Applied Linear Algebra in Data Analysis

Solution to Linear Equations

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Linear equations

- Matrices present a compact way to represent a set of linear equations. Consider the following,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 \ldots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2m}x_m &= b_2 \\ a_{31}x_1 + a_{32}x_2 \ldots + a_{3m}x_m &= b_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nm}x_m &= b_n \end{aligned}\right\} \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{n\times m}, \;\; \mathbf{x} \in \mathbb{R}^m, \;\; \mathbf{b} \in \mathbb{R}^n$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nm} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Linear equations in control problems

$\mathbf{x}$: Input $\quad \mathbf{b}$: Output $\quad \mathbf{A}$: System dynamics

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Linear equations in estimation problems

$\mathbf{x}$: Parameter $\mathbf{b}$: Measurements $\mathbf{A}$: System characteristics

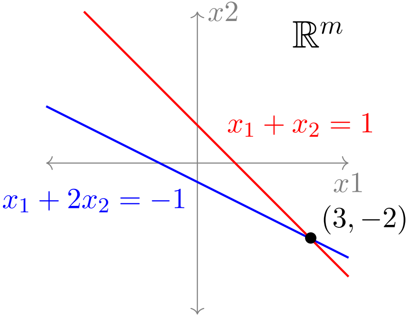
$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Geometry of linear equations

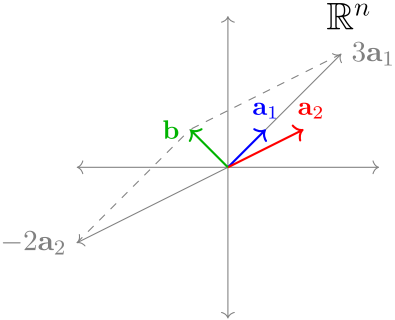
$$\left.\begin{aligned} x_1 + 2x_2 &= -1 \\ x_1 + x_2 &= 1 \end{aligned}\right\} \longrightarrow \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Two ways to view this: **row view** and the **column view**.

Traditional (row) view



Column view



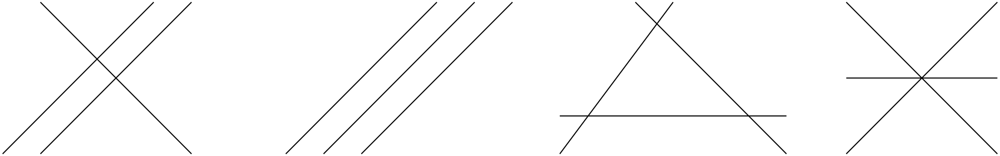
Solutions of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{m \times n}, \;\; \mathbf{x} \in \mathbb{R}^{n}, \;\; \mathbf{b} \in \mathbb{R}^{m}$$

- **Three possible situations:** No solution, Infinitely many solutions, or Unique Solution.

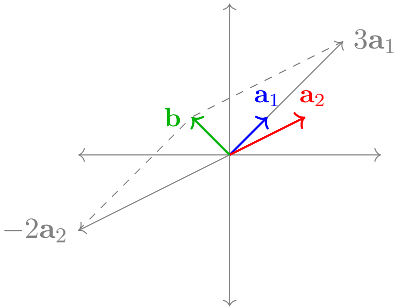
- When do we have infinitely many or no solutions? In $\mathbb{R}^3$, we can visualize the different situations.

Two parallel planes | Three parallel planes | No intersection | Line intersection



Understanding $\mathbf{Ax} = \mathbf{b}$: Unique solution

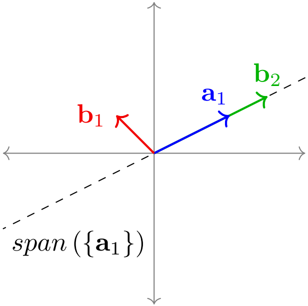
$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



- Square matrix
- Linearly independent set of columns $\{\mathbf{a}_1, \mathbf{a}_2\}$
- $\mathbf{b} \in span\left(\{\mathbf{a}_1, \mathbf{a}_2\}\right)$.
- Always solvable, and give an unique solution.

