9.11 Problems

1. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is $\frac{1}{2}$ versus the alternative that the probability is not $\frac{1}{2}$. The test rejects if either 0 or 10 heads are observed.

a. What is the significance level of the test?
b. If in fact the probability of heads is .1, what is the power of the test?

2. Which of the following hypotheses are simple, and which are composite?

a. X follows a uniform distribution on [0, 1].
b. A die is unbiased.
c. X follows a normal distribution with mean 0 and variance $\sigma^2 > 10$.
d. X follows a normal distribution with mean $\mu = 0$.

3. Suppose that $X \sim \text{bin}(100, p)$. Consider the test that rejects H_0: $p = .5$ in favor of H_A: $p \neq .5$ for $|X - 50| > 10$. Use the normal approximation to the binomial distribution to answer the following:

a. What is α?
b. Graph the power as a function of p.

4. Let X have one of the following distributions:

X	H_0	H_A
x_1	.2	.1
x_2	.3	.4
x_3	.3	.1
x_4	.2	.4

a. Compare the likelihood ratio, Λ, for each possible value X and order the x_i according to Λ.
b. What is the likelihood ratio test of H_0 versus H_A at level $\alpha = .2$? What is the test at level $\alpha = .5$?
c. If the prior probabilities are $P(H_0) = P(H_A)$, which outcomes favor H_0?
d. What prior probabilities correspond to the decision rules with $\alpha = .2$ and $\alpha = .5$?

5. True or false, and state why:

a. The significance level of a statistical test is equal to the probability that the null hypothesis is true.
b. If the significance level of a test is decreased, the power would be expected to increase.
c. If a test is rejected at the significance level α, the probability that the null hypothesis is true equals α.
d. The probability that the null hypothesis is falsely rejected is equal to the power of the test.
e. A type I error occurs when the test statistic falls in the rejection region of the test.

f. A type II error is more serious than a type I error.
g. The power of a test is determined by the null distribution of the test statistic.
h. The likelihood ratio is a random variable.

6. Consider the coin tossing example of Section 9.1. Suppose that instead of tossing the coin 10 times, the coin was tossed until a head came up and the total number of tosses, X, was recorded.

a. If the prior probabilities are equal, which outcomes favor H_0 and which favor H_1?
b. Suppose $P(H_0)/P(H_1) = 10$. What outcomes favor H_0?
c. What is the significance level of a test that rejects H_0 if $X \geq 8$?
d. What is the power of this test?

7. Let $X_1, \ldots, X_n$ be a sample from a Poisson distribution. Find the likelihood ratio for testing H_0: $\lambda = \lambda_0$ versus H_A: $\lambda = \lambda_1$, where $\lambda_1 > \lambda_0$. Use the fact that the sum of independent Poisson random variables follows a Poisson distribution to explain how to determine a rejection region for a test at level α.

8. Show that the test of Problem 7 is uniformly most powerful for testing H_0: $\lambda = \lambda_0$ versus H_A: $\lambda > \lambda_0$.

9. Let $X_1, \ldots, X_{25}$ be a sample from a normal distribution having a variance of 100. Find the rejection region for a test at level $\alpha = .10$ of H_0: $\mu = 0$ versus H_A: $\mu = 1.5$. What is the power of the test? Repeat for $\alpha = .01$.

10. Suppose that $X_1, \ldots, X_n$ form a random sample from a density function, $f(x|\theta)$, for which T is a sufficient statistic for θ. Show that the likelihood ratio test of H_0: $\theta = \theta_0$ versus H_A: $\theta = \theta_1$ is a function of T. Explain how, if the distribution of T is known under H_0, the rejection region of the test may be chosen so that the test has the level α.

11. Suppose that $X_1, \ldots, X_{25}$ form a random sample from a normal distribution having a variance of 100. Graph the power of the likelihood ratio test of H_0: $\mu = 0$ versus H_A: $\mu \neq 0$ as a function of μ, at significance levels .10 and .05. Do the same for a sample size of 100. Compare the graphs and explain what you see.

12. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with the density function $f(x|\theta) = \theta \exp[-\theta x]$. Derive a likelihood ratio test of H_0: $\theta = \theta_0$ versus H_A: $\theta \neq \theta_0$, and show that the rejection region is of the form $\{\overline{X} \exp[-\theta_0 \overline{X}] \leq c\}$.

13. Suppose, to be specific, that in Problem 12, $\theta_0 = 1$, $n = 10$, and that $\alpha = .05$. In order to use the test, we must find the appropriate value of c.

a. Show that the rejection region is of the form $\{\overline{X} \leq x_0\} \cup \{\overline{X} \geq x_1\}$, where x_0 and x_1 are determined by c.
b. Explain why c should be chosen so that $P(\overline{X} \exp(-\overline{X}) \leq c) = .05$ when $\theta_0 = 1$.
c. Explain why $\sum_{i=1}^{10} X_i$ and hence $\overline{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?

d. Suppose that you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer (simulation).

14. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = 2 \times P(H_1)$. As in Section 9.1, the hypothesis H_0 will be chosen if $P(H_0|x) > P(H_1|x)$. For $\sigma^2 = 0.1, 0.5, 1.0, 5.0$:

a. For what values of X will H_0 be chosen?
b. In the long run, what proportion of the time will H_0 be chosen if H_0 is true $\frac{2}{3}$ of the time?

15. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = P(H_1)$. For $\sigma = 1$ and $x \in [0, 3]$, plot and compare (1) the p-value for the test of H_0 and (2) $P(H_0|x)$. Can the p-value be interpreted as the probability that H_0 is true? Choose another value of σ and repeat.

16. In the previous problem, with $\sigma = 1$, what is the probability that the p-value is less than 0.05 if H_0 is true? What is the probability if H_1 is true?

17. Let $X \sim N(0, \sigma^2)$, and consider testing $H_0 : \sigma_1 = \sigma_0$ versus $H_A : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$. The values σ_0 and σ_1 are fixed.

a. What is the likelihood ratio as a function of x? What values favor H_0? What is the rejection region of a level α test?
b. For a sample, $X_1, X_2, \ldots, X_n$ distributed as above, repeat the previous question.
c. Is the test in the previous question uniformly most powerful for testing $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$?

18. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a double exponential distribution with density $f(x) = \frac{1}{2}\lambda \exp(-\lambda|x|)$. Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1$, where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?

19. Under H_0, a random variable has the cumulative distribution function $F_0(x) = x^2$, $0 \le x \le 1$; and under H_1, it has the cumulative distribution function $F_1(x) = x^3$, $0 \le x \le 1$.

a. If the two hypotheses have equal prior probability, for what values of x is the posterior probability of H_0 greater than that of H_1?
b. What is the form of the likelihood ratio test of H_0 versus H_1?
c. What is the rejection region of a level α test?
d. What is the power of the test?

20. Consider two probability density functions on [0, 1]: $f_0(x) = 1$, and $f_1(x) = 2x$. Among all tests of the null hypothesis $H_0 : X \sim f_0(x)$ versus the alternative $X \sim f_1(x)$, with significance level $\alpha = 0.10$, how large can the power possibly be?

21. Suppose that a single observation X is taken from a uniform density on $[0, \theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

a. Find a test that has significance level $\alpha = 0$. What is its power?
b. For $0 < \alpha < 1$, consider the test that rejects when $X \in [0, \alpha]$. What is its significance level and power?
c. What is the significance level and power of the test that rejects when $X \in [1 - \alpha, 1]$?
d. Find another test that has the same significance level and power as the previous one.
e. Does the likelihood ratio test determine a unique rejection region?
f. What happens if the null and alternative hypotheses are interchanged—$H_0 : \theta = 2$ versus $H_1 : \theta = 1$?

