Applied Linear Algebra in Data Analysis

Tutorial

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CONTENTS

1 CONCEPTS IN VECTOR SPACES

1. Is a this set of vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ independent? Explain your answer.

2. Consider a set of finite duration discrete-time real signals,

$$X_N = \left\{ x[n] \, \middle| \, x[n] \in \mathbb{R}, \forall 0 \leqslant n \leqslant N-1 \right\}$$

. Does this set form a vector space? Explain your answer. Would X_N still be a vector spaces if the signals were binary signals? i.e. $x[n] \in \mathbb{B}$, where $\mathbb{B} = \{0, 1\}$ with the binary addition and multiplication operations defined as the following,

a	b	$a+b$	$a \times b$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

3. Prove the following for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

   (a) **Triangle Inequality**:

   $$\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$$

   (b) **Backward Triangle Inequality**:

   $$\|\mathbf{x}-\mathbf{y}\| \geqslant \left| \|\mathbf{x}\| - \|\mathbf{y}\| \right|$$

   (c) **Parallelogram Identity**:

   $$\frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

4. Consider a set of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. When is $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}+\mathbf{y}\|$? What can you say about the geometry of the vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x}-\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$?

5. If $S_1, S_2 \subseteq V$ are subspaces of V, the is $S_1 \cap S_2$ a subspace? Demonstrate your answer.

6. Consider two sets of vectors,

$$V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} \quad \text{and} \quad W = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \mathbf{u}\}$$

   Prove that if $span(V) = span(W)$, then $\mathbf{u} \in span(V)$.

7. Prove that the sum of two subspaces $S_1, S_2 \subseteq V$ is a subspace.

8. Consider a vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$. Express the following in-terms of inner product between a constant vector $\mathbf{u}$ and the given vector $\mathbf{v}$, and in each case specify the vector $\mathbf{u}$.

   (a) $\sum_{i=1}^{n} v_i$
   (b) $\frac{1}{n} \sum_{i=1}^{n} v_i$
   (c) $\sum_{i=1}^{n} v_i a^{(n-i)}$, where $a \in \mathbb{R}$
   (d) $\frac{1}{n-1} \sum_{i=1}^{n} \left(v_i - \frac{1}{n} \sum_{i=1}^{n} v_i\right)^2$
   (e) $\frac{1}{5} \sum_{i=3}^{5} v_i$
   (f) $\sum_{i=1}^{n-1} (v_{i+1} - v_i)$

9. Which of the following are linear functions of $\{x_1, x_2, \ldots, x_n\}$?

   (a) $\min_i \{x_i\}_{i=1}^{n}$
   (b) $\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$
   (c) x_6

10. Consider a linear function $f : \mathbb{R}^n \to \mathbb{R}$. Prove that every linear function of this form can be represented in the following form.

$$y = f(\mathbf{x}) = \mathbf{w}^{\mathsf{T}}\mathbf{x} = \sum_{i=1}^{n} w_i x_i, \quad \mathbf{x}, \mathbf{w} \in \mathbb{R}^n$$

11. An *affine* function f is defined as the sum of a linear function and a constant. It can in general be represented in the form,

$$y = f(\mathbf{x}) = \mathbf{w}^{\mathsf{T}}\mathbf{x} + \beta, \quad \mathbf{x}, \mathbf{w} \in \mathbb{R}^n, \ \beta \in \mathbb{R}$$

    Prove that affine functions are not linear. Prove that any affine function can be represented in the form $\mathbf{w}^{\mathsf{T}}\mathbf{x} + \beta$.

12. Consider a function $f : \mathbb{R}^3 \to \mathbb{R}$, such that,

$$f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = 2; \ f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = -3; \ f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = 1;$$

    Can you determine the following values of $f(\mathbf{x})$, if you are told that f is linear?

$$f\left(\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}\right) = ?; \ f\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) = ?; \ f\left(\begin{bmatrix} 0.5 \\ 0.6 \\ -0.1 \end{bmatrix}\right) = ?;$$

    Can you find out these values if you are told that f is affine?

13. For the previous question, (a) assume that f is linear and find out $w \in \mathbb{R}^3$, such that $f(\mathbf{x}) = \mathbf{w}^{\mathsf{T}}\mathbf{x}$; and (b) assume f is affine and find out $\mathbf{w}, \beta$ such that $f(\mathbf{x}) = \mathbf{w}^{\mathsf{T}}\mathbf{x} + \beta$.

14. Consider the weighted norm of vector $\mathbf{v}$, defined as,

$$\|\mathbf{v}\|_{\mathbf{w}}^2 = \sum_{i=1}^{n} w_i v_i^2; \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Is this a valid norm?

15. Prove that the following modified version of the Cauchy-Bunyakovski-Schwartx Inequality is true.

$$\left| \sum_{i=1}^{n} u_i v_i w_i \right| \leqslant \|\mathbf{u}\|_{\mathbf{w}} \|\mathbf{v}\|_{\mathbf{w}}$$

16. Consider a basis $B = \{\mathbf{b}_i\}_{i=1}^{n}$ of R^n. Let the vector $\mathbf{x}$ with the following representations in the standard and B basis.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{n} x_i \mathbf{e}_i \quad \text{and} \quad \mathbf{x}_b = \begin{bmatrix} x_{b1} \\ x_{b2} \\ \vdots \\ x_{bn} \end{bmatrix} = \sum_{i=1}^{n} x_{bi} \mathbf{b}_i$$

Evaluate the $\|\mathbf{x}\|_2^2$ and $\|\mathbf{x}_b\|_2^2$. Determined what happens to $\|\mathbf{x}_b\|_2^2$ under the following conditions on the basis vectors:

(a) $\|\mathbf{b}_i\| = 1, \forall i$

(b) $\left\|\mathbf{b}_i^T \mathbf{b}_j\right\| = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

