

Applied Linear Algebra in Data Analysis

Tutorial

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1 CONCEPTS IN VECTOR SPACES

1. Which of the following sets forms a vector space?
 - a) $\{\mathbf{x} \mid x_1, x_2 \in \mathbb{R} \text{ and } a_1 x_1 + a_2 x_2 = 0\}$, where $a_1, a_2 \in \mathbb{R}$ are fixed constants.
 - b) $\{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{a}^\top \mathbf{x} = b\}$, where $\mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ are fixed constants.
 - c) $\{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^\top \mathbf{x} = 1\}$.
 - d) $\{(x[0], x[1], x[2], \dots, x[N-1]) \mid x[i] \in \mathbb{R}, 0 \leq i < N\}$.

(The set of all real-valued time-domain signals of length N . $x[i]$ is the value of the signal at time instant i .)

2. Consider the vector space of polynomials of order n or less.

$$\mathcal{P} = \left\{ \sum_{k=0}^n a_k x^k \mid a_k \in \mathbb{R} \right\}, \text{ where, } x \in [0, 1]$$

Show that polynomials of order strictly lower than n form subspaces of \mathcal{P} .

3. Is the following function a valid norm of the vector space \mathcal{P} ?

$$\|\mathbf{p}(x)\| = \sqrt{\sum_{k=0}^n a_k^2}, \quad \mathbf{p} = \sum_{k=0}^n a_k x^k \in \mathcal{P}$$

4. Consider the following function, which is often called the *zero-norm* of a vector $\mathbf{x} \in \mathbb{R}^n$.

$$\|\mathbf{x}\|_0 = \sum_{i=1}^n \mathbb{I}(x_i \neq 0), \text{ where, } \mathbb{I}(A) = \begin{cases} 1 & A \text{ is true.} \\ 0 & A \text{ is false.} \end{cases}$$

Is the *zero-norm*, which is often used for quantifying the *sparsity* of a vector, a proper norm?

5. Is the following set of vectors linear independent?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

What is the span of this set? Does this set form the basis for its span? Does it form an orthonormal basis?

6. Consider the following function,

$$f(\mathbf{x}) = \sum_{i=1}^n w_i |x_i|, \quad \mathbf{x} \in \mathbb{R}^n, w_i > 0$$

Is f a norm? If not, what properties does it lack?

7. Find the norm of the following vectors using the 1-norm, 2-norm and the ∞ -norm.

- a) $\mathbf{x} = [1 \ 2 \ 3]^\top$
- b) $\mathbf{x} = [1 \ -1 \ 0]^\top$
- c) \mathbf{e}_i , where $1 \leq i \leq n$
- d) $\mathbf{o} \in \mathbb{R}^n$

e) $\mathbf{1} \in \mathbb{R}^n$

8. For any given $\mathbf{x} \in \mathbb{R}^n$, show that,

$$\|\mathbf{x}\|_1 \geq \|\mathbf{x}\|_2 \geq \|\mathbf{x}\|_3 \cdots \geq \|\mathbf{x}\|_\infty$$

9. Consider the linear function $f : \mathbb{R}^3 \mapsto \mathbb{R}$. We know the output of the function for the following inputs,

$$f\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top\right) = -2, \quad f\left(\begin{bmatrix} -1 & 2 & -1 \end{bmatrix}^\top\right) = 1, \quad f\left(\begin{bmatrix} -1 & 1 & 2 \end{bmatrix}^\top\right) = 0$$

Find an input $\mathbf{x} \in \mathbb{R}^3$ such that $f(\mathbf{x}) = 0$.

10. Find the representation of $\mathbf{x} = \begin{bmatrix} 2 & 1 \end{bmatrix}^\top$ in the following bases.

a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

b) $\frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

c) $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

11. Consider the function $f_i : \mathbb{R}^n \mapsto \mathbb{R}$ that selects the i^{th} element of a given vector $\mathbf{x} \in \mathbb{R}^n$.

$$f(\mathbf{x}) = x_i, \text{ where } \mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^\top$$

Is this function linear? If so, what is the vector \mathbf{w} associated with this function, such that $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$?

