Econometrics A (Econ 210)

Virtual Midterm

Mohsen Mirtaher - Fall 2015

1. **(15 Points)** Consider the following discrete joint distribution in which X takes values on $\{-1, 0, 1\}$, Y takes values on $\{1, 2, 3\}$, and Z takes on values on $\{0, 1\}$.

$Z = 0 \downarrow$ X Y$\rightarrow$	1	2	3
-1	$\frac{1}{8}$	0	$\frac{1}{16}$
0	0	$\frac{1}{16}$	0
1	$\frac{1}{8}$	0	0

$Z = 1 \downarrow$ X Y$\rightarrow$	1	2	3
-1	0	$\frac{1}{8}$	0
0	$\frac{3}{16}$	0	0
1	$\frac{1}{4}$	$\frac{1}{16}$	0

(a) Find $\mathbb{E}[X]$ **(5 Points)**.

(b) Find $\mathrm{Cov}[X, Y]$ **(5 Points)**.

(c) Verify that $\mathbb{E}[\mathbb{E}[Y|X, Z = 1]|Z = 1] = \mathbb{E}[Y|Z = 1]$ **(5 Points)**.

2. **(25 Points)** We are going to estimate the true area of a square. To do so, we measure X_i, the length of the square's edge, n times, i.e. $i = 1..n$. Assume that $\{X_i\}_{i=1}^n$ are i.i.d. draws from a distribution with mean μ and variance σ^2. Consider the following two estimators for the unknown area

$$A_n^1 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$
$$A_n^2 = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

(a) Show that for $\forall i = 1..n$, $\mathbb{E}\left[X_i^2\right] = \mu^2 + \sigma^2$ **(5 Points)**.

(b) Is A_n^1 an unbiased estimator for the true area, i.e. μ^2? Prove your answer **(5 Points)**.

(c) Is A_n^2 an unbiased estimator for the true area, i.e. μ^2? Prove your answer **(5 Points)**.

(d) Is A_n^1 a consistent estimator for the true area, i.e. μ^2? Prove your answer **(5 Points)**.

(e) Is A_n^2 a consistent estimator for the true area, i.e. μ^2? Prove your answer **(5 Points)**.

3. **(15 Points)** Consider the model $y_i = \beta x_i + U_i$ where $\mathbb{E}\left[U_i | x_i\right] = 0$. A researcher has proposed the following estimator for β,

$$\tilde{\beta}_n = \frac{\sum_{i=1}^n x_i^3 y_i}{\sum_{i=1}^n x_i^4}$$

The researcher is asking you to derive the asymptotic distribution of $\tilde{\beta}_n$. Follow the following steps to do so.

(a) Show that $\sqrt{n}(\tilde{\beta}_n - \beta) = \dfrac{\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i^3 U_i}{\frac{1}{n}\sum_{i=1}^n x_i^4}$ **(2.5 Points)**. (Hint: plug in y_i in the expression for $\tilde{\beta}_n$).

(b) Find the probability limit of $\dfrac{1}{n}\sum_{i=1}^{n} x_i^4$ **(2.5 Points)**.

(c) Show that $\mathbb{E}\left[x_i^3 U_i\right] = 0$ **(2.5 Points)**.

(d) Find the asymptotic distribution of $\dfrac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i^3 U_i$ **(2.5 Points)**.

(e) Put parts (a) through (d) together to conclude that $\sqrt{n}(\tilde{\beta}_n - \beta) \xrightarrow{d} N\left(0, \dfrac{\mathbb{E}\left[x_i^6 U_i^2\right]}{\left(\mathbb{E}\left[x_i^4\right]\right)^2}\right)$ **(2.5 Points)**.

(f) Consider the following hypothesis testing,

$$\begin{cases} H_0: & \beta = 0 \\ H_1: & \beta \neq 0 \end{cases}$$

Remember to run any hypothesis testing, we need two ingredients: A test statistics and a critical value (cutoff). Use the part (e) result to propose the appropriate test statistics and critical value at α significance level for the test **(2.5 Points)**.

4. **(45 Points)** Consider the following linear regression model estimated using an i.i.d sample of $n = 20$ observations. We are going to interpret the model as a linear conditional expectation

$$Y = \alpha + \beta X + U$$

Assume that $\mathrm{Var}[U|X] = \mathrm{Var}[U]$. We have observed the following data,

$$\sum_{i=1}^{n} Y_i = 20 \quad \sum_{i=1}^{n} X_i = 10 \qquad \sum_{i=1}^{n} X_i Y_i = 100$$
$$\sum_{i=1}^{n} X_i^2 = 20 \quad \sum_{i=1}^{n} Y_i^2 = 700 \qquad SSR = 170$$

(a) Interpret the coefficient β **(2.5 Points)**.

(b) Does β capture the causal effect of X on Y **(2.5 Points)**?

(c) Show that $\mathrm{Cov}[X, U] = 0$ **(5 Points)**.

(d) Derive, $\hat{\beta}$, the OLS estimate of β **(2.5 Points)**.

(e) Derive, $\hat{\alpha}$, the OLS estimate of α **(2.5 Points)**.

(f) Estimate the variance of U **(2.5 Points)**.

(g) Calculate the standard error of $\hat{\beta}$ **(2.5 Points)**.

(h) Calculate the standard error of $\hat{\alpha}$ **(5 Points)**.

(i) Estimate the conditional mean value of Y corresponding to $X = 10$ **(5 Points)**.

(j) Compute the R^2 of the regression **(5 Points)**.

(k) Is the coefficient β significant at 95% significance level **(5 Points)**?

(l) Find a 95% confidence interval for $\hat{\alpha}$ **(5 Points)**.

5. **(15 Points)** Consider the model $y_i = \beta x_i + U_i$ where $\mathbb{E}[U_i|x_i] = 0$. A researcher has proposed the following estimator for β,

$$\tilde{\beta}_n = \frac{\sum_{i=1}^{n} x_i^3 y_i}{\sum_{i=1}^{n} x_i^4}$$

The researcher is asking you to derive the asymptotic distribution of $\tilde{\beta}_n$. Follow the following steps to do so.

(a) Show that $\sqrt{n}(\tilde{\beta}_n - \beta) = \dfrac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i^3 U_i}{\frac{1}{n} \sum_{i=1}^{n} x_i^4}$ **(2.5 Points)**. (Hint: plug in y_i in the expression for $\tilde{\beta}_n$).

(b) Find the probability limit of $\frac{1}{n} \sum_{i=1}^{n} x_i^4$ **(2.5 Points)**.

(c) Show that $\mathbb{E}\left[x_i^3 U_i\right] = 0$ **(2.5 Points)**.

(d) Find the asymptotic distribution of $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i^3 U_i$ **(2.5 Points)**.

(e) Put parts (a) through (d) together to conclude that $\sqrt{n}(\tilde{\beta}_n - \beta) \xrightarrow{d} N\left(0, \dfrac{\mathbb{E}\left[x_i^6 U_i^2\right]}{(\mathbb{E}\left[x_i^4\right])^2}\right)$ **(2.5 Points)**.

(f) Consider the following hypothesis testing,

$$\begin{cases} H_0: & \beta = 0 \\ H_1: & \beta \neq 0 \end{cases}$$

Remember to run any hypothesis testing, we need two ingredients: A test statistics and a critical value (cutoff). Use the part (e) result to propose the appropriate test statistics and critical value at α significance level for the test **(2.5 Points)**.