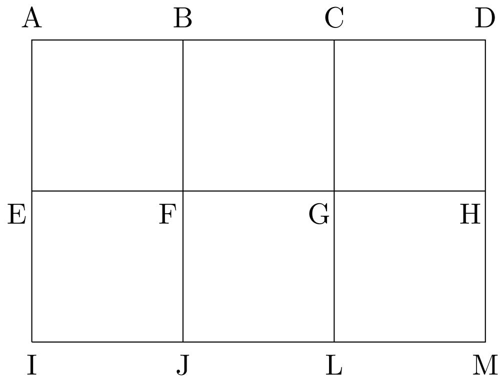
# QUIZ 2

JINGGUO LAI

## Question 2

Given a $2 \times 3$ grid with 6 blocks and 17 edges




Assuming edge length is 1, we want to find the shortest route to visit all edges. This is a variant of the famous *Seven Bridges of Königsberg Problem.* In the original paper of Euler, he proved that

**Theorem** (Euler)**.** (i) *A finite graph $G$ contains an Euler circuit if and only if $G$ is connected and contains no vertices of odd degree.*

(ii) *A finite graph $G$ contains an Euler path if and only if $G$ is connected and contains at most two vertices of odd degree.*

In this problem, we can see that nodes $B, C, E, H, J, L$ are vertices of odd degrees. So we need to connect nodes $B, C$ and nodes $J, L$ to guarentee the existence of an Euler path. Once we have these done, we know the shortest route to visit all edges should be with length $17 + 2 = 19$. A possible shortest route would be

$$E - A - B - C - D - H - G - C - B - F - E - I - J - L - G - F - J - L - M - H$$

## Question 6

(i) Let $Y$ be the number of people who select their own hats. To compute $\mathbb{E}[Y]$, we introduce i.i.d. random variables

$$Y_i = \begin{cases} 1 & \text{the i-th person select his own hat} \\ 0 & \text{otherwise} \end{cases}$$

So, we can compute

$$\mathbb{E}[Y] = \sum_{i=1}^{N} \mathbb{E}[Y_i] = N \cdot \frac{1}{N} = 1.$$

(ii) To compute $var(Y)$, again we have

$$\begin{aligned} var(Y) = var\left(\sum_{i=1}^{N} Y_i\right) &= \sum_{i=1}^{N} var(Y_i) + \sum_{i \neq j} cov(Y_i, Y_j) \\ &= N \cdot \frac{N-1}{N^2} + N(N-1) \cdot \frac{1}{N^2(N-1)} = 1. \end{aligned}$$

(iii) Let $R(N)$ be the number of rounds that are run.
Let $S(N)$ be the total number of selections made by these $N$ individuals,
Let $F(N)$ be the number of false selections made by these $N$ individuals.
To make the calculation rigorously, we need the following facts

$$\begin{aligned} \sum_{n=0}^{N} \mathbb{P}(Y = n) &= 1, \\ \mathbb{E}[Y] = \sum_{n=0}^{N} n\mathbb{P}(Y = n) &= 1, \\ \mathbb{E}[Y^2] = \sum_{n=0}^{N} n^2\mathbb{P}(Y = n) &= 2. \end{aligned}$$

(iv) Prove by induction that $\mathbb{E}[R(N)] = N$. Trivially, $\mathbb{E}[R(0)] = 0$. Assume that $\mathbb{E}[R(n)] = n$ for all $0 \leq n < N$, we have

$$\begin{aligned} \mathbb{E}[R(N)] &= \sum_{n=0}^{N} \mathbb{E}[R(N)|Y = n]\mathbb{P}(Y = n) \\ &= \sum_{n=0}^{N} \left(\mathbb{E}[R(N-n)] + 1\right)\mathbb{P}(Y = n) \\ &= 1 + \mathbb{E}[R(N)]\mathbb{P}(Y = 0) + \sum_{n=1}^{N} (N-n)\,\mathbb{P}(Y = n) \\ &= 1 + \mathbb{E}[R(N)]\mathbb{P}(Y = 0) + N\sum_{n=0}^{N} \mathbb{P}(Y = n) - N\mathbb{P}(Y = 0) - \sum_{n=0}^{N} n\mathbb{P}(Y = n) \\ &= \mathbb{E}[R(N)]\mathbb{P}(Y = 0) + N(1 - \mathbb{P}(Y = 0)) \end{aligned}$$

Hence, using the fact that $\mathbb{P}(Y = 0) > 0$, we can solve exactly $\mathbb{E}[R(N)] = N$.

