Proof Playground

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This document serves as my playground for practising the art of writing mathematical proofs. As such, do not expect to find a proof of the Riemann hypothesis here, or indeed anything else that is original. Instead, you will find a range of proofs related to mostly standard undergraduate material that I *hope* are correct.

Algebra

Theorem 1 (Quadratic Formula). *The solutions of the quadratic equation of the form* $ax^2 + bx + c = 0$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. Let $ax^2 + bx + c = 0$. Complete the square to obtain

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$

Solve for x to obtain

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as desired. □

Theorem 2 (Logarithm of a Product). $\log_b(xy) = \log_b(x) + \log_b(y)$.

Proof. Let $x := b^m$ and $y := b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

By substituting b^m and b^n for x and y in $\log_b(xy)$,

$$\begin{aligned} \log_b(xy) &= \log_b(b^m b^n) \\ &= \log_b(b^{m+n}) \\ &= m + n. \end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b(xy) = \log_b(x) + \log_b(y),$$

which is the desired identity. $\square$

Theorem 3 (Logarithm of a Quotient)**.** $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y).$

Proof. Let $x := b^m$ and $y := b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

By substituting b^m and b^n for x and y in $\log_b(x/y)$,

$$\begin{aligned}\log_b\left(\frac{x}{y}\right) &= \log_b\left(\frac{b^m}{b^n}\right)\\ &= \log_b(b^{m-n})\\ &= m - n.\end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y),$$

which is the desired identity. $\square$

Theorem 4 (Logarithm of a Power)**.** $\log_b(x^y) = y\log_b(x).$

Proof. Let $x := b^n$. From the definition of a logarithm, it follows that $\log_b(x) = n$.

By substituting b^n for x in $\log_b(x^y)$,

$$\begin{aligned}\log_b(x^y) &= \log_b((b^n)^y)\\ &= \log_b(b^{yn})\\ &= yn.\end{aligned}$$

Since $n = \log_b(x)$, the above can be written as

$$\log_b(x^y) = y\log_b(x),$$

which is the desired identity. $\square$

Theorem 5 (Change of Base Formula)**.** *Any logarithm can be rewritten in terms of another base with the formula*

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}.$$

Proof. Let $c := \log_b(a)$. From the definition of a logarithm, it follows that $b^c = a$. By taking the base-x logarithm of both sides,

$$\log_x(b^c) = \log_x(a).$$

Therefore,

$$\begin{aligned} c\log_x(b) &= \log_x(a) \\ c &= \frac{\log_x(a)}{\log_x(b)}. \end{aligned}$$

Since $c = \log_b(a)$, the above can be written as

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)},$$

which is the change of base formula. $\square$

Number Theory

Definition 1 (Natural Number)**.** A natural number is a member of the set $\mathbb{N} = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Result 1. *The number $\sqrt{2}$ is irrational.*

Proof. Suppose that $\sqrt{2}$ is rational, and so it can be expressed as a ratio of two integers. Hence, let $\sqrt{2} = a/b$, where a and b are coprime. By squaring both sides,

$$2 = \frac{a^2}{b^2} \iff 2b^2 = a^2.$$

Therefore, a^2 is even, and in turn, a is also even, since squares of odd integers are never even. Hence, there exists some integer k such that $a = 2k$. By substituting this in,

$$2b^2 = (2k)^2 \iff b^2 = 2k^2.$$

Using the same reasoning as with a, it must be that b is also even. However, since a and b are both even, a contradiction arises. Two even integers cannot be coprime, so $\sqrt{2}$ cannot be expressed as a ratio of two integers. Thus, $\sqrt{2}$ is irrational. $\square$

Theorem 1. *There are infinitely many prime numbers.*

Proof. Suppose there are only finitely many prime numbers. Let $p_1, p_2, \ldots, p_n$ be a list of all primes and let $m := p_1 p_2 \cdots p_n + 1$. Note that m is not divisible by p_1 since dividing m by p_1 gives a remainder of 1. Similarly, m is not divisible by any other number in the list. Because m is larger than 1, m is either a prime or a product of primes.

If m is a prime, then we have found a prime not in our list, which contradicts the assumption that it was a list of all prime numbers.

If m is a product of primes, then it must be divisible by one of the primes in our list. However, we have shown m is not divisible by any number in the list. Thus the assumption that the list was a list of all prime numbers is again contradicted.

Since the assumption that there are only finitely many prime numbers has led to a contradiction, there must be infinitely many prime numbers. □

Theorem 2. *The sum of the first n natural numbers is equal to*

$$\frac{n(n+1)}{2}.$$

Proof. We proceed by induction. If $n = 1$, then the theorem is clearly true:

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1.$$

So, the theorem holds for the base case of $n = 1$.

For the inductive hypothesis, assume the formula is true for all $k > 1$. Hence,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

For the inductive step, let $n = k + 1$. By the properties of summation,

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1).$$

By using the inductive hypothesis,

$$\begin{aligned}
\sum_{i=1}^{k} i + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}.
\end{aligned}$$

So, the theorem holds when $n = k + 1$.

Since the base case and inductive step have been shown, the theorem holds for all natural numbers by the principle of mathematical induction. □

Theorem 3. *The sum of the squares of the first n natural numbers is equal to*

$$\frac{n(n+1)(2n+1)}{6}.$$

Proof. We proceed by induction. If $n = 1$, then the theorem is clearly true:

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = 1.$$

So, the theorem holds for the base case of $n = 1$.

For the inductive hypothesis, assume the formula is true for all $k > 1$. Hence,

$$\sum_{i=1}^{k} i = \frac{k(k+1)(2k+1)}{6}.$$

For the inductive step, let $n = k + 1$. By the properties of summation,

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2.$$

By using the inductive hypothesis,

$$\begin{aligned}
\sum_{i=1}^{k} i^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1) \\
&= \frac{k(k+1)(2k+1) + 6(k+1)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.
\end{aligned}$$

So, the theorem holds when $n = k + 1$.

