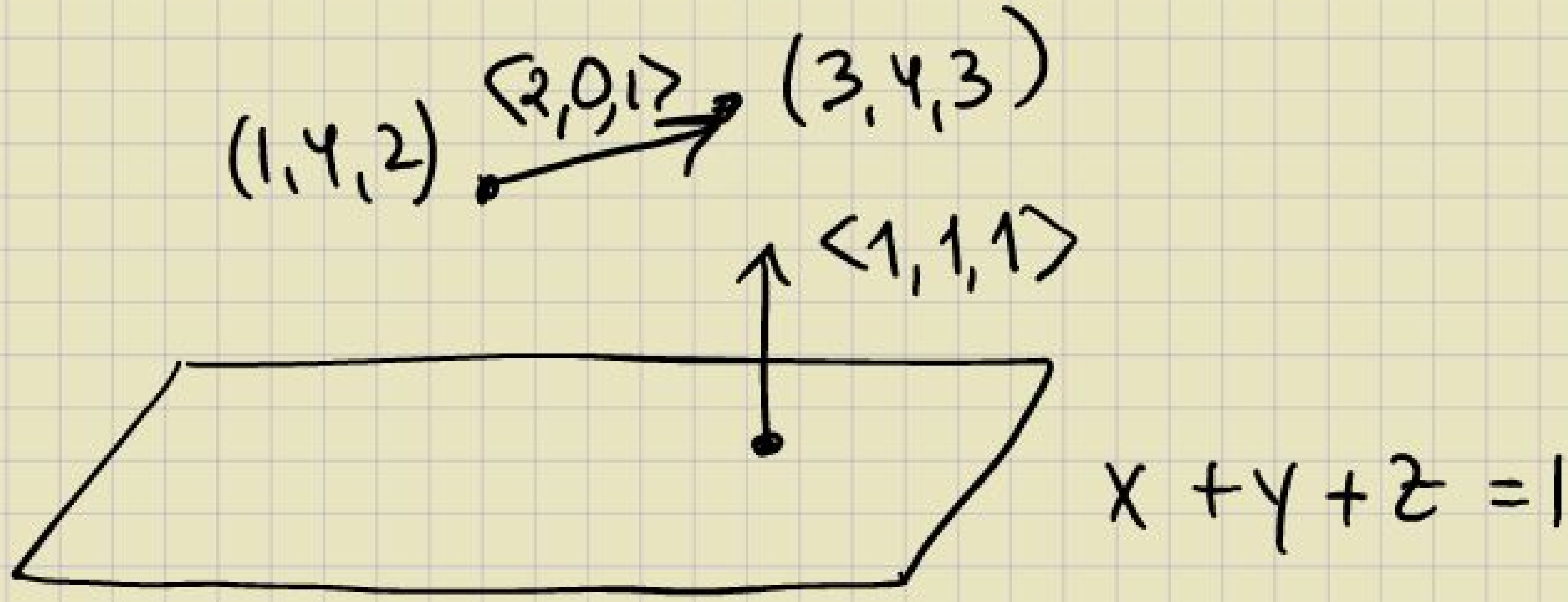
1)



Since the plane is perpendicular to the plane $x+y+z=1$, it is parallel to the vector $\langle 1,1,1\rangle$.

So our plane is parallel to two vectors, $\langle 1,1,1\rangle$ and the vector starting at $\langle 1,4,2\rangle$ and ending at $\langle 3,4,3\rangle$, namely the vector $\langle 3-1, 4-4, 3-2\rangle =$

$= \langle 2,0,1\rangle$

Therefore we can calculate a normal vector to our plane as

$\langle 1,1,1\rangle \times \langle 2,0,1\rangle =$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \hat{i} + \hat{j} - 2\hat{k}$$

$$= \langle 1,1,-2\rangle$$

Therefore we know a normal vector $\langle 1,1,-2\rangle$ to our plane and a point $(1,4,2)$ which lies in this plane.

Therefore the equation of our plane is

$$1(x-1)+1(y-4)-2(z-2)=0$$

$$x+y-2z-1=0$$

2) By Chain Rule we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= e^{x+y+z}\,\frac{1}{t} + e^{x+y+z}\left(-\frac{t}{s^2}\right) + e^{x+y+z}\,t$$

$$= e^{x+y+z}\left(\frac{1}{t} - \frac{t}{s^2} + t\right)$$

$s = t = 1$

$x = 1, y = 1, z = 1$

$$\frac{\partial f}{\partial s} = e^{1+1+1}(1 - 1 + 1) = e^3$$

3]
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$= \langle 2x - 5y, -5x + 4y \rangle$$

