Stat 24400 Homework 7 Solution

Mar 1, 2016

Total points: 100

1. **[10pts]** For the hypothesis test $H_0 : \theta = 0$ vs $H_A : \theta = 6$, the liklihood ratios for each possible value of X is summarized in the following table.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p(x\|\theta = 0)$	0	0	.02	.03	.02	.03	.02	.02	.08	.12	.02	.22	.02	.23	.02	.15
$p(x\|\theta = 6)$	0	.03	.01	.13	.08	.12	0	0	.02	.20	.01	.17	.04	.11	0	.08
$\Lambda(\theta)$	-	0	2	$\frac{3}{13}$	.25	.25	∞	∞	4	.6	2	$\frac{22}{17}$	.5	$\frac{23}{11}$	∞	$\frac{15}{8}$

According to the Neiman Pearson Lemma, the bet test should have the rejection region of the form $\Lambda(\theta) \leq k^*$, where k^* should be the largest value that satisfies $\mathbb{P}(\Lambda(\theta) \leq k^*|\theta = 0) \leq 0.10$.In order to figure out the k^*, we need to order the $\Lambda(\theta)$ in increasing order as follows.

x	2	4	5	6	13	10	12	16	14	3	11	9	7	8	15
$p(x\|\theta = 0)$	0	.03	.02	.03	.02	.12	.22	.15	.23	.02	.02	.08	.02	.02	.02
$p(x\|\theta = 6)$	.03	.13	.08	.12	.04	.20	.17	.08	.11	.01	.01	.02	0	0	0
$\Lambda(\theta)$	0	$\frac{3}{13}$	.25	.25	.5	.6	$\frac{22}{17}$	$\frac{15}{8}$	$\frac{23}{11}$	2	2	4	∞	∞	∞

From the table, we can figure out $k^* = .5$ and hence we reject H_0 if $X = 2, 4, 5, 6, 13$. The power of the test is

$$\mathbb{P}(X = 2, 4, 5, 6, 13|\theta = 6) = 0.03 + 0.13 + 0.08 + 0.12 + 0.04 = 0.4$$

Grading Schemes: 2 pts for the computation of likelihood ratio and 4 pts for finding the decision rule and 4pts for the power calculation. Marks are assigned based on the progress made through.

2. **[15pts]**

   (a) Because the means are specified, both the null and alternative hypotheses are simple hypotheses. Therefore, we can apply the Neyman Pearson Lemma in an attempt to find the most powerful test. The lemma tells us that the ratio of the likelihoods under the null and alternative must be less than some constant k:

$$\frac{L(6)}{L(9)} = \frac{\frac{1}{\sqrt{8\pi}}\exp(-\frac{(X-6)^2}{8})}{\frac{1}{\sqrt{8\pi}}\exp(-\frac{(X-9)^2}{8}} \leq k$$

   Simplifying, we get:

$$\exp\left[-\frac{1}{8}[(X-6)^2 - (X-9)^2]\right] = \exp\left[-\frac{6X-45}{8}\right] \leq k$$

We get $X \geq \frac{45-8\log k}{6}$.

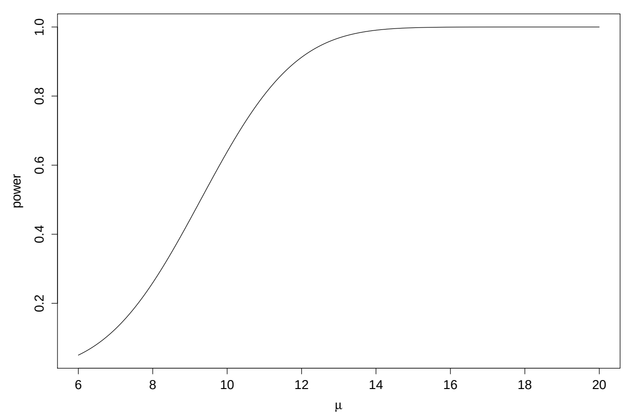
The Neiman Pearson Lemma tells us that the rejection region for the most powerful test for testing $H_0 : \mu = 6$ against $H_A : \mu = 9$, under the normal probability model, is of the form $X \geq k^*$, where k^* is selected so that the size of the critical region is $\alpha = 0.05$. Under the null hypothesis, $X \sim N(6, 4)$. Therefore, $k^* = 2z_{\frac{\alpha}{2}} + 6 = 9.29$. In other words, the most powerful test is to reject H_0 when $X \geq 9.29$.

