Question 1

Solution Let us place each point A, B, and C on the successively on the circle. For the first placement, A can be anywhere. After A has been placed, there are two semicircles, divided by the line going through A and the center of the circle. The placement of B and C can be on both sides of the circle with equal probability $1/2$, and the probability for B and C on the same semicircle is therefore $1/4$.

Since we can place either A or B or C at the very first, the probability of these three points on the circle is therefore $3/4$. ◀

Question 2

Solution Let $f(n)$ denotes the expected number of steps needed. Here n is the number of steps the starting position is away from the destination. Therefore, once we place the ant on a vertex, this vertex is vertex 3 and the destination will be 0. Using the conditional expectation, we have

$$f(3) = 1 + f(2) \tag{1}$$

$$f(2) = 1 + \frac{2}{3}f(1) + \frac{1}{3}f(3) \tag{2}$$

$$f(1) = 1 + \frac{1}{3}f(0) + \frac{2}{3}f(2) \tag{3}$$

Obviously $f(0) = 0$, solving this set of equations, yielding

$$f(3) = 10 \tag{4}$$

◀

Question 3

Solution This question resembles the American option pricing. Denote (b, r) denotes the number of blue and red cards left in the deck, respectively. At each (b, r), we face the decision whether to sop or keep playing. If we stop, the payoff is $r - b$. If we keep going, there is $\frac{b}{b+r}$ probability that the next card will be black, and we switch to the state $(b - 1, r)$, and $\frac{r}{b+r}$ probability that the next card will be black, and we switch to the state $(b, r-1)$. We will stop if the expected payoff of drawing more cards is less than $r-b$. This gives us the system equation

$$\mathbb{E}\left[f(b,r)\right] = \max\left\{r-b, \frac{b}{b+r}\mathbb{E}\left[f(b-1,r)\right] + \frac{r}{b+r}\mathbb{E}\left[f(b,r-1)\right]\right\} \tag{5}$$

Obviously, the boundary conditions are

$$f(0,r) = r \quad \forall r = 0,1,\ldots,26 \tag{6}$$

$$f(b,0) = 0 \quad \forall b = 0,1,\ldots,26 \tag{7}$$

Running the system equation (5) with the boundary conditions (6) and (7), we get the expected payoff at the beginning of the game is

$$\mathbb{E}\left[f(26,26)\right] = \$2.62 \tag{8}$$

i.e., we would pay 2.62 dollar for this game. ◀

Question 4

Solution ◀

Question 5

Solution Since the life of the light bulbs has memoryless property, we assume it follows the exponential distribution, and denote $X_i, i = 1, 2, \cdots, 5$ be i.i.d. exponential distribution with $\lambda = \frac{1}{100}$ and $Y_i, i = 1, 2, \cdots, 5$ be i.i.d. exponential distribution with $\lambda = \frac{1}{200}$. We want to compute

$$\mathbb{E}\left[\min(X_1,\ldots,X_5,Y_1,\ldots,Y_5)\right] \tag{9}$$

Denote $Z = \min(X_1,\ldots,X_5,Y_1,\ldots,Y_5)$, its pmf can be computed as

$$\begin{aligned}
\mathbb{P}\left(Z \le z\right) &= 1 - \mathbb{P}\left(Z > z\right) \\
&= 1 - \prod_{i=1}^{5}\mathbb{P}\left(X_i > z\right)\mathbb{P}\left(Y_i > z\right) \\
&= 1 - \left(\int_z^\infty \frac{1}{100}e^{-\frac{1}{100}t}dt\right)\left(\int_z^\infty \frac{1}{200}e^{-\frac{1}{200}t}dt\right) \\
&= 1 - \int_z^\infty e^{-\frac{3}{40}t}dt
\end{aligned} \tag{10}$$

Therefore, the pdf of Z is

$$f(z) = \frac{d}{dz}\mathbb{P}\left(Z \leq z\right) = \frac{3}{40}e^{-\frac{3}{40}z} \tag{11}$$

The expectation can be computed as

$$\begin{aligned}
\mathbb{E}\left[\min(X_1, \ldots, X_5, Y_1, \ldots, Y_5)\right] &= \int_0^\infty zf(z)dz \\
&= \int_0^\infty z\frac{3}{40}e^{-\frac{3}{40}z}dz \\
&= \frac{40}{3} \qquad (12)
\end{aligned}$$

◀

Question 6

Solution The total number of coin toss is

$$N = 60 \times 60 \times 24 \times 365 \times 100 = 3.1536 \times 10^{10} \tag{13}$$

