Università Bocconi - Master of Science in Finance (CLEFINLS)

CLEFINLS 20188: Quantitative Finance and Derivatives I

EXERCISES

The One-period Financial Market Model

EXERCISE 1

In a one-period financial market, the following securities are available for trade: a risk-free asset, yielding a riskless rate $r = 0.25$; two risky stocks, S_1 and S_2, with time-1 payoffs given by

$$\begin{array}{cc} S_1(1)(\omega_1) = 2 & S_2(1)(\omega_1) = 5 \\ S_1(1)(\omega_2) = 1.2 & S_2(1)(\omega_2) = 2 \\ S_1(1)(\omega_3) = 4 & S_2(1)(\omega_3) = 1.5 \end{array}$$

1. State whether the market is complete and arbitrage-free if the two risky securities, S_1 and S_2, have time-0 prices $S_1(0) = 1.92$ and $S_2(0) = 3$.
2. Answer to the same questions of the previous point in case $S_1(0) = 0.88$ and $S_2(0) = 3.5$. If the market is not arbitrage-free, provide an example of arbitrage.
3. If security S_2 takes the following values at time $t = 1$

$$\begin{array}{c} S_2(1)(\omega_1) = 6.75 \\ S_2(1)(\omega_2) = 3.55 \\ S_2(1)(\omega_3) = 14.75 \end{array}$$

   and has time-0 price $S_2(0) = 6.68$, while S_1 is as at point 1, is the market arbitrage-free and complete?
4. If the security S_2 of point 3 has time-0 price $S_2(0) = 8$, is the market arbitrage-free? Is it complete?

Solution

1. We employ the First Fundamental Theorem of Asset Pricing to check whether the market is arbitrage-free. We have to see if there exists $z \in \Re^3$, $z >> 0$, such that

$$A^T z = \begin{bmatrix} 1 \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

where

$$A^T = \begin{bmatrix} 1+r & 1+r & 1+r \\ S_1(1)(\omega_1) & S_1(1)(\omega_2) & S_1(1)(\omega_3) \\ S_2(1)(\omega_1) & S_2(1)(\omega_2) & S_2(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 1.25 & 1.25 & 1.25 \\ 2 & 1.2 & 4 \\ 5 & 2 & 1.5 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ S_1(0) \\ S_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1.92 \\ 3 \end{bmatrix}$$

Since $\det(A) = 11 \neq 0$, the system has a unique solution z. For the market to be arbitrage-free, z must have all positive components. Hence, we solve the system

$$\begin{bmatrix} 1.25 & 1.25 & 1.25 \\ 2 & 1.2 & 4 \\ 5 & 2 & 1.5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.92 \\ 3 \end{bmatrix}$$

obtaining $z = \begin{bmatrix} 0.5 \\ 0.1 \\ 0.2 \end{bmatrix} >> 0$. The market is thus arbitrage-free.

In order to verify completeness, we verify that $rank(A) = K = 3$. Indeed, since $\det(A) = 11 \neq 0$, we have $rank(A) = 3$.

2. Matrix A is the same as at point 1, hence the market is complete. How about NA? NA holds if and only if there exists $z \in \Re^3,\ z \gg 0$ such that

$$A^T z = \begin{bmatrix} 1 \\ 0.88 \\ 3.5 \end{bmatrix}$$

Since $\det(A) = 11 \neq 0$, the system admits a unique solution, given by

$$z = \begin{bmatrix} 0.6 \\ 0.4 \\ -0.2 \end{bmatrix}$$

Since $z_3 < 0$, the market is NOT arbitrage-free.

To set up an arbitrage strategy, we look for a strategy $\vartheta = [\vartheta_0; \vartheta_1; \vartheta_2]^T$ replicating the Arrow-Debreu security associated with the state ω_3, i.e. $V_\vartheta(1) = [0; 0; 1]^T > 0$. For such a strategy ϑ, we have

$$V_\vartheta(0) = \sum_{k=1}^{3} V_\vartheta(1) \cdot z_k = 0 \cdot 0.6 + 0 \cdot 0.4 + 1 \cdot (-0.2) = -0.2 < 0$$

i.e. an arbitrage of the second type. Solving the system

$$V_\vartheta(1) = A\vartheta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get $\vartheta_0 = -\frac{8}{44}$; $\vartheta_1 = \frac{15}{44}$ and $\vartheta_2 = -\frac{4}{44}$. One can further verify that

$$V_\vartheta(0) = -\frac{8}{44} \cdot 1 + \frac{15}{44} \cdot 0.88 + \left(-\frac{4}{44}\right) \cdot 3.5 = -0.2,$$

as expected.

3. Matrix A is in this case given by

$$A^T = \begin{bmatrix} 1+r & 1+r & 1+r \\ S_1(1)(\omega_1) & S_1(1)(\omega_2) & S_1(1)(\omega_3) \\ S_2(1)(\omega_1) & S_2(1)(\omega_2) & S_2(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 1.25 & 1.25 & 1.25 \\ 2 & 1.2 & 4 \\ 6.75 & 3.55 & 14.75 \end{bmatrix}$$

To see whether NA holds, we analyze the system

$$A^T z = \begin{bmatrix} 1 \\ 1.92 \\ 6.68 \end{bmatrix}$$

In this case, $\det(A^T) = 0$ and $rank(A^T) = 2$, since, for example

$$\det \begin{bmatrix} 1.25 & 1.25 \\ 2 & 1.2 \end{bmatrix} \neq 0$$

To apply the Theorem of Rouché-Capelli, we compute the rank of the matrix augmented with the vector of the known terms. We know that the third column of A^T is a linear combination of the first two, so that we can directly compute

$$\det \begin{bmatrix} 1.25 & 1.25 & 1 \\ 2 & 1.2 & 1.92 \\ 6.75 & 3.55 & 6.68 \end{bmatrix} = 0,$$

so that the rank of the augmented matrix is itself 2. As a consequence, the system admits infinite solutions, because the number of unknowns, 3, is greater than the number of linearly independent equations, 2. We can solve

$$\begin{cases} 1.25z_1 + 1.25z_2 + 1.25z_3 = 1 \\ 2z_1 + 1.2z_2 + 4z_3 = 1.92 \end{cases},$$

to obtain

$$\begin{cases} z_1 = 0.64 - 1.4z_2 \\ z_2 \in \Re \\ z_3 = 0.4z_2 + 0.16 \end{cases}$$

