Decision Theory Theorems

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Chapter 2: Utility Theory

Definition 1 (Utility function)**.** A real values function $u : X \to \mathbb{R}$ is a Paretian utility function for $\succsim$ iff $x \succsim y \iff u(x) \geq u(y)$.

Definition 2 (Quasi-Concave)**.** A function $f : C \to \mathbb{R}$ defined on a convex set of a vector space is quasi-concave if $f(\alpha v + (1-\alpha)w) \geq min\{f(v), f(w)\}$.

Theorem 1. *Concavity implies quasi-concavity.*

Theorem 2 (Quasiconcavity and Upper level sets)**.** *Let $f : C \to \mathbb{R}$ be defined on a convex set. The following are equivalent:*
a) f is quasiconcave.
b) the upper level sets $(f \geq t)$ are convex $\forall t \in \mathbb{R}$.
c) the strict upper level sets $(f > t)$ are strictly convex $\forall t \in \mathbb{R}$.

Theorem 3 (Irreducible quasiconcavity)**.** *Let $f : C \to \mathbb{R}$ and $\phi : D \subset \mathbb{R} \to \mathbb{R}$ an increasing function defined on a convex set such that $Im(f) \subset D$. If f is quasiconcave, then so is $\phi \circ f$.*

Theorem 4 (Existence of a utility function)**.** *A preference relation has a utility representation if it is complete and transitive.*

Theorem 5 (Cardinality of $X/\sim$)**.** *A preference relation on a set X of alternatives has a utility representation only if $card(X/\sim) \leq card(\mathbb{R})$.*

Theorem 6. *A preference relation on a set X with $X/\sim$ at most countable is complete and transitive iff it has a utility representation.*

Theorem 7 (Cantor Debreu)**.** *Let $\succsim$ be a complete and transitive preference on X. The follwoing conditions are equivalent:*
a) $\succsim$ has an at most countable $\succsim$-order dense subsets Z in X;
b) $\succsim$ admits a utility function $u : X \to \mathbb{R}$.

Theorem 8 (Defining a Paretian utility function)**.** *Let $\succsim$ be a preference on $\mathbb{R}^n_+$. The following conditions are equivalent:*
a) $\succsim$ is transitive, complete, strongly monotone and Archimedean;
b) There exists a strongly monotone and continuous function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that, $\forall x, y \in \mathbb{R}^n_+$,

$$x \succsim y \iff u(x) \geq u(y).$$

Moreover, $\succsim$ is (strictly) convex iff u is (strictly) quasiconcave.

Theorem 9 (Lexicographic preferences)**.** *Lexicographic preferences admit no utility function.*

Chapter 3: Rational Choice

Optimal Choice

Definition 3 (Correspondence)**.** $F : X \rightrightarrows Y$ associates to every $x \in X$ a subset $B \subset Y$.
The inverse correspondence: $F^{-1} : Y \rightrightarrows X$, $F^{-1}(y) = \{x \in X : F(x) = y\}$
The graph of F: $GrF = \{(x, y) \in X \times Y : y \in F(x)\}$

Definition 4 (Decision stuff)**.** $\mathbf{X}$ is an all-inclusive choice set. $\mathcal{X}$ is a nonempty collection of subsets (choice sets) $X \in \mathbf{X}$. Choice sets are alternatives among which the DM might have to choose. $P : X \to \succsim_X$ is the preference map. ($\mathbf{X}$, $\mathcal{X}$) is a decision framework and ($\mathbf{X}$, $\mathcal{X}$, P) is a decision environment, (X, $\succsim_X$) is a decision problem.

Definition 5. In a decision problem $(X, \succsim_X)$, $\hat{x}$ is an optimal alternative if $\nexists x \in X : x \succ \hat{x}$.

