Applied Linear Algebra in Data Analaysis
Orthogonality & Matrix Inverses Assignment

Marks: 23

1. Consider an orthonormal set of vectors,

$$V = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_r\} \quad \mathbf{v}_i \in \mathbb{R}^n \quad \forall i \in \{1, 2, \dots r\}$$

   If there is a vector $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v}_i^T \mathbf{w} = 0 \ \forall i \in \{1, 2, \dots r\}$. Prove that $\mathbf{w} \notin span\,(V)$. [**Marks: 1**]

2. For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, prove that $C(\mathbf{A}) \perp N\left(\mathbf{A}^T\right)$ and $C\left(\mathbf{A}^T\right) \perp N(\mathbf{A})$. [**Marks: 2**]

3. If the columns of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are orthonormal, prove that $\mathbf{A}^{-1} = \mathbf{A}^T$. What is $\mathbf{A}^T\mathbf{A}$ when $\mathbf{A}$ is rectangular ($\mathbf{A} \in \mathbb{R}^{m \times n}$) with orthonormal columns? [**Marks: 2**]

4. What will happen when the Gram-Schmidt procedure is applied to: (a) orthonormal set of vectors; and (b) orthogonal set of vectors? If the set of vectors are columns of a matrix $\mathbf{A}$, then what are the corresponding $\mathbf{Q}$ and $\mathbf{R}$ matrices for the orthonormal and orthogonal cases? [**Marks: 2**]

5. Prove that the rank of an orthogonal projection matrix $\mathbf{P}_S = \mathbf{U}\mathbf{U}^T$ onto a subspace $\mathcal{S}$ is equal to the dim $\mathcal{S}$, where the columns of $\mathbf{U}$ form an orthonormal basis of $\mathcal{S}$. [**Marks: 1**]

6. If the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ represent a basis for the subspace $\mathcal{S} \subset \mathbb{R}^m$. Find the orthogonal projection matrix $\mathbf{P}_\mathcal{S}$ onto the subspace $\mathcal{S}$. Hint: Gram-Schmidt orthogonalization. [**Marks: 1**]

7. Consider two orthogornal matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Is the $\mathbf{Q}_2^T\mathbf{Q}_1$ an orthogonal matrix? If yes, prove that it is so, else provide a counter-example showing $\mathbf{Q}_2^T\mathbf{Q}_1$ is not orthogonal. [**Marks: 1**]

8. Consider a 1 dimensional subspace spanned by the vector $\mathbf{u} \in \mathbb{R}^n$. What kind of a geometric operation does the matrix $\mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^T}{\mathbf{u}^T\mathbf{u}}$ represent? [**Marks: 1**]

9. Prove that when a triangular matrix is orthogonal, it is diagonal. [**Marks: 1**]

10. When does the following diagnoal matrix have an inverse? [**Marks: 1**]

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

   Write down an expression for $\mathbf{D}^{-1}$.

11. Consider a 2×2 block matrix, $\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{bmatrix}$, where $\mathbf{A} \in \mathbb{R}^{m \times m}$. Find an expression for the inverse $\mathbf{A}^{-1}$ interms of the block components and their inverses of $\mathbf{A}$. Hint: Consider $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}$, and solve $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. [**Marks: 1**]

12. Consider a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ with linearly independent columns. Prove that the Gram matrix $\mathbf{A}^T\mathbf{A}$ is invertible. [**Marks: 1**]

13. Consider the scalar equation, $ax = ay$. Here we can cancel a from the equation when $a \neq 0$. When can we carry out similar cancellations for matrcies? [**Marks: 2**]

    (a) $\mathbf{AX} = \mathbf{AY}$. Prove that here $\mathbf{X} = \mathbf{Y}$ only when $\mathbf{A}$ is left invertible.

    (b) $\mathbf{XA} = \mathbf{YA}$. Prove that here $\mathbf{X} = \mathbf{Y}$ only when $\mathbf{A}$ is right invertible.

14. Consider two non-singular matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Explain whether or not the following matrices are invertible. If they are, then provide an expression for it inverse. [**Marks: 4**]

    (a) $\mathbf{C} = \mathbf{A} + \mathbf{B}$

    (b) $\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$

    (c) $\mathbf{C} = \begin{bmatrix} \mathbf{A} & \mathbf{A} + \mathbf{B} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$

    (d) $\mathbf{C} = \mathbf{ABA}$

15. For a square matrix $\mathbf{A}$ with non-signular $\mathbf{I} - \mathbf{A}$, prove that, [**Marks: 1**]

$$\mathbf{A}\left(\mathbf{I} - \mathbf{A}\right)^{-1} = \left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{A}$$

16. Consider the non-singular matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{A} + \mathbf{B}$. Prove that, [**Marks: 1**]

$$\mathbf{A}\left(\mathbf{A} + \mathbf{B}\right)^{-1}\mathbf{B} = \mathbf{B}\left(\mathbf{A} + \mathbf{B}\right)^{-1}\mathbf{A} = \left(\mathbf{A}^{-1} + \mathbf{B}^{-1}\right)^{-1}$$