# Set Theory Cheat Sheet, Sergei Volodin

## → Axioms of ZFC

0 Existence: $\exists x\, x = x$

1 Extensionality: $\forall x \forall y\ (\forall z\,(z \in x \to z \in y) \to x = y)$

2 Comprehension: $\forall z \forall \vec{w} \exists y \exists x\,(x \in y \leftrightarrow (x \in z \wedge \varphi(x, z, \vec{w})))$

3 Pairing: $\forall x \forall y \exists z\ (x \in z \wedge y \in z)$

4 Union: $\forall a \exists b \forall x \forall y\,((x \in y \wedge y \in a) \to x \in b)$

5 Infinity: $\exists x\,(\exists y(y \in x \wedge \forall z\, z \notin y) \wedge \forall z(z \in x \to z \cup \{z\} \in x))$

6 Power: $\forall x \exists y \forall z\,(\forall u(u \in z \to u \in z) \to z \in y)$

7 Foundation: $\forall x\,(\exists y\, y \in x \to \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)))$

8 Replacement: $\forall A \forall \vec{w}\,[\forall x(x \in A \to \exists! y \varphi) \to \exists B \forall x(x \in A \to \exists y(y \in B \wedge \varphi))]$

9 Choice: $\forall x\,\{\forall y(y \in x\, \exists z z \in y) \wedge \forall y \forall y'\,[y \in x \wedge y' \in x \wedge \exists z(z \in y \wedge z \in y')] \to y = y'\} \to \exists c \forall y(y \in x \to \exists! z(z \in y \wedge z \in c))$

## → Well-orderings

$(A, R)$ ordering: $\forall x, y, z \in A\,((xRy \wedge yRz) \to xRz)$, $\forall x \in A \neg xRx$.

Total: $+\forall x, y \in A\,(x \neq y \to (xRy \vee yRx))$. Isomorph.: preserv. order

$(A, <)$ WO: tot$+\forall B \subseteq A\,(B \neq \varnothing \to \exists x \in B \forall y \in B\, y \not< x)$

$\lceil x \rceil_A^< = \{y \in A | y < x\} \not\approx (A, <)$ is WO.

$(A, <_A)$, $(B, <_B)$ WO, then $\approx$, $\lceil x \rceil_A^< \approx B$ or $\lceil x \rceil_B^< \approx A$

## → Ordinals

A trans.: $\forall x \in A \to x \subseteq A$

Ordinal $\Leftrightarrow$ WO+trans.

$x \in \alpha \Rightarrow x \in \mathbf{On}$, $x = \{x > y \in \alpha\}$

$\approx \Rightarrow =$, $=, \in$ or $\ni$. Trans. class. Min in set.

$\mathbf{On}$ is not a set. $(A, <) \approx !(\alpha, <_\alpha) = type(A, <)$

$S\alpha = \alpha \cup \{\alpha\}$, otherw. (nonempt.) limit

Class: def. by formula.

Transf. ind: least in $C \subseteq \mathbf{On}$

Transf. rec: $\forall F \exists G\, G(\alpha) = F(G \upharpoonright \alpha)$

$\alpha + 0 = \alpha$, $\alpha + S\beta = S(\alpha + \beta)$, $\alpha + \beta = \sup\{\alpha + \gamma | \gamma < \beta\}$ Same for $\cdot$, power.

not comm., assoc. $= type(\alpha \times \{0\} \cup \beta \times \{1\})$

## → Extension by Def. & Cons. Extension

Added symbols.

Ext. by def.: seq. of langs, adding a symbol and formula.

Cons. ext.: $F' \vdash \varphi \Leftrightarrow F \vdash \varphi$, $\phi$ from L

Ext. by def $\Rightarrow$ Cons. ext.

## → Axiom of Choice

Choice function: $f\colon I \to \bigcup\limits_{i \in I} A_i$, $A_i \neq \varnothing$, $f(i) \in A_i$

Part. Ord.: $X \neq \varnothing$, $x \leqslant x$, $x \leqslant y \leqslant x \Rightarrow y = x$, trans.

