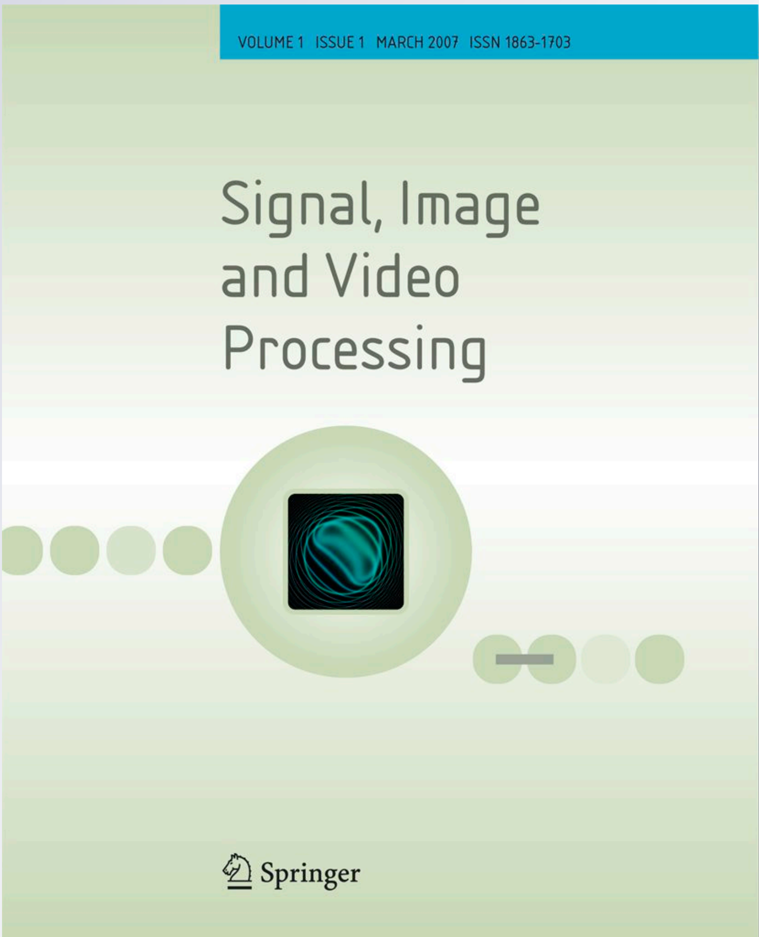
Alfredo Restrepo & Julián Quiroga

Signal, Image and Video Processing

ISSN 1863-1703
Volume 5
Number 3

SIViP (2011) 5:363-378
DOI 10.1007/s11760-010-0160-1



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ORIGINAL PAPER

Root and pre-constant signals of the 1D Teager-Kaiser operator

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Received: 31 October 2008 / Revised: 11 March 2010 / Accepted: 12 March 2010 / Published online: 18 April 2010

Abstract We give an input/output characterization of the Teager-Kaiser operator, then we classify the *root signals* (those that remain invariant) and the *pre-constant signals* (those that produce a constant output) of the operator. With the gained knowledge, a simple detector of sinusoidal signals and of their corresponding frequencies is designed. The methods used allow for a certain amount of generalization providing a paradigm of sorts for the design of quadratic and other polynomial filters.

Keywords Teager-Kaiser operator · Root signals · Pre-constant signals · Invariant conics · Nonlinear difference equations · Sinusoid detection · Polynomial filters · Volterra filters

1 Introduction

Once a nonlinear filter is known to solve a particular problem, there remains the task of studying its effect on signals in general: unlike linear invariant filters, nonlinear filters have to be analyzed rather on a case-by-case basis. "If the class of linear filters is well defined, that is no longer true in the nonlinear case because a negative property cannot be used as a definition" [9].

There are two main classes of filter of the moving-window type (see definitions below in Sect. 2), that of (locally) piecewise-linear filters, such as the median filter, and that of (locally) polynomial filters, like the Teager-Kaiser operator. The Teager-Kaiser (TK, for short) operator is of relevance to the engineering community. On the one hand, the operator has important applications in energy estimation, [1,3,4], in time-frequency estimation [5] and in contrast enhancement [8] and more general image processing [17]. On the other hand, the operator is a simple, yet nontrivial, case of a quadratic filter [9]; in fact, it is a finite-order Volterra filter [16,18], and its analysis may shed light on possible lines of research for the design and applications of filters in this important class.

It is a well-known fact [7] that the output of the Teager-Kaiser operator, when applied to a sinusoidal signal, is a constant signal of a value related to the frequency and amplitude of the input. In fact, the original application of the operator was the estimation of the so-called spring-mass energy from oversampled sinusoidal signals. It is therefore appropriate to study the signals that produce constant outputs; we say that such signals are *pre-constant,* a particular case being that of the *pre-null* signals, i.e. the signals that produce a constant output of value zero. Exponential signals such as $\{e^{j\omega n}\}$, for example, and $\{(-1)^n\}$ in particular, are pre-null signals (for this reason alone, when no restriction on the type of signals being filtered is made, and if by *energy* it is meant an intrinsic property of the signal, the operator should not be used blindly as a mere energy estimator.) Likewise, a classification of the root signals of the operator (those that remain unchanged when filtered), in addition to being of theoretical interest e.g. in functional analysis, and of being a common step in the analysis of nonlinear filters in engineering, certainly sheds light on the behavior of the operator.

For comparison purposes, we refer in Sects. 4 and 5 to the root and pre-constant signals of convolution filters. Let H be the transfer function of a convolution filter i.e. the bilateral z-transform of its impulse response (together with the region

A. Restrepo (✉)
Universidad de los Andes, Bogotá, Colombia
e-mail: arestrep@uniandes.edu.co

J. Quiroga
Pontificia Universidad Javeriana, Bogotá, Colombia
e-mail: quiroga.j@javeriana.edu.co

of convergence). The exponential signals, i.e. the signals of the form $s_n = e^{(\sigma+j\omega)n}$, with complex frequency $\sigma + j\omega$ in the region of convergence of the transfer function, are *eigensignals* i.e. signals that are merely *scaled* by the filter, by the eigenvalue $H(e^{\sigma+j\omega})$; a 0-scale being a particularly important case. The set of the linear combinations of exponential signals with a given eigenvalue together with the limits of such linear combinations (i.e. series) give the *eigenspace* of signals corresponding to that eigenvalue; note that the null signal belongs to each eigenspace.

In Sect. 2, in addition to giving a list of responses to signals of standard importance in engineering, we give an algebraic input/output characterization of the filter: we derive formulas for the responses of the operator to linear and algebraic combinations of input signals. In Sect. 3 we consider the main reasons the operator is used in engineering; we briefly explain the application of the operator as a component in a contrast enhancer and we extend the results of [5] regarding the time-frequency properties of the continuous Teager Filter to the discrete case of the TK operator.

In Sect. 4 we consider the root signals (or fixed points) of the operator. The roots can be segmented into *blocks* of two main types: *determinate blocks* (which includes the sinusoidal segments on an appropriate DC level and also hyperbolic sine and cosine segments) and *undeterminable blocks* (a certain type of binary segment), according to whether or not they meet a certain recurrence equation. Such a characterization of the root signals is made with the help of two main tools: a partition of phase space into a family of conics, and a linear equation that is met by each determinate segment. Analogously, in Sect. 5, we classify the pre-constant and pre-null signals of the operator. Interestingly, it is found that with the addition of the appropriate DC level, each determinate pre-constant segment becomes a root segment.

An application is presented in Sect. 6 that makes use of the results presented in Sect. 5. We take this opportunity to present several results previously advanced in conferences [11–13]; they are presented here in a more complete, panoramic and corrected form. In Restrepo [14], possible applications of the tools used here are indicated, for the solution of nonlinear, recursive, rational difference equations.

2 Input/output characterization

We consider the Teager-Kaiser operator from the general standpoint of a *discrete system,* i.e. the function $\mathrm{TK} : \mathbf{C}^{\mathbf{Z}} \to \mathbf{C}^{\mathbf{Z}}$ ($\mathbf{C}$ stands for the complex numbers and $\mathbf{Z}$ stands for the integers) that maps each bilateral complex sequence (discrete input signal) $\mathbf{x}$ into a bilateral complex sequence (discrete output signal) $\mathbf{y} = \mathrm{TK}(\mathbf{x})$ via the formula $y_n = x_n^2 - x_{n-1}x_{n+1}$.

2.1 Periodicity

Definition 1 If $\mathbf{x} : \mathbf{Z} \to \mathbf{C}$ is a discrete signal and N is an integer, we say that N is a *repetition time* of $\mathbf{x}$ if ${}_N\mathbf{x} = \mathbf{x}$ i.e. $\forall n \in \mathbf{Z} x_n = x_{n-N}$.

Note that 0 is a repetition time of each signal and that, if N is a repetition time of a signal, so is $-N$.

Definition 2 $\mathbf{x} : \mathbf{Z} \to \mathbf{C}$ is is said to be a *periodic* signal if it has positive repetition times. The *period* of a periodic signal is given by its minimal positive repetition time m and $\mathbf{x}$ is said to be *m-periodic*.

Note that a signal is constant if and only if it is 1-periodic.

Definition 3 A discrete signal $\mathbf{x}$ is said to be *almost periodic* if for each real $\varepsilon > 0$, there is a positive natural number N such that for each integer n, $|x_n - x_{n-N}| < \varepsilon$.

Note that each periodic signal is almost periodic. Also, note that for t irrational, $x_n = e^{2\pi tn}$ is almost periodic but not periodic.

2.2 Sliding-window filters, shift invariance and quadratic filters

Definition 4 A *sliding-window filter* is a system $\mathrm{F} : \mathbf{C}^{\mathbf{Z}} \to \mathbf{C}^{\mathbf{Z}}$ such that, for some integers M and $W \geq 2$, a function $f : \mathbf{C}^W \to \mathbf{C}$ is locally applied to the input signal $\mathbf{x}$ in the sense that

$$F(\mathbf{x})_n = f(x_{n+M}, \ldots, x_{n+M+W-1})$$

W is said to be the *size of the window* and f is said to be the *local function of the filter*.

The M in the definition determines the *causality* of the system; for $M \leq 1 - W$ the sliding-window filter is said to be *causal*.

With the appending of a pre-subscript to a signal we indicate a shift, i.e. $\mathbf{y} = {}_k\mathbf{x}$ means $y_n = x_{n-k}$, $n \in \mathbf{Z}$. Thus we write

$$\mathrm{TK}(\mathbf{x}) = \mathbf{x}^2 - ({}_1\mathbf{x})({}_{-1}\mathbf{x}) \tag{1}$$

where the product is pointwise i.e., for any signals $\mathbf{v}$ and $\mathbf{w}$, $\mathbf{vw} = \{v_n w_n\}$.

Definition 5 The system F is said to be *shift-invariant* if for each integer k and for each signal $\mathbf{x}$ in its domain of definition, $F({}_k\mathbf{x}) = {}_kF(\mathbf{x})$.

Clearly, sliding-window filters are shift-invariant.

Shift-invariant systems S preserve repetition times since, whenever k is a repetition time of $\mathbf{x}$, $S({}_k\mathbf{x}) = S(\mathbf{x})$ and by invariance, $S({}_k\mathbf{x}) = {}_k S(\mathbf{x})$, then ${}_kS(\mathbf{x}) = S(\mathbf{x})$ and thus k is also a repetition time of $F(\mathbf{x})$. Invariant systems preserve periodicity, even though the period may decrease to a divisor; thus, the period of a shift-invariant filtered version of a a prime-periodic signal is either left unchanged or changed to 1.

