Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188

a.y. 2021/22, 31st January 2022

General Exam

THEORY QUESTIONS (20 points out of 100)

1. **(6 points)** In the one-period market model state the Second Fundamental Theorem of Asset Pricing.

2. **(6 points)** Let π be the *hedging portfolio* in the derivation of the Black-Scholes Partial Differential Equation for a European derivative. The portfolio π...

   (Mark all the appropriate answers)

   - A involves positions on the riskless asset B and the risky asset S
   - B involves positions on the European derivative and the risky asset S
   - C is self-financing
   - D is locally riskless

3. **(4 points)** Consider a European derivative whose final payoff at T is $X(T)$. The no-arbitrage price at t of the European derivative satisfies the Black-Scholes Partial Differential Equation if $X(T)$ is

(Mark all the appropriate alternatives)

$\boxed{\text{A}}$ $X(T) = \max(K - S(T); 0)$ with $K \in \Re$

$\boxed{\text{B}}$ $X(T) = \sqrt{\frac{S(T)}{S(\frac{T}{2})}}$

$\boxed{\text{C}}$ $X(T) = S(T)$

$\boxed{\text{D}}$ $X(T) = \max(K - \min_{0 \le t \le T} S(t); 0)$ with $K \in \Re$

4. **(4 points)** Let $N(a, b^2)$ denote a normal random variable with mean a and variance b^2. In the Black Scholes model and with respect to the risk neutral measure $\mathbb{Q}$ the log-price of the risky security S is

(Mark the only appropriate alternative)

$\boxed{\text{A}}$ $\ln \frac{S(t)}{S(0)} \overset{\mathbb{Q}}{\sim} N(\delta t, \sigma^2 t)$

$\boxed{\text{B}}$ $\ln \frac{S(t)}{S(0)} \overset{\mathbb{Q}}{\sim} N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$

$\boxed{\text{C}}$ $\ln \frac{S(t)}{S(0)} \overset{\mathbb{Q}}{\sim} N\left(\left(\delta - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$

$\boxed{\text{D}}$ $\ln \frac{S(t)}{S(0)} \overset{\mathbb{Q}}{\sim} N\left(\left(\delta + \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$

EXERCISE 1 (45 points out of 100)
Consider a one period market with the riskless asset B yielding a risk-free rate $r = 5\%$, and a risky security S whose prices at time $T = 1$ are

$$S(1)(\omega_1) = 12.6$$
$$S(1)(\omega_2) = 10.5$$
$$S(1)(\omega_3) = 8.4$$

1. **(3 points)** Is the market complete? Justify your answer.

2. **(4 points)** Suppose that the risky security S trades at $t = 0$ at the price

$$S(0) = 9.4.$$

Let $\mathbb{Q}(\omega_k)$ for $k = 1, 2, 3$ denote a *risk-neutral probability*. Determine $\mathbb{Q}(\omega_1)$ and $\mathbb{Q}(\omega_2)$ in terms of $\mathbb{Q}(\omega_3) = q_3$.
(Mark the only appropriate alternative)

A $\quad \mathbb{Q}(\omega_1) = -\frac{3}{20} - q_3$ and $\mathbb{Q}(\omega_2) = \frac{13}{10} - q_3$

B $\quad \mathbb{Q}(\omega_1) = -\frac{3}{20} + 2q_3$ and $\mathbb{Q}(\omega_2) = \frac{13}{5} - q_3$

C $\quad \mathbb{Q}(\omega_1) = -\frac{3}{10} - q_3$ and $\mathbb{Q}(\omega_2) = \frac{13}{20} - q_3$

D $\quad \mathbb{Q}(\omega_1) = -\frac{3}{10} + q_3$ and $\mathbb{Q}(\omega_2) = \frac{13}{10} - 2q_3$

3. **(2 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_3) = q_3$ must be strictly larger than

4. **(2 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_3) = q_3$ must be strictly smaller than

5. **(4 points)** Is the market free of arbitrage opportunities? Justify your answer.

6. **(6 points)** A *European call option* on S with maturity $T = 1$ and strike $K_L = 9.45$ is introduced in the market.

   Is it possible to replicate this derivative with S and B? $\boxed{\text{YES}}$ $\boxed{\text{NO}}$

   Determine the set of no-arbitrage prices at $t = 0$ for this derivative.

7. **(4 points)** Formally describe the set of all the strategies whose final value super-replicates $X(1)$, where $X(1)$ is the payoff of the derivative introduced in Point 6. Namely, describe the set of all the $\vartheta = (x, y)$ such that

$$V_{\vartheta}\left(1\right)\left(\omega_k\right) \geq X\left(1\right)\left(\omega_k\right)$$

   for all $k = 1, ..., 3$.

8. **(2 points)** Among the strategies found in Point 7, let $\vartheta^* = (x^*, y^*)$ be the one whose initial value $V_{\vartheta^*}(0)$ is minimum. The riskless component x^* of this strategy is...

$$x^* = \boxed{}$$

9. **(2 points)** The risky component y^* of the strategy found in the previous Question is...

$$y^* = \boxed{}$$

10. **(3 points)** Suppose the call option of Point 6 trades at the initial price of 0.9. The extended market (the one in which B, S and the call option of Point 6 are traded) is...

