1. **A Binomial Experiment?** (10 pts) In each of the following cases, we describe an experiment. Please state whether or not each experiment can be considered to be a Binomial experiment. **If and only if it is a Binomial experiment**, please also state (i) what are A (success) and A^c (failure), (ii) what are n and θ, and (iii) what is $P(X = 2)$? Write the answers to all parts in the space after the question, **clearly labeling** which part you are answering and subparts (i), (ii), and (iii) if applicable.

   (a) A coin that is weighted (with heads twice as likely as tails) is flipped 6 times. We count the number of heads and denote it by X.

   (b) A properly balanced European roulette wheel has 37 slots. One slot is green, 18 slots are red, and 18 are black. The wheel is spun 10 times and we count the number of times the ball lands in a slot that is **not red**, and denote that number by X.

   (c) You have a perfect memory and the ability to do probability calculations in your head. You are one of four players playing Blackjack against the dealer. The dealer uses a shuffled deck of 104 cards (two 52 card decks shuffled together) and deals 5 hands before reshuffling. In each hand, the dealer and every player are dealt one card face down and one face up. Face cards are ten points, numbered cards their numeric value, and an ace can be worth eleven or one (player's choice). After receiving their initial two cards, players have the option of getting a "hit," or taking an additional card face up. In a given round, the player or the dealer wins by having a score of 21 or by having the highest score that is less than 21. Scoring higher than 21 (called "busting" or "going bust") results in a loss. A player may win by having any final score equal to or less than 21 if the dealer busts. A player also wins by getting 21 points on the initial deal. Equal points result in a tie, or "push." In each hand, each player bets \$50. You decide to "hit" or not by making the choice with the highest probability of winning, making use of the fact that the probabilities of being dealt a particular card are conditioned by those cards that have already appeared face up since the last shuffle. Denote the number of wins in the five rounds you play by X.

2.[**10pts**] **Binomial experiment–Solution**

1.Binomial.

(a)$A=$ {Coin lands on Head}, $A^c =$ {Coin lands on Tail}.

(b)$n = 6$, if H is twice as likely as T then $\theta = P(A) = \frac{2}{3}$.

(c)$P(X = 2) = \binom{6}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 \approx .082$.

2.Binomial.

(a)$A =$ {Slot is not red}, $A^c =$ {Slot is red}.

(b)$n = 10$. Because the roulette wheel is properly balanced, each slot is equally likely and $\theta = P(A) = 19/37$.

(c)$P(X = 2) = \binom{10}{2}\left(\frac{19}{37}\right)^2\left(\frac{18}{37}\right)^8 \approx .037$.

3. Not binomial. The probability of winning differs in each hand because it is conditioned by which cards have appeared face up.

Grading Scheme: 4 pts each for part (a) and part (b), with (i) and (ii) each worth a point, (iii) worth 2pts; 2 pts for part (c)

2. **The Asymptotic Distribution of an MLE** (30 pts)
The density of the Weibull distribution is

$$f(x) = \frac{\beta}{\alpha^\beta} x^{\beta-1} \exp\{-(x/\alpha)^\beta\} \text{ for } x > 0$$

Assume that β is known, and we want to estimate α.

(a) Show that if W follows the Weibull distribution with parameters α and β, then $X = W^\beta$ follows an exponential distribution with parameter λ. Write an equation that gives λ in terms of α.

(b) Find the MLE for α in terms of n iid transformed observations (that is, in terms of the X_i's, where $X_i = W_i^\beta$ and the W_i's are the original observations).

(c) State the distribution of the MLE for large n, and calculate its mean and variance.

4. **Asymptotic Distribution of MLE—Solution**

(a) W follows the Weibull distribution with parameters α and β and we consider $X = W^\beta$. So, the density of X is

$$\begin{aligned} f_X(x) &= f_W(x^{1/\beta}) \left| \frac{\partial}{\partial x} x^{1/\beta} \right| \\ &= \frac{\beta}{\alpha^\beta} x^{1-1/\beta} \exp\{-x/\alpha^\beta\} \frac{1}{\beta} x^{1/\beta-1} \text{ for } \{x > 0\} \\ &= \frac{1}{\alpha^\beta} \exp\{-x/\alpha^\beta\} \text{ for } \{x > 0\} \end{aligned}$$

Clearly, X follows an exponential distribution with mean parameter α^β. In other words, $X \sim$ Exponential (λ) where $\lambda = \alpha^{-\beta}$.