12. Given a set of real numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ which are used to the n -vector \mathbf{x} . Can you express the mean \bar{x} and variance σ_x^2 of this set of data using the standard inner product in \mathbb{R}^n ? Note the following,

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \quad \sigma_x^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

2 MATRICES

1. Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 1 \\ -3 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

- Find the product of the two matrices $\mathbf{C} = \mathbf{AB}$ using the four views of matrix multiplication.
 - If we change $b_{23} = 0$. Can you compute the new matrix \mathbf{C} without performing the entire matrix multiplication again?
 - If we increase the value of the elements of the 3rd column of \mathbf{A} by 1, how can we compute the new \mathbf{C} without performing the entire matrix multiplication again?
 - If we insert a new row $\mathbf{1}^\top$ in \mathbf{A} after the 2nd row of \mathbf{A} , how can we compute the new \mathbf{C} without performing the entire matrix multiplication again?
2. Consider a matrix $\mathbf{A} \in \mathbb{R}^{10^6 \times 5}$, and we are interested in computing the product $\mathbf{A}^\top \mathbf{A} \mathbf{A}^\top$. Should you compute the product as $(\mathbf{A}^\top \mathbf{A}) \mathbf{A}^\top$ or $\mathbf{A}^\top (\mathbf{A} \mathbf{A}^\top)$? Why?
3. Consider an orthogonal, square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$. We generate a new matrix $\mathbf{C} = \mathbf{AB}$. What can we say about the following questions about this product?
- How are the columns of \mathbf{C} related to the columns of \mathbf{A}
 - How is the 2-norm of the i^{th} column of \mathbf{C} related that of the columns of \mathbf{B} ?
4. Show that the matrix product \mathbf{ABC} can be written as a weighted sum of the outer products of the columns of $\mathbf{A} \in \mathbb{R}^{n \times p}$ and rows of $\mathbf{C} \in \mathbb{R}^{q \times n}$, with the weights coming from the matrix $\mathbf{B} \in \mathbb{R}^{p \times q}$.

$$\mathbf{ABC} = \sum_{i=1}^p \sum_{j=1}^q b_{ij} \mathbf{a}_i \tilde{\mathbf{c}}_j^\top$$

5. Prove the following for the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$.

$$(\mathbf{A}_1, \mathbf{A}_2 \mathbf{A}_3 \dots \mathbf{A}_n)^\top = \mathbf{A}_n^\top \mathbf{A}_{n-1}^\top \dots \mathbf{A}_2^\top \mathbf{A}_1^\top$$

6. **Matrix Inversion Lemma.** Consider an invertible matrix \mathbf{A} . The matrix $\mathbf{A} + \mathbf{uv}^\top$ is invertible if and only if the two vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, and $\mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u} \neq -1$. Then, the inverse is given by,

$$(\mathbf{A} + \mathbf{uv}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1} \mathbf{uv}^\top \mathbf{A}^{-1}}{1 + \mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u}}$$

7. Prove that $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$, where $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{B} \in \mathbb{R}^{d \times n}$.
8. Show that the diagonal elements of a square matrix \mathbf{A} , such that $\mathbf{A}^\top = -\mathbf{A}$ are zero. These are *skew-symmetric* matrices.
9. Show that $\mathbf{x}^\top \mathbf{Ax} = 0$ if $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix.

Additional problems

1. Consider the following 5×4 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 2 & 2 \\ 1 & -2 & 1 & 3 & 9 \\ -2 & 0 & 2 & -1 & -2 \\ 3 & 1 & 1 & -5 & 0 \end{bmatrix}$$

Compute the following:

