

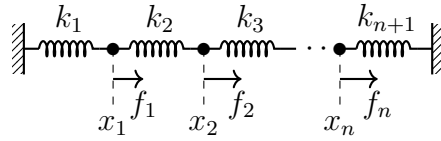
Applied Linear Algebra in Data Analysis

Solutions to Linear Equations Assignment

Marks: 31

Note: The questions you need to answer are written down in red for clarity for some of the problems.

- Derive force and displacement relationship for a series of $n + 1$ springs (with spring constants k_i) connected in a line. There are n nodes, with f_i and x_i representing the force applied and resulting displacement at the i^{th} node.



- (a) Represent the relationship in the following form, **[Marks: 2]**

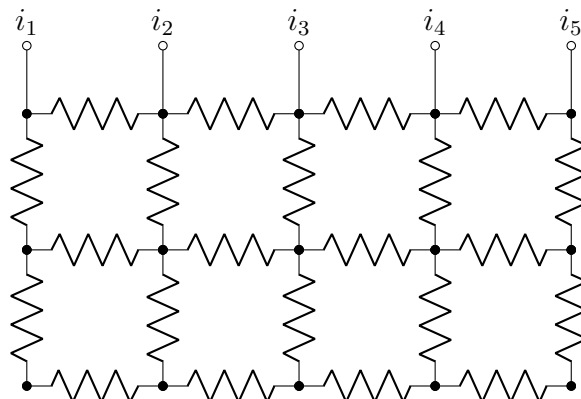
$$\mathbf{f} = \mathbf{K}\mathbf{x}; \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- (b) What kind of a pattern does \mathbf{K} have? **[Marks: 1]**

- (c) **[Programming]** Consider a specific case where $n = 4$ and $k = 1.5N.m^{-1}$. What should be forces applied at the four nodes in order to displace the spring

$$\mathbf{x} = \begin{bmatrix} 0.5 \\ -0.5 \\ 0 \\ 0 \end{bmatrix} m. \quad \mathbf{[Marks: 4]}$$

- Consider the following electrical circuit with rectangular grid of resistors R . The input to this grid is a set of current injected at the top node as shown in the figure, such that $\sum_{k=1}^5 i_k = 0$.



Express the relationship between the voltages at the different nodes (represented by • in the figure) and the net current flowing in/out of the node in the following form, $\mathbf{G}\mathbf{v} = \mathbf{i}$. Where, \mathbf{G} is the conductance matrix, \mathbf{v} is the vector of node voltages, and \mathbf{i} is the vector representing the net current flow in/out of the different node. [Marks: 3]

3. Consider the system of equation, $\mathbf{Ax} = \mathbf{b}$, such that a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$. Are the following statements true? Explain your answer. [Marks: 3]

- (a) $\text{rank}(\mathbf{A}) \leq \min(m, n)$
- (b) The system is consistent if $\text{rank}(\mathbf{A}) = m$.
- (c) The system has a unique solution if $\text{rank}(\mathbf{A}) = n$.

4. **Two point boundary problem.** $\mathbf{Ax} = \mathbf{b}$ is often encountered in many practical applications. One such application is the numerical solution of differential equations of the following form,

$$\sum_{i=0}^M a_i(x) y^{(i)}(x) = f(x)$$

where, $x \in [a, b]$ and $y(a) = \alpha, y(b) = \beta$.

Numerical methods are often employed for obtaining an approximate estimate of $y(x)$ at discrete points in the interval $[a, b]$. The interval is divided into subintervals of width Δx . The derivate of $y(x)$ at the different nodes (points between two subintervals) can be approximated as the following,

$$y'(x_i) = \frac{y(x_i + \Delta x) - y(x_i - \Delta x)}{\Delta x}$$

$$y''(x_i) = \frac{y(x_i + \Delta x) + 2y(x_i) - y(x_i - \Delta x)}{\Delta x^2}$$

where, $x_i = a + i\Delta x$, $0 \leq i \leq N + 1$, and $b - a = (N + 1)\Delta x$. Addition and subtracting the above two equations and neglecting terms involving higher orders of Δx , we get the following approximations for the derivatives of $y(x)$ at x_i .

