Functions of Random Variables

Suppose we have $Y = h(X)$.

Know the distribution of X

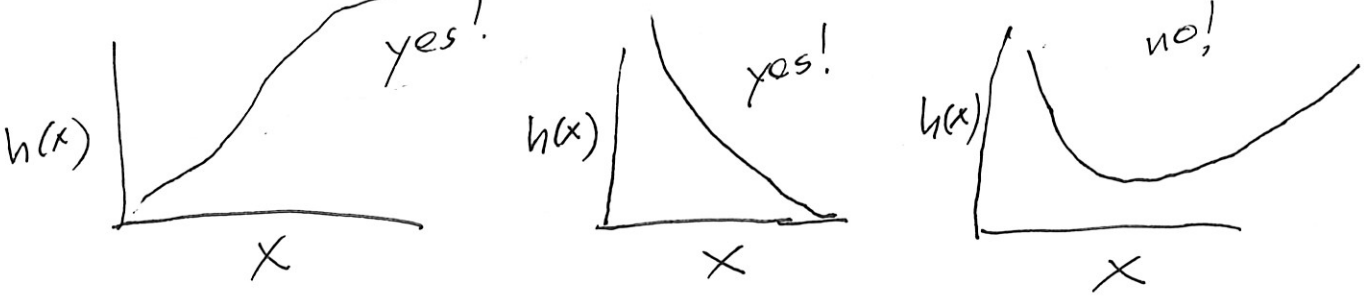
Want the distribution of Y

e.g. I X binomial, want dist of $Y = 2X = h(X)$

II X exponential, want dist of $Y = e^{-\theta x} = h(X)$, etc

$h(x)$ is a coordinate transformation, essentially. Things are most straightforward when:

$h(X)$ monotone (case I)



$Y = \log X$ yes!
$Y = e^x$ yes!

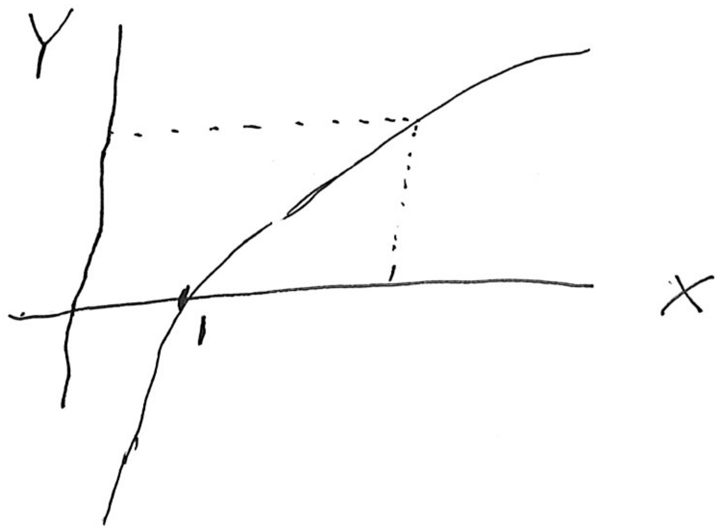
$Y = X^2$ No! $(-\infty < X < \infty)$

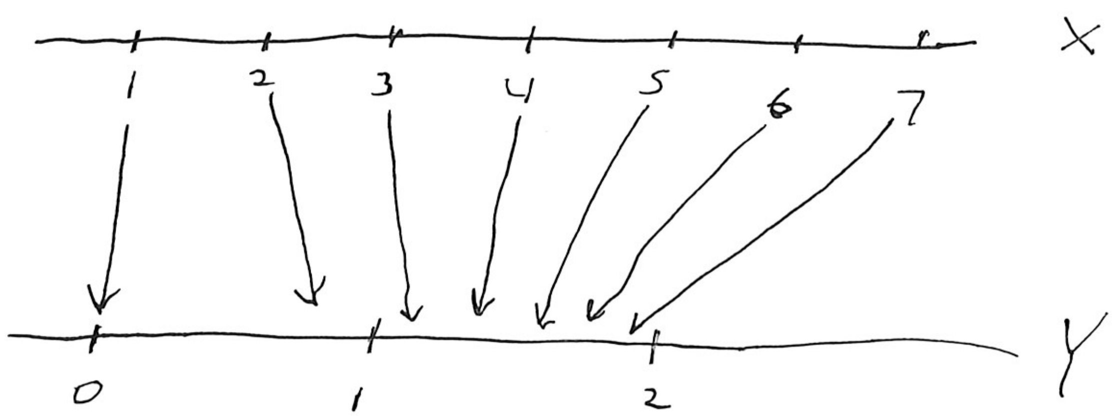
[yes if x restricted $(0 < x < \infty)$]

