Applied Linear Algebra in Data Analysis
Concepts in Vector Spaces

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

Vectors

▶ **Vectors** are ordered list of numbers (scalars). $\mathbf{v} = \begin{bmatrix} 1.2 \\ -0.1 \\ \vdots \\ -1.24 \end{bmatrix}$.

   Note: Small bold letter will represent vectors. e.g. $\mathbf{a}, \mathbf{x}, \ldots$

▶ Scalars can be any *field* $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{Q}$. Scalars will be represented using lower case normal font, e.g. $x, y, \alpha, \beta, \ldots$

▶ Addition/multiplication operations performed on vectors will follow the rules of addition/multiplication of the corresponding scalar fields.

▶ We will typically encounter only $\mathbb{R}$ and $\mathbb{C}$ in this course.

Vectors

▶ Individual elements of a vector $\mathbf{v}$ are indexed. The ith element of $\mathbf{v}$ is referred to as v_i.

▶ *Dimension* or *size* of a vector is number of elements in the vector.

▶ Set of n-real vectors is denoted by $\mathbb{R}^n$ (similarly, $\mathbb{C}^n$)

▶ Vectors $\mathbf{a}$ and $\mathbf{b}$ are equal, if
  ▶ both have the same size; and

  ▶ $a_i = b_i, i \in \{1, 2, 3, \ldots n\}$

Vectors

▶ **Unit vector** $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **Zero vector** $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **One vector** $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$
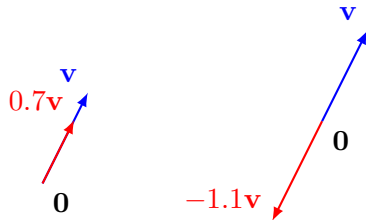
▶ Geometrically, real n-vectors can be thought of as points in $\mathbb{R}^n$ space.
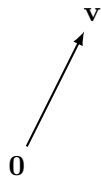
Vectors

▶ **Vector scaling**: Multiplication of a scalar and a vector.

$$\mathbf{w} = a\mathbf{v} = a \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \\ av_3 \\ \vdots \\ av_n \end{bmatrix} \quad a \in \mathbb{R}; \ \mathbf{w}, \mathbf{v} \in \mathbb{R}^n$$

Properties

▶ Scalar multiplication is *commutative*.

$$\alpha\mathbf{v} = \mathbf{v}\alpha$$

▶ Scalar multiplication is *associative*.

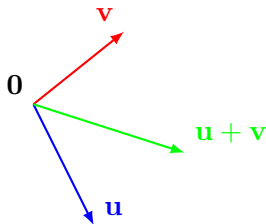$$(\alpha\beta)\,\mathbf{v} = \alpha\,(\beta\mathbf{v})$$

▶ Scalar multiplication is *distributive*.

$$(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

Vectors

▶ **Vector addition**: Adding two vectors of the same dimension, element by element.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$$

Properties

▶ Vector addition is *commutative*.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

▶ Vector addition is *associative*.

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

▶ Zero vector has no effect.

$$\mathbf{a} + \mathbf{0} = \mathbf{a}$$

▶ Subtraction of vectors.

$$\mathbf{a} + (-1)\mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}$$

Vector spaces

▶ A set of vectors V that is closed under **vector addition** and **vector scaling**.

$$\forall \mathbf{x}, \mathbf{y} \in V, \quad \mathbf{x} + \mathbf{y} \in V$$

$$\forall \mathbf{x} \in V, \text{ and } \alpha \in F, \quad \alpha \mathbf{x} \in V$$

▶ For a set to be a vector space, it must satisfy the followng properties: $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$
  ▶ *Commutativity*: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
  ▶ *Associativity of vector addition*: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$
  ▶ *Additive identity*: $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x} \ (\mathbf{0} \in V)$
  ▶ *Additive inverse*: $\exists -\mathbf{x} \in V, \mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  ▶ *Associativity of scalar multiplication*: $\alpha (\beta \mathbf{x}) = (\alpha \beta \mathbf{x})$
  ▶ *Distributivity of scalar sums*: $(\alpha + \beta) \mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{x}$
  ▶ *Distributivity of vector sums*: $\alpha (\mathbf{x} + \mathbf{y}) = \alpha \mathbf{x} + \alpha \mathbf{y}$
  ▶ *Scalar multiplication identity*: $1\mathbf{x} = \mathbf{x}$

▶ We will mostly deal with $\mathbb{R}^n$ and $\mathbb{C}^n$ vectors spaces in this course.

