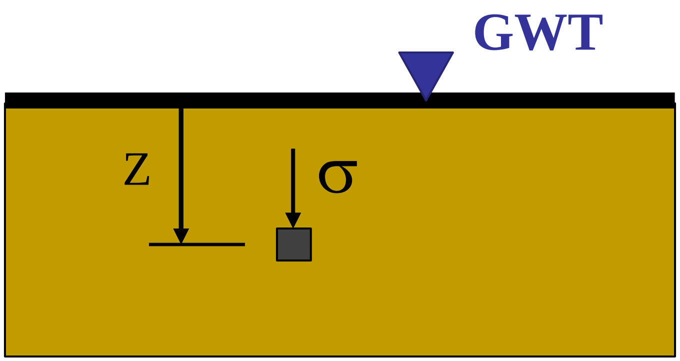
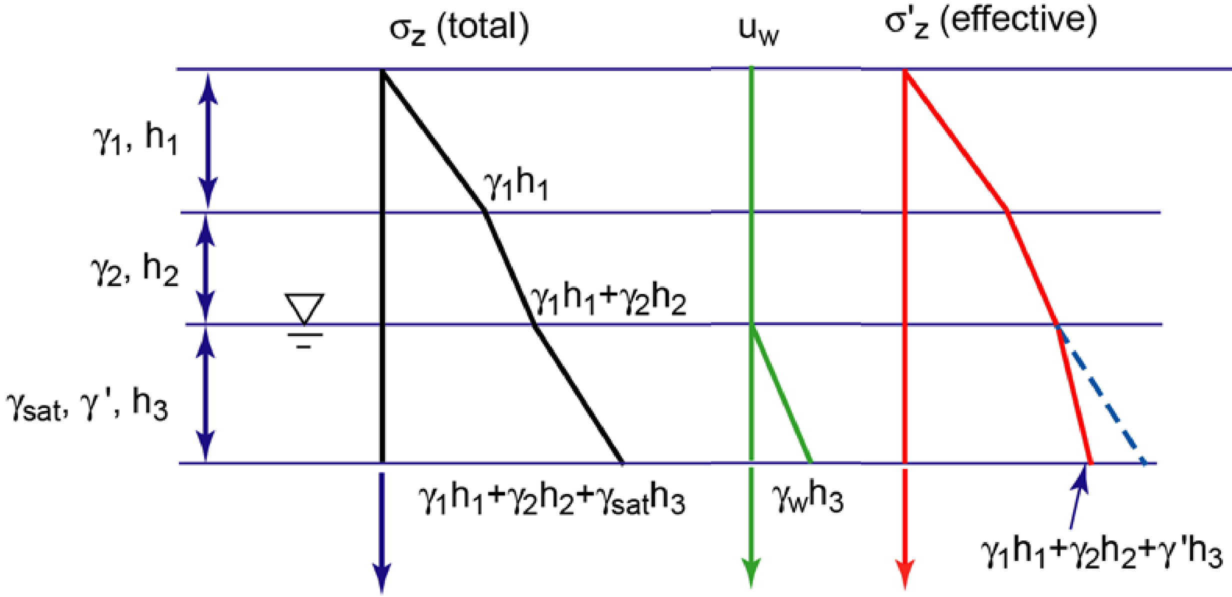
Geostatic Vertical Stress

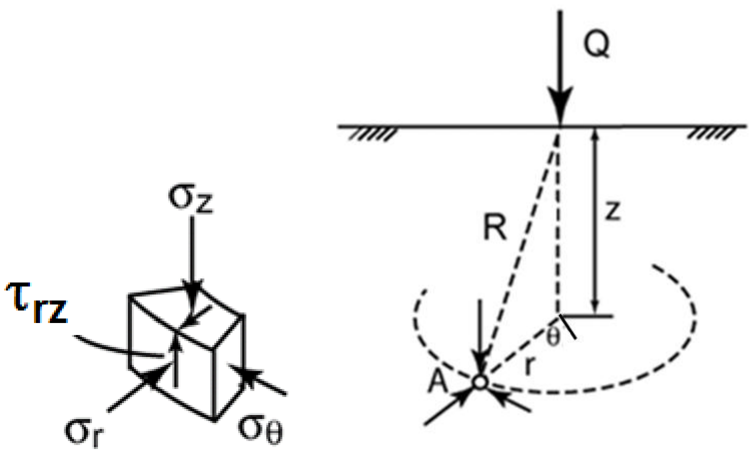
$$\sigma = \sigma' + u_w \Rightarrow \sigma = \gamma_{sat} z,\ u_w = \gamma_w z,$$
$$\sigma' = (\gamma_{sat} - \gamma_w) z = \gamma' z$$



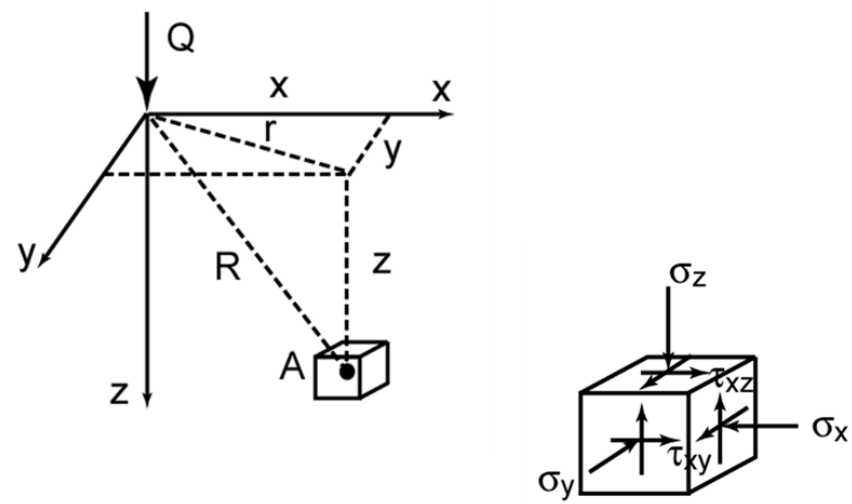


Point Load (Boussinesq Solution)