Basic Idea

Example: $Y = h(X) = \ln X$

$\quad$ so $g(Y) = h^{-1}(Y) = e^{Y} = X$





$\ln(1) = 0$

$\ln(2) = 0.7$

$\ln(3) = 1.1$

$\ln(4) = 1.4$

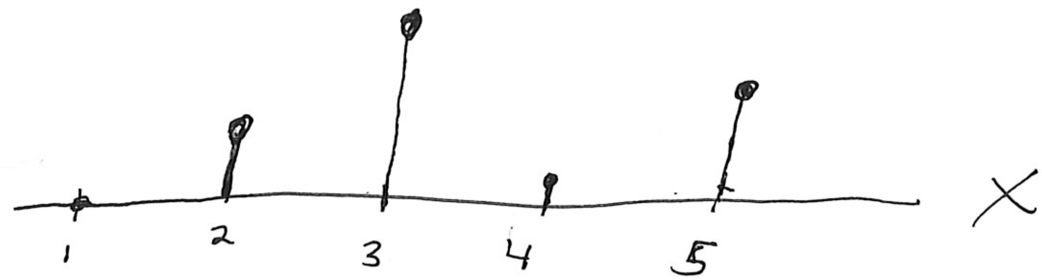
$\ln(5) = 1.6$

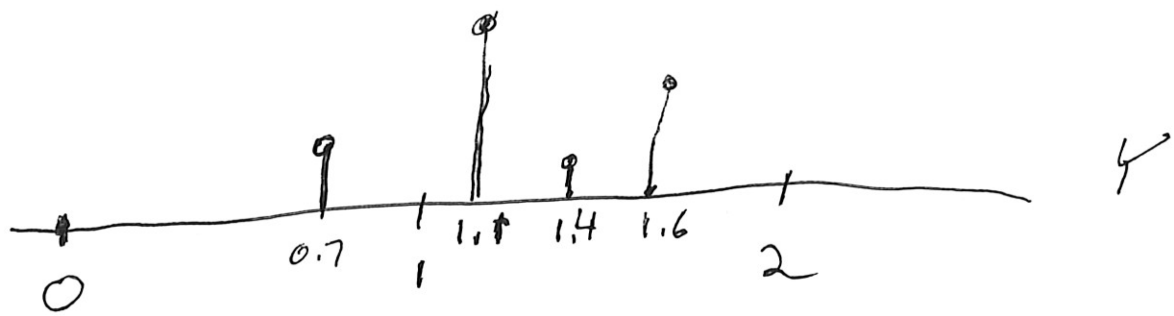
$\ln(6) = 1.8$

$\ln(7) = 1.9$

Discrete Case

$$Y = \ln X \quad , \quad X \in \{1, 2, 3, \ldots\}$$

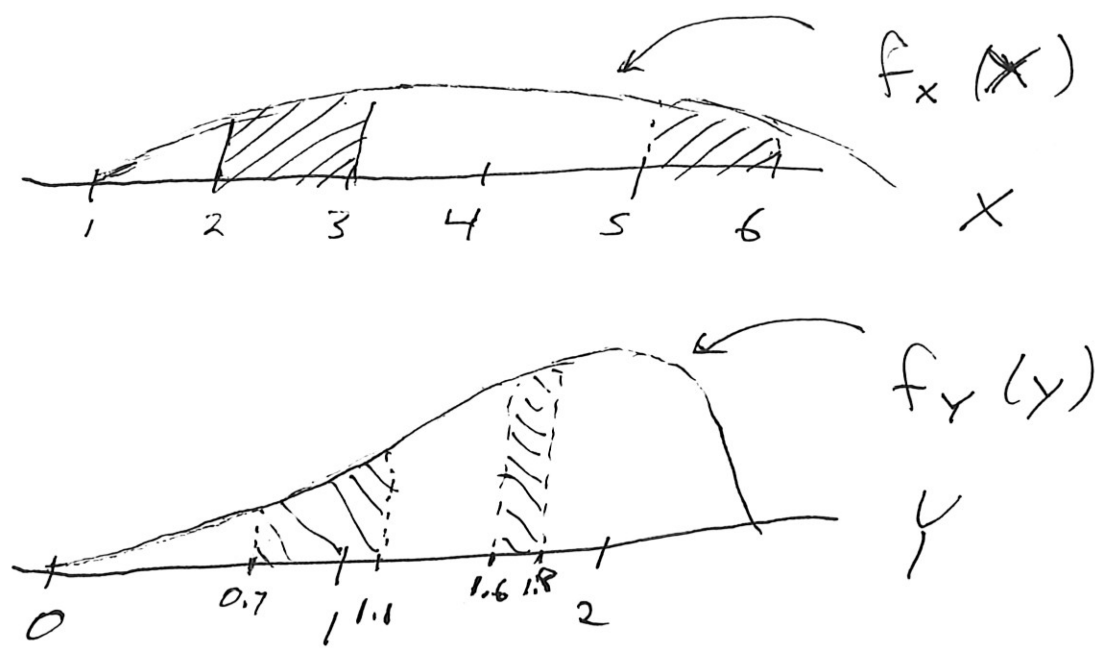




In the discrete case, the only effect of the transformation is to rearrange the spikes. The height of each spike is unchanged.

Continuous Case

$Y = \ln X$ $\quad X$ positive real



