Applied Linear Algebra in Data Analaysis

Linear Dynamical Systems & Positive Definite Matrices Assignment

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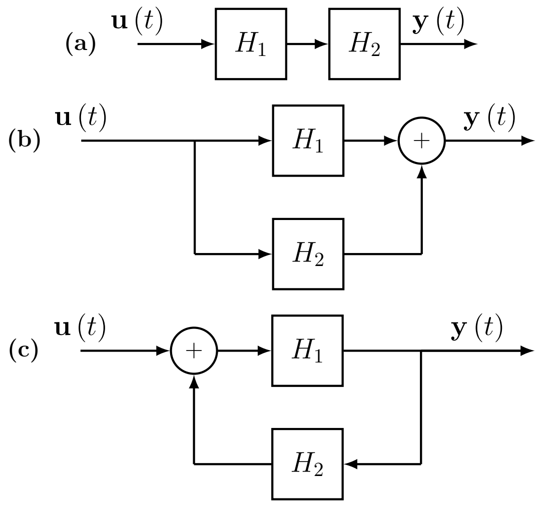
Marks: 26

1. Derive the state and measurement equations for the following composite systems, assuming the system H_i to have the parameters $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{C}_i, \mathbf{D}_i)$. **[Marks: 6]**



2. [**Programming**] Write a python program to simulate a continuous-time mass, spring, damper system, described by the follloing differential equation.

$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = u(t)$$

Assuming the states of the system to be $\mathbf{x}(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$, find out the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and $\mathbf{D}$. **[Marks: 2]**

Assuming that the input $u(t) = 0, \forall t \geq 0$, and assuming an initial condition of $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, numerically solve the state compute the evolution of the state and the output of the system using the following procedure. Let Δ be the time step used for the integration, then the time is divided into discrete time instants $n\Delta$, where $n \in \mathbb{Z}_{\geq 0}$. Assuming that we know the value of the state at time $n\Delta$, the rate of change of the state $\dot{\mathbf{x}}$ and the output $\mathbf{y}(n\Delta)$ at a time $n\Delta$ are given by,

$$\dot{\mathbf{x}}(n\Delta) = \mathbf{A}\mathbf{x}(n\Delta) + \mathbf{B}\mathbf{u}(n\Delta)$$
$$\dot{\mathbf{y}}(n\Delta) = \mathbf{C}\mathbf{x}(n\Delta) + \mathbf{D}\mathbf{u}(n\Delta)$$

We can compute the state at time $(n+1)\Delta$ from $\dot{\mathbf{x}}(n\Delta)$,

$$\mathbf{x}((n+1)\Delta) \approxeq \mathbf{x}(n\Delta) + \mathbf{x}(n\Delta) \cdot \Delta$$

Starting from the value of the start at time 0, $\mathbf{x}(0)$, we can numerically compute the evolution of the state for a given input $\mathbf{u}(t)$.

Compute the states and the output of the system from time $t = 0s$ to $t = 10s$ for following values of the parameters M, B, K, **[Marks: 3]**

(a) M = 1, B = 3, K = 1

(b) M = 1, B = 1, K = 1

(c) M = 0, B = 0, K = 1

Carry out the simulations for different values of $\Delta = 0.1, 0.01, 0.001$. Compute the states and plot them as function of time. [**Marks: 2**]

What differences do you find for the three systems for the different parameters and when using different step times? What do you think is the reason for the differences? [**Marks: 2**]

3. Prove that $\mathbf{A}^T\mathbf{A}$ is positive semi-definite for any matrix $\mathbf{A}$. When is $\mathbf{A}^T\mathbf{A}$ guaranteed to be positive definite? [**Marks: 2**]

4. If $\mathbf{A}$ is positive definite, then prove that $\mathbf{A}^{-1}$ is also positive definite. [**Marks: 1**]

5. Is the function $f(x_1, x_2, x_3) = 12x_1^2 + x_2^2 + 6x_3^2 + x_1x_2 - 2x_2x_3 + 4x_3x_1$ positive definite? [**Marks: 3**]

6. Prove the following for $\mathbf{A} \in \mathbb{R}^{m\times n}$: [**Marks: 4**]

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{a}_1^T} \\ \tilde{\mathbf{a}_1^T} \\ \vdots \\ \tilde{\mathbf{a}_m^T} \end{bmatrix}$$

   (a) $\|\mathbf{A}\|_1 = \max_{1\le i\le n} \|\mathbf{a}_i\|_1$

   (b) $\|\mathbf{A}\|_\infty = \max_{1\le i\le m} \|\tilde{\mathbf{a}}_i\|_1$

   (c) $\|\mathbf{A}\|_2 = \max_{1\le i\le n} |\lambda_i|$, where λ_i are the eigenvalues of $\mathbf{A}^T\mathbf{A}$.

   (d) $\|\mathbf{A}\|_F = trace\left(\mathbf{A}^T\mathbf{A}\right)$

7. Prove that the induced norm of a matrix product is bounded: [**Marks: 1**]

$$\|\mathbf{AB}\| \le \|\mathbf{A}\| \, \|\mathbf{B}\|$$

8. Verify the following inequalities on vector and matrix norms ($\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m\times n}$): [**Marks: 4**]

   (a) $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2$

   (b) $\|\mathbf{x}\|_2 \le \sqrt{m}\,\|\mathbf{x}\|_\infty$

   (c) $\|\mathbf{A}\|_\infty \le \sqrt{n}\,\|\mathbf{A}\|_2$

   (d) $\|\mathbf{A}\|_2 \le \sqrt{m}\,\|\mathbf{A}\|_\infty$