Solution

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1) What is the probability that three points on a circle will be on a semi-circle?

   Solution: Denote the acute angle between the first two points as θ which is uniformly distributed on $[0, \pi]$. Given the θ, P(the third point is within a semicircle which includes the first two points) $= \frac{2\pi-\theta}{2\pi}$. So
$$P = \int_0^{\pi} \frac{2\pi-\theta}{2\pi}\frac{1}{\pi}d\theta = \frac{3}{4}.$$

2) An ant walks randomly on the edges of a cube. It starts from a vertex, and each step it has equal probability to choose one of the three edges and walk to the other vertex of this edge. What is the expectation of the number of steps for the ant to walk from one vertex to the opposite vertex.

   Solution: Say the ant locates at $(0,0,0)$. Let E_i denote the expectation of steps needed to reach $(1,1,1)$, if the ant starts from a vertex with distance i to the $(1,1,1)$. Use condition expectation, we can get the following,
$$E_3 = 1 + E_2,$$
$$E_2 = \frac{1}{3}(1+E_3) + \frac{2}{3}(1+E_1),$$
$$E_1 = \frac{1}{3} + \frac{2}{3}(1+E_2).$$
   Then we can get $E_3 = 10$.

3) From a deck of 52 cards, you can pick one card each time without replacement. If the card color is black, you win 1. If the card color is red, you lose 1. You can stop the game whenever you want. Questions: Will you play the game? If you want, how much would you pay to play this game?

   Solution: Yes, there exists risk-free strategy, for example, whenever I win 1 dollar I quit or play to the end.

4) Given a coin with probability p of landing on heads after a flip, what is the probability that the number of heads will ever equal the number of tails assuming an in
   nite number of fips?

   Solution: Let's only consider the case $0 < p < 1$, otherwise the answer is trival. Denote state i as the number of heads minus number of tails. Let state 0 be an absorbed state. Then the answer to the problem is equal to the probability of the absorbing probability start from 0. Denote P_i as the absorbing probability start from i. Then we have
$$P_1 = (1-p) + pP_2,$$
$$P_2 = P_1^2,$$

the last equality comes from the fact that if we start from state 2, if it is absorbed by state 0, it is must first goes back to 1 then goes back to 0, if we regard 1 as the absorbing state in the first process, the probability of the first process is also P_1. We can get

$$P_1 = \frac{1-\sqrt{1-4p(1-p)}}{2p},$$

similarly we can get,

$$P_{-1} = \frac{1-\sqrt{1-4p(1-p)}}{2(1-p)},$$

since $P_0 = pP_1 + (1-p)P_{-1}$, then

$$P_0 = 1-\sqrt{1-4p(1-p)}.$$

5) You have ten light bulbs. Five have an average life of 100 hours, and the other fi

ve have a average life of 200 hours. These light bulbs have a memoryless property in that their current age (measured in how long they have already been on) has no bearing on their future life expectancy. Assuming they are all already on what is the expected number of hours before the fi

rst one burns out?

Solution: The answer is $E(Y)$, where $Y = \min_i(X_1, X_2, \cdots, X_{10})$, X_1 to X_5 are i.i.d exponential distribution with parameter $1/100$, X_6 to X_10 are i.i.d exponential distribution with parameter $2/200$. Then after computing the cdf of Y, we know Y is also exponential distribution with parameter $3/40$. Then $EY = 40/3$.

6) If a person tosses a coin once per second and he tosses 100 years,ask whether the following statement is correct or not: the probability of tossing 100 consecutive heads is less than 1 percent?

Solution:

$$\begin{aligned} & P(\text{there are 100 consecutive heads}) = P(\cup_i \text{there is such a sequence start from i}) \\ \leq & \sum_{i=1}^{N} P(\text{from i to i+99 the flips are all heads}) = N\frac{2^{N-100}}{2^N} = \frac{100*365*24*3600}{2^100} < 0.0001 \end{aligned} \quad (1)$$

7) Given a stick, if randomly cut into N pieces, what's the probability that the N pieces can form an N sided polygon?

