Now that $\mathbb{E}^0\left[e^{\lambda}\cdot\left(\frac{1}{pe^{\lambda}+qe^{-\lambda}}\right)^{h_1}\cdot \mathbb{1}_{\{h_1<\infty\}}\right]=1$ for all $\lambda>0$

Another use of Lebesgue DCT gives

$$\lim_{\lambda\downarrow 0^+}\mathbb{E}^0\left[e^{\lambda}\cdot\left(\frac{1}{pe^{\lambda}+qe^{-\lambda}}\right)^{h_1}\cdot \mathbb{1}_{\{h_1<\infty\}}\right]=\boxed{\mathbb{P}^0(h_1<\infty)=1}$$

For $\mathbb{P}^0(h_{-1}<\infty)$ the previous martingale fails because we cannot use Lebesgue DCT anymore. Instead, we consider the martingale $\{M_n\}_{n\geq 0}$

$$M_n=e^{-\lambda S_n}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^n.$$

① Since $p\geq q$ we only have $pe^{-\lambda}+qe^{\lambda}>1$ for all $\lambda>\log(p/q)$

Consider the stopping time h_{-1} we know $\{M_{n\wedge h_{-1}}\}$ is also a martingale

In particular, $\mathbb{E}^0[M_{n\wedge h_{-1}}]=\mathbb{E}^0\left[e^{-\lambda S_{n\wedge h_{-1}}}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^{n\wedge h_{-1}}\right]=1$

since $e^{-\lambda S_{n\wedge h_{-1}}}\leq e^{\lambda}<\infty$ and $pe^{-\lambda}+qe^{\lambda}>1$ for $\lambda>\log(p/q)$

Apply Lebesgue DCT, we have

$$\lim_{n\to\infty}\mathbb{E}^0\left[e^{-\lambda S_{n\wedge h_{-1}}}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^{n\wedge h_{-1}}\right]=\mathbb{E}^0\left[e^{\lambda}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^{h_{-1}}\cdot\mathbb{1}_{\{h_{-1}<\infty\}}\right]=1$$

now that $\mathbb{E}^0\left[e^{\lambda}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^{h_{-1}}\cdot\mathbb{1}_{\{h_{-1}<\infty\}}\right]=1$ for all $\lambda>\log(p/q)$

Another use of Lebesgue DCT gives

$$\lim_{\lambda\downarrow\log(p/q)^+}\mathbb{E}^0\left[e^{\lambda}\cdot\left(\frac{1}{pe^{-\lambda}+qe^{\lambda}}\right)^{h_{-1}}\cdot\mathbb{1}_{\{h_{-1}<\infty\}}\right]=p/q\cdot\mathbb{P}^0(h_{-1}<\infty)=1$$

Hence $\boxed{\mathbb{P}^0(h_{-1}<\infty)=q/p}$

Therefore, if $p\geq 1/2\geq q$ then we have

$$\mathbb{P}^0(T<\infty)=p\cdot 1+q\cdot q/p=\boxed{\frac{p^2+q^2}{p}}$$

should be p*(q/p) + q = 2q = 2(1-p).

On the other hand, if $q\geq 1/2\geq p$ the we have

$$\mathbb{P}^0(T<\infty)=\boxed{\frac{p^2+q^2}{q}}$$

should be 2p.