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1 Stochatic Processes in Discrete Time

In discrete time stochastic processes, we consider three fundamental objects: a set of dates $\mathcal{T} = \{0, 1, \ldots, T\}$, finite sets of the world at time T $\Omega = \{\omega_1, \ldots, \omega_K\}$ and a probability distribution over those states $\mathbb{P}$ with $\mathbb{P}(w_k) > 0, \ \forall k$.

Definition 1 (Parition)**.** A partition of Ω is a collection $\mathcal{A} = \{\mathcal{A}_1, \ldots, \mathcal{A}_\ell\}$ such that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ and $\cup_i \mathcal{A}_i = \Omega$.

Definition 2 (Finer and Coarser Partitions)**.** Given two partitions $\mathcal{A}$ and $\mathcal{A}'$, we say that $\mathcal{A}'$ is finer than $\mathcal{A}$ if every element of $\mathcal{A}'$ is contained in $\mathcal{A}$.

Definition 3 (Information Structure)**.** We call the family of partitions $\mathcal{P} = \{\mathcal{P}_t\}_{t=0}^T$ an information structure on $\mathcal{T}$ if it satisfies:

- $\mathcal{P}_0 = \Omega$,
- $\mathcal{P}_{t+1}$ is finer than $\mathcal{P}_t$ for all t.
- $\mathcal{P}_T = \{\{\omega_1\}, \{\omega_2\}, \ldots, \{\omega_K\}\}$.

We denote by s_t the number of cells/cardinality of $\mathcal{P}_t$. We denote by f_h^t for $h = 1, \ldots, s_t$, the generic element of the partition $\mathcal{P}_t$. Figure 1 illustrates this idea.

$$
\begin{array}{ccccc}
t=0 & & t=1 & & t=2 \\
 & & & \nearrow & f_1^2 = \{\omega_1\} \\
 & & f_1^1 = \{\omega_1, \omega_2, \omega_3\} & \longrightarrow & f_2^2 = \{\omega_2\} \\
 & & & \searrow & f_3^2 = \{\omega_3\} \\
 & \nearrow & & \nearrow & f_4^2 = \{\omega_4\} \\
f_1^0 = \{\Omega\} & \longrightarrow & f_2^1 = \{\omega_4, \omega_5\} & \searrow & f_5^2 = \{\omega_5\} \\
 & \searrow & & \nearrow & f_6^2 = \{\omega_6\} \\
 & & f_3^1 = \{\omega_6, \omega_7\} & \searrow & f_7^2 = \{\omega_7\}
\end{array}
$$

Figure 1: Illustrative Example of an Information Structure.

Definition 4 (Measurable Random Variable)**.** The function $X(t) : \Omega \to \mathbb{R}$ is measurable wrt $\mathcal{P}_t$ iff for any $w', w'' \in f_h^t$ we have that $X(t)(w') = X(t)(w'')$. That is X is constant on any f_h^t.

Definition 5 (Adapted Stochastic Process)**.** We call $X = \{X(t)\}_{t=0}^T$ a stochastic process adapted to information structure $\mathcal{P}$ if $X(t)$ is measurable wrt $\mathcal{P}_t$ for all t.

We finally denote by $L = \sum_t s_t$ the number of cells in a given information structure, so that the whole stochastic process X can be described by a vector in $\mathbb{R}^L$.

Remark (Alternative definition of Stochastic Process)**.** *A stochastic process X is a map $X : \mathbb{T} \times \Omega \to \mathbb{R}$ such that for any f_h^t and any $w', w'' \in f_h^t$ we have that $X(t)(w') = X(t)(w'')$.*

1.1 Conditional Expectation and Independence

Definition 6 (Conditional Expectation)**.** The conditional expected value of X given $\mathcal{P}_t$ and under the probability Q is:

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{P}_t](f_h^t) = \sum_{\omega \in f_h^t} X(\omega)\mathbb{Q}[\omega|f_h^t] = \sum_{\omega \in f_h^t} X(\omega)\frac{\mathbb{Q}[w]}{\mathbb{Q}[f_h^t]}.$$