(Mark the only appropriate alternative)

$\boxed{\text{A}}$	arbitrage-free AND complete
$\boxed{\text{B}}$	not arbitrage free AND complete
$\boxed{\text{C}}$	arbitrage-free AND incomplete
$\boxed{\text{D}}$	not arbitrage free AND incomplete

11. **(4 points)** Suppose a new *European call option* on S with strike price $K_H \geq 9.45$ and maturity $T = 1$ is introduced in the new extended market (the one in which B, S and the call option of Point 6 with initial price equal to 0.9 are traded).Write its terminal payoff as a function of K_H for $9.45 \leq K_H < 10.5$

$c_{K_H}(1)(\omega_1) = \boxed{}$ $\quad c_{K_H}(1)(\omega_2) = \boxed{}$ $\quad c_{K_H}(1)(\omega_3) = \boxed{}$

and for $10.5 < K_H \leq 12.6$

$c_{K_H}(1)(\omega_1) = \boxed{}$ $\quad c_{K_H}(1)(\omega_2) = \boxed{}$ $\quad c_{K_H}(1)(\omega_3) = \boxed{}$

12. **(4 points)** Compute the initial no arbitrage price of this call option as a function of K_H

$c_{K_H}(0) = \boxed{}$ for $9.45 \leq K_H < 10.5$

$c_{K_H}(0) = \boxed{}$ for $10.5 < K_H \leq 12.6$

13. **(2 points)** If the initial no-arbitrage price of the European call option with strike price K_H is 0.2, then

$$K_H = \boxed{}$$

14. **(3 points)** A *European call bull spread* on S with maturity $T = 1$ is introduced in the market. Its payoff at $T = 1$ is equal to the difference between the final payoff of the call option of Point 6 with strike price K_L and the final payoff of the European call option with strike K_H determined in the previous Point 13. Namely, the payoff at $T = 1$ of the call bull spread is

$$X_{BS}(1) = (S(1) - K_L)^+ - (S(1) - K_H)^+,$$

Find the initial no-arbitrage price of the European call bull spread

$$X_{BS}(0) = \boxed{}$$

EXERCISE 2 (35 points out of 100).
Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 1$, $\delta = 2\%$, $\mu = 5\%$, $\sigma = 15\%$, and $T = 2$.
(*Express your results in terms of the distribution function $N(\cdot)$ of a standard Normal random variable, whenever it is appropriate*).

1. **(6 points)** Compute the *historical probability* that a *European call option* on S with maturity $T = 2$ and strike price $K = 0.8$ closes at maturity T *in the money.*

2. **(6 points)** Write the no-arbitrage price of the call option of Point 1 for all $t \in [0, T]$.

3. **(5 points)** Consider a *European call option* on S with maturity $T = 2$ and strike price $K = S(0)$, and compute the *risk neutral probability* that this call option closes at maturity T *in the money* as a function of σ, namely in terms of σ, the volatility of S. Find the set of values of σ such that this probability is greater or equal than 50%.

4. **(6 points)** For the remaining questions assume now that $\sigma = 15\%$. Recalling that $T = 2$, compute the *risk neutral* expected value and variance of

$$\ln\left(S\left(\frac{T}{2}\right)S(T)\right)$$

5. **(5 points)** Consider the European derivative on S whose payoff at maturity $T = 2$ is equal to

$$X(T) = \sqrt{S\left(\frac{T}{2}\right) S(T)}.$$

Compute its no arbitrage price at $t = 0$.

6. **(4 points)** Find $S_X(t)$, the no arbitrage price of the derivcative in Point 5 at any $t \in \left[\frac{T}{2}, T\right]$.

7. **(3 points)** Find $S_X(t)$, the no arbitrage price of the derivative in Point 5 at any $t \in \left[0, \frac{T}{2}\right]$.

Solution of the Exercises

These are the detailed soultions of the exam.

The expected answers of the exam are written in italics.

Solution of EXERCISE 1

1. *The market is incomplete, because the number of scenarios $3 > 2$, which is the number of traded securities. Since there are fewer independent securities than scenarios, the market cannot be complete.*

2. Since the market is incomplete, the risk-neutral measures (if any) cannot be unique. Denoting by $q_i = \mathbb{Q}(\omega_i) > 0$ for $i = 1, ..., 3$, we have that

$$\frac{1}{1.05}\left(12.6\underbrace{(1-q_2-q_3)}_{q_1} + 10.5q_2 + 8.4q_3\right) = 9.4$$

$$\begin{aligned} 12\,(1-q_2-q_3) + 10q_2 + 8q_3 &= 9.4 \\ -2q_2 &= 4q_3 - 2.6 \\ q_2 &= 1.3 - 2q_3 \end{aligned}$$

Then,

$$\begin{aligned} q_1 &= 1 - q_2 - q_3 \\ &= 1 - (1.3 - 2q_3) - q_3 \\ &= -0.3 + q_3 \end{aligned}$$

Therefore, assuming the appropriate bounds x and y on q_3, we get

$$\begin{cases} q_1 = -0.3 + q_3 \\ q_2 = 1.3 - 2q_3 \\ q_3 \in (x, y) \end{cases}$$