Theorem 10 (Existence of $\hat{x}$)**.** *In a decision problem (X, $\succsim_x$), optimal alternatives exist if X is finite.*

Definition 6 (Rational Choice Correspondence)**.** $\sigma : \mathcal{D} \rightrightarrows \mathbf{X}$, $\sigma(X) = \{\hat{x} \in X\}$. (the set of optimal alternatives). *Note: $\mathcal{D}$ is the set of the choice sets X that admit at least one $\hat{x}$, which implies $\emptyset \neq \sigma(X)$.*

Theorem 11 (Nature of optimal alternatives)**.** *If $\succsim_X$ is a preorder, then optimal alternatives are either incomparable or indifferent. If $\succsim$ is a weak order, then optimal alternatives are indifferent.*

Theorem 12 (Nature of $\sigma(X)$)**.** *Let X be a convex choice set and $\succsim_X$ a weak order.*
a) If $\succsim_X$ is convex, then $\sigma(X)$ is convex.
b) If $\succsim_X$ is strictly convex, then $\sigma(X)$ is a singleton.

Universal Analysis

Definition 7 (Universal preference)**.** A universal preference $\succsim$ does not depend on the choice set X. That is: $\forall\ X \in \mathcal{X}$, $x' \succsim_X x \iff x' \succsim x$, $\forall x, x' \in X$. *We assume that $\succsim$ is universal in this subsection.*

Theorem 13 (Arrow-Uzawa)**.** *Let $\succsim$ be a weak order. For each $X, Y \in \mathcal{D}$, we have:*
1) $X \subset Y$ implies $X - \sigma(X) \subset Y - \sigma(Y)$.
2) $X \subset Y$ implies $\sigma(Y) \cap X \subset \sigma(X)$ with equality if $\sigma(Y) \cap X \neq \emptyset$.
3) if $x \in \sigma(X)$ and $y \in X - \sigma(X)$, then there is no $Y \in \mathcal{D}$ such that $x \in Y$ and $y \in \sigma(Y)$. (WARP).

Definition 8 (Menu preference)**.** The menu preference $\succeq$ over the choice sets in $\mathcal{D}$, induced by a preference $\succsim$, is defined by:
$X \succeq Y \iff \forall y \in Y, \exists x \in X, x \succsim y, \quad \forall X, Y \in \mathcal{D}$.

Theorem 14 (Kreps)**.** *If $\succsim$ is a weak order, so is the menu preference. Moreover, given any $X, Y \in \mathcal{D}$, we have:*
1) $X \sim \sigma(X)$. Note this is the equivalence for the menu preference.
2) $Y \subset X$ implies $X \succeq Y$.
3) $X \succeq Y$ implies $X \sim X \cup Y$.

Utility Analysis

In this section, it is assume that the universal weak order $\succsim$ has a utility representation u. The optimization problem becomes $max_x u(x)\ sub\ x \in X$ and $\sigma(X) = argmax_{x \in X} u(x)$.

Definition 9 (Value function)**.** $v : \mathcal{D} \to \mathbb{R}$ such that $v(X) = max_{x \in X} u(x)$. Then the menu preference becomes $X \succeq Y \iff v(X) \geq v(Y)$.

Contextualized Analysis

In this section we allow for context (framing effects).

Definition 10 (contextualized alternatives)**.** (x, X) with $x \in X$ is a contextualized alternative. $\mathcal{C}$ is the collection of all contextualized alternatives. A preference on $\mathcal{C}$ has the form $(x, X) \succsim (y, Y)$.

Definition 11 (contextualized universal preference)**.** $\succsim$ is a CUP over $\mathcal{C}$ if $\forall\, X \in \mathcal{X}$:
$(x, X) \succsim (x', X) \iff x \succeq_X x',\ \forall x', x \in X$ with $\succsim_X = P(X)$.

Definition 12 (Menus with context)**.** Assume $\succsim$ is a CUP, then $X \succeq Y \iff \forall y \in Y, \exists x \in X, (x, X) \succsim (y, Y)$.