Replacing the derivatives of $y(x)$ by the above approximations and evaluating the equation at the different nodes x_i s, we arrive a set of N linear equations with N unknowns $y(x_1), y(x_2), \dots, y(x_N)$.

Using this approach, compute an approximate solution for $y(x)$ for the following differential equations over the interval $x \in [0, 1]$.

- (a) $y''(x) = -x$
- (b) $y''(x) + y'(x) = x$

[Programming] Solve these equations for different values of Δx , and compare the resulting approximate solution for $y(x)$ with the exact solution. Present your results as a plot the solution $y(x_i)$ versus x_i . [Marks: 4]

Comment on the dependence of the solution (x) on Δx . What is the best value for Δx to use in solving these equations? [Marks: 2]

5. **Ill-conditioned systems.** A system $\mathbf{Ax} = \mathbf{b}$ is said to be ill-conditioned when small changes in the components of \mathbf{A} or \mathbf{b} can produce large changes in the solution \mathbf{x} . Consider the following system,

$$\begin{aligned}x - y &= 100 \\10 + (9 + \Delta)y &= 0\end{aligned}$$

[Programming] Find the solutions of the system for different values of $\Delta = -2, -1, 0, 1, 2$. How do the solutions change with Δ . **[Marks: 2]**

Now consider the following system,

$$\begin{aligned}x - y &= 100 \\10 - (9 + \Delta)y &= 0\end{aligned}$$

[Programming] Find the solutions of the system for different values of $\Delta = -2, -1, 0, 1, 2$. How do the solutions change with Δ . **[Marks: 2]**

The second system is an example of an ill-conditioned system. What can you say about the geometries of these two systems? **[Marks: 2]**

6. **Computed Tomography (CT)** is a medical imaging technique that is used to reconstruct the internal structure of an object from a set of X-ray measurements. The object is placed between an X-ray source and a detector. The X-ray source and the detector are rotated around the object, and the X-ray measurements are recorded at different angles. The X-ray measurements are then used to reconstruct the internal structure of the object.

The x-ray attenuation equation is given by,

$$I_o = I_i \exp(-\mu l)$$

where, I_i is the intensity of the x-ray entering an object with fixed attenuation coefficient μ , I_o is the intensity of the x-ray existing the object, and l is the path length of the x-ray in the object.

In general, the attenuation coefficient is a function of the position within the object, $\mu = \mu(x, y)$. The goal of CT is to reconstruct the spatial map of the attenuation coefficient $\mu(x, y)$.

Let L represent the line segment of the x-ray within the object as shown in Figure 1, and the attenuation outside the object is assumed to be zero. If I_s is the intensity of the x-ray leaving the source, the intensity of the x-ray reaching the detector I_d is given by,

$$I_d = I_s \exp\left(-\int_L \mu(x, y) dl\right)$$

where, dl is the differential length along the line segment L , which is a function of x, y . The integral in the above equation is the line integral of the attenuation coefficient $\mu(x, y)$ along the line segment L .

We wish to solve this problem using a computer by posing it as a set of linear equations relating the attenuation coefficient $\mu(x, y)$ to the x-ray intensity measurements