22. In Example A of Section 8.5.3 a confidence interval for the variance of a normal distribution was derived. Use Theorem B of Section 9.3 to derive an acceptance region for testing the hypothesis H_0: $\sigma^2 = \sigma_0^2$ at the significance level α based on a sample $X_1, X_2, \ldots, X_n$. Precisely describe the rejection region if $\sigma_0 = 1, n = 15, \alpha = .05$.

23. Suppose that a 99% confidence interval for the mean μ of a normal distribution is found to be $(-2.0, 3.0)$. Would a test of H_0: $\mu = -3$ versus H_A: $\mu \neq -3$ be rejected at the .01 significance level?

24. Let X be a binomial random variable with n trials and probability p of success.

a. What is the generalized likelihood ratio for testing H_0: $p = .5$ versus H_A: $p \neq .5$?
b. Show that the test rejects for large values of $|X - n/2|$.
c. Using the null distribution of X, show how the significance level corresponding to a rejection region $|X - n/2| > k$ can be determined.
d. If $n = 10$ and $k = 2$, what is the significance level of the test?
e. Use the normal approximation to the binomial distribution to find the significance level if $n = 100$ and $k = 10$.

This analysis is the basis of the **sign test,** a typical application of which would be something like this: An experimental drug is to be evaluated on laboratory rats. In n pairs of litter mates, one animal is given the drug and the other is given a placebo. A physiological measure of benefit is made after some time has passed. Let X be the number of pairs for which the animal receiving the drug benefited more than its litter mate. A simple model for the distribution of X if there is no drug effect is binomial with $p = .5$. This is then the null hypothesis that must be made untenable by the data before one could conclude that the drug had an effect.

25. Calculate the likelihood ratio for Example B of Section 9.5 and compare the results of a test based on the likelihood ratio to those of one based on Pearson's chi-square statistic.

26. True or false:

a. The generalized likelihood ratio statistic Λ is always less than or equal to 1.
b. If the p-value is .03, the corresponding test will reject at the significance level .02.

c. If a test rejects at significance level .06, then the p-value is less than or equal to .06.
d. The p-value of a test is the probability that the null hypothesis is correct.
e. In testing a simple versus simple hypothesis via the likelihood ratio, the p-value equals the likelihood ratio.
f. If a chi-square test statistic with 4 degrees of freedom has a value of 8.5, the p-value is less than .05.

27. What values of a chi-square test statistic with 7 degrees of freedom yield a p-value less than or equal to .10?

28. Suppose that a test statistic T has a standard normal null distribution.

a. If the test rejects for large values of $|T|$, what is the p-value corresponding to $T = 1.50$?
b. Answer the same question if the test rejects for large T.

29. Suppose that a level α test based on a test statistic T rejects if $T > t_0$. Suppose that g is a monotone-increasing function and let $S = g(T)$. Is the test that rejects if $S > g(t_0)$ a level α test?

30. Suppose that the null hypothesis is true, that the distribution of the test statistic, T say, is continuous with cdf F and that the test rejects for large values of T. Let V denote the p-value of the test.

a. Show that $V = 1 - F(T)$.
b. Conclude that the null distribution of V is uniform. (*Hint:* See Proposition C of Section 2.3.)
c. If the null hypothesis is true, what is the probability that the p-value is greater than .1?
d. Show that the test that rejects if $V < \alpha$ has significance level α.

31. What values of the generalized likelihood ratio Λ are necessary to reject the null hypothesis at the significance level $\alpha = .1$ if the degrees of freedom are 1, 5, 10, and 20?

32. The intensity of light reflected by an object is measured. Suppose there are two types of possible objects, A and B. If the object is of type A, the measurement is normally distributed with mean 100 and standard deviation 25; if it is of type B, the measurement is normally distributed with mean 125 and standard deviation 25. A single measurement is taken with the value $X = 120$.

a. What is the likelihood ratio?
b. If the prior probabilities of A and B are equal ($\frac{1}{2}$ each), what is the posterior probability that the item is of type B?
c. Suppose that a decision rule has been formulated that declares the object to be of type B if $X > 125$. What is the significance level associated with this rule?
d. What is the power of this test?
e. What is the p-value when $X = 120$?

