MTH 343 Numerical Analysis Lecture 7:

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1 Method of False Position

Bisection/Secant Method that constructs approximating lines similar to those of secant method but always brackets the root in the manner of the bisection method.

$$P_{n+1} = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$$

Example: Solve $x^2 - 2 = 0$ for x. $f(x) = x^2 - 2, a_1 = 1, b_1 = 2$.

n	a_n	b_n	P_{n+1}	$f(P_{n+1})$
1	1	2	1.3333	-0.22222
2	1.3333	2	1.4	-0.04
3	1.4	2	1.4117	-0.00692

Although the method of False Position may appear superior to the Secant Method, it generally converges more slowly.

2 Newton's Method

One of the most widely used methods.

Technique

1. Start from a single initial point (estimate) x_0 that is close to the root, then move along the tangent line to its intersection with the x-axis & take that point as the first approximation.

2. The procedure is continued until either the successive x-values are sufficiently close or the values of the function is sufficiently near zero.

$$
\begin{aligned}
\frac{f(x_0) - 0}{x_0 - x_1} &= f'(x_0) \\
\frac{f(x_0)}{f'(x_0)} &= x_0 - x_1 \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}
\end{aligned}
$$

This method is of order two, which means that on average it converges to a solution at roughly two additional decimal points of precision per iteration.

It fails when it encounters a maximum or minimum (as $f'(x_n)$ would be 0 and the tangent would be parallel to the x-axis), which means it's not as suitable for functions that oscillate.

Remarks

1. Widely used, it converges rapidly (quadratic convergence). The number of decimal places accuracy nearly doubles at each iteration.
2. Method may converge to a root different from the expected one.
3. Method may diverge if the starting value is not close enough to the root.
4. If we reach the max or min of a curve we fly off to infinity.

Theorem

Newton's method is quadratically convergent.

Proof

$$
f(x) = f(a) + f'(x)(x - x_0) + \frac{f''(x)(x - x_0)^2}{2!} + \frac{f^{(3)}(x)(x - x_0)^3}{3!} + ...
$$

$x = P \; a = P_n$

$$
\begin{aligned}
f(P) &= f(P_n) + f'(P_n)(P-P_n) + \frac{f''(\xi)}{2!}(P-P_n)^2 + ... \\
0 &= \frac{f(P_n)}{f'(P_n)} + P - P_n + \frac{f''(\xi)}{2f'(P_n)}(P-P_n)^2 + ... \\
P - P_n + \frac{f(P_n)}{f'(P_n)} &= -\frac{f''(\xi)}{2f'(P_n)}(P-P_n)^2 \\
|P - P_{n+1}| &= \frac{f'(\xi)}{2|f'(P_n)|}(P-P_n)^2 \\
|P - P_{n+1}| &\leq \frac{M}{2f'(P_n)}|(P-P_n)^2
\end{aligned}
$$