Applied Linear Algebra in Data Analysis

Application: Signal Processing

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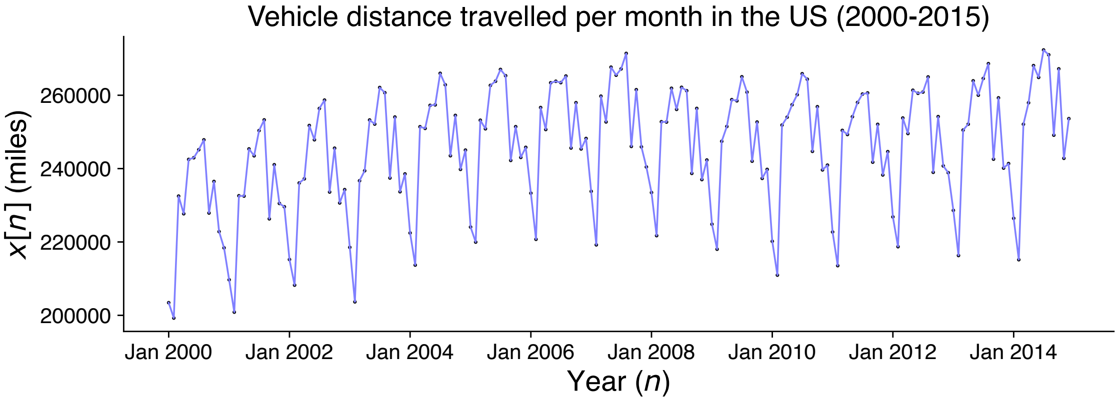
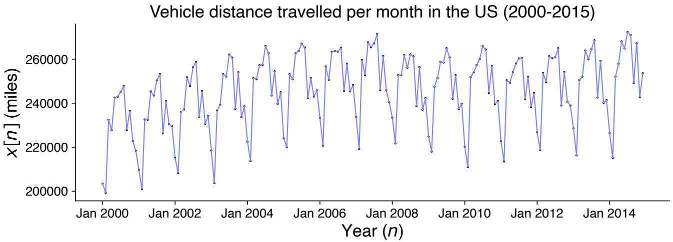
Signals as vectors

What is a signal? A *signal* is a function of an independent variable that conveys some information.



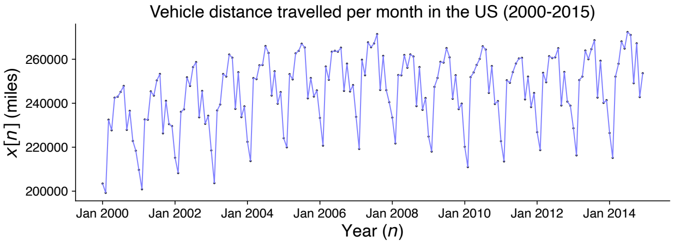
Signals as vectors



- $x[n]$ in the above figure is a finite length signal of length N, where $n \in \mathbb{Z}, 0 \leq n < N$.
- We can think of signal as a vector $\mathbf{x}$ in $\mathbb{R}^N$, i.e. this entire signal will be a point in N-dimensional space. Here, $N = 180$.

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & x[2] & \cdots & x[N-1] \end{bmatrix}^\top$$

Signals as vectors



- The above representation of $x[n]$ is in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_N\}$.

$$\mathbf{x} = x[0] \cdot \mathbf{e}_1 + x[1] \cdot \mathbf{e}_2 + \cdots + x[N-1] \cdot \mathbf{e}_N$$

- What would this signal look like in a different basis?

DFT: Fourier basis

- For rhythmic signals, the Fourier basis is often useful. We will need to switch to the complex vector space $\mathbb{C}^N$ to work with the Fourier basis.

