Quantitative Finance Cheatsheet

Klejdi Sevdari

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1 One Period Markets

Definition 1 (Value Process).

$$V_\theta(0) = S(0)^T\theta \quad V_\theta(1) = \mathcal{A}\theta$$

Definition 2 (Spaces of traded payoffs).

$$M = \{x \in \mathbb{R}^{k+1} : z = \mathcal{M}\theta\}, \quad A = \{x \in \mathbb{R}^k : z = \mathcal{A}\theta\}$$

Definition 3 (Law of One Price LOP). If $\mathcal{A}\theta_1 = \mathcal{A}\theta_2$ then $V_{\theta_1}(0) = V_{\theta_2}(0)$.

Definition 4 (Redunant Security). A redundant security is a security that can be obtained as a linear combination of other securities.

Theorem 1. *LOP holds $\iff$ all strategies with payoff 0 have 0 initial cost.*

Definition 5 (Pricing Functional).

$$\pi : z \in A \to \{S(0)^T\theta : z = \mathcal{A}\theta\}$$

If π is single-valued and linear it is a **linear pricing functional** (LPF).

Theorem 2. *LOP holds $\iff$ there exists a linear pricing functional (LPF).*

2 SDFs, SPFs, and RNPs

Definition 6 (Stochastic Discount Factor). An SDF is a vector m such that

$$\forall j :\ S_j(0) = \mathbb{E}[mS_j(1)] = \sum m_k p_k S_j(1)(w_k) \implies \mathbb{E}[m] = \frac{1}{1+r} \text{ and } V_\theta(0) = \mathbb{E}[mV_\theta(1)]$$

Theorem 3. *LOP holds $\iff$ an SDF exists.*

Remark: m is an SDF if it can be written as $m = m^* + \epsilon$ with $\epsilon \in A^\perp$ and m^* unique in A.

Definition 7 (State Price Vectors). A State Price Vector $\psi \in \mathbb{R}^k$ satisfies:

- $\psi_i > 0$.
- $V_\theta(0) = \psi^T V_\theta(1) \implies \sum_i \psi_i = \frac{1}{1+r}$.

Definition 8 (Risk Neutral Probabilities). An RNP is a probability measure $\mathcal{Q}$ with $\mathcal{Q}(w_k) = q_k > 0$ and

$$S_j(0) = \mathbb{E}^{\mathcal{Q}}\left[\frac{S_j(1)}{1+r}\right] = \sum_k \frac{q_k S_j(1)(w_k)}{1+r}$$

Remark: The relation between strictly positive SDFs, SPFs, and RNPs:

$$q_k = \psi_k(1+r), \quad m_k p_k = \psi_k, \quad q_k = m_k p_k(1+r)$$

Remark: In terms of return, one has:

$$\mathbb{E}[m(R_j - r)] = 0, \quad \mathbb{E}^{\mathcal{Q}}[R_j - r] = 0, \quad \psi^T(R_j - r) = 0$$

3 Fundamental Theorems of Asset Pricing

Definition 9 (Arbitrage Opportunity)**.** An arbitrage opportunity is a portfolio strategy such that:

- $V_\theta(0) \leq 0$.
- $V_\theta(1) \geq 0$ (note that this is a vector).
- At least one inequality above is strict.

Remark: In other words Arbitrage occurs when $\mathcal{M}\theta > 0$.

Theorem 4 (First Fundamental Theorem of Asset Pricing)**.** *TFAE:*

- *No Arbitrage (NA) holds.*
- *There exists a state price vector/ strictly positive SDF / RNP.*

Definition 10 (Complete Markets)**.** A market is complete if any $z \in \mathbb{R}^K$ is attainable ($z = \mathcal{A}\theta$).

Definition 11 (Arrow-Debreu Security)**.** $E_k(w_h) = \delta(h,k) = e_k \in \mathbb{R}^K$.

Important Remark: NA is a sufficient condition for LOP, but not necessary.

Theorem 5. *LOP + completeness* $\iff \dim M = \dim A = K$.

Corollary 5.1. *LOP + completeness* $\iff \exists!$ *SDF.*

Lemma 6. $M^\perp = span\left\{[1,\psi]^T, with\ \psi\ a\ SPV\right\}$.