Parametrized Analysis

Choice sets are usually parametrized by elements of a set Θ. Parametrization is carried out via a correspondence: $\varphi : \Theta \rightrightarrows \mathbf{X}$. Each θ is associated to a choice set $X(\theta)$. Thus, $\mathcal{X} = \{\varphi(\theta) : \theta \in \Theta\}$ and $\mathcal{D} = \{\varphi(\theta) : \theta \in\}$ Get Back to This!!

Decision problems under certainty

All inclusive set $\mathbf{A}$ of actions, all inclusive set $\mathbf{C}$ of consequences, a consequence function $\rho(a) = c$ and a collection set $\mathcal{A}$ with sets A from which the DM may have to choose from. $(\mathbf{A}, \mathcal{A}, \mathbf{C}, \rho)$ is a decision pre-framework under certainty. A preference $\succsim$ over actions and a preference $\succsim^\circ$ over consequences. $(\mathbf{A}, \mathcal{A}, \mathbf{C}, \rho, \succsim, \succsim^\circ)$ is a decision pre-environment under certainty.

Definition 13 (Outcome Consequentialism)**.** $a \succsim b \iff \rho(a) \succsim^\circ \rho(b)\ \forall a, b \in \mathbf{A}$.

Theorem 15 (Concatenation of utilities)**.** *If $\succsim^\circ$ admits a utility function $u^{\cdot}$, then $u = u^{\cdot} \circ p$ is a utility function for $\succsim$.*
Note: $(\boldsymbol{A}, \mathcal{A}, u^{\cdot} \circ p)$ is a decision environment under certainty.

Definition 14. Certainty definitions:

- Optimization problem under certainty: $max\ u^{\cdot}(\rho(a))\ sub\ a \in A$.
- Rational Choice correspondence: $\sigma(A) = argmax_{a \in A} u^{\cdot}(\rho(a))$.
- Value function: $v(A) = max_{a \in A} u^{\cdot}(\rho(a))$.

Theorem 16 (Reduced Form)**.** *Assume that $\boldsymbol{C} = Im\,\rho$ and let $c, d \in \boldsymbol{C}$. Under OC,*

$$c \succsim^\circ d \iff a \succsim b\ \forall (a, b) \in \rho^{-1}(x) \times \rho^{-1}(d)$$

Definition 15 (Consequence Stuff)**.** An action set A induces by a consequence fuction a consequence set $C = \rho(A)$. If consequence $\hat{c} \in C$ is optimal, then any action $\hat{a} \in \rho^{-1}(\hat{c}) \cap A$ is optimal in A. Hence, the decision pre-environment admits a reduced form given by the decision environment $(\mathbf{C}, \mathcal{C}, \succsim^\circ)$. $(C, \succsim^\circ)$ is a decision problem, $(\mathbf{C}, \mathcal{C}, u^{\cdot})$ and $(C, u^{\cdot})$ are respectively decision environment and decision problem under certainty.

The consumer problem

Definition 16. Budget set: $B(p, w) = \{c \in \mathbb{R}^n_+ : p \cdot c \leq w\}$. Budget correspondence $B : \mathbb{R}^{n+1}_+ \rightrightarrows \mathbb{R}^n_+$ describes how budget sets vary with prices and wealth. This is getting boring. Make a big table with all the environments.

Definition 17 (Budget line)**.** $\Delta(p, w) = \{c \in \mathbb{R}^n_+ : p \cdot c = w\}$

Definition 18 (Optimal consumption bundle)**.** Given (p, w), $\hat{c}$ is optimal if $u(\hat{c}) \geq u(c) \forall c \in B(p, w)$.

Definition 19 (Walrasian demand correspondence)**.** $d : D \rightrightarrows \mathbb{R}^n_+$ given by $d(p, w) = arg\ max_{c \in B(p,w)} u(c)$.