Definition 6 A filter with input $\mathbf{x}$ and output $\mathbf{y}$ is a (finite-order) *quadratic filter* [9,15] if, for integers M and $W \geq 2$,

$$y_n = \left[x_{n+M}, \ldots x_{n+M+W-1}\right]^T \mathbf{A} \left[x_{n+M}, \ldots x_{n+M+W-1}\right]$$

where $\mathbf{A}$ is a $W \times W$ symmetric matrix called the *matrix* of the filter.

Clearly, each quadratic filter is a sliding-window filter and is therefore shift-invariant.

2.3 The teager-kaiser operator: definition and basic properties

Definition 7 The TK operator is thus the quadratic filter with matrix

$$\begin{bmatrix} 0 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix}$$

The TK operator is thus a sliding-window filter of window-size 3 with local function $f(x, y, z) = y^2 - xz$, which is a polynomial with quadratic terms only.

Writting $x = \lambda - \Delta_1$, $y = \lambda$ and $z = \lambda + \Delta_2$, we get

$$f(\lambda - \Delta_1, \lambda, \lambda + \Delta_2) = \lambda(\Delta_1 - \Delta_2) + \Delta_1\Delta_2 \qquad (2)$$

which describes the output of the operator in terms of the *local level* λ and the *local variations* Δ_1 and Δ_2 of the signal.

The TK operator is nonlinear: it does not obey superposition nor homogeneity; it does obey however what we call *square homogeneity*.

Definition 8 An operator O (defined on a vector space of signals) is said to be *square homogeneous* if, for each constant (scalar) c and each signal (vector) $\mathbf{s}$, $O(c\mathbf{s}) = c^2 O(\mathbf{s})$.

From the definition of the operator, we have

Proposition 1 *The response of the operator to the signals* $\{x_n\}$ *and* $\{(-1)^n x_n\}$ *is the same:* $\mathrm{TK}\{(-1)^n x_n\} = \mathrm{TK}\{x_n\}$.

Definition 9 The signal $\mathbf{x}$ is said to be a *pre-constant signal for the constant* c if $\mathrm{TK}(\mathbf{x}) = \{c\}$.

By square homogeneity and from Proposition 1 we have

Proposition 2 *If* $\mathbf{x}$ *is pre-constant for the constant* c *and* k *is a constant then* $k\mathbf{x}$ *is pre-constant for the constant* k^2c *and* $\{(-1)^n x_n\}$ *is pre-constant for the constant* c.

Definition 10 A signal $\mathbf{x}$ is said to be a *pre-null signal* if $\mathrm{TK}(\mathbf{x}) = \theta$, where θ is the null signal $\{\ldots\ 0,\ 0,\ \ldots\}$; i.e. $\mathbf{x}$ is pre-null if it is pre-constant for the constant 0.

Constant signals are pre-null [7]; in particular, by Proposition 2, since $\{\ldots\ 1,\ 1,\ \ldots\}$ is pre-null, the signal $\{(-1)^n\}$ is pre-null.

Definition 11 A signal $\mathbf{x}$ is said to be a *root signal* of the operator if $\mathrm{TK}(\mathbf{x}) = \mathbf{x}$.

Below we list several responses that give handy knowledge, useful in the familiarization with the operator.

The *step* signal $\mathbf{u}$ and the *impulse* signal δ are defined as usual, as $u_n = 1$ for $n \geq 0$ and $u_n = 0$ for $n < 0$, and $\delta_n = 1$ for $n = 0$ and $\delta_n = 0$ for $n \neq 0$. Clearly, $\mathrm{TK}(\mathbf{u}) = \delta$ and $\mathrm{TK}(\delta) = \delta$; thus, the impulse signal is a root of the operator.

From (2) (we refer to equations by merely indicating their numbers in parenthesis),

Proposition 3 $\mathrm{TK}(\{a + \Delta n\}) = \left\{\Delta^2\right\}$

Thus, linear signals are pre-constant for the square of the slope of the signal. Meanwhile, for quadratic input signals,

Proposition 4 $\mathrm{TK}\left(\{c_1 + c_2 n + c_3^2 n^2\}\right) = \left\{c_2^2 - c_3^2 + 2c_2c_3 (n-1) + 2c_3^2 n^2\right\}$

And the output is a quadratic signal as well. We also have

Proposition 5 *A 2-periodic signal* $\mathbf{x} = \{\ldots a, b, a, b, \ldots\}$ *is a pre-null signal if and only if* $a^2 = b^2$*, otherwise,* $\mathrm{TK}(\mathbf{x}) = \left\{\left(a^2 - b^2\right)(-1)^n\right\}$ *has also period 2 (and no DC level).*

Exponential signals are pre-null and sinusoidal signals are pre-constant [7]: $\mathrm{TK}\{c^n\} = \theta$ and $\mathrm{TK}\{\sin(\omega n + \phi)\} = \left\{\sin^2\omega\right\}$, independently of the values of ω and ϕ, which may also be complex. By square homogeneity, the response to the signal $\{c \sin(\omega n + \phi)\}$ is the constant signal $\left\{c^2 \sin^2\omega\right\}$; see Fig. 1a. The response of the operator to an exponentially modulated sinusoidal signal $\{ce^{kn}\sin(\omega n + \phi)\}$ is the exponential signal $\{ce^{2kn}\sin^2\omega\}$. Independently of the value of ϕ, the responses of the operator to the hyperbolic signals $\{\sinh(\eta n + \phi)\}$ and $\{\cosh(\eta n + \phi)\}$ are, respectively the constant signals $sinh^2\eta = 0.5\{\cosh(2\eta) - 1\}$ and $-sinh^2\eta = 0.5\{1 - \cosh(2\eta)\}$; η and ϕ, may be real or complex. See Fig. 1b and c.

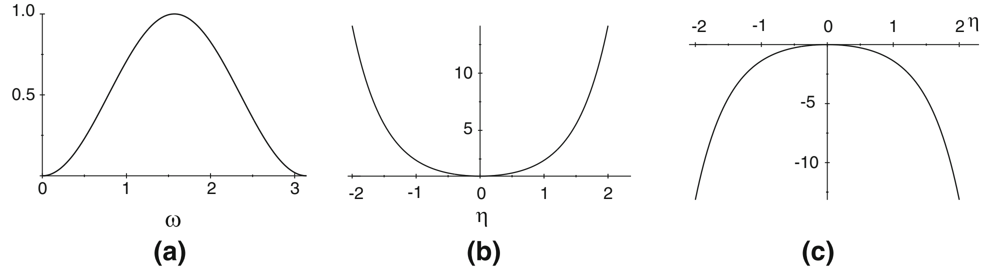
Definition 12 For each pair of signals $\mathbf{x}$ and $\mathbf{y}$, their *TK product* $\mathbf{x} \otimes \mathbf{y}$ is given by the signal $2\mathbf{xy} - ({}_{-1}\mathbf{x})({}_{1}\mathbf{y}) - ({}_{1}\mathbf{x})({}_{-1}\mathbf{y})$.

Thus we have

Proposition 6 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be signals and* a *and* b *be constants then*

$$\mathrm{TK}(a\mathbf{x} + b\mathbf{y}) = a^2\mathrm{TK}(\mathbf{x}) + b^2\mathrm{TK}(\mathbf{y}) + ab(\mathbf{x} \otimes \mathbf{y})$$

Fig. 1 **a** $\sin^2\omega$ versus ω, and, **b** $0.5[1+\cosh(2\eta)]$ and **c** $0.5[1-\cosh(2\eta)]$ versus η



In the proposition above, considering $\mathbf{x}$ to be a signal to be filtered, $\mathbf{y}$ to be a noise signal and a a small value, we conclude that a small additive perturbation of the input signal likewise produces a small change in the output signal; this provides a characterization of the stability of the operator under addition.

Also, from Proposition 6, if $\mathbf{x}$ is a signal and c is a constant

$$\mathrm{TK}\ (\mathbf{x}+\{c\}) = \mathrm{TK}\ (\mathbf{x}) + c(\mathbf{x}*\mathbf{h}) \tag{3}$$

where $\mathbf{h}=\{\ldots 0,-1,2,-1,0,\ldots\}$ and $*$ is the convolution operator, that is,

$$\mathrm{TK}\ (\mathbf{x}+\{c\}) = \mathrm{TK}\ (\mathbf{x}) + c(-\{x_{n-1}\}+2\{x_n\}-\{x_{n+1}\})$$

The transfer function corresponding to the impulse response h in the convolution term in (3) is given by $H(z)=z+1+z^{-1}$, therefore $H(e^{j\omega})=2[1-cos(2\omega)]$. Consequently, if $\mathbf{x}=\{k^n\}$ is an exponential signal and c is a constant, $\mathrm{TK}\ (\mathbf{x}+\{c\}) = c(\mathbf{x}*\mathbf{h})$: a high-pass filtered version of $\{k^n\}$.

From Proposition 6, if $\mathbf{x}$ and $\mathbf{y}$ are root signals then

$$\mathrm{TK}\ (\mathbf{x}+\mathbf{y}) = \mathbf{x}+\mathbf{y}+\mathbf{x}\otimes\mathbf{y}$$

Also from Proposition 6, if $\mathbf{x}$ and $\mathbf{y}$ are pre-constant signals for the constants c and d, respectively, then

$$\mathrm{TK}\ (\mathbf{x}+\mathbf{y}) = \{c+d\}+\mathbf{x}\otimes\mathbf{y}$$

in particular, if $\mathbf{x}$ and $\mathbf{y}$ are exponential signals, namely $x_n = a\gamma_1^n$ and $y_n = b\gamma_2^n$ then

$$\begin{aligned}\mathrm{TK}\ (\mathbf{x}+\mathbf{y}) &= -ab\frac{(\gamma_1-\gamma_2)^2}{\gamma_1\gamma_2}(\gamma_1\gamma_2)^n \\ &= -\frac{(\gamma_1-\gamma_2)^2}{\gamma_1\gamma_2}\,\mathbf{xy}\end{aligned}$$

which, in a sense, translates a sum of signals into a scaled product of signals and, also, explains the response of the operator to trigonometric and hyperbolic signals in terms of the (null) response to exponentials. In particular, if $\gamma_1\gamma_2=-1$, the output is given by

$$ab\left(2+\gamma_1^2+\frac{1}{\gamma_1^2}\right)(-1)^n$$

More generally, for a linear combination of exponential signals, we get

$$\begin{aligned}\mathrm{TK}\left\{\sum_{r=0}^{m} c_r\gamma_r^n\right\}_n &= \sum_{r=1}^{m}\sum_{s=1}^{m} c_r c_s\,(\gamma_r\gamma_s)^n \\ &\quad -\sum_{r=1}^{m}\sum_{s=1}^{m} c_r c_s \gamma_r^n\gamma_s^n\left(\frac{\gamma_s}{\gamma_r}\right) \\ &= \sum_{r=1}^{m}\sum_{s=r+1}^{m} c_r c_s\,(\gamma_r\gamma_s)^n\left(2-\frac{\gamma_r}{\gamma_s}-\frac{\gamma_s}{\gamma_r}\right)\end{aligned}$$

If the input $\mathbf{x}$ is periodic and $N\geq 2$ is one of its repetition times, with corresponding Fourier sum

$$x_n = \sum_{r=0}^{N-1} c_r e^{j\left(\frac{2\pi}{N}\right)rn}$$

then the output $\mathbf{y}$ is given by

$$y_n = \sum_{r=1}^{m}\sum_{s=r+1}^{m} c_r c_s\left[2-2\cos\left(\frac{2\pi}{N}(r-s)\right)\right]e^{j\frac{2\pi}{N}(r+s)n}$$

which permits the computation of the Fourier coefficients of the output.

