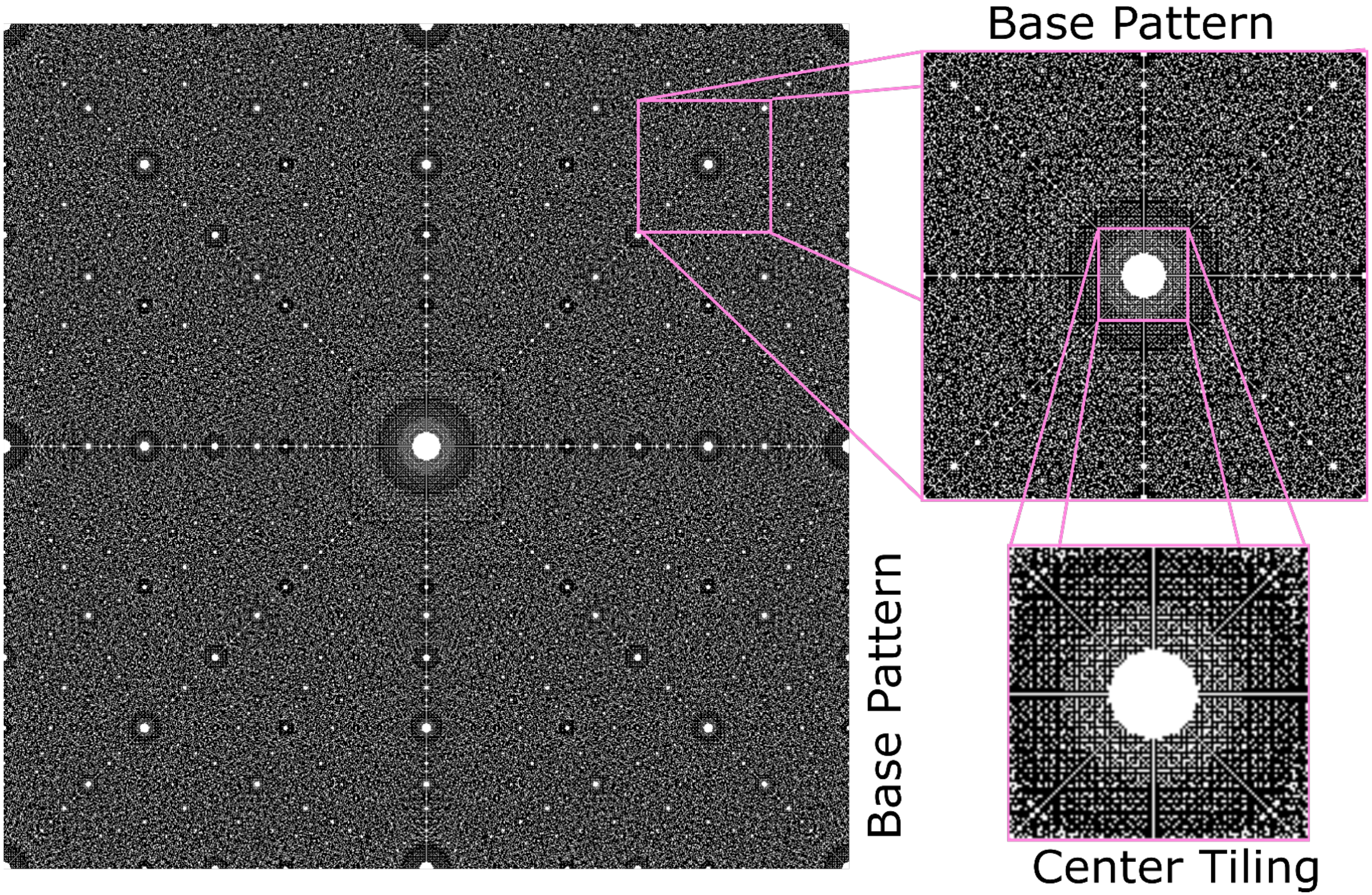

Chaotic Sensing [1]

Shekhar “Shakes” Chandra

THE UNIVERSITY OF QUEENSLAND AUSTRALIA

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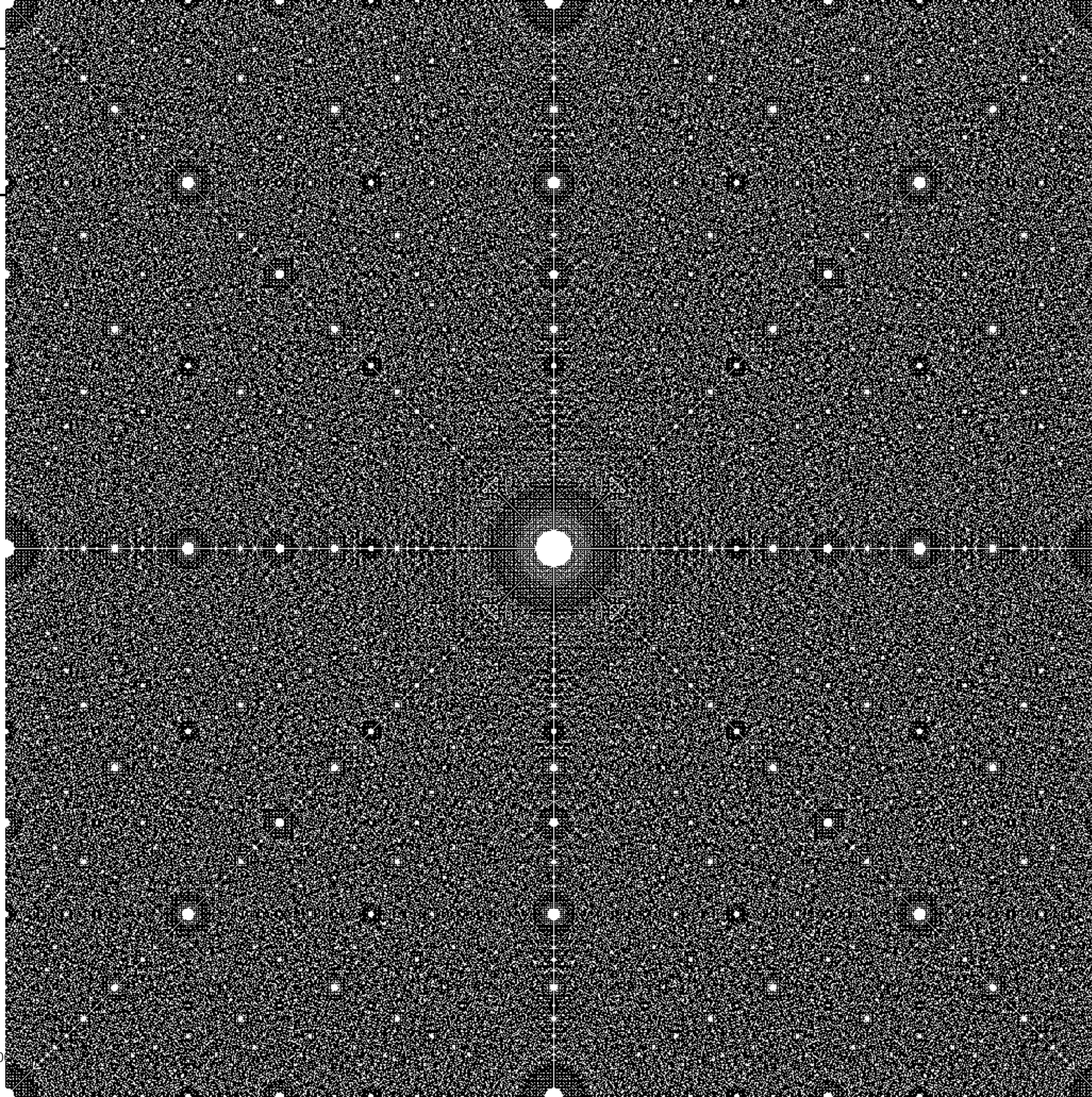
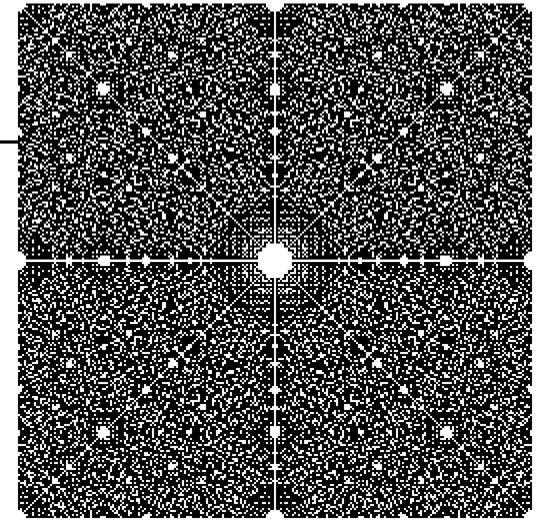
[1] S. S. Chandra *et al.*, “Chaotic Sensing,” *IEEE Transactions on Image Processing*, vol. 27, no. 12, pp. 6079–6092, **2018**.
DOI: https://doi.org/10.1109/TIP.2018.2864918 Website: https://shakes76.github.io/ChaoS/

Fractal of the Discrete Fourier Transform



N = 1031

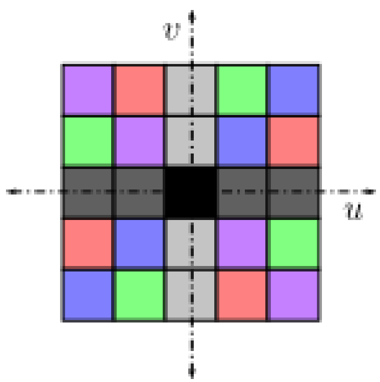
Talk Roadmap



The Amazing Fractals!

↓

Discrete Slice Theorem



↓

Turbulent Image Reconstruction

- Sparse Fractal Sampling
- Turbulent Image Artefacts
- Maximum Likelihood Reconstruction
- 4X Faster MRI!

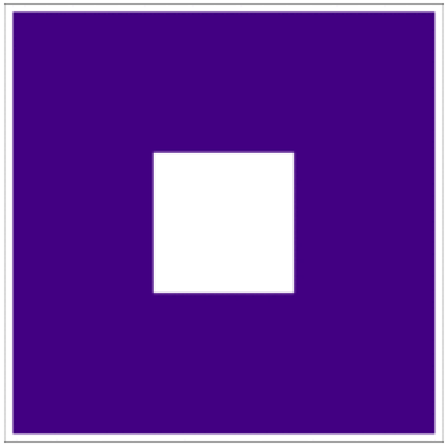
Fractal Theorems

Amazing Fractals

A Gift of Finite Geometry

Fractals

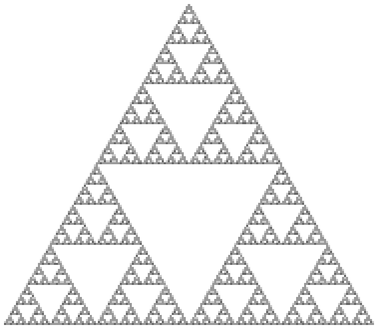
Fractals are self-similar patterns, i.e. repeat themselves, at multiple scales that are created using simple deterministic rules.



Sierpinski carpet

"The construction of the Sierpinski carpet begins with a square. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, *ad infinitum*."

- Wikipedia



Fractal Antenna

An example of a fractal antenna design that is typically used in mobile devices for compact multi-band antennas.

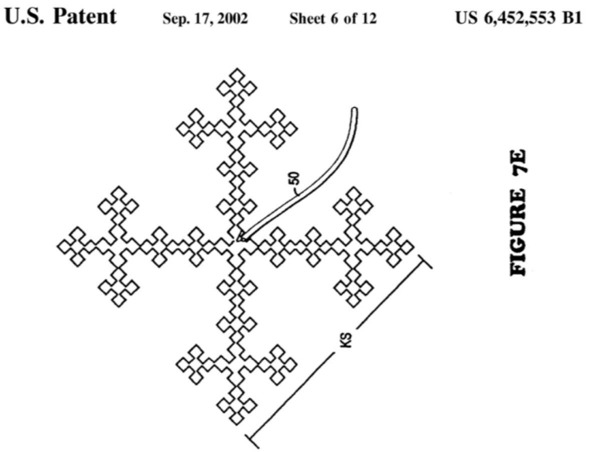
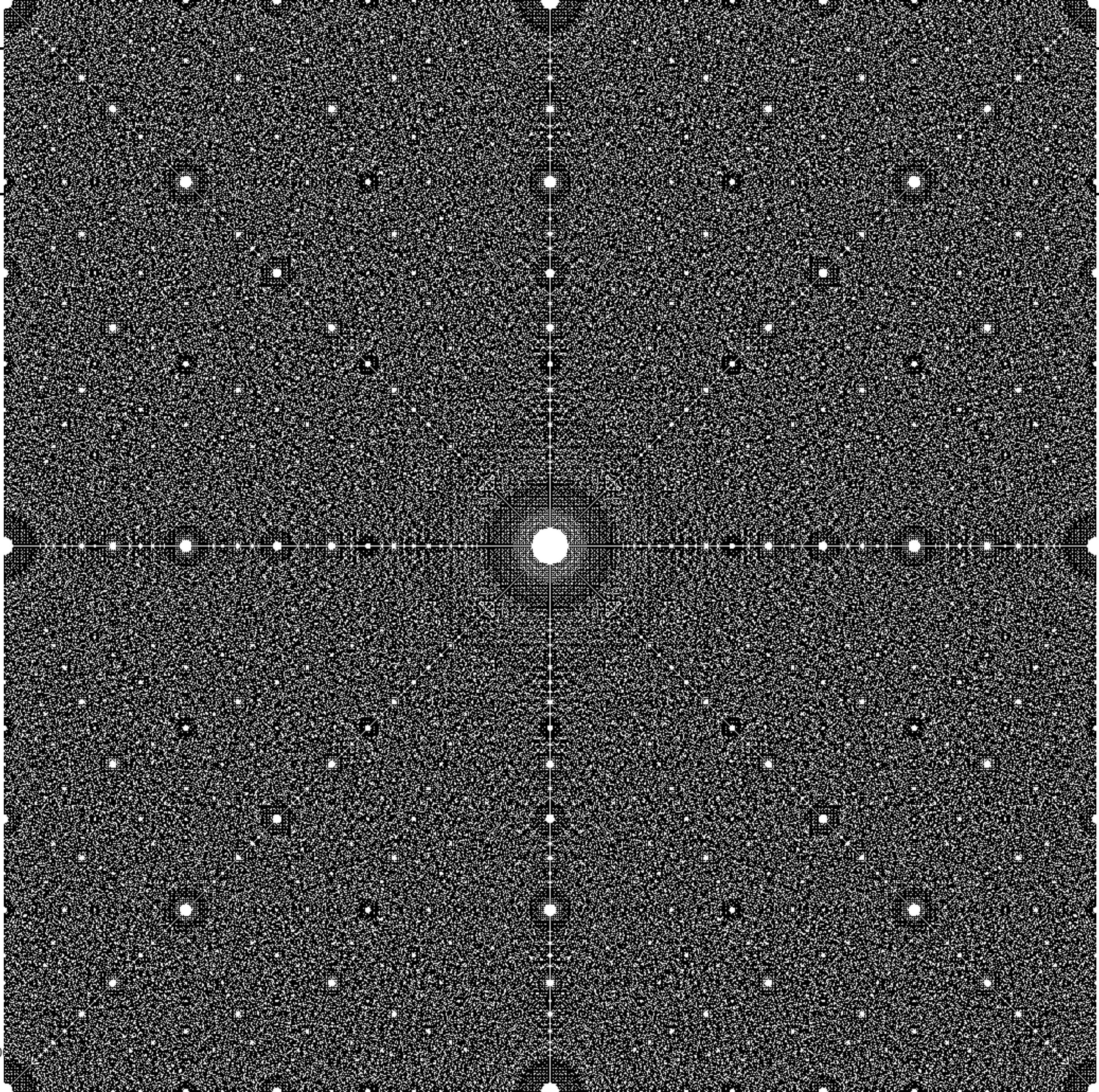


