

# Matlab and RREF

Math 45 — Linear Algebra

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## Abstract

In this exercise you will learn how to use Matlab's `rref` command to place matrices in *reduced row echelon form*. A number of applications are also discussed. *Prerequisites: Some familiarity with pencil and paper calculations required to place a matrix in reduced row echelon form.*

## 1 Introduction

This is an interactive document designed for online viewing. We've constructed this onscreen documents because we want to make a conscientious effort to cut down on the amount of paper wasted at the College. Consequently, printing of the onscreen document has been purposefully disabled. However, if you are extremely uncomfortable viewing documents onscreen, we have provided a print version. If you click on the Print Doc button, you will be transferred to the print version of the document, which you can print from your browser or the Acrobat Reader. We respectfully request that you only use this feature when you are at home. Help us to cut down on paper use at the College.

Much effort has been put into the design of the onscreen version so that you can comfortably navigate through the document. Most of the navigation tools are evident, but one particular feature warrants a bit of explanation. The section and subsection headings in the onscreen and print documents are interactive. If you click on any section or subsection header in the onscreen document, you will be transferred to an identical location in the print version of the document. If you are in the print version, you can make a return journey to the onscreen document by clicking on any section or subsection header in the print document.

Finally, the table of contents is also interactive. Clicking on an entry in the table of contents takes you directly to that section or subsection in the document.

### 1.1 Working with Matlab

This document is a working document. It is expected that you are sitting in front of a computer terminal where the Matlab software is installed. You are not supposed to read this document as if it were a short story. Rather, each time you are presented with a Matlab command, it is expected that you will enter the command, then hit the Enter key to execute the command and view the result. Furthermore, it is expected that you will ponder the result. Make sure that you completely understand why you got the result you did before you continue with the reading.

## 2 Solving Systems With Matlab

By now, you have probably spent quite some time solving systems in your linear algebra class. By now you realize that the pencil and paper calculations required to solve a system of linear equations is both time consuming and prone to error. It is time to let Matlab assist in these calculations.

First, let's recall some fundamental definitions. The first nonzero element in each row of a matrix is called a **pivot**. A matrix is said to be in *row echelon form* if it satisfies the following properties:

- Rows of all zeros appear at the bottom of the matrix.
- Each pivot is a 1.
- Each pivot occurs in a column that is strictly to the right of the pivot above it.

A matrix is said to be in *reduced row echelon form* if it satisfies one additional property.

- Each pivot is the only nonzero entry in its column.

When solving a linear system of equations, such as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n, \end{aligned} \tag{1}$$

one of three things can happen. The system can have

- a unique solution,
- no solution (the system is said to be *inconsistent*), or
- an infinite number of solutions.

We will investigate each case in this activity.

### 2.1 Unique Solution

Consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 6, \\ x_1 - 2x_3 &= 4, \\ x_2 + x_3 &= 2. \end{aligned} \tag{2}$$

The augmented matrix for this system is

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & 2 \end{pmatrix} \tag{3}$$

This augmented matrix is easily entered in Matlab's workspace.

```
>> A=[1 1 1 6;1 0 -2 4;0 1 1 2]
A =
     1     1     1     6
     1     0    -2     4
     0     1     1     2
```

Matlab's `rref` command will now be used to place matrix A in reduced row echelon form. That's right! The letters in the Matlab command `rref` stand for "reduced row echelon form."

```
>> R=rref(A)
R =
     1     0     0     4
     0     1     0     2
     0     0     1     0
```

Thus, the reduced row echelon form of the augmented **matrix 3** is

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (4)$$

Of course, this matrix represents the equivalent<sup>1</sup> system

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 2 \\ x_3 &= 0 \end{aligned} \quad (5)$$

Therefore, **system 2** has the *unique* solution, (4, 2, 0).

It is interesting to view the geometry<sup>2</sup> in this case. Each of the equations in **system 2** represents a plane in 3-space. As you can see in **Figure 1**, the three equations of **system 2** lead to three planes. Further, note that the intersection of the three planes in **Figure 1** is a single point, agreeing with our analysis.