For an input that is a product of signals we have

Proposition 7 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be signals; then,*

$$\begin{aligned}\mathrm{TK}(\mathbf{xy}) &= (\mathbf{xy})^2 - {}_{-1}(\mathbf{xy})\,{}_{1}(\mathbf{xy}) \\ &= \mathbf{x}^2\mathrm{TK}(\mathbf{y})+\mathbf{y}^2\mathrm{TK}(\mathbf{x})-\mathrm{TK}(\mathbf{x})\mathrm{TK}(\mathbf{y})\end{aligned}$$

from which we get

Proposition 8 *Let* $\mathbf{p}_1,\ldots,\mathbf{p}_n$ *be n pre-null signals, then,*

$$\mathrm{TK}\left(\prod_{i=1}^{n}\mathbf{p}_i\right)=0$$

More generally, for an input that is an algebraic combination of signals we have

Proposition 9 *Let* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$ *and* $\mathbf{y}$ *be signals; then,*

$$\begin{aligned}&\mathrm{TK}(\mathbf{vx}+\mathbf{wy}) \\ &\quad= \mathbf{v}^2\mathrm{TK}(\mathbf{x})+\mathbf{x}^2\mathrm{TK}(\mathbf{v})+\mathbf{w}^2\mathrm{TK}(\mathbf{y})+\mathbf{y}^2\mathrm{TK}(\mathbf{w}) \\ &\qquad -\mathrm{TK}(\mathbf{v})\mathrm{TK}(\mathbf{x})-\mathrm{TK}(\mathbf{w})\mathrm{TK}(\mathbf{y})+(\mathbf{vx}\otimes\mathbf{wy})\end{aligned}$$

In particular, if $\mathbf{v} = \{\alpha^n\}$ and $\mathbf{w} = \{\beta^n\}$

$$\text{TK}\ \{\mathbf{vx} + \mathbf{wy}\}_n = \left\{\alpha^{2n}\right\}\ \text{TK}\ (\mathbf{x}) + \left\{\beta^{2n}\right\}\ \text{TK}\ (\mathbf{y}) \\ + 2\left\{x_n y_n - \frac{\beta}{\alpha} x_{n-1} y_{n+1} - \frac{\alpha}{\beta} x_{n+1} y_{n-1}\right\}\left\{(\alpha\beta)^n\right\}$$

Using this last formula we obtain the response to a sinusoid with a phase signal; let ϕ be a signal and ω a constant, then

$$\text{TK}\ \{\cos(\omega n + \phi_n)\}_n = 2\left[1 - \cos(2\omega + \phi_{n+1} - \phi_{n-1})\right. \\ \left. -2\sin(\phi_n - \Phi_n)\sin(2\omega n + \phi_n + \Phi_n)\right]$$

where $\Phi_n = (\phi_{n+1} + \phi_{n-1})/2$, so the output is a sum of sinusoidal terms, one depending on the frequency ω and the phase signal $\{\phi_n\}$ alone, the other more similar to the input, but with a doubled frequency, a smoothed phase and modulated by yet another sinusoidal term that depends on a high-pass filtered version of the phase signal.

We end this section with two responses to other two (approximately) sinusoidal signals. Let c, k and ω be constants, from (3) we get

$$\text{TK}\ \{c\sin(\omega n) + k\} = c^2\sin^2\omega + 2kc[1 - \cos(2\omega)]\sin(\omega n)$$

For a frequency modulated signal,

$$\text{TK}\ \left\{\cos\left(k_1 n + k_2 n^2\right)\right\}_n \\ = \sin^2(k_1 + 2k_2 n) + 2\sin(k_2)\sin\left(k_2 + 2k_1 n + 2k_2 n^2\right)$$

for small k_2 and small $k_1 + 2k_2 n$, the output is approximately $(k_1 + 2k_2 n)^2$ which, in a sense, is the instantaneous energy of the input.

3 Contrast enhancement and time-frequency

3.1 Contrast enhancement

We perceive as *enhanced* the transitions at the edges between Mach bands (see Fig. 2); correspondingly, an approximate "sensorial response" [6] to a local abrupt change between two constant levels is as shown in Fig. 3c. On the other hand, for arbitrary constants c and d, the response of the TK operator to the *edge signal* $\{au_{-n-1} + bu_n\}$ is the signal $\{-a\Delta\delta_{n+1} + b\Delta\delta_n\}$, where $\Delta = b - a$, see Fig. 3a and b; thus, for an edge that is positive on both sides, with $\epsilon \in (0, 1)$, the second-order Volterra filter given by $\text{CE}\ (\mathbf{x}) := \epsilon\ \text{TK}\ (\mathbf{x}) + \mathbf{x}$ is an contrast enhancer. It is also, in a sense, an *unsharp masking* operator.

Likewise, the 2D TK operator, given by

$$y_{nm} = 2x_{nm}^2 - x_{n-1,m}x_{n+1,m} - x_{n,m-1}x_{n,m+1}$$

Fig. 2 Mach band: even though the physical stimulus is like that in Fig. 3a, the perceived intensity is more like that in Fig. 3c

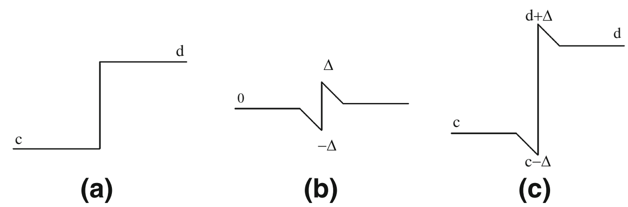


Fig. 3 **a** Schematic edge, **b** schematic TK response **c** enhanced (unsharp masked) edge: edge plus scaled version of TK response

has been successfully used as a part of a contrast enhancer for gray level images [8,17].

3.2 The TK operator in the time-frequency domain

The so-called *spring-mass energy* [7] corresponding to the signal $\{c\sin(\omega n + \phi)\}$ is $c^2\omega^2$. For $\omega < \frac{\pi}{8}$, the output $c^2\sin^2\omega$ of the TK operator approximates $c^2\omega^2$; nevertheless, for $\frac{\pi}{2} \le \omega \le \pi$, $sin^2(\omega)$ is a decreasing function. A more general characterization of the operator results in the time-frequency domain. As we show below, working in $l_1 \cap l_2$, the output of the TK operator can be expressed in terms of the ambiguity function of the input signal, as in the case of the continuous TE operator [5].

Let the *difference operator* D be given by $D(\mathbf{x})_n = x_n - x_{n-1}$; correspondingly, $D^2(\mathbf{x})_n = x_n - 2x_{n-1} + x_{n-2}$. Translating the derivative operator in terms of the operator D, the continuous TE operator $\dot{x}^2(t) - \ddot{x}(t)x(t)$ results in the causal version $x_{n-1}^2 - x_n x_{n-2}$ of the TK operator.

Let the complex Teager-Kaiser operator (CTK) be given by

$$\text{CTK}\ (\mathbf{x})_n = x_n x_n^* - \frac{1}{2}x_{n-1}^* x_{n+1} - \frac{1}{2}x_{n-1}x_{n+1}^*$$

which, for real $\mathbf{x}$, clearly restricts to the TK operator; also, let Ξ the *ambiguity function* [2] of $\mathbf{x}$. With $\gamma_{n,k} := x_{n+k}x_{n-k}^*$,

$$\Xi(\omega, k) = DFT_{\succ n}\left[\gamma_{n,k}\right](\omega) = \sum_{n=-\infty}^{\infty} \gamma_{n,k} e^{-j\omega n}$$

where $DTFT_{\succ n}$ indicates the discrete time Fourier transform, taken with respect to the variable n. Let

$$\Gamma(\omega, \tau) = DTFT_{\succ k}\left[\Xi(\omega, k)\right](\tau) = \sum_{k=-\infty}^{\infty} \Xi(\omega, k)e^{-j\tau k}$$

with coresponding inverse transform

$$\Xi(\omega,k)=DTFT^{-1}_{\succ\tau}[\Gamma(\omega,\tau)]_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)e^{j\tau k}d\tau$$

then

$$D^2_{\succ k\succ k}\Xi(\omega,k)=\frac{1}{\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)(\cos\tau-1)e^{j\tau k}d\tau$$

where $D^2_{\succ k\succ k}$ denotes the second-degree difference operator D^2, applied with respect to the variable k. We also have

$$\begin{aligned}\Gamma(\omega,\tau)&=DTFT^2_{\succ n\succ k}\left[\gamma_{n,k}\right](\omega,\tau)\\&=\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\gamma_{n,k}e^{-j\omega n}e^{-j\tau k}\end{aligned}$$

with corresponding inverse transform

$$\gamma_{n,k}=DTFT^{-2}_{\succ\tau\succ\omega}[\Gamma(\omega,\tau)]_{n,k}=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)\times e^{j\omega n}e^{j\tau k}d\tau\,d\omega$$

then

$$\begin{aligned}&D^2_{\succ k\succ k}\gamma_{n,k}\\&=\frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)(\cos\tau-1)\quad e^{j\omega n}e^{j\tau k}d\tau\quad d\omega\end{aligned}$$

Now, since

$$D^2_{\succ k\succ k}\gamma_{n,k}=x_{n+k+1}x^*_{n-k-1}-2x_{n+k}x^*_{n-k}+x_{n+k-1}x^*_{n-k+1}$$

then

$$\begin{aligned}&\mathrm{CTK}(\mathbf{x})\\&=-\frac{1}{2}\,D^2_{\succ k\succ k}\gamma_{n,k}\Big|_{k=0}\\&=-\frac{1}{2}D^2_{\succ k\succ k}\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)\quad e^{j\omega n}e^{j\tau k}d\tau\,d\omega\Bigg|_{k=0}\\&=\frac{1}{2\pi}\int_{-\pi}^{\pi}-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)(\cos\tau-1)e^{j\tau k}d\tau\Bigg|_{k=0}\quad e^{j\omega n}d\omega\end{aligned}$$

Then

$$\begin{aligned}&DTFT\,[\,\mathrm{CTK}\,(\mathbf{x})]\,(\omega)\\&=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma(\omega,\tau)(\cos\tau-1)e^{j\tau k}\,d\tau\Bigg|_{k=0}\\&=-\frac{1}{2}D^2_{\succ k\succ k}\Xi(\omega,k)\Big|_{k=0}\end{aligned}$$

or

$$\mathrm{CTK}\,(\mathbf{x})_n=-\frac{1}{2}DTFT^{-1}_{\succ\omega}\left[D^2_{\succ k\succ k}\Xi(\omega,k)\right]_n\Bigg|_{k=0}\tag{5}$$

4 Root signals of the TK operator

Root signals do not play a fundamental role in the study of convolution filters; that role is played there by the exponential signals. For each $k\in\mathbf{Z}$ the exponential signals e^{zn} with frequency z such that $H(e^z)=e^{kz}$, together with their linear combinations, give the space of *limit-cycle* signals for which k is a *repetition time.* The set of the roots of a convolution filter is given by the eigenspace of signals corresponding to the eigenvalue 1.

