30560 - Mathematical Modelling for Finance
Mock Exam

EXERCISE 1 (30 points out of 100)

Consider a single-period financial market ($t = 0$, $T = 1$) modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and

$$\mathbb{P}(\omega_1) = \frac{1}{4}, \ \ \mathbb{P}(\omega_2) = \frac{1}{4} \ \text{ and } \ \mathbb{P}(\omega_3) = \frac{1}{2}.$$

Three securities are traded. The first one is a riskless asset B which provides the riskless rate $r = 25\%$ whereas the other two are the risky securities S_1, S_2, such that

$$S_1(0) = 4.5, \ \ S_2(0) = \beta$$

$$\begin{bmatrix} S_1(\omega_1) \\ S_1(\omega_2) \\ S_1(\omega_3) \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ \alpha \end{bmatrix}, \qquad \begin{bmatrix} S_2(\omega_1) \\ S_2(\omega_2) \\ S_2(\omega_3) \end{bmatrix} = \begin{bmatrix} 5 \\ 2\alpha \\ \alpha \end{bmatrix}.$$

where $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$.

a) Determine the set of values of $\alpha \in \mathbb{R}$ for which the financial market is complete/incomplete.

b) Determine the set of couples $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ for which the LOP hold.

c) Assume $\alpha = 10$ and $\beta = 15$. Is the market free of arbitrage opportunities? If so, determine the set of state price vectors for the market. If not, find an arbitrage opportunity.

d) From now on assume $\alpha = 10$ and $\beta = 11.5$. Is the market free of arbitrage opportunities? If so, determine the set of state price vectors for the market. If not, find an arbitrage opportunity.

e) A new contingent claim is introduced in the market. Its payoff at time $T = 1$ is equal to

$$X(1)(\omega) = \begin{cases} 10 & \text{if } \min\{S_1(1)(\omega), S_2(1)(\omega)\} \geq 5 \\ 0 & \text{else} \end{cases}$$

Compute its terminal payoff $X(1)$ and the set of its no arbitrage prices at $t = 0$.

f) Assume that the contingent claim in the previous point trades at $t = 0$ at the price of 4. Is the extended market (the one with the three primary assets and the contingent claim) arbitrage free and complete? Determine the set of stochastic discount factors for the extended market.

EXERCISE 2 (20 points out of 100)

Consider a single-period financial market ($t = 0$, $T = 1$) modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and

$$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = \frac{1}{3}.$$

Only two risky securities S_1 and S_2 are traded. The securities are such that

$$S_1(0) = 5, \ S_2(0) = 2$$

$$S_1(1) = \begin{bmatrix} S_1(1)(\omega_1) \\ S_1(1)(\omega_2) \\ S_1(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \qquad S_2(1) = \begin{bmatrix} S_2(1)(\omega_1) \\ S_2(1)(\omega_2) \\ S_2(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}.$$

The market is free of arbitrage opportunities. As in the Lecture Notes, let $x = [S_1(1) \;\; S_2(2)]$. It holds

$$\left(\mathbb{E}\left[xx^T\right]\right)^{-1} = \begin{bmatrix} 1.4 & -3.4 \\ -3.4 & 8.4 \end{bmatrix}.$$

a) Determine the (only) traded stochastic discount factor m^* for this market.

b) Determine R^* and its replicating strategy.

c) Knowing that

$$R^{e^*} = \frac{1}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix},$$

find the constant mimicking portfolio return, R^{CMR}, and its replicating strategy.

EXERCISE 3 (50 points out of 100)

Consider a multiperiod discrete market with $t = 0, 1, 2$ and with the following information structure:

$$\begin{array}{ccccc} & & & & \omega_1 \\ & & f_1^1 & \begin{matrix} \nearrow \\ \searrow \end{matrix} & \\ & \nearrow & & & \omega_2 \\ f_1^0 & & & & \\ & \searrow & & & \omega_3 \\ & & f_2^1 & \begin{matrix} \nearrow \\ \searrow \end{matrix} & \\ & & & & \omega_4 \end{array}$$