- Isotropic, homogeneous

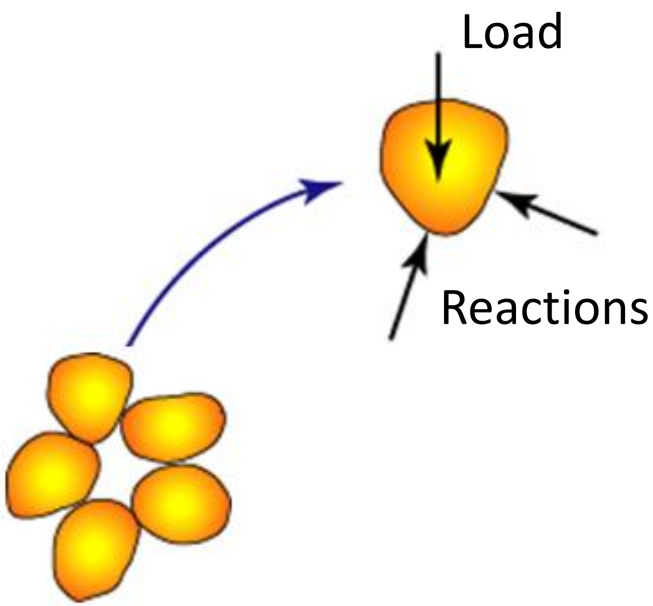


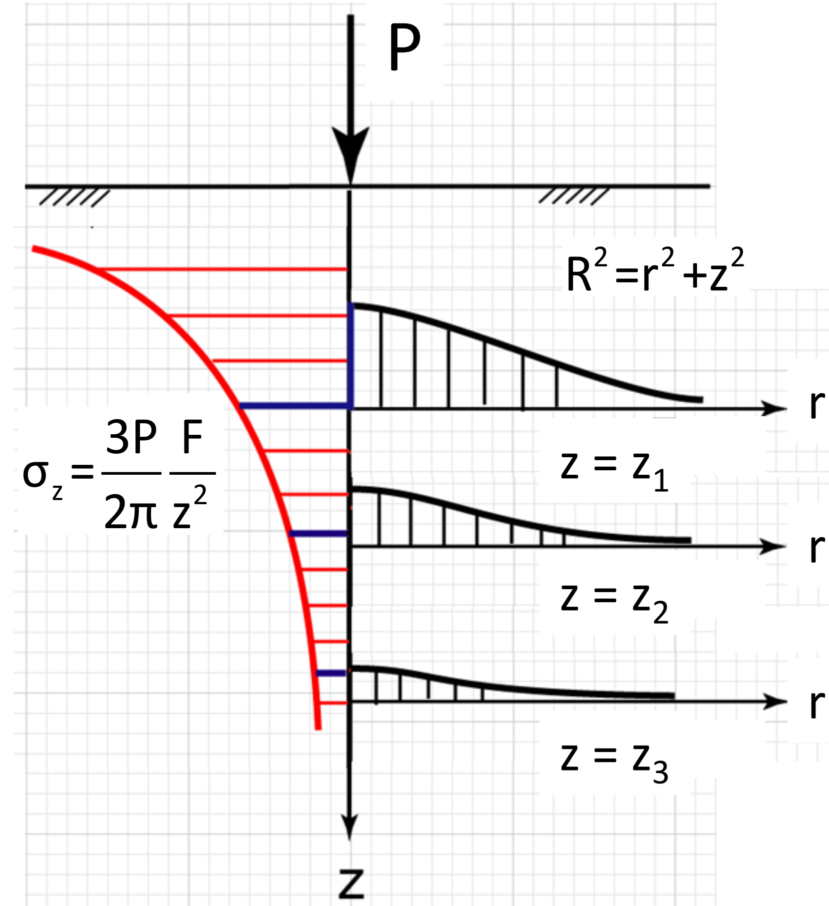
Cylindrical



Cartesian

Point load (Boussinesq Solution)



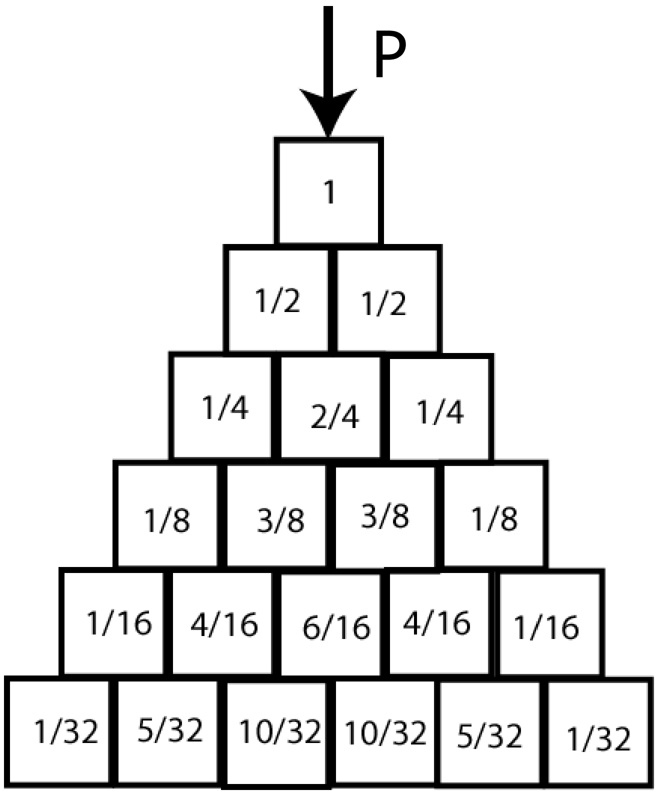


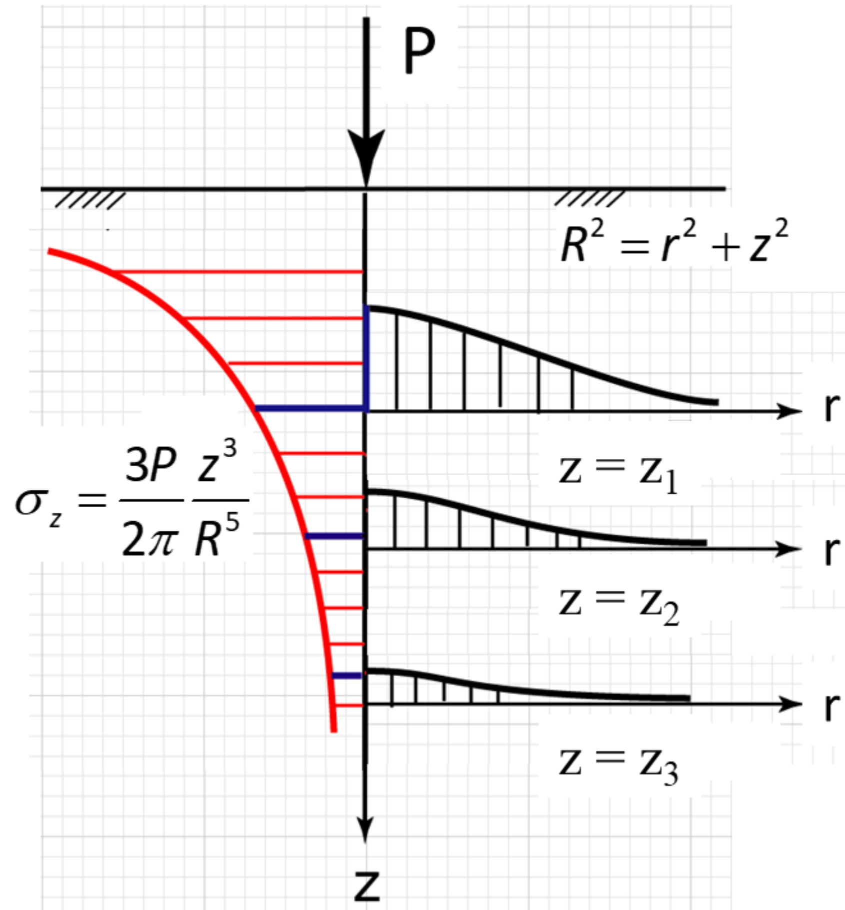
$$\sigma_{zz}=\frac{3P}{2\pi z^2}\left(\frac{1}{1+(r/z)^2}\right)^{5/2}$$

$$=\frac{3Pz^3}{2\pi R^5}=\frac{Q}{z^2}\mathrm{F}\left(\mathrm{z,R}\right)$$

Independent of material properties

Point load (Particle solution)