Hence the right answer is

$$\boxed{\text{D}} \qquad \mathbb{Q}(\omega_1) = -\tfrac{3}{10} + q_3 \text{ and } \mathbb{Q}(\omega_2) = \frac{13}{10} - 2q_3$$

3. In order to find the appropriate bounds x and y on q_3 we impose positivity constraints on q_1 and q_3

$$\begin{cases} -\tfrac{3}{10} + q_3 > 0 \\ \dfrac{13}{10} - 2q_3 > 0 \end{cases}$$

that are satisfied by

$$\begin{cases} q_3 > \tfrac{3}{10} = 0.3 \\ q_3 < \tfrac{13}{20} = 0.65 \end{cases}$$

Therefore, we get

$$q_3 \in (0.3, 0.65) = \left(\frac{3}{10}, \frac{13}{20}\right).$$

Therefore, in order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_3) = q_3$ must be strictly larger than $0.3 = \frac{3}{10}$

4. *...and $\mathbb{Q}(\omega_3) = q_3$ must be strictly smaller than $\frac{13}{20} = 0.65$.*

5. *Since there exist risk neutral probabilities, the market is arbitrage-free by the 1st FTAP.*

6. The *European call option* on S with maturity $T = 1$ and strike $K_L = 9.45$ has final payoff $c_L(1) = (S(1) - K_L)^+$ equal to

$$\begin{gathered} c_L(1)(\omega_1) = (12.6 - 9.45)^+ = 3.15 \\ c_L(1)(\omega_2) = (10.5 - 9.45)^+ = 1.05 \\ c_L(1)(\omega_3) = (8.4 - 9.45)^+ = 0 \end{gathered}$$

The derivative cannot be replicated with B, S because the system

$$A\vartheta = c_L(1)$$

does not admit solutions, since $2 = rk\,(A) \neq rk\,[A|c_L(1)] = 3$ as

$$\det \begin{bmatrix} 1.05 & 12.6 & 3.15 \\ 1.05 & 10.5 & 1.05 \\ 1.05 & 8.4 & 0 \end{bmatrix} = 2.3153.$$

Hence: *Is it possible to replicate this call option?* $\boxed{\text{NO}}$

Tthe set of no-arbitrage prices at $t = 0$ for this derivative is

$$\begin{aligned} \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c_L(1)\right] &= \frac{1}{1.05}\left(3.15\left(-\frac{3}{10} + q_3\right) + 1.05\left(\frac{13}{10} - 2q_3\right) + 0q_3\right) \\ &= q_3 + 0.4 \end{aligned}$$

for $q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)$. The bounds of the no arbitrage price interval for this derivative are

$$\inf_{q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)} \left(\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c_L(1)\right]\right) = \inf_{q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)} (q_3 + 0.4) = \frac{3}{10} + 0.4 = 0.7$$

$$\sup_{q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)} \left(\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[c_L(1)\right]\right) = \sup_{q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)} (q_3 + 0.4) = \frac{13}{20} + 0.4 = 1.05$$

7. The set of all the strategies whose final value super-replicates $X(1) = c_L(1)$ is set of all the $\vartheta = (x, y)$ such that

$$V_\vartheta\,(1)\,(\omega_k) \geq X\,(1)\,(\omega_k)$$

for all $k = 1, 2, 3$. We get

$$\begin{cases} V_\vartheta(1)(\omega_1) = 1.05\vartheta_0 + 12.6\vartheta_1 \geq 3.15 = X(1)(\omega_1) \\ V_\vartheta(1)(\omega_2) = 1.05\vartheta_0 + 10.5\vartheta_1 \geq 1.05 = X(1)(\omega_2) \\ V_\vartheta(1)(\omega_3) = 1.05\vartheta_0 + 8.4\vartheta_1 \geq 0 = X(1)(\omega_3) \end{cases}$$