- Consider the following complex exponential signals of length N,

$$\begin{aligned} w_k[n] &= e^{j\frac{2\pi k}{N}n}, \ 0 \leq n, k < N \\ &= \cos\left(\frac{2\pi k}{N}n\right) + j \sin\left(\frac{2\pi k}{N}n\right) \end{aligned}$$

We can represent this as a vector $\mathbf{w}_k \in \mathbb{C}^N$, where

$$\mathbf{w}_k = \begin{bmatrix} w_k[0] & w_k[1] & w_k[2] & \cdots & w_k[N-1] \end{bmatrix}^\top, \quad 0 \leq k < N-1$$

There are N such $\mathbf{w}_k$ vectors in $\mathbb{C}^N$.

DFT: Fourier basis

- The $\mathbf{w}_k$ vectors satisfy the following property,

$$\mathbf{w}_i^* \mathbf{w}_k = \begin{cases} N & , i = k \\ 0 & , i \neq k \end{cases}$$

- We define na orthonomgal basis for $\mathbb{C}^N$ as $\mathcal{F} = \left\{ \frac{1}{\sqrt{N}} \mathbf{w}_k \right\}_{k=0}^{N-1}$.

- Using this orthonormal basis, we define the **Fourier matrix** as the following,

$$\mathbf{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_1 & \cdots & \mathbf{w}_{N-1} \end{bmatrix}$$

- It can be verified that $\mathbf{F}_N$ is a unitary matrix, i.e. $\mathbf{F}_N^H \mathbf{F}_N = \mathbf{I}_N$.

DFT: Fourier basis

- The representation of a signal $\mathbf{x}$ in the Fourier basis is given by,

$$\mathbf{x}_{\mathcal{F}} = \mathbf{F}_N^{-1}\mathbf{x} = \mathbf{F}_N^H\mathbf{x}$$

  $\mathbf{x}_{\mathcal{F}}$ representation is called the **Discrete Fourier Transform** (DFT) of $\mathbf{x}$.

- The inverse DFT, i.e. obtaining the $\mathbf{x}$ from $\mathbf{X}_{\mathcal{F}}$, is given by,

$$\mathbf{x} = \mathbf{F}_N\mathbf{x}_{\mathcal{F}}$$

- $\mathbf{x}$ is the called the *time domain* representation of the signal, while $\mathbf{x}_{\mathcal{F}}$ is the *frequency domain* representation of the signal.

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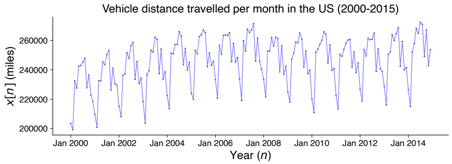
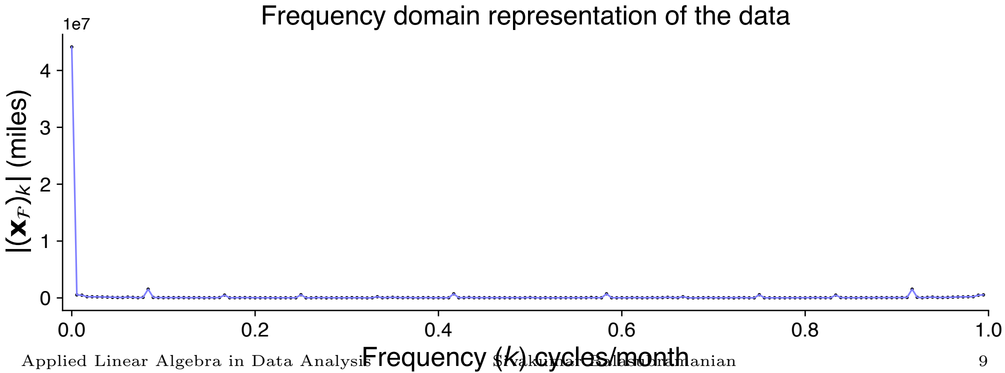
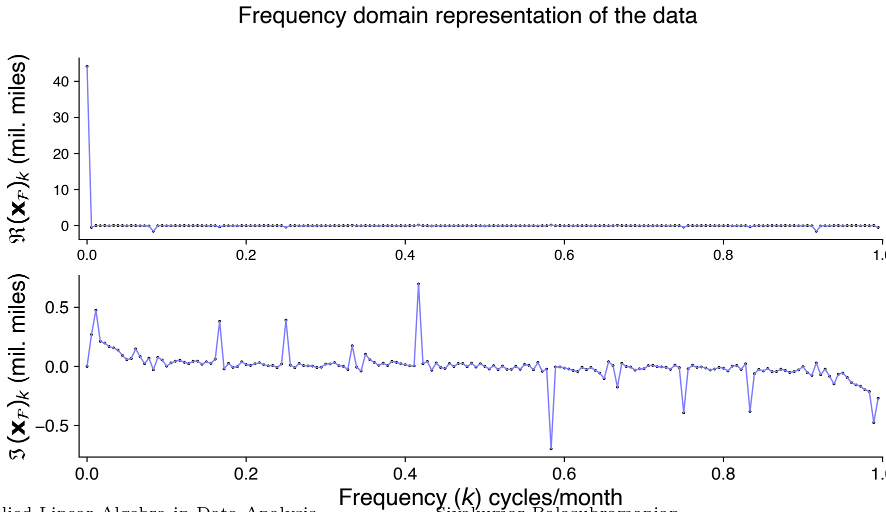
Frequency domain representation of $x[n]$

