Analysis Formula Booklet

Klejdi Sevdari

May 2021

Chapter 1: The Euclidan Space

The basics

- Dot (inner) product: if $\overline{x}, \overline{y} \in \mathbb{R}^d$, then $\overline{x} \cdot \overline{y} = \sum_{i=1}^{d} x_i y_i$
- Scalar product properties:
  - Symmetry: $x \cdot y = y \cdot x$
  - Linearity: $(ax + by)z = a(x \cdot z) + b(x \cdot z) \quad (a, b \in \mathbb{R},\ x, y, z \in \mathbb{R}^d\ )$
  - Positive definite: $x \cdot x \geq 0$ with equality if and only if (iff) $x = 0$
- Orthogonality: x, y orthogonal iff $x \cdot y = 0$
- Norm: $\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_i x_i^2}$. *Remark:* $\|x\|^2 = x \cdot x$
- Distance between two vectors $x, y \in \mathbb{R}^d = \|x - y\|$
- Cauchy-Schwarz inequality: $|x \cdot y| \leq \|x\|\|y\|$ with equality if x, y are linearly dependent.
- Norm properties:
  - Homogeneity: $\|ax\| = |a|\|x\|$ where a is a scalar.
  - Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ with equality if $x = \lambda y$ with $\lambda > 0$
  - Positivity: $\|x\| \geq 0$ with equality iff $x = 0$
- Angle between vectors: $x \cdot y = \|x\|\|y\| cos(\theta)$
- Law of cosine: $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| cos(\theta)$
- Cross product: $x \times y = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$ for $x, y \in \mathbb{R}^3$
- Determinants and Cross Product: $det(x, y, z) = (x \times y) \cdot z$ for $x, y, z \in \mathbb{R}^3$
- Cross product properties:
  - Orthogonality: $x \times y$ is orthogonal to x and y.
  - Area: $\|x \times y\| = \|x\|\|y\| \sin(\theta) =$ Area of parallelogram built from x and y
  - Positive orientation: $det(x, y, x \times y) \geq 0$
  - Anti-Symmetry: $x \times y = -y \times x$
  - Linearity: $x \times (ay + bz) = a(x \times y) + b(x \times z)$
  - Zero vector: $x \times y = 0$ iff x and y are linearly dependent.
  - Application: Given 3 points in plane, one can find the plane equation.

Domains of $\mathbb{R}^d$

- Lines: $L = \{x \in \mathbb{R}^d : \exists\, t \in \mathbb{R},\ x = x_0 + tu\}$ where $x_0, u \in \mathbb{R}^d$ (u is the direction).
- (Hyper)planes: $D = \{x \in \mathbb{R}^d : (x - x_0) \cdot n = 0\}$ where $x_0, n \in \mathbb{R}^d$ (n is the normal to the plane).
- Speheres/circles: $D = \{x \in \mathbb{R}^d : \|x - x_0\| = r\}$ (where $x_0 \in \mathbb{R}^d$ is the origin and $r \in \mathbb{R}$ is the radius).
  Circle parametrization: $D = \{(x_0 + r\cos(t), y_0 + rsin(t)) : t \in \mathbb{R}\}$
- Ellipses (centred at the origin): $D = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$
  Parametrization: $D = \{(a\cos(t), b\sin(t)) : t \in \mathbb{R}\}$
- Cylinders: $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = r^2\}$
- Cones: $D = \left\{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} = |z|\right\}$
- A line in $\mathbb{R}^3$ can be thought as the intersection of two planes which are not parallel. The direction of the line will be the cross product of the normal vectors of the plane.

Remark:
Practical knowledge for calculating rotations of vectors by an angle θ. We define the rotation matrix to be $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. Given a direction vector $u = \begin{pmatrix} a \\ b \end{pmatrix}$, one can find the direction vector u', which is obtained by a rotation of u in the counterclockwise direction by an angle θ_0 using the following formula: $u' = R(\theta_0)u$.

Chapter 2: Parametric curves

- Definition: $\gamma : I \subset \mathbb{R} \to \mathbb{R}^d$. $\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \dots \\ \gamma_d(t) \end{pmatrix}$

  where each γ_i is a real valued function callled the i^{th} coordinate function.

