1 Exercises: Multi-period Markets

Solutions from page 16.

Exercise 1 *Consider a multiperiod discrete market with* $t = 0, 1, 2$ *with the following information structure:*

$$
\begin{array}{ccccc}
 & & & \nearrow & \omega_1 \\
 & & f_1^1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f_1^0 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f_2^1 & & \\
 & & & \searrow & \omega_4
\end{array}
$$

Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$r(0) = 1\%, \quad r(1)(f_1^1) = 1\% \text{ and } r(1)(f_2^1) = 0\%.$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 100,$$

with time 1 prices

$$S(1)(f_1^1) = 105 \text{ and } S(1)(f_2^1) = 95$$

and with time 2 prices

$$S(2)(\omega_1) = 110.25, \quad S(2)(\omega_2) = 99.75, \quad S(2)(\omega_3) = 104.5, \quad S(2)(\omega_4) = 85.5.$$

1. Is the market dynamically complete?
2. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. Consider the *European derivative* X with maturity $T = 2$, whose terminal payoff at $T = 2$ is

$$X(2) = \begin{cases} S(2) & \text{if } S(2) < 100 \\ 100 & \text{otherwise} \end{cases}$$

   Compute the no-arbitrage prices at $t = 0$ and at $t = 1$ of the derivative X.

4. Consider the derivative Y whose *cashflow* is

$$Y(t) = \begin{cases} S(t) & \text{if } S(t) < 100 \\ 100 & \text{otherwise} \end{cases}$$

   for $t = 1$ *and* $T = 2$. Compute the no-arbitrage prices at $t = 0$ and at $t = 1$ of the derivative Y.

5. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *zero-coupon bond* with maturity $T = 2$, whose terminal payoff is 1 in any state of the world.
6. Determine a *buy-and-hold strategy* ϑ that replicates a *European put option on* S with maturity $T = 2$ and strike 100 using only **two** financial securities in the market extended to all the derivatives of previous points, that is in the market constituted by B, S, the derivatives X of Point 3.,Y of Point 4., and the zero coupon bond of Point 5.

Exercise 2 *Consider a multiperiod discrete market with* $t = 0, 1, 2$ *with the following information structure:*

$$
\begin{array}{ccccc}
 & & f_1^1 & \begin{array}{c}\nearrow\\ \searrow\end{array} & \begin{array}{c}\omega_1\\ \omega_2\end{array}\\
f_1^0 & \begin{array}{c}\nearrow\\ \searrow\end{array} & & & \\
 & & f_2^1 & \begin{array}{c}\nearrow\\ \searrow\end{array} & \begin{array}{c}\omega_3\\ \omega_4\end{array}
\end{array}
$$

The *historical probability* is *uniform* on Ω, i.e. $\mathbb{P}(\omega_k) = 0.25$ for $k = 1, ..., 4$.

Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$r(0) = 5\%, \quad r(1)(f_1^1) = 6\% \text{ and } r(1)(f_2^1) = 0\%.$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 11 \text{ and } S(1)(f_2^1) = 9$$

and with time 2 prices

$$S(2)(\omega_1) = 13.2, \quad S(2)(\omega_2) = 8.8, \quad S(2)(\omega_3) = 10.35, \quad S(2)(\omega_4) = 8.55.$$

1. Is the market dynamically complete?
2. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. Compute the no-arbitrage price at $t = 0$ and at $t = 1$ of a *European digital option* on S with maturity $T = 2$ and strike $K = 10$. Recall that the final payoff of the digital option is

$$X(2) = \begin{cases} 1 & \text{if } S(2) > K \\ 0 & \text{otherwise} \end{cases}$$

4. Is the final payoff $X(2)$ of the digital option of the previous point independent of $\mathcal{P}_1$ with respect to *the risk neutral probability* $\mathbb{Q}$? Is $X(2)$ independent of $\mathcal{P}_1$ with respect to the *historical probability* $\mathbb{P}$?

Exercise 3 *Consider a multiperiod discrete market with $t = 0, 1, 2$ with the following information structure:*

$$
\begin{array}{ccccc}
 & & f_1^1 & \begin{array}{c}\nearrow\\ \searrow\end{array} & \begin{array}{c}\omega_1\\ \omega_2\end{array}\\
f_1^0 & \begin{array}{c}\nearrow\\ \searrow\end{array} & & & \\
 & & f_2^1 & \begin{array}{c}\nearrow\\ \searrow\end{array} & \begin{array}{c}\omega_3\\ \omega_4\end{array}
\end{array}
$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 3\% \text{ and } r(1)(f_2^1) = 0\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 100,$$

with time 1 prices

$$S(1)(f_1^1) = 104 \text{ and } S(1)(f_2^1) = 99$$

and with time 2 prices

$$S(2)(\omega_1) = 108.16, \quad S(2)(\omega_2) = S(2)(\omega_3) = 102.96, \quad S(2)(\omega_4) = 98.01.$$

1. Is the market dynamically complete?
2. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. A zero coupon bond with maturity $T = 2$ has terminal payoff at $T = 2$
$$X(2) = 100.$$
At $t = 1$ there are no intermediate cashflows. Compute S_X , the no-arbitrage price process of zero coupon bond, specifying it at $t = 0$ and at $t = 1$.
4. Consider a *swap* with maturity $T = 2$, starting on date $t = 0$, with notional principal $N = 100$. Let Part A be the *fixed leg* of the swap contract and Part B be the *floating leg*. Denote with F the swap rate. In the swap contract Part A *pays* $N \cdot F$ at $t = 1, 2$ to Part B, and *receives* $N \cdot r(0)$ at $t = 1$, and $N \cdot r(1)$ at $T = 2$ from Part B. The swap rate F is settled at time $t = 0$ such that the *initial* no-arbitrage value of the swap contract is 0. Compute the swap rate F.
5. Consider a *forward-starting* call option on S *settled at time* $t = 0$ and *starting* at $t = 1$ with maturity $T = 2$ and strike price $K = S(1)$. Compute the terminal payoff at $T = 2$ of the forward starting call option. Find the no-arbitrage price of the forward starting call option at $t = 0$ and at $t = 1$.

Exercise 4 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc}
 & & & \nearrow & \omega_1 \\
 & & f_1^1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f_1^0 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f_2^1 & & \\
 & & & \searrow & \omega_4
\end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate
$$r(0) = 2\%, \quad r(1)(f_1^1) = 2\% \quad \textit{and } r(1)(f_2^1) = 0\%.$$
The second security is a risky asset S, with time 0 price
$$S(0) = 10,$$
with time 1 prices
$$S(1)(f_1^1) = 11 \textit{ and } S(1)(f_2^1) = 9$$
and with time 2 prices
$$S(2)(\omega_1) = 12.1, \quad S(2)(\omega_2) = 9.9, \quad S(2)(\omega_3) = 9.9, \quad S(2)(\omega_4) = 8.1.$$

1. Is the market dynamically complete?
2. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
3. Consider the *digital European derivative on S* with maturity $T = 2$, whose terminal payoff is
$$X(2) = \begin{cases} 1 & \text{if } S(2) < 9.1 \\ 0 & \text{otherwise.} \end{cases}$$
Compute the no-arbitrage price process of the digital option, specifying it at $t = 0$ and at $t = 1$.
4. Consider the European digital derivative of the previous point. Can the derivative be dynamically replicated in the original market? If your answer is positive, find its replication cost at $t = 0$ and at $t = 1$.

5. Find the probability measure $\mathbb{Q}^S$ on Ω such that the process $M = \{M(t)\}_{t=0,1,2}$ defined as

$$M(t) = \frac{B(t)}{S(t)}$$

is a *martingale* with respect to $\mathbb{Q}^S$, that is

$$M(0) = \mathbb{E}^{\mathbb{Q}^S}[M(1)] \text{ and } M(1) = \mathbb{E}^{\mathbb{Q}^S}[M(2)|\mathcal{P}_1].$$

The probability measure $\mathbb{Q}^S$ is the measure associated to the *numeraire* S.

6. Verify that the *replication cost* of the derivative of Point 4 *discounted by* S is a martingale with respect to $\mathbb{Q}^S$.

Exercise 5 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 3\% \quad \textit{and } r(1)(f_2^1) = 1\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 10.3 \textit{ and } S(1)(f_2^1) = 9.8$$

and with time 2 prices

$$S(2)(\omega_1) = 10.712, \quad S(2)(\omega_2) = 9.682, \quad S(2)(\omega_3) = 10.094, \quad S(2)(\omega_4) = 9.604.$$

1. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?

2. A *futures contract* with maturity $T = 2$ on the risky security S is characterized by a sequence of *futures prices* $f(0), f(1)$ where $f(0)$ is settled at time $t = 0$ while $f(1)$ is settled at $t = 1$ based on $\mathcal{P}_1$. A long position in a *futures contract* at time $t = 0$ generates the cashflow

$$X_{fut}(1) = f(1) - f(0)$$

at time $t = 1$, and

$$X_{fut}(2) = S(2) - f(1)$$

at time $t = 2$. The futures prices $f(0), f(1)$ must satisfy the condition that, under no-arbitrage, the value of a long position in the futures contract is 0 *both* at time $t = 0$ *and* a time $t = 1$. Determine $f(0)$ and $f(1)$.

3. Let $f(2) = S(2)$. Is the *futures prices process* $f = \{f(t)\}_{t=0,1,2}$ a $\mathbb{Q}-$martingale?

4. A barrier derivative X on S with maturity $T = 2$ pays a coupon $C = 100$ at $t \in \{1, 2\}$ if $S(t) > S(t-1)$ and then terminates. Its cashflow $X = \{X(t)\}_{t=1,2}$ is hence

$$X(1) = \begin{cases} C & \text{if } S(1) > S(0) \\ 0 & \text{otherwise} \end{cases}$$

at $t = 1$ and

$$X(2) = \begin{cases} 0 & \text{if } S(1) > S(0) \\ C & \text{if } S(1) \leq S(0) \text{ and } S(2) > S(1) \\ 0 & \text{if } S(1) \leq S(0) \text{ and } S(2) \leq S(1) \end{cases}$$

at $T = 2$. Compute the cashflow $X = \{X(t)\}_{t=1,2}$. Determine its no-arbitrage price process $S_X = \{S_X(t)\}_{t=0,1,2}$.

Exercise 6 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 4\% \quad \textit{and} \quad r(1)(f_2^1) = 2\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 100,$$

with time 1 prices

$$S(1)(f_1^1) = 112 \textit{ and } S(1)(f_2^1) = 96$$

and with time 2 prices

$$S(2)(\omega_1) = 125.44, \quad S(2)(\omega_2) = 107.52, \quad S(2)(\omega_3) = 107.52, \quad S(2)(\omega_4) = 92.16.$$

1. Compute the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.

2. Is the market dynamically complete?

3. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?

4. A European *call option* with maturity $T = 2$ on the risky security S and strike price $K = 100$ is introduced in the market. Compute $c(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European call option at $t = 1$ and at $t = 0$.

5. A path-dependent derivative X on the risky security S with maturity $T = 2$ is introduced in the market. Denoting with $R(1)$ the realized return of S over the first period, i.e.

$$R(1) = \frac{S(1) - S(0)}{S(0)},$$

the derivative provides the following cashflow at $t = 1$

$$X(1) = \begin{cases} 1 - 10 \cdot (-0.01 - R(1)) & \text{if } R(1) < -0.01 \\ 0 & \text{otherwise} \end{cases}$$

and at $T = 2$

$$X(2) = \begin{cases} 0 & \text{if } X(1) \neq 0 \\ 1 + r(0) + C & \text{otherwise} \end{cases}$$

where C is determined in such a way that the *initial no-arbitrage price of the derivative is equal to* 1. Determine C.

Exercise 7 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

In this market two securities are traded. The first is a locally riskless security B that provides the following one-period interest rates:

$$\begin{aligned} r(0) &= 10\% \\ r(1)(f_1^1) &= 10\% \\ r(1)(f_2^1) &= 20\% \end{aligned}$$

The second security is a risky security S whose price is $S(0) = 1$ at time $t = 0$, while at time $t = 1$ we have

$$\begin{array}{c} S(1)(f_1^1) = S(0) \cdot u \\ S(1)(f_2^1) = S(0) \cdot d \end{array}$$

and finally at time $T = 2$ we have

$$\begin{array}{c} S(2)(\omega_1) = S(1)(f_1^1) \cdot u \\ S(2)(\omega_2) = S(1)(f_1^1) \cdot d \\ S(2)(\omega_3) = S(1)(f_2^1) \cdot u \\ S(2)(\omega_4) = S(1)(f_2^1) \cdot d \end{array}$$

where $u = 1.4$ and $d = 0.8$.

1. Write explicitly the prices for B, S at $t = 1, 2$.
2. Is the multi-period market dynamically complete?
3. Is the multi-period market arbitrage-free? If the answer is positive find the risk neutral probabilities $\mathbb{Q}$ for the multi-period market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$.
4. Assume that a *European call* on S with maturity $T = 2$ and strike price $K = 1.46$ is introduced in the market. Write the final payoff of the call option. Determine its no-arbitrage price at $t = 1$ and at $t = 0$.
5. The payoff at maturity $T = 2$ of a *European lookback call* on S is

   $$X(2) = \max_{t=0,1,2} S(t) - S(2).$$

   This derivative delivers no intermediate cashflow at $t = 1$. Determine the no-arbitrage price of this lookback call at $t = 1$ and at $t = 0$.

Exercise 8 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{tikzcd}[row sep=small] & & \omega_1 \\ & f_1^1 \arrow[ur] \arrow[dr] & \\ & & \omega_2 \\ f_1^0 \arrow[uur] \arrow[ddr] & & \\ & & \omega_3 \\ & f_2^1 \arrow[ur] \arrow[dr] & \\ & & \omega_4 \end{tikzcd}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 4\% \quad \text{and} \quad r(1)(f_2^1) = 2\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 12 \quad \text{and } S(1)(f_2^1) = 9$$

and with time 2 prices

$$S(2)(\omega_1) = 15.6, \quad S(2)(\omega_2) = S(2)(\omega_3) = 10.8, \quad S(2)(\omega_4) = 7.2.$$

1. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?

2. A *European call option* with maturity $T = 2$ on the risky security S and strike price $K = 11$ is introduced in the market. Compute $c(2)$, the final payoff of the call option at $T = 2$. Determine the no-arbitrage price of the European call option at $t = 1$ and at $t = 0$.

3. A *zero coupon bond* with maturity $T = 2$ has terminal payoff at $T = 2$ equal to

$$ZCB(2)(\omega_i) = 1 \quad i = 1, \ldots, 4.$$

   At $t = 1$ there are no intermediate cash flows. Compute S_{ZCB} the no-arbitrage price process of the security, specifying it at $t = 0$ and at $t = 1$.

4. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K = 11$ is introduced in the market. Compute $p(2)$, the final payoff of the put option at $T = 2$.

   Find the buy and hold strategy that replicates the final put payoff with the underlying risky security S, the call option of Point 2, and the zero coupon bond of Point 3. Compute the no-arbitrage price of the put option at $t = 0$.

5. A *straddle* is an investment strategy consisting in a long position on both a call and a put option with the same maturity and the same strike price. Compute $straddle(K)(2)$, the final payoff of a straddle on the risky security S with maturity $T = 2$ and $K = 11$. Compute also its the no-arbitrage price at $t = 0$.

6. Suppose the historical probability $\mathbb{P}$ is uniform on Ω, namely suppose $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \mathbb{P}(\omega_4) = 0.25$. Given the following *target payoff*

$$X(2) = \begin{cases} 4.1 & \text{on } \omega_1 \text{ and } \omega_4 \\ 0.5 & \text{on } \omega_2 \text{ and } \omega_3 \end{cases}$$

   find $k \in (11, 15)$ such that the straddle on the risky security S with maturity $T = 2$ and strike price k minimizes the *historical mean squared error* in the replication of the *target payoff*. Namely, find

$$k \in (11, 15) \text{ such that } \mathbb{E}^{\mathbb{P}}\left[(X(2) - straddle(k)(2))^2\right] \text{ is minimum.}$$

Exercise 9 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 5\%, \quad r(1)(f_1^1) = 8\% \quad \textit{and } r(1)(f_2^1) = 5\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 15 \quad \textit{and} \quad S(1)(f_2^1) = 9$$

and with time 2 prices

$$S(2)(\omega_1) = 22.5, \quad S(2)(\omega_2) = 13.5, \quad S(2)(\omega_3) = 13.5, \quad S(2)(\omega_4) = 8.1.$$

1. Compute the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.

2. Is the market dynamically complete?

3. Determine the set of *risk neutral probabilities* $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?

4. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K = 18.5$ is introduced in the market. Compute $p\,(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European put option at $t = 1$ and at $t = 0$.

5. An *exotic derivative* on the risky security S has the following cashflow at $t = 1$

$$X\,(1) = (S\,(1) - S\,(0))^{+}$$

and at $T = 2$

$$X\,(2) = \left(K - \min_{t=0,1,2} S\,(t)\right)^{+} \quad \text{where } K = 18.5$$

Compute the cashflow process $X = \{X(t)\}_{t=1,2}$. Determine the no-arbitrge price of the derivative at $t = 1$ and at $t = 0$.

Exercise 10 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{tikzcd}
 & f_1^1 \arrow[r] \arrow[dr] & \omega_1 \\
 & & \omega_2 \\
f_1^0 \arrow[uur] \arrow[ddr] & & \\
 & f_2^1 \arrow[r] \arrow[dr] & \omega_3 \\
 & & \omega_4
\end{tikzcd}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 6\%, \quad r(1)(f_1^1) = 8\% \quad \textit{and } r(1)(f_2^1) = 5\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 1,$$

with time 1 prices

$$S(1)(f_1^1) = 1.25 \quad \textit{and} \quad S(1)(f_2^1) = 0.85$$

and with time 2 prices

$$S(2)(\omega_1) = 1.5625, \qquad S(2)(\omega_2) = 1.0625, \qquad S(2)(\omega_3) = 1.0625, \quad S(2)(\omega_4) = 0.7225.$$

1. Compute the price process of the locally riskless security $B = \{B\,(t)\}_{t=0,1,2}$.

2. Determine the set of *risk neutral probabilities* $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?

3. A *European call option* with maturity $T = 2$ on the risky security S and strike price $K = 1.0125$ is introduced in the market. Compute $c\,(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European call option at $t = 1$ and at $t = 0$.

4. An *exotic derivative* on the risky security S has the following cashflow at $t = 1, 2$

$$X\,(t) = \begin{cases} 1 & \text{if } \frac{S(t)-S(t-1)}{S(t-1)} > \frac{B(t)-B(t-1)}{B(t-1)} \\ 0 & \text{otherwise} \end{cases}$$

Compute the cashflow process $X = \{X(t)\}_{t=1,2}$. Determine the no-arbitrge price of the derivative at $t = 1$ and at $t = 0$.

Exercise 11 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$
\begin{array}{ccccc}
 & & & \nearrow & \omega_1 \\
 & & f^1_1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f^0_1 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f^1_2 & & \\
 & & & \searrow & \omega_4
\end{array}
$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 4\%, \quad r(1)(f^1_1) = 6\% \quad \textit{and} \quad r(1)(f^1_2) = 2\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f^1_1) = 12.2 \quad \textit{and} \quad S(1)(f^1_2) = 8.2$$

and with time 2 prices

$$S(2)(\omega_1) = 14.884, \qquad S(2)(\omega_2) = 10.004, \qquad S(2)(\omega_3) = 10.004, \quad S(2)(\omega_4) = 6.724.$$

1. A *forward-starting call option* with maturity $T = 2$ on the risky security S is introduced in the market at date $t = 0$. The strike price K_{FS} of the forward-starting call option is settled at the at-the money level at $t = 1$, i.e.

   $$K_{FS} = S(1).$$

   Compute $c_{FS}(2)$, the final payoff of the forward-starting call option at $T = 2$. Determine the no-arbitrge price of the option $c_{FS}(1)$ at $t = 1$ and $c_{FS}(0)$ at $t = 0$.

2. Suppose the historical probability is uniform on Ω, i.e. $\mathbb{P}(\omega_k) = 0.25$ for $k = 1, ..., 4$. Among the buy-and-hold strategies $\vartheta = (\vartheta_0, \vartheta_1)$ with $\vartheta_0(1) = \vartheta_0(0)$ and $\vartheta_1(1) = \vartheta_1(0)$ whose initial cost is

   $$V_\vartheta(0) = c_{FS}(0),$$

   find the one that replicates at maturity $c_{FS}(2)$ on average with respect to the historical probability, i.e. such that

   $$\mathbb{E}^{\mathbb{P}}[V_\vartheta(2) - c_{FS}(2)] = 0.$$

   Compute the historical and the risk neutral probability that $V_\vartheta(2)$, the terminal value of this buy and hold strategy, dominates $c_{FS}(2)$, the final payoff of the forward starting option.

Exercise 12 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$
\begin{array}{ccccc}
 & & & \nearrow & \omega_1 \\
 & & f^1_1 & & \\
 & \nearrow & & \searrow & \omega_2 \\
f^0_1 & & & & \\
 & \searrow & & \nearrow & \omega_3 \\
 & & f^1_2 & & \\
 & & & \searrow & \omega_4
\end{array}
$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 3\%, \quad r(1)(f^1_1) = 6\% \quad \textit{and} \; r(1)(f^1_2) = 4\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 *prices*

$$S(1)(f_1^1) = 13 \quad and\ S(1)(f_2^1) = 7$$

and with time 2 prices

$$S(2)(\omega_1) = 18.20, \qquad S(2)(\omega_2) = 7.15, \qquad S(2)(\omega_3) = 11.2, \quad S(2)(\omega_4) = 6.3.$$

1. Is the market dinamically complete?

2. Determine the set of *risk neutral probabilities* $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?$K_{call} = 7.15$

3. A *European call option* with maturity $T = 2$ on the risky security S and strike price $K_{call} = 7.15$ is introduced in the market. Compute $c\,(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European call option at $t = 1$ and at $t = 0$.

4. A *European put option* with maturity $T = 2$ on the risky security S and strike price K_{put} is introduced in the market. Its terminal payoff on ω_2 is

$$p\,(2)\,(\omega_2) = 4.05.$$

   Find K_{put}, the terminal payoff of the put option $p\,(2)\,(\omega)$ for $\omega = \omega_1$, ω_3, and ω_4 and the initial no-arbitrage price of the put option.

5. The *realized variance* at $T = 2$ of the risky asset S is defined as

$$RV\,(2) = \left(\ln \frac{S\,(1)}{S\,(0)}\right)^2 + \left(\ln \frac{S\,(2)}{S\,(1)}\right)^2.$$

   Compute $RV\,(2)$ for any ω_k with $k = 1, ..., 4$.