- a) \mathbf{a}_3
 - b) \mathbf{a}_1^\top
 - c) $\tilde{\mathbf{a}}_2^\top$
 - d) $\tilde{\mathbf{a}}_4$
 - e) $\mathbf{a}_1 \mathbf{a}_2^\top$
 - f) $\tilde{\mathbf{a}}_3 \mathbf{a}_2^\top$
 - g) $\tilde{\mathbf{a}}_1 \tilde{\mathbf{a}}_2^\top$
 - h) $\tilde{\mathbf{a}}_1^\top \tilde{\mathbf{a}}_2$
 - i) $\tilde{\mathbf{a}}_1 \tilde{\mathbf{a}}_1^\top + \tilde{\mathbf{a}}_2 \tilde{\mathbf{a}}_2^\top$
 - j) $\mathbf{a}_3^\top \mathbf{a}_1 + \mathbf{a}_2^\top \mathbf{a}_4$
2. Which of the following statements true about a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$?
- a) $\sum_{i=1}^m \|\mathbf{a}_i\|_2^2 = \sum_{i=1}^n \|\tilde{\mathbf{a}}_i\|_2^2$
 - b) $\sum_{i=1}^m \|\mathbf{a}_i\|_2^2 = \text{tr}(\mathbf{A}^\top \mathbf{A})$
 - c) If the matrix $\mathbf{A} = \mathbf{V} \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbf{V}^{-1}$, Then

$$\mathbf{A}^n = \mathbf{V} \begin{bmatrix} d_1^n & 0 \\ 0 & d_2^n \end{bmatrix} \mathbf{V}^{-1}$$

3. Consider the matrix $\mathbf{P} = [\mathbf{e}_1 \quad \mathbf{e}_3 \quad \mathbf{e}_2]$. What does this P matrix do to a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ in the following operations? Try to compute these without performing the matrix multiplication and by using you understanding of the row and column views of matrix multiplication.
- a) \mathbf{PA}
 - b) \mathbf{AP}
 - c) $\mathbf{P}^2 \mathbf{A}$
 - d) \mathbf{AP}^2
 - e) \mathbf{PAP}

3 SOLUTION TO LINEAR EQUATIONS

1. Consider a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ with $n > m$. Consider a linearly independent set of vectors $\mathbf{x}_1, \mathbf{x}_2$. Is the set of $\mathbf{Ax}_1, \mathbf{Ax}_2$ linearly independent?
2. Find the bases for the four fundamental subspaces of the following matrices
 - a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.
 - b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.
 - c) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.
 - d) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.
 - e) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
3. Show that the rank $(\mathbf{AB}) = \text{rank}(\mathbf{A})$, when \mathbf{B} is square and full rank.
4. Let \mathbf{A} is a full rank matrix. Show that the *Gram matrix* of the column space, $\mathbf{A}^\top \mathbf{A}$ is invertible.
5. Draw the four fundamental subspaces of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.
6. Find the complete set of solutions for the following system of equations,

$$\mathbf{Ax} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{bmatrix}.$$

7. Is the following set of equations solvable? If yes, then find the complete set of solutions.

$$\mathbf{Ax} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{where } \mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & -1 \\ 3 & 3 & 1 \end{bmatrix}.$$

4 ORTHOGONALITY

1. If \mathbf{A} is an orthogonal matrix, show that $\mathbf{A}^{-1} = \mathbf{A}^\top$.
2. If \mathbf{P}_S is the orthogonal projection matrix onto the subspace S , then what is the corresponding orthogonal projection matrix onto S^\perp – the orthogonal complement of S ?
3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be an orthonormal basis for \mathbb{R}^n . Show that the following holds,

$$\mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n (\mathbf{x}^\top \mathbf{u}_i) \cdot (\mathbf{u}_i^\top \mathbf{y})$$

4. Consider the following set of vectors, $S = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n\}$, where $\mathbf{a}_i \in \mathbb{R}^n$. The set S is linearly independent. Find the orthogonal components of a vector $\mathbf{b} \in \mathbb{R}^n$ in the subspace spanned by the sets of vectors $S_1 = \{\mathbf{a}_i\}_{i=1}^m$ and S_1^\perp .
5. Consider the set of $n \times n$ orthogonal matrices,

$$\mathcal{Q} = \left\{ \mathbf{Q} \mid \mathbf{Q} \in \mathbb{R}^{n \times n}, \mathbf{Q}^\top \mathbf{Q} = \mathbf{Q} \mathbf{Q}^\top = \mathbf{I}_n \right\}$$

Is this set a subspace of $\mathbb{R}^{n \times n}$?

Show that the set is closed under matrix multiplication.

