Applied Linear Algebra in Data Analysis

Tutorial

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CONTENTS

1 CONCEPTS IN VECTOR SPACES

1. Which of the following sets forms a vector space?
   a) $\{a_1 x_1 + a_2 x_2 = 0 \,|\, x_1, x_2 \in \mathbb{R}\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.
   b) $\left\{\mathbf{a}^\top \mathbf{x} = b \,|\, \mathbf{x} \in \mathbb{R}^n\right\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.
   c) $\left\{\mathbf{x}^\top \mathbf{x} = 1 \,|\, \mathbf{x} \in \mathbb{R}^n\right\}$.
   d) $\left\{a_0 + a_1 x + a_2 x^2 \,|\, a_0, a_1, a_2 \in \mathbb{R}\right\}$, where $x \in [0, 1]$.

      (The set of all polynomials of degree 2 or less.)
   e) $\{(x[0], x[1], x[2], \dots x[N-1]) \,|\, x[i] \in \mathbb{R}, 0 \leqslant i < N\}$.

      (The set of all real-valued time-domain signals of length N. $x[i]$ is the value of the signal at time instant i.)

2. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^{n} a_k x^k \,\middle|\, a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0, 1]$$

   Show that polynomails of order strictly lower than n form subspaces of $\mathcal{P}$.

3. Is the following function a valid norm of the vector space $\mathcal{P}$?

$$\|\mathbf{p}(x)\| = \sqrt{\sum_{k=0}^{n} a_k^2}, \; \mathbf{p} = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}$$

4. Consider the following function, which is often called the *zero-norm* of a vector $\mathbf{x} \in \mathbb{R}^n$.

$$\|\mathbf{x}\|_0 = \sum_{i=1}^{n} \mathbb{I}(x_i \neq 0), \text{ where, } \mathbb{I}(A) = \begin{cases} 1 & A \text{ is true.} \\ 0 & A \text{ is false.} \end{cases}$$

   Is the *zero-norm*, which is often used for quantifying the *sparsity* of a vector, a proper norm?

5. Is the following set of vectors linear independent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

   What is the span of this set? Does this set form the basis for its span? Does it form an orthonormal basis?

2 MATRICES

1. Conisder the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

   Find the product of the two matrices $\mathbf{C} = \mathbf{AB}$ using the four views of matrix muliplication.

   If we change $b_{23} = 0$. Can you compute the new matrix $\mathbf{C}$ without performing the matrix muliplication?

   If we increase the value of the elements of the 3^{rd} column of $\mathbf{A}$ by 1, how can we compute the new $\mathbf{C}$ without performing the matrix multiplication?

   If we insert a new row $\mathbf{1}^\top$ in $\mathbf{A}$ after the 2^{nd} row, how can we compute the new $\mathbf{C}$ without performing the matrix multiplication?

2. Show that the matrix product $\mathbf{ABC}$ can be written as a weighted sum of the outer products of the columns of $\mathbf{A}$ and rows of $\mathbf{C}$, with the weights coming from the matrix $\mathbf{B}$.

3. Prove the following for the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots \mathbf{A}_n$.

$$(\mathbf{A}_1, \mathbf{A}_2\mathbf{A}_3 \ldots \mathbf{A}_n)^\top = \mathbf{A}_n^\top \mathbf{A}_{n-1}^\top \ldots \mathbf{A}_2^\top \mathbf{A}_1^\top$$

4. Show that two polynomial functions of a square matrix $\mathbf{A}$ commutate.

5. **Nilpotent matrices**. Show that a strictly triangular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{A}^n = \mathbf{0}$.

6. **Matrix Inversion Lemma**. Consider an invertible matrix $\mathbf{A}$. The matrix $\mathbf{A} + \mathbf{uv}^\top$ is invertible if and only if the two vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u} \neq -1$. Then, the inverse is given by,

$$\left(\mathbf{A} + \mathbf{uv}^\top\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{uv}^\top\mathbf{A}^{-1}}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}}$$

7. Prove that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$, where $\mathbf{A} \in \mathbb{R}^{n\times d}$ and $\mathbf{B} \in \mathbb{R}^{d\times n}$.

8. Effect of matrix operation on matrix rank. Let $\mathbf{A} \in \mathbb{R}^{n\times d}$ and $\mathbf{B} \in \mathbb{R}^{d\times n}$, with ranks a and b respectively. What is the rank of the following matrices?
   a) $\mathbf{A} + \mathbf{B}$
   b) $\mathbf{AB}$

9. Show that the $\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{A})$, when $\mathbf{B}$ is square and full rank.

10. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, $\mathbf{AB}$ is non-singular if and only if both $\mathbf{A}$ and $\mathbf{B}$ are non-singular.

11. Let $\mathbf{A}$ is a full rank matrix. Show that the *Gram matrix* of the column space, $\mathbf{A}^\top\mathbf{A}$ is invertible.