Solution: The largest side being smaller than the sum of the other sides is necessary and sufficient for a given sides to form a polygon. Assume length of the stick is 1. Let x_i denote the i-th part length. The necessary and sufficient condition of making a polygon using x_i is, $x_i < \frac{1}{2}$ for all i.

$$P(x_i < 0.5 \text{ for all i}) = 1 - P(x_i > 0.5 \text{ for some i}) = 1 - NP(x_1 > 0.5).$$

To have $x_1 > 0.5$ all $N-1$ cuts have to be on the right half of the stick, the probability is $1/2^{N-1}$. So the answer is $1 - N/2^{N-1}$.

8) Suppose in a trading environment, to describe 20 mins prices movement, should we choose moving median or moving average? Why?

Solution: Moving median is better since it is more robust compared to the moving average, which is sensitive to the outliers.

9) What is the average number of local maxima of a permutation of 1, ..., n, over all permutations? Maxima at ends also count.

   Solution: X_i denote the values at location i. I_i is the indicator function of location i being a maxima, then $E\sum_i I_i = \sum_i EI_i$. $EI_1 = 1$ and $EI_n = 0.5$ since the maxima at ends count and $I_0 = 1$ if $X_1 > X_2$ which has probability $1/2$ due to symmetry. For any middle indicator m, it is a maxima iff X_m is the maximum in (X_{m-1}, X_m, X_{m+1}) which has probability $1/3$ due to the symmetry. So $E\sum_i I_i = 1/2 + \frac{1}{3}(n-2) + 1/2 = (n+1)/3$.

10) Give a one-line C expression to test whether a number is a power of 2.

   Solution: $x\&(x-1) == 0$.

11) Implement a smartpointer in C++.

```
#include<memory>
unique_ptr<className> smartPonterName(new className(...));
```

12) Reverse a linked list from position m to n. Do it in-place and in one-pass.

```
ListNode * reverse(ListNode * head, int m, int n){
ListNode dummy=ListNode();
dummy.next=head;
ListNode *prev=&dummy;
for(int i=1; i<m; ++i){
        prev=prev−next;  //Find the (m−1)th node and denote it as prev.
}
ListNode *p1=prev->next; *p2=p1->next->next; *p3=(p2==0)?0:p2->next;
for(int i=m; i<n;++i){
        p2->next=p1;
        p1=p2;
        p2=p3;
        p3=(p3==0)?0:p3->next;
}
prev->next->next=p2;
prev->next=p1;
return dummy->next;
}
```

13) Implement a program to find out whether there exist M days within the last N(N>=M) trading days that the average closing price of these M days is at most P. Assume we have collected the history of the closing prices of the last N trading days for a stock. Requirements: Inputs are positive integer M and N, M<= N; An array of N float elements containing the closing prices of the last N trading days; And a float P. Please design and implement the program in C, C++, Java or Python to produce the answer in most time and space efficient way.

   Solution: Assum M is less than $N/2$. Order all prices increasingly then we get $x_{(i)}$. The program returns true iff $3P$ is greater than the sum of the first M $x_{(i)}$. There are many standard algorithms to get the first M $x_{(i)}$ such as bubble sort, quick sort and so on.

14) Implement a string indexOf method that returns index of matching string.

    Solution: KMP algorithm.

15) Write a function to calculate exp(x).

    Solution: Expand the taylor series of e^x.

16) Given streaming data, design an algorithm to get approximate median of all previous data, use constant memory.

    Solution:

17) Say you have an array for which the i-th element is the price of a given stock on day i. Design an algorithm to

    find the maximum profit. You may complete as many transactions as you like (i.e. buy one and sell one share of the stock multiple times). However, you may not engage in multiple transactions at the same time (i.e. you must sell the stock before you buy again).

```
int profit(int A[], int n){
for(int i=1; i<n; ++i){
        A[i]=A[i]-A[i-1];
}
int profit=0;
for(int i=1; i<n; ++i){
        profit+=max(A[i],0);
}
return profit;
}
```