

Applied Linear Algebra in Data Analysis

Application: Signal Processing

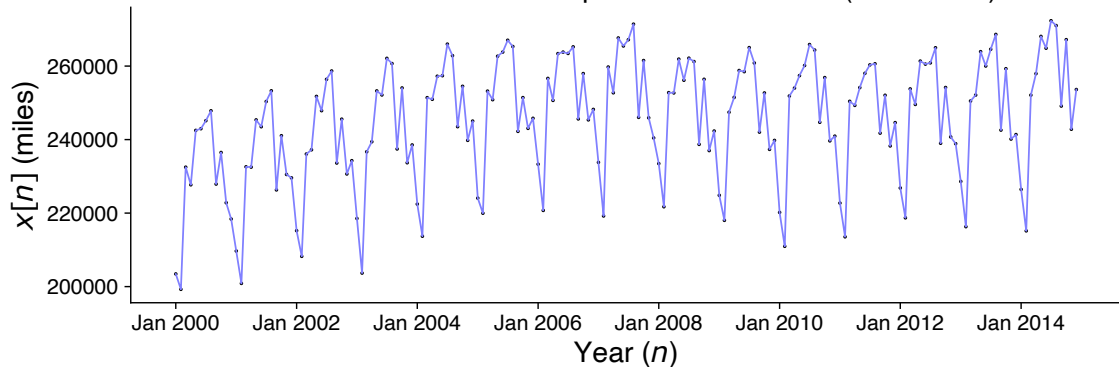
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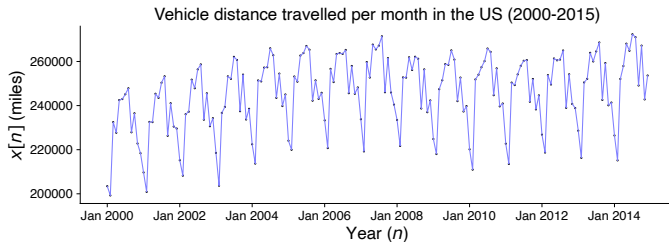
Signals as vectors

What is a signal? A *signal* is a function of an independent variable that conveys some information.

Vehicle distance travelled per month in the US (2000-2015)



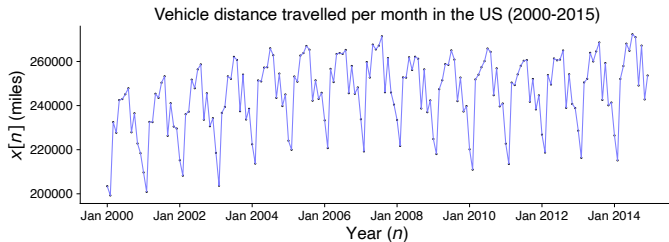
Signals as vectors



- ▶ $x[n]$ in the above figure is a finite length signal of length N , where $n \in \mathbb{Z}, 0 \leq n < N$.
- ▶ We can think of signal as a vector \mathbf{x} in \mathbb{R}^N , i.e. this entire signal will be a point in N -dimensional space. Here, $N = 180$.

$$\mathbf{x} = [x[0] \quad x[1] \quad x[2] \quad \cdots \quad x[N-1]]^\top$$

Signals as vectors



- The above representation of $x[n]$ is in the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\}$.

$$\mathbf{x} = x[0] \cdot \mathbf{e}_1 + x[1] \cdot \mathbf{e}_2 + \dots + x[N-1] \cdot \mathbf{e}_N$$

- What would this signal look like in a different basis?

