Quantitative Finance and Derivatives I

code 20188

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EXERCISE 1 (45 points out of 100).

Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\mathcal{P}_1 = \{f_1^1, f_2^1\}$, with $f_1^1 = \{\omega_1, \omega_2\}$ and $f_2^1 = \{\omega_3, \omega_4\}$, and at the final date $T = 2$ with $\mathcal{P}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$r(0) = 4\%, \quad r(1)(f_1^1) = 10\% \text{ and } r(1)(f_2^1) = 0\%.$$

The second security is a *risky asset* S, with time $t = 0$ price $S(0) = 21$, with time $t = 1$ prices

$$S(1)(f_1^1) = 31.2 \text{ and } S(1)(f_2^1) = 12.48$$

and with time $T = 2$ prices

$$S(2)(\omega_1) = 57.2, \quad S(2)(\omega_2) = 28.6, \quad S(2)(\omega_3) = 15.6, \quad S(2)(\omega_4) = 10.4.$$

1. **(3 points)** Compute the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.
2. **(3 points)** Is the market dynamically complete?
3. **(8 points)** Determine the set of risk neutral probabilities $\mathbb{Q}$ for the market, specifying $\mathbb{Q}(\omega_k)$ for $k = 1, ..., 4$. Is the market free of arbitrage opportunities?
4. **(7 points)** A *zero coupon bond* with maturity $T = 2$ has terminal payoff at $T = 2$ equal to $ZCB(2)(\omega_k) = 1$ for all $k = 1, \dots, 4$. At $t = 1$ there are no intermediate cash flows. Compute $S_{ZCB}(0)$ and $S_{ZCB}(1)$, the no-arbitrage price process of the zero coupon bond, specifying it at $t = 0$ and at $t = 1$.
5. **(9 points)** A forward contract on S with maturity $T = 2$ and delivery price, or *forward price*, $F_{0,2}$ is introduced in the market at $t = 0$. The final payoff of the forward contract is $S(T) - F_{0,2}$. Under no-arbitrage, the forward price at time $t = 0$, $F_{0,2}$, must be such that the initial value of the forward contract is zero. Determine $F_{0,2}$. At $t = 1$ a new forward contract on S with maturity $T = 2$ and final payoff $S(T) - F_{1,2}$ is introduced in the market. Likewise, the forward prices $F_{1,2}(f_1^1)$ and $F_{1,2}(f_2^1)$ are settled so that the no-arbitrage value of this new forward contract at $t = 1$ is zero. Determine $F_{1,2}(f_1^1)$ and $F_{1,2}(f_2^1)$.
6. (**5 points)** By definition, $F_{2,2}(\omega_k) = S(2)(\omega_k)$ for any $k = 1, \dots, 4$. Compute $\mathbb{E}^{\mathbb{Q}}[F_{1,2}]$ and show that the *process of forward prices* $\{F_{t,T}\}_{t=0}^{T} = \{F_{0,2}, F_{1,2}, F_{2,2}\}$ is not a martingale under the risk neutral probability found in Point 3.
7. (**5 points)** Determine the set of probabilities $\mathbb{Q}^F$ under which the process of forward prices $\{F_{t,T}\}_{t=0}^{T}$ is a martingale, specifying $\mathbb{Q}^F(\omega_k)$ for $k = 1, ..., 4$.

8. **(5 points)** By construction, $S_{ZCB}(2) = ZCB(2)$. Verify that the prices of S discounted by the zero coupon bond of Point 4, namely the process

$$\left\{ \frac{S(0)}{S_{ZCB}(0)}, \frac{S(1)}{S_{ZCB}(1)}, \frac{S(2)}{S_{ZCB}(2)} \right\},$$

is a martingale under the probability measure $\mathbb{Q}^F$ found at Point 7.

EXERCISE 2 (40 points out of 100).

Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S_0 = 2, \delta = 3\%, \mu = 8\%, \sigma = 10\%, T = 2$ and $K = 2e^{0.07} = 2.145$.

Express your results in terms of the distribution function $N(\cdot)$ of a standard normal random variable, whenever it is appropriate.

1. **(5 points)** Find the *risk neutral probability* $\mathbb{Q}$ that a put option on S with strike price K closes at maturity in the money.