Two securities are traded in the market. The first is a *locally risk-free asset* B that provides the locally riskless interest rate

$$r(0) = 3\%, \quad r(1)(f_1^1) = 5\% \ \text{ and } r(1)(f_2^1) = 2\%.$$

The second security is a *risky asset* S, with time 0 price

$$S(0) = 10,$$

with time 1 prices

$$S(1)(f_1^1) = 14.5 \text{ and } S(1)(f_2^1) = 7.5$$

and with time 2 prices

$$S(2)(\omega_1) = 16.24, \quad S(2)(\omega_2) = 14.21 \quad S(2)(\omega_3) = 8.25 \quad S(2)(\omega_4) = 6.75.$$

1. Determine the price process of the locally riskless security $B = \{B(t)\}_{t=0,1,2}$.
2. Find the set of risk neutral probabilities $\mathbb{Q}$ for the market.
3. Is the market arbitrage free?
4. A *European put option* with maturity $T = 2$ on the risky security S and strike price $K = 9.4$ is introduced in the market. Find its payoff $p(2)$ at maturity $T = 2$.
5. Find the no arbitrage prices at $t = 1$ of the put option of the previous Question.
6. Find the no arbitrage price at $t = 0$ of the put option of the previous Question.
7. Suppose the option of point 4 is now of American type. Is there any optimal early exercise opportunity? If your answer is positive, find the early exercise premium of the option.
8. Suppose the historical probability on $\Omega = \{\omega_1, ..., \omega_4\}$ is

   $$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 25\%, \; \mathbb{P}(\omega_3) = 35\% \text{ and } \mathbb{P}(\omega_4) = 15\%$$

   Suppose that at $t = 1$ on f_2^1 and investor with *log-utility* and wealth $v > 0$ wants to maximize her expected utility from terminal wealth. Find her optimal asset allocation at time $t = 1$ on f_2^1, and her optimal terminal wealth on ω_3 and ω_4.

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Mock Exam - Solutions

SOLUTION OF EXERCISE 1

a) The payoff matrix of this financial market is

$$\mathcal{A}_\alpha = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 2\alpha \\ 1.25 & \alpha & \alpha \end{bmatrix}$$

and its determinant is equal to

$$\begin{aligned} \det \mathcal{A}_\alpha &= 1.25 \det \begin{bmatrix} 1 & 15 & 5 \\ 1 & 0 & 2\alpha \\ 1 & \alpha & \alpha \end{bmatrix} \\ &= 1.25(-2\alpha^2 - 20\alpha). \end{aligned}$$

It holds

$$\det \mathcal{A}_\alpha = 0 \Longleftrightarrow \alpha(\alpha - 10) = 0$$

Therefore, if $\alpha \notin \{0, 10\}$, $rk(\mathcal{A}_\alpha) = 3 = |\Omega|$ and the market is complete. If $\alpha \in \{0, 10\}$, as the minor given by the first two rows and the first two columns is different from zero and $\det \mathcal{A}_\alpha = 0$, we have $rk(\mathcal{A}_\alpha) = 2 < 3 = |\Omega|$ and the market is not complete.

b) For $\alpha \notin \{0, 10\}$, there are no redundant securities and, therefore, the LOP holds.

If $\alpha = 0$ the payoff matrix becomes

$$\mathcal{A}_0 = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 0 \\ 1.25 & 0 & 0 \end{bmatrix}.$$

In this case, $S_2(1) = 3S_1(1)$ and the LOP holds if and only if $S_2(0) = 3S_1(0)$. This constraint delivers

$$4.5 = 3\beta$$

or $\beta = 1.5$. Therefore, if $\alpha = 0$ the LOP holds if and only if $\beta = 1.5$.

