Special multiplication of two digit numbers

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1 Introduction

The aim of this document is to explain to the reader a fast way to multiply two, two digit numbers, given that they meet certain criteria.

We'll be going over the algorithm [1] and also the correctness of the algorithm [2].

2 Example

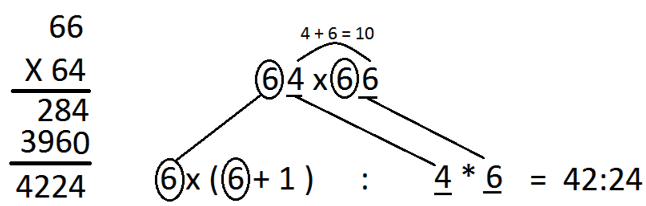
Let us first look at examples [1] , to understand what is happening.

2.1 Case 1

The first case is when the first(tens') digits are same and the last ones add to ten. If we take the numbers to be as $10x + y$ and $10x + z$ and $y + z = 10$ In such case product becomes[2]

$$x(x + 1) : yz$$

The figure given below will clear the scheme of things:

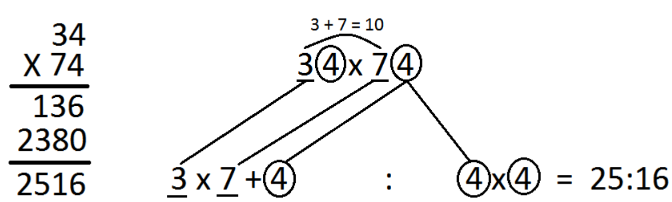


2.2 Case 2

The second case is when the first digits add to ten and the last ones are same. If we take the numbers to be as $10x + y$ and $10z + y$ and $x + z = 10$ In such case product becomes

$$xz + y : y^2$$

The figure given below will clear the scheme of things:



[1]Pretty Amazing, Eh?

[2] $x : y$ means $100x + y$

3 Why this Rocks

The above method is simply amazing because we have to spend very less time and effort in calculating the product of two 2-digit numbers, given that they satisfy one of the criteria. This is very clear from the examples given in the Section 2.

4 Proof of Correctness

4.1 Case 1

We highlight the proof for Case 1.
Say we have two numbers, one is xy and the other is xz such that $y + z = 10$. Therefore the result of multiplication of these numbers, say a, is going to be:

$$\begin{aligned} a &= (10x + y) * (10x + z) \\ &= 100x^2 + 10xz + 10xy + yz \\ &= 100x^2 + 10xy + 10x(10 - y) + y(10 - y) \\ &= 100x^2 + 10xy + 100x - 10xy + 10y - y^2 \\ &= 100x^2 + 100x + 10y - y^2 \\ a &= 100x(x + 1) + yz \end{aligned}$$

This is precisely what the algorithm gives. It says take the first part as $x(x + 1)$ and take the second part as yz. That is, the product is $x(x + 1) : yz$.

4.2 Case 2

Now, the proof for Case 2.
Let the two numbers be such that one is xy and the other is zy. Let the product of these numbers be a. Then,

$$\begin{aligned} a &= (10x + y)(10z + y) \\ &= 100xz + 10xy + 10zy + y^2 \\ &= 100xz + 10y(x + z) + y^2 \\ &= 100xz + 10y(10) + y^2 && \text{[Substituting } x + z = 10\text{]} \\ &= 100xz + 100y + y^2 \\ &= 100(xz + y) + y^2 \end{aligned}$$

This is exactly what the algorithm tells us to do. It says take the first part as $xz + y$ and take the second part as y^2. That is, the product is $xz + y : y^2$.

5 Pseudocode

$n1$ and $n2$ are the two 2-digit numbers that we are going to multiply.

```
int Multiply2(int n1,int n2){
x_1 ← n1/10;
y_1 ← n1%10;
x_2 ← n2/10;
y_2 ← n2%10;

if x_1 == x_2, y_1 + y_2 == 10 then                    ▷ Case1
    f1 ← x_1 * (x_1 + 1);
    f2 ← y_1 * y_2;
    prod ← 100 * f1 + f2;
end if

if x_1 + x_2 == 10, y_1 == y_2 then                    ▷ Case2
    f1 ← x_1 * x_2 + y_1;
    f2 ← y_1^2;
    prod ← 100 * f1 + f2;
end if
}
```

6 Generalization

We can generalize this. Let us look at how to do it. The arithmetic proofs remain similar to the 2-digit case. We consider two r-digit numbers.

Case	What is same	What adds to what	Product
Case1	The first digit	Last $r-1$ add to 10^{r-1}	$10^{2r-2}x(x+1)+y_1*y_2$
Case2	The last digit	First $r-1$ add to 10^{r-1}	$10^{2r-2}(y_1*y_2+x)+x^2$
Case3	First r digits	Last digits add to 10	$10^{2r-2}x(x+1)+yz$
Case4	Last r digits	First digits add to 10	$10^{2r-2}(xz+y)+y^2$

6.1 Case 1

Let the first digit(the one at r_{th} place) be x. Let the two remaining parts be y_1 and y_2. Then the product is $10^{2r-2}x(x+1)+y_1*y_2$

6.2 Case 2

Let the last digit(the one at ones' place) be x. Let the two remaining parts be y_1 and y_2. Then the product is $10^{2r-2}(y_1*y_2+x)+x^2$

6.3 Case 3

Let the first $r-1$ digits be x. The last digits of two numbers be y and z. Then the product is $10^{2r-2}x(x+1)+yz$

6.4 Case 4

Let the last $r-1$ digits be y. The first digits of two numbers be x and z. Then the product is $10^{2r-2}(xz+y)+y^2$

7 Miscellaneous

You can find an amazing tool to visualize and understand this kind of fast multiplication at my site[3] : `home.iitk.ac.in/~sidm/`

References

[1] Brahma. Atharveda, 1100 BC.

[2] Siddharth Mittal. Faster multiplication techniques, 1997.

[3] Just Kidding.