

# Applied Linear Algebra in Data Analysis

## Solution to Linear Equations

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# Linear equations

- Matrices present a compact way to represent a set of linear equations. Consider the following,

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 \dots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 \dots + a_{2m}x_m = b_2 \\ a_{31}x_1 + a_{32}x_2 \dots + a_{3m}x_m = b_3 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 \dots + a_{nm}x_m = b_n \end{array} \right\} \longrightarrow \mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{b} \in \mathbb{R}^m$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

# Linear equations in control problems

$\mathbf{x}$  : Input    $\mathbf{b}$  : Output    $\mathbf{A}$  : System dynamics

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

# Linear equations in estimation problems

**x** : Parameter   **b** : Measurements   **A** : System characteristics

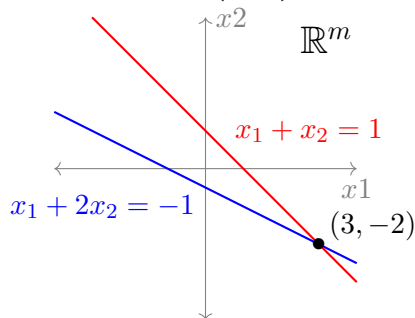
$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

# Geometry of linear equations

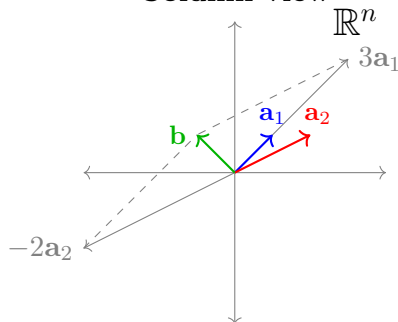
$$\left. \begin{array}{l} x_1 + 2x_2 = -1 \\ x_1 + x_2 = 1 \end{array} \right\} \longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two ways to view this: **row view** and the **column view**.

**Traditional (row) view**



**Column view**

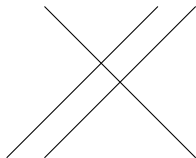


# Solutions of linear equations

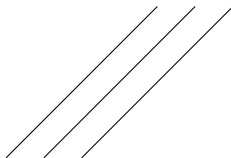
$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{b} \in \mathbb{R}^m$$

- ▶ **Three possible situations:** NO SOLUTION, INFINITELY MANY SOLUTIONS, or UNIQUE SOLUTION.
- ▶ When do we have infinitely many or no solutions? In  $\mathbb{R}^3$ , we can visualize the different situations.

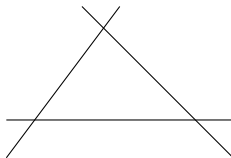
Two parallel planes



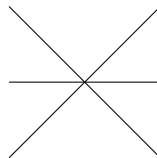
Three parallel planes



No intersection



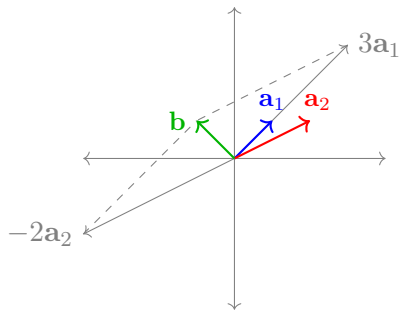
Line intersection



# Understanding $Ax = b$ : Unique solution

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- ▶ Square matrix
- ▶ Linearly independent set of columns  $\{\mathbf{a}_1, \mathbf{a}_2\}$
- ▶  $\mathbf{b} \in \text{span}(\{\mathbf{a}_1, \mathbf{a}_2\})$ .
- ▶ Always solvable, and give an unique solution.