The shape of the density changes, but the probability is unchanged. Area represents probability so we must take care to preserve area in the transformation.

If $Y = h(X)$ monotone increasing or decreasing, we just need to solve for x to get

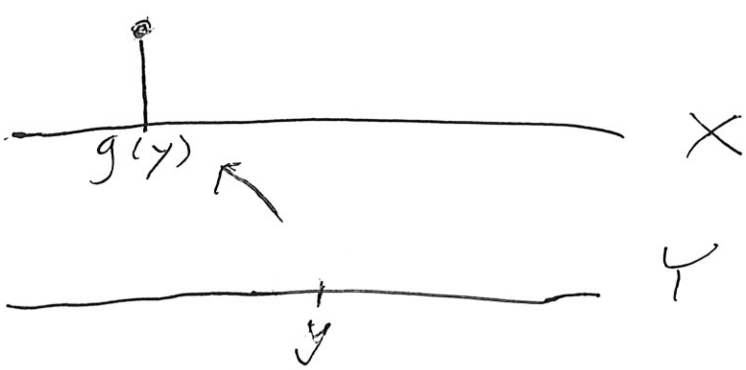
$$X = g(Y)$$

Discrete case: $P_Y(y) = P_X(g(y))$

because:

$$\begin{aligned} P_Y(y) &= P(Y = y) \\ &= P(h(X) = y) \\ &= P(X = g(y)) = P_X(g(y)) \end{aligned}$$

so for each y, to find $P_Y(y)$, find the $x = g(y)$ that led to this g and use its probability.

