**An Algorithm for Finding $A^{-1}$**

In the previous section, we listed some conditions that if true, can guarantee the existence of an inverse matrix. Notice that, you do not need to check every single condition of the Invertible Matrix Theorem but only one; since as the theorem states, if one is true then all of the statements are true and vice versa.

Suppose now, that the statements are in fact true hence the $n \times n$ matrix $A$ does have an inverse. Then the next step is to find that inverse matrix $A^{-1}$. To do that, we place $A$ and $I$ (the $n \times n$ identity matrix) side-by-side to form an augmented matrix $[A \quad I]$. Row operations on the augmented matrix $[A \quad I]$ produce identical operations on $A$ and on $I$.

Therefore, to find $A^{-1}$, you need to row reduce the augmented matrix $[A \quad I]$. If $A$ is row equivalent to $I$ (remember condition (b) in the Invertible Matrix Theorem), then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$.

*Example*

Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, if it exists.

The first step, is to row reduce matrix $A$, to figure out if it is invertible; if there exists an inverse of $A$.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 4 & -3 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The coefficient matrix in linearly independent (condition (e) of the Invertible Matrix Theorem) hence the inverse does exist.

$$[A \quad I] = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$$

Therefore $A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$

It is always a good idea to verify the answer;

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Although the procedure to find the inverse might take some respectful amount of time for higher dimensions, things are rather easy when it comes to computing the inverse of a $2 \times 2$ matrix. The rule is that the inverse exists if and only if the determinant of the matrix is non-zero.

In particular,

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $det\ A = ad - bc \neq 0$, then $A$ is invertible and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $det\ A = ad - bc = 0$ then $A$ is not invertible.