## 2.2 No Solutions

Consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 6, \\ x_1 - 2x_3 &= 4, \\ 2x_1 - x_3 &= 8. \end{aligned} \quad (6)$$

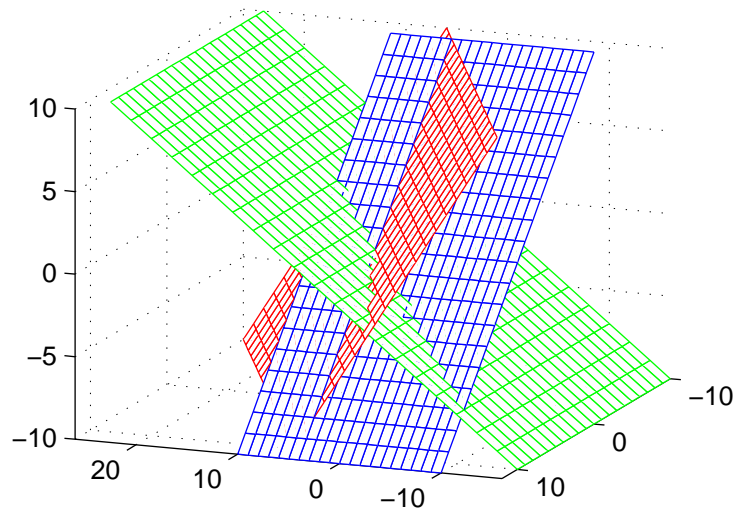
The augmented matrix for this system is

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 0 & -2 & 4 \\ 2 & 0 & -1 & 8 \end{pmatrix} \quad (7)$$

This augmented matrix is easily entered in Matlab's workspace.

<sup>1</sup> Recall that two systems are equivalent if and only if they have the same solution sets.

<sup>2</sup> **Figure 1**, **Figure 2**, and **Figure 3** were produced with a Matlab program called `sys34`. It is available for download in the same locations as this article.



**Figure 1** A system with a unique solution. Three planes meet in a single point.

```
>> A=[1 1 1 -6;1 0 -2 4;2 1 -1 18]
A =
     1     1     1    -6
     1     0    -2     4
     2     1    -1    18
```

Matlab's `rref` command will now be used to place matrix A in reduced row echelon form.

```
>> R=rref(A)
R =
     1     0    -2     0
     0     1     3     0
     0     0     0     1
```

Thus, the reduced row echelon form of the augmented **matrix 7** is

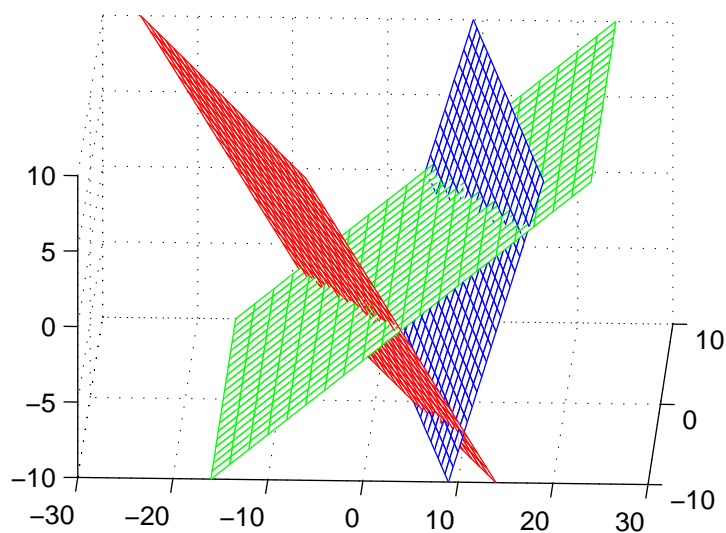
$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (8)$$

We need only concentrate on the last row of **matrix 8**, as it represents the equation

$$0x_1 + 0x_2 + 0x_3 = 1. \quad (9)$$

Clearly, **equation 9** has no solution. Therefore, **system 6** has no solution. We say that **system 6** is *inconsistent*.

Again, the geometry provides insight. Each of the equations in **system 6** represents a plane in 3-space. As you can see in **Figure 2**, any two planes intersect in a line, but that line runs parallel to the third plane. Thus, there are no points in common to all three planes, which agrees with our analysis.