If $\alpha = 10$ the payoff matrix becomes

$$\mathcal{A}_{10} = \begin{bmatrix} 1.25 & 15 & 5 \\ 1.25 & 0 & 20 \\ 1.25 & 10 & 10 \end{bmatrix}.$$

As we know that one of the three securities is redundant, say S_2, we could solve the linear system

$$[B(1) \quad S_1(1)] \begin{bmatrix} \vartheta_0 \\ \vartheta_1 \end{bmatrix} = S_2(1)$$

which is equivalent to

$$\begin{cases} 1.25\vartheta_0 + 15\vartheta_1 = 5 \\ 1.25\vartheta_0 + 0\vartheta_1 = 20 \\ 1.25\vartheta_0 + 10\vartheta_1 = 10 \end{cases}$$

$$\vartheta_0 = 16 \quad \text{and} \quad \vartheta_1 = -1.$$

Now, as $S_2(1) = 16B(1) - S_1(1)$, the LOP holds if and only if this relationship is verified also at $t = 0$, namely if

$$\begin{aligned} S_2(0) &= 16B(0) - S_1(0) \\ \beta &= 16 - 4.5 = 11.5. \end{aligned}$$

Therefore, if $\alpha = 10$, the LOP holds if and only if $\beta = 11.5$.

c) When $\alpha = 10$ and $\beta = 15$, the LOP does not hold and the market cannot be arbitrage free. In particular, we know that the only initial price of S_2 which is compatible with the LOP is $S_2(0) = 11.5$. Therefore, if S_2 trades at the initial price of 15, it is actually too expensive and an arbitrage strategy consists on (short) selling it while buying its replicating strategy computed above. Formally, the strategy underlying this arbitrage opportunity is given by

$$\underbrace{\vartheta_0 = 16, \quad \vartheta_1 = -1}_{\text{replication of } S_2} \quad \text{and} \quad \underbrace{\vartheta_2 = -1}_{\text{short position on } S_2}.$$

d) When $\alpha = 10$ and $\beta = 11.5$, the LOP holds. To check whether NA holds in the market as well, we look for state price vectors, solving the system

$$(\mathcal{A}_{10})^T \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} B(0) \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

which is equivalent to

$$\begin{cases} 1.25\psi_1 + 1.25\psi_2 + 1.25\psi_3 = 1 \\ 15\psi_1 + 0\psi_2 + 10\psi_3 = 4.5 \\ 5\psi_1 + 20\psi_2 + 10\psi_3 = 11.5 \end{cases}$$

which is solved by

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} - \frac{2}{3}\psi_3 \\ \frac{1}{2} - \frac{1}{3}\psi_3 \\ \psi_3 \end{bmatrix}.$$

These state price vectors have strictly positive component as long as

$$\psi_1 = \frac{3}{10} - \frac{2}{3}\psi_3 > 0 \Longleftrightarrow \psi_3 < \frac{9}{20} = 0.45$$

$$\psi_2 = \frac{1}{2} - \frac{1}{3}\psi_3 > 0 \Longleftrightarrow q_3 < \frac{3}{2} = 1.5.$$

If $0 < \psi_3 < \frac{9}{20}$, the market admits infinitely many strictly positive state price vectors and, according to the FFTAP, the market is arbitrage free (and incomplete).

e) As

$$\min\{S_1(1), S_2(1)\} = \begin{cases} \min\{15,5\} = 5 & \text{on } \omega_1 \\ \min\{0,20\} = 0 & \text{on } \omega_2 \\ \min\{10,10\} = 10 & \text{on } \omega_3 \end{cases}$$

we have

$$X(1) = \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix}.$$

The no arbitrage price at $t = 0$ is equal to

$$\begin{aligned} X(0) &= \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix} X(1) \\ &= 10\psi_1 + 10\psi_3 \\ &= 10\left(\frac{3}{10} - \frac{2}{3}\psi_3\right) + 10\psi_3 \\ &= \frac{10}{3}\psi_3 + 3. \end{aligned}$$

As NA is preserved as long as $0 < \psi_3 < \frac{9}{20}$, the set of admissible no arbitrage prices at $t = 0$ is

$$X(0) \in (3, 4.5)$$

as $\frac{10}{3} \cdot 0 + 3 = 3$ and $\frac{10}{3}\frac{9}{20} + 3 = 4.5$.

