

# Applied Linear Algebra in Data Analysis

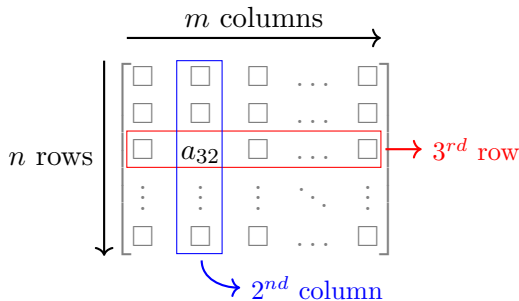
## Matrices

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# Matrices

- **Matrices** are rectangular array of numbers.  $\begin{bmatrix} 1.1 & -24 & \sqrt{2} \\ 0 & 1.12 & -5.24 \end{bmatrix}$



- Consider a matrix  $A$  with  $n$  rows and  $m$  columns.  $\begin{cases} \text{Tall/Skinny} & n > m \\ \text{Square} & n = m \\ \text{Wide/Fat} & n < m \end{cases}$

# Matrices

- ▶  $n$ -vectors can be interpreted as  $n \times 1$  matrices. These are called *column vectors*.
- ▶ A matrix with only one row is called a *row vector*, which can be referred to as  $n$ -row-vector.  $\mathbf{x} = [1.45 \quad -3.1 \quad 12.4]$
- ▶ **Block matrices & Submatrices:**  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$ . What are the dimensions of the different matrices?

# Matrices

- ▶ Matrices are also compact way to give a set of indexed column  $n$ -vectors,  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \dots \mathbf{x}_m$ .

$$\mathbf{X} = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3 \quad \dots \quad \mathbf{x}_m]$$

- ▶ **Zero matrix**  $= \mathbf{0}_{n \times m} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

- ▶ **Identity matrix** is a square  $n \times n$  matrix with all zero elements, except the diagonals where all elements are 1.

$$i_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$$

# Matrices

- **Diagonal matrices** is a square matrix with non-zero elements on its diagonal.

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & -11 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 9.3 \end{bmatrix} = \mathbf{diag}(0.4, -11, 21, 9.3)$$

- **Triangular matrices:** Are square matrices. *Upper triangular*  $a_{ij} = 0, \forall i > j$ ;  
*Lower triangular*  $a_{ij} = 0, \forall i < j$ .

# Matrix operations: Transpose

- **Transpose** switches the rows and columns of a matrix.  $\mathbf{A}$  is a  $n \times m$  matrix, then its transpose is represented by  $\mathbf{A}^\top$ , which is a  $m \times n$  matrix.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \implies \mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{13} & a_{23} \end{bmatrix}$$

Transpose converts between column and row vectors.

What is the transpose of a block matrix?  $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$

# Matrix operations: Matrix Addition

- ▶ **Matrix addition** can only be carried out with matrices of same size. Like vectors we perform element wise addition.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

- ▶ **Properties of matrix addition:**

- ▶ *Commutative:*  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- ▶ *Associative:*  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- ▶ *Addition with zero matrix:*  $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
- ▶ *Transpose of sum:*  $(\mathbf{A} + \mathbf{B})^{\top} = \mathbf{A}^{\top} + \mathbf{B}^{\top}$

# Matrix operations: Scalar multiplication

- **Scalar multiplication** Each element of the matrix gets multiplied by the scalar.

$$\alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$$

- We will mostly only deal with matrices with real entries. Such matrices are elements of the set  $\mathbb{R}^{n \times m}$ .
- Given the aforementioned matrix operations and their properties, is  $\mathbb{R}^{n \times m}$  a vector space?



# Matrix operations: Matrix multiplication

- ▶ A useful multiplication operation can be defined for matrices.
- ▶ It is possible to *multiply* two matrices  $\mathbf{A} \in \mathbb{R}^{n \times p}$  and  $\mathbf{B} \in \mathbb{R}^{p \times m}$  through this *matrix multiplication* procedure.
- ▶ The product matrix  $\mathbf{C} := \mathbf{AB} \in \mathbb{R}^{n \times m}$ , if the number of columns of  $\mathbf{A}$  is equal to the number of rows of  $\mathbf{B}$ .

$$c_{ij} := \sum_{k=1}^p a_{ik} b_{kj} \quad \forall i \in \{1, \dots, n\}, j \in \{1 \dots m\}$$