- Continuity: γ is continuous at $t \in I$ if all γ_i are continuous at $t \in I$.

- Differentiability: γ is differentiable at $t \in I$ if all γ_i are differentiable at $t \in I$. $\gamma'(t) = \begin{pmatrix} \gamma_1'(t) \\ \dots \\ \gamma_d'(t) \end{pmatrix}$

  The speed of the curve at t is defined as $\|\gamma'(t)\|$. Useful equality: $g(t) = \|\gamma(t)\|^2$, then $g'(t) = 2\gamma'(t)\cdot\gamma(t)$

- Class C^k: γ is of class C^k if all γ_i are of class C^k over I.

- Tangent of (the image of) a curve at t: It is line that passes through $\gamma(t)$ with direction $\gamma'(t)$.

- Taylor Expansion: If γ is k times differentiable at t_0 then:

$$\gamma(t_0 + h) = \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0) + \frac{h^3}{6}\gamma^{(3)}(t_0) \dots + \frac{h^k}{k!}\gamma^{(k)}(t_0) + o(h^k)$$

  where $o(h^k) : J \subset \mathbb{R} \to \mathbb{R}^d$ where J is a neighborhood of 0 and each $g_i = o(h^k)$.
  The idea is that the function behaves locally as the principal part of the Taylor expansion.

- Drawing curves and local approximation by Taylor expansion:
  - Regular points: $\gamma'(t_0) \neq 0$. The image of the curve can be approximated by the tangent line.
  - Biregular points: $\gamma'(t_0) \neq 0$ and $\gamma''(t_0)$ is **not** colinear to $\gamma'(t_0)$, then:

$$\gamma(t_0 + h) = \gamma(t_0) + h\gamma'(t_0) + \frac{h^2}{2}\gamma''(t_0) + o(h^2)$$

    Thus in the frame $(\gamma'(t_0), \gamma''(t_0))$ centred at $\gamma(t_0)$ the image of γ is close to the parabola $y = \frac{1}{2}x^2$
  - Inflection point: $\gamma''(t_0) = \lambda\gamma'(t_0)$ (colinear) but $\gamma''(t)$ not colinear to $\gamma'''(t_0)$, then

$$\gamma(t_0 + h) = \gamma(t_0) + \left(h + \frac{h^2}{2}\lambda\right)\gamma'(t_0) + \frac{h^3}{6}\gamma'''(t_0) + o(h^3)$$

    Thus in the frame $(\gamma'(t_0), \gamma'''(t_0))$ centred at $\gamma(t_0)$ the image of γ is close to $y = \frac{1}{6}x^3$
  - Cusp: $\gamma'(t_0) = 0$ but $\gamma''(t_0), \gamma'''(t_0)$ are not colinear, then:

$$\gamma(t_0 + h) = \gamma(t_0) + \frac{h^2}{2}\gamma''(t_0) + \frac{h^3}{6}\gamma'''(t_0) + o(h^3)$$

    Thus in the frame $(\gamma''(t_0), \gamma'''(t_0))$ centred at $\gamma(t_0)$ the image of γ is close to $(x, y) = (h^2/2, h^3/6)$

- Orthogonality of derivatives: $\gamma'(t) \perp\!\!\!\perp \gamma''(t)$ for all $t \in I$ iff the function $t \to \|\gamma'(t)\|$ is constant.

- Polar coordinates: Let $g : \mathbb{R} \to [0, \infty)$. The curve $r = g(\theta)$ is defined as:

$$\gamma(\theta) = g(\theta)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} = g(\theta)e_r(\theta)$$

- Derivatives in polar coordinates: $e_r' = e_\theta$ and $e_\theta' = -e_r$, then $\gamma' = g'e_r + ge_\theta$ and $\gamma'' = (g'' - g)e_r + 2g'e_\theta$

- Easy check for cusps in Polar coordinates: $\|\gamma'(t)\| = \sqrt{g'(\theta)^2 + g(\theta)^2}$