33. It has been suggested that dying people may be able to postpone their death until after an important occasion, such as a wedding or birthday. Phillips and King

(1988) studied the patterns of death surrounding Passover, an important Jewish holiday, in California during the years 1966–1984. They compared the number of deaths during the week before Passover to the number of deaths during the week after Passover for 1919 people who had Jewish surnames. Of these, 922 occurred in the week before Passover and 997, in the week after Passover. The significance of this discrepancy can be assessed by statistical calculations. We can think of the counts before and after as constituting a table with two cells. If there is no holiday effect, then a death has probability $\frac{1}{2}$ of falling in each cell. Thus, in order to show that there is a holiday effect, it is necessary to show that this simple model does not fit the data. Test the goodness of fit of the model by Pearson's X^2 test or by a likelihood ratio test. Repeat this analysis for a group of males of Chinese and Japanese ancestry, of whom 418 died in the week before Passover and 434 died in the week after. What is the relevance of this latter analysis to the former?

34. Test the goodness of fit of the data to the genetic model given in Problem 55 of Chapter 8.

35. Test the goodness of fit of the data to the genetic model given in Problem 58 of Chapter 8.

36. The National Center for Health Statistics (1970) gives the following data on distribution of suicides in the United States by month in 1970. Is there any evidence that the suicide rate varies seasonally, or are the data consistent with the hypothesis that the rate is constant? (*Hint:* Under the latter hypothesis, model the number of suicides in each month as a multinomial random variable with the appropriate probabilities and conduct a goodness-of-fit test. Look at the signs of the deviations, $O_i - E_i$, and see if there is a pattern.)

Month	Number of Suicides	Days/Month
Jan.	1867	31
Feb.	1789	28
Mar.	1944	31
Apr.	2094	30
May	2097	31
June	1981	30
July	1887	31
Aug.	2024	31
Sept.	1928	30
Oct.	2032	31
Nov.	1978	30
Dec.	1859	31

37. The following table gives the number of deaths due to accidental falls for each month during 1970. Is there any evidence for a departure from uniformity in the rate over time? That is, is there a seasonal pattern to this death rate? If so, describe its pattern and speculate as to causes.

Month	Number of Deaths
Jan.	1668
Feb.	1407
Mar.	1370
Apr.	1309
May	1341
June	1338
July	1406
Aug.	1446
Sept.	1332
Oct.	1363
Nov.	1410
Dec.	1526

38. Yip et al. (2000) studied seasonal variations in suicide rates in England and Wales during 1982–1996, collecting counts shown in the following table:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Male	3755	3251	3777	3706	3717	3660	3669	3626	3481	3590	3605	3392
Female	1362	1244	1496	1452	1448	1376	1370	1301	1337	1351	1416	1226

Do either the male or female data show seasonality?

39. There is a great deal of folklore about the effects of the full moon on humans and other animals. Do animals bite humans more during a full moon? In an attempt to study this question, Bhattacharjee et al. (2000) collected data on admissions to a medical facility for treatment of bites by animals: cats, rats, horses, and dogs. 95% of the bites were by man's best friend, the dog. The lunar cycle was divided into 10 periods, and the number of bites in each period is shown in the following table. Day 29 is the full moon. Is there a temporal trend in the incidence of bites?