Theorem 7 (Second Fundamental Theorem of Asset Pricing)**.** *TFAE:*

- *NA + completeness holds.*
- *There exists a unique SPV/ unique strictly positive SDF/ unique RNP.*

4 The Mean-Variance Frontier

Definition 12 (Gross Returns)**.** $R_j = \frac{S_1(j)}{S_0(j)}$ $R = [R_1, \ldots, R_N]^T$.

Remark: We assume that all $S_j(0) > 0$, $\mathbb{E}[R] = \mu \neq k\mathbb{1}, \det(\Sigma) \neq 0$, where $\Sigma = \mathbb{E}[(R-\mu)(R-\mu)^T]$.

Definition 13. A portfolio $w \in \mathbb{R}^N$ is defined by its weights on the N securities. We have that $w^T\mathbb{1} = 1$. The return of the portfolio is w^TR, its expected value is $w^T\mu$ and its variance is $w^T\Sigma w$.

Definition 14 (The Markowitz Approach)**.** If c is the target return, then we want to

$$\min_w w^T\Sigma w \ \textbf{s.t}\ w^T\mathbb{1} = 1 \text{ and } w^T\mu = c.$$

4.1 Hansen-Richard Approach

Let S be the vector containing the prices of the securities at time 0: $S = [S_1(0), \ldots, S_N(0)]$. Define $x \in \mathbb{R}^N$ to be the random vector containing the prices of the securities in time 1 and let $A =< x1, ..., x_n >$. Assume that LOP holds, which implies the existence of an SDF $m^* \in A$. We have that $\mathbb{E}[m^*x_j] = S_j\ \forall j$.

Theorem 8. *Assuming that* $\mathbb{E}[xx^T]$ *is nonsingular, the only traded SDF* $m^* \in A$ *is given by:*

$$m^* = S^T\left(\mathbb{E}[xx^T]\right)^{-1}x.$$

Definition 15 (Linear Space of Returns)**.**

$$A_1 = \{R \in A : \pi(R) = 1\}$$

Definition 16 (Linear Space of Excess Returns)**.**

$$A_0 = \{R^e \in A : \pi(R^e) = 0\}$$

Definition 17.

$$R^* = \frac{m^*}{\pi(m^*)} \qquad R^{e^*} = \text{proj}[1|A_0]$$

Theorem 9. *The following properties of R^{e^*} and R^* hold:*

- $\mathbb{E}[R^*] = \mathbb{E}[m^*]/\mathbb{E}\left[(m^*)^2\right]$,
- $m^* = R^*/\mathbb{E}\left[(R^*)^2\right]$,
- $\mathbb{E}[R^*R] = \mathbb{E}\left[(R^*)^2\right]$ *for any* $R \in A_1$,
- $\mathbb{E}[R^*R^e] = 0$ *for any* $R^e \in A_0$,
- $\mathbb{E}[R^{e^*}n] = \mathbb{E}[n]$ *for any* $n \in A_0$ *and in particular,* $\mathbb{E}\left[(R^{e^*})^2\right] = \mathbb{E}\left[R^{e^*}\right]$.

Lemma 10. *The space A of traded payoffs orthogonally decomposes as follows,*

$$A = \langle R^* \rangle \oplus A_0.$$

Lemma 11. *The space A_0 of traded payoffs orthogonally decomposes as follows,*

$$A_0 = \langle R^{e^*} \rangle \oplus \{n \in A_0 : \mathbb{E}[n] = 0\}.$$

Theorem 12. *The space A can be further decomposed as:*

$$A = \langle R^* \rangle \oplus \langle R^{e^*} \rangle \oplus \{n \in A_0 : \mathbb{E}[n] = 0\}.$$

Therefore if $R \in A_1$, we get that $R = R^* \oplus wR^{e^*} \oplus n$.

Theorem 13 (Characterization of the Mean Variance Frontier)**.** $R^{MV} \in A_1$ *is on the MV frontier iff:*

$$R^{MV} = R^* + wR^{e^*}$$

Theorem 14 (Two Fund Separation)**.** R^{MV} *is on the MV frontier* $\iff$ *there exist* R^1, R^2 *on the MV frontier and* $\alpha \in \mathbb{R}$ *such that:*

$$R^{MV} = \alpha R^1 + (1-\alpha)R^2.$$

4.2 Three proxies for the risk-free

Three alternative proxies for the risk-free return when the riskless security is not traded. The three proxies are:

- The global minimum variance portfolio return.
- The zero-beta portfolio on the MV frontier that displays a zero covariance with a fixed starting portfolio on the frontier.
- The constant mimicking portfolio return, which is the normalized projection of the riskless asset onto A_1.

