Applied Linear Algebra in Data Analysis

Tutorial

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CONTENTS

1 CONCEPTS IN VECTOR SPACES

1. Which of the following sets forms a vector space?
   a) $\{\mathbf{x} \,|\, x_1, x_2 \in \mathbb{R} \text{ and } a_1x_1 + a_2x_2 = 0\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.
   b) $\left\{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{a}^\top \mathbf{x} = b\right\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.
   c) $\left\{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^\top \mathbf{x} = 1\right\}$.
   d) $\{(x[0], x[1], x[2], \ldots x[N-1]) \,|\, x[i] \in \mathbb{R}, 0 \leqslant i < N\}$.

   (The set of all real-valued time-domain signals of length N. $x[i]$ is the value of the signal at time instant i.)

2. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^{n} a_k x^k \,\middle|\, a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0,1]$$

   Show that polynomails of order strictly lower than n form subspaces of $\mathcal{P}$.

3. Is the following function a valid norm of the vector space $\mathcal{P}$?

$$\|\mathbf{p}(x)\| = \sqrt{\sum_{k=0}^{n} a_k^2}, \; \mathbf{p} = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}$$

4. Consider the following function, which is often called the *zero-norm* of a vector $\mathbf{x} \in \mathbb{R}^n$.

$$\|\mathbf{x}\|_0 = \sum_{i=1}^{n} \mathbb{I}(x_i \neq 0), \text{ where, } \mathbb{I}(A) = \begin{cases} 1 & A \text{ is true.} \\ 0 & A \text{ is false.} \end{cases}$$

   Is the *zero-norm*, which is often used for quantifying the *sparsity* of a vector, a proper norm?

5. Is the following set of vectors linear independent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

   What is the span of this set? Does this set form the basis for its span? Does it form an orthonormal basis?

6. Consider the following function,

$$f(\mathbf{x}) = \sum_{i=1}^{n} w_i |x_i|, \;\; \mathbf{x} \in \mathbb{R}^n, w_i > 0$$

   Is f a norm? If not, what properties does it lack?

7. Find the norm of the following vectors using the the 1-norm, 2-norm and the ∞-norm.
   a) $\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^\top$
   b) $\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^\top$
   c) $\mathbf{e}_i$, where $1 \leqslant i \leqslant n$
   d) $\mathbf{0} \in \mathbb{R}^n$

e) $\mathbf{1} \in \mathbb{R}^n$

8. For any given $\mathbf{x} \in \mathbb{R}^n$, show that,

$$\|\mathbf{x}\|_1 \geqslant \|\mathbf{x}\|_2 \geqslant \|\mathbf{x}\|_3 \cdots \geqslant \|\mathbf{x}\|_\infty$$

9. Consider the linear function $f : \mathbb{R}^3 \mapsto \mathbb{R}$. We know the output of the function for the following inputs,

$$f\left(\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top\right) = -2, \quad f\left(\begin{bmatrix}-1 & 2 & -1\end{bmatrix}^\top\right) = 1, \quad f\left(\begin{bmatrix}-1 & 1 & 2\end{bmatrix}^\top\right) = 0$$

Find an input input $\mathbf{x} \in \mathbb{R}^3$ such that $f(\mathbf{x}) = 0$.

10. Find the representation of $\mathbf{x} = \begin{bmatrix}2 & 1\end{bmatrix}^\top$ in the following bases.

a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

b) $\frac{1}{\sqrt{2}}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right\}$

c) $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

11. Consider the function $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ that selects the i^{th} element of a given vector $\mathbf{x} \in \mathbb{R}^n$.

$$f(\mathbf{x}) = x_i, \text{where} \quad \mathbf{x} = \begin{bmatrix}x_1 & x_2 \cdots & x_n\end{bmatrix}^\top$$

Is this function linear? If so, what is the vector $\mathbf{w}$ asssociated with this function, such that $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$?