The study of the root signals of a nonlinear system is a crack tool that allows to peek inside the filter; it provides a first set of basic properties for the filter.

We characterize the roots of the TK operator both in time domain and in phase space. We consider only real roots. For $\mathbf{x}$ to be a root signal of the TK operator it is necessary and sufficient that it be a solution of the nonlinear difference equation

$$\mathbf{x}=\mathbf{x}^2-({}_1\mathbf{x})({}_{-1}\mathbf{x})\tag{6}$$

whenever $x_{n-1}\neq 0$ we have

$$x_{n+1}=\frac{x_n\,(x_n-1)}{x_{n-1}}\tag{7a}$$

also from (6), we have

$$(x_n=0,\vee,x_n=1)\Leftrightarrow(x_{n-1}=0,\vee,x_{n+1}=0)\tag{7b}$$

Definition 13 A *segment of finite length* $r\in\mathbf{N}$ of a signal $\mathbf{x}$ is an r-tuple $[x_n,\ldots,x_{n+r-1}]$ while a *segment of infinite length* is either a one-sided subsequence of the form $\{x_n : n\geq M\}$ or $\{x_n : n\leq M\}$, for some integer M, or the whole signal $\mathbf{x}$.

Definition 14 A *binary segment* is a segment with values in the set $\{0,1\}$.

Definition 15 A *block* of a signal $\mathbf{x}$ is a longest segment satisfying a given property; e.g. a binary block.

Given values x_{n-1} and x_n, so long as $x_{n-1}\neq 0$, the iteration of (7.a) determines a segment of a root signal. On the other hand, for a binary root segment, from (7.b), it must be the case that $\forall n\in\mathbf{Z}x_n=0,\vee,x_{n+2}=0$. Each block of zeros (a 0-block, for short) of a root is of length at least two and each 1-block is of length at most two. The only points on the axes of the plane of points (x_{n-1},x_n) a root can take, are (0, 0), (0, 1) and (1, 0); we adopt as *phase space* for the

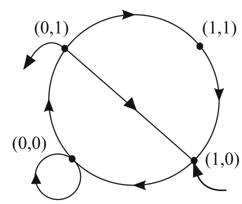


Fig. 4 Binary roots obey the rule described by this graph, in phase space. The nodes with open ended arrows connect with the rest of phase space

roots of the operator, the plane with axes x_{n-1} and x_n, minus the axes, plus the set $\{(0,0), (0,1)$ and $(1,0)\}$.

A binary segment is a root segment if and only if it stays in the graph in Fig. 4, in phase space. Unlike the segments governed by (7a), two consecutive values x_{n-1} and x_n of a binary root do not determine uniquely a third one x_{n+1}; in this sense, binary segments are *undeterminable* while the segments governed by (7a) are said to be *determinate*. The segments $[1, 0]$ and $[0, 1]$ serve as *transition segments* from nonbinary segments to binary segments and from binary segments to nonbinary segments, respectively, as in the signal $\{\ldots 0, 1, 1, 0, 0, 0, 1, 2, 2, 1, 0, 0, 1, 0, 0, \ldots\}$; see also Fig. 5. Root segments of the forms $[b, a, 1, 0]$ and $[0, 1, a, b]$ obey $b = a(a-1)$.

For the study of determinate root segments, we define the function $f : \mathbf{R}^{2-} \to \mathbf{R}^2$ ($\mathbf{R}^{2-}$ stands for the plane minus its horizontal axis) via

$$f(a, b) = \left(b, \frac{b(b-1)}{a}\right)$$

f has inverse $f^{-1} : \mathbf{R}^{2|} \to \mathbf{R}^2$ ($\mathbf{R}^{2|}$ stands for the plane minus its vertical axis) given by

$$f^{-1}(a, b) = \left(\frac{a(a-1)}{b}, a\right)$$

4.1 A family of conics and characterizations of roots

We use two main facts in the classification of the roots of the TK operator: phase space can be partitioned into an $\mathbf{R}$-indexed, f-invariant family of conics, and, determinate roots obey a certain linear equation.

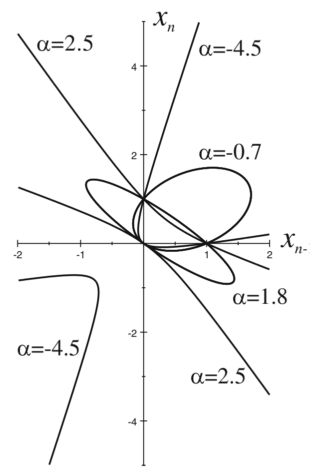


Fig. 6 A collection of α-conics

For each real α, let

$$C_\alpha(a, b) = a^2 + b^2 + \alpha ab - a - b \tag{8}$$

and call the set of points (a, b) of $\mathbf{R}^2$ meeting $C_\alpha(a, b) = 0$, an α-conic. The points $(0, 0)$, $(1, 0)$ and $(0, 1)$ belong to each α-conic while the remaining points on the axes belong to no α-conic. Each point of $\mathbf{R}^2$ off the axes belongs to one and only one conic, that with α given by

$$\alpha = \frac{(1-a)}{b} + \frac{(1-b)}{a} \tag{9}$$

see Fig. 7. Equation (9) determines a continuous surjection $h : \mathbf{R}^{2+} \to \mathbf{R}^1$ ($\mathbf{R}^{2+}$ stands for the plane minus its axes); h assigns to each point the value of α of the conic it belongs to. (The collection of sets $\left\{h^{-1}(t), t \in \mathbf{R}\right\}$ is a *cover* of $\mathbf{R}^{2+}$ by conics and h is a *quotient map*.)

For $|\alpha| < 2$ the conics are ellipses ($\alpha = 0$ corresponds to a circle); for $|\alpha| > 2$, the conics are hyperbolas; for $\alpha = 2$ the (degenerate) conic consists of the union of two lines ($a+b = 0$ and $a + b = 1$) and, for $\alpha = -2$, the (degenerate) conic is a parabola; see Figs. 6, 7, 8, 9, 10. The conics are symmetric about the line $a = b$ and the ellipses are centered at the

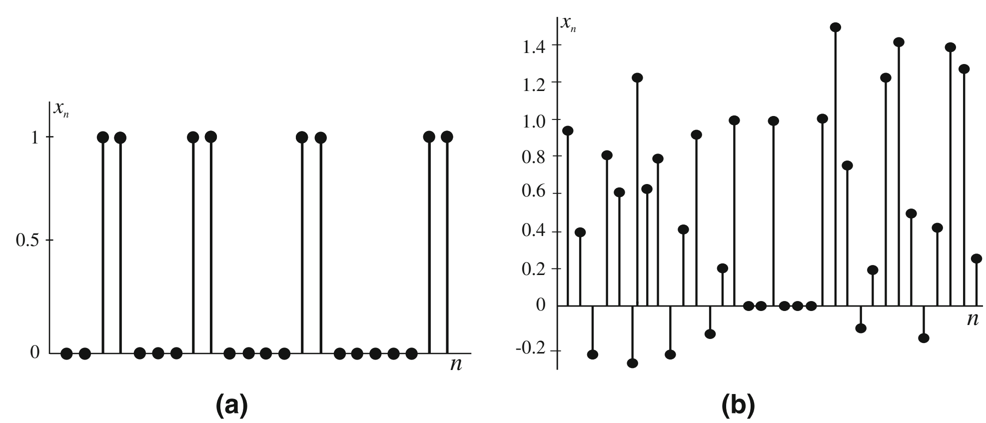


Fig. 5 **a** A (nonperiodic) binary root signal. Notice that in phase space, the graph in Fig. 4 is obeyed. **b** A binary segment sandwiched between determinate segments

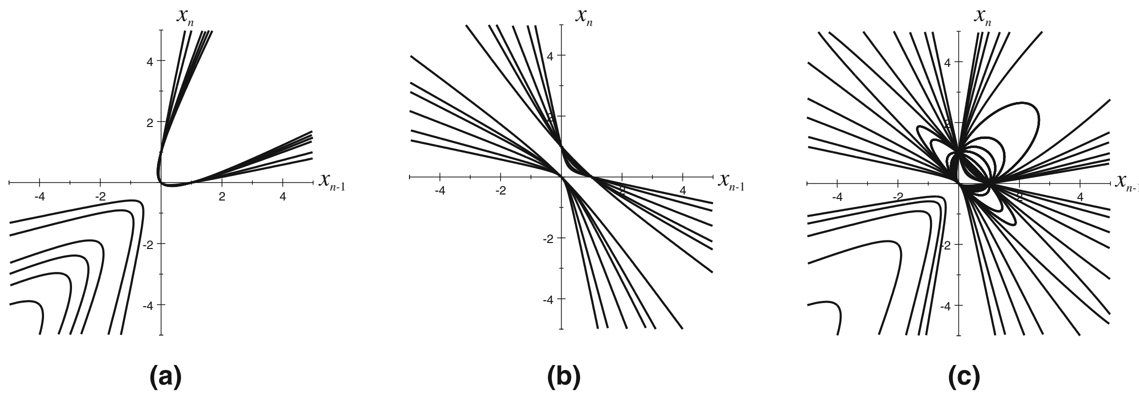


Fig. 7 **a** Hyperbolas with a branch in the 3rd quadrant ($\alpha < -2$), **b** hyperbolas disjoint from the third quadrant ($\alpha > 2$), **c** an indication of the way the phase space is covered by the α-conics

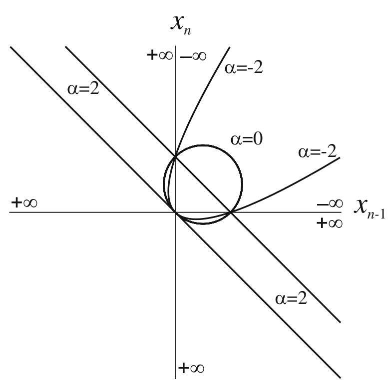


Fig. 8 Conics corresponding to the critical values $0, \pm 2, \pm\infty$, of the parameter α. The regions in between correspond to intermediate value of α as well

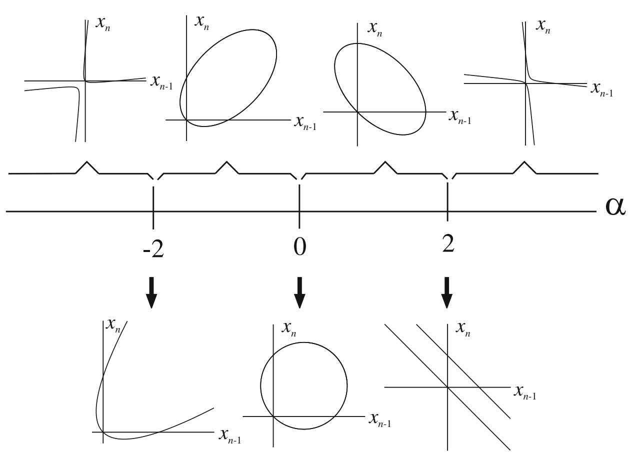


Fig. 9 Types of conic as a function of α

point $(1/(2+\alpha), 1/(2+\alpha))$. Binary root segments live in phase space on the subset $\{(0, 0), (0, 1), (1, 1), (1, 0)\}$ of the 0-conic, which we call the *finite conic*, see Figs. 4 and 8. As α tends to $+\infty$, the conics approach the axes of the plane, in the first and third quadrants and, as α tends $-\infty$, the sheets of the hyperbolas approach the positive semi-axes of the plane, and the "point $(-\infty, -\infty)$". See Figs. 7 and 8.