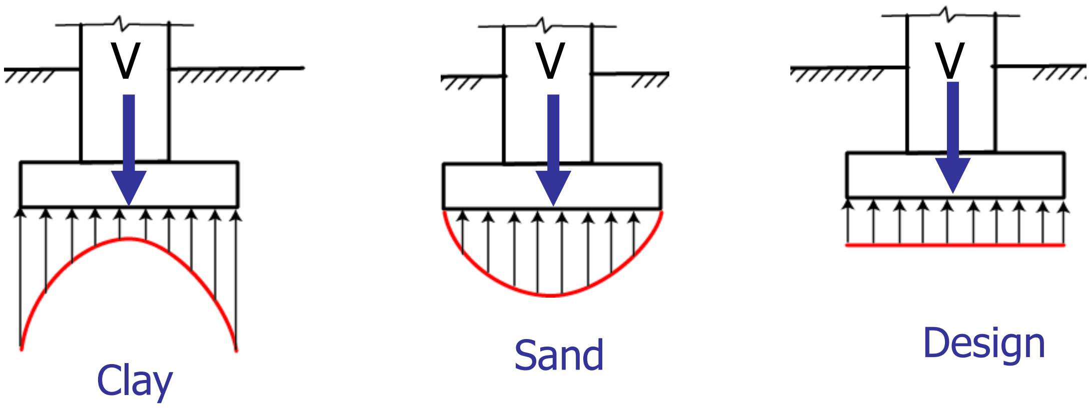




Particulate medium

$$\sigma_z = \frac{P}{2\nu\pi}\exp\left(-\frac{r^2}{2\nu z^2}\right)$$

Pressure Distributions Below Rigid Footing



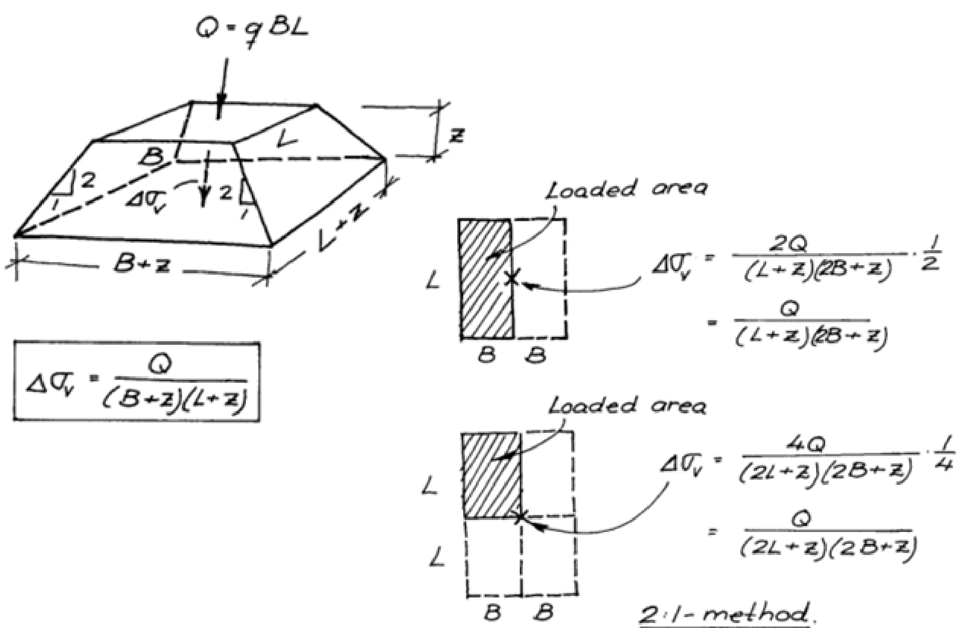
2:1 Method – At centerline

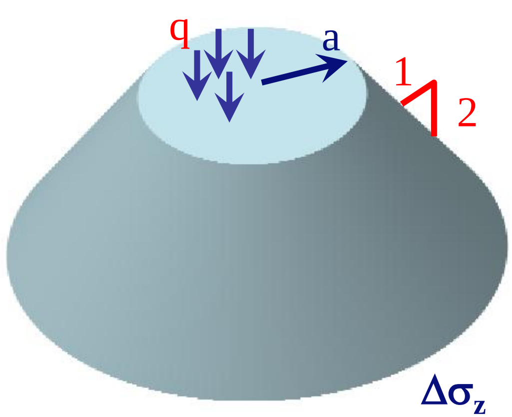
Rectangular loading area

$$\Delta\sigma_z = \frac{qBL}{(B+z)(L+z)}$$

Circular loading area

$$\Delta\sigma_z = \frac{qa^2}{(a+z/2)^2}$$

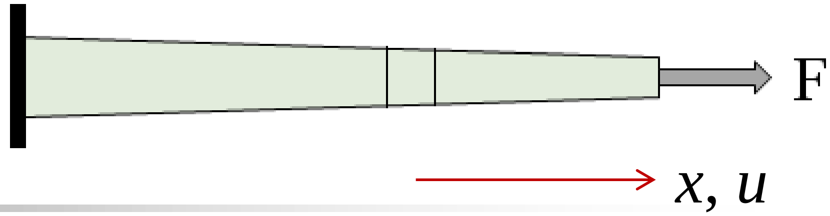




Field Equations for Elasticity Equilibrium Problems

For traditional soil mechanics compression is taken as being positive. I will deviate here by using the solid mechanics convention where tension is positive

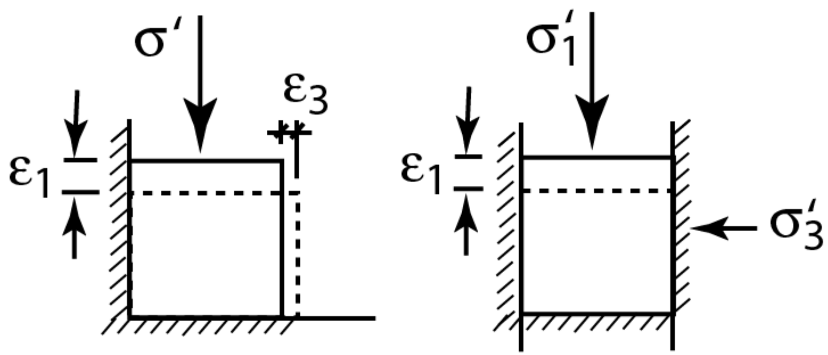
Example - Axial Model



- Kinematic – $\varepsilon = \dfrac{du}{dx}$
- Constitutive – $\sigma = E\varepsilon$
- Equilibrium – $\dfrac{d(\sigma A)}{dx} + \gamma A = 0$ (Not directly required)
- Compatibility – continuity of 'u'
- Boundary Conditions – $u(0) = 0; \quad P(L) = F$

These equations/conditions must be combined to solve a specific, axial, boundary-valued problem.

Elastic Coefficients – Kinematic & Constitutive



$$\varepsilon_1 = \frac{\sigma_1}{E}$$

$$\varepsilon_3 = -\mu\varepsilon_1$$

$$\varepsilon_1 = \frac{\sigma_1}{M}$$

$$\sigma_3' = K_0\sigma_1'$$

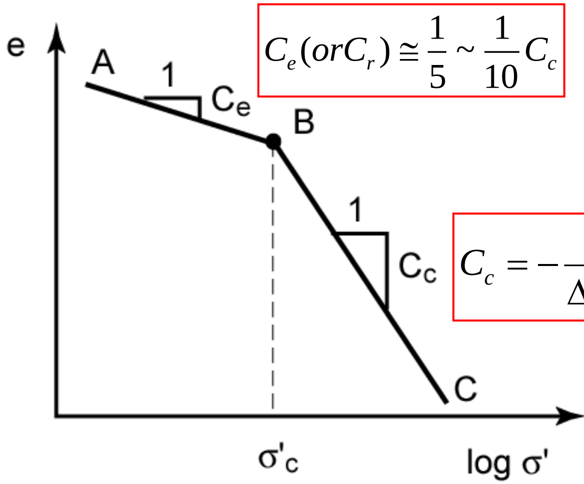
E = elastic modulus

μ = Poisson's ratio

M = compression modulus

K_0 = coefficient of earth pressure at rest

K_o – Compression – Compressibility:



$$C_e(or C_r) \cong \frac{1}{5} \sim \frac{1}{10} C_c$$

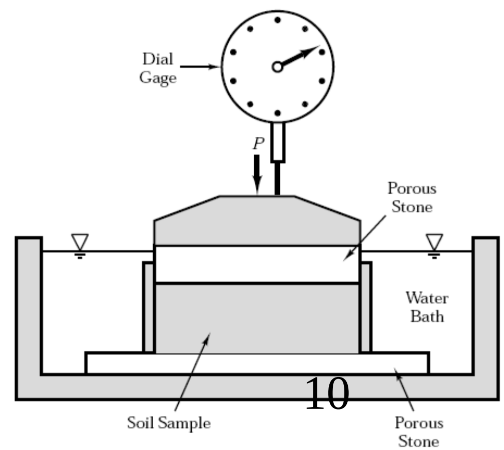
$$C_c = -\frac{\Delta e}{\Delta \log \sigma'}$$

OC Preconsolidation NC
Pressure

$$\Delta\varepsilon_z = \frac{\Delta H}{H} \rightarrow \Delta\varepsilon_z = \frac{\Delta V}{V} = \frac{\Delta e}{1+e}$$

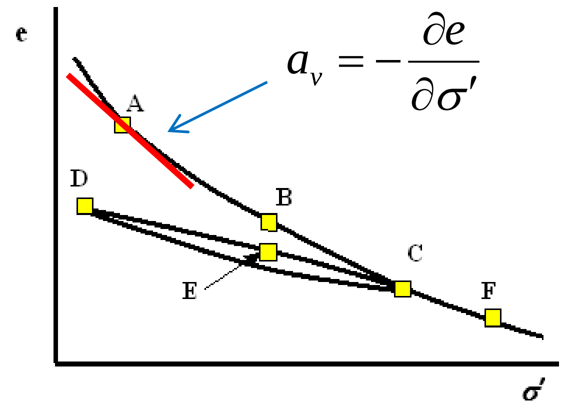
Compression/recompression Index

$$C_c = -\frac{\Delta e}{\Delta \log \sigma'} \qquad C_r = -\frac{\Delta e}{\Delta \log \sigma'}$$



Coefficient of Volume Change : m_v

Coefficient of compressibility:



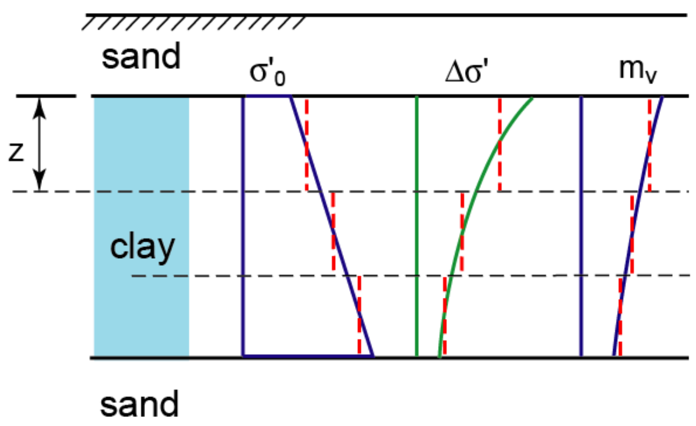
$$a_v = -\frac{\partial e}{\partial \sigma'}$$

Coefficient of volume change:

$$m_v \approx -\frac{\Delta e}{\Delta \sigma'}\frac{1}{1+e} = -\frac{\Delta e}{1+e}\frac{1}{\Delta \sigma'}$$

$$\downarrow$$

$$\frac{\Delta H}{H} = \frac{\Delta e}{(1+e)} = -m_v \Delta \sigma'$$

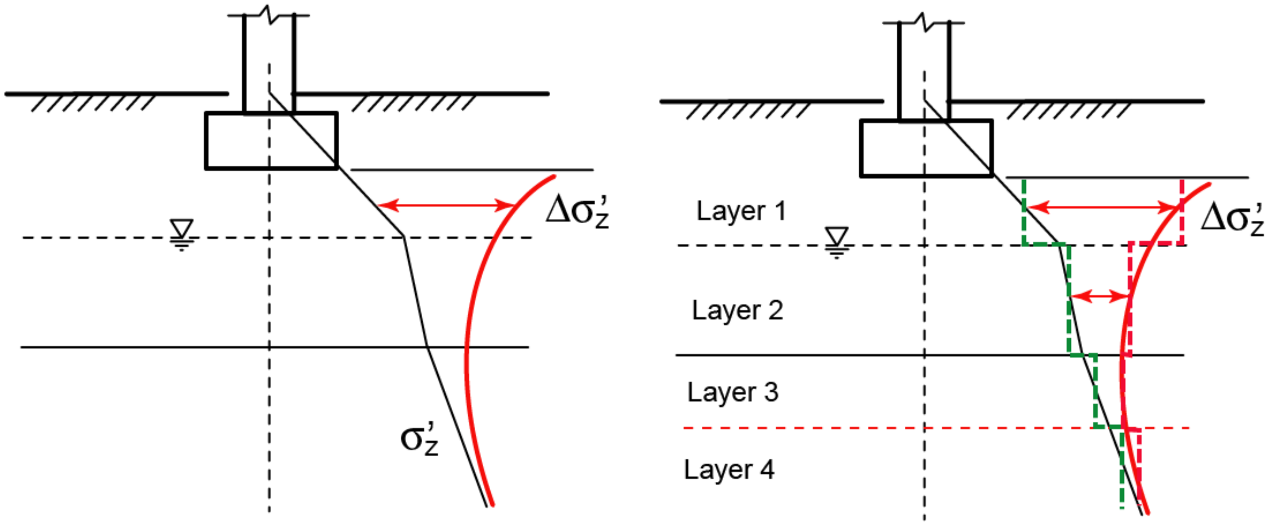


$$s_c(t) = s_c = \Delta H; \leftarrow \Delta H = \frac{\Delta e}{1+e} H$$

Work horse

From consolidation test assuming K_o conditions

Settlement Determination



- Divide soil layer into several sub-layers
- Calculate change of stress at the center of each sub-layer
- Assume uniform stress distribution in each sub-layer
- Calculate the settlement of each sub-layer $\Delta H_i = H_i\ \Delta\sigma^{(\prime)}$
- $\Delta H = \Sigma \Delta H_i$

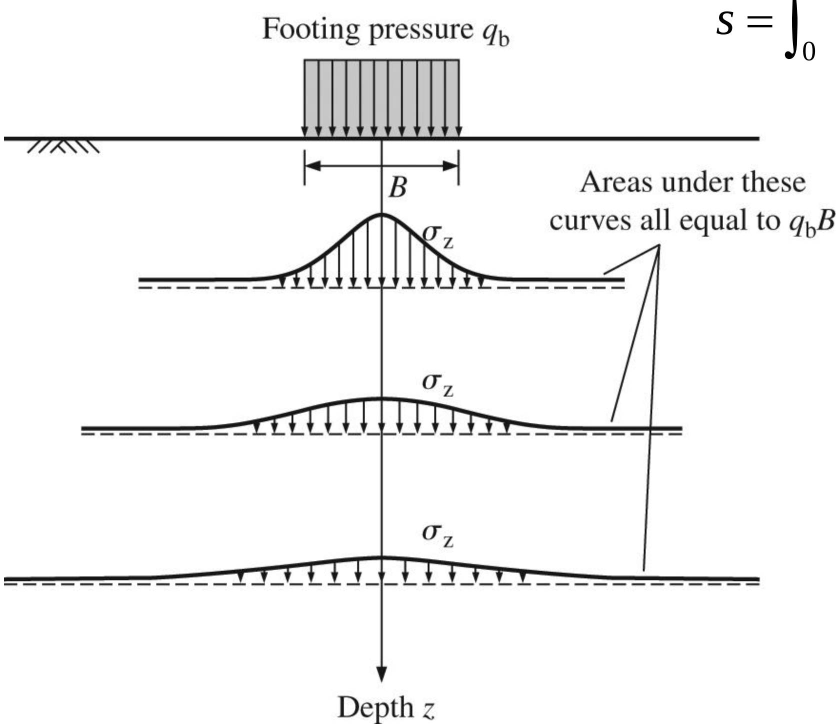
Immediate Settlement

$$s_i = \sum_i (\varepsilon_z H)_i = \sum_i \left[\frac{H}{E} \left(\Delta\sigma_z - \nu \underline{\left(\Delta\sigma_x + \Delta\sigma_y \right)} \right) \right]_i$$

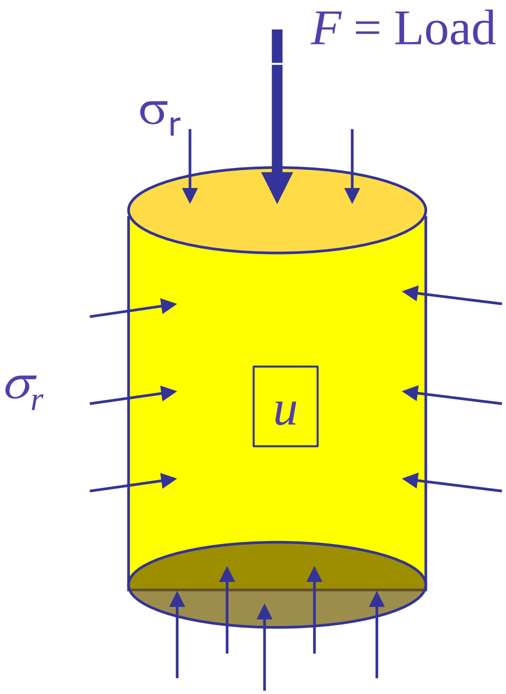
$$\Rightarrow \quad \varepsilon = \frac{\Delta\sigma_z}{E_s}$$

$$\Downarrow$$

$$s = \int_0^H \varepsilon dz = \sum_{i=1}^{n} \frac{H_i \Delta\sigma_{zi}}{E_{si}}$$



Triaxial Test



σ_r = Radial stress
(cell pressure)

σ_a = Axial stress $\Rightarrow$ $\sigma_a = \sigma_r + \overbrace{\frac{F}{A}}^{q}$

Triaxial Test

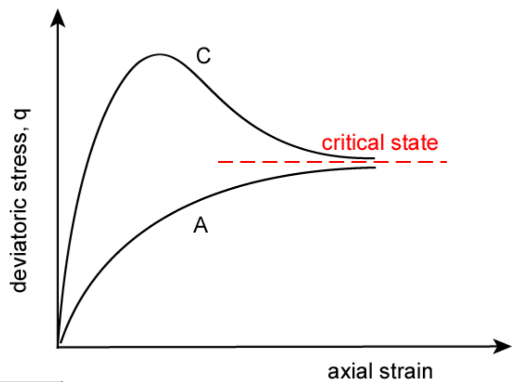
There are many test variations. Those used most in practice are:

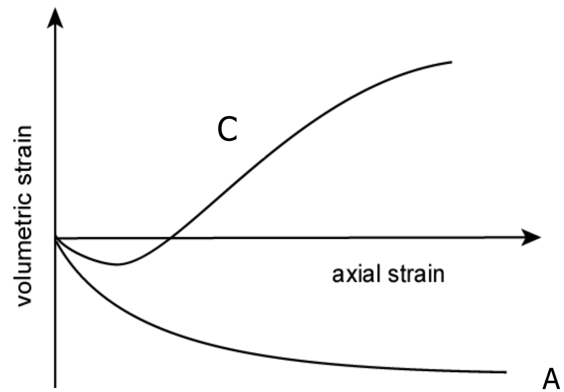
- UU (unconsolidated undrained) test – Cell pressure applied without allowing drainage. Then keeping cell pressure constant increase deviator load to failure without drainage.