Image from https://en.wikipedia.org/wiki/Sierpinski_carpet and https://en.wikipedia.org/wiki/Fractal_antenna

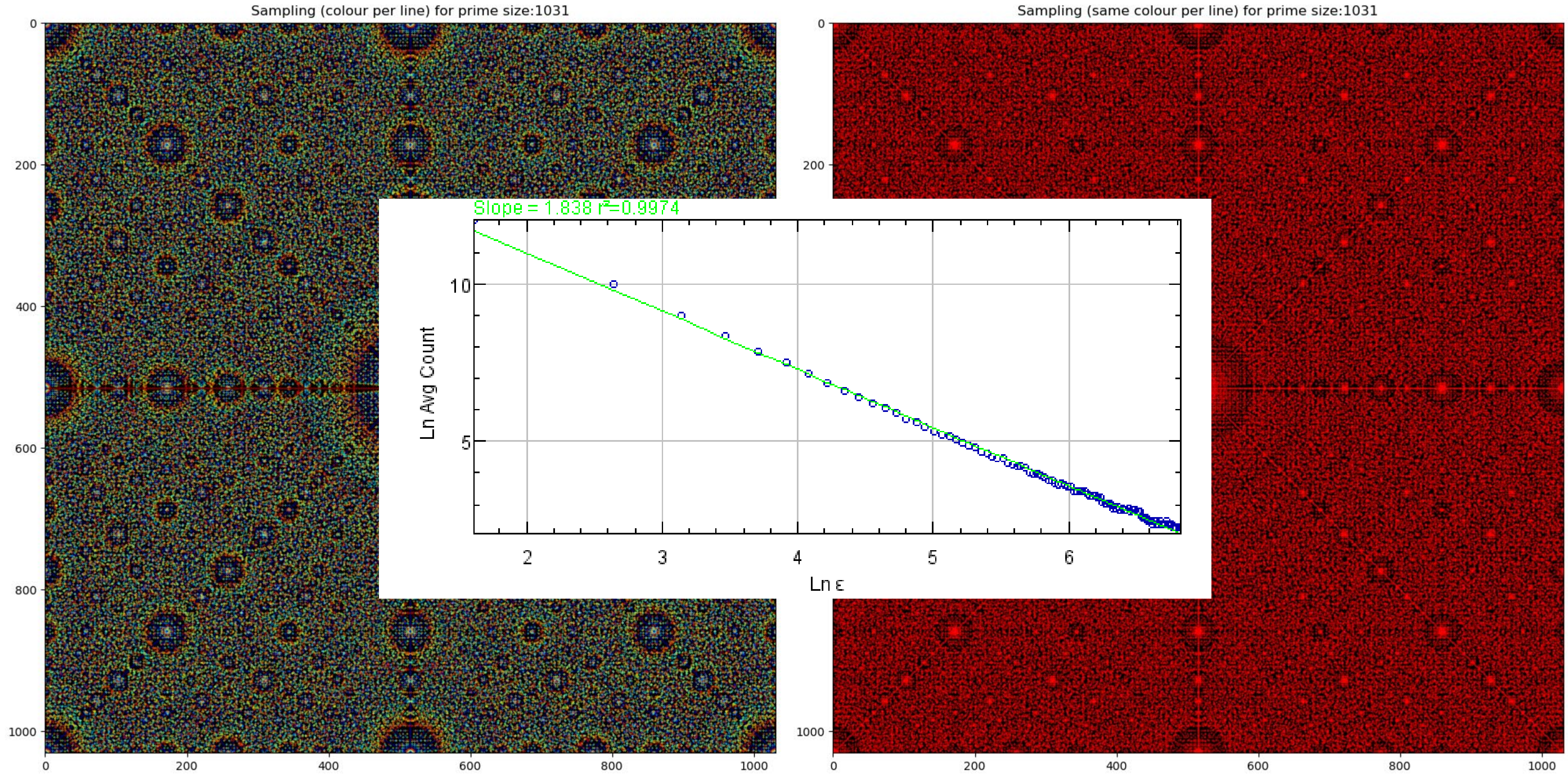
Fractal of the Discrete Fourier Transform



N = 1031

Discrete Fourier Transform of Fractal

Fractal of the Discrete Fourier Transform

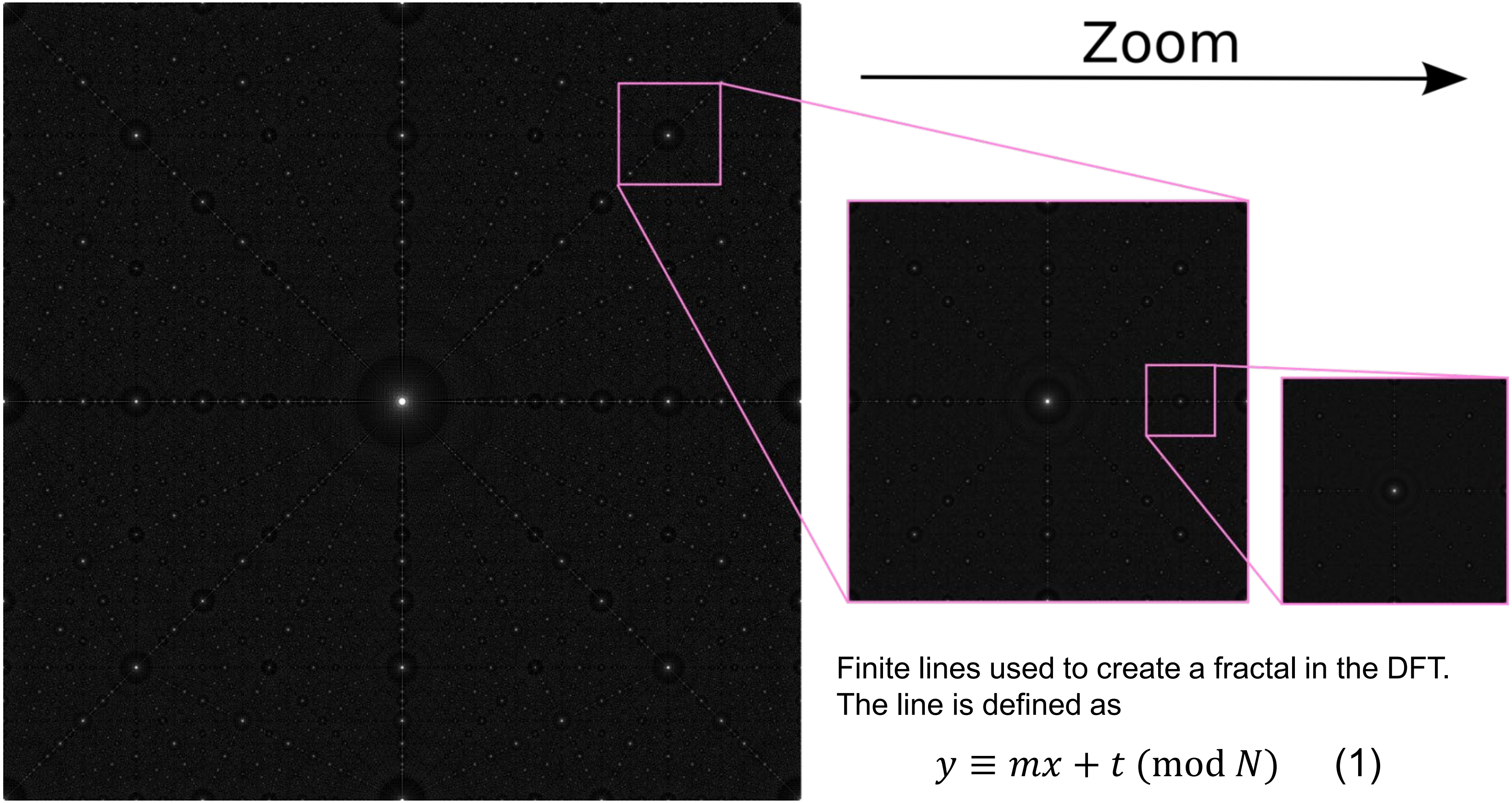


N = 1031

Fractal Dimension 1.78 – 1.89

Finite Fractal

Using discrete periodic lines as slices to create a new fractal in the DFT.



Finite lines used to create a fractal in the DFT.
The line is defined as

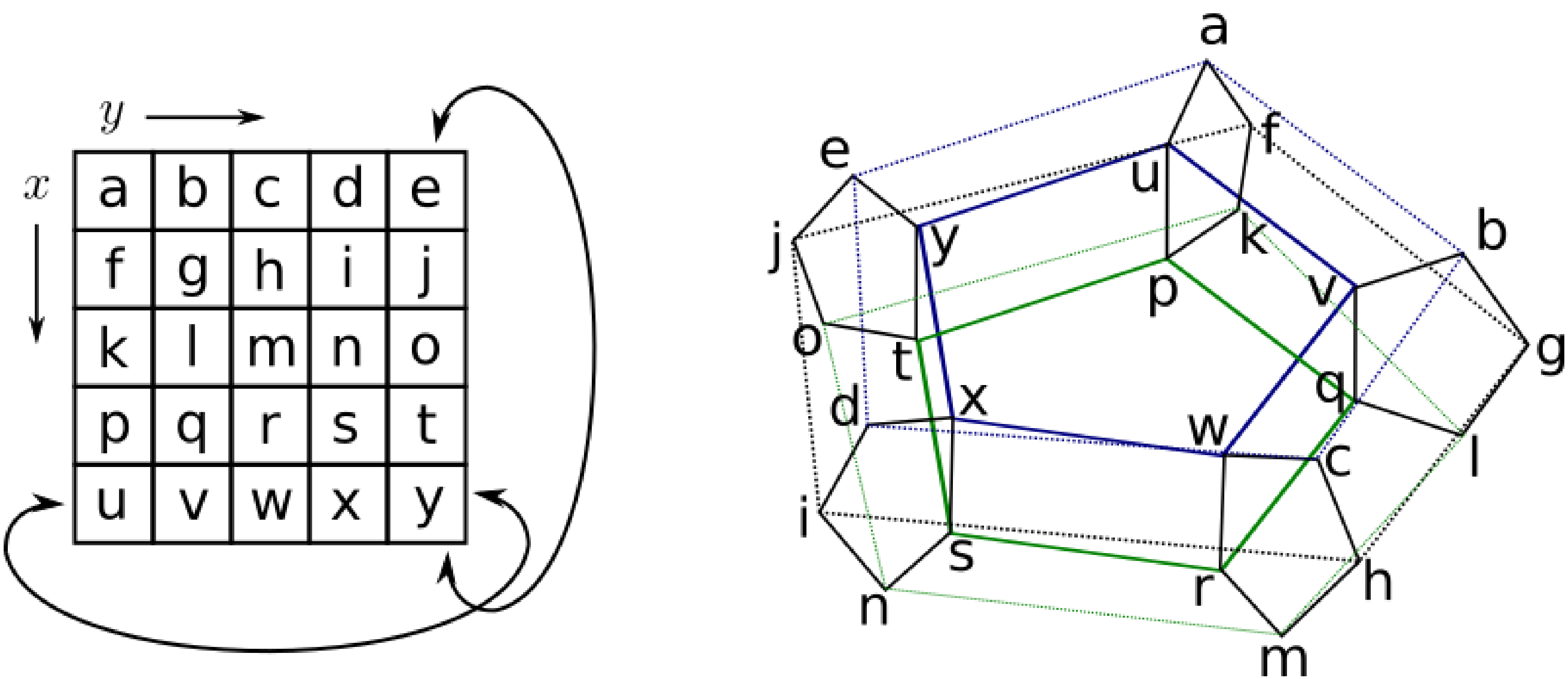
$$y \equiv mx + t \pmod{N} \qquad (1)$$

Discrete Slice Theorem

Tiling within Finite Geometry

Finite Geometry

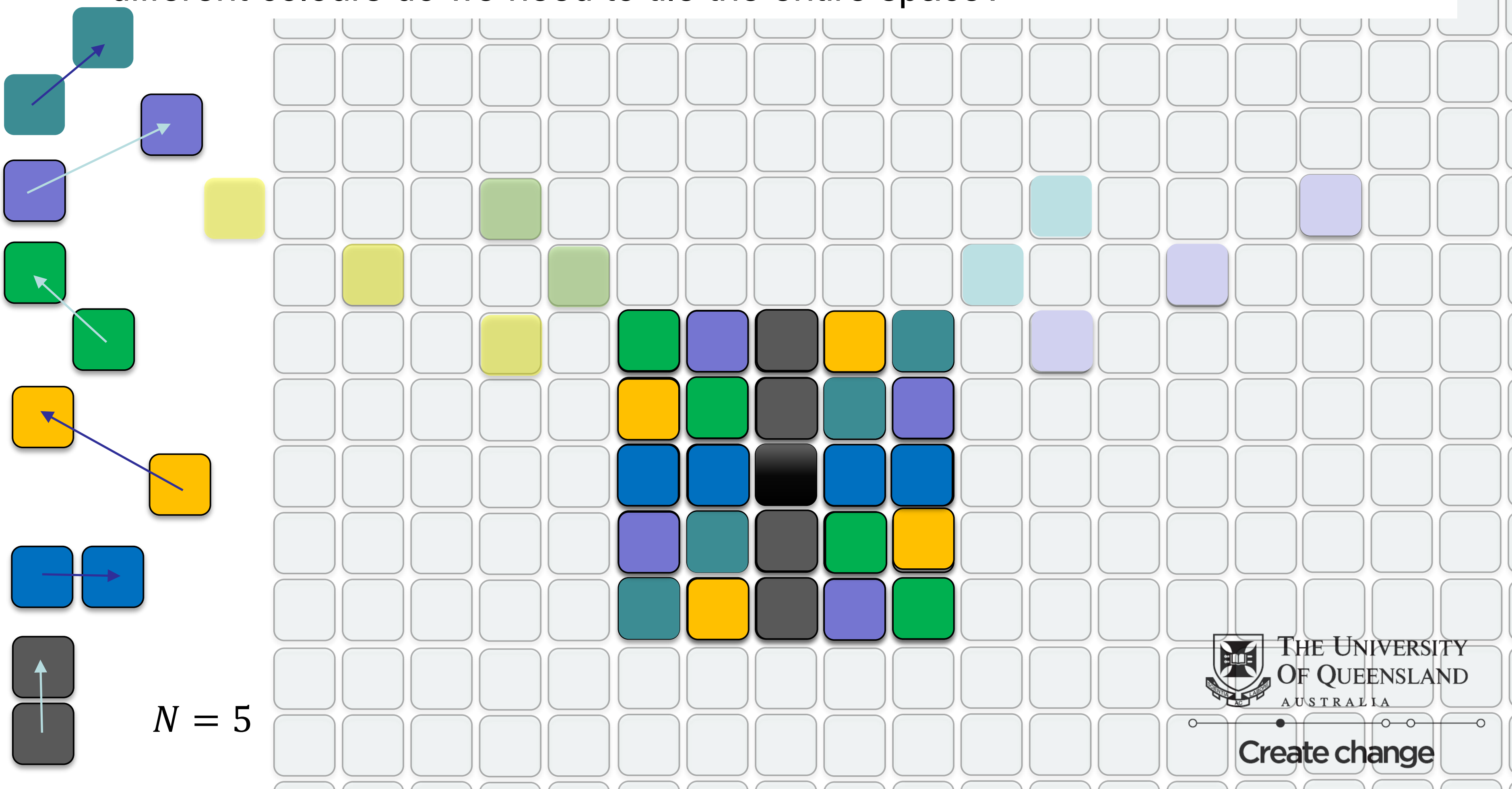
Let's start with an $N \times N$ space of discrete pixels with periodic boundary conditions.



