Problem Set 1 Solutions

ECON 210 Econometrics A

Evan Zuofu Liao*

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Question 1

(a) Using the definition of the expectation operator we have

$$\mathbb{E}[D] = \Pr(D = 1) \times 1 + \Pr(D = 0) \times 0 = p$$

(b) Using the definition of variance and the linearity of expectations we have

$$\begin{aligned}\operatorname{Var}[D] = \mathbb{E}\left[(D - \mathbb{E}[D])^2\right] &= \mathbb{E}\left[D^2\right] - \mathbb{E}[D]^2 \\ &= p - p^2 \\ &= p(1-p)\end{aligned}$$

Side note: you can check that $\mathbb{E}[D^2] = \mathbb{E}[D]$ for a binary random variable.

(c) Let us start first with the LHS of the equation

$$\operatorname{Cov}[D, Z] = \mathbb{E}\left[(D - \mathbb{E}[D])(Z - \mathbb{E}[Z])\right] = \mathbb{E}[DZ] - \mathbb{E}[D]\mathbb{E}[Z]$$

Using the Law of Iterated Expectations we have

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}[Z|D = 1]\Pr(D = 1) + \mathbb{E}[Z|D = 0]\Pr(D = 0) \\ \mathbb{E}[DZ] &= \mathbb{E}[DZ|D = 1]\Pr(D = 1) + \mathbb{E}[DZ|D = 0]\Pr(D = 0) \\ &= \mathbb{E}[Z|D = 1]\Pr(D = 1)\end{aligned}$$

*Comments and questions to evanliao@uchicago.edu. This solution draws from answers provided by previous TAs.

Plugging these results into the covariance equation and simplify we get

$$\begin{aligned}\operatorname{Cov}[D,Z] =& \mathbb{E}[Z|D=1]\Pr(D=1) - \mathbb{E}[Z|D=1]\Pr(D=1)^2 - \\ & \mathbb{E}[Z|D=0]\Pr(D=1)\Pr(D=0) \\ =& p(1-p)\Big(\mathbb{E}[Z|D=1] - \mathbb{E}[Z|D=0]\Big)\end{aligned}$$

From part (b) we know $\operatorname{Var}[D] = p(1-p)$. Thus,

$$\frac{\operatorname{Cov}[D,Z]}{\operatorname{Var}[D]} = \frac{p(1-p)(\mathbb{E}[Z|D=1] - \mathbb{E}[Z|D=0])}{p(1-p)} = \mathbb{E}[Z|D=1] - \mathbb{E}[Z|D=0]$$

Alternatively, we can start from the RHS of the property, and show the result without invoking the Law of Iterated Expectations.

$$\begin{aligned}\mathbb{E}[Z|D=1] - \mathbb{E}[Z|D=0] =& \Big(\Pr(Z=1|D=1)\times 1 + \Pr(Z=0|D=1)\times 0\Big) - \\ & \Big(\Pr(Z=1|D=0)\times 1 + \Pr(Z=0|D=0)\times 0\Big) \\ =& \Pr(Z=1|D=1) - \Pr(Z=1|D=0) \\ =& \frac{\Pr(Z=1,D=1)}{\Pr(D=1)} - \frac{\Pr(Z=1,D=0)}{\Pr(D=0)}\end{aligned}$$

Now note that

$$\begin{aligned}\mathbb{E}[DZ] =& 1\times\Pr(D=1,Z=1) + 0\times\Pr(D=0,Z=1) + \\ & 0\times\Pr(D=1,Z=0) + 0\times\Pr(D=0,Z=0) \\ =& \Pr(D=1,Z=1)\end{aligned}$$

Thus plugging in we find

$$\begin{aligned}\mathbb{E}[Z|D=1] - \mathbb{E}[Z|D=0] &= \frac{\mathbb{E}[DZ]}{\mathbb{E}[D]} - \frac{\mathbb{E}[(1-D)Z]}{1-\mathbb{E}[D]} \\ &= \frac{\mathbb{E}[DZ](1-\mathbb{E}[D]) - \mathbb{E}[(1-D)Z]\mathbb{E}[D]}{\mathbb{E}[D] - \mathbb{E}[D]^2} \\ &= \frac{\mathbb{E}[DZ] - \mathbb{E}[D]\mathbb{E}[Z]}{\mathbb{E}[D^2] - \mathbb{E}[D]^2} \\ &= \frac{\operatorname{Cov}[D,Z]}{\operatorname{Var}[D]}\end{aligned}$$

(d) $\mathbb{E}[D] = 0$ implies that $D = 0$ with probability 1. Then $\mathbb{E}[DZY] = 0$ holds trivially.