Understanding $\mathbf{Ax} = \mathbf{b}$: Unique solution or No solution

1. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \mathbf{b}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \mathbf{b}_2 = \begin{bmatrix} 3 \\ 1.5 \end{bmatrix}$



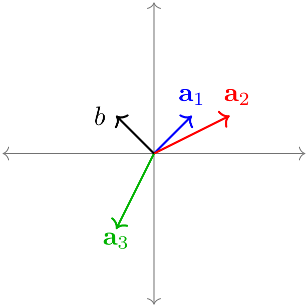
- Tall matrix
- Linearly independent set of columns $\{\mathbf{a}_1\}$

$\mathbf{b}_1 \notin span\left(\{\mathbf{a}_1\}\right) \implies$ Not solvable.

$\mathbf{b}_2 \in span\left(\{\mathbf{a}_1\}\right) \implies$ Solvable with unqiue solution.

Understanding $\mathbf{Ax} = \mathbf{b}$: Infinitely many solution

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



- Fat matrix
- Linearly dependent set of columns $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$
- $\mathbf{b} \in span\left(\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}\right)$.
- Always solvable, with infinitely many solutions.

Understanding Ax = b: Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{n \times m}, \, \mathbf{x} \in \mathbb{R}^{m}, \, \mathbf{b} \in \mathbb{R}^{n}$$

Full rank A:

- $rank\,(\mathbf{A}) = n \implies$ **Always solvable**

$$\begin{cases} n = m & \implies \text{Unique solution} \\ n < m & \implies \text{Infinitely many solutions} \end{cases}$$

- $rank\,(\mathbf{A}) = m \implies$ **No infinite solutions**

$$\begin{cases} m = n & \implies \text{Unique solution} \\ m < n & \to \begin{cases} \mathbf{b} \in span\,(\mathbf{a}_1, \ldots \mathbf{a}_m) \implies \text{Unique solution} \\ \mathbf{b} \notin span\,(\mathbf{a}_1, \ldots \mathbf{a}_m) \implies \text{No solution} \end{cases} \end{cases}$$

Understanding $\mathbf{Ax} = \mathbf{b}$: Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{n \times m}, \, \mathbf{x} \in \mathbb{R}^{m}, \, \mathbf{b} \in \mathbb{R}^{n}$$

Rank deficient $\mathbf{A}$:

- $rank\left(\mathbf{A}\right) < \min\left(n, m\right) \implies$ **No unique solution**

$$\begin{cases} \mathbf{b} \in span\left(\mathbf{a}_1, \ldots \mathbf{a}_m\right) \implies \text{Infinitely many solutions} \\ \mathbf{b} \notin span\left(\mathbf{a}_1, \ldots \mathbf{a}_m\right) \implies \text{No solution} \end{cases}$$

Understanding $\mathbf{Ax} = \mathbf{b}$: Conditions for different types of solutions

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} \in \mathbb{R}^{n \times m}, \, \mathbf{x} \in \mathbb{R}^{m}, \, \mathbf{b} \in \mathbb{R}^{n}$$

- $\mathbf{b} \notin span\left(\mathbf{a}_1, \ldots \mathbf{a}_m\right) \implies$ No solution
- $\mathbf{b} \in span\left(\mathbf{a}_1, \ldots \mathbf{a}_m\right) \implies \begin{cases} rank\left(\mathbf{A}\right) = m \implies \text{Unique} \\ rank\left(\mathbf{A}\right) < m \implies \text{Infinitely many solutions} \end{cases}$

General solution of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{n \times m}, \, \mathbf{x} \in \mathbb{R}^m, \, \mathbf{b} \in \mathbb{R}^n$$

- Assuming that this system can be solved, the most general form of the solution is,

  $$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$$

  where, $\mathbf{x}_p$ is called the particular solution, and $\mathbf{x}_h$ is the homogenous solution.

- **Homogenous solution**: Solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$.

- The set of all homogenous solutions of $\mathbf{A}$ – $\{\mathbf{x}_h \,|\, \mathbf{A}\mathbf{x}_h = \mathbf{0}\}$ – form a subspace of $\mathbb{R}^m$.