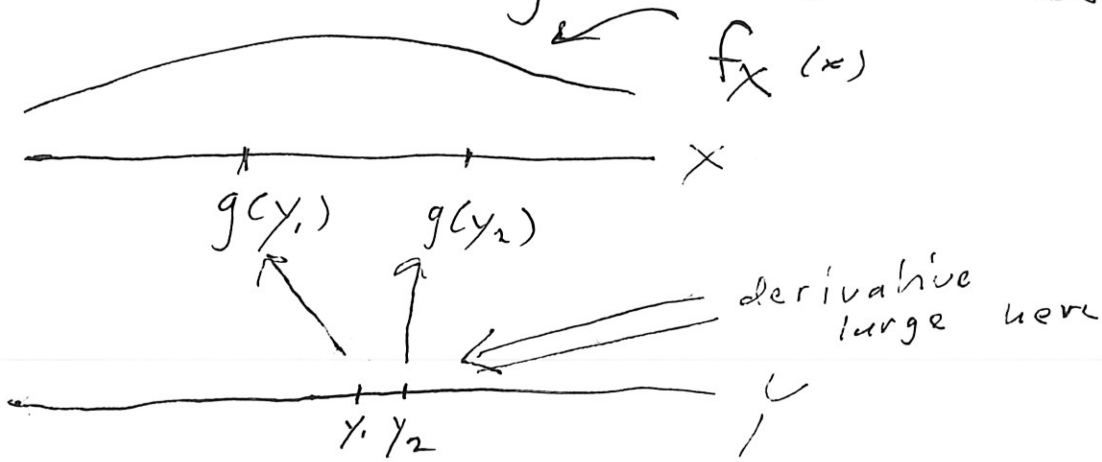


Continuous Case (h monotone)

$$f_Y(y) = f_X(g(y)) \cdot \left| \frac{dg(y)}{dy} \right|$$

rescaling factor to match areas (the "Jacobian")

For each y, to find $f_Y(y)$, "look back" to find the preimage $x = g(y)$ that led to that y, find the density $f_X(g(y))$ at that point, then rescale by $g'(y)$ to take account of how fast g deforms areas.



$$F_Y(a) = P(Y \leq a) = \int_{-\infty}^{a} f_Y(y)\,dy \quad (1)$$

We can also write

$$P(Y \leq a) = P(h(X) \leq a)$$

$$= P(X \leq g(a))$$

$$= \int_{-\infty}^{g(a)} f_X(x)\,dx \quad \left[\begin{array}{l}\text{change:} \\ x = g(y) \\ dx = |g'(y)|\,dy\end{array}\right]$$

$$= \int_{-\infty}^{a} f_X(g(y))\,|g'(y)|\,dy \quad (2)$$

Note the (1) and (2) are equal. Differentiate each side to get

$$f_Y(y) = f_X(g(y))\,|g'(y)|$$

Discrete Example:

X

$$b(x; 3, 0.5) = \binom{3}{x}(.5)^x(.5)^{3-x}$$

$$= \binom{3}{x}(.5)^3 = \frac{\binom{3}{x}}{8}$$

$$\left(= \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right) \qquad x \in \{0, 1, 2, 3\}$$

Now let's transform by

$$Y = X^2 \quad \left(\text{monotone} - \underline{\underline{X \geq 0}}\right)$$

$$h(x) = X^2 \qquad g(Y) = +\sqrt{Y}$$

$$p_Y(y) = p_X(g(y)) = b(g(y); 3, 0.5)$$

$$= \begin{cases} \frac{1}{8} & y = 0 \\ \frac{3}{8} & y = 1 \\ \frac{3}{8} & y = 4 \\ \frac{1}{8} & y = 9 \\ 0 & \text{all other } y \end{cases}$$

Continuous Examples:

I. Exponential dist, $\theta > 0$

$$X: f_X(x) = \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Suppose X is the time to failure, Y the cost of replacing the part,

$$Y = \frac{1}{1+X}$$

$$h(x) = \frac{1}{1+x}, \quad 1 + x = \frac{1}{y}, \quad x = \frac{1}{y} - 1$$

so:

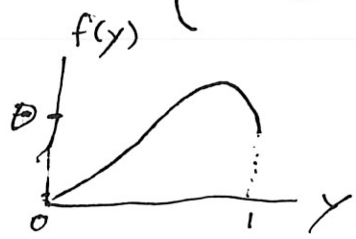
$$g(y) = y^{-1} - 1$$
$$g'(y) = -y^{-2}$$
$$|g'(y)| = y^{-2}$$