Now let's drop the assumption that $\mathbb{E}[D] = 0$, and instead assume that $Y \perp\!\!\!\perp (D, Z)$, then we can prove $\mathbb{E}[DZY] = 0$ as follows

$$\begin{aligned}
\mathbb{E}[DZY] &= \int \Big[\sum_{d \in \{0,1\}} \sum_{z \in \{0,1\}} (dzy) \Pr(d, z, y) \Big] dy \\
&= \int \Pr(D = 1, Z = 1, y) dy \\
&= \Pr(D = 1, Z = 1) \int y \Pr(y) dy \\
&= \mathbb{E}[DZ] \mathbb{E}[Y] \\
&= 0
\end{aligned}$$

Note that in general, $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Z$ do not necessarily imply that $X \perp\!\!\!\perp (Y, Z)$.

(e) Using definitions we have

$$\begin{aligned}
\mathbb{E}[W] &= (3 - 1) \Pr(D = 1) + (1 - 1) \Pr(D = 0) = 2p \\
\mathbb{E}[W^2] &= (3 - 1)^2 \Pr(D = 1) + (1 - 1)^2 \Pr(D = 0) = 4p \\
\mathrm{Var}[W] &= \mathbb{E}[W^2] - \mathbb{E}[W]^2 = 4p - 4p^2 = 4p(1 - p)
\end{aligned}$$

Question 2

(a) Using the definition of the expected value we have

$$\mathbb{E}[X] = \int_0^\infty x e^{-x} dx = 1$$

(b) Note first that

$$\mathbb{E}[X^2] = \int_0^\infty x^2 e^{-x} dx = 2$$

Thus

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1$$

(c) Yes. X, Y, Z all have the same distribution thus they must have the same expected value.

(d)

$$\begin{aligned}
\Pr(\min\{X,Y,Z\} \geq 1) &= \Pr(X \geq 1, Y \geq 1, Z \geq 1) \\
&= \Pr(X \geq 1)^3 \quad \text{by independence assumption} \\
&= \Big(\int_1^\infty e^{-x} dx\Big)^3 \quad \text{by definition} \\
&= (-e^{-x}|_1^\infty)^3 \\
&= e^{-3}
\end{aligned}$$

An alternative way is to note that since X, Y, Z follow an exponential distribution so $\min\{X, Y, Z\}$ also follows an exponential distribution with the density function $f(m) = 3e^{-3m}$. Then you can simply use the definition of a density function to compute

$$\Pr(\min\{X,Y,Z\} \geq 1) = \int_1^\infty 3e^{-3m} dm = e^{-3}$$

(e) We know that

$$\begin{aligned}
\Pr(\max\{X,Y,Z\} \geq \text{sum of the other two}) &= \Pr(X \text{ max}) \Pr(X \geq Y + Z | X \text{ max}) \\
&+ \Pr(Y \text{ max}) \Pr(Y \geq X + Z | Y \text{ max}) \\
&+ \Pr(Z \text{ max}) \Pr(Z \geq X + Y | Z \text{ max}) \\
&= 3 \cdot \Pr(X \text{ max}) \Pr(X \geq Y + Z | X \text{ max}) \\
&= 3 \cdot \Pr(X \geq Y + Z, X \text{ max}) \\
&= 3 \cdot \Pr(X \geq Y + Z, X \geq Y, X \geq Z) \\
&= 3 \cdot \Pr(X \geq Y + Z)
\end{aligned}$$

the second equality holds because of the symmetry of the problem, and the last line holds because X, Y, Z are non-negative. So it suffices to compute the probability that X is greater than $Y + Z$

$$\begin{aligned}
\Pr(X \geq Y + Z) &= \int_0^\infty \int_0^x \int_0^{x-y} f_X(x) f_Y(y) f_Z(z) dz dy dx \\
&= \int_0^\infty \int_0^x \int_0^{x-y} e^{-x} e^{-y} e^{-z} dz dy dx \\
&= \int_0^\infty \int_0^x (1 - e^{y-x}) e^{-y} e^{-x} dy dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-2x}(e^x - 1 - x)dx \\
&= \frac{1}{4}
\end{aligned}
$$

Therefore the desired probability is

$$\Pr(\max\{X, Y, Z\} \geq \text{sum of the other two}) = 3 \times \frac{1}{4} = \frac{3}{4}$$