To ensure that $z \gg 0$, we impose

$$\begin{cases} z_1 = 0,64 - 1,4z_2 > 0 \\ z_2 > 0 \\ z_3 = 0,4z_2 + 0,16 > 0 \end{cases},$$

i.e. $z_2 \in]0;\ 0.457143[$. Hence, the market is arbitrage-free (i.e., there exists a $z \gg 0$). We can get to the same result by employing risk-neutral probabilities. Setting $q_k = \mathbb{Q}(\omega_k) = z_k(1+r)$ for $k = 1, 2, 3$, we obtain

$$\begin{cases} q_1 = 0.8 - 1.4q_2 \\ q_2 \in \left(0; \frac{4}{7}\right) \\ q_3 = 0.4q_2 + 0.2 \end{cases}$$

As far as completeness is concerned, the market is incomplete, since $rank(A) = 2 < K = 3$. We can get to the same result by applying the Second Fundamental Theorem of Asset Pricing. We know that the market is arbitrage-free because there exists $z >> 0$, but since z is not unique, the market is incomplete.

4. In this case, the system

$$A^T z = \begin{bmatrix} 1 \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

does not admit solutions. Indeed, while $rank(A^T) = 2$, as in the previous point, the rank of the matrix augmented with the vector of initial prices is in this case equal to 3, since

$$\det \begin{bmatrix} 1.25 & 1.25 & 1 \\ 2 & 1.2 & 1.92 \\ 6.75 & 3.55 & 8 \end{bmatrix} = -1.32 \neq 0$$

Thus, by the Theorem of Rouché-Capelli, the system does not admit solutions. As a result, the market is not arbitrage-free. Furthermore, because the matrix of terminal payoffs, A, is not changed from point 3, the market is again incomplete.

We note that we could have used Proposition 19 in Chapetr 4 in the classnotes to verify the absence of arbitrage. Indeed, the market made of the risk-free asset and the risky security S_1 is arbitrage-free (look at the state price vectors z obtained at point 3 and use the First Fundamental Theorem of Asset Pricing) and the security S_2 can be replicated by investing in the risk-free asset and in the stock S_1. To show this, we must solve the system

$$V_\vartheta(1) = A\vartheta = S_2(1),$$

from which we obtain the vector $\vartheta = [\vartheta_0, \vartheta_1]^T$, with $\vartheta_0 = -1$ and $\vartheta_1 = 4$. For the extended market (i.e., the market including the risk-free asset and the risky securities S_1 and S_2) to be arbitrage-free, we would need that $8 = S_2(0) = V_\vartheta(0) = -1 + 4 \cdot 1.92 = 6.68$, by Proposition 9. But this is manifestly not true, hence the extended market leads to arbitrage.

EXERCISE 1.3*

Consider the market of Exercise 1, point 3, and introduce a new contingent claim X.

1. Suppose that the terminal payoff of X is

$$X(\omega_1) = 7.5$$
$$X(\omega_2) = 3.5$$
$$X(\omega_3) = 17.5$$

   Prove that the claim X is attainable and determine its unique no-arbitrage price.

2. If the terminal payoff of X is instead given by

$$X(\omega_1) = 3$$
$$X(\omega_2) = 2$$
$$X(\omega_3) = 5$$

show that the contingent claim is non-attainable. Determine the interval of no-arbitrage prices for X, the minimum super-replication cost of X and a minimum cost super-replicating strategy.

Solution.

1. The security X is redundant if there exists a strategy $\vartheta = [\vartheta_0; \vartheta_1; \vartheta_2]^T$ such that $A\vartheta = X$.

Matrix A has rank 2. The system admits solutions if and only if $rank(A|X) = 2$. Since

$$\det \begin{bmatrix} 1.25 & 2 & 7.5 \\ 1.25 & 1.2 & 3.5 \\ 1.25 & 4 & 17.5 \end{bmatrix} = 0,$$

the rank of A augmented with the column X is actually 2. The system then admits *infinite solutions,* because the number of unknowns, 3, is greater than the number of linearly independent equations. We can thus arbitrarily fix a component of ϑ and set, for example, $\vartheta_2 = 0$. By solving the system

$$\begin{cases} 1.25\vartheta_0 + 2\vartheta_1 = 7.5 \\ 1.25\vartheta_0 + 1.2\vartheta_1 = 3.5 \end{cases}$$

we obtain $\vartheta_0 = -2$ and $\vartheta_1 = 5$. As a result, *the* no-arbitrage price of X at time $t = 0$ is $S_X(0) = V_\vartheta(0) = -2 \cdot 1 + 5 \cdot 1.92 = 7.6$. Such price would be unchanged if we had left ϑ_2 undetermined, i.e. if we had considered *any* other strategy replicating X. We can verify this by determining $\vartheta = [\vartheta_0; \vartheta_1; \vartheta_2]^T$ such that $A\vartheta = X$. We obtain

$$\begin{cases} \vartheta_0 = -2 + \vartheta_2 \\ \vartheta_1 = 5 - 4\vartheta_2 \\ \vartheta_2 \in \Re \end{cases},$$

from which $V_\vartheta(0) = (-2 + \vartheta_2) \cdot 1 + (5 - 4\vartheta_2) \cdot 1.92 + \vartheta_2 \cdot 6.68 = 7.6$, independently of the strategy considered.

