TMA4240 Innlevering 4

Oppgave 1)

$M_x(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$ (Fra tabell)

MGF for X er:

$$\int_0^\infty e^{xt} \cdot \frac{\theta^4}{6} x^3 \cdot e^{-\theta x} \, dx$$

$$= \frac{\theta^4}{6} \int_0^\infty x^3 e^{x(t-\theta)} \, dx$$

$$= \frac{\theta^4}{6}\left(\left[x^3 \cdot \frac{e^{x(t-\theta)}}{(t-\theta)^3}\right]_0^\infty - \int_0^\infty 3x^2 \cdot \frac{e^{x(t-\theta)}}{(t-\theta)} dx\right)$$

$$= \frac{\theta^4}{6}\left(-\left[3x^2 \cdot \frac{e^{x(t-\theta)}}{(t-\theta)^2}\right]_0^\infty + \int_0^\infty 6x \cdot \frac{e^{x(t-\theta)}}{(t-\theta)^2} dx\right)$$

$$= \frac{\theta^4}{6}\left(\left[6x \cdot \frac{e^{x(t-\theta)}}{(t-\theta)^3}\right]_0^\infty - \int_0^\infty 6 \cdot \frac{e^{x(t-\theta)}}{(t-\theta)^3} dx\right)$$

$$= -\frac{\theta^4}{6} \cdot 6 \cdot \left[\frac{e^{x(t-\theta)}}{(t-\theta)^4}\right]_0^\infty = \theta^4 \cdot \frac{1}{(t-\theta)^4} = \left(\frac{\theta}{t-\theta}\right)^4$$

$$= \left(\frac{1}{\frac{t}{\theta}-1}\right)^4 = \left(1-\frac{t}{\theta}\right)^4$$

Forventningsverdien er $E[X] = \alpha\beta$

$\alpha = 4$, $\beta = \frac{1}{\theta}$ $= \underline{\underline{\frac{4}{\theta}}}$

Oppgave 2

a) $$\int_{\theta}^{\infty} c\, e^{\theta - x}\, dx = 1$$

$$c\int_{\theta}^{\infty} e^{\theta - x}\, dx$$, setter $u = \theta - x$ og $du = -dx$

$$-c\int_{0}^{-\infty} e^{u}\, du = c\int_{-\infty}^{0} e^{u}\, du = c\left[e^{u}\right]_{-\infty}^{0}$$

$$= c \cdot e^{-0} - c e^{-\infty} = c \cdot 1 = \underline{\underline{c}}$$

$$c = \underline{\underline{1}}$$

$$P(X > \theta + 1) = 1 - P(X < \theta + 1) = 1 - \int_{\theta}^{\theta+1} e^{\theta - x}\, dx$$

$$1 - \int_{\theta}^{\theta+1} -1 \cdot e^{u}\, du = 1 - \left[-e^{\theta - x}\right]_{\theta}^{\theta+1}$$

$$= 1 - \left[\lim_{x \to \theta^+} -e^{-\theta - x} - \left(\lim_{x \to \theta + 1} \left(-e^{-\theta - x}\right)\right)\right]$$

$$= 1 - \left(-\frac{1}{e} + 1\right) = \underline{\underline{\frac{1}{e}}}$$

b) $Y = 2x + 3$

$$x = \frac{Y-3}{2} = w(y)$$

$$g(Y) = f(w(y)) \cdot |w'(y)|$$

$$= e^{-\left(\frac{y-3}{2} - \theta\right)} \cdot \frac{1}{2} = \frac{e^{\theta - \frac{(y-3)}{2}}}{2}$$

$P(Y > \theta + 1)$:

$Y = 2x + 3$, $x \geq \theta$. Setter inn $x = \theta$

$Y = 2\theta + 3$. Dette er større enn $\theta + 1$, altså er $P(Y > \theta + 1) = \underline{\underline{1}}$

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c)

$W = \min(X_1, X_2 \ldots X_{10})$

Den kumulative fordelingen til W er gitt ved $F_W(x) = 1-(1-F_X(x))^n$

$$F_X(x) = \int_\theta^x e^{\theta - x}\,dx = 1 - e^{\theta - x}$$

$$F_W = 1-(1-(1-e^{\theta-x}))^{10}$$
$$= 1-(e^{\theta - x})^{10} = \underline{\underline{1 - e^{10\theta - 10x}}}$$

