Stat 24400 Homework 1 Solution

Jan 14, 2016

Total points: 100

1. **[10 pts] Rice 1.20**

   Answer: $\frac{1}{5^2 \cdot 13 \cdot 17} = \frac{4! \cdot 49!}{52!}$.

   The trick is to treat the four aces as a single unit in shuffling the whole deck, and then multiply by the number of possible permutations of the aces. This gives us $4! \cdot 49!$, and we have

   $$\Pr(\text{four aces are next to each other}) = \frac{4! \cdot 49!}{52!} = \frac{4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50} = \frac{1}{5^2 \cdot 13 \cdot 17}.$$

   Grading Scheme: 9 pts for the set up, 1 pt for the correct numerical result.

2. **[10 pts] Rice 1.60**

   Answer: $\frac{8}{3}\%$; $\frac{5}{8}$.

   Let D denote the event that an item is defective. We have

   $$\Pr(\text{1st Shift}) = \Pr(\text{2nd Shift}) = \Pr(\text{3rd Shift}) = \tfrac{1}{3}.$$

   $$\begin{aligned}\Pr(D) &= \Pr(D \cap \text{1st Shift}) + \Pr(D \cap \text{2nd Shift}) + \Pr(D \cap \text{3rd Shift}) \\ &= \tfrac{1}{3}\big(\Pr(D \mid \text{1st Shift}) + \Pr(D \mid \text{2nd Shift}) + \Pr(D \mid \text{3rd Shift})\big) \\ &= \tfrac{1}{3}\left(\tfrac{1}{100} + \tfrac{2}{100} + \tfrac{5}{100}\right) \\ &= \tfrac{8/3}{100}.\end{aligned}$$

   To compute the probability that an item was produced by the third shift given that it is defective, use the identity

   $$\Pr(\text{3rd Shift} \mid D)\Pr(D) = \Pr(D \mid \text{3rd Shift})\Pr(\text{3rd Shift})$$

   so that

   $$\Pr(\text{3rd Shift} \mid D) = \frac{\Pr(D \mid \text{3rd Shift})\Pr(\text{3rd Shift})}{\Pr(D)} = \frac{\frac{5}{100} \cdot \frac{1}{3}}{\frac{8}{300}} = \tfrac{5}{8}.$$

   Grading Scheme: 4 pts each for the set ups for the two parts, 1 pt each for the correct numerical results.

3. **[10 pts] Rice 1.72**

   Answer: $(1 - p^2)^n$; $(1 - .05^2)^{10} \approx .975$.

   Since the components are connected in series and all the units are independent,

   $$\Pr(\text{system works}) = \prod_{i=1}^{n} \Pr(i\text{th component works}).$$

Since the event that a component works is the complement of the event that both the main unit and its backup fail, it has probability $1 - p^2$. Therefore,

$$\Pr(\text{system works}) = \prod_{i=1}^{n}(1 - p^2) = (1 - p^2)^n.$$

For $n = 10$ and $p = .05$, we have $\Pr(\text{system works}) = (1 - .05^2)^{10} \approx .975$. This is better than the naive serial architecture and worse than the parallel architecture.

Grading Scheme: 7 pts for the derivation, 2 pts for the correct numerical result, and 1 pt for the comparison with the example from the text.

4. **[10 pts] Rice 1.78**

   (a) *Answer*: $\Pr(AA) = \Pr(Aa) = .5$.

	A	a
A	AA	Aa
A	AA	Aa

   and each of the four squares has probability .25.