$$x = y = 1$$

$$\nabla f = \langle -3, -1 \rangle$$

this is the direction of the maximum rate of change

Value of max rate of change is

$$|\nabla f| = |\langle -3, -1 \rangle| = \sqrt{(-3)^2 + (-1)^2}$$

$$= \sqrt{10}$$

4] $f_x = f_y = 0$

$$\begin{cases} 3x^2 - 12y = 0 \\ -12x + 24y^2 = 0 \end{cases}$$

from 1st equation we get

$y = \frac{1}{4}x^2$

from 2nd equation we get

$x = 2y^2 = 2\frac{1}{16}x^4$

$-8x + x^4 = 0$

$x(-8 + x^3) = 0$

$x = 0$ or $x = 2$

if $x = 0$ then $y = \frac{1}{4}x^2 = 0$

so we get $(0,0)$

if $x = 2$ then $y = \frac{1}{4}x^2 = 1$

so we get $(2,1)$

Now we have to calculate the Hessian at each critical point.

The Hessian is

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$$= 6x \cdot 48y - (-12)^2 =$$

$$= 24x \cdot 12y - (12)^2$$

$$= (12)^2 (2xy - 1)$$

$(x,y) = (0,0)$ $\quad D < 0$ $\quad$ saddle point

$(x,y) = (2,1)$ $\quad D > 0$ $\quad$ local minimum

$f_{xx} = 12 > 0$

5] $f(x,y,z) = yz + xy$

$g(x,y,z) = xy = 1$

$h(x,y,z) = y^2 + z^2 = 1$

$f_x = y, \ f_y = z + x, \ f_z = y$

$g_x = y, \ g_y = x, \ g_z = 0$

$h_x = 0, \ h_y = 2y, \ h_z = 2z$

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ xy = 1 \\ y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} y = \lambda y \\ z + x = \lambda x + 2\mu y \\ y = 2\mu z \\ xy = 1 \\ y^2 + z^2 = 1 \end{cases}$$

From first equation $y(\lambda - 1) = 0$

$\Rightarrow y = 0$ or $\lambda = 1$

But y cannot be 0 because $xy = 1$

So $\lambda = 1$

$$\begin{cases} z + x = x + 2\mu y \\ y = 2\mu z \\ y^2 + z^2 = 1 \end{cases} \qquad \begin{cases} z = 2\mu y \\ y = 2\mu z \\ y^2 + z^2 = 1 \end{cases}$$

We see that if $y=0$ then $z=0$ because $z=2\mu y$; similarly if $z=0$ then $y=0$. But $y=z=0$ do not satisfy the third equation. Therefore $y\neq 0$ and $z\neq 0$. From the first two equations we get

$$2\mu = \frac{y}{z} = \frac{z}{y} = \frac{1}{2\mu} \text{ ; so } \quad 4\mu^2 = 1, \ \mu = \pm\frac{1}{2}$$

If $\mu = \frac{1}{2}$ then $z=y$ so from the third equation we get $z=\frac{1}{\sqrt{2}}, y=\frac{1}{\sqrt{2}}$ or $z=-\frac{1}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}}$

If $\mu=-\frac{1}{2}$ then $z=-y$ so from the third equation we get $z=\frac{1}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}}$ or $z=-\frac{1}{\sqrt{2}}, y=-\frac{1}{\sqrt{2}}$

Since $x=\frac{1}{y}$ we get 4 critical points

$$\left(\sqrt{2}, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(\sqrt{2}, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \left(-\sqrt{2}, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)$$

We calculate the value of the function

$$f(x,y,z) = yz + xy$$

at these points and get

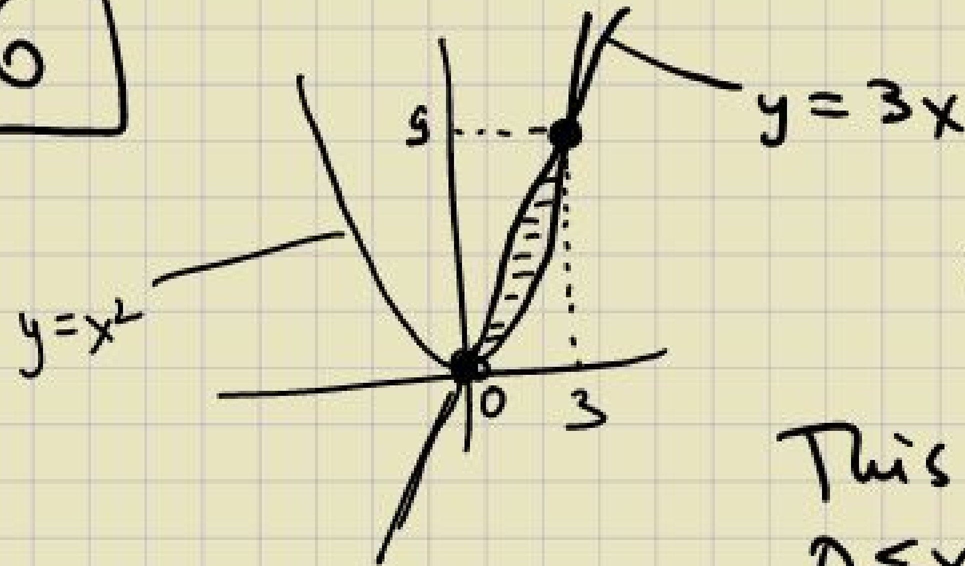
$$\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}$$

respectively.

