Problem Set 2 Solutions

ECON 210 Econometrics A

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October 8, 2015

Question 1

(a) Let $\{Y_1, ..., Y_n\}$ be a random sample drawn from a distribution with mean μ and variance σ^2. $\bar{Y} = \frac{1}{n}\sum Y_i$ is the sample mean.

$$\begin{aligned}
\mathbb{E}[W_1] &= \mathbb{E}\left[\frac{n-1}{n}\bar{Y}\right] = \mathbb{E}\left[\frac{n-1}{n}\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \frac{n-1}{n}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[Y_i] = \frac{n-1}{n}\mu \\
\text{bias}(W_1) &= \frac{n-1}{n}\mu - \mu = -\frac{1}{n}\mu \\
\mathbb{E}[W_2] &= \mathbb{E}\left[\frac{1}{2}\bar{Y}\right] = \frac{1}{2}\mu \\
\text{bias}(W_2) &= -\frac{1}{2}\mu
\end{aligned}$$

So $\text{bias}(W_1) \to 0$ as $n \to \infty$ but $\text{bias}(W_2)$ remains the same. Thus as sample size goes to infinity W_1 is unbiased whereas W_2 is not.

(b) Using the Continuous Mapping Theorem we have

$$\begin{aligned}
\text{plim}(W_1) &= \text{plim}\left(\frac{n-1}{n}\right)\text{plim}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = 1 \cdot \mathbb{E}[Y_i] = \mu \\
\text{plim}(W_2) &= \frac{1}{2}\text{plim}\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{2}\mu
\end{aligned}$$

So W_1 is consistent whereas W_2 is not.

*Comments and questions to evanliao@uchicago.edu. This solution draws from answers provided by previous TAs.

(c) We know Y_i's are i.i.d. so

$$\begin{aligned}
\mathrm{Var}\left(W_1\right) &= \left(\frac{n-1}{n}\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) \\
&= \left(\frac{n-1}{n}\frac{1}{n}\right)^2 \left(\sum_{i=1}^{n} \mathrm{Var}\left(Y_i\right)\right) \\
&= \frac{(n-1)^2}{n^3}\sigma^2 \\
\mathrm{Var}\left(W_2\right) &= \frac{1}{4n^2}\mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) \\
&= \frac{1}{4n^2} n\sigma^2 \\
&= \frac{1}{4n}\sigma^2
\end{aligned}$$

(d) We can compare the MSE of the two estimators.

$$\begin{aligned}
\mathrm{MSE}(\bar{Y}) &= \mathrm{Var}\left(\bar{Y}\right) = \frac{1}{n}\sigma^2 \\
\mathrm{MSE}(W_1) &= \mathrm{Var}\left(W_1\right) + \mathrm{bias}(W_1)^2 = \frac{(n-1)^2}{n^3}\sigma^2 + \frac{1}{n^2}\mu^2
\end{aligned}$$

When μ is arbitrarily small, W_1 has a smaller MSE and thus is a better estimator. However, note that as $n \to \infty$ there is almost no difference in the two estimators.

Question 2

(a) We have

$$\mathbb{E}\left[Z\right] = \mathbb{E}\left[\mathbb{E}\left[Z|X\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{Y}{X}\middle|X\right]\right] = \mathbb{E}\left[\frac{1}{X}\mathbb{E}\left[Y|X\right]\right] = \mathbb{E}\left[\frac{1}{X}\theta X\right] = \theta$$

(b) Using the results in part (a) we have

$$\mathbb{E}\left[W\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{Y_i}{X_i}\right] = \mathbb{E}\left[Z\right] = \theta$$

(c) Computing the estimator you will find that $W \approx 0.418$

Question 3

(a) Suppose the random variable X has a symmetric distribution about zero with a support $[-a, a]$, and suppose for any odd number j the jth moment exists. Then we know

$$\begin{aligned}
\mathbb{E}\left[X^j\right] &= \int_{-a}^{a} x^j f_X(x)dx \\
&= \int_{-a}^{0} x^j f_X(x)dx + \int_{0}^{a} x^j f_X(x)dx \\
&= -\int_{a}^{0} x^j f_X(x)(-dx) + \int_{0}^{a} x^j f_X(x)dx \\
&= -\int_{0}^{a} x^j f_X(x)dx + \int_{0}^{a} x^j f_X(x)dx \\
&= 0
\end{aligned}$$

Notice that in the third equality I used u-substitution. The result also applies for $a \to \infty$.

