Little Theories for Projectile

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The inspiration for the following “little theories” version of projectile motion is Farmer’s little theories formalization in simple type theory. An example of this can be found in the paper “Monoid Theory in Alonzo” by Farmer and Zvigelsky (2023).

The intention of this document is to start a discussion of the best way to envision projectile motion using the little theories approach. This is an informal document where the notation is not rigorous. For instance, given that the author doesn’t understand exactly how one theory is transported to another, this issue is glossed over by adding a field called “Extends” to informally communicate this notion.

1 Questions

- How to combine theory with documentation of theory?
- How to bring in the context information (typical values, problem specific constraints etc.)?
- Where does the rationale go?

2 Documentation Theories?

Name: Concept

Constants: definition : $\mathbb{S}$, symbol : $\mathbb{S}$, unit : Unit

3 Time Theory

Name: Time

Extends: Concept, Reals

Variable: $t : \mathbb{R}$ (time)

Axioms:

1. definition = “time”

2. symbol = t (not necessary for the symbol to be the same as t)

3. unit = s

4 Position Theory

Name: Position

Extends: Concept, Reals

Variable: $p : \mathbb{R} \to \mathbb{R}$ (position)

Axioms:

1. definition = “location in space”

2. symbol = p (not necessary for the symbol to be the same as p)

3. unit = m

4. $p \in C^2$ (continuous for the function and the first and second derivatives)

5 Velocity Theory

Name: Velocity

Extends: Concept, Reals, Position

Variable: $v : \mathbb{R} \to \mathbb{R}$ (velocity)

Axioms:

1. definition = “rate of change of position in space”

2. symbol = v

3. unit = m/s

4. $v = \frac{dp}{dt}$

Definition and theorems:

Thm1: $v \in C^1$

6 Acceleration Theory

Name: Acceleration

Extends: Concept, Reals, Velocity

Constants: $a : \mathbb{R} \to \mathbb{R}$ (acceleration)

Axioms:

1. definition = "rate of change of velocity"
2. symbol = a
3. unit = $\frac{\mathrm{m}}{\mathrm{s}^2}$
4. $a = \frac{dv}{dt}$

Definition and theorems:

Thm1: $a \in C^0$

7 1D Kinematic Theory

Name: Kin1D

Extends: Position, Velocity, Acceleration

8 1D Kinematic Constant Acceleration Theory

Name: Kin1DConstAccel

Extends: Kin1D

Variables: $t : \mathbb{R}$ (time)

Constants: $a^c : \mathbb{R}$ (constant accel.), $p^i : \mathbb{R}$ (initial pos.), $v^i : \mathbb{R}$ (initial velo.)

Axioms:

1. $t \geq 0$ (A:timeStartZero)
2. $p\ 0 = p^i$ (initial position)
3. $v\ 0 = v^i$ (initial velocity)
4. $\forall t : \mathbb{R} | t \geq 0\ .\ a\ t = a^c$ (A:constAccel)

Definitions and theorems:

Thm1: $v\ t = v^i + a^c t$

Thm2: $p\ t = p^i + v^i t + \frac{a^c t^2}{2}$

9 Context Theory

Name: Context

Extends: Kin1DConstAccel

Axioms:

1. $t \geq 0$ (A:timeStartZero)
2. constraints on p^i
3. constraints on v^i
4. constraints on a^c

10 nD Kinematic Constant Acceleration Theory

Do this using a product of 1D theories.

Name: KinnDConstAccel (*This section is not complete; it is currently a mix of nD and 2D.*)

Extends: Kin1DConstAccel, n Dimensional Euclidean space. Somehow p, v and a have to be mapped to both x and y components of each.

Variables: $n : \mathbb{N}$ (dimension)

Constants: $a^c_{x\mathbb{R}}$ (constant accel. x direction), $a^c_{y\mathbb{R}}$ (constant accel. y direction), $p^i_{x\mathbb{R}}$ (initial pos. x direction), $p^i_{y\mathbb{R}}$ (initial pos. y direction),$v^i_{x\mathbb{R}}$ (initial velo. x direction), $v^i_{y\mathbb{R}}$ (initial velo. y direction)

Axioms:

1. $\forall k : \mathbb{N} | 0 \leq k \leq (n-1) \ . \ p_k \ 0 = p^i_k$ (initial position vector)
2. $\forall k : \mathbb{N} | 0 \leq k \leq (n-1) \ . \ v_k \ 0 = v^i_k$ (initial velocity vector)
3. $\forall k : \mathbb{N} | 0 \leq k \leq (n-1) \ . \ (\forall t : \mathbb{R} | t \geq 0 \ . \ a_k \ t = a^c_k)$ (constant acceleration vector)

Definitions and theorems:

Thm1:

$$\mathbf{v}(t) = \begin{bmatrix} v_{\mathrm{x}} \\ v_{\mathrm{y}} \end{bmatrix} = \begin{bmatrix} {v_{\mathrm{x}}}^{\mathrm{i}} + {a_{\mathrm{x}}}^{\mathrm{c}} t \\ {v_{\mathrm{y}}}^{\mathrm{i}} + {a_{\mathrm{y}}}^{\mathrm{c}} t \end{bmatrix}$$

Thm2:

$$\mathbf{p}(t) = \begin{bmatrix} p_{\mathrm{x}} \\ p_{\mathrm{y}} \end{bmatrix} = \begin{bmatrix} {p_{\mathrm{x}}}^{\mathrm{i}} + {v_{\mathrm{x}}}^{\mathrm{i}} t + \frac{{a_{\mathrm{x}}}^{\mathrm{c}} t^2}{2} \\ {p_{\mathrm{y}}}^{\mathrm{i}} + {v_{\mathrm{y}}}^{\mathrm{i}} t + \frac{{a_{\mathrm{y}}}^{\mathrm{c}} t^2}{2} \end{bmatrix}$$

11 Next Steps

1. Projectile motion specific via $p_x = 0$, $p_y = 0$, $v_x = v^{\text{launch}} \cos\theta$, $v_y = v^{\text{launch}} \sin\theta$, $a_x^c = 0$, $a_y^c = -g$
2. Add constraint axioms $v^{\text{launch}} > 0$, $0 < \theta < \frac{\pi}{2}$
3. Add theorem for calculation of landing time
4. Add theorem for calculation of landing distance

12 Questions

- How should units be handled? For instance, position has units of length (m), velocity has units of length per unit time (m/s) and acceleration has units of length^2 per unit time (m^2/2).
- How to capture context information? The typical values for projectiles used in games are much different than the values for projectiles in ballistics.
- How to document that context-specific rationale information? For instance, the assumption that the Earth is flat leads to only a tiny error for sports-related projectiles.