6. Consider the linear map, $\mathbf{y} = \mathbf{A}\mathbf{x}$, such that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let us assume that \mathbf{A} is full rank. What conditions must \mathbf{A} satisfy for the following statements to be true,
 - a) $\|\mathbf{y}\|_2 = \|\mathbf{x}\|_2$, for all \mathbf{x}, \mathbf{y} such that $\mathbf{y} = \mathbf{A}\mathbf{x}$.
 - b) $\mathbf{y}_1^\top \mathbf{y}_2 = \mathbf{x}_1^\top \mathbf{x}_2$, for all $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ such that $\mathbf{y}_1 = \mathbf{A}\mathbf{x}_1$ and $\mathbf{y}_2 = \mathbf{A}\mathbf{x}_2$.

Note: A linear map \mathbf{A} with the aforementioned properties preserves lengths and angle between vectors. Such maps are encountered in rigid body mechanics.

7. Find the QR decomposition of the following matrices.

a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

c) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 \end{bmatrix}$

d) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

e) $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

5 MATRIX INVERSES

1. Find a left inverse for the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Find the set of all possible left inverses.
2. Consider an upper triangular matrix $\mathbf{R} \in \mathbb{R}^{n \times n}$. We are interested in solving the following set of n linear equations,

$$\mathbf{R}\mathbf{x} = \mathbf{e}_i$$

$\mathbf{x} = [x_1 \ x_2 \ x_3 \ \dots \ x_n]^\top$ is the solution to the above equation. Show that $x_{i+1} = x_{i+2} = \dots = x_n = 0$.

3. Find the pseudo-inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. Show that the matrix $\mathbf{A}\mathbf{A}^\dagger$ is the orthogonal projection matrix onto the column space of \mathbf{A} .

6 EIGENVALUES AND EIGENVECTORS

1. Find the eigenvalues and eigenvectors of the following matrices.

a) $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

b) $\mathbf{A} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

c) $\mathbf{A} = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

d) $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -3 & -1 \end{bmatrix}$

2. Let $\mathbf{A} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$. What is the value of the following?

a) \mathbf{A}^2

b) \mathbf{A}^{100}

c) \mathbf{A}^∞

3. For a matrix \mathbf{A} with eigenvalues $\{\lambda_i\}_{i=1}^n$, verify for the following matrices that $\prod_{i=1}^n \lambda_i = \det(\mathbf{A})$ and $\sum_{i=1}^n \lambda_i = \text{trace}(\mathbf{A})$.

a) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

b) $\frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$

4. Let $\{\lambda_i, \mathbf{v}_i\}_{i=1}^n$ are the eignepairs of a matrix \mathbf{A} . What are the eigenpairs of the following?

a) $2\mathbf{A}$

b) $\mathbf{A} - 2\mathbf{I}$

c) $\mathbf{I} - \mathbf{A}$

7 POSITIVE DEFINITE MATRICES AND MATRIX NORMS

1. Prove that $\mathbf{A}^\top \mathbf{A}$ is positive semi-definite for any matrix \mathbf{A} . When is $\mathbf{A}^\top \mathbf{A}$ guaranteed to be positive definite?
2. If \mathbf{A} is positive definite, then prove that \mathbf{A}^{-1} is also positive definite.
3. Is the function $f(x_1, x_2, x_3) = 12x_1^2 + x_2^2 + 6x_3^2 + x_1x_2 - 2x_2x_3 + 4x_3x_1$ positive definite?
4. Prove the following for $\mathbf{A} \in \mathbb{R}^{m \times n}$:

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n] = \begin{bmatrix} \tilde{\mathbf{a}}_1^\top \\ \tilde{\mathbf{a}}_2^\top \\ \vdots \\ \tilde{\mathbf{a}}_m^\top \end{bmatrix}$$

- a) $\|\mathbf{A}\|_F = \text{trace}(\mathbf{A}^\top \mathbf{A})$
5. Verify the following inequalities on vector and matrix norms ($\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$):
 - a) $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2$
 - b) $\|\mathbf{x}\|_2 \leq \sqrt{m} \|\mathbf{x}\|_\infty$