Definition 20. Value function: $v : D \to \mathbb{R}$ given by $v(p, w) = max_{c \in B(p,w)} u(c)$

Lemma 17. *If $p \gg 0$, then the budget set $B(p, w)$ is compact.*

Theorem 18. *If u is continuous, then $\mathbb{R}^n_{++} \times \mathbb{R}_+ \subseteq D$. Equality holds if u is strictly monotone.*

Theorem 19 (Homogeneity of the demand)**.** $\forall (p, w) \in D,\ d(p, w) = d(\alpha p, \alpha w)\ \forall \alpha > 0$.

Theorem 20 (Walras' law)**.** *If u is strictly monotone, then $\forall (p, w) \in D,\ p \cdot \hat{c} = w\ \ \forall \hat{c} \in d(p, w)$.*

Theorem 21 (Topology of demand correspondence)**.** *$d(p, w)$ is:*

- *convex-valued if u is quasiconcave.*
- *single-valued if u is strictly quasiconcave.*

Theorem 22 (Homogeneity of the value function)**.** $v(p, w) = v(\alpha p, \alpha w)\ \forall \alpha > 0$

Theorem 23 (Monotonicity of the value function)**.** *The value function(indirect utility function) is:*

- *increasing in w: $w \geq w' \Rightarrow v(p, w) \geq v(p, w')$.*
- *decreasing in p: $p \geq p' \Rightarrow v(p, w) \leq v(p', w)$.*

Remark: if u is strongly monotone, the monotonicities above are strict.

Theorem 24. *The value function is quasiconvex.*

Theorem 25. *The value function is (strictly) concave in w if u is (strictly) concave.*

Price Theory

In this chapter all utility functions are continuous, strongly monotone and strictly quasiconcave.

Definition 21 (Type of goods)**.** A good k is:

- normal if, for all $(p, w'), (p, w) \in D$, $w > w' \to d_k(p, w) \geq d_k(p, w')$
- inferior if, for all $(p, w'), (p, w) \in D$, $w > w' \to d_k(p, w) \leq d_k(p, w')$

Definition 22 (Locality)**.** We say that a good k at (p, w), is locally:

- normal: $\frac{\partial d_k(p,w)}{\partial w} \geq 0$
- inferior: $\frac{\partial d_k(p,w)}{\partial w} \leq 0$

Definition 23 (Giffen Good)**.** A good k is Giffen if there exists two price vectors p and p' with $p'_k > p_k$ and $p'_i = p_i \; \forall i \neq k$, such that $d_k(p', w) > d_k(p, w)$.

Definition 24 (Slutsky's Wealth Adjustment)**.** Suppose that prices increase from $p \to p'$. Slutsky's Wealth Adjustmenet is the amount of wealth that makes $\hat{c} \in (p, w)$ still feasible. That is $p' \cdot \hat{c} = w'$.

Theorem 26 (p)**.** *Let $(p', w'), (p, w) \in D$. Then*

$$p' \cdot \hat{c} \leq w' \text{ and } \hat{c} \neq \hat{c}' \to p \cdot \hat{c}' > w$$

Theorem 27 (p, Slutksy Law of Demand)**.** *Let $(p', w'), (p, w) \in D$ be such that $p' \cdot \hat{c} = w'$. Then:*

$$(p' - p)(\hat{c}' - \hat{c}) \leq 0$$

with strict inequality when $\hat{c} \neq \hat{c}'$.