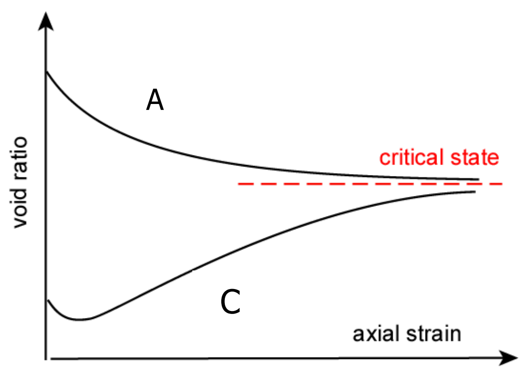
- CU (isotropically consolidated undrained) test – Drainage allowed during cell pressure application. Then without allowing further drainage increase q keeping σ_r constant as for UU test.

- CD (isotropically consolidated drained) test – Similar to CU except that as deviator stress is increased drainage is permitted.

Summary of CD Stress-Strain Response







Drained behaviour

A: Loose sand or NC clay
C: dense sand or OC clay

Total Settlement of Foundation

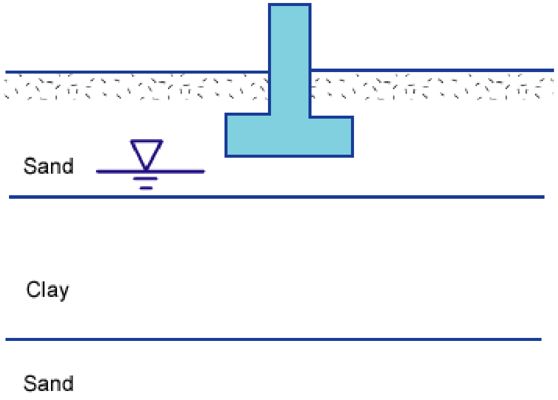
$$s_c(t) = s_c = \Delta H; \leftarrow \Delta H = \frac{\Delta e}{1+e} H$$

Primary consolidation settlement

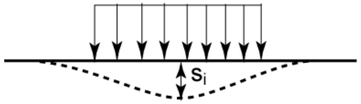
$$s = s_i + s_c(t) + s_s$$

Immediate settlement

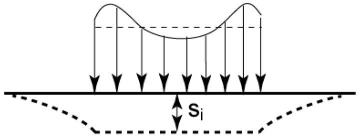
Secondary compression settlement



$$s_i = \frac{qB}{E}(1-\nu^2)I_s$$



Flexible foundation



Rigid foundation

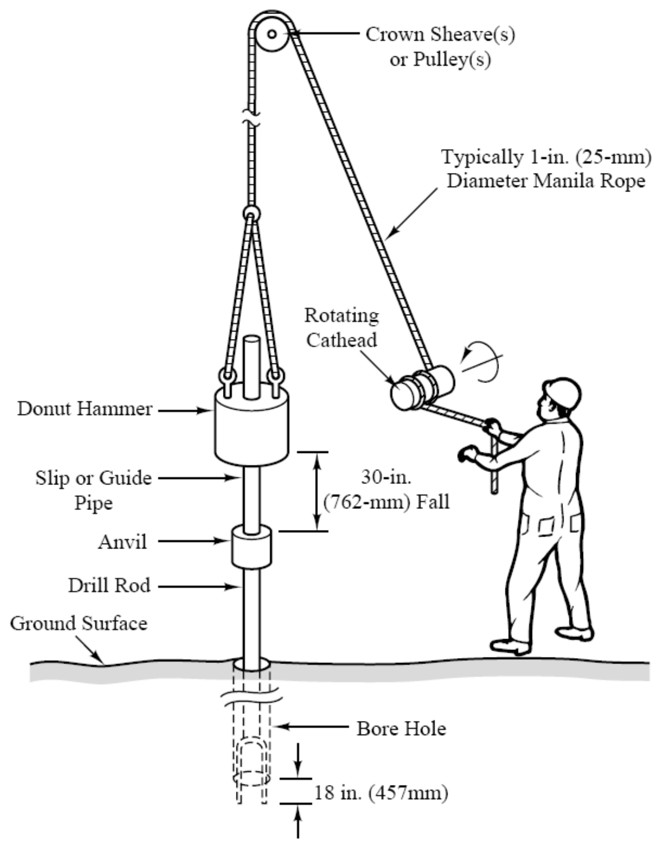
Correlations for C_c

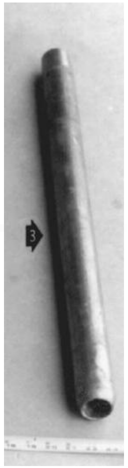
Correlation equations for soil compressibility/consolidation

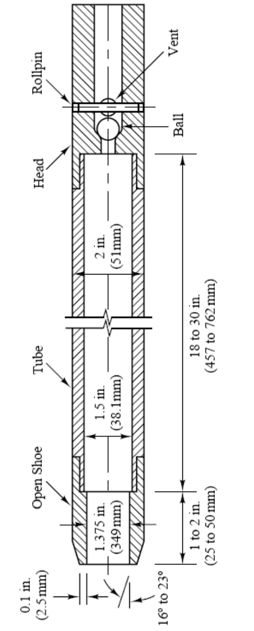
Compression index, C_c	Comments	Source/Reference
$C_c = 0.009(w_L - 10)$ ($\pm 30\%$ error)	Clays of moderate S_t	Terzaghi and Peck (1967)
$C_c = 0.37(e_o + 0.003w_L + 0.0004w_N - 0.34)$	678 data points	Azzouz et al. (1976)
$C_c = 0.141G_s\left(\dfrac{\gamma_{sat}}{\gamma_{dry}}\right)^{2.4}$	All clays	Rendon-Herrero (1983)
$C_c = 0.0093w_N$	109 data points	Koppula (1981)
$C_c = -0.0997 + 0.009w_L + 0.0014I_P + 0.0036w_N + 0.1165e_o + 0.0025C_P$	109 data points	Koppula (1981)
$C_c = 0.329[w_N G_s - 0.027w_P + 0.0133I_P(1.192 + C_P/I_P)]$	All inorganic clays	Carrier (1985)
$C_c = 0.046 + 0.0104I_P$	Best for $I_P < 50\%$	Nakase et al. (1988)
$C_c = 0.00234w_L G_s$	All inorganic clays	Nagaraj and Srinivasa Murthy (1985, 1986)
$C_c = 1.15(e_o - 0.35)$	All clays	Nishida (1956)
$C_c = 0.009w_N + 0.005w_L$	All clays	Koppula (1986)
$C_c = -0.156 + 0.411e_o + 0.00058w_L$	72 data points	Al-Khafaji and Andersland (1992)

Standard Penetration Test (SPT)

- ASTM D-1586
- Standardized SPT: 140 lb (63.5 kg) hammer drops 30″ onto upper end of drill rod
- Keep track of number of blows to drive sampler 6″ increments
- SPT index *N* = # of blows for split spoon sampler to be driven last 12″
- *N* is sensitive to equipment used and release mechanism, length of rod. Necessary corrections are needed.