Global Minimum Variance Portfolio return

The global minimum variance portfolio return solves the followin optimization problem:

$$\min_{w\in\mathbb{R}} \mathbb{V}ar\left[R^* + wR^{e^*}\right] = \min_{w\in\mathbb{R}} \mathbb{V}ar[R^*] + w^2\mathbb{V}ar\left[R^{e^*}\right] + 2w\mathbb{C}ov[R^*, R^{e^*}]$$

One can show that $\mathbb{C}ov[R^*, R^{e^*}] = -\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right]$. Taking the first derivative of the previous optimization problem we obtain:

$$w^{MIN} = \frac{\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right]}{\mathbb{V}ar[R^{e^*}]} = \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}\left[R^{e^*}\right]}.$$

Thus:

$$R^{MIN} = R^* + w^{MIN}R^{e^*} = R^* + \frac{\mathbb{E}[R^*]}{1 - \mathbb{E}\left[R^{e^*}\right]}R^{e^*}$$

Note that $\mathbb{E}[R^{MIN}] = w^{MIN}$. R^{MIN} is the only return on MV whose expectation is equal to its weight on R^{e^*}.

Definition 18 (Efficient Return)**.** We say that a mena variance return R^{MV} is efficient if $\mathbb{E}\left[R^{MV}\right] > \mathbb{E}\left[R^{MIN}\right]$.

Zero-beta portfolio return
Given any return $R \neq R^{MIN}$ on the MV frontier, there exists a unique uncorrelated portfolio return $zc[R]$ on the frontier. Given $R = R^* + wR^{e^*}$, then $zc[R] = R^* + w^{zc[R]}R^{e^*}$ with

$$w^{zc[R]} = \frac{w\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right] - \mathbb{V}ar[R^*]}{w\mathbb{V}ar[R^{e^*}] - \mathbb{E}[R^*]\mathbb{E}[R^{e^*}]}$$

One can check that $\mathbb{C}ov[R, zc[R]] = 0$.

Constant Mimicking Portfolio

Using the fact that $R^{e^*} = proj[1|A_0]$ and $A = \langle R^* \rangle \oplus A_0$, we can say that:

$$proj[1|A] = proj[1|R^*] + R^{e^*} = \frac{\mathbb{E}[R^*]}{\mathbb{E}[(R^*)^2]}R^* + R^{e^*}$$

One can easily see that:

$$\pi(proj[1|A]) = \frac{\mathbb{E}[(R^*)^2]}{\mathbb{E}[R^*]}$$

Then, we can define:

$$R^{CMR} = \frac{proj[1|A]}{\pi\left[proj[1|A]\right]} = R^* + \frac{\mathbb{E}[(R^*)^2]}{E[R^*]}R^{e^*} \in \text{MV Frontier.}$$

One can check that $\mathbb{E}[zc(R^{CMR})] = 0$.

4.3 The frontier with the risk-free asset

Assume now that a risk-free asset is traded and denote its gross return by R_f. We define

$$\tilde{A} = \langle 1, A \rangle, \quad \tilde{\pi}[1] = \frac{1}{R_f}, \quad R_f = \frac{1}{\mathbb{E}[\tilde{m}^*]}, \quad \text{and so on.}$$

Theorem 15. *R_f is on the MV frontier and can be written as $R_f = \tilde{R}^* + R_f\tilde{R}^e$. The two fund separation theorem then implies that*

$$R^{MV} = (1-\alpha)R_f + \alpha\tilde{R}^* \implies \mathbb{E}[R^{MV}] = R_f + \alpha\left(\mathbb{E}[\tilde{R}^*] - R_f\right) \quad \textit{and} \quad \mathbb{V}ar[R^{MV}] = \alpha^2\mathbb{V}ar[\tilde{R}^*]$$

Lemma 16. $\mathbb{E}[R^{MV}] - R_f > 0 \iff \alpha < 0$.