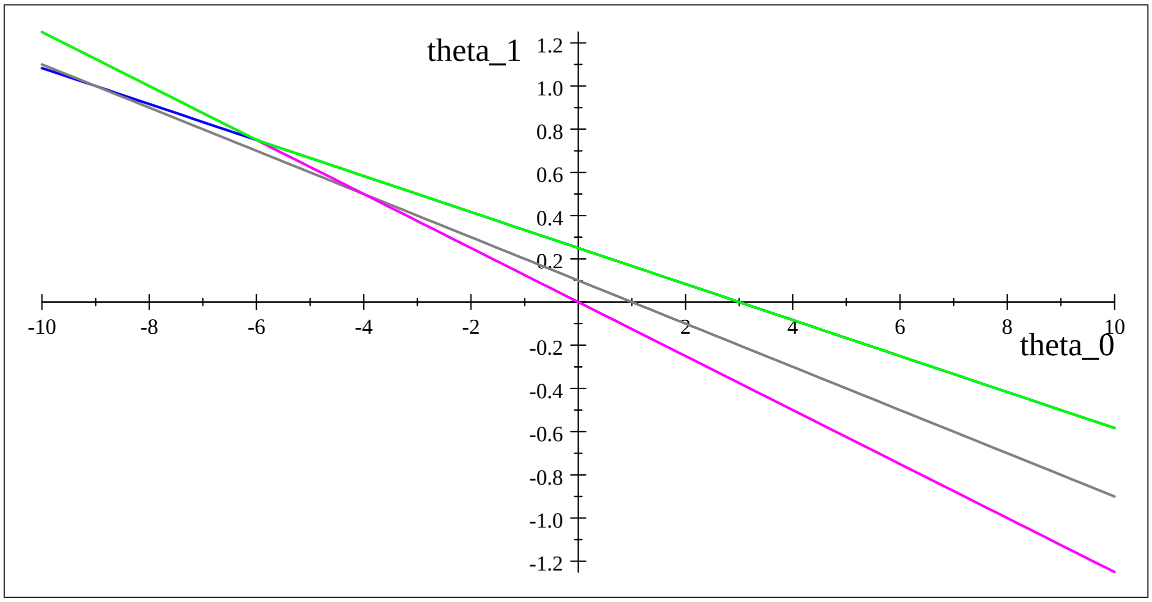
that is

$$\begin{cases} \vartheta_0 + 12\vartheta_1 \geq 3 \\ \vartheta_0 + 10\vartheta_1 \geq 1 \\ \vartheta_0 + 8\vartheta_1 \geq 0 \end{cases}$$

The system leads to the equation of the three lines

$$\begin{aligned} r_1 &: \vartheta_1 = -\frac{1}{12}\vartheta_0 + \frac{3}{12} \text{ blue in the graph below } \textit{(replication in } \omega_1\textit{)} \\ r_2 &: \vartheta_1 = -\frac{1}{10}\vartheta_0 + \frac{1}{10} \text{ gray in the graph below } \textit{(replication in } \omega_2\textit{)} \\ r_3 &: \vartheta_1 = -\frac{1}{8}\vartheta_0 \text{ magenta in the graph below } \textit{(replication in } \omega_3\textit{)} \end{aligned}$$

The super-replication region is the set of $(\vartheta_0, \vartheta_1)$ bounded by below by the green contour in the graph plotted below.



Such a green contour coincides with the maximum between r_1, r_2 and r_3. The green contour in particular is made of r_1 and r_3 whose intersection is

$$-\frac{1}{8}\vartheta_0 = -\frac{1}{12}\vartheta_0 + \frac{3}{12}$$

leading to

$$\begin{aligned} \vartheta_0 &= -6 \\ \vartheta_1 &= -\frac{1}{8}(-6) = 0.75 \end{aligned}$$

Hence the right solution is: *the set of all the strategies $\vartheta = (x, y)$ whose final value super-replicates $X(1)$ is described by*

$$y > \max_x \left\{ -\frac{1}{12}x + \frac{3}{12}, -\frac{1}{10}x + \frac{1}{10}, -\frac{1}{8}x \right\}$$

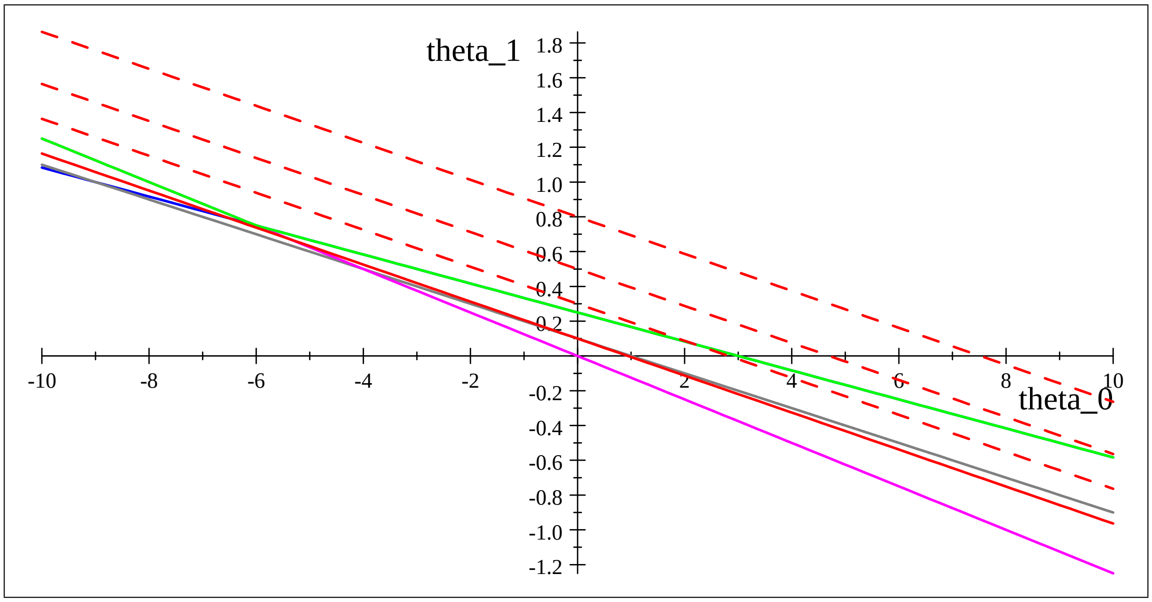
which is equivalent to the following system

$$\begin{cases} y > -\frac{1}{12}x + \frac{3}{12} & x \le -6 \\ y > -\frac{1}{8}x & x > -6 \end{cases}$$

