Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188

a.y. 2021/22, Midterm, 20$^{\text{th}}$ October 2021

EXERCISE 1 (35 points out of 100).
Consider a one period market with the riskless asset B yielding a risk-free rate $r = 4\%$, and a risky security S whose prices at time $T = 1$ are

$$S(1)(\omega_1) = 20.8$$
$$S(1)(\omega_2) = 15.6$$
$$S(1)(\omega_3) = 5.2$$

1. **(3 points)** Is the market complete? Justify your answer.

2. **(7 points)** Suppose that the risky security S trades at $t = 0$ at the price

$$S(0) = 11.$$

Let $\mathbb{Q}(\omega_k)$ for $k = 1, 2, 3$ denote a *risk-neutral probability*. Determine $\mathbb{Q}(\omega_1)$ and $\mathbb{Q}(\omega_3)$ in terms of $\mathbb{Q}(\omega_2) = q_2$.

(Mark the only appropriate alternative)

A $\quad \mathbb{Q}(\omega_1) = \frac{3}{5} - \frac{1}{3}q_2$ and $\mathbb{Q}(\omega_3) = \frac{2}{5} - \frac{2}{3}q_2$

B $\quad \mathbb{Q}(\omega_1) = \frac{2}{5} - \frac{2}{3}q_2$ and $\mathbb{Q}(\omega_3) = \frac{3}{5} - \frac{1}{3}q_2$

C $\quad \mathbb{Q}(\omega_1) = \frac{5}{2} - \frac{3}{2}q_2$ and $\mathbb{Q}(\omega_3) = \frac{5}{2} - \frac{3}{2}q_2$

D $\quad \mathbb{Q}(\omega_1) = \frac{2}{5} - \frac{1}{3}q_2$ and $\mathbb{Q}(\omega_3) = \frac{3}{5} - \frac{2}{3}q_2$

3. **(4 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_2) = q_2$ must be strictly larger than $\boxed{}$

4. **(4 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_2) = q_2$ must be strictly smaller than $\boxed{}$

5. **(4 points)** Is the market free of arbitrage opportunities? Justify your answer.

6. **(8 points)** A European *put option* on S with maturity $T = 1$ and strike price $K = 20.8$ is introduced in the market.

   Is it possible to replicate this put option? $\boxed{\text{YES}}$ $\boxed{\text{NO}}$

   If your answer is positive, determine the set of no-arbitrage prices at $t = 0$ for this put option.

7. **(5 points)** Suppose the put option of Point 6 trades at the initial price of 10. The extended market (the one in which B, S and the put option are traded) is...

   (Mark the only appropriate alternative)

   $\boxed{\text{A}}$ arbitrage-free AND complete

   $\boxed{\text{B}}$ not arbitrage free AND complete

   $\boxed{\text{C}}$ arbitrage-free AND incomplete

   $\boxed{\text{D}}$ not arbitrage free AND incomplete

EXERCISE 2 (45 points out of 100).

Consider a one period market with the riskless asset B yielding a zero riskless interest rate (namely, $r = 0\%$), and two risky securities S_1 and S_2 whose prices at time $T = 1$ are

$$\begin{matrix} S_1(1)(\omega_1) = 8 \\ S_1(1)(\omega_2) = 10 \\ S_1(1)(\omega_3) = 5 \end{matrix} \quad \text{and} \quad \begin{matrix} S_2(1)(\omega_1) = 11 \\ S_2(1)(\omega_2) = 15 \\ S_2(1)(\omega_3) = 5 \end{matrix}$$

leading to the matrix $\mathcal{A}$ of $T = 1$ prices

$$\mathcal{A} = \begin{bmatrix} 1 & 8 & 11 \\ 1 & 10 & 15 \\ 1 & 5 & 5 \end{bmatrix}.$$

1. **(3 points)** Is the market complete? Justify your answer.

2. **(4 points)** Suppose that the risky securities S_1 and S_2 trade at $t = 0$ at the prices

$$S_1(0) = 15 \quad \text{and} \quad S_2(0) = 25.$$

   Which of the following linear systems must be satisfied by a state price vector $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$?

(Mark the only appropriate alternative)

A $\quad \mathcal{A}\psi = \begin{bmatrix} 1 \\ 15 \\ 25 \end{bmatrix}$ B $\quad \mathcal{A}^T\psi = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

C $\quad \mathcal{A}^T\psi = \begin{bmatrix} 1 \\ 15 \\ 25 \end{bmatrix}$ D $\quad \mathcal{A}^T\psi = \begin{bmatrix} -1 \\ -15 \\ -25 \end{bmatrix}$

3. **(4 points)** Does the (correct) linear system defining ψ in the previous point admit any solution?
(Mark the only appropriate alternative)

A Yes, the system admits infinitely many solutions in $\mathbb{R}^3$, whose components are all strictly positive

B Yes, the system admits infinitely many solutions in $\mathbb{R}^3$

C Yes, the system admits one and only one solution in $\mathbb{R}^3$

D No, the system does not admit any solution

4. **(3 points)** Is the market free of arbitrage opportunities? Justify your answer.

