Algebra and Geometry (Cod. 30544)

General Exam – December 10, 2020

Time: 2 hours. Total: 150 points.

Multiple choice questions (total: 24 points)

Each question has a single correct answer: write the correct answer in the box on the right. If you want to change your response cancel it and write another answer next to the box. 6 points are assigned for a correct answer, 0 points for a missing answer, -2 point for an incorrect answer.

1. Given an integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$. Then
   (A) $\{a + b \cdot x : a, b \in \mathcal{P}_n\}$ is a vector space of dimension $n+1$ (B) $\mathcal{P}_n$ is isomorphic to $\mathbf{R}^n$
   (C) $\mathcal{P}_n$ is isomorphic to $\mathbf{R}^{n+1}$ (D) none of the others

2. Let V be a vector space and fix a subspace $W \subseteq V$. Then
   (A) V/W is a vector space contained in V (B) $(x + W) \cap W = \emptyset$ for all $x \neq 0$
   (C) V/W is isomorphic to $V + W$ (D) none of the others

3. Fix a linear operator $T \in \mathcal{L}(V, W)$ and denote by T' its transpose, where V, W are finite dimensional. Then
   (A) $T' : W' \to \mathbf{F}$ (B) $\mathrm{rank}(T) = \mathrm{rank}(T')$
   (C) $T' : V' \to W'$ (D) none of the others

4. Let σ be a permutation of $\{1, 2, \ldots, n\}$ and fix a multilinear alternate form $f \in \mathcal{A}(V)$. Then
   (A) σ be can be always decomposed as product of transpositions (B) $f(x, 2x, 3x, ..., nx) = n!$
   (C) $\sigma f = f$ if σ is a transposition (D) none of the others

True/False questions (total: 24 points)

Each statement can be either true or false: write T for true or F for false in the box on the right. If you want to change your response cancel it and write another answer next to the box. 4 points are assigned for a correct answer, 0 points for a missing answer, -1 point for an incorrect answer.

1. Fix $T \in \mathcal{L}(V, W)$ and let A be a subspace of V. Then $T(A)$ is a subspace of W.
2. Fix $T \in \mathcal{L}(V, W)$ where the field is $\mathbf{R}$ and $\mathrm{Ker}(T)$ is a finite set. Then T is injective.
3. Let p a prime number. Then $(\mathbf{Z}_{p^2}, \cdot)$ is an abelian group.
4. For every $A \in \mathscr{M}_2(\mathbf{R})$ there exists an invertible P and a diagonal D such that $A = P^{-1}DP$.
5. If the columns of $A \in \mathscr{M}_n(\mathbf{F})$ are linearly dependent then $\mathrm{rank}(A) \leq n - 1$.
6. If 3 is an eigenvalue of $A \in \mathscr{M}_3(\mathbf{R})$ then 12 is an eigenvalue of $A^2 + A$.

Open answer questions (total: 102 points)

Answers must be written in the corresponding spaces. Each of the six questions will be assigned from 0 to 17 points.
Answers must be adequately justified.

Question 1. Consider the vector space $\mathcal{P}$ of polynomials $p : \mathbf{R} \to \mathbf{R}$ with real coefficients and and define the following subsets of $\mathcal{P}$:

(a) $\mathcal{U}_1 := \{p \in \mathcal{P} : 2p(0) = p(1)\}$;

(b) $\mathcal{U}_2 := \{p \in \mathcal{P} : p(2) + p(0) + p(1) + p(9) = 2020\}$;

(c) $\mathcal{U}_3 := \{p \in \mathcal{P} : p(x) \geq 0 \text{ for all } 0 \leq x \leq 1\}$;

(d) $\mathcal{U}_4 := \{p \in \mathcal{P} : p(x) = p(1 - x) \text{ for all } x\}$.

Decide, in each case, whether $\mathcal{U}_i$ is a vector space (with $i \in \{1, 2, 3, 4\}$).

Question 2. Consider the vector space $\mathscr{M}_2(\mathbf{R})$ and define

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

Show that the set $S = \{A, B, C, D\}$ is a basis for $\mathscr{M}_2(\mathbf{R})$.

Question 3. Let V and W two finitely generated real vector spaces and let $T : V \to W$ be a linear operator. Prove that T is injective if and only if, given any $x_1, \ldots, x_n$ linearly independent in V, then $T(x_1), \ldots, T(x_n)$ are linearly independent in W.

Question 4. Solve the following system

$$\begin{cases} x + 2y + 3z = 2 \\ -x + 4y + 6z = 3 \\ x + 8y + 12z = 4 \end{cases}$$

Question 5. 1. Provide the definition of eigenvalue of a matrix A.
2. Fix $A \in \mathscr{M}_n(\mathbf{R})$ with n (possibly equal) real eigenvalues. Prove that $\det(A^2 + A + I) \geq \frac{1}{2^n}$.

Question 6. 1. Provide the definition of dual basis.
2. Let M be a vector subspace of a finite dimensional vector space V, and recall that the annihilator of M is $M^\circ = \{f \in V' : \forall x \in M, f(x) = 0\}$. Show that $M^\circ = V'$ if and only if $M = \{0\}$.

