STAT 24400
Lecture 2
1/7/16

Random Variables

(But first:)

I. HW0 solution

Three dice, chance of "3" as the sum of 1, 2, or 3 dice.

There are $6^3 = 216$ possible outcomes.

There are $5^3 = 125$ outcomes without the "3" face.

Hence there are $216 - 125 = \boxed{91}$ outcomes with the "3" face.

The other winning outcomes are:

1 1 1	happens 1 way
1 1 2	" 3 ways
1 2 2	" 3 "
1 2 4	" 6 "
1 2 5	" 6 "
1 2 6	" 6 "
	25

$9 + 25 = 116$ (out of 216)

Chance of "4":

As before, $6^3 - 5^3 = 216 - 125 = \boxed{91}$ outcomes show "4".

Other winning outcomes:

$\underbrace{2\ \ 2\ \ 2}$ happens 1 way

$1\ \underbrace{1\ \ 3}$ " 3 ways

$\underbrace{1\ \ 3}\ \ 3$ " 3 ways

$\underbrace{1\ \ 1\ \ 2}$ " 3 "

$\underbrace{1\ \ 3}\ \{2, 5, 6\}$ " $3 \cdot 6 = 18$ "

$\underbrace{2\ \ 2}\ \{1, 3, 5, 6\}$ " $4 \cdot 3 = 12$ "

Total ways to get "4": $91 + 1 + 3 + 3 + 3 + 18 + 12 = \boxed{131}$ out of 216.

Random Variables

Functions that take numerical values at each point in the sample space.

e.g. flip two coins, so

$$S = \{HH, HT, TH, TT\}$$

Let X be the number of heads, so

$$X = x = 0, 1, 2$$

A <u>Probability distribution</u> $p_X(x)$ or $p(x)$ is a list of possible values of X and their probabilities.

For the example of flipping two coins, you have:

X	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$

(if coins are fair!)

<u>Discrete</u> random variables: countable number of values

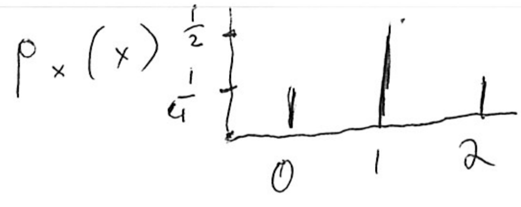
<u>Continuous</u> random variables: values form an interval

Cumulative Distribution Function (cdf)

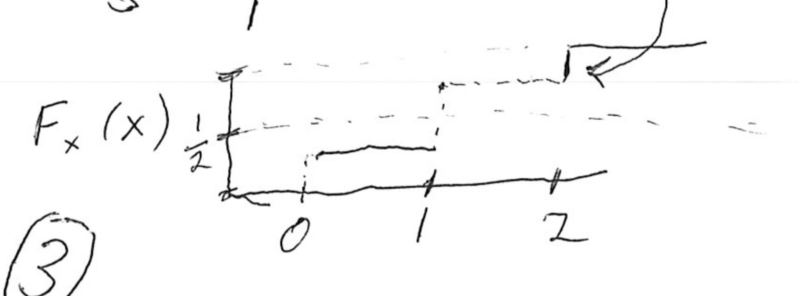
$$F_X(x) \text{ or } F(x) = P(X \leq x)$$
$$= \sum_{a \leq x} p(a)$$

Example: (same two coins)

X	0	1	2
$p_X(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$F_X(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1



size of jump $p_X(x)$



Some Important Examples

1. (very simple) Bernoulli random var: $S = \{0, 1\}$ $\quad p(1) = p \quad p(0) = 1 - p$

2. Binomial random var.

This describes the Binomial Experiment:

n independent trials

$\theta = P(\text{"success"})$ each trial... ie each trial is described by an iid (independently identically distributed) Bernoulli random var

X = number of successes

Example: Flip 3 coins, count heads.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$P(X=2) = P(HHT) + P(HTH) + P(THH)$

$P(HHT) = \theta\theta(1-\theta) = P(HTH) = P(THH)$

$\Rightarrow P(X=2) = 3\theta^2(1-\theta)$

In general:

n idep trials, $A =$ "success"
$A^c =$ "failure"

→ in each trial $P(A) = \theta$

X is # of A's

S has 2^n points. For a point in S with x A's and $n-x$ A^c's,

$$P(\underbrace{AA\ldots A}_{x}\underbrace{A^cA^c\ldots A^c}_{n-x}) = \theta^x(1-\theta)^{n-x}$$

$X \in \{0, 1, \ldots, n\}$, a binomial random var.