(b) The power of this test is $\mathbb{P}(X \geq 9.29|\mu = 9) = \mathbb{P}(Z \geq \frac{9.29-9}{2}) = \mathbb{P}(Z \geq 0.145) = 0.4423$.

(c) For $H_1 : \mu = \mu_1, \sigma^2 = 4(\mu_1 > 6)$, a similar argument with the previous part indicates that the rejection region is also in the form of $X \geq k^*$. The k^* is also determined in the same way, i.e. $k^* = 9.29$. The power of this test is

$$\mathbb{P}(X \geq 9.29|\mu = \mu_1) = \mathbb{P}(Z \geq \frac{9.29 - \mu_1}{2}) = 1 - \Phi(\frac{9.29 - \mu_1}{2})$$

The graph of the power function is give in Figure 2.



3. **[10pts]**

(a) The marginal distribution for the multinomial random variables is binomial. If we think of each trial as resulting in outcome i or not, then clearly we have a sequence of n Bernoulli trials with success parameter θ_i. Random variable X_i is the number of successes in the n trials. For similar reason, $X_i + X_j \sim B(n, \theta_i + \theta_j)$. Therefore, we have

$$Var(X_i) = n\theta_i(1-\theta_i), Var(X_j) = n\theta_j(1-\theta_j), Var(X_i+X_j) = n(\theta_i+\theta_j)(1-\theta_i-\theta_j)$$

Using the fact that $Var(X_i + X_j) = Var(X_i) + Var(X_j) + 2Cov(X_i, X_j)$, we can easily get

$$Cov(X_i, X_j) = -n\theta_i\theta_j$$

$$\rho_{X_i,X_j} = \frac{Cov(X_i,X_j)}{\sqrt{Var(X_i)Var(X_j)}} = \frac{-n\theta_i\theta_j}{n\sqrt{\theta_i\theta_j(1-\theta_i)(1-\theta_j)}} = -\sqrt{\frac{\theta_i\theta_j}{(1-\theta_i)(1-\theta_j)}}$$

(b)

$$Cov(X_i+X_j, X_i-X_j) = Var(X_i) - Var(X_j) = \theta_i - \theta_j = 0$$

Grading Scheme: For part (a), 2 pts each for the marginal distributions and distribution of $X_i + X_j$, 2 pts for calculating $Cov(X_i - X_j)$, 1 pt for ρ_{X_i,X_j}, 3 pts for part (b).

4. **[10pts] Rice 9.12**
The likelihood function is $L(\theta) = \prod_{i=1}^n \theta\exp(-\theta X_i)$. Therefore, the log likelihood function is

$$\ell(\theta) = nlog(\theta) - \theta\sum_{i=1}^n X_i$$

Setting $\ell'(\theta) = 0$, we can get the MLE.

$$\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n X_i = 0 \Rightarrow \hat{\theta} = \frac{1}{\bar{X}}$$

and $\ell''(\theta) = -\frac{n}{\theta^2} < 0$. Therefore, the MLE of θ is $\bar{X}$. The likelihood ratio is

$$\begin{aligned}\Lambda &= \frac{\prod_{i=1}^n \theta_0\exp(-\theta_0 X_i)}{\prod_{i=1}^n 1/\bar{X}\exp(-1/\bar{X}X_i)} \\ &= \frac{\theta_0^n\exp(\theta_0 n\bar{X})}{(1/\bar{X})^n\exp(-n)} \\ &= (e\theta_0\bar{X}\exp(-\theta_0\bar{X}))^n\end{aligned}$$