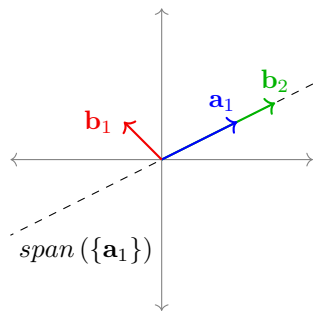
# Understanding $Ax = b$ : Unique solution or No solution

1.  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} [x_1] = \mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2.  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} [x_1] = \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}$

- ▶ Tall matrix
- ▶ Linearly independent set of columns  $\{\mathbf{a}_1\}$

$\mathbf{b}_1 \notin \text{span}(\{\mathbf{a}_1\}) \implies$  Not solvable.

$\mathbf{b}_2 \in \text{span}(\{\mathbf{a}_1\}) \implies$  Solvable with unique solution.

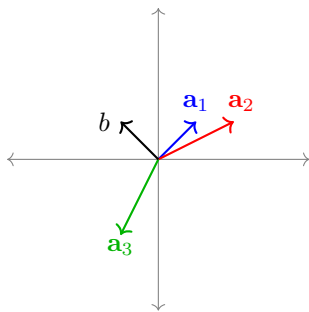




# Understanding $Ax = b$ : Infinitely many solution

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Fat matrix
- Linearly dependent set of columns  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$
- $\mathbf{b} \in \text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\})$ .
- Always solvable, with infinitely many solutions.



# Understanding $\mathbf{Ax} = \mathbf{b}$ : Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{b} \in \mathbb{R}^n$$

**Full rank  $\mathbf{A}$ :**

►  $\text{rank}(\mathbf{A}) = n \implies$  **Always solvable**

$$\begin{cases} n = m & \implies \text{Unique solution} \\ n < m & \implies \text{Infinitely many solutions} \end{cases}$$

►  $\text{rank}(\mathbf{A}) = m \implies$  **No infinite solutions**

$$\begin{cases} m = n & \implies \text{Unique solution} \\ m < n & \rightarrow \begin{cases} \mathbf{b} \in \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies \text{Unique solution} \\ \mathbf{b} \notin \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies \text{No solution} \end{cases} \end{cases}$$

# Understanding $\mathbf{Ax} = \mathbf{b}$ : Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{b} \in \mathbb{R}^n$$

**Rank deficient  $\mathbf{A}$ :**

►  $\text{rank}(\mathbf{A}) < \min(n, m) \implies$  **No unique solution**

$$\begin{cases} \mathbf{b} \in \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies \text{Infinitely many solutions} \\ \mathbf{b} \notin \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies \text{No solution} \end{cases}$$

# Understanding $\mathbf{Ax} = \mathbf{b}$ : Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{b} \in \mathbb{R}^n$$

- ▶  $\mathbf{b} \notin \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies$  No solution
- ▶  $\mathbf{b} \in \text{span}(\mathbf{a}_1, \dots, \mathbf{a}_m) \implies \begin{cases} \text{rank}(\mathbf{A}) = m \implies \text{Unique} \\ \text{rank}(\mathbf{A}) < m \implies \text{Infinitely many solutions} \end{cases}$

# General solution of linear equations

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{b} \in \mathbb{R}^n$$

- ▶ Assuming that this system can be solved, the most general form of the solution is,

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

where,  $\mathbf{x}_p$  is called the particular solution, and  $\mathbf{x}_h$  is the homogenous solution.

- ▶ **Homogenous solution:** Solution of the equation  $\mathbf{Ax} = \mathbf{0}$ .
- ▶ The set of all homogenous solutions of  $\mathbf{A} - \{\mathbf{x}_h \mid \mathbf{Ax}_h = \mathbf{0}\}$  – form a subspace of  $\mathbb{R}^m$ .