Chapter 3: Topology

- Limit: $(x_n) \in \mathbb{R}^d$ converges to $a \in \mathbb{R}^d$ iff $\forall \epsilon > 0,\ \exists N \in \mathbb{N}, \forall n \geq N, \|x_n - a\| \leq \epsilon$
- Coordinate wise convergence: $(x_n) \to a$ iff $\forall i \in \{1, \dots d\}, (x_{i,n}) \to a_i$.
- Operations on limits: Let $(x_n) \to a$ and $(y_n) \to b$. Then:
  - $(x_n \pm y_n) \to a \pm b$
  - $(x_n \cdot y_n) \to a \cdot b$ and if $x_n, y_n \in \mathbb{R}^3, (x_n \times y_n) \to a \times b$
  - $\|x_n\| \to \|a\|$

  Remark: The norm, dot product and the cross product are continuous functions.
- Open and Closed Ball: $B_o(a, r) = \{x \in \mathbb{R}^d : \|x - a\| < r\}$ and $B_c(a, r) = \{x \in \mathbb{R}^d : \|x - a\| \leq r\}$
- Neighborhood: V is a neighborhood of x if $\exists \epsilon > 0 : B_c(x, \epsilon) \subset V$
- Interior: $x \in V^{\text{o}}$ iff V is a neighborhood of x.
- Properties of interior: $V^{\text{o}} \subset V$, and if $V \subset W$ then $V^{\text{o}} \subset W^{\text{o}}$
- Closure: $x \in \overline{V}$ iff $\exists\ (y_n) \in V : (y_n) \to x$
- Properties of closure: $V \subset \overline{V}$, and if $V \subset W$ then $\overline{V} \subset \overline{W}$
- Boundary: $\partial V = \overline{V} \setminus V^{\text{o}}$.
- Link between interior and closure: $\overline{(V^c)} = (V^{\text{o}})^c$ and $\left(\ \overline{V}\ \right)^c = (V^c)^{\text{o}}$
- Open and closed sets: V is open if $V^{\text{o}} = V$ and is closed if $\overline{V} = V$.
- Link between open and closed sets: if V is open then V^c is closed and vice versa.
- Interior open, Closure closed: V^{o} is open and it is the largest open set contained in V. $\overline{V}$ is closed and is the largest closed set containing V.

Chapter 4: functions from $\mathbb{R}^2 \to \mathbb{R}$.

In this chapter by f we denote a function from $D \subset \mathbb{R}^2 \to \mathbb{R}$ and γ is a parametric curve from $I \subset \mathbb{R} \to \mathbb{R}^2$.

Continuity

- Continuity: f continuous at x_0 if $\forall \epsilon > 0,\ \exists\, \delta > 0 : \|y - x_0\| \leq \delta \implies |f(y) - f(x_0)| \leq \epsilon$
- Sequential characterization of continuity: f continuous at x_0 iff $\forall y_n \to x_0, \lim_{n\to\infty} f(y_n) = f(\lim_{n\to\infty} y_n) = f(x_0)$
- If f, g continuous, then $f \circ g, \frac{f}{g}, f+g, f-g$ are continuous. A linear function is always continuous.

Differentiability

- Partial Derivatives: $\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$.

  $\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$
- Gradient: $\nabla f(\mathbf{x}) = \begin{bmatrix} f_x(\mathbf{x}) \\ f_y(\mathbf{x}) \end{bmatrix}$
- The differential: $D_{f_{x_0}}(h)$ is a linear map. $D_{f_{x_0}}(h) = \nabla f(x_0, y_0) \cdot \mathbf{h}$. $D_{f_{x_0}}(e_1) = f_x(x_0)$, $D_{f_{x_0}}(e_2) = f_y(x_0)$.
- Differentiability: $f(x_0 + h, y_0 + k) = f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \mathbf{h} + o(\mathbf{h})$. where $o(h) = \|h\|\, w(h)$, $w(0) = 0$ and w continuous at 0.
- Implications: Differentiability implies continuity and the existence of partial derivatives.
- Compositions: if $g = f \circ \gamma$, then $g'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)$
- Directional Derivatives: $\frac{\partial f}{\partial u}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot u$
- Tangent Plane at a point (x_0, y_0): $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
- Sufficient Condition for Differentiability: If the partial derivatives exist and are continuous everywhere in an open neighborhood V of a point (x_0, y_0), then the function is differentiable in the whole V.
- Level sets and gradient: if γ is a function which takes values on a level set of f, then $\nabla f(\gamma(t)) \cdot \gamma'(t) = 0$ (orthogonality). This follows from the fact that $f \circ \gamma = c$.
  Note: The gradient denotes the direction in which the function increases the most locally, and the direction orthogonal to it denotes the direction of no increase.
- Class C^1: A function is of class C^1 if the partial derivatives exists everywhere and are continuous. If f, g are of class C^1 then so are $f \pm g, fg, f/g$(if g does not vanish) and $f \circ g$.