2. **(5 points)** Consider the put option introduced in Point 1. Compute its no-arbitrage price at $t = 0$.

3. **(5 points)** The payoff of a European *digital option* on S is

$$Y(T) = \begin{cases} 1 & \text{if } S(T) < K \\ 0 & \text{else} \end{cases}$$

Compute the no-arbitrage price at $t = 0$ of this digital option.

4. **(3 points)** Compute the final payoff of a portfolio constitued by a long position on K units of the digital option of Point 3 and a short position on 1 unit of the put option of Point 1 in terms of $S(T)$.

5. **(5 points)** Consider a European derivative on S with payoff

$$X(T) = \begin{cases} S(T) & \text{if } S(T) < K \\ 0 & \text{else} \end{cases}$$

Using the results from Points 2, 3 and 4, compute the no-arbitrage price at $t = 0$ of this derivative.

6. **(5 points)** Compute the no-arbitrage price of the derivative of Point 5 at any $t \in (0, T)$.

7. **(5 points)** Compute the replicating strategy $(\vartheta_0^R(t), \vartheta_1^R(t))$ of the derivative of Point 5 at any $t \in [0, T]$.

8. **(3 points)** Is the replicating strategy of the derivative of Point 5 long or short on B, S at $t = 0$?

9. **(4 points)** Consider the buy-and-hold portfolio $\vartheta^{BH} = \left(\vartheta_0^{BH}, \vartheta_1^{BH}\right)$ with $\vartheta_i^{BH}(t) = \vartheta_i^R(0)$ for all $t \in [0, T]$ and $i = 0, 1$. Compute the *historical probability* that the value of this portfolio exceeds the value of the derivative of Point 5 at T, that is

$$\mathbb{P}[V_{\vartheta^{BH}}(T) > X(T)].$$

QUESTION (15 points out of 100)

State and *derive* the Black-Scholes Partial Differential Equation.

SOLUTIONS TO EXERCISES

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.04$$

and at the final date $T = 2$

$$B(2)(\omega_1) = B(2)(\omega_2) = 1.144 \quad B(2)(\omega_3) = B(2)(\omega_4) = 1.04.$$

2. The market is dynamically complete, because each one-period submarket is complete (in your exam check explicitly that the rank of the *terminal* payoff matrix of each one-period submarket has rank 2).

3. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \{S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1]\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases} \tag{1}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \{S(2)(\omega_1)\ \mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1]\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases} \tag{2}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)} \{S(2)(\omega_3)\ \mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases} \tag{3}$$

System (1) can be rewritten as

$$\begin{cases} 21 = \frac{1}{1.04} \{31.2 \cdot \mathbb{Q}[f_1^1] + 12.48 \cdot \mathbb{Q}[f_2^1]\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.5 \\ \mathbb{Q}[f_2^1] &= 0.5 \end{aligned}$$

System (2) can be rewritten as

$$\begin{cases} 31.2 = \frac{1}{1.1} \{57.2 \cdot \mathbb{Q}[\omega_1|f_1^1] + 28.6 \cdot \mathbb{Q}[\omega_2|f_1^1]\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned}\mathbb{Q}[\omega_1|f_1^1] &= 0.2\\ \mathbb{Q}[\omega_2|f_1^1] &= 0.8\end{aligned}$$

and System (3) can be rewritten as

$$\begin{cases} 12.48 = \frac{1}{1.00}\left\{15.6\cdot\mathbb{Q}[\omega_3|f_2^1] + 10.4\cdot\mathbb{Q}[\omega_4|f_2^1]\right\}\\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1\\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0\end{cases}$$

and is solved by

$$\begin{aligned}\mathbb{Q}[\omega_3|f_2^1] &= 0.4\\ \mathbb{Q}[\omega_4|f_2^1] &= 0.6\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{Q}[\omega_1] &= 0.5\cdot 0.2 = 0.1\\ \mathbb{Q}[\omega_2] &= 0.5\cdot 0.8 = 0.4\\ \mathbb{Q}[\omega_3] &= 0.5\cdot 0.4 = 0.2\\ \mathbb{Q}[\omega_4] &= 0.5\cdot 0.6 = 0.3\end{aligned}$$

Since there exists a unique risk neutral probability measure, the market is arbitrage free and complete (by the 2^{nd} FTAP).

