Algebra and Geometry (Cod. 30544)

General Exam Mock– December 03, 2020

Time: 2 hours. Total: 150 points.

Multiple choice questions (total: 24 points)

Each question has a single correct answer: write the correct answer in the box on the right. If you want to change your response cancel it and write another answer next to the box. 6 points are assigned for a correct answer, 0 points for a missing answer, -2 point for an incorrect answer.

1. Given an integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$. In the following, $\setminus$ represents difference of sets, that is $A \setminus B = A \cap B^{\complement}$. Then

   (A) $\mathcal{P}_4$ is a vector space of dimension 4 (B) $\mathcal{P}_4 \setminus \mathcal{P}_3$ is a vector space of dimension 1
   (C) $\mathcal{P}_4/\mathcal{P}_3$ is a vector space of dimension 1 (D) none of the others

2. Fix $A \in \mathscr{M}_n(\mathbf{F})$ and $c \in \mathbf{F}$. Then

   (A) $\det(cA) = c\det(A)$ (B) $\det(cA) = \det(A)$
   (C) $\det(cA) = c^n\det(A)$ (D) none of the others

3. Let $(G, \cdot)$ be a group, and fix $x, y, z \in G$. Then

   (A) $x \cdot y = y \cdot x$ (B) $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$
   (C) $(x \cdot y) \cdot z = (y \cdot z) \cdot x$ (D) none of the others

4. Fix matrices $A, B \in \mathscr{M}_n(\mathbf{F})$. Then

   (A) $\mathrm{tr}(A) = \mathrm{tr}(B)$ if $\det(A) = \det(B)$ (B) $\mathrm{tr}(AB - BA) = 1$
   (C) $\det(A) = \det(B)$ if $\mathrm{tr}(A) = \mathrm{tr}(B)$ (D) none of the others

True/False questions (total: 24 points)

Each statement can be either true or false: write T for true or F for false in the box on the right. If you want to change your response cancel it and write another answer next to the box. 4 points are assigned for a correct answer, 0 points for a missing answer, -1 point for an incorrect answer.

1. Let V a vector subspace of $\mathbf{R}^{\mathbf{N}}$. Then V is not finitely generated.
2. Two vector spaces V, W over the same field are isomorphic only if $\dim(V) = \dim(W)$
3. Every field has infinitely many elements.
4. Eigenvectors associated to distinct eigenvalues of $A \in \mathscr{M}_n(\mathbf{F})$ are linearly independent.
5. Let V, W be two finite dimensional v.s.; then $T \in \mathcal{L}(V, W)$ is bijective if and only if it is injective.
6. Let V be a finite dimensional v.s. and fix $S, T \in \mathcal{L}(V)$. Then $(ST)^\wedge = S^\wedge T^\wedge$.

Open answer questions (total: 102 points)

Answers must be written in the corresponding spaces. Each of the four questions will be assigned from 0 to 17 points.
Answers must be adequately justified.

Question 1. 1. Is $A := \{p \in \mathcal{P}_1 : p(p(1)) = 0\}$ a vector space?

2. Is $B := \{p \in \mathcal{P}_2 : p(0) + p(1) = 0\}$ a vector space?

Question 2. 1. Provide the definition of isomorphism between vector spaces

2. Prove that $\{p \in \mathcal{P}_3 : p(1) = 0\}$ is isomorphic to the vector space of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$.

Question 3. Let V be a vector space and $W \subseteq V$ a vector subspace. Provide the definition of coset in the quotient vector space V/W. Then, prove that

$$\forall x, y \in V, \quad (x + W) + (y + W) = x + y + W.$$

Question 4. 1. Let V be a vector space. Provide the definition of generating set $A \subseteq V$.

2. Fix linear operators $S, T \in \mathcal{L}(V, W)$, and let A be a generating subset of V. Show that if $Sx = Tx$ for all $x \in A$ then $S = T$.

Question 5. 1. What is the meaning that a matrix A is diagonalizable?

2. Diagonalize the following real-valued matrix

$$A := \begin{bmatrix} -5 & 6 & -21 \\ 2 & -1 & 7 \\ 2 & -2 & 8 \end{bmatrix}.$$

Question 6. 1. Provide the definition of alternate multilinear form $f \in \mathcal{A}(V)$.

