The Derivative

Recommended Reading: Chapter 3.

1. (a) Let g be a real function, and a a real number. By writing the definition of the limit and making a change of variables, explain why

$$\lim_{x\to a} g(x) = L \qquad \text{if and only if} \qquad \lim_{h\to 0} g(a+h) = L.$$

   (b) Let f be a real function. We defined the derivative of f as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

   Taking $g(x) = \dfrac{f(x) - f(a)}{x - a}$, use part (a) to show that

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

   This is sometimes given as the definition of the derivative.

2. The idea of this problem is to prove the following theorem: Suppose that f is a function defined on an interval I, and that c is a point in I. If f is differentiable at the point a, then f is continuous at the point a.

   Suppose that f is differentiable at a real number a. By Question 1,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

   (a) Explain how we know that the limit $\lim_{x\to a} f(x) - f(a)$ exists, and evaluate it.

   Hint: Write $f(x) - f(a) = \left(\dfrac{f(x) - f(a)}{x - a}\right)(x - a)$, *and use the properties of limits.*

   (b) Explain how we know that the limit $\lim_{x\to a} f(x)$ exists, and evaluate it.

   Hint: Write $f(x) = \Big(f(x) - f(a)\Big) + f(a)$, *and use the properties of limits. Remember that* $f(a)$ *is a constant.*

   (c) Conclude that the function f is continuous at the point a.

3. **Section 3.3, Exercises 28, 29.**

The Mean Value Theorem

Recommended Reading: Chapter 4.1

4. State the Mean Value Theorem.
5. **Section 4.1, Exercise 5.**
6. **Section 4.1, Exercise 15.**
7. **Section 4.1, Exercise 20.**
8. **Section 4.1, Exercise 39(a).**

Applications of the Derivative

Recommended Reading: Chapter 4.2, 4.3.

9. **Section 4.2, Exercises 38, 39.**
10. **Section 4.2, Exercise 42.**
11. **Section 4.3, Exercise 2, 15, 20.**
12. **Section 4.3, Exercise 29.**
13. Let f and g be differentiable functions. Suppose that, on some interval $[a, b]$, the vertical separation between the graphs of these two functions is greatest at the point c. Show that the tangent line to the graph of f at the point $x = c$ is parallel to the tangent line to the graph g at the point $x = c$. *Hint: Think about the difference of the functions.*