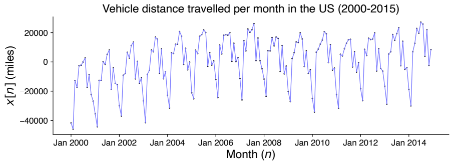


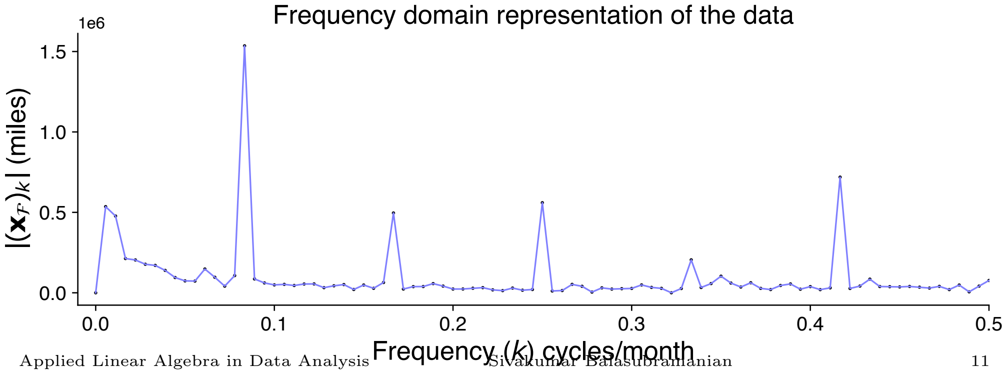


Real and Imaginary components



Frequency domain representation of mean subtracted $x[n]$





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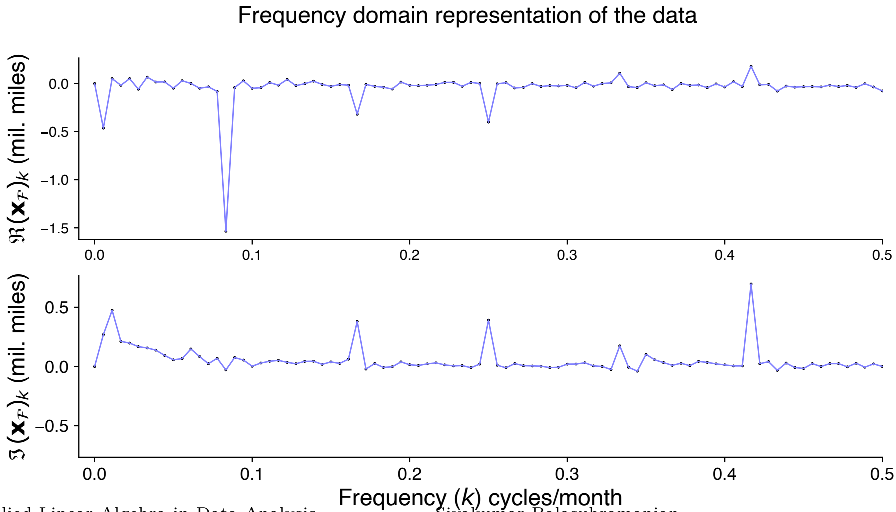
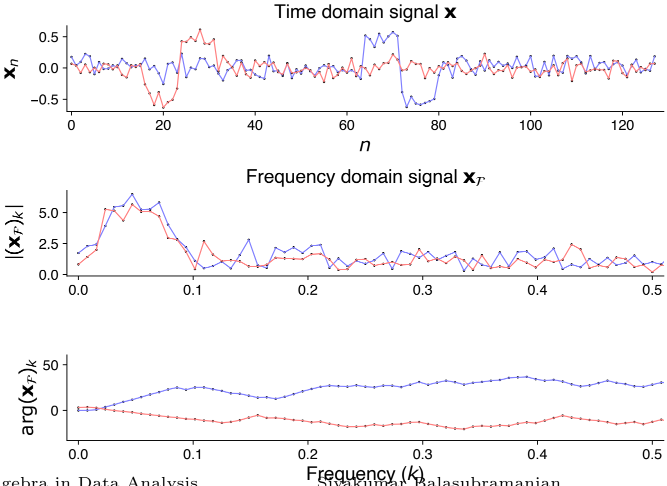


Problems with the Fourier basis

Fourier basis is not suitable for representing transient signals. They are not localized in time.



Wavelet basis

Wavelet basis are localized in time and frequency, making them suitable for transient signals.
The **Haar wavelet** is the simplest wavelet basis. Consider a the vector space $\mathbb{R}^8$, the Haar wavelet basis $\mathcal{W} = \{\mathbf{h}_k\}_{k=1}^{8}$ for this space is given by,

