Algebra and Geometry (Cod. 30544)

General Exam – December 10, 2021

Time: 2 hours. Total: 150 points.

Multiple choice questions (total: 24 points)

Each question has a single correct answer: write the correct answer in the box on the right. If you want to change your response cancel it and write another answer next to the box. 6 points are assigned for a correct answer, 0 points for a missing answer, -2 point for an incorrect answer.

1. Given an integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$ and set $\mathcal{P} := \bigcup_{n\geq 0} \mathcal{P}_n$. Then:

   (A) $\mathcal{P}$ is a finitely generated vector space
   (B) $\mathcal{P}_4$ contains 6 L.I. polynomials
   (C) $\mathcal{P}_6$ contains 4 linearly dependent polynomials
   (D) none of the others

2. Let $V = \mathbf{Z}_2^{\mathbf{N}}$ be the vector space of binary sequence over the field $\mathbf{Z}_2$. Then

   (A) The set $\{x \in V : nx = 0_V \text{ for some } n \geq 1\}$ is a vector space
   (B) $\dim(V) = 1$
   (C) Every $T \in \mathcal{L}(V)$ is injective
   (D) none of the others

3. Let $T : V \to W$ be a linear operator, where V, W are finite dimensional real vector spaces. Then

   (A) The transpose operator is a linear function $V' \to W'$
   (B) $\dim(T(V)) \leq \dim(V)$
   (C) $\dim(\mathrm{Ker}(T)) \geq \dim(V)$
   (D) none of the others

4. Let V be a vector space over a field $\mathbf{F}$ and fix a subspace $W \subseteq V$. Then

   (A) $V + 2W := \{x + 2y : x \in V, y \in W\}$ is a vector space
   (B) $|V/W| = \infty$ if $W \neq V$
   (C) $V + 2W^2 := \{x + 2y^2 : x \in V, y \in W\}$ is a vector space
   (D) none of the others

True/False questions (total: 24 points)

Each statement can be either true or false: write T for true or F for false in the box on the right. If you want to change your response cancel it and write another answer next to the box. 4 points are assigned for a correct answer, 0 points for a missing answer, -1 point for an incorrect answer.

1. Every permutation of $\{1, \ldots, n\}$ can be written uniquely as product of transpositions.
2. $\mathbf{Z}_3 \times \mathbf{Z}_3$ is vector space over the field $\mathbf{Z}_3$.
3. Suppose that 3 is an eigenvalue of $A \in \mathscr{M}_3(\mathbf{R})$. Then $A^2 - 3A$ is not invertible.
4. $\mathcal{A}(V)$ is a vector space of dimension $\dim(V)$, provided that V is a finite dimensional vector space.
5. Every square matrix is diagonalizable.
6. For each integer $n \in [0, \dim(V)]$, there exists a subspace M such that $\dim(V/M) = n$.

Open answer questions (total: 102 points)

Answers must be written in the corresponding spaces. Each of the six questions will be assigned from 0 to 17 points.
Answers must be adequately justified.

Question 1. (a) Provide the definition of field $\mathbf{F}$.
(b) Does there exist a field $\mathbf{F}$ such that $5x = 0$ for all $x \in F$?

Question 2. Given an integer $n \geq 0$, let $\mathcal{P}_n$ be the set of polynomials with real coefficients and degree $\leq n$. Define $X_0 := \{0\}$ and $X_n := \{p_1^2 + \cdots + p_n^2 : p_1, \ldots, p_n \in \mathcal{P}_n\}$ if $n \geq 1$. Show that X_n is a vector space over $\mathbf{R}$ if and only if $n = 0$.

Question 3. Let V be a finite dimensional vector space over a field $\mathbf{F}$, and fix a vector subspace $M \subseteq V$.
(a) Give the definition of annihilator M° and show that it is vector space.
(b) Prove that $\dim(M^\circ) + \dim(M) = \dim(V)$.

Question 4. Let V, W be finite dimensional vector spaces and $T \in \mathcal{L}(V, W)$. Show that

$$\forall x \in V, \quad (T(x))'' = T''x''.$$

(Here, $T''x'' := T'' \circ x''$ and, for each $x \in V$, x'' stands for the element of the bidual V'' associated with x through the canonical isomorphism.)

Question 5. Let V be a finite dimensional vector space over $\mathbf{R}$ and fix $T \in \mathcal{L}(V)$.
(a) Provide the definition of $T^\frown$.
(b) State the theorem which gives you the relationship between $T^\frown$ and the determinant of T.