f) If the contingent claim trades at $4 \in (3, 4.5)$, the extended market is arbitrage free. Moreover,

$$\frac{10}{3}\psi_3 + 3 = 4$$

delivers $\psi_3 = \frac{3}{10}$ and

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{10} - \frac{2}{3}\frac{3}{10} = \frac{1}{10} = 0.1 \\ \frac{1}{2} - \frac{1}{3}\frac{3}{10} = \frac{4}{10} = 0.4 \\ \frac{3}{10} = 0.3 \end{bmatrix}.$$

As there exists one and only one strictly positive state price vector, the market is arbitrage free and complete (according to the SFTAP). From the relationships

$$\psi_k = \frac{q_k}{1+r} \quad \text{and} \quad m_k = \frac{q_k}{p_k(1+r)}$$

we get

$$m_k = \frac{q_k}{p_k(1+r)} = \frac{\psi_k(1+r)}{p_k(1+r)} = \frac{\psi_k}{p_k}$$

that delivers

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \frac{0.1}{0.25} = 0.4 \\ \frac{0.4}{0.25} = 1.6 \\ \frac{0.3}{0.5} = 0.6 \end{bmatrix}.$$

Solution of EXERCISE 2

a) We know from the Lecture Notes that it holds

$$\begin{aligned} \vartheta_{m^*} &= \left(\mathbb{E}\left[xx^T\right]\right)^{-1} S \\ m^* &= \mathcal{A}\left(\mathbb{E}\left[xx^T\right]\right)^{-1} S \end{aligned}$$

where

$$\mathcal{A} = \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore, we get

$$\begin{aligned} \vartheta_{m^*} &= \begin{bmatrix} 1.4 & -3.4 \\ -3.4 & 8.4 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} m^* &= \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} \\ &= \begin{bmatrix} 1.0 \\ 0.8 \\ 0.4 \end{bmatrix}. \end{aligned}$$

b) We know from the Lecture Notes that

$$R^* = \frac{m^*}{\pi\left[m^*\right]}.$$

The no-arbitrage price $\pi\left[m^*\right]$ of the traded stochastic discount factor m^* is

$$\begin{aligned} \pi\left[m^*\right] &= \mathbb{E}\left[\left(m^*\right)^T m^*\right] \\ &= \frac{1}{3}\left(1^2 + 0.8^2 + 0.4^2\right) \\ &= 0.6. \end{aligned}$$

As m^* can be replicated by 0.2 units of S_1 and -0.2 units of S_2 we have

$$\vartheta_{R^*} = \begin{bmatrix} \frac{0.2}{0.6} \\ \frac{-0.2}{0.6} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

and

$$\begin{aligned} R^* &= \underbrace{\frac{1}{0.6}}_{\frac{1}{\pi[m^*]}} \underbrace{\begin{bmatrix} 1.0 \\ 0.8 \\ 0.4 \end{bmatrix}}_{m^*} \\ &= \begin{bmatrix} \frac{5}{3} \\ \frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}. \end{aligned}$$

c) From the Lecture Notes we know that

$$R^{CMR} = R^* + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} R^{e^*}.$$

Then,

$$\begin{aligned} \mathbb{E}\left[R^*\right] &= \frac{1}{3}\left(\frac{1}{3}(5+4+2)\right) = \frac{11}{9} \\ \mathbb{E}\left[(R^*)^2\right] &= \frac{1}{3}\left(\frac{1}{3^2}\left(5^2+4^2+2^2\right)\right) = \frac{5}{3} \\ \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} &= \frac{\frac{5}{3}}{\frac{11}{9}} = \frac{15}{11}. \end{aligned}$$

Therefore, we have

$$R^{CMR} = \underbrace{\frac{1}{3}\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}}_{R^*} + \underbrace{\frac{15}{11}}_{\frac{\mathbb{E}[(R^*)^2]}{\mathbb{E}[R^*]}} \underbrace{\frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}_{R^{e^*}} = \begin{bmatrix} \frac{15}{11} \\ \frac{18}{11} \\ \frac{9}{11} \end{bmatrix}$$

$$\frac{1}{3}\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} + \frac{15}{11}\frac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

