An Extension of Romberg Integration Procedures to N-Variables

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Abstract. The Romberg method of numerical integration is extended to multidimensional integration. The elements of the mth column of the Romberg array are shown to approximate up to $O(h^{2m+2})$ provided the integrand has $2m+2$ bounded partial derivatives in each variable.

Introduction

In 1955 a method for approximating an integral,

$$J = \int_a^b f(x)\,dx,$$

was proposed by Romberg [9]. Stiefel, Rutishauser and Bauer [2, 3, 10, 11] have made improvements, particularly in the notation, thus easing computation and error analysis.

Let

$$T_{0,k} = \frac{b-a}{2^k}\left[\frac{1}{2}f(a) + \sum_{p=1}^{2k-1} f\left(a + \frac{b-a}{2^k}p\right) + \frac{1}{2}f(b)\right], \quad k = 0, 1, \cdots,$$

$$T_{m,k} = \frac{4^m T_{m-1,k+1} - T_{m-1,k}}{4^m - 1}, \qquad m = 1, 2, \cdots, \quad k = 0, 1, \cdots, \quad (1)$$

and form the triangular array:

$$\begin{array}{ccc} T_{0,0} & & \\ T_{0,1} & T_{1,0} & \\ T_{0,2} & T_{1,1} & T_{2,0} \\ \vdots & \vdots & \vdots \end{array} \qquad (2)$$

Although the array is an innovation of Stiefel, Rutishauser and Bauer, Romberg showed that the order of the difference of J and the elements comprising the first column of (2) is $O(h^2)$, where h is the increment of integration, and that the order increases in even powers to the right.

We define $f(h_1, \cdots, h_n) = O(h^m)$ to mean that there exists a constant C for which $|f(h_1, \cdots, h_n)| < Ch_1^{\beta_1} \cdots h_n^{\beta_n}$ holds as $h_1, h_2, \cdots, h_n$ tend simultaneously to zero in such a way that the ratios h_i/h_j are bounded $(i, j = 1, \cdots, n)$, where the β_i are any nonnegative real numbers satisfying $\beta_1 + \cdots + \beta_n = m$.

In this paper the triangular array (2) is applied to approximate the integral

$$J = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \cdots, x_n)\,dx_1 \cdots dx_n,$$

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where the first column is an approximation to the integral by n-dimensional hyper-parallelopipeds and

$$T_{m,k} = \frac{4^m T_{m-1,k+1} - T_{m-1,k}}{4^m - 1}, \qquad m = 1, 2, \cdots; \quad k = 0, 1, \cdots .$$

The error in each column is shown to be $O(h^{2m+2})$, assuming f has $2m+2$ bounded partial derivatives in each variable.

In what follows, superscripts on f are used to represent partial derivatives, e.g.,

$$f^{1,1}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y} \mid x_0, y_0 .$$

Further abbreviations are

$$\sum_{\partial_i=0}^{\beta}\Big]_1^n \quad \text{for} \quad \sum_{\partial_1=0}^{\beta} \cdots \sum_{\partial_n=0}^{\beta},$$

$$\sum_{\partial_i=0}^{\beta} \cdot \sum_{\lambda_i=0}^{\delta}\Big]_1^n \quad \text{for} \quad \sum_{\partial_1=0}^{\beta} \cdots \sum_{\partial_n=0}^{\beta} \sum_{\lambda_1=0}^{\delta} \cdots \sum_{\lambda_n=0}^{\delta},$$

and

$$f^{(\partial_i)}(\beta_i)]_1^n \quad \text{for} \quad f^{\partial_1, \cdots, \partial_n}(\beta_1, \cdots, \beta_n).$$