DFT: Fourier basis

- For rhythmic signals, the Fourier basis is often useful. We will need to switch to the complex vector space \mathbb{C}^N to work with the Fourier basis.
- Consider the following complex exponential signals of length N ,

$$\begin{aligned}w_k[n] &= e^{j\frac{2\pi k}{N}n}, \quad 0 \leq n, k < N \\&= \cos\left(\frac{2\pi k}{N}n\right) + j \sin\left(\frac{2\pi k}{N}n\right)\end{aligned}$$

We can represent this as a vector $\mathbf{w}_k \in \mathbb{C}^N$, where

$$\mathbf{w}_k = [w_k[0] \quad w_k[1] \quad w_k[2] \quad \cdots \quad w_k[N-1]]^\top, \quad 0 \leq k < N-1$$

There are N such \mathbf{w}_k vectors in \mathbb{C}^N .

DFT: Fourier basis

- ▶ The \mathbf{w}_k vectors satisfy the following property,

$$\mathbf{w}_i^* \mathbf{w}_k = \begin{cases} N & , i = k \\ 0 & , i \neq k \end{cases}$$

- ▶ We define an orthonormal basis for \mathbb{C}^N as $\mathcal{F} = \left\{ \frac{1}{\sqrt{N}} \mathbf{w}_k \right\}_{k=0}^{N-1}$.
- ▶ Using this orthonormal basis, we define the **Fourier matrix** as the following,

$$\mathbf{F}_N = \frac{1}{\sqrt{N}} \begin{bmatrix} \mathbf{w}_0 & \mathbf{w}_1 & \cdots & \mathbf{w}_{N-1} \end{bmatrix}$$

- ▶ It can be verified that \mathbf{F}_N is a unitary matrix, i.e. $\mathbf{F}_N^H \mathbf{F}_N = \mathbf{I}_N$.

DFT: Fourier basis

- ▶ The representation of a signal \mathbf{x} in the Fourier basis is given by,

$$\mathbf{x}_{\mathcal{F}} = \mathbf{F}_N^{-1} \mathbf{x} = \mathbf{F}_N^H \mathbf{x}$$

$\mathbf{x}_{\mathcal{F}}$ representation is called the **Discrete Fourier Transform** (DFT) of \mathbf{x} .

- ▶ The inverse DFT, i.e. obtaining the \mathbf{x} from $\mathbf{X}_{\mathcal{F}}$, is given by,

$$\mathbf{x} = \mathbf{F}_N \mathbf{x}_{\mathcal{F}}$$

- ▶ \mathbf{x} is called the *time domain* representation of the signal, while $\mathbf{x}_{\mathcal{F}}$ is the *frequency domain* representation of the signal.

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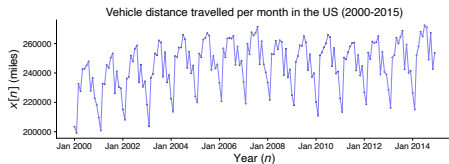
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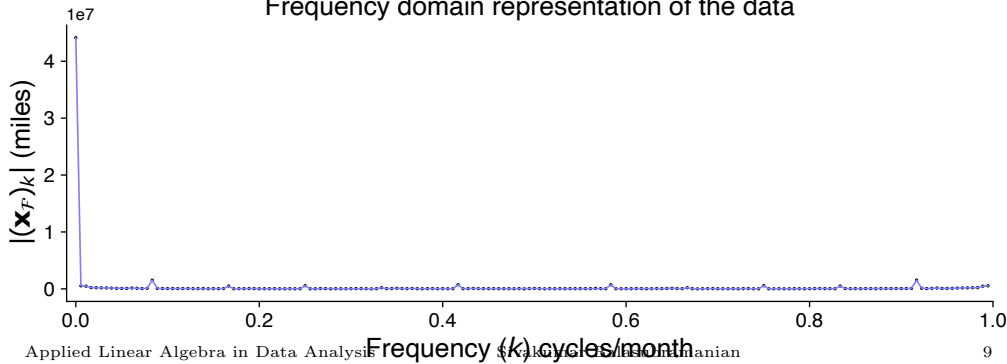
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Frequency domain representation of $x[n]$