   (b) For the second generation,

$$\begin{aligned}
\Pr(AA) &= \Pr(AA \mid AA, AA)\Pr(AA, AA) + \Pr(AA \mid AA, Aa)\Pr(AA, Aa) \\
&\quad + \Pr(AA \mid Aa, AA)\Pr(Aa, AA) + \Pr(AA \mid Aa, Aa)\Pr(Aa, Aa) \\
&= 1 \cdot p^2 + \tfrac{1}{2} \cdot 2pq + \tfrac{1}{2} \cdot 2pq + \tfrac{1}{4} \cdot 4q^2 \\
&= (p+q)^2, \\
\Pr(Aa) &= \Pr(Aa \mid AA, Aa)\Pr(AA, Aa) + \Pr(Aa \mid AA, aa)\Pr(AA, aa) \\
&\quad + \Pr(Aa \mid Aa, AA)\Pr(Aa, AA) + \Pr(Aa \mid Aa, Aa)\Pr(Aa, Aa) \\
&\quad + \Pr(Aa \mid Aa, aa)\Pr(Aa, aa) + \Pr(Aa \mid aa, AA)\Pr(aa, AA) \\
&\quad + \Pr(Aa \mid aa, Aa)\Pr(aa, Aa) \\
&= \tfrac{1}{2} \cdot 2pq + 1 \cdot pr + \tfrac{1}{2} \cdot 2pq + \tfrac{1}{2} \cdot 4q^2 + \tfrac{1}{2} \cdot 2qr + 1 \cdot pr + \tfrac{1}{2} \cdot 2qr \\
&= 2(p+q)(q+r), \\
\Pr(aa) &= \Pr(aa \mid Aa, Aa)\Pr(Aa, Aa) + \Pr(aa \mid Aa, aa)\Pr(Aa, aa) \\
&\quad + \Pr(aa \mid aa, Aa)\Pr(aa, Aa) + \Pr(aa \mid aa, aa)\Pr(aa, aa) \\
&= \tfrac{1}{4} \cdot 4q^2 + \tfrac{1}{2} \cdot 2qr + \tfrac{1}{2} \cdot 2qr + 1 \cdot r^2 \\
&= (q+r)^2.
\end{aligned}$$

   Thus,

$$\Pr(AA) = (p+q)^2, \quad \Pr(Aa) = 2(p+q)(q+r), \quad \Pr(aa) = (q+r)^2.$$

   Put $p' = \Pr(AA)$, $q' = \Pr(Aa)/2$, and $r' = \Pr(aa)$. Substituting p', q', and r' in place of p, q, and r, respectively, and using the identity $p + 2q + r = 1$, the probabilities for the third generation are

$$\Pr(AA) = (p'+q')^2 = ((p+q)^2 + (p+q)(q+r))^2 = ((p+q)(p+2q+r))^2 = (p+q)^2,$$

$$
\begin{aligned}
\Pr(Aa) &= 2(p' + q')(q' + r') \\
&= 2((p+q)^2 + (p+q)(q+r))((p+q)(q+r) + (q+r)^2) \\
&= 2(p+q)(p+2q+r)(p+2q+r)(q+r) = 2(p+q)(q+r),
\end{aligned}
$$

and

$$
\Pr(aa) = (q'+r')^2 = ((p+q)(q+r)+(q+r)^2)^2 = ((p+2q+r)(q+r))^2 = (q+r)^2.
$$

So the probabilities are unchanged from the second to the third generation.

(c) For the second generation,

$$
\begin{aligned}
\Pr(AA) &= \left(\frac{up+vq}{up+2vq+rw}\right)^2, \\
\Pr(Aa) &= \frac{2(up+vq)(vq+rw)}{(up+2vq+wr)^2}, \\
\Pr(aa) &= \left(\frac{wr+vq}{up+2vq+wr}\right)^2.
\end{aligned}
$$

Denote these probabilities a, $2b$, and c, respectively. For the third generation,

$$
\begin{aligned}
\Pr(AA) &= \left(\frac{ua+vb}{ua+2vb+wc}\right)^2, \\
\Pr(Aa) &= \frac{2(ua+vb)(vb+wc)}{(ua+2vb+wc)^2}, \\
\Pr(aa) &= \left(\frac{wc+vb}{ua+2vb+wc}\right)^2.
\end{aligned}
$$

Grading Scheme: 2 pt for (a), 8 pts for (b). Since (c) was mistakenly omitted from the latest Chalk version of the assignment, we give 8 points extra credit, while holding the maximum number of points on the entire homework to 100.

