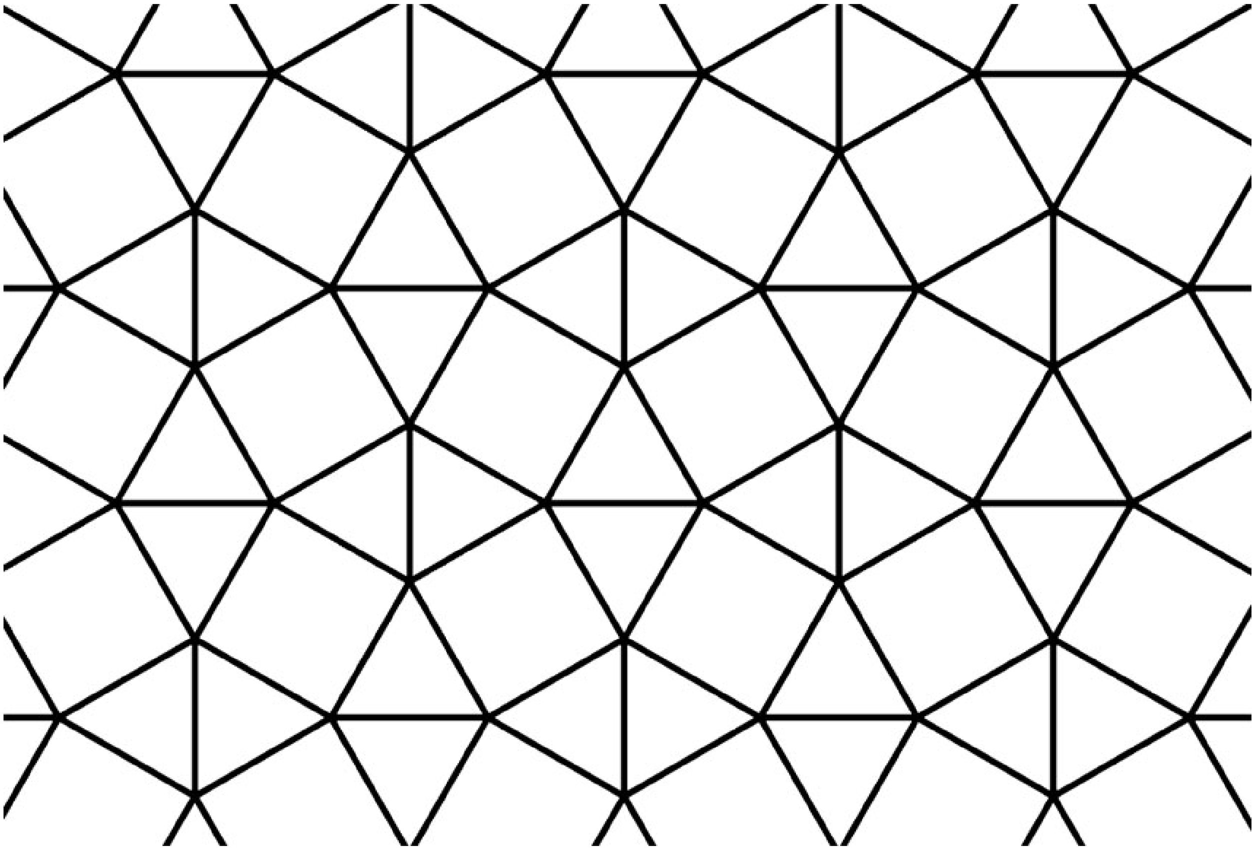

Tessellation

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Introduction

Tilings, the arrangement of shapes to fill a plane without overlapping or gaps, have captivated mathematicians for centuries. Beyond their aesthetic appeal, tilings serve as fascinating puzzles that challenge our understanding of patterns, symmetry, and geometry. In this essay, we delve into the world of rewrite tilings, exploring their historical significance, problem complexity, types, decidability, and relationships.

Historical Significance

Tilings have been a subject of interest throughout history, with examples found in ancient cultures like the Egyptians and Greeks. However, it was the discovery of quasicrystals in the 1980s that sparked renewed interest in the field. These aperiodic tilings exhibited non-repeating patterns, defying traditional periodic tiling theories. This breakthrough led to the exploration of various tilings problems, including rewrite tilings.

Different Types of Tilings Problems

Tiling problems can be classified into various categories based on the types of shapes used, constraints imposed, or goals to achieve. Some prominent examples include:

- Regular Tilings: These involve using regular polygons, such as squares or equilateral triangles, to completely cover the plane. Regular tilings have a high degree of symmetry and are governed by well-defined mathematical rules. For example, it is known that any regular polygon can tile the plane, and the number of possible solutions is infinite.
- Penrose Tilings: Proposed by Sir Roger Penrose, these are aperiodic tilings formed using specific kite and dart shapes, creating intricate patterns with five-fold symmetry. The Penrose tiling problem is undecidable, meaning there is no general algorithm that can determine whether a given shape tiles the plane. Despite this, specific instances of Penrose tilings can be proven solvable.
- Domino Tilings: In this problem, a rectangle is divided into smaller squares, and the goal is to cover it completely using rectangular dominos. This problem has connections to various areas of mathematics, including combinatorics and graph theory. Determining whether a given rectangle can be tiled with dominos is a well-studied problem with known algorithms.