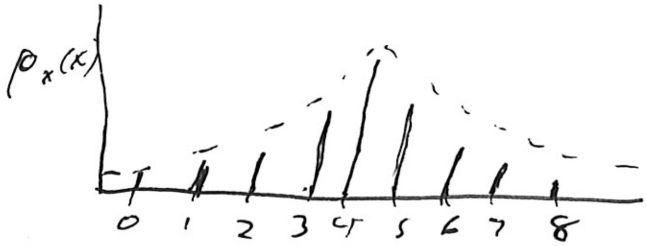
We need to count the number of ways x successes can be chosen from n trials... but that is just $\binom{n}{x}$. The probability of x success is then

$$b(x; n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$$

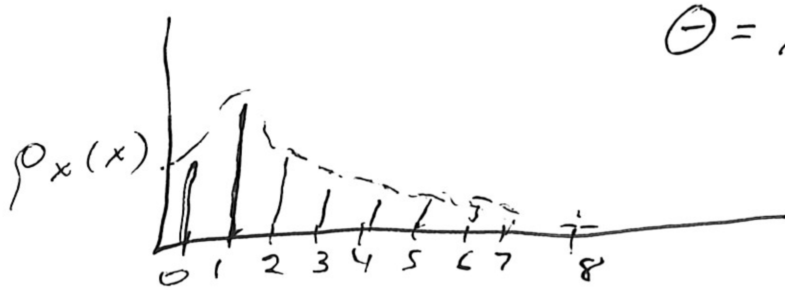
The Binomial Distribution

$\theta = \frac{1}{2}$

$n = 8$



$\theta = 1/8$



$\theta = 3/4$

$Bin(n, \theta)$ — n and θ are <u>parameters</u>. specifying them determines the distribution.

The Negative Binomial Distribution

Perform Bernoulli trials with prob of success θ until there are r successes and X failures

(Neg Binomial, continued)

Example for $r=1$ (this case has a special name: the "geometric distribution")

S	X	Prob
A	0	θ
A^cA	1	$\theta(1-\theta)$
A^cA^cA	2	$\theta(1-\theta)^2$
$A^cA^cA^cA$	3	$\theta(1-\theta)^3$
$\vdots$	$\vdots$	$\vdots$
	k	$\theta(1-\theta)^k$

Negative Binomial, in general:

probability of r successes: θ^r

$\underbrace{\text{" of } k \text{ failures before the } r^{th} \text{ success}: (1-\theta)^k}_{\text{each such outcome}}$

To find the prob that $X=k$, we

① must count these outcomes:

$$\underbrace{A \ldots A}_{\substack{r-1 \ A\text{'s} \\ k \ A^c\text{'s}}} A$$

② multiply by $(1-\theta)^k \theta^r$

Negative Binomial, continued

We are counting outcomes with r successes and k failures before before the r^{th} success, that is

Events: $\underbrace{A \ldots A^c \ldots A A^c}_{\substack{r-1\ A\text{'s} \\ k\ A^c\text{'s}}}\ \underset{\substack{\uparrow \\ \text{the } r^{th} \\ \text{success}}}{A}$ prob: $(1-\theta)^k \theta^r$

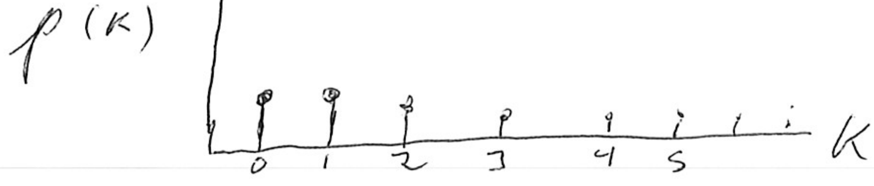
there are $(r-1)+k$ positions total in this string of A's and A^c's. We want to choose $r-1$ A's from this string.

