

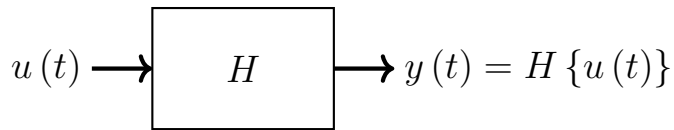
Applied Linear Algebra for Data

Application: Linear Dynamical Systems

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Linear System



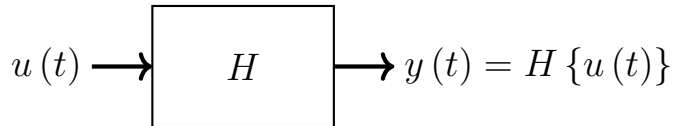
Behavior of dynamic systems can be described mathematically through differential equations (continuous-time systems), or difference equations (discrete-time systems).

A system is **linear** iff,

$$y_1(t) = H\{u_1(t)\} \text{ and } y_2(t) = H\{u_2(t)\}$$

$$\begin{aligned} \implies H\{a_1x_1(t) + a_2u_2(t)\} &= a_1H\{u_1(t)\} + a_2H\{u_2(t)\} \\ &= a_1y(t) + a_2y_2(t), \quad a_1, a_2 \in \mathbb{C} \end{aligned}$$

Time-Invariant System

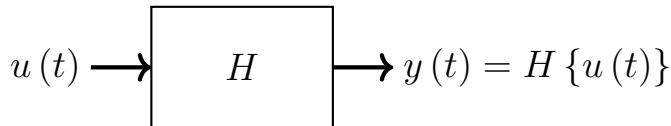


- A system is **time-invariant** if,

$$y(t) = H\{u(t)\} \implies H\{u(t - \tau)\} = y(t - \tau)$$

- Characteristics of the system do not change with time. Time-shifted inputs produce correspondingly time-shifted output.

Linear Time-Invariant System



LTI systems: both **linear** and **time-invariant**. These are described through constant coefficient linear differential (or difference) equations.

Continuous-time:

$$\frac{d^n}{dt^n}y(t) + a_1\frac{d^{n-1}}{dt^{n-1}}y(t) + \dots + a_ny(t) = u(t) + b_1\frac{d}{dt}u(t) + \dots + b_m\frac{d^m}{dt^m}u(t)$$

Discrete-time:

$$y[k-n] + a_1y[k-n+1] + \dots + a_ny[k] = u[k] + b_1u[k-1] + \dots + b_mu[k-m]$$

State space representation of linear systems

- In the case of a linear system, the equations representing the dynamics takes a simpler form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t) \mathbf{x}(t) + \mathbf{B}(t) \mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t) \mathbf{x}(t) + \mathbf{D}(t) \mathbf{u}(t)$$

where,

- $\mathbf{A}(t) \in \mathbb{R}^{n \times n}$ is the *system* matrix.
- $\mathbf{B}(t) \in \mathbb{R}^{n \times p}$ is the *input* matrix.
- $\mathbf{C}(t) \in \mathbb{R}^{m \times n}$ is the *output* matrix.
- $\mathbf{D}(t) \in \mathbb{R}^{m \times p}$ is the *feedforward* matrix.

State space representation of linear systems

- In the case of time-invariant system, the matrices are constant.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- These two equations represent how the states and the measured outputs of the system are affected by the current states and inputs. The individual terms in these matrices indicate how a particular state/input affects another state/output.

State space representation of discrete-time linear systems

- Discrete-time linear system,

$$\mathbf{x}[k+1] = \mathbf{A}[k] \mathbf{x}[k] + \mathbf{B}[k] \mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k] \mathbf{x}[k] + \mathbf{D}[k] \mathbf{u}[k]$$

where, $k \in \mathbb{Z}$ correspond to time index.

- $\mathbf{A}[k] \in \mathbb{R}^{n \times n}$ is the *system* matrix.
 - $\mathbf{B}[k] \in \mathbb{R}^{n \times p}$ is the *input* matrix.
 - $\mathbf{C}[k] \in \mathbb{R}^{m \times n}$ is the *output* matrix.
 - $\mathbf{D}[k] \in \mathbb{R}^{m \times p}$ is the *feedforward* matrix.
- In the case of time-invariant system, the matrices are constant.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

Zero-input solution for $\mathbf{x}(t)$

- **Zero-Input Solution:** We will start by assuming $\mathbf{u}(t) = \mathbf{0}$.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

And that the initial value of the state at time $t = 0^-$ is $\mathbf{x}(0^-) = \mathbf{x}_0$.

- The solution to this differential equation is given by,

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}_0, \quad t \geq 0$$

where, $e^{t\mathbf{A}}$ is the matrix exponential of \mathbf{A} .

- Note the interesting similarity to the scalar case: $\dot{x}(t) = ax(t)$, we know the solution to be the following, $x(t) = e^{at}x(0^-)$.

What is the matrix exponential $e^{t\mathbf{A}}$?

- Functions of matrices are often defined to have properties consistent with that of their scalar counterparts.

$$e^{t\mathbf{A}} = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}t^k\mathbf{A}^k \implies \frac{d}{dt}e^{t\mathbf{A}} = \mathbf{A}e^{t\mathbf{A}}$$

- But we do not have to compute all the powers of \mathbf{A} to compute the matrix exponential. Thanks to the Cayley-Hamilton theorem,

Cayley-Hamilton Theorem

Every square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfies its own characteristic equation $p(\lambda) = 0$, i.e. $p(\mathbf{A}) = \mathbf{0}$.

$$p(\mathbf{A}) = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \dots + a_n\mathbf{I} = \mathbf{0}$$

What is the matrix exponential $e^{t\mathbf{A}}$?

