

MATH223 - Linear Algebra (class notes)

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January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra.**

Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in R^3 . Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

Complex numbers

Def: Let i be a symbol. We declare $i^2 = -1$.

Now, what we'd like to do is take this symbol i and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} 3i \\ i - 4 \\ 3i - \pi \\ \sqrt{i} + 21 \end{aligned}$$

Def: The field of complex numbers C consists of all expressions of the form $a + bi$, where $a, b \in R$.

Def: Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Notation:

- $0 + bi = bi$
- $a + 0i = a$ (a *real* number)
- $0 + 0i = 0$

Ex: If $z_1 = 2 - i$, $z_2 = 5i$, then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

Def: Let $z = a + bi \in C$

- (i) $\bar{z} = a - bi$, called the *complex conjugate* of z
- (ii) $|z| = \sqrt{a^2 + b^2}$, called the *absolute value* or *modulus*

Def: If $z = a + bi \in C$ and $z \neq 0$ (ie $z \neq 0 + 0i$), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \end{aligned}$$

is called the (multiplicative) inverse of z . It has the property $zz^{-1} = 1 = z^{-1}z$.

Proof. We have

$$\begin{aligned} zz^{-1} &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

Note: Since $z \neq 0 + 0i$, $a^2 + b^2 \neq 0$

□

Def: If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

Ex: If $z = 1 + 2i$, $w = 3 - i$ then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3 - i)\left(\frac{1}{5} - \frac{2}{5}i\right) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5}i - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned} \frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3-6i-i+2i^2}{1-2i+2i-4i^2} \\ &= \frac{1-7i}{5} \end{aligned}$$

January 9th 2019

Complex numbers as points in R^2

You can view $a + bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in R^2 , and they will conserve the same properties (addition of complex numbers corresponds to vector addition in R^2). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

Equations with complex numbers

Fact: Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$

- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned} (\pm i\sqrt{|a|})^2 &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a \quad (\text{since } a < 0) \end{aligned}$$

Fact: Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in C .

Ex: Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.

Sol:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

Note: If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say z , its other root is \bar{z} ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.

Caution: -1 has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} \quad (\text{this step doesn't quite work}) \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Theorem 1 (Fundamental Theorem of Algebra). *If*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 n^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \dots, a_0 \in C$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \dots \cdot (x - r_n)$$

for some complex numbers r_1, r_2, \dots, r_n . Some r_i 's may be equal.

Corollary 1.1. Every such polynomial has at least one root, and at most n distinct roots.

Note: Finding the roots is, in general, quite difficult.

Ex: Factor $2x^3 + 2x$ (over C).

Sol:

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

Ex: Solve $x^2 - i = 0$

Sol: $x^2 = i$ so $x = \pm\sqrt{i}$. Want \sqrt{i} in format $a + bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \\ 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} \quad (\text{so } a=b \text{ both + or both -}) \\ a^2 &= \frac{1}{2} \\ a &= \pm \frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Vector spaces (Ch 4)

Def. The sets R and C (and also Q , rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of K " means that K is either R or C .

January 11th 2019

Last time: Field K is R or C (for this class).

Geometric vectors ('arrows')

You can add two vectors (arrows) (see figure 1)

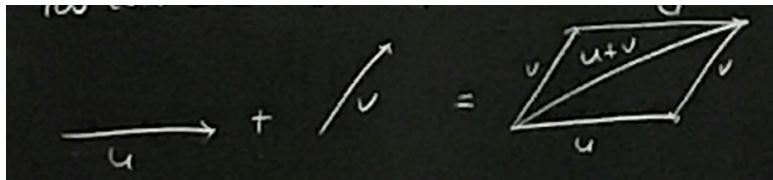


Figure 1: Vector addition

Observation: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

You can rescale a vector (see figure 2) **Observation:** $a(b\vec{u}) = (ab)\vec{u}$.

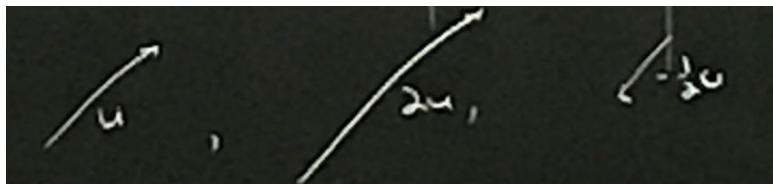


Figure 2: Vector rescaling

Also: $1\vec{u} = \vec{u}$

Question: What properties are interesting? What other objects obey the same properties?

Abstraction: Focus on properties more than on the objects.

Definition of a vector space

Let V be a set, called set of "vectors", and let K be a field (R or C) (elements of K called *scalars*). Assume that we have already defined two operations:

- (1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.
- (2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then V is called a *vector space* (over K).

- (A1) $u + v = v + u$ (commutative laws)
- (A2) There exists a vector in V , named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$
- (A3) For each $u \in V$, there is a vector in V , called the (additive) inverse of u and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)
- (A4) $(u + v) + w = u + (v + w)$

(SM1) $a(u + v) = au + av$ (distributive laws)

(SM2) $(a + b)u = au + bu$

(SM3) $a(bu) = (ab)u$

(SM4) $1u = u$ ($1 \in R$ or C)

These are called the vector space *axioms*.

Examples of vector spaces

Some examples:

(1) $K^n = \{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$$

where $c \in K$ (and K = set of scalar).

Proof that K^n is a vector space

Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.

Proof concept: To prove "for all $x \in A$, something", say "let $x \in A$ " (means x is an arbitrary element of A , ie you only know $x \in A$). Then, prove something for that x .

Proof: Let $u, v \in K^n$. This means $u = (a_1, a_2, \dots, a_n)$, $v = (b_1, b_2, \dots, b_n)$ for some $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, \dots, a_n + b_n) \quad (\text{definition of addition in } K^n) \\ &= (b_1 + a_1, \dots, b_n + a_n) \quad (\text{since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \dots, b_n) + (a_1, \dots, a_n) \quad (\text{definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define $0 = (0, 0, \dots, 0)$ (which is in K^n). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \dots, a_n)$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) \\ &= u \end{aligned}$$

- (2) In the vector space C^2 , $(2 + 3i, 5 - 7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

January 14th 2019

Problem: Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that J is not a vector space.

Solution: Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned} ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3) \end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.

Actual proof: Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned} u + (v + w) &= (1, 1) + ((2, 2) + (3, 3)) \\ &= (1, 1) + (-2 - 3, 5) \\ &= (1, 1) + (-5, 5) \\ &= (-1 + 5, 6) \\ &= (4, 6) \end{aligned}$$

Whereas,

$$\begin{aligned}
 (u + v) + w &= ((1,1) + (2,2)) + (3,3) \\
 &= (-1 - 2, 3) + (3,3) \\
 &= (-3,3) + (3,3) \\
 &= (-(-3) - 3, 6) \\
 &= (0,6)
 \end{aligned}$$

Hence, the axiom does not hold.

More examples of vector spaces

- (1) K^n (ie R^n or C^n). See before
- (2) $P(K)$ = polynomials, where coefficients are in K . Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}
 f(x) &= x^2 + 2ix - 4 \in P(C) \\
 g(x) &= -x^2 + ix \in P(C) \quad (\text{and also in } P(R))
 \end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned}
 2if(x) &= 2ix^2 + 4i^2x - 8i \\
 &= 2ix^2 - 4x - 8i
 \end{aligned}$$

- (3) $P_n(K)$ = polynomials of degree n or less, coefficient from K . For example,

$$\begin{aligned}
 x^2 - 2x + 2 &\in P_2(R) \\
 x^2 - 2x + 2 &\in P_3(R) \\
 x^2 - 2x + 2 &\in P_2(C) \\
 x^2 - 2x + 2 &\notin P_1(R)
 \end{aligned}$$

Note: In $P(K), P_n(K)$ the "vectors" are polynomials.

- (4) $M_{m \times n}(K)$ = $m \times n$ matrices with entries from K . Scalars are K ,

addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2 \times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2 \times 2}(C) \\ A + B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in $M_{m \times n}(K)$ is the $m \times n$ matrix with all entries 0.

- (5) Let X be any set (think $x = R$ or C , but not required). Define $F(X, K) = \{f : X \rightarrow K\}$ = all functions from X to K .
Ex: $f(x) = x^2 \in F(R, R)$.
Ex: Let $x = \{1, 2\}$. Then g defined by

$$\begin{aligned} g(1) &= 3 \\ g(2) &= \sqrt{2} \end{aligned}$$

Addition in this space is defined by:

If $f, g \in F(X, K)$ then $f + g$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Note that $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f + g)$ is the addition of vectors f and g , while the other $+$ is scalar addition.

Scalar multiplication in this space is defined by: if $f \in F(X, K), c \in K$ then cf is the function defined by

$$(cf)(x) = cf(x)$$

Note that cf is the name of the function, that "multiplication" is scalar multiplication $F(X, K)$ and $cf(x)$ is the multiplication of two scalars (numbers).

The fact that $F(X, K)$ is a vector space and the axioms are followed is not so obvious.

Prove (A2) true for $F(X, K)$. Define $z \in F(X, K)$ by

$$z(x) = 0 \quad (\text{for all } x \in X)$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence, $f + z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.

Exercise: Try (A3).

January 16th 2019

Theorem 2 (Cancellation Law). Suppose v is a vector space over K . For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.

Note: To prove "for all" you say let $u \in V$ (means u is an arbitrary vector).

To prove "if p then q ", denoted $p \rightarrow q$, assume p is true and use it to prove q .

Proof. Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned}(u + w) + (-w) &= (v + w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) \quad (\text{by A1}) \\ u + \vec{0} &= v + \vec{0} \quad (\text{by A3}) \\ u &= v \quad (\text{by A2})\end{aligned}$$

□

Theorem 3. Two points:

1. The zero vector is unique
2. For each $u \in V$, $-u$ is unique

Note: To prove something is unique, suppose you have two of them and show they are the same.

Proof. 1) Assume 0 and z both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned}z &= z + 0 && (\text{by } *, \text{ with } u = z) \\ &= 0 + z && (\text{by A4}) \\ z &= 0 && (\text{by } **, \text{ with } u = 0)\end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

Theorem 4. $\forall u \in V, c \in K,$

$$1) \quad c\vec{0} = \vec{0}$$

$$2) \quad 0u = \vec{0}$$

$$3) \quad -(cu) = ((-c)u)$$

Proof. Of 2). Let $u \in V$. Then,

$$\begin{aligned} 0u + 0u &= (0 + 0)u && \text{(By SM2)} \\ 0u + 0u &= 0u && \text{(by R addition)} \\ 0u + 0u &= 0u + \vec{0} && \text{(by A2)} \\ 0u + 0u &= \vec{0} + 0u && \text{(by A4)} \\ 0u &= \vec{0} && \text{(by cancellation law)} \end{aligned}$$

□

Note: $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

Linear combinations and spans

Def: Let $u, v_1, v_2, \dots, v_n \in V$. If there are scalars $a_1, a_2, \dots, a_n \in K$ such that $u = a_1v_1, a_2v_2 \dots a_nv_n$ then u is said to be a linear combination of v_1, v_2, \dots, v_n .

Ex: In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.

Important problem: Given vectors u, v_1, v_2, \dots, v_n , determine if u is a linear combination of v_1, v_2, \dots, v_n and if so find a_1, a_2, \dots, a_n .

Ex: Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

Sol. Are there a_1, a_2, a_3 s.t.

$$\begin{aligned} 2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\ &= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3) \end{aligned}$$

Equating coefficients,

$$a_1 - 2a_2 + 2a_3 = 2$$

$$2a_1 - 4a_2 = 6$$

$$a_1 - 2a_2 - 3a_3 = 8$$

Solve the linear system:

$$\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array}$$

↓

$$\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}$$

(row reduce)

\therefore No solution, because of the last row. f is not a linear combination of g_1, g_2, g_3 .

Def: Let $S \subseteq V$ (S is a subset of V) and assume $s \neq 0$. The span of s , denoted $\text{span}(s)$ is the set of all linear combinations of vectors from S , ie

$$\begin{aligned} \text{span}(s) = \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

January 18th 2019

Last class

$$S \subseteq V$$

$$\begin{aligned} \text{span}(s) = \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

Ex: $S = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\} \subseteq R^2$. Prove $\text{span}(S) = R^2$.