The same result can be obtained more quickly by employing Proposition 19 in Chapter 4 of the classnotes and the risk-neutral probability measures. For the extended market to be arbitrage-free, we must have

$$\begin{aligned} S_X(0) &= E^{\mathbf{Q}}\left[\frac{X}{1+r}\right] = \\ &= \sum_{k=1}^{3} \frac{X(\omega_k)}{1+r} q_k = \\ &= \frac{7.5}{1.25} \cdot (0.8 - 1.4q_2) + \frac{3.5}{1.25} \cdot q_2 + \frac{17.5}{1.25} \cdot (0.4q_2 + 0.2) = \\ &= 7.6 \end{aligned}$$

2. The claim X is non-attainable. Indeed, while matrix A has rank 2, the augmented matrix $[A|X]$ has rank 3, because

$$\det \begin{bmatrix} 1.25 & 2 & 3 \\ 1.25 & 1.2 & 2 \\ 1.25 & 4 & 5 \end{bmatrix} = 0.5 \neq 0$$

and thus the system $A\vartheta = X$ does not admit solutions. By Proposition 20 in Chapter 4 of the classnotes, the interval of no-arbitrage prices for X has the following endpoints:

$$\begin{aligned} \inf_{\mathbf{Q}} E^{\mathbf{Q}}\left[\frac{X}{1+r}\right] &= \inf_{q_2 \in \left(0;\frac{4}{7}\right)} \left(\frac{3}{1.25}\cdot(0.8-1.4q_2)+\frac{2}{1.25}\cdot q_2+\frac{5}{1.25}\cdot(0.4q_2+0.2)\right) = \\ &= \inf_{q_2 \in \left(0;\frac{4}{7}\right)} (2.72-0.16q_2) = 2.72-0.16\cdot\frac{4}{7} = 2.628 \end{aligned}$$

and

$$\begin{aligned} \sup_{\mathbf{Q}} E^{\mathbf{Q}}\left[\frac{X}{1+r}\right] &= \sup_{q_2 \in \left(0;\frac{4}{7}\right)} \left(\frac{3}{1.25}\cdot(0.8-1.4q_2)+\frac{2}{1.25}\cdot q_2+\frac{5}{1.25}\cdot(0.4q_2+0.2)\right) = \\ &= \sup_{q_2 \in \left(0;\frac{4}{7}\right)} (2.72-0.16q_2) = 2.72-0.16\cdot 0 = 2.72 \end{aligned}$$

As a result, the interval of NA prices is given by $(2.628; 2.72)$. The minimum cost of super-replication of X coincides with the right endpoint and is thus equal to 2.72. In order to find a super-replicating strategy with minimum cost, we must solve the following problem

$$\begin{cases} \min \ V_\vartheta(0) \\ \text{with } \vartheta \text{ such that } V_\vartheta(1) \geq X \end{cases}$$

The set of strategies super-replicating X is solution to the system of inequalities $V_\vartheta(1) \geq X$. For simplicity, we consider strategies involving the risk-free asset and stock S_1, since we have seen that S_2 is redundant. The super-replicating strategies $\vartheta = [\vartheta_0; \vartheta_1]^T$ solve the system

$$\begin{cases} 1.25\vartheta_0 + 2\vartheta_1 \geq 3 \\ 1.25\vartheta_0 + 1.2\vartheta_1 \geq 2 \\ 1.25\vartheta_0 + 4\vartheta_1 \geq 5 \end{cases},$$

whose solution can be found graphically. We draw the equations

$$\begin{cases} r_1 : 1,25\vartheta_0 + 2\vartheta_1 = 3 \\ r_2 : 1,25\vartheta_0 + 1,2\vartheta_1 = 2 \\ r_3 : 1,25\vartheta_0 + 4\vartheta_1 = 5 \end{cases}$$

and the strategies super-replicating X are found by taking the points $(\vartheta_0; \vartheta_1)$ lying above the three straight lines r_1, r_2 and r_3. The resulting region is shaded in Figure 1.

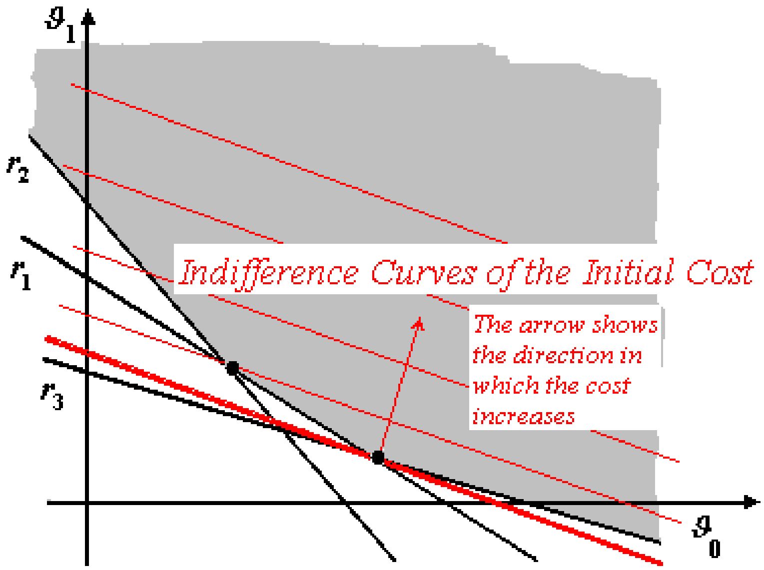

Figure 1: Figure 1. *Super-replication of X at minimal cost. The shaded region represents all strategies super-replicating X. We can identify the minimal cost super-replicating strategy by means of the level curves of the initial cost function of the super-replicating strategies.*

To determine the minimum cost strategy among all those super-replicating X, we need to study the initial cost function

$$V_\vartheta(0) = \vartheta_0 + \vartheta_1 \cdot 1.92$$

The level curve c of such function is identified by the straight line of equation

$$\vartheta_0 + \vartheta_1 \cdot 1.92 = c,$$

which is depicted in red in Figure 1. The level curves are hence parallel lines having intercept (i.e., the intersection with the axis ϑ_1) increasing with c (i.e. with the cost of the strategy). The minimum cost super-replicating strategy is given by the couple $(\vartheta_0;\vartheta_1)$ belonging to the intersection of r_1 and r_3. We can thus solve the system

$$\begin{cases} 1.25\vartheta_0 + 2\vartheta_1 = 3 \\ 1.25\vartheta_0 + 4\vartheta_1 = 5 \end{cases}$$

to obtain the minimum cost super-replicating strategy

$$\begin{cases} \vartheta_0 = 0.8 \\ \vartheta_1 = 1, \end{cases}$$

from which

$$V_\vartheta(0) = 0.8 \cdot 1 + 1 \cdot 1.92 = 2.72,$$

a value coinciding with that found through $\sup_{\mathbf{Q}} E^{\mathbf{Q}}\left[\frac{X}{1+r}\right]$.

