Calculus Notes

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Chapter 1

Limits and Continuity

A *limit* is the value that a function (or sequence) "approaches" as the input (or index) "approaches" some value. Limits form the foundation of modern calculus and are used to define continuity, derivatives, and integrals.

Formally, the limit of a function is written as

$$\lim_{x \to c} f(x) = L,$$

and is read as "the limit of f of x as x approaches c is equal to L". Occasionally, a limit may be denoted by a right arrow, as in

$$f(x) \to L \text{ as } x \to c.$$

Continuity is the property of a function where *sufficiently small changes in the input result in sufficiently small changes in the output.* A function that satisfies this property is *continuous*, and a function that does not is *discontinuous.*

For example, consider the function $h(t)$, which describes the height of a growing flower at time t. This function is continuous. By contrast, if $a(t)$ describes the amount of money in a bank account at time t, the function jumps at each point in time when money is deposited or withdrawn, so the function a is discontinuous.

1.1 Limits at Infinity

A *limit at infinity* is the value a function approaches as the input approaches positive or negative infinity (that is, as the input increases or decreases without bound). Formally, we write that

$$\lim_{x \to \infty} f(x) = \text{ or } \lim_{x \to -\infty} f(x) = L.$$

When computing limits at infinity, the following two facts are often useful:

Fact 1. If $n \in \mathbb{Q}^+$ and $x \in \mathbb{R}$, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0.$$

Fact 2. If $n \in \mathbb{Q}^+$, $x \in \mathbb{R}$, and x^n is defined for $x < 0$, then

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

1.2 The Squeeze Theorem

The *squeeze theorem*, or *sandwich theorem*, is a theorem regarding the limit of a function as it is bounded by two other functions. It is useful when attempting to determine a limit of a function that is either difficult to determine or cannot be computed at a given point.

Theorem (Squeeze Theorem)**.** *Let a be a point on the interval I. Let g, f, and h be functions defined at all points on I, except possibly at a itself.*

Suppose that for every x in I not equal to a, we have

$$g(x) \leq f(x) \leq h(x),$$

and also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then,

$$\lim_{x \to a} f(x) = L.$$

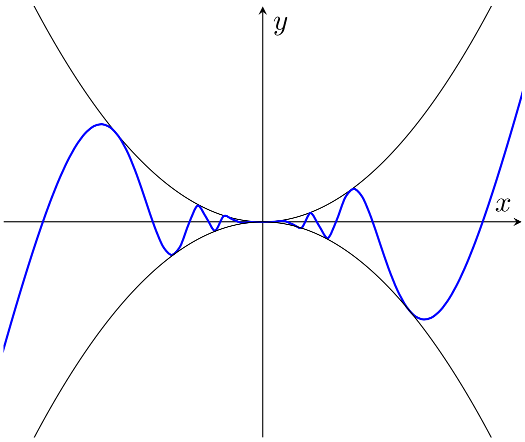

Figure 1.1: Illustration of $x^2 \sin(\frac{1}{x})$ being squeezed as $x \to 0$.

Example. The limit of $x^2 \sin(\frac{1}{x})$ as $x \to 0$ (see Figure 1.1) cannot be determined through direct substitution because $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

However, by the definition of the sine function,

$$-1 \leq \sin(\frac{1}{x}) \leq 1.$$

Therefore, it follows that

$$-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2.$$

Since

$$\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0,$$

by the squeeze theorem, $\lim_{x\to 0}[x^2 \sin(\frac{1}{x})]$ must also be zero.

1.3 Continuity at a Point

A function f is said to be continuous at c if it is both defined at c and its value at c equals the limit of f as x approaches c:

$$\lim_{x\to c} f(x) = f(c)$$

Chapter 2

Differentiation

Differential calculus is the area of calculus that is concerned with the study of the rates at which quantities change.

The primary object of study in differential calculus is the *derivative of a function*, which describes the rate of change of a given function. The process of finding a derivative is called *differentiation*.