The Theorem

THEOREM 1. *Let $f(x_1, \cdots, x_n)$ have, in each variable, $2r+2$ partial derivatives which are defined and bounded on $[a_i, b_i]$, $i = 1, \cdots, n$. Let the elements of the triangular array (2) be used to approximate the integral*

$$J = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n ,$$

where $T_{0,k}$ is defined by

$$T_{0,k} = \frac{b_1 - a_1}{2^{k+1}} \cdots \frac{b_n - a_n}{2^{k+1}} \sum_{\lambda_i=0}^{2^k-1}\Big]_1^n [f(a_1+\lambda_1 h_1, \cdots, a_n+\lambda_n h_n)$$
$$+ f(a_1+(\lambda_1+1)h_1, \cdots, a_n+(\lambda_n+1)h_n)], \quad k = 0, 1, \cdots ,$$

and the $T_{m,k}$ are defined exactly as in (1). Then $J - T_{m,k}$ is $O(h^{2m+2})$, $m = 0, 1, \cdots, r$, where $h_i = (b_i - a_i)/2^k$, $i = 1, \cdots, n$; $k = 0, 1, \cdots$.

PROOF. Since we can make a change of coordinates

$$x_1' = \alpha_1 x_1 + \beta_1$$
$$\vdots$$
$$x_n' = \alpha_n x_n + \beta_n$$

where the α_i and β_i are determined so as to map the interval $a_i \leq x_i \leq b_i$ onto the interval $0 \leq x_i' \leq 1$, $i = 1, \cdots, n$, yielding $h_1 = h_2 = \cdots = h_n = 1/2^k$, there is no loss in generality in assuming $a_i = 0$ and $b_i = 1$, $i = 1, \cdots, n$. Let $x_{\lambda_i} = \lambda_i h + (h/2)$, $\lambda_i = 0, \cdots, 2^k-1$, $i = 1, \cdots, n$. We prove, by induction on m, that for each $m = 0, \cdots, r$, there exists a bounded function B_m and constants $C_{q_1, \cdots, q_n, m}$, $(q_1 + \cdots + q_n \geq m + 1$, $q_i = 0, \cdots, r$, $i = 1, \cdots, n)$, such that

$$J - T_{m,k} = \sum_{\lambda_1=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Big]_1^n C_{q_1, \cdots, q_n, m} \left\{ \prod_{i=1}^{n} h^{2q_i+1} \right\} f^{(2q_i)}(x_{\lambda_i})]_1^n + h^{2r+2} B_m(h).$$

First, consider the case $m = 0$. Whenever $x_{\lambda_i} \in [x_{\lambda_i} - (h/2), x_{\lambda_i} + (h/2)]$, $(i = 1, \cdots, h)$, one has

$$f(x_1, \cdots, x_n) = \sum_{q_i=0}^{2r+1}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{(x_i - x_{\lambda_i})^{q_i}}{q_i!} \right\} f^{(q_i)}(x_{\lambda_i})]_1^n + R(\bar{x}_{\lambda_i}, \cdots, \bar{x}_{\lambda_n}),$$

where

$$R(\bar{x}_{\lambda_i}, \cdots, \bar{x}_{\lambda_n}) = \sum_{q_i=0}^{2r+2}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{(x_i - x_{\lambda_i})^{q_i}}{q_i!} \right\} f^{(q_i)}(\bar{x}_{\lambda_i})]_1^n - \sum_{q_i=0}^{2r+1}\Bigg]_1^n$$

$$\cdot \left\{ \prod_{i=1}^{n} \frac{(x_i - x_{\lambda_i})^{q_i}}{q_i!} \right\} f^{(q_i)}(\bar{x}_{\lambda_i}) \Bigg]_1^n, \quad x_{\lambda_i} - \frac{h}{2} \le \bar{x}_{\lambda_i} \le x_{\lambda_i} + \frac{h}{2}, \quad i = 0, 1, \cdots, n.$$

Term-by-term integration yields

$$\int_{\lambda_1 h}^{(\lambda_1+1)h} \cdots \int_{\lambda_n h}^{(\lambda_n+1)h} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n$$

$$= \sum_{q_i=0}^{r}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i+1)!} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n$$

$$+ \int_{\lambda_1 h}^{(\lambda_1+1)h} \cdots \int_{\lambda_n h}^{(\lambda_n+1)h} R(\bar{x}_{\lambda_1}, \cdots, \bar{x}_{\lambda_n})\, dx_1 \cdots dx_n .$$

