30560 - Mathematical Modelling for Finance
Mock Exam

EXERCISE 1 (30 points out of 100)

Consider a single-period financial market ($t = 0$, $T = 1$) modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and

$$\mathbb{P}(\omega_1) = \frac{1}{4}, \quad \mathbb{P}(\omega_2) = \frac{1}{4} \quad \text{and} \quad \mathbb{P}(\omega_3) = \frac{1}{2}.$$

Three securities are traded. The first one is a riskless asset B which provides the riskless rate $r = 25\%$ whereas the other two are the risky securities S_1, S_2, such that

$$S_1(0) = 4.5, \quad S_2(0) = \beta$$

$$\begin{bmatrix} S_1(\omega_1) \\ S_1(\omega_2) \\ S_1(\omega_3) \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ \alpha \end{bmatrix}, \qquad \begin{bmatrix} S_2(\omega_1) \\ S_2(\omega_2) \\ S_2(\omega_3) \end{bmatrix} = \begin{bmatrix} 5 \\ 2\alpha \\ \alpha \end{bmatrix}.$$

where $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$.

a) Determine the set of values of $\alpha \in \mathbb{R}$ for which the financial market is complete/incomplete.

b) Determine the set of couples $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ for which the LOP hold.

c) Assume $\alpha = 10$ and $\beta = 15$. Is the market free of arbitrage opportunities? If so, determine the set of state price vectors for the market. If not, find an arbitrage opportunity.

d) From now on assume $\alpha = 10$ and $\beta = 11.5$. Is the market free of arbitrage opportunities? If so, determine the set of state price vectors for the market. If not, find an arbitrage opportunity.

e) A new contingent claim is introduced in the market. Its payoff at time $T = 1$ is equal to

$$X(1)(\omega) = \begin{cases} 10 & \text{if } \min\{S_1(1)(\omega), S_2(1)(\omega)\} \geq 5 \\ 0 & \text{else} \end{cases}$$

Compute its terminal payoff $X(1)$ and the set of its no arbitrage prices at $t = 0$.

f) Assume that the contingent claim in the previous point trades at $t = 0$ at the price of 4. Is the extended market (the one with the three primary assets and the contingent claim) arbitrage free and complete? Determine the set of stochastic discount factors for the extended market.

EXERCISE 2 (20 points out of 100)

Consider a single-period financial market ($t = 0$, $T = 1$) modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and

$$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \frac{1}{3}.$$

Only two risky securities S_1 and S_2 are traded. The securities are such that

$$S_1(0) = 5, \;\; S_2(0) = 2$$

$$S_1(1) = \begin{bmatrix} S_1(1)(\omega_1) \\ S_1(1)(\omega_2) \\ S_1(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \qquad S_2(1) = \begin{bmatrix} S_2(1)(\omega_1) \\ S_2(1)(\omega_2) \\ S_2(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

The market is free of arbitrage opportunities. As in the Lecture Notes, let $x = [S_1(1) \;\; S_2(2)]$. It holds

$$\left(\mathbb{E}\left[xx^T\right]\right)^{-1} = \begin{bmatrix} 1.4 & -3.4 \\ -3.4 & 8.4 \end{bmatrix}.$$

a) Determine the (only) traded stochastic discount factor m^* for this market.

b) Determine R^* and its replicating strategy.

c) Knowing that

$$R^{e^*} = \frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

find the constant mimicking portfolio return, R^{CMR}, and its replicating strategy.

EXERCISE 3 (50 points out of 100)

Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:

$$\begin{array}{ccccc}
 & & & \nearrow & \omega_1 \\
 & & f_1^1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f_1^0 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f_2^1 & & \\
 & & & \searrow & \omega_4
\end{array}$$

Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$r(0) = 3\%, \quad r(1)(f_1^1) = 5\% \;\; \text{and} \; r(1)(f_2^1) = 2\%.$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 14.5 \text{ and } S(1)(f_2^1) = 7.5$$

and with time 2 prices

$$S(2)(\omega_1) = 16.24, \quad S(2)(\omega_2) = 14.21 \quad S(2)(\omega_3) = 8.25 \quad S(2)(\omega_4) = 6.75.$$

1. Determine the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.
2. Find the set of risk neutral probabilities $\mathbb{Q}$ for the market.
3. Is the market arbitrage free?
4. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K = 9.4$ is introduced in the market. Find its payoff $p(2)$ at maturity $T = 2$.
5. Find the no arbitrage prices at $t = 1$ of the put option of the previous Question.
6. Find the no arbitrage price at $t = 0$ of the put option of the previous Question.
7. Suppose the option of point 4 is now of American type. Is there any optimal early exercise opportunity? If your answer is positive, find the early exercise premium of the option.
8. Suppose the historical probability on $\Omega = \{\omega_1, ..., \omega_4\}$ is

   $$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 25\%, \ \mathbb{P}(\omega_3) = 35\% \text{ and } \mathbb{P}(\omega_4) = 15\%$$

   Suppose that at $t = 1$ on f_2^1 and investor with *log-utility* and wealth $v > 0$ wants to maximize her expected utility from terminal wealth. Find her optimal asset allocation at time $t = 1$ on f_2^1, and her optimal terminal wealth on ω_3 and ω_4.