5. **(3 points)** Consider the strategy $\vartheta = (\vartheta_0, \vartheta_1)$ where ϑ_0 denotes the number of units of the riskless asset B and ϑ_1 the number of units of the risky security S_1. The strategy does not involve S_2 at all. Assume $\vartheta_0 = 1$ and $\vartheta_1 = -0.1$. The value at $T = 1$ of this strategy on the three scenarios is equal to...

$$V_\vartheta(1)(\omega_1) = \boxed{}$$

$$V_\vartheta(1)(\omega_2) = \boxed{}$$

$$V_\vartheta(1)(\omega_3) = \boxed{}$$

6. **(4 points)** The strategy ϑ introduced in the previous point is...
*(Mark **ALL** the appropriate alternatives)*

A an arbitrage of the first type

B an arbitrage of the second type

C a strategy that does not constitute an arbitrage

7. **(10 points)** Consider now a strategy $\vartheta = (\vartheta_0, \vartheta_1)$ such that $V_\vartheta(1)(\omega_k) \geq 0$ for $k = 1, 2, 3$. This happens if and only if ϑ_1 satisfies the following inequalities in terms of $\vartheta_0 \in \mathbb{R}$

$\vartheta_1 \geq \boxed{}$ for $\vartheta_0 \geq 0$

$\vartheta_1 \geq \boxed{}$ for $\vartheta_0 < 0$

8. **(3 points)** Consider a strategy $\vartheta = (\vartheta_0, \vartheta_1)$ such that $V_\vartheta(1)(\omega_k) \geq 0$ for $k = 1, 2, 3$ as above. What additional constraint ϑ has to satisfy in order to be an arbitrage opportunity?

$$\vartheta_1 \leq \boxed{} \quad \text{for } \vartheta_0 \in \mathbb{R}$$

9. **(5 points)** Describe formally the set of arbitrage opportunities that you have retrieved in this market.

10. **(6 points)** Assume now that at $t = 0$ you cannot short more than 0.2 units of S_1. Among the arbitrage strategies previously found, determine the strategy $\vartheta^* = (\vartheta_0^*, \vartheta_1^*)$ that obeys the short-selling constraint $\vartheta_1^* \geq -0.2$ and maximizes the initial cashflow $-V_{\vartheta^*}(0)$.

$$\vartheta_0^* = \boxed{} \qquad \text{and} \qquad \vartheta_1^* = \boxed{}$$

THEORY QUESTIONS (20 points out of 100)

1. **(4 points)** State the Second Fundamental Theorem of Asset Pricing.

2. For the remaining questions, consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure $\mathcal{P} = \{\mathcal{P}_t\}_{t=0,1,2}$:

$$\begin{tikzcd}
 & & \omega_1 \\
 & f_1^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_2 \\
f_1^0 \arrow[uur] \arrow[ddr] & & \\
 & & \omega_3 \\
 & f_2^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_4
\end{tikzcd}$$

and assume the probability is given by

$$\mathbb{P}[\omega_1] = \frac{3}{20} \quad \mathbb{P}[\omega_2] = \frac{7}{20} \quad \mathbb{P}[\omega_3] = \frac{7}{20} \quad \mathbb{P}[\omega_4] = \frac{3}{20}$$

Consider the stochastic process X defined as follows:

$$X(0) = 8.5$$
$$X(1)(\omega) = \begin{cases} 9 & \omega = \omega_1, \omega_2 \\ 7 & \omega = \omega_3, \omega_4 \end{cases}$$
$$X(2)(\omega) = \begin{cases} 5 & \omega = \omega_1 \\ 10 & \omega = \omega_2 \\ 10 & \omega = \omega_3 \\ 5 & \omega = \omega_4 \end{cases}$$

(4 points) Consider the random variable $X(2)$: is it $\mathcal{P}_1$-measurable? Justify your answer.

3. **(4 points)** Compute $\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]$.

$\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\left(f_1^1\right) = \boxed{}$ and $\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\left(f_2^1\right) = \boxed{}$

4. **(4 points)** Consider the random variable $X(2)$: is it independent of $\mathcal{P}_1$ with respect to $\mathbb{P}$? Justify your answer.

5. **(4 points)** Given the previous computations...

*(Mark **ALL** the appropriate alternatives)*

- A $\{X(t)\}_{t=0,1,2}$ is a martingale with respect to $\mathbb{P}$
- B $\{X(t)\}_{t=0,1,2}$ is not a martingale with respect to $\mathbb{P}$
- C $\mathbb{E}^{\mathbb{P}}\left[X(1)|\,\mathcal{P}_1\right] = X(1)$
- D $\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\right] = \mathbb{E}^{\mathbb{P}}\left[X(2)\right]$

Solution of the Exercises

These are the detailed soultions of the exam.

The expected answers of the exam are written in italics.

