CS510 Linear Systems with the Hilbert Matrix October 3, 2012

The n by n **Hilbert matrix** is defined by

$$H_n = \begin{pmatrix} 1 & 1/2 & \dots & 1/n \\ 1/2 & 1/3 & \dots & 1/(n+1) \\ \cdot & \cdot & \dots & \cdot \\ 1/n & 1/(n+1) & \dots & 1/(2n-1) \end{pmatrix} ;$$

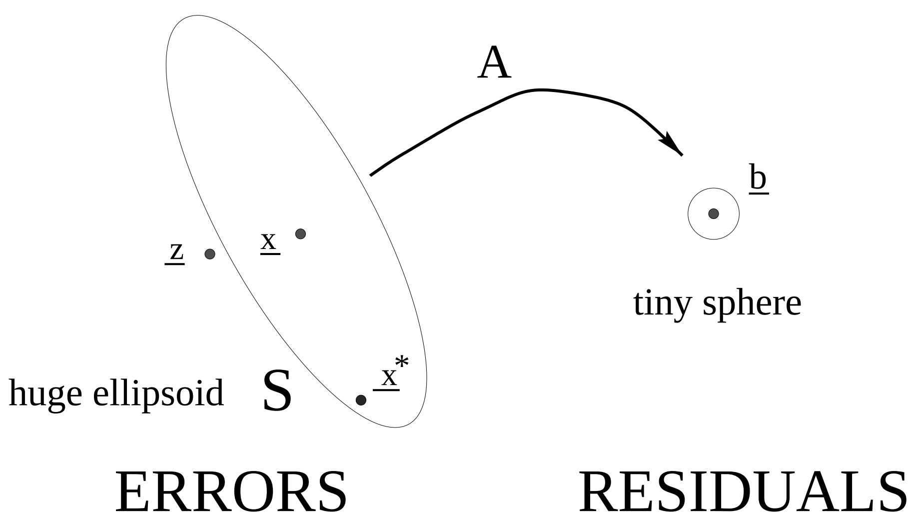
the row i, column j entry is $h_{ij} = 1/(i+j-1)$. This matrix arises naturally in several contexts, e.g., in approximation theory.

Consider the system $A\underline{x} = \underline{b}$ where $A = H_n$ and $\underline{b}$ is the vector with $b_i = \sum_{j=1}^{n} h_{ij}$. Clearly this system has solution vector $\underline{x}$ whose components are all ones.

The matrix $A = H_n$ is notorious for propagating roundoff errors in Gaussian elimination (we will see some reasons for this later). The following outputs are from my program to compute the LUP factorization of H_n using partial pivoting, and then, the solution to $A\underline{x} = \underline{b}$ by forward-substitution (solution of $L\underline{y} = \underline{b}(\underline{p})$) and backsolving $U\underline{x} = \underline{y}$. We denote the computed solution by $\underline{x}^*$ and the **error** vector as $\underline{e} \equiv \underline{x} - \underline{x}^*$. Finally the **residual** vector is defined to be

$$\underline{r} = A\underline{e} = A(\underline{x} - \underline{x}^*) = \underline{b} - A\underline{x}^*.$$

The following pages of output show that **even though $\underline{e}$ may be large (i.e., its length $\|\underline{e}\| = \sqrt{e_1^2 + \cdots + e_n^2}$ is large), the residual vector is small. This seems to be a property of the <u>kind</u> of errors committed by Gaussian elimination.**



The picture is meant to convey the idea that for certain matrices A (like H_n, e.g.) (i) the points $\underline{z}$ that A maps "near" to $\underline{b}$ can lie in a large set S (with points FAR from $\underline{x} = A^{-1}\underline{b}$); (ii) Gaussian-Elimination computes a solution $\underline{x}^* \in S$ (so it has small residual, even if it may have a large error); (iii) there are points $\underline{z} \notin S$ that are close to $\underline{x} = A^{-1}\underline{b}$ (small error) but still have large residuals (Gaussian elimination does not seem to find such points).

```
 p     compact LU         |             L                      U
                          |
 1 1.0000 0.5000 0.3333   |    1.0000 0.0000 0.0000   1.0000 0.5000 0.3333
 2 0.5000 0.0833 0.0833   |    0.5000 1.0000 0.0000   0.0000 0.0833 0.0833
 3 0.3333 1.0000 0.0056   |    0.3333 1.0000 1.0000   0.0000 0.0000 0.0056


      y              x*             e              r

  0.183333E+01   0.999999E+00   0.953674E-06   0.298023E-07
  0.166667E+00   0.100001E+01  -0.548363E-05   0.298023E-07
  0.555554E-02   0.999995E+00   0.536442E-05   0.447035E-07
                              |e| = 0.773024E-05
```

```
 p           compact LU

 1 1.0000 0.5000 0.3333 0.2500 0.2000
 2 0.5000 0.0833 0.0833 0.0750 0.0667
 5 0.2000 0.8000 0.0095 0.0150 0.0178
 3 0.3333 1.0000 0.5833 -.0004 -.0008
 4 0.2500 0.9000 0.8750 0.6428 0.0000

      y              x*             e              r

  0.228333E+01   0.100007E+01  -0.749826E-04   0.149012E-07
  0.308333E+00   0.998680E+00   0.132036E-02   0.298023E-07
  0.423016E-01   0.100548E+01  -0.548434E-02  -0.447035E-07
 -0.126313E-02   0.991939E+00   0.806147E-02  -0.447035E-07
 -0.113896E-04   0.100386E+01  -0.386322E-02  -0.149012E-07
                               |e|= 0.105707E-01
```

