Quantitative Finance and Derivatives I
Finanza Quantitativa e Derivati I

code 20188

a.y. 2021/22, 14th January 2022

General Exam

THEORY QUESTIONS (20 points out of 100)

1. **(6 points)** In the one-period market model state the First Fundamental Theorem of Asset Pricing.

2. **(6 points)** Let $\mathbb{Q}$ denote any risk-neutral probability for an arbitrage-free discrete multiperiod market. Consider the redundant cashflow $X = \{X(t)\}_{t=1}^{T}$ and let $S_X = \{S_X(t)\}_{t=0}^{T}$ be its price process. The extended market is arbitrage free if and only if...

   (Mark all the appropriate answers)

   A $\quad S_X(t) = V_{\vartheta^X}(t)$ for all $t = 0, ..., T$ and for every replicating strategy ϑ^X

   B $\quad S_X(t) = \sum_{\tau=t+1}^{T} \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{B(t)}{B(\tau)} X(\tau)\right| \mathcal{P}_t\right]$ for all $t = 0, ..., T-1$

   C $$\begin{cases} S_X(t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(t+1)}{1+r(t)}\right| \mathcal{P}_t\right] & \text{for } t = 0, 1, ..., T-2 \\ S_X(T-1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(T)}{1+r(T-1)}\right| \mathcal{P}_{T-1}\right] & \text{for } t = T-1 \end{cases}$$

   D $$\begin{cases} S_X(t) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(t+1)+S_X(t+1)}{1+r(t)}\right| \mathcal{P}_t\right] & \text{for } t = 0, 1, ..., T-2 \\ S_X(T-1) = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X(T)}{1+r(T-1)}\right| \mathcal{P}_{T-1}\right] & \text{for } t = T-1 \end{cases}$$

3. **(4 points)** Consider a diffusion process $\{X(t)\}_{t\in[0,T]}$ satisfying the stochastic differential equation $dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t)$ with $X(0) = X_0$. Assume $\varphi : [0;T] \times \Re \to \Re$ be continuously differentiable, once with respect to the first variable, denoted by t, twice with respect to the second, denoted by x. Let $Y(t) = \varphi(t; X(t))$. Then the process Y satisfies

   (Mark the only appropriate alternative)

   - [A] $dY(t) = \left[\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x} \cdot a(t, X(t)) + \frac{\partial^2\varphi(t;X(t))}{\partial x^2} b^2(t, X(t))\right] dt + \frac{\partial\varphi(t;X(t))}{\partial x} b(t, X(t)) dW(t)$
   - [B] $dY(t) = \left[\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x} \cdot a(t, X(t)) + \frac{1}{2}\frac{\partial^2\varphi(t;X(t))}{\partial x^2} b^2(t, X(t))\right] dt + \frac{\partial\varphi(t;X(t))}{\partial x} b(t, X(t)) dW(t)$
   - [C] $dY(t) = \left[\left(\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x}\right) \cdot a(t, X(t)) + \frac{1}{2}\frac{\partial^2\varphi(t;X(t))}{\partial x^2} b^2(t, X(t))\right] dt + \frac{\partial\varphi(t;X(t))}{\partial x} b(t, X(t)) dW(t)$
   - [D] $dY(t) = \left[\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x} + \frac{1}{2}\frac{\partial^2\varphi(t;X(t))}{\partial x^2}\right] dt + \frac{\partial\varphi(t;X(t))}{\partial x} dW(t)$

4. **(4 points)** Let $N(a, b^2)$ denote a normal random variable with mean a and variance b^2. In the Black Scholes model and with respect to the historical measure $\mathbb{P}$ the log-price of the risky security S is

   (Mark the only appropriate alternative)

   - [A] $\ln \frac{S(t)}{S(0)} \overset{\mathbb{P}}{\sim} N(\mu t, \sigma^2 t)$
   - [B] $\ln \frac{S(t)}{S(0)} \overset{\mathbb{P}}{\sim} N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$
   - [C] $\ln \frac{S(t)}{S(0)} \overset{\mathbb{P}}{\sim} N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma t\right)$
   - [D] $\ln \frac{S(t)}{S(0)} \overset{\mathbb{P}}{\sim} N\left(\left(\mu + \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$

EXERCISE 1 (45 points out of 100).
Consider a one period market with the riskless asset B yielding a risk-free rate $r = 2\%$, and a risky security S whose prices at time $T = 1$ are

$$S(1)(\omega_1) = 15.3$$
$$S(1)(\omega_2) = 5.1$$
$$S(1)(\omega_3) = 10.2$$

1. **(4 points)** Is the market complete? Justify your answer.