The SPT sampler
(ASTM D1586)

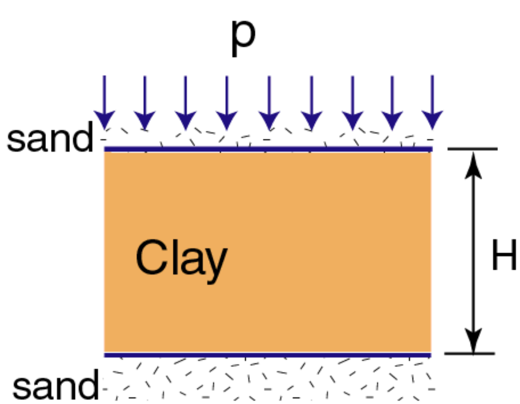
Correlations for E

Soil Elastic Moduli from *In Situ* Test Data

Soil	**SPT**	**CPT**
Sand (normally consolidated)	$E_s = 500(N + 15)$	$E_s = (2\text{–}4)q_u$
	$= 7{,}000\sqrt{N}$	$= 8{,}000\sqrt{q_c}$
	$= 6{,}000N$	—
	—	$E_s = 1.2(3D_r^2 + 2)q_c$
	$\ddagger E_s = (15{,}000\text{–}22{,}000)\ln N$	$^*E_s = (1 + D_r^2)q_c$
Sand (saturated)	$E_s = 250(N + 15)$	$E_s = Fq_c$
		$e = 1.0,\ F = 3.5$
		$e = 0.6,\ F = 7.0$
Sands, all (norm. consol.)	¶$E_s = (2{,}600\text{–}2{,}900)N$	
Sand (overconsolidated)	$\dagger E_s = 40{,}000 + 1{,}050N$	$E_s = (6\text{–}30)q_c$
	$E_{s(OCR)} \approx E_{s,nc}\sqrt{OCR}$	
Gravelly sand	$E_s = 1{,}200(N + 6)$	
	$= 600(N + 6) \quad N < 15$	
	$= 600(N + 6) + 2{,}000 \quad N > 15$	
Clayey sand	$E_s = 320(N + 15)$	$E_s = (3\text{–}6)q_c$
Silts, sandy silt, or clayey silt	$E_s = 300(N + 6)$	$E_s = (1\text{–}2)q_c$
	If $q_c < 2{,}500$ kPa use	
	$2{,}500 < q_c < 5{,}000$ use where	
	$E_s^t = \text{constrained modulus} = \dfrac{E_3(1-\mu)}{(1+\mu)(1-2\mu)} = \dfrac{1}{m_\mu}$	
Soft clay or clayey silt		$E_s = (3\text{–}8)q_c$

Notes: E_s in kPa for SPT and units of q_c for CPT; divide kPa by 50 to obtain ksf. The N values should be estimated as N_{55} and not N_{70}.

Consolidation: (Terzaghi)



Time-dependent deformation of soil due to expelling pore fluid (hydraulic lag)

Consolidation equation:

$$c_v \frac{\partial u_w^2}{\partial z^2} = \frac{\partial u_w}{\partial t} - \cancel{\frac{\partial \sigma}{\partial t}}$$

Coefficient of consolidation:

$$c_v = \frac{k}{m_v \gamma_w}$$

Degree of consolidation:

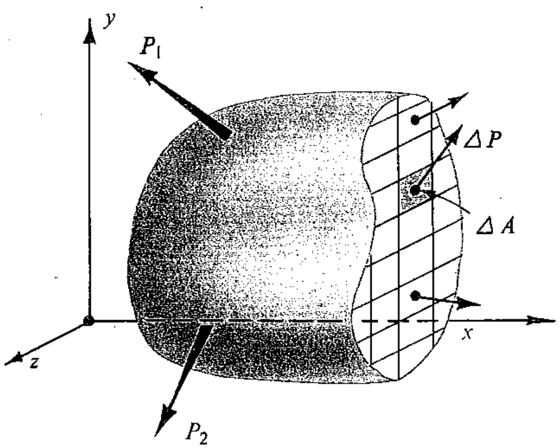
$$U = \frac{\Delta H(t)}{\Delta H_{\max}} \Rightarrow \Delta H(t) = U(t)\Delta H_{\max}$$

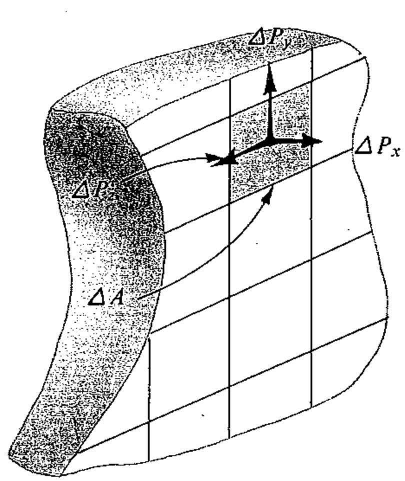
The End

Extras

Stress on Plane with Normal in x-Direction

$$\vec{\sigma}_x = \lim_{\Delta A_x \to 0} \frac{\Delta \vec{P}}{\Delta A_x} \quad \Rightarrow \quad \vec{\sigma}_x = \underbrace{\sigma_{xx}\vec{i}}_{\text{Normal}} + \underbrace{\sigma_{xy}\vec{j} + \sigma_{xz}\vec{k}}_{\text{Shear}}$$





$$\sigma_{xy}$$

Orientation Normal to Plane

Orientation Force Component

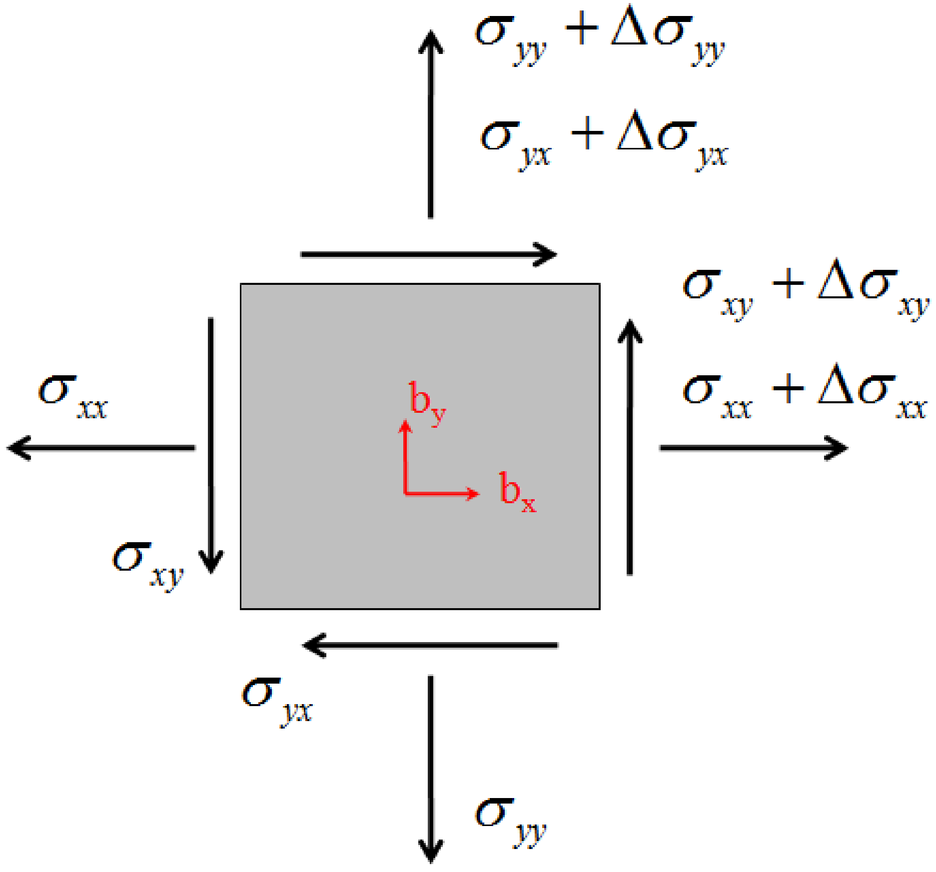
2^D Equilibrium at a Point

- Assume unit width
- Equilibrium of forces of $\Delta x \times \Delta y \times 1$ slice

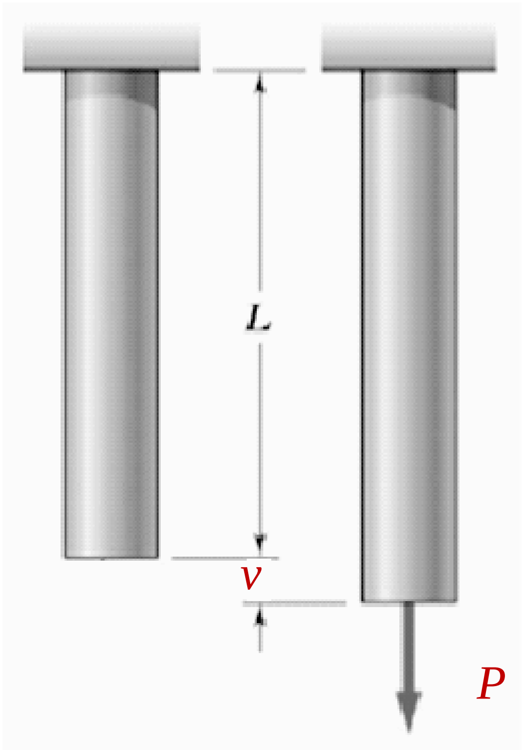
$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = 0$$