2. Show that if τ is a transposition and $f \in \mathcal{A}(V)$ then $\tau f = -f$.

0.1 Solutions Multiple choices / True-False

Multiple choices:

1	2	3	4
C	C	B	D

True/False:

1	2	3	4	5	6
F	T	F	T	F	F

0.2 Open question

1 1. No. If A were a vector space, it would be closed under scalar multiplications. This is false. Indeed, set $p(x) = 2x - 4/3$. Then

$$p(p(x)) = 2(2x - 4/3) - 4/3 = 4(x-1) \implies p(p(1)) = 0 \implies p \in A.$$

However, at the same time $g := 3p \notin A$. Indeed $g(x) = 3 \cdot p(x) = 6x - 4$ and

$$g(g(x)) = 6(6x-4) - 4 = 36x - 28 \implies g(g(1)) \neq 0.$$

2. Yes. Since B is a nonempty subset of the vector space $\mathcal{P}_2$, it is sufficient to show that B is closed under linear combinations. Indeed, for all $\alpha, \beta \in \mathbb{R}$ and $p, q \in B$, we have

$$(\alpha p + \beta q)(0) + (\alpha p + \beta q)(1) = \alpha(p(0) + p(1)) + \beta(q(0) + q(1)) = 0,$$

which implies that $\alpha p + \beta q \in B$.

2 1. See lecture notes.
2. Recall the result the two finite dimensional vector spaces are isomorphic if and only if they have the same dimension. At this point, the dimension of $C := \{p \in \mathcal{P}_3 : p(1) = 0\}$ is 3 because a basis is $\{x^3 - 1, x^2 - 1, x - 1\}$ (which is a L.I. set in C which generates C itself). Moreover, the dimension of the vector space of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$ is 3 because (considering $d = a + b - c$) a possible basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

As before, this is a L.I. set which generates the whole vector space.

3 Fix $a \in x + W$ and $b \in y + W$. Then there exists $w_a, w_b \in W$ such that $a = x + w_a$ and $b = y + w_b$. Therefore

$$a + b = (x + w_a) + (y + w_b) = (x + y) + (\underbrace{w_a + w_b}_{\in W}) \in x + y + W.$$

This proves that $(x + W) + (y + W) \subseteq x + y + W$.

Viceversa, fix $z \in x + y + W$. Then there exists $w \in W$ such that $z = x + y + w$. Since $0 \in W$, then

$$z = x + y + w = (x + 0) + (y + w) \in (x + W) + (y + W).$$

This proves, conversely, that $x + y + W \subseteq (x + W) + (y + W)$.

4 1. See lecture notes.

2. Fix $v \in V$. Since A is generating, there exists $\alpha_1, \ldots, \alpha_n \in \mathbf{F}$ and $x_1, \ldots, x_n \in A$ such that $v = \sum_{i=1}^n \alpha_i x_i$. Since both S and T are linear it follows that

$$Sv = S(\sum_i \alpha_i x_i) = \sum_i \alpha_i S(x_i) = \sum_i \alpha_i T(x_i) = T(\sum_i \alpha_i x_i) = Tv.$$

Since v is arbitrary, that $S = T$.

5 1. See lecture notes.

2. The eigenvalues of A are the roots of its characteristic polynomial $p(\lambda) = \det(\lambda I - A) = \lambda(\lambda - 1)^2$, i.e., $\lambda = 0$ or $\lambda = 1$ (with algebraic multiplicity 2). Then A has to be rewritten as PDP^{-1}, where D is the diagonal matrix

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and P can be calculated so that the columns are the associated eigenvectors (to each eigenvalue). For $\lambda = 0$ you get that $\{(-3, 1, 1)\}$ is a base for the vector space of vectors x such that $Ax = 0x$. For $\lambda = 1$ you get that $\{(1, 1, 0), (-7/2, 0, 1)\}$ is a base for the vector space of vectors x such that $Ax = x$. It follows that

$$P = \begin{bmatrix} -3 & 1 & -7/2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Numerically we find also that

$$P^{-1} = \begin{bmatrix} -2 & 2 & -7 \\ 2 & -1 & 7 \\ 2 & -2 & 8 \end{bmatrix}.$$

Hence $A = PDP^{-1}$.

6 See lecture notes.