12. Given a set of real numbers $x_1, x_2, \ldots x_n \in \mathbb{R}$ which are used to the n-vector $\mathbf{x}$. Can you express the mean $\bar{x}$ and variance σ_x^2 of this set of data using the the standard inner product in $\mathbb{R}^n$? Note the following,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

2 MATRICES

1. Conisder the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

   a) Find the product of the two matrices $\mathbf{C} = \mathbf{AB}$ using the four views of matrix muliplication.

   b) If we change $b_{23} = 0$. Can you compute the new matrix $\mathbf{C}$ without performing the entire matrix muliplication again?

   c) If we increase the value of the elements of the 3^{rd} column of $\mathbf{A}$ by 1, how can we compute the new $\mathbf{C}$ without performing the entire matrix multiplication again?

   d) If we insert a new row $\mathbf{1}^\top$ in $\mathbf{A}$ after the 2^{nd} row of $\mathbf{A}$, how can we compute the new $\mathbf{C}$ without performing the entire matrix multiplication again?

2. Consider a matrix $\mathbf{A} \in \mathbb{R}^{10^6 \times 5}$, and we are interested in computing the product $\mathbf{A}^\top \mathbf{A} \mathbf{A}^\top$. Should you compute the product as $\left(\mathbf{A}^\top \mathbf{A}\right) \mathbf{A}^\top$ or $\mathbf{A}^\top \left(\mathbf{A}\mathbf{A}^\top\right)$? Why?

3. Consider an orthogonal, square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. We generate a new matrix $\mathbf{C} = \mathbf{AB}$. What can we say about the following questions about this product?

   a) How are the columns of $\mathbf{C}$ related to the columns of $\mathbf{A}$

   b) How is the 2-norm of the i^{th} column of $\mathbf{C}$ related that of the columns of $\mathbf{B}$?

4. Show that the matrix product $\mathbf{ABC}$ can be written as a weighted sum of the outer products of the columns of $\mathbf{A} \in \mathbb{R}^{n \times p}$ and rows of $\mathbf{C} \in \mathbb{R}^{q \times n}$, with the weights coming from the matrix $\mathbf{B} \in \mathbb{R}^{p \times q}$.

$$\mathbf{ABC} = \sum_{i=1}^{p} \sum_{j=1}^{q} b_{ij} \mathbf{a}_i \tilde{\mathbf{c}}_j^\top$$

5. Prove the following for the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots \mathbf{A}_n$.

$$\left(\mathbf{A}_1, \mathbf{A}_2 \mathbf{A}_3 \ldots \mathbf{A}_n\right)^\top = \mathbf{A}_n^\top \mathbf{A}_{n-1}^\top \ldots \mathbf{A}_2^\top \mathbf{A}_1^\top$$

6. **Matrix Inversion Lemma**. Consider an invertible matrix $\mathbf{A}$. The matrix $\mathbf{A} + \mathbf{u}\mathbf{v}^\top$ is invertible if and only if the two vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u} \neq -1$. Then, the inverse is given by,

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^\top\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{u} \mathbf{v}^\top \mathbf{A}^{-1}}{1 + \mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u}}$$

7. Prove that $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$, where $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$.

8. Show that the diagonal elements of a square matrix $\mathbf{A}$, such that $\mathbf{A}^\top = -\mathbf{A}$ are zero. These are *skew-symmetric* matrices.

9. Show that $\mathbf{x}^\top \mathbf{A} \mathbf{x} = 0$ if $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix.

Additional problems

1. Consider the following 5×4 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 2 \\ 1 & -2 & 1 & 3 & 9 \\ -2 & 0 & 2 & -1 & -2 \\ 3 & 1 & 1 & -5 & 0 \end{bmatrix}$$

