# Applied Linear Algebra for Data

## Application: Linear Dynamical Systems

Sivakumar Balasubramanian

Department of Bioengineering
Christian Medical College, Bagayam
Vellore 632002

# Linear System

$$u(t) \longrightarrow \boxed{\quad H \quad} \longrightarrow y(t) = H\left\{u(t)\right\}$$

Behavior of dynamic systems can be described mathematically through differential equations (continuous-time systems), or difference equations (discrete-time systems).

A system is **linear** iff,

$$y_1(t) = H\left\{u_1(t)\right\} \text{ and } y_2(t) = H\left\{u_2(t)\right\}$$

$$\begin{aligned} \implies H\left\{a_1 x_1(t) + a_2 u_2(t)\right\} &= a_1 H\left\{u_1(t)\right\} + a_2 H\left\{u_2(t)\right\} \\ &= a_1 y(t) + a_2 y_2(t), \; a_1, a_2 \in \mathbb{C} \end{aligned}$$

# Time-Invariant System

$$u(t) \longrightarrow \boxed{\quad H \quad} \longrightarrow y(t) = H\{u(t)\}$$

- A system is **time-invariant** if,

$$y(t) = H\{u(t)\} \implies H\{u(t-\tau)\} = y(t-\tau)$$

- Characteristics of the system do not change with time. Time-shifted inputs produce correspondingly time-shifted output.

# Linear Time-Invariant System

$$u(t) \longrightarrow \boxed{\quad H \quad} \longrightarrow y(t) = H\{u(t)\}$$

LTI systems: both **linear** and **time-invariant**. These are described through constant coefficient linear differential (or difference) equations.

**Continuous-time**:

$$\frac{d^n}{dt^n} y(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots + a_n y(t) = u(t) + b_1 \frac{d}{dt} u(t) + \ldots + b_m \frac{d^m}{dt^m} y(t)$$

**Discrete-time**:

$$y[k-n] + a_1 y[k-n+1] + \ldots + a_n y[k] = u[k] + b_1 u[k-1] + \ldots + b_m u[k-m]$$

# State space representation of linear systems

- In the case of a linear system, the equations representing the dynamics takes a simpler form,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t)$$

  where,
  - $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ is the *system* matrix.
  - $\mathbf{B}(t) \in \mathbb{R}^{n\times p}$ is the *input* matrix.
  - $\mathbf{C}(t) \in \mathbb{R}^{m\times n}$ is the *output* matrix.
  - $\mathbf{D}(t) \in \mathbb{R}^{m\times p}$ is the *feedforward*matrix.

# State space representation of linear systems

- In the case of time-invariant system, the matrices are constant.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

- These two equations represent how the states and the measured outputs of the system are affected by the current states and inputs. The individual terms in these matrices indicate how a particular state/input affects another state/output.

# State space representation of discrete-time linear systems

- Discrete-time linear system,

$$\mathbf{x}[k+1] = \mathbf{A}[k]\,\mathbf{x}[k] + \mathbf{B}[k]\,\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}[k]\,\mathbf{x}[k] + \mathbf{D}[k]\,\mathbf{u}[k]$$

  where, $k \in \mathbb{Z}$ correspond to time index.
  - $\mathbf{A}[k] \in \mathbb{R}^{n \times n}$ is the *system* matrix.
  - $\mathbf{B}[k] \in \mathbb{R}^{n \times p}$ is the *input* matrix.
  - $\mathbf{C}[k] \in \mathbb{R}^{m \times n}$ is the *output* matrix.
  - $\mathbf{D}[k] \in \mathbb{R}^{m \times p}$ is the *feedforward* matrix.
- In the case of time-invariant system, the matrices are constant.

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

# Zero-input solution for $\mathbf{x}(t)$

- **Zero-Input Solution**: We will start by assuming $\mathbf{u}(t) = \mathbf{0}$.

  $$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

  And that the initial value of the state at time $t = 0^-$ is $\mathbf{x}(0^-) = \mathbf{x}_0$.

- The solution to this differential equation is given by,

  $$\mathbf{x}(0) = e^{t\mathbf{A}}\mathbf{x}_0, \;\; t \geq 0$$

  where, $e^{t\mathbf{A}}$ is the matrix exponential of $\mathbf{A}$.

- Note the interesting similarity to the scalar case: $\dot{x}(t) = ax(t)$, we know the solution to be the following, $x(t) = e^{at}x(0^-)$.

# What is the matrix exponential $e^{t\mathbf{A}}$?

- Functions of matrices are often defined to have properties consistent with that of their scalar coutnerparts.

$$e^{t\mathbf{A}} = \mathbf{I} + t\mathbf{A} + \frac{1}{2!}t^2\mathbf{A}^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k \implies \frac{d}{dt} e^{t\mathbf{A}} = \mathbf{A} e^{t\mathbf{A}}$$

- But we do not have to compute all the powers of $\mathbf{A}$ to compute the matrix exponential. Thanks to the Cayley-Hamilton theorm,

**Cayley-Hamilton Theorem**

> Every square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfies it own characteristic equation $p(\lambda) = 0$, i.e. $p(\mathbf{A}) = \mathbf{0}$.
>
> $$p(\mathbf{A}) = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + a_2\mathbf{A}^{n-2} + \ldots + a_n\mathbf{I} = \mathbf{0}$$