Lunar Day	16,17,18	19,20,21	22,23,24	25,26,27	28,29,1	2,3,4	5,6,7	8,9,10	11,12,13	14,15
Number of Bites	137	150	163	201	269	155	142	146	148	110

40. Consider testing goodness of fit for a multinomial distribution with two cells. Denote the number of observations in each cell by X_1 and X_2 and let the hypothesized probabilities be p_1 and p_2. Pearson's chi-square statistic is equal to

$$\sum_{i=1}^{2} \frac{(X_i - np_i)^2}{np_i}$$

Show that this may be expressed as

$$\frac{(X_1 - np_1)^2}{np_1(1 - p_1)}$$

Because X_1 is binomially distributed, the following holds approximately under the null hypothesis:

$$\frac{X_1 - np_1}{\sqrt{np_1(1 - p_1)}} \sim N(0, 1)$$

Thus, the square of the quantity on the left-hand side is approximately distributed as a chi-square random variable with 1 degree of freedom.

41. Let $X_i \sim \text{bin}(n_i, p_i)$, for $i = 1, \dots, m$, be independent. Derive a likelihood ratio test for the hypothesis

$$H_0\colon p_1 = p_2 = \cdots = p_m$$

against the alternative hypothesis that the p_i are not all equal. What is the large-sample distribution of the test statistic?

42. Nylon bars were tested for brittleness (Bennett and Franklin 1954). Each of 280 bars was molded under similar conditions and was tested in five places. Assuming that each bar has uniform composition, the number of breaks on a given bar should be binomially distributed with five trials and an unknown probability p of failure. If the bars are all of the same uniform strength, p should be the same for all of them; if they are of different strengths, p should vary from bar to bar. Thus, the null hypothesis is that the p's are all equal. The following table summarizes the outcome of the experiment:

Breaks/Bar	Frequency
0	157
1	69
2	35
3	17
4	1
5	1

a. Under the given assumption, the data in the table consist of 280 observations of independent binomial random variables. Find the mle of p.
b. Pooling the last three cells, test the agreement of the observed frequency distribution with the binomial distribution using Pearson's chi-square test.
c. Apply the test procedure derived in the previous problem.

43. **a.** In 1965, a newspaper carried a story about a high school student who reported getting 9207 heads and 8743 tails in 17,950 coin tosses. Is this a significant discrepancy from the null hypothesis $H_0\colon p = \frac{1}{2}$?
b. Jack Youden, a statistician at the National Bureau of Standards, contacted the student and asked him exactly how he had performed the experiment (Youden

1974). To save time, the student had tossed groups of five coins at a time, and a younger brother had recorded the results, shown in the following table:

Number of Heads	Frequency
0	100
1	524
2	1080
3	1126
4	655
5	105

Are the data consistent with the hypothesis that all the coins were fair ($p = \frac{1}{2}$)?

c. Are the data consistent with the hypothesis that all five coins had the same probability of heads but that this probability was not necessarily $\frac{1}{2}$? (*Hint:* Use the binomial distribution.)

44. Derive and carry out a likelihood ratio test of the hypothesis H_0: $\theta = \frac{1}{2}$ versus H_1: $\theta \neq \frac{1}{2}$ for Problem 58 of Chapter 8.

45. In a classic genetics study, Geissler (1889) studied hospital records in Saxony and compiled data on the gender ratio. The following table shows the number of male children in 6115 families with 12 children. If the genders of successive children are independent and the probabilities remain constant over time, the number of males born to a particular family of 12 children should be a binomial random variable with 12 trials and an unknown probability p of success. If the probability of a male child is the same for each family, the table represents the occurrence of 6115 binomial random variables. Test whether the data agree with this model. Why might the model fail?

Number	Frequency
0	7
1	45
2	181
3	478
4	829
5	1112
6	1343
7	1033
8	670
9	286
10	104
11	24
12	3

46. Show that the transformation $Y = \sin^{-1}\sqrt{\hat{p}}$ is variance-stabilizing if $\hat{p} = X/n$, where $X \sim \text{bin}(n, p)$.

47. Let X follow a Poisson distribution with mean λ. Show that the transformation $Y = \sqrt{X}$ is variance-stabilizing.

48. Suppose that $E(X) = \mu$ and $\text{Var}(X) = c\mu^2$, where c is a constant. Find a variance-stabilizing transformation.