Chandra, S. S. Circulant Theory of the Radon Transform, *School of Physics, Monash University*, **2010**

Finite Geometry

Let’s start with an $N \times N$ space of discrete pixels with periodic boundary conditions. Starting from the center, using N pixels per colour, how many different colours do we need to tile the entire space?



Finite Geometry

Finite lines tile all of image space exactly once when N is prime!

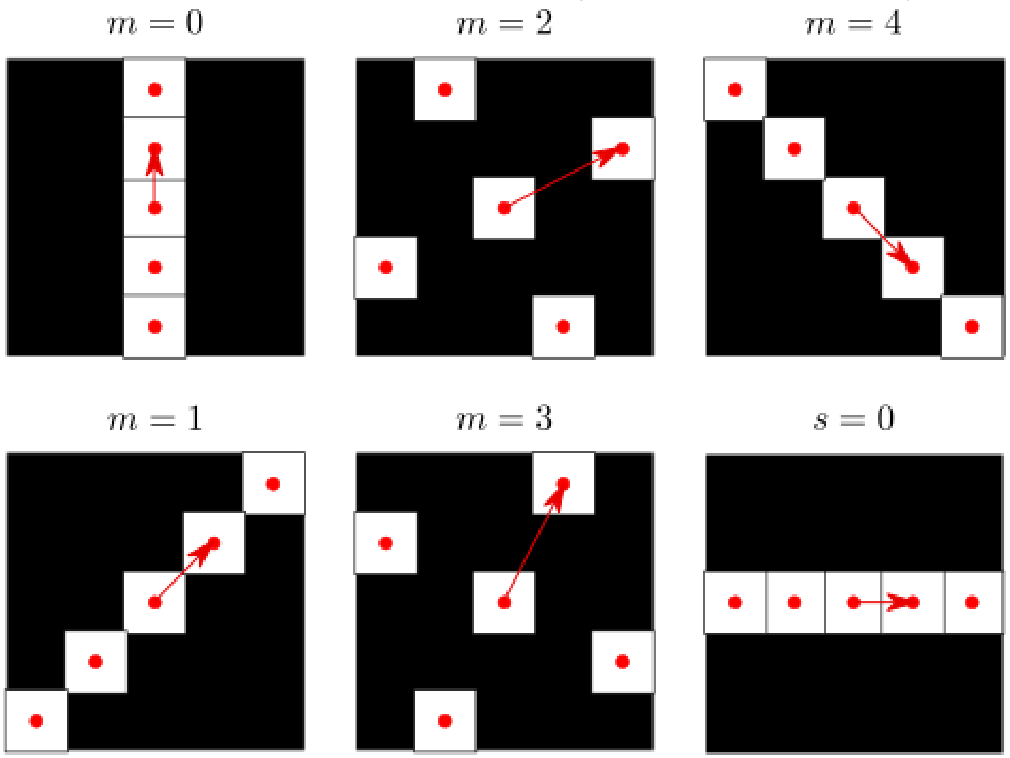


Figure: The tiling defined by equation (1) for a prime-sized image space, where $N = 5$ and $t = 0$. The origin is centered and each white square (and red dot) represents a sample of the line. The lines are generated with increasing m values.

[4] Matúš, F. & Flusser, J., Image Representation via a Finite Radon Transform
Pattern Analysis and Machine Intelligence, IEEE Transactions on, **1993**, *15*, 996-1006

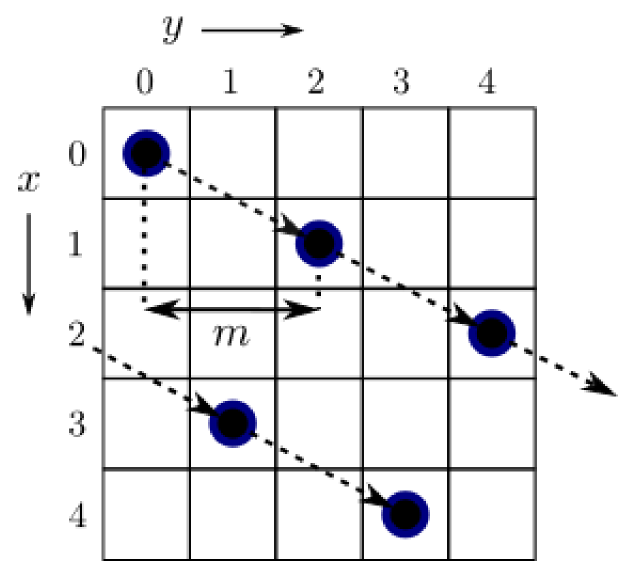


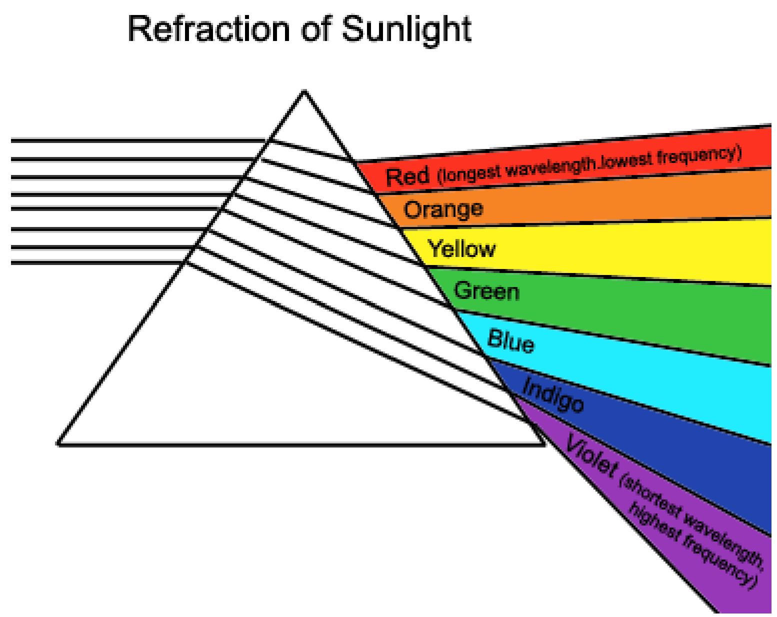
Figure: A finite line for image $N = 5$ and $m = 2$. The line is defined as

$$y \equiv mx + t \pmod N \qquad (1)$$

Discrete Fourier Transform

Refraction Analogy

White light
(all wavelengths)



Colour
spectrum

$f = 1/\lambda$

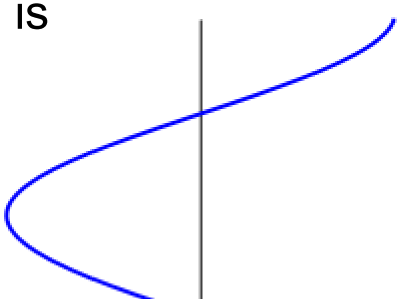
Think of a Fourier Transform like a prism:
"Destructs a source signal into its constituent frequencies"

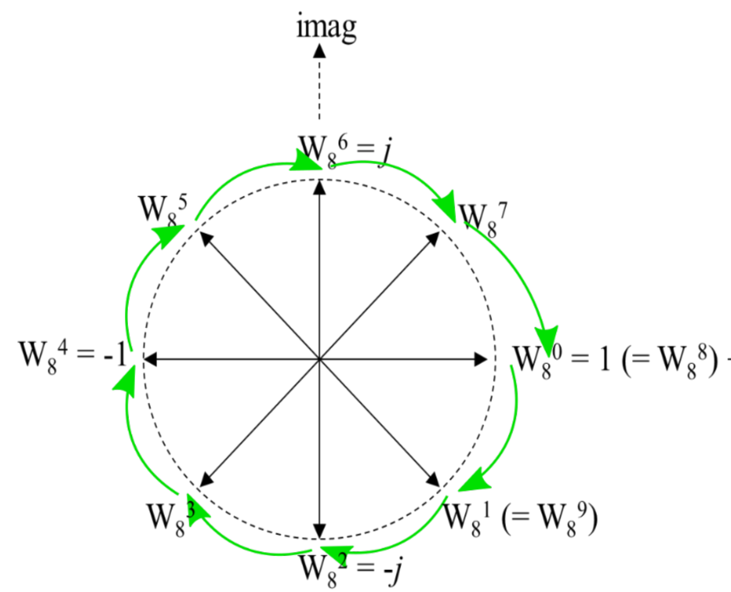
Discrete Fourier Transform

Given a finite sequence $x[n]$ of length N, its DFT $X[k]$ is

$$\hat{X}(k\Delta\omega) = X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-jnk2\pi}{N}\right)$$

where $0 \leq n, k \leq N-1$





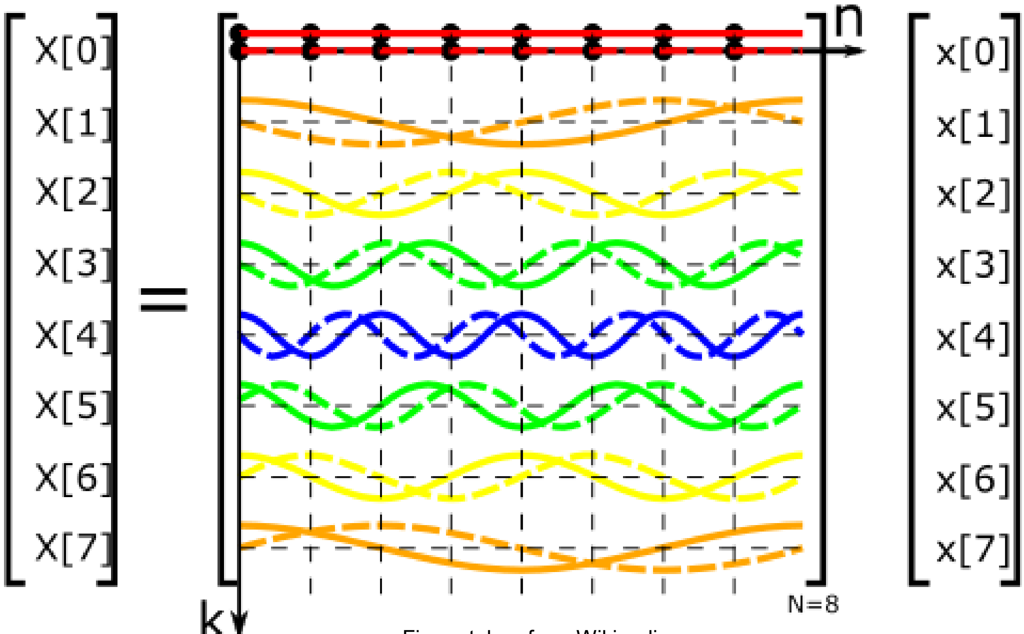
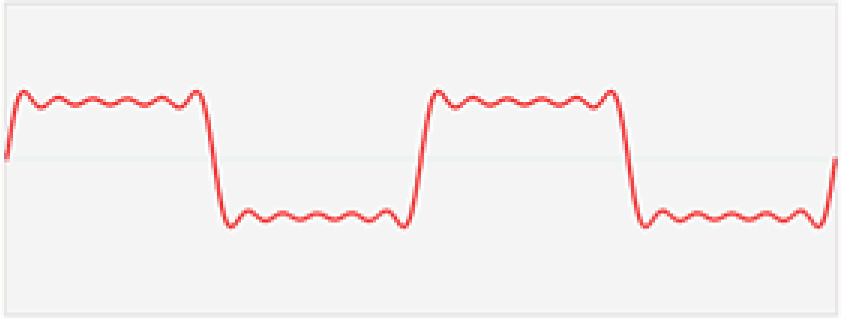


Figure taken from Wikipedia

We are taking the inner product of our sequence with harmonics.

In MRI, we measure the 2D DFT space directly



By: Andrés Cabrera and Karl Yerkes.
http://w2.mat.ucsb.edu/201A/nb/Sinusoids%20and%20Phasors.html

Discrete Fourier Transform

When applied to an image $x[n, m]$ of length N, its 2D DFT $X[k, l]$ is a 2D image of Fourier coefficients. In other words, each 'pixel' of the 2D DFT is a magnitude of a particular sine wave required to form the 2D image

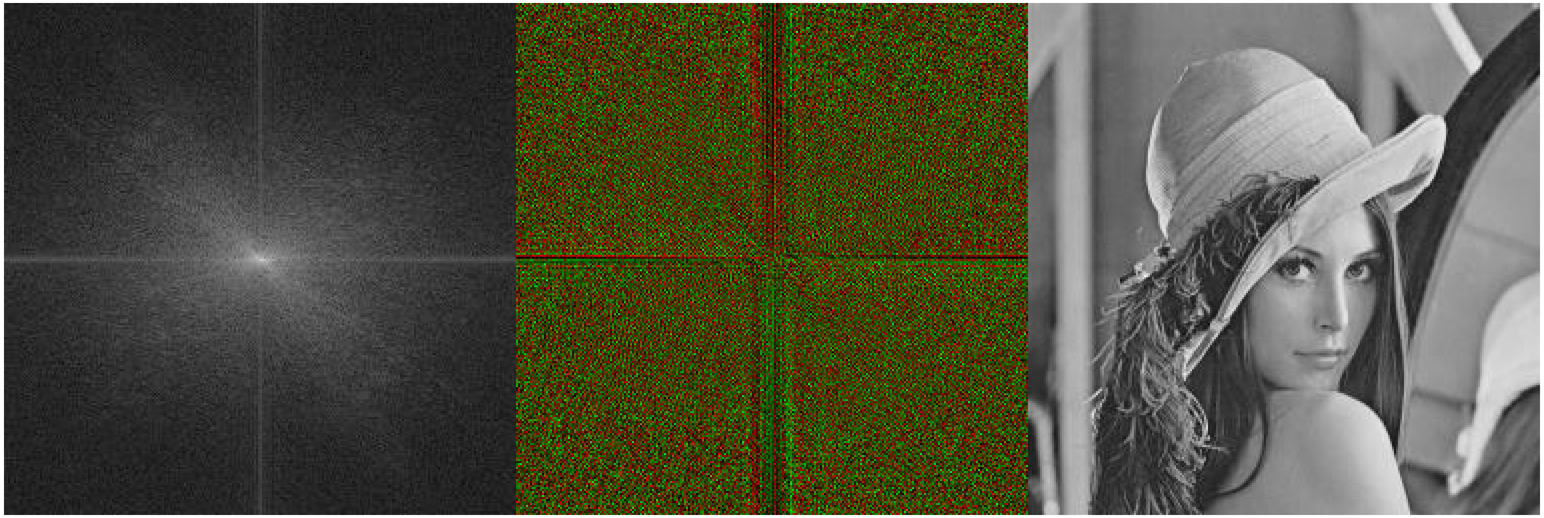


Figure: The magnitude and phase of the 2D DFT of the image of Lena (far right).