This can be done in $\binom{r+k-1}{r-1}$ ways.

Hence: The <u>negative</u> <u>binomial</u> distribution

$$nb(k; r, \theta) = Pr(X=k) = \binom{r+k-1}{r-1}(1-\theta)^k \theta^r$$

$$k = 0, 1, 2, \ldots$$



$r=1$

$\theta = \frac{1}{2}$

Poisson Distribution

Let's take the limit of the Binomial Distribution as $n \to \infty$ while $p \to 0$, but $np = \lambda$

$$p(k) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

Let $\lambda = np$, so $p = \frac{\lambda}{n}$

$$p(k) = \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \frac{n!}{(n-k)!} \frac{1}{n^k} \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

but as $n \to \infty$,

$$\frac{\lambda}{n} \to 0, \quad \frac{n!}{(n-k)!\, n^k} \to 1,$$

and: $\left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda}$

$\left(1 - \frac{\lambda}{n}\right)^{-k} \to 1$

$\therefore$

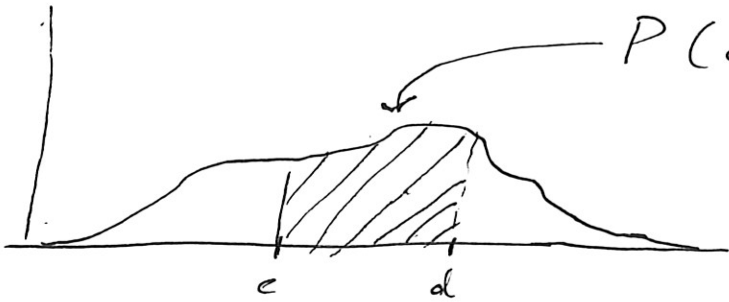
$$p(k) \longrightarrow \frac{\lambda^k e^{-\lambda}}{k!}$$

Continuous Random Variables

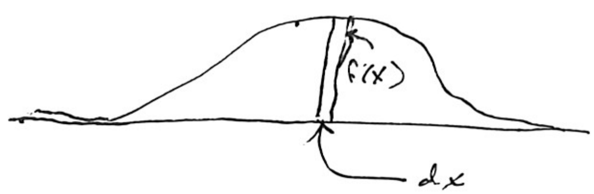
Definition: $f_X(x) = f(x)$ is the <u>probability density</u> of cont. random var X if (a) $f(x) \geq 0$

(b) $\forall c, d \quad c < d$

$$P(c < X \leq d) = \int_c^d f(x)\,dx$$



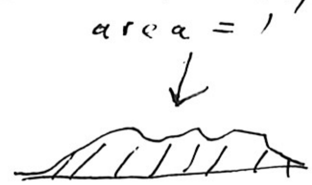
Neglecting math rigor, we can write



$$f(x)\,dx = P(x < X \leq x + dx)$$

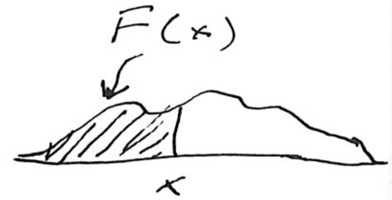
From property (b), the density function f obeys:

$$1 = P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx$$



the <u>cumulative distribution function</u>

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(s)\,ds$$



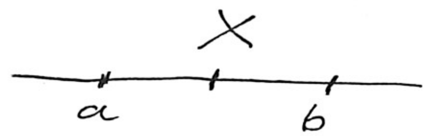
hence

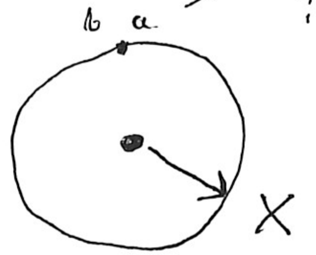
$$\frac{d}{dx} F(x) = f(x)$$

F determines f

f determines F

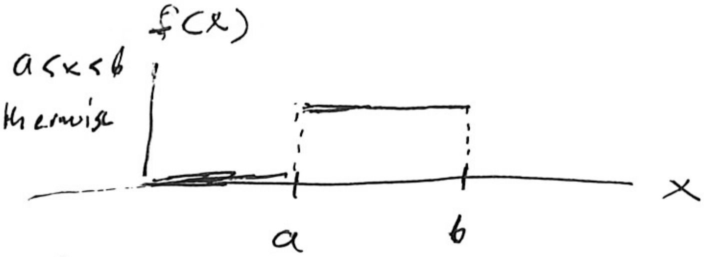
Example: A spinner (uniform distribution)