Sannsynlighetstettheten blir da $F'_W(x)$

Som er gitt ved $f_W(x) = n(1-F_X(x))^{n-1} f_X(x)$

$$= 10(1-(1-e^{\theta - x}))^9 \cdot e^{\theta - x}$$
$$= 10 \cdot (e^{\theta - x})^9 \cdot e^{\theta - x}$$
$$= 10(e^{\theta - x})^{10}$$
$$= \underline{\underline{10 e^{10\theta - 10x}}}$$

$$P(W > \theta + 1) = 1 - F_W(\theta + 1)$$
$$= 1-(1-e^{10\theta - 10x})$$
$$= e^{10\theta - 10(\theta + 1)} = \underline{\underline{e^{-10}}}$$

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Oppgave 3)

De to egenskapene som kjennetegner en god estimator er at den er forventningsrett ($E(\hat{\mu}) = \mu$) og at den har så lav varians som mulig.

Jeg vil finne den som er forventningsrett og har lavest varians:

$E[\hat{\mu}] = E[Y] = \underline{\mu}$ Forventningsrett!

$Var[\hat{\mu}] = Var[Y] = 0,1^2 = \underline{\frac{1}{100}}$

$E[\tilde{\mu}] = E[\frac{1}{2}x + \frac{1}{2}y] = \frac{1}{2}E[X] + \frac{1}{2}E[Y] = \underline{\mu}$ Forventningsrett!

$Var[\tilde{\mu}] = (\frac{1}{2})^2 Var[X] + (\frac{1}{2})^2 Var[Y] = \frac{0,2^2}{4} + \frac{0,1^2}{4} = \underline{\frac{1}{80}}$

$E[\mu^*] = \frac{1}{5}E[X] + \frac{4}{5}E[Y] = \underline{\mu}$ forventningsrett!

$Var[\mu^*] = (\frac{1}{5})^2 Var[X] + (\frac{4}{5})^2 Var[Y] = \frac{0,2^2}{25} + \frac{16 \cdot 0,1^2}{25} = \underline{\underline{\frac{1}{125}}}$

Jeg ser at alle er forventningsrette, men μ^* har lavest varians, så jeg ville brukt denne.

Oppgave 4)

a) $P(X>2) = 1 - F(2) = 1 - (1 - e^{-0{,}04 \cdot 2^2}) = \underline{\underline{0{,}852}}$

Må finne $P(X>5 \mid X>2)$:

$$P(X>5 \mid X>2) = \frac{P(X>2 \mid X>5) \cdot P(X>5)}{P(X>2)}$$

$$= \frac{1 \cdot e^{-0{,}04 \cdot 5^2}}{0{,}852} = \underline{\underline{0{,}437}}$$

Sannsynlighetstettheten er gitt ved

$$f(x) = F'(x) = (1 - e^{-\alpha \cdot x^2})'$$

$$= (1 - e^u)' \cdot u'$$

$$= -e^u \cdot (-2x\alpha) = \underline{\underline{2\alpha x e^{-\alpha x^2}}}$$

b) $f(x; \alpha) = 2\alpha x e^{-\alpha x^2}$

$$f(x_1, x_2 \dots x_n ; \alpha) = \prod_{i=1}^{n} f(x_i ; \alpha)$$

$$= \prod_{i=1}^{n} 2\alpha x_i e^{-\alpha x_i^2}$$

$$= (2\alpha)^n \cdot e^{-\alpha \sum_{i=1}^{n} x_i^2} \cdot \prod_{i=1}^{n} x_i$$ = Rimelighetsfunk.

finner log-Rimelighetsfunksjonen:

$$\ln(L(\alpha)) = \ln\left((2\alpha)^n \cdot e^{-\alpha \sum_{i=1}^{n} x_i^2} \cdot \prod_{i=1}^{n} x_i\right)$$

$$= \ln((2\alpha)^n) + \ln\left(e^{-\alpha \sum_{i=1}^{n} x_i^2}\right) + \ln\left(\prod_{i=1}^{n} x_i\right)$$

$$= n \ln(2\alpha) - \alpha \sum_{i=1}^{n} x_i^2 + \sum \ln(x_i)$$

Deriverer M.H.P α og setter lik 0:

$$\left(n \ln(2\alpha) - \alpha \sum_{i=1}^{n} x_i^2 + \sum \ln(x_i)\right)' = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i^2 + 0$$

$$\frac{n}{\alpha} - \sum_{i=1}^{n} x_i^2 = 0 \Rightarrow \underline{\underline{\alpha = \frac{n}{\sum_{i=1}^{n} x_i^2}}}$$

