1. Given a fifty-seat room where each seat is numbered, people entered the room in order. The first one is drunk and he will take a seat randomly. The rest people will take their own seat as long as it is not taken. Calculate the probability that last two people can take their own seats.

Solution Let $P(50, 2)$ be the probability of the last two people will take their own seats. When the first one randomly takes his seat, there are three possible situations as follows:

1) he takes his own seat with probability $\frac{1}{50}$.

2) he takes the seats that belong to one of the last two people with probability $\frac{2}{50}$.

3) he takes the seats of the mth person with probability $\frac{1}{50}$ for each $2 \leq m \leq 48$.

In case 1, everyone takes their own seat naturally. In case 2, it is impossible that last two people still take their own seats. In case 3, everyone before mth person takes their seats naturally, whereas the mth person have to randomly choose one seat from $51 - m$ seats and the seat assigned to the first person can be viewed as his newly assigned seat. So the probability of the last two people taking their own seats is $P(51 - m, 2)$.

To sum up,

$$P(50,2) = \frac{1}{50} + \frac{1}{50} \sum_{m=2}^{48} P(51 - m, 2) \tag{1}$$

It is easily transformed to

$$Q(50,2) = \frac{1}{50} \sum_{m=2}^{48} Q(51 - m, 2), \quad \text{where } Q(50,2) \equiv P(50,2) - \frac{1}{2+1} \tag{2}$$

It is easy to see that $P(3, 2) = \frac{1}{3}$, namely, $Q(3, 2) = 0$. By induction, we have $Q(n, 2) = 0$ for any $n \geq 2$. So

$$P(50,2) = Q(50,2) + \frac{1}{2+1} = \frac{1}{3} \tag{3}$$

◀

2. There are 5 girls who need to guard a store from Mon-Fri. Calculate the number of possible arrangements given the constraint: (a) Everyone works twice a week. (b) Two girls guard the store in a single day and no two girls can work together twice.

Solution Suppose the five girls are labeled as A, B, C, D and E. First consider how many combinations of such five pairs exist. Without loss of generality, we can initially focus on A. The number of possible two pairs involving A (e.g., A-B and A-C) is $\binom{4}{2}$. Then for a girl who forms a pair with A, the number of remaining choices to be with that girl for her second time is $\binom{2}{1}$ (in the above example, since A already pairs with B and C, respectively, B can only pair with D or E in B's second pair, because if B pairs with C, D and E will have to pair with each other twice). It follows that the other girl pairing with A would have only one choice left to pair with and so do the rest of the girls (in the above example, if B pairs with D for the second time, then C has no other choices but to pair with E, and D will have to pair with E, which leads to a combination of five pairs A-B, A-C, B-D, C-E, D-E). Hence, the number of combinations of five pairs that satisfy the constrants is

$$\binom{4}{2} \times \binom{2}{1} = 12 \tag{4}$$

Now for each combination of pairs, we can assign them to any day from Mon-Fri, so the total number of such arrangements is

$$5! \times 12 = 1440 \tag{5}$$

◀

3. Follow up, what if there are 7 girls and 7 days schedule to be set up, what is the number of possible arrangement?

Solution Let the 7 girls be labeled as A, B, C, D, E, F and G. First consider how many combinations of specified seven pairs exist. This problem is equivalent to how many different graphs like Figure 1 or Figure 2 exist, because other divisions such as two and five girls cannot form polygons corresponding to pairs that satisfy the constraints.

The number of possible graphs like Figure 1 is

$$\begin{aligned}
&\text{\# of different triangles } \times \text{ \# of different squares with different relative vertex arrangements} && (6)\\
&= \binom{7}{3}\binom{3}{2} && (7)\\
&= 105 && (8)
\end{aligned}$$

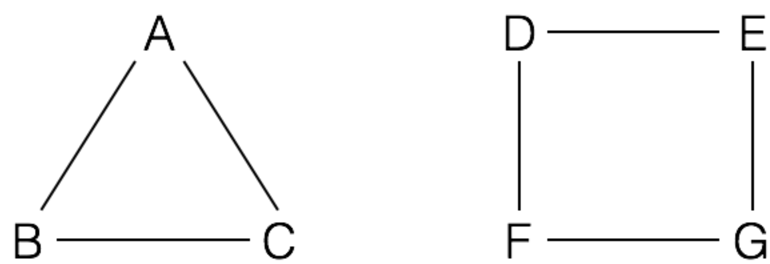

Figure 1: combinations formed by two groups

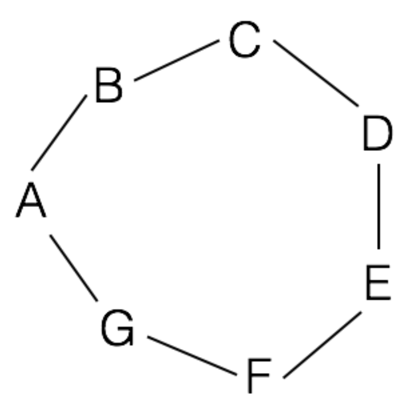

Figure 2: combinations formed by one group

The number of possible graphs like Figure 2 is

$$\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1} = 360 \tag{9}$$

Hence, the total number of such combinations is $105 + 360 = 465$. It follows that the number of possible arrangements for pairs of girls in 7 days is

$$7! \times 465 = 2343600 \tag{10}$$

◀