Chain: all comparable, Antichain: $x \leqslant y \Rightarrow x = y$

Ind. PO: $\exists x \in X \forall y \in C y \leqslant x$

Zorn: for X-Ind.PO: $\exists x \in X \forall y \in X x \leqslant y \to x = y$

Zeremelo: every set can be WO

**AC**$\Leftrightarrow$ Zorn $\Leftrightarrow$ Zerem.

## → Cardinals

$\simeq$: bijection, $\lesssim$: inj., $\lnsim$: $A \lesssim B \wedge B \not\lesssim A$

Cantor: $A \lesssim B \wedge B \lesssim A \Rightarrow A \simeq B$

A can be w.o. $\Rightarrow$ $|A| = \inf\{\alpha \in \mathbf{On} | \alpha \simeq A\}$. Cardinal $\Leftrightarrow \alpha = |\alpha|$

$|\alpha| \leqslant \beta \leqslant \alpha \to |\alpha| = |\beta|$. $n \not\approx n + 1$.

Infinite: $|A| \not< \omega$. $\sup A$ card if $\alpha \in A$ card.

$|\alpha| = \alpha > \omega$ limit ordinal

$P(A) = \{B | B \subseteq A\}$ set by axiom

Cantor: $A \lnsim P(A)$.

Succ. card: $\forall \alpha \exists \alpha^+ = \beta(\alpha < \beta \wedge |\beta| = \beta)$

Limit card: not succ and $\kappa > \omega$.

$\aleph_0 = \omega$, $\aleph_{\alpha+1} = \aleph_\alpha^+$, $\aleph_\alpha = \sup\{\aleph_\beta | \beta < \alpha\}$. $\omega_\alpha \overset{\text{def}}{=} \aleph_\alpha$

$< \Rightarrow \aleph_\alpha < \aleph_\beta$, inf.cards=all $\aleph$s

$\aleph_\alpha$ lim/succ card $\Leftrightarrow$ $\alpha$ limit/succ ord.

$\kappa + \lambda = |\kappa \times \{0\} \cup \lambda \times \{1\}|$

$\kappa \cdot \lambda = |\kappa \times \lambda|$. $\kappa^\lambda = |{}^\lambda\kappa|$. $\kappa \cdot \kappa = \kappa$

int. on $\omega$. Othws. $\lambda + \kappa = \max\{\lambda, \kappa\}$,

$\lambda \cdot \kappa = 0$ or $\max\{\lambda, \kappa\}$. **ZFC**: inj $\Rightarrow |A| \leqslant |B|$. Surj. $\Rightarrow |A| \geqslant |B|$.

$\lambda^\kappa = 2^\kappa = |P(\kappa)|$.

$|\bigcup\limits_{\alpha \leqslant \kappa} A_\alpha| \leqslant \kappa$, $|A_\alpha| \leqslant \kappa$

$A$ unb. in $\alpha \Leftrightarrow \forall \gamma \in \alpha \exists \delta \in A \delta \geqslant \gamma \Leftrightarrow S\alpha \in A$ or $\sup A = \alpha$

$f\colon \beta \to \alpha$ cof. if $ran(f)$ unb. in $\alpha$

$cof(\alpha)$ least $\beta\ \exists f \overset{cof.}{\to} \alpha$

$\kappa$ reg. $cof(\kappa) = \kappa$ oth. sing.

$\kappa^+$ reg., $\kappa^{cof(\kappa)} > \kappa$, $cof(2^\kappa) > \kappa$. **CH**: $2^{\aleph_0} = \aleph_1$. **GCH**: $\forall \alpha 2^{\aleph_\alpha} = \aleph_{\alpha+1}$

$\kappa$ weak. incs. if reg. and limit; str. if reg and $\forall \lambda < \kappa 2^\lambda < \kappa$

$|\mathbb{R}| = 2^{\aleph_0} = \beth_1$. **GCH** $\Rightarrow$ str. inacc. $\Leftrightarrow$ weak. inacc.

## → Well-Founded Sets

$W(0) = \varnothing$, $W(\alpha + 1) = P(W(\alpha))$, $W(\alpha) = \bigcup\limits_{\xi < \alpha} W(\xi)$,

$\mathbf{WF} = \bigcup W(\alpha)$

$rk(x) =$ least $\alpha\colon x \in W(\alpha{+}1)$. $W(\alpha) = \{x \in \mathbf{WF} | rk(x) < \alpha\}$.

$\forall x \in y(x \in \mathbf{WF} \wedge rk(x) < rk(y)$

$rk(y) = \sup\{rk(x) + 1 | x \in y\}$. Sequence: $s\colon \alpha \to A$, $s^\frown t\colon \alpha + \beta \to A \cup B$. DC: $\forall A \forall T \subseteq A^{<\omega} T n.e.p. \to \exists b \in A^\omega \forall n \in \omega b \upharpoonright n \in T$. **AC** $\to DC \to CC$

$\nexists x_0 \ni x_1 \ni x_2 ...$

$\mathbf{On} \subseteq \mathbf{WF}$, $rk(\alpha) = \alpha$, $W(\alpha) \cap \mathbf{On} = \alpha$. $y \in \mathbf{WF} \to y \subseteq \mathbf{WF}$.