8. The strategies $(\vartheta_0, \vartheta_1)$ whose initial value c are on the line r_c given by the equation

$$\begin{aligned} \vartheta_0 + 9.4\vartheta_1 &= c \\ \vartheta_1 &= -\frac{1}{9.4}\vartheta_0 + \frac{c}{9.4} \end{aligned}$$

The slope of the line r_c is negative. As c decreases, the lines r_c (red and dashed in the graph below) move downwards in the plane $(\vartheta_0, \vartheta_1)$. The smallest value of c within the super-replication region of X (which is identified by the green contour) is thus reached at the intersection of r_1 and r_3 (the line r_{c^*} is red and solid in the graph).



This intersection is from our previous computations

$$\begin{aligned}\vartheta_0 &= -6 \\ \vartheta_1 &= -\frac{1}{8}(-6) = 0.75\end{aligned}$$

Its initial cost is

$$\begin{aligned}V_{\vartheta^*}(0) &= -6 \cdot 1 + 0.75 \cdot 9.4 \\ &= 1.05 \\ &= \sup_{q_3 \in \left(\frac{3}{10}, \frac{13}{20}\right)} \left(\frac{1}{1+r} \mathbb{E}^{\mathbb{Q}} \left[c_L(1) \right] \right)\end{aligned}$$

previoiusly computed.

Hence: *The riskless component* x^* *of this strategy is* $x^* = -6$.

9. *The risky component* x^* *of this strategy is* $y^* = 0.75$.

10. The initial price for the call option of 0.9 is inside the interval of no-arbitrage prices $(0.7, 1.05)$ previously computed. Moreover the market is completed by the introduction of the call option, because

$$rk \begin{bmatrix} 1.05 & 12.6 & 3.15 \\ 1.05 & 10.5 & 1.05 \\ 1.05 & 8.4 & 0 \end{bmatrix} = 3$$

Hence the extended market (the one in which B, S and the call option of Point 6 are traded) is...

$\boxed{\text{A}}$ arbitrage-free AND complete

The unique risk-neutral probability measure in the extended market is such that

$$\frac{1}{1+r} \mathbb{E}^{\mathbb{Q}} \left[c_L(1) \right] = q_3 + 0.4 = 0.9$$

leading to

$$q_3 = 0.5$$

and $\mathbb{Q}(\omega_1) = -\frac{3}{10} + 0.5 = 0.2$ and $\mathbb{Q}(\omega_2) = \dfrac{13}{10} - 2 \cdot 0.5 = 0.3$.

11. The terminal payoff of the new *European call option* on S with strike price $K_H \geq 9.45$ is

$$\begin{gathered} c_H(1)(\omega_1) = (12.6 - K_H)^+ \\ c_H(1)(\omega_2) = (10.5 - K_H)^+ \\ c_H(1)(\omega_3) = (8.4 - K_H)^+ = 0 \end{gathered}$$

since the call is out of the money in ω_3 for any $K_H \geq 9.45$. In particular, for $9.45 \leq K_H < 10.5$ the payoff is in the money on ω_2 and ω_1

$$\begin{gathered} c_H(1)(\omega_1) = 12.6 - K_H > 0 \\ c_H(1)(\omega_2) = 10.5 - K_H > 0 \\ c_H(1)(\omega_3) = (8.4 - K_H)^+ = 0 \end{gathered}$$

and for $10.5 < K_H \leq 12.6$ the payoff is in the money only on ω_1:

$$\begin{gathered} c_H(1)(\omega_1) = 12.6 - K_H \geq 0 \\ c_H(1)(\omega_2) = (10.5 - K_H)^+ = 0 \\ c_H(1)(\omega_3) = (8.4 - K_H)^+ = 0 \end{gathered}$$

12. The initial no arbitrage price of this call option as a function of K_H is

$$c_{K_H}(0) = \frac{12.6 - K_H}{1.05} 0.2 + \frac{10.5 - K_H}{1.05} 0.3 + 0 = 5.4 - 0.4762 K_H \quad \text{for } 9.45 \leq K_H < 10.5$$

and

$$c_{K_H}(0) = \frac{12.6 - K_H}{1.05} 0.2 + 0 + 0 = 2.4 - 0.190\,48 K_H \quad \text{for } 10.5 < K_H \leq 12.6$$

13. If the initial no-arbitrage price of the European call option with strike price K_H is 0.2, then we have two possible equations

$$\begin{aligned} 5.4 - 0.4762 K_H &= 0.2 \quad \text{for } 9.45 \leq K_H < 10.5 \\ \text{delivering } K_H &= 10.9202 > 10.5 \text{ and hence not acceptable} \end{aligned}$$

and

$$\begin{aligned} 2.4 - 0.190\,48 K_H &= 0.2 \quad \text{for } 10.5 < K_H \leq 12.6 \\ \text{delivering } K_H &= 11.55 \text{ that is acceptable} \end{aligned}$$