$$\mathbf{h}_1 = \frac{1}{\sqrt{8}}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \mathbf{h}_2 = \frac{1}{\sqrt{8}}\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix} \mathbf{h}_3 = \frac{1}{2}\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix} \mathbf{h}_4 = \frac{1}{2}\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix} \mathbf{h}_5 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix} \mathbf{h}_6 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix} \mathbf{h}_7 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix} \mathbf{h}_8 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

$$\mathbf{W}_8 = \begin{bmatrix}\mathbf{h}_1 & \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 & \mathbf{h}_5 & \mathbf{h}_6 & \mathbf{h}_7 & \mathbf{h}_8\end{bmatrix}$$

Wavelet basis

The wavelet basis is a an orthonormal basis for $\mathbb{R}^8$.

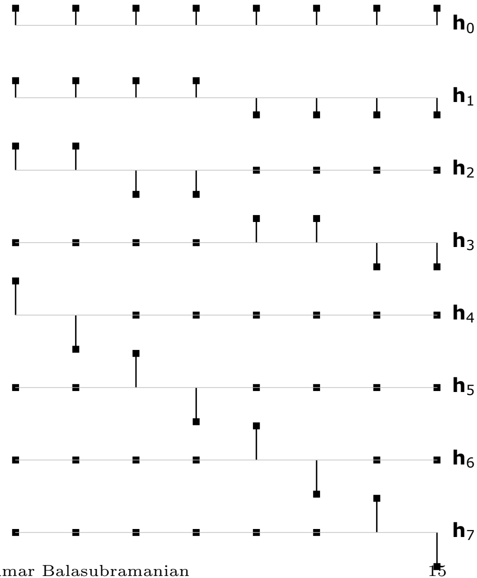
$$\mathbf{W}_8^H \mathbf{W}_8 = \mathbf{I}$$