$<$ **WF** on $A \Leftrightarrow \forall B \subseteq AB \neq \varnothing \to \exists y \in B \forall x \in Bx \not< y$.

$A \in \mathbf{WF} \Rightarrow \in$ w.f. on A. $A$ trans. and $\in$-w.f. $\Rightarrow A \in \mathbf{WF}$.

$\bigcup^0 A = A, \bigcup^{n+1} A = \bigcup \bigcup^n A, tc(A) = \bigcup\{\bigcup^n A | n \in \omega\}$ is $\subseteq$-least trans. cont. A.

## → Axiom of Foundation

Foundation $\Leftrightarrow \in$ w.f. $\forall x \Leftrightarrow \mathbf{V} = \mathbf{WF}$

## → Relativization from inside a class

$(\exists x \varphi)^M = \exists x \in M(\varphi)^M$, $(\forall x \varphi)^M = \forall x \in M(\varphi)^M$. $\vdash \varphi \Rightarrow \vdash \varphi^M$

## → The Mostowski Collapse

Goal: making class $\mathbf{M}$ transitive.

$\mathbf{R}$ set-like on $\mathbf{M} \Leftrightarrow \forall y \in \mathbf{M} \mathbf{R}^{-1}[y] = \{x \in \mathbf{M} | x\mathbf{R}y\}$ set.

$cl^0(\lceil x \rceil_{\mathbf{X}}^{\mathbf{R}}) = \mathbf{X} \cap \mathbf{R}^{-1}[z]$, $cl^{n+1} = \bigcup\{\lceil z \rceil_{\mathbf{X}}^{\mathbf{R}} | z \in cl^n\}$, $cl = \bigcup\{cl^n | n \in \omega\}$

$\mathbf{R}$ w.f. on $\mathbf{M} \Leftrightarrow \forall \underline{X} \subseteq \mathbf{M}\mathbf{X} \neq \varnothing \to \exists y \in x \forall z \in x \neg z\mathbf{R}y$. $\forall \mathbf{X} \subseteq \mathbf{M}\mathbf{X} \neq \varnothing \to \exists y \in \mathbf{X} \forall x \in \mathbf{X} \neg x\mathbf{R}y$

$\mathbf{G}(x) = \{\mathbf{G}(y) | y \in \mathbf{M} \wedge y\mathbf{R}x\}$. Most. coll. $\mathbf{G}[\mathbf{M}]$. $x\mathbf{R}y \to \mathbf{G}(x) \in \mathbf{G}(y)$, $N$ trans., $N \subseteq \mathbf{WF}$

$\mathbf{R}$ extensional iff $\forall x \in \mathbf{M} \forall y \in \mathbf{M}\,(x \neq y \to \lceil x \rceil_{\mathbf{M}}^{\mathbf{R}} \neq \lceil y \rceil_{\mathbf{M}}^{\mathbf{R}})$

(**ZF**-AF) $\mathbf{R}$ w.f. and set-like, ext. $\Rightarrow (\mathbf{M}, \mathbf{R}) \overset{!\mathbf{G}}{\approx} (\mathbf{G}[\mathbf{M}], \in)$

## → Preservation under Relativization

Prop. of class $\Rightarrow$ axioms in class.

$\mathbf{M}$ trans. $\Rightarrow$ Extensionality$^M$, $\{x \in y | \varphi^{\mathbf{M}}\} \in \mathbf{M} \Rightarrow$ Compr.$^M$, $P(x) \cap \mathbf{M} \subseteq y \in \mathbf{M} \Rightarrow$ Pow$^{\mathbf{M}}$, $\mathbf{M} \subseteq \mathbf{WF} \Rightarrow$ Found$^{\mathbf{M}}$

## → Absoluteness

$\varphi$ abs. for $\mathbf{M} \subseteq \mathbf{N} \Leftrightarrow \forall x_1 \in M ... \forall x_n \in M \varphi^M \leftrightarrow \varphi^N$.