(b) Since $X_i - \mu \sim N(0, \sigma^2)$ has a symmetric distribution around 0, so we can directly apply result in part (a) to find $\mathbb{E}\left[(X_i - \mu)^3\right] = 0$

(c) Writing out the terms we find

$$\begin{aligned}
\mathbb{E}\left[(X_i - \mu)^3\right] &= \mathbb{E}\left[(X_i^3 - 3X_i^2\mu + 3X_i\mu^2 - \mu^3)\right] \\
&= \mathbb{E}\left[X_i^3\right] - 3\mu\mathbb{E}\left[X_i^2\right] + 2\mu^3 \\
&= \mathbb{E}\left[X_i^3\right] - 3\mu(\sigma^2 + \mu^2) + 2\mu^3 \\
&= \mathbb{E}\left[X_i^3\right] - 3\mu\sigma^2 - \mu^3 = 0 \\
\mathbb{E}\left[X_i^3\right] &= 3\mu\sigma^2 + \mu^3
\end{aligned}$$

(d) Let's first check whether V_n^1 is unbiased

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^3\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[X_i^3\right] = 3\mu\sigma^2 + \mu^3 > \mu^3$$

So V_n^1 is positively biased with $\text{bias}(V_n^1) = 3\mu\sigma^2$.

For V_n^2, we have

$$\mathbb{E}\left[\Big(\frac{1}{n}\sum_{i=1}^{n}X_i\Big)^3\right] = \frac{1}{n^3}\mathbb{E}\left[\Big(\sum_{i=1}^{n}X_i\Big)^3\right] = \frac{1}{n^3}\mathbb{E}\left[\Big(\underbrace{\sum_{i=1}^{n}X_i - n\mu}_{\equiv A} + n\mu\Big)^3\right]$$

$$= \frac{1}{n^3}\mathbb{E}\left[A^3 + 3A^2 n\mu + 3A(n\mu)^2 + (n\mu)^3\right]$$

Since $A \sim N(0, n\sigma^2)$, so using the results in part (a), we know that $\mathbb{E}\left[A^3\right] = \mathbb{E}\left[A\right] = 0$. Notice also that $E[A^2] = n\sigma^2$. Then we can compute

$$\mathbb{E}\left[V_n^2\right] = \frac{1}{n^3}(3n^2\mu\sigma^2 + n^3\mu^3) = \frac{3\mu\sigma^2}{n} + \mu^3 > \mu^3$$

Thus V_n^2 is biased in the positive direction with $\text{bias}(V_n^2) = \frac{3\mu\sigma^2}{n}$.

Another way to compute this is to realize that the third moment of a normal distribution is given by $\mu^3 + 3\mu\sigma^2$. Now we know that $\overline{X}_n \sim N(\mu, \sigma^2/n)$ thus plugging in will also give us the answer.

The third way to compute this is to do a brute force polynomial expansion. Note that

$$\mathbb{E}\left[\Big(\frac{1}{n}\sum_{i=1}^{n}X_i\Big)^3\right] = \frac{1}{n^3}\Big(\sum_{i=1}^{n}\mathbb{E}\left[X_i^3\right] + \sum_{i\neq j}3\mathbb{E}\left[X_i^2X_j\right] + \sum_{i\neq j\neq k}6\mathbb{E}\left[X_iX_jX_k\right]\Big)$$

Simplify this will give us the answer.