49. An English naturalist collected data on the lengths of cuckoo eggs, measuring to the nearest .5 mm. Examine the normality of this distribution by (a) constructing a histogram and superposing a normal density, (b) plotting on normal probability paper, and (c) constructing a hanging rootogram.

Length	Frequency
18.5	0
19.0	1
19.5	3
20.0	33
20.5	39
21.0	156
21.5	152
22.0	392
22.5	288
23.0	286
23.5	100
24.0	86
24.5	21
25.0	12
25.5	2
26.0	0
26.5	1

50. Burr (1974) gives the following data on the percentage of manganese in iron made in a blast furnace. For 24 days, a single analysis was made on each of five casts. Examine the normality of this distribution by making a normal probability plot and a hanging rootogram. (As a prelude to topics that will be taken up in later chapters, you might also informally examine whether the percentage of manganese is roughly constant from one day to the next or whether there are significant trends over time.)

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7	Day 8	Day 9	Day 10	Day 11	Day 12
1.40	1.40	1.80	1.54	1.52	1.62	1.58	1.62	1.60	1.38	1.34	1.50
1.28	1.34	1.44	1.50	1.46	1.58	1.64	1.46	1.44	1.34	1.28	1.46
1.36	1.54	1.46	1.48	1.42	1.62	1.62	1.38	1.46	1.36	1.08	1.28
1.38	1.44	1.50	1.52	1.58	1.76	1.72	1.42	1.38	1.58	1.08	1.18
1.44	1.46	1.38	1.58	1.70	1.68	1.60	1.38	1.34	1.38	1.36	1.28

(*Continued*)

Day 13	Day 14	Day 15	Day 16	Day 17	Day 18	Day 19	Day 20	Day 21	Day 22	Day 23	Day 24
1.26	1.52	1.50	1.42	1.32	1.16	1.24	1.30	1.30	1.48	1.32	1.44
1.50	1.50	1.42	1.32	1.40	1.34	1.22	1.48	1.52	1.46	1.22	1.28
1.52	1.46	1.38	1.48	1.40	1.40	1.20	1.28	1.76	1.48	1.72	1.10
1.38	1.34	1.36	1.36	1.26	1.16	1.30	1.18	1.16	1.42	1.18	1.06
1.50	1.40	1.38	1.38	1.26	1.54	1.36	1.28	1.28	1.36	1.36	1.10

51. Examine the probability plot in Figure 9.6 and explain why there are several sets of horizontal bands of points.

52. The following table gives values of two abundance ratios for different isotopes of potassium from several samples of minerals (H. Ku, private communication). Examine whether each of the ratios appears normally distributed by first making histograms and superposing normal densities and then making probability plots.

$^{39}K/^{41}K$	$^{41}K/^{40}K$	$^{39}K/^{41}K$	$^{41}K/^{40}K$	$^{39}K/^{41}K$	$^{41}K/^{40}K$
13.8645	576.369	13.8689	578.277	13.8724	576.017
13.8695	578.012	13.8593	574.708	13.8665	574.881
13.8659	575.597	13.8742	573.630	13.8566	578.508
13.8622	575.244	13.8703	576.069	13.8555	576.796
13.8696	575.567	13.8472	575.637	13.8534	580.394
13.8604	576.836	13.8555	575.971	13.8685	576.772
13.8672	576.236	13.8439	576.403	13.8694	576.501
13.8598	575.291	13.8646	576.179	13.8599	574.950
13.8641	576.478	13.8702	575.129	13.8605	577.614
13.8673	576.992	13.8606	577.084	13.8619	574.506
13.8597	578.335	13.8622	576.749	13.9641	576.317
13.8604	576.767	13.8588	576.669	13.8597	575.665
13.8591	576.571	13.8547	575.869	13.8617	575.815
13.8472	576.617	13.8597	577.793	13.861	576.109
13.863	575.885	13.8663	577.770	13.8615	576.144
13.8566	576.651	13.8597	577.697	13.8469	576.820
13.8503	575.974	13.8604	576.299	13.8582	576.672
13.8553	577.255	13.8634	575.903	13.8645	576.169
13.8642	574.664	13.8658	574.773	13.8713	575.390
13.8613	576.405	13.8547	577.391	13.8593	575.108
13.8706	574.306	13.8519	577.057	13.8522	576.663
13.8601	577.095	13.863	577.286	13.8489	578.358
13.866	576.957	13.8581	575.510	13.8609	575.371
13.8655	576.434	13.8644	576.509	13.857	575.851
13.8612	575.211	13.8665	574.300	13.8566	575.644
13.8598	576.630	13.8648	575.846	13.864	574.462