$\varphi$ abs. for $M$ if abs. for $\mathbf{M}, \mathbf{V}$. $\mathbf{M}$ trans. then $\Delta_0^0$-rud. form. $(\exists x(x \in y \wedge \psi)$ or $\forall x(x \in y \to \psi))$ abs.

## → Arithmetic and Recursivity

$f\colon \mathbb{N}^p \supseteq dom_f \to \mathbb{N}$. Part. and tot.

Part. rec: const., proj., succ., compos., rec., min. $\Leftrightarrow$ Tur.-comp. Prim. rec: same but min. Robinson: $L = \{0, S, +, \cdot\}$, 1 $\forall x Sx \neq 0$, 2 $\forall x \exists y(x \neq 0 \to Sy = x)$, 3 $\forall x \forall y(Sx = Sy \to x = y)$, 4 $\forall x\, x + 0 = x$, 5 $\forall x \forall y(x + Sy = S(x + y))$, 6 $\forall x\, x \cdot 0 = 0$, 7 $\forall x \forall y\, x \cdot Sy = (x \cdot y) + x$. $\phi$ repr. func. $f \Leftrightarrow$ Rob$\vdash f(\vec{x}) = x_0 \Leftrightarrow \phi(x_0, \vec{x})$. $\phi$ repr. set $\Leftrightarrow \vdash \phi$ or $\vdash \neg\phi$.

Coding seqs of ints. Coding formulas. Coding proofs (seqnt calc.). Ex. set $\mathcal{F}_{\checkmark !free}$ is prim. rec.

→ **Undecidability Results**

$T$ rec. if $\{\ulcorner\varphi\urcorner|\varphi\in T\}$ is rec. Dec. if $thms(T)=\{\ulcorner\varphi\urcorner|T\vdash\varphi\}$ is rec.

$\{(\ulcorner P\urcorner,\ulcorner\phi\urcorner)|P\, is\, proof\, of\, T\vdash\phi\}$

prim. rec if T prim. rec., rec. if T rec

T rec $\exists A$: $fin.time$, $\forall x A(x)=[x\in T]$

$\{\ulcorner\phi\urcorner|\vdash\phi\}$ not rec.

$T$ compl. $T\vdash P$ or $T\vdash\neg P$

$T$ compl. $\Rightarrow$ $T$ dec.

$T\supseteq Rob$ cons. $\Leftrightarrow$ undec.; cons+rec $\Rightarrow$ incompl. (1st incompl.)

Peano: +Ind: $\forall\vec{x}\phi_{[0/x_0]}\wedge\forall x_0(\phi\to\phi_{[Sx_0/x_0]})\to\forall x_0\phi$

Arithm. hier. $\Delta^0_0$: atom., conj-d-n., bound. quant. $<t$

$\Sigma^0_1$: $\exists\vec{x}\phi$, $\phi\in\Delta^0_0$

$I(ind)\Sigma^0_1$. $\exists y(y+x=z\wedge x\neq z)$ $f$ tot. rec. $\Rightarrow$ repr. by $\Sigma^0_1$ form.

**2nd inc.** $diag(n)=\ulcorner\phi_{\ulcorner\phi\urcorner/x_0}\urcorner$, $\phi$ 1 free var. $\Rightarrow\phi_{diag}(x_0,x_1)$, $\Sigma^0_1\ni\Xi(x_0)=\exists x_1,x_2\phi_{proof_T}(x_1,x_2)\wedge\phi_{diag}(x_2,x_1)$. $\mathbb{N}\models\Xi(n)\Leftrightarrow Rob.\vdash\Xi(n)$

$T\supseteq Rob$ cons., rec. $T\vdash\Xi_{\ulcorner\Xi/x_0\urcorner}\to\exists x_1\phi_{proof_{Rob}}(x_1,\ulcorner\Xi_{\ulcorner\Xi/x_0\urcorner}\urcorner)$ then $T\nvdash\mathrm{cons}(T)$

$\mathrm{cons}(\mathbf{ZF}/C)\Rightarrow\mathbf{ZF}\nvdash\mathrm{cons}(\mathbf{ZF}/C)$

→ **Gödel's Constructible Universe**

$X$ def over $Y$: $X=\{y\in Y|\varphi(x,a_1/x_1,...,a_n/x_n)^Y, a_i\in Y\}$

Definability with codes in 1st order. Defin. subsets Def($Y$)

Constructible sets **L**: $\mathbf{L}(0)=\varnothing$, $\mathbf{L}(\alpha+1)=\mathrm{Def}(\mathbf{L}(\alpha))$, $\mathbf{L}(\alpha)=\bigcup\limits_{\xi<\alpha}\mathbf{L}(\xi)$

$Y\in\mathrm{Def}(Y)$, $P_{fin}\in\mathrm{Def}(Y)$, Y trans $\Rightarrow$ $Y\subseteq\mathrm{Def}(Y)$, **AC**, $|Y|\geqslant\omega\Rightarrow|\mathrm{Def}(Y)|=|Y|$.