Question 6. Let V be a finite dimensional vector space and fix $T \in \mathcal{L}(V)$.
(a) Provide the definition of eigenvalue of T.
(b) Let c be an eigenvalue of $T \in \mathcal{L}(V)$, and let $M = \mathrm{Ker}(T - cI)$. If $S \in \mathcal{L}(V)$ and $ST = TS$, show that $S(M) \subseteq M$.

0.1 Solutions Multiple choices / True-False

Multiple choices:

1	2	3	4
C	A	B	A

True/False:

1	2	3	4	5	6
F	T	T	F	F	T

0.2 Open question

1 (a) A field $\mathbf{F}$ is a set F endowed with two operations $+ : F^2 \to F$ and $\times : F^2 \to F$ such that: (i) $(F, +)$ is an abelian group; (ii) $(F^\star, \times)$ is an abelian group; (iii) $a \times (b + c) = a \times b + a \times c$ for all $a, b, c \in F$.

(b) Yes, $\mathbf{F} = \mathbf{Z}_5$.

Note: The hypothesis that $(F, +)$ is an abelian group requires the existence of an additivity identity, let us call it 0, hence $|F| \geq 1$. Moreover, the hypothesis that $(F^\star, \times)$ is an abelian group, where $F^\star := F \setminus \{0\}$, requires the existence of a multiplicative identity, so that $|F^\star| \geq 1$, and this implies that $|F| \geq 2$. In particular, $F = \{0\}$ cannot be a field!

2 If $n = 0$ then $X_0 = \{0\}$ is a vector space. Conversely, if $n \geq 1$, choose $p_1 = \cdots = p_n = 1$, hence $g := \sum_{i=1}^n p_i^2 \in X_n$, where g is the constant function n. If X_n were a vector space then $\alpha g \in X_n$ for all $\alpha \in \mathbf{R}$. But this this impossible since $-g \notin X_n$: indeed if $p \in X_n$ then necessarily $p(x) \geq 0$ for all $x \in \mathbf{R}$.

3 (a) $M^\circ := \{f \in V' : f(x) = 0 \text{ for all } x \in M\}$. In addition, if $f, g \in M^\circ$ and $\alpha \in \mathbf{F}$, then $(f+g)(x) = f(x)+g(x) = 0$ for all $x \in M$ and $(\alpha f)(x) = \alpha f(x) = 0$ for all $x \in M$. Moreover, $0 \in M^\circ \subseteq V'$, hence it is a vector subspace of the algebraic dual V'.

(b) Let $\{x_1, \ldots, x_k\}$ be a basis of M. Extend it to a basis $\{x_1, \ldots, x_n\}$ of V. Consider the dual basis $\{f_1, \ldots, f_n\}$ of V' such that $f_i(x_j) = 1$ if $i = j$ and 0 otherwise (in particular $\dim(V') = \dim(V)$). Now it is enough to check that $f_1, \ldots, f_k \notin M^\circ$ and $f_{k+1}, \ldots, f_n \in M^\circ$. Therefore $\dim(M^\circ) = n - k = \dim(V) - \dim(M)$.

4 Fix $x \in V$, so that $Tx \in W$. The associated linear function $(Tx)''$ in the bidual W'' is, hence, a linear form $W' \to \mathbf{F}$.

Note also that T'' is the bitranspose of $T : V \to W$, hence it is a function $V'' \to W''$. Considering that $x'' \in V''$, the composite linear operator $T''x''$ is a function in W'', that is, a linear operator $W' \to \mathbf{F}$.

Fix a function $f \in W'$. Then, on the one hand,

$$(T(x))''(f) = f(Tx)$$

since $Tx \in W$ and $(Tx)'' \in W''$.

On the other hand,

$$(T''x'')(f) = (x''T')(f) = x''(T'f) = (T'f)(x) = f(Tx).$$

Therefore these two functions are equal.

5 See lecture notes.

6 (a) See lecture notes.

(b) M is composed by the vectors $x \in V$ such that $Tx = c \cdot Ix$, i.e., $Tx = cx$. At this point, if $ST = TS$ and $x \in M$ then Sx belongs to M because

$$T(Sx) = (TS)(x) = (ST)(x) = S(Tx) = S(cx) = c(Sx),$$

which completes the proof.