Thus, we have

$$\begin{aligned} J &= \sum_{\lambda_i=0}^{2^k-1}\Bigg]_1^n \int_{\lambda_1 h}^{(\lambda_1+1)h} \cdots \int_{\lambda_n h}^{(\lambda_n+1)h} f(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\ &= \sum_{\lambda_1=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i+1)!} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n \\ &\qquad + \sum_{\lambda_i=0}^{2^k-1}\Bigg]_1^n \int_{\lambda_1 h}^{(\lambda_1+1)h} \cdots \int_{\lambda_n h}^{(\lambda_n+1)h} R(\bar{x}_{\lambda_1}, \cdots, \bar{x}_{\lambda_n})\, dx_1 \cdots dx_n . \end{aligned} \tag{3}$$

Approximating J by $T_{0,k}$,

$$\begin{aligned} T_{0,k} &= \frac{h^n}{2^{n'}} \sum_{\lambda_i=0}^{2^k-1}\Bigg]_1^n [f(\lambda_1 h, \cdots, \lambda_n h) + f((\lambda_1 + 1)h, \cdots, (\lambda_n + 1)h)] \\ &= \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n + R'(\bar{\bar{x}}_{\lambda_1}, \cdots, \bar{\bar{x}}_{\lambda_n}), \end{aligned} \tag{4}$$

where

$$R'(\bar{\bar{x}}_{\lambda_1}, \cdots, \bar{\bar{x}}_{\lambda_n}) = \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{q_i=0}^{r+1}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \right\} f^{(2q_i)}(\bar{\bar{x}}_{\lambda_i}) \Bigg]_1^n$$

$$- \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \right\} f^{(2q_i)}(\bar{\bar{x}}_{\lambda_i}) \Bigg]_1^n,$$

$$x_{\lambda_i} - \frac{h}{2} \le \bar{\bar{x}}_{\lambda_i} \le x_{\lambda_i} + \frac{h}{2}, \quad i = 1, 2, \cdots, n.$$

Subtracting (4) from (3) gives

$$J - T_{0,k} = \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i} - (2q_i+1)!} - \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \Bigg\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n + B_1(h)h^{2r+2},$$

where $q_1 + \cdots + q_n \neq 0$ and

$$B_1(h) = \frac{1}{h^{2r+2}} \left[\int_{\lambda_1 h}^{(\lambda_1+1)h} \cdots \int_{\lambda_n h}^{(\lambda_n+1)h} R(\bar{x}_{\lambda_1}, \cdots, \bar{x}_{\lambda_n})\, dx_1 \cdots dx_n - R'(\bar{\bar{x}}_{\lambda_1}, \cdots, \bar{\bar{x}}_{\lambda_n}) \right].$$

Denote by M a number such that

$$\begin{gathered} |f^{p_1,\cdots,p_n}(x_1, \cdots, x_n)| < M, \\ p_i = 0, 1, \cdots, 2r+2, \quad 0 \leq x_i \leq 1, \quad i = 1, 2, \cdots, n. \end{gathered} \tag{5}$$

Noting that every term in R and R' contains at least one q_i such that $q_i = 2r+2$, $b_i - a_i = 1$ and $h \leq 1$, $i = 1, 2, \cdots, n$, we obtain

$$\begin{aligned} | B_1(h) | \leq \sum_{\lambda_i=0}^{2^k-1}\Bigg]_1^n \Bigg[&\sum_{q_i=0}^{r+1}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i+1)!} \Bigg\} \frac{M}{h^{2r+2}} \\ &- \sum_{q_i=0}^{r}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i+1)!} \Bigg\} \frac{M}{h^{2r+2}} + \sum_{q_i=0}^{r+1}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \Bigg\} \frac{M}{h^{2r+2}} \\ &- \sum_{q_i=0}^{r}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i}(2q_i)!} \Bigg\} \frac{M}{h^{2r+2}} \Bigg] \leq 2M[(r+2)^n - (r+1)^n]. \end{aligned}$$

Thus the proposition is true in case $m = 0$, and the proof of the theorem is complete if $r = 0$.