Properties of Conditional Expectation:

- $\mathbb{E}\left[\mathbb{E}[X|\mathcal{P}_{t_2}]|\mathcal{P}_{t_1}\right] = \mathbb{E}[X|\mathcal{P}_{t_1}]$
- $\mathcal{P}_t$-measurable random variables can be taked out of $\mathcal{P}_t$-conditional expectations:

$$\mathbb{E}\left[hX|\, \mathcal{P}_t\right] = h\mathbb{E}[X|\mathcal{P}_t] \qquad \text{if } h \text{ is } \mathcal{P}_t\text{-measurable.}$$

- Linearity: $\mathbb{E}[hX + kY|\mathcal{P}_t] = h\mathbb{E}[X|\mathcal{P}_t] + k\mathbb{E}[Y|\mathcal{P}_t]$.

Definition 7 (Independence)**.** A random variable X with values $x_1, \dots, x_M$ on Ω is independent of the information $\mathcal{P}_t$ wrt probability Q if:

$$\mathbb{Q}\left[\{\omega : X(\omega) = x_m\} \cap f_h^t\right] = \mathbb{Q}\left[\{\omega : X(\omega) = x_m\}\right] \cdot Q\left[f_h^t\right], \quad \forall f_h^t \text{ and for all } m = 1, \dots, M.$$

Lemma 1. *If X is independent of $\mathcal{P}_t$ wrt $\mathbb{Q}$, then $\mathbb{E}[X|\mathcal{P}_t] = \mathbb{E}[X]$.*

Lemma 2. *If X is independent of $\mathcal{P}_t$ and Y is measurable under $\mathcal{P}_t$, then $\mathbb{E}[f(X,Y)|\mathcal{P}_t] = \mathbb{E}[f(X,Y)]$.*

Definition 8 (Martingale)**.** Let $M = \{M\}_{t=0}^T$ be a process adapted to $\mathcal{P}$. Then, under probability $\mathbb{Q}$, M is a martingale given $\mathcal{P}$ if $\forall t_1, t_2 \in \{0, \dots, T\}$ such that $t_2 > t_1,$ we have that:

$$\mathbb{E}^{\mathbb{Q}}\left[M(t_2)|\mathcal{P}_{t_1}\right] = M(t_1).$$

Lemma 3. *For all t, we have that $\mathbb{E}^{\mathbb{Q}}[M(t)] = M(0)$.*

Lemma 4. *M is a martingale iff for all t, we have that $\mathbb{E}^{\mathbb{Q}}[M(t+1)|\mathcal{P}_t] = M(t)$.*

2 Multi-period Markets

In this section we consider $N+1$ securities indexed by j which are traded at dates $t = 0, \dots, T$. We also assume that there is an information structure $\mathcal{P} = \{\mathcal{P}_t\}_{t=0}^T$. We will denote by $S_j(t)(f_h^t)$ the price of security j at node f_h^t of the information structure. The security indexed by $j = 0$ is denoted by $B = \{B(t)\}_{t=0}^T$, and we have $B(0) = 1$. We also define

$$r(t) = \frac{B(t+1) - B(t)}{B(t)} \qquad \text{which represents the net return from investing in the riskless bond at time } t.$$

We assume that $\{r(t)\}_{t=0}^{T-1}$ is a stochastic process adapted to the information structure $\mathcal{P}$.

Observing that $B(t+1) = B(t)(1 + r(t))$ and using the fact that $B(0) = 1$, we can see that:

$$B(t+1) = B(t)(1 + r(t)) = B(t-1)(1 + r(t-1))(1 + r(t)) = \prod_{\tau=0}^{t}(1 + r(\tau)).$$

2.1 Dynamic Investment Strategies

We let the positions taken on the $N+1$ securities by described by $N+1$ stochastic processes denoted by $\theta_j = \{\theta_j(t)\}_{t=0}^{T-1}$. We denote by $\theta = (\theta_0, \dots, \theta_N)$ the dynamic investment strategy.