   The terminal payoff at $T = 2$ of a *European exotic derivative* is

$$X\,(2) = \begin{cases} \frac{1}{3}\sum_{t=0,1,2} S\,(t) & \text{if } RV\,(2) \geq 0.25 \\ 0 & \text{otherwise} \end{cases}$$

   Determine the no-arbitrge price of this exotic at $t = 0$.

6. Assume that the historical probability is $\mathbb{P}(\omega_1) = 0.2$, $\mathbb{P}(\omega_2) = 0.1$, $\mathbb{P}(\omega_3) = 0.2$, and $\mathbb{P}(\omega_4) = 0.5$. Find the *buy and hold strategy* that exploits S,the call option of Point 3 and the put option of Point 4 and that replicates the exotic derivative of Point 5 with 90% historical probability.

   Compute the terminal value of this buy and hold strategy for any ω_k with $k = 1, ..., 4$.

   Compute the initial price of this buy and hold strategy.

Exercise 13 *Consider a multiperiod discrete market with* $t = 0, 1, 2$ *and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 8\% \quad and\ r(1)(f_2^1) = 2\%.$$

The second security is a risky asset S, with time 0 *price*

$$S(0) = 20,$$

with time 1 *prices*

$$S(1)(f_1^1) = 21 \quad \textit{and } S(1)(f_2^1) = 18$$

and with time 2 *prices*

$$S(2)(\omega_1) = 31.5, \quad S(2)(\omega_2) = 18.9, \quad S(2)(\omega_3) = 21.6, \quad S(2)(\omega_4) = 14.4.$$

1. Compute the price process of the locally riskless security $B = \{B\,(t)\}_{t=0,1,2}$.

2. Is the market dynamically complete?

3. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?

4. A *cliquet call option* with maturity $T = 2$ on the risky security S is a portfolio of one-period at the money European call options on S. In particular, the first call option begins at $t = 0$, expires at $t = 1$ and has strike price equal to $S(0)$. The second call option begins at $t = 1$, expires at $T = 2$ and has strike price equal to $S(1)$. Determine the cashflow of the cliquet option at $t = 1, 2$. Find the no-arbitrage price of the cliquet option at $t = 1$ and at $t = 0$.

5. A *forward contract* on S with maturity $T = 2$ and delivery price $F_{0,2}$ is introduced in the market at $t = 0$. The final payoff of the forward contract is $S\,(T) - F_{0,2}$. The delivery price $F_{0,2}$, set at time $t = 0$, is such that the initial no-arbitrage price of the forward contract is zero. Determine $F_{0,2}$.

   At $t = 1$ a new forward contract on S with maturity $T = 2$ is introduced in the market. The payoff of this new forward contract is $S\,(T) - F_{1,2}$. The delivery price $F_{1,2}$ is set at $t = 1$ in f_1^1 and f_2^1 in such a way that the no-arbitrate price at $t = 1$ of this new forward contract is zero. Determine $F_{1,2}$.

6. Compute the *conditional risk neutral probability at* $t = 1$ that $S(2) \geq F_{0,2}$, namely compute

   $$\mathbb{Q}\,[\,S(2) \geq F_{0,2}|\,\mathcal{P}_1]\,.$$

   Compute also the unconditional risk neutral probability that $S(2) \geq F_{0,2}$, namely compute

   $$\mathbb{Q}\,[S(2) \geq F_{0,2}]\,.$$

Exercise 14 *Consider a multiperiod discrete market with* $t = 0, 1, 2$ *and with the following information structure:*

$$\begin{array}{ccccc} & & & & \omega_1 \\ & & & \nearrow & \\ & & f_1^1 & \rightarrow & \omega_2 \\ & \nearrow & & \searrow & \\ & & & & \omega_3 \\ f_1^0 & & & & \\ & & & & \omega_4 \\ & \searrow & & \nearrow & \\ & & f_2^1 & & \\ & & & \searrow & \\ & & & & \omega_5 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 1\% \quad \textit{and } r(1)(f_2^1) = 3\%.$$

The second security is a risky asset S, *with time* $t = 0$ *price* $S(0) = 10$, *with time* $t = 1$ *prices*

$$S(1)(f_1^1) = 12 \quad \textit{and } S(1)(f_2^1) = 8.4$$

and with time $T = 2$ *prices*

$$S(2)(\omega_1) = 19.392, \quad S(2)(\omega_2) = 14.544, \quad S(2)(\omega_3) = 9.696,$$

$$S(2)(\omega_4) = 14.544, \quad S(2)(\omega_5) = 7.179.$$

1. Compute the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.

2. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 5$. Is the market free of arbitrage opportunities?

3. Is the market dynamically complete?

4. A *European call option* with maturity $T = 2$ on the risky security S and strike price $K_{call} = 9.696$ is introduced in the market. Compute $c(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrage price of the European call option at $t = 1$ and at $t = 0$.

5. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K_{put} = 14.544$ is introduced in the market. Compute $p(2)$, the final payoff of the option at $T = 2$. Determine the set of no-arbitrage prices of the European call option at $t = 0$.

6. Assume that the put option of Point 5 at $t = 0$ trades at 4.2159. Is the market extended with this put option aribtrage-free? Compute the set of risk-neutral probabilities $\mathbb{Q}$ for the market extended with the put option of Point 5, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 5$.

7. Assume that the historical probability $\mathbb{P}$ is uniform on Ω, namely assume that $\mathbb{P}(\omega_k) = 0.2$ for all $k = 1, \ldots, 5$. Consider a buy and hold strategy $\bar{\vartheta} = (\bar{\vartheta}_C, \bar{\vartheta}_P)$ in the call option of point 4 and in the put option of point 5. The initial price of this strategy is $V_{\bar{\vartheta}}(0) = 2.8058$ and its historical expected net profit under $\mathbb{P}$ is $\mathbb{E}^{\mathbb{P}}[V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0)] = 7.6081$. Find $\bar{\vartheta}_C$ and $\bar{\vartheta}_P$.

8. Compute the historical expected loss of the strategy $\bar{\vartheta}$ of point 7 conditioning on actually having a loss, namely compute
$$\mathbb{E}^{\mathbb{P}}[V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) | V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) < 0].$$

Exercise 15 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{tikzcd}[row sep=small] & & \omega_1 \\ & f_1^1 \arrow[ur] \arrow[dr] & \\ & & \omega_2 \\ f_1^0 \arrow[uur] \arrow[ddr] & & \\ & & \omega_3 \\ & f_2^1 \arrow[ur] \arrow[dr] & \\ & & \omega_4 \end{tikzcd}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate
$$r(0) = 2\%, \quad r(1)(f_1^1) = 5\% \quad \textit{and } r(1)(f_2^1) = 4\%.$$
The second security is a risky asset S, with time $t = 0$ price $S(0) = 10$, with time $t = 1$ prices
$$S(1)(f_1^1) = 12 \quad \textit{and } S(1)(f_2^1) = 9$$
and with time $T = 2$ prices
$$S(2)(\omega_1) = 16.8, \quad S(2)(\omega_2) = 10.8, \quad S(2)(\omega_3) = 10.8, \quad S(2)(\omega_4) = 7.2.$$

1. A *European call option* with maturity $T = 2$ on the risky security S and strike price $K_{call} = 12.3375$ is introduced in the market. Compute $c(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge prices of the European call option at $t = 1$ and at $t = 0$.

2. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K_{put} = 9.41$ is introduced in the market. Compute $p(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge prices of the European put option at $t = 1$ and at $t = 0$.

3. Consider the portfolio constituted by one unit of the call option of Point 1 and one unit of the put option of Point 2. Find its final payoff $X(2)$. Determine the no-arbitrage prices of this portfolio at $t = 1$ and at $t = 0$.

4. Consider a buy-and-hold strategy with $\alpha \in \Re$ units of the risky asset S. Find the number of units of the locally riskless asset B to be held in the buy-and-hold strategy so that the initial cost of the strategy is equal to the intial no-arbitrage price of the portfolio of Point 3.

5. Assume that the historical probability $\mathbb{P}$ is as follows: $\mathbb{P}(\omega_1) = 0.125$, $\mathbb{P}(\omega_2) = 0.5$, $\mathbb{P}(\omega_3) = 0.25$, and $\mathbb{P}(\omega_4) = 0.125$. Among the strategies found in Point 4, find the set of all strategies ϑ^α that maximize the historical probability of superreplicating $X(2)$, the final payoff of the portfolio of point 6 at maturity $T = 2$. Namely, find all strategies ϑ^α that maximize

$$\mathbb{P}\left[V_{\vartheta^\alpha}(2) \geq X(2)\right].$$

6. For the strategies ϑ^α found in Point 5, compute their shortfall in the super-replication of $X(2)$, namely compute

$$\max\left(X(2) - V_{\vartheta^\alpha}(2); 0\right)$$

for all ω_k for $k = 1, ..., 4$.

Compute then the *historical expected* shortfall, namely

$$\mathbb{E}^{\mathbb{P}}\left[\max\left(X(2) - V_{\vartheta^\alpha}(2); 0\right) \middle| X(2) - V_{\vartheta^\alpha}(2) > 0\right].$$

Finally, among these strategies ϑ^α, compute the one that *minimizes* the *historical expected* shortfall.

Exercise 16 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{array}{ccccc} & & & \nearrow & \omega_1 \\ & & f_1^1 & & \\ & \nearrow & & \searrow & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & \nearrow & \omega_3 \\ & & f_2^1 & & \\ & & & \searrow & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 5\%, \quad r(1)(f_1^1) = 10\% \quad \textit{and} \quad r(1)(f_2^1) = 5\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 11.5 \textit{ and } S(1)(f_2^1) = 9.5$$

and with time 2 prices

$$S(2)(\omega_1) = 13.225, \quad S(2)(\omega_2) = 10.925, \quad S(2)(\omega_3) = 10.925, \quad S(2)(\omega_4) = 9.025.$$

1. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?

2. A European *put option* with maturity $T = 2$ on the risky security S and strike price $K = 11.555$ is introduced in the market. Compute $p(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European put option at $t = 1$ and at $t = 0$.

3. The holder of an *American put option* with maturity $T = 2$ on the risky security S and strike price $K = 11.555$ is allowed to exercise at $t = 1$ *or* at $T = 2$. The holder exercises optimally the option at $t = 1$ if $\max(K - S(1); 0)$, the immediate payoff at $t = 1$, exceeds the value at $t = 1$ of keeping the option until the next period, which is $\mathbb{E}^{\mathbb{Q}}\left[\frac{\max(K - S(2);0)}{1+r(1)} \middle| \mathcal{P}_1\right]$. Hence the option is optimally exercised at $t = 1$ if

$$\max(K - S(1); 0) \geq \mathbb{E}^{\mathbb{Q}}\left[\frac{\max(K - S(2); 0)}{1 + r(1)} \middle| \mathcal{P}_1\right],$$

and the option is optimally exercised at $T = 2$ otherwise.

Determine the scenarios f_h^1, with $h = 1, 2$ in which it is optimal to exercise the option at $t = 1$. Compute then the cash-flow of the optimally exercised American put option for $t = 1, 2$.

Compute the no-arbitrage price at $t = 0$ of the optimally exercised American put option.

The difference between the initial no-arbitrage prices of the optimally exercised American put option and the European put option is called the early exercise premium. Compute the early exercise premium in this case.

Exercise 17 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{tikzcd}[row sep=tiny]
 & & \omega_1 \\
 & f_1^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_2 \\
f_1^0 \arrow[uur] \arrow[ddr] & & \\
 & & \omega_3 \\
 & f_2^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_4
\end{tikzcd}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 5\% \text{ and } r(1)(f_2^1) = 2\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 12 \text{ and } S(1)(f_2^1) = 9$$

and with time 2 prices

$$S(2)(\omega_1) = 14.4, \quad S(2)(\omega_2) = 10.8, \quad S(2)(\omega_3) = 10.8, \quad S(2)(\omega_4) = 8.1.$$

1. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities? Is it complete?

2. A European *call option* with maturity $T = 2$ on the risky security S and strike price $K = 10.4$ is introduced in the market. Compute $c(2)$, the final payoff of the option at $T = 2$. Determine the no-arbitrge price of the European call option at $t = 1$ and at $t = 0$.

3. A forward-starting European call option c_{FS} with maturity $T = 2$ on the risky security S and strike price $K = 10.4$ is introduced in the market. The option starts at $t = 1$ if it is *in* or *at the money at* $t = 1$. The option is not activated if it is *out of the money* at $t = 1$. The final payoff of the forward-starting European call option c_{FS} is therefore

$$c_{FS}(2) = \begin{cases} c(2) & if\ S(1) \geq K \\ 0 & otherwise \end{cases}$$

Determine the no-arbitrge price of the forward-starting European call option c_{FS} at $t = 1$ and at $t = 0$.

Exercise 18 *Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:*

$$\begin{tikzcd}[row sep=tiny]
 & & \omega_1 \\
 & f_1^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_2 \\
f_1^0 \arrow[uur] \arrow[ddr] & & \\
 & & \omega_3 \\
 & f_2^1 \arrow[ur] \arrow[dr] & \\
 & & \omega_4
\end{tikzcd}$$

Two securities are traded in the market. The first is a locally risk-free asset B that provides the locally riskless interest rate

$$r(0) = 2\%, \quad r(1)(f_1^1) = 4\% \quad \text{and } r(1)(f_2^1) = 1\%.$$

The second security is a risky asset S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 10.4 \text{ and } S(1)(f_2^1) = 9.9$$

and with time 2 prices

$$S(2)(\omega_1) = 10.92, \quad S(2)(\omega_2) = 9.88, \quad S(2)(\omega_3) = 10.296, \quad S(2)(\omega_4) = 9.306.$$

1. Compute the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.

2. Is the market dynamically complete?

3. Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?

4. Consider the following three *Zero Coupon Bonds:* the first is issued at time $t = 0$ and matures at time $t = 1$, the second is also issued at time $t = 0$ and matures at time $t = 2$, the third is issued at time $t = 1$ and matures at time $t = 2$. By definition of *Zero Coupon Bond*, these three bonds deliver a cashflow equal to 1 at their maturity and 0 otherwise. Determine the *no-arbitrage price processes* $p_{0,1} = \{p_{0,1}(t)\}_{t=0,1}$, $p_{0,2} = \{p_{0,2}(t)\}_{t=0,1,2}$ and $p_{1,2} = \{p_{1,2}(t)\}_{t=1,2}$ of these three *Zero Coupon Bonds.*

5. Consider a *swap* contract between Party I and Party II written on S, with maturity $T = 2$ and starting at date $t = 0$. The swap contract requires Party I *to pay to* Party II *a fixed amount* F_{swap}, called the swap price, *at the times* $t = 1, 2$, while Party II is required *to pay to* Party I *the amounts* $S(1)$ at $t = 1$, and $S(2)$ at $T = 2$

   Write the cashflow of *Party I* at $t = 1, 2$.

   Is it possible to replicate the *fixed outflow* of Party I in the swap contract at $t = 1, 2$ with a portfolio of the zero coupon bonds defined in the previous point?

   The swap price F_{swap} is settled at time $t = 0$ in such a way that the *initial* no-arbitrage value of the swap contract is 0. Use this fact to compute the swap price F_{swap}.

   Compute $S_{swap} = \left\{S_{swap}(t)_{t=0,1,2}\right\}$ the no-arbitrage values of the swap contract (i.e. Party I cashflow) at the times $t = 0, 1, 2$.

6. Consider a *swaption* on the swap contract defined in the previous point. The swaption is settled at time 0, matures at time 1, and it gives to the holder the right to enter the swap contract on S at the maturity date 1. The holder of the swaption enters the swap contract at $t = 1$ if and only if the time $t = 1$ market value of the swap contract is positive. Thus the payoff of the swaption at maturity $t = 1$ is

   $$\max(S_{swap}(1); 0)$$

   Determine the no-arbitrage value of the swaption at $t = 0$.

2 Solutions: Multi-period Markets

Solution of Exercise 1

1. The prices of security B are
$$B(1)(f_1^1) = B(1)(f_2^1) = 1.01$$
and at the final date $T = 2$
$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.01 \cdot 1.01 = 1.0201 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.01. \end{aligned}$$
The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds
$$\det \begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det \begin{bmatrix} 1.01 & 105 \\ 1.01 & 95 \end{bmatrix} = -10.1 \neq 0,$$
for $m_{1,1}$ we have
$$\det \begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det \begin{bmatrix} 1.0201 & 110.25 \\ 1.0201 & 99.75 \end{bmatrix} = -10.711 \neq 0$$
and for $m_{1,2}$ we have
$$\det \begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det \begin{bmatrix} 1.01 & 104.5 \\ 1.01 & 85.5 \end{bmatrix} = -19.19 \neq 0.$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems
$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$
for m_0,
$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
for $m_{1,1}$, and
$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$
for $m_{1,2}$. The first system can be rewritten as
$$\begin{cases} 100 = \frac{1}{1.01} \left\{ 105 \cdot \mathbb{Q}[f_1^1] + 95 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$
and it is solved by
$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.6 \\ \mathbb{Q}[f_2^1] &= 0.4. \end{aligned}$$
The second system can be rewritten as
$$\begin{cases} 105 = \frac{1}{1.01} \left\{ 110.25 \cdot \mathbb{Q}[\omega_1|f_1^1] + 99.75 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
and it is solved by
$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.6 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.4. \end{aligned}$$

The third system can be rewritten as

$$\begin{cases} 95 = \frac{1}{1+r(1)(f_2^1)} \left\{104.5 \cdot \mathbb{Q}[\omega_3|f_2^1] + 85.5 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.5 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.5. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.6 \cdot 0.6 = 0.36 \\ \mathbb{Q}[\omega_2] &= 0.6 \cdot 0.4 = 0.24 \\ \mathbb{Q}[\omega_3] &= 0.4 \cdot 0.5 = 0.2 \\ \mathbb{Q}[\omega_4] &= 0.4 \cdot 0.5 = 0.2 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The terminal payoff

$$X(2) = \begin{cases} S(2) \text{ if } S(2) < 100 \\ 100 \text{ otherwise} \end{cases}$$

is equal to

$$\begin{gathered} X(2)(\omega_1) = 100 \\ X(2)(\omega_2) = 99.75 \\ X(2)(\omega_3) = 100 \\ X(2)(\omega_4) = 85.5 \end{gathered}$$

The no-arbitrage prices at $t = 1$ of this derivative are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{100 \cdot 0.6 + 99.75 \cdot 0.4}{1.01} = 98.911 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{100 \cdot 0.5 + 85.5 \cdot 0.5}{1.0} = 92.75 \end{aligned}$$

$$\begin{aligned} S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{98.911 \cdot 0.6 + 92.75 \cdot 0.4}{1.01} = 95.492. \end{aligned}$$

4. The terminal payoff of the derivative Y coincides with the terminal payoff of the derivative X. Therefore from $Y(2) = X(2)$ we get that their no-arbitrage prices at $t = 1$ coincide too, namely

$$\begin{aligned} S_Y(1)(f_1^1) &= S_X(1)(f_1^1) = 98.911 \\ S_Y(1)(f_2^1) &= S_X(1)(f_2^1) = 92.75. \end{aligned}$$

At $t = 1$ the cashflow of the derivative Y is

$$\begin{gathered} Y(1)(f_1^1) = 100 \\ Y(1)(f_2^1) = 95. \end{gathered}$$

The initial no-arbitrage price of the derivative Y is given by

$$\begin{aligned} S_Y(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{Y(1)+S_Y(1)}{1+r(0)}\right] \\ &= \frac{(100+98.911)\cdot 0.6+(95+92.75)\cdot 0.4}{1.01}=192.52. \end{aligned}$$

5. The no-arbitrage price at $t=0$ and at $t=1$ of a *zero-coupon bond* with maturity $T=2$, whose terminal payoff is 1 in any state of the world is at date $t=1$

$$\begin{aligned} ZCB(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{1}{1.01}=0.9901 \\ ZCB(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{1}{1.0}=1 \end{aligned}$$

At $t=0$ the price is

$$ZCB(0)=\mathbb{E}^{\mathbb{Q}}\left[\frac{ZCB(1)}{1+r(0)}\right]=\frac{0.9901\cdot 0.6+1\cdot 0.4}{1.01}=0.98422.$$

6. The terminal payoff of a *European put option on* S with maturity $T=2$ and strike 100 is

$$put(2)=(100-S(2))^+=\begin{cases} 100-S(2) \text{ if } S(2)<100 \\ 0 \text{ otherwise}\end{cases}$$

This payoff is equal to the constant amount 100 plus the final payoff of a short position on X, because

$$-X(2)=\begin{cases} -S(2) \text{ if } S(2)<100 \\ -100 \text{ otherwise}\end{cases}$$

and

$$100-X(2)=\begin{cases} 100-S(2) \text{ if } S(2)<100 \\ 100-100=0 \text{ otherwise}\end{cases}$$

In the extended market this final payoff $100-X(2)$ is obtained by buying at the initial date 100 units of the zero coupon bond of point 5. and by selling 1 unit of the derivative X. More formally consider the *buy-and-hold strategy* in the extended market

$$\begin{aligned} \vartheta_0(t) &= \vartheta_0=0 \quad \text{units of } B \\ \vartheta_1(t) &= \vartheta_1=0 \quad \text{units of } S \\ \vartheta_X(t) &= \vartheta_X=-1 \quad \text{units of } X \\ \vartheta_Y(t) &= \vartheta_Y=0 \quad \text{units of } Y \\ \vartheta_{ZCB}(t) &= \vartheta_{ZCB}=100 \quad \text{units of } ZCB \end{aligned}$$

for $t=0,1$. Then

$$\begin{aligned} C_\vartheta(2) &= V_\vartheta(2)=-1\cdot X(2)+100\cdot ZCB(2)=-1\cdot X(2)+100\cdot 1 \\ &= \begin{cases} 100-S(2) \text{ if } S(2)<100 \\ 100-100=0 \text{ otherwise}\end{cases} =(100-S(2))^+=put(2). \end{aligned}$$

At $t=1$ the cashflow of the strategy $C_\vartheta(1)=0$, because ϑ is buy-and-hold. Therefore, the cashflow process of ϑ coincides with the cashflow process of the European put option: hence ϑ replicates the put option.