4. The zero coupon bond payoff is $ZCB\left(2\right)=1.$ Its no arbitrage price is therefore at $t=1$

$$\begin{aligned}S_{ZCB}\left(1\right) &= \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{ZCB\left(2\right)}{1+r\left(1\right)}\right|\mathcal{P}_1\right] = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{1+r\left(1\right)}\right|\mathcal{P}_1\right] =\\ &= \frac{1}{1+r\left(1\right)} = \begin{cases}\frac{1}{1.1} = 0.9091 & if\ f_1^1\\ \\ \frac{1}{1.00} = 1 & if\ f_2^1\end{cases}\end{aligned}$$

and at $t=0$

$$\begin{aligned}S_{ZCB}\left(0\right) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{S_{ZCB}\left(1\right)}{1+r\left(0\right)}\right]\\ &= \frac{0.9091}{1.04}\cdot 0.5 + \frac{1}{1.04}\cdot 0.5\\ &= 0.9178.\end{aligned}$$

5. The delivery price $F_{0,2}$ of the forward contract with maturity $T=2$ is such that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S\left(2\right)-F_{0,2}}{B\left(2\right)}\right] = 0$$

that yields

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{F_{0,2}}{B\left(2\right)}\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{S\left(2\right)}{B\left(2\right)}\right] = S\left(0\right),$$

as $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}-$martingale. Then

$$F_{0,2}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] = S(0)$$

We therefore compute

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(2)}\right] &= \frac{1}{B(2)(f_1^1)}\cdot\mathbb{Q}[f_1^1] + \frac{1}{B(2)(f_2^1)}\cdot\mathbb{Q}[f_2^1] \\ &= \frac{1}{1.144}\cdot 0.5 + \frac{1}{1.04}\cdot 0.5 = 0.91783\end{aligned}$$

and obtain

$$F_{0,2} = \frac{21}{0.917\,83} = 22.88.$$

For the new forward contract on S with maturity $T = 2$ introduced in the market in $t = 1$ with delivery price $F_{1,2}$ we have to determine $F_{1,2}$ such that

$$\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2) - F_{1,2}}{1 + r(1)}\right|\mathcal{P}_1\right] = 0,$$

that is

$$\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{F_{1,2}}{1 + r(1)}\right|\mathcal{P}_1\right] = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S(2)}{1 + r(1)}\right|\mathcal{P}_1\right] = S(1),$$

as $\frac{S(t)}{B(t)}$ is a $\mathbb{Q}-$martingale. Moreover $\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{F_{1,2}}{1+r(1)}\right|\mathcal{P}_1\right] = \frac{F_{1,2}}{1+r(1)}$, as $r(1)$ is known at date $t = 1$. Hence we get

$$\frac{F_{1,2}}{1 + r(1)} = S(1)$$

and finally,

$$F_{1,2} = S(1)(1 + r(1)) = \begin{cases} 31.2 \cdot 1.1 = 34.32 & if\ f_1^1 \\ \\ 12.48 \cdot 1.00 = 12.48 & if\ f_2^1 \end{cases}$$

6. As

$$\mathbb{E}^{\mathbb{Q}}[F_{1,2}] = 0.5 \cdot 34.32 + 0.5 \cdot 12.48 = 23.4,$$

we have that

$$F_{0,2} = 22.88 \neq \mathbb{E}^{\mathbb{Q}}[F_{1,2}] = 23.4,$$

and therefore $\{F_{t,T}\}_{t=0}^{T} = \{F_{0,2}, F_{1,2}, F_{2,2}\}$ is not a martingale under the risk neutral probability.