Definition 9 (Value Process)**.**

$$V_\theta(t) = \begin{cases} \theta_0(t)B(t) + \sum_{j=1}^N \theta_j(t)S_j(t) & t = 0, \dots, T-1 \\ \theta_0(T-1)B(T) + \sum_{j=1}^N \theta_j(T-1)S_j(T) & t = T. \end{cases}$$

Definition 10 (Cashflow Process)**.**

$$C_\theta(t) = \begin{cases} -V_\theta(0) & t = 0 \\ \theta_0(t-1)B(t) + \sum_{j=1}^N \theta_j(t-1)S_j(t) - V_\theta(t) & t = 1, \dots, T-1 \\ V_\theta(T) & t = T. \end{cases}$$

Definition 11 (Self-financing Dynamic Strategies)**.** A dynamic strategy is said to be self-financing if $C_\theta(t) = 0$ for all $t = 1, \dots, T-1$. This implies that the intermediate cash flow is zero.

Definition 12 (Discounted Gain Process)**.**

$$G_\theta(t) = \begin{cases} \frac{V_\theta(t)}{B(t)} + \sum_{\tau=0}^t \frac{C_\theta(\tau)}{B(\tau)}, & t = 0, \dots, T-1 \\ \sum_{\tau=0}^T \frac{C_\theta(\tau)}{B(\tau)} & t = T. \end{cases}$$

Definition 13 (Violation of the LOP)**.** A multi-period financial market gives rise to violations of the law of one price if there exist two dynamic strategies generating the same cashflow in $t > 1$ but having different initial values. That is:

$$\theta \neq \theta', \quad C_\theta(t)(f_h^t) = C_{\theta'}(t)(f_h^t), \quad V_\theta(0) \neq V_{\theta'}(0).$$

Definition 14 (Arbitrage Opportunities of the 1^{st} type)**.** A multi-period financial market gives rise to arbitrage opportunities of the first type if there exists a dynamic strategy θ having the following properties:

$$\begin{aligned} & V_\theta \leq 0 \\ & C_\theta(t)(f_h^t) \geq 0, \quad \text{for all } h = 1, \dots, s_t, \ t = 1, \dots, T. \\ & C_\theta(\tau)(f_\ell^\tau) > 0, \quad \text{for some } \tau \in \{1, \dots, T\}, \ \ell \in \{1, \dots, s_\tau\}. \end{aligned}$$

Definition 15 (Arbitrage Opportunities of the 2^{nd} type)**.** A multi-period financial market gives rise to arbitrage opportunities of the second type if the following hold:

$$\begin{aligned} & V_\theta < 0 \\ & C_\theta(t)(f_h^t) \geq 0, \quad \text{for all } h = 1, \dots, s_t, \ t = 1, \dots, T. \end{aligned}$$

Lemma 5. *No Arbitrage implies LOP.*

3 Characterization of No Arbitrage

Definition 16 (State Price Vectors)**.** We say that $\psi = \big(\psi(f_1^0), \psi(f_1^1), \psi(f_2^1), \dots, \psi(f_1^T), \psi(f_2^T), \psi(f_K^T)\big) \in \mathbb{R}^L$ is a state price vector if

- $\psi > 0$,
- $\psi(f_1^0) = 1$,
- $$\frac{1}{1 + r(t)(f_h^t)} = \sum_{f_\ell^{t+1} \subset f_h^t} \frac{\psi(f_\ell^{t+1})}{\psi(f_h^t)}, \qquad t = 1, \dots, T, \ h = 1, \dots, s_t.$$

-
$$S_j(t)(f_h^t) = \sum_{f_\ell^{t+1} \subset f_h^t} \frac{\psi(f_\ell^{t+1})}{\psi(f_h^t)} S_j(t+1)(f_\ell^{t+1}), \qquad \begin{array}{l} j = 1, \dots, N \\ h = 1, \dots, s_t \\ t = 1, \dots, T. \end{array}$$