(v) Prove by induction that $\mathbb{E}[S(N)] = (N+2)N/2$. Trivially, $\mathbb{E}[S(0)] = 0$. Assume that $\mathbb{E}[S(n)] = (n+2)n/2$ for all $0 \leq n < N$, we have

$$\begin{aligned}
\mathbb{E}[S(N)] &= \sum_{n=0}^{N} \mathbb{E}[S(N)|Y=n]\mathbb{P}(Y=n) = \sum_{n=0}^{N} \left(\mathbb{E}[S(N-n)] + N\right)\mathbb{P}(Y=n) \\
&= N + \mathbb{E}[S(N)]\mathbb{P}(Y=0) + \sum_{n=0}^{N} \frac{(N-n+2)(N-n)}{2}\mathbb{P}(Y=n) - \frac{(N+2)N}{2}\mathbb{P}(Y=0) \\
&= N + \mathbb{E}[S(N)]\mathbb{P}(Y=0) + \frac{(N+2)N}{2}\sum_{n=0}^{N}\mathbb{P}(Y=n) - N\sum_{n=0}^{N} n\mathbb{P}(Y=n) \\
&+ \frac{1}{2}\sum_{n=0}^{N} n^2\mathbb{P}(Y=n) - \sum_{n=0}^{N} n\mathbb{P}(Y=n) - \frac{(N+2)N}{2}\mathbb{P}(Y=0) \\
&= \mathbb{E}[S(N)]\mathbb{P}(Y=0) + \frac{(N+2)N}{2}(1-\mathbb{P}(Y=0))
\end{aligned}$$

Hence, using the fact that $\mathbb{P}(Y=0) > 0$, we can solve exactly $\mathbb{E}[R(N)] = (N+2)N/2$.

(vi) Again prove by induction that $\mathbb{E}[F(N)] = N^2/2$. Trivially, $\mathbb{E}[F(0)] = 0$. Assume that $\mathbb{E}[F(n)] = n^2/2$ for all $0 \leq n < N$, we have

$$\begin{aligned}
\mathbb{E}[F(N)] &= \sum_{n=0}^{N} \mathbb{E}[F(N)|Y=n]\mathbb{P}(Y=n) = \sum_{n=0}^{N} \left(\mathbb{E}[F(N-n)] + N - n\right)\mathbb{P}(Y=n) \\
&= \mathbb{E}[F(N)]\mathbb{P}(Y=0) + N\mathbb{P}(Y=0) + \sum_{n=0}^{N} \frac{(N-n+2)(N-n)}{2}\mathbb{P}(Y=n) \\
&- \frac{(N+2)N}{2}\mathbb{P}(Y=0) \\
&= \mathbb{E}[F(N)]\mathbb{P}(Y=0) + \frac{N^2}{2}(1-\mathbb{P}(Y=0))
\end{aligned}$$

Hence, using the fact that $\mathbb{P}(Y=0) > 0$, we can solve exactly $\mathbb{E}[F(N)] = N^2/2$. In terms of the expected number of false selections made by 1 person, we have by symmetry the answer equals $N/2$.

## Question 10

The code:

**const int *const fun(const int *const& p) const;**

means that this is a const member function named fun that takes a reference to a const pointer to a const int and returns a const pointer to a const int.

Department of Mathematics and Computer Science, Brown University, Providence, RI 02912
*E-mail address*: jglai@math.brown.edu