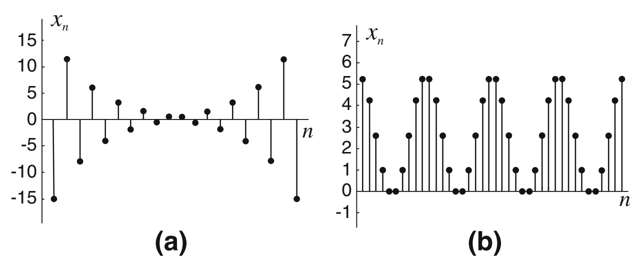


Fig. 10 Determinate root signals. **a** Unbounded signal and **b** a periodic root signal

Corresponding to the transition segments [0, 1] and [1, 0], we have in phase space the *switch points* between conics (0, 1) and (1, 0). A root can leave the 0-conic at the point (0, 1) enter it back at the point (1, 0); in the rest of phase space, α-conics are f-invariant as Theorem 1 shows.

Theorem 1 (Characterization in phase space) *A signal* $\boldsymbol{x} = \{x_n\}$ *is a root signal of the TK operator if and only if, for each n, one of the following conditions holds*:

(i) (x_{n-1}, x_n) *and* (x_n, x_{n+1}) *belong to the finite conic and there is an arrow in Fig. 4 pointing from* (x_{n-1}, x_n) *to* (x_n, x_{n+1}).

(ii) *Neither* (x_{n-1}, x_n) *nor* (x_n, x_{n+1}) *lie on the axes of phase space, they lie on the same conic, and satisfy* (7a).

(iii) $(x_{n-1}, x_n) = (0, 1)$ *and* x_{n+1} *is any number, or,* $(x_n, x_{n+1}) = (1, 0)$ *and* x_{n-1} *is any number.*

Proof Regarding necessity, it is clear that if any of the conditions (i)-(iii) is met, the output of the operators has value x_n. As for sufficiency,

(i) It follows from Condition (7b).

(ii) Let (x_{n-1}, x_n) be as indicated; since $\mathbf{x}$ is a root, $(x_n, x_{n+1}) = f(x_{n-1}, x_n)$; denote (x_{n-1}, x_n) as (a, b) and let α be the real number for which $C_\alpha(a, b) = 0$, now, since $a \neq 0$,

$$\begin{aligned} &C_\alpha (f(a, b)) \\ &= \left(1/a^2\right)\left[\left(b^2 - a\right)\left(a^2 + b^2 + \alpha ab - a - b\right)\right] \\ &= \left(1/a^2\right)\left[\left(b^2 - a\right) C_\alpha (a, b)\right] \\ &= 0 \end{aligned}$$

(iii) It follows directly from the definitions. □

We therefore have

Theorem 2 (Characterization in the time domain) *Each root of the TK operator is a concatenation* $\{\ldots S_{\alpha_i},\ S_{B_j},\ S_{\alpha_{i+1}},\ S_{B_{j+1}},\ \ldots\}$ *of determinate blocks* S_{α_i}*, that live on a unique* α*-conic, and of undeterminable blocks* S_{B_j} *that live on the finite conic and begin with* $[1, 0]$ *and end with* $[0, 1]$.

4.2 A linear equation and explicit expressions for determinate blocks

Root segments that live on an ellipse are bounded and, if determinate, in the time domain they correspond to sinusoids (on a DC level that corresponds to the coordinates of the center of the ellipse); the frequency of the sinusoid is determined by the α-conic they live on. Determinate root segments that live on hyperbolas and on degenerate conics are unbounded, when of infinite length. This and other properties of the root segments are consequences of Lemma 1 below. In addition to obeying 7a, determined root segments obey as well a certain linear equation, the solution of which gives explicit expressions for the determinate blocks of the roots.

Lemma 1 *Each determined root segment obeys the linear equation*

$$x_{n+1} + \alpha x_n + x_{n-1} = 1 \tag{10}$$

Proof Consider two consecutive points (x_{n-1}, x_n) and (x_n, x_{n+1}) in phase space, corresponding to a determinate segment; from Theorem 1 and (9), we have

$$\alpha = \frac{1 - x_{n-1}}{x_n} + \frac{1 - x_n}{x_{n-1}} \tag{10a}$$

then

$$\alpha x_n = 1 - x_{n-1} - \frac{x_n(x_n - 1)}{x_{n-1}} \tag{10b}$$

and the lemma follows. □

Not every solution of (10) is a root, however; in phase space, the solutions of (10) live on conics of the type $C_\alpha(x_{n-1}, x_n) = \kappa$, with κ possibly different from 0. After giving the solution of the linear equation, we give conditions for it to be a root.

Solution of the linear equation. (10) is a linear difference equation with characteristic equation $1+\alpha\gamma+\gamma^2 = 0$, which has roots (see Fig. 11)

$$\gamma_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1}$$

Thus, for $\alpha \neq \pm 2$, the homogeneous part of the solution of (10) is given by $x_n = c_1\gamma_1^n + c_2\gamma_2^n$; for $\alpha = -2$ and $\alpha = 2$, the homogeneous part of the solution is given by $x_n = c_1(1)^n + c_2 n(1)^n$ and $x_n = c_1(-1)^n + c_2 n(-1)^n$, respectively. In each case, $c_1, c_2 \in \mathbf{C}$. The following proposition checks out easily.

Proposition 10 *The product* $\gamma_1\gamma_2$ *of the roots of the characteristic equation is one and their sum* $\gamma_1 + \gamma_2$ *is* $-\alpha$.

For $|\alpha| < 2$, γ is a complex number of magnitude one and, for a real solution, the homogeneous part of the solution of (10) is the sinusoid $A\cos(\omega n + \phi)$ of frequency

$$\omega = \text{angle}(\gamma) = -\arctan\sqrt{\left(\frac{2}{\alpha}\right)^2 - 1} + \frac{0}{\pi} \tag{10c}$$

(with an offset of value π if $\text{Re}(\gamma) < 0$ i.e. if $\alpha > 0$), with the understanding that for $\alpha = 0$ we get $\omega = \pi/2$ and the signal is periodic with period 4, see Fig. 12. For $\alpha = 2$, (10c) would predict period 2 while for $\alpha = -2$, (10c) would predict a constant signal; however, see Lemmas 2 and 3 below. If ω is not a rational multiple of 2π, the signal is almost periodic but not periodic.

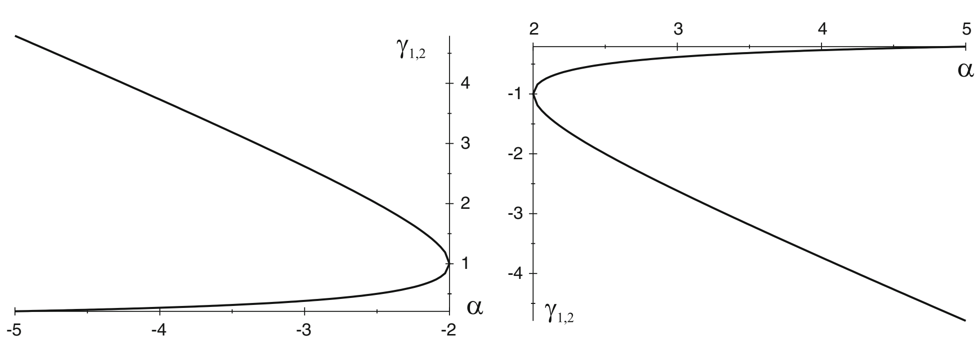


Fig. 11 Behavior of the roots $\gamma_{1,2}$ of the characteristic equation of (10), as a function of α, when $|\alpha| > 2$

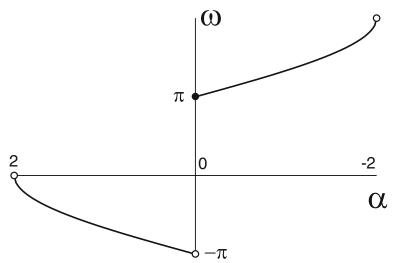


Fig. 12 ω versus α

For $|\alpha| > 2$, γ_1 and γ_2 are real and the solution is a sum of exponentials.

For $\alpha \neq -2$, a particular solution of (10) is the constant signal $x_n = 1/(2+\alpha)$; for $\alpha \neq \pm 2$ the complete solution is of the form

$$x_n = c_1\gamma_1^n + c_2\gamma_2^n + 1/(2+\alpha) \tag{11a}$$

Note that $1/(2+\alpha)$ is the DC level in the case of sinusoidal segments. For $\alpha = 2$, the complete solution is of the form

$$x_n = c_1(-1)^n + c_2 n(-1)^n + 1/4 \tag{11b}$$

For $\alpha = -2$, $n(n+1)/2$ is a particular solution and the complete solution is of the form

$$x_n = c_1 + c_2 n + \frac{n(n+1)}{2} \tag{11c}$$

Lemma 2 *A necessary and sufficient condition for a segment of a solution of (10) to be a root segment of the TK operator is that*

(i) $c_1c_2 = 1/[(2+\alpha)^2(2-\alpha)]$, *if* $\alpha \neq \pm 2$, *and the root has the form* (11a).
(ii) $c_1 \in \boldsymbol{R}$, *if* $\alpha = 2$, *and the root is of the form* $c_1(-1^n) + 0.5n(-1)^n + 0.25$.
(iii) $c_2 \in \boldsymbol{R}$, *if* $\alpha = -2$, *and the root is of the form* $(n+c_2)(n+c_2+1)/2$.