Geometrically, the derivative of a function can be viewed as the function that gives the slope of a tangent line to the graph of the function at any given point.

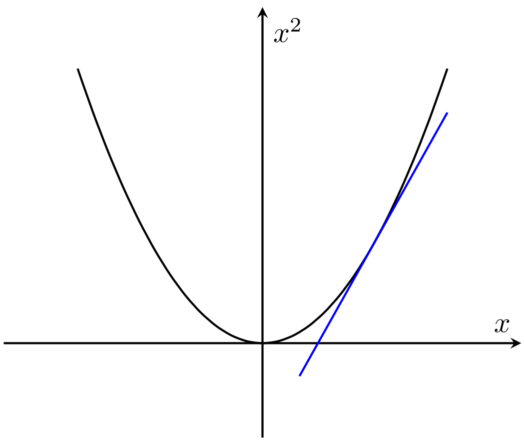

Figure 2.1: Graph of x^2 and a tangent line at $x = 3$.

2.1 Notation

Differential calculus has no single uniform notation for differentiation. Several different notations for the derivative of a function or variable have been proposed by different mathematicians, the most common of which are Langrange's[1] and Leibniz's[2].

[1] Joseph Lewis Lagrange was an Italian Enlightenment Era mathematician and astronomer.

[2] Gottfried Wilhelm von Leibniz was a German Enlightenment Era polymath most notably known as (one of) the inventors of differential and integral calculus.

While this may appear to be a mess at first, the usefulness of each notation varies with context, as it is sometimes advantageous to use more than one notation in a given context.

2.1.1 Lagrange's Notation

Lagrange's notation makes use of a prime mark to denote a derivative.

For example, if f is a function, then its derivative evaluated at x is written as $f'(x)$ and pronounced as "f prime of x". Similarly, if a function is written in the form $y = f(x)$, the derivative can be expressed as y'.

Higher derivatives are indicated with additional prime marks, as in f'' for the second derivative and f''' for the third derivative. Since this quickly becomes unwieldy, some authors continue by either employing Roman numerals (f^{IV}) or Arabic numerals in brackets ($f^{(4)}$).

2.1.2 Leibniz's Notation

Leibniz's notation makes use of…

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

2.1.3 Newton's Notation

Newton's notation makes use of a dot to denote a derivative.

For example, if f is a function, then its derivative evaluated at x is written as $\dot{f}(x)$ and pronounced as "f dot of x". Similarly, if a function is written in the form $y = f(x)$, the derivative can be expressed as $\dot{y}$.

Newton's notation is mostly used in physics and other sciences where calculus is applied in a real-world context, and consequently is not seen often in pure mathematics.

2.2 Implicit Differentiation

Implicit differentiation is an application of the chain rule to *implicit functions* in order to make such functions easier (or, in some cases, possible at all) to differentiate.

Definition (Implicit Function)**.** An *implicit equation* is a relation of the form

$$R(x_1, \dots, x_n) = 0,$$

where R is a function of several variables.

An *implicit function* is a function that is defined implicitly by an implicit equation, by associating one of the variables (the value) with the others (the arguments).

Example. The implicit equation of the unit circle is

$$x^2 + y^2 - 1 = 0.$$

Therefore, an implicit function for y in the context of the unit circle is defined implicitly by $x^2 + y(x)^2 - 1 = 0$. With this particular equation, we can rewrite it in the form

$$y = \pm\sqrt{1 - x^2},$$

which *explicitly* defines the two functions that were previously defined implicitly.

Implicit differentiation is a useful tool because not every implicit equation, and therefore not every implicit function, can be made explicit and differentiated using normal means. For example, no amount of algebra will yield an explicit solution to the equation $y^5 + 2y^4 - 7y^3 + 3y^2 - 6y - x = 0$. In this situation, implicit differentiation is required.

In implicit differentiation, we differentiate each side of an equation with two variables (usually x and y) by treating one of the variables as a function of the other.