   Compute the following:

   a) $\mathbf{a}_3$
   b) $\mathbf{a}_1^\top$
   c) $\tilde{\mathbf{a}}_2^\top$
   d) $\tilde{\mathbf{a}}_4$
   e) $\mathbf{a}_1 \mathbf{a}_2^\top$
   f) $\tilde{\mathbf{a}}_3 \mathbf{a}_2^\top$
   g) $\tilde{\mathbf{a}}_1 \tilde{\mathbf{a}}_2^\top$
   h) $\tilde{\mathbf{a}}_1^\top \tilde{\mathbf{a}}_2$
   i) $\tilde{\mathbf{a}}_1 \tilde{\mathbf{a}}_1^\top + \tilde{\mathbf{a}}_2 \tilde{\mathbf{a}}_2^\top$
   j) $\mathbf{a}_3^\top \mathbf{a}_1 + \mathbf{a}_2^\top \mathbf{a}_4$

2. Which of the following statements true about a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$?

   a) $\sum_{i=1}^{m} \|\mathbf{a}_i\|_2^2 = \sum_{i=1}^{n} \|\tilde{\mathbf{a}}_i\|_2^2$
   b) $\sum_{i=1}^{m} \|\mathbf{a}_i\|_2^2 = \text{tr}\left(\mathbf{A}^\top \mathbf{A}\right)$
   c) If the matrix $\mathbf{A} = \mathbf{V} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbf{V}^{-1}$, Then

$$\mathbf{A}^n = \mathbf{V} \begin{bmatrix} d_1^n & 0 \\ 0 & d_2^n \end{bmatrix} \mathbf{V}^{-1}$$

3. Consider the matrix $\mathbf{P} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_3 & \mathbf{e}_2 \end{bmatrix}$. What does this P matrix do to a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ in the following operations? Try to compute these without performing the matrix multiplication and by using you understaning of the row and column views of matrix multiplication.

   a) $\mathbf{PA}$
   b) $\mathbf{AP}$
   c) $\mathbf{P}^2\mathbf{A}$
   d) $\mathbf{AP}^2$
   e) $\mathbf{PAP}$

3 SOLUTION TO LINEAR EQUATIONS

1. Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ with $n > m$. Consider a linearly independent set of vectors $\mathbf{x}_1, \mathbf{x}_2$. Is the set of $\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2$ linearly independent?

2. Find the bases for the four fundamental subspaces of the following matrices

   a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

   b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

   c) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

   d) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

   e) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

3. Show that the $\mathrm{rank}(\mathbf{AB}) = \mathrm{rank}(\mathbf{A})$, when $\mathbf{B}$ is square and full rank.

4. Let $\mathbf{A}$ is a full rank matrix. Show that the *Gram matrix* of the column space, $\mathbf{A}^\top\mathbf{A}$ is invertible.

5. Draw the four fundamental subspaces of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

6. Find the complete set of solutions for the following system of equations,

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

   where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{bmatrix}$.

7. Is the following set of equations solvable? If yes, then find the complete set of solutions.

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   where $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{bmatrix}$.

4 ORTHOGONALITY

1. If $\mathbf{A}$ is an orthogonal matrix, show that $\mathbf{A}^{-1} = \mathbf{A}^\top$.

2. If $\mathbf{P}_{\mathcal{S}}$ is the orthogonal projection matrix onto the subspace $\mathcal{S}$, then what is the corresponding orthogonal projection matrix onto $\mathcal{S}^\perp$ – the orthogonal complement of $\mathcal{S}$?

3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Show that the following holds,
$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{n} \left(\mathbf{x}^\top \mathbf{u_i}\right) \cdot \left(\mathbf{u_i}^\top \mathbf{y}\right)$$

4. Consider the following set of vectors, $\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots \mathbf{a}_n\}$, where $\mathbf{a}_i \in \mathbb{R}^n$. The set $\mathcal{S}$ is linearly independent. Find the orthogonal components of a vector $\mathbf{b} \in \mathbb{R}^n$ in the subspace spanned by the sets of vectors $\mathcal{S}_1 = \{\mathbf{a}_i\}_{i=1}^m$ and $\mathcal{S}_1^\perp$.