0.1 Solutions Multiple choices / True-False

Multiple choices:

1	2	3	4
C	D	B	A

True/False:

1	2	3	4	5	6
T	T	F	F	T	T

0.2 Open question

1 (a) Yes: it is sufficient to prove that if $p, q \in \mathcal{U}$ and $\alpha \in \mathbf{R}$, then αp and $p+q$ belong to $\mathcal{U}$. To this aim, note that

$$2(\alpha p)(0) = \alpha(2p(0)) = \alpha(p(1)) = (\alpha p)(1)$$

and

$$2(p+q)(0) = 2(p(0)+q(0)) = 2p(0)+2q(0) = p(1)+q(1) = (p+q)(1).$$

(b) No because the polynomial (constantly equal to) 0 does not belong to $\mathcal{Q}$.
(c) No because $p(x) := x \in \mathcal{U}$ but $(-1) \cdot p(x) \notin \mathcal{U}$.
(d) Yes: it is sufficient to prove that if $p, q \in \mathcal{U}$ and $\alpha \in \mathbf{R}$, then αp and $p+q$ belong to $\mathcal{U}$. To this aim, note that

$$\forall x \in \mathbf{R}, \ \ (\alpha p)(x) = \alpha(p(x)) = \alpha(p(1-x)) = (\alpha p)(1-x)$$

and

$$\forall x \in \mathbf{R}, \ \ (p+q)(x) = p(x)+q(x) = p(1-x)+q(1-x) = (p+q)(1-x).$$

2 As $dim(V) = 4$ and S is made up of 4 elements, it is enough to prove that S is linearly independent. In order to do this, recall that V can be identified with $\mathbf{R}^4$ through the isomorphism

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \longleftrightarrow x = (x_1, x_2, x_3, x_4).$$

Hence, we can work in $\mathbf{R}^4$ and consider the set $\mathcal{S} = \{a, b, c, d\}$ made up of vectors

$$a = (2,1,0,0), \quad b = (1,0,0,1), \quad c = (0,0,2,1), \quad d = (0,2,1,0).$$

Now it is easy to check that, for any real numbers $\alpha, \beta, \gamma, \delta$,

$$\alpha(2,1,0,0) + \beta(1,0,0,1) + \gamma(0,0,2,1) + \delta(0,2,1,0) = (0,0,0,0)$$

implies that $\alpha = \beta = \gamma = \delta = 0$. Consequently, $\mathcal{S}$ is linearly independent and so is S.

3 If part. Let us assume for the sake of contradiction that there exist linearly independent vectors $x_1, \ldots, x_n$ in V such that $T(x_1), \ldots, T(x_n)$ are not linearly independent in W, i.e.,

$$\sum_{i=1}^{n} \alpha_i T(x_i) = T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = 0$$

for some scalar $\alpha_1, \ldots, \alpha_n$ not all equal to 0. Now $T(0) = 0$ and $T(\sum_{i=1}^{n} \alpha_i x_i) = 0$ and, on the other hand, $\sum_{i=1}^{n} \alpha_i x_i \neq 0$ because $x_1, \ldots, x_n$ are linearly independent. Hence T is not injective.

Only if part. Assume that T is injective and $x_1, \ldots, x_n$ are linearly independent in V. Suppose also that

$$\sum_{i=1}^{n} \alpha_i T(x_i) = T\left(\sum_{i=1}^{n} \alpha_i x_i\right) = 0$$

for some scalar $\alpha_1, \ldots, \alpha_n$. Since T is injective and $T(0) = 0$, then $\sum_{i=1}^{n} \alpha_i x_i$ has to be equal to 0, therefore $\alpha_1 = \cdots = \alpha_n = 0$.

4 The rank of the matrix of coefficients (i.e., 2) is not equal to the rank of the augmented matrix (i.e., 3). Therefore the system is impossible.

Alternatively, observe that if you compute $2R_1 + R_2 - R_3$, you get $0 = 3$.

5 1. See lecture notes

2. Suppose that $\{\lambda_1, \ldots, \lambda_n\}$ are the eigenvalues of A. Then the eigenvalues of $A^2 + A + I$ are $\{\lambda_1^2 + \lambda_1 + 1, \ldots, \lambda_n^2 + \lambda_n + 1\}$. It follows that

$$\det(A^2 + A + I) = \prod_{i=1}^{n} \left(\lambda_i^2 + \lambda_i + 1\right) = \prod_{i=1}^{n} \left((\lambda_i + 1/2)^2 + \frac{3}{4}\right) \geq \prod_{i=1}^{n} \frac{3}{4} \geq \prod_{i=1}^{n} \frac{1}{2} = \frac{1}{2^n}$$

6 1. See lecture notes.

2. Since $M^\circ = \{f \in V' : \mathrm{Ker}(f) \supseteq M\}$, then $M^\circ = V'$ means that every function $f \in V'$ verifies the inclusion $M \subseteq \mathrm{Ker}(f)$. This is equivalent to $M = \{0\}$.

This is true if $V = \{0\}$. Indeed, in such case $M = \{0\}$ and $M^\circ = V' = \{0_{V'}\}$. Otherwise, let us suppose hereafter that $|V| \geq 2$.

Of course, $0 \in M$. Suppose for the sake of contradiction that there exists a nonzero $c \in M$. Let $\{x_1, \ldots, x_{n-1}, c\}$ be a basis of V and let f be the unique linear form such that $f(x_1) = \cdots = f(x_{n-1}) = f(c) = [1]$. In particular, this implies that $c \notin \mathrm{Ker}(f)$.