Solution of EXERCISE 1

1. *The market is incomplete, because the number of scenarios* $3 > 2$, *which is the number of traded securities. Since there are fewer independent securities than scenarios, the market cannot be complete.*

2. Since the market is incomplete, the risk-neutral measures (if any) cannot be unique. Denoting by $q_i = \mathbb{Q}(\omega_i) > 0$ for $i = 1, ..., 3$, we have that

$$\begin{aligned} \frac{1}{1.04}\left(20.8q_1 + 15.6q_2 + 5.2\underbrace{(1-q_1-q_2)}_{q_3}\right) &= 11 \\ 20q_1 + 15q_2 + 5(1-q_1-q_2) &= 11 \\ 15q_1 &= 6 - 10q_2 \\ q_1 &= \frac{6}{15} - \frac{10}{15}q_2 \\ q_1 &= \frac{2}{5} - \frac{2}{3}q_2 \end{aligned}$$

Then,

$$\begin{aligned} q_3 &= 1 - q_1 - q_2 \\ &= 1 - \frac{2}{5} + \frac{2}{3}q_2 - q_2 \\ &= \frac{3}{5} - \frac{1}{3}q_2. \end{aligned}$$

Therefore, assuming the appropriate bounds x and y on q_2, we get

$$\begin{cases} q_1 = \frac{2}{5} - \frac{2}{3}q_2 \\ q_2 \in (x, y) \\ q_3 = \frac{3}{5} - \frac{1}{3}q_2 \end{cases}$$

Hence the right answer is

$$\boxed{\text{B}} \qquad \mathbb{Q}(\omega_1) = \frac{2}{5} - \frac{2}{3}q_2 \text{ and } \mathbb{Q}(\omega_3) = \frac{3}{5} - \frac{1}{3}q_2$$

3. In order to find the appropriate bounds x and y on q_2 we impose positivity constraints on q_1 and q_3

$$\begin{cases} \frac{2}{5} - \frac{2}{3}q_2 > 0 \\ \frac{3}{5} - \frac{1}{3}q_2 > 0 \end{cases}$$

that are satisfied by

$$\begin{cases} q_2 < \frac{3}{5} = 0.6 \\ q_2 < \frac{9}{5} = 1.8 \end{cases}$$

Since $q_2 \in (0, 1)$ and since the first inequality of the previous system is more restrictive than the second one, we get

$$q_2 \in (0, 0.6) = \left(0, \frac{3}{5}\right).$$

Therefore, in order for $Q(\omega_1)$, $Q(\omega_2)$, $Q(\omega_3)$ *to define a risk-neutral probability,* $Q(\omega_2) = q_2$ *must be strictly larger than* 0

4. *...and* $Q(\omega_2) = q_2$ *must be strictly smaller than* $\frac{3}{5} = 0.6$.

5. *Since there exist risk neutral probabilities, the market is arbitrage-free by the 1^{st} FTAP.*

6. The payoff at $T = 1$ of the European *put option* with strike price $K = 20.8$ is

$$\begin{array}{r}p(1)(\omega_1) = (20.8 - 20.8)^+ = 0 \\ p(1)(\omega_2) = (20.8 - 15.6)^+ = 5.2 \\ p(1)(\omega_3) = (20.8 - 5.2)^+ = 15.6\end{array}$$

   It can be replicated with B, S_1 and S_2 because the system

$$A\vartheta = p(1)$$

   admits solutions, as $rk\,(A) = rk(\,A|\,p(1)) = 2$ as

$$\det \begin{bmatrix} 1.04 & 20.8 & 0 \\ 1.04 & 15.6 & 5.2 \\ 1.04 & 5.2 & 15.6 \end{bmatrix} = 0.$$

   Hence: *Is it possible to replicate this put option?* $\boxed{\text{YES}}$

   Thus the put option does admit a unique no-arbitrage price at $t = 0$ given by the risk-neutral expected value of the put's discounted payoff:

$$\begin{aligned} p(0) &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[p(1)\right] \\ &= \frac{1}{1.04}\left(0\cdot\left(\frac{2}{5}-\frac{2}{3}q_2\right)+5.2q_2+15.6\cdot\left(\frac{3}{5}-\frac{1}{3}q_2\right)\right) \\ &= \frac{1}{1.04}\left(5.2q_2+9.36-5.2q_2\right) \\ &= \frac{9.36}{1.04} = 9. \end{aligned}$$

   This expected value is indeed independent of q_2, and the initial no arbitrage price of the put option is unique (and equal to 9). For sake of completeness we notice that the put option be replicated by the strategy $\vartheta^p = (\vartheta^p_0, \vartheta^p_1)$ given by

$$\vartheta^p_0 = 20 \quad \text{and} \quad \vartheta^p_1 = -1$$

   as

$$\begin{cases} \vartheta^p_0\cdot(1+r)+\vartheta^p_1\cdot S(1)(\omega_1) = 20\cdot 1.04 - 1\cdot 20.8 = 0 = p(1)(\omega_1) \\ \vartheta^p_0\cdot(1+r)+\vartheta^p_1\cdot S(1)(\omega_2) = 20\cdot 1.04 - 1\cdot 15.6 = 5.2 = p(1)(\omega_2) \\ \vartheta^p_0\cdot(1+r)+\vartheta^p_1\cdot S(1)(\omega_3) = 20\cdot 1.04 - 1\cdot 5.2 = 15.6 = p(1)(\omega_3) \end{cases}$$

   The initial value of this strategy is $\vartheta^p_0 B(0) + \vartheta^p_1 S(0) = 20 - 11 = 9 = p(0)$.