5. Consider the set of $n \times n$ orthogonal matrices,
$$\mathcal{Q} = \left\{\mathbf{Q} \,\middle|\, \mathbf{Q} \in \mathbb{R}^{n \times n}, \mathbf{Q}^\top \mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}_n\right\}$$
Is this set a subspace of $\mathbb{R}^{n \times n}$?
Show that the set is closed under matrx multiplication.

6. Consider the linear map, $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let us assume that $\mathbf{A}$ is full rank. What conditions must $\mathbf{A}$ satisfy for the following statements to be true,
   a) $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$.
   b) $\mathbf{y}_1^\top \mathbf{y}_2 = \mathbf{x}_1^\top \mathbf{x}_2$, for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$.

   Note: A linear map $\mathbf{A}$ with the aforementioned properties preserves lengths and angle between vectors. Such maps are encountered in rigid body mechanics.

7. Find the QR demcomposition of the following matrices.
   a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
   b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
   c) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}$
   d) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$
   e) $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

5 MATRIX INVERSES

1. Find a left inverse for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the set of all possible left inverses.

2. Consider an upper triangular matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$. We are interested in solving the following set of n linear equations,

$$\mathbf{R}\mathbf{x} = \mathbf{e}_i$$

$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}^\top$ is the solution to the above equation. Show that $x_{i+1} = x_{i+2} = \dots = x_n = 0$.

3. Find the pseudo-inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. Show that the matrix $\mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projection matrix onto the column space of $\mathbf{A}$.

6 EIGENVALUES AND EIGENVECTORS

1. Find the eigenvalues and eigenvectors of the following matrices.

   a) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

   b) $\mathbf{A} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

   c) $\mathbf{A} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

   d) $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -3 & -1 \end{bmatrix}$

2. Let $\mathbf{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$. What is the value of the following?

   a) A^2

   b) A^{100}

   c) A^{∞}

3. For a matrix $\mathbf{A}$ with eigenvalues $\{\lambda_i\}_{i=1}^{n}$, verify for the following matrices that $\prod_{i=1}^{n} \lambda_i = \det(\mathbf{A})$ and $\sum_{i=1}^{n} \lambda_i = \mathrm{trace}(\mathbf{A})$.

   a) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

   b) $\frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

4. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^{n}$ are the eignepairs of a matrix $\mathbf{A}$. What are the eigenpairs of the folliowing?

   a) $2\mathbf{A}$

   b) $\mathbf{A} - 2\mathbf{I}$

   c) $\mathbf{I} - \mathbf{A}$

7 POSITIVE DEFINITE MATRICES AND MATRIX NORMS

1. Prove that $\mathbf{A}^T\mathbf{A}$ is positive semi-definite for any matrix $\mathbf{A}$. When is $\mathbf{A}^T\mathbf{A}$ guaranteed to be positive definite?
2. If $\mathbf{A}$ is positive definite, then prove that $\mathbf{A}^{-1}$ is also positive definite.
3. Is the function $f(x_1, x_2, x_3) = 12x_1^2 + x_2^2 + 6x_3^2 + x_1x_2 - 2x_2x_3 + 4x_3x_1$ positive definite?
4. Prove the following for $\mathbf{A} \in \mathbb{R}^{m \times n}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^{\tilde{T}} \\ \mathbf{a}_1^{\tilde{T}} \\ \vdots \\ \mathbf{a}_m^{\tilde{T}} \end{bmatrix}$$

   a) $\|\mathbf{A}\|_F = \mathit{trace}\left(\mathbf{A}^T\mathbf{A}\right)$
5. Verify the following inequalities on vector and matrix norms ($\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$):
   a) $\|\mathbf{x}\|_\infty \leqslant \|\mathbf{x}\|_2$
   b) $\|\mathbf{x}\|_2 \leqslant \sqrt{m}\,\|\mathbf{x}\|_\infty$