Solution of Exericse 2

1. The prices of security B are
$$B(1)(f_1^1) = B(1)(f_2^1) = 1.05$$
and at the final date $T = 2$
$$B(2)(\omega_1) = B(2)(\omega_2) = 1.05 \cdot 1.06 = 1.113$$
$$B(2)(\omega_3) = B(2)(\omega_4) = 1.05$$
The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds
$$\det \begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det \begin{bmatrix} 1.05 & 11 \\ 1.05 & 9 \end{bmatrix} = -2.1 \neq 0,$$
for $m_{1,1}$ we have
$$\det \begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det \begin{bmatrix} 1.113 & 13.2 \\ 1.113 & 8.8 \end{bmatrix} = -4.8972 \neq 0$$
and for $m_{1,2}$ we have
$$\det \begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det \begin{bmatrix} 1.02 & 10.35 \\ 1.02 & 8.55 \end{bmatrix} = -1.836 \neq 0.$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems
$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$
for m_0,
$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
for $m_{1,1}$, and
$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$
for $m_{1,2}$. The first system can be rewritten as
$$\begin{cases} 10 = \frac{1}{1.05} \left\{ 11 \cdot \mathbb{Q}[f_1^1] + 9 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$
and it is solved by
$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.75 \\ \mathbb{Q}[f_2^1] &= 0.25. \end{aligned}$$
The second system can be rewritten as
$$\begin{cases} 11 = \frac{1}{1.06} \left\{ 13.2 \cdot \mathbb{Q}[\omega_1|f_1^1] + 8.8 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$
and it is solved by
$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.65 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.35. \end{aligned}$$

The last system can be rewritten as

$$\begin{cases} 9 = \frac{1}{1+r(1)(f_2^1)}\left\{10.35 \cdot \mathbb{Q}[\omega_3|f_2^1] + 8.55 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.25 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.75. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.75 \cdot 0.65 = 0.4875 \\ \mathbb{Q}[\omega_2] &= 0.75 \cdot 0.35 = 0.2625 \\ \mathbb{Q}[\omega_3] &= 0.25 \cdot 0.25 = 0.0625 \\ \mathbb{Q}[\omega_4] &= 0.25 \cdot 0.75 = 0.1875 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The terminal payoff of *European digital option* on S with maturity $T = 2$ and strike $K = 10$ is

$$\begin{aligned} X(2)(\omega_k) &= 1 \text{ for } k = 1, 3 \\ X(2)(\omega_k) &= 0 \text{ for } k = 2, 4 \end{aligned}$$

The no arbitrage prices of the *digital* option at $t = 0$ and in the nodes f_1^1 and f_2^1 at $t = 1$ are

$$\begin{aligned} S_X(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_1^1) \\ &= \frac{1 \cdot 0.65 + 0 \cdot 0.35}{1.06} = 0.613\,21 \\ S_X(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right](f_2^1) \\ &= \frac{1 \cdot 0.25 + 0 \cdot 0.75}{1.0} = 0.25 \\ S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{0.613\,21 \cdot 0.75 + 0.25 \cdot 0.25}{1.05} = 0.497\,53 \end{aligned}$$

4. The final payoff $X(2)$ of the digital option of the previous point is *not* independent of $\mathcal{P}_1$ with respect to the risk neutral probability $\mathbb{Q}$, because

$$\mathbb{Q}\left((X(2) = 1) \cap f_1^1\right) = \mathbb{Q}(\omega_1) = 0.487\,5$$

is different from the product of the two probabilities

$$\begin{aligned} \mathbb{Q}(X(2) = 1) \cdot \mathbb{Q}\left(f_1^1\right) &= \mathbb{Q}(\omega_1 \cup \omega_3) \cdot \mathbb{Q}\left(f_1^1\right) \\ &= (0.487\,5 + 0.062\,5) \cdot 0.75 = 0.412\,5. \end{aligned}$$

On the contrary, with respect to the uniform historical probability, the probability of the following intersections

$$\begin{aligned} \mathbb{P}\left((X(2) = 1) \cap f_1^1\right) &= \mathbb{P}(\omega_1) = 0.25 \\ \mathbb{P}\left((X(2) = 1) \cap f_2^1\right) &= \mathbb{P}(\omega_3) = 0.25 \\ \mathbb{P}\left((X(2) = 0) \cap f_1^1\right) &= \mathbb{P}(\omega_2) = 0.25 \\ \mathbb{P}\left((X(2) = 0) \cap f_2^1\right) &= \mathbb{P}(\omega_4) = 0.25 \end{aligned}$$

do concide (resp.) with the product of the probabilities:

$$
\begin{aligned}
\mathbb{P}\left(X\left(2\right)=1\right)\cdot\mathbb{P}\left(f_1^1\right) &= \mathbb{P}\left(\omega_1\cup\omega_3\right)\cdot\mathbb{P}\left(f_1^1\right)=\left(0.25+0.25\right)\cdot 0.5=0.25\\
\mathbb{P}\left(X\left(2\right)=1\right)\cdot\mathbb{P}\left(f_2^1\right) &= \mathbb{P}\left(\omega_1\cup\omega_3\right)\cdot\mathbb{P}\left(f_2^1\right)=\left(0.25+0.25\right)\cdot 0.5=0.25\\
\mathbb{P}\left(X\left(2\right)=0\right)\cdot\mathbb{P}\left(f_1^1\right) &= \mathbb{P}\left(\omega_2\cup\omega_4\right)\cdot\mathbb{P}\left(f_1^1\right)=\left(0.25+0.25\right)\cdot 0.5=0.25\\
\mathbb{P}\left(X\left(2\right)=0\right)\cdot\mathbb{P}\left(f_2^1\right) &= \mathbb{P}\left(\omega_2\cup\omega_4\right)\cdot\mathbb{P}\left(f_2^1\right)=\left(0.25+0.25\right)\cdot 0.5=0.25
\end{aligned}
$$

Thus the payoff $X\left(2\right)$ is independent of $\mathcal{P}_1$ with respect to historical ptobability $\mathbb{P}$.

Solution of Exercise 3

1. The prices of security B are

$$B(1)(f_1^1)=B(1)(f_2^1)=1.02$$

and at the final date $T=2$

$$
\begin{gathered}
B(2)(\omega_1)=B(2)(\omega_2)=1.02\cdot 1.03=1.0506\\
B(2)(\omega_3)=B(2)(\omega_4)=1.02\cdot 1.00=1.02.
\end{gathered}
$$

The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det\begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1)\\ B(1)(f_2^1) & S(1)(f_2^1)\end{bmatrix}=\det\begin{bmatrix}1.02 & 104\\ 1.02 & 99\end{bmatrix}=-5.1\neq 0,$$

for $m_{1,1}$ we have

$$\det\begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1)\\ B(2)(\omega_2) & S(2)(\omega_2)\end{bmatrix}=\det\begin{bmatrix}1.0506 & 108.16\\ 1.0506 & 102.96\end{bmatrix}=-5.4631\neq 0$$

and for $m_{1,2}$ we have

$$\det\begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3)\\ B(2)(\omega_4) & S(2)(\omega_4)\end{bmatrix}=\det\begin{bmatrix}1.02 & 102.96\\ 1.02 & 98.01\end{bmatrix}=-5.049\neq 0.$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0)=\frac{1}{1+r(0)}\left\{S(1)(f_1^1)\mathbb{Q}[f_1^1]+S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\}\\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1\\ \mathbb{Q}[f_1^1],\mathbb{Q}[f_2^1]>0\end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1)=\frac{1}{1+r(1)(f_1^1)}\left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1]+S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\}\\ \mathbb{Q}[\omega_1|f_1^1]+\mathbb{Q}[\omega_2|f_1^1]=1\\ \mathbb{Q}[\omega_1|f_1^1],\mathbb{Q}[\omega_2|f_1^1]>0\end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1)=\frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1]+S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\}\\ \mathbb{Q}[\omega_3|f_2^1]+\mathbb{Q}[\omega_4|f_2^1]=1\\ \mathbb{Q}[\omega_3|f_2^1],\mathbb{Q}[\omega_4|f_2^1]>0\end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 100=\frac{1}{1.02}\left\{104\cdot\mathbb{Q}[f_1^1]+99\cdot\mathbb{Q}[f_2^1]\right\}\\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1\\ \mathbb{Q}[f_1^1],\mathbb{Q}[f_2^1]>0\end{cases}$$

and it is solved by

$$
\begin{aligned}
\mathbb{Q}[f_1^1] &= 0.6\\
\mathbb{Q}[f_2^1] &= 0.4.
\end{aligned}
$$

The second system can be rewritten as

$$\begin{cases} 104 = \frac{1}{1.03}\left\{108.16 \cdot \mathbb{Q}[\omega_1|f_1^1] + 102.96 \cdot \mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.8 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.2. \end{aligned}$$

The last system can be rewritten as

$$\begin{cases} 99 = \frac{1}{1+0}\left\{102.96 \cdot \mathbb{Q}[\omega_3|f_2^1] + 98.01 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.2 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.8. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.6 \cdot 0.8 = 0.48 \\ \mathbb{Q}[\omega_2] &= 0.6 \cdot 0.2 = 0.12 \\ \mathbb{Q}[\omega_3] &= 0.4 \cdot 0.2 = 0.08 \\ \mathbb{Q}[\omega_4] &= 0.4 \cdot 0.8 = 0.32 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The no-arbitrage prices of a zero coupon bond with maturity $T = 2$ whose terminal payoff is

$$X(2) = 100$$

are

$$\begin{aligned} S_X(1)(f_1^1) &= \frac{\mathbb{E}^{\mathbb{Q}}\left[X(2)|\mathcal{F}_1\right](f_1^1)}{1 + r(1)(f_1^1)} = \frac{100 \cdot 0.8 + 100 \cdot 0.2}{1.03} = \frac{100}{1.03} = 97.087 \\ S_X(1)(f_2^1) &= \frac{\mathbb{E}^{\mathbb{Q}}\left[X(2)|\mathcal{F}_1\right](f_2^1)}{1 + r(1)(f_2^1)} = \frac{100 \cdot 0.2 + 100 \cdot 0.8}{1} = \frac{100}{1} = 100 \end{aligned}$$

at $t = 1$, and at $t = 0$

$$S_X(0) = \frac{97.087 \cdot 0.6 + 100 \cdot 0.4}{1.02} = 96.326.$$

4. The cashflow of the fixed leg (Part A) is at $t = 1$

$$X_A(1)(f_1^1) = X_A(1)(f_2^1) = N \cdot r(0) - N \cdot F = N \cdot (0.02 - F)$$

and at $T = 2$

$$\begin{aligned} X_A(2)(\omega_k) &= N \cdot r(1)(f_1^1) - N \cdot F = N \cdot (0.03 - F) \text{ for } k = 1, 2 \\ X_A(2)(\omega_k) &= N \cdot r(1)(f_2^1) - N \cdot F = N \cdot (0 - F) \text{ for } k = 3, 4. \end{aligned}$$

The no-arbitrage price of the cashflow of the swap contract (Part A) at $t = 0$ is

$$\begin{aligned} S_A(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X_A(1)}{1 + r(0)} + \frac{X_A(2)}{(1 + r(0))(1 + r(1))}\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{N \cdot (r(0) - F)}{1 + r(0)} + \frac{N \cdot (r(1) - F)}{(1 + r(0))(1 + r(1))}\right] \end{aligned}$$

Imposing

$$S_A(0) = 0$$

we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{r(0)}{1+r(0)}+\frac{r(1)}{(1+r(0))(1+r(1))}\right] &= F\cdot\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{1+r(0)}+\frac{1}{(1+r(0))(1+r(1))}\right] \\
\frac{r(0)}{1+r(0)}+\frac{1}{1+r(0)}\mathbb{E}^{\mathbb{Q}}\left[\frac{r(1)}{1+r(1)}\right] &= \frac{F}{1+r(0)}\cdot\mathbb{E}^{\mathbb{Q}}\left[1+\frac{1}{(1+r(1))}\right] \\
r(0)+\mathbb{E}^{\mathbb{Q}}\left[\frac{r(1)}{1+r(1)}\right] &= F\cdot\mathbb{E}^{\mathbb{Q}}\left[1+\frac{1}{(1+r(1))}\right] \\
F = \frac{r(0)+\mathbb{E}^{\mathbb{Q}}\left[\frac{r(1)}{1+r(1)}\right]}{1+\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{(1+r(1))}\right]} &= \frac{0.02+\left(\frac{0.03}{1.03}\cdot 0.6+\frac{0}{1}\cdot 0.4\right)}{1+\left(\frac{1}{1+0.03}\cdot 0.6+\frac{1}{1+0}\cdot 0.4\right)} = 0.0189
\end{aligned}$$

5. The terminal payoff of the forward-starting call option on S settled at time $t=0$ and *starting* at $t=1$ with maturity $T=2$ and strike price $K=S(1)$ is

$$\begin{aligned}
Y(2)(\omega_1) &= \left(S(2)(\omega_1)-S(1)\left(f_1^1\right)\right)^+ = (108.16-104)^+ = 4.16 \\
Y(2)(\omega_2) &= \left(S(2)(\omega_2)-S(1)\left(f_1^1\right)\right)^+ = (102.96-104)^+ = 0 \\
Y(2)(\omega_3) &= \left(S(2)(\omega_3)-S(1)\left(f_2^1\right)\right)^+ = (102.96-99)^+ = 3.96 \\
Y(2)(\omega_4) &= \left(S(2)(\omega_4)-S(1)\left(f_2^1\right)\right)^+ = (98.01-99)^+ = 0
\end{aligned}$$

The no-arbitrage price of the forward starting call option at $t=1$ is

$$\begin{aligned}
S_Y(1)(f_1^1) &= \frac{\mathbb{E}^{\mathbb{Q}}\left[Y(2)|\mathcal{F}_1\right](f_1^1)}{1+r(1)(f_1^1)} = \frac{4.16\cdot 0.8+0\cdot 0.2}{1.03} = 3.2311 \\
S_Y(1)(f_2^1) &= \frac{\mathbb{E}^{\mathbb{Q}}\left[Y(2)|\mathcal{F}_1\right](f_2^1)}{1+r(1)(f_2^1)} = \frac{3.96\cdot 0.2+0\cdot 0.8}{1} = 0.792
\end{aligned}$$

and at $t=0$ and

$$S_Y(0) = \frac{\mathbb{E}^{\mathbb{Q}}\left[S_Y(1)\right]}{1+r(0)} = \frac{3.2311\cdot 0.6+0.792\cdot 0.4}{1.02} = 2.2112.$$

Solution of Exercise 4

1. The prices of security B are

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.02$$

and at the final date $T=2$

$$\begin{gathered}
B(2)(\omega_1) = B(2)(\omega_2) = 1.02\cdot 1.02 = 1.0404 \\
B(2)(\omega_3) = B(2)(\omega_4) = 1.02.
\end{gathered}$$

The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det\begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1)\end{bmatrix} = \det\begin{bmatrix} 1.02 & 11 \\ 1.02 & 9\end{bmatrix} = -2.04 \neq 0,$$

for $m_{1,1}$ we have

$$\det\begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2)\end{bmatrix} = \det\begin{bmatrix} 1.0404 & 12.1 \\ 1.0404 & 9.9\end{bmatrix} = -2.2889 \neq 0$$

and for $m_{1,2}$ we have

$$\det\begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4)\end{bmatrix} = \det\begin{bmatrix} 1.02 & 9.9 \\ 1.02 & 8.1\end{bmatrix} = -1.836 \neq 0.$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)}\left\{S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)}\left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 10 = \frac{1}{1.02}\left\{11 \cdot \mathbb{Q}[f_1^1] + 9 \cdot \mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.6 \\ \mathbb{Q}[f_2^1] &= 0.4. \end{aligned}$$

The second system can be rewritten as

$$\begin{cases} 11 = \frac{1}{1.02}\left\{12.1 \cdot \mathbb{Q}[\omega_1|f_1^1] + 9.9 \cdot \mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.6 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.4. \end{aligned}$$

and System (**??**) can be rewritten as

$$\begin{cases} 9 = \frac{1}{1+r(1)(f_2^1)}\left\{9.9 \cdot \mathbb{Q}[\omega_3|f_2^1] + 8.1 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and it is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.5 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.5. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.6 \cdot 0.6 = 0.36 \\ \mathbb{Q}[\omega_2] &= 0.6 \cdot 0.4 = 0.24 \\ \mathbb{Q}[\omega_3] &= 0.4 \cdot 0.5 = 0.2 \\ \mathbb{Q}[\omega_4] &= 0.4 \cdot 0.5 = 0.2 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The *European digital option on* S with maturity $T=2$ has terminal payoff equal to

$$
\begin{gathered}
X(2)(\omega_1)=0 \text{ since } 12.1 \geq 9.1 \\
X(2)(\omega_2)=0 \text{ since } 9.9 \geq 9.1 \\
X(2)(\omega_3)=0 \text{ since } 9.9 \geq 9.1 \\
X(2)(\omega_4)=1 \text{ since } 8.1 < 9.1.
\end{gathered}
$$

Hence its no-arbitrage prices at $t=1$ are

$$
\begin{aligned}
S_X(1)(f_1^1) &= \frac{0\cdot 0.6+0\cdot 0.4}{1.02}=0 \\
S_X(1)(f_2^1) &= \frac{0\cdot 0.5+1\cdot 0.5}{1.0}=0.5
\end{aligned}
$$

and at $t=0$

$$
S_X(0)=\frac{0\cdot 0.6+0.5\cdot 0.4}{1.02}=0.19608.
$$

4. The European digital option can be replicated by a dynamic investment strategy ϑ because the market is complete. The cost of replication V_ϑ equals the price of the option S_X by no-arbitrage. Hence

$$
\begin{aligned}
V_\vartheta(1)(f_1^1) &= S_X(1)(f_1^1)=0 \\
V_\vartheta(1)(f_2^1) &= S_X(1)(f_2^1)=0.5
\end{aligned}
$$

and at $t=0$

$$
V_\vartheta(0)=S_X(0)=0.196\,08
$$

5. The process M is a martingale with respect to $\mathbb{Q}^S$ if

$$
M(0)=\mathbb{E}^{\mathbb{Q}^S}\left[M(1)\right]
$$

and

$$
\begin{aligned}
M(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}^S}\left[M(2)|\mathcal{P}_1\right](f_1^1) && (*) \\
M(1)(f_2^1) &= \mathbb{E}^{\mathbb{Q}^S}\left[M(2)|\mathcal{P}_1\right](f_2^1). && (**)
\end{aligned}
$$

From the first equation we obtain

$$
\begin{aligned}
\frac{1}{S(0)} &= M(0)=\mathbb{E}^{\mathbb{Q}^S}\left[M(1)\right]=\frac{1+r(0)}{S(1)(f_1^1)}\mathbb{Q}^S[f_1^1]+\frac{1+r(0)}{S(1)(f_2^1)}\mathbb{Q}^S[f_2^1] \\
\frac{1}{10} &= \frac{1.02}{11}\mathbb{Q}^S[f_1^1]+\frac{1.02}{9}\left(1-\mathbb{Q}^S[f_1^1]\right)
\end{aligned}
$$

that delivers $\mathbb{Q}^S[f_1^1]=0.64706$ and $\mathbb{Q}^S[f_2^1]=1-\mathbb{Q}^S[f_1^1]=1-0.64706=0.352\,94$.

Conditions $(*)$ and $(**)$ deliver two different equations to determine (resp.) $\mathbb{Q}^S[\omega_1|f_1^1]$ and $\mathbb{Q}^S[\omega_3|f_2^1]$. In fact, from equation $(*)$ we get

$$
\frac{1.02}{11}=\left\{\frac{1.0404}{12.1}\cdot\mathbb{Q}^S[\omega_1|f_1^1]+\frac{1.0404}{9.9}\cdot\left(1-\mathbb{Q}^S[\omega_1|f_1^1]\right)\right\}
$$

that is solved by

$$
\begin{aligned}
\mathbb{Q}^S[\omega_1|f_1^1] &= 0.647\,06 \\
\mathbb{Q}^S[\omega_2|f_1^1] &= 1-0.647\,06=0.35294.
\end{aligned}
$$

From equation $(**)$ we get

$$
\frac{1.02}{9}=\left\{\frac{1.02}{9.9}\cdot\mathbb{Q}^S[\omega_3|f_2^1]+\frac{1.02}{8.1}\cdot\left(1-\mathbb{Q}^S[\omega_3|f_2^1]\right)\right\}
$$

leading to

$$\begin{aligned}\mathbb{Q}^S[\omega_3|f_2^1] &= 0.55\\ \mathbb{Q}^S[\omega_4|f_2^1] &= 1-0.55=0.45.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{Q}^S[\omega_1] &= 0.64706\cdot 0.64706=0.41869\\ \mathbb{Q}^S[\omega_2] &= 0.64706\cdot 0.35294=0.22837\\ \mathbb{Q}^S[\omega_3] &= 0.35294\cdot 0.55=0.19412\\ \mathbb{Q}^S[\omega_4] &= 0.35294\cdot 0.45=0.15882.\end{aligned}$$

6. To verify that V_ϑ, the replication cost of the derivative of Point 4, discounted by S is a martingale with respect $\mathbb{Q}^S$ we have to check that

$$\begin{aligned}\frac{V_\vartheta(0)}{S(0)} &= \mathbb{E}^{\mathbb{Q}^S}\left[\frac{V_\vartheta(1)}{S(1)}\right]\\ \frac{V_\vartheta(1)}{S(1)} &= \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right] \qquad (\diamondsuit)\end{aligned}$$

We can simply verify the equalities as follows

$$\begin{aligned}\frac{V_\vartheta(0)}{S(0)} &= \frac{0.196\,08}{10}=1.9608\times 10^{-2}\\ \mathbb{E}^{\mathbb{Q}^S}\left[\frac{V_\vartheta(1)}{S(1)}\right] &= \frac{0}{11}\cdot 0.647\,06+\frac{0.5}{9}\cdot 0.352\,94=1.9608\times 10^{-2}\end{aligned}$$

for the first one. Equality $(\diamondsuit)$ delivers at f_1^1

$$\begin{aligned}\frac{V_\vartheta(1)}{S(1)}(f_1^1) &= \frac{0}{11}=0\\ \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right](f_1^1) &= 0\end{aligned}$$

and at f_2^1

$$\begin{aligned}\frac{V_\vartheta(1)}{S(1)}(f_2^1) &= \frac{0.5}{9}=5.5556\times 10^{-2}\\ \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right](f_2^1) &= \frac{0}{9.9}\cdot 0.55+\frac{1}{8.1}\cdot 0.45=5.5556\times 10^{-2}\end{aligned}$$