Chapter 5: functions from $\mathbb{R}^d \to \mathbb{R}^p$.

Continuity definitions and theorems are the same as in chapter 4, so we only concern ourselves with differentiability in this chapter. Below we will denote by $f : D \subset \mathbb{R}^d \to \mathbb{R}^p$. The coordinate functions are denoted by f_j for $1 \leq j \leq p$.

- A word on continuity: f is continuous at x_0 iff $\forall j \in \{1, \ldots, d\}$ f_j is continuous at x_0.
- Parital derivatives: Let $i \in \{1, \ldots, d\}$. The partial derivative in the direction x_i at a point $x_0 \in R^d$ is the vector $(\frac{\partial f_j}{\partial x_i}(x_0))_{1 \leq j \leq p}$.
- Jacobian Matrix: $J_f(x_0)$ is a matrix M of dimensions $p \times d$, where $M_{ji} = \frac{\partial f_j}{\partial x_i}$.
- Differentiability: $f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + J_f(\mathbf{x}) \cdot \mathbf{h} + o(\mathbf{h})$, where $o(\mathbf{h}) = \|h\| \, w(h)$.
- Tangent to a surface: Let $f : (u, v) \in \mathbb{R}^2 \to (x, y, z) \in \mathbb{R}^3$. Let $(u_0, v_0) \in \mathbb{R}^2$ and assume that f is differentiable at that point. The the tangent plane at (u_0, v_0) to the surface is the image of the principle part of the Taylor expansion: $(h, k) \in \mathbb{R}^2 \to f(u_0, v_0) + h\frac{\partial f}{\partial u}(u_0, v_0) + k\frac{\partial f}{\partial v}(u_0, v_0)$.

  A point $\mathbf{z} = (x, y, z)$ belongs to the the tanget plane at (u_0, v_0) iff:

  $$(\mathbf{z} - f(u_0, v_0)) \cdot (\frac{\partial f}{\partial u}(u_0, v_0) \times \frac{\partial f}{\partial v}(u_0, v_0)) = 0$$

  Note that $\frac{\partial f}{\partial u}(u_0, v_0)$ and $\frac{\partial f}{\partial v}(u_0, v_0)$ are vectors.
- Sufficient condition for Differentiability: Same as in chapter 4, but we require all partial derivatives of the form $\frac{\partial f_j}{\partial x_i}$ (in total $p \times d$ partial derivatives) to exist and be continuous in a neighborhood of the point of interest.
- Chain rule: Let $f : \mathbb{R}^d \to \mathbb{R}^p$ and $g : \mathbb{R}^p \to \mathbb{R}^q$. Denote by $h = g \circ f$. Then:

  $$J_h(x_0) = J_g(f(x_0))J_f(x_0)$$
- Compositions of class C^1 functions: if f, g are of class C^1, then so is $g \circ f$.

Chapter 6: Higher order derivatives

Below we denote by $f : D \subset \mathbb{R}^d \to \mathbb{R}$. We will sometimes denote $\frac{\partial f}{\partial x}$ by f_x.

- Definition: We say that f is of class C^2 if it is of class C^1 and all its partial derivatives $\frac{\partial f}{\partial x_i} : D \to R$ for $i \in \{1, 2, \dots d\}$ are of class C^1. Then we define:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

- Schwarz's Theorem: if f is of class C^2 then: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ (the order is irrelevant).

- Necessary condition for a vector field to be a gradient: In order for $g(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$ to be a gradient (that is $g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \nabla f$), the following condition should be satisfied:

$$(*)\ \frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}$$

  Note: The converse implication holds if (*) holds and the domain of definition of g is the whole space.

