Markov chain methods: WLOG $p \geq \frac{1}{2} \geq q$.

Markov chain methods is easier and more general, but more abstract.

For $\mathbb{P}^0(h_1<\infty)$, we have

$$\mathbb{P}^0(h_1<\infty) = p + q \cdot \mathbb{P}^0(h_1<\infty \mid S_1=-1)$$

Let $h_2 = \inf\{n>0 : S_n = 2\}$ ← hit 2.

$$\mathbb{P}^0(h_1<\infty \mid S_1=-1) \overset{\text{Markov property}}{=} \mathbb{P}^{S_0=-1}(h_1<\infty) \overset{\text{homogeneity}}{=} \mathbb{P}^0(h_2<\infty)$$
$$= \mathbb{P}^0(h_2<\infty \mid h_1<\infty)\cdot \mathbb{P}^0(h_1<\infty)$$
$$\overset{\text{strong Markov property}}{=} \mathbb{P}^{S_0=1}(h_2<\infty)\cdot \mathbb{P}^0(h_1<\infty)$$
$$\overset{\text{homogeneity}}{=} \mathbb{P}^0(h_1<\infty)\cdot \mathbb{P}^0(h_1<\infty) = \left(\mathbb{P}^0(h_1<\infty)\right)^2$$

Hence $\mathbb{P}^0(h_1<\infty) = p + q\cdot\left(\mathbb{P}^0(h_1<\infty)\right)^2$

So $\mathbb{P}^0(h_1<\infty) = \min\left(1, \frac{p}{q}\right) = \boxed{1}$

For $\mathbb{P}^0(h_{-1}<\infty)$ similarly we have

$$\mathbb{P}^0(h_{-1}<\infty) = p\cdot \mathbb{P}^0(h_{-1}<\infty \mid S_1=1) + q$$

Let $h_{-2} = \inf\{n>0 : S_n=-2\}$ ← hit -2

$$\mathbb{P}^0(h_{-1}<\infty \mid S_1=1) \overset{\text{M.P.}}{=} \mathbb{P}^{S_0=1}(h_{-1}<\infty) \overset{\text{homo}}{=} \mathbb{P}^0(h_{-2}<\infty)$$
$$= \mathbb{P}^0(h_{-2}<\infty \mid h_{-1}<\infty)\cdot \mathbb{P}^0(h_{-1}<\infty)$$
$$\overset{\text{S.M.P.}}{=} \mathbb{P}^{S_0=-1}(h_{-2}<\infty)\cdot \mathbb{P}^0(h_{-1}<\infty)$$
$$\overset{\text{homo}}{=} \mathbb{P}^0(h_{-1}<\infty)\cdot \mathbb{P}^0(h_{-1}<\infty) = \left(\mathbb{P}^0(h_{-1}<\infty)\right)^2$$

Hence $\mathbb{P}^0(h_{-1}<\infty) = p\cdot\left(\mathbb{P}^0(h_{-1}<\infty)\right)^2 + q$.

So $\mathbb{P}^0(h_{-1}<\infty) = \min\left(1, \frac{q}{p}\right) = \boxed{\frac{q}{p}}$

The rest is now trivial