7. The probability $\mathbb{Q}^F$ has to satisfy

$$\begin{cases} 22.88 = 34.32 \cdot \mathbb{Q}^F[f_1^1] + 12.48 \cdot \mathbb{Q}^F[f_2^1] \\ \mathbb{Q}^F[f_1^1] + \mathbb{Q}^F[f_2^1] = 1 \\ \mathbb{Q}^F[f_1^1], \mathbb{Q}^F[f_2^1] > 0 \end{cases}$$

and is solved by,

$$\begin{aligned}\mathbb{Q}^F[f_1^1] &= 0.4762 \\ \mathbb{Q}^F[f_2^1] &= 0.5238\end{aligned}$$

System (2) can be rewritten as

$$\begin{cases} 34.32 = 57.2 \cdot \mathbb{Q}^F[\omega_1|f_1^1] + 28.6 \cdot \mathbb{Q}^F[\omega_2|f_1^1] \\ \mathbb{Q}^F[\omega_1|f_1^1] + \mathbb{Q}^F[\omega_2|f_1^1] = 1 \\ \mathbb{Q}^F[\omega_1|f_1^1], \mathbb{Q}^F[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}^F[\omega_1|f_1^1] &= 0.2 \\ \mathbb{Q}^F[\omega_2|f_1^1] &= 0.8 \end{aligned}$$

and System (3) can be rewritten as

$$\begin{cases} 12.48 = \left\{15.6 \cdot \mathbb{Q}^F[\omega_3|f_2^1] + 10.4 \cdot \mathbb{Q}^F[\omega_4|f_2^1]\right\} \\ \mathbb{Q}^F[\omega_3|f_2^1] + \mathbb{Q}^F[\omega_4|f_2^1] = 1 \\ \mathbb{Q}^F[\omega_3|f_2^1], \mathbb{Q}^F[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}^F[\omega_3|f_2^1] &= 0.4 \\ \mathbb{Q}^F[\omega_4|f_2^1] &= 0.6 \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{Q}^F[\omega_1] &= 0.4762 \cdot 0.2 = 0.095\,24 \\ \mathbb{Q}^F[\omega_2] &= 0.4762 \cdot 0.8 = 0.380\,96 \\ \mathbb{Q}^F[\omega_3] &= 0.5238 \cdot 0.4 = 0.209\,52 \\ \mathbb{Q}^F[\omega_4] &= 0.5238 \cdot 0.6 = 0.314\,28 \end{aligned}$$

8. Since

$$\mathbb{E}^{\mathbb{Q}^F}\left[\left.\frac{S(2)}{ZCB(2)}\right|\mathcal{P}_1\right] = \mathbb{E}^{\mathbb{Q}^F}\left[\left.\frac{S(2)}{1}\right|\mathcal{P}_1\right]$$

$$= \begin{cases} S(2)(\omega_1) \cdot \mathbb{Q}^F[\omega_1|f_1^1] + S(2)(\omega_2) \cdot \mathbb{Q}^F[\omega_2|f_1^1] \\ S(2)(\omega_3) \cdot \mathbb{Q}^F[\omega_3|f_2^1] + S(2)(\omega_4) \cdot \mathbb{Q}^F[\omega_4|f_2^1] \end{cases}$$

$$= \begin{cases} 57.2 \cdot 0.2 + 28.6 \cdot 0.8 = 34.32 \\ 15.6 \cdot 0.4 + 10.4 \cdot 0.6 = 12.48 \end{cases}$$

$$= \begin{cases} \frac{S(1)}{S_{ZCB}(1)}(f_1^1) = \frac{31.2}{0.9091} = 34.32 \\ \frac{S(1)}{S_{ZCB}(1)}(f_2^1) = \frac{12.48}{1} = 12.48 \end{cases}$$

and

$$\mathbb{E}^{\mathbb{Q}^F}\left[\left.\frac{S(1)}{ZCB(1)}\right|\mathcal{P}_0\right] = \mathbb{E}^{\mathbb{Q}^F}\left[\frac{S(1)}{ZCB(1)}\right]$$

$$= \frac{S(1)(f_1^1)}{ZCB(1)(f_1^1)} \cdot \mathbb{Q}^F[f_1^1] + \frac{S(2)(f_1^1)}{ZCB(1)(f_2^1)} \cdot \mathbb{Q}^F[f_2^1]$$

$$= \frac{31.2}{0.9091} \cdot 0.4762 + \frac{12.48}{1} \cdot 0.5238 = 22.88$$

$$= \frac{21}{0.9178} = \frac{S(0)}{ZCB(0)}$$

we can conclude that $\frac{S}{ZCB}$ is a martingale under $\mathbb{Q}^F$.