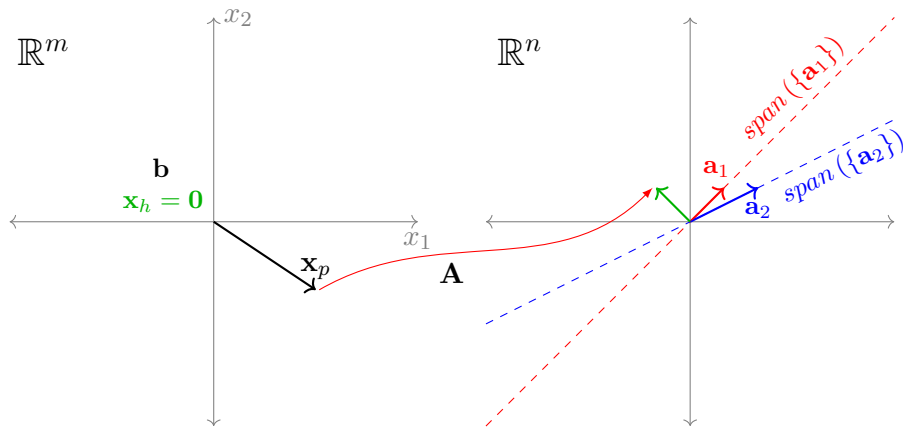
# Geometry of the general solution

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{b} \in \mathbb{R}^n$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

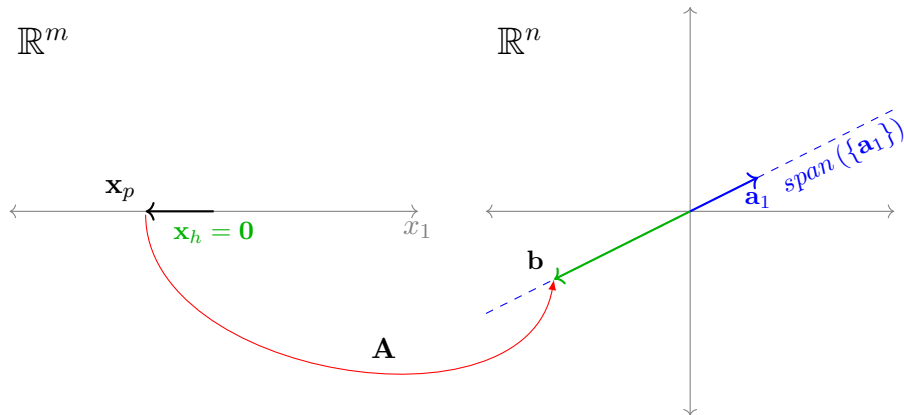
# Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



# Geometry of the general solution

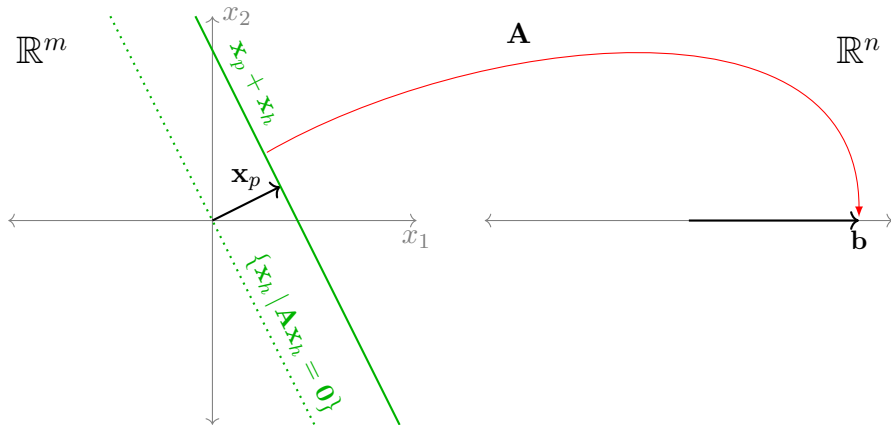
$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$