2. **(6 points)** Suppose that the risky security S trades at $t = 0$ at the price

$$S(0) = 7.5.$$

Let $\mathbb{Q}(\omega_k)$ for $k = 1, 2, 3$ denote a *risk-neutral probability*. Determine $\mathbb{Q}(\omega_2)$ and $\mathbb{Q}(\omega_3)$ in terms of $\mathbb{Q}(\omega_1) = q_1$.
(Mark the only appropriate alternative)

A $\quad \mathbb{Q}(\omega_2) = q_1$ and $\mathbb{Q}(\omega_3) = 1 - 2q_1$

B $\quad \mathbb{Q}(\omega_2) = q_1 - \frac{1}{2}$ and $\mathbb{Q}(\omega_3) = \frac{3}{2} - q_1$

C $\quad \mathbb{Q}(\omega_2) = q_1 + \frac{1}{2}$ and $\mathbb{Q}(\omega_3) = \frac{1}{2} - 2q_1$

D $\quad \mathbb{Q}(\omega_2) = q_1 - \frac{1}{2}$ and $\mathbb{Q}(\omega_3) = \frac{1}{2} - 2q_1$

3. **(3 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_1) = q_1$ must be strictly larger than $\boxed{}$

4. **(3 points)** In order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_1) = q_1$ must be strictly smaller than $\boxed{}$

5. **(4 points)** Is the market free of arbitrage opportunities? Justify your answer.

6. **(7 points)** Let $R(\omega_k) = \frac{S(1)(\omega_k) - S(0)}{S(0)}$ for $k = 1, .., 3$ be the linear one-period return of S. A *European derivative* on R with maturity $T = 1$ is introduced in the market. Its payoff at $T = 1$ is equal to

$$X(1)(\omega_k) = \begin{cases} 1.02 & \text{if } R(\omega_k) > 0 \\ 2.04 & \text{else} \end{cases}$$

Is it possible to replicate this derivative with S and B? YES NO

Determine the set of no-arbitrage prices at $t = 0$ for this derivative.

7. **(4 points)** Formally describe the set of all the strategies whose final value super-replicates $-X(1)$, where $X(1)$ is the payoff of the derivative introduced in Point 6. Namely, describe the set of all the $\vartheta = (x, y)$ such that

$$V_{\vartheta}(1)(\omega_k) \geq -X(1)(\omega_k)$$

for all $k = 1, ..., 3$.

8. **(2 points)** Among the strategies found in Point 7, let $\vartheta^* = (x^*, y^*)$ be the one whose initial cashflow $C_{\vartheta^*}(0)$ is maximum. The riskless component x^* of this strategy is...

$$x^* = \boxed{}$$

9. **(2 points)** The risky component y^* of the strategy found in the previous Question is...

$$y^* = \boxed{}$$

10. **(5 points)** Suppose the derivative of Point 6 trades at the initial price of 1.6. The extended market (the one in which B, S and the derivative of Point 6 are traded) is...

(Mark the only appropriate alternative)

A	arbitrage-free AND complete
B	not arbitrage free AND complete
C	arbitrage-free AND incomplete
D	not arbitrage free AND incomplete

11. **(5 points)** Suppose a new *European derivative* on R is introduced in the extended market described in Point 10. Its payoff at $T = 1$ is equal to

$$Y(1)(\omega_k) = \begin{cases} K & \text{if } R(\omega_k) > 0 \\ 2K & \text{else} \end{cases}$$

The initial no arbitrage price of this derivative is equal to 8. In order to have an arbitrage-free market, K has to be equal to...

$$K = \boxed{}$$

EXERCISE 2 (35 points out of 100).
Consider a Black-Scholes market with the riskless security $B(t) = e^{\delta t}$ and the lognormal risky security S with drift μ and volatility σ under the historical probability $\mathbb{P}$. Assume the following values for the parameters: $S(0) = 10$, $\delta = 3\%$, $\mu = 7\%$, $\sigma = 20\%$, and $T = 2$.
(*Express your results in terms of the distribution function $N(\cdot)$ of a standard Normal random variable, whenever it is appropriate*).

1. **(5 points)** Let $R(t) = \ln \frac{S(t)}{S(0)}$ be the continuously compounded return on the risky asset. Compute the *historical probability* that this return is greater than $T\delta$ at $T = 2$. Namely, compute

$$\mathbb{P}\left[R(T) > T\delta\right].$$

Is this probability larger or smaller than 50%?

2. **(5 points)** A *forward contract* on R is introduced in the market. The payoff at $T = 2$ of this forward is equal to

$$X(2) = R(2) - F$$

where $F \in \Re$ is the *delivery price* of the forward. F is settled at $t = 0$ in such a way that the no aribtrage price of the forward contract at $t = 0$ is equal to zero. Determine the numerical value of F.

3. **(5 points)** Compute the *historical probability* that the payoff at $T = 2$ of the forward introduced in Point 2 is positive. Namely, compute

$$\mathbb{P}\left[X(2) > 0\right].$$

Is this probability increasing or decreasing with respect to the volatility of the risky asset σ? Justify your answer.

4. **(4 points)** Find $S_X(t)$, the no arbitrage price of the forward in Point 2 at any $t \in [0, T]$.

5. **(3 points)** Find $\vartheta(t) = (\vartheta_0(t), \vartheta_1(t))$, the *replicating portfolio* of the forward in Point 2 at any $t \in [0, T]$. Recall that ϑ_0 denotes the number of units of the riskless asset B and ϑ_1 the number of units of the risky security S.