We can rewrite $\sigma[R^{MV}]$ by substituting α:

$$\sigma[R^{MV}] = \left|\frac{\mathbb{E}[R^{MV}] - R_f}{\mathbb{E}[\tilde{R}^*] - R_f}\right| \sigma\left[\tilde{R}^*\right]$$

if $\mathbb{E}[R^{MV}] - R_f > 0$,

$$\sigma[R^{MV}] = -\frac{\mathbb{E}[R^{MV}] - R_f}{\mathbb{E}[\tilde{R}^*] - R_f}, \qquad \mathbb{E}[R^{MV}] = R_f + \frac{-\left(\mathbb{E}[\tilde{R}^*] - R_f\right)}{\sigma[\tilde{R}^*]}\sigma[R^{MV}].$$

Definition 19 (Optimal Risky Portfolio)**.** Assume $R_f < \mathbb{E}[R^{MIN}]$. The Optimal Risky Portfolio is the point that lies both in the MV frontier with the risk-free security and the one without it. It is depicted graphically in figure 1.

Lemma 17. *Assume that $R_f < \mathbb{E}[R^{MIN}]$. Then the efficient frontier when a risk-free asset is traded coincides with the tangent line from (0, R_f) to the MV frontier without the risk-free asset. As a result, R^{ORP} is unique and well-defined.*

Theorem 18. *As a consequence of the Two-Fund Separation Theorem, $R \in \tilde{A}_1 \in$ MV frontier iff:*

$$R = (1-\alpha)R_f + \alpha R^{ORP}$$

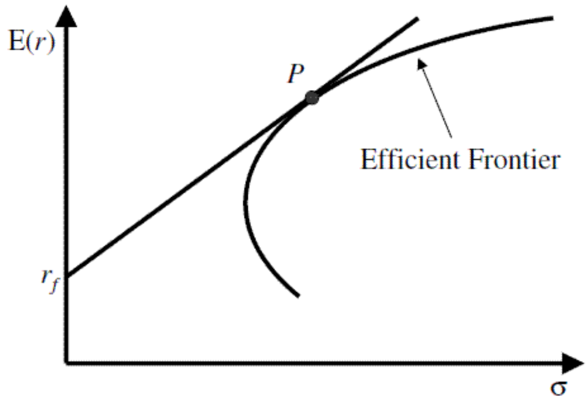

Figure 1: The Optimal Risky Portfolio (denoted by P).

5 Single Beta Pricing Equations

Assuming that a risk-free rate is not traded, a general beta-pricing equation takes the form:

$$\mathbb{E}[R] = \gamma + \sum_{l=1}^{L} \beta_{R,l}\lambda_l, \quad \forall R \in A_1$$

where γ shadows a risk-free rate, and $\beta_{R,l}$ represents what proportion of the total risk of factor f_l is loaded on each security R, while λ_l dictates the right remuneration in terms of expected return to carry a unit of factor f_l risk. In this chapter, we set $L = 1$, and the equation boils down to

$$\mathbb{E}[R] = \gamma + \beta_{R,f}\lambda, \forall R \in A_1 \text{ with } \beta_{R,f} = \frac{\mathbb{C}ov[R, f]}{\mathbb{V}ar[f]}$$

Theorem 19. *Assume $\mathbb{V}ar[f] \neq 0$. There exists $\gamma, \lambda \in \mathbb{R}, \gamma \neq 0$ such that*

$$\mathbb{E}[R] = \gamma + \beta_{R,f}\lambda, \forall R \in A_1 \textit{ with } \beta_{R,f} = \frac{\mathbb{C}ov[R, f]}{\mathbb{V}ar[f]}$$

iff there exists $a, b \in \mathbb{R}$ with $a + b\mathbb{E}[f] \neq 0$ such that:

$$m = a + bf \textit{ is a SDF.}$$

Assume now that the factor is a traded return, $\hat{R} \in A_1$, in a market that does not trade the risk-free rate R_f.