   The initial price of the put option is $p(0) = 9$.

7. If the put option of the previous question trades at an initial price of $10 \neq 9 = p(0)$, the market is not arbitrage free. For sake of completeness, we point out that in this case the LOP is violated (the put option and its replicating strategy ϑ^p have the same terminal payoff but different initial prices as the put option trades at $t = 0$ at 10 and its cost of replication is $V_{\vartheta^p}(0) = 9$).

   Hence the right answer is

   $\boxed{\text{D}}$ not arbitrage free AND incomplete

Solution of EXERCISE 2

1. The payoff matrix for this market is

$$\mathcal{A} = \begin{bmatrix} 1 & 8 & 11 \\ 1 & 10 & 15 \\ 1 & 5 & 5 \end{bmatrix}.$$

To check whether the market is complete or not we compute $\det[\mathcal{A}]$ which turns out to be equal to 0.

Therefore, by the completeness' test, the market is incomplete because $rk[\mathcal{A}] = 2 < 3 = K$, *i.e. we have fewer linearly independent securities than scenarios.*

2. Let $\psi = [\psi_1 \ \ \psi_2 \ \ \psi_3]'$. We look for state price vectors solving

$$\mathcal{A}'\psi = \begin{bmatrix} B(0) \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

Hence the right alternative is

$$\boxed{\text{C}} \qquad \mathcal{A}^T\psi = \begin{bmatrix} 1 \\ 15 \\ 25 \end{bmatrix}$$

3. The previous linear system is

$$\begin{cases} \psi_1 + \psi_2 + \psi_3 = 1 \\ 8\psi_1 + 10\psi_2 + 5\psi_3 = 15 \\ 11\psi_1 + 15\psi_2 + 5\psi_3 = 25 \end{cases}$$

$$\begin{cases} \psi_1 = 1 - \psi_2 - \psi_3 \\ 8(1 - \psi_2 - \psi_3) + 10\psi_2 + 5\psi_3 = 15 \\ 11(1 - \psi_2 - \psi_3) + 15\psi_2 + 5\psi_3 = 25 \end{cases}$$

$$\begin{cases} \psi_1 = 1 - \psi_2 - \psi_3 \\ 2\psi_2 - 3\psi_3 = 7 \\ 4\psi_2 - 6\psi_3 = 14 \end{cases}$$

The second and the third equation are linearly depedent (the third is two times the second one), which is consistent with the fact that the market is incomplete. Getting rid of one of the two redundant equation we get

$$\begin{cases} \psi_1 = 1 - \psi_2 - \psi_3 \\ \psi_2 = \frac{7}{2} + \frac{3}{2}\psi_3 \end{cases}$$

$$\begin{cases} \psi_1 = 1 - \left(\frac{7}{2} + \frac{3}{2}\psi_3\right) - \psi_3 \\ \psi_2 = \frac{7}{2} + \frac{3}{2}\psi_3 \end{cases}$$

$$\begin{cases} \psi_1 = -\frac{5}{2} - \frac{5}{2}\psi_3 \\ \psi_2 = \frac{7}{2} + \frac{3}{2}\psi_3 \end{cases}$$

Therefore, our linear system does have solutions and they are given by

$$\begin{cases} \psi_1 = -\frac{5}{2} - \frac{5}{2}\psi_3 \\ \psi_2 = \frac{7}{2} + \frac{3}{2}\psi_3 \\ \psi_3 \end{cases}$$

However, when looking for $\psi >> 0$ we get

$$\begin{cases} \psi_1 = -\frac{5}{2} - \frac{5}{2}\psi_3 > 0 \\ \psi_2 = \frac{7}{2} + \frac{3}{2}\psi_3 > 0 \\ \psi_3 > 0 \end{cases}$$

which delivers

$$\begin{cases} \psi_3 < -1 \\ \psi_3 > -\frac{7}{3} \\ \psi_3 > 0 \end{cases}$$

and that has no solution. Therefore, there is no way to get a striclty positive state price vector. *Hence the right answer is*

$\boxed{\text{B}}$ Yes, the system admits infinitely many solutions in $\mathbb{R}^3$

4. *Because there is no state price vector* $\psi >> 0$ *such that* $A^T\psi = \begin{bmatrix} 1 \\ 15 \\ 25 \end{bmatrix}$, *the market admits arbitrage opportunities by the 1*st *FTAP.*