One can further verify that the minimum cost super-replicating strategy generates a terminal cashflow equal to

$$V_\vartheta(1) = \begin{bmatrix} 3 \\ 2.2 \\ 5 \end{bmatrix} > X$$

As a result, we have perfect replication of X only in scenarios ω_1 and ω_3, while in ω_2 the strategy ϑ strictly super-replicates the contingent claim X. This is not surprising at all, since we know that X is non-attainable.

EXERCISE 2

In a one-period financial market the following securities are available for trade: the risk-free asset, yielding a riskless rate $r = 0.05$, and two risky stocks, S_1 and S_2, with the following payoffs at time $t = 1$

$$\begin{array}{ll} S_1(1)(\omega_1) = 4 & S_2(1)(\omega_1) = 10 \\ S_1(1)(\omega_2) = 2 & S_2(1)(\omega_2) = 5 \\ S_1(1)(\omega_3) = 8 & S_2(1)(\omega_3) = 1 \end{array} .$$

1. State whether the market is complete and arbitrage-free if the two risky securities have time-0 prices $S_1(0) = 6.0190$ and $S_2(0) = 3.6023$. If your answer is affirmative, determine all the risk-neutral probabilities $\mathbb{Q}$.

2. If S_2 takes the following values at time $t = 1$

$$\begin{array}{c} S_2(1)(\omega_1) = 21.9 \\ S_2(1)(\omega_2) = 9.9 \\ S_2(1)(\omega_3) = 45.9 \end{array}$$

   and has time-0 price $S_2(0) = 34.1142$, while S_1 is as in point 1, is the market arbitrage-free? Is it complete? Is there any redundant security? In case there is, how can it be replicated by investing in the securities available in the market?

3. Suppose that the securities S_1 and S_2 have time-1 payoffs as in point 2 and time-0 prices $S_1(0) = 7,8115$ and $S_2(0) = 44,8690$. Discuss arbitrage and completeness.

Solution

1. We apply the First Fundamental Theorem of Asset Pricing to see whether the market is arbitrage-free. We must verify that there exists $z \in \Re^3$, $z >> 0$ such that

$$A^T z = \begin{bmatrix} 1.05 & 1.05 & 1.05 \\ 4 & 2 & 8 \\ 10 & 5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ S_1(0) \\ S_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 6.0190 \\ 3.6023 \end{bmatrix}$$

Since $\det(A^T) = 39.9 \neq 0$, the system admits a unique solution. We can solve it to obtain

$$z = \begin{bmatrix} 0.25 \\ 0.1 \\ 0.60238 \end{bmatrix} >> 0$$

so that the market is arbitrage-free. Since z is unique, the market is complete by the Second Fundamental Theorem of Asset Pricing. From

$$z(1+r) = \mathbb{Q},$$

we get the unique risk-neutral probability $\mathbb{Q} = [0.2625;\ 0.105;\ 0.6325]$.

2. Matrix A is in this case given by

$$A^T = \begin{bmatrix} 1+r & 1+r & 1+r \\ S_1(1)(\omega_1) & S_1(1)(\omega_2) & S_1(1)(\omega_3) \\ S_2(1)(\omega_1) & S_2(1)(\omega_2) & S_2(1)(\omega_3) \end{bmatrix} = \begin{bmatrix} 1.05 & 1.05 & 1.05 \\ 4 & 2 & 8 \\ 21.9 & 9.9 & 45.9 \end{bmatrix}$$

Since $\det A = 0$, we have $rank(A) < 3$ and the market is thus incomplete. To see whether NA holds, we analyze the system

$$A^T z = \begin{bmatrix} 1 \\ 6.0190 \\ 34.1142 \end{bmatrix}$$

In this case we have $\det(A^T) = 0$ (compute it, to be sure), and $rank(A^T) = 2$. Moreover

$$\det \begin{bmatrix} 1,05 & 1.05 & 1 \\ 4 & 2 & 6.0190 \\ 21.9 & 9.9 & 34.1142 \end{bmatrix} = 0,$$

so that the rank of the augmented matrix is strictly lower than 3. In particular, such rank is 2. The system admits solutions, because the ranks of A^T and of the augmented matrix are equal. Since the number of unknowns, 3, is greater than the number of linearly independent equations, 2, there are infinite solutions. We can solve

$$\begin{cases} 1.05z_1 + 1.05z_2 + 1.05z_3 = 1 \\ 4z_1 + 2z_2 + 8z_3 = 6.0190 \end{cases}$$

to obtain

$$\begin{cases} z_1 = 2,0571 - 3z_3 \\ z_2 = 2z_3 - 1.10473 \\ z_3 \in \Re \end{cases}$$

In order to have $z >> 0$, we impose

$$\begin{cases} 2.0571 - 3z_3 > 0 \\ 2z_3 - 1.10473 > 0 \\ z_3 > 0 \end{cases},$$

i.e. $z_3 \in]0.55236; 0.6857[$. For example, $z = [0.25\ ; 0.10\ ; 0.60238]$, obtained for $z_3 = 0.60238$, is *a* state price vector.