----

p	compact LU						
1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429
2	0.5000	0.0833	0.0833	0.0750	0.0667	0.0595	0.0536
7	0.1429	0.6429	0.0099	0.0161	0.0195	0.0213	0.0221
3	0.3333	1.0000	0.5600	-.0007	-.0014	-.0020	-.0024
5	0.2000	0.8000	0.9600	0.6429	0.0000	-.0001	-.0001
4	0.2500	0.9000	0.8400	0.9643	0.8439	0.0000	0.0000
6	0.1667	0.7143	1.0000	0.2976	0.6257	-.5555	0.0000

y	x*	e	r
0.259286E+01	0.100090E+01	-0.903249E-03	0.745058E-07
0.421429E+00	0.967215E+00	0.327848E-01	0.149012E-07
0.888072E-01	0.129571E+01	-0.295714E+00	0.447035E-07
-0.647800E-02	-0.945102E-01	0.109451E+01	0.447035E-07
-0.260744E-03	0.293098E+01	-0.193098E+01	0.745058E-07
0.351676E-05	-0.617661E+00	0.161766E+01	0.000000E+00
-0.165790E-06	0.151771E+01	-0.517712E+00	0.372529E-07

|e| = 0.281070E+01

----

p	compact LU								
1	1.0000	0.5000	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111
2	0.5000	0.0833	0.0833	0.0750	0.0667	0.0595	0.0536	0.0486	0.0444
7	0.1429	0.6429	0.0099	0.0161	0.0195	0.0213	0.0221	0.0223	0.0222
3	0.3333	1.0000	0.5600	-.0007	-.0014	-.0020	-.0024	-.0028	-.0030
9	0.1111	0.5333	0.9503	-.4242	0.0000	0.0001	0.0002	0.0003	0.0004
4	0.2500	0.9000	0.8400	0.9643	-.6159	0.0000	0.0000	0.0000	0.0000
5	0.2000	0.8000	0.9600	0.6429	-.7298	0.8505	0.0000	0.0000	0.0000
6	0.1667	0.7143	1.0000	0.2976	-.4566	0.3754	0.1487	0.0000	0.0000
8	0.1250	0.5833	0.9800	-.2386	0.5076	-.1102	-.3953	-.3756	0.0000

y	x*	e	r
0.282897E+01	0.998807E+00	0.119287E-02	0.178814E-06
0.514484E+00	0.104947E+01	-0.494668E-01	-0.596046E-07
0.133351E+00	0.522337E+00	0.477663E+00	0.119209E-06
-0.122726E-01	0.258982E+01	-0.158982E+01	0.000000E+00
0.104370E-02	0.210264E+00	0.789736E+00	0.596046E-07
0.619291E-04	-0.551675E+01	0.651675E+01	-0.298023E-07
0.134884E-05	0.161007E+02	-0.151007E+02	0.596046E-07
0.257875E-06	-0.119670E+02	0.129670E+02	0.178814E-06
0.845820E-07	0.501332E+01	-0.401332E+01	0.894070E-07

|e| = 0.214040E+02

A similar phenomenon occurs in the computation of A^{-1}. When A is the Hilbert matrix we make large errors in the computation of (A^{-1}). Here is the true inverse of H_7, the 7 by 7 Hilbert matrix: $H_7^{-1} =$

```
[ 49        -1176       8820        -29400       48510        -38808          12012]
[                                                                                   ]
[-1176       37632      -317520      1128960     -1940400      1596672      -504504]
[                                                                                   ]
[8820       -317520     2857680     -10584000     18711000     -15717240     5045040]
[                                                                                   ]
[-29400      1128960   -10584000     40320000    -72765000      62092800   -20180160]
[                                                                                   ]
[48510      -1940400    18711000    -72765000    133402500    -115259760   37837800]
[                                                                                   ]
[-38808      1596672   -15717240     62092800   -115259760     100590336   -33297264]
[                                                                                   ]
[12012      -504504     5045040     -20180160     37837800     -33297264    11099088]
```

and here is what I computed by following the efficient algorithm using Gauss-Jordan reduction on the LUP factorization, answers rounded to the nearest integer (A^* is the computed inverse of H_7). $A^* =$

```
      37.       -680.       3997.     -10499.      13635.      -8513.       2021.
    -682.      17701.    -123698.     369784.    -540135.     380627.    -103545.
    4031.    -124267.     979226.   -3228815.    5148214.   -3941198.    1162847.
  -10673.     373653.   -3244812.   11590284.  -19798362.   16111001.   -5023300.
   14014.    -549598.    5200422.  -19887368.   35931044.  -30652272.    9951588.
   -8880.     390447.   -4002181.   16249666.  -30765518.   27254454.   -9128023.
    2152.    -107212.    1187148.   -5085478.   10019309.   -9154415.    3142867.
```

Though the relative errors in A^* are enormous, the product $(H_7)(A^*)$ is not far from the identity matrix.