- Hessian Matrix: It is the collection of the second order derivatives. For a general function $f : \mathbb{R}^d \to \mathbb{R}$ it is a $d \times d$ matrix, whose entry in the i^{th} row and j^{th} column is $\frac{\partial^2 f}{\partial x_i \partial x_j}$. If $d = 2$:

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

  Note: H_f is a symmetric matrix (wrt the diagonal) due to Schwarz's Theorem.

- Taylor Expansion up to order k: Let $x_0 \in D$ and $r > 0$ such that $B_c(x_0, r) \subset D$. Then for $\mathbf{h} = (h_i)_{1 \le i \le d} \in B_c(0, r)$ there holds:

$$f(x_0 + \mathbf{h}) = f(x_0) + \sum_{j=1}^{k} \frac{1}{j!} \left(\sum_{i_1, i_2, \dots, i_j} h_{i_1} h_{i_2} \dot{h}_{i_j} \frac{\partial^j f}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}}(x_0) \right) + o(\|\mathbf{h}\|^k)$$

  If k = 2: $f(x_0 + \mathbf{h}) = f(x_0) + \nabla f(x_0) \cdot \mathbf{h} + \frac{1}{2}\mathbf{h} \cdot (H_f(x_0)\mathbf{h}) + o(\|\mathbf{h}\|^2)$. Written explicitly:

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + hf_x + kf_y + \frac{h^2}{2} f_{xx} + hkf_{xy} + \frac{k^2}{2} f_{yy} + o(h^2 + k^2)$$

  where all partial derivatives are evaluated at (x_0, y_0).

- Optimization of functions $f : \mathbb{R}^2 \to \mathbb{R}$: The tangent plane at the local optimal points is horizontal. To determine if a point is a maximum, minimum or a "saddle", we need to study the eigenvalues of the 2×2 Hessian Matrix:
  - 2 positive eigenvalues: minimum
  - 2 negative eigenvalues: maximum
  - otherwise: saddle

Chapter 7: Path Integrals

Integrals of a scalar field

Below we denote by $f : D \subset \mathbb{R}^d \to \mathbb{R}$ continuous and by $\gamma : I = [a,b] \subset \mathbb{R} \to \mathbb{R}^d$ of class C^1.

- Path integral of a scalar field: $\int_\gamma f ds = \int_a^b f(\gamma(t))\|\gamma'(t)\|dt$.
  If γ is piecewise defined on segments $[t_{i-1}, t_i]$, then we apply the above definition to every segment and sum them up.
- Diffeomorphism: $\varphi : I \subset \mathbb{R} \to J \subset \mathbb{R}$ is a C^k diffeomorphism, if φ is of class C^k, bijective and its inverse φ^{-1} is also of class C^k.
  Equivalent characterization: $\varphi : I \to J$ bijective and of class C^k is a C^k diffeomorphism iff its derivative φ' does not vanish.
- C^k oriented curve: Γ is the data of (I, γ) where $I \subset R$ and $\gamma : I \to \mathbb{R}^d$ is a C^k function. γ is called a parametrization of the curve.
  (I, γ) and (J, ω) (both of class C^k) represent the same **oriented** curve Γ if there exists $\varphi : I \to J$ an **increasing** diffeomorphism of class C^k such that $\gamma = \omega \circ \varphi$ (that is $\gamma(t) = \omega(\varphi(t))\ \forall t \in I$).
- Independence of the parametrization: If $(I, \gamma) = (J, \omega)$ (see point above), then $\int_\gamma f ds = \int_\omega f ds$.
- Length of a curve: $\text{Length}(\gamma) = \int_\gamma ds = \int_a^b \|\gamma'(t)\|ds$. If $(I, \gamma) = (J, \omega)$, then $\text{Length}(\gamma) = \text{Length}(\omega)$.
- Normal parametrization: (I, γ) is a normal parametrization of Γ if $\|\gamma'(t)\| = 1\ \forall t \in I$.
- Normalizing parametrizations: If (I, γ) is a parametrization of Γ and $\gamma'(t) \neq 0\ \forall t \in I$, then define:

  $$\varphi(t) = \int_{t_0}^t \|\gamma'(t)\|dt.$$

  Then $w = \gamma \circ \varphi^{-1}$ is a normal parametrization of Γ.