53. Hoaglin (1980) suggested a "Poissonness plot"—a simple visual method for assessing goodness of fit. The expected frequencies for a sample of size n from a Poisson distribution are

$$E_k = nP(X = k) = ne^{-\lambda}\frac{\lambda^k}{k!}$$

or

$$\log E_k = \log n - \lambda + k \log \lambda - \log k!$$

Thus, a plot of $\log(O_k) + \log k!$ versus k should yield nearly a straight line with a slope of $\log \lambda$ and an intercept of $\log n - \lambda$. Construct such plots for the data of Problems 1, 2, and 3 of Chapter 8. Comment on how straight they are.

54. A random variable X is said to follow a lognormal distribution if $Y = \log(X)$ follows a normal distribution. The lognormal is sometimes used as a model for heavy-tailed skewed distributions.

a. Calculate the density function of the lognormal distribution.

b. Examine whether the lognormal roughly fits the following data (Robson 1929), which are the dorsal lengths in millimeters of taxonomically distinct octopods.

110	15	60	54	19	115	73
190	57	43	44	18	37	43
55	19	23	82	175	50	80
65	63	36	16	10	17	52
43	70	22	95	20	41	17
15	12	11	29	29	61	22
40	17	26	30	16	116	28
32	33	29	27	16	55	8
11	49	82	85	20	67	27
44	16	6	35	17	26	32
76	150	21	5	6	51	75
23	29	64	22	47	9	10
28	18	84	52	130	50	45
12	21	73				

55. a. Generate samples of size 25, 50, and 100 from a normal distribution. Construct probability plots. Do this several times to get an idea of how probability plots behave when the underlying distribution is really normal.

b. Repeat part (a) for a chi-square distribution with 10 df.

c. Repeat part (a) for $Y = Z/U$, where $Z \sim N(0, 1)$ and $U \sim U[0, 1]$ and Z and U are independent.

d. Repeat part (a) for a uniform distribution.

e. Repeat part (a) for an exponential distribution.

f. Can you distinguish between the normal distribution of part (a) and the subsequent nonnormal distributions?

56. Suppose that a sample is taken from a symmetric distribution whose tails decrease more slowly than those of the normal distribution. What would be the qualitative shape of a normal probability plot of this sample?

57. The Cauchy distribution has the probability density function

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \qquad -\infty < x < \infty$$

What would be the qualitative shape of a normal probability plot of a sample from this distribution?

58. Show how probability plots for the exponential distribution, $F(x) = 1 - e^{-\lambda x}$, may be constructed. Berkson (1966) recorded times between events and fit them to an exponential distribution. (The times between events in a Poisson process are exponentially distributed.) The following table comes from Berkson's paper. Make an exponential probability plot, and evaluate its "straightness."