Note: $\begin{pmatrix} a \\ b \end{pmatrix}$ means (a, b) .

Proof note: To prove two sets A, B are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.

Sol:

- (1) Prove $\text{span}(S) \subseteq R^2$. Trivial, since any linear combination of vectors in R^2 is still in R^2 .

- (2) Prove $R^2 \subseteq \text{span}(S)$. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all a, b). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{span}(S)$ so $R^2 \subseteq \text{span}(S)$. So by (1), (2), $\text{span}(S) = R^2$. \square

Note: $Ax = b, A_{n \times n}$ if A inv, $x = A^{-1}b$.

Theorem 5. Let $S \subseteq V, S \neq \emptyset$ ($\emptyset = \text{empty set}$). Then,

- (1) If $u, v \in \text{span}(S)$ then $u + v \in \text{span}(S)$
- (2) If $u \in \text{span}(S)$ and $c \in K$, then $cu \in \text{span}(S)$
- (3) $\vec{0} \in \text{span}(S)$

Proof. By direct proof.

- (1) (Note, "if $u, v \in \text{span}(S)$ " means for all $u, v \in \text{span}(S)$).

Let $u, v \in \text{span}(S)$. Then,

$$u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

$$v = b_1 v_1 + b_2 v_2 + \dots + b_m v_m \text{ where } v_1, \dots, v_m \in S, b_1, \dots, b_m \in K$$

Then $u + v = a_1 u_1 + \dots + a_n u_n + b_1 v_1 + \dots + b_m v_m$ which is in $\text{span}(S)$ since $u_1, \dots, u_n, v_1, \dots, v_m \in S$.

- (2) Let $u \in \text{span}(S), c \in K$. Then,

$$u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \dots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_n)u_n \end{aligned}$$

Note: If you want to be very formal, you need to write down all of the vector space axioms. Which is in $\text{span}(S)$ since it is a linear combination of a_1, \dots, a_n which are in S .

- (3) (Prove $\vec{0} \in \text{span}(S)$) Let $u \in S$. **Note:** This is possible only because $S \neq \emptyset$.

Then $u = 1u$, so $u \in \text{span}(S)$. Then using $c = 0$ and (2) and fact that $u \in \text{span}(S)$,

$$cu = 0u = \vec{0}$$

is also in $\text{span}(S)$. **Note:** Since $u = 1u$, $S \subseteq \text{span}(S)$.

□

Subspaces

Def. Let V be a vector space and $W \subseteq V$ (subset). If W , using addition and scalar multiplication as defined in V , satisfies the definition of vector space, then W is called a subspace of V , denoted $W \leq V$ (less than equal sign, read as "subspace").

Note: Main issue is that addition and scalar multiplication with vector from W produce vectors which are still in W .

Theorem 6. Let $W \subseteq V$. Then, if the following three properties hold, $W \leq V$ (subspace).

- (SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")
- (SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")
- (SS3) $\vec{0} \in W$.

These are the same properties we just proved for spans; in other words, we proved earlier that $\text{span}(S)$ is a subspace.

Proof. For W to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

- (A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in V ,
- $$u + (v + w) = (u + v) + w.$$

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let $w \in W$.

Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \quad (\text{thm from last class})$$

is in W , as needed.

(A4) Still true because it is true in V .

(SM1-SM4) All hold because they hold in V .

□

January 21st 2019

A note on logic

Let P, Q be statements that are true or false.

- (1) "If P then Q ", also written symbolically as " $P \Rightarrow Q$ " (P implies Q) means if P is true, then Q is also true. To prove " $P \Rightarrow Q$ ", assume P and prove Q is true. If you know that " $P \Rightarrow Q$ " is true, you can use it: if you can establish that P is true, you may conclude Q is true.

Ex: Let A be an $n \times n$ matrix:

$$P : \det(A) = 1 \quad Q : "A \text{ is invertible}"$$

Thm: $P \Rightarrow Q$

- (2) The *converse* of " $P \Rightarrow Q$ " is " $Q \Rightarrow P$ ". This is a (logically) different statement.

Ex: With P and Q as above, " $Q \Rightarrow P$ " is not true because $A_{inv} \neq \det(A) = 1$.

- (3) The *contrapositive* of " $P \Rightarrow Q$ " is " $\neg Q \Rightarrow \neg P$ " ie "if Q false, then P also false". Logically, this is the same as " $P \Rightarrow Q$ ".

- (4) The *equivalence* " P if and only if Q ", written " $P \iff Q$ " means " $P \Rightarrow Q$ and also $Q \Rightarrow P$ " is true. Also means that either both P and Q are true or both are false.

Ex: $\det(A) \neq 0 \iff A \text{ is invertible}$.

To prove " $P \iff Q$ ", need to prove " $P \Rightarrow Q$ " and " $Q \Rightarrow P$ ".

Note: $\neg P \Rightarrow \neg Q$ is the same as $Q \Rightarrow P$.

Subspaces (cont'd)

Thm (last class): Let $W \subseteq V$ (subset). If

1. For all $u, v \in W$, $u + v \in W$
2. For all $u \in W$, $c \in K$, $cu \in W$
3. $\vec{0} \in W$

then $W \leq V$ (subspace). (ie: (1), (2), (3) are true $\Rightarrow W \leq V$)

Theorem 7. Let $W \subseteq V$. Then

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).

Proof. Exercise.

Theorem 8. Let $W \subseteq V$. Then

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

Examples of subspaces and non-subspaces

Is each subset a subspace?

- (a) $W = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq R^2$. Not a subspace, since the zero vector is not in W . The others are also false, but it's enough to prove that one of the statements does not hold. But $\text{span}(W) = R^2$ (so $\text{span}(W) \leq R^2$)

- (b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x + y - z = 0 \right\}$. Need to check (1), (2), (3):

- (i) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$. Then we know $x + y - z = 0$ and $x' + y' - z' = 0$. Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in W .

(2) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ (means $x + y - z = 0$), let $c \in K$. To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here, $cx + cy - cz = c(x + y - z) = c(0) = 0$. So $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

(3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$, since $0 + 0 - 0 = 0$

Since (1), (2), (3) true, $W \leq R^2$ (subspace)

(c) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x + y - z = 1 \right\}$. This is *not* a subspace. (3) is false.

(d) $W = \{A \in M_{2 \times 2} \mid A_{ij} \geq 0 \forall i, j\}$, where A_{ij} is the entry of A in row i , column j . (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

(2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since,

for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ but

$$(-1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

Two special subspaces

Let V be a vector space.

(1) $V \leq V$ is true

(2) $\{\vec{0}\} \leq V$ is true ("zero subspace")

A refinement on the definition of span

Def. If $S = \emptyset$ (emptyset), define $\text{span}(S) = \{\vec{0}\}$ (if $S \neq \emptyset$, $\text{span}(S)$ defined as before).

Theorem 9. $\text{span}(S) \leq V$.

Proof Two cases :

1. If $S = \emptyset$, $\text{span}(S) = \{\vec{0}\} \leq V$
2. If $S \neq \emptyset$, you already proved $\text{span}(S)$ satisfies (1), (2), (3).
So $\text{span}(S) \leq V$.

Theorem 10. (improved version of subspace conditions) Let $W \subseteq V$. Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W$$

Proof We will actually prove (1), (2), (3) \iff RHS (right-hand side). Two parts to proof.

(1) " $(1), (2), (3) \Rightarrow \text{RHS}$ " or " \Rightarrow "

January 23rd 2019

Recap:

- (1) If $u, v \in W$ then $u + v \in W$
- (2) if $u \in W, c \in K$ then $cu \in W$
- (3) $\vec{0} \in W$

Theorem 11. Let $W \subseteq V$. Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall u, v \in W, c \in K \text{ we have } cu + v \in W$$

Proof: Suffices to prove (1), (2), (3) \iff RHS.

1. \Rightarrow Assume (1), (2), (3) (prove right-hand side). Two things to prove:

- (1) Since $\vec{0} \in W$ (by (3)), $W \neq \emptyset$
- (2) Let $u, v \in W$ and $c \in K$. Since (2) holds, $cu \in W$. Since (1) holds, $cu \in W$ and $v \in W$, so $cu + v \in W$.

2. \Leftarrow Assume RHS, prove (1), (2), (3).

- (1) Let $u, v \in W$. Apply RHS with \Leftarrow to get

$$cu + v = 1u + v = u + v \in W$$

(2) (Prove $\vec{0} \in W$) Since $W \neq \emptyset$, there is a vector $w \in W$. Apply right-hand side with $u = w, v = w, c = -1$. So $cu + v = (-1)w + w = -w + w = \vec{0} \in W$.

(3) Let $u \in W, c \in K$. Apply RHS ($cu + v \in W$) with $u = u, c = c, v = \vec{0}$ (note: $\vec{0} \in W$ by (3) above). Then $cu + v = cu + \vec{0} = cu \in W \quad \square$

Ex: In $F(R, R) = V$ (functions $f : R \rightarrow R$), prove that

$$W = \{f \in V | f(3) = 0\}$$

is a subspace. Eg: $f(x) = (x - 3)e^x \in W$.

Solution: (1), (2) together (by last thm). Let $f, g \in W, c \in R$ (prove $cf + g \in W$). We know $f(3) = 0$ and $g(3) = 0$. Then, check $(cf + g)(3) = cf(3) + g(3) = 0 + 0 = 0$. So $cf + g \in W$.

Also, prove $w \neq \emptyset$. $f(x) = x - 3 \in W$, since $f(3) = 0$ (or, $z(3) = 0$ satisfies $z(3) = 0$ so $z \in W$. Note that z is the zero vector of $F(R, R)$).

Theorem 12. Let $A \in M_{m \times n}(K), b \in K^m$. Define

$$S = \{x \in K^n | Ax = b\}$$

i.e S = solution set to linear system $Ax = b$. Then,

$$S \leq K^n \iff b = \vec{0} \text{ (ie system is homogeneous)}$$

Proof

(i) \Rightarrow Assume $S \leq K^n$. Then $\vec{0}_n \in S$ (by (3)). So $A\vec{0} = b$ but $A\vec{0}_n = \vec{0}_m$ so $\vec{0} = b$.

(ii) \Leftarrow Assume $b = \vec{0}_m$ (prove $S \leq K^n$). Then $A\vec{0}_n = \vec{0}_m$, so $\vec{0}_n \in S$.

Next, let $u, v \in S, c \in K$. So $u, v \in K^n$ and $Au = b, Av = b$. Verify $cu + v$ is a solution.

$$\begin{aligned} A(cu + v) &= A(cu) + Av && \text{(prop of matrix multiplication)} \\ &= c(Au) + Av && \text{(prop of matrix multiplication)} \\ &= cb + b \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \\ &= b \quad \square \end{aligned}$$

Ex: Equation $ax + by + cz = d$ describes a plane in R^3 (eg $x + y + z = 1$) (and also, every plane can be described this way). That is,

$$\{(x, y, z) \in R^3 | ax + by + cz = d\}$$

is a plane.

By last thm,

$$\begin{aligned} P \text{ is a subspace} &\iff ax + by + cz = d \text{ is a homogeneous system} \\ &\iff d = 0 \\ &\iff P \text{ passes through origin } (0, 0, 0) \end{aligned}$$

Theorem 13. Let $S \subseteq V$. Then,

- (1) $\text{span}(S) \leq V$ and $S \subseteq \text{span}(S)$
- (2) If $S \subseteq W$, and $W \leq V$ (subspace) then $\text{span}(S) \subseteq W$ (actually, $\text{span}(S) \leq W$, subspace by (1))

Proof:

- (1) \leq We know already. Let $u \in S$. Then $u = 1u$, so $u \in \text{span}(S)$
- (2) Assume $S \subseteq W$, and $W \leq V$. Let $v \in \text{span}(S)$. Then $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$ for some scalars and vectors $u_1, u_2, \dots, u_n \in S$. Since $S \subseteq W$, $u_1, u_2, \dots, u_n \in W$. But W subspace. So $a_1u_1, a_2u_2, \dots, a_nu_n \in W$ (by prop (2) subspace) then $a_1u_1 + a_2u_2 \in W$ (by prop (1) of subspaces). So then $(a_1u_1 + a_2u_2) + a_3u_3 \in W$ (etc). So $a_1u_1 + a_2u_2 + \dots + a_nu_n \in W$.

