Review problems from an old exam

8. The Intel Pentium Processor chip has been discovered to make small errors occasionally; that is, errors of +1 or –1 (in 10^{-4} units) in a small fraction of its calculations. Suppose that these occur in a single calculation with probability 0.0005 each, and that otherwise the results are correct. Suppose further that positive and negative errors are equally frequent. An Intel engineer discovers a partial "fix" that would eliminate the chance of negative errors by making all errors positive (so the chance of a positive error would become 0.001). Which chip would have smaller Mean Squared Error for a single calculation, old or new?

9. Suppose (X, Y) are bivariate random variables with joint density

$$f(x, y) = x^2 + 2y^2 \qquad 0 < x < 1, 0 < y < 1$$
$$= 0 \qquad \text{otherwise.}$$

(a) Find the marginal density of X.

(b) Find the conditional density of Y given X = 0.7.

10. Inference about an extreme value distribution. Suppose that $X_1, X_2, X_3, \ldots, X_n$ are independent random variables, each having the continuous probability distribution with cumulative distribution function given by:

$$F(x) = P\{X \le x\} = x^\theta \quad \text{for } 0 \le x \le 1$$
$$= 1 \quad \text{for } x \ge 1$$
$$= 0 \quad \text{for } x \le 0.$$

Here θ is a parameter, $\theta > 0$, and the problem is concerned with the distribution described by F(x) and with making inferences about the parameter θ based on either a single X (parts (a) – (e) or based on $X_1, X_2, X_3, \ldots, X_n$ (parts (f) – (j)).

[Motivation: (For background information only; it may help your intuition but is **not essential** to doing the problem) One way that the distribution described by F(x) can arise is as the distribution of the maximum of θ components $\{U_1, U_2, \ldots, U_\theta\}$ each independently and uniformly distributed on the interval [0,1]. For example, if a safety system fails only if all components fail, and each component has a uniformly distributed failure time, independent of the others, and there are θ components, then we could think of X_1 as the time the system fails. Naturally, the more components there are (i.e. the larger θ is) the longer the time to failure tends to be. This motivation would have θ being an integer, but we will be treating θ as any positive real number, possibly fractional. This will simplify the analysis.]

For parts (a) – (e), consider only a single X with cumulative distribution function F as described above.

(a) [3 pts] Find the probability density of X, namely f(x).

(b) [3 pts] Find the expectation of X, E(X).

(c) [3 pts] Find the variance of X, Var(X).

(d) [4 pts] Find the probability density of $Y = -\log_e(X)$.

(e) (Bayesian inference) Suppose that it is known *a priori* that θ is either 1 or 2; furthermore it is known that $P\{\theta = 1\} = .4$ and $P\{\theta = 2\} = .6$. A single X is observed to be equal to 0.75. What is the posterior probability that $\theta = 1$? That is, what is
$P\{\theta = 1|X = 0.75)$?

For parts (f) – (j), consider the data as consisting of the "list" $X_1, X_2, X_3, ..., X_n$.

(f) Find the likelihood function, $L(\theta)$.

(g) Find the Maximum Likelihood Estimator (MLE) $\hat{\theta}$.

(h) Consider the hypothesis testing problem of testing

H_0: $\theta = 1$ vs.
H_1: $\theta = 2$.

Find the form of the Most Powerful test (do not attempt to specify the cutoff numerically).

For the special case n=1 specify the test completely, giving the cutoff numerically, at the $\alpha = .05$ level, and find the power of the test vs. H_1.

(i) Consider the hypothesis testing problem of testing

H_0: $\theta = 1$ vs.
H_1: $\theta \neq 1$.

Find the form of the Likelihood Ratio test (do not attempt to specify the cutoff numerically).

(j) Use Fisher's theorem to find the approximate distribution of the MLE of part (g)

5. Consider the following testing problem:

Data: X, a single value

H_0: X has a uniform (0,1) distribution with density $f(x) = 1$ for $0 \le x \le 1$

H_1: X has a uniform (0,4) distribution with density $f(x) = .25$ for $0 \le x \le 4$

Test: Reject H_0 if $X > 0.8$

Find: The probability of the Type I error, α = _________

The probability of the Type II error, β = _________

The power of the test, π = _________

2. Let the data be denoted X, modeled as having the continuous probability distribution with density function f(x).

(a) [10 points] Consider the hypothesis testing problem of testing

H_0: f(x) has a Pareto (α=1,θ=1) density

vs.

H_1: f(x) has a Pareto (α=20,θ=1) density

Find the best test at level $\alpha = .05$.

Show work above, then fill in the answer in ONLY ONE of these two blanks):

Reject if $X \le$ ______

Reject if $X \ge$ ______

(b) [5 points] Find the power of the test. Power = __________

3. [10 points] Here are data on n = 400 randomly selected students, cross-classified by sex and self-declared political affiliation.

Affiliation:	Democrat	Republican	Independent
Male	100	60	30
Female	130	40	40

Based upon these data, test at the 5% level the hypothesis that sex and political affiliation are independent.