Image taken from https://boofcv.org/index.php?title=Example_Discrete_Fourier_Transform

Discrete Fourier Slice Theorem

Classical Fourier (AKA central or projection) slice theorem of the continuous Fourier transform (FT) exists within the DFT.

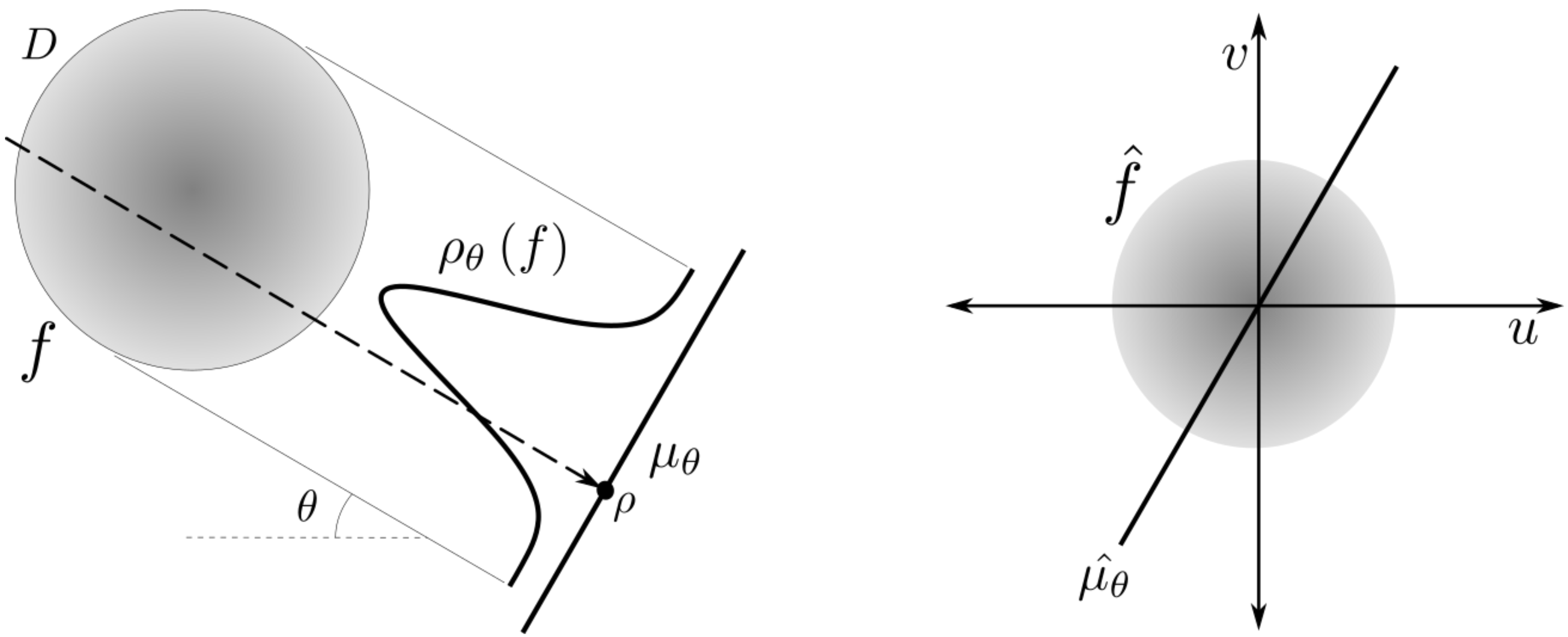


Figure: The (FT of the) projection of f (μ_1 at angle θ) is actually a slice of the 2D FT (at angle theta + 90 degrees) of the same function. Here, the projection is defined as a set of line integrals of f at the same angle θ.

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[4] Matúš, F. & Flusser, J., Image Representation via a Finite Radon Transform
Pattern Analysis and Machine Intelligence, IEEE Transactions on, **1993**, *15*, 996-1006

Discrete Fourier Slice Theorem

Thus, the discrete slices of the DFT, and hence the fractal, map to projections of their corresponding images. This is referred to as the discrete (or Finite) Radon transform.

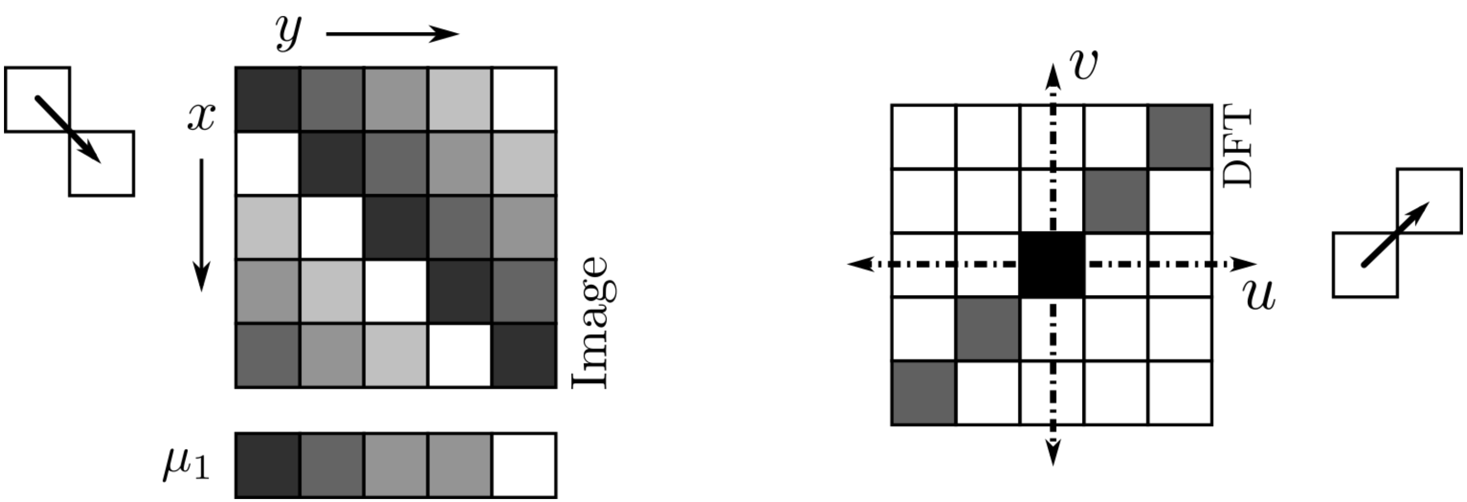

Figure: The (DFT of the) projection of the image (μ_1 at angle m) is actually a slice of the 2D DFT (at angle m + 90 degrees) of the same image. Here, the projection is defined as a discrete pixel sums of the image along the rational vector [1, m].

[4] Matúš, F. & Flusser, J., Image Representation via a Finite Radon Transform
Pattern Analysis and Machine Intelligence, IEEE Transactions on, **1993**, *15*, 996-1006

Discrete Fourier Slice Theorem

Thus, the discrete slices of the DFT, and hence the fractal, map to projections of their corresponding images. This is referred to as the discrete (or Finite) Radon transform (DRT).

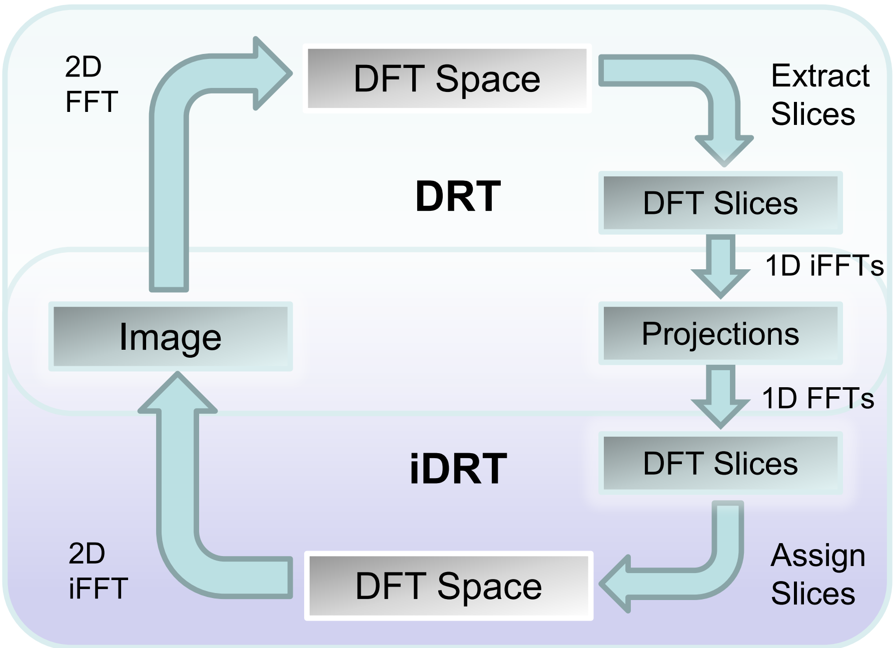

Figure: Computing the DRT and the inverse DRT via the discrete Fourier slice theorem.

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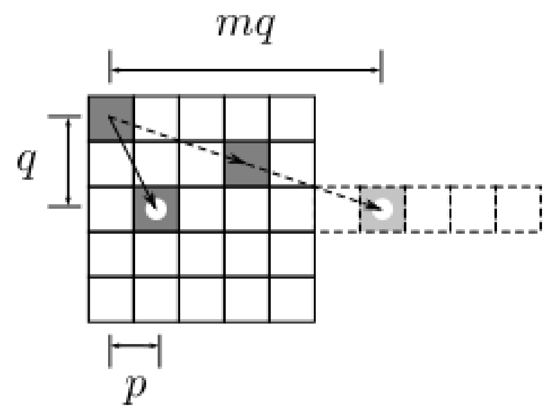
[4] Matúš, F. & Flusser, J., Image Representation via a Finite Radon Transform
Pattern Analysis and Machine Intelligence, IEEE Transactions on, **1993**, *15*, 996-1006

How to get the fractal?

- We choose the lines with the shortest vectors from the origin by using the rational slopes ${}^{q}/_{p}$.
- These rational slopes are known as the Farey sequence when greatest common divisor (GCD) is unity, i.e. p and q have no common factor.
- The Farey sequence has amazing mathematical properties in itself forming structures like the Ford circles.



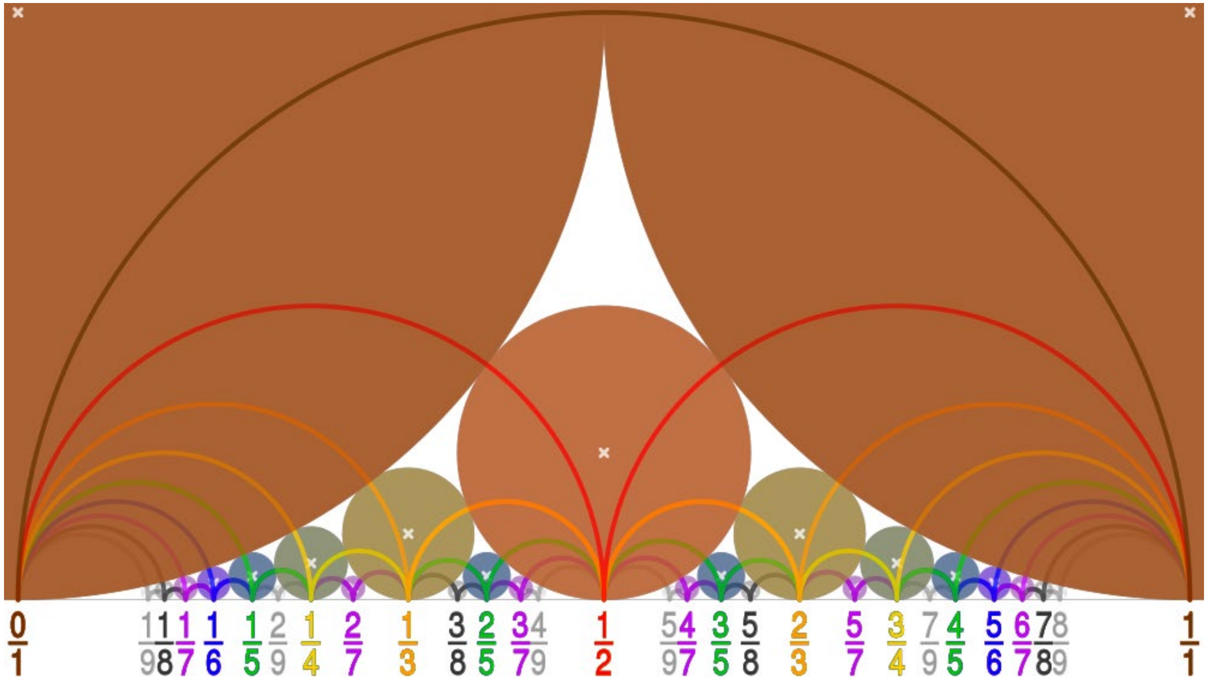


Figure taken from Wikipedia

How to get the fractal?