Note: "etc" here is actually a proof by mathematical induction.

Omit for now.

January 25th 2019

Interlude : Symbolic logic (briefly)

Let P, Q be statements that could be true (T) or false (F). Define:

- (1) $\neg P$, "not P ", is F when P is T , T when P is F
- (2) $P \wedge Q$, " P and Q ", is T exactly when P, Q both T
- (3) $P \vee Q$, " P or Q " is T when P, Q both F
- (4) $P \Rightarrow Q$, " P implies Q ", is T unless P is T and Q is F . Hence, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$. We will write $P \Rightarrow Q \equiv \neg P \vee Q$.
- (5) $P \iff Q$, " P if and only if Q ", is T if both T or both F .

De Morgan's Laws

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Quantifiers

- \forall means "for all"
- \exists means "there exists"

Ex. (A4) (commutativity) $\forall u, v \in V \ u + v = v + u$.

Ex. 2 (A2) (zero vector) $\exists z \in V \ \forall u \in V \ (u + z = u) \wedge (z + u = u)$
(textbook version)

Negating quantifiers

- $\neg \forall u \in V P(u) \equiv \exists u \in V \neg P(u)$
- $\neg \exists u \in V P(u) \equiv \forall u \in V \neg P(u)$

Ex.

$$\begin{aligned} \neg(A2) &\equiv \neg \exists z \in V \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \neg \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \exists u \in V \ \neg(u + z = u \wedge z + u = u) \\ &\equiv \forall z \in V \exists u \in V \ (u + z \neq u \vee z + u \neq u) \end{aligned}$$

Proof by contradiction

You want to prove some statement P . Proof by contradiction works this way:

- (1) Assume $\neg P$
- (2) Derive a contradiction (hard part)
- (3) Conclude P is true

Ex. Outline of how to prove (A2) *does not hold* in some vector space.

You want to prove $\neg(A2)$.

$$\begin{aligned} \neg(A2) &\equiv \neg \exists z \in V \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \neg \forall u \in V \ u + z = u \wedge z + u = u \end{aligned}$$

Let $z \in V$. Prove the right-hand part ($\neg \forall u \in V \ u + z = u \wedge z + u = u$) by contradiction. Assume (for contradiction) that

$$\forall u \in V \ u + z = u \wedge z + u = u \tag{1}$$

Use (1) by substituting $u = \text{some specific vector}$ (derive a contradiction). Conclude that ($\neg \forall u \in V \ u + z = u \wedge z + u = u$) is true.

Last time

Theorem 14. If $S \subseteq W, W \leq V$ then $\text{span}(S) \subseteq W$.

Note. This means if you "promote" a subset to a subspace, adding in only what's necessary, what you get is $\text{span}(S)$. Or, $\text{span}(S)$ is the "smallest" subspace containing S .

Fact. Subspaces are "closed under taking linear combinations". Ie if $W \leq V, w_1, \dots, w_n \in W$ and $a_1, \dots, a_n \in K$ then

$$a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$$

Caution. Linear combinations are *finite* sums by definition. So you can't sum up infinitely many vectors.

Illustration of this theorem

Let $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\} \subseteq W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} | x, y \in R \right\}$. Then $\text{span}(S) \subseteq W$ ie $\text{span}(S)$ is in xy plane. In fact, $\text{span}(S) = W$.

Def. If $W = \text{span}(S)$, we say that S spans W or is a spanning set for W .

Ex. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\text{span}(S)$ = xy -plane in R^3 . So S spans the xy -plane.

Ex. 2. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\text{span}(S) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} | x \in R \right\}$ = line.

Intersection of two subspaces

Theorem 15. Let $W_1 \leq V, W_2 \leq V$. Then $W_1 \cap W_2 \leq V$ (ie intersection of two subspaces is a subspace).

Proof. $W_1 \cap W_2 = \{w \in V | w \in W_1 \wedge w \in W_2\}$.

- (1) $\vec{0} \in W_1, \vec{0} \in W_2$ (because subspace). So $\vec{0} \in W_1 \cap W_2$.
- (2) Let $u, v \in W_1 \cap W_2, c \in K$. So $u, v \in W_1$ and $W_1 \leq V$ so $cu + v \in W_1$ and $u, v \in W_2$ and $W_2 \leq V$ so $cu + v \in W_2$. Hence $cu + v \in W_1 \cap W_2$. \square

January 28th 2019

Last time: $W_1 \leq V$ and $W_2 \leq V \Rightarrow W_1 \cap W_2 \leq V$.

Corollary 15.1. *The intersection of any number of subspaces is a subspace.*

Problem. Prove that $W = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.

Sol #1: Directly from subspace properties (omit)

Sol #2: We saw an example proving that $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(3) = 0\}$ is a subspace. The "3" is not important, so similarly:

$$W_1 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0\}$$

$$W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(2) = 0\}$$

both subspaces of $F(\mathbb{R}, \mathbb{R})$. Then $W_1 \cap W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace.

Q: Is union of two subspaces also a subspace?

A: Not in general.

Eg: $W_1 = \text{x-axis} = \{\begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \mathbb{R}\} \leq \mathbb{R}^2$

$W_2 = \text{y-axis} = \{\begin{pmatrix} 0 \\ y \end{pmatrix} | y \in \mathbb{R}\} \leq \mathbb{R}^2$

$W_1 \cup W_2 = \text{xy-axis} = \{\begin{pmatrix} x \\ y \end{pmatrix} | x = 0 \vee y = 0\}$, which, importantly, is not \mathbb{R}^2 . Not a subspace, since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2$, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$.

Note: To promote $W_1 \cup W_2$ to a subspace, you form $\text{span}(W_1 \cup W_2)$.

Def: Let $W_1 \leq V$ and $W_2 \leq V$. The sum of W_1 and W_2 is

$$W_1 + W_2 = \{v \in V | \exists w_1 \in W_1, w_2 \in W_2, \text{ such that } v = w_1 + w_2\}$$

Ex:

$$W_1 = \{ax^2 | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

$$W_2 = \{ax | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

We have,

$$W_1 + W_2 = \{ax^2 + bx | a, b \in \mathbb{R}\}$$

Theorem 16. Let $W_1 \leq V$, $W_2 \leq V$. Then

(a) $W_1 + W_2 = \text{span}(W_1 \cup W_2)$ (hence $W_1 + W_2$ is a subspace)

(b) $W_1 \leq W_1 + W_2$, $W_2 \leq W_1 + W_2$

Proof:

- (a)(1) Prove $W_1 + W_2 \subseteq \text{span}(W_1 \cup W_2)$. Let $v \in W_1 + W_2$, so $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1, w_2 \in W_1 \cup W_2$ so $v \in \text{span}(W_1 \cup W_2)$

(2) " \supseteq ". Let $v \in \text{span}(W_1 \cup W_2)$. Means $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$, $u_1, u_2, \dots, u_n \in W_1 \cup W_2$ and $a_1, a_2, \dots, a_n \in K$. Each u_i is in $W_1 \cup W_2$. Separate into two groups and relabel, so that:

- Those in W_1 , call these

$$u_1, u_2, \dots, u_l$$

So $0 \leq l \leq n$, $l = 0$ means none in W_1 .

- Those in $W_2 \setminus W_1 = \{w \in W_2 | w \notin W_1\}$ ("set difference"), call these

$$u_{l+1}, \dots, u_n$$

So $l = 0$ means all in $W_2 \setminus W_1$, $l = n$ means all in W_1 .

Then, let $w_1 = a_1u_1 + a_2u_2 + \dots + a_lu_l$ (or $w_1 = \vec{0}$ if $l = 0$), $w_2 = a_{l+1}u_{l+1} + \dots + a_nu_n$ (or $w_2 = \vec{0}$ if $l = n$).

Then $w_1 \in W_1$ since W_1 is a subspace, similarly $w_2 \in W_2$. So

$$\begin{aligned} v &= a_1u_1 + \dots + a_nu_n \\ &= w_1 + w_2 \in W_1 + W_2 \text{ as required} \end{aligned}$$

(b) $W_1 \subseteq W_1 + W_2$, $W_2 \subseteq W_1 + W_2$. Follows from (a), since $S \subseteq \text{span}(S)$ \square .

Linear independence

Def: Vectors $u_1, u_2, \dots, u_n \in V$ (all distinct) are said to be *linearly dependent* if \exists scalars $a_1, a_2, \dots, a_n \in K$ not all 0 such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$$

Above equation called a *dependence relation*.

Note: If $a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$ and $a_1 \neq 0$, then you can solve for u_1 :

$$u_1 = \frac{-a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \dots - \frac{a_n}{a_1}u_n$$

ie u_1 = linear combination of others, "depends on" others.

Ex: $\{x^2 + x, 2x^2, \frac{x}{10}\}$ is a dependent set of vectors in $P(\mathbb{R})$ since

$$(x^2 + x) - \frac{1}{2}(2x^2) - 10\left(\frac{x}{10}\right) = 0$$

Def: A set of vectors $S \subseteq V$ (possibly infinite) is dependent if \exists a finite subset $\{v_1, v_2, \dots, v_n\} \subseteq S$ of it which is dependent.

Def: Vectors v_1, v_2, \dots, v_n are linearly independent if they are *not* dependent. That is,

$$\neg \exists a_1, \dots, a_n \in K \quad (a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

$$\forall a_1, \dots, a_n \in K \quad \neg(a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

$$\forall a_1, \dots, a_n \in K \quad (\neg(a_1u_1 + \dots + a_nu_n = \vec{0}) \vee (a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

Note that $P \implies Q \equiv \neg P \vee Q$. In other words, u_1, u_2, \dots, u_n are linearly independent if

$$\forall a_1, \dots, a_n \in K (a_1 u_1 + \dots + a_n u_n = \vec{0} \implies a_1 = 0 \wedge \dots \wedge a_n = 0)$$

Which is to say that the only solution to $a_1 u_1 + \dots + a_n u_n = \vec{0}$ is the trivial solution $a_1 = 0, a_2 = 0, \dots, a_n = 0$.

January 30th 2019

Last class

v_1, v_2, \dots, v_n independent if $x_1 v_1 + \dots + x_n v_n = \vec{0}$ has only trivial solution $x_1 = x_2 = \dots = x_n = 0$.

Ex: Prove that $\{1+x^2, x+x^2, 1+x+x^2\}$ is independent.

Solution: Consider equation

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0 = 0(1) + 0x + 0x^2$$

Want to show $a = b = c = 0$ is the only solution.

Equation means for all $x \in K$ (\mathbb{R} or \mathbb{C}),

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0$$

So, substitute any scalar for x :

$$\begin{aligned} x = 0 & \quad a + c = 0 \\ x = 1 & \quad 2a + 2b + 2c = 0 \\ x = -1 & \quad 2a + 0b + c = 0 \end{aligned}$$

Can translate into linear system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Row-reduce:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} & \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ & \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Only solution is $a = 0, b = 0, c = 0$ so vectors are independent.

If we obtain infinitely many, then you can find dependent set so dependent.

Some important cases

- (i) $S = \emptyset$ is linearly independent since there are no vectors with which to form a dep. relation.
- (ii) If $\vec{0} \in S$, then dependent (since $1\vec{0} = \vec{0}$ is a dep. relation)
- (iii) $\{u\}$ is independent $\iff u \neq \vec{0}$.

Note: $u + (-1)u = \vec{0}$ is not a dep. relation, since u is repeated. But, $\{u, -u\}$ is dependent since

$$u + (-u) = \vec{0}$$

is a dep. relation.

Proposition 17. Let $A, B \subseteq V$ where $A \subseteq B$.

- (i) If A is dependent, B is also dependent
- (ii) If B is independent, A is also independent (contrapositive)

Proof:

- (i) If A dep, we have a dep relation

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in A)$$

which is also a dependence relation in B since $v_1, \dots, v_n \in B$.

