Kilonovae Inference

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1 Derivation of the Model Posterior

We model the luminosity evolution of kilonovae using a power law, where the absolute magnitude in any band (M) is given as a function of time

$$M(t) = M_o \left(\frac{t_0}{t}\right)^{\gamma} \tag{1}$$

We can see that the two parameters (M_o, γ) completely define the evolution at all times. Note, one can define different and more complicated functions to incorporate different models that maybe motivated from numerical modelling but the idea here is to present a derivation of the inference to constrain parameters of interest.

Before we begin with our derivation, we will state our notation as follows:

1. N_E : The total number of events that were followed up, indexed by i
2. N_F : The total number of fields for which EM observations have been recorded, indexed by f
3. t_j^f : The time of observation,indexed by j for each field f over the duration of follow-up of the event. j would run over the total number of observations for each field
4. $P^i(A)$, $P^i(T)$: The probabilities of the i^{th} event being of astrophysical and terrestrial origin, respectively. This is an estimate provided by LIGO for the associated gravitational-wave (GW) event. It can either be a low-latency estimate or an update provided after a refined analysis.
5. $\{data^i\}$: The set of EM data associated with all events, indexed by i. We will further sub-index this data by the field index f and time of observation index j, in our derivation below. We assume our data (for now) to be the limiting apparent magnitudes m_l in each field at the given time of observation.

We are now ready to begin our derivation of the model posterior. We start with the basic equation of Bayes' law:

$$p(M_0, \gamma | \{data^i\}) = \frac{p(\{data^i\} | M_0, \gamma) p(M_0, \gamma)}{p(\{data^i\})} \tag{2}$$

Since the evidence $p(\{data^i\})$ is a constant of normalization, and does not affect the inference of posterior parameters directly, we can choose to neglect it. Further, we also make a reasonable assumtion to treat each event and its associated data to be independent. The likelihood $p(\{data^i\}|M_0, \gamma)$ can then be written as a product over events.

$$p(M_0, \gamma|\{data^i\}) \propto \prod_{i=1}^{N_e} \Bigg[p(data^i|M_0, \gamma) \Bigg] p(M_0, \gamma) \tag{3}$$

There are two possibilities for any given event - either the event is astrophysical or it is of the terrestrial kind. We can thus split the likelihood into two terms

$$p(M_0, \gamma|\{data^i\}) \propto \prod_{i=1}^{N_e} \Bigg[p(data^i|M_0, \gamma, A)P^i(A) + p(data^i|T)P^i(T) \Bigg] p(M_0, \gamma) \tag{4}$$

The assumption in the last term (terrestrial) in the parentheses is that the contribution to the likelihood cannot depend on the parameters of the kilonovae distribution if the event is of terrestrial origin.This is easy to check because in the case of a purely terrestrial event ($p^i(A) = 0$) we must recover the prior when performing inference.

We now use the fact that EM observations are spread over a number (N_F) of fields and each field with its associated limiting magnitude (m_l) will contribute to the inference separately. Furthermore, for any given field there are two possibilities - either the kilonova is contained within the field (hypothesis f) or it is not (hypothesis $\bar{f}$). An important quantity to compute then, is the probability that the kilonova is within a field f i.e. $P(f)$

When information about a GW candidate event is released to the public, it contains the 3D probability distribution (including sky location and distance) of the location of the event [1, 2]. Using this, it is straightforward to compute the probability for a kilonova to be present in a given field. Referring to eqn.(3) in [1], for the sum of probabilities over the sky

$$P(f) = \sum_{p=0}^{N_{pix}^f} \rho_p \tag{5}$$

where the sum is over the N_{pix}^f pixels that are contained within field f. Our posterior then becomes

$$
\begin{aligned}
p(M_0,\gamma|\{data^i\}) \propto \prod_{i=1}^{N_e}\Bigg[\Bigg(\sum_{f=1}^{N_F} p(m_l^{i,f}|M_0,\gamma,A,f)\prod_{\bar{f}} p(m_l^{i,\bar{f}}|A,f)P(f) \\
+\prod_{f} p(m_l^{i,f}|A,\bar{f})\Bigg(1-\sum_{f=1}^{N_F}P(f)\Bigg)\Bigg)p^i(A) \\
+\prod_{f} p(m_l^{i,f}|T)p^i(T)\Bigg]p(M_0,\gamma)
\end{aligned} \tag{6}
$$

Since the observations in each field will have observation times associated with them, each field would thus constrain the model independently. Thus, the likelihood term for each field can be written as a product over the observation times corresponding to that field.

$$
\begin{aligned}
p(M_0,\gamma|\{data^i\}) \propto \prod_{i=1}^{N_e}\Bigg[\Bigg(\sum_{f=1}^{N_F}\Big(\prod_{j=t_1^f}^{t_{N_f}^f} p(m_l^{i,f,j}|M_0,\gamma,A,f)\Big)\prod_{\bar{f}} p(m_l^{i,\bar{f}}|A,f)P(f) \\
+\prod_{f} p(m_l^{i,f}|A,\bar{f})\Bigg(1-\sum_{f=1}^{N_F}P(f)\Bigg)\Bigg)p^i(A) \\
+\prod_{f} p(m_l^{i,f}|T)p^i(T)\Bigg]p(M_0,\gamma)
\end{aligned} \tag{7}
$$