The market is thus arbitrage-free (a $z >> 0$ exists). We have already observed that the market is incomplete. Security S_2 can be replicated by investing in S_1 and in the risk-free asset. Indeed, the system

$$\begin{bmatrix} 1.05 & 4 \\ 1.05 & 2 \\ 1.05 & 8 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 21.9 \\ 9.9 \\ 45.9 \end{bmatrix}$$

admits a unique solution (because the rank of the matrix of the only coefficients coincides with the rank of the matrix augmented with the column vector of the known terms and is equal to 2): $\theta_0 = -2$ and $\theta_1 = 6$.

3. Since the matrix A is the same as in the previous point, we know that the market is incomplete. As far as NA is concerned, we note that the matrix augmented with the vector of known terms of the system

$$A^T z = \begin{bmatrix} 1 \\ 7,8115 \\ 44,8690 \end{bmatrix}$$

has rank equal to 2. Indeed,

$$\det \begin{bmatrix} 1.05 & 1.05 & 1 \\ 4 & 2 & 7,8115 \\ 21.9 & 9.9 & 44,8690 \end{bmatrix} = 0,$$

and the first two columns are linearly independent. As a result, the system admits solutions. Since there is one degree of freedom, they are infinite. We can solve the system and obtain

$$\begin{cases} z_1 = 2.95337 - 3z_3 \\ z_2 = 2z_3 - 2.00099 \\ z_3 \in \Re \end{cases}$$

In order to have $z >> 0$, we require

$$\begin{cases} 2.95337 - 3z_3 > 0 \\ 2z_3 - 2.00099 > 0 \\ z_3 > 0 \end{cases},$$

which has no solution, because the first inequality is verified if $z_3 < 0.9845$ and the second one if $z_3 > 1.0005$. By the First Fundamental Theorem of Asset Pricing, there is arbitrage.

EXERCISE 3

In a one-period financial market the following securities are available for trade: the risk-free asset, yielding $r = 8\%$; a risky security, S_1, with time-0 price $S_1(0) = 9.0740$ and payoff at time $t = 1$ given by

$$
\begin{gathered}
S_1(1)(\omega_1) = 12 \\
S_1(1)(\omega_2) = 8 \\
S_1(1)(\omega_3) = 10
\end{gathered}
$$

1. State whether the market is complete and arbitrage-free.
2. Suppose that a new security is introduced in the market, namely a *forward* contract on S_1, requiring the long position to buy S_1 at time $t = 1$ at a delivery price F specified at time $t = 0$. Is the new security, denoted by S_2, redundant? Does it complete the market?
3. If the delivery price is $F' = 9$, state whether the market gives rise to violations of the Law of One Price.

   Solution

1. The market is clearly incomplete, since the number of securities (2) is strictly less than the number of scenarios (3). As a further proof, the matrix

$$
A = \begin{bmatrix} 1.08 & 12 \\ 1.08 & 8 \\ 1.08 & 10 \end{bmatrix}
$$

has clearly rank $2 < 3$. To see whether the market is arbitrage-free, we employ the First Fundamental Theorem of Asset Pricing. We have to verify the existence of a vector $z \in \Re^3$, $z >> 0$, or equivalently of a probability $\mathbb{Q} = [q_1 \; q_2 \; q_3] >> 0$, such that

$$
(1+r)^{-1} A^T [q_1 \; q_2 \; q_3]^T = \begin{bmatrix} 1 \\ S_1(0) \end{bmatrix},
$$

where the system

$$
\begin{bmatrix} 1 & 1 & 1 \\ 12(1+0.08)^{-1} & 8(1+0.08)^{-1} & 10(1+0.08)^{-1} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9.0740 \end{bmatrix}
$$

is solved by:

$$
\begin{cases} q_1 \in \Re \\ q_2 = q_1 + 0.1 \\ q_3 = 0.9 - 2q_1 \end{cases}
$$

For $\mathbb{Q}$ to be positive, we require

$$
0 < q_1 < 0.45
$$

so that $q_k > 0$ $k = 1, 2, 3$ and NA holds.

2. The new security is redundant if there exists a strategy $\theta = (\theta_0, \theta_1)^T$ such that

$$V_\theta(1) = A\theta = S_2(1)$$

$$\begin{bmatrix} 1.08 & 12 \\ 1.08 & 8 \\ 1.08 & 10 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 12 - F \\ 8 - F \\ 10 - F \end{bmatrix}$$

Computing

$$\det \begin{bmatrix} 1.08 & 12 & 12 - F \\ 1.08 & 8 & 8 - F \\ 1.08 & 10 & 10 - F \end{bmatrix} = 0$$

we conclude that the rank of matrix A augmented with the coefficients $S_2(1)$ is equal to the rank of A and hence by the Theorem of Rouché-Capelli the system admits a solution (the solution is unique, because the number of unknowns is equal to the number of linearly independent equations). As a result, the security S_2 is redundant and in particular we find that $\theta_0 = -F(1.08)^{-1}$ and $\theta_1 = 1$. The forward contract does not complete the market, because the rank of the augmented matrix remains equal to $2 < 3$.

3. The delivery price F is set so as to have a time-0 price equal to zero. The strategy replicating the forward contract has been found at the previous point: $\theta_0 = -F(1.08)^{-1}$ and $\theta_1 = 1$. In order to have $V_\theta(0) = 0$, we must set

$$F = S_1(0)(1.08) = 9.8$$

To avoid violations of the Law of One Price, the delivery price of S_1 should be 9.8. Since the price actually set at time 0 is $F' = 9$, there are such violations. Indeed, the strategy replicating the forward with $F' = 9$ is given by $\theta_0 = -9(1.08)^{-1} = -8.33333$ and $\theta_1 = 1$, and costs $V_\theta(0) = -8.33333 + 9.07407 = 0.74074$, while a *long* position on the forward contract provides the very same payoff at $t = 1$, but with zero initial cost.

EXERCISE 4

In the one-period financial market considered in exercise 3, point 1, with the risk-free asset yielding $r = 8\%$ and a risky security S_1 with time-0 price $S_1(0) = 9.0740$ and payoff at time $t = 1$ given by

$$\begin{array}{c} S_1(1)(\omega_1) = 12 \\ S_1(1)(\omega_2) = 8 \\ S_1(1)(\omega_3) = 10 \end{array}$$

we introduce a *put* option on S_1 with strike price $K = 11$ and maturity $T = 1$. We denote this put option by S_2 in the sequel.