- (ii) This is the contrapositive of (i). \square

Note: Converse is false, B dep $\not\Rightarrow A$ dep.

Extending an independent set

Theorem 18. Let $S \subseteq V$ be linearly independent and suppose $u \notin S$. Then, $S \cup \{u\}$ independent $\iff u \notin \text{span}(S)$.

Proof:

- (i) " \rightarrow " We will prove this as the contrapositive, ie $u \in \text{span}(S) \rightarrow$ dep. Assume $u \in \text{span}(S)$. So,

$$\begin{aligned} u &= a_1v_1 + \dots + a_nv_n \quad \text{where } v_1, v_2, \dots, v_n \in S \\ \vec{0} &= (-1)u + a_1v_1 + \dots + a_nv_n \end{aligned}$$

Which is a linear combination of vectors from $S \cup \{u\}$, not all coefficients 0 since first is -1 . Also, the vectors u, v_1, v_2, \dots, v_n are all distinct, since $u \notin S$. So this is a dependence relation on $S \cup \{u\}$, so the set is dependent.

(ii) " \leftarrow " Also by contrapositive. Assume $S \cup \{u\}$ dep, want to show that $u \in \text{span}(S)$. So there is a dependence relation on $S \cup \{u\}$.

Two cases:

- **Case 1:** Dependence relation does not involve u (or, involves u but with coefficient 0), ie we have

$$a_1v_1 + \dots + a_nv_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in S)$$

But this contradicts independence of S , so case 1 does not occur.

- **Case 2:** Dependence relation involves u (with coeff not 0), so

$$au + a_1v_1 + \dots + a_nv_n = \vec{0} \quad v_1, \dots, v_n \in S$$

and $a \neq 0$. Rewrite:

$$u = \frac{-a_1}{a}v_1 - \frac{a_2}{a}v_2 - \dots - \frac{a_n}{a}v_n \quad (a \neq 0)$$

Hence $u \in \text{span}(S)$. \square

Note: Conclusion can be restated as

$$S \cup \{u\} \text{ dependent} \iff u \in \text{span}(S)$$

Basis and dimension

Fact: If W is subspace, then $\text{span}(W) = W$. (Exercise)

So every subspace is a span. But thinking of W as $\text{span}(W)$ is excessive. Would like to find the *smallest* S such that

$$\text{span}(S) = W$$

Def: Let $W \leq V$. A *basis* of W is a set $B \subseteq V$ such that

- (i) $\text{span}(B) = W$ ("enough vectors to produce W ")
- (ii) B is linearly independent ("no extra vectors in B ")

Examples:

- (i) Let $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \leftarrow (\text{row } i)$. Then,

$$\{e_1, e_2, \dots, e_n\}$$

is a basis for K^n . For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for K^3 .

More next class.

February 1st 2019

Recall: B is a basis of W if $\text{span}(B) = W$ and B is linearly independent.

Examples:

- (1) $P_n(K)$ has basis $\{1, x, x^2, \dots, x^n\}$
- (2) $P(K)$ has basis $\{1, x, x^2, x^3, \dots\}$ (infinitely many)
- (3) $M_{m \times n}(K)$ has basis $\{E^{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ where $E^{ij} = m \times n$ matrix of 0s except 1 in row i , column j . eg: $M_{2 \times 2}(\mathbb{R})$ has basis

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- (4) $W = \{\vec{0}\}$ has basis \emptyset since

- (i) $\text{span } \emptyset = \{\vec{0}\}$ (by special def)
- (ii) \emptyset is independent

Two important questions

- (1) Does W always have basis? (spoiler: yes)
- (2) How to find a basis?

Theorem 19 (Bases exist). Let V be vector space and S a finite set with $\text{span}(S) = V$. Then there is a subset $B \subseteq S$ which is a basis of V .

Proof. Algorithm to produce B .

- (1) If $V = \{\vec{0}\}$, use $B = \emptyset$.
- (2) Take one vector, $u_1 \in S (u_1 \neq \vec{0})$. Consider $\text{span}\{u_1\}$
- (3) If $\text{span}\{u_1\} = V$, done. $B = \{u_1\}$ is a basis (set of one non-zero vector is independent)
- (4) If $\text{span}\{u_1\} \neq V$, there must be a vector $u_2 \in S$ where $u_2 \notin \text{span}\{u_1\}$ (Why? If not, $S \subseteq \text{span}\{u_1\} \leq V$, then $\text{span}(S) \subseteq \text{span}\{u_1\}$, but $\text{span}(S) = V$ contradicts $V \neq \text{span}\{u_1\}$). By previous theorem, since $u_2 \notin \text{span}\{u_1\}$, $\{u_1, u_2\}$ is linearly independent.
- (5) Consider $\{u_1, u_2\}$. If $\text{span}\{u_1, u_2\} = V$, done: $B = \{u_1, u_2\}$. Else, continue as before, finding $u_3 \in S, u_3 \notin \text{span}\{u_1, u_2\}$ (etc)

Since S is finite, this must stop and at that point you have basis $B \subseteq S$. □

Illustration of this thm

Find basis of \mathbb{R}^3 that is a subset of

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Following this algorithm,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Theorem 20. Let V be a vector space, $L \subseteq V$ a linearly independent set, and $S \subseteq V$ a spanning set (ie $V = \text{span}(S)$). Then \exists a subset $E \subseteq S$ such that $L \cup E$ is a basis of V (ie you can always extend it to a basis)

Proof Omitted.

Theorem 21. Suppose V has a finite spanning set S . Then V has a basis and all bases have the same size, which is at most $|S|$.

Proof Omitted.

Def If V has a finite basis B , then the *dimension* of V is

$$\dim V = |B|$$

If V does not have a finite basis, it is called *infinite dimensional*.

Ex:

(1) $\dim K^n = n$.

$$\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right\} \right)$$

(2) $\dim P_n(K) = n + 1$ (basis $\{1, x, x^2, \dots, x^n\}$)

(3) $P(K)$ is infinite dimensional (A#1, proved a finite set of polynomials cannot span $P(K)$)

(4) $\dim M_{m \times n}(K) = mn$ (see basis E^{ij} , defined above)

Theorem 22. Every vector space (including the infinite dimensional ones) has a basis.

Proof Uses Axiom of Choice. Difficult.

Theorem 23. Suppose $\dim V = n$. Let $A \subseteq V$. Then,

- (1) If $\text{span}(A) = V$, then $|A| \geq n$ (or, if $|A| < n$ then A does not span V) and if also $|A| = n$ then A is linearly independent, hence basis.
- (2) If A is linearly independent, then $|A| \leq n$ (or, if $|A| > n$ then A dep) and if also $|A| = n$ then $\text{span}(A) = V$ hence A is a basis.

Proof Omitted.

Note: If you have *correct number* of vectors, you need only check spanning or independent, not both, to check if basis.

Ex: If you have 7 matrices in $M_{3 \times 2}(K)$, they *will be* dependent. If you have 5, it's *not* a basis.

February 4th 2019

Last class

Suppose $\dim V = n$, $S \subseteq V$, $|S| = n$. Then S spans $V \iff S$ linearly independent (only in case $|S| = \dim V$).

Lagrange Interpolation

Problem Given "data points" $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ where all a_i are different. Find a polynomial $p(x)$ of degree $n-1$, $p(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1x + c_0$ whose graph $y = p(x)$ passes through all the points.

Sol #1 Substitute (a_1, b_1) into $y = p(x)$:

$$b_1 = c_{n-1}a_1^{n-1} + \dots + c_1a_1 + c_0 \quad (\text{for each } i = 1, \dots, n)$$

Which is a system of n linear equations ($\text{vars} = c_{n-1}, \dots, c_0$) in n variables.

We'll do something different.

Def For scalars a_1, a_2, \dots, a_n (all different), define the *Lagrange polynomials* for each $i = 1, 2, \dots, n$ set

$$\begin{aligned} l_i(x) &= \prod_{k=1, k \neq i}^n \frac{(x - a_k)}{(a_i - a_k)} \\ &= \frac{(x - a_1)}{(a_i - a_1)} \cdot \frac{(x - a_2)}{(a_i - a_2)} \cdot \dots \cdot \frac{(x - a_n)}{(a_i - a_n)} \quad (\text{omitting } \frac{(x - a_i)}{(a_i - a_i)}) \end{aligned}$$

Ex For $a_1 = 2, a_2 = 4, a_3 = 6$ we would have

$$l_1(x) = \frac{(x-4)}{2-4} \cdot \frac{(x-6)}{(2-6)}$$

$$l_2(x) = \frac{(x-2)}{4-2} \cdot \frac{(x-6)}{(4-6)}$$

$$l_3(x) = \frac{(x-2)}{6-2} \cdot \frac{(x-4)}{(6-4)}$$

Note: All degree 2, $l_1(4) = 0, l_1(6) = 0, l_1(2) = 1$.

Fact $l_i(a_j) = 0$ if $i \neq j$ and 1 if $i = j$.

Proof If $i \neq j$, there is a factor $\frac{x-a_j}{a_i-a_j}$, so at $x = a_j, \frac{a_j-a_j}{a_i-a_j} = 0$. If $i = j$,

$$l_i(a_i) = \prod_{k=1, k \neq i}^n \frac{(a_i - a_k)}{(a_i - a_i)} = 1$$

Proposition 24. Lagrange polynomials $l_1(x), \dots, l_n(x)$ form a basis of $P_{n-1}(\mathbb{R})$.

Proof We have n polynomials (they are distinct), $\dim P_{n-1}(\mathbb{R}) = n - 1 + 1 = n$. So correct number. Suffices to prove span or lin independence. We'll prove independence. Suppose

$$d_1 l_1(x) + d_2 l_2(x) + \dots + d_n l_n(x) = 0 \quad (\text{note: for all } x \in \mathbb{R})$$

Substitute $x = a_1, x = a_2$, etc into the above. At $x = a_1, l_1(a_1) = 1$ but $l_j(a_1) = 0$ for $j \neq 1$ so

$$d_1 1 + d_2 0 + \dots + d_n 0 = 0$$

so $d_1 = 0$. Similarly, $d_j = 0$ for all j . More formally, for any $j = 1, 2, \dots, n$ we have at $x = a_j$

$$\sum_{i=1}^n d_i l_i(a_j) = 0$$

but all terms are 0 except when $i = j$. Set

$$d_j = d_j(1) = d_j l_j(a_j) = 0$$

Hence Lagrange polynomials for a basis.

Problem Find poly degree $n - 1$ through points $(a_1, b_1), \dots, (a_n, b_n)$.

Sol: Set $p(x) = b_1 l_1(x) + b_2 l_2(x) + \dots + b_n l_n(x)$ (it has degree $n - 1$).

Then

$$\begin{aligned} p(a_1) &= b_1 l_1(a_1) + b_2 l_2(a_1) + \dots + b_n l_n(a_1) \\ &= b_1(1) + 0 + 0 + \dots + 0 \\ &= b_1 \end{aligned}$$

For each $i = 1, 2, \dots, n$,

$$\begin{aligned} p(a_i) &= \sum_{j=1}^n b_j l_j(a_i) \\ &= 0 + 0 + \dots + b_i l_i(a_i) + \dots + 0 \\ &= b_i \end{aligned}$$

Dimension of subspaces

Theorem 20. Let $W \leq V$, V finite-dimensional. Then

- (i) $\dim W \leq \dim V$
- (ii) $\dim W = \dim V \iff W = V$

Proof

- (i) Similar to proof that V has basis. Use W as a spanning set for W . Pick out vectors one at a time (similar to before) to build a basis. You cannot put more than $\dim V$ vectors into your basis, as this would give an independent set in V of size *more than* $\dim V$ (impossible). So this process has to stop, and it produces a basis for W .
- (ii) " \rightarrow " Assume $\dim W = \dim V = n$. Take basis B of W . It is a size n linearly independent set inside V , hence B also basis for V , hence,

$$V = \text{span } B = W$$

" \leftarrow " If $W = V$, clearly $\dim W = \dim V$. \square

Subspaces of \mathbb{R}^3 If $W \leq \mathbb{R}^3$, $\dim W = 0, 1, 2$ or 3 .