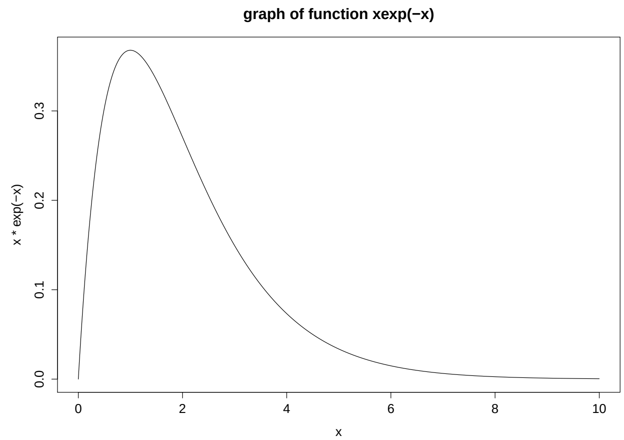
Since n, e and θ_0 are all positive, Λ is small precisely when $\bar{X}\exp(-\theta_0\bar{X})$ is small. Since the generalized likelihood ratio test rejects when $\Lambda < c'$, we see that this is equivalent to rejecting when $\bar{X}\exp(-\theta_0\bar{X}) \le c$.

Grading Scheme: 3 pts for MLE (it is acceptable to directly give the MLE since they did the MLE computation from last homework.) and 4 pts for calculating the likelihood ratio and 3 pts for the final form of decision rule. Marks are assigned based on the progress made through.

5. **[20pts] Rice 9.13**

(a) Define $f(x) = xe^{-x}$. Based on 9.12, we see that our test will be to reject H_0 when $f(\bar{X})$ is small. From Figure 1 it is clear that this is precisely when $\bar{X} < x_0(c)$ or $\bar{X} > x1(c)$.
To show this a bit more analytically, observe that $f_0(x) = \exp(-x)(1-x)$, which is positive for $x \in (0,1)$ and negative for $x > 1$. Thus $f(x)$ is strictly increasing

on $(0,1)$ and strictly decreasing on $(1,\infty)$. Further, $f(0)=0$ and $f(x)\to 0$ as $x\to 1$. This, plus continuity of f, implies that for any $c\in(0,f(1))$, there are exactly two solutions to $f(x)=c$, and that $x: f(x)\le c$ is of the desired form. (See Figure 1: there are two solutions b/c the line $y=c$ cuts the graph of the function f exactly twice if $c<1/e$. Note that $1/e$ is the maximum of f.)



(b) In the Neyman-Pearson framework, the rejection region should be such that $P_0(rejectH_0)=\alpha$. From part a, we see that choosing a rejection region for Λ is equivalent to choosing a rejection region of the form $[0,c]$ for $f(\bar{X})$. We should now choose this c to give the correct probability, and here $\alpha=.05$.

(c) On page 147, we see that for independent $X\sim\Gamma(\alpha_1,\lambda)$ and $Y\sim Gamma(\alpha_2,\lambda)$, the sum $X+Y$ is distributed as $\Gamma(\alpha_1+\alpha_2,\lambda)$. It is also true that, for a positive constant κ,$\kappa X\sim\Gamma(\alpha_1,\frac{\lambda}{\kappa})$. (Apply Proposition B on page 60 with $g(x)=\kappa x$.) Observe also that $X\sim Exp(1)$, $X\sim Gamma(1,1)$. Thus under H_0, $\sum_i X_i\sim\Gamma(n,1)$ and $X\sim\Gamma(n,n)$.
Let F be the CDF for $\Gamma(n,n)$. Then for any $c\in(0,e)$, we can numerically solve $f(x)=c$ to get $x_0(c)$ and $x_1(c)$ and

$$\begin{aligned}\alpha(c)&=\mathbb{P}_0(\bar{X}\exp(-\bar{X})\le c)\\&=\mathbb{P}_0(\bar{X}\in[0,x_0(c)]\cup\bar{X}\in[x_1(c),\infty))\\&=F(x_0(c))+1-F(x_1(c))\end{aligned}$$

To obtain a specific α, we'd now need to have the computer try a range of values for c until $\alpha(c)$ came out close to the desired value.