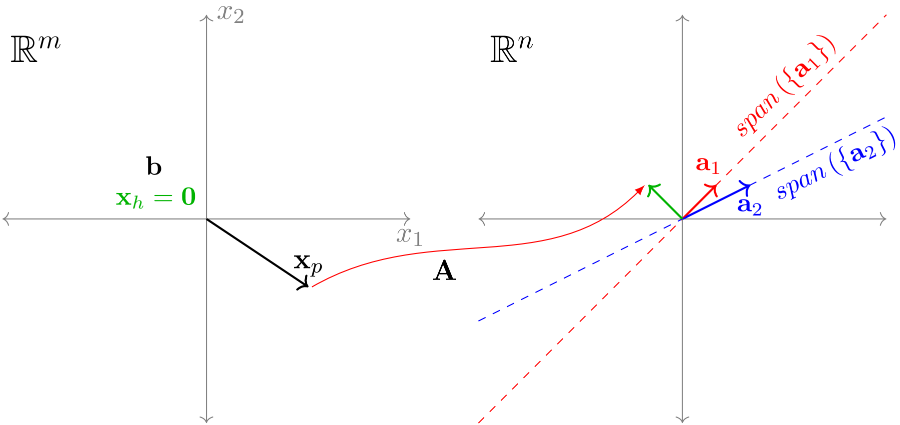
Geometry of the general solution

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \;\; \mathbf{A} \in \mathbb{R}^{n \times m}, \, \mathbf{x} \in \mathbb{R}^{m}, \, \mathbf{b} \in \mathbb{R}^{n}$$

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix}$$

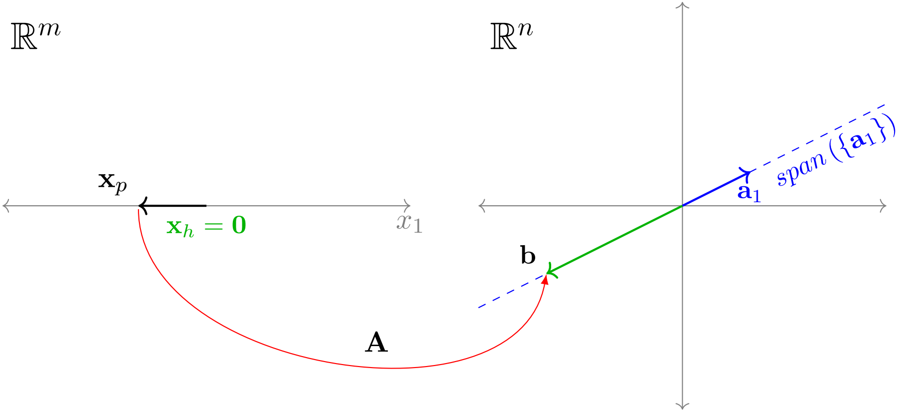
Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



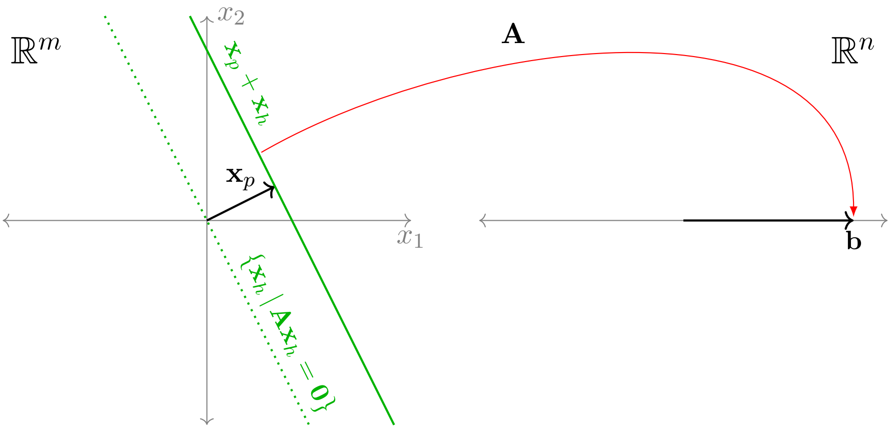
Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$