Hence $f_Y(y) = f_X\left(\frac{1}{y} - 1\right)\frac{1}{y^2}$

$$= \begin{cases} \theta e^{-\theta\left(\frac{1}{y} - 1\right)}\left(\frac{1}{y^2}\right) & (*) \\ 0 & (**) \end{cases}$$

$*$: when $x>0$ or $0 < y < 1$

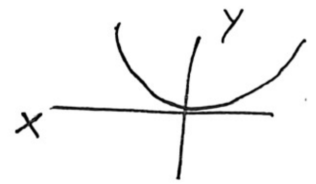
$**$: otherwise – $y \le 0$, $y \ge 1$



II Standard Normal $N(0,1)$

$$X : f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

$Y = X^2$ not monotone!! ($-\infty < x < \infty$)



$y = x^2$

But:

$h(x) = x^2$, has two monotone pieces:

monotone decreasing for $-\infty < x < 0$

monotone increasing for $0 < x < \infty$

Each range has an inverse:

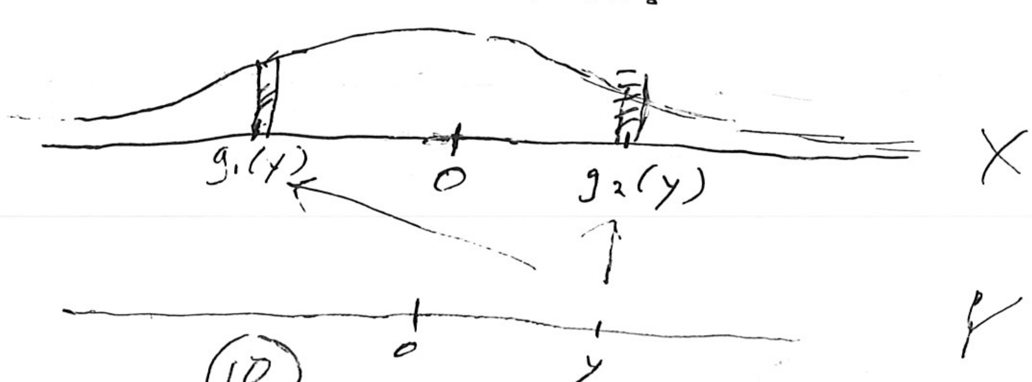
$$x = g_1(y) = -\sqrt{y} \qquad -\infty < x < 0$$

$$x = g_2(y) = +\sqrt{y} \qquad 0 < x < \infty$$

transform with:

$$f_Y(y) = f_X(g_1(y))\,|g_1'(y)| + f_X(g_2(y))\,|g_2'(y)|$$

Why? The probability at y came from two different x's, $g_1(y) = -\sqrt{y}$ and $g_2(y) = +\sqrt{y}$. Need to add both densities:



X standard normal (cont)

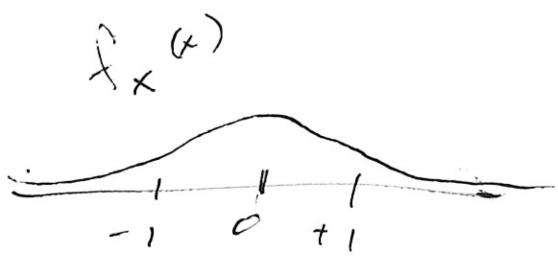
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

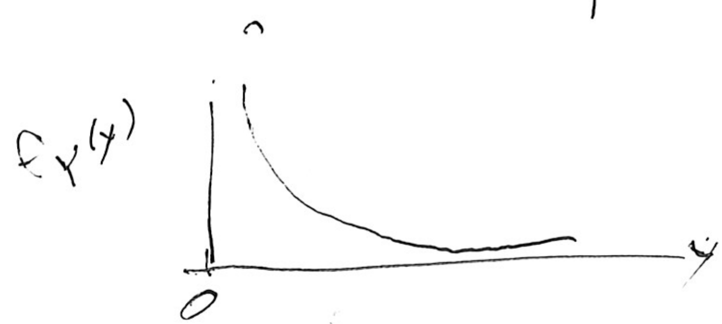
$$Y = X^2$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} \qquad y > 0$$

$$= 0 \qquad y \leq 0$$

This is the density function of the chi-square distribution with 1 degree of freedom