We now focus on the first term in the likelihood for each field $p(m_l^{i,f,j}|M_0,\gamma,A,f)$. In order to simplify this term and derive an expression for the same, we note that in reality a telescope measures an apparent magnitude instead of an absolute magnitude.The relationship between the apparent magnitude (m), absolute magnitude (M) and luminosity distance (d) of an astrophysical transient is given as

$$
m = M + 5log_{10}\left(\frac{d}{10pc}\right) \tag{8}
$$

where we neglect the effects of extinction [1]. Thus, one can rewrite this likelihood term as

$$
p(m_l^{i,f,j}|M_0,\gamma,A,f) = \int_{-\infty}^{\infty} p(m_l^{i,f,j}|m_j)p(m_j|M_0,\gamma,A,f)dm_j \tag{9}
$$

The intrinsic parameters of the kilonova (M_o,γ) along-with the observation time (t_j) uniquely determine the absolute magnitude ($M_j(M_o,\gamma)$) at any given time. Using eqn.(8) above , one

[1] It should be straightforward to include extinction.

can relate the apparent magnitude distribution ($p(m_j|M_0, \gamma, A, f)$) in the integral above to the marginal distance distribution ($p_f(d)$) for each field, given by LIGO as

$$p(m_j|M_0, \gamma, A, f) = p_f(d)\frac{dd}{dm_j}(M_0, \gamma) \tag{10}$$

We can thus rewrite the integral in eqn.(9) as

$$p(m_l^{i,f,j}|M_0, \gamma, A, f) = \int_0^\infty p(m_l^{i,f,j}|m_j(d, M_0, \gamma))p_f(d)dd \tag{11}$$

In order to simplify eqn.(11) further, we focus once again on eqn.(8). Since our data are comprised of limiting magnitudes, we can conclude that our kilonova model must generate apparent magnitudes greater than these limiting magnitudes for each observation. Any apparent magnitude greater than the corresponding limiting magnitude, would be supported equally. In other words,

$$p(m_l^{i,f,j}|m_j)\begin{cases} = 0; m_l^{i,f,j} \geq m_j, \\ = k; m_l^{i,f,j} < m_j \end{cases} \tag{12}$$

The constant k can be derived by normalizing the likelihood between the limits m_l^{low}, m_l^{high} of the limiting magnitude distribution of the survey. The boundary case in the equation above gives us a unique limiting distance (d_{lim}) for a given limiting magnitude ($m_l^{i,f,j}$) and absolute magnitude (M_j) [2]

$$d_{lim} = 10^{\left(\frac{m_l^{i,f,j} - M_j}{5}\right)} 10pc \tag{13}$$

Any model value of M_o, γ, that gives a permissible value of the apparent magnitude for a given time of observation will be favored over those that do not. This corresponds to $d_{lim} \leq d$. Combining eqns. (12-13) we can write the likelihood in eqn.(11) as

$$p(m_l^{i,f,j}|M_0, \gamma, A, f) = \int_{d_{lim(m_l^{i,f,j}, M_j)}}^\infty kp_f(d)dd \tag{14}$$

where the value of the proportionality constant k is given by the likelihood normalization as

$$k = \frac{1}{\int_{m_l^{low}}^{m_l^{high}} \left(\int_{d_{lim(m_l^{i,f,j}, M_j)}}^\infty p_f(d)dd \right) dm_l^{i,f,j}} \tag{15}$$

[2]We can also account for uncertainties in the measurement of $m_l^{i,f,j}$ and change the limit appropriately

This normalization also ensures that we account for selection effects based on the limiting magnitude limits of the survey. Since the likelihood is a sum of probability densities, care must be taken to normalize each term in the sum. This implies, that the remaining terms in the field (f), non-field (f_{bar}) and terrestrial (T) hypotheses must be normalized. We assume that each of these terms follows a uniform distribution between the survey limits m_l^{low}, m_l^{high}. This gives us a normalized density as

$$p(m_l^{i,\bar{f}}|A, f) = p(m_l^{i,f}|A, \bar{f}) = p(m_l^{i,f}|T) = \frac{1}{\left(m_l^{high} - m_l^{low}\right)} \tag{16}$$

Using the likelihood definitions as in eqns. (14-16) and the posterior defined in eqn. (7), we carry out inference using an MCMC frame-work such as emcee [3]

References

[1] Leo P. Singer et al. GOING THE DISTANCE: MAPPING HOST GALAXIES OF LIGO AND VIRGO SOURCES IN THREE DIMENSIONS USING LOCAL COSMOGRAPHY AND TARGETED FOLLOW-UP. *The Astrophysical Journal*, 829(1):L15, sep 2016.

[2] Leo P. Singer et al. SUPPLEMENT: "GOING THE DISTANCE: MAPPING HOST GALAXIES OF LIGO AND VIRGO SOURCES IN THREE DIMENSIONS USING LOCAL COSMOGRAPHY AND TARGETED FOLLOW-UP" (2016, ApJL, 829, l15). *The Astrophysical Journal Supplement Series*, 226(1):10, sep 2016.

[3] Daniel Foreman-Mackey, David W. Hogg, Dustin Lang, and Jonathan Goodman. emcee: The MCMC hammer. *Publications of the Astronomical Society of the Pacific*, 125(925):306–312, mar 2013.