Proof For a (segment of a) solution of (10) to be a root segment of the TK operator, it must produce the same signal when applied to the operator. Consider (11a); since $c_1\gamma_1^n + c_2\gamma_2^n$ is pre-constant for the constant $-c_1c_2(\gamma_1-\gamma_2)^2$ (see Proposition 10 and the response to a combination of exponential signals in Sect. 2), using the formula for the response to pre-constant signal plus constant, we have that the response to $x_n = c_1\gamma_1^n + c_2\gamma_2^n + 1/(2+\alpha)$ is given by $y_n = -c_1c_2(\gamma_1-\gamma_2)^2 + [1/(2+\alpha)](c_1H(\gamma_1)\gamma_1^n + c_2H(\gamma_2)\gamma_2^n)$ where $H(z) = (-z+2-z^{-1})$, also, by Proposition 10 $(\gamma_1-\gamma_2)^2 = \alpha^2-4$ and $\frac{1}{\gamma_2} = \gamma_1$; thus $y_n = -c_1c_2(\alpha^2-4) + [1/(2+\alpha)]\left(c_1(2+\alpha)\gamma_1^n + c_2(2+\alpha)\gamma_2^n\right) = -c_1c_2(\alpha^2-4) + c_1\gamma_1^n + c_2\gamma_2^n$ therefore it is necessary (and sufficient) that $-c_1c_2(\alpha^2-4) = 1/(2+\alpha)$ that is

$$c_1c_2 = 1/[(2+\alpha)^2(2-\alpha)] \tag{12}$$

ii. Consider now (11b); note first that TK $\{c_1(-1)^n + c_2n(-1)^n\} =$ TK $\{c_1 + c_2n\} = \{c_2^2\}$ next, using the fact that $\{c_1(-1)^n + c_2n(-1)^n\}$ is pre-constant for the constant c_2^2, TK $\{c_1(-1)^n + c_2n(-1)^n + 1/4\} = \{c_2^2 + (1/4)[c_1H(-1)(-1)^n + c_2 4n(-1)^n]\} = \{c_2^2 + c_1(-1)^n + c_2n(-1)^n]\}$ thus, it is necessary and sufficient that $c_2 = 1/2$, while the value of c_1 is arbitrary; the signal is therefore of the form

$$x_n = c_1(-1)^n + 0.5n(-1)^n + 0.25 \tag{13a}$$

iii.Consider (11c); using the response of the operator to quadratic signals of Proposition 4, we get TK $\{c_1 + c_2n + n(n+1)/2\} = \{(c_2+1/2)^2 - 1/4 - c_1 + c_2n + n(n+1)/2\}$ thus, it is necessary and sufficient that $c_1 = c_2(c_2+1)/2$ in which case, with arbitrary c_2, the signal has the general form

$$x_n = (n+c_2)(n+c_2+1)/2 \tag{13b}$$

For sinusoidal roots, $|\alpha| < 2$ and $c_1 = c_2^*$; the general form of a root is then

$$x_n = A\cos(\omega n + \phi) + 1/(2+\alpha) \tag{13c}$$

with $A = 2|c_1| = 2/[(2+\alpha)\sqrt{2-\alpha}]$ and ϕ arbitrary. For each $\alpha, |\alpha| < 2$ the orbits of the (homogeneous parts of) solutions are either (all) finite (when periodic) or (all) dense (when not periodic but almost periodic), in the corresponding ellipse. The 4-periodic binary signal $\{\ldots 0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$ and the 3-periodic signal $\{\ldots 0, 0, 1, 0, 0, 1, \ldots\}$ are peculiar in the sense that even though they are solutions of (10), for $\alpha = 0$ and $\alpha = 1$, respectively, they are not determinate signals since they do not obey (7a). □

Summarizing, we have

Theorem 3 (Explicit Expressions for Determinate Blocks) *Each of the determinate blocks* S_{α_i} *mentioned in Theorem 2, has one of the following forms, depending on the value of* α*:*

(i) ***Elliptic case***, *for* $|\alpha| < 2$; $x_n = A\cos(\omega n + \phi) + 1/(2+\alpha)$, *where* $A = 2|c_1| = 2/[(2+\alpha)\sqrt{2-\alpha}]$, $\omega =$ *angle* $(\gamma) = -\arctan\sqrt{\left(\frac{2}{\alpha}\right)^2 - 1} + \begin{matrix}0\\ \pi\end{matrix}$ *(with an offset of value* π *only in case* $\alpha > 0$*), and* ϕ *is arbitrary.*
(ii) ***Hyperbolic case***, *for* $|\alpha| > 2$; $x_n = c_1\gamma^n + c_2\gamma^{-n} + 1/(2+\alpha)$ *with* $\gamma = -\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1}$ *and* $c_1c_2 = 1/[(2+\alpha)^2(2-\alpha)]$.

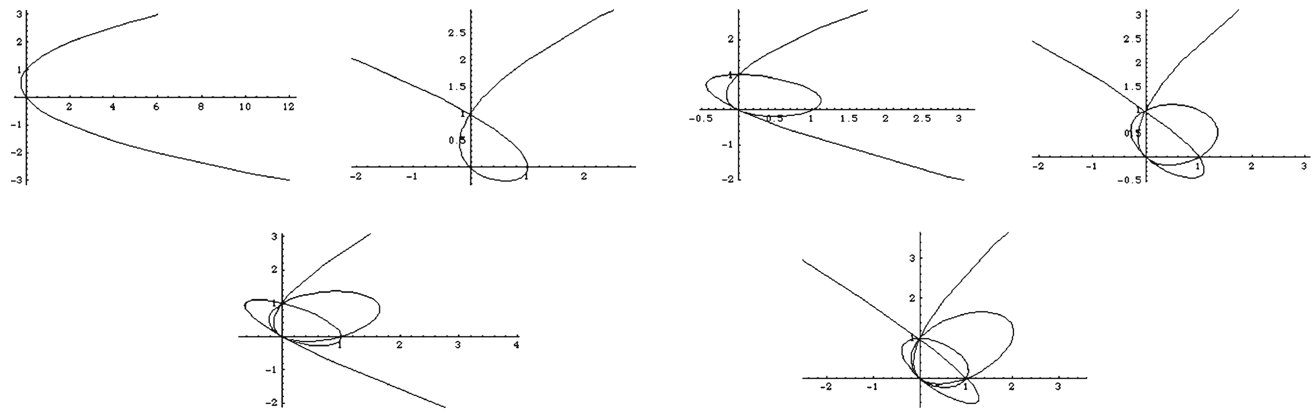


Fig. 13 N-fold f-inverses, $N = 1, 2, \ldots 6$, of the line $\{(x, 1) : x \in \mathbf{R}^1\}$

(iii) ***Linear case***, *for* $|\alpha| = 2$; $x_n = c(-1)^n + 0.5n(-1)^n + 0.25$, *c arbitrary.*

(iv) ***Parabolic case***, *for* $|\alpha| = 2$; $x_n = (n + c)(n + c + 1)/2$, *c arbitrary.* □

Each determinate segment of period 4 lives on the 0-conic and each determinate segment of period 3 lives on the 1-conic. Binary root segments live on the finite conic and, in general, are not periodic. Among the binary root signals, the only ones that are solutions of (10) are the 3-periodic signal $\{\ldots 0, 0, 1, \ldots\}$ and the 4-periodic signal $\{\ldots 0, 0, 1, 1, \ldots\}$. Other binary periodic signals are for example the 5-periodic signal $\{\ldots 0, 0, 0, 0, 1, \ldots\}$, the 6-periodic signal $\{\ldots 0, 0, 0, 0, 0, 1, \ldots\}$, etc.

Lemma 3 *There are no periodic roots of period 2.*

Proof Such a root could be written as $\{\ldots c, d, c, d \ldots\}$ then, since a root, $c = c^2 - d^2$ and $d = d^2 - c^2$ then $c = -d$ then the signal can be written also as $\{\ldots d, -d, d, -d \ldots\}$ then this root is pre-null then $d = 0$, then it is 1-periodic, which is a contradiction. (See also the response to 2-periodic signals, in Sect. 2.) □

If a root never takes the value 0, it lives on a unique α-conic and consists of a unique determinate block; likewise if it never takes the value 1. On the other hand, given a starting point (x_{n-1}, x_n) in phase space, off the finite conic, it is hard to predict whether or not the resulting orbit will contain a point of the form (a, 1). We learn something regarding such starting points by taking the n-fold f-inverse of the line $b = 1$ in phase space; such curve sets contain the points that eventually end up at the crossroads point (1, 0). The curves are rather circuitous; several of them are shown in Fig. 13; they live in the quadrants I, II and IV of phase space and therefore the corresponding starting segments never contain two consecutive negative values.

5 Pre-constant and pre-null signals

The TK operator is perhaps best known for producing a constant output corresponding to a sinusoidal input; the question arises, what signals produce constant outputs? It seems convenient to study them in complete generality. It turns out that the pre-constant and root signals of the operator are closely related. Also, they provide new applications for the operator.

For comparison purposes, with a well-known class of linear filters, we have that the pre-null signals of a convolution filter are given by the eigenspace of signals corresponding to the eigenvalue zero (the stop bands of the filter) and determine the *null eigenspace* (or *kernel*) of the filter. A constant signal is either a (nonzero) scaled exponential of zero frequency, or the null signal, which is not an exponential; since two signals produce the same signal if and only if they differ by a pre-null signal and, other than pre-null exponentials, the only exponential that produces a constant output is the exponential of zero frequency (times any non-null constant), the pre-constant signals are of the form constant signal plus pre-null signal. (More generally, an exponential output is possible only if the input is an exponential of the same complex frequency plus a pre-null signal.)

The signal $\mathbf{x} = \{x_n\}$ is pre-constant for the constant κ if and only if

$$\mathbf{x}^2 - ({}_1\mathbf{x})({}_{-1}\mathbf{x}) = \{\kappa\} \tag{14}$$

We consider only real pre-constant signals. If $\mathbf{x}$ is a pre-constant signal for the constant κ, we also have

$$(x_{n-1} = 0, \vee, x_{n+1} = 0) \Rightarrow x_n^2 = \kappa \tag{15a}$$

$$x_n = 0 \Rightarrow x_{n-1}x_{n+1} = -\kappa \tag{15b}$$

If $\mathbf{x}$ is a pre-constant signal for κ, and $x_{n-1} \neq 0$, it follows from (24) that

$$x_{n+1} = \frac{x_n^2 - \kappa}{x_{n-1}} \tag{16}$$

For each real κ, let the function $g_\kappa : \mathbf{R}^{2-} \to \mathbf{R}^2$ be given by

$$g_\kappa(a, b) = \left(b, \frac{b^2 - \kappa}{a}\right) \tag{17}$$

Notice that, for non-negative κ, g_κ sends the lines $b = \pm\sqrt{\kappa}$ of the a-b plane, to the points $(\pm\sqrt{\kappa}, 0)$, respectively.

For κ positive, the pre-constant segments taking values from the set $\{0, \sqrt{\kappa}, -\sqrt{\kappa}\}$ are said to be *trinary* segments; they are *undeterminable* in the sense that, even though they obey the graph in phase space shown in Fig. 14 (in Fig. 15, a signal that obeys this graph is shown), there are several ways to *continue* a given trinary segment on the right and on the left. On the other hand, the pre-constant segments governed by (16) are said to be *determinate*. Between a determinate segment on the left and a trinary segment on the right, there is the *transition segment* $[\pm\sqrt{\kappa}, 0]$ and, between a trinary segment on the left and a determinate segment on the right there is the transition segment $[0, \pm\sqrt{\kappa}]$. Between two determinate segments, corresponding to different conics (see below) there is the transition segment $[\pm\sqrt{\kappa}, 0, \pm\sqrt{\kappa}]$. After a 0, a $\pm\sqrt{\kappa}$ must come; after a segment $[0, \pm\sqrt{\kappa}]$, any nonzero element can be appended and then, with two nonzero consecutive points, (16) can be used to continue the signal.

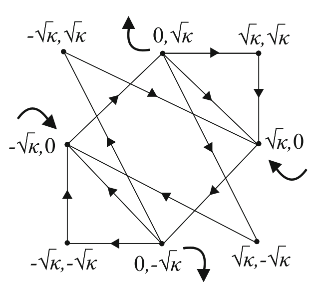


Fig. 14 Graph law for trinary pre-constant signals. The *open-ended arrows* indicate changes from trinary to other determinate pre-constant signals

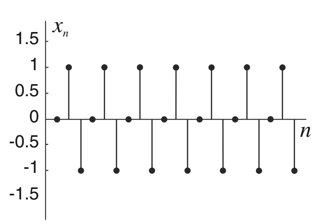


Fig. 15 A periodic trinary pre-constant signal

5.1 A family of conics and a linear equation

As with the roots of the operator, we classify the pre-constant signals in phase space with the help of a family of conics and, in time domain, with the help of a linear equation.

