Proofs

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1 About This Document

This document contains various assorted proofs that I have written down in order to assist my understanding. Nothing in here is novel or new, these are all simply retellings of that which has already been proven.

2 The Quadratic Formula

Theorem 1. *The solutions of a quadratic equation of the form $ax^2 + bx + c = 0$ can be found with the formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a \neq 0$.

Proof. Completing the square on the general quadratic equation $ax^2 + bx + c = 0$ gives

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} - \frac{c}{a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2}{4a^2} - \frac{c}{a}.
\end{aligned}
$$

This allows for isolating x, giving

$$
\begin{aligned}
(x + \frac{b}{2a})^2 &= \frac{b^2}{2a^2} - \frac{c}{a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\end{aligned}
$$

which is the quadratic formula. □

3 Irrationality of $\sqrt{2}$

Theorem 2. *The square root of two, $\sqrt{2}$, can not be represented as a ratio of two integers, and therefore is irrational.*

Proof. Suppose that $\sqrt{2}$ is rational and thus can be written as the ratio of two integers $\frac{a}{b}$, where a is prime to b ($a \perp b$). Therefore, we can state that

$$\begin{aligned}\sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2}.\end{aligned}$$

Rearranging for a^2 gives the equation

$$a^2 = 2b^2,$$

which shows that a^2 is even, since $2b^2$ is necessarily even because it is a multiple of 2.

It follows that a must also be even, as squares of odd integers are never even. Therefore, there exists some integer k that satisfies the equation $a = 2k$.

Substituting $2k$ for a leads to

$$\begin{aligned}(2k)^2 &= 2b^2 \\ 4k^2 &= 2b^2 \\ 2k^2 &= b^2,\end{aligned}$$

which demonstrates that b^2 and therefore b itself must be even, using the same reasoning as with a^2.

Since a and b are both even, the assumption that $\sqrt{2}$ is rational and a is prime to b no longer holds, as they share a common factor of 2. Therefore, $\sqrt{2}$ must be **irrational**. □

4 Logarithms

4.1 The Product Law

Theorem 3. *A logarithm of the form* $\log_b(xy)$ *can be expressed as* $\log_b(x) + \log_b(y)$.

Proof. Let $x = b^m$ and $y = b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

Substituting b^m and b^n for x and y in the expression $\log_b(xy)$ gives the equation

$$\begin{aligned} \log_b(xy) &= \log_b(b^m \cdot b^n) \\ &= \log_b(b^{m+n}) \\ &= m + n. \end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

with the use of substitution. □

4.2 The Quotient Law

Theorem 4. *A logarithm of the form* $\log_b(\frac{x}{y})$ *can be expressed as* $\log_b(x) - \log_b(y)$.

Proof. Let $x = b^m$ and $y = b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

Substituting b^m and b^n for x and y in the expression $\log_b(\frac{x}{y})$ gives the equation

$$\begin{aligned} \log_b\left(\frac{x}{y}\right) &= \log_b\left(\frac{b^m}{b^n}\right) \\ &= \log_b(b^{m-n}) \\ &= m - n. \end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

with the use of substitution. □

4.3 The Power Law

Theorem 5. *A logarithm of the form* $\log_b(x^y)$ *can be expressed as* $y \cdot \log_b(x)$.

Proof. Let $x = b^c$. From the definition of a logarithm, it follows that $\log_b(x) = c$.

Substituting b^c for x in the expression $\log_b(x^y)$ gives the equation

$$\begin{aligned} \log_b(x^y) &= \log_b((b^c)^y) \\ &= \log_b(b^{cy}) \\ &= cy \end{aligned}$$

Since $x = \log_b$, the above can be written as

$$\begin{aligned} \log_b(x^y) &= \log_b(x) \cdot y \\ &= y \cdot \log_b(x) \end{aligned}$$

with the use of substitution. □

4.4 The Change of Base Formula

Theorem 6. *Any logarithm* $\log_b(a)$ *can be re-expressed in terms of another base with the formula*

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}.$$

Proof. Let $c = \log_b(a)$. From the definition of a logarithm, it follows that $b^c = a$.

Taking the base-x logarithm on both sides gives the equation

$$\log_x(b^c) = \log_x(a).$$

Applying the logarithmic power law and solving for c leads to the equation

$$\begin{aligned} c\log_x(b) &= \log_x(a) \\ c &= \frac{\log_x(a)}{\log_x(b)}. \end{aligned}$$

Since $c = \log_b(a)$, the above can be written as

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}$$

with the use of substitution. $\square$