$rk_L(x)=\mu\alpha\in\mathbf{On}\, x\in\mathbf{L}(\alpha+1)$, $\mathbf{AC}|\mathbf{L}(\alpha)|=|\alpha|$, $\mathbf{L}(\omega)=\mathbf{V}(\omega)$, $\mathbf{L}\cap\mathbf{On}=\mathbf{On}$

$\mathbf{V}=\mathbf{L}$: $\forall x\exists\alpha\, x\in\mathbf{L}(\alpha)$, $\mathbf{V}=\mathbf{L}\Rightarrow P(\kappa)\subseteq\mathbf{L}(\kappa^+)$, $<_\mathbf{L}:\mathbf{L}\times\mathbf{L}$

Refl.: $\mathbf{ZF}\vdash\forall\alpha\exists\beta>\alpha\,\phi_0,...,\phi_n\, abs.\, for\,\mathbf{L}(\beta)$, **L**

$X\subseteq M$. $X\prec M\Leftrightarrow\forall\varphi,\forall a_i\in X\ \varphi(\vec{a})^X\leftrightarrow\varphi(\vec{a})^M$

$\omega<\alpha$ limit, $X\prec\mathbf{L}(\alpha)\Rightarrow\exists\beta,(X,\in)\approx(\mathbf{L}(\beta),\in)$

$\forall\alpha$, $X\subseteq\mathbf{L}(\alpha)$, $\exists M$, $|M|=\sup\{\aleph_0,|X|\}$, $X\subseteq M$, $M\prec\mathbf{L}(\alpha)$

Inner model **M**: trans., $\mathbf{On}\cap\mathbf{M}=\mathbf{On}\cap\mathbf{V}$, $(\mathbf{ZFC})^\mathbf{M}$

→ **Forcing**

**Want prove:** $\mathrm{cons}(\mathbf{ZFC})\Rightarrow\mathrm{cons}(\mathbf{ZFC}+\neg\mathbf{CH})$. Method: $\mathbf{ZFC}+\neg\mathbf{CH}\vdash\bot\Rightarrow\exists\Phi_1,...,\Phi_n$ in $+\neg\mathbf{CH}$, $\Phi_1,...,\Phi_n\vdash\Phi\wedge\neg\Phi$.

Forcing $\Rightarrow\mathbf{ZFC}\vdash\exists\mathbf{N}(\Phi_1^\mathbf{N}\wedge...\wedge\Phi_n^\mathbf{N})$, then $\mathbf{ZFC}\vdash\Phi^\mathbf{N}\wedge\neg\Phi^\mathbf{N}$, then $\mathbf{ZFC}\wedge\bot$

Forcing: $\Psi_1,...,\Psi_k\vdash\exists\mathbf{N}(\Phi_1^\mathbf{N}\wedge...\wedge\Phi_n^\mathbf{N})$.

$\forall\{\Psi_1,...,\Psi_k\}$ consider c.t.m. **M** (refl.) $\mathbf{M}\models\Psi_1\wedge...\wedge\Psi_k$, forcing: $\mathbf{N}=\mathbf{M[G]}\models\Phi_1\wedge...\wedge\Phi_n$

**Montague's refl. princ.:** $\phi_0,...,\phi_n\in L_{ST}$. $\mathbf{ZF}\vdash\forall\alpha\in\mathbf{On}\,\exists\beta>\alpha$ "$\phi_0,...,\phi_n\, abs.\, for\,\mathbf{V}(\beta)$. count. trans. mod. of "**ZFC**": c.t.m. of suff. large n. of axioms of **ZFC**

**Forcing:** poset $(\mathbb{P},\leqslant)$ or $(\mathbb{P},\leqslant,\mathbb{1})$. $p\in\mathbb{P}$ condition, $p\leqslant q$ p stronger q.