1. State whether the put option is redundant or not. Does the put option complete the market?
2. Find the interval of prices consistent with NA in the extended market. If the put has time-0 price 1.75925, are there arbitrage opportunities?
3. Under the same assumptions of point 2, state whether the extended market gives rise to violations of the Law of One Price.
4. State whether the payoff

$$\begin{array}{c} S_3(1)(\omega_1) = 0 \\ S_3(1)(\omega_2) = 0 \\ S_3(1)(\omega_3) = 10 \end{array}$$

is redundant in the extended market and, if that is the case, find the replication cost of the security. Does the introduction of S_3 in the market generate violations of the Law of One Price? Does it generate arbitrage opportunities?

Solution

1. The payoff of the put option at maturity is given by

$$\begin{array}{c} S_2(1)(\omega_1) = 0 \\ S_2(1)(\omega_2) = 3 \\ S_2(1)(\omega_3) = 1 \end{array},$$

and to see if it is redundant we look for a strategy $\theta = (\theta_0, \theta_1)^T$ such that

$$V_\theta(1) = A\theta = S_2(1),$$

i.e.

$$\begin{bmatrix} 1.08 & 12 \\ 1.08 & 8 \\ 1.08 & 10 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

The matrix obtained by augmenting A with the column $S_2(1)$, denoted by $\widetilde{A}$, has rank 3. Indeed, we have

$$\det \widetilde{A} = \det \begin{bmatrix} 1.08 & 12 & 0 \\ 1.08 & 8 & 3 \\ 1.08 & 10 & 1 \end{bmatrix} = 2.16 \neq 0$$

and hence the system cannot be solved, i.e. the put option cannot be replicated. Since the rank of the matrix including the put has rank equal to the number of scenarios at time $t = 1$, the market is completed by introducing the put.

2. Recalling from the previous exercise that the risk-neutral measure in the original market, $\mathbb{Q}$, is given by

$$\begin{cases} q_1 \in \Re \\ q_2 = q_1 + 0.1 \\ q_3 = 0.9 - 2q_1 \end{cases}$$

with

$$0 < q_1 < 0.45$$

the NA price interval for the put is given by:

$$\begin{aligned} p &= (0 \cdot q_1 + 3 \cdot (q_1 + 0.1) + 1 \cdot (0.9 - 2q_1)) \cdot (1 + 0.08)^{-1} = \\ &= (q_1 + 1.2) \cdot (1 + 0.08)^{-1} \end{aligned}$$

for q_1 in $]0; 0,45[$. We thus find $p \in]1,111; 1,5277[$. Hence, if the put has time-0 price 1.75925, the extended market is not arbitrage-free.

3. To see whether violations of the Law of One Price are possible in the extended market, we try to characterize the extended market strategies $\theta = (\theta_0, \theta_1, \theta_2)^T$ and $\theta' = (\theta'_0, \theta'_1, \theta'_2)^T$ providing the same payoff at maturity, i.e. $V_\theta(1) = V_{\theta'}(1)$. By using the notation previously introduced, we have

$$\widetilde{A}\theta = \widetilde{A}\theta'$$

or, equivalently,

$$\widetilde{A}(\theta - \theta') = 0,$$

which admits the null solution $\theta - \theta' = 0$ only, because the matrix $\widetilde{A}$ is nonsingular (see point 1). Hence,

$$\theta_n = \theta'_n \quad \text{for } n = 0, 1, 2$$

and thus $V_\theta(0) = V_{\theta'}(0)$: the Law of One Price holds even if the introduction of the put with time-0 price 1.75925 gives rise to arbitrage.

4. The security S_3 can be replicated by employing the risk-free asset, S_1 and S_2, because the extended market is complete. By solving the system

$$\begin{bmatrix} 1.08 & 12 & 0 \\ 1.08 & 8 & 3 \\ 1.08 & 10 & 1 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

we get

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 166.6667 \\ -15 \\ -20 \end{bmatrix},$$

and hence the time-0 price is given by $V_\theta(0) = -4.62944 < 0$. We already know from the previous point that violations of the Law of One Price are not possible in the market with the risk-free asset, S_1 and S_2. The introduction of the new redundant security S_3 does not generate violations of the Law of One Price if and only if it has price equal to the replication cost $S_3(0) = V_\theta(0) = -4.62944 < 0$. The strategy replicating θ is in any case an arbitrage strategy of the second type.

EXERCISE 5

In a one-period financial market only two risky securities are available for trade, S_1 and S_2, with payoff at time $t = 1$ given by

$$\begin{array}{ll} S_1(1)(\omega_1) = 5 & S_2(1)(\omega_1) = 0 \\ S_1(1)(\omega_2) = 4 & S_2(1)(\omega_2) = 0,5 \\ S_1(1)(\omega_3) = 0 & S_2(1)(\omega_3) = 2.5 \end{array}$$

1. Is the market complete? Is it possible to replicate a riskless contingent claim, that is X such that $X(\omega_k) = \pi$ for $k = 1, 2, 3$, using the two securities S_1 and S_2?

2. If the two securities S_1 and S_2 have time-0 prices $S_1(0) = 2.407$ and $S_2(0) = 1.11$, what is the riskless return on any riskless contingent claim? What is the share invested in S_1? Is the market arbitrage-free?

3. Introduce an option *best of two assets* on the two stocks, i.e. an option with payoff at time $t = 1$ equal to $\max(S_1(1); S_2(1))$. Is the option redundant? What is its no-arbitrage price at time $t = 0$?

4. Does the option introduced in the previous point complete the market? If the option has time-0 price 3.333, is there arbitrage? If that is not the case, determine $\mathbb{Q}$, the risk-neutral probability measure for the extended market.