6. **(3 points)** Is the replicating portfolio found in the previous Point long or short on S?

7. **(5 points)** Compute the *historical mean and variance* of $\Delta R = R(2) - R(1)$.

8. **(5 points)** Express $X(2)$ defined in Point 2 in terms of ΔR and $R(1)$ and compute the following conditional *historical probability*

$$\mathbb{P}\left[\, X(2) > 0 \middle|\, \mathcal{F}_1 \right].$$

Determine the set of values of the underlying $S(1)$ such that

$$\mathbb{P}\left[\, X(2) > 0 \middle|\, \mathcal{F}_1 \right] \geq 0.5$$

Solution of the Exercises

These are the detailed soultions of the exam.
The expected answers of the exam are written in italics.

Solution of EXERCISE 1

1. *The market is incomplete, because the number of scenarios $3 > 2$, which is the number of traded securities. Since there are fewer independent securities than scenarios, the market cannot be complete.*

2. Since the market is incomplete, the risk-neutral measures (if any) cannot be unique. Denoting by $q_i = \mathbb{Q}(\omega_i) > 0$ for $i = 1, ..., 3$, we have that

$$\begin{aligned}
\frac{1}{1.02}\left(15.3q_1 + 5.1q_2 + 10.2\underbrace{(1 - q_1 - q_2)}_{q_3}\right) &= 7.5 \\
15q_1 + 5q_2 + 10\,(1 - q_1 - q_2) &= 7.5 \\
5q_2 &= 5q_1 + 2.5 \\
q_2 &= q_1 + \frac{1}{2}
\end{aligned}$$

Then,

$$\begin{aligned}
q_3 &= 1 - q_1 - q_2 \\
&= 1 - q_1 - q_1 - \frac{1}{2} \\
&= \frac{1}{2} - 2q_1
\end{aligned}$$

Therefore, assuming the appropriate bounds x and y on q_1, we get

$$\begin{cases} q_1 \in (x, y) \\ q_2 = q_1 + \frac{1}{2} \\ q_3 = \frac{1}{2} - 2q_1 \end{cases}$$

Hence the right answer is

$$\boxed{\text{C}} \qquad \mathbb{Q}(\omega_2) = q_1 + \frac{1}{2} \;\text{ and }\; \mathbb{Q}(\omega_3) = \frac{1}{2} - 2q_1$$

3. In order to find the appropriate bounds x and y on q_1 we impose positivity constraints on q_2 and q_3

$$\begin{cases} q_1 + \dfrac{1}{2} > 0 \\ \dfrac{1}{2} - 2q_1 > 0 \end{cases}$$

that are satisfied by

$$\begin{cases} q_1 > -\frac{1}{2} = -0.5 \\ q_1 < \frac{1}{4} = 0.25 \end{cases}$$

Since by definition $q_1 \in (0, 1)$, we get

$$q_2 \in (0, 0.25) = \left(0, \frac{1}{4}\right).$$

Therefore, in order for $\mathbb{Q}(\omega_1)$, $\mathbb{Q}(\omega_2)$, $\mathbb{Q}(\omega_3)$ to define a risk-neutral probability, $\mathbb{Q}(\omega_1) = q_1$ must be strictly larger than 0

4. *...and $\mathbb{Q}(\omega_1) = q_1$ must be strictly smaller than $\frac{1}{4} = 0.25$.*

5. *Since there exist risk neutral probabilities, the market is arbitrage-free by the 1st FTAP.*

6. Since

$$\begin{aligned} R(1)(\omega_1) &= \frac{15.3-7.5}{7.5} = 1.04 > 0 \\ R(1)(\omega_2) &= \frac{5.1-7.5}{7.5} = -0.32 < 0 \\ R(1)(\omega_3) &= \frac{10.2-7.5}{7.5} = 0.36 > 0 \end{aligned}$$

the payoff of the European derivative is

$$\begin{aligned} X(1)(\omega_1) &= 1.02 \\ X(1)(\omega_2) &= 2.04 \\ X(1)(\omega_3) &= 1.02 \end{aligned}$$

The derivative cannot be replicated with B, S because the system

$$A\vartheta = p(1)$$

does not admit solutions, since $2 = rk\,(A) \neq rk\,[A|X(1)] = 3$ as

$$\det \begin{bmatrix} 1.02 & 15.3 & 1.02 \\ 1.02 & 5.1 & 2.04 \\ 1.02 & 10.2 & 1.02 \end{bmatrix} = 5.31.$$

Hence: *Is it possible to replicate this put option?* $\boxed{\text{NO}}$

Therefore, the derivative does not admit a unique no-arbitrage price at $t = 0$. Indeed, it holds:

$$\begin{aligned} X(0) &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X(1)] \\ &= \frac{1}{1.02}(1.02q_1 + 2.04q_2 + 1.02q_3) \\ &= q_1 + 2q_2 + q_3 \\ &= q_1 + 2\left(q_1 + \frac{1}{2}\right) + \left(\frac{1}{2} - 2q_1\right) \\ &= q_1 + \frac{3}{2} \end{aligned}$$

for $q_1 \in (0, 0.25)$. The bounds of the no arbitrage price interval for this derivative are

$$\inf_{q_1 \in (0,0.25)} \left(\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X(1)]\right) = \inf_{q_1 \in (0,0.25)} \left(q_1 + \frac{3}{2}\right) = 0 + \frac{3}{2} = \frac{3}{2} = 1.5$$

$$\sup_{q_1 \in (0,0.25)} \left(\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X(1)]\right) = \sup_{q_1 \in (0,0.25)} \left(q_1 + \frac{3}{2}\right) = 0.25 + \frac{3}{2} = \frac{7}{4} = 1.75$$

