4. If a rod with length L is cut randomly into N pieces, what is the distribution of the longest piece?

Solution Without loss of generality, suppose $L = 1$. Let $X_1, X_2, \cdots, X_{n-1}$ denote the ordered distances of cut positions from the origin. Let X_0 and X_n denote the beginning and the end of the rod, respectively, such that $X_0 = 0$ and $X_n = 1$. Let $V_i = X_i - X_{i-1}$, then the V_i's are the lengths of the pieces of rod. Note that the probability that any particular k of the V_i's simultaneously have lengths longer than $c_1, c_2, \cdots, c_k$, respectively (where $\sum_{i=1}^{k} c_i \leq 1$), is

$$\mathsf{Pr}(V_1 > c_1, V_2 > c_2, \cdots, V_k > c_k) = (1 - c_1 - c_2 - \cdots - c_k)^{n-1} \tag{1}$$

This is proved formally in David and Nagaraja's *Order Statistics*, p. 135 (see below). Intuitively, the idea is that in order to have pieces of size at least $c_1, c_2, \cdots, c_k$, all $n-1$ of the cuts have to occur in intervals of the rod of total length $1 - c_1 - c_2 - \cdots - c_k$. To see this, we can look at the range of each single point by drawing a graph. Take the first cut position X_1 for example. X_1 must be located in a range such that $V_1 > c_1$ and $1 - V_1 > c_2 + c_3 + \cdots + c_k$ (see Figure 1 for illustration). Hence, the range of X_1 (enclosed by square brackets in Figure 1) is $1 - c_1 - c_2 - \cdots - c_k$. Similarly, the range of other cut positions $X_2, X_3, \cdots, X_{n-1}$ is also $1 - c_1 - c_2 - \cdots - c_k$. Together, these facts culminate in equation (1).

Here is a formal proof of equation (1):

Denote the positions on the rod where the cuts are made (unordered) by $U_i (i = 1, \cdots, n-1)$. Since all cuts are random, we have $U_i \overset{iid}{\sim} \mathsf{Unif}(0, 1)$, for $i = 1, 2, \cdots, n-1$. Then we have the joint pdf of order statistics $X_1, X_2, \cdots, X_{n-1}$ over the simplex $0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1$:

$$\begin{aligned} & f(X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1}) && (2) \\ = & (n-1)! f(U_1 = x_1, U_2 = x_2, \cdots, U_{n-1} = x_{n-1}) && (3) \\ = & (n-1)! f(U_1 = x_1) f(U_2 = x_2) \cdots f(U_{n-1} = x_{n-1}) && (4) \\ = & (n-1)! && (5) \end{aligned}$$

Since $V_i = X_i - X_{i-1}$, the determinant of the Jacobian matrix J that maps from X_i's to V_i's is:

$$\det J = \begin{vmatrix} \frac{\partial V_1}{\partial X_1} & \cdots & \frac{\partial V_1}{\partial X_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial V_{n-1}}{\partial X_1} & \cdots & \frac{\partial V_{n-1}}{\partial X_{n-1}} \end{vmatrix} \tag{6}$$

$$= \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} \tag{7}$$

$$= 1 \tag{8}$$

By change of variable, the joint pdf of V_i's is

$$f(V_i = v_1, V_2 = v_2, \cdots, V_{n-1} = v_{n-1}) = \frac{f(X_1 = x_1, X_2 = x_2, \cdots, X_{n-1} = x_{n-1})}{\det J} \tag{9}$$

$$= (n-1)! \tag{10}$$

where $v_i \geq 0$ and $\sum_{i=1}^{n-1} v_i \leq 1$. It follows that the joint pdf of $V_1, \cdots, V_k$ is, for $\sum_{i=1}^{k} v_i \leq 1$,

$$f(V_1 = v_1, \cdots, V_k = v_k) = (n-1)! \int_0^{1-v_1-\cdots-v_k} \cdots \int_0^{1-v_1-\cdots-v_{n-2}} \mathrm{d}v_{n-1} \cdots \mathrm{d}v_{k+1} \tag{11}$$

Note

$$\int_0^{1-v_1-\cdots-v_k} \cdots \int_0^{1-v_1-\cdots-v_{n-2}} \mathrm{d}v_{n-1} \cdots \mathrm{d}v_{k+1} \tag{12}$$

$$= \int_0^{1-v_1-\cdots-v_k} \cdots \int_0^{1-v_1-\cdots-v_{n-3}} (1 - v_1 - \cdots - v_{n-2}) \mathrm{d}v_{n-2} \cdots \mathrm{d}v_{k+1} \tag{13}$$