Subspaces

▶ A **subspace** S of a vector space V is a subset of V and is itself a vector space.

$$S \subset V, \quad \forall \mathbf{x}, \mathbf{y} \in S, \alpha \mathbf{x} + \beta \mathbf{y} \in S, \quad \alpha, \beta \in F$$

▶ The zero vector is called the **trivial subspace** of a vector space V.

▶ For example, in $\mathbb{R}^3$ all planes and lines passing through the origin are subspaces of $\mathbb{R}^3$.
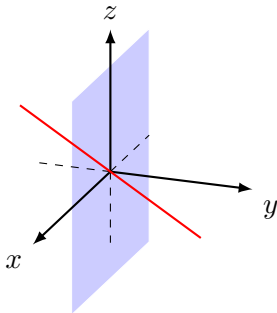
Linear independence

▶ A collection of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots \mathbf{x}_n\}, \ \mathbf{x}_i \in \mathbb{R}^m \ i \in \{1, 2, 3, \ldots n\}$ is called *linearly dependent* if,

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0, \text{ hold for some } \alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}, \text{ such that } \exists \alpha_i \neq 0$$

▶ Another way to state this: A collection of vectors is *linearly dependent* if at least one of the vectors in the collection can be expressed as a linear combination of the other vectors in the collection, i.e.

$$\mathbf{x}_i = -\sum_{j=1, j \neq i}^{n} \left(\frac{\alpha_j}{\alpha_i} \right) \mathbf{x}_j$$

Linear independence

▶ A collection of vectors is *linearly independent* if it is **not** *linearly dependent.*

$$\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_n = 0$$

Span of a set of vectors

▶ Consider a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots \mathbf{v}_r\}$ where $\mathbf{v}_i \in \mathbb{R}^n, 1 \leq i \leq r$.

▶ The **span** of the set S is defined as the set of all linear combinations of the vectors $\mathbf{v}_i$,

$$span\,(S) = \{\alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r\},\ \alpha_i \in \mathbb{R}$$

▶ Is $span\,(S)$ a subspace of $\mathbb{R}^n$?

Span of a set of vectors

▶ We say that the subspace $span(S)$ is spanned by the *spanning set S*. $\longrightarrow$ *S spans span (S)*.

▶ **Sum of subspaces** X, Y is defined as the sum of all possible vectors from X and Y.

$$X + Y = \{\mathbf{x} + \mathbf{y} \,|\, \mathbf{x} \in X, \mathbf{y} \in Y\}$$

▶ Sum of two subspace is also a subspace.

Inner Product

▶ **Standard inner product** is defined as the following,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \ \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

For complex vectors: $\mathbf{x}^* \mathbf{y} = \sum_{i=1}^{n} \overline{x}_i y_i, \ \ \mathbf{x}, \mathbf{y} \in \mathbb{C}^n$

▶ **Properties**

▶ $\mathbf{x}^\top \mathbf{x} > 0, \ \forall \mathbf{x} \neq 0$ and $\mathbf{x}^\top \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = 0$

▶ *Commutative*: $\mathbf{x}^\top \mathbf{y} = \mathbf{y}^\top \mathbf{x}$

▶ *Associativity with scalar multiplication*: $(\alpha \mathbf{x})^\top \mathbf{y} = \alpha \left(\mathbf{x}^\top \mathbf{y} \right)$

▶ *Distributivity with vector addition*: $(\mathbf{x} + \mathbf{y})^\top \mathbf{z} = \mathbf{x}^\top \mathbf{z} + \mathbf{y}^\top \mathbf{z}$
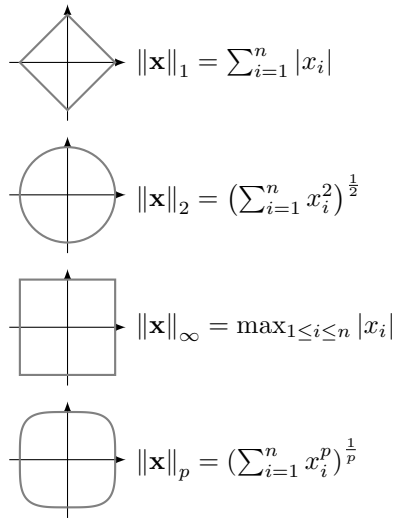
Norm

- ▶ Norm is a measure of the size of a vector.
- ▶ *Euclidean norm* of a n-vector $\mathbf{x} \in \mathbb{R}^n$ is defined as, $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$.
- ▶ $\|\mathbf{x}\|_2$ is a measure of the length of the vector $\mathbf{x}$.
- ▶ Any function of the form $\|\bullet\| : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ is a valid norm, provided it satisfies the following properties.