(d) We could repeatedly generate sets of $n=10$ independent $Exp(1)$ variables, and for each such set compute $W_i=\bar{X}_i\exp(-\bar{X}_i)$. Given B such W_i, let $i^*=\lfloor\alpha B\rfloor$. Then $W_{(i^*)}$ (basically our estimate of the α quantile of our test statistic) provides a good approximation to c, particularly for very large B.

Grading Scheme: 5 pts for each part. For (a), it is enough to provide the graph of the function $f(x)=x\exp(-x)$ to justify the answer. For (b), they need to mention

Type-I Error. For (c), full grades are assigned if they explain why X has the gamma distribution. If the answer is $Gamma(10, \frac{1}{10})$, it is still acceptable. For (d), it is enough to provide any reasonable numerical methods. Marks are assigned based on the progress made through.

6. **[15 pts] Rice 9.24**

(a) In this case, we can write out the likelihood under the null hypothesis explicitly:

$$L(1/2) = \binom{n}{X}(1/2)^n$$

Furthermore we know that the maximium likelihood estimator of p for the binomial distribution is X/n. This means we can write the generalized likelihood ratio as

$$\Lambda = \frac{\binom{n}{X}(1/2)^n}{\binom{n}{X}(X/n)^X(1-X/n)^{n-X}} = \frac{n^n(1/2)^n}{n^X(X/n)^X n^{n-X}(1-X/n)^{n-X}} = \frac{(n/2)^n}{X^X(n-X)^{n-X}}$$

(b) First, let $g(x) = x^x(n-x)^{n-x}$. Notice that this is the reciprocal of the non-constant part of the generalized likelihood ratio. Notice also that $g(x) = g(n-x)$ (try plugging in $n-x$ to see why this is true), which means that $g(x)$ is symmetric about $n/2$. This means that letting $y = x - n/2$ we have $g(n/2+y) = g(n/2-y)$. Without loss of generality, we will consider the function $g(n/2+y)$ for $y \geq 0$. For the argument we will make, an equivalent argument will hold for $g(n/2-y)$.

Call $h(y) = log(g(n/2+y))$. We want to show that h is non-decreasing. This will mean that as y gets bigger $h(y)$ gets bigger. Note that this is what we want to show since, as y gets bigger, so does $x - n/2$, and as h gets bigger, our likelihood ratio statistic gets smaller. Since a symmetric argument holds for $g(n/2-y)$(which will be non-decreasing as y gets big, assuming $y \geq 0$), we know that, if h is indeed non-decreasing for $y \geq 0$, as $|y|$ gets big, our likelihood ratio gets small which corresponds to rejecting the null hypothesis.

To show why $h(y)$ is non-decreasing, we see first that

$$log(g(n/2+y)) = (n/2+y)log(n/2+y) + (n/2-y)log(n/2-y) = h(y)$$

and therefore that

$$h^{'}(y) = log(n/2+y) - log(n/2-y) \geq 0$$

since $y \geq 0$ and log is an increasing function. This means that $h(y)$ is a non-decreasing function of y, as we wanted to show.

Note that this test statistic and rejection region make intuitive sense. Under the null hypothesis, $E(X) = n/2$, so the farther X deviates from $n/2$, the more intuitive "evidence" we have against the null, using the weak law of large numbers.

(c) To find the significance level for a test corresponding to a rejection region $|X - n/2| > k$, we want to compute $\alpha = \mathbb{P}_0(|X - n/2| > k) = \mathbb{P}_0((X - n/2) < -k \quad or \quad (X - n/2) > k) = \mathbb{P}_0(X < -k + n/2) + \mathbb{P}_0(X > k + n/2)$. The last equality holds because the events $\{X < -k + n/2\}$ and $\{X > k + n/2\}$ are disjoint. Since our null distribution is $B(n, 1/2)$, we can compute these probabilities explicitly for any values of n and k.