14. The final payoff of the European call option with strike K_H determined in the previous Point 13 is

$$X_{BS}(1) = (S(1) - K_L)^+ - (S(1) - K_H)^+,$$

The initial no-arbitrage price of the European call bull spread is therefore

$$X_{BS}(0) = c_L(0) - c_{K_H}(0) = 0.9 - 0.2 = 0.7$$

Solution of EXERCISE 2

1. The *historical probability* that a *European call option* on S with maturity $T = 2$ and strike price $K = 0.8$ closes at maturity T *in the money* is

$$\begin{aligned}
\mathbb{P}\left[S(2) > K\right] &= \mathbb{P}\left[S(0)e^{\left(\left(\mu-\frac{\sigma^2}{2}\right)2+\sigma\sqrt{2}Z\right)} > K\right] \quad \text{with } Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1) \\
&= \mathbb{P}\left[Z > \left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)2\right)\frac{1}{\sigma\sqrt{2}}\right] \\
&= \mathbb{P}\left[Z < -\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)2\right)\frac{1}{\sigma\sqrt{2}}\right] \\
&= N\left(-\left(\ln\frac{K}{S(0)} - \left(\mu - \frac{\sigma^2}{2}\right)2\right)\frac{1}{\sigma\sqrt{2}}\right) \\
&= N\left(-\left(\ln\frac{0.8}{1} - \left(0.05 - \frac{0.15^2}{2}\right)2\right)\frac{1}{0.15\sqrt{2}}\right) \\
&= N\left(1.4172\right)
\end{aligned}$$

2. Write the no-arbitrage price of the call option of Point 1 for all $t \in [0, T]$.

$$c\left(t\right) = S\left(t\right)N\left(d_1\right) - Ke^{-\delta(T-t)}N\left(d_2\right),$$

where $N(z)$ is the distribution function of a standard normal random variable, i.e.

$$N\left(y\right) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz,$$

while

$$\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{(T-t)}}\left(\ln\left(\frac{S\left(t\right)}{K}\right) + \left(\delta + \frac{1}{2}\sigma^2\right)(T-t)\right) \\
&= \frac{1}{0.15\sqrt{(2-t)}}\left(\ln\left(\frac{S\left(t\right)}{0.8}\right) + \left(0.02 + \frac{1}{2}0.15^2\right)(2-t)\right) \\
&= \frac{1}{0.15\sqrt{(2-t)}}\left(\ln\left(\frac{S\left(t\right)}{0.8}\right) + 0.03125\left(2-t\right)\right)
\end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} = d_1 - 0.15\sqrt{(T-t)}$$

3. Consider a *European call option* on S with maturity $T = 2$ and strike price $K = S(0)$, and compute the *risk neutral probability* that this call option closes at maturity T *in the money* as a function of σ, namely in terms of σ, the volatility of S. Find the set of values of σ such that this probability is greater or equal than 50%.

$$\begin{aligned}
\mathbb{Q}\left[S(2) > K\right] &= \mathbb{Q}\left[S(0)e^{\left(\left(\delta-\frac{\sigma^2}{2}\right)2+\sigma\sqrt{2}Z\right)} > S(0)\right] \quad \text{with } Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1) \\
&= \mathbb{Q}\left[Z > \left(-\left(\delta - \frac{\sigma^2}{2}\right)2\right)\frac{1}{\sigma\sqrt{2}}\right] \\
&= \mathbb{Q}\left[Z < \left(\delta - \frac{\sigma^2}{2}\right)\sqrt{2}\frac{1}{\sigma}\right] \geq 0.5 \\
&= N\left(\left(0.02 - \frac{0.15^2}{2}\right)2\frac{1}{0.15\sqrt{2}}\right) \\
&= N\left(1.4172\right)
\end{aligned}$$

if and only if

$$\begin{aligned}\left(\delta-\frac{\sigma^2}{2}\right)\sqrt{2}\frac{1}{\sigma} &\geq 0\\ \delta-\frac{\sigma^2}{2} &\geq 0\\ \sigma^2 &\leq 2\delta\end{aligned}$$

that holds for $-\sqrt{2\delta}\leq\sigma\leq\sqrt{2\delta}$. Since $\sigma>0$, the solution is $0<\sigma\leq\sqrt{2\delta}=\sqrt{2\cdot 0.02}=0.2=20\%$.

4. For the remaining questions assume now that $\sigma=15\%$. Recalling that $T=2$, compute the *risk neutral* expected value and variance of

$$\ln\left(S\left(\frac{T}{2}\right)S(T)\right)=\ln\left(S\left(\frac{T}{2}\right)\right)+\ln\left(S(T)\right)$$
$$=2\ln S(0)+\left(\delta-\frac{\sigma^2}{2}\right)\frac{T}{2}+\sigma W^{\mathbb{Q}}\left(\frac{T}{2}\right)+\left(\delta-\frac{\sigma^2}{2}\right)T+\sigma W^{\mathbb{Q}}(T)$$
$$=2\ln S(0)+\left(\delta-\frac{\sigma^2}{2}\right)\frac{3T}{2}+2\sigma W^{\mathbb{Q}}\left(\frac{T}{2}\right)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}\left(\frac{T}{2}\right)\right)$$