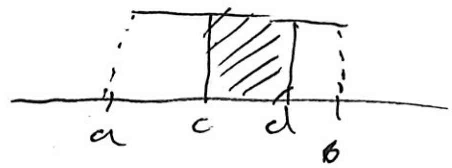


$b-a$ is one period

$$f(x) = f(x; a, b) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$



$$P(c < X < d) = \int_c^d f(x)\,dx$$
$$= \frac{d-c}{b-a}$$



Note that

$$P(c < x < d) = P(c \leq x < d)$$
$$= P(c \leq x \leq d) \quad \text{etc etc}$$

so $P(X = c) = 0$, X continuous!

Example: Time before "next event" when events occur with constant probability λ, independently

↳ could be:
failure
molecular collision
radioactive decay
etc

Recall from last time we found the Poisson distribution

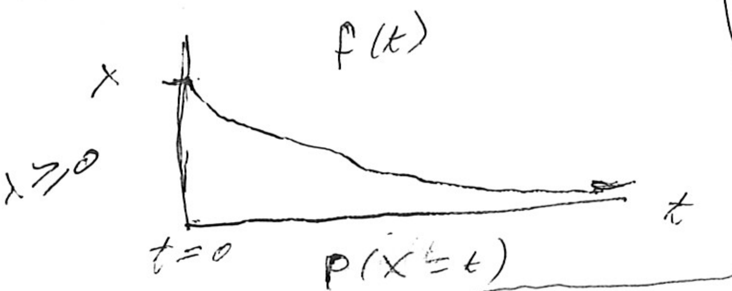
$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!}. \quad \text{Then } p(0) = e^{-\lambda}$$

(recall $0! = 1$)

Pick units so that λ is the probability of an event happening between t and $t+1$. Then λ describes a <u>Poisson process</u> with parameter λ. Let an event happen at t_0, and let X be the time to the next event. Then

density
$$P(X=t) = f(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

cumulative
$$P(X<t) = F(t) = \int_{-\infty}^{t} \lambda e^{-\lambda x}\, dx$$



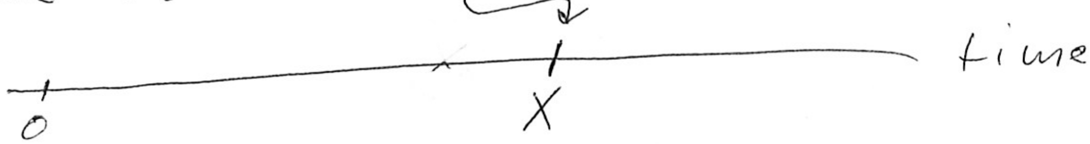
$$= \begin{cases} 1 - e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

hence time to next event in a Poisson process is governed by the <u>Exponential distribution</u>

Each event at t_0 is <u>independent</u>, so the system (and the exponential distribution) are said to be "memoryless"

What do we mean by "memoryless"?

For concreteness, let X be "time before failure" (say of a lightbulb)



$F(t)$  $F(t) = P(X \le t)$ = Prob. fail by t

$S(t)$  $S(t) = 1 - F(t)$ = Prob. alive at t ("survival function")
$= P(X > t)$

Assume no memory, so

$$P(X > t+s \mid X > s) = P(X > t)$$

$$\frac{P(X > t+s \cap X > s)}{P(X > s)} = P(X > t)$$

$$\frac{P(X > t+s)}{P(X > s)} = P(X > t)$$

$$\frac{S(t+s)}{S(t)} = S(s)$$

$$S(t+s) = S(t)\,S(s)$$

Obviously, $S(t) = e^{-\lambda t}$ has this property.