Assume that $r \geq 1$ and that the proposition is true for each integer $m - 1 < r$. Thus we assume

$$J - T_{m-1,k} = \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n C_{q_1,\cdots,q_n,m-1} \Bigg\{ \prod_{i=1}^{n} h^{2q_i+1} \Bigg\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n + h^{2r+2} B_{m-1}(h),$$

where $| B_{m-1}(h)| < B_1$ and $\sum_{i=1}^{n} q_i \geq m$. Now substituting $k+1$ for k and $h/2$ for h, we obtain

$$J - T_{m-1,k+1} = \sum_{\lambda_i=0}^{2^{k+1}-1} \cdot \sum_{q_i=0}^{r}\Bigg]_1^n C_{q_1,\cdots,q_n,m-1} \Bigg\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i+1}} \Bigg\} f^{(2q_i)}(x'_{\lambda_i}) \Bigg]_1^n + \frac{h^{2r+2}}{2^{2r+2}} B_{m-1}\left(\frac{h}{2}\right),$$

where $x'_{\lambda_i} = \lambda_i(h/2) + (h/4)$. We see that

$$\begin{aligned} \sum_{\lambda_i=0}^{2^{k+1}-1}\Bigg]_1^n f^{(2q_i)}(x'_{\lambda_i}) \Bigg]_1^n &= \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{e_i=0}^{1}\Bigg]_1^n f^{(2q_i)}\left(x_{\lambda_i} + \frac{h}{4} - e_i\frac{h}{2}\right)\Bigg]_1^n \\ &= 2^n \sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{s_i=q_i}^{r}\Bigg]_1^n \Bigg\{ \prod_{i=1}^{n} \frac{h^{2s_i-2q_i}}{4^{2s_i-2q_i}(2s_i-2q_i)!} \Bigg\} f^{(2s_i)}(x_{\lambda_i}) \Bigg]_1^n \\ &\qquad + R_{q_1,\cdots,q_n}(h), \end{aligned}$$

where

$$R_{q_1,\cdots,q_n}(h) = 2^n \sum_{\lambda_i=0}^{2k-1} \cdot \sum_{s_i=q_i}^{r+1} \Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2s_i-2q_i}}{4^{2s_i-2q_i}(2s_i - 2q_i)!} \right\} f^{(2s_i)}(x''_{\lambda_i}) \Bigg]_1^n$$

$$- 2^n \sum_{\lambda_i=0}^{2k-1} \cdot \sum_{s_i=q_i}^{r} \Bigg]_1^n \left\{ \prod_{i=1}^{n} \frac{h^{2s_i-2q_i}}{4^{2s_i-2q_i}(2s_i - 2q_i)!} \right\} f^{(2s_i)}(x''_{\lambda_i}) \Bigg]_1^n,$$

and $x_{\lambda_i} - h/4 \leq x''_{\lambda_i} \leq x_{\lambda_i} + h/4$, $i = 1, 2, \cdots, n$. Taking

$$\bar{B}_{m-1}(h) = \frac{B_{m-1}(h/2)}{2^{2r+2}} + \sum_{q_i=0}^{r} \Bigg]_1^n \frac{C_{q_1,\cdots,q_n,m-1}}{h^{2r+2}} \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{2^{2q_i+1}} \right\} R_{q_1,\cdots,q_n}(h),$$

where $\sum_{i=1}^{n} q_i \geq m$, one sees that $\bar{B}_{m-1}(h)$ is bounded, say $|\bar{B}_{m-1}(h)| < B_2$. Thus one obtains

$$J - T_{m-1,k+1} = \sum_{\lambda_i=0}^{2k-1} \cdot \sum_{q_i=0}^{r} \cdot \sum_{s_i=0}^{q_i} \Bigg]_1^n C_{s_1,\cdots,s_n,m-1} \left\{ \prod_{i=1}^{n} \frac{h^{2q_i+1}}{4^{2q_i-s_i}(2q_i - 2s_i)!} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n + h^{2r+2}\bar{B}_{m-1}(h).$$

