Applied Linear Algebra in Data Analysis

Tutorial

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4.1 Product of orthogonal matrices

Show that the product of a set of orthogonal matrices is orthogonal.

$$\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_l^\top$$

where, $\mathbf{Q}_i \in \mathbb{R}^{n\times n}$ and $\mathbf{Q}_i^\top\mathbf{Q}_i = \mathbf{I}$.

Proof: The matrix $\mathbf{Q}$ is orthogonal if and only if $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$.
We know that,

$$\begin{aligned} \mathbf{Q}^\top &= (\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_l)^\top = \mathbf{Q}_l^\top\mathbf{Q}_{l-1}^\top\cdots\mathbf{Q}_2^\top\mathbf{Q}_1^\top \\ \implies \mathbf{Q}^\top\mathbf{Q} &= \mathbf{Q}_l^\top\mathbf{Q}_{l-1}^\top\cdots\mathbf{Q}_2^\top\mathbf{Q}_1^\top\mathbf{Q}_1\mathbf{Q}_2\cdots\mathbf{Q}_{l-1}\mathbf{Q}_l \end{aligned}$$

Since, $\mathbf{Q}_i^\top\mathbf{Q}_i = \mathbf{I}$, we have,

$$\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$$

We can similarly show that $\mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$. Thus, the product of a set of orthogonal matrices is orthogonal. ■

4.2 Orthogonal projections of vectors

Let $\mathbf{u} \in \mathbf{R}^n$ be a unit vector. Show that the orthogonal projections on x – $\mathbf{x_u} = \left(\mathbf{u}\mathbf{u}^\top\right)\mathbf{x}$ and $\mathbf{x}_{\mathbf{u}^\perp} = \left(\mathbf{I} - \mathbf{u}\mathbf{u}^\top\right)\mathbf{x}$ – are orthogonal complements.

Proof: The orthogonal projection matrices $\mathbf{u}\mathbf{u}^\top$ and $\mathbf{I} - \mathbf{u}\mathbf{u}^\top$ project any vector onto the $\mathrm{span}\{\mathbf{u}\}$ and the orthogonal complement of $\mathrm{span}\{\mathbf{u}\}$ respectively.
Its easy that $\mathbf{x_u}$ and $\mathbf{x}_{\mathbf{u}^\perp}$ are complementary components of $\mathbf{x}$,

$$\mathbf{x} = \mathbf{x_u} + \mathbf{x}_{\mathbf{u}^\perp} = \left(\mathbf{u}\mathbf{u}^\top\right)\mathbf{x} + \left(\mathbf{I} - \mathbf{u}\mathbf{u}^\top\right)\mathbf{x}$$

They are also orthogonal complements because of the following,

$$\begin{aligned} \mathbf{x_u}^\top\mathbf{x}_{\mathbf{u}^\perp} &= \left(\mathbf{u}\mathbf{u}^\top\mathbf{x}\right)^\top\left(\left(\mathbf{I} - \mathbf{u}\mathbf{u}^\top\right)\mathbf{x}\right) \\ &= \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\left(\mathbf{I} - \mathbf{u}\mathbf{u}^\top\right)\mathbf{x} \\ &= \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\mathbf{x} - \mathbf{x}^\top\mathbf{u}\left(\mathbf{u}^\top\mathbf{u}\right)\mathbf{u}^\top\mathbf{x} \\ &= \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\mathbf{x} - \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\mathbf{x} \\ &= 0 \end{aligned}$$

Using the fact that, $\mathbf{u}^\top\mathbf{u} = 1$,

$$\mathbf{x_u}^\top\mathbf{x}_{\mathbf{u}^\perp} = \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\mathbf{x} - \mathbf{x}^\top\mathbf{u}\mathbf{u}^\top\mathbf{x} = 0$$

Thus, $\mathbf{x_u}$ and $\mathbf{x}_{\mathbf{u}^\perp}$ are orthogonal to each other. ■

5 MATRIX INVERSES