Solution

1. A, the matrix collecting the payoffs of the two risky securities S_1 and S_2 at time $t = 1$, is equal to

$$A = \begin{bmatrix} 5 & 0 \\ 4 & 0,5 \\ 0 & 2,5 \end{bmatrix}$$

and has rank $2 < K = 3$. The market is thus incomplete. To see whether it is possible to replicate a riskless contingent claim, we check if it is possible to set up a strategy $\theta = (\theta_1, \theta_2)^T$ such that $V_\theta(1)$ is constant, i.e. $V_\theta(1) = \pi \in \Re$. We thus have to study the system

$$\begin{bmatrix} 5 & 0 \\ 4 & 0,5 \\ 0 & 2.5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \\ \pi \end{bmatrix},$$

which by the Theorem of Rouché-Capelli admits solutions if and only if the rank of A augmented with the column vector of the known terms is equal to 2. Hence, we compute

$$\det \begin{bmatrix} 5 & 0 & \pi \\ 4 & 0.5 & \pi \\ 0 & 2.5 & \pi \end{bmatrix} = 0,$$

and see that the system admits solutions.

2. By solving the system

$$\begin{cases} 5\theta_1 + 0 = \pi \\ 4\theta_1 + 0.5\theta_2 = \pi \end{cases}$$

setting $\pi = (1+r)V_\theta(0)$ we obtain

$$\begin{cases} \theta_1 = \frac{(1+r)V_\theta(0)}{5} \\ \theta_2 = \frac{2(1+r)V_\theta(0)}{5} \end{cases}$$

Since the value of the strategy at time $t = 0$ is given by

$$V_\theta(0) = \frac{(1+r)V_\theta(0)}{5} S_1(0) + \frac{2(1+r)V_\theta(0)}{5} S_2(0),$$

we obtain

$$r = \frac{5}{2.407 + 2 \cdot 1.11} - 1 = 0.08$$

The share invested in the first security is

$$\frac{\theta_1 S_1(0)}{V_\theta(0)} = \frac{S_1(0)}{S_1(0) + 2S_2(0)} = \frac{2.407}{2.407 + 2 \cdot 1.11} = 0.52$$

The market is arbitrage-free if it is possible to find a vector $\mathbb{Q} = [q_1; q_2; q_3]$ of risk-neutral probabilities, i.e.

$$\frac{1}{1+r}[q_1; q_2; q_3]A = [S_1(0); S_2(0)]$$

with $r = 0.08$, from which

$$\begin{cases} \frac{1}{1.08}(5q_1 + 4q_2) = 2.407 \\ \frac{1}{1.08}(0.5q_2 + 2.5q_3) = 1.11 \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

i.e.

$$\begin{cases} q_1 = 0.52 - 0.8q_2 \\ q_2 \in \Re \\ q_3 = 0.48 - 0.2q_2 \end{cases}$$

and, imposing the positivity of all components of $\mathbb{Q}$, we get $q_2 \in]0; 0,65[$. The market is thus arbitrage-free.

3. The payoff of the *best of two assets* option at time $t = 1$ is

$$\begin{bmatrix} \max(S_1(1); S_2(1))(\omega_1) \\ \max(S_1(1); S_2(1))(\omega_2) \\ \max(S_1(1); S_2(1))(\omega_3) \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2.5 \end{bmatrix}$$

The option is redundant if there exists $\theta = [\theta_1, \theta_2]^T$ such that

$$A\theta = \begin{bmatrix} 5 & 0 \\ 4 & 0.5 \\ 0 & 2.5 \end{bmatrix} \theta = \begin{bmatrix} 5 \\ 4 \\ 2.5 \end{bmatrix}$$

For the matrix obtained by augmenting A with the column vector of known terms, we have

$$\det \begin{bmatrix} 5 & 0 & 5 \\ 4 & 0.5 & 4 \\ 0 & 2.5 & 2.5 \end{bmatrix} = 6.25 \neq 0$$

and hence it has maximum rank. The system cannot be solved, by the Theorem of Rouché-Capelli. As a consequence, the option cannot be replicated. We find an interval of NA prices for the option, with left endpoint equal to

$$\begin{gathered} \inf_{q_2 \in]0;0.65[} \frac{1}{1{,}08} \left(5 \cdot (0.52 - 0.8 q_2) + 4 \cdot q_2 + 2.5 \cdot (0.48 - 0.2 q_2)\right) = \\ = \inf_{q_2 \in]0;0.65[} (3.5188 - 0.4629 q_2) = 3.2179 \end{gathered}$$

and right endpoint given by

$$\begin{gathered} \sup_{q_2 \in]0;0.65[} \frac{1}{1.08} \left(5 \cdot (0.52 - 0.8 q_2) + 4 \cdot q_2 + 2.5 \cdot (0.48 - 0.2 q_2)\right) = \\ \sup_{q_2 \in]0;0.65[} (3.5188 - 0.4629 q_2) = 3.5188 \end{gathered}$$

4. The option introduced completes the market, because the rank of the matrix collecting the time-1 payoffs of S_1, S_2 and the option is equal to $3 = K$, as from the previous point. Morevoer, since the time-0 price of the option is $3.333 \in]3.2179; 3.5188[$, there is no arbitrage in the extended market. By the Second Fundamental Theorem of Asset Pricing, there exists a unique risk-neutral probability measure for the extended market, which can be obtained by solving the equation

$$3.5188 - 0.4629 q_2 = 3.333,$$

from which $q_2 = 0.40$ and thus

$$\begin{cases} q_1 = 0.2 \\ q_2 = 0.4 \\ q_3 = 0.4 \end{cases}$$

EXERCISE 6

In a one-period financial market the following securities are available for trade: the risk-free asset yielding $r = 0.04$, and two risky securities, S_1 and S_2, with payoffs at time $t = 1$ given by

$$\begin{array}{cc} S_1(1)(\omega_1) = 8 & S_2(1)(\omega_1) = 5 \\ S_1(1)(\omega_2) = 12 & S_2(1)(\omega_2) = 15 \\ S_1(1)(\omega_3) = 10 & S_2(1)(\omega_3) = 2 \end{array}$$

1. State whether the market is complete and arbitrage-free if the two stocks S_1 and S_2 have time-0 prices $S_1(0) = 11.346$ and $S_2(0) = 7.0192$. Is the Law of One Price violated? Is it possible to set up an arbitrage strategy of the second type?