So maximum value is $\frac{3}{2}$ at the first and the fourth of the above critical points

The minimum value is $\frac{1}{2}$ at the second and the third of the critical points above.

6



$x^2 = 3x$: $x = 0$ or $x = 3$

This is type I region

$0 \le x \le 3$, $x^2 \le y \le 3x$

$$\iint_D xy \, dA = \int_0^3 \int_{3x}^{x^2} xy \, dy\, dx$$

$$= \int_0^3 \frac{xy^2}{2}\Big]_{y=x^2}^{y=3x} dx = \int_0^3 \left(\frac{9x^3}{2} - \frac{x^5}{2}\right) dx$$

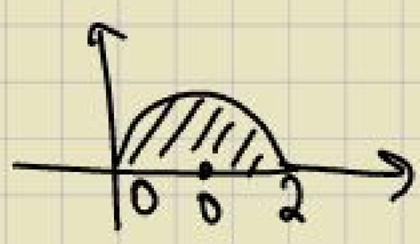
$$= \left(\frac{9x^4}{8} - \frac{x^6}{12}\right)\Big]_0^3 = \frac{9 \cdot 3^4}{8} - \frac{3^6}{12} = 3^6\left(\frac{1}{8} - \frac{1}{12}\right)$$

$$= 3^6 \cdot \frac{1}{24} = \frac{3^5}{8} = \frac{243}{8}$$

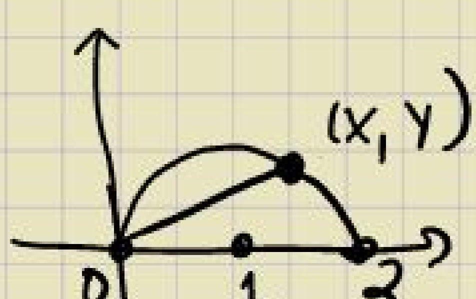
7] The region over which we are integrating is type 1 region
$0 \leq x \leq 2$, $0 \leq y \leq \sqrt{2x-x^2}$.
Since $y = \sqrt{2x-x^2}$ can be written as

$$y^2 = 2x - x^2, y \geq 0$$
$$y^2 + x^2 = 2x, y \geq 0$$
$$y^2 + x^2 - 2x + 1 = 1, y \geq 0$$
$$y^2 + (x-1)^2 = 1, y \geq 0$$

this is a region enclosed by upper semiscircle of the circle $y^2+(x-1)^2=0$ and the x-axis, namely a half-disk:



We need to write this region in polar coordinates.



$$y^2 + (x-1)^2 = 1$$
$$(r \sin\theta)^2 + (r\cos\theta - 1)^2 = 1$$
$$r^2 - 2r\cos\theta = 0 \qquad r = 2\cos\theta$$

So $r = 2\cos\theta$ is the equation of the semicircle in polar coordinates

So the polar region is

$0 \leq \theta \leq \frac{\pi}{2}$ (region is in 1st quadrant)

$0 \leq r \leq 2\cos\theta$

So we can write

$\sqrt{x^2+y^2} = r$

$$\iint_D \sqrt{x^2+y^2}\, dA = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r \, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^{2\cos\theta} r^2 \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{2} \Big]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\pi/2} 8(\cos\theta)^3 d\theta$$

$$= \int_0^{\pi/2} 8\left(1-\sin^2\theta\right) d(\sin\theta)$$

$$= 8\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Big]_0^{\pi/2}$$

$$= 8\left(1-\frac{1}{3}\right) = \frac{16}{3}$$

8) The surfaces intersect by the curve $z^2+x^2 = 8-z^2-x^2$, $y = x^2+z^2$

$x^2+z^2=2,\ y=2$

which is a circle centered at 0 of radius 2 in the plane $y=2$.

Our region is described by

$$\begin{cases} (x,z) \text{ in } D = \text{circle of radius 2 centered at 0 in the x-z plane} \\ x^2+z^2 \le y \le 8-x^2-z^2 \end{cases}$$

So we get

$$\text{vol}(E) = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} 1 \, dy \right) dA$$

$$= \iint_D (8-2x^2-2z^2) \, dA$$

We switch to polar coordinates:
D is given by $0 \le \theta \le 2\pi,\ 0 \le r \le 2$
so we get

$$\iint_D (8-2x^2-2z^2) \, dA = \int_0^{2\pi} \int_0^2 (8-2r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[4r^2 - \frac{r^4}{2} \right]_0^2 d\theta = \int_0^{2\pi} (16-8) \, d\theta = 16\pi.$$