The probability of getting 100 heads in a row can be computed as

$$\mathbb{P}\left(100 \text{ consecutive heads}\right) = \mathbb{P}\left(A_1 \cap A_2 \cap \cdots A_{N-100}\right) \tag{14}$$

where A_i denotes the 100 consecutive sequence starts from toss i. Clearly

$$\mathbb{P}(A_i) = \frac{1}{2^{100}} \tag{15}$$

Therefore

$$\mathbb{P}\left(100 \text{ consecutive heads}\right) \leq \sum_{i=1}^{N-100}\mathbb{P}(A_i) = \frac{N-100}{2^{100}} < \frac{10^{11}}{1024^{10}} < \frac{10^{11}}{10^{30}} < 0.01\% \tag{16}$$

The statement is correct. ◀

Question 7

Solution If we randomly cut a stick into N pieces and denote the length of each piece as $X_i, i = 1, \ldots, N$. Without loss of generality, we assume the length of the stick to be 1, therefore

$$\sum_{i=1}^{N} X_i = 1 \tag{17}$$

In order for this N pieces to form a polygon, we must have

$$X_i < \frac{1}{2} \quad \forall i = 1, 2, \ldots, N \tag{18}$$

This reminds us of Question 1. The problem is then equivalent to that not all N points are on the same semicircle of a circle with the circumference of 1. Following the same convention as of question, we calculate the probability that all N points are on the same semicircle. Given a cut point i, we define an event E_i as the next $N - 1)$ point are in the clockwise semicircle and obviously $P(E_i) = \frac{1}{2^{N-1}}$. Thus the probability of all N points are on the same clockwise semicircle is

$$\mathbb{P}\left(\bigcup_{i=1}^{N} E_i\right) = \bigcup_{i=1}^{N} \mathbb{P}\left(E_i\right) = \frac{N}{2^{N-1}} \tag{19}$$

since E_i is mutually exclusive because there must be a point when we start from j that has a distance to the point starting from i greater than a clockwise semicircle when $i \neq j$. Thus the probability to be calculated is

$$\mathbb{P} = 1 - \frac{N}{2^{N-1}} \tag{20}$$

◀

Question 8

Solution The moving average and moving median are the same provided the distribution is not skewed. However, this is definitely not the case observed in the market. The famous market model assumes a lognormal distribution of the price. The moving median and moving average capture different aspects of the price process. More importantly, moving median is not sensitive to a sudden jump of the price price, while moving average does. If we want to have a sense of the upward and downward moving of the stock market, the moving median is better quantity to use. However, if we have a sense of mean return, the moving average is better to use. ◀

Question 9

Solution Denote

$$X_i = \begin{cases} 1 & \text{if } i \text{ is a local maxima} \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Obviously, if i is at the end, the probability of being a local maxima is $\frac{1}{2}$, while it is $\frac{1}{3}$ if it is not a the end. Thus, the expected number of local maxima of the permutation can be calculated as

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] &= \frac{1}{3} \times (n-2) + \frac{1}{2} \times (2) \\ &= \frac{n+1}{3} \end{aligned} \tag{22}$$

◀

Question 10

Solution `cout << ((n & n-1) ==0 ?  "True" :  "False")` ◀

Question 11

Solution A smart pointer is an object that stores a pointer to a heap allocated object. If you use a smart pointer correctly, you no longer have to remember when to delete the new created memory. The implementation looks like this

```
template<class T>
```

```
class SmartPointer {
public:
SmartPointer(T* DumbPtr = 0); // Default initialized to NULL
SmartPointer(const SmartPoint& rhs); //Copy constructor
~SmartPoint(); //Destructor

SmartPointer& operator= (const SmartPoint& rhs); //Assignment operator
T& operator* () ; //Dereference operator
const T& operator* () const; //Dereference operator
T* operator -> () const; //Address-of operator

private:
T* pointee; };
```
◀

Question 12

Solution ◀

Question 13

Solution ◀

Question 14

Solution ◀

Question 15

Solution We use Taylor series to calculate the exponent. Scan over each term of the expansion, we have $O(n)$ time complexity and $O(1)$ space complexity.

```
double exp(double x, double delta) // x is the power exponent, delta
is the precision
{ double term = 1; // first term
double sum = 0; // initial value
```

```
for (double i = 1; term >= delta; ++i) {
sum = sum + term;
term = term*x/i;
}
return exp;
}
```

◀

Question 16

Solution ◀

Question 17

Solution ◀