5. **[10 pts] Rice 2.8** This follows immediately from an application of Proposition B in Chapter 1 that says

$$
(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.
$$

Take $a = p$ and $b = p - 1$. Then,

$$
\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \big(p + (1-p)\big)^n = 1.
$$

6. **[10 pts] Rice 2.28**

$p_0 = q^n$ is obvious. For $k \in \{1, \dots, n\}$

$$\begin{aligned}
\frac{(n-k+1)p}{kq} p_{k-1} &= \frac{(n-k+1)p}{kq} \binom{n}{k-1} p^{k-1} q^{n-k+1} \\
&= \frac{(n-k+1)p}{kq} \cdot \frac{n!}{(k-1)!(n-k+1)!} p^{k-1} q^{n-k+1} \\
&= \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= p_k.
\end{aligned}$$

$\Pr(X \le 4) = 0.532$.

Grading Scheme: 8 pts for the derivation, 2 pts for the correct numerical result.

7. **[20 pts] Texas hold 'em**

(a) (Two players)
Let a = total number of combination s.t. the four cards are all of different ranks
b = total number of combination s.t. two players have cards of the same rank (1 or 2 pairs) and neither player is dealt a pair.
Then the

$$\Pr(\text{no pair}) = \frac{a+b}{\text{number of combinations for the four cards}} = \frac{a+b}{\binom{52}{2} \cdot \binom{50}{2}}$$

By simple counting, we know
a = (possibilities of ranks)(possibilities of suits)(ways assigned to the players) = $\binom{13}{4} 4^4 \binom{4}{2}$

b = (poss. of players having cards of the form $\{1, 2\}$ and $\{1, 2\}$)
+(poss. of players having cards of the form $\{1, 2\}$ and $\{1, 3\}$)
$= \binom{13}{2}(4)(3)(4)(3) + \binom{13}{1}(4)(3)(48)(44)$
Substituting into above yields $\Pr(\text{no pair}) = 0.8859$.

(b) (Three players)
Notice that there is ordering of player.
Let A denote the event that player 1 has a pair; let B denote the event that player 2 has a pair and; C denote the event that player 3 has a pair.
Then $\Pr(\text{at least a pair}) = \Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(B \cap C) - \Pr(C \cap A) + \Pr(A \cap B \cap C) = 3 \cdot \Pr(A) - 3 \cdot \Pr(A \cap B) + \Pr(A \cap B \cap C)$, by Inclusion-Exclusion Principle and symmetry of the probabilities.

$$\Pr(A) = \frac{(\text{no. of ways to choose the rank})(\text{no. of possible suits combination})}{\binom{52}{2}} = \frac{(13)\binom{4}{2}}{\binom{52}{2}}$$

$$\Pr(A \cap B) = \frac{\text{no. of poss. of form } \{1, 1\} \text{ and } \{1, 1\} + \text{no. of poss. of form } \{1, 1\} \text{ and } \{2, 2\}}{\binom{52}{2}\binom{50}{2}}$$

$$= \frac{\binom{13}{1}\binom{4}{2} + 13\binom{4}{2}12\binom{4}{2}}{\binom{52}{2}\binom{50}{2}}$$

$\Pr(A \cap B \cap C) = ($no. of form $\{1,1\}$ and $\{1,1\}$ and $\{2,2\}$
+no. of form $\{1,1\}$ and $\{2,2\}$ and $\{1,1\}$
+no. of form $\{2,2\}$ and $\{1,1\}$ and $\{1,1\}$
+no. of form $\{1,1\}$ and $\{2,2\}$ and $\{3,3\})$
$/(\binom{52}{2}\binom{50}{2}\binom{48}{2}) = \frac{(3)(13)\binom{4}{2}\binom{2}{2}(12)\binom{4}{2}+(13)(12)(11)\binom{4}{2}^3}{\binom{52}{2}\binom{50}{2}\binom{48}{2}}$
In total, $\Pr(\text{no pair}) = 1 - \Pr(\text{at least a pair}) = 0.83$.

Grading Scheme: 10 pts for each part. Marks are assigned based on the number of correct counting made in each part.