5. The terminal value at $T = 1$ of the strategy $\vartheta = (\vartheta_0, \vartheta_1)$ with $\vartheta_0 = 1$ and $\vartheta_1 = -0.1$ is

$$\begin{cases} V_\vartheta(1)(\omega_1) = \vartheta_0 \cdot 1 + \vartheta_1 \cdot S_1(1)(\omega_1) = 1 - 0.1 \cdot 8 = 0.2 \\ V_\vartheta(1)(\omega_2) = \vartheta_0 \cdot 1 + \vartheta_1 \cdot S_1(1)(\omega_2) = 1 - 0.1 \cdot 10 = 0 \\ V_\vartheta(1)(\omega_3) = \vartheta_0 \cdot 1 + \vartheta_1 \cdot S_1(1)(\omega_3) = 1 - 0.1 \cdot 5 = 0.5 \end{cases}$$

Thus

$$\begin{aligned} V_\vartheta(1)(\omega_1) &= 0.20 \\ V_\vartheta(1)(\omega_2) &= 0 \\ V_\vartheta(1)(\omega_3) &= 0.5 \end{aligned}$$

6. The inital value of the strategy introduced in the previous point is

$$\begin{aligned} V_\vartheta(0) &= \vartheta_0 \cdot 1 + \vartheta_1 \cdot S_1(0) \\ &= 1 - 0.1 \cdot 15 = -0.5. \end{aligned}$$

Therefore, this strategy is both an arbitrage of the first type and an arbitrage of the second type. *Hence the appropriate alternatives are*

$\boxed{\text{A}}$ an arbitrage of the first type

$\boxed{\text{B}}$ an arbitrage of the second type

7. The strategies $\vartheta = (\vartheta_0, \vartheta_1)$ whose final value is positive on all the scenarios are given by

$$V_\vartheta(1)(\omega_k) \geq 0$$

for all $k = 1, ..., 3$. We get

$$\begin{cases} V_\vartheta(1)(\omega_1) = \vartheta_0 + 8\vartheta_1 \geq 0 \\ V_\vartheta(1)(\omega_2) = \vartheta_0 + 10\vartheta_1 \geq 0 \\ V_\vartheta(1)(\omega_3) = \vartheta_0 + 5\vartheta_1 \geq 0 \end{cases}$$

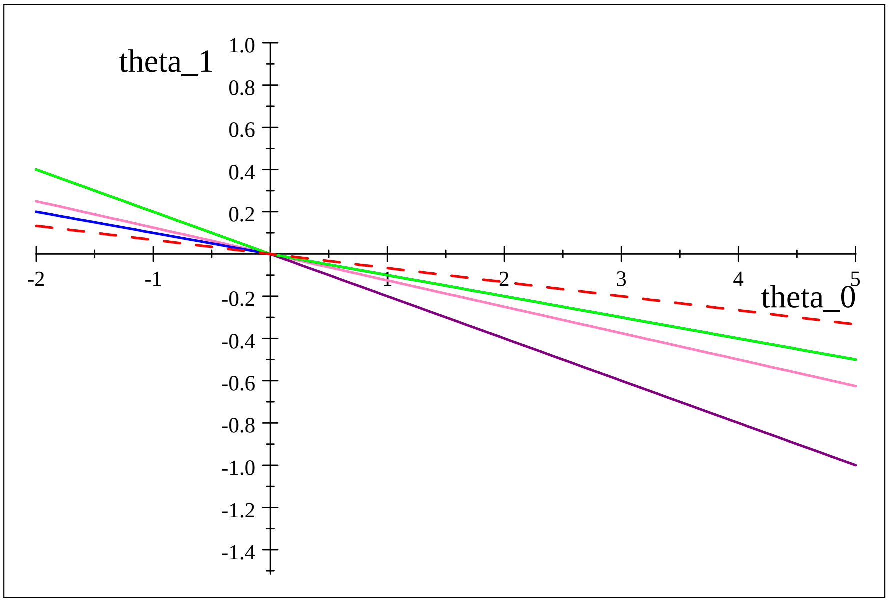
i.e. the super-replication at $T = 1$ of a constant *zero* contingent claim:

$$\begin{cases} \vartheta_0 + 8\vartheta_1 \geq 0 \\ \vartheta_0 + 10\vartheta_1 \geq 0 \\ \vartheta_0 + 5\vartheta_1 \geq 0 \end{cases}$$

The system leads to the equation of the three lines

$$\begin{aligned} r_1 &: \vartheta_1 = -\frac{1}{8}\vartheta_0 = -0.125\vartheta_0 \quad \text{pink in the graph below} \\ r_2 &: \vartheta_1 = -\frac{1}{10}\vartheta_0 = -0.1\vartheta_0 \quad \text{blue in the graph below} \\ r_3 &: \vartheta_1 = -\frac{1}{5}\vartheta_0 = -0.2\vartheta_0 \quad \text{purple in the graph below} \end{aligned}$$

The region we are interested in is the set of $(\vartheta_0, \vartheta_1)$ bounded by below by the green contour in the graph plotted below.



Such a green contour coincides with the maximum between r_1, r_2 and r_3.

If we restrict our analysis to the cases $\vartheta_0 \geq 0$ (i.e., to the right half plane of the graph), we see that the strategies that deliver a non negative terminal payoff on each scenario are the ones such that $\vartheta_1 \geq -\frac{1}{10}\vartheta_0$ as the green contour coincides with the blue line (r_2) if $\vartheta_0 \geq 0$.