Let $x[n],\ 0 \leq n < 8$ be a signal of length 8, which can represented in the standard basis of $\mathbb{R}^8$ as,

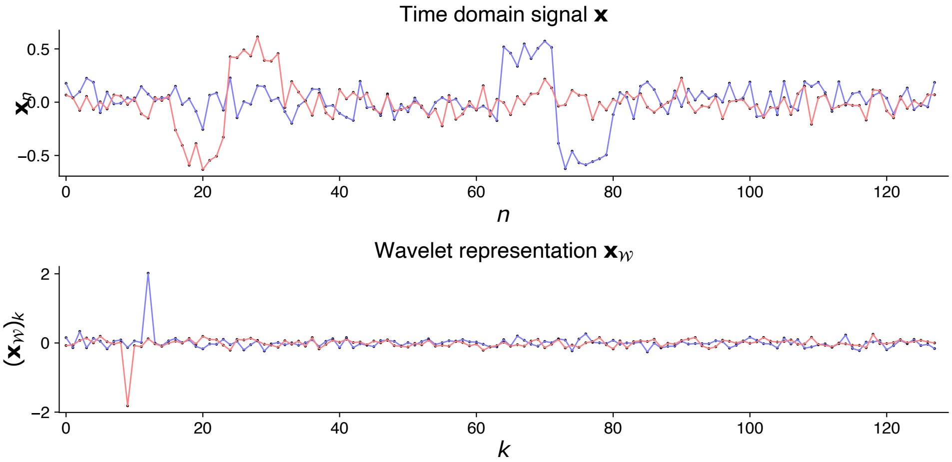
$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & x[2] & \cdots & x[7] \end{bmatrix}^\top$$

The resentation of this signal is the wavelet basis is given by,

$$\mathbf{x}_{\mathcal{W}} = \mathbf{W}_8^{-1}\mathbf{x} = \mathbf{W}_8^\top \mathbf{x}$$

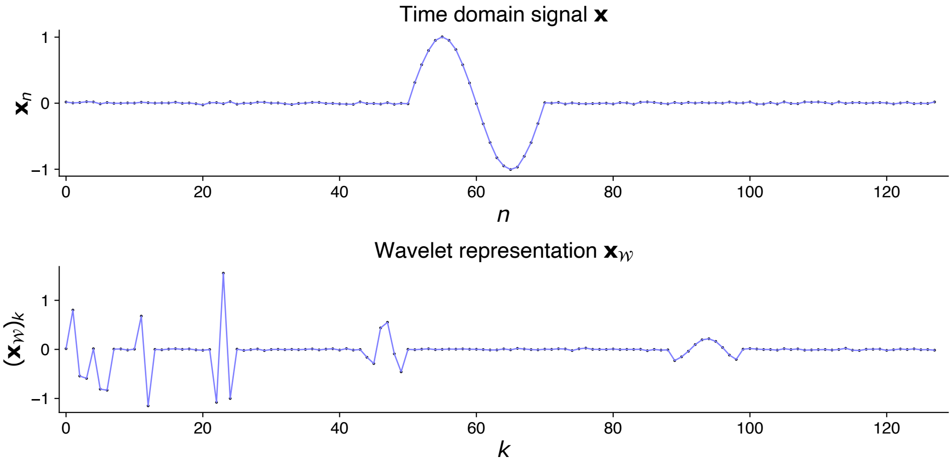


Represention in the wavelet basis



The Haar wavelet provides a sparse representation of the red and blue signals, because they are well matched with the Haar bases.

Represention in the wavelet basis



When the signal is not well matched with the wavelet basis, the representation is not sparse or less sparse.