**Figure 2** Any two planes intersect in a line that runs parallel to the third plane. Thus, there are no points of intersection.

### 2.3 An Infinite Number of Solutions

For our final example, consider the system

$$\begin{aligned}x_1 + x_2 + x_3 &= 6, \\x_1 - 2x_3 &= 4, \\2x_1 + x_1 - x_3 &= 10.\end{aligned}\tag{10}$$

The augmented matrix for this system is

$$\left(\begin{array}{cccc}1 & 1 & 1 & 6 \\1 & 0 & -2 & 4 \\2 & 1 & -1 & 10\end{array}\right)\tag{11}$$

This augmented matrix is easily entered in Matlab's workspace.

```
>> A=[1 1 1 6;1 0 -2 4;2 1 -1 10]
A =
     1     1     1     6
     1     0    -2     4
     2     1    -1    10
```

Matlab's `rref` command will now be used to place matrix A in reduced row echelon form.

```
>> R=rref(A)
R =
     1     0    -2     4
     0     1     3     2
     0     0     0     0
```

Thus, the reduced row echelon form of the augmented **matrix 11** is

$$\begin{pmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

Note that we have a row of all zeros at the bottom of our matrix, Furthermore, note that we only have two pivots. It is essential, at this point, that we identify the *pivot variables* and the *free variables*. Note that columns one and two have pivots. Therefore,  $x_1$  and  $x_2$  are pivot variables. Column three has no pivot. Therefore,  $x_3$  is a free variable.

Because the last row of the matrix represents the equation

$$0x_1 + 0x_2 + 0x_3 = 0, \quad (13)$$

which is satisfied by any values of  $x_1$ ,  $x_2$  and  $x_3$ , we need only find the values of  $x_1$ ,  $x_2$  and  $x_3$  that satisfy the equations represented by the first two rows of **matrix 12**.

$$\begin{aligned} x_1 - 2x_3 &= 4 \\ x_2 + 3x_3 &= 2 \end{aligned} \quad (14)$$

At this point, the rule is simple and direct. Solve each equation for its pivot variable in terms of the free variable. Consequently,

$$\begin{aligned} x_1 &= 4 + 2x_3, \\ x_2 &= 2 - 3x_3. \end{aligned} \quad (15)$$

It is common to pick a favorite letter to represent the free variable. In this case, let's let  $x_3 = \alpha$ . Thus, **system 10** has an infinite number of solutions, described by

$$\begin{aligned} x_1 &= 4 + 2\alpha, \\ x_2 &= 2 - 3\alpha, \\ x_3 &= \alpha, \end{aligned} \quad (16)$$

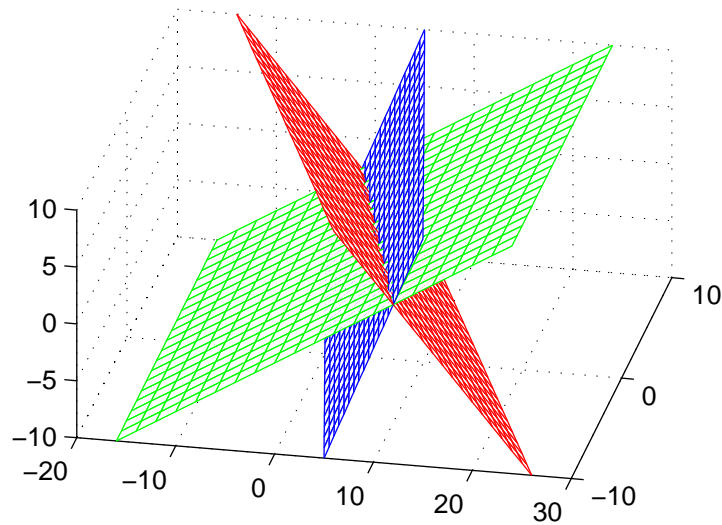
where  $\alpha$  is any real number. Each time you select another real number for  $\alpha$ , you get a new solution. For example, if  $\alpha = 0$ , then you get the point  $(4, 2, 0)$ . If  $\alpha = 1$ , you get the point  $(6, -1, 1)$ , and so on.