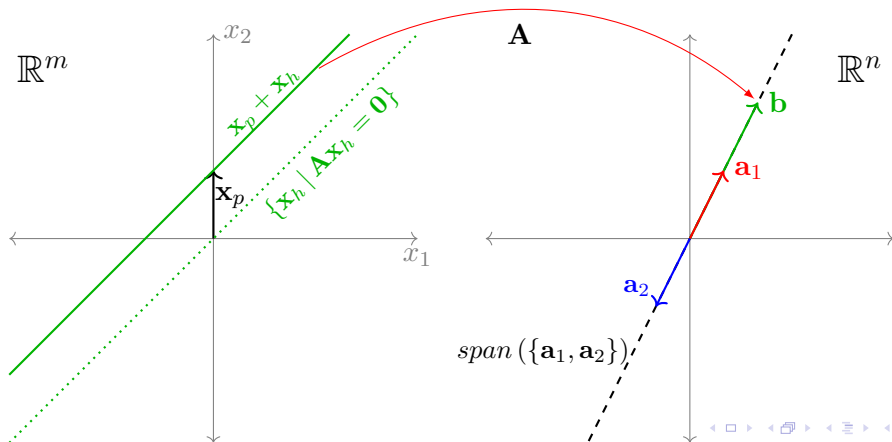
# Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \end{bmatrix}$$



# Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



## Four Fundamental Subspaces of $\mathbf{A} \in \mathbb{R}^{n \times m}$

- $\mathcal{C}(\mathbf{A})$ : **Column Space of  $\mathbf{A}$**  – the span of the columns of  $\mathbf{A}$ .

$$\mathcal{C}(\mathbf{A}) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

- $\mathcal{N}(\mathbf{A})$ : **Nullspace of  $\mathbf{A}$**  – the set of all  $\mathbf{x} \in \mathbb{R}^m$  that are mapped to zero by  $\mathbf{A}$ .

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

- $\mathcal{C}(\mathbf{A}^\top)$ : **Row Space of  $\mathbf{A}$**  – the span of the rows of  $\mathbf{A}$ .

$$\mathcal{C}(\mathbf{A}^\top) = \{\mathbf{A}^\top \mathbf{y} \mid \mathbf{y} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

- $\mathcal{N}(\mathbf{A}^\top)$ : **Nullspace of  $\mathbf{A}^\top$**  – the set of all  $\mathbf{y} \in \mathbb{R}^n$  that are mapped to zero by  $\mathbf{A}^\top$ .

$$\mathcal{N}(\mathbf{A}^\top) = \{\mathbf{y} \mid \mathbf{A}^\top \mathbf{y} = \mathbf{0}\} \subseteq \mathbb{R}^n$$

This is also called the **left nullspace** of  $\mathbf{A}$ .

# Linear Independence

- ▶ Given a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ ,  $\mathbf{v}_i \in \mathbb{R}^n$ , how can we determine if this set is linear independent?
- ▶ We need to verify,  $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_m\mathbf{v}_m = \mathbf{0}$

$$\left[ \mathbf{v}_1 \quad \dots \quad \mathbf{v}_m \right] \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{V}\mathbf{a} = \mathbf{0} \left\} \mathcal{N}(\mathbf{V}) = \{\mathbf{0}\}, \text{ rank}(\mathbf{V}) = n$$

- ▶ This is also equivalent to saying that when the  $\text{rank}(\mathbf{A}) = n \implies$  the columns of  $\mathbf{A}$  form an independent set of vectors.
- ▶ When do the rows of  $\mathbf{A}$  form an independent set?
- ▶ What about both rows and columns? When does that happen?

# Dimension of the four fundamental subspaces

- ▶ **Column space**  $C(\mathbf{A})$ 
  - ▶  $\dim C(\mathbf{A}) = \text{rank}(\mathbf{A}) = r$
- ▶ **Nullspace**  $N(\mathbf{A})$ 
  - ▶  $\dim N(\mathbf{A}) = n - r$
- ▶ **Row space**  $C(\mathbf{A}^\top)$ 
  - ▶  $\dim C(\mathbf{A}^\top) = \text{rank}(\mathbf{A}^\top) = \text{rank}(\mathbf{A}) = r$
- ▶ **Left Nullspace**  $N(\mathbf{A}^\top)$ 
  - ▶  $\dim N(\mathbf{A}^\top) = m - r$