7. The strategies $\vartheta = (\vartheta_0, \vartheta_1)$ whose final value super-replicates $-X(1)$ are such that

$$V_\vartheta\,(1)\,(\omega_k) \geq -X(1)\,(\omega_k)$$

for all $k = 1, 2, 3$. We get

$$\begin{cases} V_\vartheta(1)(\omega_1) = 1.02\vartheta_0 + 15.3\vartheta_1 \geq -1.02 = -X(1)(\omega_1) \\ V_\vartheta(1)(\omega_2) = 1.02\vartheta_0 + 5.1\vartheta_1 \geq -2.04 = -X(1)(\omega_2) \\ V_\vartheta(1)(\omega_3) = 1.02\vartheta_0 + 10.2\vartheta_1 \geq -1.02 = -X(1)(\omega_3) \end{cases}$$

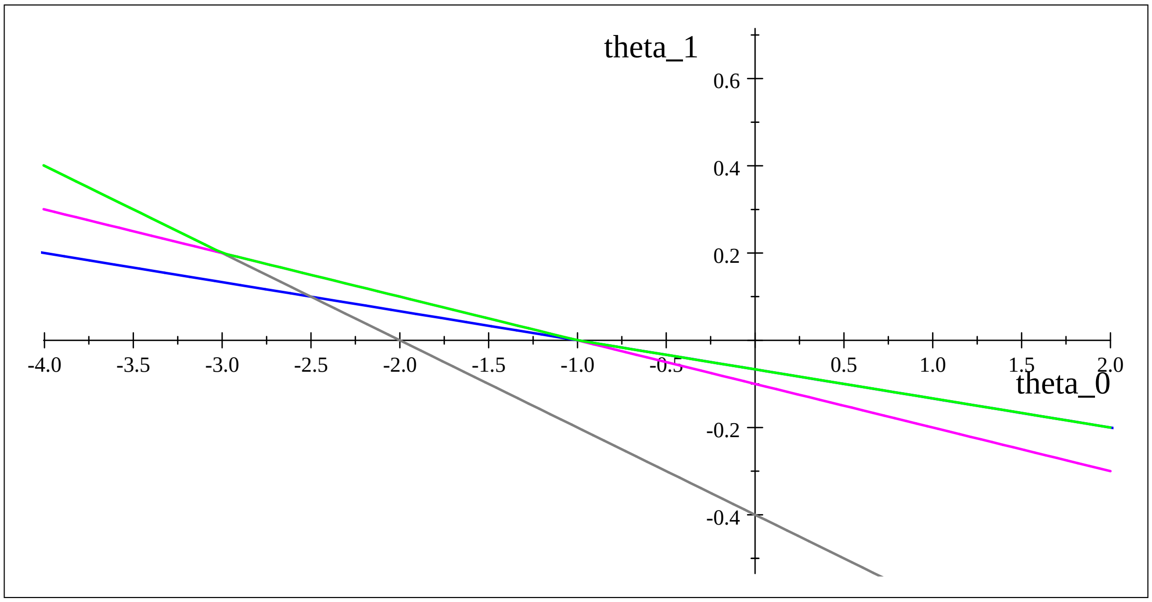
that is

$$\begin{cases} \vartheta_0 + 15\vartheta_1 \geq -1 \\ \vartheta_0 + 5\vartheta_1 \geq -2 \\ \vartheta_0 + 10\vartheta_1 \geq -1 \end{cases}$$

The system leads to the equation of the three lines

$$
\begin{aligned}
r_1 &: \vartheta_1 = -\frac{1}{15}\vartheta_0 - \frac{1}{15} \text{ blue in the graph below } \textit{(replication in } \omega_1\textit{)}\\
r_2 &: \vartheta_1 = -\frac{1}{5}\vartheta_0 - \frac{2}{5} \text{ gray in the graph below } \textit{(replication in } \omega_2\textit{)}\\
r_3 &: \vartheta_1 = -\frac{1}{10}\vartheta_0 - \frac{1}{10} \text{ magenta in the graph below } \textit{(replication in } \omega_3\textit{)}
\end{aligned}
$$

The super-replication region is the set of $(\vartheta_0, \vartheta_1)$ bounded by below by the green contour in the graph plotted below.



Such a green contour coincides with the maximum between r_1, r_2 and r_3.