$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$

$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$
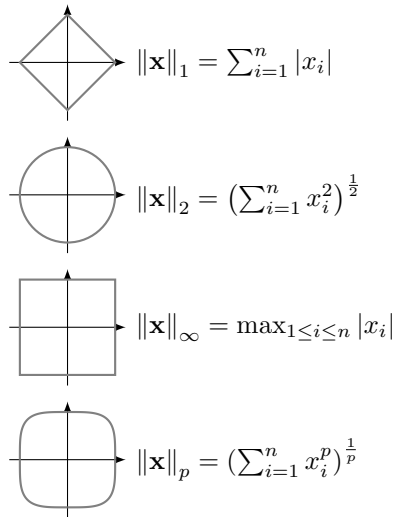
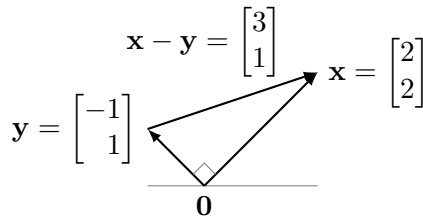$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$

Norm

- **Properties**
  - *Definiteness.* $\|\mathbf{x}\| = 0 \iff x = 0$
  - *Non-negativity.* $\|\mathbf{x}\| \geq 0$
  - *Non-negative homogeneity.* $\|\beta\mathbf{x}\| = |\beta|\,\|\mathbf{x}\|,\ \beta \in \mathbb{R}$
  - *Triangle inequality.* $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$
- p-norm: $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$
- Norm of difference between two vectors is a measure of the distance between the vectors. $d = \|\mathbf{x} - \mathbf{y}\|_2$.

$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$

$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$

$\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n x_i^p\right)^{\frac{1}{p}}$

Orthogonality

▶ Orthogonality is the idea of two vectors being perpendicular, $\mathbf{x} \perp \mathbf{y}$.
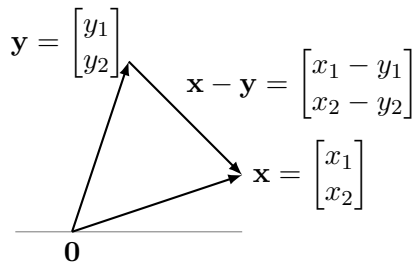

Using the Pythagonean theorem, $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x}^\top \mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \implies \mathbf{x}^\top \mathbf{y} = 0$$

▶ We extend this to the n-dimensional case and define two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ being orthogonal, if

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} x_i y_i = 0$$

Angle between vectors

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbf{x} - \mathbf{y} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

▶ Inner products are used for projecting a vector onto another vector or a subspace.

▶ It is also a measure of similarity between two vectors, $\cos(\theta) = \frac{\mathbf{x}^{\top}\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$
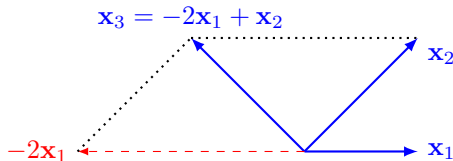
▶ **Cauchy-Bunyakovski-Schwartz Inequality**:

$$\left| \mathbf{x}^{\top}\mathbf{y} \right| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|, \ \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

Basis

Consider a vector $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. What can we say about the coefficients α_is when the collection $\{\mathbf{x}_i\}_{i=1}^{n}$ is,

▶ linearly independent $\implies$ α_is are *unique*.

▶ linearly dependent $\implies$ α_is are not *unique*.

Consider $\mathbb{R}^2$ vector space. $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
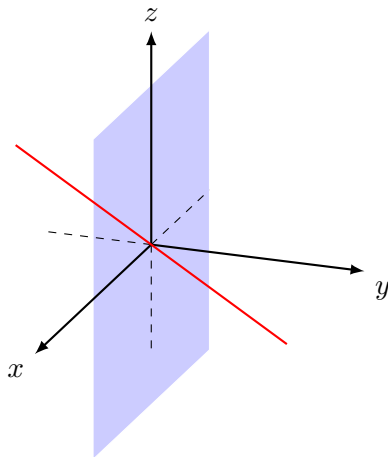

Independence-Dimension inequality: What is the maximum possible size of a linearly independent collection?