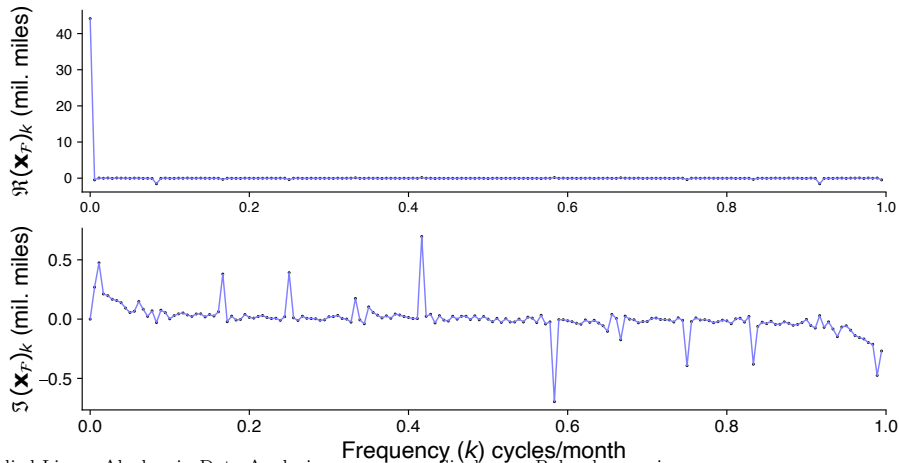
Frequency domain representation of the data



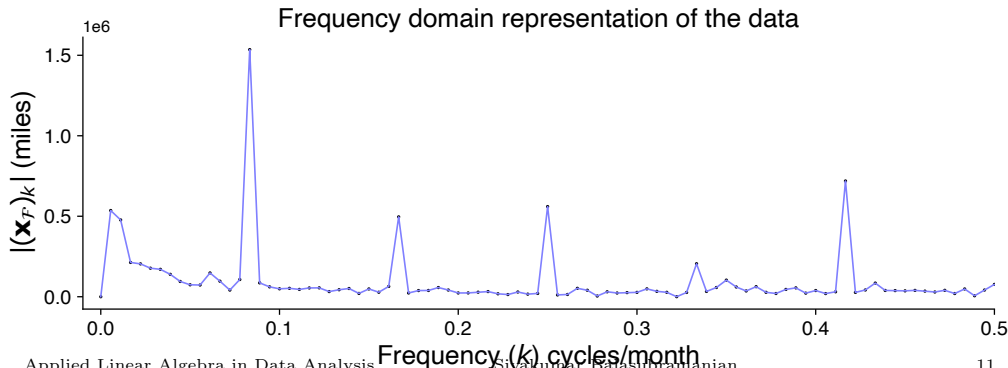
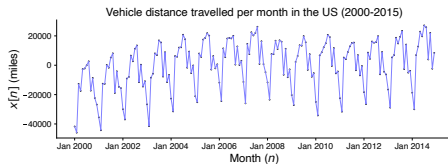
Frequency domain representation of $x[n]$