Definition 25 (Rewriting demand)**.**

$$d(p', w) - d(p, w) = d(p', w) - d(p', w') + d(p', w') - d(p, w)$$

Theorem 28 (p, Normal Law of Demand)**.** *If income and other prices do not change, an increase in the price of a normal good decreases its demand. That is: if $p' = p + \Delta_k$, then $d(p', w) \leq d(p, w)$.*

Theorem 29 (p, Local Normal Law of Demand)**.** *Let the demand function $d : D \to \mathbb{R}$ be differentiable at $(p, w) \gg \mathbf{0}$. If good k is normal at (p, w), then*

$$\frac{\partial d_k(p, w)}{\partial p_k} \leq 0$$

Theorem 30. *(p) If the demand function is differentiable, then:*

$$\frac{d_k(p, w)}{\partial p_j} = s_{k,j} - \frac{\partial d_k(p, w)}{\partial w} d_j(p, w) \quad \forall j, k = 1, \ldots, n$$

Theorem 31. *If the indirect utility function is twice differentiable at $(p, w) \gg \mathbf{0}$, with $\partial v(p, w)/\partial w \neq 0$, then $s_{k,j}(p, w) = s_{j,k}(p, w)$.*

Definition 26. Assume that the demand function is differentiable at $(p, w) \gg \mathbf{0}$

Chapter 7

Definition 27 (Convex Structure)**.** The space of lotteries **L** has a convex structure. Given two lotteries l and l' and any weight $q \in [0, 1]$, the convex combination $ql + (1 - q)l'$ is an element of **L**.

Definition 28 (Random Consequentialism)**.** Actions are ranked according to their random consequences:

$$a \succsim b \iff \rho(a) \ddot{\succsim} \rho(b)$$

for all $a, b \in \mathbf{A}$.

Definition 29. Lingo:

- optimization problem: max $\ddot{u}(\rho(a))$ sub $a \in A$.
- $\sigma(A) = \text{argmax } \ddot{u}(\rho(a))$
- $v(A) = \max \ddot{u}(\rho(a))$ sub $a \in A$.

Definition 30. In reduced form:

- optimization problem: max $\ddot{u}(l)$ sub $l \in L$.

Once we have found a solution $\hat{l}$ of this problem, any action $\hat{a} \in \rho^{-1}(\hat{l}) \cup A$ solves the original problem.

Expected utility theory

Definition 31. Axioms on lotteries:

- B.1 WEAK ORDER: $\succsim$ is complete and transitive.
- B.2 INDEPENDENCE: $l \succ l' \to pl + (1 - p)l'' \succ pl' + (1 - p)l'' \quad \forall p \in (0, 1)$.
- B.3 ARCHIMEDEAN: Let $l \succ l' \succ l''$. There exists $p, q \in (0, 1)$ such that $pl + (1-p)l'' \succ l' \succ ql + (1-q)l''$.

Definition 32. A binary relation $\succsim$ on **L** is a vN-M preference if it satisfies axioms B.1, B.2 and B.3.

Theorem 32 (vN-M)**.** *Let $\succsim$ be a preference on $\boldsymbol{L}$. TFAE:*

- *$\succsim$ satisfies B.1, B.2 and B.3.*
- *$\exists\, u : \boldsymbol{C} \to \mathbb{R}$ (vN-M) utility function such that*

$$\ddot{u}(l) = \sum_{i=1}^{n} u(c_i) p_i$$

  represents $\succsim$.

Definition 33. A binary relation on **L** is a:

- monotone preference if it satisfies strong monotonicity.
- monotone vN-M preference if it satisfies axioms B.1-B.3 and strong monotonicity.

Risk Aversion

Definition 34. A preference $\succsim$ on $\mathbf{L}$ is called:

- risk averse if, for each lottery in $\mathbf{L}$, it holds $\mathbb{E}(l) \succsim l$
- risk loving if, for each lottery in $\mathbf{L}$, it holds $l \succsim \mathbb{E}(l)$.
- risk neutral if, for each lottery in $\mathbf{L}$, it holds $l \sim \mathbb{E}(l)$.