# What is the matrix exponential $e^{t\mathbf{A}}$?

- $e^{t\mathbf{A}}$ can be computed as the following,

$$e^{t\mathbf{A}} = \sum_{k=0}^{n-1} c_k t \mathbf{A}^k$$

- When the matrix $\mathbf{A}$ is diagonalizable, then the matrix exponential can be computed as follows,

$$\begin{aligned} e^{t\mathbf{A}} &= \sum_{k=0}^{n-1} c_k t^k \mathbf{A}^k = \sum_{k=0}^{n-1} c_k t^k \left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\right)^k \\ &= \sum_{k=0}^{n-1} c_k t \mathbf{V}\mathbf{\Lambda}^k\mathbf{V}^{-1} == \mathbf{V}\left(\sum_{k=0}^{n-1} c_k t \mathbf{\Lambda}^k\right)\mathbf{V}^{-1} \\ &= \mathbf{V} e^{t\mathbf{\Lambda}} \mathbf{V}^{-1} \end{aligned}$$

# Usefulness of the $e^{t\mathbf{A}}$?

- Let $\{(\lambda_i, \mathbf{v}_i)\}_{i=1}^{n}$ be the $n$ eigenpairs of the matrix $\mathbf{A}$. Then,

$$e^{t\boldsymbol{\Lambda}} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & e^{t\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{t\lambda_n} \end{bmatrix}$$

$$e^{t\mathbf{A}} = \mathbf{V} e^{t\boldsymbol{\Lambda}} \mathbf{V}^{-1}$$

  The eigenvectors in this case are referred to the as the eigenmodes of the system, which are a characteristic of the system. The eigenvalues are the corresponding rates of exponential growth/decay of the eigenmodes.

- The reprsentation allows us to express the response to any initial condition as the linear combination of the exponential evolution of the eigenmodes.

# Usefulness of the $e^{t\mathbf{A}}$

- Let $\mathbf{x}_0$ is the initial state at time $t = 0^-$. Then the zero-input response can be expressed as the following,

$$\mathbf{x}(t) = \mathbf{V}e^{t\mathbf{\Lambda}}\mathbf{V}^{-1}\mathbf{x}_0$$

We know, $\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix}$, and let $\mathbf{V}^{-1} = \begin{bmatrix}\tilde{\mathbf{v}}_1^\top \\ \tilde{\mathbf{v}}_2^\top \\ \vdots \\ \tilde{\mathbf{v}}_n^\top\end{bmatrix}$.

Then we can expressed the above expression as,

$$\mathbf{x}(t) = \sum_{i=1}^{n} e^{t\lambda_i}\mathbf{v}_i\tilde{\mathbf{v}}_i^\top\mathbf{x}_0$$

$\mathbf{v}_i\tilde{\mathbf{v}}_i^\top\mathbf{x}_0$ is the projection of the initial state $\mathbf{x}_0$ onto the $i$-th eigenmode. The exponential term $e^{t\lambda_i}$ is the rate of exponential growth/decay of the $i$-th eigenmode.

# Solution for discrete-time LTI system

- **System equations:**

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]$$

- **Zero-input solution:**

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0]$$

- If $\mathbf{A}$ is diagonalizable, then,

$$\mathbf{x}[k] = \mathbf{V}\mathbf{\Lambda}^k\mathbf{V}^{-1}\mathbf{x}[0]$$

# Internal stability

- There are two types of stability one can associate with a system – **Internal stability** and **Input-Output stability**.

- **Internal stability**: Deals with the stability of the zero-input response of the system states.

# Internal stability

- Definition of stability in the Lyapunov sense for linear systems:
  - The zero-input response of a linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is *stable or marginally stable* if every finite initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)\ \forall t \geq 0$.

$$\|\mathbf{x}(t)\| \leq d, \quad \forall t \geq 0$$

  - The zero-input response is *asymtotically stable* if everyf initial condition $\mathbf{x}(0^-)$ results in a bounded state trajectory $\mathbf{x}(t)$ that coverges to 0 as $t \to \infty$.

$$\|\mathbf{x}(t)\| \leq d \text{ and } \lim_{t\to\infty} \|\mathbf{x}(t)\| = 0$$

# Internal stability

- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is marginally stable if and only if all eigenvales of $\mathbf{A}$ have either zero or negative real parts, and the eigenvalues with zero real parts have the same algebraic and geometric multiplicity.

- The system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is asymptotically stable if and only if all eigenvales of $\mathbf{A}$ have negative real parts.

# Internal stability

- Consider the solution, $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x}(0^-)\,, \;\; t \geq 0$, and $\mathbf{A} = \mathbf{V}\mathbf{J}\mathbf{V}^{-1}$.

$$\|\mathbf{x}(t)\| = \left\|e^{t\mathbf{A}}\mathbf{x}(0^-)\right\| \leq \left\|e^{t\mathbf{J}}\right\| \left\|\mathbf{x}(0^-)\right\|$$

- When $\mathbf{A}$ is diagonalizable ($\lambda_i$ are the eigenvalues of $\mathbf{A}$),
  - $\|\mathbf{x}(t)\| \leq e^{\sigma t}\|\mathbf{x}(0^-)\|$, where $\sigma = \max_i \Re\{\lambda_i\}$.
  - When $\sigma = 0$, $\|\mathbf{x}(t)\|$ is bounded $\forall t \geq 0$.
  - When $\sigma < 0$, $\lim_{t\to\infty}\|\mathbf{x}(t)\| = 0$.

# Internal stability – Discrete-time LTI systems

When $\mathbf{A}$ is diagonalizable ($\lambda_i$ are the eigenvalues of $\mathbf{A}$),

- $\|\mathbf{x}[k]\| \leq |\lambda|^k \|\mathbf{x}[0]\|$, where $\lambda = \max_i |\lambda_i|$.
- When $|\lambda| = 1$, $\|\mathbf{x}[k]\|$ is bounded $\forall k > 0$.
- When $|\lambda| < 1$, $\lim_{k\to\infty} \|\mathbf{x}[k]\| = 0$.