This allows us to make the following classification: **Problem** Let

$\dim W$	Classification
0	$\{\vec{0}\}$
1	$\text{span}\{u\}$ = line through origin
2	$\text{span}\{u, v\}$ = plane through origin
3	\mathbb{R}^3

$W = \{A \in M_{n \times n}(\mathbb{R}) | \text{tr}(A) = 0\}$, where $\text{tr}(A)$ = trace of A = sum of entries on diagonal = $A_{11} + A_{22} + \dots + A_{nn}$.

Exercise Prove W is a subspace.

Will do next class: Find $\dim W$ and find a basis of W .

February 6th 2019

Intuition

Solution set W to a homogeneous system $A\vec{x} = \vec{0}$ is a subspace of K^n ($n = \#$ of variables). If no equations, $W = K^n$, $\dim W = n$. For each equation, expect the dimension of W to drop by 1, unless the equation is *redundant*.

Eg: In \mathbb{R}^3 , one equation

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 && (= \text{plane}) \\ \text{add in } a_2x + b_2y + c_2z &= 0 && (\text{intersection of two planes, } = \text{line}) \\ \text{add in } a_3x + b_3y + c_3z &= 0 && (\text{intersection of three planes, } (0,0)) \end{aligned}$$

Problem: $W = \{A \in M_{n \times n}(\mathbb{R}) \mid \text{tr } A = 0\}$. Find $\dim W$, basis of W .

Solution #1: Clever way: "guess" a basis. Note: $\text{tr } A = A_{11} + A_{22} + \dots + A_{nn}$ (one linear condition). Expecting

$$\dim W = n^2 - 1$$

Observe that $\dim W \neq n^2$. This happens only if $W = M_{n \times n}(\mathbb{R})$, and obviously there are matrices which don't have trace 0. Specifically:

$$\text{tr} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \dots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

(In proofs, can choose any example, provided property holds).

Know $\dim W \leq n^2 - 1$. If you can find independent set of size $n^2 - 1$ in W_1 , it will be a basis. Try first $n = 3$. Looking for $3^2 - 1 = 8$ independent 3×3 matrices, all trace 0.

Want trace = 0. Therefore, consider all matrices which have all 0's in the diagonal:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

These 6 are obviously independent. Now, take two more which are independent:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we have 8 independent matrices (check carefully). So for $n = 3$, $\dim W = 8$, this is a basis.

General case

Two types of basis matrices:

- (I) All E^{ij} (1 in (i,j) -pos, 0 elsewhere)) where $i \neq j$. How many are there?

$$\begin{aligned}\# \text{ of non-diagonal entries} &= \text{entries} - \text{entries on diagonal} \\ &= n^2 - n\end{aligned}$$

Or, $\binom{n}{2}$ ways to choose 2 distinct values from $\{1, 2, \dots, n\}$, 2 ways to order each pair. Total:

$$\begin{aligned}\binom{n}{2}2 &= \frac{n!}{2!(n-2)!}2 \\ &= n(n-1) \\ &= n^2 - n\end{aligned}$$

- (II) Looking for $n-1$ more, since $n^2 - n + n - 1 = n^2 - 1$

$$\begin{pmatrix} 1 & -1 & & & \\ & \dots & 0 & & \\ & & & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 & & \\ & \dots & 0 & & \\ & & & 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 & & & \\ & \dots & 1 & & \\ & & & -1 & \end{pmatrix}, \dots$$

(n-1 of those)

Formally, let, for $i = 1, 2, \dots, n-1$, D_i = matrix with 1 in pos (i, i) and -1 in pos $(i+1, i+1)$, 0 elsewhere.

Verifying all matrices E^{ii} , D_i are independent; clear that suffices to check D_1, D_2, \dots, D_{n-1} independent. Suppose

$$x_1 D_1 + x_2 D_2 + \dots + x_n D_n = n \times n \text{ zero matrix}$$

The $(1, 1)$ -entry on left is x_1 , so $x_1 = 0$. The $(2, 2)$ -entry on left is $-x_1 + x_2$,

$$x_1 \begin{pmatrix} 1 & -1 & & & \\ & \dots & 0 & & \\ & & & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & -1 & & \\ & \dots & 0 & & \\ & & & 0 & \end{pmatrix} + \dots = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

but $x_1 = 0$ so $x_2 = 0$ also, etc. So similarly for all $x_i = 0$, so independent. Formally you'd do a proof by induction, but this is good enough.

Now have $n^2 - 1$ independent vectors in W_1 so $\dim W \geq n^2 - 1$. Already know $\dim W \leq n^2 - 1$. So $\dim W = n^2 - 1$, have independent set of correct size, so basis.

Solution #2: Let x_{ij} be the (i, j) -entry of A . So have n^2 variables $(x_{ij}, i, j = 1, 2, \dots, n)$ one equation,

$$x_{11} + x_{22} + \dots + x_{nn} = 0 \quad (\text{tr } A = 0)$$

Solve system. All $x_{ij}, i \neq j$ free variables, so are x_{22}, \dots, x_{nn} .

Theorem 21. Let U, W be finite dimension subspaces of V . Then,

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W$$

It's like sets, $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof Omitted.

Ex: If W is a plane in \mathbb{R}^3 (through $(0, 0)$) and L is a line in \mathbb{R} (through $(0, 0)$) and L is not in the plane, prove $W + L = \mathbb{R}^3$.

Sol: L not in plane gives $L \cap W = \{\vec{0}\}$. So

$$\begin{aligned}\dim(L + W) &= \dim L + \dim W - \dim L \cap W \\ &= 1 + 2 - 0 \\ &= 3\end{aligned}$$

Hence $L + W = \mathbb{R}^3$.

Problem: Suppose $\dim V = n$, and U, W subspaces, each of dimension more than $\frac{n}{2}$. Prove that $U \cap W \neq \{\vec{0}\}$.

Proof By contradiction. Suppose $U \cap W = \{\vec{0}\}$. So $\dim U \cap W = 0$.

Then

$$\begin{aligned}\dim(U + W) &= \dim U + \dim W - \dim U \cap W \\ &> \frac{n}{2} + \frac{n}{2} - 0 = n\end{aligned}$$

Says $U + W$ is a subspace of V of dim more than $\dim V$. Impossible, so $U \cap W \neq \{\vec{0}\}$.

END OF MIDTERM MATERIAL.

February 8th 2019

Monday: No class, office hours during class time. Tuesday night : Midterm!

Linear transformations - Definition and basic properties

(Chap. 5 in the text) **Def.** Let U, V be vector spaces, both over field K . A function $T : U \rightarrow V$ is called a *linear transformation* if

- (i) $\forall u_1, u_2 \in U \quad T(u_1 + u_2) = T(u_1) + T(u_2)$. The first '+' is in U , while the second '+' is in V . The vectors spaces need not be related in any way, except that they must be over the same field of scalars.
- (ii) $\forall u \in U, c \in K \quad T(cu) = cT(u)$. Again, the first scalar multiplication happens in U , while the second scalar multiplication happens in V .

Comment: Linear transformations are the functions that are somehow "compatible" with the vector space operations.

Ex: Prove that $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ defined by

$$T(ax^2 + bx + c) = \begin{pmatrix} a+b \\ b+c \end{pmatrix}$$

Sol:

(i) Let $p_1(x) = a_1x^2 + b_1x + c_1$, $p_2(x) = a_2x^2 + b_2x + c_2$ be in $P_2(x)$.

Then,

$$\begin{aligned} T(p_1(x) + p_2(x)) &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2) \\ &= \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ b_1 + b_2 + c_1 + c_2 \end{pmatrix} \\ T(p_1(x)) + T(p_2(x)) &= \begin{pmatrix} a_1 + b_1 \\ b_1 + c_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ b_2 + c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 + a_2 + b_2 \\ b_1 + c_1 + b_2 + c_2 \end{pmatrix} \end{aligned}$$

(ii) Let $p(x) = ax^2 + bx + c \in P_2(\mathbb{R}), d \in K$.

$$\begin{aligned} T(dp(x)) &= T(dax^2 + dbx + dc) \\ &= \begin{pmatrix} da + db \\ db + dc \end{pmatrix} \\ &= d \begin{pmatrix} a + b \\ b + c \end{pmatrix} \\ &= dT(ax^2 + bx + c) \\ &= dT(p(x)) \end{aligned}$$

So T is a linear transformation.

Ex Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (x^2, x + y)$. Show that T is not a linear transformation.

Sol Try $u = (2, 3), v = (3, 4)$.

$$\begin{aligned} T(u + v) &= T(5, 7) \\ &= (25, 12) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(2, 3) + T(3, 4) \\ &= (4, 5) + (9, 7) \\ &= (13, 12) \\ &\neq (25, 12) \end{aligned}$$

So T is *not* linear.

Ex: Define $\frac{d}{dx} : P(\mathbb{R}) \rightarrow P(\mathbb{R})$ by

$$\frac{d}{dx} p(x) = p'(x) \quad (\text{derivative})$$

Then $\frac{d}{dx}$ is a linear transformation, since we know from calculus that

$$\begin{aligned} \frac{d}{dx}(p(x) + q(x)) &= \frac{d}{dx}p(x) + \frac{d}{dx}q(x) \\ \frac{d}{dx}(cp(x)) &= c\frac{d}{dx}p(x) \quad (c \in \mathbb{R}) \end{aligned}$$

Proposition 22. Let $T : U \rightarrow V$ be a linear transformation. Then,

(i) $T(\vec{0}) = \vec{0}$ (where the first $\vec{0}$ is the zero vector of U and the second is the zero vector of V)

(ii) $\forall u_1, u_2, \dots, u_n \in U$ and $c_1, c_2, \dots, c_n \in K$,

$$\begin{aligned} T(c_1u_1 + c_2u_2 + \dots + c_nu_n) &= \\ c_1T(u_1) + c_2T(u_2) + \dots + c_nT(u_n) \end{aligned}$$

Proof. (i)

$$\begin{aligned} T(\vec{0}_U) &= T(\vec{0}_U + \vec{0}_U) \\ T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{T linear}) \\ \vec{0}_V + T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{A2}) \\ \vec{0}_V &= T(\vec{0}_V) \quad (\text{cancellation law}) \end{aligned}$$

(ii)

$$\begin{aligned} T(c_1u_1 + (c_2u_2 + \dots + c_nu_n)) &= T(c_1u_1) + T(c_2u_2 + \dots + c_nu_n) \\ &\quad (\text{T linear}) \\ &= c_1T(u_1) + T(c_2u_2 + \dots + c_nu_n) \\ &\quad (\text{T linear}) \\ &= \dots \quad (\text{proof by induction}) \\ &= c_1T(u_1) + \dots + c_nT(u_n) \end{aligned}$$

□

Proposition 23. Let $T : U \rightarrow V$ function (U, V vector spaces). Then,

T is linear transformation \iff

$$\forall u_1, u_2 \in U \text{ } c \in K, T(cu_1 + u_2) = cT(u_1) + T(u_2)$$

Proof: Exercise. □

February 15th 2019

Def ("matrix defines a linear transformation") Let $A \in M_{m \times n}(K)$.

Define a function $L_A : K^n \rightarrow K^m$ by

$$L_A(v) = Av \quad (\text{A an } m \times n \text{ matrix, v } n \times 1)$$

i.e. multiply matrix by vector.

Proposition 24. L_A is a linear transformation.

Proof. Let $u, v \in K^n, c \in K$. Then

$$\begin{aligned} L_A(cu + v) &= A(cu + v) \\ &= A(cu) + Av \quad (\text{prop of matrix multiplication}) \\ &= cAu + Av \\ &= cL_A(u) + L_A(v) \end{aligned}$$

□

Ex $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix}, L_A : R^3 \rightarrow R^2$. Calculate:

$$\begin{aligned} L_A(1, 3, -2) &= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 - 3 - 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -5 \end{pmatrix} \end{aligned}$$

Spoiler: All linear transformations between finite-dim vector spaces can be described in this way, "matrix transformation".

Two special linear transformations

- (1) **Zero transformations:** $0 : V \rightarrow W$ defined by $0(v) = \vec{0}$ ($\vec{0}$ of W) for all $v \in V$.
- (2) **Identity transformation,** $I : V \rightarrow V$ (same vector space) $I(v) = v$ for all $v \in V$

Both are linear transformations (exercise).