10. [19 points] "One-and-one foul shots". Suppose that when a basketball player has been "fouled", he is allowed a chance to score a point, and if he scores (a "hit" = H) he is given a chance to make a second score. If he fails on the first attempted shot (a "miss" = M) he is not allowed a second attempt. So some fouls lead to only one shot, some to two. And the only possible outcomes are M, HM, and HH. Suppose that $P(H) = \theta$, $P(M) = 1 - \theta$.

(a) [3 points] Suppose that a player gets one point for a H and no points for a M. So a "foul" will lead to 0, 1, or 2 points (e.g. "HH" will get 2 points). If the attempts are independent of one another, find the probability distribution of the number of points X made after one "foul" in terms of θ:

$P(X = 0)=$________; $P(X = 1)=$__________; $P(X = 2)=$_________

(b) [4 points] However, different players may have different θ's: Shaquille O'Neal's is θ_S and Kobie Bryant's is θ_K. Suppose their coach has been keeping track of the players for a few games, with the following results:

	M	HM	HH	#Fouls
Player K	25	14*	6*	45
Player S	29	4	3	36
Totals	54	18	9	81

(*So for example, K had HM 14 times for 14 Hits and HH 6 times for 12 Hits)

	Misses	Hits	Shots
Player K			
Player S			
Totals			

How many Hits did each player make? Misses? How many attempted shots? Fill in the entries of the table to the right.

(c) [5 points] Each player's attempted shots (NOT fouls) can be considered as independent Bernoulli trials. Based on this table, what are the maximum likelihood estimates of θ_S and θ_K? [No derivation is required, you may use "known facts"]

MLE of $\theta_S =$________ MLE of $\theta_K =$________

If the two players are equally accurate (i.e. $\theta_S = \theta_K$), what is the MLE of their common θ? MLE of $\theta=$__________

10. (Continued) (d) [5 points] Use a Chi-square test to test the null hypothesis that players S and K are equally skilled and all attempts are independent of one another (i.e. test if $\theta_S = \theta_K$ and the distribution of hits is that found in (a)). The expected counts under the null hypothesis are:

Expected

Player	M	HM	HH
K			
S			

Observed

Player	M	HM	HH
K	25	14	6
S	29	4	3

The test statistic is $\chi^2 =$

The degrees of freedom is df =

(e) [2 points] Use the table at the back of the exam to perform the test. State your conclusion in words, concisely. (one sentence)

11. [15 points] Automatic Exam Grading. Many exams (but not this one!) are graded by machine. Here we consider a machine-graded True-False Exam. A student is instructed to write "T" or "F" in a small circle and the machine scans that circle and judges the symbol to be "T" or "F". Because handwriting varies, and because students often write one answer and then erase it incompletely if they change their minds, the scan can be ambiguous. Based upon extensive experimentation with a wide variety of students' writing, the machine groups the scans into one of six categories, with the following being the probabilities that a student will make a symbol assigned to one of those categories. For example, a writer intending to mark "T" will make a mark that is classed in class (1) 91% of the time; a writer intending to make an "F" will make a mark classed in the class (3) 3% of the time.

Symbol Category	(1)	(2)	(3)	(4)	(5)	(6)
Intend "T"	.91	.02	.02	.01	.02	.02
Intend "F"	.04	.02	.03	.86	.01	.04

First Approach

(a) [5 points] Design a grouping method for assigning a mark as "T" or "F". The method should be such that the chance of misclassification when the mark really is a "T" is no more than 5%, and subject to that proviso, the chance of misclassifying when the mark is "F" is as small as possible. For which categories would you declare the mark a "T"? For which an "F"?
Declare a "T" if it falls in any one of these categories :__________

Declare a "F" if it falls in any one of these categories :__________

(b) [5 points] The method you found in (a) can be considered a statistical test. Find

The level of the test_______

The power of the test_______

11. (Continued)

Symbol Category	(1)	(2)	(3)	(4)	(5)	(6)
Intend "T"	.91	.02	.02	.01	.02	.02
Intend "F"	.04	.02	.03	.86	.01	.04

Second Approach

(c) [5 points] Suppose now that it has been objected that the method you first designed neglects the fact that on exams people write many more "F"s than they do "T"s. And so you start again from scratch and design a procedure that takes into account the fact that of symbols of one of these types written on exams, only 20% are "T". Specifically, with the facts as given, your new procedure should classify as "T" if the posterior probability that it is "T" is at least 50%, otherwise say "F". For this procedure,

If Category (1) is observed what would you declare? __________

What is the probability this declaration is correct? __________

If Category (5) is observed what would you declare? __________

What is the probability this declaration is correct? __________

12. A researcher locates 6 published studies of the existence of ESP. Each study reports a P-value; these were: .04, .09, .06, .37, .03, .87. (a) Use Fisher's method for combining independent tests of significance to assess the significance of the set of six investigations combined. (b) Give a reason why this conclusion is dubious.

13. A city reports 48 homicides in 2012 and 60 homicides in 2013. Use the square root transformation to make a quick assessment of the claim of a dramatic increase in homicides. What assumptions does this test require?