For each real β, and for each real κ, let

$$K_{\beta\kappa}(a, b) = a^2 + b^2 + \beta ab - \kappa \tag{18}$$

and call the set of points (a, b) of the plane for which $K_{\beta\kappa}(a, b) = 0$, a *$\beta$-$\kappa$-conic*.

For $\kappa < 0$, in a real setting, it follows that $|\beta| > 2$; this can be seen as follows. Using polar coordinates, writing $a = r\cos\phi$ and $b = r\sin\phi$, the equation of the conic is $r^2\left(1 + \frac{\beta}{2}\sin(2\phi)\right) = \kappa < 0$ therefore $\beta > \frac{-2}{\sin(2\phi)}$, if $\phi \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$, and $\beta < \frac{-2}{\sin(2\phi)}$, if $\phi \in \left(0, \frac{\pi}{2}\right) \cup \left(\pi, \frac{3\pi}{2}\right)$, and the conics can only be hyperbolas that do not intersect the axes, with sheets in the first and third quadrants ($\beta < -2$), or in the second and fourth quadrants ($\beta > 2$).

For $\kappa \geq 0$, any real value of β is possible; for $|\beta| < 2$, the conics are ellipses ($\beta = 0$ corresponding to a circumference) while for $|\beta| > 2$ the conics are hyperbolas. For $\beta = \pm 2$ the (degenerate) conic consists of the union of the lines: $y = \pm 1 - x$ and $y = \pm 1 + x$, respectively.

For each $\kappa > 0$, each β-κ-conic contains the four point set $\left\{(0, \sqrt{\kappa}), (0, -\sqrt{\kappa}), (\sqrt{\kappa}, 0), (-\sqrt{\kappa}, 0)\right\}$, which we call the finite conic (for pre-constant signals) for the constant κ; for $\kappa = 0$, each β-0-conic contains the point $(0, 0)$. The remaining points of the axes belong to no conic.

Off the axes, for each κ, each point (a, b) of the plane belongs to one and only one conic: that with β given by

$$\beta = \frac{\kappa}{ab} - \frac{a}{b} - \frac{b}{a} \tag{19}$$

If the orbit of a pre-constant segment arrives to the finite conic, at one of the points, $(\pm\sqrt{\kappa}, 0)$, then, after leaving the finite conic, it may reenter a new conic at one of the points $(0, \pm\sqrt{\kappa})$ of the phase plane; see Fig. 14. We indicate in Fig. 16b, the different regions that the conic live in (see also Fig. 17 where two determinate preconstant signals are indicated), for different values of κ and β.

Theorem 4 (Characterization in phase space) *A signal* $\mathbf{x}$ *is a pre-constant signal for the constant* κ *if and only if, for each* n*, one of the following conditions holds*

(i) *Both* (x_{n-1}, x_n) *and* (x_n, x_{n+1}) *appear as nodes in the graph in* Fig. 14 *and are connected by an arrow.*
(ii) *Neither* (x_{n-1}, x_n) *nor* (x_n, x_{n+1}) *are one of the nodes in* Fig. 14*, and they meet* 16.
(iii) *Either* $(x_{n-1}, x_n) = (0, \sqrt{\kappa})$ *or* $(x_{n-1}, x_n) = (0, -\sqrt{\kappa})$ *and* x_{n+1} *is any number, or,* $(x_n, x_{n+1}) = (\sqrt{\kappa}, 0)$ *or* $(x_n, x_{n+1}) = (-\sqrt{\kappa}, 0)$ *and* x_{n-1} *is any number.*

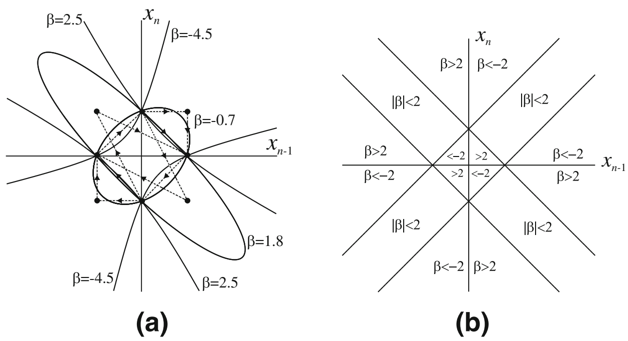


Fig. 16 **a** β-κ-conics,the finite conic is included here for reference and scale; compare with Fig. 6. **b** The regions $\beta-\kappa$-conics live on, for $\kappa > 0$

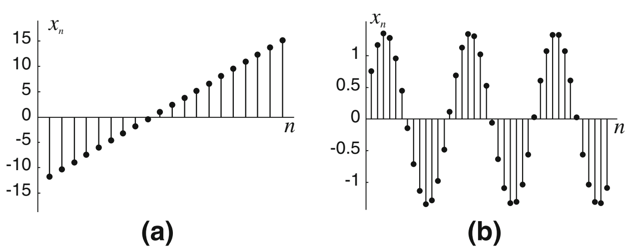


Fig. 17 Determinate pre-constant signals. **a** Linear signal and **b** an almost periodic, pre-constant signal

Regarding necessity, it is clear that if any of the conditions (i)-(iii) is met, the output of the operators has value x_n. As for sufficiency,

(i) It follows from Conditions (15a) and (15b).

(ii) Let $(a, b) = (x_{n-1}, x_n)$ be as indicated. Then, since x is pre-constant for the constant κ and $a \neq 0$, then, $(x_n, x_{n+1}) = g(x_{n-1}, x_n)$; assume $K_{\beta\kappa}(a, b) = 0$; now,

$$\begin{aligned} &K_{\beta\kappa}\left(g_\kappa\left(a, b\right)\right) \\ &\quad= \left(1/a^2\right)\left[\left(b^2-\kappa\right)\left(a^2+b^2+\beta ab-\kappa\right)\right] \\ &\quad= \left(1/a^2\right)\left[\left(b^2-\kappa\right)K_{\beta\kappa}\left(a, b\right)\right] \\ &\quad= 0 \end{aligned}$$

Thus, $K_{\beta\kappa}\left(a, b\right)$ implies $K_{\beta\kappa}\left(g_\kappa\left(a, b\right)\right) = 0$, and the conic is g_κ-invariant.

(iii) It follows directly from the definitions. □

Thus, each pre-constant signal for a constant κ can be segmented into segments that live on a unique conic and we have

Theorem 5 (Characterization in the time domain) *For each real $\kappa < 0$, each pre-constant signal for the constant κ consists of a unique determinate segment that lives on a unique $\beta-\kappa$-conic, with $|\beta| > 2$. For each real $\kappa > 0$, each pre-constant signal for the constant κ is a concatenation $\{\ldots S_{\beta_i},\ S_{A_j},\ S_{\beta_{i+1}},\ S_{A_{j+1}},\ \ldots\}$ of determinate blocks S_{β_i}, that live on a unique $\beta-\kappa$-conic, and of undeterminable blocks S_{A_j} that in phase space live on the graph in Fig. 14, begin with $[\pm\sqrt{\kappa}, 0]$ and end with $[0, \pm\sqrt{\kappa}]$.*

If a pre-constant segment takes the value 0, κ must be nonnegative; therefore pre-constant signals for a negative constant consist of a unique segment and live on a unique conic since, in phase space, they cannot reach a crossroads point. In the time domain (see Lemma 4 below) for $\kappa < 0$, pre-constant signals are unbounded.

As in the case of the root signals, for each κ, determinate pre-constant signals obey a linear equation. The situation is analogous to that of Theorem 1.

Lemma 4 (Linear Equation) *Each determinate segment of a pre-constant signal is also a segment of a solution of the linear equation*

$$x_{n+1} + \beta x_n + x_{n-1} = 0 \tag{20}$$

Proof Let κ be given and $\mathbf{x}$ be pre-constant for κ. Given two consecutive phase-space points (x_{n-1}, x_n) and $(x_n, x_{n+1}) = g_\kappa(x_{n-1}, x_n)$, not on the axes of the phase plane, from (19)

$$\beta = \frac{\kappa}{x_{n-1}x_n} - \frac{x_n}{x_{n-1}} - \frac{x_{n-1}}{x_n}$$

thus

$$\beta x_n = \left(\frac{\kappa}{x_{n-1}} - \frac{x_n^2}{x_{n-1}}\right) - x_{n-1}$$

or

$$x_{n+1} + \beta x_n + x_{n-1} = 0 \tag{21}$$

□

Equation (20) is the homogeneous version of (10); it has characteristic equation $1 + \beta\gamma + \gamma^2 = 0$ with roots

$$\gamma_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4}}{2}$$

(note that $\gamma_1\gamma_2 = 1$, hence $\beta = \gamma + 1/\gamma$ where $\gamma = \gamma_1 = \gamma_2^{-1}$). For $\beta \neq \pm 2$, it has solution

$$x_n = c_1\gamma^n + c_2\gamma^{-n}$$

The corresponding value of κ is $c_1c_2(2 - \gamma^2 - 1/\gamma^2)$

For $\beta = -2$, it has solution

$$x_n = c_1(1)^n + c_2^n(1)^n = c_1 + c_2 n,$$

With corresponding value $\kappa = c_2^2$.

For $\beta = 2$, it has solution

$$x_n = c_1(-1)^n + c_2 n(-1)^n$$

With corresponding value also $\kappa = c_2^2$.

For $|\beta| < 2$, γ is of the form $e^{j\omega}$ and, for $c_2 = c_1^*$, $\mathbf{x}$ is a sinusoid; in this case, κ must be positive. The frequency of the sinusoid is given by

$$\omega = \pi - \arctan\sqrt{\left(\frac{2}{\beta}\right)^2 - 1} \tag{22}$$

with the convention that for $\beta = 0$, $\omega = \pi/2$. The amplitude of the sinusoid can also be written as

$$A = \frac{\sqrt{\kappa}}{\sqrt{1-(\beta/2)^2}} \tag{23}$$

One also has $A = 2\,|c|$ and $\beta^2 = 4\cos^2\omega$.

Unlike the case of root signals, there are 2-periodic pre-constant signals; they are pre-null as shown in Lemma 5 below.

For $\beta \neq -2$, with the transformation $a' = a - t, b' = b - t$, and with $t = \kappa = \frac{1}{2+\beta}$, the conic $K_{\beta\kappa}(a', b') = 0$, becomes the conic $C_\beta(a, b) = 0$; likewise, for $\beta \neq -2$, with $t = \frac{1}{2+\beta} = \kappa$, the conic $C_\alpha(a, b) = 0$ is the conic $K_{\alpha\kappa}(a', b') = 0$. Therefore, for $\alpha = \beta \neq -2$, $C_\alpha(a, b) = 0$ and $K_{\beta\kappa}(a', b') = 0$ are the same type of conic, only displaced. However (for $\alpha = \beta = -2$) $C_{-2}(a, b) = 0$ is a parabola, while $K_{\{-2\kappa}(a', b') = 0$ is the union of two straight lines.

