STAT 24400 Statistics Theory and Methods I

Homework 5: Due 3:00PM Thurs, February 18, 2016.

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1. Suppose that X is the number of successes in a Binomial experiment with n trials and probability of success $\theta/(1+\theta)$, where $0 \le \theta < \infty$.

   (a) Find the MLE of θ.

   (b) Use Fisher's Theorem to find the approximate distribution of the MLE when n is large.

2. If X and Y are independent, each with a Poisson distribution, show that $Z = X + Y$ is Poisson distributed.

3. In a famous example, Bortkiewicz tabulated the number of Cavalry men kicked to death by horses in the Prussian Cavalry, for 14 Corps over 20 years (1875-1894), giving $n = 280$ observations in all. The frequency tabulation was:

Number of deaths	Frequency count
0	144
1	91
2	32
3	11
4	2
More	0
Total	280

   These represent data $X_1, \ldots, X_n$, and one model that has been considered for these data is the Poisson distribution

$$P(X_i = k \mid \theta) = \frac{e^{-\theta}\theta^k}{k!}, \qquad k = 0, 1, 2. \ldots$$

   Recall that the parameter θ equals both the expectation $E(X_i)$ and the variance $\mathrm{Var}(X_i)$ for this distribution. Thus θ is the mean number of deaths a year for a single Corps. Solve (a)-(d) algebraically (symbolically), then use the data to answer (e).

   (a) For this model, find the MLE of θ, assuming the X_i 's are independent.

   (b) Find the MSE of the MLE.

   (c) From the Central Limit Theorem find the approximate distribution of the MLE when n is large.

   (d) From Fisher's Approximation, find the approximate distribution of the MLE when n is large.

   (e) Evaluate the MLE for the given data.

(f) If θ, the mean number of deaths per Corps in a year, is really 1.0, what (approximately) is the probability that the MLE would turn out to be below 0.85?

4. Suppose the data consist of a single number X, and the model is that X has the following probability density:

$$f(x \mid \theta) = (1 + x\theta)/2 \text{ for } -1 \leq x \leq 1; \; = 0 \text{ otherwise.}$$

Supposing the possible values of θ are $0 \leq \theta \leq 1$; find the maximum likelihood estimate (MLE) of θ, $\hat{\theta}$, and find its (exact) probability distribution. Is the MLE unbiased? Find its bias and MSE. [Hint: First find the MLE for a few sample values of X, such as $X = -0.5$ and $X = .5$; that should suggest to you the general solution. Drawing a graph helps! The distribution of the MLE will of course depend upon θ.]

5. Suppose we observe $X \sim \text{Unif}(0, \theta)$, $\theta > 0$.

(a) Find the mle of θ (be careful, you can't find it by differentiating the log likelihood).

(b) Find the mean squared error of the mle $\hat{\theta}$; that is, $E_\theta\{(\hat{\theta} - \theta)^2\}$.

(c) Find the constant c that makes cX an unbiased estimate of θ. Find the mean squared error of this estimate.

(d) Among all estimates of θ of the form cX, is there a choice of c that minimizes the resulting mean squared error for all θ? If yes, find this value of c. If not, explain why not.

6. Rice 8.51: The double exponential distribution is

$$f(x \mid \theta) = \frac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty$$

For an i.i.d. sample of size $n = 2m + 1$, show that the mle of θ is the median of the sample. (The observation that half of the rest of the observations are samller and half are larger.) [Hint: The function $g(x) = |x|$ is not differentiable. Draw a picture for a small value of n to try and understand what is going on.