This verifies that $\frac{V_\vartheta}{S}$ is a $\mathbb{Q}^S-$martingale. Alternatively one can prove the martingality of $\frac{V_\vartheta}{S}$ with respect to $\mathbb{Q}^S$ for *any* self-financing strategy ϑ as follows:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right] &= \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{\vartheta_0(1)\,B(2)+\vartheta_1(1)\,S(2)}{S(2)}\right|\mathcal{P}_1\right] \text{ from the definition of } V_\vartheta(2)\\ &= \vartheta_0(1)\,\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{B(2)}{S(2)}\right|\mathcal{P}_1\right]+\vartheta_1(1)\,\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{S(2)}{S(2)}\right|\mathcal{P}_1\right]\end{aligned}$$

where the last equality follows because $\vartheta_0(1), \vartheta_1(1)$ are $\mathcal{P}_1-$measurable. Hence

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right] &= \vartheta_0(1)\frac{B(1)}{S(1)}+\vartheta_1(1)\cdot 1 \text{ because } \frac{B}{S} \text{ is a } \mathbb{Q}^S-\text{martingale}\\ &= \frac{\vartheta_0(1)\,B(1)+\vartheta_1(1)\cdot S(1)}{S(1)}=\frac{V_\vartheta(1)}{S(1)}\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^S}\left[\frac{V_\vartheta(1)}{S(1)}\right] &= \mathbb{E}^{\mathbb{Q}^S}\left[\frac{\vartheta_0(0)B(1)+\vartheta_1(0)S(1)}{S(1)}\right] \text{ because } \vartheta \text{ is self-financing} \\
&= \vartheta_0(0)\,\mathbb{E}^{\mathbb{Q}^S}\left[\frac{B(1)}{S(1)}\right]+\vartheta_1(0)\,\mathbb{E}^{\mathbb{Q}^S}\left[\frac{S(1)}{S(1)}\right] \text{ since } \vartheta_0(0),\vartheta_1(0)\in\Re \\
&= \vartheta_0(0)\frac{B(0)}{S(0)}+\vartheta_1(0)\cdot 1 \text{ because } \frac{B}{S} \text{ is a } \mathbb{Q}^S-\text{martingale} \\
&= \frac{\vartheta_0(0)B(0)+\vartheta_1(0)\cdot S(0)}{S(0)}=\frac{V_\vartheta(0)}{S(0)}
\end{aligned}$$

This concludes the solution of the exercise. We remark that, S_X , the no-arbitrage price process of the European digital derivative of Point 3, is equal to its replication cost V_ϑ. Therefore we get

$$\begin{aligned}
\frac{S_X(0)}{S(0)} &= \frac{V_\vartheta(0)}{S(0)}=\mathbb{E}^{\mathbb{Q}^S}\left[\frac{V_\vartheta(1)}{S(1)}\right]=\mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_X(1)}{S(1)}\right] \\
\frac{S_X(1)}{S(1)} &= \frac{V_\vartheta(1)}{S(1)}=\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right]=\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{S_X(2)}{S(2)}\right|\mathcal{P}_1\right],
\end{aligned}$$

i.e. $\frac{S_X}{S}$ is a $\mathbb{Q}^S-$martingale. The link between the process S_X and the terminal payoff X in terms of the probability measure $\mathbb{Q}^S$ is therefore

$$\begin{aligned}
\frac{S_X(1)}{S(1)} &= \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{S_X(2)}{S(2)}\right|\mathcal{P}_1\right] \\
&= \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{V_\vartheta(2)}{S(2)}\right|\mathcal{P}_1\right] \\
&= \mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{X(2)}{S(2)}\right|\mathcal{P}_1\right]
\end{aligned}$$

and

$$\begin{aligned}
\frac{S_X(0)}{S(0)} &= \mathbb{E}^{\mathbb{Q}^S}\left[\frac{S_X(1)}{S(1)}\right] \text{ by the martingality property} \\
&= \mathbb{E}^{\mathbb{Q}^S}\left[\mathbb{E}^{\mathbb{Q}^S}\left[\left.\frac{X(2)}{S(2)}\right|\mathcal{P}_1\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}^S}\left[\frac{X(2)}{S(2)}\right] \text{ by the law of iterated conditional expectations}
\end{aligned}$$

Solution of Exercise 5

1. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0)=\frac{1}{1+r(0)}\left\{S(1)(f_1^1)\mathbb{Q}[f_1^1]+S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1 \\ \mathbb{Q}[f_1^1],\mathbb{Q}[f_2^1]>0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1)=\frac{1}{1+r(1)(f_1^1)}\left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1]+S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1]+\mathbb{Q}[\omega_2|f_1^1]=1 \\ \mathbb{Q}[\omega_1|f_1^1],\mathbb{Q}[\omega_2|f_1^1]>0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1)=\frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1]+S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1]+\mathbb{Q}[\omega_4|f_2^1]=1 \\ \mathbb{Q}[\omega_3|f_2^1],\mathbb{Q}[\omega_4|f_2^1]>0 \end{cases}$$

for $m_{1,2}$. Solving the three systems delivers

$$\begin{matrix}\mathbb{Q}[f_1^1]=0.8 \\ \mathbb{Q}[f_2^1]=0.2\end{matrix}\ ,\quad \begin{matrix}\mathbb{Q}[f_1^1]=0.9 \\ \mathbb{Q}[f_2^1]=0.1\end{matrix}\quad \text{and}\quad \begin{matrix}\mathbb{Q}[f_1^1]=0.6 \\ \mathbb{Q}[f_2^1]=0.4\end{matrix}\ .$$

Therefore,

$$\begin{aligned}
\mathbb{Q}[\omega_1] &= 0.8 \cdot 0.9 = 0.72 \\
\mathbb{Q}[\omega_2] &= 0.8 \cdot 0.1 = 0.08 \\
\mathbb{Q}[\omega_3] &= 0.2 \cdot 0.6 = 0.12 \\
\mathbb{Q}[\omega_4] &= 0.2 \cdot 0.4 = 0.08
\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

2. We start by determining $f(1)$. To this aim we impose the condition that the value of a long position in the futures contract at time $t = 1$ is zero:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{fut}(2)}{1+r(1)}\right|\mathcal{P}_1\right] &= 0 \\
\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)-f(1)}{1+r(1)}\right|\mathcal{P}_1\right] &= 0 \\
\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)}{1+r(1)}\right|\mathcal{P}_1\right] &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{f(1)}{1+r(1)}\right|\mathcal{P}_1\right] \qquad (*) \\
S(1) &= \frac{f(1)}{1+r(1)}
\end{aligned}$$

where we exploited the definition of risk neutral measure that yields $S(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)}{1+r(1)}\right|\mathcal{P}_1\right]$ and the fact that $f(1)$ and $r(1)$ are $\mathcal{P}_1-$measurable, thus implying $\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{f(1)}{1+r(1)}\right|\mathcal{P}_1\right] = \frac{f(1)}{1+r(1)}$. Therefore,

$$f(1) = S(1)(1+r(1)) = \begin{cases} 10.3 \cdot (1+0.03) = 10.609 & \text{if } f_1^1 \\ \\ 9.8 \cdot (1+0.01) = 9.898 & \text{if } f_2^1 \end{cases}$$

We now determine $f(0)$. To this aim we impose the condition that the value of a long position in the futures contract at time $t = 0$ is zero:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X_{fut}(1)}{1+r(0)} + \frac{X_{fut}(2)}{(1+r(0))(1+r(1))}\right] = 0$$

From our previous step we know that $f(1)$ is such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{fut}(2)}{1+r(1)}\right|\mathcal{P}_1\right] = 0$$

Therefore

$$\begin{aligned}
0 &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X_{fut}(1)}{1+r(0)} + \frac{X_{fut}(2)}{(1+r(0))(1+r(1))}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{X_{fut}(1)}{1+r(0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{1+r(0)}\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{fut}(2)}{1+r(1)}\right|\mathcal{P}_1\right]\right] \quad \text{be the tower property} \\
&= \mathbb{E}^{\mathbb{Q}}\left[\frac{X_{fut}(1)}{1+r(0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{1+r(0)} \cdot 0\right]
\end{aligned}$$

Hence $f(0)$ must be settled such that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{X_{fut}(1)}{1+r(0)}\right] &= 0 \\
\mathbb{E}^{\mathbb{Q}}\left[\frac{f(1)-f(0)}{1+r(0)}\right] &= 0
\end{aligned}$$

leading to

$$\begin{aligned}
\frac{f(0)}{1+r(0)} &= \mathbb{E}^{\mathbb{Q}}\left[\frac{f(1)}{1+r(0)}\right] \\
f(0) &= \mathbb{E}^{\mathbb{Q}}[f(1)] \text{ dividing by the constant } 1+r(0) \qquad (**) \\
&= 10.609 \cdot 0.8 + 9.898 \cdot 0.2 = 10.467.
\end{aligned}$$

3. To show that the *futures prices process* $f = \{f(t)\}_{t=0,1,2}$ is a $\mathbb{Q}-$martingale, we first observe that equation $(*)$ delivers

$$\begin{aligned}
\frac{1}{1+r(1)}\mathbb{E}^{\mathbb{Q}}[S(2)|\mathcal{P}_1] &= \frac{1}{1+r(1)}\mathbb{E}^{\mathbb{Q}}[f(1)|\mathcal{P}_1] \text{ because } 1+r(1) \text{ is } \mathcal{P}_1-\text{meas.} \\
\mathbb{E}^{\mathbb{Q}}[S(2)|\mathcal{P}_1] &= \mathbb{E}^{\mathbb{Q}}[f(1)|\mathcal{P}_1] \\
\mathbb{E}^{\mathbb{Q}}[S(2)|\mathcal{P}_1] &= f(1) \text{ because } f(1) \text{ is } \mathcal{P}_1-\text{meas.}
\end{aligned}$$

since $1+r(1)$ is $\mathcal{P}_1-$measurable.

By imposing the terminal condition $f(2) = S(2)$ we observe that this is exactly the martingality requirement

$$f(1) = \mathbb{E}^{\mathbb{Q}}[S(2)|\mathcal{P}_1] = \mathbb{E}^{\mathbb{Q}}[f(2)|\mathcal{P}_1]$$

from $t = 1$ to $t = 2$. The further condition from $t = 0$ to $t = 1$ is

$$f(0) = \mathbb{E}^{\mathbb{Q}}[f(1)]$$

which is the Equation $(**)$ we imposed above.

This proves that the *futures prices process* $f = \{f(t)\}_{t=0,1,2}$ is a $\mathbb{Q}-$martingale. You can also prove the property be numerically verifying that

$$f(0) = \mathbb{E}^{\mathbb{Q}}[f(1)] \text{ and } f(1) = \mathbb{E}^{\mathbb{Q}}[f(2)|\mathcal{P}_1].$$

4. The payoff scheme of the barrier derivative is represented below

$$\begin{array}{ll}
\underset{X(1)=C}{10.3 = S(1) > S(0) = 10} & \begin{array}{l} \Rightarrow X(2) = 0 \quad \text{on } \omega_1 \\ \Rightarrow X(2) = 0 \quad \text{on } \omega_2 \end{array} \\
\underset{X(1)=0}{9.8 = S(1) \not> S(0) = 10} & \begin{array}{l} \underset{X(2)=C}{9.8 = S(1) < S(2) = 10.094} \quad \text{on } \omega_3 \\ \underset{X(2)=0}{9.8 = S(1) \not< S(2) = 9.604} \quad \text{on } \omega_4 \end{array}
\end{array}.$$

The payoff at maturity of the given barrier derivative is

$$X(2) = \begin{cases} C = 100 & \text{on } \omega_3 \\ 0 & \text{on } \omega_1, \omega_2, \omega_4 \end{cases}.$$

Its no-arbitrage price at $t = 1$ is

$$S_X(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right] = \begin{cases} 0 & \text{if } f_1^1 \\ \dfrac{100 \cdot 0.6 + 0 \cdot 0.4}{1.03} = 59.406 & \text{if } f_2^1 \end{cases}.$$

The payoff at $t = 1$ barrier derivative is

$$X(1) = \begin{cases} C = 100 & \text{if } f_1^1 \\ 0 & \text{if } f_2^1 \end{cases}.$$

Therefore, its no-arbitrage price at $t = 0$ is

$$\begin{aligned}
S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X(1) + S_X(1)}{1+r(0)}\right] \\
&= \frac{100+0}{1.02} \cdot 0.8 + \frac{0 + 59.406}{1.02} \cdot 0.2 \\
&= 90.079.
\end{aligned}$$

Solution of Exercise 6

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.02$$

and at the final date $T = 2$

$$\begin{array}{c} B(2)(\omega_1) = B(2)(\omega_2) = 1.02 \cdot 1.04 = 1.0608 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.02 \cdot 1.02 = 1.0404. \end{array}$$

2. The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det \begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det \begin{bmatrix} 1.02 & 112 \\ 1.02 & 96 \end{bmatrix} = -16.32 \neq 0,$$

for $m_{1,1}$ we have

$$\det \begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det \begin{bmatrix} 1.0608 & 125.44 \\ 1.0608 & 107.52 \end{bmatrix} = -19.010 \neq 0$$

and for $m_{1,2}$ we have

$$\det \begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det \begin{bmatrix} 1.0404 & 107.52 \\ 1.0404 & 92.16 \end{bmatrix} = -15.981 \neq 0.$$

3. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 100 = \frac{1}{1.02} \left\{ 112 \cdot \mathbb{Q}[f_1^1] + 96 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= \frac{3}{8} = 0.375 \\ \mathbb{Q}[f_2^1] &= \frac{5}{8} = 0.625. \end{aligned}$$

The second system can be rewritten as

$$\begin{cases} 112 = \frac{1}{1.04} \left\{ 125.44 \cdot \mathbb{Q}[\omega_1|f_1^1] + 107.52 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned}\mathbb{Q}[\omega_1|f_1^1] &= \frac{1}{2}=0.5\\ \mathbb{Q}[\omega_2|f_1^1] &= \frac{1}{2}=0.5.\end{aligned}$$

The third system can be rewritten as

$$\begin{cases} 96=\frac{1}{1.02}\left\{107.52\cdot\mathbb{Q}[\omega_3|f_2^1]+92.16\cdot\mathbb{Q}[\omega_4|f_2^1]\right\}\\ \mathbb{Q}[\omega_3|f_2^1]+\mathbb{Q}[\omega_4|f_2^1]=1\\ \mathbb{Q}[\omega_3|f_2^1],\mathbb{Q}[\omega_4|f_2^1]>0\end{cases}$$

and is solved by

$$\begin{aligned}\mathbb{Q}[\omega_3|f_2^1] &= \frac{3}{8}=0.375\\ \mathbb{Q}[\omega_4|f_2^1] &= \frac{5}{8}=0.625.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{Q}[\omega_1] &= \frac{3}{8}\cdot\frac{1}{2}=\frac{3}{16}=0.1875\\ \mathbb{Q}[\omega_2] &= \frac{3}{8}\cdot\frac{1}{2}=\frac{3}{16}=0.1875\\ \mathbb{Q}[\omega_3] &= \frac{5}{8}\cdot\frac{3}{8}=\frac{15}{64}=0.23438\\ \mathbb{Q}[\omega_4] &= \frac{5}{8}\cdot\frac{5}{8}=\frac{25}{64}=0.39063\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The European call option on S has terminal payoff

$$c(2)=(S(2)-K)^+=\begin{cases}(125.44-100)^+=25.44 & \text{on } \omega_1\\ (107.52-100)^+=7.52 & \text{on } \omega_2,\omega_3\\ (92.16-100)^+=0 & \text{on } \omega_4\end{cases}$$

Its no arbitrage price at $t=1$ is

$$c(1)=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c(2)}{1+r(1)}\right|\mathcal{P}_1\right]=\begin{cases}\dfrac{25.44\cdot 0.5+7.52\cdot 0.5}{1.04}=15.846 & \text{if } f_1^1\\ \\ \dfrac{7.52\cdot 0.375+0\cdot 0.625}{1.02}=2.7647 & \text{if } f_2^1\end{cases}$$

and at $t=0$

$$\begin{aligned}c(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1+r(0)}\right]=\\ &= \frac{15.846\cdot 0.375+2.7647\cdot 0.625}{1.02}=7.5198.\end{aligned}$$

5. The return of S over the first period is

$$R(1)=\frac{S(1)-S(0)}{S(0)}=\begin{cases}0.12>-0.01 & \text{at } f_1^1\\ \\ -0.04<-0.01 & \text{at } f_2^1\end{cases}$$

Thus, the cashflow of the path-dependent derivative X at $t=1$ is

$$X(1)=\begin{cases}0 & \text{at } f_1^1\\ \\ 1-10\cdot(-0.01-(-0.04))=0.7 & \text{at } f_2^1\end{cases}.$$

At $T=2$

$$X(2)=\begin{cases}1+r(0)+C & \text{on } \omega_1,\omega_2\\ 0 & \text{on } \omega_3,\omega_4\end{cases}.$$

Thus,

$$\begin{aligned}S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X(1)}{1+r(0)}\right]+\mathbb{E}^{\mathbb{Q}}\left[\frac{X(2)}{(1+r(0))(1+r(1))}\right]\\ &= \frac{0}{1+0.02}\cdot\frac{3}{8}+\frac{0.7}{1+0.02}\cdot\frac{5}{8}+\frac{(1+0.02+C)}{(1+0.02)\cdot(1+0.04)}\cdot\frac{3}{8}\\ &= 0.35351C+0.78950.\end{aligned}$$

As the initial no-arbitrage price of this derivative must be equal to one,

$$S_X(0)=1,$$

we get

$$\begin{aligned}0.35351C+0.78950 &= 1\\ C &= \frac{1-0.78950}{0.35351}\\ &= 0.59546.\end{aligned}$$

Solution of Exercise 7

1. The prices of security B are $B(0)=1$,

$$B(1)(f_1^1)=B(1)(f_2^1)=1.1$$

and at the final date $T=2$

$$\begin{aligned}B(2)(\omega_1)=B(2)(\omega_2)=1.1\cdot 1.1=1.21\\ B(2)(\omega_3)=B(2)(\omega_4)=1.1\cdot 1.2=1.32.\end{aligned}$$

The prices of the risky security S are

$$S(1)(f_1^1)=1.4 \quad\text{and}\quad S(1)(f_2^1)=0.8$$

and at the final date $T=2$

$$\begin{aligned}S(2)(\omega_1) &= 1.4\cdot 1.4=1.96\\ S(2)(\omega_2) &= 1.4\cdot 0.8=1.12\\ S(2)(\omega_3) &= 0.8\cdot 1.4=1.12\\ S(2)(\omega_4) &= 0.8\cdot 0.8=0.64.\end{aligned}$$

2. The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det\begin{bmatrix}B(1)(f_1^1) & S(1)(f_1^1)\\ B(1)(f_2^1) & S(1)(f_2^1)\end{bmatrix}=\det\begin{bmatrix}1.1 & 1.4\\ 1.1 & 0.8\end{bmatrix}=-0.66\neq 0,$$

for $m_{1,1}$ we have

$$\det\begin{bmatrix}B(2)(\omega_1) & S(2)(\omega_1)\\ B(2)(\omega_2) & S(2)(\omega_2)\end{bmatrix}=\det\begin{bmatrix}1.21 & 1.96\\ 1.21 & 1.12\end{bmatrix}=-1.0164\neq 0$$

and for $m_{1,2}$ we have

$$\det\begin{bmatrix}B(2)(\omega_3) & S(2)(\omega_3)\\ B(2)(\omega_4) & S(2)(\omega_4)\end{bmatrix}=\det\begin{bmatrix}1.32 & 1.12\\ 1.32 & 0.64\end{bmatrix}=-0.6336\neq 0.$$

3. As this market model is a binomial market model, we know that the only risk-neutral proability measure for each submarket is given by

$$\begin{aligned}\mathbb{Q}\left(S(t+1)=S(t)\cdot u\right) &= \frac{1+r-d}{u-d}\\ \mathbb{Q}\left(S(t+1)=S(t)\cdot d\right) &= 1-\frac{1+r-d}{u-d}\\ &= \frac{u-(1+r)}{u-d}.\end{aligned}$$

Therefore, for $m_{0,0}$ we get

$$\mathbb{Q}[f_1^1]=\frac{1+0.1-0.8}{1.4-0.8}=0.5 \quad \text{and} \quad \mathbb{Q}[f_2^1]=0.5.$$

For $m_{1,1}$

$$\mathbb{Q}[\omega_1|f_1^1]=\frac{1+0.1-0.8}{1.4-0.8}=0.5 \quad \text{and} \quad \mathbb{Q}[\omega_2|f_1^1]=0.5,$$

for $m_{1,2}$

$$\begin{aligned}\mathbb{Q}[\omega_3|f_2^1] &= \frac{1+0.2-0.8}{1.4-0.8}=\frac{2}{3}=0.66667\\ \mathbb{Q}[\omega_4|f_2^1] &= \frac{1}{3}=0.33333.\end{aligned}$$

Furthermore,

$$\begin{aligned}\mathbb{Q}[\omega_1] &= 0.5\cdot 0.5=0.25\\ \mathbb{Q}[\omega_2] &= 0.5\cdot 0.5=0.25\\ \mathbb{Q}[\omega_3] &= 0.5\cdot 0.66667=0.33333\\ \mathbb{Q}[\omega_4] &= 0.5\cdot 0.33333=0.16667.\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The European call option on S has terminal payoff

$$c\left(2\right)=\left(S\left(2\right)-K\right)^{+}=\begin{cases}(1.96-1.46)^{+}=0.5 & \text{on } \omega_1\\ (1.12-1.46)^{+}=0 & \text{on } \omega_2\\ (1.12-1.46)^{+}=0 & \text{on } \omega_3\\ (0.64-1.46)^{+}=0 & \text{on } \omega_4\end{cases}.$$

Its no arbitrage price is therefore at $t=1$

$$c\left(1\right)=\mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1+r\left(1\right)}\middle|\mathcal{P}_1\right]=\begin{cases}\dfrac{0.5\cdot 0.5+0\cdot 0.5}{1.1}=0.22727 & \text{if } f_1^1\\[2ex] \dfrac{0\cdot 0.66667+0\cdot 0.33333}{1.2}=0 & \text{if } f_2^1\end{cases}$$

and at $t=0$

$$\begin{aligned}c\left(0\right) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1+r\left(0\right)}\right]\\ &= \frac{0.22727}{1.1}\cdot 0.5+\frac{0}{1.1}\cdot 0.5\\ &= 0.1033.\end{aligned}$$

5. The so-called running maximum $\max_{t=0,1,2} S(t)$ at $T=2$ is equal to

$$\max_{t=0,1,2} S(t)=\begin{cases}\max\{1,1.4,1.96\}=1.96 & \text{on } \omega_1\\ \max\{1,1.4,1.12\}=1.4 & \text{on } \omega_2\\ \max\{1,0.8,1.12\}=1.12 & \text{on } \omega_3\\ \max\{1,0.8,0.64\}=1 & \text{on } \omega_4\end{cases}.$$

Therefore, the terminal payoff of the lookback call is

$$X(2) = \max_{t=0,1,2} S(t) - S(2) = \begin{cases} 1.96 - 1.96 = 0 & \text{on } \omega_1 \\ 1.4 - 1.12 = 0.28 & \text{on } \omega_2 \\ 1.12 - 1.12 = 0 & \text{on } \omega_3 \\ 1 - 0.64 = 0.36 & \text{on } \omega_4 \end{cases}.$$

Its no arbitrage price at $t = 1$ is

$$X(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right| \mathcal{P}_1\right] = \begin{cases} \dfrac{0 \cdot 0.5 + 0.28 \cdot 0.5}{1.1} = 0.12727 & \text{if } f_1^1 \\ \dfrac{0 \cdot 0.66667 + 0.36 \cdot 0.33333}{1.2} = 0.1 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X(1)}{1+r(0)}\right] \\ &= \frac{0.12727}{1.1} \cdot 0.5 + \frac{0.1}{1.1} \cdot 0.5 \\ &= 0.1033. \end{aligned}$$

Notice that, by change, the price of the standard European call option in Point 4 and the lookback call option in Point 5 coincide.