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estimator for $\hat{\mu} = \frac{\sqrt{\pi}}{2\sqrt{\frac{n-1}{\sum x_i^2}}}$

$$= \frac{1}{2} \cdot \left(\frac{\pi}{\frac{n-1}{\sum x_i^2}} \right)^{\frac{1}{2}} = \frac{1}{2} \cdot \left(\frac{\pi \sum_{i=1}^{n} x_i^2}{n-1} \right)^{\frac{1}{2}}$$

Setter inn $n=6$ og verdiene for x_i:

$$\hat{\mu} = \frac{1}{2} \cdot \left(\frac{\pi \cdot (3^2 + 4{,}5^2 + 5^2 + 7^2 + 6{,}5^2 + 5^2)}{6-1} \right)^{\frac{1}{2}} \approx \underline{\underline{5{,}2 \text{ uker}}}$$

c) $y = x^2$

$$x = \sqrt{y} = w(y)$$

$$g(y) = f(w(y)) \cdot |w'(y)|$$

$$= f(\sqrt{y}) \cdot \left| \frac{1}{2\sqrt{y}} \right|$$

$$= 2\alpha\sqrt{y} \cdot e^{-\alpha y} \cdot \frac{1}{2\sqrt{y}} = \underline{\underline{\alpha e^{-\alpha y}}}$$

Som er eksponential-fordelingen.

Må vise at $E[\hat{\alpha}] = \beta$

$$E[\hat{\alpha}] = E\left[\frac{n-1}{\sum_{i=1}^{n} y_i} \right] = E\left[\frac{n}{\sum_{i=1}^{n} y_i} - \frac{1}{\sum_{i=1}^{n} y_i} \right]$$

$$= n E\left[\frac{1}{\sum y_i} \right] - E\left[\frac{1}{\sum y_i} \right]$$

$$= n \frac{1}{\beta(n-1)} - \frac{1}{\beta(n-1)} = \frac{\cancel{(n-1)}}{\beta \cancel{(n-1)}} = \underline{\underline{\frac{1}{\beta}}}$$

$E[\hat{\alpha}] = \frac{1}{\beta} \neq \beta$, derfor er den ikke forventningsrett.

Oppgave 5)

a) $X = \{\text{vekten på fisk}\}$

$X \sim N(x; 800, 100)$

$P(X > 1000) = 1 - P(X < 1000)$

$= 1 - P\left(Z < \frac{1000 - 800}{100}\right) = 1 - P(Z < 2) = 1 - 0{,}9772 = \underline{\underline{0{,}0228}}$

$P(500 < X < 1000) = P\left(\frac{500 - 800}{100} < Z < 2\right)$

$= P(Z < 2) - P(Z < -3) = 0{,}9772 - 0{,}0013 = \underline{\underline{0{,}9759}}$

b) $X = \{\text{antall fisk}\}$

$X \sim \text{Poisson}(x; 3)$

$P(X = 0) = \frac{3^0}{0!} \cdot e^{-3} = \underline{\underline{e^{-3}}}$

$P(X > 3 \mid X > 0) = \frac{P(X > 0 \mid X > 3) \cdot P(X > 3)}{P(X > 0)} = \frac{P(X > 3)}{P(X > 0)} = \frac{1 - P(X \leq 3)}{1 - P(X \leq 0)}$

$= \frac{1 - \sum_{i=0}^{3} \frac{3^x}{x!} \cdot e^{-3}}{1 - \frac{3^0}{0!} \cdot e^{-3}} = \frac{0{,}352768}{1 - e^{-3}} = \underline{\underline{0{,}37}}$

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c) $P(X > 0) = 1 - P(X \leq 0) = 1 - P(X = 0)$

$= 0{,}5 \cdot 1 + 0{,}5 \cdot \frac{4^0}{0!} \cdot e^{-4} = \underline{\underline{0{,}44}}$

Oppgave 6)

a) $E[\hat{a}] = E[\bar{X}] = E[\frac{1}{n}\sum_{i=1}^{n} X_i] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{\cancel{n} a}{\cancel{n}} = \underline{\underline{a}}$

$Var[\hat{a}] = Var[\frac{1}{n}\sum_{i=1}^{n} X_i] = (\frac{1}{n})^2 \sum_{i=1}^{n} Var[X_i] = \frac{\cancel{n}\sigma^2}{n^{\cancel{2}}} = \underline{\underline{\frac{\sigma^2}{n}}}$

$\hat{a}$ har normalfordeling, fordi den er en funksjon av uavhengige stokastiske normalfordelte variabler.