Hence the right solution is: *the set of all the strategies $\vartheta = (x, y)$ whose final value super-replicates $-X(1)$ is described by*

$$y > \max_x \left\{ -\frac{1}{15}x - \frac{1}{15}, -\frac{1}{5}x - \frac{2}{5}, -\frac{1}{10}x - \frac{1}{10} \right\}$$

which is equivalent to the following system

$$\begin{cases} y > -\frac{1}{5}x - \frac{2}{5} & x \leq -3 \\ y > -\frac{1}{10}x - \frac{1}{10} & -3 < x \leq -1 \\ y > -\frac{1}{15}x - \frac{1}{15} & x < -1 \end{cases}$$

8. The strategies $(\vartheta_0, \vartheta_1)$ whose initial cashflow f are on the line r_f given by the equation

$$
\begin{aligned}
\vartheta_0 + 7.5\vartheta_1 &= -f\\
\vartheta_1 &= -\frac{1}{7.5}\vartheta_0 - \frac{f}{7.5}
\end{aligned}
$$

The slope of the line r_f is negative and inbetween the one of r_2 and r_3. As f increases, the lines r_f (red and dashed in the graph below) move downwards in the plane $(\vartheta_0, \vartheta_1)$. The largest value of f^* within the super-replication region of $-X$ (which is identified by the green contour) is thus reached at the intersection of r_2 and r_3 (the line r_{f^*} is red and solid in the graph). This intersection

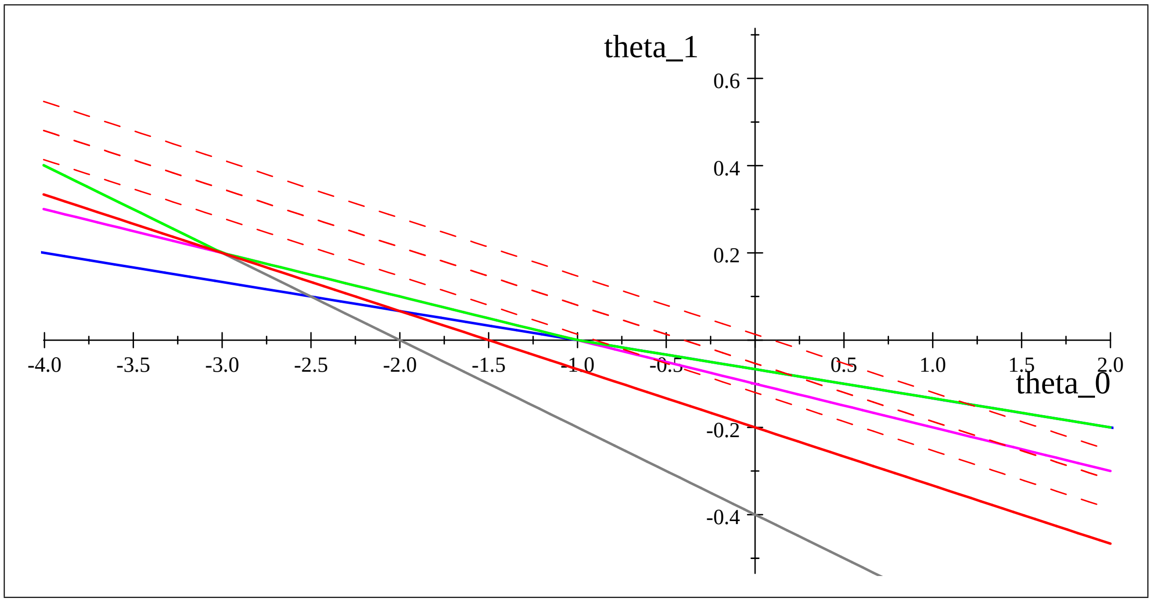
$$
\begin{aligned}
r_2 &: \vartheta_1 = -\frac{1}{5}\vartheta_0 - \frac{2}{5} \textit{ (replication in } \omega_2\textit{)}\\
r_3 &: \vartheta_1 = -\frac{1}{10}\vartheta_0 - \frac{1}{10} \textit{ (replication in } \omega_3\textit{)}
\end{aligned}
$$

can be found solving the following linear system

$$\begin{cases} y = -\frac{1}{5}x - \frac{2}{5} \\ y = -\frac{1}{10}x - \frac{1}{10} \end{cases}$$

The unique solution is $x = -3$ and $y = \frac{1}{5} = 0.2$.

Hence: *The riskless component* x^* *of this strategy is* $x^* = -3$.



9. *The risky component* y^* *of this strategy is* $y^* = \frac{1}{5}$.

   Notice that the initial cashflow derived from this strategy

$$\begin{aligned} -V_{\vartheta^*}(0) &= -(x^* + 7.5 \cdot y^*) \\ &= -\left(-3 + 7.5 \cdot \frac{1}{5}\right) \\ &= 1.5 \end{aligned}$$

   coincides with the lower bound of the interval of no arbitrage prices found in Point 6.

10. If the derivative trades at the initial price of $1.6 \in (1.5, 1.75)$, the market is arbitrage free and complete, since we already showed that the derivate was not redundant.