As for the replicating strategy of R^{CMR} we have

$$\vartheta_{R^{CMR}} = \vartheta_{R^*} + \frac{\mathbb{E}\left[(R^*)^2\right]}{\mathbb{E}\left[R^*\right]} \vartheta_{R^{e^*}}.$$

To compute this, we must find $\vartheta_{R^{e^*}}$ such that

$$R^{e^*} = \mathcal{A}\vartheta_{R^{e^*}}$$

or

$$\begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We can focus on the first two equations

$$\begin{cases} -\frac{2}{9} = 9x + 4y \\ \frac{2}{9} = 6x + 2y \end{cases}$$

solved by

$$x = \frac{2}{9} \quad \text{and} \quad y = -\frac{5}{9}$$

that (of course) satisfy also the last equation $\frac{1}{9} = 3 \cdot \frac{2}{9} + 1 \cdot \left(-\frac{5}{9}\right) = \frac{1}{9}$. Therefore,

$$\vartheta_{R^{e*}} = \begin{bmatrix} \frac{2}{9} \\ -\frac{5}{9} \end{bmatrix}$$

and

$$\begin{aligned} \vartheta_{R^{CMR}} &= \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} + \frac{15}{11} \begin{bmatrix} \frac{2}{9} \\ -\frac{5}{9} \end{bmatrix} \\ &= \begin{bmatrix} \frac{7}{11} \\ -\frac{12}{11} \end{bmatrix}. \end{aligned}$$

Solution of EXERCISE 3

1. The prices of security B are $B(0) = 1$,

$$B(1)(f_1^1) = B(1)(f_2^1) = 1.03$$

and at the final date $T = 2$

$$\begin{aligned} B(2)(\omega_1) = B(2)(\omega_2) = 1.03 \cdot 1.05 = 1.0815 \\ B(2)(\omega_3) = B(2)(\omega_4) = 1.03 \cdot 1.02 = 1.0506. \end{aligned}$$

2. We look for risk neutral probabilities $\mathbb{Q}$ for the market. We have to solve the systems

$$\begin{cases} S(0) = \frac{1}{1+r(0)} \left\{ S(1)(f_1^1)\mathbb{Q}[f_1^1] + S(1)(f_2^1)\mathbb{Q}[f_2^1] \right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases} \tag{1}$$

for m_0,

$$\begin{cases} S(1)(f_1^1) = \frac{1}{1+r(1)(f_1^1)} \left\{ S(2)(\omega_1)\mathbb{Q}[\omega_1|f_1^1] + S(2)(\omega_2)\mathbb{Q}[\omega_2|f_1^1] \right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases} \tag{2}$$

for $m_{1,1}$, and

$$\begin{cases} S(1)(f_2^1) = \frac{1}{1+r(1)(f_2^1)}\left\{S(2)(\omega_3)\mathbb{Q}[\omega_3|f_2^1] + S(2)(\omega_4)\mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases} \tag{3}$$

for $m_{1,2}$. System (1) can be rewritten as

$$\begin{cases} 10 = \frac{1}{1.03}\left\{14.5 \cdot \mathbb{Q}[f_1^1] + 7.5 \cdot \mathbb{Q}[f_2^1]\right\} \\ \mathbb{Q}[f_1^1] + \mathbb{Q}[f_2^1] = 1 \\ \mathbb{Q}[f_1^1], \mathbb{Q}[f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[f_1^1] &= 0.4 \\ \mathbb{Q}[f_2^1] &= 0.6. \end{aligned}$$

System (2) can be rewritten as

$$\begin{cases} 14.5 = \frac{1}{1.05}\left\{16.24 \cdot \mathbb{Q}[\omega_1|f_1^1] + 14.21 \cdot \mathbb{Q}[\omega_2|f_1^1]\right\} \\ \mathbb{Q}[\omega_1|f_1^1] + \mathbb{Q}[\omega_2|f_1^1] = 1 \\ \mathbb{Q}[\omega_1|f_1^1], \mathbb{Q}[\omega_2|f_1^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_1|f_1^1] &= 0.5 \\ \mathbb{Q}[\omega_2|f_1^1] &= 0.5 \end{aligned}$$

