Question 1

Solution Let us denote the number of girls as $G = 7$ and the number of boys as $B = 9$, and label the circle 1 to $G + B$ counterclockwisely. Define the indicator function $X_i = 1$i f the person at position i and the neighbor in the counterclockwise direction have different genders and $X_i = 0$ otherwise then the number of boy-girl neighbors is given by

$$X = \sum_{i=1}^{G+B} X_i \tag{1}$$

and the expectation is just given by

$$\mathbb{E}(X) = G\mathbb{E}(X_{i\in G}) + B\mathbb{E}(X_{i\in B}) \tag{2}$$

The probability of $X_{i\in G} = 1$ is the position i seats a girl and the position $i + 1$ seats a boy, and by combination

$$\mathbb{P} = \frac{G}{G+B}\frac{B}{G+B-1} \tag{3}$$

therefore

$$\mathbb{E}(X_{i\in G}) = \frac{G}{G+B}\frac{B}{G+B-1} \tag{4}$$

and similarly

$$\mathbb{E}(X_{i\in B}) = \frac{B}{G+B}\frac{G}{G+B-1} \tag{5}$$

Therefore,

$$\mathbb{E}(X) = \frac{GB}{G+B-1} \tag{6}$$

◀

Question 2

Solution Two random variables X and Y are uncorrelated when their correlation coefficient is zero:

$$\rho(X, Y) = 0 \tag{7}$$

and are independent when their joint distribution is the product of their marginal probability distribution,

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \quad \forall x,y \tag{8}$$

where f is the probability density distribution.

If X and Y are independent, then they are also uncorrelated. However, if X and Y are uncorrelated, they can still be dependent. For example, let X be uniformly distributed on the interval [-1,1], and let Y be

$$Y = \begin{cases} -X & X \leq 0 \\ X & X > 0 \end{cases} \tag{9}$$

Clearly, X, Y has zero correlation coefficient, but they are not dependent. ◀

Question 3

Solution Denote X being the number shown up for a toss. If there is only one toss, the average return is simply

$$E_1 = \frac{1}{6}\sum_{i=1}^{6} i = 3.5 \tag{10}$$

If we have two tosses, we would proceed to the second toss if the first toss is less than the average of the last toss, and otherwise stop. Specifically, we would stop when we get a toss of 4, 5, 6 on the first trial. Therefore, the average return for two tosses will be

$$E_2 = \frac{1}{6}(4+5+6) + \frac{1}{2} \times E_1 = 4.25 \tag{11}$$

Similarly, if there are three tosses, we would not proceed to the second if we get a 5 or 6 on the first trial. Subsequently, the expectation of three tosses will be

$$E_3 = \frac{1}{6}(5+6) + \frac{4}{6} \times E_2 = \frac{14}{3} \tag{12}$$

◀

Question 4

Solution ◀

Question 5

Solution Correlation is semidefinite, and therefore we have

$$A = \begin{pmatrix} 1 & r & 0 \\ r & 1 & r \\ 0 & r & 1 \end{pmatrix} \tag{13}$$

has to have all positive eigenvalues. The eigenvalue of this matrix is

$$\lambda_1 = 1 \tag{14}$$

$$\lambda_2 = 1 + \sqrt{2}r \tag{15}$$

$$\lambda_2 = 1 - \sqrt{2}r \tag{16}$$

The positive semidefinite condition is equivalent to have nonnegative eigenvalues, which set the boundary of r as

$$-\frac{\sqrt{2}}{2} \leq r \leq \frac{\sqrt{2}}{2} \tag{17}$$

◀

Question 6

Solution Let $X_i = 1$ if the drunk man takes a step left at time i and $X_i = -1$ if the drunk man takes a step right. Take the left door as $X = 0$, then the position of the drunk man at time n is

$$S_n = \sum_{i=1}^{n} X_i \tag{18}$$

with

$$S_0 = 1 \tag{19}$$

when the left door is not locked, X_i is a random walk, and

$$S_n, \quad S_n^2 - n \tag{20}$$

are obviously martingales. In addition, the time N the drunk man gets out of the door is a stop time, and a martingale at stopped time is also a martingale, i.e.

$$S_N, \quad S_N^2 - N \tag{21}$$

are still martingales. Denote the probability of getting out from the left door is p, and then the probability of getting out from the right door is $1 - p$. Therefore

$$\mathbb{E}[S_N] = p \times 0 + (1-p) \times 100 = 1 \tag{22}$$

$$\mathbb{E}[S_N^2 - N] = p \times 0 + (1-p) \times 100^2 - \mathbb{E}[N] = S_0^1 - 0 = 1 \tag{23}$$

solving this, yielding

$$p = \frac{99}{100} \tag{24}$$

$$\mathbb{E}[N] = 99 \tag{25}$$

◀

Question 7

Solution ◀

Question 8

Solution ◀

Question 9

Solution A simple expectation of this game is given by

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots = \infty \tag{26}$$

However, we really need to maximum utility rather than the expected return from an economic view because does not feel too much different to have another \$1 billion when they already have \$1 billion. In other words, the sum has to have a cutoff in some sense, and let this to be \$1 billion, or roughly 2^{30}, we have

$$2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots + 2^{30} \times \left(\frac{1}{2^{30}} + \frac{1}{2^{31}} + \cdots \right) = 31 \tag{27}$$

which is not infinite any more. ◀