Question 3

Note first that the marginal pdf of X is

$$f_X(x) = \int_2^4 f_{Y,X}(y, x)dy = \int_2^4 \frac{1}{8}(6 - x - y)dy = \frac{1}{4}(3 - x)$$

(a) Now let's compute the conditional pdf of Y given that $X = x$

$$f_{Y|X}(y|x) = \frac{f_{Y,X}(y, x)}{f_X(x)} = \frac{\frac{1}{8}(6 - x - y)}{\frac{1}{4}(3 - x)} = \frac{6 - x - y}{2(3 - x)}$$

so the expected value is simply

$$\mathbb{E}\left[Y|X = x\right] = \int_2^4 y\frac{6 - x - y}{2(3 - x)}dy = \frac{26 - 9x}{9 - 3x}$$

(b) Similarly we get

$$\mathbb{E}\left[Y^2|X = x\right] = \int_2^4 y^2\frac{6 - x - y}{2(3 - x)}dy = \frac{78 - 28x}{9 - 3x}$$

(c) Using the results above we have

$$\text{Var}\left[Y|X = x\right] = \mathbb{E}\left[Y^2|X = x\right] - \mathbb{E}\left[Y|X = x\right]^2 = \frac{26 - 18x + 3x^2}{9(x - 3)^2}$$

Question 4

(a) For the continuous case, we have

$$\mathbb{E}\left[\mathbb{E}\left[Y|X, Z\right]|X\right] = \int_Z \mathbb{E}\left[Y|X, Z\right] f_{Z|X}(z|x)dz$$

$$
\begin{aligned}
&= \int_Z \Big(\int_Y y f_{Y|X,Z}(y|x,z)dy \Big) f_{Z|X}(z|x)dz \\
&= \int_Z \int_Y y f_{Y|X,Z}(y|x,z) f_{Z|X}(z|x)dydz \\
&= \int_Z \int_Y y f_{Y,Z|X}(y,z|x)dydz \\
&= \int_Y y \Big(\int_Z f_{Y,Z|X}(y,z|x)dz \Big) dy \\
&= \int_Y y f_{Y|X}(y|x)dy \\
&= \mathbb{E}\left[Y|X\right]
\end{aligned}
$$

as desired. Note that normally we cannot change the order of integration so easily. However, we will ignore that technical detail along with the existence of the joint and marginal probability density functions. Also note that the proof in discrete time is identical replacing the integrals with summations.

(b) Again, using the definition we have

$$
\begin{aligned}
\mathbb{E}\left[Y|X\right] &= \int_Y y f_{Y|X}(y|x)dy \\
&= \int_Y y \frac{f_{Y,X}(y,x)}{f_X(x)}dy \\
&= \int_Y y \frac{f_Y(y) f_X(x)}{f_X(x)}dy \quad \text{by independence} \\
&= \int_Y y f_Y(y)dy \\
&= \mathbb{E}\left[Y\right]
\end{aligned}
$$

(c) We know that

$$
\begin{aligned}
\mathrm{Cov}\left[Y,X\right] &= \mathbb{E}\left[YX\right] - \mathbb{E}\left[Y\right]\mathbb{E}\left[X\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[YX|X\right]\right] - \mathbb{E}\left[Y\right]\mathbb{E}\left[X\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Y|X\right]X\right] - \mathbb{E}\left[Y\right]\mathbb{E}\left[X\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Y\right]X\right] - \mathbb{E}\left[Y\right]\mathbb{E}\left[X\right] \\
&= 0
\end{aligned}
$$

(d) No. Here is a canonical counter example. Let X be a standard normal and $Y = X^2$.

Now if we know X then we know Y and thus they are clearly not independent. However, if we compute

$$\begin{aligned}\operatorname{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[X^3] - 0\mathbb{E}[Y] \\ &= 0\end{aligned}$$

Question 5

(a) No. Looking at the formulas for X, Y we see that they share a common component U. Thus they cannot be independent.