Solution of Exercise 8

1. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. Solving the three systems delivers

$$\begin{matrix} \mathbb{Q}[f_1^1] = 0.4 \\ \mathbb{Q}[f_2^1] = 0.6 \end{matrix}, \quad \begin{matrix} \mathbb{Q}[f_1^1] = 0.35 \\ \mathbb{Q}[f_2^1] = 0.65 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{Q}[f_1^1] = 0.55 \\ \mathbb{Q}[f_2^1] = 0.45 \end{matrix}.$$

Furthermore,

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.40 \cdot 0.35 = 0.14 \\ \mathbb{Q}[\omega_2] &= 0.40 \cdot 0.65 = 0.26 \\ \mathbb{Q}[\omega_3] &= 0.60 \cdot 0.55 = 0.33 \\ \mathbb{Q}[\omega_4] &= 0.60 \cdot 0.45 = 0.27. \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

2. The European call option on S has terminal payoff

$$c(2) = (S(2) - K)^+ = \begin{cases} (15.6 - 11)^+ = 4.6 & \text{on } \omega_1 \\ (10.8 - 11)^+ = 0 & \text{on } \omega_2 \\ (10.8 - 11)^+ = 0 & \text{on } \omega_3 \\ (7.2 - 11)^+ = 0 & \text{on } \omega_4 \end{cases}.$$

Its no arbitrage price is therefore at $t = 1$

$$c(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1 + r(1)}\middle| \mathcal{P}_1\right] = \begin{cases} \dfrac{4.6 \cdot 0.35 + 0 \cdot 0.65}{1.04} = 1.5481 & \text{if } f_1^1 \\ \dfrac{0 \cdot 0.55 + 0 \cdot 0.45}{1.02} = 0 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} c(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1 + r(0)}\right] \\ &= \frac{1.5481}{1.02} \cdot 0.4 + \frac{0}{1.02} \cdot 0.6 \\ &= 0.6071. \end{aligned}$$

3. The ZCB no arbitrage price at $t = 1$ is

$$S_{ZCB}(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{1 + r(1)}\middle| \mathcal{P}_1\right] = \begin{cases} \dfrac{1}{1.04} = 0.9615 & \text{if } f_1^1 \\ \dfrac{1}{1.02} = 0.9804 & \text{if } f_2^1 \end{cases}$$

The ZCB no arbitrage price at $t = 0$ is

$$S_{ZCB}(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_{ZCB}(1)}{1 + r(0)}\right] = \frac{0.9615 \cdot 0.4 + 0.9804 \cdot 0.6}{1.02} = 0.9537.$$

4. The terminal payoff of the European put option on S is

$$p(2) = (K - S(2))^+ = \begin{cases} (11 - 15.6)^+ = 0 & \text{on } \omega_1 \\ (11 - 10.8)^+ = 0.2 & \text{on } \omega_2 \\ (11 - 10.8)^+ = 0.2 & \text{on } \omega_3 \\ (11 - 7.2)^+ = 3.8 & \text{on } \omega_4 \end{cases}.$$

The payoff of the put option can be replicated by a portfolio consisting of the underlying asset, the call option (on the same asset, with same maturity and strike price) and the ZCB, with weights

$$\begin{cases} \vartheta_{ZCB} = K \\ \vartheta_S = -1 \\ \vartheta_c = 1 \end{cases}$$

where ϑ_{ZCB},ϑ_S and ϑ_c are respectively the weights of the zero coupon bond, the underlying stock and the call option. Thanks to the Law of One Price, the initial price of such a portfolio has to be equal to the no-arbitrage price of the put option:

$$p(0) = KS_{ZCB}(0) - S(0) + c(0) = 1.0986.$$

5. The terminal payoff of the straddle is

$$straddle(K)(2) = c(2) + p(2) = \begin{cases} 4.6 & \text{on } \omega_1 \\ 0.2 & \text{on } \omega_2 \\ 0.2 & \text{on } \omega_3 \\ 3.8 & \text{on } \omega_4 \end{cases}.$$

The straddle can be replicated by buying one call and one put option, as its payoff suggests. Thus, the initial no-arbitrage price of the straddle is

$$straddle(K)\,(0) = c(0) + p(0) = 1.7057.$$

6. The payoff of the call and put options, depending on $k \in (11, 15)$ are

$$c\,(2)\,(k) = (S\,(2) - k)^+ = \begin{cases} (15.6 - k)^+ = 15.6 - k & \text{on } \omega_1 \\ (10.8 - k)^+ = 0 & \text{on } \omega_2 \\ (10.8 - k)^+ = 0 & \text{on } \omega_3 \\ (7.2 - k)^+ = 0 & \text{on } \omega_4 \end{cases}$$

and

$$p\,(2)\,(k) = (k - S\,(2))^+ = \begin{cases} (k - 15.6)^+ = 0 & \text{on } \omega_1 \\ (k - 10.8)^+ = k - 10.8 & \text{on } \omega_2 \\ (11 - 10.8)^+ = k - 10.8 & \text{on } \omega_3 \\ (k - 7.2)^+ = k - 7.2 & \text{on } \omega_4 \end{cases}.$$

Thus, the straddle payoff as a function of $k \in (11, 15)$ is

$$straddle(k)\,(2)\,(k) = c(2)(k) + p(2)(k) = \begin{cases} 15.6 - k & \text{on } \omega_1 \\ k - 10.8 & \text{on } \omega_2 \\ k - 10.8 & \text{on } \omega_3 \\ k - 7.2 & \text{on } \omega_4 \end{cases}.$$

The historical mean squared error in the replication of the target payoff is

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[(X(2) - straddle(k)(2))^2\right] \\ &= \frac{1}{4}\left(2(0.5 - k + 10.8)^2 + (4.1 - 15.6 + k)^2 + (4.1 - k + 7.2)^2\right) \\ &= \frac{1}{4}\left(3(11.3 - k)^2 + (-11.5 + k)^2\right) \\ &= \frac{1}{4}\left(4k^2 - 90.8k + constant\right). \end{aligned}$$

Taking the first derivative with respect to k of such a function and setting it equal to zero (namely, imposing the First Order Condition, FOC) leads to

$$\frac{\partial}{\partial k}\mathbb{E}^{\mathbb{P}}\left[(X(2) - straddle(k)(2))^2\right] = \frac{1}{4}\,(8k - 90.8) = 0$$

which is solved by

$$k^* = \frac{90.8}{8} = 11.35.$$

Since the second derivative of the objective function is always positive,

$$\frac{\partial^2}{\partial^2 k}\mathbb{E}^{\mathbb{P}}\left[(X(2) - straddle(k)(2))^2\right] = \frac{1}{4}\cdot 8 = 2,$$

$k^* = 11.35$ is the minimum we were looking for.

Solution of Exercise 9

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.05$$

and at the final date $T = 2$

$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.05 \cdot 1.08 = 1.134 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.05 \cdot 1.05 = 1.1025. \end{aligned}$$

2. The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det\begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det\begin{bmatrix} 1.05 & 15 \\ 1.05 & 9 \end{bmatrix} = -6.3 \neq 0,$$

for $m_{1,1}$ we have

$$\det\begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det\begin{bmatrix} 1.134 & 22.5 \\ 1.134 & 13.5 \end{bmatrix} = -10.206 \neq 0$$

and for $m_{1,2}$ we have

$$\det\begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det\begin{bmatrix} 1.1025 & 13.5 \\ 1.1025 & 8.1 \end{bmatrix} = -5.9535 \neq 0.$$

3. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)}\left\{S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)}\left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 10 = \frac{1}{1.05}\left\{15 \cdot \mathbb{Q}[f_1^1] + 9 \cdot \mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.25 \\ \mathbb{Q}[f_2^1] &= 0.75 \end{aligned}$$

The second one can be rewritten as

$$\begin{cases} 15 = \frac{1}{1.08}\left\{22.5 \cdot \mathbb{Q}[\omega_1|f_1^1] + 13.5 \cdot \mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.3 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.7 \end{aligned}$$

and, finally, the last system can be rewritten as

$$\begin{cases} 9 = \frac{1}{1.05}\left\{13.5 \cdot \mathbb{Q}[\omega_3|f_2^1] + 8.1 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.25 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.75. \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{Q}[\omega_1] &= 0.25 \cdot 0.3 = 0.075 \\
\mathbb{Q}[\omega_2] &= 0.25 \cdot 0.7 = 0.175 \\
\mathbb{Q}[\omega_3] &= 0.75 \cdot 0.25 = 0.1875 \\
\mathbb{Q}[\omega_4] &= 0.75 \cdot 0.75 = 0.5625
\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The European put option on S has terminal payoff

$$p(2) = (K - S(2))^+ = \begin{cases} (18.5 - 22.5)^+ = 0 & \text{on } \omega_1 \\ (18.5 - 13.5)^+ = 5 & \text{on } \omega_2, \omega_3 \\ (18.5 - 8.1)^+ = 10.4 & \text{on } \omega_4 \end{cases}.$$

Its no arbitrage price is therefore at $t = 1$

$$p(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{p(2)}{1 + r(1)}\right| \mathcal{P}_1\right] = \begin{cases} \dfrac{0 \cdot 0.3 + 5 \cdot 0.7}{1.08} = 3.2407 & \text{if } f_1^1 \\ \dfrac{5 \cdot 0.25 + 10.4 \cdot 0.75}{1.05} = 8.619 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned}
p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1 + r(0)}\right] \\
&= \frac{3.2407}{1.05} \cdot 0.25 + \frac{8.619}{1.05} \cdot 0.75 \\
&= 6.928.
\end{aligned}$$

5. The derivative cashflow is

$$X(1) = (S(1) - S(0))^+ = \begin{cases} (15 - 10)^+ = 5 & \text{if } f_1^1 \\ (9 - 10)^+ = 0 & \text{if } f_2^1 \end{cases}$$

at $t = 1$ and

$$X(2) = \left(K - \min_{t=0,1,2} S(t)\right)^+ = \begin{cases} (18.5 - 10)^+ = 8.5 & \text{on } \omega_1 \\ (18.5 - 10)^+ = 8.5 & \text{on } \omega_2 \\ (18.5 - 9)^+ = 9.5 & \text{on } \omega_3 \\ (18.5 - 8.1)^+ = 10.4 & \text{on } \omega_4 \end{cases}$$

at $T = 2$. Therefore the no arbitrage price is

$$S_X(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1 + r(1)}\right| \mathcal{P}_1\right] = \begin{cases} \dfrac{8.5 \cdot 0.3 + 8.5 \cdot 0.7}{1.08} = 7.8704 & \text{if } f_1^1 \\ \dfrac{9.5 \cdot 0.25 + 10.4 \cdot 0.75}{1.05} = 9.6905 & \text{if } f_2^1 \end{cases}$$

and

$$\begin{aligned}
S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1) + X(1)}{1 + r(0)}\right] \\
&= \frac{7.8704 + 5}{1.05} \cdot 0.25 + \frac{9.6905 + 0}{1.05} \cdot 0.75 \\
&= 9.9862.
\end{aligned}$$

Solution of Exercise 10

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.06$$

and at the final date $T = 2$

$$\begin{array}{c} B(2)(\omega_1) = B(2)(\omega_2) = 1.06 \cdot 1.08 = 1.1448 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.06 \cdot 1.05 = 1.113. \end{array}$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 1 = \frac{1}{1.06} \left\{ 1.25 \cdot \mathbb{Q}[f_1^1] + 0.85 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{array}{rcl} \mathbb{Q}[f_1^1] & = & 0.525 \\ \mathbb{Q}[f_2^1] & = & 0.475 \end{array}$$

The second system can be rewritten as

$$\begin{cases} 1.25 = \frac{1}{1.08} \left\{ 1.5625 \cdot \mathbb{Q}[\omega_1|f_1^1] + 1.0625 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{array}{rcl} \mathbb{Q}[\omega_1|f_1^1] & = & 0.575 \\ \mathbb{Q}[\omega_2|f_1^1] & = & 0.425 \end{array}$$

and the last system which can be rewritten as

$$\begin{cases} 0.85 = \frac{1}{1.05} \left\{ 1.0625 \cdot \mathbb{Q}[\omega_3|f_2^1] + 0.7225 \cdot \mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

is solved by

$$\begin{array}{rcl} \mathbb{Q}[\omega_3|f_2^1] & = & 0.5 \\ \mathbb{Q}[\omega_4|f_2^1] & = & 0.5 \end{array}$$

Therefore

$$\begin{array}{rcl} \mathbb{Q}[\omega_1] & = & 0.525 \cdot 0.575 = 0.30188 \\ \mathbb{Q}[\omega_2] & = & 0.525 \cdot 0.425 = 0.22312 \\ \mathbb{Q}[\omega_3] & = & 0.475 \cdot 0.5 = 0.2375 \\ \mathbb{Q}[\omega_4] & = & 0.475 \cdot 0.5 = 0.2375. \end{array}$$

Since there exists a unique risk neutral probability measure, the market is both arbitrage free and complete (by the 2^{nd} FTAP).

3. The European put option on S has terminal payoff

$$c(2) = (S(2) - K)^{+} = \begin{cases} (1.5625 - 1.0125)^{+} = 0.55 & \text{on } \omega_1 \\ (1.0625 - 1.0125)^{+} = 0.05 & \text{on } \omega_2, \omega_3 \\ (0.7225 - 1.0125)^{+} = 0 & \text{on } \omega_4 \end{cases}$$

Its no arbitrage price is therefore at $t = 1$

$$c(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1 + r(1)}\middle|\mathcal{P}_1\right] = \begin{cases} \dfrac{0.55 \cdot 0.575 + 0.05 \cdot 0.425}{1.08} = 0.3125 & \text{if } f_1^1 \\ \\ \dfrac{0.05 \cdot 0.5 + 0 \cdot 0.5}{1.05} = 0.02381 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} c(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1 + r(0)}\right] \\ &= \frac{0.3125}{1.06} \cdot 0.525 + \frac{0.02381}{1.06} \cdot 0.475 \\ &= 0.16545. \end{aligned}$$

4. The derivative cashflow is

$$X(t) = \begin{cases} 1 & \text{if } \dfrac{S(t) - S(t-1)}{S(t-1)} > \dfrac{B(t) - B(t-1)}{B(t-1)} \\ \\ 0 & \text{otherwise} \end{cases}$$

We observe that

$$\frac{B(t) - B(t-1)}{B(t-1)} = r(t-1).$$

Therefore we compare

$$\begin{cases} \dfrac{S(1) - S(0)}{S(0)} = 0.25 > 0.06 = r(0) & \text{if } f_1^1 \\ \\ \dfrac{S(1) - S(0)}{S(0)} = -0.15 < 0.06 = r(0) & \text{if } f_2^1 \end{cases}$$

that delivers

$$X(1) = \begin{cases} 1 & \text{if } f_1^1 \\ \\ 0 & \text{if } f_2^1 \end{cases}$$

Similarly, at $T = 2$ we get

$$\begin{cases} \dfrac{S(2) - S(1)}{S(1)} = 0.25 > 0.08 = r(1)(f_1^1) & \text{on } \omega_1 \\ \dfrac{S(2) - S(1)}{S(1)} = -0.15 < 0.08 = r(1)(f_1^1) & \text{on } \omega_2 \\ \dfrac{S(2) - S(1)}{S(1)} = 0.25 > 0.05 = r(1)(f_2^1) & \text{on } \omega_3 \\ \dfrac{S(2) - S(1)}{S(1)} = -0.15 < 0.05 = r(1)(f_2^1) & \text{on } \omega_4 \end{cases}$$

that delivers

$$X(2) = \begin{cases} 1 & \text{on } \omega_1 \\ 0 & \text{on } \omega_2 \\ 1 & \text{on } \omega_3 \\ 0 & \text{on } \omega_4 \end{cases}.$$

Therefore, the no arbitrage price is

$$S_X(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right] = \begin{cases} \dfrac{1\cdot 0.575 + 0\cdot 0.425}{1.08} = 0.53241 & \text{if } f_1^1 \\ \dfrac{1\cdot 0.5 + 0\cdot (0.5)}{1.05} = 0.47619 & \text{if } f_2^1 \end{cases}$$

and

$$\begin{aligned} S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1) + X(1)}{1+r(0)}\right] \\ &= \frac{0.53241+1}{1.06}\cdot 0.525 + \frac{0.47619+0}{1.06}\cdot(1-0.525) \\ &= 0.97236. \end{aligned}$$

Solution of Exercise 11

1. The risk-neutral probabilities for the three submarkets are

$$\begin{array}{l} \mathbb{Q}[f_1^1] = 0.55 \\ \mathbb{Q}[f_2^1] = 0.45 \end{array}, \quad \begin{array}{l} \mathbb{Q}[\omega_1|f_1^1] = 0.6 \\ \mathbb{Q}[\omega_2|f_1^1] = 0.4 \end{array} \quad \text{and} \quad \begin{array}{l} \mathbb{Q}[\omega_3|f_2^1] = 0.5 \\ \mathbb{Q}[\omega_4|f_2^1] = 0.5 \end{array}$$

that deliver

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.55\cdot 0.6 = 0.33 \\ \mathbb{Q}[\omega_2] &= 0.55\cdot 0.4 = 0.22 \\ \mathbb{Q}[\omega_3] &= 0.45\cdot 0.5 = 0.225 \\ \mathbb{Q}[\omega_4] &= 0.45\cdot 0.5 = 0.225 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

The terminal payoff of the *forward-starting call option is*

$$c_{FS}(2) = (S(2) - S(1))^+ = \begin{cases} (14.884 - 12.2)^+ = 2.684 & \text{on } \omega_1 \\ (10.004 - 12.2)^+ = 0 & \text{on } \omega_2 \\ (10.004 - 8.004)^+ = 2 & \text{on } \omega_3 \\ (6.724 - 8.004)^+ = 0 & \text{on } \omega_4 \end{cases}$$

Its no arbitrage price is therefore at $t = 1$

$$c_{FS}(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c_{FS}(2)}{1+r(1)}\right|\mathcal{P}_1\right] = \begin{cases} \dfrac{2.684\cdot 0.6 + 0\cdot 0.4}{1.06} = 1.5192 & \text{if } f_1^1 \\ \dfrac{2\cdot 0.5 + 0\cdot 0.5}{1.02} = 0.98039 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} c_{FS}(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c_{FS}(1)}{1+r(0)}\right] \\ &= \frac{1.5219}{1.04}\cdot 0.55 + \frac{0.98039}{1.04}\cdot 0.45 \\ &= 1.2275. \end{aligned}$$

2. The buy-and-hold strategies $\vartheta = (\vartheta_0, \vartheta_1)$ whose initial cost is $V_\vartheta(0) = c_{FS}(0)$ satisfy

$$\vartheta_0(0) + \vartheta_1(0)\cdot 10 = c_{FS}(0) = 1.2275$$

that yields

$$\vartheta_0(0) = c_{FS}(0) - \vartheta_1(0)\cdot 10,$$

and

$$\begin{aligned} V_\vartheta(2) &= \vartheta_0(0)B(2) + \vartheta_1(0)S(2) \\ &= \left(c_{FS}(0) - \vartheta_1(0)\cdot 10\right)B(2) + \vartheta_1(0)S(2). \end{aligned}$$

The historical expected values of $B(2)$ and $S(2)$ are

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[B(2)\right] &= \frac{1}{4}\left(2\cdot 1.102\,4 + 2\cdot 1.0608\right) = 1.0816 \\ \mathbb{E}^{\mathbb{P}}\left[S(2)\right] &= \frac{1}{4}\left(14.884 + 2\cdot 10.004 + 6.724\right) = 10.404. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[V_\vartheta(2)\right] &= \mathbb{E}^{\mathbb{P}}\left[\left(c_{FS}(0) - \vartheta_1(0)\cdot 10\right)B(2) + \vartheta_1(0)S(2)\right] \\ &= \left(c_{FS}(0) - \vartheta_1(0)\cdot 10\right)\mathbb{E}^{\mathbb{P}}\left[B(2)\right] + \vartheta_1(0)\mathbb{E}^{\mathbb{P}}\left[S(2)\right] \\ &= \left(c_{FS}(0) - \vartheta_1(0)\cdot 10\right)\cdot 1.0816 + \vartheta_1(0)\cdot 10.404. \end{aligned}$$

The historical expected value of the terminal payoff of the forward starting call is

$$\mathbb{E}^{\mathbb{P}}\left[c_{FS}(2)\right] = \frac{1}{4}\left(2.684 + 0 + 2 + 0\right) = 1.171.$$

Imposing $\mathbb{E}^{\mathbb{P}}\left[c_{FS}(2)\right] = \mathbb{E}^{\mathbb{P}}\left[V_\vartheta(2)\right]$ we get

$$1.171 = \left(1.2275 - \vartheta_1(0)\cdot 10\right)\cdot 1.0816 + \vartheta_1(0)\cdot 10.404$$

which delivers

$$\vartheta_1(0) = 0.3803.$$

From the initial constraint we recover

$$\begin{aligned} \vartheta_0(0) &= c_{FS}(0) - \vartheta_1(0)\cdot 10 \\ &= 1.227 - 0.3881\cdot 10 = -2.5801. \end{aligned}$$

The value of the riskless security at $T = 2$ is

$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.04\cdot 1.06 = 1.1024 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.04\cdot 1.02 = 1.0608. \end{aligned}$$