Again, the geometry provides insight. Each of the equations in **system 10** represents a plane in 3-space. As you can see in **Figure 3**, the three planes intersect and form a line. Therefore, there are an infinite number of points of intersection, which agrees with our analysis.

### 3 A Harder Example

Students tend to panic when the number of equations and unknowns are increased. Of course, this increase makes for a more difficult problem. However, if you follow a couple of simple rules, you will lower your frustration level considerably.

- Identify the pivot variables. These are easily identified by noting the columns that have pivots.
- Identify the free variables. These are easily identified by noting the columns that do not have pivots.
- Solve each equation for its pivot variable in terms of the free variables.
- Make your answer look nice by choosing some of your favorite letters for the free variables.



**Figure 3** All three planes meet in a line, which contains an infinite number of points.

To clarify this process, consider the system

$$\begin{aligned} -4x_1 - 2x_2 + 2x_4 - 4x_5 + 4x_6 &= 2, \\ 4x_1 + x_2 - 3x_4 + 4x_5 - 4x_6 &= -3, \\ x_1 - 2x_2 - 3x_4 + x_5 - x_6 &= -3, \\ -2x_2 - 2x_4 &= -2. \end{aligned} \tag{17}$$

As you can see, this problem is enough to make anyone's jaw drop. Don't panic! If you follow the rules, you will have no problem finding the solution. First, set up the augmented matrix.

$$\left( \begin{array}{cccccc|c} -4 & -2 & 0 & 2 & -4 & 4 & 2 \\ 4 & 1 & 0 & -3 & 4 & -4 & -3 \\ 1 & -2 & 0 & -3 & 1 & -1 & -3 \\ 0 & -2 & 0 & -2 & 0 & 0 & -2 \end{array} \right) \tag{18}$$

Enter the augmented matrix.

```
>> A=[-4 -2 0 2 -4 4 2;4 1 0 -3 4 -4 -3;1 -2 0 -3 1 -1 -3;0 -2 0 -2 0 0 -2]
A =
    -4    -2     0     2    -4     4     2
     4     1     0    -3     4    -4    -3
     1    -2     0    -3     1    -1    -3
     0    -2     0    -2     0     0    -2
```

Use the `rref` command to place this result in reduced row echelon form.

```
>> R=rref(A)
R =
     1     0     0    -1     1    -1    -1
     0     1     0     1     0     0     1
```

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Columns one and two have pivots. Therefore,  $x_1$  and  $x_2$  are pivot variables. The remaining unknowns,  $x_3$ ,  $x_4$ ,  $x_5$ , and  $x_6$ , are free variables.

The last two rows of all zeros can be ignored, because these equations are satisfied by all values of the unknowns. Thus, we need only solve

$$\begin{aligned} x_1 - x_4 + x_5 - x_6 &= -1, \\ x_2 + x_4 &= 1. \end{aligned} \tag{19}$$

Solve each equation for its pivot variable.

$$\begin{aligned} x_1 &= -1 + x_4 - x_5 + x_6 \\ x_2 &= 1 - x_4. \end{aligned} \tag{20}$$

Pick favorite letters for the free variables. We will use  $x_3 = \alpha$ ,  $x_4 = \beta$ ,  $x_5 = \gamma$ , and  $x_6 = \delta$ . Thus,

$$\begin{aligned} x_1 &= -1 + \beta - \gamma + \delta, \\ x_2 &= 1 - \beta, \\ x_3 &= \alpha, \\ x_4 &= \beta, \\ x_5 &= \gamma, \\ x_6 &= \delta, \end{aligned} \tag{21}$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ , and  $\delta$  are any real numbers. Thus, **system 17** has an infinite number of solutions. Samples of this solution set can be found by choosing arbitrary values for the free variables.

As you can see, when the number of unknowns and equations rise, the problems become more difficult. Hopefully, you also noticed that if you follow a few simple rules, the size of the problem should never be a problem.

## 4 Homework

Each of the matrices in the following exercises represents the augmented matrix of a system of linear equations. Perform each of the following tasks for each exercise.

