Algebra and Geometry (Cod. 30544)

General Exam – December 10, 2020

Time: 2 hours. Total: 150 points.

Multiple choice questions (total: 24 points)

Each question has a single correct answer: write the correct answer in the box on the right. If you want to change your response cancel it and write another answer next to the box. 6 points are assigned for a correct answer, 0 points for a missing answer, -2 point for an incorrect answer.

1. Given an integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$. Then
   (A) $\{abx : a, b \in \mathcal{P}_0\}$ is isomorphic to $\mathcal{P}_2$ (B) $\mathcal{P}_n$ has dimension n
   (C) $\mathcal{P}_n$ has a base containing the zero polynomial (D) none of the others

2. Let V be a finite dimensional vector space and fix a subspace $W \subseteq V$ with $|W| = \infty$. Then
   (A) Every element of V/W is finite (B) $(x + W) \cap (y + W) = \emptyset$ for all distinct $x, y \in V$
   (C) $\dim(V/W) < \dim(V)$ (D) none of the others

3. Let p be a prime number. Then
   (A) $(\mathbf{Z}_{p^2}, +, \cdot)$ is a field (B) $(\mathbf{Z}_{p^2}, \cdot)$ is a group
   (C) $(\mathbf{Z}_{p^2}, +)$ is a group (D) none of the others

4. Let σ be a permutation of $\{1, 2, \ldots, n\}$ and fix a multilinear alternate form $f \in \mathcal{A}(V)$. Then
   (A) σ be can be always decomposed as product of $\leq n - 2$ transpositions (B) $\sigma f = f$
   (C) $f(x, x, x, ..., x) = 1$ (D) none of the others

True/False questions (total: 24 points)

Each statement can be either true or false: write T for true or F for false in the box on the right. If you want to change your response cancel it and write another answer next to the box. 4 points are assigned for a correct answer, 0 points for a missing answer, -1 point for an incorrect answer.

1. Let $V = \mathbf{R}$ be the real vector space over $\mathbf{R}$. Then $\{\sqrt{2}, \sqrt{3}\}$ is linearly dependent.
2. Fix $T \in \mathcal{L}(V_1, V_2)$ where V_i is a vector space over a field $\mathbb{F}_i$ for $i = 1, 2$. Then $\mathbb{F}_1 = \mathbb{F}_2$.
3. Fix $T \in \mathcal{L}(V, W)$. Then T can be represented by a matrix.
4. Define $A = \begin{bmatrix} 1^2 & 2^2 \\ 3^2 & 4^2 \end{bmatrix}$. Then $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x) := Ax$ is linear.
5. There exists a matrix $A \in \mathscr{M}_{2\times 3}(\mathbf{R})$ which is diagonalizable.
6. Let V be a finite dimensional vector space. Then $\dim(V'') = \dim(V)$.

Open answer questions (total: 102 points)

Answers must be written in the corresponding spaces. Each of the six questions will be assigned from 0 to 17 points. Answers must be adequately justified.

Question 1. Let $T : V \to V$ be a linear operator.

(a) Show that $W := \{x + y : x \in \mathrm{Ker}(T), y \in \mathrm{Im}(T)\}$ is a vector space.

(b) Provide an example of a linear operator T which is neither injective nor surjective and such that $W = V$.

Question 2. 1. Provide the definition of field $\mathbb{F}$.
2. Let V be a vector space over $\mathbb{F}$. Show that $-(-x) = x$ for all $x \in V$.

Question 3. For each integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$. Moreover define

$$\mathcal{Q}_n := \{p(x)q(x) : p(x), q(x) \in \mathcal{P}_n\}$$

for each $n \geq 0$. Then prove that:

(i) $\mathcal{Q}_0$ is a real vector space;

(ii) $\mathcal{Q}_1$ is not a real vector space.

Question 4. Consider then vector space $V := \mathbf{R}^{\{1,2,\ldots,2020\}}$ and consider the functions $f, g, h \in V$ defined by

$$f(i) = i \text{ for all } i = 1, \ldots, 2020,$$

$$g(1) = 1^2, g(2) = 2^2, g(3) = 3^2, g(i) = 4^2 \text{ for all } i = 4, 5, \ldots, 2020,$$
$$h(1) = h(2) = 0, h(i) = 1 \text{ for all } i = 3, 4, \ldots, 2020.$$

Show that $\{f, g, h\}$ is linearly independent.