$p,q$ **compat.** $\exists r\in\mathbb{P}\ r\leqslant p,q$, oth. $p\perp q$

**Antichain:** all incomp., $D\subseteq\mathbb{P}$ **dense** $\forall p\in\mathbb{P}\,\exists q\in D\, q\leqslant p$

**Filter:** $G\subseteq\mathbb{P}$, $\forall p,q\in G\,\exists r\in G\, r\leqslant p,q$ and $\forall p\in G$, $q\in\mathbb{P}\, p\leqslant q\to q\in G$

$G\subseteq\mathbb{P}$ is **$\mathbb{P}$-genrc ov. M**: fltr+$\forall D\subseteq\mathbb{P}(D\,dns.\wedge D\in\mathbf{M})\to D\cap G\neq\varnothing$

**M** count., $\mathbb{P}$ poset, $p\in\mathbb{P}\Rightarrow\exists G$ $\mathbb{P}$-gen. ov. **M**, $p\in G$ L277

**M** c.t.m. "**ZFC**", $\mathbb{P}\in M\mathbf{M}$ forcing, $\forall p\in\mathbb{P}\,\exists r,q\in\mathbb{P}(q\leqslant p\wedge r\leqslant p\wedge q\perp r)$. $G$ $\mathbb{P}$-gen. ov. $\mathbf{M}\Rightarrow G\notin\mathbf{M}$

$\tau-$ **$\mathbb{P}$-name** $\tau$ bin.rel, $\forall(\sigma,p)\in\tau$, $\sigma$ $\mathbb{P}$-name, $p\in\mathbb{P}$

$\mathbf{V}^\mathbb{P}=\{\mathbb{P}-names\}$. **M** trans. "**ZFC**" $\Rightarrow\mathbf{M}^\mathbb{P}\overset{\text{def}}{=}\mathbf{M}\cap\mathbf{V}^\mathbb{P}=\{\tau\in\mathbf{M}|(\tau\, is\, a\,\mathbb{P}-name)^\mathbf{M}\}$

**Extension:** $(\tau)_G=\{(\sigma)_G|\exists p\in G(\sigma,p)\in\tau\}$.

**M** trans. "**ZFC**", $\mathbb{P}\in\mathbf{M}$, $G\subseteq\mathbb{P}$ fltr. $\mathbf{M}[G]=\{(\tau)_G|\tau\in\mathbf{M}^\mathbb{P}\}$

$\check{x}=\{(\check{y},\mathbb{1})|y\in x\}$, $\Gamma=\{(\check{p},p)|p\in\mathbb{P}\}$. $(\check{x})_G=x$, $(\Gamma)_G=G$.

$\mathbf{M}\subseteq\mathbf{M}[G]$, $G\in\mathbf{M}[G]$.

$\mathbf{M}[G]$ trans.; if **N** t.m. "**ZFC**" $\mathbf{M}\subseteq\mathbf{N}$, $G\in\mathbf{N}$, then $\mathbf{M}[G]\subseteq\mathbf{N}$. $\mathbf{On}^\mathbf{M}=\mathbf{On}^{\mathbf{M}[G]}$

$E$ is **dense below** $p\in\mathbb{P}$ iff $\forall q\leqslant p\,\exists r\in E\, r\leqslant q$

**M** c.t.m. of "**ZFC**", $\mathbb{P}\in\mathbf{M}$, $\varphi$, $\tau_1,...,\tau_n\in\mathbf{M}^\mathbb{P}$, $p\in\mathbb{P}$. p forces $\varphi$ ($p\models_{\mathbb{P},\mathbf{M}}\varphi(\tau_1,...,\tau_n)$) if $\forall G$ $\mathbb{P}$-gen. over **M** $p\in G$ holds $\mathbf{M}[G]\models\varphi((\tau_1)_G,...,(\tau_n)_G)$

**Def** $\Vdash_*$: $(p\models_{\mathbb{P},\mathbf{M}}\varphi(\tau)\Leftrightarrow(p\Vdash_*\varphi(\tau))^\mathbf{M})$

$(\pi_1,\pi_2)\prec(\tau_1,\tau_2)\Leftrightarrow\pi_1\in dom(\tau_1)$, $\pi_2\in dom(\tau_2)$ is w.f.