Theorem 33 (p)**.** *A vN-M preference $\succsim$ is risk averse iff its vN-M utility function $u : \boldsymbol{C} \to R$ is concave. (convex if risk loving, affine if risk neutral).*

Theorem 34. *A vN-M preference is risk neutral iff it ranks lotteries according to their expected value:*

$$l \succsim l' \iff \mathbb{E}(l) \geq \mathbb{E}(l)$$

Definition 35 (Comparitive Risk Aversion)**.** A preference $\succsim_1$ is more risk averse than a preference $\succsim_2$ if, for all $c \in \mathbf{C}, l \in \mathbf{L}$, both:

$$l \succsim_1 c \to l \succsim_2 c \qquad l \succ_1 c \to l \succ_2 c$$

hold.

Definition 36 (B.5 Certainty Equivalent)**.** : each lottery in $\mathbf{L}$ has a price $c_l \in \mathbf{C}$ such that $l \sim c_l$.

Theorem 35. *A monotone vN-M preference $\succsim$ satisfies B.5 iff its vN-M utility function is continuous.*

Theorem 36 (p)**.** *If $\succsim$ is a monotone (satisfying B.4 Strong Monotonicity) preorder, then every lottery has at most a unique certainty equivalent.*

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Definition 37 (Certainty equivalent function)**.** $\boldsymbol{c}(l) = c_l$

Definition 38 (Risk premia)**.** Let $\succsim$ be a preference satisfying axiom B.5. The risk premium of a lottery $l \in \mathbf{L}$ is the difference

$$\pi_l = \mathbb{E}(l) - c_l$$

between its expected value and its certainty equivalent.

Definition 39 (Risk premium function)**.** $\boldsymbol{\pi}(l) = \pi_l$

Theorem 37 (p)**.** *A monotone preorder that satisfies axiom B,4 is risk averse if and only if every lottery has a positive risk premium.*

Theorem 38 (p)**.** *A monotone preorder that satisfies the certainty equivalent axiom is risk averse iff it is more risk averse than a risk neutral one.*

Theorem 39. *Let $\succsim_1$ and $\succsim_2$ be two monotone vN-M preferences. The following conditions are equivalent:*

- *$\succsim_1$ is more risk averse than $\succsim_2$.*
- *$\exists$ a strictly increasing concave transformation $g : Imu_2 \to \mathbb{R}$ such that $u_1 = g \circ u_2$.*

Definition 40. Arrow Pratt index:

$$\lambda(c) = -\frac{u''(c)}{u'(c)}$$

Theorem 40. *Let $\succsim_1$ and $\succsim_2$ be two monotone preorders that satisfy axiom B.5. TFAE:*

- *$\succsim_1$ is more risk averse than $\succsim_2$.*
- *$\boldsymbol{c}_1 \leq \boldsymbol{c}_2$.*
- *$\boldsymbol{\pi}_1 \geq \boldsymbol{\pi}_2$.*
- *$\lambda_1 \geq \lambda_2$.*

Chapter 11: A portfolio illustration

In this chapter we consider two assets, a risk free asset that return r_f per euro invested for sure, and a risky asset with an uncertain gross return represented by the lottery:

$$l = \{r_1, p_1; \ldots; r_k, p_k\}$$

The investor invests an amount $\alpha_1 \geq 0$ in the risky asset and an amount $\alpha_2 \geq 0$ in the risk-free asset. The vector $\alpha = (\alpha_1, \alpha_2)$ represents the overall investment. The random gross return portfolio is then described by:

$$l_\alpha = \{\alpha_1 r_1 + \alpha_2 r_f, p_1; \ldots; \alpha_1 r_k + \alpha_2 r_f, p_k\} = l + \alpha_2 r_f$$

$A(w)$ is the collection of portfolios the investor with wealth w is able to trade. An investor evaluates a portfolio α:

$$u(\alpha) = (\ddot{u} \circ \rho)(\alpha) = \sum_{i=1}^{k} \dot{u}(\alpha_1 r_i + \alpha_2 r_f) p_i$$

where $\dot{u}$ is a vN-M utility function. Throughout this chapter, we assume that $\dot{u}$ is strictly increasing and strictly concave with $u'_+ > 0$. Thus investors in this chapter are strictly risk averse. The portfolio optimization problem is $\max_\alpha u(\alpha)$ sub $\alpha \in A(w)$. A portfolio is optimal if $u(\hat{\alpha}) \geq u(\alpha)\ \forall \alpha \in A(w)$.