Question 5. Fix $T \in \mathcal{L}(V, W)$.

(i) Define the meaning of the transpose operator T'.

(ii) Show that $\mathrm{Ker}T'$ is the annihilator of $T(V)$.

Question 6. Define the matrix

$$M := \begin{bmatrix} 1 & 4 & 9 \\ -1 & 0 & -2 \\ 2 & 0 & 0 \end{bmatrix}$$

Let a, b, c be its eigenvalues. Compute $a^2 + b^2 + c^2$.

0.1 Solutions Multiple choices / True-False

Multiple choices:

1	2	3	4
D	C	C	D

True/False:

1	2	3	4	5	6
T	T	F	T	F	T

0.2 Open question

1 (a) Let $A, B \subseteq V$ be vector subspaces. Then $Q := A+B$ is contained in V, it is not empty since it contains $0+0$, it is closed under sums (indeed, if $q_i = a_i+b_i \in Q$ and $a_i \in A$, $b_i \in B$ for $i=1,2$, then $q_1+q_2 = (a_1+a_2)+(b_1+b_2) \in A+B$) and it is closed under products (indeed, if $q = a+b \in Q$ and $\alpha \in \mathbb{F}$ then $\alpha q = (\alpha a) + (\alpha b) \in A+B$). In our case, set $A := \mathrm{Ker}(T)$ and $B := \mathrm{Im}(T)$.

(b) Set $V = \mathbf{R}^2$, and define $T(x,y) := (x,0)$. Then $\mathrm{Ker}(T) = \{(0,y) : y \in \mathbf{R}\}$, $\mathrm{Im}(T) := \{(x,0) : x \in \mathbf{R}\}$, and $W = \mathbf{R}^2$.

2 1 and 2. See lecture notes.

3 (i) Since $\mathcal{Q}_n$ is the set of product of polynomials with degree $\leq n$, then $\mathcal{Q}_n \subseteq \mathcal{P}_{2n}$. In particular, $\mathcal{Q}_0 \subseteq \mathcal{P}_0$. Conversely, each constant *polynomial* $p(x) \in \mathcal{P}_0$ belongs to $\mathcal{Q}_0$, indeed $p(x) = g(x) \cdot p(x)$, where $g(x)$ is the constant polynomial 1. Therefore $\mathcal{Q}_0 = \mathcal{P}_0$, which is a vector space.

(ii) It is sufficient to show that $\mathcal{Q}_1$ is not closed under addition. Indeed $x \cdot x \in \mathcal{Q}_1$ and $1 \cdot 1 \in \mathcal{Q}_1$. On the other hand, $x^2+1 \notin \mathcal{Q}_1$. Indeed, let us suppose by contradiction that there exist $a,b,c,d \in \mathbf{R}$ such that

$$x^2+1 = (ax+b)(cx+d).$$

Then $ac = bd = 1$ and $bc+ad = 0$. From the first equation we have that all a,b,c,d are different from 0, hence $c = 1/a$ and $d = 1/b$. We conclude that

$$bc + ad = \frac{b}{a} + \frac{a}{b} = 0,$$

i.e., $x + \frac{1}{x} = 0$ with $x := b/a \neq 0$. This is impossible since x and $1/x$ have the same sign.

4 Suppose that $\alpha f + \beta g + \gamma h = \mathbf{0}$. In particular,

$$\alpha f(i) + \beta g(i) + \gamma h(i) = 0 \text{ for all } i = 1,2,3.$$

Setting $i \in \{1,2\}$, we obtain that $\alpha = \beta = 0$. Setting, at this point, $i=3$, we obtain that $\gamma = 0$.

5 1 and 2. See lecture notes.

6 Denoting by I the identity matrix, the eingevalues a,b,c are exactly the roots of the third degree polynomial

$$0 := \det(\lambda I - M) = \det \begin{bmatrix} \lambda - 1 & -4 & -9 \\ 1 & \lambda & 2 \\ -2 & 0 & \lambda \end{bmatrix}$$

Equivalently

$$q(\lambda) := \lambda^3 - \lambda^2 - 14\lambda + 16 = 0.$$

It follows that

$$abc = -16, ab+bc+ca = -14, a+b+c = 1.$$

Therefore

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 1 - 2 \cdot (-14) = 29.$$