whereas System (3) can be rewritten as

$$\begin{cases} 7.5 = \frac{1}{1.02}\left\{8.25 \cdot \mathbb{Q}[\omega_3|f_2^1] + 6.75 \cdot \mathbb{Q}[\omega_4|f_2^1]\right\} \\ \mathbb{Q}[\omega_3|f_2^1] + \mathbb{Q}[\omega_4|f_2^1] = 1 \\ \mathbb{Q}[\omega_3|f_2^1], \mathbb{Q}[\omega_4|f_2^1] > 0 \end{cases}$$

and is solved by

$$\begin{aligned} \mathbb{Q}[\omega_3|f_2^1] &= 0.6 \\ \mathbb{Q}[\omega_4|f_2^1] &= 0.4. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{Q}[\omega_1] &= \mathbb{Q}[f_1^1] \cdot \mathbb{Q}[\omega_1|f_1^1] = 0.4 \cdot 0.5 = 0.2 \\ \mathbb{Q}[\omega_2] &= \mathbb{Q}[f_1^1] \cdot \mathbb{Q}[\omega_2|f_1^1] = 0.4 \cdot 0.5 = 0.2 \\ \mathbb{Q}[\omega_3] &= \mathbb{Q}[f_2^1] \cdot \mathbb{Q}[\omega_3|f_2^1] = 0.6 \cdot 0.6 = 0.36 \\ \mathbb{Q}[\omega_4] &= \mathbb{Q}[f_2^1] \cdot \mathbb{Q}[\omega_4|f_2^1] = 0.6 \cdot 0.4 = 0.24 \end{aligned}$$

3. According to the SECOND Fundamental Theorem of Asset Pricing, the market is ARBITRAGE-FREE AND COMPLETE since THERE EXISTS A UNIQUE RISK-NEUTRAL PROBABILITY MEASURE.

4. The European *put option* with maturity $T = 2$ on the risky security S and strike price $K = 9.4$ has final payoff $p(2)$ at maturity $T = 2$ equal to

$$p(2) = \begin{cases} (9.4 - 16.24)^+ = 0 & \text{on } \omega_1 \\ (9.4 - 14.21)^+ = 0 & \text{on } \omega_2 \\ (9.4 - 8.25)^+ = 1.15 & \text{on } \omega_3 \\ (9.4 - 6.75)^+ = 2.65 & \text{on } \omega_4 \end{cases}.$$

5. The no arbitrage prices at $t = 1$ of the put option of the previous Question are

$$p(1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{p(2)}{1 + r(1)}\right| \mathcal{P}_1\right] = \begin{cases} \dfrac{0 \cdot 0.5 + 0 \cdot 0.5}{1.05} = 0 & \text{if } f_1^1 \\ \\ \dfrac{1.15 \cdot 0.6 + 2.65 \cdot 0.4}{1.02} = 1.7157 & \text{if } f_2^1 \end{cases}.$$

6. At $t = 0$, the no arbitrage price of the put option of Question 4 is

$$p(0) = \mathbb{E}^{\mathbb{Q}}\left[\frac{p(1)}{1 + r(0)}\right] = \frac{0 \cdot 0.4 + 1.7157 \cdot 0.6}{1.03} = 0.99944$$

7. Suppose the option of point 4 is now of American type. Is there any optimal early exercise opportunity? If your answer is positive, find the early exercise premium of the option.

   The American option coincides with the European one at maturity.

$$p_{Am}(1) = \max\left((K - S(1))^+; p(1)\right) = \begin{cases} \max(0; 0) = 0 & \text{if } f_1^1 \\ \\ \max(1.9; 1.7157) = 1.9 > 1.7157 & \text{if } f_2^1 \end{cases}.$$

   and we see that there is an optimal early exercise opportunity at $t = 1$ on f_2^1. The value at $t = 0$ of the American option is

$$p_{Am}(0) = \max\left((K - S(0))^+; \mathbb{E}^{\mathbb{Q}}\left[\frac{p_{Am}(1)}{1 + r(0)}\right]\right) = \max(0; 1.1068) = 1.1068$$

   since

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{p_{Am}(1)}{1 + r(0)}\right] = \frac{0 \cdot 0.4 + 1.9 \cdot 0.6}{1.03} = 1.1068$$

