Applied Linear Algebra for Data Analysis

Eigenvalues and Eigenvectors

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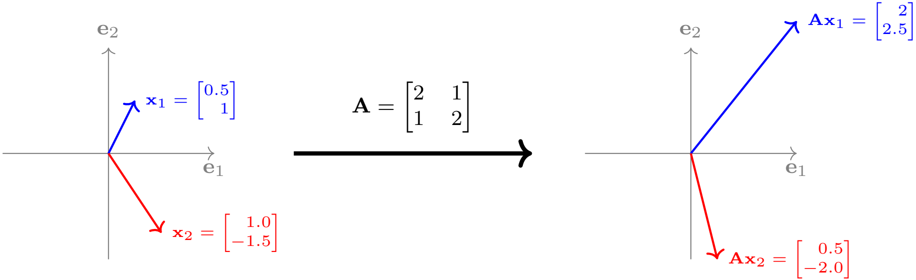
Linear transformation

- Matrices represent linear transformations, $\mathbf{A} \in \mathbb{R}^{m \times n}$ represents a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

$$\mathbf{y} = T(\mathbf{x}) = \mathbf{A}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{y} \in \mathbb{R}^m$$

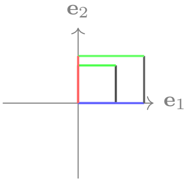
- Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.
  In general, T scales, and rotates/reflects the vector $\mathbf{x}$ to produce $\mathbf{y}$.



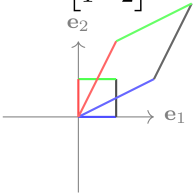
Linear transformation

An easier way is to look at what happens to the standard basis $\{\mathbf{e}_i\}_{i=1}^n$.

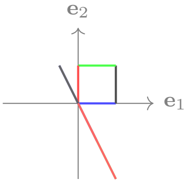
$$\mathbf{A} = \begin{bmatrix} 1.75 & 0 \\ 0 & 1.25 \end{bmatrix}$$



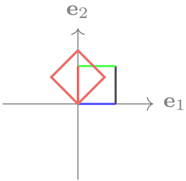
$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$



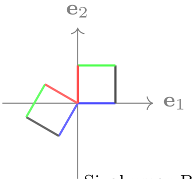
$$\mathbf{A} = \begin{bmatrix} -0.5 & 1 \\ 1 & -2 \end{bmatrix}$$



$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



Linear transformation in different basis

- Consider a basis $V = \{\mathbf{v}_i\}_{i=1}^n$ for $\mathbb{R}^n$. Let $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ be the representation of $\mathbf{x}$ in the standard basis.
  Representation of $\mathbf{x}$ in V is,

$$\mathbf{x} = \sum_{i=1}^{n} x_{vi}\mathbf{v}_i, \quad \mathbf{x}_V = \begin{bmatrix} x_{v1} & x_{v2} & \dots & x_{vn} \end{bmatrix}^\top$$

- We can go back and forth between these two representations in the following way,

$$\mathbf{x} = \mathbf{V}\mathbf{x}_V \quad \text{and} \quad \mathbf{x}_V = \mathbf{V}^{-1}\mathbf{x}; \;\; \text{where, } \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

- When V is an orthonormal basis, then the algebra gets simpler,

$$\mathbf{x} = \mathbf{V}\mathbf{x}_V \quad \text{and} \quad \mathbf{x}_V = \mathbf{V}^\top\mathbf{x}$$

Linear transformation in different basis

Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Consider a vector $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What is $\mathbf{y} = \mathbf{A}\mathbf{x}$?

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Now, consider a basis $V = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$. The representation of $\mathbf{x}, \mathbf{y}$ in V is,

$$\mathbf{x}_V = \mathbf{V}^{-1}\mathbf{x} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{y}_V = \mathbf{V}^{-1}\mathbf{y} = \frac{1}{3} \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

Now, if we apply the linear transformation T on $\mathbf{x}_V$ will we get $\mathbf{y}_V$?

$$\mathbf{A}\mathbf{x}_V = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -4 \\ 5 \end{bmatrix} \neq \mathbf{y}_V$$

Representation of a linear transformation T is basis dependent!

Similarity transformation

- Linear transformations represented in one basis represent a different transformation in another basis. This issue can be addressed by keeping track of the basis one is working in.