- Write the augmented matrix on notebook paper.
- Enter the matrix and use Matlab's `rref` command to place the augmented matrix in reduced row echelon form. Copy the reduced row echelon form of the matrix on your notebook paper.
- Identify pivot and free variables.
- Solve each equation for its pivot variable.
- Use your favorite letters for the free variables.

1.



$$\begin{pmatrix} -2 & -3 & -1 & 2 & -1 \\ 0 & -2 & -2 & 0 & -2 \\ -1 & -3 & -2 & 1 & -2 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

2.

$$\begin{pmatrix} -1 & -1 & 1 & 4 & 0 & 2 & 2 \\ -1 & -1 & 0 & 1 & -1 & 3 & 3 \\ -1 & -1 & 1 & 5 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

3.

$$\begin{pmatrix} -1 & 3 & 3 & 3 \\ 2 & -5 & -1 & -5 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix}$$

4.

$$\begin{pmatrix} 3 & -1 & 0 & -1 & -3 & -1 & -2 & -3 \\ -2 & 0 & 0 & 0 & 2 & 0 & 2 & 2 \\ 3 & 0 & 0 & -1 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 2 & -2 & -2 \\ 3 & 1 & 0 & 0 & -1 & -1 & -2 & -1 \\ 1 & -4 & 0 & -2 & -5 & 0 & -1 & -5 \end{pmatrix}$$

5.

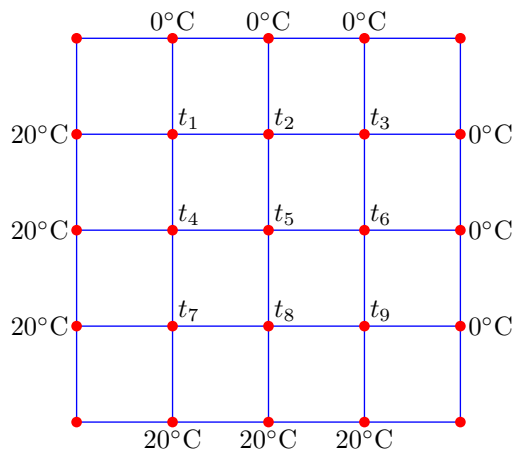
$$\begin{pmatrix} -1 & 3 & -2 & 1 & 1 & -1 & -2 \\ 3 & 3 & 0 & 3 & 3 & 3 & 0 \\ 1 & 3 & -1 & 2 & 2 & 1 & -1 \end{pmatrix}$$

6.

$$\begin{pmatrix} -2 & -2 & 2 & -1 & 1 & -2 & -1 & -1 & 0 \\ -1 & -2 & 2 & 1 & 3 & -1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3 & -2 & -1 & -1 & 0 \\ 1 & 0 & 0 & 2 & 2 & 1 & -1 & 0 & 0 \\ -2 & 1 & 0 & -1 & -2 & -1 & 0 & -1 & -2 \\ 0 & 1 & -2 & -1 & -4 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & -2 & -1 & -2 \\ -2 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & -1 \end{pmatrix}$$

7. Janelle has \$2.81 in coins, all in pennies, nickels, dimes, and quarters. The number of nickels equals the number of dimes. There are 24 coins in all. In how many ways is this possible? List all possibilities. *Note: This is not guess and check. You are expected to set up a system of equations and solve, much as you've done with the other problems in this activity.*

8. In **Figure 4** is pictured a steel plate. The temperature at each point in the plate is constant (not changing with time). The temperature at each lattice point at the edge of the plate is given in degrees Celsius.



**Figure 4** Temperature grid on a steel plate.

Let  $t_i$  represent the temperature in degrees Celsius at each lattice point in the interior of the plate. Assume that the temperature at each interior lattice point is the average of the temperatures of its four nearest neighbors. Find the temperature  $t_i$  at each interior lattice point.

9. Use `sys34` to draw the row-pictures for systems of three equations in three unknowns that
  - a. have a unique solution because the three planes intersect in a unique point.
  - b. have an infinite number of solutions because the three planes intersect in a line.
  - c. have no solutions because two of the planes are parallel and the third plane is not parallel to either of the first two.
  - d. have no solutions, yet no two of the planes are parallel.