Integrals of a vector field

Below we denote by $f : D \subset \mathbb{R}^d \to \mathbb{R}^d$ continuous and by $\gamma : I = [a,b] \subset \mathbb{R} \to \mathbb{R}^d$ of class C^1.

- Path integral of a vector field:$\int_\gamma f \cdot ds = \int_I f(\gamma(t)) \cdot \gamma'(t)dt$.
- Indpendence of parametrization: if $(I, \gamma) = (J, \omega)$, then $\int_\gamma f \cdot ds = \int_\omega f \cdot ds$
- Path integral of a gradient field: if $g : \mathbb{R}^d \to \mathbb{R}$ then: $\int_\gamma \nabla g \cdot ds = \int_a^b \nabla g(\gamma(t)) \cdot \gamma'(t)dt = [g \circ \gamma]_a^b = g(\gamma(b)) - g(\gamma(a))$. If $\gamma(b) = \gamma(a)$ (closed curve), then $\int_\gamma \nabla g \cdot ds = 0$

Properties of both integrals

$$\int_\gamma af + bg = a\int_\gamma f + b\int_\gamma g$$

Chapter 8: Integrals of functions of several variables

Topology

- Compact set: $K \subset \mathbb{R}^d$ is compact if it closed and bounded ($\exists\, C : \|x\| \leq C\ \forall x \in K$).
- Bolzano-Weierstrass in $\mathbb{R}^d$: If $K \subset \mathbb{R}^d$ is a compact set, and (x_n) is a sequence such that $x_n \in K\ \forall n$, then there exists a subsequence of x_n that converges to some point $x \in K$.
- Heine's Theorem (Continuity + Compactness = Uniform Continuity): Let $K \subset \mathbb{R}^d$ be compact and $f : K \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous:

$$\forall\, \epsilon > 0, \exists\, \delta > 0, \forall x, y \in K, \|x - y\| \leq \delta \implies |\, f(x) - f(y) \,| \leq \epsilon$$

- Extrema of continuous functions over compact sets: Let f be continuous, $K \subset \mathbb{R}^d$ compact and $f : K \to \mathbb{R}$. Then f is bounded over K and attains its extrema. That is:

$$\exists\, x_m, x_M \in K : f(x_m) = \underset{x \in K}{inf}\ f(x),\ f(x_M) = \underset{x \in K}{sup}\ f(x)$$

Definition of the Integral

- Lower and Upper Sums: Let $D = [a,b] \times [c,d] \subset \mathbb{R}^2$ a rectangle and $f : D \to R$ a bounded function over D. ($\exists\, C : f(x,y) \leq C\ , \forall (x,y) \in D$). If P_1, P_2 are partitions of $[a,b]$ and $[c,d]$ respectively, then:

$$L(f, P_1 \otimes P_2) = \sum_{R \in P_1 \otimes P_2} A(R) \left(\underset{(x,y) \in R}{inf}\ f(x,y) \right)$$

$$U(f, P_1 \otimes P_2) = \sum_{R \in P_1 \otimes P_2} A(R) \left(\underset{(x,y) \in R}{sup}\ f(x,y) \right)$$

- Riemann Integrability: Let f and D be defined as above. Then f is Riemann integrable if $\underset{P_1,P_2}{sup}\, L(f, P_1 \otimes P_2) = \underset{P_1,P_2}{inf}\, U(f, P_1 \otimes P_2) = \iint_D f$.

  Note: In order to prove that f is Riemann integrable, it is enough to show that $\forall \epsilon > 0, \exists P_1, P_2 : L(f, P_1 \otimes P_2) \geq U(f, P_1 \otimes P_2) - \epsilon$. One can use this result to prove that continuous functions over rectangles are Riemann integrable. (*Hint: Rectangles are compact sets.*)

- Integrals over arbitary sets: Let $D \subset \mathbb{R}^2$ be a bounded domain and $f : D \to \mathbb{R}$ a bounded function over D. Then define:

$$\tilde{f} = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

  Let D' be am arbitrary rectangle that contains D. Then f is Riemann integrable over D iff $\tilde{f}$ is Riemann integrable over D' and $\iint_D f = \iint_{D'} \tilde{f}$.