(d) Given that $n = 10$ and $k = 2$, our null distribution is $B(10, 1/2)$ and we want to calculate $\alpha = \mathbb{P}_0(X < 3 \quad or \quad X > 7) = \mathbb{P}_0(X \in \{0, 1, 2, 8, 9, 10\})$. We can compute this using a binomial table or R to get $\alpha = 0.0654$

Grading Scheme: 3 pts for the likelihood ratio calculation (part (a)), 5 pts for the proof in part (b)(All other reasonable proofs should be given full credits.), 2 pts for correctly expressing α in terms of the probabilities, 2 pts for describing how to compute α for different values of n and k, 3 pts for part (d).

7. **[20 pts]**

(a) The probability mass function of geometric distribution is

$$f(x|p) = (1-p)^x p$$

for $x = 0, 1, \cdots$. Note that by taking the derivative of log-likelihood function and equating it to zero, we obtain

$$0 = l^{'}(p) = \frac{d}{dp} x log(1-p) + log(p) = -\frac{x}{1-p} + \frac{1}{p}$$

which gives $\hat{p}^{MLE} = \frac{1}{x+1}$. Thus, we reject $H_0 : p = p_0$ iff

$$\Lambda := \frac{max_{p=p_0} L(p)}{max_{p\in[0,1]} L(p)} = \frac{(1-p_0)^x p_0}{(x/(x+1))^x/(x+1)}$$

for some $c > 0$.

(b) We plug in $p_0 = 0.01$ and $c = 0.1$ in the LR-test obtained from (a), which is

$$\frac{0.99^x 0.01}{(x/(x+1))^x/(x+1)} < 0.1$$

After some algebra, we get

$$\frac{(x+1)^{x+1}}{x^x} 0.99^x < 10$$

Let's take log in both side because log function is non decreasing:

$$(2) : x\log(0.99) + (x+1)\log(x+1) - x\log(x) < \log(10)$$

We will use R to find the set of x for which the inequality (2) is satisfied. The graph of the function $x\log(0.99) + (x+1)\log(x+1) - x\log(x)$ looks like following: From the graph, we see that the values of x that satisfies the inequality $x\log(0.99) + (x+1)\log(x+1) - x\log(x) < \log(10)$ must lie on both extreme side of the line. R-code for this calculation is given here:

```
x = seq(0,500,1)
> x[x*log(0.99) +(x+1)*log(x+1) - x * log(x) < log(10)]
[1] NA 1 2 3 487 488 489 490 491 492 493 494 495 496 497 498 499 500
```

Therefore, we reject $H_0 : p = 0.01$ when we observe $x \leq 3$ or $x \geq 487$.

(c) Type-I error can be calculated by

$$\alpha = \mathbb{P}(X \leq 3|p = 0.01) + P(X \geq 487|p = 0.01) \approx 0.0468$$

You can get this probability by typing the following R-command:

```
> pgeom(3,0.01,lower.tail=TRUE) + pgeom(487,0.01,lower.tail=FALSE)
[1] 0.04681667
```

or by calculating directly using the sum of geometric series. The power when $p = 0.5$ is

$$power = \mathbb{P}(X < 3|p = 0.5) + \mathbb{P}(X > 487|p = 0.5) \approx 0.9375$$

For R-command:

```
> pgeom(3,0.5,lower.tail=TRUE) + pgeom(487,0.5,lower.tail=FALSE)
[1] 0.9375
```

The power when $p = 0.001$ is

$$power = \mathbb{P}(X < 3|p = 0.001) + \mathbb{P}(X > 487|p = 0.001) \approx 0.6178$$

For R-command:

```
> pgeom(3,0.001,lower.tail=TRUE) + pgeom(487,0.001,lower.tail=FALSE)
[1] 0.617697
```

*Grading Scheme:*2 pts for MLE and 3 pts for deriving likelihood ratio in (a); 3 pts for mathematical analysis and 3 pts for the answer including computer simulation; 3 pts for type-I error and 3 pts for each power calculation. Marks are assigned based on the progress made through. If the student uses the other parameterization and gets correct answer, he/she should get full credits.