Using transformation (1), we have

$$J - T_{m,k} = \frac{4^m(J - T_{m-1,k+1}) - (J - T_{m-1,k})}{4^m - 1}$$

$$= \frac{1}{4^m - 1} \sum_{\lambda_i=0}^{2k-1} \cdot \sum_{q_i=0}^{r} \Bigg]_1^n \left\{ \sum_{s_i=0}^{q_i} \Bigg]_1^n C_{s_1,\cdots,s_n,m-1} \left\{ \prod_{i=1}^{n} \frac{4^m}{4^{2q_i-s_i}(2q_i - 2s_i)!} \right\} \right.$$

$$\left. - C_{s_1,\cdots,s_n,m-1} \right\} \left\{ \prod_{i=1}^{n} h^{2q_i+1} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n$$

$$+ \frac{h^{2r+2}}{4^m - 1} [4^m \bar{B}_{m-1}(h) - B_{m-1}(h)].$$

Let

$$C_{q_1,\cdots,q_n,m} = \sum_{s_i=0}^{q_i} \Bigg]_1^n C_{s_1,\cdots,s_n,m-1} \prod_{i=1}^{n} \frac{4^m}{4^{2q_i-s_i}(2q_i - 2s_i)!} - C_{s_1,\cdots,s_n,m-1}$$

and note that $C_{q_1,\cdots,q_n,m} = 0$ when $\sum_{i=1}^{n} q_i = m$. Letting

$$B_m(h) = \frac{4^m \bar{B}_{m-1}(h) - B_{m-1}(h)}{4^m - 1},$$

we obtain

$$J - T_{m,k} = \sum_{\lambda_i=0}^{2k-1} \cdot \sum_{q_i=0}^{r} \Bigg]_1^n C_{q_1,\cdots,q_n,m} \left\{ \prod_{i=1}^{n} h^{2q_i+1} \right\} f^{(2q_i)}(x_{\lambda_i}) \Bigg]_1^n + B_m(h),$$

where $\sum_{i=1}^{n} q_i \geq m + 1$, as was to be proved.

To prove the theorem, let

$$C = \max_{q_i,m} |C_{q_1,\cdots,q_n,m}|, \quad i = 1, 2, \cdots, n;$$

$m = 0, 1, \cdots, r$; M be as in (5); and B be an upper bound for $|B_m(h)|$. Since $b_i - a_i = 1$, $i = 1, 2, \cdots, n$, and $h \leq 1$, every element of $J - T_{m,k}$ is less in absolute value than CMh^{2m+2}; thus one has

$$\frac{|J - T_{m,k}|}{h^{2m+2}} < \left[\sum_{\lambda_i=0}^{2^k-1} \cdot \sum_{\rho_i=0}^{r}\right]_1^n CM + B = 2^{nk}(r+1)^n CM + B.$$

Thus the theorem is true.

Examples

The following examples illustrate results for two and three dimensions.

Example 1. $\displaystyle\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} x^2 y^2 \, dx \, dy = 17.3611 \times 10^{-4}.$

39.0625×10^{-4}		
21.9726×10^{-4}	16.2760×10^{-4}	
18.4631×10^{-4}	17.2932×10^{-4}	17.3610×10^{-4}

Example 2. $\displaystyle\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} x^2 y^2 z^2 \, dx \, dy \, dz = 7.2338 \times 10^{-5}.$

24.414×10^{-5}		
10.300×10^{-5}	5.595×10^{-5}	
7.496×10^{-5}	6.561×10^{-5}	6.626×10^{-5}

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