Qi Shi Quiz 1

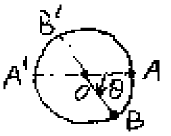
Math

1. Solution: Consider 3 points A, B, and C on the cicle

WLOG we can fix point A and move points B and C.

The angle θ between point A and point B can be

$$-\pi \le \theta \le \pi .$$



If $-\pi \le \theta \le 0$, then C cannot locate on $\overset{\frown}{A'OB'}$

So A, B, C would be on a semi-cicle with probability

$$\frac{1}{2\pi}(2\pi + \theta).$$

If $0 \le \theta \le \pi$, then C cannot locate on $\overset{\frown}{A'OB'}$

So A, B, C would be on a semi-cicle with probability

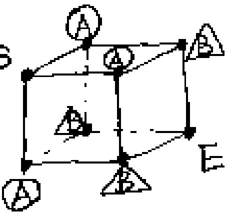
$$\frac{1}{2\pi}(2\pi - \theta).$$

Hence, the answer can be computed by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}(2\pi - |\theta|)\right]d\theta$$

$$= \frac{1}{(2\pi)^2}\int_0^{\pi}(2\pi-\theta)\,d\theta + \frac{1}{(2\pi)^2}\int_{-\pi}^{0}(2\pi+\theta)\,d\theta = \boxed{\frac{3}{4}}$$

2. Solution: We want to compute the expectation of number of steps starting from S ending at E



Identify the vertices which are 1 step away from S to Ⓐ and which are 2 steps away from S to $\triangle$B.

We have the following 4-states Markov Chain:

$$S \underset{1}{\overset{1/3}{\rightleftarrows}} A \underset{2/3}{\overset{2/3}{\rightleftarrows}} B \xrightarrow[1/3]{} E \circlearrowleft 1$$

Let μ_S, μ_A, μ_B and μ_E be the expectation of number of steps starting from S/A/B/E ending at E, then

$$\mu_A = \mu_B + 1, \quad \mu_B = \frac{1}{3}\mu_A + \frac{2}{3}\mu_C + 1 \Rightarrow \mu_A = \boxed{10}$$

$$\mu_C = \frac{1}{3}\mu_D + \frac{2}{3}\mu_B + 1, \quad \mu_D = 0$$