- $e^{t\mathbf{A}}$ can be computed as the following,

$$e^{t\mathbf{A}} = \sum_{k=0}^{n-1} c_k t \mathbf{A}^k$$

- When the matrix \mathbf{A} is diagonalizable, then the matrix exponential can be computed as follows,

$$\begin{aligned} e^{t\mathbf{A}} &= \sum_{k=0}^{n-1} c_k t^k \mathbf{A}^k = \sum_{k=0}^{n-1} c_k t^k (\mathbf{V} \mathbf{\Lambda} \mathbf{V}^{-1})^k \\ &= \sum_{k=0}^{n-1} c_k t \mathbf{V} \mathbf{\Lambda}^k \mathbf{V}^{-1} = \mathbf{V} \left(\sum_{k=0}^{n-1} c_k t \mathbf{\Lambda}^k \right) \mathbf{V}^{-1} \\ &= \mathbf{V} e^{t\mathbf{\Lambda}} \mathbf{V}^{-1} \end{aligned}$$

Usefulness of the $e^{t\mathbf{A}}$?

- Let $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^n$ be the n eigenpairs of the matrix \mathbf{A} . Then,

$$e^{t\mathbf{\Lambda}} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{t\lambda_n} \end{bmatrix}$$
$$e^{t\mathbf{A}} = \mathbf{V}e^{t\mathbf{\Lambda}}\mathbf{V}^{-1}$$

The eigenvectors in this case are referred to as the eigenmodes of the system, which are a characteristic of the system. The eigenvalues are the corresponding rates of exponential growth/decay of the eigenmodes.

- The representation allows us to express the response to any initial condition as the linear combination of the exponential evolution of the eigenmodes.

Usefulness of the $e^{t\mathbf{A}}$

- Let \mathbf{x}_0 is the initial state at time $t = 0^-$. Then the zero-input response can be expressed as the following,

$$\mathbf{x}(t) = \mathbf{V}e^{t\mathbf{A}}\mathbf{V}^{-1}\mathbf{x}_0$$

We know, $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$, and let $\mathbf{V}^{-1} = \begin{bmatrix} \tilde{\mathbf{v}}_1^\top \\ \tilde{\mathbf{v}}_2^\top \\ \vdots \\ \tilde{\mathbf{v}}_n^\top \end{bmatrix}$.

Then we can expressed the above expression as,

$$\mathbf{x}(t) = \sum_{i=1}^n e^{t\lambda_i} \mathbf{v}_i \tilde{\mathbf{v}}_i^\top \mathbf{x}_0$$

$\mathbf{v}_i \tilde{\mathbf{v}}_i^\top \mathbf{x}_0$ is the projection of the initial state \mathbf{x}_0 onto the i -th eigenmode. The exponential term $e^{t\lambda_i}$ is the rate of exponential growth/decay of the i -th eigenmode.

Solution for discrete-time LTI system

- System equations:

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

- Zero-input solution:

$$\mathbf{x}[k] = \mathbf{A}^k \mathbf{x}[0]$$

- If \mathbf{A} is diagonalizable, then,

$$\mathbf{x}[k] = \mathbf{V}\mathbf{\Lambda}^k\mathbf{V}^{-1}\mathbf{x}[0]$$

Internal stability

- ▶ There are two types of stability one can associate with a system – **Internal stability** and **Input-Output stability**.
- ▶ **Internal stability:** Deals with the stability of the zero-input response of the system states.

Internal stability

- Definition of stability in the Lyapunov sense for linear systems:
 - The zero-input response of a linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is *stable or marginally stable* if every finite initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t) \forall t \geq 0$.

$$\|\mathbf{x}(t)\| \leq d, \quad \forall t \geq 0$$

- The zero-input response is *asymptotically stable* if every initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)$ that converges to 0 as $t \rightarrow \infty$.

$$\|\mathbf{x}(t)\| \leq d \text{ and } \lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$$

Internal stability

- ▶ The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is marginally stable if and only if all eigenvalues of \mathbf{A} have either zero or negative real parts, and the eigenvalues with zero real parts have the same algebraic and geometric multiplicity.
- ▶ The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable if and only if all eigenvalues of \mathbf{A} have negative real parts.

Internal stability

- ▶ Consider the solution, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)$, $t \geq 0$, and $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$.

$$\|\mathbf{x}(t)\| = \|e^{t\mathbf{A}}\mathbf{x}(0^-)\| \leq \|e^{t\mathbf{J}}\| \|\mathbf{x}(0^-)\|$$

- ▶ When \mathbf{A} is diagonalizable (λ_i are the eigenvalues of \mathbf{A}),

- ▶ $\|\mathbf{x}(t)\| \leq e^{\sigma t} \|\mathbf{x}(0^-)\|$, where $\sigma = \max_i \Re\{\lambda_i\}$.

- ▶ When $\sigma = 0$, $\|\mathbf{x}(t)\|$ is bounded $\forall t \geq 0$.

- ▶ When $\sigma < 0$, $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$.

Internal stability – Discrete-time LTI systems

When \mathbf{A} is diagonalizable (λ_i are the eigenvalues of \mathbf{A}),

► $\|\mathbf{x}[k]\| \leq |\lambda|^k \|\mathbf{x}[0]\|$, where $\lambda = \max_i |\lambda_i|$.

► When $|\lambda| = 1$, $\|\mathbf{x}[k]\|$ is bounded $\forall k > 0$.

► When $|\lambda| < 1$, $\lim_{k \rightarrow \infty} \|\mathbf{x}[k]\| = 0$.