Since the base case and inductive step have been shown, the theorem holds for all natural numbers by the principle of mathematical induction. □

Set Theory

Definition 1 (Ordered Pair)**.** The ordered pair of two elements a and b is the set

$$(a, b) := \{\{a\}, \{a, b\}\}.$$

Definition 2 (Cartesian Product)**.** The Cartesian product of two sets A and B is the set

$$A \times B := \{\, (a, b) : a \in A, b \in B \,\}.$$

Theorem 1. *For any sets A, B, and C, the following hold:*

(*a*) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(*b*) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(c) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(d) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof of A. Let $(u, v) \in (A \cup B) \times C$. Therefore, $u \in A \cup B$ and $v \in C$. This means that $u \in A$ or $u \in B$. If $u \in A$, then $(u, v) \in A \times C$. If $u \in B$, then $(u, v) \in B \times C$. Either way, $(u, v) \in (A \times C) \cup (B \times C)$. Hence,

$$(A \cup B) \times C \subseteq (A \times C) \cup (B \times C).$$

Now, let $z := (x, y) \in (A \times C) \cup (B \times C)$. Either $z \in A \times C$ or $z \in B \times C$. In the first case, $x \in A$ and $y \in C$. In the second, $x \in B$ and $y \in C$, so $z = (x, y) \in (A \cup B) \times C$. This implies that

$$(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C.$$

Putting the two parts together completes the proof. □

Proof of B. Let $(u, v) \in (A \cap B) \times C$. Therefore, $u \in A \cap B$ and $v \in C$. This means that $u \in A$ and $u \in B$. Thus, $(u, v) \in A \times C$ and $(u, v) \in B \times C$, and consequently, $(u, v) \in (A \times C) \cap (B \times C)$. Hence,

$$(A \cap B) \times C \subseteq (A \times C) \cap (B \times C).$$

Now, let $z := (x, y) \in (A \times C) \cap (B \times C)$. Therefore, $z \in A \times C$ and $z \in B \times C$. So, $x \in A$ and $x \in B$, and likewise, $y \in C$. Thus, $z = (x, y) \in (A \cap B) \times C$. This implies that

$$(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C.$$

Putting the two parts together completes the proof. □

Real Analysis

Definition 1 (Limit)**.** Let f be a real-valued function defined on a subset D of the real numbers. Let c be a limit point of D and let L be a real number. We say that

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in D$,

$$0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Result 1. $\lim_{x \to 2} (x^2 + 1) = 5$.

Proof. Suppose $\varepsilon > 0$. Let $\delta := \min(1, \varepsilon/5)$ and $x \in \mathbb{R}$ such that $0 < |x - 2| < \delta$.
Since $|x - 2| < \delta$, it follows that

$$\begin{aligned} |x - 2| < 1 &\implies -1 < x - 2 < 1 \\ &\implies 1 < x < 3. \end{aligned}$$

In particular, this means that $|x+2| < 5$. Likewise, it follows that $|x-2| < \varepsilon/5$.

Hence,

$$\begin{aligned}
|x-2| < \delta &\implies |x-2| < \frac{\varepsilon}{5} \\
&\implies |x+2||x-2| < 5 \cdot \frac{\varepsilon}{5} \\
&\implies |x^2-4| < \varepsilon \\
&\implies |(x^2+1)-5| < \varepsilon.
\end{aligned}$$

□

Result 2. $\lim_{x\to 3}(x^2+6) = 15.$

Proof. Suppose $\varepsilon > 0$. Let $\delta := \min(1, \varepsilon/7)$ and $x \in \mathbb{R}$ such that $0 < |x-3| < \delta$.

Since $|x-3| < \delta$, it follows that

$$\begin{aligned}
|x-3| < 1 &\implies -1 < x-3 < 1 \\
&\implies 2 < x < 4.
\end{aligned}$$

In particular, this means that $|x+3| < 7$. Likewise, it follows that $|x-3| < \varepsilon/7$.

Hence,

$$\begin{aligned}
|x-3| < \delta &\implies |x-3| < \frac{\varepsilon}{7} \\
&\implies |x+3||x-3| < 7 \cdot \frac{\varepsilon}{7} \\
&\implies |x^2-9| < \varepsilon \\
&\implies |(x^2+6)-15| < \varepsilon.
\end{aligned}$$

□

Result 3. $\lim_{x\to 0} \frac{x}{x^2+1} = 0.$

Proof. Suppose $\varepsilon > 0$. Let $\delta := \min(1, 2\varepsilon)$ and $x \in \mathbb{R}$ such that $0 < |x| < \delta$.

Since $|x| < \delta$, it follows that $|x| < 1$, and thus $|x^2+1| < 2$. Likewise, it follows that $|x| < 2\varepsilon$.

Hence,

$$\begin{aligned}
|x| < 2\varepsilon &\implies \frac{|x|}{|x^2+1|} < \frac{2\varepsilon}{2} \\
&\implies \frac{|x|}{|x^2+1|} < \varepsilon \\
&\implies \left|\frac{x}{x^2+1}\right| < \varepsilon.
\end{aligned}$$

□

Result 4. $\lim_{x\to 5^+} \frac{1}{x-5} = \infty.$

Proof. Suppose $M > 0$. Let $\delta := 1/M$ and $x \in \mathbb{R}$ such that $0 < x-5 < \delta$.

Since $x-5 < \delta$, it follows that

$$x-5 < \frac{1}{M} \implies \frac{1}{x-5} > M.$$

□