Then, the terminal value of the strategy is

$$\begin{aligned} V_\vartheta(2) &= (-2.5801)\,B(2) + 1.1320\cdot S(2) = \\ &= \begin{cases} (-2.5801)\cdot 1.1024 + 1.132\cdot 14.884 = 2.8233 > 2.684 & \text{on } \omega_1 \\ (-2.5801)\cdot 1.1024 + 1.132\cdot 10.004 = 0.96504 > 0 & \text{on } \omega_2 \\ (-2.5801)\cdot 1.0608 + 1.132\cdot 10.004 = 1.0724 < 2 & \text{on } \omega_3 \\ (-2.5801)\cdot 1.0608 + 1.132\cdot 6.724 = -0.17660 < 0 & \text{on } \omega_4 \end{cases} \end{aligned}$$

Thus, $V_\vartheta(2)$ is above the final option payoff $c_{FS}(2)$ on ω_1 and ω_2, and is below is the final option payoff $c_{FS}(2)$ on ω_3 and ω_4. Finally, we can compute the historical probability of this event, namely we can compute

$$\mathbb{P}\left[V_\vartheta(2) > c_{FS}(2)\right] = \mathbb{P}(\omega_1) + \mathbb{P}(\omega_2) = 0.5.$$

Observe that the risk neutral probability of the same event is a little bit different

$$\mathbb{Q}\left[V_\vartheta(2) > c_{FS}(2)\right] = \mathbb{Q}(\omega_1) + \mathbb{Q}(\omega_2) = 0.33 + 0.22 = 0.55.$$

Solution of Exercise 12

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.03$$

and at the final date $T = 2$

$$\begin{array}{c} B(2)(\omega_1) = B(2)(\omega_2) = 1.03 \cdot 1.06 = 1.0918 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.03 \cdot 1.04 = 1.0712. \end{array}$$

The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det \begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det \begin{bmatrix} 1.03 & 13 \\ 1.03 & 7 \end{bmatrix} = -6.18 \neq 0,$$

for $m_{1,1}$ we have

$$\det \begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det \begin{bmatrix} 1.0918 & 18.2 \\ 1.0918 & 7.15 \end{bmatrix} = -12.064 \neq 0$$

and for $m_{1,2}$ we have

$$\det \begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det \begin{bmatrix} 1.0712 & 11.2 \\ 1.0712 & 6.3 \end{bmatrix} = -5.2489 \neq 0.$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 10 = \frac{1}{1.03} \left\{ 13 \cdot \mathbb{Q}[f_1^1] + 7 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.55 \\ \mathbb{Q}[f_2^1] &= 0.45. \end{aligned}$$

The second system can be rewritten as

$$\begin{cases} 13 = \frac{1}{1.06} \left\{ 18.2 \cdot \mathbb{Q}[\omega_1|f_1^1] + 7.15 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.6 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.4 \end{aligned}$$

whereas the third system can be rewritten as

$$\begin{cases} 7 = \frac{1}{1.04}\left\{11.2 \cdot \mathbb{Q}[\omega_3|f_2^1] + 6.3 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.2 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.8. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.55 \cdot 0.6 = 0.33 \\ \mathbb{Q}[\omega_2] &= 0.55 \cdot 0.4 = 0.22 \\ \mathbb{Q}[\omega_3] &= 0.45 \cdot 0.2 = 0.09 \\ \mathbb{Q}[\omega_4] &= 0.45 \cdot 0.8 = 0.36. \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

3. The European call option on S has terminal payoff

$$c(2) = (S(2) - K_{call})^+ = \begin{cases} (18.2 - 7.15)^+ = 11.05 & \text{on } \omega_1 \\ (7.15 - 7.15)^+ = 0 & \text{on } \omega_2 \\ (11.2 - 7.15)^+ = 4.05 & \text{on } \omega_3 \\ (6.3 - 7.15)^+ = 0 & \text{on } \omega_4 \end{cases}.$$

Its no arbitrage price is therefore at $t = 1$

$$c(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{c(2)}{1 + r(1)}\bigg|\mathcal{P}_1\right] = \begin{cases} \dfrac{11.05 \cdot 0.6 + 0 \cdot 0.4}{1.06} = 6.2547 & \text{if } f_1^1 \\ \\ \dfrac{4.05 \cdot 0.2 + 0 \cdot 0.8}{1.04} = 0.77885 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} c(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1 + r(0)}\right] \\ &= \frac{6.2547}{1.03} \cdot 0.55 + \frac{0.77885}{1.03} \cdot 0.45 \\ &= 3.6802. \end{aligned}$$

4. The strike price K_{put} is given by

$$p(2)(\omega_2) = (K_{put} - S(2))^+(\omega_2) = (K_{put} - 7.15)^+ = 4.05$$

that delivers

$$K_{put} = 4.05 + 7.15 = 11.2.$$

The terminal payoff of the put option is

$$p(2) = (K_{put} - S(2))^+ = \begin{cases} (11.2 - 18.2)^+ = 0 & \text{on } \omega_1 \\ (11.2 - 7.15)^+ = 4.05 & \text{on } \omega_2 \\ (11.2 - 11.2)^+ = 0 & \text{on } \omega_3 \\ (11.2 - 6.3)^+ = 4.9 & \text{on } \omega_4 \end{cases}.$$

The initial price of the put option is

$$
\begin{aligned}
p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{B(2)}\right] \\
&= \frac{p(2)}{B(2)}(\omega_1)\cdot\mathbb{Q}[\omega_1] + \frac{p(2)}{B(2)}(\omega_2)\cdot\mathbb{Q}[\omega_2] + \frac{p(2)}{B(2)}(\omega_3)\cdot\mathbb{Q}[\omega_3] + \frac{p(2)}{B(2)}(\omega_4)\cdot\mathbb{Q}[\omega_4] \\
&= \frac{0}{1.0918}\cdot 0.33 + \frac{4.05}{1.0918}\cdot 0.22 + \frac{0}{1.0712}\cdot 0.09 + \frac{4.9}{1.0712}\cdot 0.36 \\
&= 2.4628.
\end{aligned}
$$

5. The realized variance at $T=2$ is

$$
RV(2) = \begin{cases}
\left(\ln\frac{S(1)(f_1^1)}{S(0)}\right)^2 + \left(\ln\frac{S(2)(\omega_1)}{S(1)(f_1^1)}\right)^2 = \left(\ln\frac{13}{10}\right)^2 + \left(\ln\frac{18.2}{13}\right)^2 = 0.18205 & \text{on } \omega_1 \\
\left(\ln\frac{S(1)(f_1^1)}{S(0)}\right)^2 + \left(\ln\frac{S(2)(\omega_2)}{S(1)(f_1^1)}\right)^2 = \left(\ln\frac{13}{10}\right)^2 + \left(\ln\frac{7.15}{13}\right)^2 = 0.42624 & \text{on } \omega_2 \\
\left(\ln\frac{S(1)(f_2^1)}{S(0)}\right)^2 + \left(\ln\frac{S(2)(\omega_3)}{S(1)(f_2^1)}\right)^2 = \left(\ln\frac{7}{10}\right)^2 + \left(\ln\frac{11.2}{7}\right)^2 = 0.34812 & \text{on } \omega_3 \\
\left(\ln\frac{S(1)(f_2^1)}{S(0)}\right)^2 + \left(\ln\frac{S(2)(\omega_4)}{S(1)(f_2^1)}\right)^2 = \left(\ln\frac{7}{10}\right)^2 + \left(\ln\frac{6.3}{7}\right)^2 = 0.13832 & \text{on } \omega_4
\end{cases}.
$$

Therefore, the terminal payoff of the exotic derivative is

$$
X(2) = \begin{cases}
0 & \text{on } \omega_1 \\
\frac{1}{3}\left(S(0) + S(1)(f_1^1) + S(2)(\omega_2)\right) = \frac{1}{3}(10+13+7.15) = 10.05 & \text{on } \omega_2 \\
\frac{1}{3}\left(S(0) + S(1)(f_2^1) + S(2)(\omega_3)\right) = \frac{1}{3}(10+7+11.2) = 9.4 & \text{on } \omega_3 \\
0 & \text{on } \omega_4
\end{cases}
$$

The price of this exotic derivative is

$$
\begin{aligned}
X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{X(2)}{B(2)}\right] \\
&= \frac{X(2)}{B(2)}(\omega_1)\cdot\mathbb{Q}[\omega_1] + \frac{X(2)}{B(2)}(\omega_2)\cdot\mathbb{Q}[\omega_2] + \frac{X(2)}{B(2)}(\omega_3)\cdot\mathbb{Q}[\omega_3] + \frac{X(2)}{B(2)}(\omega_4)\cdot\mathbb{Q}[\omega_4] \\
&= \frac{0}{1.0918}\cdot 0.33 + \frac{10.05}{1.0918}\cdot 0.22 + \frac{9.4}{1.0712}\cdot 0.09 + \frac{0}{1.0712}\cdot 0.36 \\
&= 2.8149.
\end{aligned}
$$

6. Given the historical probabilities of the four scenarios ω_k with $k=1,...,4$, the only way to replicate the exotic derivative of Point 5 with 90% historical probability is to replicate it on $\omega_1, \omega_3, \omega_4$ (and not on ω_2).

   Let $\pi = (\vartheta_S, \vartheta_c, \vartheta_p)$ be the buy and hold portofolio composed by ϑ_S units of the underlying S, ϑ_c units of the call option of Point 3 and ϑ_p units of the put option of Point 4. The replication conditions read

$$
\begin{cases}
S(2)(\omega_1)\vartheta_S + c(2)(\omega_1)\vartheta_c + p(2)(\omega_1)\vartheta_p = X(2)(\omega_1) \\
S(2)(\omega_3)\vartheta_S + c(2)(\omega_3)\vartheta_c + p(2)(\omega_3)\vartheta_p = X(2)(\omega_3) \\
S(2)(\omega_4)\vartheta_S + c(2)(\omega_4)\vartheta_c + p(2)(\omega_4)\vartheta_p = X(2)(\omega_4)
\end{cases}
$$

   namely,

$$
\begin{cases}
18.2\vartheta_S + 11.05\vartheta_c + 0\cdot\vartheta_p = 0 \\
11.2\vartheta_S + 4.05\vartheta_c + 0\cdot\vartheta_p = 9.4 \\
6.3\vartheta_S + 0\cdot\vartheta_c + 4.9\vartheta_p = 0
\end{cases}.
$$

   The solution is

$$
\vartheta_S = 2.0753, \quad \vartheta_c = -3.4182 \quad \text{and} \quad \vartheta_p = -2.6683.
$$

   By construction the value of the portfolio on $\omega_1, \omega_3, \omega_4$ conincides with $X(2)$. On ω_2

$$
\begin{aligned}
V_\pi(2) &= S(2)(\omega_2)\vartheta_S + c(2)(\omega_2)\vartheta_c + p(2)(\omega_2)\vartheta_p \\
&= 7.15\cdot 2.0753 + 0\cdot(-3.4182) + 4.05\cdot(-2.6683) \\
&= 4.0318 < 10.05 = X(2)(\omega_2)
\end{aligned}
$$

The initial value of the portfolio is

$$\begin{aligned} V_\pi(0) &= S(0)\vartheta_S + c(0)\vartheta_c + p(0)\vartheta_p \\ &= 10 \cdot 2.0753 + 3.6802 \cdot (-3.4182) + 2.4628 \cdot (-2.6683) \\ &= 1.6019 < 2.8149 = X(0). \end{aligned}$$

Solution of Exercise 13

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.02$$

and at the final date $T = 2$

$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.02 \cdot 1.08 = 1.1016 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.02 \cdot 1.02 = 1.0404. \end{aligned}$$

2. The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det \begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1) \end{bmatrix} = \det \begin{bmatrix} 1.02 & 21 \\ 1.02 & 18 \end{bmatrix} = -3.06 \neq 0,$$

for $m_{1,1}$ we have

$$\det \begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2) \end{bmatrix} = \det \begin{bmatrix} 1.1016 & 31.5 \\ 1.1016 & 18.9 \end{bmatrix} = -13.88 \neq 0$$

and for $m_{1,2}$ we have

$$\det \begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4) \end{bmatrix} = \det \begin{bmatrix} 1.0404 & 21.6 \\ 1.0404 & 14.4 \end{bmatrix} = -7.4909 \neq 0.$$

3. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The three systems, that can be rewritten respectively as

$$\begin{cases} 20 = \frac{1}{1.02} \left\{ 21 \cdot \mathbb{Q}[f_1^1] + 18 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

$$\begin{cases} 21 = \frac{1}{1.08} \left\{ 31.5 \cdot \mathbb{Q}[\omega_1|f_1^1] + 18.9 \cdot \mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

$$\begin{cases} 18 = \frac{1}{1.02} \left\{ 21.6 \cdot \mathbb{Q}[\omega_3|f_2^1] + 14.4 \cdot \mathbb{Q}[\omega_4|f_2^1] \right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

are solved by

$$\begin{array}{l}\mathbb{Q}[f_1^1]=0.8\\ \mathbb{Q}[f_2^1]=0.2\end{array}, \quad \begin{array}{l}\mathbb{Q}[f_1^1]=0.3\\ \mathbb{Q}[f_2^1]=0.7\end{array} \quad \text{and} \quad \begin{array}{l}\mathbb{Q}[f_1^1]=0.55\\ \mathbb{Q}[f_2^1]=0.45\end{array}.$$

As a consequence,

$$\begin{array}{rcl}\mathbb{Q}[\omega_1] & = & 0.8\cdot 0.3=0.24\\ \mathbb{Q}[\omega_2] & = & 0.8\cdot 0.7=0.56\\ \mathbb{Q}[\omega_3] & = & 0.2\cdot 0.55=0.11\\ \mathbb{Q}[\omega_4] & = & 0.2\cdot 0.45=0.09\end{array}$$

that ensures also that the market is free of arbitrage opportunities (due to the 2^{nd} FTAP).

4. The cashflows of the cliquet call option are

$$X(2)=(S(2)-S(1))^{+}=\begin{cases}(31.5-21)^{+}=10.5 & \text{on } \omega_1\\ (18.9-21)^{+}=0 & \text{on } \omega_2\\ (21.6-18)^{+}=3.6 & \text{on } \omega_3\\ (14.4-18)^{+}=0 & \text{on } \omega_4\end{cases}$$

at $T=2$ and

$$X(1)=(S(1)-S(0))^{+}=\begin{cases}(21-20)^{+}=1 & \text{on } f_1^1\\ (18-20)^{+}=0 & \text{on } f_2^1\end{cases}$$

at $t=1$. The time 1 non arbitrage price is:

$$S_X(1)=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(2)}{1+r(1)}\right|\mathcal{P}_1\right]=\begin{cases}\dfrac{10.5\cdot 0.3+0\cdot 0.7}{1.08}=2.9167 & \text{if } f_1^1\\[2ex] \dfrac{3.6\cdot 0.55+0\cdot 0.45}{1.02}=1.9412 & \text{if } f_2^1\end{cases}$$

while the no-arbitrage price at inception is:

$$\begin{array}{rcl}S_X(0) & = & \mathbb{E}^{\mathbb{Q}}\left[\dfrac{S_X(1)+X(1)}{1+r(0)}\right]\\ & = & \dfrac{2.9167+1}{1.02}\cdot 0.8+\dfrac{1.9412+0}{1.02}\cdot 0.2\\ & = & 3.4525.\end{array}$$

5. The delivery price $F_{0,2}$ of the forward contract with maturity $T=2$ is such that

$$\begin{array}{rcl}\mathbb{E}^{\mathbb{Q}}\left[\dfrac{S(2)-F_{0,2}}{B(2)}\right] & = & 0\\ \mathbb{E}^{\mathbb{Q}}\left[\dfrac{S(2)}{B(2)}\right]-\mathbb{E}^{\mathbb{Q}}\left[\dfrac{F_{0,2}}{B(2)}\right] & = & 0.\end{array}$$

As $\left\{\frac{S(t)}{B(t)}\right\}_{t=0,1,2}$ is a $\mathbb{Q}-$martingale,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)}{B(2)}\right]=S(0).$$

Therefore, as the delivery price $F_{0,2}$ is a constant,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S(2)}{B(2)}\right]-\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{0,2}}{B(2)}\right]=S(0)-F_{0,2}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right],$$

that yields

$$F_{0,2}=\frac{S(0)}{\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right]}$$

Notice that, since the interest rate is not constant and, therefore, $\{B(t)\}_{t=0,1,2}$ is not deterministic,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] \neq \frac{1}{B(2)}.$$

Furthermore, notice that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] \neq \frac{1}{\mathbb{E}^{\mathbb{Q}}[B(2)]}.$$

Therefore, we have to explicitly compute

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] &= \frac{1}{B(2)(f_1^1)}\cdot\mathbb{Q}[f_1^1] + \frac{1}{B(2)(f_2^1)}\cdot\mathbb{Q}[f_2^1] \\
&= \frac{1}{1.1016}\cdot 0.8 + \frac{1}{1.0404}\cdot 0.2 = 0.91845.
\end{aligned}$$

Finally, we can get the required delivery price

$$F_{0,2} = \frac{20}{0.918\,45} = 21.776.$$

Consider the new forward contract on S with maturity $T=2$ introduced in the market in $t=1$ with delivery price $F_{1,2}$. We have to determine $F_{1,2}$ such that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)-F_{1,2}}{1+r(1)}\right|\mathcal{P}_1\right] &= 0 \\
\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)}{1+r(1)}\right|\mathcal{P}_1\right] - \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{F_{1,2}}{1+r(1)}\right|\mathcal{P}_1\right] &= 0 \\
S(1) - \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{F_{1,2}}{1+r(1)}\right|\mathcal{P}_1\right] &= 0
\end{aligned}$$

as, again, $1+r(1) = B(1)$ and $\left\{\frac{S(t)}{B(t)}\right\}_{t=0,1,2}$ is a $\mathbb{Q}-$martingale. Moreover $\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{F_{1,2}}{1+r(1)}\right|\mathcal{P}_1\right] = \frac{F_{1,2}}{1+r(1)}$, as $r(1)$ is known at date $t=1$. Hence we get

$$\frac{F_{1,2}}{1+r(1)} = S(1)$$

and finally,

$$F_{1,2} = S(1)(1+r(1)) = \begin{cases} 21\cdot 1.08 = 22.68 & \text{if } f_1^1 \\ \\ 18\cdot 1.02 = 18.36 & \text{if } f_2^1 \end{cases}.$$

6. Given the values of $S(2)$, we have that $S(2) \geq F_{0,2}$ on ω_1 only. Therefore, the conditional risk neutral probability at $t=1$ that $S(2)\geq F_{0,2}$ is

$$\mathbb{Q}\left[S(2)\geq F_{0,2}\right|\mathcal{P}_1] = \begin{cases} \mathbb{Q}[\omega_1|f_1^1] = 0.3 & \text{if } f_1^1 \\ \\ 0 & \text{if } f_2^1 \end{cases}$$

The risk neutral probability that $S(2)\geq F_{0,2}$ is

$$\begin{aligned}
\mathbb{Q}\left[S(2)\geq F_{0,2}\right] &= \mathbb{Q}[\omega_1|f_1^1]\cdot\mathbb{Q}[f_1^1] \\
&= \mathbb{Q}[\omega_1] = 0.24.
\end{aligned}$$

Solution of Exercise 14

1. The prices of security B are $B(0)=1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.02$$

and at the final date $T=2$

$$\begin{gathered}
B(2)(\omega_1) = B(2)(\omega_2) = B(2)(\omega_3) = 1.05\cdot 1.01 = 1.0605 \\
B(2)(\omega_4) = B(2)(\omega_5) = 1.05\cdot 1.03 = 1.0815.
\end{gathered}$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] + S(2)(\omega_3)\mathbb{Q}[\omega_3|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] + \mathbb{Q}[\omega_3|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1], \mathbb{Q}[\omega_3|f_1^1] > 0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \left\{ S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1] + S(2)(\omega_5)\mathbb{Q}[\omega_5|f_2^1] \right\} \\ \mathbb{Q}[\omega_4|f_2^1] + \mathbb{Q}[\omega_5|f_2^1] = 1 \\ \mathbb{Q}[\omega_4|f_2^1], \mathbb{Q}[\omega_5|f_2^1] > 0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 10 = \frac{1}{1.02} \left\{ 12 \cdot \mathbb{Q}[f_1^1] + 8.4 \cdot \mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.5 \\ \mathbb{Q}[f_2^1] &= 0.5 \end{aligned}$$

The second system can be rewritten as

$$\begin{cases} 12 = \frac{1}{1.01} \left\{ 19.393 \cdot \mathbb{Q}[\omega_1|f_1^1] + 14.544 \cdot \mathbb{Q}[\omega_2|f_1^1] + 9.696 \cdot \mathbb{Q}[\omega_3|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] + \mathbb{Q}[\omega_3|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1], \mathbb{Q}[\omega_3|f_1^1] > 0 \end{cases} .$$

As $\mathbb{Q}[\omega_3|f_1^1] = 1 - \mathbb{Q}[\omega_1|f_1^1] - \mathbb{Q}[\omega_2|f_1^1]$, from the first equation we get

$$\begin{aligned} 12 &= 19.2 \cdot \mathbb{Q}[\omega_1|f_1^1] + 14.4 \cdot \mathbb{Q}[\omega_2|f_1^1] + 9.6 \cdot \left(1 - \mathbb{Q}[\omega_1|f_1^1] - \mathbb{Q}[\omega_2|f_1^1]\right) \\ 2.4 &= 9.6 \cdot \mathbb{Q}[\omega_1|f_1^1] + 4.8 \cdot \mathbb{Q}[\omega_2|f_1^1] \\ \mathbb{Q}[\omega_2|f_1^1] &= \frac{\left(2.4 - 9.6 \cdot \mathbb{Q}[\omega_1|f_1^1]\right)}{4.8} \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.5 - 2 \cdot \mathbb{Q}[\omega_1|f_1^1]. \end{aligned}$$

Labelling $\mathbb{Q}[\omega_1|f_1^1]$ by q_1 we get..