If $\vartheta_0 < 0$, the green contour coincides with the purple line (r_3) instead. Therefore, if $\vartheta_0 < 0$ the strategies that deliver a non negative terminal payoff on each scenario are the ones such that $\vartheta_1 > -\frac{1}{5}\vartheta_0$.

Hence $V_\vartheta(1)(\omega_k) \geq 0$ for $k = 1, 2, 3$ *happens if and only if* ϑ_1 *satisfies the following inequalities in terms of* $\vartheta_0 \in R$

$$\vartheta_1 \geq -0.1\ \vartheta_0 \qquad \text{for } \ \vartheta_0 \geq 0$$

$$\vartheta_1 \geq -0.2\ \vartheta_0 \qquad \text{for } \ \vartheta_0 < 0$$

8. The strategies of the previous two questions are such that $V_\vartheta(1) \geq 0$. In order to get arbitrage strategies out of them, we must also impose the extra constraint $V_\vartheta(0) \leq 0$ ($V_\vartheta(0) < 0$ for an arbitrage of second type). This delivers

$$\begin{aligned} V_\vartheta(0) &= \vartheta_0 + 15\vartheta_1 \leq 0 \\ \vartheta_1 &\leq -\frac{1}{15}\vartheta_0 \end{aligned}$$

which are the values below the the straight line $\vartheta_1 = -\frac{1}{15}\vartheta_0 = -0.067\vartheta_0$, represented as a dashed red line in the previous graph.

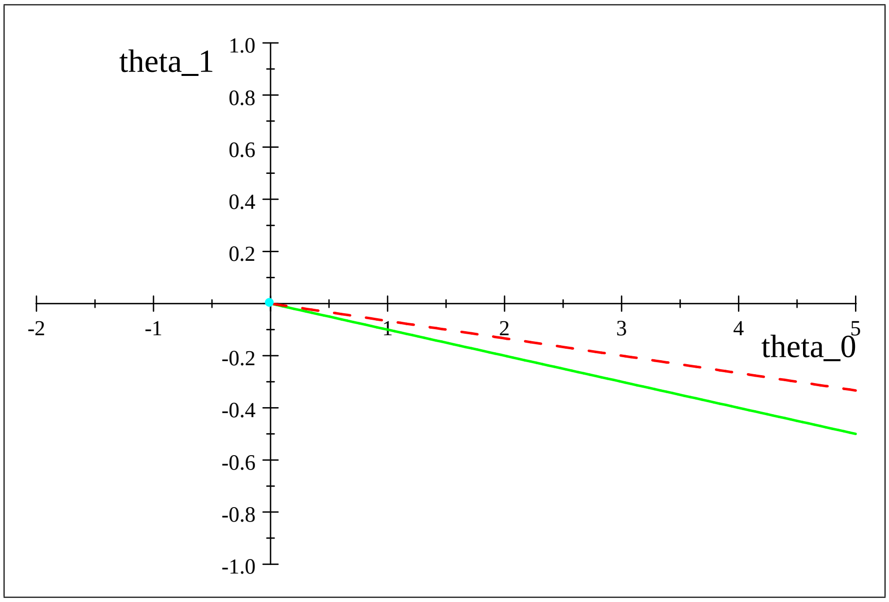
Thus the additional constraint is

$$\vartheta_1 \leq -0.067\vartheta_0 \quad \text{for } \vartheta_0 \in \mathbb{R}$$

9. In order to be an arbitrage opportunity, a strategy ϑ must

- have a non-negative terminal payoff, i.e. ϑ must be *above the green contour* in the previous graph;
- have a non-positive initial cost, i.e. a non-negative initial cashflow, i.e. ϑ must be *below the red dashed line* in the previous graph.

These strategies are only those ones in the slice of the $(\vartheta_0, \vartheta_1)$-plane, with $\vartheta_0 > 0$ and delimited by the dashed red line from above and the green contour from below. The strategy $\vartheta = (0, 0)$, marked with a dot, that does not constitute an arbitrage opportunity. Strategies on the red dashed half-line ($\vartheta_0 > 0$) are arbitrage opportunities of first type only, as their initial cost in zero, but they have a strictly positive payoff everywhere at $T = 1$. Strategies (strictly) in between the red dashed and green half-lines are arbitrages of first and second type, because they have a strictly negative initial cost and a strictly positive final payoff at $T = 1$ everywhere. Strategies on the green half-line are also arbitrages of first and second type, because they have a strictly negative initial cost, a strictly positive final payoff at $T = 1$ in ω_1 and ω_3, while their final payoff in ω_2 is zero.



