Proofs

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This document consists of various assorted proofs that I have written down, both for fun and in order to enhance my knowledge and understanding. Nothing in here is novel or new, so do not expect to find an amazing proof of the Riemann hypothesis.

1 The Quadratic Formula

Theorem. *The solutions of a quadratic equation of the form* $ax^2 + bx + c = 0$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Proof. Let $ax^2 + bx + c = 0$. Completing the square gives the equation

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} - \frac{c}{a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2}{4a^2} - \frac{c}{a}.
\end{aligned}$$

Solving for x results in

$$
\begin{aligned}
(x + \frac{b}{2a})^2 &= \frac{b^2}{2a^2} - \frac{c}{a} \\
(x + \frac{b}{2a})^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\
(x + \frac{b}{2a})^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\end{aligned}
$$

which is the quadratic formula. □

2 Irrationality of $\sqrt{2}$

Theorem. *The number $\sqrt{2}$ is irrational.*

Proof. Suppose that $\sqrt{2}$ is rational and thus can be written as the ratio of two integers a/b, where a and b are coprime. Therefore,

$$
\begin{aligned}
\sqrt{2} &= \frac{a}{b} \\
2 &= \frac{a^2}{b^2}.
\end{aligned}
$$

Solving for a^2 gives the equation

$$a^2 = 2b^2,$$

which shows that a^2 is even, since $2b^2$ is necessarily even as it is a multiple of 2.

It follows that a must also be even, since squares of odd integers are never even. Therefore, there exists some integer k that satisfies the equation $a = 2k$.

Substituting $2k$ for a yields that

$$
\begin{aligned}
(2k)^2 &= 2b^2 \\
4k^2 &= 2b^2 \\
2k^2 &= b^2,
\end{aligned}
$$

which demonstrates that b^2 and therefore b itself must be even, using the same reasoning as with a^2.

Since a and b are both even, the assumption that $\sqrt{2}$ is rational and a and b are coprime no longer holds because they share a common factor of 2. Therefore, $\sqrt{2}$ must be **irrational**. □

3 Logarithmic Identities

3.1 The Product Law

Theorem. *A logarithm of the form* $\log_b(xy)$ *is equivalent to* $\log_b(x) + \log_b(y)$.

Proof. Let $x = b^m$ and $y = b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

Substituting b^m and b^n for x and y in the expression $\log_b(xy)$ gives the equation

$$\begin{aligned} \log_b(xy) &= \log_b(b^m b^n) \\ &= \log_b(b^{m+n}) \\ &= m + n. \end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b(xy) = \log_b(x) + \log_b(y),$$

which is the desired identity. □

3.2 The Quotient Law

Theorem. *A logarithm of the form* $\log_b(x/y)$ *is equivalent to* $\log_b(x) - \log_b(y)$.

Proof. Let $x = b^m$ and $y = b^n$. From the definition of a logarithm, it follows that $\log_b(x) = m$ and $\log_b(y) = n$.

Substituting b^m and b^n for x and y in the expression $\log_b(x/y)$ gives the equation

$$\begin{aligned}\log_b(\frac{x}{y}) &= \log_b(\frac{b^m}{b^n}) \\ &= \log_b(b^{m-n}) \\ &= m - n.\end{aligned}$$

Since $m = \log_b(x)$ and $n = \log_b(y)$, the above can be written as

$$\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y),$$

which is the desired identity. □

3.3 The Power Law

Theorem. *A logarithm of the form* $\log_b(x^y)$ *is equivalent to* $y \log_b(x)$*.*

Proof. Let $x = b^n$. From the definition of a logarithm, it follows that $\log_b(x) = n$.

Substituting b^n for x in the expression $\log_b(x^y)$ gives the equation

$$\begin{aligned}\log_b(x^y) &= \log_b((b^n)^y) \\ &= \log_b(b^{ny}) \\ &= ny\end{aligned}$$

Since $n = \log_b(x)$, the above can be written as

$$\log_b(x^y) = y \log_b(x)$$

which is the desired identity. □

3.4 The Change of Base Formula

Theorem. *Any logarithm* $\log_b(a)$ *can be rewritten in terms of another base with the formula*

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)}.$$

Proof. Let $c = \log_b(a)$. From the definition of a logarithm, it follows that $b^c = a$.

Taking the base-x logarithm on both sides gives the equation

$$\log_x(b^c) = \log_x(a).$$

Applying the logarithmic power law and solving for c leads to the equation

$$\begin{aligned} c \log_x(b) &= \log_x(a) \\ c &= \frac{\log_x(a)}{\log_x(b)}. \end{aligned}$$

Since $c = \log_b(a)$, the above can be written as

$$\log_b(a) = \frac{\log_x(a)}{\log_x(b)},$$

which is the change of base formula. $\square$