(b) i

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[2U+V+1] = 2\mathbb{E}[U] + \mathbb{E}[V] + 1 = 1 \\ \mathbb{E}[Y] &= \mathbb{E}[-U+3W+3] = -\mathbb{E}[U] + 3\mathbb{E}[W] + 3 = 3\end{aligned}$$

ii

$$\begin{aligned}\operatorname{Var}[X] &= \operatorname{Var}[2U+V+1] = 4\operatorname{Var}[U] + \operatorname{Var}[V] = 5 \\ \operatorname{Var}[Y] &= \operatorname{Var}[-U+3W+3] = \operatorname{Var}[U] + 9\operatorname{Var}[W] = 10\end{aligned}$$

iii

$$\begin{aligned}\operatorname{Cov}[X,Y] &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[-2U^2 + 6UW + 6U - UV + 3VW + 3V - U + 3W + 3] - 3 \\ &= -2 + 3 - 3 \\ &= -2 \\ &\textbf{OR,} \\ \operatorname{Cov}[X,Y] &= \operatorname{Cov}[2U+V+1, -U+2W+3] \\ &= \operatorname{Cov}[2U, -U] \\ &= -2\operatorname{Var}[U] \\ &= -2 \\ \operatorname{Corr}[X,Y] &= \frac{\operatorname{Cov}[X,Y]}{\sqrt{\operatorname{Var}[X]\operatorname{Var}[Y]}}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{\sqrt{50}} \\
&= -\frac{2}{5\sqrt{2}}
\end{aligned}
$$

iv

$$
\begin{aligned}
\mathbb{E}\left[X+Y\right] &= \mathbb{E}\left[U+V+3W+4\right] = 4 \\
\mathrm{Var}\left[X+Y\right] &= \mathrm{Var}\left[U+V+3W+4\right] = 1+1+9 = 11 \\
\mathrm{Var}\left[X\right] + \mathrm{Var}\left[Y\right] + 2\mathrm{Cov}\left[X,Y\right] &= 5+10-4 = 11
\end{aligned}
$$

v

$$
\begin{aligned}
\mathrm{Cov}\left[X+Y,Y\right] &= \mathrm{Cov}\left[X,Y\right] + \mathrm{Var}\left[Y\right] = 8 \\
\mathrm{Corr}\left[X+Y,Y\right] &= \frac{\mathrm{Cov}\left[X+Y,Y\right]}{\sqrt{\mathrm{Var}\left[X+Y\right]\mathrm{Var}\left[Y\right]}} = \frac{8}{\sqrt{110}}
\end{aligned}
$$

(c) The R code used for this part will be posted online. The results of the simulation reflect the sample error inherent in any simulation. Thus the numbers we find are close but not exactly those we computed. As the sample size increases we expect our results to approach the values we calculated in part (b).

Question 6

(a)

$$
\begin{aligned}
\mathbb{E}\left[X\right] &= 0(.1+.05+.025+.025) + 1(.07+.13+.04+.06) + 2(.1+.1+.25+.05) \\
&= 1.3 \\
\mathbb{E}\left[Y\right] &= 1(.1+.07+.1) + 2(.05+.13+.1) + 3(.025+.04+.25) + 4(.025+.06+.05) \\
&= 2.315 \\
\mathrm{Var}\left[X\right] &= (0-1.3)^2(.2) + (1-1.3)^2(.3) + (2-1.3)^2(.5) = .61 \\
\mathbb{E}\left[X^2\right] &= 0(.2) + 1(.3) + 2^2(.5) = 2.3
\end{aligned}
$$

You can verify that $\mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 = .61$

(b) We can compute this by taking

$$\mathbb{E}[XY] = \sum_X \sum_Y xy \Pr(x, y) = 3.19$$

Then we know

$$\text{Cov}[X, Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 3.19 - (1.3)(2.315) = .1805$$

(c) Take $\Pr(Y = 1|X = 0)$ as an example

$$\Pr(Y = 1|X = 0) = \frac{\Pr(Y = 1, X = 0)}{\Pr(X = 0)} = \frac{.1}{.1 + .05 + .025 + .025} = .5$$

Here I don't write out all 12 conditional probabilities. With all conditional probabilities, you can compute

$$\begin{aligned}
\mathbb{E}[Y|X = 0] &= 1(.5) + 2(.25) + 3(.125) + 4(.125) = 1.875 \\
\mathbb{E}[Y|X = 1] &= 1(.233) + 2(.433) + 3(.133) + 4(.2) = 2.3 \\
\mathbb{E}[Y|X = 2] &= 1(.2) + 2(.2) + 3(.5) + 4(.1) = 2.5
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|X = x]] &= 1.875 \Pr(X = 0) + 2.298 \Pr(X = 1) + 2.5 \Pr(X = 2) \\
&= 1.875(.2) + 2.3(.3) + 2.5(.5) \\
&= 2.315 = E[Y]
\end{aligned}$$

which is what we should find via the law of iterated expectation. The outer expectation is taken with respect to X. The inner expectation is taken with respect to Y given X.