Formally, these arbitrage strategies are described by the following constraints

$$\begin{cases} \vartheta_0 > 0 \\ \vartheta_1 \geq -\frac{1}{10}\vartheta_0 \\ \vartheta_1 < -\frac{1}{15}\vartheta_0 \end{cases}$$

Thus the arbitrage strategies are formally described by

$$\begin{cases} \vartheta_0 > 0 \\ -\frac{1}{10}\vartheta_0 \leq \vartheta_1 < -\frac{1}{15}\vartheta_0 \end{cases}$$

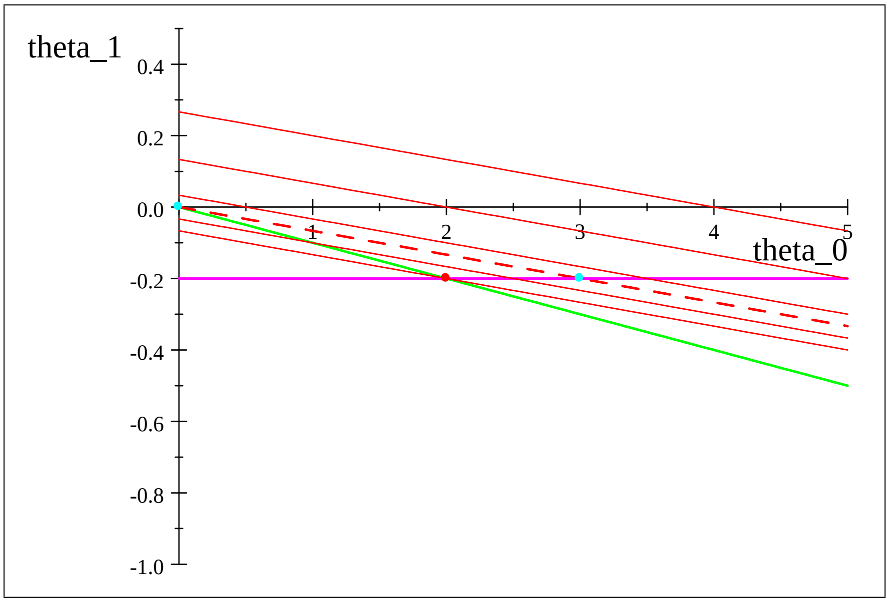
10. If we impose also the constraint $\vartheta_1 \geq -0.2$ on the strategies found in the previous point we get the system of inequalities

$$\begin{cases} \vartheta_0 > 0 \\ -\frac{1}{10}\vartheta_0 \leq \vartheta_1 < -\frac{1}{15}\vartheta_0 \\ \vartheta_1 \geq -0.2 \end{cases}$$

and they lie within the triangle delimited by the three dots in the graph below. The initial cashflow of a strategy ϑ is

$$f = -V_\vartheta(0) = -\vartheta_0 - 15\vartheta_1.$$

Strategies with initial cashflow f are described by the line $\vartheta_1 = -\frac{1}{15}\vartheta_0 - \frac{1}{15}f$, portraited for different values of f in red in the graph below. The higher f, the lower the red line in the graph. The maximum of f within the triangle is reached at the red dot in the plot below.



This red dot is the intersection between the green line $r_2 : \vartheta_1 = -0.1\vartheta_0$ and the magenta line $\vartheta_1 = -0.2$, leading to

$$(\vartheta_0^*, \vartheta_1^*) = \left(-\frac{1}{0.1}\vartheta_1^*, -0.2\right) = (2, -0.2)\,.$$

The (non required) initial cashflow of this strategy is

$$\begin{aligned} -V_{\vartheta^*}(0) &= -\vartheta_0^* - 15\vartheta_1^* \\ &= -2 - 15 \cdot (-0.2) = 1, \end{aligned}$$

where its value at $T = 1$ is still always non negative and equal to

$$\begin{cases} V_{\vartheta^*}(1)(\omega_1) = 2 + 8 \cdot (-0.2) = 0.4 \\ V_{\vartheta^*}(1)(\omega_2) = 2 + 10 \cdot (-0.2) = 0 \\ V_{\vartheta^*}(1)(\omega_3) = 2 + 5 \cdot (-0.2) = 1 \end{cases}$$

The answer to this question is:

$$\vartheta_0^* = 2 \qquad \text{and} \qquad \vartheta_1^* = -0.2$$

THEORY QUESTIONS

1. State the Second Fundamental Theorem of Asset Pricing. (the answer is provided in the Lecture Notes).

2. For the remaining questions, consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure $\mathcal{P} = \{\mathcal{P}_t\}_{t=0,1,2}$:

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

and assume the probability is given by

$$\mathbb{P}[\omega_1] = \frac{3}{20} \quad \mathbb{P}[\omega_2] = \frac{7}{20} \quad \mathbb{P}[\omega_3] = \frac{7}{20} \quad \mathbb{P}[\omega_4] = \frac{3}{20}$$

Consider the stochastic process X defined as follows:

$$\begin{aligned} X(0) &= 8.5 \\ X(1)(\omega) &= \begin{cases} 9 & \omega = \omega_1, \omega_2 \\ 7 & \omega = \omega_3, \omega_4 \end{cases} \\ X(2)(\omega) &= \begin{cases} 5 & \omega = \omega_1 \\ 10 & \omega = \omega_2 \\ 10 & \omega = \omega_3 \\ 5 & \omega = \omega_4 \end{cases} \end{aligned}$$

The random variable $X(2)$ is not P_1-measurable, because it takes different realizations on ω_1 and ω_2 that belong to the same $f_1^1 \in \mathcal{P}_1$.