Definition 17 (Equivalent Martingale Measures EMM)**.** $\mathbb{Q}$, a strictly positive probability over Ω, is an EMM if the following holds:

$$\begin{array}{l} i = 0, \dots, N \\ t = 0, \dots, T-1 \end{array}, \qquad S_i(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_i(t+1)}{1+r(t)}\right] \implies \mathbb{E}^{\mathbb{Q}}\left[\frac{S_i(t+1) - S_i(t)}{S_i(t)}\right] = r(t).$$

One can also show the following for the present value of security j given $\mathbb{Q}$:

$$\frac{S_j(t)}{B(t)} = \tilde{S}_j(t) = \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_j(t+1)|\mathcal{P}_t\right]$$

Lemma 6. *Given a strictly probability $\mathbb{Q}$, the following are equivalent:*

- *$\mathbb{Q}$ is an equivalent martingale measure,*
- *Every dynamic strategy satisfies:*

$$V_\theta(t) = \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[\frac{V_\theta(t+1)+C_\theta(t+1)}{1+r(t)}|\mathcal{P}_t\right], & t = 0, \dots, T-2 \\ \mathbb{E}^{\mathbb{Q}}\left[\frac{C_\theta(T)}{1+r(T-1)}|\mathcal{P}_{t-1}\right], & t = T-1. \end{cases}$$

- *the discounted gain process of every dynamic strategy is a martingale under $\mathbb{Q}$:*

$$G_\theta(t) = \mathbb{E}^{\mathbb{Q}}[G_\theta(t+1)|\mathcal{P}_t], \quad t = 0, 1, \dots, T-1.$$

Theorem 7 (First Fundamental Theorem of Asset Pricing)**.** *TFAE:*

- *No Arbitrage holds.*
- *There exists a state price vector ψ.*
- *There exists a risk-neutral probability $\mathbb{Q}$.*

4 Dynamically Complete Multi-period Markets

Definition 18 (Contingent Claim in a Multi-period setting)**.** A contingent claim is a sequence $X = \{X(t)\}_{t=1}^T$ of random variables adapted to the given information structure $\mathcal{P} = \{P_t\}_{t=1}^T$. We then say that X is attainable in the multi-period market financial market if there exists a dynamic investment strategy θ such that:

$$C_\theta(t) = X(t), \ t = 1, \dots, T.$$

Definition 19 (Dynamically Complete Multi-period Financial Market)**.** We say that a discrete-time multi-period financial market is dynamically complete if every contingent claim $X = \{X(t)\}_{t=0}^T$ is attainable. (typo: t=1?)

Lemma 8. *A discrete-time multi-period market is dynamically complete if every one-period submarket is complete. Conversely, if a multi-period market is dynamically complete and every one-period submarket satisfies the law of one price, then every one-period submarket is complete.*

Theorem 9 (Second Fundamental Theorem of Asset Pricing)**.** *The following statements are equivalent:*

- *The market obtains NA and is complete.*
- *There exists a unique risk-neutral probability (EMM).*

5 No-arbitrage Valuation in the Multi-period Case

We look for conditions on the price of a new security such that the extended market is arbitrage-free as well. We take as an input a multi-period market in which B and N risky securities are traded. We assume that this market is arbitrage-free. A new security is introduced with a cashflow $\{X(t)\}_{t=1}^{T}$ and available for trading at dates $t = 0, \ldots, T$ at prices $\{S_X(t)\}_{t=0}^{T}$. We assume that $X(T) = S_X(T)$.

5.1 Redunant Securities

Definition 20 (Redunant Security)**.** A security is redunant if there exists a strategy $\theta^X = \{\theta^X(t)\}_{t=0}^{T-1}$ such that $C_{\theta^X}(t) = X(t), \forall t$.