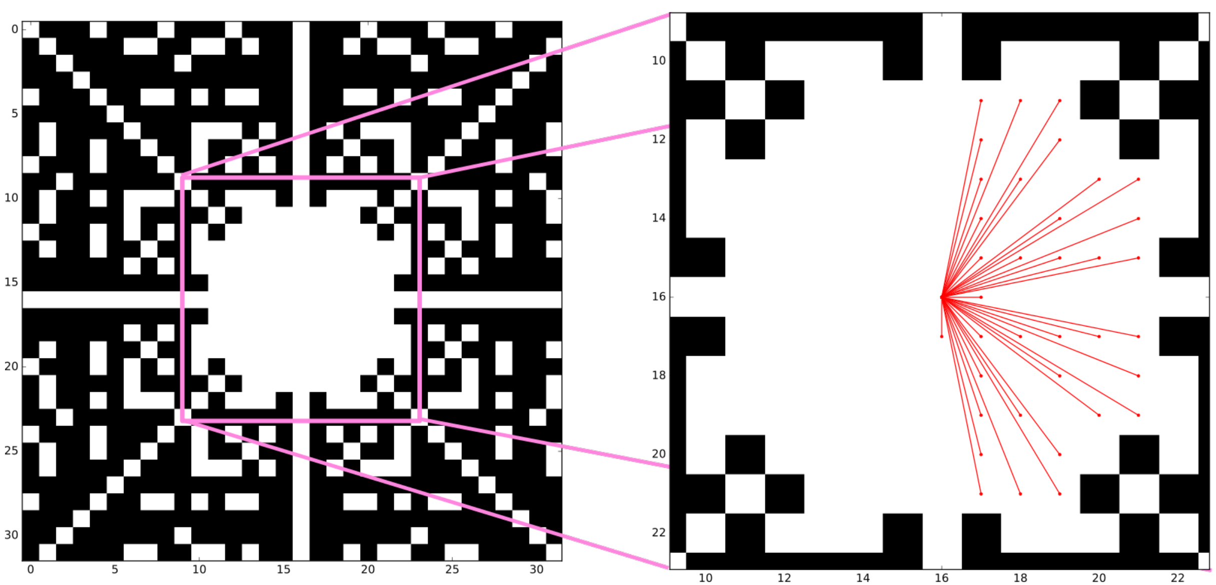
- Discrete (aperiodic) lines with Rational slopes ${}^{q}/_{p}$ are known to tile the space near the origin.
- These Farey sequence and are easily generated using the mediant property of the sequence
- If ${}^{q_1}/_{p_1}$ and ${}^{q_2}/_{p_2}$ are successive terms in the sequence of order n and a sequence begins with ${}^{0}/_{1}$ and ${}^{1}/_{n}$ until one obtains ${}^{1}/_{1}$, then the neighboring fraction ${}^{q_3}/_{p_3}$ in the sequence is

$$\frac{q_3}{p_3} = \frac{q_1+q_2}{p_1+p_2}, \qquad (2)$$

Iterate using Pascal's triangle to get all the terms required

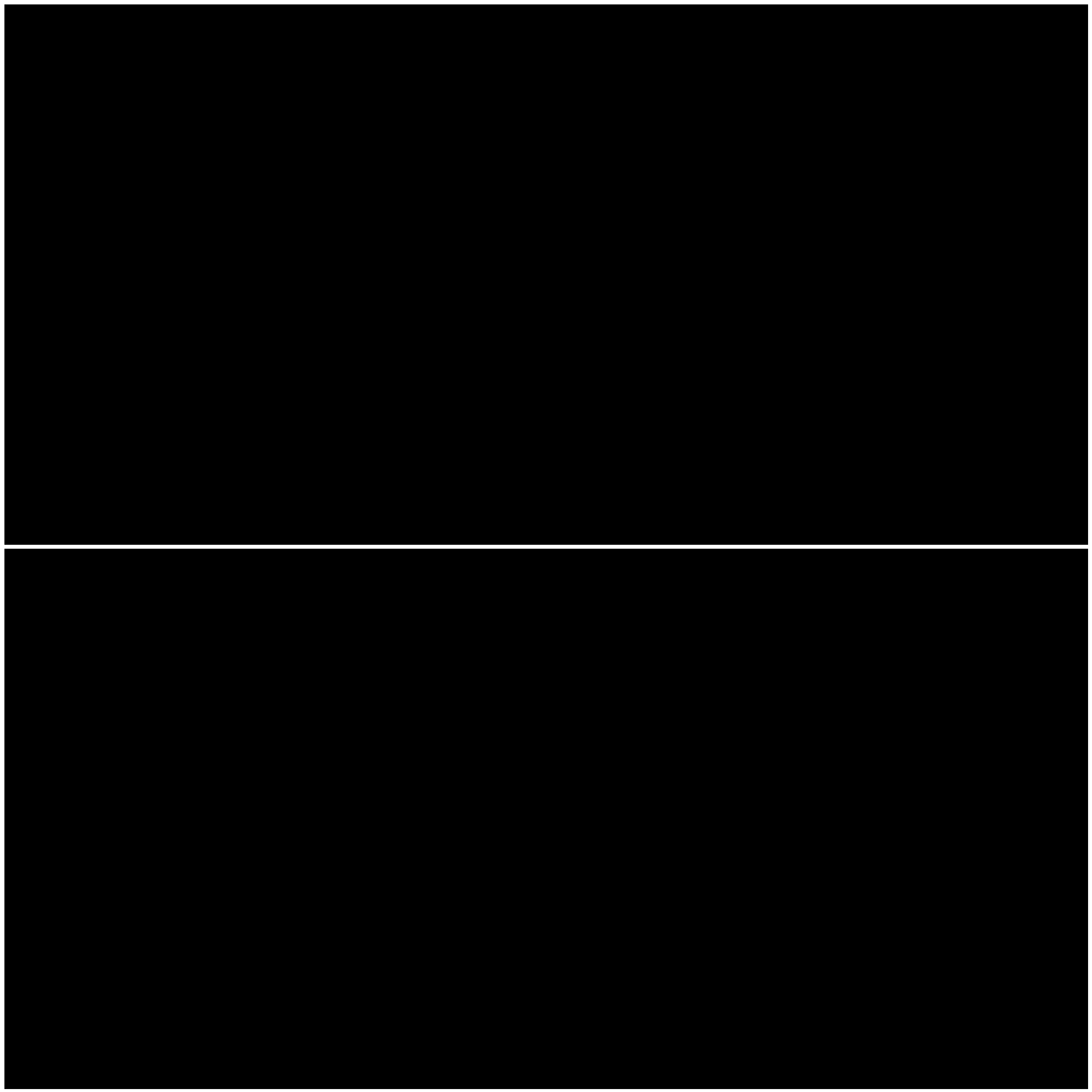
$$\mathfrak{I}_4 = \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

$$m \equiv pq^{-1} \ (mod\ M), \qquad (3)$$



[5] **Chandra, S. S.**; Normand, N.; Kingston, A.; Guedon, J. & Svalbe, I., Robust Digital Image Reconstruction via the Discrete Fourier Slice Theorem, *IEEE Signal Processing Letters*, **2014**, *21*, 682-686

Finite Fractal



Turbulent Image Reconstruction

Accelerating MR Imaging

Introduction – MR Imaging Contrast

- Hydrogen atoms behave like mini bar magnets
- Magnetic fields and radio waves are used to spatially encode the atoms, i.e they become oriented differently based on their position
- Measure the Fourier responses of the encoding to create a *k*-space in order to resolve the location of the atoms.

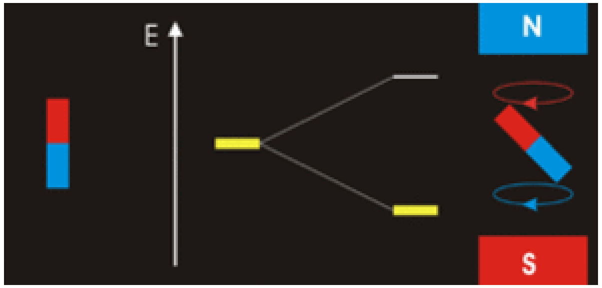
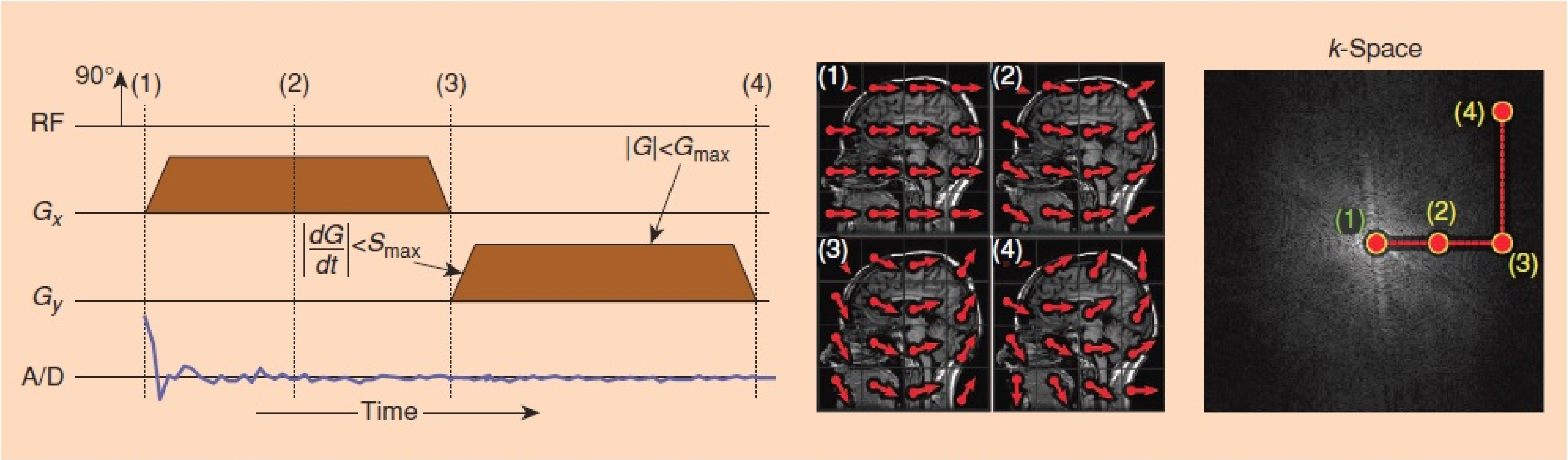


Image From https://en.wikipedia.org/wiki/Nuclear_magnetic_resonance

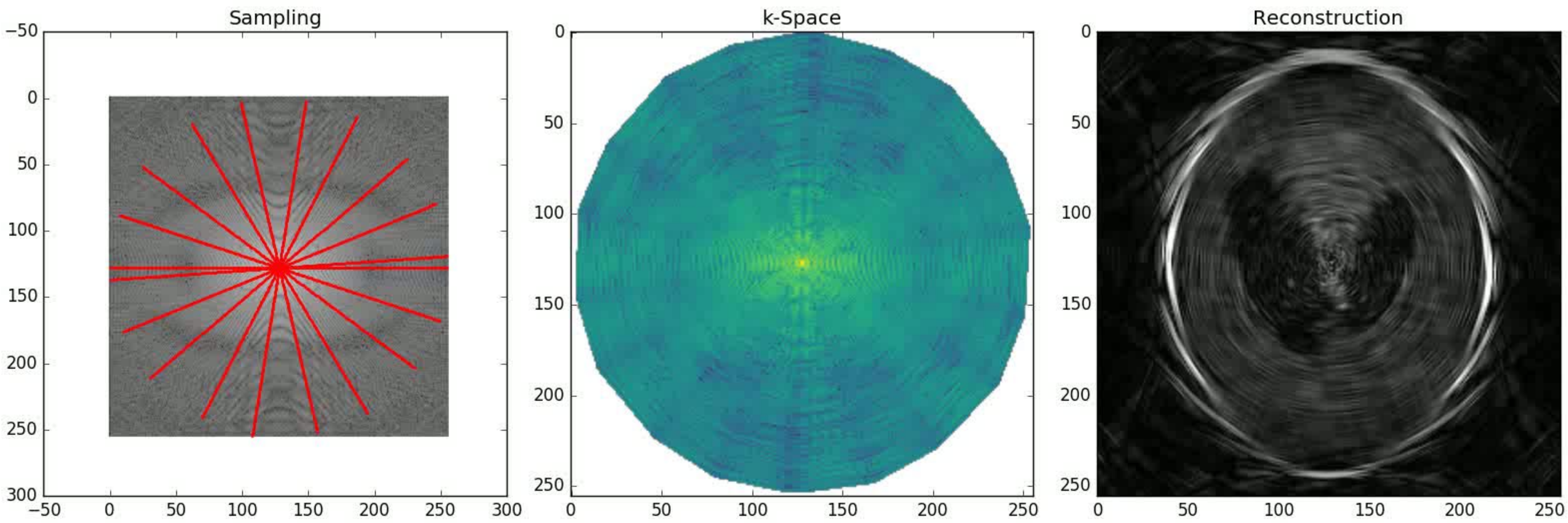


[FIG1] The temporal MRI signal directly samples the spatial frequency domain of the image. Gradient fields cause a linear frequency distribution across the image, which produces a linear phase accrual with time. The received signals are spatial frequency samples of the image. The corresponding spatial frequencies are proportional to the gradient waveform area. The gradient is limited in amplitude, G_{max}, and slew rate, S_{max}, which are both system specific.

[6] Lustig, M.; Donoho, D. L.; Santos, J. M. & Pauly, J. M., Compressed Sensing MRI, *IEEE Signal Processing Magazine*, **2008**, *25*, 72-82

Reconstruction – MR Imaging

- ➢ Directly samples an object's Fourier space, referred to also as *k*-space.
- ➢ Measurements in *k*-space need to be made repeatedly (forming a trajectory in this space) to cover this space.
- ➢ Therefore, recovering an object comes down to sufficiently sampling an object's *k*-space.



Reconstruction – MR Imaging

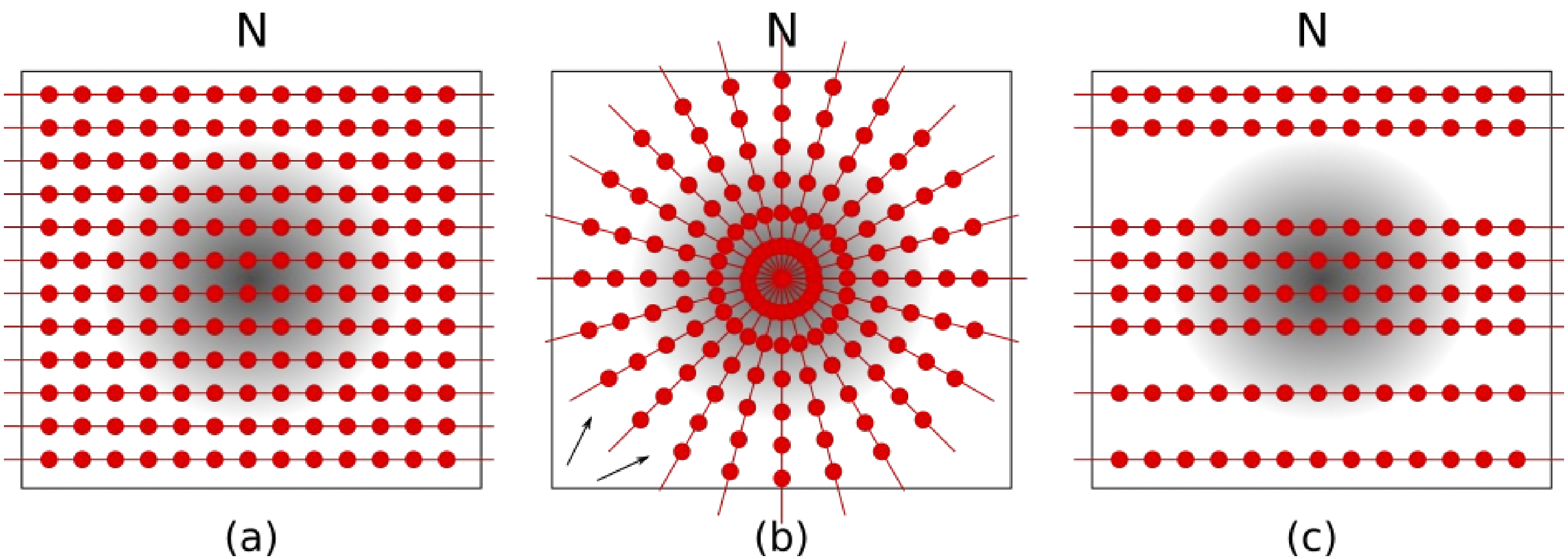


Figure: Measurements are made as k-space trajectories in MR imaging visualized using (red) connected points. (a) shows a Cartesian trajectory. (b) shows a traditional radial trajectory with arrows highlighting the gaps between the lines at high frequencies that can lead to significant artefacts in the recovered image. (c) shows the Cartesian trajectory with random phase encoding for Compressed Sensing type methods.