8. **[20 pts] Poisson Process**

(a) $N(1,5] \sim \text{Poisson}(4\lambda)$.

$$\begin{aligned}
\Pr\{N(1,5] > 1\} &= 1 - \Pr\{N(1,5] \leq 1\} \\
&= 1 - \Pr\{N(1,5] = 1\} - \Pr\{N(1,5] = 0\} \\
&= 1 - \frac{(4\lambda)^0}{0!}e^{-4\lambda} - \frac{(4\lambda)^1}{1!}e^{-4\lambda} \\
&= 1 - (1+4\lambda)e^{-4\lambda}.
\end{aligned}$$

(b) $N(1,2] \sim \text{Poisson}(\lambda)$.

$$\Pr\{N(0,1] = N(0,2]\} = \Pr\{N(1,2] = 0\} = e^{-\lambda}$$

(c) There are two ways to see this. The first of the two is to note that $N(1,2]$ and $N(3,4]$ are independent Poisson(λ). Then,

$$\begin{aligned}
\Pr\{N(1,2] + N(3,4] = 6\} &= \sum_{k=0}^{6} \Pr\{N(1,2] = k\} \Pr\{N(3,4] = 6-k\} \\
&= \sum_{k=0}^{6} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{\lambda^{6-k} e^{-\lambda}}{(6-k)!} \\
&= \lambda^6 e^{-2\lambda} \sum_{k=0}^{6} \frac{1}{k!(6-k)!} \\
&= \frac{\lambda^6 e^{-2\lambda}}{6!} \sum_{k=0}^{6} \frac{6!}{k!(6-k)!} \\
&= \frac{(2\lambda)^6 e^{-2\lambda}}{6!},
\end{aligned}$$

where in the last step, we have used the identity

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n.$$

Alternatively, one could observe that getting 6 on two disjoint intervals of length 1 is "like" getting 6 on a single interval of length 2. (This is an illustration of the more general property that if $X \sim \text{Poisson}(\mu_X)$ and $Y \sim \text{Poisson}(\mu_Y)$ are independent, then $X + Y \sim \text{Poisson}(\mu_X + \mu_Y)$.) Thus, $N(1,2] + N(3,4] \sim \text{Poisson}(2\lambda)$, from which the conclusion follows.

(d) It has been pointed that there is an ambiguity in the wording of this problem, specifically whether we mean "a particular value of m", or "any m." We give credit for either solution, which differ by the presence or absence of a second infinite series, For a "particular m:" We note that $N(0,1]$ and $N(1,2]$ are independent Poisson(λ).

$$\begin{aligned}
\Pr\{N(0,1] = N(1,2] + m\} &= \sum_{k=0}^{\infty} \Pr\{N(1,2] = k\} \Pr\{N(0,1] = k + m\} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{\lambda^{k+m} e^{-\lambda}}{(k+m)!} \\
&= e^{-2\lambda} \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{2\lambda}{2}\right)^{2k+m} \qquad (*)
\end{aligned}$$

A modified Bessel function of the first kind $I_m(x)$ is a solution to the second-order differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} - (x^2 + m^2) y = 0.$$

For our purposes, it is enough to know that when m is a non-negative integer, $I_m(x)$ has the power series expansion

$$I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}.$$

Substituting the above into $(*)$, we obtain

$$\Pr\{N(0,1] = N(1,2] + m\} = e^{-2\lambda} I_m(2\lambda).$$

(e) For "any m:": The probability is essentially $\Pr(N((0,1]) \geq N((1,2]))$, which is in turn

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \sum_{n=k}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-2\lambda} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{\lambda^{n+k}}{k!n!} = e^{-2\lambda} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\lambda^{(m+k)+k}}{k!(m+k)!}$$

$$= e^{-2\lambda} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+m+1)} \left(\frac{2\lambda}{2}\right)^{2k+m} = e^{-2\lambda} \sum_{m=0}^{\infty} I_m(2\lambda),$$

where I_m is a modified Bessel function of the first kind.

Grading Scheme: 2 pts for (a), 4 pts for (b), 6 pts for (c), and 8 pts for (d).