Lemma 10. *If the new security is redunant, TFAE:*

- *the extended market is arbitrage-free.*
- *for every θ^X replicating the new strategy, we have:*

$$S_X(t) = V_{\theta^X}(t), \quad \forall t.$$

- *for every risk-neutral probability measure, $\mathbb{Q}$, of the initial market, we have that:*

$$S_X(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(t+1)+S_X(t+1)}{1+r(t)}\middle|\mathcal{P}_t\right], \qquad t = 0, \ldots, T-2.$$

$$S_X(T-1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T)}{1+r(T-1)}\middle|\mathcal{P}_{T-1}\right], \quad t = T-1.$$

Definition 21 (Gordon's formula)**.** The following equation enables us to determine the price of a new security:

$$S_X(t) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{\tau=t+1}^{T}\frac{B(t)}{B(\tau)}X(\tau)\middle|\mathcal{P}_t\right], \quad \forall t = 0, \ldots, T-1.$$

*Non-redundant securities were skipped.

6 Binomial Model

The multi-period binomial model involves two securities. The first one is the risk-free bond B having a constant interest rate r at all times. The second security is $S(t)$ modeled in the following way:

$$\begin{array}{cccc} & \nearrow & S(t)\,u & \text{with probability } p \\ S(t) & & & \\ & \searrow & S(t)\,d & \text{with probability } 1-p \\ & & & \\ t & & t+1 & \end{array}$$

Figure 2: The risky stock in the Binomial model.

At time t, there are $t+1$ prices that S can take. The probability of each state is given by

$$\mathbb{P}\left[S(t) = Su^k d^{t-k}\right] = \binom{t}{k} p^k (1-p)^{t-k}.$$

In the binomial model, every 1-period submarket is complete and thus the whole multi-period market is dynamically complete. As long as $d < 1+r < u$ holds, then the market obtains no-arbitrage as well. The risk-neutral probability is given by:

$$\mathbb{Q}\left[S(t+1) = S(t)u\right] = q = \frac{1+r-d}{u-d}.$$

The risk-neutral probabilities do not depend on time nor the information node. For a generic date t, we get that:

$$\mathbb{Q}\left[S(t) = Su^k d^{t-k}\right] = \binom{t}{k} q^k (1-q)^{t-k}.$$

7 American and Path dependant Options

7.1 Valuation of American Options

Let X be an American option with maturity T. The option can be exercised at any intermediate date $t = 1, \ldots, T$. By properly choosing the exercise time τ, the holder can increase the value of his position. Hence the value of the American option at time t is

$$\tilde{V}(t) = \max_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}} \left[\frac{X(\tau)}{B(\tau)} \middle| \mathcal{P}_t \right].$$

τ is a random variable, $\tau : \Omega \to T$ with the property that $\{\omega : \tau(\omega) = t\} \in \mathcal{P}_t$ for every t. A random variable τ with this property is called stopping time with respect to $\mathcal{P}_t$.

Properties of $\tilde{V}$:

- $\tilde{V}$ is a super-martingale: $\tilde{V}(t_1) \geq \mathbb{E}^{\mathbb{Q}} \left[\tilde{V}(t_2) \middle| \mathcal{P}_{t_1} \right]$ for any $t_1 \leq t_2$.
- $\tilde{V}$ is the smallest supermartingale greater or equal to the discounted payoff $\tilde{X}$. This means that if Y is a supermartingale such that $Y(t) \geq \tilde{X}(t)$, then $Y(t) \geq \tilde{V}(t)$.
- The optimal exercise policy τ^* is defined as

$$\tau^*(w) = \min \left\{ t : \tilde{V}(t)(\omega) = \tilde{X}(t)(w) \right\}.$$

- The discounted value of the American option if optimally exercises is a $\mathbb{Q}-$martingale:

$$\left\{ \tilde{V}((\tau^*(\omega) \wedge t)(\omega)) \right\}_{t=0}^{T} \text{ is a } \mathbb{Q} \text{ martingale.}$$

Lemma 11 (Backward recursive formula for the American option)**.** *The discounted value of the American option $\tilde{V}$ is given by:*

$$\tilde{V}(t) = \max \left\{ \tilde{X}(t); \mathbb{E}^{\mathbb{Q}} \left[\tilde{V}(t+1) \middle| \mathbb{P}_t \right] \right\}$$