Real and Imaginary components

Frequency domain representation of the data

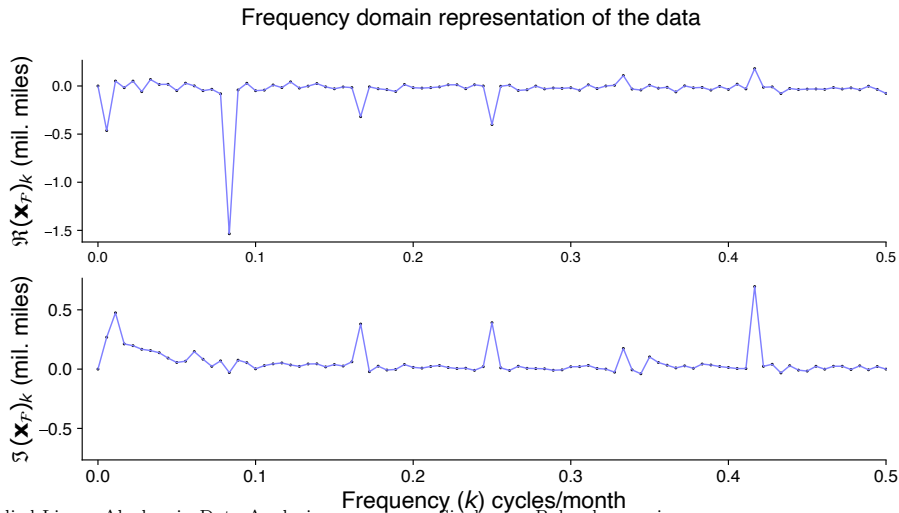


Frequency domain representation of mean subtracted $x[n]$



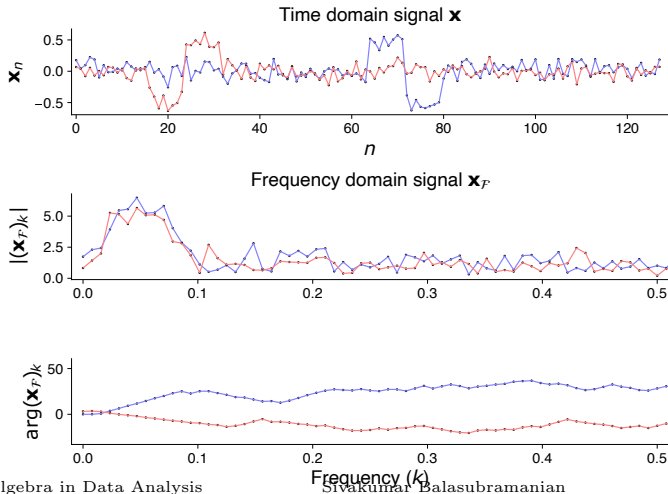
Frequency domain representation of mean subtracted $x[n]$

Real and Imaginary components



Problems with the Fourier basis

Fourier basis is not suitable for representing transient signals. They are not localized in time.



Wavelet basis

Wavelet basis are localized in time and frequency, making them suitable for transient signals.

The **Haar wavelet** is the simplest wavelet basis. Consider a the vector space \mathbb{R}^8 , the Haar wavelet basis $\mathcal{W} = \{\mathbf{h}_k\}_{k=1}^8$ for this space is given by,

$$\begin{aligned} \mathbf{h}_1 &= \frac{1}{\sqrt{8}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} & \mathbf{h}_2 &= \frac{1}{\sqrt{8}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} & \mathbf{h}_3 &= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \mathbf{h}_4 &= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \\ \mathbf{h}_5 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \mathbf{h}_6 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \mathbf{h}_7 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} & \mathbf{h}_8 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \end{aligned}$$

$$\mathbf{W}_8 = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3 \quad \mathbf{h}_4 \quad \mathbf{h}_5 \quad \mathbf{h}_6 \quad \mathbf{h}_7 \quad \mathbf{h}_8]$$

Wavelet basis

The wavelet basis is an orthonormal basis for \mathbb{R}^8 .

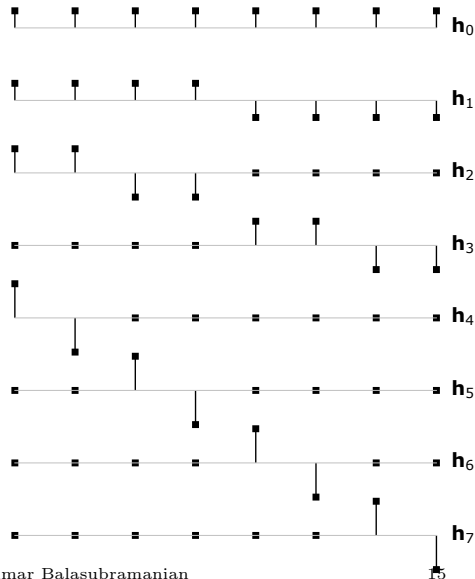
$$\mathbf{W}_8^H \mathbf{W}_8 = \mathbf{I}$$