# Matrix multiplication

- *Inner product* is a special case of matrix multiplication between a *row vector* and a *column vector*.

$$\mathbf{x}^\top \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^\top \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n x_i y_i$$

# Matrix multiplication: Post-multiplication by a column vector

- Consider a matrix  $\mathbf{A} \in \mathbb{R}^{n \times m}$  and a  $m$ -vector  $\mathbf{x} \in \mathbb{R}^m$ . We can multiply  $\mathbf{A}$  and  $\mathbf{x}$  to obtain  $\mathbf{y} = \mathbf{A}\mathbf{x} \in \mathbb{R}^n$ .

$$\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m a_{1i}x_i \\ \sum_{i=1}^m a_{2i}x_i \\ \vdots \\ \sum_{i=1}^m a_{ni}x_i \end{bmatrix} = \sum_{i=1}^m x_i \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = \sum_{i=1}^m x_i \mathbf{a}_i$$

- Post-multiplying a matrix  $\mathbf{A}$  by a column vector  $\mathbf{x}$  results in a linear combination of the columns of matrix  $\mathbf{A}$ .
- $\mathbf{x}$  provides the column mixture.

# Matrix multiplication: Pre-multiplication by a row vector

- ▶ Let  $\mathbf{x}^\top \in \mathbb{R}^n$  and  $\mathbf{A} \in \mathbb{R}^{n \times m}$ , then  $\mathbf{y} = \mathbf{x}^\top \mathbf{A}$ .

$$\mathbf{y} = [x_1 \quad \dots \quad x_n] \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} = \left[ \sum_{i=1}^n x_i a_{i1} \quad \dots \quad \sum_{i=1}^n x_i a_{im} \right] = \sum_{i=1}^n x_i \tilde{\mathbf{a}}_i^\top$$

where,  $\tilde{\mathbf{a}}_i^\top = [a_{i1} \quad \dots \quad a_{im}]$

- ▶ Pre-multiplying a matrix  $\mathbf{A}$  by a row vector  $\mathbf{x}$  results in a linear combination of the rows of  $\mathbf{A}$ .
- ▶  $\mathbf{x}^\top$  provides the row mixture.

# Matrix multiplication

- Multiplying two matrices  $\mathbf{A} \in \mathbb{R}^{n \times p}$  and  $\mathbf{B} \in \mathbb{R}^{p \times m}$  produces  $\mathbf{C} \in \mathbb{R}^{n \times m}$ ,

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{np} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & b_{pm} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1m} \\ c_{21} & c_{22} & \dots & c_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1} & c_{n2} & \dots & c_{nm} \end{bmatrix}$$

- **Four interpretations of matrix multiplication.**

1. Inner-Product interpretation
2. Column interpretation
3. Row interpretation
4. Outer product interpretation.

# Matrix multiplication: Inner-product Interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- $ij^{th}$  element of  $\mathbf{C}$  is the inner product of the  $i^{th}$  row of  $\mathbf{A}$  and the  $j^{th}$  column of  $\mathbf{B}$ .

$$c_{ij} = \sum_{k=1}^p a_{ik} b_{kj} = \tilde{\mathbf{a}}_i^\top \mathbf{b}_j$$

where,  $i \in \{1 \dots n\}, j \in \{1 \dots m\}$

# Matrix multiplication: Column interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- Columns of  $\mathbf{C}$  are the linear combinations of the columns of  $\mathbf{A}$ .

$$\mathbf{C} = \mathbf{A} [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_m] = [\mathbf{Ab}_1 \quad \mathbf{Ab}_2 \quad \dots \quad \mathbf{Ab}_m]$$

- $j^{th}$  column of  $\mathbf{C}$  is the linear combination of the columns of  $\mathbf{A}$

$$\mathbf{c}_j = \sum_{k=1}^p b_{kj} \mathbf{a}_k$$

# Matrix multiplication: Row interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \quad \mathbf{B} \in \mathbb{R}^{p \times m}, \quad \mathbf{C} \in \mathbb{R}^{n \times m}$$

- Rows of  $\mathbf{C}$  are the linear combinations of the rows of  $\mathbf{B}$ .

$$\mathbf{C} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \\ \tilde{\mathbf{a}}_2^\top \\ \vdots \\ \tilde{\mathbf{a}}_n^\top \end{bmatrix} \mathbf{B} = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \mathbf{B} \\ \tilde{\mathbf{a}}_2^\top \mathbf{B} \\ \vdots \\ \tilde{\mathbf{a}}_n^\top \mathbf{B} \end{bmatrix}$$