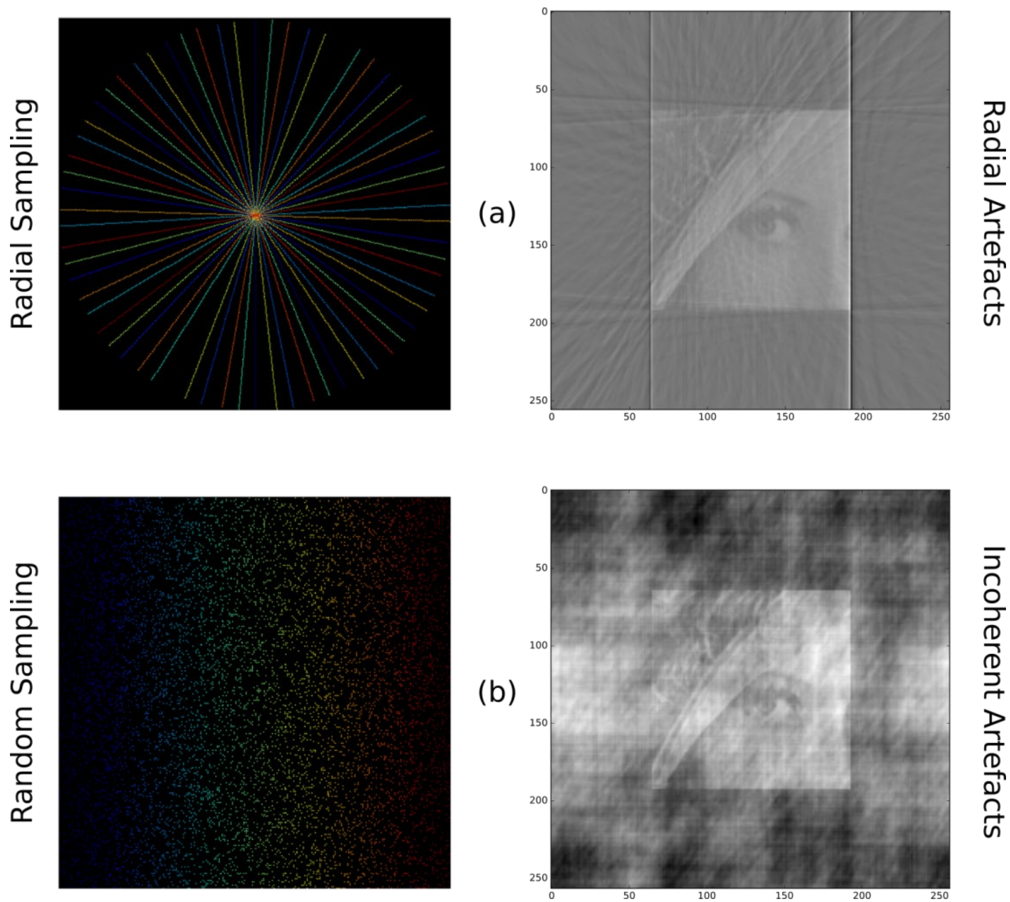
[7] Haase, A.; Frahm, J.; Matthaei, D.; Hanicke, W. & Merboldt, K. D. FLASH imaging. Rapid NMR imaging using low flip-angle pulses *Journal of Magnetic Resonance*, 1986, 67, 258-266

[8] Lustig, M.; Donoho, D. & Pauly, J. M. Sparse MRI: The application of compressed sensing for rapid MR imaging *Magnetic Resonance in Medicine*, 2007, 58, 1182-1195

[9] Lauterbur, P. C. Image Formation by Induced Local Interactions: Examples Employing Nuclear Magnetic Resonance *Nature*, 1973, 242, 190-191

Reconstruction – Ghosts

- Missing data in the DFT creates ambiguity about the image
- This manifests itself as reconstruction artefacts known as Ghosts



[10] **Chandra, S. S.**; Svalbe, I.; Guedon, J.; Kingston, A. & Normand, N., Recovering Missing Slices of the Discrete Fourier Transform Using Ghosts, *IEEE Transactions on Image Processing,* **2012***, 21*, 4431 -4441

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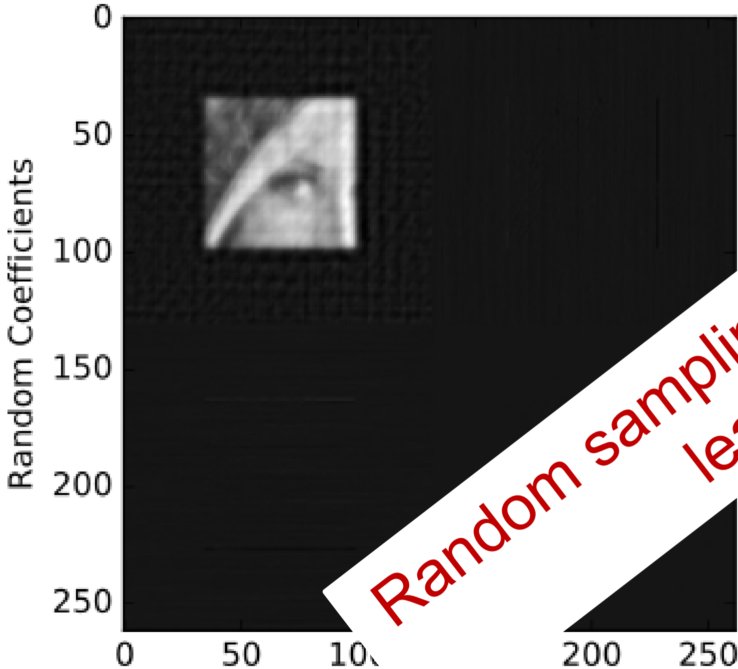
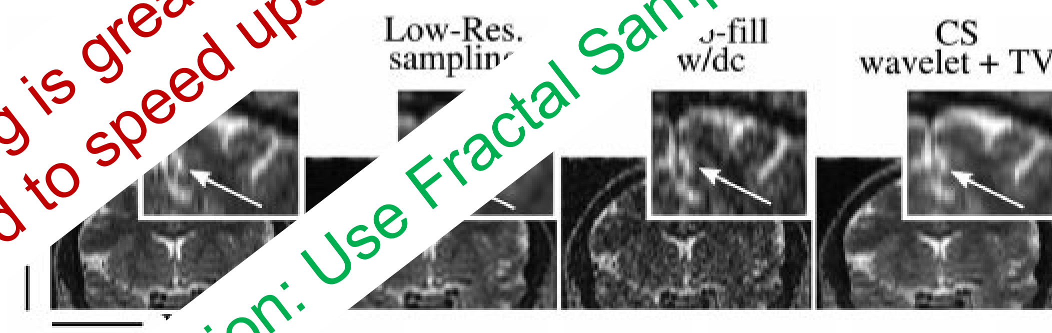
Reconstruction – Ghosts

- Random sampling creates incoherent Ghosts
- However, these Ghosts are also incoherent in an transf... ...here the image is sparse, e.g. the wavelet transform
- We can threshold the Ghosts out, recovering th...



Discrete Wavelet Trans... ...n of incoheren...

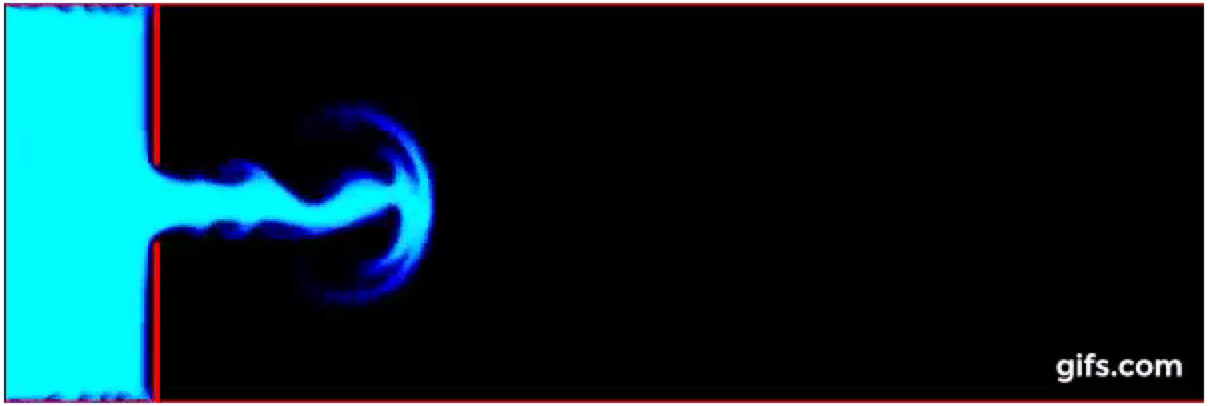


Random sampling is great theoretically, but does not lead to speed ups in practice

Solution: Use Fractal Sampling

[8] Lustig, M.; Donoho, D. & Pauly, J. M. Sparse MRI: Th... ...compressed sensing for rapid MR imaging *Magnetic Resonance in Medicine*, 2007, 58, 1182-1195

Reconstruction – Turbulence

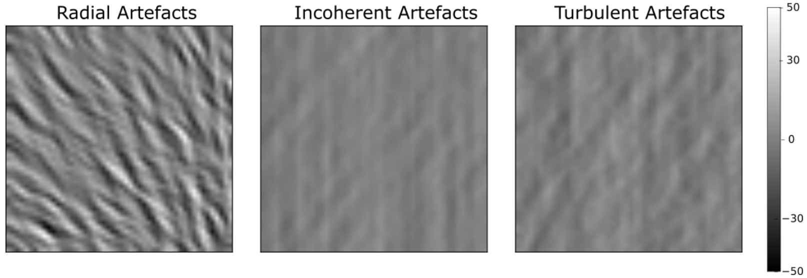


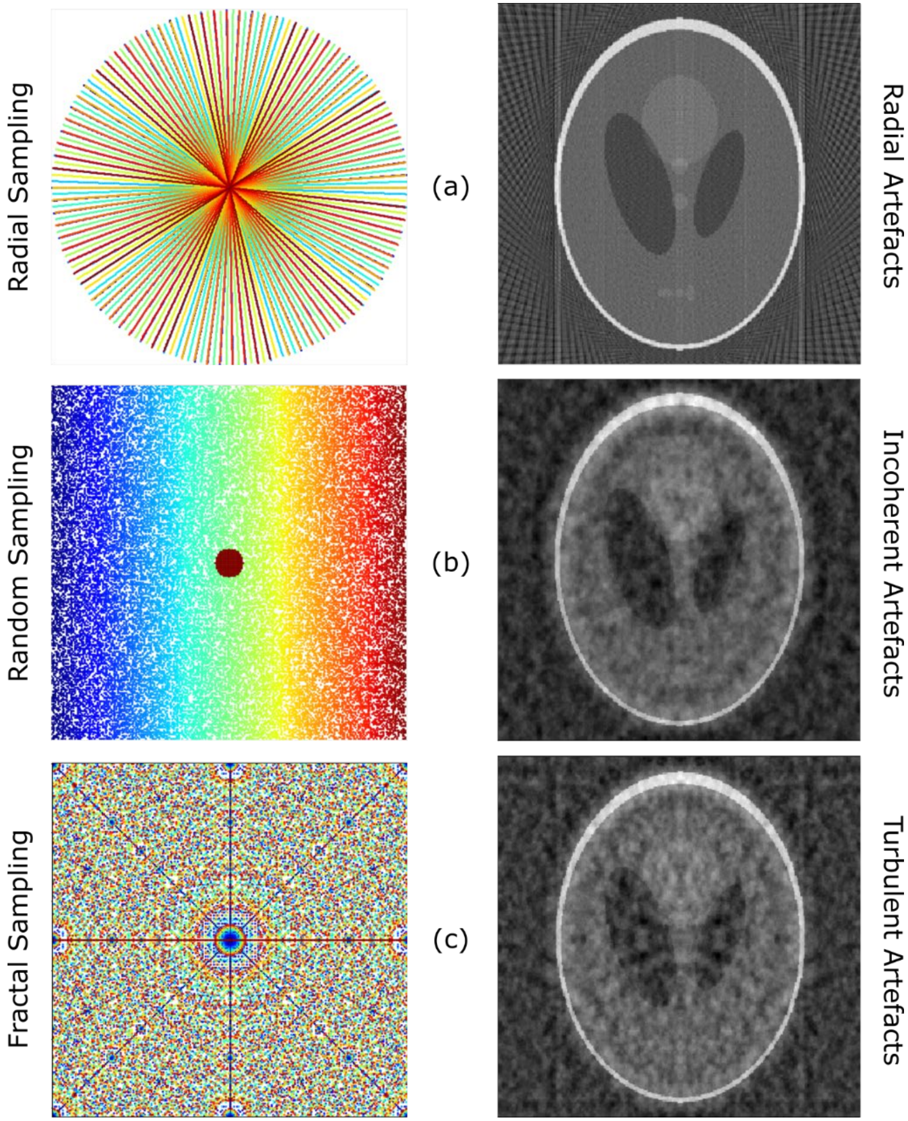
- The pattern and its process are akin to trying to see a reflection in the surface of a pond.
- When measuring completely, without trying to be efficient by getting all the information, it is if there is no disturbance in the pond and the reflection is easy to see.
- However, the conditions must be perfect and, in terms of MRI, it takes a long time to acquire all these measurements, but the image is straight forward to obtain.
- When we discard or have missing measurements, it is as if the surface of the pond is no longer still.
- In fact, the pond surface becomes disturbed with many, many ripples that is usually no way to resolving or reconciling a face on the surface anymore.
- The fractal nature of the pattern allows us to do is to ensure that these ripples interact with each other in such a precise way that they all cancel each other out.
- The cancelling out is done by producing turbulence among the ripples and amounts to a chaotic mixing of image information, which is why the work is called Chaotic Sensing.

