Econometrics A (Econ 210)

Problem Set 1

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Due: Oct 8, 2015; TA Session

1. Suppose D_i and Z_i are two binary random variables where $\Pr[D_i = 1] = p$ and $\Pr[Z_i = 1] = q$. Y_i is a continuous random variable with $\mathbb{E}[Y_i] = 0$. Show the following propositions

   (a) $\mathbb{E}[D] = p$.

   (b) $\text{Var}[D] = p(1-p)$.

   (c) $\dfrac{\text{Cov}[D, Z]}{\text{Var}[D]} = \mathbb{E}[Z|D = 1] - \mathbb{E}[Z|D = 0]$.

   (d) Assume $\mathbb{E}[D] = 0$. Show that if $Y \perp\!\!\!\perp D$ and $Y \perp\!\!\!\perp Z$ then $\mathbb{E}[DZY] = 0$.

   (e) Define $W = 3^D - 1$. Derive $\mathbb{E}[W]$, $\mathbb{E}[W^2]$, and $Var[W]$.

2. Random variable X is distributed with a distribution with density $f(X = x) = e^{-x}$. Y and Z are two other random variables but with the same distribution as X. X, Y, and Z are non-negative and independent of each other.

   (a) Find $\mathbb{E}[X]$ (Hint: use the fact that $\int_0^\infty xe^{-x}dx = 1$).

   (b) Find $\text{Var}[X]$ (Hint: use the fact that $\int_0^\infty x^2e^{-x}dx = 2$).

   (c) Is that true that $\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[Z]$?

   (d) What is the probability that the smallest random variable, $\min\{X, Y, Z\}$, to be greater than or equal to 1?

(e) What is the probability that the largest random variable, $\max\{X, Y, Z\}$, to be greater than or equal to the sum of the other two random variables?

3. Consider random vector $V = (X, Y)'$ with the joint density $f_{X,Y}(X = x, Y = y) = \frac{1}{8}(6 - x - y)$ where $0 \leq x \leq 2$ and $2 \leq y \leq 4$. Find the following conditional moments,

   (a) $\mathbb{E}[Y|X = x]$.

   (b) $\mathbb{E}[Y^2|X = x]$.

   (c) $\text{Var}[Y|X = x]$.

4. Prove the following statements:

   (a) $\mathbb{E}[\mathbb{E}[Y|X, Z]|X] = \mathbb{E}[Y|X]$.

   (b) Prove that if X and Y are independent of each other then $\mathbb{E}[Y|X] = \mathbb{E}[Y]$.

   (c) Prove that if $\mathbb{E}[Y|X] = \mathbb{E}[Y]$ then $\text{Cov}[X, Y] = 0$.

   (d) Is that true that $\text{Cov}[X, Y] = 0$ implies that X and Y are independent?

5. Let U, V, and W be independent, identically distributed random (i.i.d.) variables following a standard normal distribution, i.e. $N(0, 1)$. Let

$$X = 2U + V + 1$$

$$Y = -U + 3W + 3$$

   (a) Is $X \perp\!\!\!\perp Y$? Why or why not?

   (b) Calculate analytically:

      i. $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

      ii. $\text{Var}[X]$ and $\text{Var}[Y]$.

      iii. $\text{Cov}[X, Y]$ and $\text{Corr}[X, Y]$.

iv. $\mathbb{E}[X+Y]$ and $\text{Var}[X+Y]$ (and confirm that $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y]$).

v. $\text{Cov}[X+Y, Y]$ and $\text{Corr}[X+Y, Y]$.

(c) In R, set the random number generator seed to 210, then generate 40 values each for U, V, and W. Then, generate X and Y using those values. Show a scatterplot of X versus Y. Calculate empirically:

vi. $\widehat{\mathbb{E}[X]}$ and $\widehat{\mathbb{E}[Y]}$.

vii. $\widehat{\text{Var}[X]}$ and $\widehat{\text{Var}[Y]}$.

viii. $\widehat{\text{Cov}[X,Y]}$ and $\widehat{\text{Corr}[X,Y]}$.

ix. $\widehat{\mathbb{E}[X+Y]}$ and $\widehat{\text{Var}[X+Y]}$.

x. $\widehat{\text{Cov}[X+Y,Y]}$ and $\widehat{\text{Corr}[X+Y,Y]}$.

Comment on your "small sample" results. What happens as the number of observations goes up?

6. Consider the following discrete joint distribution:

↓ X Y→	1	2	3	4
0	0.1	0.05	0.025	0.025
1	0.07	0.13	0.04	0.06
2	0.1	0.1	0.25	0.05

(a) Find the first two moments - the mean and variance - of the marginal distributions of X and Y. Check that the identity $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ holds.

(b) Confirm that $\mathbb{E}[XY] = 3.19$. Use this in combination with your calculations in (a) above to find $\text{Cov}[X,Y]$.

(c) Find the conditional probability mass function $P(Y = y|X = x)$ and use it to calculate the conditional mean function $\mathbb{E}[Y|X = x]$ for each possible realization of X.

(d) Calculate $\mathbb{E}\left[\mathbb{E}\left[Y|X=x\right]\right]$. With respect to which distribution is the outer expectation taken? The inner expectation? What do you notice about your answer as compared to the means you calculated in (a) above?