Kernel and Image (ch. 5.4)

Def Let $T : V \rightarrow W$ be a linear transformation. Define:

- (i) **Kernel or nullspace** of T ,

$$\text{Ker}(T) = \{v \in V | T(v) = \vec{0}\}$$

Note: Always one vector which satisfies this.

- (ii) **Image** of T is

$$\text{Im}(T) = \{w \in W | \exists v \in V \ w = T(v)\}$$

Note: $\text{Ker}(T) \subseteq V$, $\text{Im}(T) \subseteq W$.

Ex Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(x, y) = (x, 0) \quad (\text{"proj onto x-axis"})$$

Then

$$\begin{aligned} \text{Ker}(T) &= \{(x, y) \in \mathbb{R}^2 | T(x, y) = (0, 0)\} \\ &= \{(0, y) | y \in \mathbb{R}\} \\ &= \text{"y-axis"} \\ \text{Im}(T) &= \{(x, y) \in \mathbb{R}^2 | (x, y) = T(x', y') \text{ some } x', y' \in \mathbb{R}\} \\ &= \{(x, 0) | x \in \mathbb{R}\} \\ &= \text{"x-axis"} \end{aligned}$$

Ex Define $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ to be derivative, $D(f(x)) = f'(x)$. Find kernel and image of D .

Sol We have

$$\begin{aligned} \text{Ker}(D) &= \{f \in P_n(\mathbb{R}) | f'(x) = 0\} \\ &= \text{const. polys} \\ &= \{a | a \in \mathbb{R}\} \\ &= P_0(\mathbb{R}) \end{aligned}$$

Claim $\text{Im}(D) = P_{n-1}(\mathbb{R})$.

Proof. Prove inclusion " \subseteq " and " \supseteq ".

- (i) " \subseteq " Let $f(x) \in \text{Im}(D)$. Then $\exists g(x) \in P_n$ s.t. $f(x) = D(g(x)) = g'(x)$. Since $\deg(g) \leq n$, $\deg(f) = \deg(g') \leq n-1$ (property of differentiation). So $f(x) \in P_{n-1}$.
- (ii) " \supseteq " Let $f(x) \in P_{n-1}$. Need to find $g(x) \in P_n$ such that $D(g(x)) = g'(x) = f(x)$. Set $g(x) = \int f(x) dx$. Know from calculus that the degree of g is one higher, ie

$$\deg(g(x)) = 1 + \deg(f(x))$$

So $\deg(g) \leq n$. So $g(x) \in P_n$ and $g'(x) = f(x)$ (calculus).

□

Theorem 25. Let $T : V \rightarrow W$ be linear transformation. Then,

$$(i) \ Ker(T) \leq V$$

$$(ii) \ Im(T) \leq W$$

Ie they are subspaces.

Proof. By direct proof.

(i) $T(\vec{0}) = \vec{0}$ always (lin transform) so $\vec{0} \in Ker(T)$. Let $v_1, v_2 \in Ker(T), c \in K$. We know $T(v_1) = \vec{0}, T(v_2) = \vec{0}$. Then

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) && (\text{T linear}) \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

Hence $cv_1 + v_2 \in Ker(T)$. So $Ker(T) \subseteq V$ (we already knew $Ker(T) \subseteq V$)

(ii) $T(\vec{0}) = \vec{0}$, hence $\vec{0}_w = T(\text{something})$, ie $\vec{0}_w \in Im(T)$. Let $w_1, w_2 \in Im(T), c \in K$. We know $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Then

$$\begin{aligned} cw_1 + w_2 &= cT(v_1) + T(v_2) \\ &= T(cv_1 + v_2) && (\text{T linear}) \end{aligned}$$

Hence $cw_1 + w_2 \in Im(T)$. So $Im(T) \leq W$.

□

Def $T : V \rightarrow W$ linear. The *nullity* of T is $\dim Ker(T)$ (\dim nullspace). The *rank* of T is $\dim Im(T)$.

Note: $Ker(T) \leq V$ so $\text{nullity}(T) \leq \dim V$, $Im(T) \leq W$ so $\text{rank}(T) \leq \dim W$.

Ex In $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, proj onto x-axis,

$$\begin{aligned} Ker(T) &= y-axis && (\text{so } \text{nullity}(T) = 1) \\ Im(T) &= x-axis && (\text{so } \text{rank}(T) = 1) \end{aligned}$$

Ex 2 For $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$, differentiation.

$$\begin{aligned} Ker D &= P_0(\mathbb{R}) && (\text{so } \text{nullity}(D) = 1) \\ Im D &= P_{n-1} && (\text{so } \text{rank}(D) = n) \end{aligned}$$

February 18th 2019

Notation For set $S = \{v_1, v_2, \dots, v_n\}$, $T : V \rightarrow W$ denotes $T(S) = \{T(v_1), T(v_2), \dots, T(v_n)\}$.

Proposition 26. $T : V \rightarrow W$ linear and $V = \text{span}(S)$. Then $\text{Im } T = \text{span}(T(S))$. In particular, if B basis of V , $T(B)$ spans $\text{Im } (T)$ (but need not be a basis).

Proof. By direct proof.

(i) " \subseteq ". Let $w \in \text{Im}(T)$, ie $w = T(v)$, some $v \in V$. Since S spans V , $v = \sum_{i=1}^n a_i v_i$, some $v_i \in S$. So

$$\begin{aligned} w &= T(v) = T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \quad (T(v_i) \in T(S), \text{ by T linear}) \end{aligned}$$

All of which is $\in \text{span}(T(S))$.

(ii) " \supseteq " Let $w \in \text{span } T(S)$. So

$$\begin{aligned} w &= \sum_{i=1}^n a_i T(v_i) \quad (\text{for some vectors } v_i \in S) \\ &= T\left(\sum_{i=1}^n a_i v_i\right) \quad (\text{T linear}) \\ &= T(\text{something}) \quad (\text{so } w \in \text{Im}(T)) \end{aligned}$$

□

Ex Define $T : P_2(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Exercise: T is linear. Find basiss for $\text{Im } T$.

Sol Take basis $\{1, x, x^2\}$ for P_2 . Calculate

$$\begin{aligned} T(1) &= \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ T(x) &= \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \\ T(x^2) &= \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So $\text{Im } T = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

Basis for $\text{Im } T$ is $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

(so $\text{Im } T = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\right\}$)

Note: The next theorem is very important!

Theorem 27. ("Dimension theorem") Let $T : V \rightarrow W$ linear with V finite-dimensional. Then,

$$\dim V = \dim \ker(T) + \dim \text{Im}(T)$$

$$\dim V = \text{nullity}(T) + \text{rank}(T)$$

Note $\dim W$ is not involved.

Proof. Let $B = \{v_1, v_2, \dots, v_k\}$ be basis $\text{Ker } T$ (so $k = \dim \text{Ker } T$). Let $n = \dim V$. Note $T(v_i) = 0$, ($i = 1, 2, \dots, k$). Let S span V .

Plan: extend B to basis of V , show $T(\text{extra vector}) = \text{basis of Im}$.

By theorem 20-1, there exists $E \subseteq S$ such that $B \cup E$ is a basis of V .

Denote

$$E = \{v_{k+1}, \dots, v_n\} \quad (\text{note } n = \dim V, |E| = n - k)$$

Claim $T(E)$ is basis for $\text{Im } T$.

(i) $T(E)$ spans $\text{Im } T$

(a) " \subseteq " is clear since $T(E) \subseteq \text{Im } T$ by definition. So $\text{span } T(E) \leq \text{Im } T$ ($\text{Im } T \leq W$)

(b) " \supseteq " Let $w \in \text{Im}(T)$, ie $w = T(v)$, some $v \in V$. Since $B \cup E$ is a basis, $v = \sum_{i=1}^n a_i v_i$. Then,

$$\begin{aligned} w &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \quad (\text{T linear}) \\ &= \sum_{i=k+1}^n a_i v_i \quad (\text{Since } T(v_i) = 0 \text{ for } i = 1, 2, \dots, k) \end{aligned}$$

Hence $w \in \text{span}(T(E))$, since $E = \{v_{k+1}, \dots, v_n\}$

(ii) $T(E)$ is linearly independent. Suppose

$$\sum_{i=k+1}^n b_i T(v_i) = \vec{0} \quad (\text{linear comb vectors in } T(E))$$

So by linearity of T ,

$$T\left(\sum_{i=k+1}^n b_i v_i\right) = \vec{0}$$

So $\sum_{i=k+1}^n b_i v_i \in \text{Ker } T$, ie is linear comb of B

$$\text{So } \sum_{i=k+1}^n b_i v_i = \sum_{i=1}^k b_i v_i$$

ie $\sum_{i=1}^k (-b_i)v_i + \sum_{i=k+1}^n b_i v_i = \vec{0}$ is linear comb of v_1, \dots, v_n (ie $B \cup E$) but these independent. So all $b_i = 0$, hence $T(E)$ independent.

Conclude $T(E)$ basis of $\text{Im } T$. So,

$$\dim \text{Im } T = |T(E)| = |E| = n - k$$

So,

$$n = k + n - k$$

$$\dim V = |B| + |T(E)| = \dim \text{Ker } T + \dim \text{Im } T$$

□

Why is $|T(E)| = |E|$? True unless

$$T(v_i) = T(v_j) \quad (\text{for some } i, j \geq k+1, i \neq j)$$

If so,

$$\begin{aligned} T(v_i) - T(v_j) &= 0 \\ T(v_i - v_j) &= 0 \quad (\text{so } v_i - v_j \in \text{Ker } T) \end{aligned}$$

Hence $v_i - v_j = \sum_{l=1}^n a_l v_l$, dep relation on v_1, \dots, v_n . Impossible. □

Problem For $T : P_2 \rightarrow \mathcal{M}_{2 \times 2}$,

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find basis for $\text{Ker } T$.

Sol Already know $\dim \text{Im } T = 2$ (last ex). So

$$\begin{aligned} \dim P_2 &= \dim \text{Ker } T + \dim \text{Im } T \\ 3 &= \dim \text{Ker } T + 2 \end{aligned}$$

So $\text{Ker } T$ is 1-dimensional. Only need to find one non-zero $f(x)$ s.t.

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ie need $f(1) = f(2)$ and $f(0) = 0$. For example, $f(x) = x^2 - 3x$ works. So $\{x^2 - 3x\}$ is a basis for $\text{Ker } T$ (or, $f(x) = ax^2 + bx + c$, $f(1) = a + b + c = f(2) = 4a + 2b + c$, $f(0) = 0 = c$, solve)

February 20th 2019

Comments on dimension theorem

$T : V \rightarrow W$, linear.

$$\dim V = \dim (\text{Im } T) + \dim (\text{Ker } T)$$

Left-hand part of the sum: Dimensions that are preserved ("saved") by T . Right-hand part: dimensions that are "lost" when you apply T .

Dimension: Subspaces are *infinite* sets (except $\{\vec{0}\}$). Dimension gives a way to compare the *sizes* of subspaces.

Injective/surjective transformation (ch. 5.5.)

Def Let $f : X \rightarrow Y$ be a *function* (X, Y sets).

(i) f is *surjective* ("onto") if

$$\forall y \in Y \ \exists x \in X \ f(x) = y$$

(equivalently, the image of f is Y)

(ii) f is called *injective* (or "on-to-one") if

$$\forall x_1, x_2 \in X (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

(equivalently, $\forall x_1, x_2 \in X (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$)

Theorem 28. ("How to check if T inj/surj") Let $T : V \rightarrow W$. Then,

(i) T injective $\iff \text{Ker}(T) = \{\vec{0}\}$ (nullity $(T) = 0$)

(ii) T surjective $\iff \dim (\text{Im } T) = \dim W$ (rank(T) = dim W)

(i) *Proof.* By direct proof.