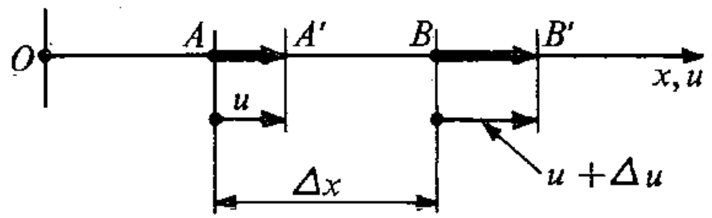
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0$$

$\sigma_{yx} = \sigma_{xy}$ (Moment equilibrium)

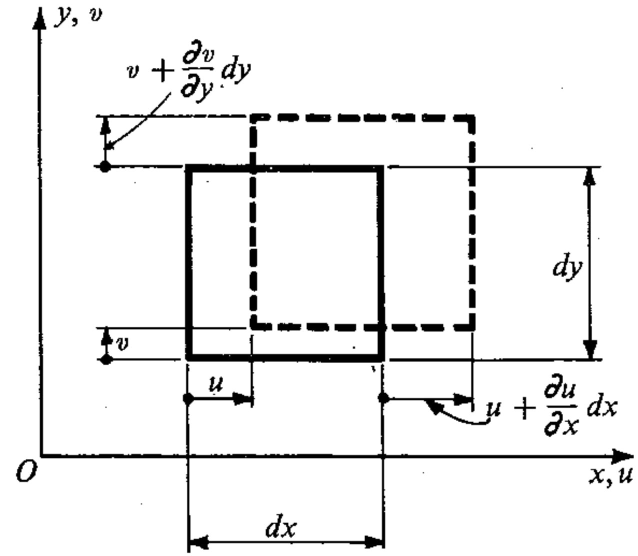


Normal Strain - Deformation per unit length of member (longitudinal):



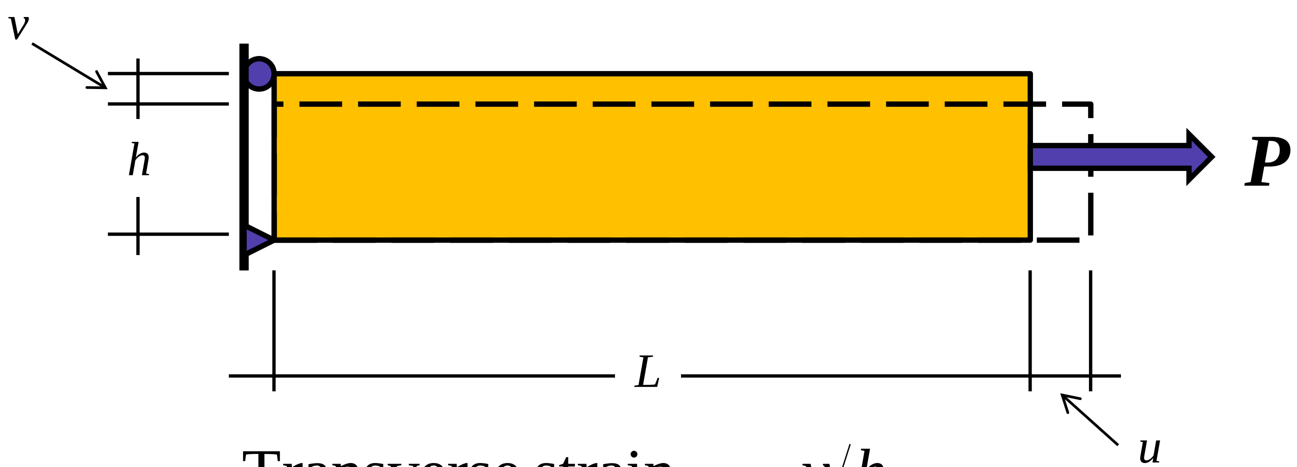


$$\varepsilon_{xx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \to \varepsilon_{xx} = \frac{\partial u}{\partial x}$$



$$\varepsilon_{yy} = \lim_{\Delta y \to 0} \frac{\Delta v}{\Delta y} \to \varepsilon_{yy} = \frac{\partial v}{\partial y}$$

Poisson's Ratio

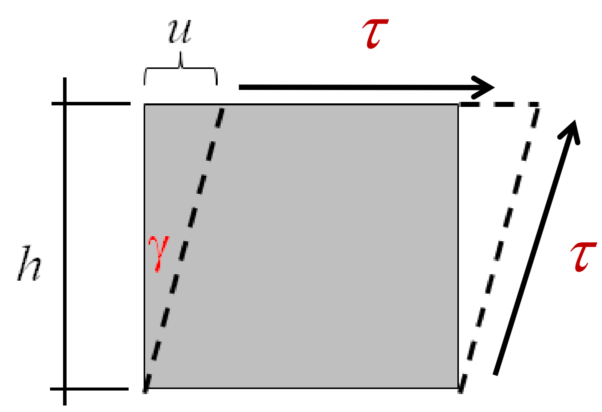


$$\nu = -\frac{\text{Transverse strain}}{\text{Axial strain}} = -\frac{v/h}{u/L}$$

In general we have, for example

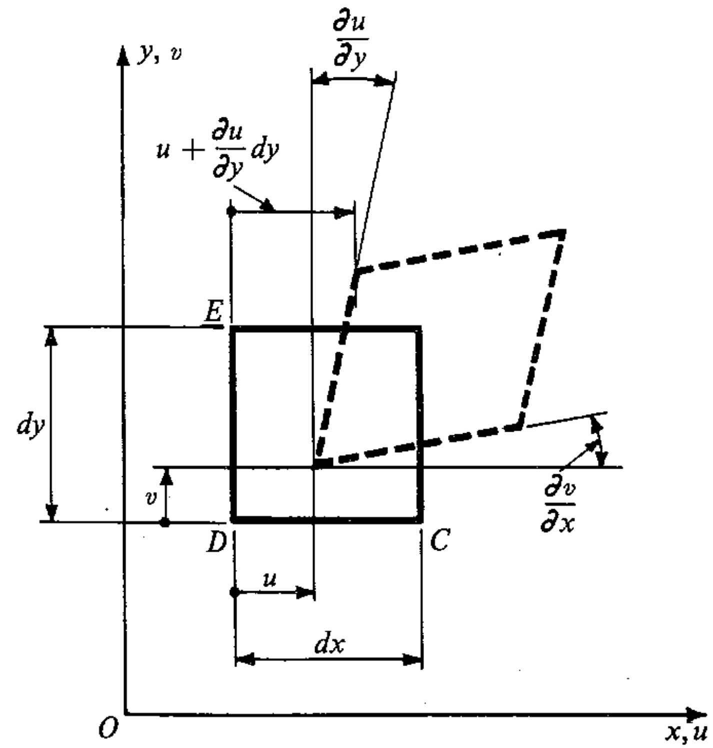
$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \left(-\frac{\varepsilon_{xx}}{\varepsilon_{yy}}, \text{ if isotropic} \right)$$

Shear Strain – Change in angle during deformation



$$\tan\gamma \approx \gamma = \frac{u}{h}$$ (distortion)

Simple Shear



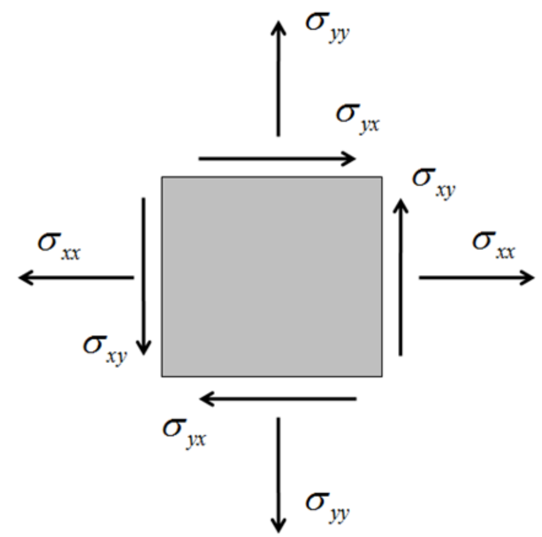
Pure Shear

$$\gamma_{xy} = \lim_{\Delta x, \Delta y \to 0}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) \to \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Generalized Hooke's Law - Direct Stresses

Consider a test on a unit cube where direct stresses are applied on all faces.