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= q_1 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.5 - 2 \cdot q_1 \\ \mathbb{Q}[\omega_3|f_1^1] &= 1 - q_1 - \mathbb{Q}[\omega_2|f_1^1] \\ &= 0.5 + q_1. \end{aligned}$$

As we also must have $\mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1], \mathbb{Q}[\omega_3|f_1^1] > 0$, the constraints on the parameter q_1 are

$$\begin{cases} q_1 > 0 \\ 0.5 - 2 \cdot q_1 > 0 \\ 0.5 + q_1 > 0 \end{cases}$$

$$\begin{cases} q_1 > 0 \\ q_1 < 0.25 \\ q_1 > -0.5 \end{cases}$$

namely $q_1 \in (0, 0.25)$ .Finally, the last system can be rewritten as

$$\begin{cases} 8.4 = \frac{1}{1.03} \left\{ 14.544 \cdot \mathbb{Q}[\omega_4 | f_2^1] + 7.179 \cdot \mathbb{Q}[\omega_5 | f_2^1] \right\} \\ \mathbb{Q}[\omega_4 | f_2^1] + \mathbb{Q}[\omega_5 | f_2^1] = 1 \\ \mathbb{Q}[\omega_4 | f_2^1], \mathbb{Q}[\omega_5 | f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_4 | f_2^1] &= 0.2 \\ \mathbb{Q}[\omega_5 | f_2^1] &= 0.8. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.5 \cdot q_1 \\ \mathbb{Q}[\omega_2] &= 0.5 \cdot (0.5 - 2 \cdot q_1) = 0.25 - q_1 \\ \mathbb{Q}[\omega_3] &= 0.5 \cdot (0.5 + q_1) = 0.25 + 0.5 \cdot q_1 \\ \mathbb{Q}[\omega_4] &= 0.5 \cdot 0.2 = 0.1 \\ \mathbb{Q}[\omega_5] &= 0.5 \cdot 0.8 = 0.4 \end{aligned}$$

is a risk neutral probability measure as long as $q_1 \in (0, 0.25)$. Since there exists a risk neutral probability measure, the market is arbitrage free (by the 1^{st} FTAP).

3. The market is dynamically incomplete by the 2^{nd} FTAP as there exists infinitely many risk neutral probability measures. Alternatively, the market cannot be complete as the LOP is satisfied (the market is arbitrage free) and the one-period submarket starting from f_1^1 is incomplete.

4. The European call option on S has terminal payoff

$$c\,(2) = (S(2) - K_{call})^+ = \begin{cases} (19.392 - 9.696)^+ = 9.696 & \text{on } \omega_1 \\ (14.544 - 9.696)^+ = 4.848 & \text{on } \omega_2 \\ (9.696 - 9.696)^+ = 0 & \text{on } \omega_3 \\ (14.544 - 9.696)^+ = 4.848 & \text{on } \omega_4 \\ (7.179 - 9.696)^+ = 0 & \text{on } \omega_5 \end{cases}$$

Let's first compute its no arbitrage price in f_1^1. We have

$$\begin{aligned} c\,(1)\left(f_1^1\right) &= \frac{9.696 \cdot \mathbb{Q}[\omega_1 | f_1^1] + 4.848 \cdot \mathbb{Q}[\omega_2 | f_1^1] + 0 \cdot \mathbb{Q}[\omega_3 | f_1^1]}{1.01} \\ &= \frac{9.696 \cdot q_1 + 4.848 \cdot (0.5 - 2 \cdot q_1)}{1.01} \\ &= \frac{9.696 \cdot q_1 + 2.424 - 9.696 \cdot q_1}{1.01} \\ &= 2.4 \end{aligned}$$

since (as it can be shown) the call option can be replicated by S and B and therefore its no-arbitrage price is unique in f_1^1. In f_2^1 we have

$$\begin{aligned} c\,(1)\left(f_2^1\right) &= \frac{4.848 \cdot \mathbb{Q}[\omega_4 | f_2^1] + 0 \cdot \mathbb{Q}[\omega_5 | f_2^1]}{1.03} \\ &= \frac{4.848 \cdot 0.2}{1.03} \\ &= 0.9414. \end{aligned}$$

Finally, the no-arbitrage price of the call option at $t = 0$ is

$$\begin{aligned} c\,(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1 + r\,(0)}\right] \\ &= \frac{2.4}{1.02} \cdot 0.5 + \frac{0.9414}{1.02} \cdot 0.5 \\ &= 1.6379. \end{aligned}$$

5. The European put option on S has terminal payoff

$$p(2) = (K_{put} - S(2))^+ = \begin{cases} (14.544 - 19.392)^+ = 0 & \text{on } \omega_1 \\ (14.544 - 14.544)^+ = 0 & \text{on } \omega_2 \\ (14.544 - 9.696)^+ = 4.848 & \text{on } \omega_3 \\ (14.544 - 14.544)^+ = 0 & \text{on } \omega_4 \\ (14.544 - 7.179)^+ = 7.365 & \text{on } \omega_5 \end{cases}$$

In f_1^1 we have

$$\begin{aligned} p(1)(f_1^1) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{1+r(1)} \middle| f_1^1 \right] \\ &= \frac{0 \cdot \mathbb{Q}[\omega_1|f_1^1] + 0 \cdot \mathbb{Q}[\omega_2|f_1^1] + 4.848 \cdot \mathbb{Q}[\omega_3|f_1^1]}{1.01} \\ &= 4.8\,(0.5 + q_1)\,. \end{aligned}$$

were $q_1 = \mathbb{Q}[\omega_1|f_1^1]$. In f_2^1 we have

$$\begin{aligned} p(1)\left(f_2^1\right) &= \frac{0 \cdot \mathbb{Q}[\omega_4|f_2^1] + 7.635 \cdot \mathbb{Q}[\omega_5|f_2^1]}{1.03} \\ &= \frac{7.365 \cdot 0.8}{1.03} \\ &= 5.7204. \end{aligned}$$

Therefore, the no-arbitrage price of the put option at $t = 0$ is

$$\begin{aligned} p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1+r(0)} \right] \\ &= \frac{4.8\,(0.5+q_1)}{1.02} \cdot 0.5 + \frac{5.7204}{1.02} \cdot 0.5 \\ &= 1.1765 + 2.3529 q_1 + 2.8041 \\ &= 3.9806 + 2.3529 q_1. \end{aligned}$$

Recalling that $\mathbb{Q}$ is a risk neutral probability measure as long as $q_1 \in (0, 0.25)$, the interval of no-arbitrage prices of the put option is

$$(3.9806 + 2.3529 \cdot 0, 3.9806 + 2.3529 \cdot 0.25) = (3.9806, 4.5688).$$

6. The price at which the put option trades at $t = 0$ falls within the interval of no-arbitrage prices of the put option computed before since

$$4.2159 \in (3.9806, 4.5688).$$

Therefore, the extended market is arbitrage-free. From the previous Point we got $p(0) = 3.9806 + 2.3529 q_1$. Setting this price equal to the traded initial price delivers

$$3.9806 + 2.3529 q_1 = 4.2159$$

which yields

$$q_1 = \frac{4.2159 - 3.9806}{2.3529} = 0.1.$$

Therefore, the only risk neautral probability measure compatible with the initial put price computed above is

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.5 \cdot 0.1 = 0.05 \\ \mathbb{Q}[\omega_2] &= 0.5 \cdot (0.5 - 2 \cdot q_1) = 0.25 - 0.1 = 0.15 \\ \mathbb{Q}[\omega_3] &= 0.5 \cdot (0.5 + q_1) = 0.25 + 0.5 \cdot 0.1 = 0.3 \\ \mathbb{Q}[\omega_4] &= 0.5 \cdot 0.2 = 0.1 \\ \mathbb{Q}[\omega_5] &= 0.5 \cdot 0.8 = 0.4 \end{aligned}$$

As there exists a unique risk neutral probability measure, the market is arbitrage free and dynamically complete by the 2^{nd} FTAP. Alternatively, one can compute the no-arbitrate price of the put option at $t = 1$ if f_1^1

$$p(1)(f_1^1) = 4.8\,(0.5 + 0.1) = 2.88$$

and observe that the put option completes the one-period submarket that begins at $t = 1$ in f_1^1 (in an arbitrage-free way), as the matrix of the payoffs at the consequent nodes at $T = 2$ has rank equal to 3, since

$$\det \begin{bmatrix} 1.0605 & 19.392 & 0 \\ 1.0605 & 14.544 & 0 \\ 1.0605 & 9.696 & 4.848 \end{bmatrix} = -24.925 \neq 0.$$

Thus all the 3 one-period submarkets of the multiperiod market are complete, and therefore the 2-period market is dynamically complete.

7. The final payoff of the strategy $\bar{\vartheta} = (\bar{\vartheta}_C, \bar{\vartheta}_P)$ is

$$V_{\bar{\vartheta}}(2) = \bar{\vartheta}_C c(2) + \bar{\vartheta}_P p(2) = \begin{cases} 9.696 \cdot \bar{\vartheta}_C + 0 \cdot \bar{\vartheta}_P = 9.696 \cdot \bar{\vartheta}_C & \text{on } \omega_1 \\ 4.848 \cdot \bar{\vartheta}_C + 0 \cdot \bar{\vartheta}_P = 4.848 \cdot \bar{\vartheta}_C & \text{on } \omega_2 \\ 0 \cdot \bar{\vartheta}_C + 4.848 \cdot \bar{\vartheta}_P = 4.848 \cdot \bar{\vartheta}_P & \text{on } \omega_3 \\ 4.848 \cdot \bar{\vartheta}_C + 0 \cdot \bar{\vartheta}_P = 4.848 \cdot \bar{\vartheta}_C & \text{on } \omega_4 \\ 0 \cdot \bar{\vartheta}_C + 7.365 \cdot \bar{\vartheta}_P = 7.365 \cdot \bar{\vartheta}_P & \text{on } \omega_5 \end{cases}$$

ans its historical expected value is equal to

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[V_{\bar{\vartheta}}(2)\right] &= \sum_{i=1}^{5} \mathbb{P}(\omega_i) \cdot V_{\bar{\vartheta}}(2)(\omega_i) \\ &= \sum_{i=1}^{5} \frac{1}{5} \cdot V_{\bar{\vartheta}}(2)(\omega_i) \\ &= \frac{1}{5} \sum_{i=1}^{5} V_{\bar{\vartheta}}(2)(\omega_i) \\ &= \frac{1}{5}\left(9.696 \cdot \bar{\vartheta}_C + 4.848 \cdot \bar{\vartheta}_C + 4.848 \cdot \bar{\vartheta}_P + 4.848 \cdot \bar{\vartheta}_C + 7.365 \cdot \bar{\vartheta}_P\right) \\ &= \frac{9.696 + 4.848 + 4.848}{5} \bar{\vartheta}_C + \frac{4.848 + 7.365}{5} \bar{\vartheta}_P \\ &= 3.8784 \bar{\vartheta}_C + 2.4426 \bar{\vartheta}_P. \end{aligned}$$

The initial price of the strategy is

$$\begin{aligned} V_{\bar{\vartheta}}(0) &= \bar{\vartheta}_C c(0) + \bar{\vartheta}_P p(0) \\ &= 1.6379 \bar{\vartheta}_C + 4.2159 \bar{\vartheta}_P. \end{aligned}$$

Therefore, the historical net profit of the strategy is

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0)] &= \mathbb{E}^{\mathbb{P}}\left[V_{\bar{\vartheta}}(2)\right] - V_{\bar{\vartheta}}(0) = 3.8784 \bar{\vartheta}_C + 2.4426 \bar{\vartheta}_P - \left(1.6379 \bar{\vartheta}_C + 4.2159 \bar{\vartheta}_P\right) \\ &= (3.8784 - 1.6379)\, \bar{\vartheta}_C + (2.4426 - 4.2159)\, \bar{\vartheta}_P \\ &= 2.2405 \bar{\vartheta}_C - 1.7733 \bar{\vartheta}_P. \end{aligned}$$

Setting $V_{\bar{\vartheta}}(0) = 2.8058$ and $\mathbb{E}^{\mathbb{P}}[V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0)] = 7.6081$ leads to the system

$$\begin{cases} 1.6379 \bar{\vartheta}_C + 4.2159 \bar{\vartheta}_P = 2.8058 \\ 2.2405 \bar{\vartheta}_C - 1.7733 \bar{\vartheta}_P = 7.6081 \end{cases}$$

solved by

$$\begin{aligned} \bar{\vartheta}_C &= 3 \\ \bar{\vartheta}_P &= -0.5. \end{aligned}$$

8. The strategy $\bar{\vartheta}$ of point 8 closes at maturity T with a loss if $V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) < 0$. To understand where it happens we compute

$$V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) = \begin{cases} 9.696 \cdot 3 + 0 \cdot (-0.5) - 2.8058 = 26.282 & \text{on } \omega_1 \\ 4.848 \cdot 3 + 0 \cdot (-0.5) - 2.8058 = 11.738 & \text{on } \omega_2 \\ 0 \cdot 3 + 4.848 \cdot (-0.5) - 2.8058 = -5.2298 & \text{on } \omega_3 \\ 4.848 \cdot 3 + 0 \cdot (-0.5) - 2.8058 = 11.738 & \text{on } \omega_4 \\ 0 \cdot 3 + 7.365 \cdot (-0.5) - 2.8058 = -6.4883 & \text{on } \omega_5 \end{cases}.$$

We see that there is a loss on ω_3 and ω_5. Thus the historical probability of having a loss is

$$\mathbb{P}\left(V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) < 0\right) = \mathbb{P}\left(\omega_3\right) + \mathbb{P}\left(\omega_5\right) = \frac{2}{5} = 0.4.$$

The historical expected loss of the strategy $\bar{\vartheta}$ of point 8 conditioning on actually having a loss is

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) | V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) \quad &< \quad 0] = \left(V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0)\right)(\omega_3) \frac{\mathbb{P}\left(\omega_3\right)}{\mathbb{P}\left(V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) < 0\right)} \\ &+ \left(V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0)\right)(\omega_5) \frac{\mathbb{P}\left(\omega_5\right)}{\mathbb{P}\left(V_{\bar{\vartheta}}(2) - V_{\bar{\vartheta}}(0) < 0\right)} \\ &= \quad (-5.2298) \frac{\frac{1}{5}}{\frac{2}{5}} + (-6.4883) \frac{\frac{1}{5}}{\frac{2}{5}} = -5.8591. \end{aligned}$$

Solution of Exercise 15

1. The risk-neutral probabilities for the market are

$$\begin{matrix} \mathbb{Q}[f_1^1] = 0.4 \\ \mathbb{Q}[f_2^1] = 0.6 \end{matrix}, \quad \begin{matrix} \mathbb{Q}[\omega_1 | f_1^1] = 0.3 \\ \mathbb{Q}[\omega_2 | f_1^1] = 0.7 \end{matrix} \quad \text{and} \quad \begin{matrix} \mathbb{Q}[\omega_3 | f_2^1] = 0.6 \\ \mathbb{Q}[\omega_4 | f_2^1] = 0.4 \end{matrix}$$

that deliver

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.4 \cdot 0.3 = 0.12 \\ \mathbb{Q}[\omega_2] &= 0.4 \cdot 0.7 = 0.28 \\ \mathbb{Q}[\omega_3] &= 0.6 \cdot 0.6 = 0.36 \\ \mathbb{Q}[\omega_4] &= 0.6 \cdot 0.4 = 0.24 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

The European call option on S has terminal payoff

$$c\,(2) = \left(S\,(2) - K_{call}\right)^+ = \begin{cases} (16.8 - 12.3375)^+ = 4.4625 & \text{on } \omega_1 \\ (10.8 - 12.3375)^+ = 0 & \text{on } \omega_2, \omega_3 \\ (7.2 - 12.3375)^+ = 0 & \text{on } \omega_4 \end{cases}.$$

At $t = 1$ we have

$$c\,(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c(2)}{1 + r\,(1)}\right| \mathcal{P}_1\right] = \begin{cases} \dfrac{4.4625 \cdot 0.3 + 0 \cdot 0.7}{1.05} = 1.275 & \text{if } f_1^1 \\ \dfrac{0 \cdot 0.6 + 0 \cdot 0.4}{1.04} = 0 & \text{if } f_2^1 \end{cases}$$

and the price at inception of the call option is

$$\begin{aligned} c\,(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1 + r\,(0)}\right] \\ &= \frac{1.275}{1.02} \cdot 0.4 + \frac{0}{1.02} \cdot 0.6 \\ &= 0.5 \end{aligned}$$

2. The European put option on S has terminal payoff

$$p(2) = (K_{put} - S(2))^+ = \begin{cases} (9.41 - 16.8)^+ = 0 & \text{on } \omega_1 \\ (9.41 - 10.8)^+ = 0 & \text{on } \omega_2, \omega_3 \\ (9.41 - 7.2)^+ = 2.21 & \text{on } \omega_4 \end{cases} .$$

At $t = 1$ we have

$$p(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{1+r(1)}\middle| \mathcal{P}_1\right] = \begin{cases} \dfrac{0 \cdot 0.3 + 0 \cdot 0.7}{1.05} = 0 & \text{if } f_1^1 \\ \\ \dfrac{0 \cdot 0.6 + 2.21 \cdot 0.4}{1.04} = 0.85 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1+r(0)}\right] \\ &= \frac{0}{1.02} \cdot 0.4 + \frac{0.85}{1.02} \cdot 0.6 \\ &= 0.5. \end{aligned}$$

3. The terminal payoff of the portfolio is

$$X(2) = c(2) + p(2) = \begin{cases} 4.4625 + 0 = 4.4625 & \text{on } \omega_1 \\ 0 + 0 = 0 & \text{on } \omega_2 \text{ and } \omega_3 \\ 0 + 2.21 = 2.21 & \text{on } \omega_4 \end{cases} .$$

Its no arbitrage price at $t = 1$ is

$$c(1) + p(1) = \begin{cases} 1.275 + 0 = 1.275 & \text{if } f_1^1 \\ \\ 0 + 0.85 = 0.85 & \text{if } f_2^1 \end{cases}$$

which coincides with

$$S_X(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(2)}{1+r(1)}\middle| \mathcal{P}_1\right] = \begin{cases} \dfrac{4.4625 \cdot 0.3 + 0 \cdot 0.7}{1.05} = 1.275 & \text{if } f_1^1 \\ \\ \dfrac{0 \cdot 0.6 + 2.21 \cdot 0.4}{1.04} = 0.85 & \text{if } f_2^1 \end{cases} .$$

At inception, namely at $t = 0$, the no-arbitrage price of the portfolio is $c(0) + p(0) = 0.5 + 0.5 = 1$, that coincides with

$$\begin{aligned} S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_X(1)}{1+r(0)}\right] \\ &= \frac{1.275}{1.02} \cdot 0.4 + \frac{0.85}{1.02} \cdot 0.6 \\ &= 1. \end{aligned}$$

4. The buy and hold strategies with the riskless security B and α units of the risky asset S, whose initial cost coincides with the initial price of the portfolio derivative of Point 3 are

$$1 = V_\vartheta(0) = \vartheta_0 \cdot 1 + \alpha \cdot 10$$

and

$$\vartheta_0 = 1 - \alpha \cdot 10.$$

Therefore the buy and hold strategies are

$$\vartheta = (1 - \alpha \cdot 10, \alpha) \text{ for } \alpha \in \Re.$$

5. At maturity $T = 2$, the prices of the riskless security are

$$B(2)(\omega_1) = B(2)(\omega_2) = 1.02 \cdot 1.05 = 1.071$$
$$B(2)(\omega_3) = B(2)(\omega_4) = 1.02 \cdot 1.04 = 1.0608.$$

For the strategies in the previous Point we find

$$V_\vartheta(2) = (1 - 10 \cdot \alpha)\, B(2) + \alpha S(2) = \begin{cases} (1 - 10 \cdot \alpha) \cdot 1.071 + \alpha \cdot 16.8 = 6.09\alpha + 1.071 & \text{on } \omega_1 \\ (1 - 10 \cdot \alpha) \cdot 1.071 + \alpha \cdot 10.8 = 0.09\alpha + 1.071 & \text{on } \omega_2 \\ (1 - 10 \cdot \alpha) \cdot 1.0608 + \alpha \cdot 10.8 = 0.192\alpha + 1.0608 & \text{on } \omega_3 \\ (1 - 10 \cdot \alpha) \cdot 1.0608 + \alpha \cdot 7.2 = 1.0608 - 3.408\alpha & \text{on } \omega_4 \end{cases}$$

So that

$$V_\vartheta(2) - X(2) = \begin{cases} 6.09\alpha + 1.071 - 4.4625 = 6.09\alpha - 3.3915 & \text{on } \omega_1 \\ 0.09\alpha + 1.071 & \text{on } \omega_2 \\ 0.192\alpha + 1.0608 & \text{on } \omega_3 \\ 1.0608 - 3.408\alpha - 2.21 = -3.408\alpha - 1.1492 & \text{on } \omega_4 \end{cases}$$

is greater than zero for

$$\begin{cases} \alpha \geq \dfrac{3.391\,5}{6.09} = 0.55690 & \text{on } \omega_1 \\ \alpha \geq -\dfrac{1.071}{0.09} = -11.9 & \text{on } \omega_2 \\ \alpha \geq -\dfrac{1.0608}{0.192} = -5.525 & \text{on } \omega_3 \\ \alpha \leq -\dfrac{1.1492}{3.408} = -0.33721 & \text{on } \omega_4 \end{cases}.$$

Therefore, the buy and hold strategies of the previous Point super-replicate X on

(a) ω_4 for $\alpha \leq -11.9$

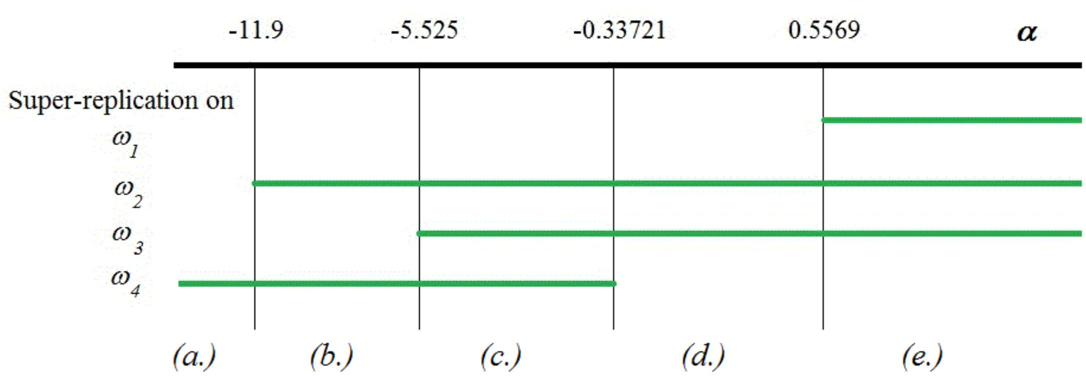
(b) $\omega_2 \cup \omega_4$ for $-11.9 \leq \alpha \leq -5.525$

(c) $\omega_2 \cup \omega_3 \cup \omega_4$ for $-5.525 \leq \alpha \leq -0.33721$

(d) $\omega_2 \cup \omega_3$ for $-0.337\,21 \leq \alpha \leq 0.55690$

(e) $\omega_1 \cup \omega_2 \cup \omega_3$ for $\alpha \geq 0.55690$

as one can see from the graph.