Set $z = 1 - v_1 - \cdots - v_{n-2}$. Then $\mathrm{d}v_{n-2} = -\mathrm{d}z$. When $v_{n-2} = 0$, $z = 1 - v_1 - \cdots - v_{n-3}$; when $v_{n-2} = 1 - v_1 - \cdots - v_{n-3}$, $z = 0$. Hence, by change of variable,

$$(14) = \int_0^{1-v_1-\cdots-v_k} \cdots \int_{1-v_1-\cdots-v_{n-3}}^{0} (-z)\mathrm{d}z \cdots \mathrm{d}v_{k+1} \tag{14}$$

$$= \int_0^{1-v_1-\cdots-v_k} \cdots \int_0^{1-v_1-\cdots-v_{n-4}} \frac{(1-v_1-\cdots-v_{n-3})^2}{2} \mathrm{d}v_{n-3} \cdots \mathrm{d}v_{k+1} \tag{15}$$

$$= \cdots = \frac{(1-v_1-\cdots-v_k)^{n-k-1}}{(n-k-1)!} \tag{16}$$

Therefore,

$$(11) = \frac{(n-1)!}{(n-k-1)!}(1-v_1-\cdots-v_k)^{n-k-1} \tag{17}$$

For constants $c_i \geq 0$ $(i = 1, \cdots, k)$ with $\sum_{i=1}^{k} c_i \leq 1$, we have

$$\mathsf{Pr}(V_1 > c_1, \cdots, V_k > c_k) \tag{18}$$

$$= \int_{c_1}^{1} \int_{c_2}^{1-v_1} \cdots \int_{c_k}^{1-v_1-\cdots-v_{k-1}} f(v_1, \cdots, v_k)\mathrm{d}v_k \cdots \mathrm{d}v_1 \tag{19}$$

$$= (1-c_1-\cdots-c_k)^{n-1} \tag{20}$$

Let $V_{(n)}$ denote the longest piece of rod, then

$$\mathsf{Pr}(V_{(n)} > x) = \mathsf{Pr}(V_1 > x \text{ or } V_2 > x \text{ or } \cdots \text{ or } V_n > x) \tag{21}$$

According to the principle of inclusion/exclusion, namely

$$\mathsf{Pr}\left(\bigcup_{i=1}^{n} \mathrm{A}_i\right) = \sum_{i=1}^{n} \mathsf{Pr}(\mathrm{A}_i) - \sum_{i<j} \mathsf{Pr}(\mathrm{A}_i \cap \mathrm{A}_j) + \sum_{i<j<k} \mathsf{Pr}(\mathrm{A}_i \cap \mathrm{A}_j \cap \mathrm{A}_k) - \cdots + (-1)^{n-1} \mathsf{Pr}\left(\bigcap_{i=1}^{n} \mathrm{A}_i\right) \tag{22}$$

we then have the distribution of $V_{(n)}$ as follows:

$$\mathsf{Pr}(V_{(n)} > x) = n(1-x)^{n-1} - \binom{n}{2}(1-2x)^{n-1} + \cdots + (-1)^{k-1}\binom{n}{k}(1-kx)^{n-1} + \cdots \tag{23}$$

where $0 < x < 1$ and the sum continues until $kx > 1$. ◀

1 Reference

https://books.google.com/books?id=bdhzFXg6xFkC&pg=PA133&lpg=PA133&dq=david+and+nagaraja+order+statistics+division+of+random+interval&source=bl&ots=OaK_l7ycZk&sig=BADI1rALE7hGm3-tvR6du1XKKzk&hl=en&ei=G68GTcqPEIeusAOZmdC4Bw&sa=X&oi=book_result&ct=result#v=onepage&q=david%20and%20nagaraja%20order%20statistics%20division%20of%20random%20interval&f=false

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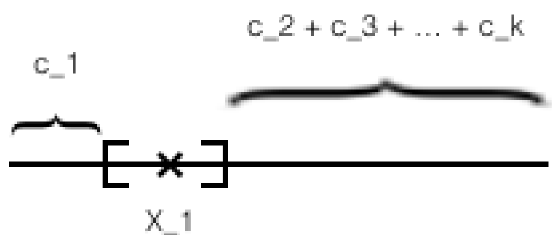

Figure 1: Range of the first cut position on the rod