Let $x[n]$, $0 \leq n < 8$ be a signal of length 8, which can be represented in the standard basis of \mathbb{R}^8 as,

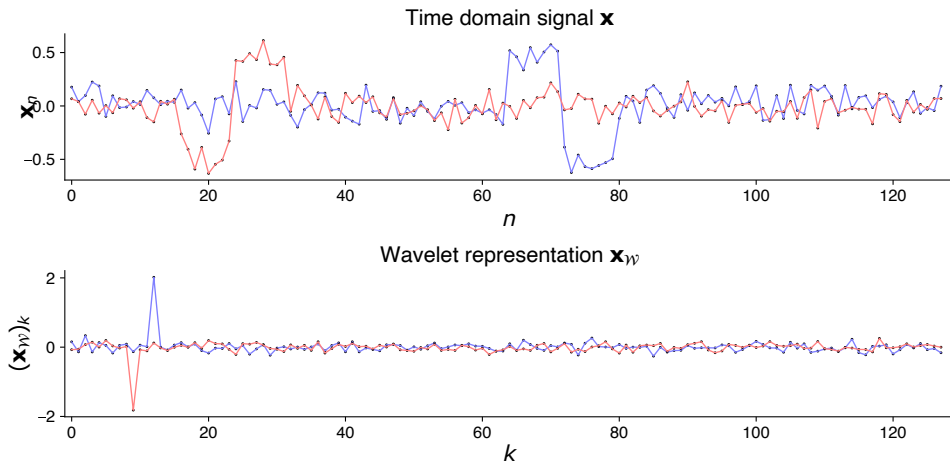
$$\mathbf{x} = [x[0] \ x[1] \ x[2] \ \cdots \ x[7]]^\top$$

The representation of this signal in the wavelet basis is given by,

$$\mathbf{x}_W = \mathbf{W}_8^{-1} \mathbf{x} = \mathbf{W}_8^\top \mathbf{x}$$

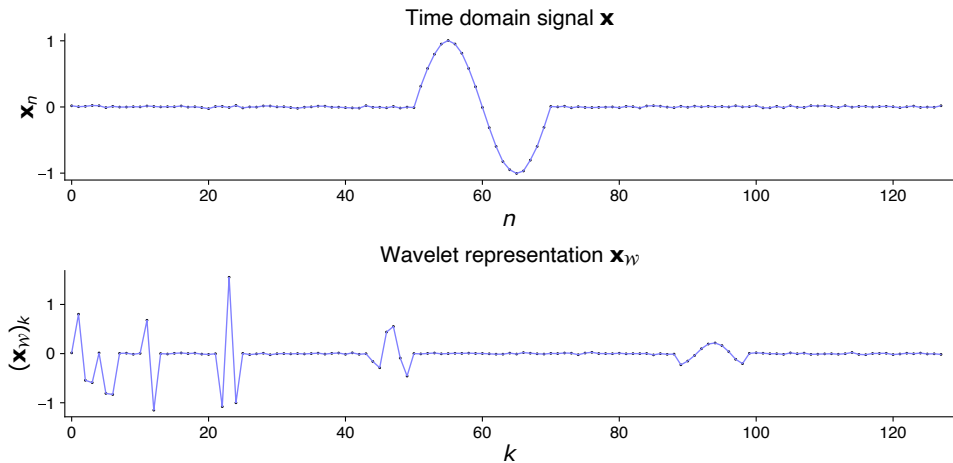


Representation in the wavelet basis



The Haar wavelet provides a sparse representation of the red and blue signals, because they are well matched with the Haar bases.

Representation in the wavelet basis



When the signal is not well matched with the wavelet basis, the representation is not sparse or less sparse.