A linearly independent collection of n-vectors can at most have n vectors.

Basis

How many vectors can we choose from the following vectors before the set becomes linearly dependent?

Basis

▶ A linearly independent set of n-vectors from $\mathbb{R}^n$, of size n, is called a *basis* for $\mathbb{R}^n$.

▶ Any n-vector from $\mathbb{R}^n$ can be represented as a *unique* linear combination of the elements of the basis.

▶ Consider the basis $\{\mathbf{x}_i\}_{i=1}^n$, $\mathbf{x}_i \in \mathbb{R}^n$. Any vector $\mathbf{y} \in \mathbb{R}^n$ can be represented as a linear combination of $\mathbf{x}_i$s, $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$. This is called the *expansion of* $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^n$ basis.

Basis

▶ The numbers α_i are called the *coefficients* of the expansion of $\mathbf{y}$ in the $\{\mathbf{x}_i\}_{i=1}^{n}$ basis.

▶ **Orthogonal vectors**: A set of vectors $\{\mathbf{x}_i\}_{i=1}^{n}$ is *(mutually) orthogonal* if $\mathbf{x}_i \perp \mathbf{x}_j$ for all $i, j \in \{1, 2, 3, \dots n\}$ and $i \neq j$.

▶ This set is called **orthonormal** if its elements are all of unit length $\|\mathbf{x}_i\|_2 = 1$ for all $i \in \{1, 2, 3, \dots n\}$.

$$\mathbf{x}_i^\top \mathbf{x}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Representing a Vector in an Orthonormal Basis

▶ An orthonormal collection of vectors is linearly independent.

▶ Consider an orthonormal basis $\{\mathbf{x}_i\}_{i=1}^n$. The expansion of a vector $\mathbf{y}$ is given by,
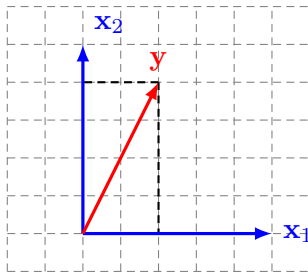
$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 + \ldots + \alpha_n \mathbf{x}_n$$

$$\mathbf{x}_i^\top \mathbf{y} = \alpha_1 \mathbf{x}_i^\top \mathbf{x}_1 + \alpha_2 \mathbf{x}_i^\top \mathbf{x}_2 + \alpha_3 \mathbf{x}_i^\top \mathbf{x}_3 + \ldots + \alpha_n \mathbf{x}_i^\top \mathbf{x}_n = \alpha_i$$

Representing a Vector in an Orthonormal Basis

▶ Thus, we can rewrite this as,

$$\mathbf{y} = \left(\mathbf{y}^\top \mathbf{x}_1\right) \mathbf{x}_1 + \left(\mathbf{y}^\top \mathbf{x}_2\right) \mathbf{x}_2 + \left(\mathbf{y}^\top \mathbf{x}_3\right) \mathbf{x}_3 + \ldots + \left(\mathbf{y}^\top \mathbf{x}_n\right) \mathbf{x}_n$$

Dimension of a Vector Space

▶ There an infinite number of bases for a vector space.

▶ There is one thing that is common among all these bases – the number of bases vectors.

▶ This number is a property of the vector space, and represents the "degrees of freedom" of the space. This is called the **dimension** of the vector space.

Dimension of a Vector Space

▶ A subspace of dimension m can have at most m independent vectors.

▶ Notice that the word "dimension" of a vector space is different from the "dimension" of a vector.

▶ E.g. Vectors from $\mathbb{R}^3$ are three dimensional vectors. But the yz-plane in $\mathbb{R}^3$ is a 2 dimensional subspace of $\mathbb{R}^3$.

Linear Functions

▶ Let f be a function which maps vectors from $\mathbb{R}^n$ to scalar real numbers. It can be represented as the following,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}; \quad y = f(\mathbf{x}) = f(x_1, x_2, x_3, \ldots x_n)$$

▶ Criteria for f to be a linear function:

$$\textbf{Superposition} : f(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Linear Functions

▶ **Inner product** is a linear function in one of the arguments.

$$f(x) = \mathbf{w}^\top \mathbf{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots + w_n x_n$$

▶ Any linear function can be represented in the form $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$ with an appropriately chosen $\mathbf{w}$.