Decidability of Tilings Problems

Determining whether a given tiling problem is solvable, known as the decidability of the problem, varies depending on the problem type. Some problems, such as regular tilings and dominos, are easily solvable and have well-defined algorithms. The solvability of regular tilings stems from the inherent symmetries and regularity of the shapes involved.

However, for aperiodic tilings like Penrose tilings and domino problem, decidability is more complex. The Penrose tiling problem and domino problem is undecidable, meaning there is no general algorithm that can determine whether a given shape tiles the plane. Nonetheless, specific instances of Penrose tilings can be proven solvable. For example, Penrose's original kite and dart shapes can be used to create a tiling pattern. Therefore Tilings problem is undecidable.

Undecidability Of The Domino Problem

To prove the undecidability of the domino problem, we will show a reduction from a well-known undecidable problem called the Post Correspondence Problem (PCP) to the domino problem.

The Post Correspondence Problem is defined as follows: Given a finite set of dominoes, each containing two strings, is there a way to arrange some or all of these dominoes in a sequence such that the concatenation of the top strings matches the concatenation of the bottom strings?

To reduce PCP to the domino problem, we construct a set of dominoes based on an instance of PCP. Each domino in the constructed set will represent a pair of strings from the PCP instance.

Given an instance of PCP with a set of dominoes, we construct a set of corresponding dominoes for the domino problem as follows:

For each domino in the PCP instance, create a corresponding domino with the top string being the same as the top string of the PCP domino and the bottom string being the same as the bottom string of the PCP domino.

Now, we can see that if there exists a sequence of dominoes that can be arranged in the domino problem such that the top and bottom strings of each domino match, it directly corresponds to a solution to the PCP instance.

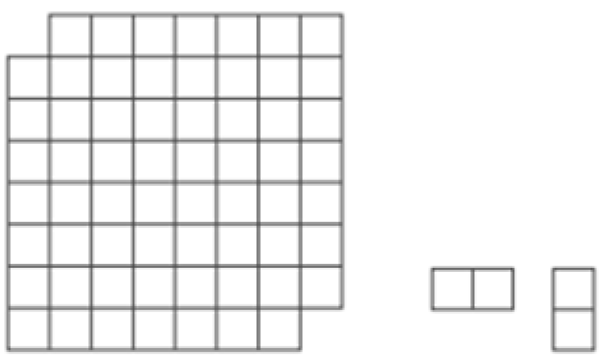
The reduction above shows that if we have an algorithm that can decide the domino problem, we can use it to solve the PCP. Since the undecidability of PCP is well-established, this implies that the domino problem is also undecidable.

Therefore, we can conclude that the domino problem, which asks whether a given set of dominoes can tile the plane, is undecidable.

Some Decidable Examples

1. Reject:

   Suppose we remove two opposite corners of an 8 * 8 chessboard, and we ask: Is it possible to tile the resulting figure with 31 dominoes ?



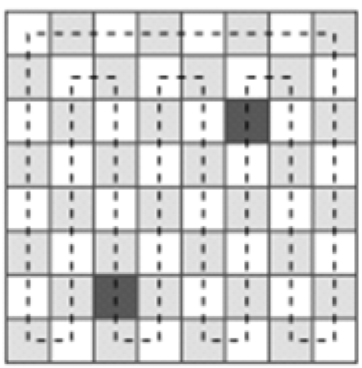
   Notice that, regardless of where it is placed, a domino will cover one dark and one white square of the board.

   Therefore, 31 dominoes will cover 31 dark squares and 31 white squares. However, the board has 32 dark squares and 30 white squares in all, so a tiling does not exist. This is an example of a coloring argument; such arguments are very common in showing that certain tilings are impossible.

2. Accept:

   A natural variation of this problem is to now remove one dark square and one white square from the chessboard, as shaded above. Now the resulting board has the same number of dark squares and white squares; is it possible to tile it with dominoes?

   The answer is yes, regardless of which dark square and which white square we remove. Consider any closed path that covers all the cells of the chessboard, like the following one:



   Now start traversing the path, beginning with the point immediately after the dark hole of the chessboard. Cover the first and second cell of the path with a domino; they are white and dark, respectively.

   Then cover the third and fourth cells with a domino; they are also white and dark, respectively. Continue in this way, until the path reaches the second hole of the chessboard. Fortunately, this second hole is white, so there is no gap between the last domino placed and this hole. We can, therefore, skip this hole and continue covering the path with successive dominoes. When the path returns to the first hole, there is again no gap between the last domino placed and the hole. Therefore, the board is entirely tiled with dominoes. We now illustrate this procedure.