Exercise 2

1. The *risk neutral probability* $\mathbb{Q}$ that the put option closes in the money is

$$\begin{aligned}
\mathbb{Q}\left[S(T)<K\right] &= \mathbb{Q}\left[\left(\delta-\frac{\sigma^2}{2}\right)\cdot T+\sigma W^{\mathbb{Q}}(T)<\ln\frac{K}{S(0)}\right] \\
&= \mathbb{Q}\left[W^{\mathbb{Q}}(T)<\left(\ln\frac{K}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma}\right] \\
&= \mathbb{Q}\left[\sqrt{T}Z^{\mathbb{Q}}<\left(\ln\frac{K}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma}\right] \quad \text{where } Z^{\mathbb{Q}}\overset{\mathbb{Q}}{\sim}\mathcal{N}(0,1) \\
&= \mathbb{Q}\left[Z^{\mathbb{Q}}<\left(\ln\frac{K}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma\sqrt{T}}\right] \\
&= N\left(\underbrace{\left(\ln\frac{K}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma\sqrt{T}}}_{-d_2}\right) \\
&= N(-d_2)
\end{aligned}$$

with

$$\begin{aligned}
d_2 &= \left(\ln\frac{S(0)}{K}+\left(\delta-\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma\sqrt{T}} \\
&= \left(\ln\frac{2}{2e^{0.07}}+\left(0.03-\frac{0.1^2}{2}\right)\cdot 2\right)\frac{1}{0.1\sqrt{2}} \\
&= (-0.07+0.05)\frac{1}{0.1\sqrt{2}} \\
&= \frac{-0.02}{0.1\sqrt{2}}=-0.1414.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{Q}\left[S(T)<K\right] &= N(-d_2) \\
&= N(0.1414)=\text{NormalDist}\,(0.1414)=0.556.
\end{aligned}$$

2. According to the Black-Scholes formula, the initial no-arbitrage price of the put option is

$$S_{put}(0)=Ke^{-\delta T}N(-d_2)-S_0N(-d_1)$$

where

$$\begin{aligned}
d_1 &= \left(\ln\frac{S(0)}{K}+\left(\delta+\frac{\sigma^2}{2}\right)\cdot T\right)\frac{1}{\sigma\sqrt{T}} \\
&= \left(\ln\frac{2}{2e^{0.07}}+\left(0.03+\frac{0.1^2}{2}\right)\cdot 2\right)\frac{1}{0.1\sqrt{2}} \\
&= (-0.07+0.07)\frac{1}{0.1\sqrt{2}} \\
&= 0
\end{aligned}$$

and d_2 has been already defined in Point 1. Since $N(-d_1) = N(0) = \frac{1}{2}$, we have

$$\begin{aligned} S_{put}(0) &= Ke^{-\delta T} N(-d_2) - S_0 N(-d_1) \\ &= 2e^{0.07} e^{-0.03 \cdot 2} N(0.1414) - \frac{2}{2} \\ &= 2.0201 N(0.1414) - 1 \\ &= 0.1181. \end{aligned}$$

3. Let $\mathbf{I}_{S(T)<K}$ be the indicator function of the event $S(T) < K$, namely

$$\mathbf{I}_{S(T)<K}(\omega) = \begin{cases} 1 & \text{if } S(T)(\omega) < K \\ 0 & \text{else} \end{cases}$$

The initial no-arbitrage price of the digital option is

$$\begin{aligned} S_Y(0) &= e^{-\delta T} \mathbb{E}^{\mathbb{Q}} \left[\mathbf{I}_{(S(T)<K)} \right] \\ &= e^{-\delta T} \mathbb{Q}(S(T) < K) \\ &= e^{-\delta T} N(-d_2) \\ &= e^{-0.06} N(0.1414) = 0.5238. \end{aligned}$$

4. The final payoff of a portfolio constitued by a long position on K units of the digital option of Point 3 and a short position on 1 unit of the put option of Point 1 is

$$\begin{aligned} K \cdot Y(T) - 1 \cdot (K - S(T))^+ &= \begin{cases} K - K + S(T) & \text{if } S(T) < K \\ 0 - 0 & \text{else} \end{cases} \\ &= \begin{cases} S(T) & \text{if } S(T) < K \\ 0 & \text{else} \end{cases} . \end{aligned}$$