The five scenarios from *(a.)* to *(e.)* have the following historical probabilities:

(a) $\mathbb{P}(\omega_4) = 0.125$ for $\alpha \leq -11.9$,

(b) $\mathbb{P}(\omega_2 \cup \omega_4) = 0.5 + 0.125 = 0.625$ for $-11.9 \leq \alpha \leq -5.525$,

(c) $\mathbb{P}(\omega_2 \cup \omega_3 \cup \omega_4) = 0.5 + 0.25 + 0.125 = 0.875$ for $-5.525 \leq \alpha \leq -0.33721$

(d) $\mathbb{P}(\omega_2 \cup \omega_3) = 0.5 + 0.25 = 0.75$ for $-0.337\,21 \leq \alpha \leq 0.55690$,

(e) $\mathbb{P}(\omega_1 \cup \omega_2 \cup \omega_3) = 0.125 + 0.5 + 0.25 = 0.875$ for $\alpha \geq 0.55690$.

We see that scenarios $(c.)$ and $(e.)$ have the largest historical probability, as $\mathbb{P}(\omega_2 \cup \omega_3 \cup \omega_4) = \mathbb{P}(\omega_1 \cup \omega_2 \cup \omega_3) = 0.875$. Thus, the buy and hold strategies that have the largest historical probability of super-replicating $X(2)$ are those with $-5.525 \leq \alpha \leq -0.33721$, or $\alpha \geq 0.55690$. More precisely, these strategies are are

$$\vartheta^{\alpha} = (1 - \alpha \cdot 10, \alpha) \text{ for } -5.525 \leq \alpha \leq -0.33721 \text{ or } \alpha \geq 0.55690.$$

6. For $-5.525 \leq \alpha \leq -0.33721$ the shortfall is experienced on ω_1 only, as these strategies super-replicate the terminal payoff in all the other scenarios. Thus, for $-5.525 \leq \alpha \leq -0.33721$ the shortfall is

$$(X(2) - V_{\vartheta^{\alpha}}(2))^{+} = \begin{cases} -6.09\alpha + 3.3915 & \text{on } \omega_1 \\ 0 & \text{on } \omega_2,\ \omega_3,\ \omega_4 \end{cases}$$

that coincides with the expected shorfall for $-5.525 \leq \alpha \leq -0.33721$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[(X(2) - V_{\vartheta^{\alpha}}(2))^{+} \middle| V_{\vartheta}(2) - X(2) < 0 \right] &= (X(2) - V_{\vartheta^{\alpha}}(2))(\omega_1) \\ &= -6.09\alpha + 3.3915. \end{aligned}$$

This shortfall is minimum for $\alpha = -0.33721$ and it is equal to $-6.09 \cdot (-0.33721) + 3.3915 = 5.4451$.

For $\alpha \geq 0.55690$ the shortfall is experienced on ω_4 only, as these strategies super-replicate the terminal payoff in all the other scenarios. Thus the shortfall is for $\alpha \geq 0.55690$ is

$$(X(2) - V_{\vartheta^{\alpha}}(2))^{+} = \begin{cases} 3.408\alpha + 1.1492 & \text{on } \omega_4 \\ 0 & \text{on } \omega_2,\ \omega_3,\ \omega_1 \end{cases}$$

and the expected shorfall for $\alpha \geq 0.55690$ is

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}\left[(X(2) - V_{\vartheta^{\alpha}}(2))^{+} \middle| V_{\vartheta^{\alpha}}(2) - X(2) < 0 \right] &= (X(2) - V_{\vartheta^{\alpha}}(2))(\omega_4) \\ &= +3.408\alpha + 1.1492. \end{aligned}$$

This shortfall is minimum for $\alpha = 0.556\,90$ and it is equal to $+3.408 \cdot (0.55690) + 1.1492 = 3.0471$.

Comparing the two minimal expected shortfall, we see that the lowest value is reached when $\alpha = 0.556\,90$.

Thus the strategy that has the maximal historical probability of superreplication and the minimal historical expected shortfall is

$$\vartheta^{\alpha} = (1 - 0.556\,90 \cdot 10, 0.556\,90) = (-4.569, 0.55690).$$

Solution of Exercise 16

1. The risk-neutral probabilities for the market are

$$\begin{array}{c} \mathbb{Q}[f_1^1] = 0.5 \\ \mathbb{Q}[f_2^1] = 0.5 \end{array}, \quad \begin{array}{c} \mathbb{Q}[\omega_1 | f_1^1] = 0.75 \\ \mathbb{Q}[\omega_2 | f_1^1] = 0.25 \end{array} \quad \text{and} \quad \begin{array}{c} \mathbb{Q}[\omega_3 | f_2^1] = 0.5 \\ \mathbb{Q}[\omega_4 | f_2^1] = 0.5 \end{array}$$

that deliver

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.5 \cdot 0.75 = 0.375 \\ \mathbb{Q}[\omega_2] &= 0.5 \cdot 0.25 = 0.125 \\ \mathbb{Q}[\omega_3] &= 0.5 \cdot 0.5 = 0.25 \\ \mathbb{Q}[\omega_4] &= 0.5 \cdot 0.5 = 0.25 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).We look for risk neutral probabilities $\mathbb{Q}$ for the market.

2. The European put option on S has terminal payoff

$$p(2) = (K - S(2))^{+} = \begin{cases} (11.555 - 13.225)^{+} = 0 & \text{on } \omega_1 \\ (11.555 - 10.925)^{+} = 0.63 & \text{on } \omega_2, \omega_3 \\ (11.555 - 9.025)^{+} = 2.53 & \text{on } \omega_4 \end{cases}$$

Its no arbitrage at $t = 1$ is, therefore,

$$p(1) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p(2)}{1+r(1)}\middle|\mathcal{P}_1\right] = \begin{cases} \dfrac{0 \cdot 0.75 + 0.63 \cdot 0.25}{1.1} = 0.14318 & \text{if } f_1^1 \\ \dfrac{0.63 \cdot 0.5 + 2.53 \cdot 0.5}{1.05} = 1.5048 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$\begin{aligned} p(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1+r(0)}\right] \\ &= \frac{0.14318}{1.05} \cdot 0.5 + \frac{1.5048}{1.05} \cdot 0.5 \\ &= 0.78475. \end{aligned}$$

3. The American put option is optimally exercised at $t = 1$ if

$$(K - S(1))^+ \geq \mathbb{E}_1^{\mathbb{Q}}\left[\frac{(K - S(2))^+}{1+r(1)}\right].$$

Since the immediate payoff of the put option at $t = 1$ reads

$$(K - S(1))^+ = \begin{cases} (11.555 - 11.5)^+ = 0.055 & \text{on } f_1^1 \\ (11.555 - 9.5)^+ = 2.055 & \text{on } f_2^1 \end{cases}$$

while its continuation value at $t = 1$ is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{(K - S(2))^+}{1+r(1)}\middle|\mathcal{P}_1\right] = p(1) = \begin{cases} 0.14318 & \text{if } f_1^1 \\ 1.5048 & \text{if } f_2^1 \end{cases},$$

it is optimal to exercise the American option at $t = 1$ for f_2^1, and $T = 2$ otherwise. The cashflow produced by the optimal exercised American option is therefore

$$X(1) = \begin{cases} 0 & \text{if } f_1^1 \\ 2.055 & \text{if } f_2^1 \end{cases}$$

and

$$X(2) = \begin{cases} (11.555 - 13.225)^+ = 0 & \text{on } \omega_1 \\ (11.555 - 10.925)^+ = 0.63 & \text{on } \omega_2 \\ 0 & \text{on } \omega_3, \omega_4 \end{cases}$$

Its initial no-arbitrage price at $t = 0$ is

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X(1)}{1+r(0)}\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{X(2)}{(1+r(0))(1+r(1))}\right].$$

Since

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X(1)}{1+r(0)}\right] = \frac{0 + 2.055 \cdot 0.5}{1 + 0.05} = 0.97857$$

and

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{X(2)}{(1+r(0))(1+r(1))}\right] = \frac{0.63 \cdot 0.125}{(1+0.05)(1+0.1)} = 0.06818$$

we get that the initial no arbitrage price is

$$0.978\,57 + 0.06818 = 1.0468.$$

Finally, the early exercise premium of the American put option is

$$1.0468 - 0.78475 = 0.26205.$$

Solution of Exercise 17

1. The risk-neutral probabilities for the market are

$$\begin{array}{l}\mathbb{Q}[f_1^1]=0.4\\ \mathbb{Q}[f_2^1]=0.6\end{array},\quad \begin{array}{l}\mathbb{Q}[\omega_1|f_1^1]=0.5\\ \mathbb{Q}[\omega_2|f_1^1]=0.5\end{array}\quad\text{and}\quad \begin{array}{l}\mathbb{Q}[\omega_3|f_2^1]=0.4\\ \mathbb{Q}[\omega_4|f_2^1]=0.6\end{array}$$

that deliver

$$\begin{aligned}\mathbb{Q}[\omega_1] &= 0.4\cdot 0.5=0.2\\ \mathbb{Q}[\omega_2] &= 0.4\cdot 0.5=0.2\\ \mathbb{Q}[\omega_3] &= 0.6\cdot 0.4=0.24\\ \mathbb{Q}[\omega_4] &= 0.6\cdot 0.6=0.36\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

2. The European call option on S has terminal payoff

$$c\,(2)=(S\,(2)-K)^{+}=\begin{cases}(14.4-10.4)^{+}=4 & \text{on } \omega_1\\ (10.8-10.4)^{+}=0.4 & \text{on } \omega_2,\omega_3\\ (8.1-10.4)^{+}=0 & \text{on } \omega_4\end{cases}$$

Its no arbitrage price at $t=1$ is

$$c\,(1)=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c(2)}{1+r\,(1)}\right|\mathcal{P}_1\right]=\begin{cases}\dfrac{4\cdot 0.5+0.4\cdot 0.5}{1.05}=2.0952 & \text{if } f_1^1\\[2ex] \dfrac{4\cdot 0.4+0\cdot 0.6}{1.02}=1.5686 & \text{if } f_2^1\end{cases}$$

and at $t=0$

$$\begin{aligned}c\,(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c(1)}{1+r\,(0)}\right]\\ &= \frac{2.0952}{1.02}\cdot 0.4+\frac{1.5686}{1.02}\cdot 0.6\\ &= 1.7444.\end{aligned}$$

3. Since

$$S\,(1)-K=\begin{cases}12-10.4>0 & \text{on } f_1^1\\ 9-10.4<0 & \text{on } f_2^1\end{cases},$$

the forward-starting European call option c_{FS} with maturity $T=2$ on the risky security S and strike price $K=10.4$ is activated at $t=1$ in f_1^1 only. Consequently, the final payoff of the forward-starting European call option c_{FS} is

$$c_{FS}\,(2)=\begin{cases}c\,(2) & \text{on } \omega_1,\omega_2\\[2ex] 0 & \text{on } \omega_3,\omega_4\end{cases}$$

The no-arbitrge price of the forward-starting European call option c_{FS} at $t=1$ is

$$c_{FS}\,(1)\left(f_1^1\right)=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c_{FS}(2)}{1+r\,(1)}\right|\mathcal{P}_1\right]\left(f_1^1\right)=\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c(2)}{1+r\,(1)}\right|\mathcal{P}_1\right]\left(f_1^1\right)=2.0952$$

and

$$\begin{aligned}c_{FS}\,(1)\left(f_2^1\right) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{c_{FS}(2)}{1+r\,(1)}\right|\mathcal{P}_1\right]\left(f_2^1\right)\\ &= \frac{0\cdot 0.4+0\cdot 0.6}{1.02}=0,\end{aligned}$$

yielding

$$
\begin{aligned}
c_{FS}(0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{c_{FS}(1)}{1+r(0)}\right] \\
&= \frac{2.0952}{1.02}\cdot 0.4+\frac{0}{1.02}\cdot 0.6 \\
&= 0.82165
\end{aligned}
$$

at $t=0$.

Solution of Exercise 18

1. The prices of security B are $B(0)=1$,

$$B(1)(f_1^1)=B(1)(f_2^1)=1.02$$

and at the final date $T=2$

$$
\begin{aligned}
B(2)(\omega_1)=B(2)(\omega_2)=1.02\cdot 1.04=1.0608\\
B(2)(\omega_3)=B(2)(\omega_4)=1.02\cdot 1.01=1.0302.
\end{aligned}
$$

2. The market is dynamically complete, because each one-period submarket is complete. Indeed, for m_0 it holds

$$\det\begin{bmatrix} B(1)(f_1^1) & S(1)(f_1^1) \\ B(1)(f_2^1) & S(1)(f_2^1)\end{bmatrix}=\det\begin{bmatrix} 1.02 & 10.4 \\ 1.02 & 9.9\end{bmatrix}=-0.51\neq 0,$$

for $m_{1,1}$ we have

$$\det\begin{bmatrix} B(2)(\omega_1) & S(2)(\omega_1) \\ B(2)(\omega_2) & S(2)(\omega_2)\end{bmatrix}=\det\begin{bmatrix} 1.0608 & 10.92 \\ 1.0608 & 9.88\end{bmatrix}=-1.1032\neq 0$$

and for $m_{1,2}$ we have

$$\det\begin{bmatrix} B(2)(\omega_3) & S(2)(\omega_3) \\ B(2)(\omega_4) & S(2)(\omega_4)\end{bmatrix}=\det\begin{bmatrix} 1.0302 & 10.296 \\ 1.0302 & 9.206\end{bmatrix}=-1.1229\neq 0.$$

3. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0)=\frac{1}{1+r(0)}\left\{S(1)(f_1^1)\mathbb{Q}[f_1^1]+S(1)(f_2^1)\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1 \\ \mathbb{Q}[f_1^1],\mathbb{Q}[f_2^1]>0 \end{cases}$$

for m_0,

$$\begin{cases} S(1)(f_1^1)=\frac{1}{1+r(1)(f_1^1)}\left\{S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1]+S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1]+\mathbb{Q}[\omega_2|f_1^1]=1 \\ \mathbb{Q}[\omega_1|f_1^1],\mathbb{Q}[\omega_2|f_1^1]>0 \end{cases}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1)=\frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1]+S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1]+\mathbb{Q}[\omega_4|f_2^1]=1 \\ \mathbb{Q}[\omega_3|f_2^1],\mathbb{Q}[\omega_4|f_2^1]>0 \end{cases}$$

for $m_{1,2}$. The first system can be rewritten as

$$\begin{cases} 10=\frac{1}{1.02}\left\{10.4\cdot\mathbb{Q}[f_1^1]+9.9\cdot\mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1]+\mathbb{Q}[f_2^1]=1 \\ \mathbb{Q}[f_1^1],\mathbb{Q}[f_2^1]>0 \end{cases}$$

and is solved by

$$
\begin{aligned}
\mathbb{Q}[f_1^1] &= 0.6 \\
\mathbb{Q}[f_2^1] &= 0.4.
\end{aligned}
$$

The second system can be rewritten as

$$\begin{cases} 10.4 = \frac{1}{1.04}\left\{10.92 \cdot \mathbb{Q}[\omega_1|f_1^1] + 9.88 \cdot \mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.9 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.1, \end{aligned}$$

whereas the last system can be rewritten as

$$\begin{cases} 9.9 = \frac{1}{1.01}\left\{10.296 \cdot \mathbb{Q}[\omega_3|f_2^1] + 9.306 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.7 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.3. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{Q}[\omega_1] &= 0.6 \cdot 0.9 = 0.54 \\ \mathbb{Q}[\omega_2] &= 0.6 \cdot 0.1 = 0.06 \\ \mathbb{Q}[\omega_3] &= 0.4 \cdot 0.7 = 0.28 \\ \mathbb{Q}[\omega_4] &= 0.4 \cdot 0.3 = 0.12 \end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The *Zero Coupon Bond* $p_{0,1}$ settled at $t = 0$ with maturity 1 has a time $t = 1$ the payoff $p_{0,1}(1) = 1$. Its initial no-arbitrage price is

$$p_{0,1}(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p_{0,1}(1)}{1 + r(0)}\right] = \frac{1}{1 + 0.02} = 0.980\,39.$$

The *Zero Coupon Bond* $p_{0,2}$ settled at $t = 0$ with maturity 2 has a time $T = 2$ the payoff $p_{0,2}(2) = 1$. Its no-arbitrage prices at $t = 1$ are

$$p_{0,2}(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1 + r(1)}\right|\mathcal{P}_1\right] = \begin{cases} \dfrac{1}{1 + 0.04} = 0.961\,54 & \text{if } f_1^1 \\ \\ \dfrac{1}{1 + 0.01} = 0.990\,10 & \text{if } f_2^1 \end{cases}$$

and at $t = 0$

$$p_{0,2}(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p_{0,2}(1)}{1 + r(0)}\right] = \frac{0.6 \cdot 0.961\,54 + 0.4 \cdot 0.990\,10}{1.02} = 0.953\,89.$$

The *Zero Coupon Bond* $p_{1,2}$ settled at $t = 1$ with maturity 2 has a time $T = 2$ the payoff $p_{1,2}(2) = 1$. Its no-arbitrage prices at $t = 1$ coincide with those of $p_{0,2}$ at $t = 1$:

$$p_{1,2}(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1 + r(1)}\right|\mathcal{P}_1\right] = \begin{cases} \dfrac{1}{1 + 0.04} = 0.961\,54 & \text{if } f_1^1 \\ \\ \dfrac{1}{1 + 0.01} = 0.990\,10 & \text{if } f_2^1 \end{cases}$$

5. The cashflow of Party I *(floating minus fixed)* at $t = 1, 2$ is $X_{swap}(t) = S(t) - F_{swap}$.

   The *fixed outflow of* part I of the *swap* contract on S with maturity $T = 2$ is

$$F_{swap} \quad \text{for } t = 1, 2.$$

   The portfolio of zero coupon bonds defined in the previous point that replicates this fixed cashflow is

$$\pi \text{ constituted of } F_{swap} \text{ units of } p_{0,1} \text{ and } F_{swap} \text{ units of } p_{0,2}.$$

   In fact, the cashflow provided by π is

$$\begin{aligned} X_\pi(1) &= F_{swap} \cdot p_{0,1}(1) + 0 = F_{swap} \cdot 1 = F_{swap} \text{ at } t = 1 \\ X_\pi(2) &= 0 + F_{swap} \cdot p_{0,2}(1) = F_{swap} \cdot 1 = F_{swap} \text{ at } t = 2. \end{aligned}$$

   The swap price F_{swap} is settled at time $t = 0$ such that the *initial* no-arbitrage value of the swap contract is 0, i.e.

$$S_{swap}(0) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{X_{swap}(t)}{B(t)}\right] = 0$$

   Then we get

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{(S(t) - F_{swap})}{B(t)}\right] &= 0 \\ \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{S(t)}{B(t)}\right]}_{t=0 \text{ no-arbitrage value of the floating leg}} &= \underbrace{\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{F_{swap}}{B(t)}\right]}_{t=0 \text{ no-arbitrage value of the fixed leg}} \qquad (*) \end{aligned}$$

   Therefore, the initial no-arbitrage value of the fixed leg coincides with the initial no-arbitrage value of the floating leg. Since π and the fixed leg have the same cashflow, by no-arbitrage their initial price must coincide as well:

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{F_{swap}}{B(t)}\right] = \pi(0).$$

   By our previous computations

$$\begin{aligned} \pi(0) &= F_{swap} \cdot p_{0,1}(0) + F_{swap} \cdot p_{0,2}(0) \\ &= F_{swap} \cdot (p_{0,1}(0) + p_{0,2}(0)) \end{aligned}$$

   The initial value of the floating leg in $(*)$ is

$$\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1,2} \frac{S(t)}{B(t)}\right] = S(0) + S(0) = 2S(0) \quad \text{because } \frac{S}{B} \text{ is a } \mathbb{Q}-\text{martingale.}$$

   By equation $(*)$ we get

$$\underbrace{F_{swap} \cdot (p_{0,1}(0) + p_{0,2}(0))}_{t=0 \text{ no-arbitrage value of the fixed leg}} = \underbrace{2S(0)}_{t=0 \text{ no-arbitrage value of the floating leg}}$$

   And therefore

$$\begin{aligned} F_{swap} &= \frac{2S(0)}{p_{0,1}(0) + p_{0,2}(0)} \\ &= \frac{2 \cdot 10}{0.980\,39 + 0.953\,89} = 10.340 \end{aligned}$$

The cashflow of Party I of the swap contract *(floating minus fixed)* at $t = 1$ is

$$X_{swap}(1) = S(1) - F_{swap} = \begin{cases} 10.4 - 10.340 = 0.06 & \text{if } f_1^1 \\ \\ 9.9 - 10.340 = -0.44 & \text{if } f_2^1 \end{cases}$$

and $t = 2$

$$X_{swap}(2) = S(2) - F_{swap} = \begin{cases} 10.92 - 10.340 = 0.58 & \text{if } \omega_1 \\ 9.88 - 10.340 = -0.46 & \text{if } \omega_2 \\ 10.296 - 10.340 = -0.044 & \text{if } \omega_3 \\ 9.306 - 10.340 = -1.034 & \text{if } \omega_4 \end{cases}$$

Its no-arbitrage prices at at $t = 2$ are

$$S_{swap}(2) = X_{swap}(2)$$

and at $t = 1$ the risk-neutral valuation formula

$$S_{swap}(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{swap}(2)}{1 + r(1)}\right| \mathcal{P}_1\right]$$

delivers

$$\begin{aligned} S_{swap}(1)\left(f_1^1\right) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{swap}(2)}{1 + r(1)}\right| \mathcal{P}_1\right]\left(f_1^1\right) \\ &= \frac{0.58 \cdot 0.9 - 0.46 \cdot 0.1}{1.04} = 0.457\,69 \end{aligned}$$

and

$$\begin{aligned} S_{swap}(1)\left(f_2^1\right) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_{swap}(2)}{1 + r(1)}\right| \mathcal{P}_1\right]\left(f_2^1\right) \\ &= \frac{-0.044 \cdot 0.7 - 1.034 \cdot 0.3}{1.01} = -0.33762 \end{aligned}$$

with $S_{swap}(0) = 0$ by definition.

6. The payoff of the swaption at maturity $t = 1$ is

$$\max\left(S_{swap}(1); 0\right) = \begin{cases} 0.45769 & \text{if } f_1^1 \\ \\ 0 & \text{if } f_2^1 \end{cases}$$

The no-arbitrage value of the swaption at $t = 0$ is

$$\mathbf{E}^{\mathbf{Q}}\left[\frac{\max\left(S_{swap}(1); 0\right)}{1 + r(0)}\right] = \frac{0.6 \cdot 0.457\,69 + 0.4 \cdot 0}{1.02} = 0.26923.$$