Algebra and Geometry (Cod. 30544)

General Exam Mock– September 15, 2020

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Rules of conduct during exams or other tests:
I hereby undertake to respect the regulations described in the *Honor Code* and sign my presence at the exam.

Signature:

Exercise 1 *Consider a inner product vector space $(V, \| \ \|)$ and let x and y be two elements of V both different from 0.*

1. *Prove that if x and y are such that $\|x + ky\| \geq \|x\|$ for all $k \in \mathbb{R}$, then x and y are linearly independent.*
2. *Prove that the converse does not hold.*

Solution 1. and 2. You will need to report both your answers in **one single sheet for both points (you can use both sides if necessary), but no more than one sheet which must include both points**. **At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. Assume that $\lambda x + \mu y = 0$. If $\lambda = 0$, then $\mu y = 0$. Since $y \neq 0$, this implies that $\mu = 0$. If $\lambda \neq 0$, it follows that $x + \gamma y = 0$ where $\gamma = \mu/\lambda$. This implies that

$$0 \leq \|x\| \leq \|x + \gamma y\| = \|0\| = 0,$$

yielding that $x = 0$, a contradiction. Thus, we have that $\lambda = 0 = \mu$, proving that x and y are linearly independent.

2. Consider for example $(V, \| \ \|) = \left(\mathbb{R}^2, \| \ \|_\infty\right)$. Let $x = (1, 0)$ and $y = (1, 1/2)$. Both vectors are clearly different from 0. Moreover, if $\lambda x + \mu y = 0$, then $(\lambda + \mu, \mu/2) = 0$, yielding that $\mu = 0$ and $\lambda = 0$ and proving that $\{x, y\}$ are linearly independent. If $k = -1$, then $x + ky = (0, -1/2)$, yielding that $\|x + ky\|_\infty = \frac{1}{2} < 1 = \|x\|_\infty$. ■

Exercise 2 *Let X be a nonempty set and $+$ be an operation $X \times X \to X$.*

1. *Provide the definition of abelian group $(X, +)$.*

2. *Mark the sentences which are true:*

- *For each integer $n > 0$, there exists an abelian group $(X, +)$ of cardinality n.*
- *There exists an abelian group with two distinct identities.*
- *If $(X, +)$ is an abelian group then it is also a monoid.*
- *There exists a group $(X, +)$ with identity e such that $e + x \neq x + e$ for some $x \in X$.*

Solution 1. You will need to report your answer in **one sheet (you can use both sides if necessary), but no more than one sheet**. **At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. See the lecture notes.
2. Via BBoard, you will mark the correct answers. The correct answers are "For each integer $n > 0$, there exists an abelian group $(X, +)$ of cardinality n" and "If $(X, +)$ is an abelian group then it is also a monoid." **For this type of questions, no justification will be needed**.

$\blacksquare$

Exercise 3 **1**. *Fix a nonnegative integer n. Show that the vector space $\mathbf{R}^{\mathbf{N}}$ has a subspace of dimension n.*

2. *Is it true that if V is a subspace of $\mathbf{R}^{\mathbf{N}}$ and $V \neq \mathbf{R}^{\mathbf{N}}$ then the dimension of V is finite?*

Solution 1. You will need to report your answer in **one sheet (you can use both sides if necessary), but no more than one sheet**. **At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. If $n = 0$ then $\{0\}$ is a subspace of $\mathbf{R}^{\mathbf{N}}$ of dimension 0. Here and after, suppose that $n \geq 1$. Let e_i be the function such that $e_i(i) = 0$ and $e_i(j) = 0$ for all $j \neq i$. Also, let V be set of functions $f : \mathbf{N} \to \mathbf{R}$ such that $f(i) = 0$ for all $i \geq n+1$. Then it is easily seen that V is a vector space, and that $\{e_1, \dots, e_n\}$ is a linearly independent subset of V which generates V itself. Therefore the dimension of V is n.

