Solutions to Exercises in *How to Prove It* by Daniel J. Velleman

Third Edition

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Contents

Introduction

Exercise 1.

(a) Factor $2^{15} - 1 = 32,767$ into a product of two smaller positive integers.

(b) Find an integer x such that $1 < x < 2^{32,767} - 1$ and $2^{32,767} - 1$ is divisible by x.

Solution. The proof of Theorem 1 shows that $2^n - 1 = xy$, where $x := 2^b - 1$ and $y := 1 + 2^b + 2^{2b} + \cdots + 2^{(a-1)b}$, and that $2^n - 1 = 2^{ab} - 1$, which implies $n = ab$.

(a) If $n = 15 = 3 \cdot 5$, we can take $a = 3$ and $b = 5$. Therefore, $x = 2^5 - 1 = 31$ and $y = 1 + 2^5 + 2^{2 \cdot 5} = 1057$, and thus $2^{15} - 1 = 31 \cdot 1057 = 32,767$.

(b) If $n = 32,767 = 7 \cdot 4,681$, we can take $b = 7$. Therefore, $x = 2^7 - 1 = 127$, which is clearly a divisor of $2^{32,767} - 1$ and necessarily less than $2^{32,767} - 1$. □

Exercise 2. Make some conjectures about the values of n for which $3^n - 1$ is prime or the values of n for which $3^n - 2^n$ is prime.

Solution. By looking at Table 1, we can make the following conjecture about $3^n - 1$:

Conjecture. *If n is an integer greater than or equal to 1, then $3^n - 1$ is even.*

By looking at Table 2, we can make the following two conjectures about $3^n - 2^n$:

Conjecture. *If n is an integer greater than 1 and is prime, then $3^n - 2^n$ is prime.*

Conjecture. *If n is an integer greater than 1 and is not prime, then $3^n - 2^n$ is not prime.* □

n	Is n prime?	$3^n - 1$	Is $3^n - 1$ prime?
1	no	2	yes
2	yes	8	no
3	yes	26	no
4	no	80	no
5	yes	242	no
6	no	728	no
7	yes	2186	no
8	no	6560	no
9	no	19682	no
10	no	59048	no

Table 1: Values of $3^n - 1$.

n	Is n prime?	$3^n - 2^n$	Is $3^n - 2^n$ prime?
1	no	1	no
2	yes	5	yes
3	yes	19	yes
4	no	65	no
5	yes	211	yes
6	no	665	no
7	yes	2059	yes
8	no	6305	no
9	no	19171	no
10	no	58025	no

Table 2: Values of $3^n - 2^n$.

1 Sentential Logic

1.1 Deductive Reasoning and Logical Connectives

Exercise 1. Analyse the logical forms of the following statements:

(a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.

(b) You won't go skiing, or you will and there won't be any snow.

(c) $\sqrt{7} \nleq 2$.

Solution. The logical forms are:

(a) $(R \vee H) \wedge \neg(H \wedge T)$, where R stands for "we will have a reading assignment", H for "we will have homework problems", and T for "we will have a test".

(b) $\neg G \vee (G \wedge \neg S)$, where G stands for "you will go skiing" and S for "there will be snow".

(c) $\neg((\sqrt{7} < 2) \vee (\sqrt{7} = 2))$. □

Exercise 2. Analyse the logical forms of the following statements:

(a) Either John and Bill are both telling the truth, or neither of them is.

(b) I'll have either fish or chicken, but I won't have both fish and mashed potatoes.

(c) 3 is a common divisor of 6, 9, and 15.

Solution. The logical forms are:

(a) $(J \wedge B) \vee \neg(A \vee B)$, where J stands for "John is telling the truth" and B for "Bill is telling the truth".

(b) $(F \vee C) \wedge \neg(F \wedge M)$, where F stands for "I will have fish", C for "I will have chicken", and M for "I will have mashed potatoes".

(c) $A \wedge B \wedge C$, where A stands for "3 divides 6", B for "3 divides 9", and C for "3 divides 15". □

Exercise 3. Analyse the logical forms of the following statements:

(a) Alice and Bob are not both in the room.

(b) Alice and Bob are both not in the room.

(c) Either Alice or Bob is not in the room.

(d) Neither Alice nor Bob is in the room.

Solution. Let A stand for "Alice is in the room" and B for "Bob is in the room". Then, the logical forms are:

(a) $\neg(A \wedge B)$.

(b) $\neg A \wedge \neg B$.

(c) $\neg A \vee \neg B$.

(d) $\neg A \wedge \neg B$. □

1.2 Truth Tables

Exercise 1. Make truth tables for the following formulae:

(a) $\neg P \vee Q$.

(b) $(S \vee G) \wedge (\neg S \vee \neg G)$.

Solution. The truth tables are:

(a)

P	Q	$\neg P \vee Q$
T	T	T
T	F	F
F	F	T
F	T	T

(b)

S	G	$(S \vee G) \wedge (\neg S \vee \neg G)$
T	T	T
T	F	F

□

2 Quantificational Logic

2.1 Quantifiers

Exercise 1. Analyse the logical forms of the following statements:

(a) Anyone who has forgiven at least one person is a saint.

(b) Nobody in the calculus class is smarter than everybody in the discrete maths class.

(c) Everyone likes Mary, except Mary herself.

(d) Jane saw a police officer, and Roger saw one too.

(e) Jane saw a police officer, and Roger saw him too.

Solution. The logical forms are:

(a) Todo.

(b) Todo.

(c) Todo.

(d) Todo.

(e) Todo. □

3 Proofs

3.1 Proof Strategies

Exercise 1. Consider the following theorem. (This theorem was proved in the introduction.)

Theorem. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

(a) Identify the hypotheses and conclusion of the theorem. Are the hypotheses true when $n = 6$? What does the theorem tell you in this instance? Is it right?

(b) What can you conclude from the theorem in the case $n = 15$? Check directly that this conclusion is correct.

(c) What can you conclude from the theorem in the case $n = 11$?

Solution.

(a) Todo.

(b) Todo.

(c) Todo. □