- (1) " \Rightarrow " Assume T inj. (know $\{0\} \leq \text{Ker } T$). Let $v \in \text{Ker } (T)$. So $T(v) = \vec{0}$. But also $T(\vec{0}) = \vec{0}$, so $T(v) = T(\vec{0})$ hence $v = \vec{0}$ since T is injective.
- (2) " \Leftarrow " Assume $\text{Ker } T = \{\vec{0}\}$. Let $v_1, v_2 \in V$. Suppose $T(v_1) = T(v_2)$ (prove $v_1 = v_2$).

$$\begin{aligned} T(v_1) - T(v_2) &= \vec{0} \\ T(v_1 - v_2) &= \vec{0} \end{aligned} \quad (\text{linear})$$

So $v_1 - v_2 \in \text{Ker } T = \{\vec{0}\}$. So $v_1 - v_2 = \vec{0}, v_1 = v_2$.

□

(ii) *Proof.* By direct proof.

- (1) " \Rightarrow " Assume T is surjective, that is $\text{Im } T = W$. Hence $\dim \text{Im } T = \dim W$.
- (2) " \Leftarrow " Assume $\dim \text{Im } T = \dim W$. But $\text{Im } T \leq W$, hence $\text{Im } T = W$ (by thm 2o-2)

□

Problem Define $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$T(f(x)) = \int_0^1 f(x)dx$$

(Exercise: T is linear). Is T injective? Surjective?

Sol Dim Thm:

$$\begin{aligned} \dim P_2 &= \dim \text{Im } T + \dim \text{Ker } T \\ 3 &= \dim \text{Im } T + \dim \text{Ker } T \end{aligned}$$

Hence $\text{Im } T \leq \mathbb{R}^1$, so $\text{Im } T = \{\vec{0}\}$ or \mathbb{R} . It is *not* $\{\vec{0}\}$ since $\int_0^1 1dx = 1 \neq 0$, $T(1) \neq 0$. Hence $\text{Im } T = \mathbb{R}$ so

$$3 = 1 + \dim \text{Ker } T$$

So $\dim \text{Ker } T = 2$. $\text{Ker } T \neq \{\vec{0}\}$ not injective. $\text{Im } T = \mathbb{R}$ is surjective.

Theorem 29. ("shortcut when dim same") $T : V \rightarrow W$ linear, and $\dim V = \dim W$. Then,

$$T \text{ injective} \iff T \text{ surjective}$$

Proof. Dim Thm:

$$\dim W = \dim V = \dim \text{Im } T + \dim \text{Ker } T$$

If T inj, $\dim \text{Ker } T = 0$. So

$$\dim W = \dim \text{Im } T + 0$$

So T surjective (thm 28). If T surj, $\dim \text{Im } T = \dim W$ (thm 28), so

$$\dim W = \dim W + \dim \text{Ker } T$$

So $\dim \text{Ker } T = 0$ so $\text{Ker } T = \{\vec{0}\}$

□

Problem $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$, defined by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$$

Is T injective? Surjective?

Sol Same $\dim (= 3)$. Check only one. Check surjective directly from def surj:

Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Is $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T(f(x))$, some $f(x) \in P_2$?

That is, given $a, b, c \in \mathbb{R}$, is there a degree 2 polynomial such that $f(0) = a, f(1) = b, f(2) = c$? By Lagrange Interpolation, $f(x)$ exists ($\deg = 1$, less than # of points). So T surj, so also inj.

Isomorphism and coordinates (ch 5.5, 4.11 and 4.12)

Def: (Isomorphism)

- (1) If $T : V \rightarrow W$ (linear) is injective and surjective, it is called an *isomorphism*.
- (2) If V, W vector spaces and *there exists* an isomorphism $T : V \rightarrow W$, we say V and W are *isomorphic* and write $V \simeq W$

Note A function that is injective and surjective is called *bijective*.

Ex $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$, $T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$ is an isomorphism (last ex.)

so $P_2(\mathbb{R}) \simeq \mathbb{R}^3$

Ex Prove that

$$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is isomorphism $P_2 \rightarrow \mathbb{R}^3$.

Sol T is linear : let $f(x), g(x) \in P_2(\mathbb{R}), d \in \mathbb{R}$. Then,

$$\begin{aligned} T(df + g) &= T(c(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) \\ &= T((da_1 + a_2)x^2 + (db_1 + b_2) + (dc_1 + c_2)) \\ &= \begin{pmatrix} da_1 + a_2 \\ db_1 + b_2 \\ dc_1 + c_2 \end{pmatrix} \\ &= d \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \\ &= dT(f) + T(g) \end{aligned}$$

So T linear. Same $\dim (= 3)$. Check surj. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Then

$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, hence surj., hence inj., hence *isomorphism*.

February 22nd 2019

Notes about functions

- (1) If $f : X \rightarrow Y$, then f injective and surjective $\iff f$ is invertible,
ie $\exists f^{-1} : Y \rightarrow X$ such that $\forall x \in X, y \in Y$ $f^{-1}(f(x)) = x$ and
 $f(f^{-1}(y)) = y$
- (2) If $g : Y \rightarrow Z$, you can compose f and g to get $g \cdot f : X \rightarrow Z$, defined
by $(g \cdot f)(x) = g(f(x))$ $x \xrightarrow{f} y \xrightarrow{g} z$

Theorem 30. Let $T : V \rightarrow W$ be an isomorphism (ie T linear, inj, surj.).

Then T has an inverse $T^{-1} : W \rightarrow V$ which is also a linear transformation.

Proof. Fact that T^{-1} exists is since T inj and surj. Prove T^{-1} is linear.

Let $w_1, w_2 \in W, c \in K$. Since T surjective, $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Also, $T^{-1}(w_1) = T^{-1}(T(v_1)) = v_1$ and $T^{-1}(w_2) = v_2$. Then

$$\begin{aligned} T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\ &= T^{-1}(T(cv_1 + v_2)) \quad (\text{T linear}) \\ &= cv_1 + v_2 \\ &= cT^{-1}(w_1) + T^{-1}(w_2) \end{aligned}$$

So T^{-1} linear. \square

Ex

$$\begin{aligned} T : P_2(\mathbb{R}) &\rightarrow \mathbb{R}^3, T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ T^{-1} : \mathbb{R}^3 &\rightarrow P_2(\mathbb{R}), T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (ax^2 + bx + c) \end{aligned}$$

Point Once you know $V \simeq W$ (isomorphic) you can go back and forth between them, do vector space operations in either V or W . That is, V and W have exactly the same *structure* (as far as addition and scalar multiplication are concerned), even though "vectors" look different.

Proposition 31. If $V \simeq W$, both finite-dimensional, then $\dim V = \dim W$

Proof. $V \simeq W$ so $\exists T : V \rightarrow W$, T inj and surj (bijective), linear. So Dim Thm,

$$\dim V = \dim \text{Im } T + \dim \text{Ker } T$$

and T inj., so $\dim \text{Ker } T = 0$, and T surj., so $\text{Im } T = W$, so

$$\dim V = \dim W + 0$$

□

Theorem 32. Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V . For any $v \in V$, you can write

$$v = \sum_{i=1}^n a_i v_i$$

Then,

- (a) The numbers (a_1, a_2, \dots, a_n) are unique and are called the coordinates of v relative to B , denoted

$$[v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- (b) The function $C_B : V \rightarrow K^n$ defined by

$$C_B(v) = [v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad (\text{"find coordinates"})$$

is an isomorphism

Hence, if $\dim V = n$ then $V \simeq K^n$

Proof. By direct proof.

- (a) Assume v can also be written as

$$v = \sum_{i=1}^n b_i v_i \quad (\text{as well as } \sum a_i v_i = v)$$

Then

$$\begin{aligned}\vec{0} &= v - v = \left(\sum_{i=1}^n a_i v_i \right) - \left(\sum_{i=1}^n b_i v_i \right) \\ \vec{0} &= \sum_{i=1}^n (a_i - b_i) v_i\end{aligned}$$

Since $\{v_1, \dots, v_n\}$ independent (B = basis) all $a_i - b_i = 0$ ($i = 1, 2, \dots, n$) so $a_1 = b_1$. Hence representation is *unique*.

(b) Let $v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i$ be in $V, c \in K$. Then,

$$\begin{aligned}C_B(cv + u) &= C_B\left(\sum_{i=1}^n (ca_i + b_i)v_i\right) \\ &= \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix} \\ &= c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= C_B(v) + C_B(u)\end{aligned}$$

Hence C_B is linear. To check C_B inj. and surj., since $\dim V = n = \dim K^n$, need only check on (other will follow). We will prove surj.

Let $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in K^n$. Then let $v = \sum_{i=1}^n a_i v_i$, so $C_B(v) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

□

Remarks

- (1) We need the coords to be *unique* in order for $C_B : V \rightarrow K^n$ to be a (well-defined) function.
- (2) If you use a different basis, or even same basis but in different order, you get different coords and also different isomorphism.

Always infinitely many isomorphisms

Lemma 33. Let $T : V \rightarrow W, S : W \rightarrow U$ be a linear transformation. Then

- (a) $S \cdot T : V \rightarrow U$ ($Vt \rightarrow Ws \rightarrow U$) is linear
- (b) If T, S both injective (surjective), then $S \cdot T$ is also injective (surjective)

Proof. Exercise. □

Theorem 34. Let V, W be finite-dimensional vector spaces over field K .

Then,

$$V \simeq W \iff \dim V = \dim W$$

That is, as far as vector space ops go, only the dimension really matters.

Proof. By direct proof.

- " \Rightarrow " Prop 31.
- " \Leftarrow " $\dim V = \dim W = n$. By Thm 32, $V \simeq K^n, W \simeq K^n$, using $C_{B_1} : V \rightarrow K^n, C_{B_2} : W \rightarrow K^n$. Then $C_{B_2}^{-1} : K^n \rightarrow W$ is an isomorphism (Thm 30), so

$$C_B^{-1} \cdot C_B : V \rightarrow W \quad (V \xrightarrow{C_{B_1}} K^n \xrightarrow{C_{B_2}^{-1}} W)$$

is linear, injective, surjective by lemma 33 so it is an isomorphism.

□

February 25th 2019

Recall $V \simeq W \iff \dim V = \dim W$ (proved for finite-dim vector spaces only).

Note: If $T : V \rightarrow W$ isomorphism, $T^{-1} : W \rightarrow V$ is also an isomorphism.

Examples of isomorphisms:

- $P_n(K) \simeq K^{n+1}$
- $\mathcal{M}_{m \times n} \simeq K^{mn}$
- $K^n \simeq K^m \iff n = m$

Question If $n = \dim V$, then $V \simeq K^n$, why bother studying vector spaces other than K^n ?

Answer If you only want to know about addition and scalar multiplication, only K^n matters but the "vectors" $P_n, \mathcal{M}_{n \times m}$ etc... have other properties not always related to vector space operations.

For example, in $P_2(\mathbb{R})$ we can evaluate polynomials $f(x)$ at say $x = 3$,

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(3) &= 9a + 3b + c \end{aligned}$$

If we consider $P_2(\mathbb{R}) \simeq \mathbb{R}^3$, "eval at $x = 3$ " is a linear transformation:

$$\begin{aligned} T : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ T(a, b, c) &= 9a + 3b + c \end{aligned}$$

Computations related to linear transformation

Theorem 35 (T is determined by its value on a basis). Let V, W be vector spaces, $\{v_1, v_2, \dots, v_n\}$ basis V .

Let $w_1, w_2, \dots, w_n \in W$ be any vectors (need not be distinct). Then there is one linear transformation $T : V \rightarrow W$ s.t. $T(v_i) = w_i$

Idea of proof If you want to calculate $T(v)v \in V$ (arbitrary element), write v uniquely in terms of basis

$$v = \sum_{i=1}^n a_i v_i$$

Then since T is supposed to be linear, compute

$$\begin{aligned} T(v) &= T(\sum a_i v_i) \\ &= \sum a_i T(v_i) \\ &= \sum a_i w_i \end{aligned}$$

Problem Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear and

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution $\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$ = basis \mathbb{R}^2 , should have enough info to know what T is. Need to find

$$\begin{aligned} \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 4 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} T\begin{pmatrix} 3 \\ 4 \end{pmatrix} &= T\left(\frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\ &= \frac{7}{2}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{7}{2}\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

Row, column, nullspace of a matrix

Def $A \in \mathcal{M}_{m \times n}(K)$

1. The row space, $\text{row}(A)$ is the span of the rows of A . Subspace of K^n
2. The column space, $\text{col}(A)$ is span of columns. Subspace of K^n
3. Nullspace(\ker), is the solution set to the homogeneous system $Ax = \vec{0}$. Subspace of K^n

Proposition 36. Let $A \in \mathcal{M}_{m \times n}(K)$. Then

- (1) $A_{ei} = \text{column } i \text{ of } A$
- (2) If $B \in \mathcal{M}_{n \times p}(K)$ then column i of AB is Ab_i , $b_i = \text{column } i \text{ of } B$

Proof. Proof by picture!