   Hence the early exercise premium is

$$p_{Am}(0) - p(0) = 1.1068 - 0.99944 = 0.10736$$

8. The historical probability on $\Omega = \{\omega_1, ..., \omega_4\}$ is

$$\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 25\%, \;\; \mathbb{P}(\omega_3) = 35\% \text{ and } \mathbb{P}(\omega_4) = 15\%$$

This implies that

$$\mathbb{P}(\omega_1 | \, f_1^1) = \mathbb{P}(\omega_2 | \, f_1^1) = 50\%, \;\; \mathbb{P}(\omega_3 | \, f_2^1) = 70\% \text{ and } \mathbb{P}_1(\omega_4 | \, f_2^1) = 30\%$$

In our setting we have to look for $\vartheta\left(1\right)\left(f_2^1\right)$ such that

$$\begin{aligned}
\vartheta_0\left(1\right) B\left(1\right) + \vartheta_1\left(1\right) S\left(1\right) &= V\left(1\right) = v \\
x = \frac{\vartheta_1\left(1\right) S\left(1\right)}{v} \quad &\text{and} \quad 1 - x = \frac{\vartheta_0\left(1\right) B\left(1\right)}{v} \\
\vartheta_1\left(1\right) = x\frac{v}{S\left(1\right)} \quad &\text{and} \quad \vartheta_0\left(1\right) = \left(1-x\right)\frac{v}{B\left(1\right)} \\
V\left(2\right) &= \vartheta_1\left(1\right) S\left(2\right) + \vartheta_0\left(1\right) B\left(2\right) \\
&= x\frac{v}{S\left(1\right)} S\left(2\right) + \left(1-x\right)\frac{v}{B\left(1\right)} B\left(2\right) \\
&= v\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\frac{B\left(2\right)}{B\left(1\right)}\right) \\
&= v\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\left(1+r\left(1\right)\right)\right)
\end{aligned}$$

and

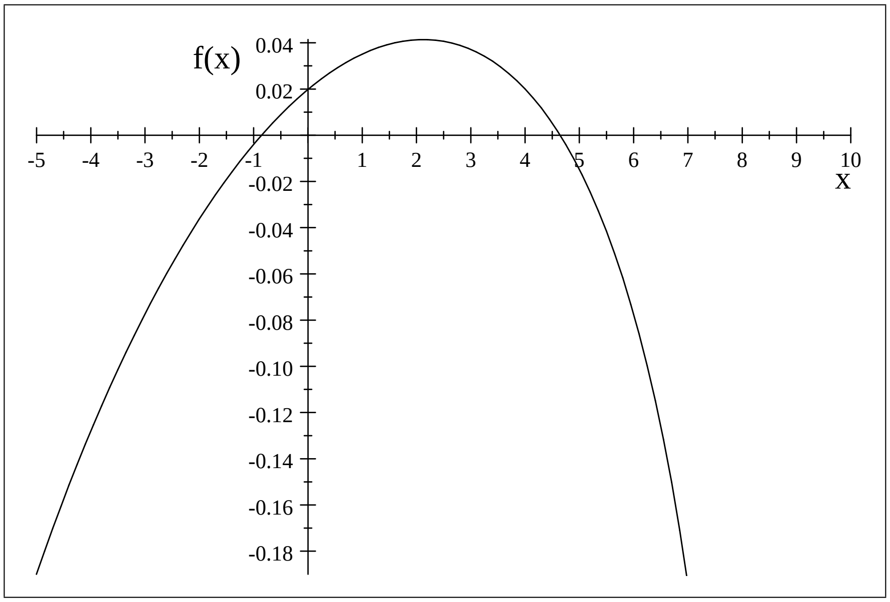
$$\begin{aligned}
F\left(1, v\right) &= \max_x \left(\mathbb{E}_1\left[F\left(2, V\left(2\right)\right)\right]\right) \\
&= \max_x \left(\mathbb{E}_1\left[\ln\left(v\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\left(1+r\left(1\right)\right)\right)\right)\right]\right) \\
&= \ln v + \max_x \left(\mathbb{E}_1\left[\ln\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\left(1+r\left(1\right)\right)\right)\right]\right)
\end{aligned}$$