The two random variables $W^{\mathbb{Q}}\left(\frac{T}{2}\right)$ and $\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}\left(\frac{T}{2}\right)\right)$ are independent of each other and both normally distributed with mean 0 and variance $\frac{T}{2}$. Therefore

$$\ln\left(S\left(\frac{T}{2}\right)S(T)\right)\overset{\mathbb{Q}}{\sim}\mathcal{N}\left(2\ln S(0)+\left(\delta-\frac{\sigma^2}{2}\right)\frac{3T}{2},4\sigma^2\frac{T}{2}+\sigma^2\frac{T}{2}\right)=$$
$$=\mathcal{N}\left(2\ln 1+\left(0.02-\frac{0.15^2}{2}\right)\frac{6}{2},0.15^2\frac{5}{2}2\right)$$
$$=\mathcal{N}\left(0.026\,25,0.112\,5\right)$$

5. The no-arbitrage price at $t=0$ of the European derivative on S whose payoff at maturity $T=2$ is equal to

$$X(T)=\sqrt{S\left(\frac{T}{2}\right)S(T)}.$$

is

$$\begin{aligned}S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T}X(T)\right]\\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T}\sqrt{S\left(\frac{T}{2}\right)S(T)}\right]\end{aligned}$$

There are various ways to compute this expectation. Since we have already computed the risk-neutral distribution of $Y=\ln\left(S\left(\frac{T}{2}\right)S(T)\right)$, we observe that

$$\begin{aligned}S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T}\sqrt{S\left(\frac{T}{2}\right)S(T)}\right]\\ &= e^{-\delta T}\mathbb{E}^{\mathbb{Q}}\left[e^{\frac{1}{2}Y}\right]\\ &= e^{-0.02\cdot 2}\mathbb{E}^{\mathbb{Q}}\left[e^{\mathcal{N}\left(\frac{0.026\,25}{2},\frac{0.112\,5}{4}\right)}\right]\end{aligned}$$

and recall that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\mathcal{N}\left(a,b^2\right)}=e^{a+\frac{b^2}{2}}\right]\tag{2}$$

we get

$$\begin{aligned}S_X(0) &= e^{-0.02\cdot 2}\exp\left(\frac{0.026\,25}{2}+\frac{1}{2}\frac{0.1125}{4}\right)\\ &= 0.98727\end{aligned}$$

6. The no arbitrage price of the derivative in Point 5 at any $t \in \left[\frac{T}{2}, T\right]$

$$
\begin{aligned}
S_X(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)S(T)}\right] \\
&= e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S(T)}\right] \quad \text{as } S\left(\frac{T}{2}\right) \text{ is } \mathcal{F}_t - \text{measurable}
\end{aligned}
$$

We now compute

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S(T)}\right] &= e^{-\delta(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S(t)\exp\left(\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t)\right)\right)}\right] \\
&= e^{-\delta(T-t)}\sqrt{S(t)}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{\exp\left(\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t)\right)\right)}\right] \quad \text{as } S(t) \text{ is } \mathcal{F}_t - \text{measurable} \\
&= e^{-\delta(T-t)}\sqrt{S(t)}\mathbb{E}^{\mathbb{Q}}\left[\sqrt{\exp\left(\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t)\right)\right)}\right] \quad \text{as } W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t) \text{ is } \mathcal{F}_t - \\
&= e^{-\delta(T-t)}\sqrt{S(t)}\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\frac{1}{2}\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t)\right)\right)\right] \\
&= e^{-\delta(T-t)}\sqrt{S(t)}\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\mathcal{N}\left(\frac{1}{2}\left(\delta-\frac{\sigma^2}{2}\right)(T-t), \frac{1}{4}\sigma^2(T-t)\right)\right)\right] \\
&= e^{-\delta(T-t)}\sqrt{S(t)}\exp\left(\frac{1}{2}\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\frac{1}{2}\frac{1}{4}\sigma^2(T-t)\right) \quad \text{using (2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
S_X(t) &= e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S(T)}\right] \\
&= e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)}\sqrt{S(t)}\exp\left(\frac{1}{2}\left(\delta-\frac{\sigma^2}{2}\right)(T-t)+\frac{1}{2}\frac{1}{4}\sigma^2(T-t)\right) \\
&= e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)}\sqrt{S(t)}\exp\left(\frac{1}{8}(T-t)\left(4\delta-\sigma^2\right)\right) \\
&= e^{-0.02(2-t)}\sqrt{S\left(\frac{T}{2}\right)}\sqrt{S(t)}\exp\left(\frac{1}{8}(2-t)\left(4\cdot 0.02-0.15^2\right)\right) \\
&= \sqrt{S\left(\frac{T}{2}\right)}\sqrt{S(t)}\exp\left(-0.02(2-t)+1.4375\times 10^{-2}-7.1875\times 10^{-3}t\right) \\
&= \sqrt{S\left(\frac{T}{2}\right)}\sqrt{S(t)}\left(1.2813\times 10^{-2}t-2.5625\times 10^{-2}\right)
\end{aligned}
$$