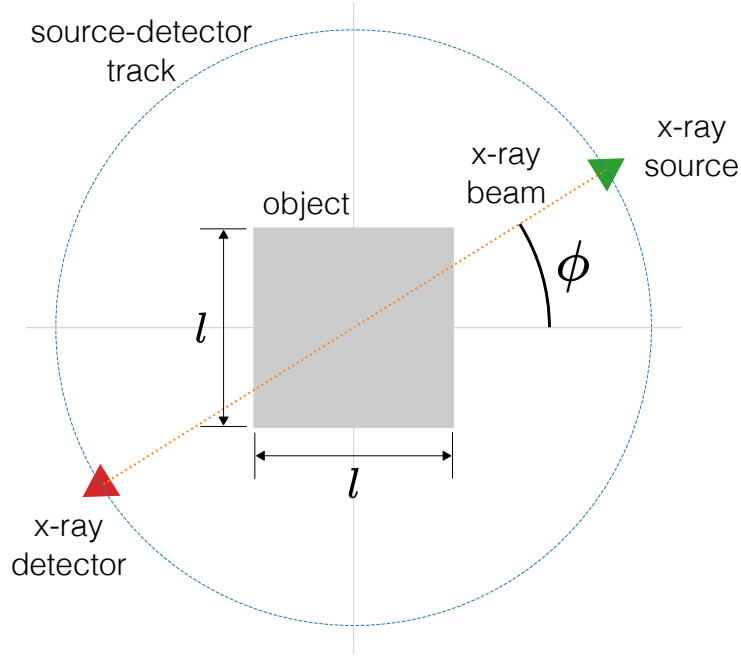


Figure 1: A simplified CT set-up with a single X-ray source and a single detector, that are located diametrically opposite to each other and can rotate to any scan angle ϕ . The object is placed between the X-ray source and the detector, which is depicted by the gray square of side l . The x-ray originates from the green triangle (x-ray source), passes through the object, and is detected by the red triangle (detector). The x-ray undergoes attenuation as it passes through the object. Different points in the objects are most likely to have different attenuation coefficients, and the goal of CT is to reconstruct a spatial map of the attenuation within the object, which provides a measure of the internal structure of the object.

I_d . For simplicity, we will assume $I_s = 1$. First, we simplify the above integration equation by taking the log on both sides, which results in,

$$\ln I_d = - \int_L \mu(x, y) dl$$

Explain why the above step is valid. Do we need to worry about the case where $I_d = 0$? [Marks: 2]

Discretize the object into $n \times n$ grid of pixels, and assume that attenuation coefficient within each pixel to be constant. For any given scan angle ϕ you need to find out the pixels the line segment L passes through, along with the path length of the x-ray line segment within each pixel. **Derive the discrete expression relating the x-ray intensity measurements I_d to the attenuation coefficient $\mu(x, y)$ for a given scan angle ϕ , assuming a discretized object $n \times n$. Write down the above expression in the following form [Marks: 3],**

$$\tilde{\mathbf{a}}(\phi)^\top \mathbf{x} = y(\phi)$$

where, $\mathbf{x} \in \mathbb{R}^{n^2}$ is the vector of unknown attenuation coefficients of the pixels in the image, $\tilde{\mathbf{a}}(\phi) \in \mathbb{R}^{n^2}$ is the vector relating the attenuation coefficients to the detector measurement for the given source-detector angle ϕ and $y(\phi) \in \mathbb{R}$ is the log of the x-ray intensity measured by the detector.

If we make a set of such measurements for m different angles $\phi_1, \phi_2, \dots, \phi_m$, we can write the above equation in the following matrix form,

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

$$\mathbf{A} = [\tilde{\mathbf{a}}_1 \quad \tilde{\mathbf{a}}_2 \quad \dots \quad \tilde{\mathbf{a}}_m]^\top \in \mathbb{R}^{m \times n^2} \quad \mathbf{y} = [y_1 \quad y_2 \quad \dots \quad y_m]^\top \in \mathbb{R}^m$$

where, $\tilde{\mathbf{a}}_i$ is the vector relating the attenuation coefficients to the detector measurement for the source-detector angle ϕ_i and y_i is the log of the x-ray intensity measured by the detector for the angle ϕ_i .

Forward problem for CT. Once you've derived the above matrix linear equation, we can use it to simulate a CT scan by computing \mathbf{y} for a given \mathbf{x} and scan angle ϕ , we can compute intensity of the x-ray that will be measured by the detector. **[Programming]** Write a python function that will compute the measured detector intensities \mathbf{y} for a given \mathbf{x} and a given set of scan angles $\{\phi_i\}_{i=0}^{N-1}$. You will be provided with a iPython notebook template for this purpose where you will only need to focus on writing the function that computes \mathbf{y} . After completing this, you need to use the data given to you to compute the measured detector intensities \mathbf{y} for a given set of \mathbf{x} and scan angles $\{\phi_i\}_{i=0}^{N-1}$. **[Marks: 6]**