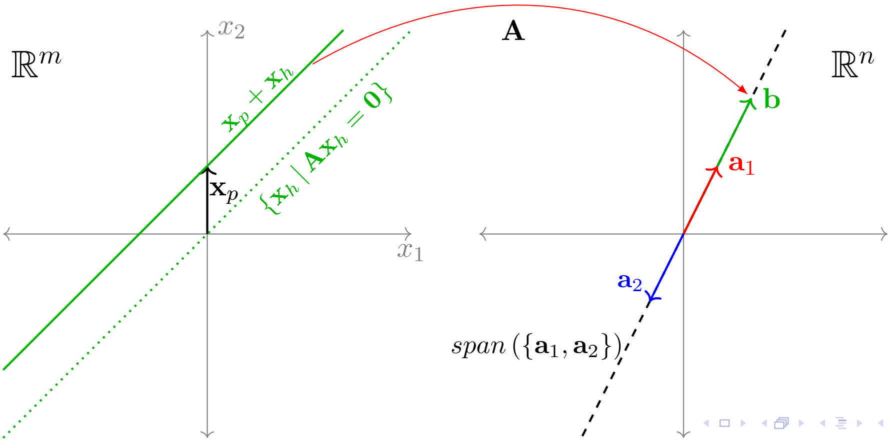
Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 5 \end{bmatrix}$$



Geometry of the general solution

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



Four Fundamental Subspaces of $\mathbf{A} \in \mathbb{R}^{n\times m}$

- $\mathcal{C}(\mathbf{A})$: **Column Space of A** – the span of the columns of $\mathbf{A}$.

$$\mathcal{C}(\mathbf{A}) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

- $\mathcal{N}(\mathbf{A})$: **Nullspace of A** – the set of all $\mathbf{x} \in \mathbb{R}^m$ that are mapped to zero by $\mathbf{A}$.

$$\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^m$$

- $\mathcal{C}(\mathbf{A}^\top)$: **Row Space of A** – the span of the rows of $\mathbf{A}$.

$$\mathcal{C}\left(\mathbf{A}^\top\right) = \left\{\mathbf{A}^\top\mathbf{y} \mid \mathbf{y} \in \mathbb{R}^n\right\} \subseteq \mathbb{R}^m$$

- $\mathcal{N}(\mathbf{A}^\top)$: **Nullspace of $\mathbf{A}^\top$** – the set of all $\mathbf{y} \in \mathbb{R}^n$ that are mapped to zero by $\mathbf{A}^\top$.

$$\mathcal{N}\left(\mathbf{A}^\top\right) = \left\{\mathbf{y} \mid \mathbf{A}^\top\mathbf{y} = \mathbf{0}\right\} \subseteq \mathbb{R}^n$$

  This is also called the **left nullspace** of $\mathbf{A}$.

Linear Independence

- Given a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}, \quad \mathbf{v}_i \in \mathbb{R}^n$, how can we determine if this set is linear independent?
- We need to verify, $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_m\mathbf{v}_m = 0$

$$\left.\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_m\end{bmatrix}\begin{bmatrix}a_1\\ \vdots \\ a_m\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix} = \mathbf{Va} = \mathbf{0}\right\} \mathcal{N}\left(\mathbf{V}\right) = \{\mathbf{0}\}, \quad rank\left(\mathbf{V}\right) = n$$

- This is also equivalent to saying that when the $rank\left(\mathbf{A}\right) = n \implies$ the columns of $\mathbf{A}$ form an independent set of vectors.
- When do the rows of $\mathbf{A}$ form an independent set?
- What about both rows and columns? When does that happen?

Dimension of the four fundamental subspaces

- **Column space** $C(\mathbf{A})$
  - $\dim C(\mathbf{A}) = rank\,(\mathbf{A}) = r$
- **Nullspace** $N(\mathbf{A})$
  - $\dim N(\mathbf{A}) = n - r$
- **Row space** $C(\mathbf{A}^\top)$
  - $\dim C(\mathbf{A}^\top) = rank\left(\mathbf{A}^\top\right) = rank\,(\mathbf{A}) = r$
- **Left Nullspace** $N(\mathbf{A}^\top)$
  - $\dim N(\mathbf{A}^\top) = m - r$