- Indicator function: Let $D \subset \mathbb{R}^2$. Then define I_D to be the indicator function of D:

$$I_D = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases}$$

- Jordan Measurability and Area: Let $D \subset \mathbb{R}^2$ be an arbitrary set and D' a rectangle containing D. D is **Jordan measurable** if I_D is Riemann integrable over D'. In this case, **area** of D is: $A(D) = \iint_{D'} I_D$.
- Sufficient condition for Jordan Measurability: Let $\psi, \varphi : I = [a,b] \to \mathbb{R}$ such that $\psi(x) \leq \varphi(x)\ \forall x \in I$. Define $D = \{(x,y) \in \mathbb{R}^2 : x \in I,\ \psi(x) \leq y \leq \varphi(x)\}$. Then ψ, φ continuous $\implies D$ Jordan measurable.

- Sharp criterion for Jordan Measurability (JM): D is JM if its topological boundary has zero measure.
- Continuous functions over Jordan Measurable set are Riemann integrable.
- Properties of Riemann Integrable functions: Let f, g Riemann Integrable over $D \subset \mathbb{R}^2$. Then:
  - Linearity: $\iint_D (af + bg) = a \iint_D f + b \iint_D g$
  - Positivity: $f(x) \geq 0 \; \forall x \in D \implies \iint_D f \geq 0$
  - Monotinicity: $f(x) \leq g(x) \; \forall x \in D \implies \iint_D f \leq \iint_D g$
- Fubini's theorem for rectangles: Let $D = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle and $f : D \to \mathbb{R}$ a continuous function. Then:

$$\iint_D f = \int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$$

- Lemma for Fubini: Let f, D defined as above. Then the functions $y \in [c, d] \to \int_a^b f(x, y) dx$ and $x \in [a, b] \to \int_c^d f(x, y) dy$ are continuous.
- Independent functions: Let f, D be defined as above. If $f(x, y) = g(x)h(y)$, then:

$$\iint_D f = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right)$$

- Fubini for general domains: Let $\psi, \varphi : I = [a, b] \to \mathbb{R}$ such that $\psi(x) \leq \varphi(x) \; \forall x \in I$. Define $D = \{(x, y) \in \mathbb{R}^2 : x \in I, \; \psi(x) \leq y \leq \varphi(x)\}$. Then:

$$\iint_D f = \int_a^b \left(\int_{\psi(x)}^{\varphi(x)} f(x, y) dy \right) dx$$

Change of Variables

- Change of variables for integral of 2 variables: Let U be a Jordan measurable domain and $\varphi : U' \to \mathbb{R}^2$ be a function defined on an open set U' containing U. We assume that φ is injective, of class C^1 and that det $D_\varphi (D_\varphi = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u} & \frac{\partial \varphi_1}{\partial v} \\ \frac{\partial \varphi_2}{\partial u} & \frac{\partial \varphi_2}{\partial v} \end{bmatrix})$ does not vanish. We define $D = \varphi(U)$ and take $f : D \to R$ a continuous function. Then D is Jordan measurable and

$$\iint_D f(x, y) dx dy = \iint_U f(\varphi(u, v)) \left| det D_\varphi(u, v) \right| du dv$$

If $\varphi(u, v) = M \begin{pmatrix} u \\ v \end{pmatrix}$, where M is a 2×2 matrix, then

$$\iint_D f(x, y) dx dy = |det M| \iint_U f \left(M \begin{pmatrix} u \\ v \end{pmatrix} \right) du dv$$

- Change of variables in polar coordinates: Let D be a Jordan measurable domain of $\mathbb{R}^2$ and define $U = \left\{ (r, \theta) \in [0, \infty) \times [0, 2\pi) : \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \end{pmatrix} \in D \right\}$. Then U is Jordan Measurable and:

$$\iint_D f(x, y) dx dy = \iint_U f(rcos(\theta), rsin(\theta)) r dr d\theta$$