1. [10 points]

   (a) $P(X > 2) = 1 - \Phi((2+4)/4) = \Phi(-1.5) = 0.0668$

   (b) $P(0 < X < 4) = \Phi((4+4)/4) - \Phi(4/4) = 0.1359$

   (c) $P(|X+3| \geq 3) = P(X < -6 \cup X > 0) = 1 + \Phi((-6+4)/4) - \Phi(4/4) = 0.4672$

   (d) $P(X \leq 0 \text{ or } X \geq 3) = 1 - \Phi((3+4)/4) + \Phi(4/4) = 0.8814$

2. [30 points] The prior distribution for the true ability, X is $N(70, 8^2)$. Let the midterm score be Y. We view the midterm as our datum, and conditional on the true ability, the midterm has a $N(X, 6^2)$ distribution. Using results from the notes or from section 4.5 in the required reading, the posterior distribution of the students true ability after receiving a midterm score of 90 is normal with mean and variance given below.

$$E(X \mid Y = 90) = \frac{6^2.70 + 8^2.90}{6^2 + 8^2} = 82.8$$

$$Var(X \mid Y = 90) = \frac{6^2.8^2}{6^2 + 8^2} = 23.04$$

$$P(X > 85 \mid Y = 90) = 1 - \Phi\left(\frac{85 - 82.8}{\sqrt{23.04}}\right) \approx 0.3234$$

$$P(X > 90 \mid Y = 90) = 1 - \Phi\left(\frac{90 - 82.8}{\sqrt{23.04}}\right) \approx 0.0668$$

3. [30 points] Let $P(ESP)$ be p, and $P(no\ ESP)$ be $1 - p$.

   Let X be the number of correct guesses among the five trials. Then,

$$X|_{ESP} \sim Bin(5, 0.5) \qquad X|_{no\ ESP} \sim Bin(5, 0.2).$$

$$\begin{aligned} P(X = 3) &= P(X = 3|ESP)P(ESP) + P(X = 3|no\ ESP)P(no\ ESP) \\ &= p\binom{5}{3}(0.5)^5 + (1-p)\binom{5}{3}(0.2)^3(0.8)^2. \end{aligned}$$

$$P(ESP/X = 3) = \frac{p\binom{5}{3}(0.5)^5}{p\binom{5}{3}(0.5)^5 + (1-p)\binom{5}{3}(0.2)^3(0.8)^2}.$$

   So the minimum value of the a priori probability, for the posteriori probability to be at least 0.7, is $\frac{.7\times(0.2)^3(0.8)^2}{0.3\times(0.5)^5 + 0.7\times(0.2)^3(0.8)^2} = 0.2766$.

4. [30 points] Let σ_i^2 be the variance of estimator $\hat{\theta}_i$, $i = 1, 2, 3$.

   (a) $E(\hat{\theta}_3) = cE(\hat{\theta}_1) + (1-c)E(\hat{\theta}_2) = c\theta + (1-c)\theta = \theta.$

(b) $\sigma_3^2 = 2c^2\sigma_2^2 + (1-c)^2\sigma_2^2 = (3c^2 - 2c + 1)\sigma_2^2$.
Minimizing this quantity we get that $c = 1/3$ gives a minimum.

(c) Surely for $c = 1/3$ the MSE is better than both $\hat{\theta}_1$ and $\hat{\theta}_2$ since in this case $\sigma_3^2 = (2/3)\sigma_2^2 < \sigma_2^2 < \sigma_1^2$.
Now for other values of c, we need $(3c^2 - 2c + 1) < 1$. This gives $0 < c < 2/3$.