Hence: $X(2)$ *is not* P_1*-measurable, because* $X(2)(\omega_1) = 5 \neq X(2)(\omega_2) = 10$ *and* $\omega_1, \omega_2 \in f_1^1 \in \mathcal{P}_1$

3. To compute $\mathbb{E}^{\mathbb{P}}[X(2)|\mathcal{P}_1]$ we first compute

$$\mathbb{P}(f_1^1) = \mathbb{P}[\omega_1] + \mathbb{P}[\omega_2] = \frac{3}{20} + \frac{7}{20} = \frac{1}{2} \text{ that implies } \mathbb{P}(f_2^1) = 1 - \mathbb{P}(f_1^1) = \frac{1}{2}$$

Thus

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[X(2)|\mathcal{P}_1](f_1^1) &= 5\frac{\frac{3}{20}}{\frac{1}{2}} + 10\frac{\frac{7}{20}}{\frac{1}{2}} = 8.5 \\ \mathbb{E}^{\mathbb{P}}[X(2)|\mathcal{P}_1](f_2^1) &= 10\frac{\frac{7}{20}}{\frac{1}{2}} + 5\frac{\frac{3}{20}}{\frac{1}{2}} = 8.5 \end{aligned}$$

The answer is

$$\mathbb{E}^{\mathbb{P}}[X(2)|\mathcal{P}_1](f_1^1) = 8.5 \quad \text{and} \quad \mathbb{E}^{\mathbb{P}}[X(2)|\mathcal{P}_1](f_2^1) = 8.5$$

4. The random variable $X(2)$ satisfies the necessary conditions to be independent, but we need to check also the sufficient ones, i.e. the definition. To this aim we compute

$$\mathbb{P}[X(2) = 10] = \mathbb{P}[\omega_2] + \mathbb{P}[\omega_3] = \frac{7}{20} + \frac{7}{20} = \frac{14}{20}$$

and

$$\mathbb{P}[X(2) = 5] = \mathbb{P}[\omega_1] + \mathbb{P}[\omega_4] = \frac{3}{20} + \frac{3}{20} = \frac{6}{20}.$$

Hence $X(2)$ *is independent of* $\mathcal{P}_1$ *with respect to* $\mathbb{P}$ *because*

$$
\begin{aligned}
\mathbb{P}\left[(X(2)=5)\cap f_1^1\right] &= \mathbb{P}[\omega_1]=\frac{3}{20}=\frac{6}{20}\cdot\frac{1}{2}=\mathbb{P}[X(2)=5]\cdot\mathbb{P}\left[f_1^1\right]\\
\mathbb{P}\left[(X(2)=10)\cap f_1^1\right] &= \mathbb{P}[\omega_2]=\frac{7}{20}=\frac{14}{20}\cdot\frac{1}{2}=\mathbb{P}[X(2)=10]\cdot\mathbb{P}\left[f_1^1\right]\\
\mathbb{P}\left[(X(2)=5)\cap f_2^1\right] &= \mathbb{P}[\omega_4]=\frac{3}{20}=\frac{6}{20}\cdot\frac{1}{2}=\mathbb{P}[X(2)=5]\cdot\mathbb{P}\left[f_2^1\right]\\
\mathbb{P}\left[(X(2)=10)\cap f_2^1\right] &= \mathbb{P}[\omega_3]=\frac{7}{20}=\frac{14}{20}\cdot\frac{1}{2}=\mathbb{P}[X(2)=10]\cdot\mathbb{P}\left[f_2^1\right]
\end{aligned}
$$

5. Given the previous computations $\{X(t)\}_{t=0,1,2}$ is not a martingale with respect to $\mathbb{P}$, because

$$X(1)\left(f_1^1\right)=9\neq 8.5=\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\left(f_1^1\right)$$

implies that

$$X(1)\neq\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right],$$

thus $\{X(t)\}_{t=0,1,2}$ does not qualify to be a martingale with respect to $\mathbb{P}$.

The equation

$$\mathbb{E}^{\mathbb{P}}\left[X(1)|\,\mathcal{P}_1\right]=X(1)$$

is valid because $X(1)$ is $\mathcal{P}_1-$measurable and hence $\mathbb{E}^{\mathbb{P}}\left[X(1)|\,\mathcal{P}_1\right]=X(1)$ by the properties of the conditional expectation.

The tower property implies that

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\right]=\mathbb{E}^{\mathbb{P}}\left[X(2)\right].$$

Thus the appropriate alternatives are

$\boxed{\text{B}}$	$\{X(t)\}_{t=0,1,2}$ is not a martingale with respect to $\mathbb{P}$
$\boxed{\text{C}}$	$\mathbb{E}^{\mathbb{P}}\left[X(1)|\,\mathcal{P}_1\right]=X(1)$
$\boxed{\text{D}}$	$\mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[X(2)|\,\mathcal{P}_1\right]\right]=\mathbb{E}^{\mathbb{P}}\left[X(2)\right]$