- $i^{th}$  row of  $\mathbf{C}$  is the linear combination of the rows of  $\mathbf{B}$

$$\tilde{\mathbf{c}}_i^\top = \sum_{k=1}^p a_{ik} \tilde{\mathbf{b}}_k^\top$$



# Matrix multiplication: Outer product interpretation

- **Outer product:** Product between a column vector and a row vector. Let  $\mathbf{x} \in \mathbb{R}^n$  and  $\mathbf{y} \in \mathbb{R}^m$ . The *outer product* is defined as,

$$\mathbf{xy}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \dots & y_m \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_m \\ x_2y_1 & x_2y_2 & \dots & x_2y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_m \end{bmatrix} \in \mathbb{R}^{n \times m}$$

# Matrix multiplication: Outer product interpretation

$$\mathbf{C} = \mathbf{AB}, \quad \mathbf{A} \in \mathbb{R}^{n \times p}, \mathbf{B} \in \mathbb{R}^{p \times m}, \mathbf{C} \in \mathbb{R}^{n \times m}$$

- $\mathbf{C}$  can be written as the sum of  $p$  outer products of columns of  $\mathbf{A}$  and rows of  $\mathbf{B}$ .

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \dots & \mathbf{a}_p \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{b}}_1^\top \\ \tilde{\mathbf{b}}_2^\top \\ \tilde{\mathbf{b}}_3^\top \\ \vdots \\ \tilde{\mathbf{b}}_p^\top \end{bmatrix} = \sum_{i=1}^p \mathbf{a}_i \tilde{\mathbf{b}}_i^\top$$

# Properties of matrix multiplication

- ▶ **Not commutative:**  $\mathbf{AB} \neq \mathbf{BA}$

The product of two matrices might not always be defined. When it is defined,  $\mathbf{AB}$  and  $\mathbf{BA}$  need not match.

- ▶ **Distributive:**  $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$  and  $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$

- ▶ **Associative:**  $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$

- ▶ **Transpose:**  $(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$

- ▶ **Scalar product:**  $\alpha(\mathbf{AB}) = (\alpha\mathbf{A})\mathbf{B} = \mathbf{A}(\alpha\mathbf{B})$

# Linear transformations

- ▶ Linear functions  $f : \mathbb{R}^m \mapsto \mathbb{R}$ ,

$$y = f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}; \quad \mathbf{w}, \mathbf{x} \in \mathbb{R}^m, \quad y \in \mathbb{R}$$

- ▶ Generalization of the linear function is when its range  $\mathbb{R}^n$ :

$$\mathbf{y} = f(\mathbf{x}); \quad \mathbf{x} \in \mathbb{R}^m, \quad \mathbf{y} \in \mathbb{R}^n$$

- ▶ These can be represented as,  $\mathbf{y} = \mathbf{A}\mathbf{x}$ ,  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .

- ▶ Matrices can be thought of as representing a particular linear transformation.

# Why does matrix multiplication have this strange definition?

Consider the following two functions,

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\mathbf{x} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{v} = g(\mathbf{u}) = \mathbf{B}\mathbf{u} \longrightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = g\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{aligned} \mathbf{z} = h(\mathbf{u}) = f(g(\mathbf{u})) &= f\left(\begin{bmatrix} \alpha u_1 + \beta u_2 \\ \gamma u_1 + \delta u_2 \end{bmatrix}\right) = \begin{bmatrix} a\alpha u_1 + a\beta u_2 + b\gamma u_1 + b\delta u_2 \\ c\alpha u_1 + c\beta u_2 + d\gamma u_1 + d\delta u_2 \end{bmatrix} \\ &= \begin{bmatrix} (a\alpha + b\gamma)u_1 + (a\beta + b\delta)u_2 \\ (c\alpha + d\gamma)u_1 + (c\beta + d\delta)u_2 \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

$$\mathbf{z} = \mathbf{A}(\mathbf{B}\mathbf{u}) = (\mathbf{A}\mathbf{B})\mathbf{u} \implies \mathbf{A}\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{bmatrix}$$

Matrix multiplication represents the composition of linear transformations.

