

Applied Linear Algebra in Data Analysis

Tutorial

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1 CONCEPTS IN VECTOR SPACES

1. Which of the following sets forms a vector space?

- a) $\{a_1x_1 + a_2x_2 = 0 \mid x_1, x_2 \in \mathbb{R}\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.
- b) $\{\mathbf{a}^\top \mathbf{x} = b \mid \mathbf{x} \in \mathbb{R}^n\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.
- c) $\{\mathbf{x}^\top \mathbf{x} = 1 \mid \mathbf{x} \in \mathbb{R}^n\}$.
- d) $\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$, where $x \in [0, 1]$.
- e) $\{(x[0], x[1], x[2], \dots, x[N-1]) \mid x[i] \in \mathbb{R}, 0 \leq i < N\}$.

(The set of all polynomials of degree 2 or less.)

(The set of all real-valued time-domain signals of length N . $x[i]$ is the value of the signal at time instant i .)

2. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^n a_k x^k \mid a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0, 1]$$

Show that polynomials of order strictly lower than n form subspaces of \mathcal{P} .

3. Is the following function a valid norm of the vector space \mathcal{P} ?

$$\|\mathbf{p}(x)\| = \sqrt{\sum_{k=0}^n a_k^2}, \quad \mathbf{p} = \sum_{k=0}^n a_k x^k \in \mathcal{P}$$

4. Consider the following function, which is often called the *zero-norm* of a vector $\mathbf{x} \in \mathbb{R}^n$.

$$\|\mathbf{x}\|_0 = \sum_{i=1}^n \mathbb{I}(x_i \neq 0), \text{ where, } \mathbb{I}(A) = \begin{cases} 1 & A \text{ is true.} \\ 0 & A \text{ is false.} \end{cases}$$

Is the *zero-norm*, which is often used for quantifying the *sparsity* of a vector, a proper norm?

5. Is the following set of vectors linear independent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

What is the span of this set? Does this set form the basis for its span? Does it form an orthonormal basis?

2 MATRICES

- 1.
2. Show that the matrix product \mathbf{ABC} can be written as a weighted sum of the outer products of the columns of \mathbf{A} and rows of \mathbf{C} , with the weights coming from the matrix \mathbf{B} .
3. Prove the following for the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$.

$$(\mathbf{A}_1, \mathbf{A}_2 \mathbf{A}_3 \dots \mathbf{A}_n)^\top = \mathbf{A}_n^\top \mathbf{A}_{n-1}^\top \dots \mathbf{A}_2^\top \mathbf{A}_1^\top$$

4. Show that two polynomial functions of a square matrix \mathbf{A} commute.
5. **Nilpotent matrices.** Show that a strictly triangular matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{A}^n = \mathbf{0}$.
6. **Matrix Inversion Lemma.** Consider an invertible matrix \mathbf{A} . The matrix $\mathbf{A} + \mathbf{uv}^\top$ is invertible if and only if the two vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u} \neq -1$. Then, the inverse is given by,

$$(\mathbf{A} + \mathbf{uv}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{uv}^\top \mathbf{A}^{-1}}{1 + \mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u}}$$

7. Prove that $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, where $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$.
8. Effect of matrix operation on matrix rank. Let $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$, with ranks a and b respectively. What is the rank of the following matrices?
 - a) $\mathbf{A} + \mathbf{B}$
 - b) \mathbf{AB}
9. Show that the rank $(\mathbf{AB}) = \text{rank}(\mathbf{A})$, when \mathbf{B} is square and full rank.
10. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, \mathbf{AB} is non-singular if and only if both \mathbf{A} and \mathbf{B} are non-singular.
11. Let \mathbf{A} is a full rank matrix. Show that the *Gram matrix* of the column space, $\mathbf{A}^\top \mathbf{A}$ is invertible.

3 SOLUTION TO LINEAR EQUATIONS

1.

4 ORTHOGONALITY

1. If \mathbf{A} is an orthogonal matrix, show that $\mathbf{A}^{-1} = \mathbf{A}^\top$.
2. If \mathbf{P}_S is the orthogonal projection matrix onto the subspace S , then what is the corresponding orthogonal projection matrix onto S^\perp – the orthogonal complement of S ?
3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be an orthonormal basis for \mathbb{R}^n . Show that the following holds,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n (\mathbf{x}^\top \mathbf{u}_i) \cdot (\mathbf{u}_i^\top \mathbf{y})$$

4. Consider the following set of vectors, $S = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n\}$, where $\mathbf{a}_i \in \mathbb{R}^n$. The set S is linearly independent. Find the orthogonal components of a vector $\mathbf{b} \in \mathbb{R}^n$ in the subspace spanned by the sets of vectors $S_1 = \{\mathbf{a}_i\}_{i=1}^m$ and S_1^\perp .
5. Show that the set of orthogonal matrices $\{\mathbf{Q} \mid \mathbf{Q} \in \mathbb{R}^{n \times n}, \mathbf{Q}^\top \mathbf{Q} = \mathbf{Q} \mathbf{Q}^\top = \mathbf{I}_n\}$ is closed under matrix multiplication.
6. Consider the linear map, $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let us assume that \mathbf{A} is full rank. What conditions must \mathbf{A} satisfy for the following statements to be true,
 - a) $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, for all \mathbf{x}, \mathbf{y} such that $\mathbf{y} = \mathbf{A}\mathbf{x}$.
 - b) $\mathbf{y}_1^\top \mathbf{y}_2 = \mathbf{x}_1^\top \mathbf{x}_2$, for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$.

Note: A linear map \mathbf{A} with the aforementioned properties preserves lengths and angle between vectors. Such maps are encountered in rigid body mechanics.

5 MATRIX INVERSES

1. Find a left inverse for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the set of all possible left inverses.
2. Show that if the product of two square $d \times d$ matrices \mathbf{A} and \mathbf{B} is the identity matrix \mathbf{I} , then $\mathbf{BA} = \mathbf{I}$.
3. Consider an upper triangular matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$. We are interested in solving the following set of n linear equations,

$$\mathbf{R}\mathbf{x} = \mathbf{e}_i$$

$\mathbf{x} = [x_1 \ x_2 \ x_3 \ \dots \ x_n]^\top$ is the solution to the above equation. Show that $x_{i+1} = x_{i+2} = \dots = x_n = 0$.

Show that the solution to this equation is equal to the i^{th} column of the inverse of \mathbf{R} .

4. Find the