Q3. From a deck of 52 cards, you can pick one card each time without replacement. If the card color is black, you win 1\$. If the card color is red, you lose 1\$. You can stop the game whenever you want. Questions: Will you play the game? If you want, how much would you pay to play this game?

Let (b, r) represent the number of black and red cards left in the deck, respectively. At each (b, r), we face the decision whether to stop or keep on playing. If we ask the dealer to stop at (b, r), the payoff is $b - r$, If we keep on going, there is $\frac{b}{b+r}$ probability that the next card will be black– in which case the state changes to $(b - 1, r)$ – and $\frac{r}{b+r}$ probability that the next card will be red – in which case the state changes to $(b, r - 1)$. That also gives us the system equation:

$$E\left[f(b,r)\right] = max\left(b - r, \frac{b}{b+r}E\left[f(b-1,r)\right] + \frac{r}{b+r}\left[f(b,r-1)\right]\right) \tag{1}$$

using the boundary conditions $f(0, r) = 0, f(b, 0) = b, \forall b, r = 0, 1, \cdots, 26$, and the system equation for $E[f(b, r)]$, we can recursively calculate $E[f(b, r)]$ for all pairs of b and r.

The expected payoff at the beginning of the game is $E[f(26, 26)] = \$2.62$

Q9. What is the average number of local maxima of a permutation of 1,...,n, over all permuta- tions? Maxima at ends also count. (Putnam problem.)

By the linearity of expectation, the average number of local maxima is equal to the sum of the probability of having a local maximum at k over k = 1,...,n. For k = 1, this probability is 1/2: given the permutation pair {p(1),p(2)}, it is equally likely that p(1) or p(2) is bigger. Similarly, for k = n, the probability is 1/2. For 1 ¡ k ¡ n, the probability is 1/3: given the pair {p(k-1), p(k), p(k + 1)}, it is equally likely that any of the three is the largest. Thus the average number of local maxima is $2 \cdot \frac{1}{2} + (n - 2) \cdot \frac{1}{3} = (n + 1)/3$