2. Answer to the same questions of the previous point, this time with $S_1(0) = 10.192$ and $S_2(0) = 8.75$.

3. Determine the no-arbitrage price of a *Margrabe option,* i.e. an *option to exchange two risky securities.* Specifically, such option provides the holder with the *right* to exchange S_2 with S_1 at time $t = 1$ and has payoff equal to $\max(S_1(1) - S_2(1); 0)$. State whether the option can be replicated by employing *only* the two risky securities.

Solution

1. To see if the market is arbitrage-free, we apply the First Fundamental Theorem of Asset Pricing. We must verify the existence of $\mathbb{Q} \in \Re^3$, $\mathbb{Q} >> 0$ such that

$$\frac{1}{1+r} A^T \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ S_1(0) \\ S_2(0) \end{bmatrix}$$

i.e.

$$\frac{1}{1.04} \begin{bmatrix} 1.04 & 1.04 & 1.04 \\ 8 & 12 & 10 \\ 5 & 15 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 11.346 \\ 7.0192 \end{bmatrix}$$

Since

$$\det A^T = \det \begin{bmatrix} 1.04 & 1.04 & 1.04 \\ 8 & 12 & 10 \\ 5 & 15 & 2 \end{bmatrix} \neq 0$$

the system admits one and only one solution and we have

$$\begin{cases} q_1 = -0.4 < 0 \\ q_2 = 0.5 > 0 \\ q_3 = 0.9 > 0 \end{cases}$$

so that the market is allows for arbitrage opportunities. The market is however complete, because the rank of matrix A is 3.

Although there are rabitrage opportunities, the Law of One Price holds, because if $\theta = (\theta_0, \theta_1, \theta_2)^T$ and $\theta' = (\theta'_0, \theta'_1, \theta'_2)^T$ are such that $V_\theta(1) = V_{\theta'}(1)$, then

$$A\theta = A\theta'$$

or, equivalently,

$$A(\theta - \theta') = 0,$$

which admits as unique solution the null solution $\theta - \theta' = 0$, because matrix A is non-singular ($\det A \neq 0$). Hence,

$$\theta_n = \theta'_n \quad \text{for } n = 0, 1, 2$$

from which

$$V_\theta(0) = V_{\theta'}(0).$$

Since the market is complete, the contingent claim having payoff $[1, 0, 0]^T$ at maturity can be replicated by investing in the risk-free asset, in S_1 and in S_2. The time-0 prices of the securities available in the market are such that the first component of the *state price vector*, z_1 (i.e. the first component of $\mathbb{Q}$, q_1), is negative. That means the time-0 value of the strategy replicating the *Arrow-Debreu security* on ω_1 is negative. As a result, to set up an arbitrage of the second type, we look for the strategy $\theta = (\theta_0, \theta_1, \theta_2)^T$ with value at time $t = 1$ equal to

$$V_\theta(1) = A\theta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

By solving the system, we find

$$\begin{cases} \theta_0 = 3.78605 \\ \theta_1 = -0.40625 \\ \theta_2 = 0.0625 \end{cases}$$

whose time-0 price is

$$V_\theta(0) = \theta_0 + \theta_1 S_1(0) + \theta_2 S_2(0) = -0.38461 < 0$$

Hence, θ is an arbitrage of the second type.

2. The market is again complete, because matrix A is unchanged with respect to the previous point. As far as NA is concerned, the system

$$\frac{1}{1.04} \begin{bmatrix} 1.04 & 1.04 & 1.04 \\ 8 & 12 & 10 \\ 5 & 15 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10.192 \\ 8.75 \end{bmatrix}$$

admits a unique solution, as in point 1, equal to

$$\begin{cases} q_1 = 0.2 > 0 \\ q_2 = 0.5 > 0 \\ q_3 = 0.3 > 0 \end{cases}$$

and thus the market is arbitrage-free by the First Fundamental Theorem of Asset Pricing. The Law of One Price clearly holds and it is not possible to set up any arbitrage strategy of the first nor of the second type.

3. The Margrabe option has payoff at maturity equal to

$$\begin{bmatrix} \max(S_1(1) - S_2(1); 0)(\omega_1) \\ \max(S_1(1) - S_2(1); 0)(\omega_2) \\ \max(S_1(1) - S_2(1); 0)(\omega_3) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

and hence its no-arbitrage price is given by

$$\frac{1}{1,04}(3 \cdot q_1 + 0 \cdot q_2 + 8 \cdot q_3) = \frac{1}{1.04}(3 \cdot 0.2 + 0 \cdot 0.5 + 8 \cdot 0.3) = 2.88$$

The option can be replicated by investing in the *only* two risky securities if the strategy $\theta = (\theta_0, \theta_1, \theta_2)^T$ with time-1 value equal to

$$V_\theta(1) = A\theta = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

is such that $\theta_0 = 0$. We just have to take a look at the determinant of the matrix collecting the time-1 payoffs of the two risky securities and of the option

$$\det \begin{bmatrix} 8 & 5 & 3 \\ 12 & 15 & 0 \\ 10 & 2 & 8 \end{bmatrix} = 102 \neq 0$$

to understand that this is not possible, since the system

$$\begin{bmatrix} 8 & 5 \\ 12 & 15 \\ 10 & 2 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix}$$

has no solution by the Theorem of Rouché-Capelli. We can get to the same conclusion by noting that the (unique) market strategy replicating the option,

$$\begin{bmatrix} 1.04 & 8 & 5 \\ 1.04 & 12 & 15 \\ 1.04 & 10 & 2 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 8 \end{bmatrix},$$

is given by

$$\begin{cases} \theta_0 = -3.0649 \neq 0 \\ \theta_1 = 1.28125 \\ \theta_2 = -0.8125 \end{cases}$$