    Hence the right answer is

    $\boxed{\text{A}}$ arbitrage-free AND complete

11. To compute the initial no arbitrage price of the new derivative in the extended market we first need to compute the unique risk neutral probability. Recalling from Point 6 that the initial no arbitrage price of the European derivative as a function of the risk neutral probability is $q_1 + \dfrac{3}{2}$ we get

$$q_1 + \frac{3}{2} = 1.6$$

   that delivers $q_1 = 1.6 - 1.5 = 0.1$. Then, $q_2 = q_1 + \frac{1}{2} = 0.1 + 0.5 = 0.6$ and $q_3 = \frac{1}{2} - 2q_1 = 0.5 - 2 \cdot 0.1 = 0.3$. Therefore,

$$\begin{cases} q_1 = 0.1 \\ q_2 = 0.6 \\ q_3 = 0.3 \end{cases}$$

is the unique risk neutral measure for the extended market. The no arbitrage price of the new derivative is

$$\begin{aligned}
8 &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}\left[Y(1)\right] \\
&= \frac{1}{1.02}\left(K \cdot q_1 + 2K \cdot q_2 + K \cdot q_3\right) \\
&= \frac{1}{1.02}\left(0.1K + 1.2K + 0.3K\right) \\
&= \frac{1.6}{1.02}K
\end{aligned}$$

that delivers

$$K = \frac{8 \cdot 1.02}{1.6} = 5.1.$$

Alternatively we observe that the final payoff of the new derivative is $\frac{K}{1.02}$ the payoff od the derivative of Point 6. Hence the initial price of the new derivative is $\frac{K}{1.02}$ times the initial price of the derivative of Point 6. Thus

$$\frac{K}{1.02}\left(q_1 + \frac{3}{2}\right) = \frac{K}{1.02} \cdot 1.6 = 8$$

and

$$K = \frac{8 \cdot 1.02}{1.6} = 5.1$$

Hence, the answer is $K = 5.1$.

Solution of EXERCISE 2

1. First of all, notice that, under $\mathbb{P}$,

$$\begin{aligned}
R(t) &= \ln \frac{S(t)}{S(0)} \\
&= \ln \frac{S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W^{\mathbb{P}}(t)\right)}{S(0)} \\
&= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W^{\mathbb{P}}(t) \\
&\overset{\mathbb{P}}{\sim} \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z
\end{aligned}$$

with $Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$, for any $t \in [0,T]$. Therefore,

$$\begin{aligned}
\mathbb{P}\left[R(T) > T\delta\right] &= \mathbb{P}\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z > T\delta\right] \\
&= \mathbb{P}\left[\sigma\sqrt{T}Z > T\delta - \left(\mu - \frac{\sigma^2}{2}\right)T\right] \\
&= \mathbb{P}\left[Z > \sqrt{T}\frac{\delta - \mu + \frac{\sigma^2}{2}}{\sigma}\right] \\
&= 1 - N\left(\sqrt{2}\frac{0.03 - 0.07 + \frac{0.2^2}{2}}{0.2}\right) \\
&= 1 - N(-0.1414) \\
&= N(0.1414)
\end{aligned}$$

Since $N(0.1414) > N(0) = 50\%$, the probability of interest is larger than 50%.

2. From the initial computation in the previous Point, we also get

$$R(T) = \left(\delta - \frac{\sigma^2}{2}\right)T + \sigma W^{Q}(T) \overset{\mathbb{Q}}{\sim} \left(\delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z$$

with $Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$.

The initial no arbitrage price of the forward contract is

$$\begin{aligned}
S_X(0) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\delta T}X(T)\right] \\
&= e^{-\delta T}\mathbb{E}^{\mathbb{Q}}\left[R(T) - F\right].
\end{aligned}$$

It holds $S_X(0) = 0$ if and only if

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[R(T) - F\right] &= 0 \\
\mathbb{E}^{\mathbb{Q}}\left[R(T)\right] - F &= 0
\end{aligned}$$

as F is constant. Therefore, we get

$$\begin{aligned}
F &= \mathbb{E}^{\mathbb{Q}}\left[R(T)\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left(\delta - \frac{\sigma^2}{2}\right)T + \sigma W^{Q}(T)\right] \\
&= \left(\delta - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\underbrace{\mathbb{E}^{\mathbb{Q}}\left[Z\right]}_{=0} \\
&= \left(\delta - \frac{\sigma^2}{2}\right)T \\
&= \left(0.03 - \frac{0.2^2}{2}\right)2 = 0.02
\end{aligned}$$

3. The *historical probability* that the payoff at $T=2$ of the forward introduced in Point 2 is positive is

$$\begin{aligned}
\mathbb{P}\left[X(T)>0\right] &= \mathbb{P}\left[R(T)-F>0\right] \\
&= \mathbb{P}\left[\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}Z-F>0\right] \\
&= \mathbb{P}\left[Z>\frac{F-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \\
&= \mathbb{P}\left[Z>\frac{\left(\delta-\frac{\sigma^2}{2}\right)T-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right] \\
&= \mathbb{P}\left[Z>\frac{(\delta-\mu)\,T}{\sigma\sqrt{T}}\right] \\
&= N\left(-\frac{(\delta-\mu)\sqrt{T}}{\sigma}\right) \\
&= N(0.2828).
\end{aligned}$$

