LAB #5

POPULATION MODELS

Goal: Compare various population models for the population of New York over the last 200 years.

Required tools: MATLAB routines **plot**, **norm**, **fplot**; separable differential equations.

<u>DISCUSSION</u>

This lab compares three models of population growth:

$$(*) \qquad \frac{dP}{dt} = r \quad \textbf{(Linear Growth)}$$

$$(**) \qquad \frac{dP}{dt} = rP \quad \textbf{(Exponential Growth)}$$

$$(***) \quad \frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \quad \textbf{(Logistic Growth)}$$

$P(t)$ is the population at time t (r and K are positive constants).

The following table lists the population of New York State in 10 year intervals from the year 1790 to 1990:

Population of New York (in thousands)

Year	1790	1800	1810	1820	1830	1840	1850	1860	1870	1880
Pop.	**379**	**423**	**472**	**523**	**610**	**738**	**995**	**1231**	**1457**	**1783**

Year	1890	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990
Pop.	**2239**	**2805**	**3366**	**3852**	**4250**	**4317**	**4691**	**5149**	**5689**	**5737**	**6016**

<u>ASSIGNMENT</u>

(1) Use MATLAB to plot the population given in the table as a function of T where T is measured in decades. Thus $T = 0$ corresponds to the year 1790 and $T = 20$ corresponds to 1990.

(i) Data in MATLAB is stored in matrices (in fact MATLAB is short for "MATrix LABoratory"). Enter the time matrix T and the actual population matrix A into MATLAB as follows:

$>>$ T $= [0, 1, 2, \cdots, 20]$;

$>>$ A $= [379, 423, 472, \cdots, 6016]$;

(The semicolons ";" prevent MATLAB from displaying the entries on the screen.) A quick way to produce the matrix T is

$>> T = 0 : 1 : 20$;

(In general $a : c : b$ produces a (row) matrix whose entries start at a and ends at b in increments of c.)

Plot this data as follows:

$>>$ grid on (displays grid lines)

$>>$ hold on (prevents MATLAB from erasing grid lines)

$>>$ plot(T,A,'r') (plots A vertical vs T horizontal, in red : 'r' = red)

To get other options, type "help plot".

(ii) The plot appears to be a smooth curve, but it is not. MATLAB connects the data with straight lines. This may be seen by zooming in on a part of this curve by using the "Zoom In" button $\oplus$. If you want to return to the original figure select the "Zoom Out" button $\ominus$ repeatedly until the original plot is shown. Type "help zoom" for more information on this feature.

(2) Find the solution $P(t)$ to the equation in ($*$) satisfying $P(0) = 379$ and $P(1) = 423$. You should get the formula below. Use this formula to predict the population of New York in the year 2020.

$$P(t) = 44t + 379 \qquad\qquad \text{(A)}$$

(3) Find the solution $P(t)$ to the equation in ($**$) satisfying $P(0) = 379$ and $P(1) = 423$. You should get the formula below. Use this formula to predict the population of New York in the year 2020.

$$P(t) = 379\, e^{0.11t} \qquad\qquad \text{(B)}$$

(4) The solution to ($***$) with $P(0) = P_0$ is

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \qquad \text{(C)}$$

Use these formulas from page 112 of your book to approximate r and K, using $h = 10$:

$$K = \frac{P_0 P_1 (1 - e^{-rh})}{P_0 - P_1 e^{-rh}} \quad \text{and} \quad K = \frac{P_1 P_2 (1 - e^{-rh})}{P_1 - P_2 e^{-rh}} \qquad (1.18)$$

$$r = \frac{1}{h} \ln \left(\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} \right). \qquad (1.19)$$

Now use the formula in (C) to predict the population in the year 2020.

(5) Use **fplot** to plot the functions given by (A) , (B) and (C) on the same graph as the plot of the actual population data in Part (1). As long as "hold on" is still on, that graphs should automatically appear in the same window. Which seems the most accurate (A), (B) or (C) ?
(Recall that $>>$ `fplot('44*T+379',[0,20,0,7000],'g')` will plot the function in (A) in green.)

(6) If $A = [a_1, a_2, \cdots, a_n]$ is a data matrix and $B = [b_1, b_2, \cdots, b_n]$ is an approximation to A, then one measure of the error in this approximation is

$$\epsilon = \frac{|A - B|}{n} = \frac{1}{n}\left(|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|\right)$$

In MATLAB this is computed simply by typing

$>>$ `norm(A − B, 1)/n`

Let $A = [a_1, a_2, \cdots, a_n]$ be the population data given in the table on Page 1 and let $B = [b_1, b_2, \cdots, b_n]$ be the approximation to the population data at times $t = 0, 1, 2, \cdots, 20$ obtained from formulas (A), (B), (C). Compute ϵ for each of these approximations. Do the values of ϵ support your empirical observations in (5) ? Explain.

Remark : In MATLAB, if A is a (row) matrix, then **exp(A)** is a (row) matrix obtained by exponentiating each entry of the matrix A. Thus the matrix corresponding to the values of $P(t)$ given by equation (B) is described in MATLAB as
$>>$ `B=379*exp(0.11*T);`
while the matrix corresponding to the values of $P(t)$ given by (C) is
$>>$ `B=K*P0./(P0+(K-P0)*exp(-r*T));`
(**Warning**: Note the period "**.**" in the above expression. This is necessary. Basically it is telling MATLAB to perform entry-by-entry operations on the same size matrix as the matrix T. For more information why, consult the MATLAB manual.)

(7) Use the closest decade to $(1830 + \textbf{seed})$ as your $t = 0$. Find the solution of $(**)$ that agrees with the actual population data at $t = 0$ and $t = 1$ decade. Discuss the accuracy of this approximation both for $t > 0$ and $t < 0$. Use this model to predict the population in the year 2020.

(8) Show that $P = 0$ and $P = K$ are equilibrium solutions to the differential equation $(***)$. Sketch by hand a rough picture of the solution curves for the differential equation $(***)$. You should find that if $0 < P(0) < K$, then $0 < P(t) < K$ for all t and that $\lim_{t \to \infty} P(t) = K$. This K represents the maximum population that the system can support. What, according to your work in (4), will be the maximum population of New York ?