2. In BBoard, you'll be able to cross the appropriate true/false box. The correct answer is "False." **For this type of questions, no justification will be needed**. For the interested student, I report afterwards the correct reasoning leading to the answer. Let V be the set of functions $f : \mathbf{N} \to \mathbf{R}$ with limit 0. Then V is a proper vector subspace of $\mathbf{R}^{\mathbf{N}}$ with dimension infinity.

■

Exercise 4 *Consider the real vector space V of continuous functions $\mathbf{R} \to \mathbf{R}$. Define $f(x) = \sin(x)$ and $g(x) = x$ and $h(x) = e^{x+1}$ for all $x \in \mathbf{R}$.*

1. *Prove that $\{f, g, h\}$ is a linearly independent set in V.*

2. *Find the dimension of V.*

Solution 1. and 2. You will need to report both your answers in **one single sheet for both points (you can use both sides if necessary), but no more than one sheet which must include both points. At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. Let α, β, γ be real such that $\alpha f(x) + \beta g(x) + \gamma h(x)$ is the constant function 0. By setting $x = 0$ we obtain $\gamma = 0$. Hence

$$\forall x \in \mathbf{R}, \quad \alpha \sin(x) + \beta x = 0.$$

Set $x = \pi$ to get $\beta = 0$ and, lastly, $x = \pi/2$ to conclude that $\alpha = 0$. Hence $\{f, g, h\}$ is a linearly independent subset of V.

2. We know that $\{1, x, \ldots, x^n\}$ is a linearly independent subset of V for each integer $n \geq 0$. Since n is arbitrary, it follows that the dimesion of V is infinite. ■

Exercise 5 **1**. *Let $f : V \to W$ be a linear operator between real vector spaces. Prove that f is injective if and only if* $\mathrm{Ker}(f) = \{0\}$.

2. *Suppose that V, W are two finite dimensional vector spaces with the same dimesion. Then the linear operator f is injective if and only if the matrix which represents f has a nonzero determinant. True or False?*

Solution 1. and 2. You will need to report both your answers in **one single sheet for both points (you can use both sides if necessary), but no more than one sheet which must include both points**. **At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. See lecture notes.

2. In BBoard, you'll be able to cross the appropriate true/false box. The correct answer is "True."
For this type of questions, no justification will be needed. For the interested student, the answer is "True". ■

Exercise 6 1. *Provide the definition of orthonormal basis.*

2. *Let $\mathcal{P}_2$ be the real vector space of polynomials with degree ≤ 2. Find an orthonormal basis of $\mathcal{P}_2$, where the inner product is given by*

$$\langle ax^2 + bx + c, a'x^2 + b'x + c' \rangle := 2aa' + 7bb' + cc'.$$

3. *Every finite-dimensional inner product space has an orthonormal basis. True or False?*

Solution 1. and 2. You will need to report both your answers in **one single sheet for both points (you can use both sides if necessary), but no more than one sheet**. **At the end of the exam you will scan all the 6 sheets (back and front, one for each question) and upload them.** For the actual answer:

1. See lecture notes.
2. We will apply the Gram–Schmidt Procedure to the basis $\{1, x, x^2\}$ of $\mathcal{P}_2$. Note that

$$\forall a, b, c \in \mathbf{R}, \quad \|ax^2 + bx + c\|^2 = 2a^2 + 7b^2 + c^2.$$

At this point, it follows that $\|1\| = 1$, hence

$$e_1 = 1.$$

The numerator in the expression of e_2 is

$$x - \langle x, e_1 \rangle e_1 = x - \langle x, 1 \rangle = x.$$

At the same time, $\|x\|^2 = 7$, hence

$$e_2 = \frac{x}{\sqrt{7}}.$$

Lastly, the numerator in the expression of e_3 is

$$x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2 = x^2,$$

from which we conclude that

$$e_3 = \frac{x^2}{\|x^2\|} = \frac{x^2}{\sqrt{2}}.$$

Therefore $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathcal{P}_2$.

3. In BBoard, you'll be able to cross the appropriate true/false box. The correct answer is "True." **For this type of questions, no justification will be needed**. For the interested student, the answer is "True". ■