

Applied Linear Algebra in Data Analysis

Tutorial

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1 CONCEPTS IN VECTOR SPACES

1. Which of the following sets forms a vector space?

- a) $\{\mathbf{x} \mid x_1, x_2 \in \mathbb{R} \text{ and } a_1 x_1 + a_2 x_2 = 0\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.
- b) $\{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{a}^\top \mathbf{x} = b\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.
- c) $\{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^\top \mathbf{x} = 1\}$.
- d) $\{(x[0], x[1], x[2], \dots, x[N-1]) \mid x[i] \in \mathbb{R}, 0 \leq i < N\}$.

(The set of all real-valued time-domain signals of length N . $x[i]$ is the value of the signal at time instant i .)

2. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^n a_k x^k \mid a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0, 1]$$

Show that polynomials of order strictly lower than n form subspaces of \mathcal{P} .

3. Is the following function a valid norm of the vector space \mathcal{P} ?

$$\|\mathbf{p}(x)\| = \sqrt{\sum_{k=0}^n a_k^2}, \quad \mathbf{p} = \sum_{k=0}^n a_k x^k \in \mathcal{P}$$

4. Consider the following function, which is often called the *zero-norm* of a vector $\mathbf{x} \in \mathbb{R}^n$.

$$\|\mathbf{x}\|_0 = \sum_{i=1}^n \mathbb{I}(x_i \neq 0), \text{ where, } \mathbb{I}(A) = \begin{cases} 1 & A \text{ is true.} \\ 0 & A \text{ is false.} \end{cases}$$

Is the *zero-norm*, which is often used for quantifying the *sparsity* of a vector, a proper norm?

5. Is the following set of vectors linear independent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

What is the span of this set? Does this set form the basis for its span? Does it form an orthonormal basis?

2 MATRICES

1. Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Find the product of the two matrices $\mathbf{C} = \mathbf{AB}$ using the four views of matrix multiplication.

If we change $b_{23} = 0$. Can you compute the new matrix \mathbf{C} without performing the matrix multiplication?

If we increase the value of the elements of the 3rd column of \mathbf{A} by 1, how can we compute the new \mathbf{C} without performing the matrix multiplication?

If we insert a new row $\mathbf{1}^\top$ in \mathbf{A} after the 2nd row, how can we compute the new \mathbf{C} without performing the matrix multiplication?

2. Show that the matrix product \mathbf{ABC} can be written as a weighted sum of the outer products of the columns of \mathbf{A} and rows of \mathbf{C} , with the weights coming from the matrix \mathbf{B} .
3. Prove the following for the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$.

$$(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n)^\top = \mathbf{A}_n^\top \mathbf{A}_{n-1}^\top \dots \mathbf{A}_2^\top \mathbf{A}_1^\top$$

4. **Nilpotent matrices.** Show that a strictly triangular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{A}^n = \mathbf{0}$.
5. **Matrix Inversion Lemma.** Consider an invertible matrix \mathbf{A} . The matrix $\mathbf{A} + \mathbf{uv}^\top$ is invertible if and only if the two vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u} \neq -1$. Then, the inverse is given by,

$$(\mathbf{A} + \mathbf{uv}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{uv}^\top \mathbf{A}^{-1}}{1 + \mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u}}$$

6. Prove that $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, where $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$.
7. Effect of matrix operation on matrix rank. Let $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$, with ranks a and b respectively. What is the rank of the following matrices?
 - a) $\mathbf{A} + \mathbf{B}$
 - b) \mathbf{AB}
8. Show that the rank $(\mathbf{AB}) = \text{rank}(\mathbf{A})$, when \mathbf{B} is square and full rank.
9. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, \mathbf{AB} is non-singular if and only if both \mathbf{A} and \mathbf{B} are non-singular.
10. Let \mathbf{A} is a full rank matrix. Show that the *Gram matrix* of the column space, $\mathbf{A}^\top \mathbf{A}$ is invertible.