$\underline{p\Vdash_*\tau_1=\tau_2}$ iff 1 $\forall(\pi_1,s_1)\in\tau_1$ $D_\alpha(\pi_1,s_1,\tau_2)=\{q\in\mathbb{P}|q\leqslant s_1\to\exists(\pi_2,s_2)\in\tau_2\, q\leqslant s_2\wedge q\Vdash_*\pi_1=\pi_2\}$ dense below $p$, 2 $\forall(\pi_2,s_2)\in\tau_2$ $D_\beta(\pi_2,s_2,\tau_1)$ dense below $p$. $\underline{p\Vdash_*\tau_1\in\tau_2}$ iff $\{q\in\mathbb{P}|\exists(\pi,s)\in\tau_2\, q\leqslant s\wedge q\Vdash_*\pi=\tau_1\}$. $\underline{p\Vdash_*\varphi\wedge\psi}$ iff $p\Vdash_*\varphi$ and $p\Vdash_*\psi$. $\underline{p\Vdash_*\neg\varphi}$ iff $\forall q\leqslant p\, q\nVdash_*\varphi$. $\underline{p\Vdash_*\exists x\varphi(x,...)}$ iff $\{q\in\mathbb{P}|\exists\sigma\in\mathbf{V}^\mathbb{P},q\Vdash_*\varphi(\sigma,...)\}$ dense below $\underline{p.\ p\Vdash_*\varphi(\tau_1,...,\tau_n)}\Leftrightarrow\forall r\leqslant p\, r\Vdash_*\varphi(...)\Leftrightarrow\{r\in\mathbb{P}|r\Vdash_*\varphi(...)\}$ dense below $p$.

**Truth lemma:** $\varphi(x_1,...,x_n)\in L_{ST}$ **M** is c.t.m. of "**ZFC**", $\mathbb{P}$ forc. on **M**, $\tau_1,...,\tau_n\in\mathbf{M}^\mathbb{P}$, $G$ $\mathbb{P}$-gen. over **M** $\mathbf{M}[G]\models\varphi((\tau_1)_G,...,(\tau_n)_G)\Leftrightarrow\exists p\in G(p\Vdash_*\varphi(\tau_1,...,\tau_n))^M$

also $\mathbf{M}[G]\models$ "**ZFC**".

→ ¬**CH**

Define $\mathbb{P}\in\mathbf{M}$ c.t.m. "**ZFC**", $\forall G$ $\mathbb{P}$-gen. over **M**, $\mathbf{M}[G]\models\neg\mathbf{CH}$. $\mathbb{P}_{\omega_2}=\{f:(\omega_2)^\mathbf{M}\times\omega\to\{0,1\}|dom(f)\, is\, finite\}$, $f\leqslant g\Leftrightarrow f\supseteq g$, $\mathbb{1}=\varnothing$. $\mathcal{F}=\bigcup G:(\omega_2)^\mathbf{M}\times\omega\to\{0,1\}$ function, $G\notin\mathbf{M}$. $G\in\mathbf{M}[G]\Rightarrow\mathcal{F}\in\mathbf{M}[G]$,

$D_{\alpha\beta}=\{p\in\mathbb{P}|\exists n\in\omega(\alpha,n),(\beta,n)\in dom(p),p(\alpha,n)\neq p(\beta,n)\}\in\mathbf{M}$, dense in $\mathbb{P}$. $\alpha<\beta<(\omega_2)^\mathbf{M}\Rightarrow D_{\alpha\beta}\cap G=\varnothing\Rightarrow\exists n\mathcal{F}(\alpha,n)\neq\mathcal{F}(\beta,n)\Rightarrow(|P(w)|\geqslant(\omega_2))^{\mathbf{M}[G]}$

Str. antich. $\forall p,q\in A\, p\neq q\to p\perp q$

$(\lambda\, card)^\mathbf{M}$ $\mathbb{P}-\lambda\, c.c.\Leftrightarrow\forall A\, ant.\, ch.\,(|A|<\lambda)^\mathbf{M}$

$(\lambda\, reg)^\mathbf{M}$, $(\mathbb{P}-\lambda\, c.c.)^\mathbf{M}\Rightarrow\mathbb{P}$ preserv. card. $\geqslant\lambda$

$\mathbb{P}-\lambda-closed\Leftrightarrow\forall\gamma<\lambda\forall(p_\xi)\subseteq\mathbb{P}\,\exists p\in\mathbb{P}\, p\leqslant p_\xi$

$p\Vdash\exists x\in\sigma\varphi(x,\vec{\tau})\Rightarrow\exists q\leqslant p\,\eta\in dom\sigma: q\Vdash\varphi(\eta,\vec{\tau})$

$(\lambda-card)^\mathbf{M}$, $(\mathbb{P}-\lambda-cl.)^\mathbf{M}\Rightarrow\mathbb{P}$ pres. card. $\leqslant\lambda$