(e) Assuming n is sufficiently large we see that V_n^1 is more biased in that $\text{bias}(V_n^1)$ is larger. As $n \to \infty$ we see that V_n^2 is unbiased since the bias term goes to 0.

(f) Using the "analog principle", we can construct the following estimator to correct the bias

$$\widetilde{V}_n^1 = \frac{1}{n}\sum_{i=1}^{n}X_i^3 - 3\Big(\frac{1}{n}\sum_{i=1}^{n}X_i\Big)\Big(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2\Big)$$

Now let's check if this estimator is unbiased

$$
\begin{aligned}
\mathbb{E}\left[\widetilde{V}_n^1\right] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i^3\right] - 3\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\left(\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2\right)\right] \\
&= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i^3\right] - 3\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\right]\mathbb{E}\left[\left(\frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2\right)\right] \\
&= \mu^3 + 3\mu\sigma^2 - 3\mu\sigma^2 \\
&= \mu^3
\end{aligned}
$$

so this is unbiased! Note that the second equality holds because the sample mean and the sample variance are actually independent. I leave it to you to prove this independence property.

Question 4

(a) Using the Weak Law of Large Numbers we have

$$
\operatorname{plim}\left(\hat{\mu}_n\right) = \operatorname{plim}\left(\frac{1}{n}\sum_{i=1}^n y_i\right) = \mathbb{E}\left[y_i\right] = \mu
$$

so $\hat{\mu}_n$ is consistent.

(b) Using the Central Limit Theorem we know

$$
\begin{aligned}
\sqrt{n}(\hat{\mu}_n - \mu) &\xrightarrow{d} \mathcal{N}(0, \sigma^2) \\
\hat{\mu}_n - \mu &\xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{n}\right) \\
\hat{\mu}_n &\xrightarrow{d} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \to \mu \text{ as } n \to \infty
\end{aligned}
$$

Thus the asymptotic distribution collapses to a single point μ as $n \to \infty$.

(c) We first show that $\tilde{\mu}_n$ is (asymptotically) unbiased.

$$
\mathbb{E}\left[\tilde{\mu}_n\right] = \sum_{i=1}^n w_i \mathbb{E}\left[y_i\right] = \mathbb{E}\left[y_i\right] = \mu
$$

We also know from part (d) that the asymptotic variance of $\tilde{\mu}_n$ is equal to 0. Thus it must be consistent.

(d) The variance of $\tilde{\mu}_n$ is given by

$$\text{Var}\left(\tilde{\mu}_n\right) = \sum_{i=1}^{n} \text{Var}\left(w_i y_i\right) = \sum_{i=1}^{n} w_i^2 \text{Var}\left(y_i\right) = \sigma^2 \frac{2(2n+1)}{3n(n+1)} \to 0 \text{ as } n \to \infty$$

(e) Code will be posted on Chalk

Question 5

(a) Given $\mu = .49$ we have

$$\Pr(y_1 < .5|\mu = .49) = \Pr(\varepsilon_1 < .01) = 60\%$$
$$\Pr(y_1 > .5|\mu = .49) = \Pr(\varepsilon_1 > .01) = 40\%$$

So Gore will be declared the winner with probability .6 and thus Bush will be declared the winner with probability .4.

(b) An obvious estimator is $\hat{\mu} = \frac{1}{2}(y_1 + y_2)$

(c) $\mathbb{E}\left[\hat{\mu}\right] = \frac{1}{2}(\mathbb{E}\left[y_1\right] + \mathbb{E}\left[y_2\right]) = \mu$

(d) $\text{Var}\left(\hat{\mu}\right) = \frac{1}{4}(\text{Var}\left(y_1\right) + \text{Var}\left(y_2\right)) = \frac{1}{2}\text{Var}\left(\varepsilon_i\right) < \text{Var}\left(\varepsilon_2\right) = \text{Var}\left(y_2\right)$