- Let $\mathbf{x}, \mathbf{y}$ be representations in the standard basis. Changing basis to V, gives us $\mathbf{x}_V, \mathbf{y}_V$.

$$\mathbf{y}_V = \mathbf{V}^{-1}\mathbf{y} = \mathbf{V}^{-1}\mathbf{A}\mathbf{x} = \mathbf{V}^{-1}\mathbf{A}\mathbf{V}\mathbf{x}_V = \mathbf{A}_V\mathbf{x}_V$$

Similarity transformation

- Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ are called *similar* matrices, if there exists a non-singular matrix $\mathbf{Q}$, such that,
$$\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$$
- The transformation represented by $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is called the *similarity transformation*.
- *Similar* matrices represent the same linear transformation in different basis.
- When $\mathbf{Q}$ is an orthogonal matrix, we have $\mathbf{B} = \mathbf{Q}^{\top}\mathbf{A}\mathbf{Q}$.

Eigenvectors and Eigenvalues

- Any linear transformation represented by $\mathbf{A} \in \mathbb{C}^{n \times n}$ has vectors that satisfy the following property,
$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \quad \mathbf{x} \in \mathbb{C}^n, \lambda \in \mathbb{C}, \;\; \mathbf{x} \neq \mathbf{0}$$
where, λ and $\mathbf{x}$ are called the eigenvalue and the associated eigenvector of $\mathbf{A}$.

- Any such pair $(\lambda, \mathbf{x})$ is called the eigenpair of $\mathbf{A}$.

- These are important for understanding and solving linear differential and difference equations:
$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) \quad \text{and} \quad \mathbf{x}[n+1] = \mathbf{A}\mathbf{x}[n]$$

Eigenvectors and Eigenvalues

Consider the differential equation, $\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) = \begin{bmatrix} -5 & 11 \\ 0 & -6 \end{bmatrix} \mathbf{x}(t)$. Let us assume that the solution is of the form, $\mathbf{x} = e^{\lambda t}\hat{\mathbf{x}}$. Then we have,

$$\frac{d\mathbf{x}(t)}{dt} = e^{\lambda t}\mathbf{A}\hat{\mathbf{x}} = e^{\lambda t}\lambda\hat{\mathbf{x}} \implies \mathbf{A}\hat{\mathbf{x}} = \lambda\hat{\mathbf{x}}$$

$$\begin{bmatrix} -5 & 11 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \lambda\hat{x}_1 \\ \lambda\hat{x}_2 \end{bmatrix} \implies \begin{bmatrix} -5-\lambda & 11 \\ 0 & -6-\lambda \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where, $\hat{\mathbf{x}} \in N(\mathbf{A} - \lambda\mathbf{I})$.

Eigenvalues and Eigenvectors

- We can find the eigenpairs using the same approach for $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0 = p\left(\lambda\right)$.
- $p\left(\lambda\right)$ is the characteristic polynomial of $\mathbf{A}$, and $p\left(\lambda\right) = 0$ is the characteristic equation.
- The eigenvalues are the roots of the polynomial $p\left(\lambda\right)$, and the $\mathbf{x}$ in $\left(\mathbf{A} - \lambda \mathbf{I}\right)\mathbf{x} = 0$ for the different λs are the corresponding eigenvectors.

Eigenvalues and Eigenvectors

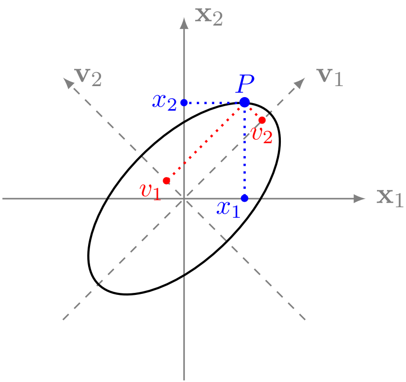
Compute the eigenpairs for the following matrices: $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Eigenvalues and Eigenvectors

Compute the eigenpairs for the following matrices: $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Diagonalization of a matrix

Often the right choice of basis can simplify an equation or the analysis of a problem. For example,



The equation of the ellipse in standard basis is:

$$3x_1^2 + 3x_2^2 - 2x_1x_2 = 1$$

This has a much simpled representation in the dashed coordinate frame.

$$4v_1^2 + 2v_2^2 = 1$$

The use of similarity transformation to simplify a matrix is at the heart of diagonalization.