Video excerpt from https://www.youtube.com/watch?v=KS-GSLrkf30

Reconstruction – Turbulent Ghosts

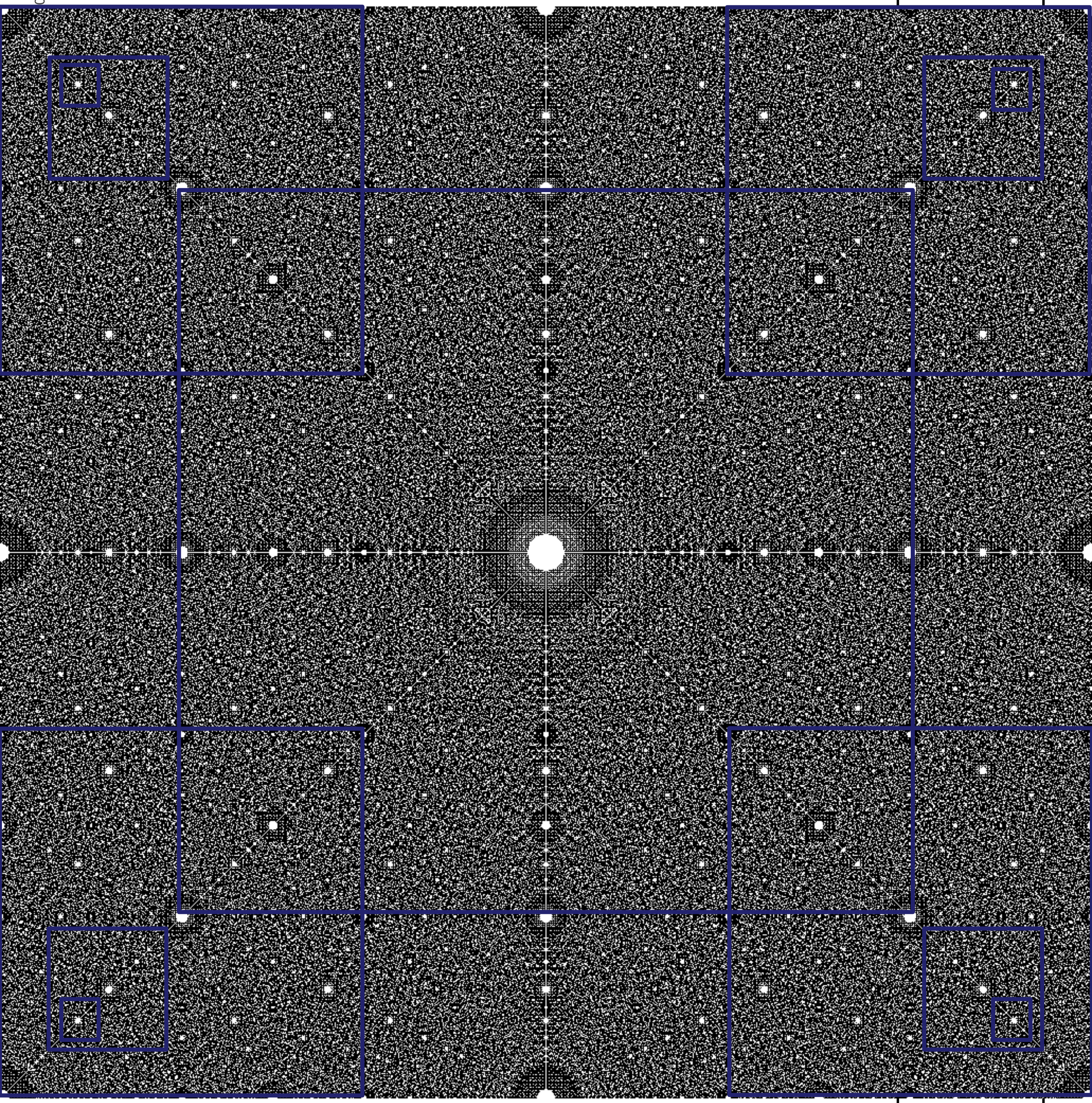
- This fractal is important because it produces “turbulent” artefacts in the reconstruction
- The term ``turbulent‘‘ is used because the artefacts are analogous to the dissipation of energy at large scales and low frequencies into many small scales and high frequencies
- It produces a multi-band response in image space, in the same way as fractal antennas were designed [11].





[11] Z. Baharav, “Fractal arrays based on iterated functions system (IFS),” in IEEE Antennas and Propagation Society International Symposium, 1999, 1999, vol. 4, pp. 2686–2689 vol.4.

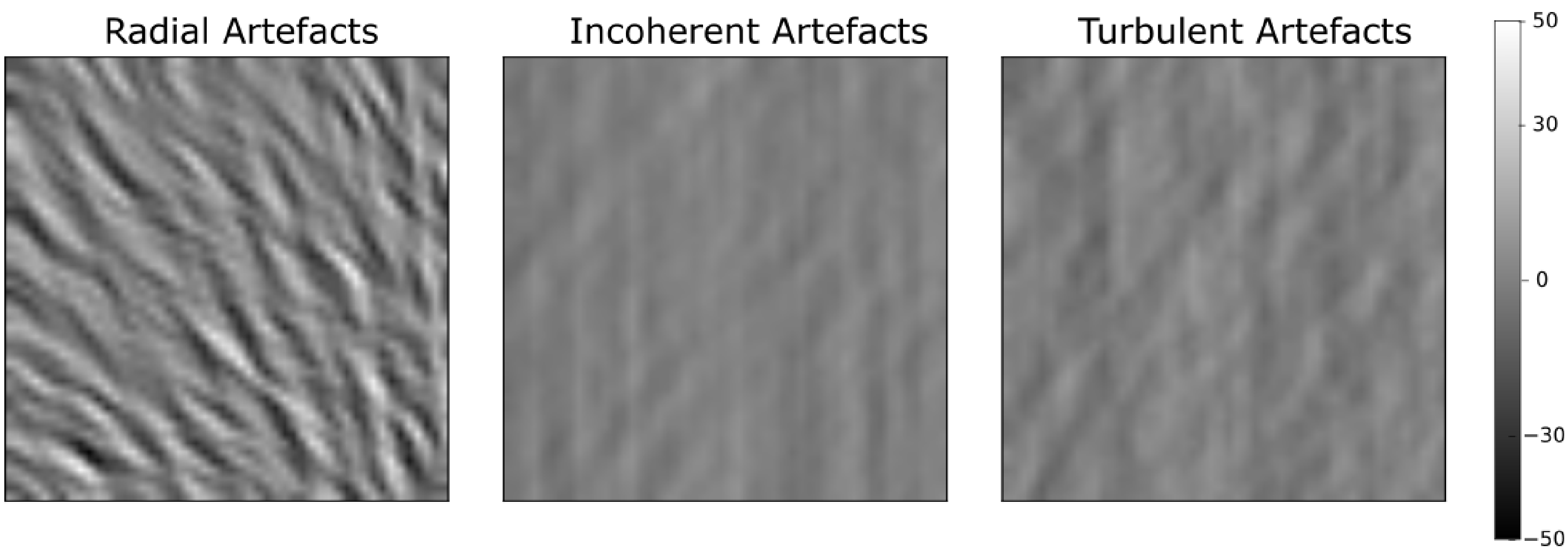
Fractal of the Discrete Fourier Transform



N = 1031

Reconstruction – Turbulent Ghosts

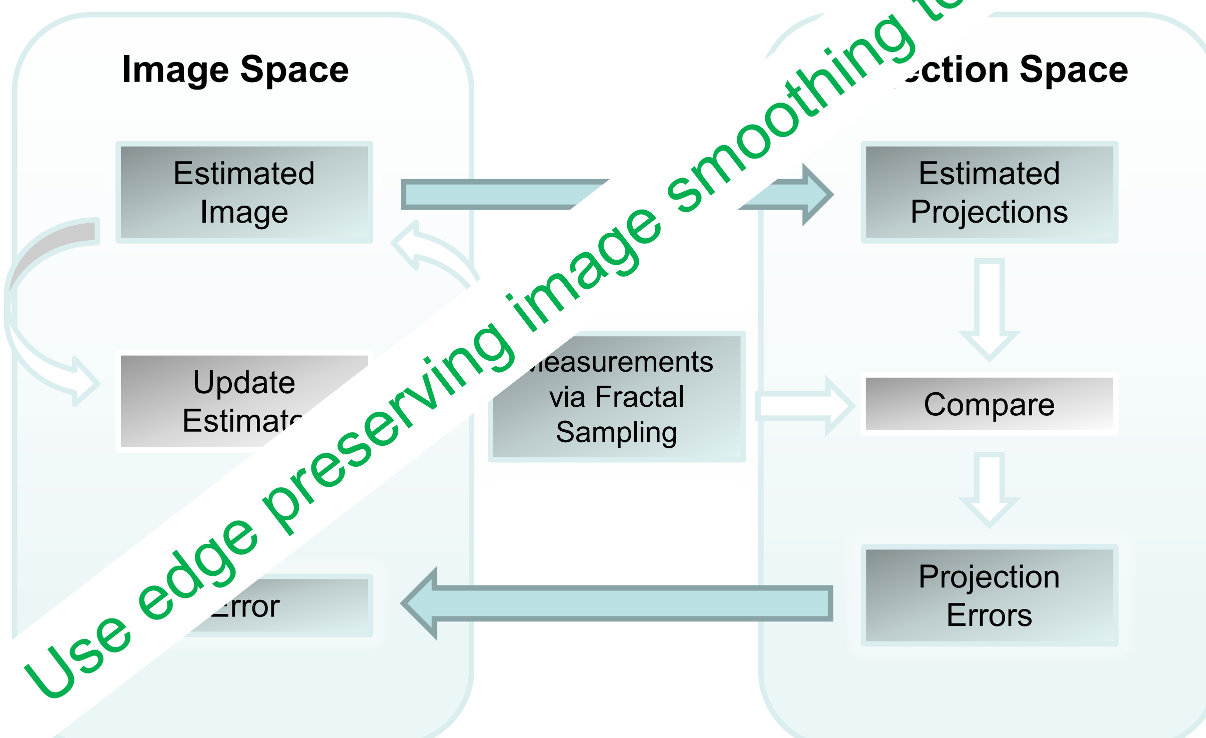
- Using the measure of turbulence called turbulent intensity τ (where $\tau = \sigma(\vec{v})$, $\vec{v} = g - \bar{g}$, $g = \nabla I$, $\vec{v}$ is known as the velocity fluctuation, ∇I is the gradient of the image and σ is the standard deviation)
- The incoherent and turbulent artefacts have similar τ values, while both being a few orders of magnitude different to the radial artefacts



Reconstruction – Fractal Decomposition

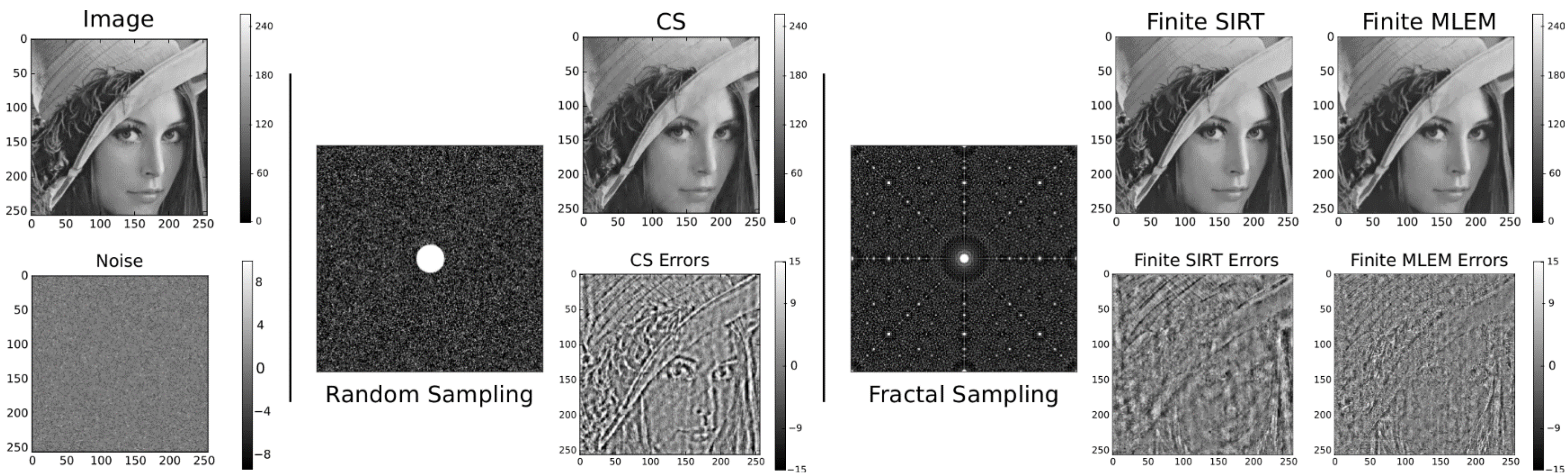
- The fractal maps to slices of the DFT and periodic projections
- We create finite versions of the Maximum Likelihood Expectation Maximisation (MLEM) and Simultaneous Iterative Reconstruction Technique (SIRT) algorithms from tomography for robust noise tolerant limited data reconstruction.