We assume that linear superposition is applicable and strains are small.



$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{zz}}{E}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{zz}}{E} - \nu \frac{\sigma_{xx}}{E}$$

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

For 2D it is not necessary to consider the out of plane direction, assumed to be the *z*-direction here. For plane strain, the strain in the out-of-plane direction is zero.

For a linear elastic isotropic body we also have:

$$\sigma_{xy} = G\gamma_{xy} \quad \Longrightarrow \quad G = \frac{E}{2(1+\nu)}$$

Compatibility – 2^D Problems

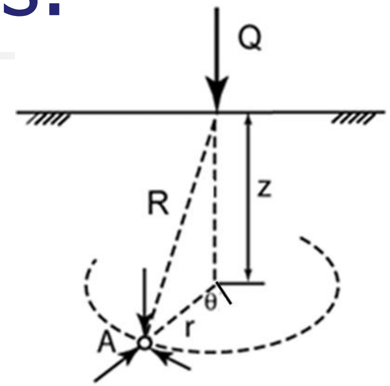
For planar problems we have 3 strains but two displacements, which means that the strains are somehow related. The relation is referred to as the compatibility constraint

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Boundary Conditions:

- Tractions
- Displacements

Assumptions for Induced Stresses:

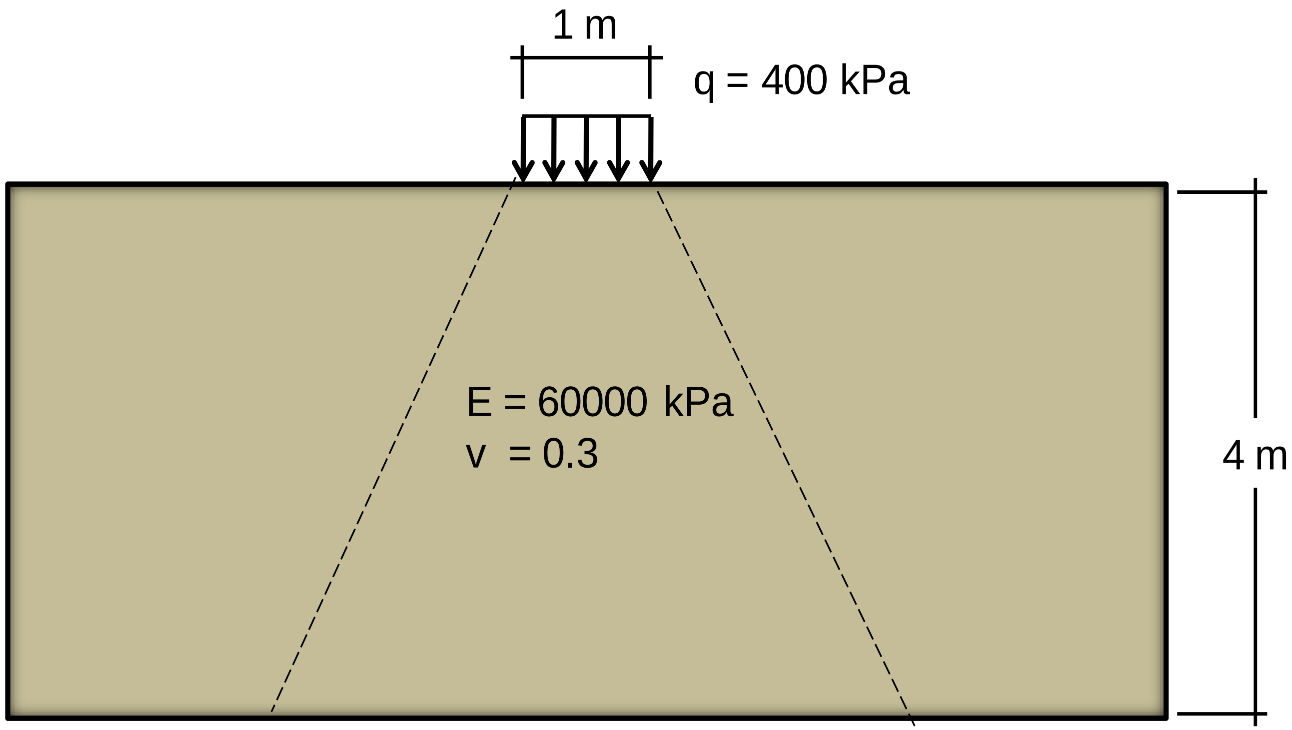


Take compression as positive

- Soil is a semi-infinite, homogeneous, linear, isotropic, elastic material
- A semi-infinite mass is bounded on one side and extends infinitely in all other directions; this is also called an "elastic half-space".

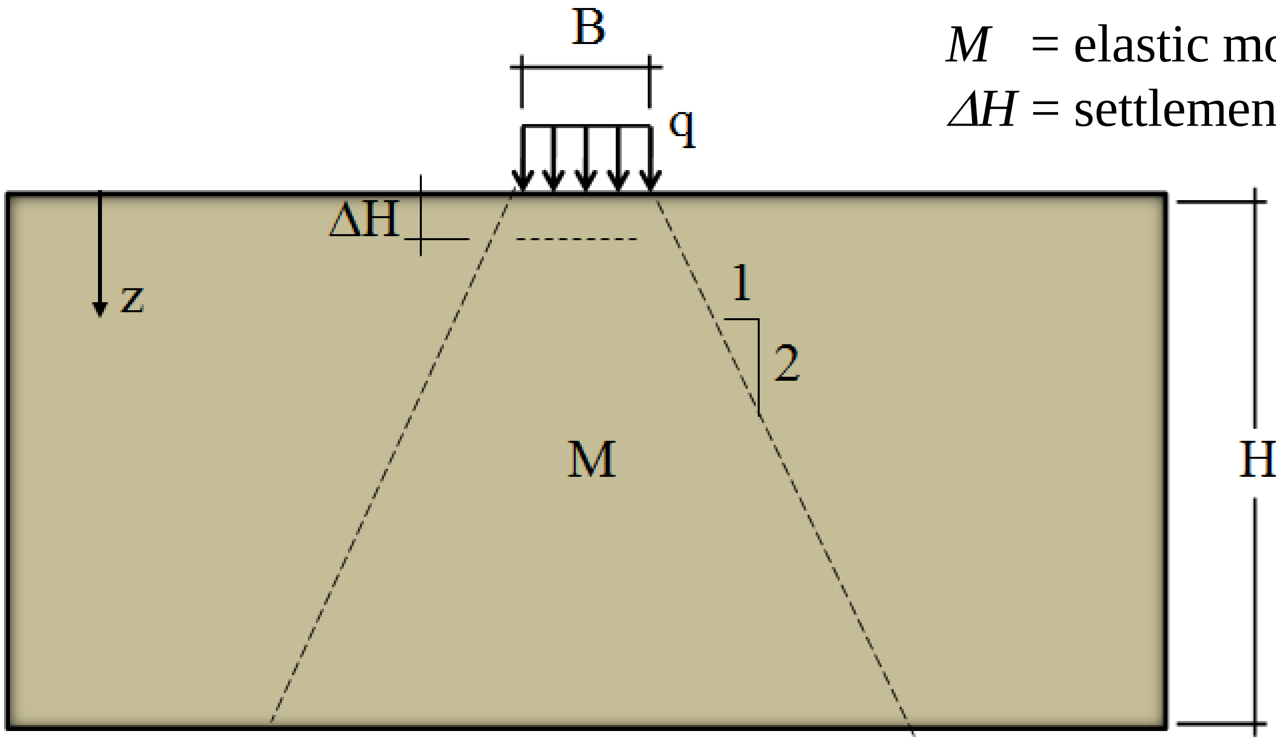
Example – Circular Footing Analysis (1)

Find elastic settlement and vertical stress increase under circular footing using the 2:1 rule and compare to 2D solution



Caution: Not for gravity loading

Example – Footing Analysis (2)



M = elastic modulus
ΔH = settlement

$$\Delta\sigma_{zz} = \frac{qB^2}{(B+z)^2}$$ (approximate)

$$\Delta H = \int \frac{\Delta\sigma_{zz}}{M} dz + C$$

$$\Delta H = \frac{qB}{M}\frac{H}{B+H}$$

Example – Settlement & Stresses (3)