(b) Now, we have to find the mle for α based on n iid transformed observations X_i, which can be thought to be coming from exponential distribution with parameter $\lambda = \alpha^{-\beta}$. We have already seen in the lecture that $\hat{\lambda} = 1/\bar{X}$. So, substituting original values, we can get that

$$\boxed{\hat{\alpha}_{\text{mle}} = \left(\frac{1}{n} \sum_{i=1}^n w_i^\beta \right)^{1/\beta}}$$

(c) Finally, we have to find the asymptotic distribution of these mle's for large n. So, we consider the statistic

$$\hat{\alpha} = \left(\frac{1}{n} \sum_{i=1}^n X_i \right)^{1/\beta}$$

Now, recall that if $\hat{\theta}_n$ is the mle of θ and the true value is θ_0, then

$$\sqrt{nI(\hat{\theta}_0)}(\hat{\theta}_n - \theta_0)$$

converges to $N(0,1)$ random variable. Here, if $X_1, \ldots, X_n$ are coming from an Exponential(λ) distribution, then, the information is

$$
\begin{aligned}
I(\alpha) &= \mathbb{E}\left(-\frac{\partial}{\partial\alpha^2}\log L(\alpha; X_1)\right) \\
&= \mathbb{E}\left(-\frac{\partial}{\partial\alpha^2}\left[-\beta\log\alpha - \frac{X_1}{\alpha^\beta}\right]\right) \\
&= \mathbb{E}\left(-\frac{\partial}{\partial\alpha}\left[-\frac{\beta}{\alpha} + X_1\beta\alpha^{-\beta-1}\right]\right) \\
&= \mathbb{E}\left(-\frac{\beta}{\alpha^2} + X_1\beta(\beta+1)\alpha^{-\beta-2}\right) \\
&= -\frac{\beta}{\alpha^2} + \frac{\beta(\beta+1)}{\alpha^2} = \frac{\beta^2}{\alpha^2}
\end{aligned}
$$

Hence, the asymptotic distribution should be of the form

$$
\frac{\sqrt{n}(\hat{\alpha}-\alpha)}{(\alpha/\beta)} \Rightarrow N(0,1)
$$

3. **Test for Homogeneity** (20 pts) In the top tennis matches, a player may challenge a referee's line call (e.g. that the serve was 'in' or 'out'). If so, the official will review the output of an advanced technology camera called "Hawk-Eye." Here are the results of 119 challenges made by some top players at the US Open in 2014.

Player	Won	Lost
Murray	6	19
Ferrer	4	16
Azarenka	9	9
Nadal	5	12
Djokivic	4	12
Isner	4	9
Federer	1	9

Based upon these data, test at the 10% level whether players all have the *same* chance of winning a challenge.

5.[**15pts**] **Testing for homogeneity—Solution**

Under the null hypothesis that all players have the same chance of winning a challenge θ, the MLE can be estimated as $\hat{\theta} = \dfrac{33}{33+86} \approx 0.277$.

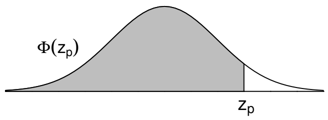
Player	Observed Won	Expected Won	Observed Lost	Expected Lost
Murray	6	6.933	19	18.067
Ferrer	4	5.546	16	14.454
Azarenka	9	4.992	9	13.008
Nadal	5	4.714	12	12.286
Djokivic	4	4.437	12	11.563
Isner	4	3.605	9	9.395
Fedderer	1	2.773	9	7.227

The test statistics is

$$X = \sum_{i=1}^{7} \sum_{j=1}^{2} \frac{(E_{i,j} - O_{i,j})^2}{E_{i,j}} \approx 6.937$$

with $(7-1) \times (2-1) = 6$ degrees of freedom. Therefore, the pvalue is ≈ 0.33, so we cannot reject the null hypothesis: the chance of winning a challenge is the same for all players.

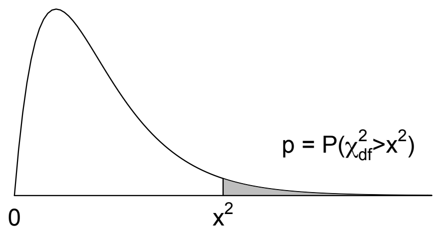
Grading Scheme: 5 pts for calculating the expected count, 5 pts for test statistics and degree of freedom, 5 pts for p-value and conclusion.



Cumulative Normal Distribution

z	**.00**	**.01**	**.02**	**.03**	**.04**	**.05**	**.06**	**.07**	**.08**	**.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Also for $z_p = 4.0$, 5.0, and 6.0, the values of p are 0.99997, 0.9999997, and 0.999999999.



χ^2 **Distribution**

df	Tail probability p											
	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
1	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.52	22.11
6	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.78	46.80	49.01
22	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.64	52.22	55.48	57.86
28	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.09
50	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2