To investigate the sensibility of this probability to σ we study the sign of

$$\frac{\partial}{\partial\sigma}\mathbb{P}\left[R(T)-F>0\right]=\frac{\partial}{\partial\sigma}N\left(-\frac{(\delta-\mu)\sqrt{T}}{\sigma}\right).$$

As $N(\cdot)$ is increasing in its argument, σ appears only at the denominator, and the coefficient $-(\delta-\mu)\sqrt{T}>0$, we have that $\mathbb{P}\left[R(T)-F>0\right]$ is decreasing in σ. More formally,

$$\frac{\partial}{\partial\sigma}\left(-\frac{(\delta-\mu)\sqrt{T}}{\sigma}\right)=\frac{\partial}{\partial\sigma}\left(\frac{1}{\sigma^2}(\delta-\mu)\sqrt{T}\right)<0$$

Therefore, the probability of interest is decreasing with respect to the volatility of the risky asset σ.

4. $S_X(t)$, the no arbitrage price of the forward in Point 2 at any $t\in[0,T]$ is

$$\begin{aligned}
S_X(t) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\delta(T-t)}X(T)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\delta(T-t)}\left(\left(\delta-\frac{\sigma^2}{2}\right)T+\sigma W^Q(T)-F\right)\right] \\
&= e^{-\delta(T-t)}\left(\left(\delta-\frac{\sigma^2}{2}\right)T+\sigma W^Q(t)-F\right) \quad \text{as } W^Q \text{ is a } Q-\text{martingale} \\
&= e^{-\delta(T-t)}\left(\left(\delta-\frac{\sigma^2}{2}\right)T+\sigma W^Q(t)-\left(\delta-\frac{\sigma^2}{2}\right)T\right) \\
&= e^{-\delta(T-t)}\sigma W^Q(t) \\
&= e^{-\delta(T-t)}\sigma\underbrace{\left(\ln\frac{S(t)}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)t\right)\frac{1}{\sigma}}_{W^Q(t)} \\
&= e^{-\delta(T-t)}\left(\ln\frac{S(t)}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)t\right)
\end{aligned}$$

5. $\vartheta(t)=(\vartheta_0(t),\vartheta_1(t))$, the *replicating portfolio* of the forward in Point 2 at any $t\in[0,T]$ is

$$\vartheta_1(t)=\frac{\partial}{\partial S(t)}S_X(t)=e^{-\delta(T-t)}\frac{1}{S(t)}$$

and

$$\vartheta_0(t)e^{\delta t}=e^{-\delta(T-t)}\left(\ln\frac{S(t)}{S(0)}-\left(\delta-\frac{\sigma^2}{2}\right)t\right)-e^{-\delta(T-t)}\frac{1}{S(t)}S(t)$$

6. The replicating portfolio found in the previous Point is long on S

7. The *historical mean and variance* of

$$\begin{aligned}\Delta R &= R(2)-R(1)=\ln\frac{S(0)\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)2+\sigma W^{\mathbb{P}}(2)\right)}{S(0)}-\ln\frac{S(0)\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)1+\sigma W^{\mathbb{P}}(1)\right)}{S(0)}\\ &= \left(\mu-\frac{\sigma^2}{2}\right)(2-1)+\sigma\left(W^{\mathbb{P}}(2)-W^{\mathbb{P}}(1)\right)\overset{\mathbb{P}}{\sim}N\left(\left(\mu-\frac{\sigma^2}{2}\right)1,\sigma^2\sqrt{1}\right)\end{aligned}$$

   Hence the mean is $\mu-\frac{\sigma^2}{2}=0.07-\frac{0.20^2}{2}=0.05$ and variance $\sigma^2=0.2^2=0.04$.

8. The *historical probability* conditional at $t=1$ that the final payoff at $T=2$ of the forward introduced in Point 2 is positive is

$$\mathbb{P}_1\left[X(2)>0\right]=\mathbb{P}_1\left[R(2)-F>0\right]$$

   We observe that

$$R(2)=\Delta R+R(1)\ \text{ with } \Delta R \text{ is } \mathcal{F}_1-\text{independent and } \Delta R \text{ is } \mathcal{F}_1-\text{measurable.}$$

   Hence

$$\begin{aligned}\mathbb{P}_1\left[X(2)>0\right] &= \mathbb{P}_1\left[R(2)-F>0\right]=\mathbb{P}_1\left[\Delta R+R(1)-F>0\right]\\ &= \mathbb{P}_1\left[\Delta R>F-R(1)\right]\\ &= \mathbb{P}\left[N\left(\left(\mu-\frac{\sigma^2}{2}\right)1,\sigma^2\sqrt{1}\right)>F-x\right]\quad\text{where } x=R(1)\\ &= \mathbb{P}\left[\left(\mu-\frac{\sigma^2}{2}\right)1+\sigma Z>F-x\right]\\ &= \mathbb{P}\left[Z>\frac{F-x-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma}\right]\\ &= N\left(-\frac{F-x-\left(\mu-\frac{\sigma^2}{2}\right)}{\sigma}\right)\geq 50\%\end{aligned}$$