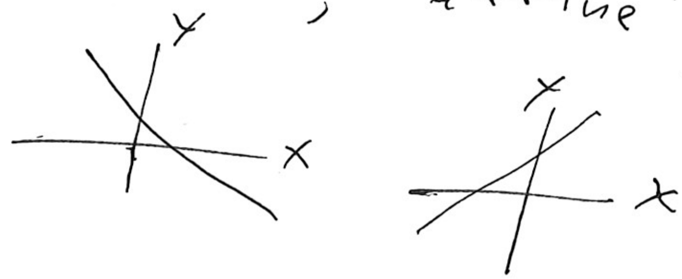


Simple VERY IMPORTANT Example

$Y = aX + b$, $a \neq 0$, b constants

("change of scale", "affine")

monotone:



$$x = \frac{y-b}{a} = g(y), \quad g'(y) = \frac{1}{a}$$

$$|g'(y)| = \frac{1}{|a|}$$

<u>Continuous case</u>:

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{|a|}$$

<u>Discrete case</u>:

$$p_Y(y) = p_X\left(\frac{y-b}{a}\right)$$

Example: X standard normal $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$Y = \sigma X + \mu, \quad \sigma > 0$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

The "general normal" $N(\mu, \sigma^2)$

Example: Suppose X is the time to failure of a light bulb, and we believe X has an exponential (θ) distribution with density

$$f_X(x) = \theta e^{-\theta x} \quad x \geq 0$$
$$= 0 \quad x < 0$$

When the light bulb fails, we replace it with a second one with the same characteristics. The probability the first survives beyond time t, $P(X > t) = e^{-\theta t}$. What is the probability that

$$\boxed{\text{The second bulb survives longer than the first?}}$$

That will be $Y = e^{-\theta X} = h(X)$

$$\ln(Y) = -\theta X$$

So $g(Y) = -\dfrac{\ln Y}{\theta}$, the inverse of h

$h(x) = e^{-\theta x}$ $\quad g(y) = \dfrac{-\ln y}{\theta}$

both monotone decreasing.

$g(y)$ is only defined for $y > 0$, but in fact it must be true that $0 < y \leq 1$.

$g'(y) = \dfrac{-1}{\theta} \cdot \dfrac{1}{y}$, and for $y > 0$

$$|g'(y)| = \frac{1}{\theta y}$$

$$f_Y(y) = f_X(g(y))\,|g'(y)|$$

$f_X(g(y)) = 0$ if $y \leq 0$ or $y > 1$, so

$$f_Y(y) = \begin{cases} \theta e^{-\theta\left(\frac{-\ln y}{\theta}\right)} \cdot \dfrac{1}{\theta y} & 0 < y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

but $\theta e^{-\theta\left(-\frac{\ln y}{\theta}\right)} = \theta y$, so

$$f_Y(y) = 1 \quad 0 < y < 1$$
$$= 0 \quad \text{otherwise.}$$

This is the uniform $(0, 1)$ distribution.

(The probability Integral Transform)

More generally, if X is a continuous random var, and $Y = \underbrace{F}_{\text{the cdf of } X}(X)$, then

the cdf of Y is

$$P(Y \le y) = P(F(X) \le y)$$
$$= P(F^{-1}(F(X)) \le F^{-1}(y))$$
$$= P(X \le F^{-1}(y))$$
$$= F(F^{-1}(y))$$
$$= y \qquad (0 \le y \le 1)$$

$F(y) = y$, the cdf of the uniform distribution

because

$$f_Y(y) = \frac{dF}{dy} = 1 \qquad 0 \le y \le 1$$

See Rice, pp 62-63; very useful for generating random deviates.