# Rank of a matrix $\mathbf{A}$

- **Rank of a matrix  $\mathbf{A}$ :** dimension of the subspace spanned by the columns of  $\mathbf{A}$  or the rows of  $\mathbf{A} \in \mathbb{R}^{n \times m}$ .

$$\begin{aligned} \text{rank}(\mathbf{A}) &= \dim \text{span}(\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}) \rightarrow \text{Column rank} \\ &= \dim \text{span}\left(\left\{\tilde{\mathbf{a}}_1^\top, \tilde{\mathbf{a}}_2^\top, \dots, \tilde{\mathbf{a}}_n^\top\right\}\right) \rightarrow \text{Row rank} \end{aligned}$$

- Column Rank is always equal to the row rank.
- Rank tells us the number of independent columns/rows in the matrix.
- **Full rank matrix  $\mathbf{A}$ :**  $\text{rank}(\mathbf{A}) = \min(n, m)$   
**Rank deficient matrix  $\mathbf{A}$ :**  $\text{rank}(\mathbf{A}) < \min(n, m)$

# Matrix Inverse

- ▶ Consider the square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ .  $\mathbf{B} \in \mathbb{R}^{n \times n}$  is the inverse of  $\mathbf{A}$ , if  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$ , and  $\mathbf{B}$  is represented as  $\mathbf{A}^{-1}$ .
- ▶ Not all matrices have inverses. A matrix with an inverse is called **non-singular**, otherwise it is called **singular**.
- ▶ For a non-singular matrix  $\mathbf{A}$ ,  $\mathbf{A}^{-1}$  is unique.  $\mathbf{A}^{-1}$  is both the left and right inverse.
- ▶ A matrix  $\mathbf{A}$  has an inverse, if and only if  $\mathbf{A}$  is full rank, i.e.  $\text{rank}(\mathbf{A}) = n$
- ▶  $\mathbf{Ax} = \mathbf{b}$  can be solved as follows,  $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ . *It is never solved like this in practice.*
- ▶ Inverse of product of matrices,  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ .
- ▶  $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$  and  $(\mathbf{A}^{-1})^\top = (\mathbf{A}^\top)^{-1}$

# Complex Vectors and Matrices

- ▶ Similar to  $\mathbb{R}^n$ , we can have  $\mathbb{C}^n$ .  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{r1} + jx_{i1} \\ x_{r2} + jx_{i2} \\ \vdots \\ x_{rn} + jx_{in} \end{bmatrix}$
- ▶ Vector addition and scalar multiplication are the same. The scalar is a complex number.
- ▶ Additive identity, and scalar multiplication identity are the same. So is the **standard basis**  $\{\mathbf{e}_i\}_{i=1}^n$
- ▶ **Linear independence:** The set  $\{\mathbf{v}_i\}_{i=1}^n$  with  $\mathbf{v}_i \in \mathbb{C}^n$  is linearly independent, if  $\sum_{i=1}^n c_i \mathbf{v}_i = 0, \implies c_i = 0, \forall 1 \leq i \leq n, c_i \in \mathbb{C}$
- ▶ **Inner product:**  $\mathbf{x}^* \mathbf{y} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^n \bar{x}_i y_i$



# Complex Vectors and Matrices

- ▶ **Length:**  $\|\mathbf{x}\|_2^2 = \mathbf{x}^* \mathbf{x} = \sum_{i=1}^n \bar{x}_i x_i = \sum_{i=1}^n |x_i|^2$
- ▶ **Orthogonality:**  $\mathbf{x}^* \mathbf{y} = 0$
- ▶ Complex matrices have complex entries.  $\mathbf{A} \in \mathbb{C}^{m \times n}$  such that  $a_{ij} \in \mathbb{C}, \forall 1 \leq i \leq m, 1 \leq j \leq n$
- ▶ The transpose operation is generalized to conjugate transpose known as the *Hermitian*.  $\mathbf{A}^* = \overline{\mathbf{A}}^\top$ .
- ▶ The idea of symmetric matrices  $\mathbb{R}^{n \times n}$  are now generalized to  $\mathbb{C}^{n \times n}$  as  $\mathbf{A} = \mathbf{A}^*$ . Such matrices are called **Hermitian** matrices.
- ▶ Orthogonal matrices in the complex case are called **Unitary** matrices,  $\mathbf{U}^* \mathbf{U} = \mathbf{I} \implies \mathbf{U}^{-1} = \mathbf{U}^*$ .