The same change of variables transforms (10) into (20): for $\alpha = \beta \neq -2$, in $x_{n+1} + \alpha x_n + x_{n-1} = 1$, let $x_n = x'_n + 1/(2+\alpha)$, then $x'_{n+1} + \beta x'_n + x'_{n-1} = 0$. The Equations (10) and (20) differ in that one is the homogeneous version of the other; therefore, their solutions differ in the particular-solution term. In particular, for $\alpha \neq -2$, the solutions of (10) and (20), differ in the constant term $\frac{1}{\alpha+2}$ and, for $\alpha = -2$, to the root signal $(n + c_2)(n + c_2 + 1)/2$ there corresponds the pre-constant signal $n(2c_2 + 1)/2 + c_2$.

Thus we have

Theorem 6 (Relationship between root and pre-constant segments.) *To each determinate root segment S with $\alpha \neq -2$, there corresponds the determinate pre-constant segment $S - \left\{\frac{1}{\alpha+2}, \dots \frac{1}{\alpha+2}\right\}$, for the constant $\frac{1}{2+\alpha}$. For $\alpha = -2$, to a root segment of the form $(n + c_2)(n + c_2 + 1)/2$ there corresponds the pre-constant segment $n(2c_2+1)/2+c_2$, for the constant c_2^2.*

For example, for $\alpha = 1$, to the root signal $\{(2/3)\cos[(2\pi/3)n] + 1/3\} = \{\dots 1, 0, 0, 1, 0, 0 \dots\}$ there corresponds the pre-constant signal $\{(2/3)\cos[(2\pi/3)n]\}$, for the constant $1/3$. For $\alpha = 0$, to the root $\{\dots 1, 1, 0, 0, 1, 1, 0, 0, \dots\}$ there corresponds the pre-constant signal $\{\dots 0.5, 0.5, -0.5, -0.5, 0.5, 0.5, -0.5, -0.5, \dots\}$, for the constant $1/2$. For $\alpha = 2$, to the root $\{(-1)^n(3/4+n/2)+1/4\}$ there corresponds the pre-constant $\{(-1)^n(3/4 + n/2)\}$ for the constant $1/4$.

Not every solution of (10) is a root segment but each solution of (20) is a pre-constant segment, for some constant κ. Each determinate root segment becomes a determinate pre-constant segment with the addition of an appropriate constant or quadratic signal, but not viceversa.

Lemma 5 *If a pre-constant segment is 2-periodic then it is a pre-null segment.*

Proof A signal of period 2 has the form $\{\dots c, d, c, d, \dots\}$ and, if a pre-constant, then for some $\kappa, \kappa = c^2 - d^2$ and $\kappa = d^2 - c^2$ then $\kappa = -\kappa$ then $\kappa = 0$. Then it is pre-null. □

Several other aspects of the pre-null signals, are better treated separately.

5.2 Pre-null signals

The pre-null signals of the TK operator are the signals $\mathbf{x}$ that meet the equation

$$\mathbf{x}^2 = ({}_{+1}\mathbf{x})({}_{-1}\mathbf{x}) \tag{24}$$

Thus, if $\mathbf{x}$ is pre-null, $x_n = 0 \Rightarrow x_{n-1} = 0, \wedge, x_{n+1} = 0$ and we have that if a pre-null signal is zero valued anywhere, it is the constant null signal. Thus, if $\mathbf{x} \neq \theta$ (θ is the null signal) is pre-null, from (16),

$$x_{n+1} = \frac{x_n^2}{x_{n-1}} \tag{25}$$

from which we get that non-null pre-null signals are of the form $\{\dots, a^2/b,\ a,\ b,\ b^2/a\ , \dots\}$, which in turn are of the forms $\{\dots,\ ar^{-2},\ ar^{-1},\ br^{-1}, ar,\ br,\ br^2,\ \dots\}$ or $\{\dots\ c, cr,\ cr^2,\ cr^3,\ cr^4,\ \dots\}$ with $r = b/a$ and $c = a^2/b$. Such signals obey the linear equation $x_n = r x_{n-1}$ and have the general form $x_n = Cr^n$. See e.g. Fig. 18.

In phase space, they live on rays and lines through the origin; from (17)

$$g_0(a, b) = \left(b, \frac{b^2}{a}\right) \tag{26}$$

For each real β, as κ tends to 0, the curves $\mathbf{K}_{\beta,\kappa}(a, b) = 0$ approximate the β-0-conic $a^2 + b^2 + \beta ab = 0$. See Fig. 19. Off the axes, each point (a, b) of the phase plane belongs to one and only one β-0-conic, that with β given by

$$\beta = -a/b - b/a = -(r + 1/r)$$

For $\beta = 0$, $\mathrm{K}_{0,0}(a, b) = a^2 + b^2 = 0$ is a point, corresponding to the null signal; for $\beta = 0$ there are no other

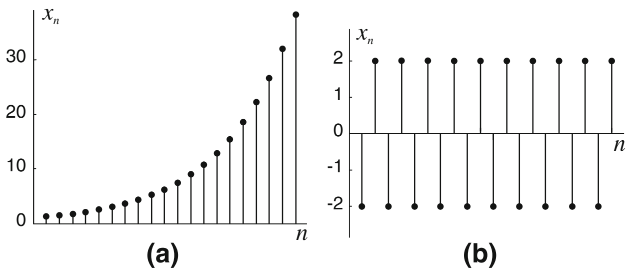


Fig. 18 Pre-null signals

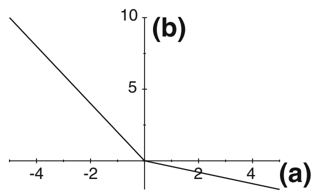


Fig. 19 Two β-0-conics

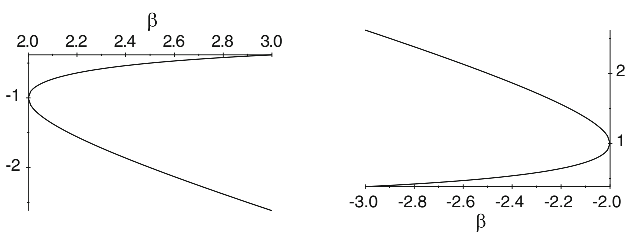


Fig. 20 Slopes of the rays versus β between 2 and 3 and -3 and -2

real pre-null signals. For $\beta = \pm 2$, we get $b = \mp a$. More generally, we have

$$b = -a\frac{\beta}{2} \pm |a|\sqrt{\left(\frac{\beta}{2}\right)^2 - 1}$$

which correspond to rays with slopes $-\beta/2 \pm \sqrt{(\beta/2)^2 - 1}$; see Figs. 19 and 20.

6 An application

We check whether a given (discrete) system is oscillating in sinusoidal regime, by feeding its output signal to the algorithm depicted in Fig. 21.

Whenever the signal is locally a sinusoid with no DC level and with at least two consecutive nonzero values, the system turns ON a flag SINUSOID in which case the values at the outputs AMPLITUDE and FREQUENCY can be read. Initially, the signal is fed to the TK operator; if the output of the operator does not change, the input is locally a pre-constant segment; after estimating β with the help of (19), it is decided whether the input signal is a sinusoid; in such a case, its amplitude and frequency are estimated with the help of (23) and (22), respectively.

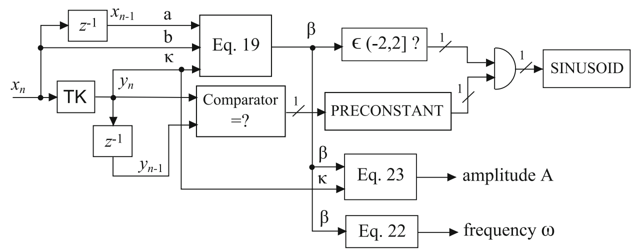


Fig. 21 The system shown checks whether **x** is locally a sinusoid, in which case, gives its amplitude and its frequency

Algorithm
input: **x**, n
outputs: **y**, A, ω (possibly),
flags: *PRE-CONSTANT*, *SINUSOID* Apply the signal **x** to the TK operator; as the window moves along the input signal,

(i) Check if the output of the operator remains unchanged: $y_n = y_{n-1}$; if so, turn ON the flag *PRE-CONSTANT*, and let $\kappa \leftarrow y_n$; otherwise, turn OFF the flag *PRE-CONSTANT*.

(ii) If *PRE-CONSTANT*, with κ assigned as above and with the two corresponding consecutive input values $x_n = b$ and $x_{n-1} = a$, using (19) (so long as none of the input values is zero), compute β; if $-2 < \beta \leq 2$,turn ON the flag *SINUSOID*; otherwise, turn OFF the flag *SINUSOID*.

(iii) If *SINUSOID*, compute A and ω using (23) and (22)

end of algorithm

The minimal length of an output segment for which it makes sense to state that the output is locally constant is 2, in which case input segment has minimal length of 4. For the computation of β we may however need a longer input segment, long enough to contain two nonzero consecutive values; this is possible except for the 4-periodic signal $\{\ldots\ 0,\ a,\ 0\ , -a\ , 0\ , \ldots\}$.

7 Conclusion

The TK operator, advanced about 20 years ago, gives a simple yet nontrivial case of a polynomial filter for which several interesting applications have been found. It is a quadratic filter, known mainly for being an energy estimator and a component of a contrast enhancer.

We have characterized the operator algebraically by giving expressions of the outputs corresponding to several types of *combination* (e.g. linear, algebraic) of input signals, we have provided as well a characterization of the operator in the time-frequency domain and we have classified the root and pre-constant signals of the operator. Finally, we have given a new application of the operator based on the classification of the pre-constant signals.

The TK operator was proposed as a discrete version of a formula for the energy stored in a second-order, mechanical, oscillating, mass-spring system [7]; the energy in the oscillator increasing not only with the amplitude of the oscillation but also with its frequency. Even though the instantaneous energy of chirp signals could be estimated with the operator as well, in principle, the operator was meant to be applied to *monochromatic* (i.e. of a single Fourier frequency) continuous signals, sampled at least at a rate four times the Nyquist rate. Nevertheless, the operator is often applied on signals arising from as yet not well-modeled systems e.g. [1,3,4,18] and, of course, the operator can be applied to discrete signals not necessarily resulting from the sampling of continuous signals.

Among the tools used in the analysis of the operator, we have the fact that solutions of certain nonlinear difference equations are also solutions of linear equations; this may be further explored in the theory of nonlinear difference equations. Several other paths for further research remain open; there remains for example, to give statistical characterizations of the operator, such as finding the distribution of the output for white input noise, briefly considered in [11] and cases of input signals with additive deterministic and stochastic components. Likewise, we are currently studying the root and pre-constant signals of the 2D version of the operator [10].

A class of signals broader than that of the root signals of an operator is that of its *limit cycles*; the study of the limit cycles of quadratic filters is surely important.

A quadratic filter that estimates local energy contents in a simple sense is the sliding-window filter with local function $f(x, y, z) = x^2 + y^2 + z^2$. Two other simple but nontrivial quadratic filters, related to the TK operator that were briefly considered in [11], are the sliding-window filters with local functions $f(x, y, z) = y^2$ and $f(x, y, z) = xz$.

The classification of the limit-cycle, pre-constant and other types of signal, for the class of quadratic filters may well provide parameters for their design as well as applications for them. We think the class of the quadratic filters to be a good starting point for further analysis finite-order Volterra filters. The finite-order Volterra filters provide a class of analytically tractable nonlinear filters with yet-unknown applications. Filters with given types of root or pre-constant signals could be searched for; such requirements could be part of a set of constraints for the design of nonlinear filters.

Acknowledgments We thank the reviewers of the original manuscript for their many corrections and suggestions.

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