On f_2^1 the value function $F\left(1, v\right)$ becomes

$$\begin{aligned}
F\left(1, v\right) &= \ln v + \max_x \left(\mathbb{E}_1\left[\ln\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\left(1+r\left(1\right)\right)\right)\right]\right) \\
&= \max_x \left(\mathbb{E}_1\left[\ln\left(v\left(x\frac{S\left(2\right)}{S\left(1\right)} + \left(1-x\right)\left(1+r\left(1\right)\right)\right)\right)\right]\right) \\
&= \ln v + \max_x \left(\mathbb{E}_1\left[F\left(1, v\right)\right]\right) \\
&= \ln v + \max_x \left(0.7\ln\left(x1.1 + \left(1-x\right)1.02\right) + 0.3\ln\left(x0.9 + \left(1-x\right)1.02\right)\right)
\end{aligned}$$

The function we have to maximize is

$$f\left(x\right) = 0.7\ln\left(0.08x + 1.02\right) + 0.3\ln\left(1.02 - 0.12x\right)$$



We compute

$$
\begin{gathered}
f'(x) = 0.7\,(0.08x + 1.02)^{-1}\,0.08 + 0.3\,(1.02 - 0.12x)^{-1}\,(-0.12) = 0 \\
f'(x) = 0.056\,(0.08x + 1.02)^{-1} - 0.036\;(1.02 - 0.12x)^{-1} \geq 0 \\
f'(x) \geq 0 \;\text{ if } 1.02 - 0.12x \leq 0 \text{ i.e. for } x \geq \frac{1.02}{0.12} = 8.5 \\
\text{For } x < \frac{1.02}{0.12} = 8.5 \text{ we have } 1.02 - 0.12x > 0 \text{ and} \\
f'(x) \geq 0 \text{ if } \frac{0.056}{0.08x + 1.02} \geq \frac{0.036}{1.02 - 0.12x} \\
\frac{1.02 - 0.12x}{0.08x + 1.02} \geq \frac{0.036}{0.056} = 0.642\,86 \\
f'(x) \geq 0 \text{ if} \\
1.02 - 0.12x \geq 0.642\,86\,(0.08x + 1.02) \\
x \leq x^* = 2.125
\end{gathered}
$$

Hence $f'(x) \geq 0$ for $x \leq x^*$, and $f'(x) \leq 0$ for $x \geq x^*$. This implies that $x^* = 2.125$ is a maximizing point of f. Hence

$$
\begin{aligned}
x^* &= 2.125 \;\text{ and } 1 - x^* = 1 - 2.125 = -1.125 \\
\vartheta_1(1)\left(f_2^1\right) &= x^* \frac{v}{S(1)\left(f_2^1\right)} = \frac{2.125}{7.5} v = 0.283\, v \\
\text{and }\; \vartheta_0(1)\left(f_2^1\right) &= (1 - x^*) \frac{v}{B(1)\left(f_2^1\right)} = \frac{-1.125}{1.03} v = -1.092\, v \\
F(1, v) &= \ln v + f(x^*) \quad \text{on } f_2^1
\end{aligned}
$$

and the optimal wealth at maturity is

$$\begin{aligned}
V_\vartheta(2)(\omega_3) &= \vartheta_0(1)\left(f_2^1\right)B(2)\left(f_2^1\right)+\vartheta_1(1)\left(f_2^1\right)S(2)(\omega_3)=v\left(-1.092\cdot 1.0506+0.283\cdot 8.25\right)=1. \\
V_\vartheta(2)(\omega_4) &= \vartheta_0(1)\left(f_2^1\right)B(2)\left(f_2^1\right)+\vartheta_1(1)\left(f_2^1\right)S(2)(\omega_4)=v\left(-1.092\cdot 1.0506+0.283\cdot 6.75\right)=0.
\end{aligned}$$