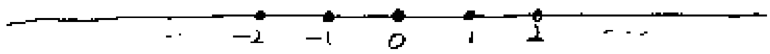
4. Solution :



Let $X_i = \begin{cases} +1, & \text{with probability } p \\ -1, & \text{with probability } q = 1-p \end{cases}$, $i = 1, 2, 3, \cdots$

Let $S_0 = 0$ $\quad S_n = X_1 + X_2 + \cdots + X_n$

Let $\tau = \inf\{n > 0 : S_n = 0\}$ $\quad$ 1^{st} – return time

The problem requires to compute : $\mathbb{P}^0(\tau < \infty) = \mathbb{P}(\tau < \infty \mid S_0 = 0)$

Since $\mathbb{P}^0(\tau < \infty) = p \cdot \mathbb{P}^0(\tau < \infty \mid S_1 = 1) + q \cdot \mathbb{P}^0(\tau < \infty \mid S_1 = -1)$

Define hitting times : $h_1 = \inf\{n > 0 : S_n = 1\}$ ← hit 1
$\qquad h_{-1} = \inf\{n > 0 : S_n = -1\}$ ← hit -1

Since this (asymmetric) random walk is a homogeneous Markov chain

$$\mathbb{P}^0(\tau < \infty \mid S_1 = 1) = \mathbb{P}^{S_0 = 1}(\tau < \infty) = \mathbb{P}^0(h_{-1} < \infty)$$
$$\mathbb{P}^0(\tau < \infty \mid S_1 = -1) = \mathbb{P}^{S_0 = -1}(\tau < \infty) = \mathbb{P}^0(h_1 < \infty)$$

Hence $\underline{\mathbb{P}^0(\tau < \infty) = p \cdot \mathbb{P}^0(h_{-1} < \infty) + q \cdot \mathbb{P}^0(h_1 < \infty)}$

Next we evaluate $\mathbb{P}^0(h_{-1} < \infty)$ and $\mathbb{P}^0(h_1 < \infty)$ using two different methods : Martingales vs Markov chains

<u>Martingale methods</u>: WLOG $p \geq \frac{1}{2} \geq q$

For $\mathbb{P}^0(h_1 < \infty)$. consider the following martingale $\{M_n\}_{n \geq 0}$

$$M_n = e^{\lambda S_n} \cdot \left(\frac{1}{pe^{\lambda} + qe^{-\lambda}} \right)^n$$

⊛ since $p \geq q$ we have $pe^{\lambda} + qe^{-\lambda} > 1$ for all $\lambda > 0$.

Consider the stopping time h_1. we know $\{M_{n \wedge h_1}\}_{n \geq 0}$ is also a martingale

In particular, $\mathbb{E}^0[M_{n \wedge h_1}] = \mathbb{E}^0\left[e^{\lambda S_{n \wedge h_1}} \cdot \left(\frac{1}{pe^{\lambda} + qe^{-\lambda}} \right)^{n \wedge h_1}\right] = 1$

Since $e^{\lambda S_{n \wedge h_1}} \leq e^{\lambda} < \infty$ and $pe^{\lambda} + qe^{-\lambda} > 1$ for all $\lambda > 0$

Lebesgue DCT gives

$$\lim_{n \to \infty} \mathbb{E}^0\left[e^{\lambda S_{n \wedge h_1}} \left(\frac{1}{pe^{\lambda} + qe^{-\lambda}} \right)^{n \wedge h_1}\right] = \mathbb{E}^0\left[e^{\lambda} \cdot \left(\frac{1}{pe^{\lambda} + qe^{-\lambda}} \right)^{h_1} \cdot \mathbb{1}_{\{h_1 < \infty\}}\right] = 1$$