Use edge preserving image smoothing to regularise

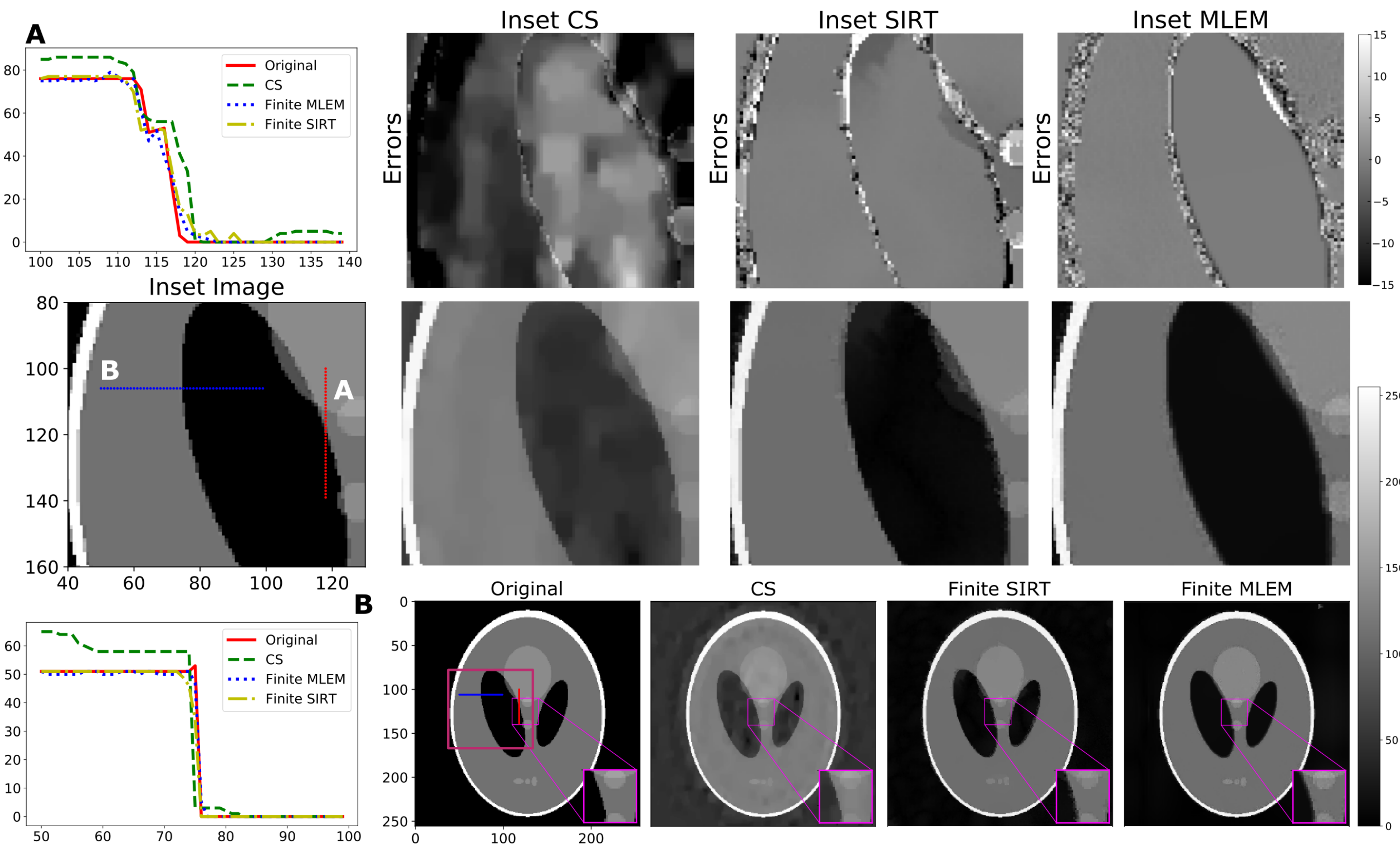


Reconstruction – Results

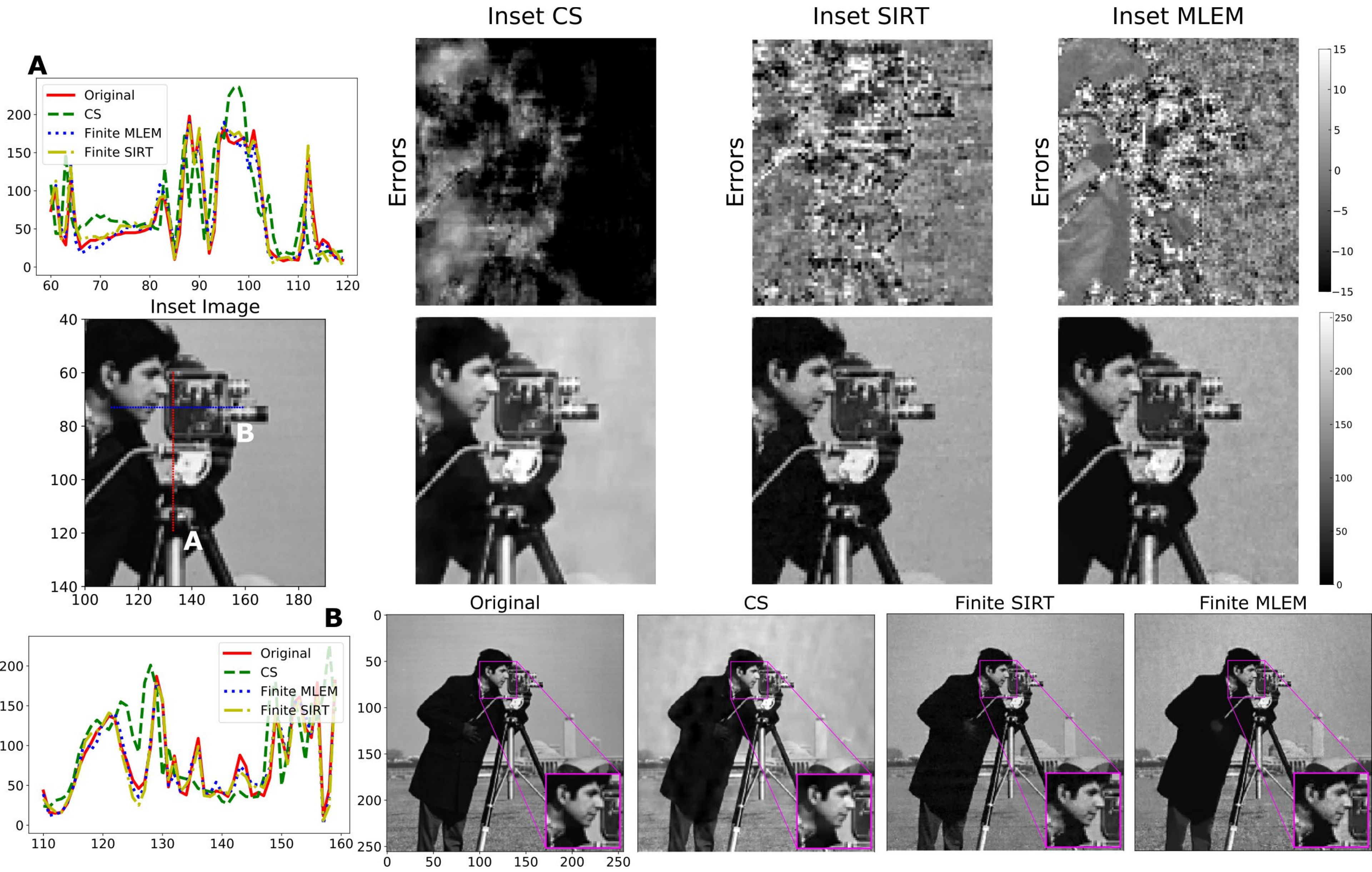
- The fractal maps to slices of the DFT and periodic projections
- We create finite versions of the Maximum Likelihood Expectation Maximisation (MLEM) and Simultaneous Iterative Reconstruction Technique (SIRT) algorithms from tomography for robust, noise tolerant limited data reconstruction.



[1] Chandra, S. S.; Ruben, G.; Jin, J.; Li, M.; Kinston, A. M.; Svalbe, I. & Crozier, S., Chaotic Sensing, *Image Processing, IEEE Transactions on,* **2018**, *27*, 6079-6092



[1] Chandra, S. S.; Ruben, G.; Jin, J.; Li, M.; Kinston, A. M.; Svalbe, I. & Crozier, S., Chaotic Sensing, *Image Processing, IEEE Transactions on,* **2018**, *27*, 6079-6092



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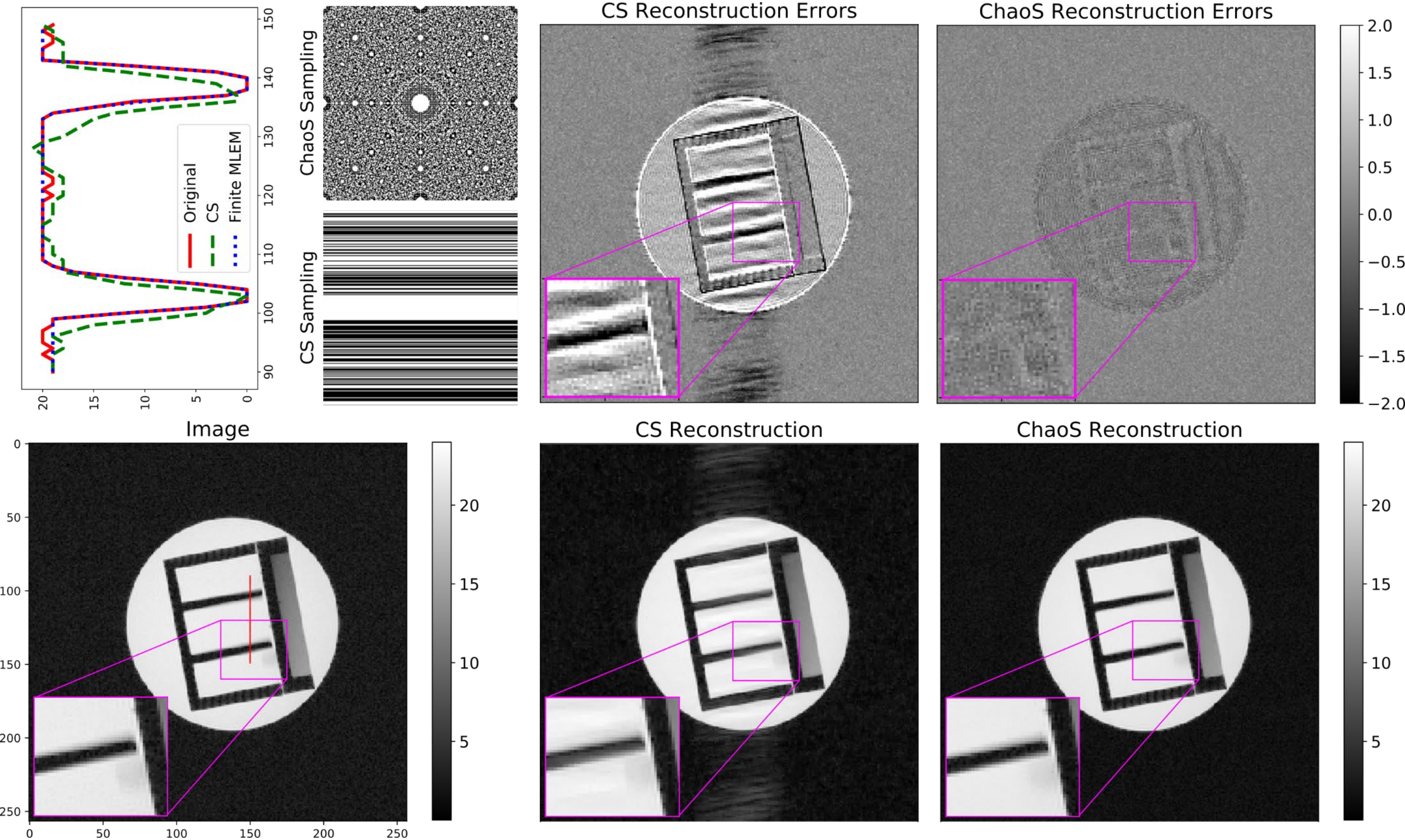

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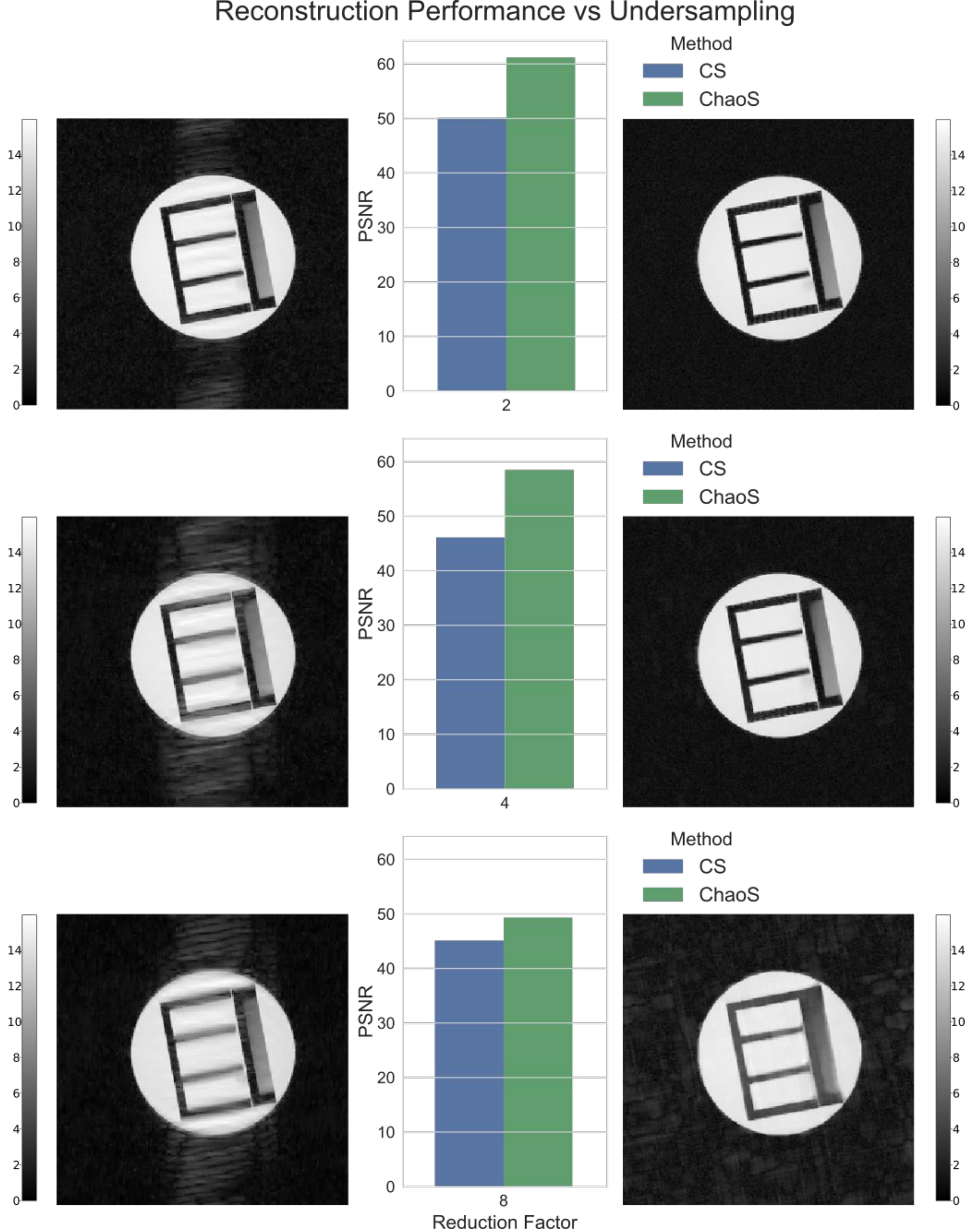
- A factor of 4 reduction in number of measurements is possible without any significant loss of imaging information
- A factor of 8 although possible, would likely have poor signal-to-noise ratio.
- We hope to implement this scheme for small animals and later clinical pilots in the near future.



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Conclusion

- The discrete slice theorem is an important property of the DFT
- The periodicity of the DFT space can produce fractals
- We learnt that fractals could produce turbulence and turbulent artefacts
- A new finite fractal in the DFT allows for speed ups in MR image acquisition through the use of limited imaging data and turbulence.
- Further work is required in understanding the fractal and if other applications of the turbulent nature of DFT can be found

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To my family…