5. According to Point 4, the derivative can be replicated by a long position on K units of the digital option and a short position on the put option. As a consequence, its initial no-arbitrage price is

$$\begin{aligned} S_X(0) &= K S_Y(0) - S_{put}(0) \\ &= Ke^{-\delta T} N(-d_2) - (Ke^{-\delta T} N(-d_2) - S_0 N(-d_1)) \\ &= S_0 N(-d_1) \\ &= \frac{S_0}{2} = 1. \end{aligned}$$

6. The initial no-arbitrage price of the derivative in Point 5 is

$$S_X(0) = S_0 N(-d_1) = S_0 N\left(- \left(\ln \frac{S_0}{K} + \left(\delta + \frac{\sigma^2}{2} \right) \cdot T \right) \frac{1}{\sigma \sqrt{T}} \right).$$

As the payoff of the derivative depends only on the value of the underlying at maturity, we can retrieve its no-arbitrage price at any $t \in (0, T)$ substituting T by the time to maturity $T - t$ and

S_0 by $S(t)$. Therefore, we obtain

$$\begin{aligned}
S_X(t) &= S(t)N\left(-\left(\ln\frac{S(t)}{K} + \left(\delta + \frac{\sigma^2}{2}\right)\cdot(T-t)\right)\frac{1}{\sigma\sqrt{T-t}}\right) \\
&= S(t)N\left(-\left(\ln\frac{S(t)}{2e^{0.07}} + \left(0.03 + \frac{0.1^2}{2}\right)\cdot(2-t)\right)\frac{1}{0.1\sqrt{2-t}}\right) \\
&= S(t)N\left(-\left(\ln\frac{S(t)}{2} - 0.07 + 0.035\cdot(2-t)\right)\frac{1}{0.1\sqrt{2-t}}\right).
\end{aligned}$$

7. Let

$$\alpha := -\left(\ln\frac{S(t)}{2} - 0.07 + 0.035\cdot(2-t)\right)\frac{1}{0.1\sqrt{2-t}}$$

so that $S_X(t) = S(t)N(\alpha)$. The risky component of the replicating strategy of the derivative in Point 4 is

$$\begin{aligned}
\vartheta_1(t) &= \frac{\partial}{\partial S(t)}S_X(t) \\
&= N(\alpha) + S(t)f_N(\alpha)\cdot\frac{\partial}{\partial S(t)}\alpha \\
&= N(\alpha) + S(t)f_N(\alpha)\cdot\left(-\frac{1}{S(t)}\frac{1}{\sigma\sqrt{T-t}}\right) \\
&= N(\alpha) - \frac{f_N(\alpha)}{0.1\sqrt{2-t}}
\end{aligned}$$

with

$$f_N(\alpha) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\alpha^2}{2}\right).$$

The riskless component of the replicating strategy is given by

$$\begin{aligned}
\vartheta_0(t) &= e^{-\delta t}\left[S_X(t) - S(t)\vartheta_1(t)\right] \\
&= e^{-\delta t}\left[S(t)N(\alpha) - S(t)\left(N(\alpha) - \frac{f_N(\alpha)}{0.1\sqrt{2-t}}\right)\right] \\
&= e^{-\delta t}S(t)\left[N(\alpha) - N(\alpha) + \frac{f_N(\alpha)}{0.1\sqrt{2-t}}\right] \\
&= e^{-\delta t}S(t)\frac{f_N(\alpha)}{0.1\sqrt{2-t}}.
\end{aligned}$$

8. At $t = 0$ we have

$$\begin{aligned}
\alpha &= -\left(\ln\frac{2}{2} - 0.07 + 0.035\cdot(2-0)\right)\frac{1}{0.1\sqrt{2-0}} \\
&= 0
\end{aligned}$$

and

$$f_N(0) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{0^2}{2}\right) = \frac{1}{\sqrt{2\pi}}.$$

Therefore, the components of the replicating strategy are

$$\begin{aligned}\vartheta_0(0) &= S(0)\frac{f_N(0)}{0.1\sqrt{2}} \\ &= 2\frac{1}{\sqrt{2\pi}}\frac{1}{0.1\sqrt{2}} \\ &= 5.6419\end{aligned}$$

and

$$\begin{aligned}\vartheta_1(0) &= \frac{S_X(0)-\vartheta_0(0)}{S(0)} \\ &= \frac{1-5.6419}{2} \\ &= -2.3210.\end{aligned}$$

Therefore, we the replicating strategy at $t=0$ is long on B and short on S.

