Applied Linear Algebra in Data Analaysis
Matrices Assignment

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1. Elements of the matrix $\mathbf{C} \in \mathbb{R}^{m\times n}$ obtained as the product of two matrices $\mathbf{A} \in \mathbb{R}^{m\times p}$ and $\mathbf{B} \in \mathbb{R}^{p\times n}$ is given by,
$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$
We had discussed four different ways to think of matrix multiplication. By algebraically manipulating the previous equation arrive at these four views (inner product view, column view, row view and outer product view)?

2. **Computational cost of different operations.** What is computational cost of the following matrix operations? Computational cost refers to the number of arithmetic operations required to carry out a particular matrix operation. Computational cost is a measure of the efficiency of an algorithm. For example, the consider the operation of vector addition, $\mathbf{a}+\mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. This requires n addition/subtraction operations and zero multiplication/division operations.
   (a) Matrix multiplication: $\mathbf{AB}$, where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$
   (b) Inner product: $\mathbf{u}^T\mathbf{v}$

   Report the counts for the addition/subtraction and multiplication/division operations separately.

3. Prove $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

4. Consider the following matrix,
$$\mathbf{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$
Find out the expression for $\mathbf{A}_n = \mathbf{A}^n$. What is $\mathbf{A}_\infty = \lim_{n\to\infty} \mathbf{A}^n$?

5. Prove that a matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ can always be written as a sum a symmetric matrix $\mathbf{S}$ and a skew-symmetric matrix $\mathbf{A}$.
$$\mathbf{M} = \mathbf{S} + \mathbf{A}, \ \ \mathbf{S}^T = \mathbf{S} \ \text{ and } \ \mathbf{A}^T = -\mathbf{A}$$
Does this property also hold for a complex matrix $\mathbf{M} \in \mathbb{C}^{n\times n}$?

6. The trace of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as, $\mathit{trace}\,(\mathbf{A}) = \sum_{i=1}^{n} a_{ii}$. Prove the following,
   (a) $\mathit{trace}\,(\mathbf{A})$ is a linear function of $\mathbf{A}$.
   (b) $\mathit{trace}\,(\mathbf{AB}) = \mathit{trace}\,(\mathbf{BA})$
   (c) $\mathit{trace}\left(\mathbf{A}^T\mathbf{A}\right) = 0 \implies \mathbf{A} = 0$

7. Prove that the rank of an outer product $\mathbf{x}\mathbf{y}^T$ is 1, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$.

8. Is there a relationship between the space of solutions to the following two equations?

$$\mathbf{y}^T\mathbf{A} = \mathbf{c}^T \quad \text{and} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

If so, how are they related?

9. Consider an upper triangular and lower triangular matrices $\mathbf{U}$ and $\mathbf{L}$, respectively.
   (a) Is the product of two upper triangular matrices $\mathbf{U}_1\mathbf{U}_2$ upper triangular?
   (b) Is the product of two lower triangular matrices $\mathbf{L}_1\mathbf{L}_2$ upper triangular?
   (c) What is the $trace\,(\mathbf{LU})$?

10. For a $n \times n$ square matrix $\mathbf{A}$, prove that if $\mathbf{AX} = \mathbf{I}$, then $\mathbf{XA} = \mathbf{I}$ and $\mathbf{X} = \mathbf{A}^{-1}$.

11. Prove the following for the non-singular square matrices $\mathbf{A}$ and $\mathbf{B}$:
    (a) $\mathbf{AB}$ is non-singular.
    (b) $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$.
    (c) $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
    (d) $\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T$

12. Derive the inverse of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

13. Consider the following upper-triangular matrix,

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

where, $u_{ii} \neq 0, \;\; 1 \leq i \leq n$. Do the columns of this matrix form a linearly independent set? Explain your answer.

14. Verify that $\mathbf{A}$ and $\mathbf{B}$ are inverses of each other,
    (a) $\mathbf{A} = \mathbf{I} - \mathbf{u}\mathbf{v}^T$ and $\mathbf{B} = \mathbf{I} + \mathbf{u}\mathbf{v}^T / \left(1 - \mathbf{v}^T\mathbf{u}\right)$
    (b) $\mathbf{A} = \mathbf{C} - \mathbf{u}\mathbf{v}^T$ and $\mathbf{B} = \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{C}^{-1} / \left(1 - \mathbf{v}^T\mathbf{C}^{-1}\mathbf{u}\right)$
    (c) $\mathbf{A} = \mathbf{I} - \mathbf{UV}$ and $\mathbf{B} = \mathbf{I}_n + \mathbf{U}\left(\mathbf{I}_m - \mathbf{VU}\right)^{-1}\mathbf{V}$
    (d) $\mathbf{A} = \mathbf{C} - \mathbf{U}\mathbf{D}^{-1}\mathbf{V}$ and $\mathbf{B} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{U}\left(\mathbf{D} - \mathbf{V}\mathbf{A}^{-1}\mathbf{U}\right)^{-1}\mathbf{V}\mathbf{A}^{-1}$

    where, $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{U} \in \mathbb{R}^{n\times m}$, $\mathbf{V} \in \mathbb{R}^{m\times n}$ and $\mathbf{D} \in \mathbb{R}^{m\times m}$.

15. Consider the matrices $\mathbf{A} \in \mathbb{R}^{m\times m}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{C} \in \mathbb{R}^{m\times n}$. Verify the following,
    (a) $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{bmatrix}$