   or alternatively

$$\begin{aligned}\mathbb{P}_{\frac{T}{2}}\left[X(2)>0\right] &= \mathbb{P}_{\frac{T}{2}}\left[\ln\frac{S(T)}{S(\frac{T}{2})}\frac{S(\frac{T}{2})}{S(0)}-F>0\right]\\ &= \mathbb{P}_{\frac{T}{2}}\left[\ln\frac{S(T)}{S(\frac{T}{2})}>F-\ln\frac{S(\frac{T}{2})}{S(0)}\right]\\ &= \mathbb{P}\left[\ln\frac{S(T)}{S(\frac{T}{2})}>F-x\right]\quad\text{with } x=\ln\frac{S(\frac{T}{2})}{S(0)},\ \text{as } \frac{S(T)}{S(\frac{T}{2})} \text{ is } \mathcal{F}_{\frac{T}{2}}\text{-independent}\\ &= \mathbb{P}\left[\left(\mu-\frac{\sigma^2}{2}\right)\frac{T}{2}+\sigma\sqrt{\frac{T}{2}}Z>F-x\right]\\ &= \mathbb{P}\left[Z>\frac{F-x-\left(\mu-\frac{\sigma^2}{2}\right)\frac{T}{2}}{\sigma\sqrt{\frac{T}{2}}}\right]\end{aligned}$$

In any case

$$
\begin{gathered}
-\frac{F - x - \left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{2}}{\sigma\sqrt{\frac{T}{2}}} \geq 0 \\
F - x - \left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{2} \leq 0 \\
x \geq F - \left(\mu - \frac{\sigma^2}{2}\right)\frac{T}{2} \\
= 0.02 - \left(0.07 - \frac{0.2^2}{2}\right) = -0.03
\end{gathered}
$$

Hence if

$$
\begin{aligned}
\ln\frac{S(\frac{T}{2})}{S(0)} &\geq -0.03 \\
S(\frac{T}{2}) &\geq S(0)\exp(-0.03) = 10\exp(-0.03) = 9.70
\end{aligned}
$$

then $\mathbb{P}_{\frac{T}{2}}\left[X(T) > 0\right] \geq 50\%$.

Solution to the THEORY QUESTIONS

1. In the one-period market model state the First Fundamental Theorem of Asset Pricing: see the lecture notes

2. Let $\mathbb{Q}$ denote any risk-neutral probability for an arbitrage-free discrete multiperiod market. Consider the redundant cashflow $X = \{X(t)\}_{t=1}^{T}$ and let $S_X = \{S_X(t)\}_{t=0}^{T}$ be its price process. The extended market is arbitrage free if and only if *(mark all appropriate answers)*

   $\boxed{\text{A}}$ $S_X(t) = V_{\vartheta^X}(t)$ for all $t = 0, ..., T$ and for every replicating strategy ϑ^X

   $\boxed{\text{B}}$ $\frac{S_X(t)}{B(t)} = \sum_{\tau=t+1}^{T} \mathbf{E}^{\mathbb{Q}}\left[\left.\frac{X(\tau)}{B(\tau)}\right|\mathcal{P}_t\right]$ for all $t = 0, ..., T-1$

   $\boxed{\text{D}}$ $\begin{cases} S_X(t) = \mathbf{E}^{\mathbb{Q}}\left[\left.\frac{X(t+1)+S_X(t+1)}{1+r(t)}\right|\mathcal{P}_t\right] & \text{for } t = 0, 1, ..., T-2 \\ S_X(T-1) = \mathbf{E}^{\mathbb{Q}}\left[\left.\frac{X(T)}{1+r(T-1)}\right|\mathcal{P}_{T-1}\right] & \text{for } t = T-1 \end{cases}$

3. Consider a diffusion process $\{X(t)\}_{t\in[0,T]}$ satisfying the stochastic differential equation $dX(t) = a(t, X(t))\ dt + b(t, X(t))\ dW(t)$ with $X(0) = X_0$. Assume $\varphi : [0;T] \times \Re \to \Re$ be continuously differentiable, once with respect to the first variable, denoted by t, twice with respect to the second, denoted by x. Let $Y(t) = \varphi(t; X(t))$. Then the process Y satisfies

   $\boxed{\text{B}}$ $dY(t) = \left[\frac{\partial\varphi(t;X(t))}{\partial t} + \frac{\partial\varphi(t;X(t))}{\partial x} \cdot a(t, X(t)) + \frac{1}{2}\frac{\partial^2\varphi(t;X(t))}{\partial x^2} b^2(t, X(t))\right] dt + \frac{\partial\varphi(t;X(t))}{\partial x} b(t, X(t)) dW(t)$

4. Let $N(a, b^2)$ denote a normal random variable with mean a and variance b^2. In the Black Scholes model and with respect to the historical measure $\mathbb{P}$ the log-price of the risky security S is

   $\boxed{\text{B}}$ $\ln\frac{S(t)}{S(0)} \overset{\mathbb{P}}{\sim} N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$