Demonstrating That a Tiling Does Not Exist

Philip Hall showed that in any region that cannot be tiled with dominoes, one can find such a demonstration of impossibility.

More precisely, one can find k cells of one color which have fewer than k neighbors. Therefore, to demonstrate to someone that tiling the region is impossible, we can simply show them those k cells and their neighbors!

Hall’s statement is more general than this and is com_monly known as the marriage theorem.

The name comes from thinking of the dark cells as men and the white cells as women. These men and women are not very adventurous:

They are only willing to marry one of their neighbors. We are the matchmakers; we are trying to find an arrangement in which everyone can be happily mar_ried. The marriage theorem tells us exactly when such an arrangement exists.

Tiling Rectangles with Rectangles

When can an m * n rectangle be tiled with a * b rectangles?

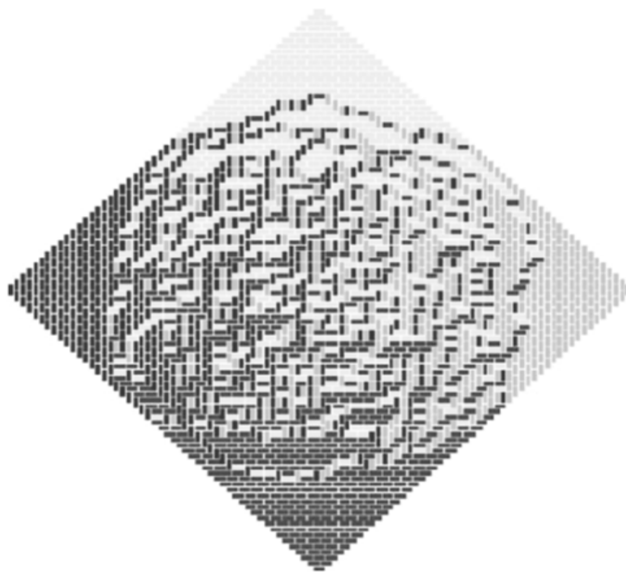
The full answer to our question was given by de Bruijn and by Klarner .

They proved that an m * n rectangle can be tiled with a * b rectangles if and only if:

- mn is divisible by ab
- The first row and column can be covered; i.e., both m and n can be written as sums of a s and b s, and
- Either m or n is divisible by a, and either m or n is divisible by b.

What Does a Typical Tiling Look Like?

Here, there are two shades of horizontal dominoes and two shades of vertical dominoes, assigned according to a certain rule not relevant here. These pictures were created by Jim Propp's Tilings Research Group.



This very nice picture suggests that something inter_esting can be said about random tilings. The tiling is clearly very regular at the corners, and gets more chaotic as we move away from the edges.

There is a well defined region of regularity, and we can predict its shape. Jock_usch, Propp and Shor showed that for very large n, and for "most" domino tilings of the Aztec diamond AZ(n), the region of regularity "approaches" the outside of a circle tangent to the four limiting sides. Sophisticated probability theory is needed to make the terms "most" and "approaches" precise, but the intuitive meaning should be clear.

This result is known as the Arctic Circle theorem.

The tangent circle is the Arctic Circle; the tiling is “frozen” outside of it. Many similar phenomena have since been observed and (in some cases) proved for other tiling problems.

Applications of Tilings

Tilings find applications in various fields, including:

- Architecture : Tilings inspire architects and designers in creating visually appealing structures, facades, and interior designs. The principles of symmetry and balance derived from tilings can guide architectural concepts and layout planning

- Computer Graphics and Animation : : Tilings serve as a source of inspiration for generating intricate and visually appealing patterns in computer graphics and animation. Algorithms based on tilings can be used to create realistic textures, backgrounds, and special effects.

- Material Science : Understanding the arrangement of atoms and molecules in crystalline structures relies on principles similar to those used in tilings. The study of tilings provides insights into the properties and behaviors of materials, aiding in the development of new materials with desired characteristics.

Inspirations from Tilings

Tilings have inspired artists, mathematicians, and researchers, providing fertile ground for creativity and innovation.

The intricate patterns and symmetries found in tilings have influenced various artistic forms, from traditional Islamic art to contemporary geometric art movements.

Additionally, tilings have inspired mathematical explorations in fields like topology, graph theory, and group theory, advancing our understanding of abstract mathematical concepts

Conclusion

Tilings have captivated mathematicians for centuries, providing a creative playground for exploring patterns, symmetries, and mathematical beauty. Rewrite tilings, with their rich history and varied problem types, continue to challenge our understanding of the fundamental principles of geometry and combinatorics. Beyond their mathematical significance, tilings find applications in architecture, computer graphics, material science, and inspire artistic expressions. The intricate relationships among different types of tilings highlight the interconnectedness of mathematical concepts and deepen our appreciation for the elegance and complexity of the mathematical world.

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