Applied Linear Algebra in Data Analaysis

Signal Processing & Eigenvalues/Eigenvectors Assignment

Marks: 27

1. Prove that the Fourier and the Haar basis are orthogonal for the space $\mathbb{R}^{2^n}$.[**Marks: 2**]

2. [**Programming**] Write a python function to that will return the i^{th} Haar wavelet basis vector for a the given space $\mathbb{R}^{2^n}$. [**Marks: 2**]

3. Explain why an eigenvector cannot be associated with two eigenvalues. [**Marks: 1**]

4. What are the eigenspaces associated with the diagonal matrix $\mathbf{D}$? [**Marks: 1**]

$$\mathbf{D} = \text{diag}\left(d_1, d_2, \ldots d_n\right)$$

5. If a matrix $\mathbf{A}$ has zero as one of its eigenvalues, explain why $\mathbf{A}$ must be singular. [**Marks: 1**]

6. For a matrix $\mathbf{A}$ with eigenvalues $\{\lambda_i\}_{i=1}^{n}$, verify for the following matrices that $\Pi_{i=1}^{n}\lambda_i = \det\left(\mathbf{A}\right)$ and $\sum_{i=1}^{n} \lambda_i = trace\left(\mathbf{A}\right)$. [**Marks: 2**]

   (a) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

   (b) $\frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

7. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^{n}$ be the eigenpairs of a matrix $\mathbf{A}$. Then prove that, [**Marks: 2**]

   (a) $\left\{\lambda_i^k, \mathbf{v}_i\right\}_{i=1}^{n}$ are the eigenpairs of $\mathbf{A}^k$.

   (b) $\{p\left(\lambda_i\right), \mathbf{v}_i\}_{i=1}^{n}$ are the eigenpairs of $p\left(\mathbf{A}\right)$, where $p\left(\mathbf{A}\right) = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \cdots + \alpha_k\mathbf{A}^k$.

8. Consider the matrices $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{v}$ is an eigenvector $\mathbf{B}$, underwhat condition will $\mathbf{v}$ also be the eignevector of $\mathbf{AB}$. Under these conditions, what will be corresponding eigenvalue of $\mathbf{v}$? How do your answers change in the case of $\mathbf{BA}$? [**Marks: 2**]

9. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^{n}$ are the eignepairs of a matrix $\mathbf{A}$. What are the eigenpairs of the folliowing? [**Marks: 3**]

   (a) $2\mathbf{A}$

   (b) $\mathbf{A} - 2\mathbf{I}$

   (c) $\mathbf{I} - \mathbf{A}$

10. Let $\mathbf{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$. What is the value of: [**Marks: 3**] (a) A^2 (b) A^{100} (c) A^{∞}?

11. Show that $\mathbf{u} \in \mathbb{R}^2$ is an eigenvector of $\mathbf{A} = \mathbf{u}\mathbf{v}^T$. What are the two eigenvalues of $\mathbf{A}$? [**Marks: 2**]

12. **Left eigenvectors**: Consider a matrix $\mathbf{A}$ with eigenpairs $\{\lambda_1, \mathbf{v}_i\}_{i=1}^n$. The left eigenvectors of the matrix $\mathbf{A}$ are the vectors that satisfy the equation, $\mathbf{A}^T\mathbf{w} = \mu\mathbf{w}$ (or $\mathbf{w}^T\mathbf{A} = \mu\mathbf{w}^T$), and let $\{\mu_i, \mathbf{w}_i\}_{i=1}^n$ be the left eigenpairs of $\mathbf{A}$. Show the following, [**Marks: 3**]

   (a) The eigenvalues of both $\mathbf{A}$ and $\mathbf{A}^T$ are the same.

   (b) $\mathbf{v}_i^T\mathbf{w}_j = 0$. The eigenvector $\mathbf{v}_i$ corresponding to the eigenvalue λ_i and the left eigvenvector $\mathbf{w}_j$ corresponding to the eigenvalue λ_j are orthogonal, when $\lambda_i \neq \lambda_j$.

   (c) The matrix A can be expressed as a sum of rank-one matrices,

$$\mathbf{A} = \lambda_1\mathbf{v}_1\mathbf{w}_1^T + \lambda_2\mathbf{v}_2\mathbf{w}_2^T + \cdots + \lambda_n\mathbf{v}_n\mathbf{w}_n^T$$

13. Prove that $\mathbf{A}\mathbf{A}^T$ has real and positive eigenvalues, and that the eigenvectors corresponding to distinct eigenvalues of $\mathbf{A}\mathbf{A}^T$ are orthogonal. [**Marks: 2**]

14. If $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ are the eigenpairs of a non-singular matrix $\mathbf{A}$, the prove that the eigen pairs of $\mathbf{A}^{-1}$ are $\left\{\lambda_i^{-1}, \mathbf{v}_i\right\}_{i=1}^n$. [**Marks: 1**]