Time Interval (sec)	Observed Frequency
0–60	115
60–120	104
120–181	99
181–243	106
243–306	113
306–369	104
369–432	101
432–497	106
497–562	104
562–628	96
628–698	512
689–1130	524
1130–1714	468
1714–2125	531
2125–2567	461
2567–3044	526
3044–3562	506
3562–4130	509
4130–4758	520
4758–5460	540
5460–6255	542
6255–7174	499
7174–8260	494
8260–9590	500
9590–11,304	550
11,304–13,719	465
13,719–14,347	104
14,347–15,049	97
15,049–15,845	101
15,845–16,763	104
16,763–17,849	92
17,849–19,179	102
19,179–20,893	103
20,893–23,309	110
23,309–27,439	112
27,439+	100

59. Construct a hanging rootogram from the data of the previous problem in order to compare the observed distribution to an exponential distribution.

60. The exponential distribution is widely used in studies of reliability as a model for lifetimes, largely because of its mathematical simplicity. Barlow, Toland, and Freeman (1984) analyzed data on the strength of Kevlar 49/epoxy, a material used in the space shuttle. The times to failure (in hours) of 76 strands tested at a stress level of 90% are given in the following table.

Times to Failure at 90% Stress Level				
.01	.01	.02	.02	.02
.03	.03	.04	.05	.06
.07	.07	.08	.09	.09
.10	.10	.11	.11	.12
.13	.18	.19	.20	.23
.24	.24	.29	.34	.35
.36	.38	.40	.42	.43
.52	.54	.56	.60	.60
.63	.65	.67	.68	.72
.72	.72	.73	.79	.79
.80	.80	.83	.85	.90
.92	.95	.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10
1.11	1.15	1.18	1.20	1.29
1.31	1.33	1.34	1.40	1.43
1.45	1.50	1.51	1.52	1.53
1.54	1.54	1.55	1.58	1.60
1.63	1.64	1.80	1.80	1.81
2.02	2.05	2.14	2.17	2.33
3.03	3.03	3.24	4.20	4.69
7.89				

a. Construct a probability plot of the data against the quantiles of an exponential distribution to assess qualitatively whether the exponential is a reasonable model. Can you explain the peculiar appearance of the plot?

b. Compare the data to the exponential distribution by means of a hanging rootogram.

61. The files `haliburton` and `macdonalds` give the monthly returns on the stocks of these two companies from 1975 through 1999.

a. Make histograms of the returns and superimpose fitted normal densities. Comment on the quality of the fit. Which stock is more volatile?

b. Make normal probability plots and again comment on the quality of the fit.

62. Apply the Poisson dispersion test to the data on gamma-ray counts—Problem 42 of Chapter 8. You will have to modify the development of the likelihood ratio test in Section 9.5 to take account of the time intervals being of different lengths.

63. Construct a gamma probability plot for the data of Problem 46 of Chapter 8.

64. The file `bodytemp` contains normal body temperature readings (degrees Fahrenheit) and heart rates (beats per minute) of 65 males (coded by 1) and 65 females (coded by 2) from Shoemaker (1996).

a. Assess the normality of the male and female body temperatures by making normal probability plots. In order to judge the inherent variability of these plots, simulate several samples from normal distributions with matching means and standard deviations, and make normal probability plots. What do you conclude?

b. Repeat the preceding problem for heart rates.

c. For the males, test the null hypothesis that the mean body temperature is 98.6° versus the alternative that the mean is not equal to 98.6°. Do the same for the females. What do you conclude?

65. This problem continues the analysis of the chromatin data from Problem 45 of Chapter 8 and is concerned with further examining goodness of fit.

a. Goodness of fit can also be examined via probability plots in which the quantiles of a theoretical distribution are plotted against those of the empirical distribution. Following the discussion in Section 9.8, show that it is sufficient to plot the observed order statistics, $X_{(k)}$, versus the quantiles of the Rayleigh distribution with $\theta = 1$. Construct three such probability plots and comment on any systematic lack of fit that you observe. To get an idea of what sort of variability could be expected due to chance, simulate several sets of data from a Rayleigh distribution and make corresponding probability plots.

b. Formally test goodness of fit by performing a chi-squared goodness of fit test, comparing histogram counts to those predicted from the Rayleigh model. You may need to combine cells of the histograms so that the expected counts in each cell are at least 5.