$F$-fam. of fin. sets $|F|=\aleph_1\Rightarrow\exists F'\subseteq F$, $r$-fin., $|F'|=\aleph_1$, $\forall a,b\in F'\, a\cap b=r$ ($\Delta$-system)

$\mathbb{P}_{\omega_2}$ has c.c.c. $\Rightarrow$ pres. all cards: $\aleph_\alpha^{\mathbf{M}[G]}=\aleph_\alpha^\mathbf{M}$

$(\mathbb{P}-c.c.c)^\mathbf{M}$, $(\lambda-card)^\mathbf{M}\Rightarrow|P(\lambda)|^{\mathbf{M}[G]}\leqslant(|\mathbb{P}|^\lambda)^\mathbf{M}$

**M** c.t.m. $\mathbf{ZFC}+\mathbf{CH}$, $\mathbb{P}=\mathbb{P}^\mathbf{M}_{\omega_2}$. $\mathbf{CH}^M\to(2^{\aleph_0}=\aleph_2)^{\mathbf{M}[G]}$

→ ¬**GCH**

$\alpha\in\mathbf{On}$, $\mathbb{P}_{\aleph_\alpha}=\{f:\aleph_\alpha^\mathbf{M}\times\omega\to 2|dom(f)\, is\, fin.\}$, $f\leqslant g\leftrightarrow f\supseteq g$, $\mathbb{1}=\varnothing$

$\mathbb{P}_{\aleph_\alpha}$ has c.c.c., $\mathbb{P}=\mathbb{P}^\mathbf{M}_{\aleph_{\alpha+1}}$, $(\mathbf{GCH})^\mathbf{M}\to(2^{\aleph_0}=\aleph_{\alpha+1})^{\mathbf{M}[G]}$

→ ¬**AC**

$\pi:\mathbb{P}\to\mathbb{P}$ autom.: bij, $p\leqslant q\Leftrightarrow\pi(p)\leqslant\pi(q)$, $\pi(\mathbb{1})=\mathbb{1}$

$\pi\in\mathbf{M}$ autom. $G-\mathbb{P}-gen\Leftrightarrow\pi[G]-\mathbb{P}-gen.$

$\hat{\pi}:\mathbf{M}^\mathbb{P}\to\mathbf{M}^\mathbb{P}$, $\hat{\pi}(\tau)=\{(\hat{\pi}(\sigma),\pi(p))|(\sigma,p)\in\tau\}$

$\pi$ autom. $\Rightarrow\mathbf{M}[\pi[G]]=\mathbf{M}[G]$, $\forall x\hat{\pi}(\check{x})=\check{x}$, $\forall\vec{\tau}\forall p\in\mathbb{P}p\Vdash\varphi(\vec{\tau})\Leftrightarrow\pi(p)\Vdash\varphi(\hat{\pi}(\vec{\tau}))$

$b\in OD(A)\Leftrightarrow b=\{z\in\mathbf{V}_\alpha|a_i\in A,\varphi(z,\vec{\alpha},\vec{a},A)\}$

$b\in HOD(A)\Leftrightarrow b\in OD(A)$, $tc(b)\subseteq OD(A)$

$(\mathbf{ZF})^{HOD(A)}$,

$\forall\mathbf{M}$ c.t.m. "**ZFC**" $\mathbb{P}_{\aleph_0}\in\mathbf{M}\,\exists A\in\mathbf{M}\,((\neg\mathbf{AC})^{HOD(A)})^{\mathbf{M}[G]}$

→ **Set/class models**

$\mathbf{ZF}\setminus\mathbf{AF}\vdash(\mathbf{ZF}\setminus\mathbf{Inf.})^{\mathbf{W}(\omega)}$, $(\mathbf{ZF}\setminus\mathbf{Inf.})^\mathbf{WF}$, $(\mathbf{ZF})^\mathbf{WF}$

$\mathbf{ZF}\vdash(\mathbf{AC})^\mathbf{L}$, $\mathbf{ZF}\vdash\mathbf{V}=\mathbf{L}\to\mathbf{AC},\mathbf{GCH}$.

$\mathbf{ZFC}\vdash\mathrm{cons}(\mathbf{ZFC})\to\mathrm{cons}(\mathbf{ZF}+\neg\mathbf{AC})$

$\mathbf{ZFC}\vdash\mathrm{cons}(\mathbf{ZFC})\to\mathrm{cons}(\mathbf{ZFC}+\neg\mathbf{CH})$