If the portfolio is constrained in the following way $\alpha_1 + \alpha_2 = w$, then we can focus only on the investment on the risky asset $alpha_1 = a$ and denote by $w - a = \alpha_2$, reducing a 2D problem into a 1D one. The uncertain payoff of a is then:

$$l_{a,w} = \{ar_1 + (w-a)r_f, p1; \ldots; ar_k + (w-a)r_f, p_k \qquad u_w(a) = \sum_i \dot{u}(ar_i + (w-a)r_f)p_i$$

Theorem 41. *The porfolio has a unique solution $\hat{a}_w$ for each wealth level $w \geq 0$.*

Definition 41 (p, Risky asset demand function)**.** a$(w) = a_w$. It associates to each wealth level the corresponding optimal risky investment.

Definition 42 (Expected excess asset return)**.** $\bar{E}(l) = E(l) - r_f$

Theorem 42 (p)**.** $\bar{E}(l) \leq 0$ *if and only if* $a(w) = 0\ \forall w \geq 0$.

Theorem 43. *If an analytical investor $\succsim_1$ is more risk averse than an analytical investor $\succsim_2$, then $a_2 \geq a_1$.*

Chapter 12: Wealth effects

In this chapter we consider the DM's wealth as a factor in determining a choice between lotteries. That is:

$$l = \{c_1, p_1; \ldots; c_n, p_n\} \text{ translates to } l^w = \{c_1 + w, p_1; \ldots; c_n + w, p_n\}$$

$\succsim$ induces a preference $\succsim_w$ as follows:

$$l \succsim_w \tilde{l} \iff l^w \succsim \tilde{l}^w$$

Definition 43 (DARA)**.** A preference on L is DARA (decreasingly absolute risk averse) if $w' \geq w$, $\succsim_{w'}$ is less risk averse than $\succsim_w$.

Definition 44 (Certainty Equivalent)**.** The wealth adjusted certainty equivalent is defined as $c_l^w \sim_w l$. For a weak order satisfying certainty equivalent, the certainty equivalent function $\mathbf{c} : [0, \infty) \times L \to \mathbb{R}$ is defined as $\mathbf{c}(w, l) = c_l^w$. The risk premium function is $\pi(w, l) = E(l) - \mathbf{c}(w, l)$.

Theorem 44. *Let $\succsim$ be a monotone weak order satisfying $B.5$ on L. The following conditions are equivalent:*

- *$\succsim$ is DARA.*
- *$\mathbf{c}_l$ is increasing in w for each $l \in L$.*
- *π_l is decreasing in w for each $l \in L$.*

Theorem 45 (Arrow Pratt)**.** *Let $\succsim$ be a vN-M preference with a twice continosly differentiable vN-M utility function $u : (a, \infty) \to \mathbb{R}$ with $u' > 0$ and $a \in \mathbb{R}$. TFAE:*

- *$\succsim$ is DARA.*
- *its Arrow-Pratt index $\lambda : (a, \infty) \to \mathbb{R}$ is decreasing.*

Theorem 46 (Uniqueness of CARA)**.** *A twice differentiable vN-M utility function with $u' > 0$ is CARA iff up to a affine transformation:*

$$u(c) = \begin{cases} -e^{\lambda c} & \text{if } \lambda > 0 \\ c & \text{if } \lambda = 0 \\ e^{\lambda c} & \text{if } \lambda < 0 \end{cases}$$

Theorem 47. *The risky asset demand $a : [0, \infty) \to \mathbb{R}$ is increasing in w if an investor is DARA.*