7. The no arbitrage price of the derivative in Point 5 at any $t \in \left[0, \frac{T}{2}\right]$.is

$$
S_X(t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)S(T)}\right]
$$

We rewrite $\sqrt{S\left(\frac{T}{2}\right)S(T)}$ in terms of $S(t)$ and the risk-neutral Brownian motion:

$$\sqrt{S\left(\frac{T}{2}\right)S(T)} =$$
$$= \sqrt{S(t)\exp\left(\left(\delta - \frac{\sigma^2}{2}\right)\left(\frac{T}{2} - t\right) + \sigma\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right)\right)}\sqrt{S(t)\exp\left(\left(\delta - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right)}$$
$$= S(t)\exp\left(\frac{1}{2}\left(\delta - \frac{\sigma^2}{2}\right)\left(\frac{T}{2} - t\right) + \frac{1}{2}\sigma\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \frac{1}{2}\left(\delta - \frac{\sigma^2}{2}\right)(T-t) + \frac{1}{2}\sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right)\right)$$

We focus on the risk-neutral Brownian motion:

$$\frac{1}{2}\sigma\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \frac{1}{2}\sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right) =$$
$$= \frac{1}{2}\sigma\left[\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \left(W^{\mathbb{Q}}(T) \pm W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right)\right]$$
$$= \frac{1}{2}\sigma\left[\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}\left(\frac{T}{2}\right)\right) + \left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right)\right]$$
$$= \frac{1}{2}\sigma\left[2\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}\left(\frac{T}{2}\right)\right)\right]$$

The two increments are now independent of each other, as $t < \frac{T}{2} < T$ and independent of $\mathcal{F}_t$

$$\frac{1}{2}\sigma\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \frac{1}{2}\sigma\left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}(t)\right) = \frac{1}{2}\sigma\left[2\left(W^{\mathbb{Q}}\left(\frac{T}{2}\right) - W^{\mathbb{Q}}(t)\right) + \left(W^{\mathbb{Q}}(T) - W^{\mathbb{Q}}\left(\frac{T}{2}\right)\right)\right]$$
$$= \frac{1}{2}\sigma\mathcal{N}\left(0, 4\left(\frac{T}{2} - t\right) + \left(T - \frac{T}{2}\right)\right)$$
$$= \mathcal{N}\left(0, \frac{\sigma^2}{4}\left(\frac{5}{2}T - 4t\right)\right)$$

Hence:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S\left(\frac{T}{2}\right)S(T)}\right] =$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[S(t)\exp\left[\frac{1}{2}\left(\delta - \frac{\sigma^2}{2}\right)\left(\frac{T}{2} - t\right) + \frac{1}{2}\left(\delta - \frac{\sigma^2}{2}\right)(T-t) + \mathcal{N}\left(0, \frac{\sigma^2}{4}\left(\frac{5}{2}T - 4t\right)\right)\right]\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[S(t)\exp\left(\frac{1}{8}\left(2\delta - \sigma^2\right)(3T - 4t) + \frac{\frac{\sigma^2}{4}\left(\frac{5}{2}T - 4t\right)}{2}\right)\right] \text{ using (2)}$$
$$= S(t)\exp\left(-\frac{1}{16}T\sigma^2 + \frac{3}{4}T\delta - t\delta\right)$$

$$
\begin{aligned}
S_X(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\delta(T-t)}\sqrt{S\left(\frac{T}{2}\right)S(T)}\right] \\
&= \exp(-\delta\,(T-t))\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{S\left(\frac{T}{2}\right)S(T)}\right] \\
&= \exp(-\delta\,(T-t))S(t)\exp\left(-\frac{1}{16}T\sigma^2+\frac{3}{4}T\delta-t\delta\right) \\
&= S(t)\exp\left(-\delta T-\frac{1}{16}T\sigma^2+\frac{3}{4}T\delta\right) \\
&= S(t)\exp\left(-\frac{1}{16}T\sigma^2-\frac{1}{4}T\delta\right) \\
&= S(t)\exp\left(-\frac{1}{16}2\cdot 0.15^2-\frac{1}{4}2\cdot 0.02\right) \\
&= 0.987\cdot S(t)
\end{aligned}
$$

Solution to the THEORY QUESTIONS

1. In the multi-period market model state the Second Fundamental Theorem of Asset Pricing: see the lecture notes

2. Let π be the *hedging portfolio* in the derivation of the Black-Scholes Partial Differential Equation for a European derivative. The portfolio π...

   $\boxed{\text{B}}$ involves positions on the European derivative and the risky asset S

   $\boxed{\text{C}}$ is self-financing

   $\boxed{\text{D}}$ is locally riskless

3. Consider a European derivative whose final payoff at T is $X(T)$. The no-arbitrage price at t of the European derivative satisfies the Black-Scholes Partial Differential Equation if $X(T)$ is

   $\boxed{\text{A}}$ $X(T) = \max(K - S(T); 0)$ with $K \in \Re$

   $\boxed{\text{C}}$ $X(T) = S(T)$

4. Let $N(a, b^2)$ denote a normal random variable with mean a and variance b^2. In the Black Scholes model and with respect to the risk neutral measure $\mathbb{Q}$ the log-price of the risky security S is

   $\boxed{\text{C}}$ $\ln \frac{S(t)}{S(0)} \overset{\mathbb{Q}}{\sim} N\left(\left(\delta - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$