Theorem 20. *Given any return $\hat{R} = R^* + \hat{w}R^{e^*} + \hat{n} \in A_1$, $\exists a, b \in \mathbb{R}$ such that $m = a + b\hat{R}$ is a SDF if and only if*

$$\hat{n} = 0 \textit{ and } \hat{w} \neq \frac{\mathbb{E}\left[(R^*)^2\right]}{E[R^*]} = w^{CMR}$$

Theorem 21. *Given any return $\hat{R} = R^* + \hat{w}R^{e^*} + \hat{n}, \in A_1$,*

$$E[\mathbb{R}] = \mathbb{E}[zc[\hat{R}]] + \beta_{R,\hat{R}}\left(\mathbb{E}[\hat{R}] - \mathbb{E}[zc[\hat{R}]]\right), \forall R \in A_1 \qquad \textit{with } \beta_{R,\hat{R}} = \frac{\mathbb{C}ov[R, \hat{R}]}{\mathbb{V}ar[\hat{R}]}$$

iff

$$\hat{n} = 0 \quad \textit{and} \quad \hat{w} \neq \frac{E[R^*]}{1 - \mathbb{E}[R^{e^*}]} \neq R^{MIN}.$$

Theorem 22. *Given any return $\hat{R} \in A_1$, TFAE:*

- $\exists a, b \in \mathbb{R}$ *such that* $a + b\mathbb{E}(\hat{R}) \neq 0$ *and* $m = a + b\hat{R}$ *is a SDF.*
- $\hat{R}$ *is a MV return with* $\hat{R} \neq R^{CMR}$ *and* $R \neq R^{MIN}$.
- *The frontier return uncorrelated with* $\hat{R}$ *has a non-vanishing mean, and*

$$\mathbb{E}[R] = \mathbb{E}[zc[\hat{R}]] + \beta_{R,\hat{R}} \left(\mathbb{E}[\hat{R}] - \mathbb{E}[zc[\hat{R}]] \right), \forall R \in A_1 \quad \textit{where } \beta_{R,\hat{R}} = \frac{\mathbb{C}ov[R, \hat{R}]}{\mathbb{V}ar[\hat{R}]}$$

6 CAPM Model

The formulation of the CAPM model relies on the following assumptions: investors are price takers, no market friction (taxes, transaction costs), investors rank portfolios to the mean-variance criterion, all investors agree on the estimates $\mathbb{E}[R_j], \sigma_j^2, \rho_{ij}$, the risk-free rate R_f is the same for both borrowing and lending and the risky securities are in fixed supply.

We assume that the market is populated with m investors, indexed with $1, \ldots, m$. We let $W^i(0) > 0$ be the wealth of investor i at time 0. We let θ_j^i be units invested by investor i in security j. We have:

$$W^i(0) = \theta_0^i + \sum_{j=1}^{n} \theta_j^i S_j(0) \quad \text{and} \quad W^i(1) = \theta_0^i R_f + \sum_{j=1}^{n} \theta_j^i x_j$$

Using $R_j = x_j / S_j(0)$ and $w_j^i = \theta_j^i S_j(0)/W^i(0)$, we have that:

$$W^i(1) = W^i(0) \sum_{j=0}^{n} w_j^i R_j$$

The investor wants to choose the optimal:

$$\tilde{R}_i = \frac{W^i(1)}{W^i(0)} \in \tilde{A}_1 = \{\tilde{R} \in \tilde{A} | \tilde{\pi}[\tilde{R}] = 1\} \quad \text{as measured by} \quad U^i(\mathbb{E}[\tilde{R}_i], \sigma[\tilde{R}_i]).$$

Assumptions on U_i :

- U^i is well defined and continuously differentiable on the set $\tilde{A}_1$, as many times as we need.
- $\frac{\partial U^i}{\partial \mathbb{E}[\tilde{R}_i]} > 0, \frac{\partial U^i}{\partial \sigma[\tilde{R}_i]} < 0$: the investor likes higher returns and lower risk.
- For any given $k \in \mathbb{R}$, let $E[\tilde{R}_i] = \varphi \left(\sigma[\tilde{R}_i] \right)$ be the function defined by

$$U^i \left(\mathbb{E}[\tilde{R}_i], \sigma[\tilde{R}_i] \right) = k \quad \text{with} \quad \frac{\partial \varphi \left(\mathbb{E}[\tilde{R}_i] \right)}{\partial \sigma[\tilde{R}_i]} > 0, \frac{\partial^2 \varphi \left(\mathbb{E}[\tilde{R}_i] \right)}{\partial (\sigma[\tilde{R}_i])^2} > 0$$

In other words, the indifference curves are increasing and convex in the plane $(\sigma, \mathbb{E})$.