9. Set $\vartheta_0^{BH} = \vartheta_0(0)$, and $\vartheta_1^{BH} = \vartheta_1(0)$.

$$\begin{aligned}\mathbb{P}\left[\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > X(T)\right] &= \mathbb{P}\left[\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > X(T)\right] \\ &= \mathbb{P}\left[\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\mathbf{I}_{(S(T)<K)}\right] \\ &= \mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)>K\}\right]+ \\ &\quad +\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)\le K\}\right].\end{aligned}$$

Starting from the first probability we get

$$\begin{aligned}&\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)>K\}\right] \\ =\ &\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\cdot 0\right\}\cap\{S(T)>K\}\right] \\ =\ &\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > 0\right\}\cap\{S(T)>K\}\right] \\ =\ &\mathbb{P}\left[\left\{5.6419e^{0.03\cdot 2}-2.3210S(T)>0\right\}\cap\left\{S(T)>2e^{0.07}\right\}\right] \\ =\ &\mathbb{P}\left[\left\{S(T)<\frac{5.6419e^{0.03\cdot 2}}{2.3210}=2.5811\right\}\cap\left\{S(T)>2e^{0.07}=2.145\right\}\right] \\ =\ &\mathbb{P}\left[\{2.145<S(T)<2.5811\}\right].\end{aligned}$$

The other probability reads

$$\begin{aligned}&\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)\le K\}\right] \\ =\ &\mathbb{P}\left[\left\{\vartheta_0^{BH}B(T)+\vartheta_1^{BH}S(T) > S(T)\cdot 1\right\}\cap\{S(T)\le K\}\right] \\ =\ &\mathbb{P}\left[\left\{5.6419e^{0.03\cdot 2}-2.3210S(T)>S(T)\right\}\cap\left\{S(T)\le 2e^{0.07}\right\}\right] \\ =\ &\mathbb{P}\left[\left\{5.6419e^{0.03\cdot 2}-3.3210(T)>0\right\}\cap\left\{S(T)\le 2e^{0.07}\right\}\right] \\ =\ &\mathbb{P}\left[\left\{S(T)<\frac{5.6419e^{0.03\cdot 2}}{3.3210}=1.8039\right\}\cap\left\{S(T)\le 2e^{0.07}=2.145\right\}\right] \\ =\ &\mathbb{P}\left[\{S(T)<2.145\}\right].\end{aligned}$$

Summing up

$$
\begin{aligned}
\mathbb{P}\left[\vartheta_0^{BH} B(T)+\vartheta_1^{BH} S(T)>X(T)\right] &= \mathbb{P}\left[\left\{\vartheta_0^{BH} B(T)+\vartheta_1^{BH} S(T)>S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)>K\}\right]+ \\
&\quad +\mathbb{P}\left[\left\{\vartheta_0^{BH} B(T)+\vartheta_1^{BH} S(T)>S(T)\mathbf{I}_{(S(T)<K)}\right\}\cap\{S(T)\leq K\}\right] \\
&= \mathbb{P}\left[\{2.145<S(T)<2.5811\}\right]+\mathbb{P}\left[\{S(T)<2.145\}\right] \\
&= \mathbb{P}\left[\{S(T)<2.5811\}\right] \\
&= \mathbb{P}\left[S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)\cdot T+\sigma W^{\mathbb{P}}(T)}<2.5811\right] \\
&= \mathbb{P}\left[Z^{\mathbb{P}}<\left(\ln\frac{2.5811}{2}-\left(0.05-\frac{0.1^2}{2}\right)\cdot 2\right)\frac{1}{0.1\sqrt{2}}\right] \\
&= N(1.1672)=\mathrm{NormalDist}(1.1672)=0.87844.
\end{aligned}
$$