17. Consider a set of measurements made from adult male subjects, where their height, weight and BMI (body moass index) were recorded as stored as vectors of length three; the first element is the height in cm, second is the weight in Kg, and the alst the the BMI. Consider the following four subjects,

$$\mathbf{s}_1 = \begin{bmatrix} 167 \\ 102 \\ 36.6 \end{bmatrix}; \ \mathbf{s}_2 = \begin{bmatrix} 180 \\ 87 \\ 26.9 \end{bmatrix}$$

$$\mathbf{s}_3 = \begin{bmatrix} 177 \\ 78 \\ 24.9 \end{bmatrix}; \ \mathbf{s}_4 = \begin{bmatrix} 152 \\ 76 \\ 32.9 \end{bmatrix}$$

You can use the distance between these vectors $\left\|\mathbf{s}_i - \mathbf{s}_j\right\|_2$ as a measure of the similiarity between the the four subjects. Generate a 4×4 table comparing the distance of each subject with respect to another subject; the diagonal elements of this table will be zero, and it will be symmetric about the main diagonal.

(a) Based on this table, how do the different subjects compare to each other?

(b) How do the similarities change if the height had been measured in m instead of cm? Can you explain this difference?

(c) Is there a way to fix this problem? Consider the weighted norm presented in one of the earlier problems.

$$\|x\|_{\mathbf{w}} = \left(w_1 x_1^2 + w_2 x_2^2 + \ldots + w_n x_n^2\right)^{\frac{1}{2}}$$

(d) What would be a good choice for $\mathbf{w}$ to address the problems with comparing distance between vectors due to change in units?

(e) Can the angle between two vectors be used as a measure of similarity between vectors? Does this suffer from the problem of $\|\mathbf{x}\|_2$?

2 | ORTHOGONALITY

18. Consider an orthonormal set of vectors,

$$V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_r\}, \;\; \mathbf{v}_i \in \mathbb{R}^n \;\; \forall i \in \{1, 2, \ldots r\}$$

If there is a vector $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v}_i^T \mathbf{w} = 0 \;\; \forall i \in \{1, 2, \ldots r\}$. Prove that $\mathbf{w} \notin \text{span}\,(V)$.

19. Consider the following set of vectors in $\mathbb{R}^4$.

$$V = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

Find the set of all vectors that are orthogonal to V?

20. For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, prove that $C(\mathbf{A}) \perp N\left(\mathbf{A}^T\right)$ and $C\left(\mathbf{A}^T\right) \perp N(\mathbf{A})$.

21. If the columns of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ are orthonormal, prove that $\mathbf{A}^{-1} = \mathbf{A}^T$. What is $\mathbf{A}^T\mathbf{A}$ when $\mathbf{A}$ is rectangular $\left(\mathbf{A} \in \mathbb{R}^{m \times n}\right)$ with orthonormal columns?

22. What will happen when the Gram-Schmidt procedure is applied to: (a) orthonormal set of vectors; and (b) orthogonal set of vectors? If the set of vectors are columns of a matrix **A**, then what are the corresponding **Q** and **R** matrices for the orthonormal and orthogonal cases?

23. Consider the linear map, $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let us assume that **A** is full rank. What conditions must **A** satisfy for the following statements to be true,

    (a) $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, for all $\mathbf{x}, \mathbf{y}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$.

    (b) $\mathbf{y}_1^T\mathbf{y}_2 = \mathbf{x}_1^T\mathbf{x}_2$, for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$.

    Note: A linear map **A** with the aforementioned properties preserves lengths and angle between vectors. Such maps are encountered in rigid body mechanics.

24. Prove that the rank of an orthogonal projection matrix $\mathbf{P}_S = \mathbf{U}\mathbf{U}^T$ onto a subspace $\mathcal{S}$ is equal to the dim $\mathcal{S}$, where the columns of **U** form an orthonormal basis of $\mathcal{S}$.

25. If the columns of $\mathbf{A} \in \mathbb{R}^{m \times n}$ represent a basis for the subspace $\mathcal{S} \subset \mathbb{R}^m$. Find the orthogonal projection matrix $\mathbf{P}_{\mathcal{S}}$ onto the subspace $\mathcal{S}$.

    Hint: Gram-Schmidt orthogonalization.

26. Consider two orthogornal matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Is the $\mathbf{Q}_2^\top \mathbf{Q}_1$ an orthogonal matrix? If yes, prove that it is so, else provide a counter-example showing $\mathbf{Q}_2^\top \mathbf{Q}_1$ is not orthogonal.

27. Let $\mathbf{P}_{\mathcal{S}}$ represent an orthogonal projection matrix onto to the subspace $\mathcal{S} \subset \mathbb{R}^n$. How can we obtain an orthonormal basis for $\mathcal{S}$ from $\mathbf{P}_{\mathcal{S}}$.

28. Consider a 1 dimensional subspace spanned by the vector $\mathbf{u} \in \mathbb{R}^n$. What kind of a geometric operation does the matrix $\mathbf{R} = \mathbf{I} - 2\frac{\mathbf{u}\mathbf{u}^\top}{\mathbf{u}^\top \mathbf{u}}$ represent?

    Show that $\mathbf{R}$ satisfies the following properties:

    (a) $\mathbf{R}^2 = \mathbf{I}$

    (b) Consider a vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top \in \mathbb{R}^n$ such that $x_1 \neq 0$. If we choose $\mathbf{u} =$

29. Prove that when a triangular matrix is orthogonal, it is diagonal.

30. If an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ is to be partitioned such that, $\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}$, then prove that $C(\mathbf{Q}_1) \perp C(\mathbf{Q}_2)$.

31. Find an orthonormal basis for the subspace spanned by the following set,

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$$