The investor's optimal portfolio is then the one that maximizes the utility function:

$$\max_{\alpha^i} \ U^i \left(\mathbb{E}\left[\tilde{R}_i\right], \sigma\left[\tilde{R}_i\right] \right)$$
$$s.t. \ \tilde{R}_i = (1 - \alpha^i) R_f + \alpha^i R^{ORP}$$

Simple calculus shows that the necessary and sufficient condition for an interior optimum is:

$$-\frac{\partial U^i / \partial \sigma \left[\tilde{R}_i\right]}{\partial U^i / \partial \mathbb{E}\left[\tilde{R}_i\right]} = \frac{\mathbb{E}[R^{ORP}] - R_f}{\sigma[R^{ORP}]}$$

We now introduce the following notation:

- $\bar{\theta}_j$: fixed supply of security j.
- $\bar{\theta}_j S_j(0)$: market capitalization of security j.

Definition 20 (Market Portfolio)**.** We call the market portfolio the one defined by the following weights:

$$w_j^M = \frac{\bar{\theta}_j S_j(0)}{\sum_{j=1}^n \bar{\theta}_j S_j(0)}$$

In equilibrium, market clearing conditions require that the aggregate quantity supplied of each security in the market is equal to the aggregate quantity demanded of each security in the market. We assume that the risk-free return is in zero net supply, meaning that the long positions are equal to the short positions. If $(1-\alpha_i)W^i$ is investor i's investment in R_f, we must have: $\sum_i (1-\alpha^i)W^i = 0$. Let W denote the total wealth in the economy. Then it must be that $W = W^i\alpha^i$. For each risky security, market clearing requires that:

$$\bar{\theta}_j S_j(0) = \left(\sum_{i=1}^m W^i\alpha^i\right) w_j^{ORP} \implies \sum_{j=1}^n \bar{\theta}_j S_j(0) = \sum_{j=1}^n w_j^{ORP} \sum_{i=1}^m W^i\alpha^i = \sum_{i=1}^m W^i\alpha^i$$

This also implies that:

$$w_j^M = \frac{\bar{\theta}_j S_j(0)}{\sum_{j=1}^n \bar{\theta}_j S_j(0)} = \frac{\bar{\theta}_j S_j(0)}{\sum_{i=1}^m W^i\alpha^i} = w_j^{ORP}$$

Theorem 23 (The Classical CAPM)**.** *In equilibrium and under the CAPM assumptions, we have that*

$$\mathbb{E}[R_i] = R_f + \beta_i[E[R^M] - R_f], \forall R_i \in \tilde{A}_1 \quad \textit{where } \beta_i = \frac{\mathbb{C}ov[R_i, R^M]}{\mathbb{V}ar[R^M]}$$

This follows from theorem 22 and the fact that $R^M = R^{ORP} \in$ MV frontier.

6.1 Optimal Portfolio Selection without a risk-free rate

In the absence of a risk-free security the optimal portfolio of an investor solves

$$\max_{R_i} U^i(\mathbb{E}[R_i], \sigma(R_i))$$
$$s.t.\ R_i = R^* + wR^{e^*}, w > \frac{\mathbb{E}[R^*]}{1-\mathbb{E}[R^{e^*}]} = \mathbb{E}[R^{MIN}]$$

Hence the maximization problem becomes

$$\max_{w^i} U(\mathbb{E}[R_i], \sigma[R_i])\ s.t.$$
$$\mathbb{E}[R_i] = \mathbb{E}[R^*] + w^i\mathbb{E}\left[R^{e^*}\right]$$
$$\sigma[R_i] = \left[\mathbb{V}ar[R^*] + (w^i)^2\mathbb{V}ar\left[R^{e^*}\right] - 2w^i\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right]\right]^{1/2}$$

This problem has a unique solution w^i, which is obtained by setting equal to zero the total derivative of the function U with respect to w^i:

$$\frac{w^i\left(1-\mathbb{E}\left[\mathbb{R}^*\right]\right) - \mathbb{E}[R^*]}{[\mathbb{V}ar[R^*] + (w^i)^2\mathbb{V}ar[R^{e^*}] - 2w^i\mathbb{E}[R^*]\mathbb{E}\left[R^{e^*}\right]]^{1/2}} = -\frac{\partial U^i/\partial\mathbb{E}}{\partial U^i/\partial\sigma}$$

6.2 Equilibrium without a risk-free rate and Black's Zero-Beta CAPM

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