Diagonalization of a matrix

- Consider a matrix $\mathbf{A}$ with n eigenpairs $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{n}$.

$$\mathbf{A}\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \dots & \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\mathbf{x}_1 & \lambda_2\mathbf{x}_2 & \dots & \lambda_n\mathbf{x}_n\end{bmatrix}$$

$$\mathbf{AX} = \mathbf{X}\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_n\end{bmatrix} = \mathbf{X\Lambda}$$

- If the eignevectors are linearly independent, then we have $\mathbf{X}^{-1}\mathbf{AX} = \mathbf{\Lambda}$

Diagonalization of a matrix

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by $\mathbf{A} = \begin{bmatrix} 8 & 1 \\ 2 & 7 \end{bmatrix}$. Diagonalize this matrix. What does $\mathbf{A}$ do to $\mathbf{x} = \begin{bmatrix} 3 & 4 \end{bmatrix}^\top$?

Diagonalization of a matrix

What about $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$?

Diagonalization of a matrix: Eigenpairs of special matrices

- A square matrices with a complete set of eigenvectors, i.e. a linearly independent set of n eigenvectors, can be decomposed into the following,

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$$

Diagonalization of a matrix: Eigenpairs of special matrices

- When $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, i.e. $\mathbf{A} = \mathbf{A}^\top$,
  - All eigenvalues are real.
  - The matrix poses a complete set of eigenvectors, i.e. they form a linearly independent set.
  - The eigenvectors can be chosen to be orthogonal to each other. When the eigenvalues are distinct, the eigenvectors are orthogonal. But when the eigenvalues are not distinct, we can choose them to be orthogonal.

This gives us, $\mathbf{A} = \mathbf{A}^\top = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^\top$.

Diagonalization of a matrix: Eigenpairs of special matrices

Diagonalize $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Diagonalization of a matrix

- A change of basis to $\mathbf{X}$ simplifies $\mathbf{A}$ to a diagonal matrix, the simplest possible form.
- If a matrix $\mathbf{A}$ has n distinct eigenvalues, then $\mathbf{A}$ can always be diagonalized.
- When there are repeated eigenvalues, we might not always be able to diagonalize a matrix. This happens when there aren't enough eigenvectors. These are called *defective* matrices.

$$\text{Algebraic multiplicity} \neq \text{Geometric multiplicity}$$

where, *algebraic multiplicity* is the number of times the eigenvalue λ is repeated, and *geometric multiplicity* is $\dim N\left(\mathbf{A} - \lambda\mathbf{I}\right)$.

Diagonalization of a matrix

Diagonalize $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Jordan Form

- If $\mathbf{A}$ cannot be diagonalized, the next best thing is the *Jordan form*.

- Let $\mathbf{A}$ have eigenvalues $(\lambda_1, \lambda_2, \dots \lambda_k)$. We can find a similarity transformation, such that,

$$\mathbf{A} = \mathbf{P}\mathbf{J}\mathbf{P}^{-1}, \;\; \mathbf{J} = \begin{bmatrix} \mathbf{J}(\lambda_1) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}(\lambda_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{J}(\lambda_k) \end{bmatrix}$$

Jordan Form

Each $\mathbf{J}(\bullet)$ is associated with an eigenvalue and an eigenvector, and is called a Jordan block, and has the form

$$\mathbf{J}(\lambda_l) = \begin{bmatrix} \lambda_l & 1 & 0 & \dots & 0 \\ 0 & \lambda_l & 1 & \dots & 0 \\ 0 & 0 & \lambda_l & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_l \end{bmatrix}$$

- $\mathbf{J} \in \mathbb{C}^{r \times r}$. $r =$ the algebraic multiplicity of the eigenvalue λ_l.
- $1 =$ the geometric multiplicity of the eigenvalue $\lambda_l = \dim N(\mathbf{A} - \lambda_l \mathbf{I})$.
- A 1-by-1 Jordan block is simply $[\lambda_l]$, corresponding to a eigenvalue with an associated eigenvector.

Jordan Form

- Jordan form of a diagonalizable matrix $\mathbf{A} \to \mathbf{J} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

$$\lambda = -2 \ (\text{AM}^1 = 1, \ \text{GM}^2 = 1), \ \& \ \lambda = 11 \ (\text{AM} = 2, \ \text{GM} = 1) \to \mathbf{J} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 11 & 1 \\ 0 & 0 & 11 \end{bmatrix}$$

[1] AM: Algebraic multiplicity

[2] GM: Geometric multiplicity

Jordan Form

Write down the Jordan form.

$\lambda_1 = 1$ (AM $= 2$, GM $= 1$)

$\lambda_2 = 11$ (AM $= 3$, GM $= 2$)

$\lambda_3 = 0$ (AM $= 3$, GM $= 1$)

$\lambda_4 = -1$ (AM $= 2$, GM $= 2$).