□

i)
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

ii)
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Proposition 37. Let $A \in \mathcal{M}_{m \times n}(K)$, so $L_A : K^n \rightarrow K^m$.

- (1) $\ker(A) = \text{Ker}(L_A)$
- (2) $\text{col}(A) = \text{Im}(L_A)$
- (3) $\text{row}(A) = \text{Im}(L_{A^T})$

Proof. By direct proof.

(1)

$$\begin{aligned} \text{Ker}(A) &= \{x \in K^n | A_x = \vec{0}\} \\ &= \{x \in K^n | L_A(x) = \vec{0}\} \\ &= \text{Ker}(A) \end{aligned}$$

(2) Take basis $\{e_1, e_2, \dots, e_n\}$ for K^n . Then by prop 26,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots$$

$$L_A(e_1) \dots L_A(e_n) \text{ spans } \text{Im}(L_A)$$

But $L_A(e_1) = A_{ei} = \text{column } i \text{ of } A$, ie columns of A span $\text{Im}(L_A)$
hence $\text{col}(A) = \text{Im}(L_A)$

(3) $\text{col}(A) = \text{col}(A^T) = \text{Im}(L_{A^T})$ by (2)

□

Def: Rank of $A \in \mathcal{M}_{m \times n}(K)$ is number of non-zero rows in RREF.**Proposition 38.** Let $A \in \mathcal{M}_{m \times n}(K), R = \text{RREF}(A)$. Then,

- (i) $\text{rank}(A) = \text{rank}(A^T)$
- (ii) $\text{rank}(A) = \dim \text{row}(A)$
- (iii) $\dim \text{row}(A) = \dim \text{col}(A)$
- (iv) There is an invertible matrix $B \in \mathcal{M}_{m \times n}(K)$ s.t. $BA = R$

Proof. (iii) We have:

$$\begin{aligned} \dim \text{row}(A) &= \text{rank}(A) && \text{(by (ii))} \\ &= \text{rank}(A^T) && \text{(by (i))} \\ &= \dim \text{row}(A^T) && \text{(by (ii))} \\ &= \dim \text{col}(A) && \text{(by (iii))} \end{aligned}$$

□

February 27th 2019

Theorem 39 (computing bases). Let $A \in \mathcal{M}_{m \times n}(K)$, let R be the reduced non-echelon form of A . Then,

- (i) The non-zero rows of R form a basis of $\text{row}(A)$.
- (ii) The columns of A which correspond to the pivot columns (columns containing a leading 1) form a basis of $\text{col}(A)$.
- (iii) The “basic solutions” obtained when solving $Ax = \vec{0}$ form a basis for nullspace (\ker) of A .

Proof. By direct proof.

- (i) Elementary row ops do not change the row space so $\text{row}(A) = \text{row}(R)$. Non-zero rows form basis because of form of R .
- (ii) Let w_1, w_2, \dots, w_r be the columns of R containing leading 1’s (pivot columns). Because of form of R , no other non-zero entries above/below a leading 1, so w_1, w_2, \dots, w_r are standard basis vectors (ie in $\{e_1, e_2, \dots, e_m\}$). So, $\{w_1, \dots, w_r\}$ are linearly independent. Let v_1, v_2, \dots, v_r be corresponding columns.

Note $r = \text{rank}(A) = \dim \text{row}(A) = \dim \text{col}(A)$.

Prove v_1, v_2, \dots, v_r are linearly independent. Suppose

$$\sum_{i=1}^n a_i v_i = \vec{0}$$

By proposition 38, \exists invertible M s.t. $MA = R$. Multiply by M :

$$\begin{aligned} M\left(\sum_{i=1}^r a_i v_i\right) &= M\vec{0} \\ &= \vec{0} \end{aligned}$$

So $\sum_{i=1}^r a_i Mv_i = \vec{0}$, but M (column i of A) = col i of MA ie of R (prop 36). So,

$$\sum_{i=1}^r a_i w_i = \vec{0}$$

But $\{w_1, \dots, w_r\}$ are independent. So all $a_i = 0$, so $\{v_1, \dots, v_r\}$ independent so basis.

- (iii) Solve $Ax = 0$, obtain general solution,

$$\begin{aligned} \vec{x} &= x_1 v_1 + x_2 v_2 + \dots + x_s v_s \\ &= x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_s \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

Where x_1, x_2, \dots, x_s free variables. Claim is that v_1, v_2, \dots, v_s form a basis for $\ker(A)$. They clearly span. Independent? In the x_1 position, only v_i has a non-zero entry, so they are independent.

Comment The dimension of $\ker(A)$ is therefore the number of free variables.

□

Basis-finding problems

Problem Let $W \subseteq \mathcal{M}_{2 \times 2}(\mathbb{R})$, where W consists of all A such that sum of entries in each row and column is the same. Find basis of W .

Solution Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$. So

$$a + b = c + d$$

$$a + c = b + d$$

$$a + b = a + c \quad (a + b = b + d \text{ etc are not needed})$$

Write as linear system:

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{So } a = d, b = c, c = c \text{ and } d = d. \text{ ie, } \vec{x} = c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

General solution,

$$\begin{aligned} A &= \begin{pmatrix} d & c \\ c & d \end{pmatrix} \\ &= c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Linearly independent by Thm 39 (kernel basis case). So

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (\text{basis})$$

Problem: Let

$$\begin{aligned}P_1(x) &= 1 + 2x + 3x^2 - x^3 \\P_2(x) &= -1 + 3x + x^2 + x^3 \\P_3(x) &= 3 - 4x + x^2 - 3x^3 \\P_4(x) &= 1 + 7x + 7x^2 - x^3 \\P_5(x) &= 2 + 2x - x^2 - x^3\end{aligned}$$

Let $W = \text{span}\{P_1(x), \dots, P_5(x)\} \leq P_3(\mathbb{R})$. Find:

- (i) basis of W that is a subset of $\{P_1(x), \dots, P_5(x)\}$
- (ii) basis of W consisting of polys of different degree.

Sol Isomorphism $T : P_3 \rightarrow \mathbb{R}^4$,

$$T(d + cx + bx^2 + ax^3) = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \quad (\text{or } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix})$$

- (i) Put the vectors as columns of a matrix,

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 3 \\ 2 & 3 & -4 & 7 & 2 \\ 3 & 1 & 1 & 7 & -1 \\ -1 & 1 & -3 & -1 & -1 \end{pmatrix}$$

Find basis $\text{col}(A)$. Row-reduce to

$$R = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So columns 1, 2 and 5 of A form a basis for $\text{col}(A)$, which corresponds (using isomorphism T) to W , so

$$\{P_1(x), P_2(x), P_5(x)\} \quad (\text{basis})$$

- (ii) Basis all diff degree. Use row space of a matrix. Put P_1, \dots, P_5 as rows. But use isomorphism

$$d + cx + bx^2 + ax^3 \iff \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So

$$A = \begin{pmatrix} -1 & 3 & 2 & 1 \\ 1 & 1 & 3 & -1 \\ -3 & 1 & -4 & 3 \\ -1 & 7 & 7 & 1 \\ -1 & -1 & 2 & 2 \end{pmatrix} \quad (\text{So } W \text{ corresponds to row space.})$$

$$\rightarrow = \begin{pmatrix} 1 & 0 & 0 & \frac{-27}{20} \\ 0 & 1 & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

First three rows form basis $\text{row}(A)$. As polynomials, we get

$$x^3 - \frac{27}{20}, x^2 - \frac{1}{4}, x + \frac{1}{5}$$

Which is basis of W , all of different degree. The choice of order was relevant since we knew in advance the general form the reduced form would take.

March 1st 2019

Problem Let

$$\begin{aligned} v_1 &= (1, 3, -1, 2, 0, 2) \\ v_2 &= (3, 3, 5, -4, -7, -5) \\ v_3 &= (2, 2, -1, 1, 2, 1) \\ w_1 &= (3, 1, -1, 0, 4, 0) \\ w_2 &= (3, 3, 1, 1, 1, -1) \\ w_3 &= (1, 1, -1, 2, 3, 1) \end{aligned}$$

Let $V = \text{span}\{v_1, v_2, v_3\}$, $W = \text{span}\{w_1, w_2, w_3\}$. Find bases (and dimensions of) $V + W$, $V \cap W$.

Solution Check that $\{v_1, v_2, v_3\}$, $\{w_1, w_2, w_3\}$ both independent (put into matrix as either rows or columns, verify $\text{rank} = 3$)

$V + W = \text{span}\{V \cup W\} = \text{span}\{v_1, v_2, v_3, w_1, w_2, w_3\}$. For basis, put vectors as rows or columns, solve for row space or col space. I used columns, matrix reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis = cols 1, 2, 3, 5 of original matrix. So $\{v_1, v_2, v_3, w_2\}$ so $\dim(V + W) = 4$.

Formula:

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$$

$$4 = 3 + 3 - \dim(V \cap W)$$

So $\dim(V \cap W) = 2$.

$V \cap W$ is all $u = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{R}^6$ such that $u = x_1 v_1 + x_2 v_2 + x_3 v_3$ (*) (ie $u \in V$) and $u = y_1 w_1 + y_2 w_2 + y_3 w_3$ (**) (ie $u \in W$) for some $x_1, x_2, x_3, y_1, y_2, y_3$. This is linear system. 12 variables, 12 equations (2 for each of 6 components):

$$z_1 = x_1 + 3x_2 + 2x_3 \quad (z_1\text{-component of } *)$$

$$z_2 = 3x_1 + 3x_2 + 2x_3 \quad (z_2\text{-component of } *)$$

...

And

$$z_1 = 3y_1 + 3y_2 + y_3 \quad (z_1\text{-component of } **)$$

...

$$z_6 = 0y_1 - y_2 + y_3 \quad (z_6\text{-component of } **)$$

Goal is to solve the system, need only $u = (z_1, \dots, z_6)$. Remember that:

$$\begin{pmatrix} z_1 \\ \dots \\ z_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 3 \\ 5 \\ \dots \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 2 \\ \dots \end{pmatrix}$$

Rewrite as

$$z_1 - x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 = 0$$

$$z_2 - 3x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 = 0$$

...

Coefficient matrix: see fig 12

The form is

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6 \times 3} \\ I_6 & 0_{6 \times 3} & -w_1 - w_2 - w_3 \end{pmatrix}$$

(Coeff matrix) $(-v_1)(-v_2)(-v_3)$

$$\left(\begin{array}{cccccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -3 & -3 & -2 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & & & \\ & & & & & & 0 & 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \end{array} \right)$$

\sim
 \sim
 \sim
 \sim
 \sim
 \sim

$-w_1 \quad -w_2 \quad -w_3$

Figure 3: Coefficient matrix

Row-reduce, find basic solutions, each solution is in \mathbb{R}^{12} (12 variables), you only need first 6 components $((z_1, z_2, \dots, z_6)) = u \in V \cap W$.

Obtain basis

$$\begin{aligned} u_1 &= (3, 1, -1, 0, 4, 0) & (= w_1) \\ u_2 &= (-1, -1, -5/3, 4/3, 7/3, 5/3) & (= \frac{-1}{3} v_2) \end{aligned}$$

Shortcut When you row-reduce, after 6 ops, get

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6 \times 3} \\ 0_{6 \times 6} & v_1 + v_2 + v_3 & -w_1 - w_2 - w_3 \end{pmatrix}$$

Another viewpoint. Had

$$u = x_1 v_1 + x_2 v_2 + x_3 v_3$$

$$u = y_1 w_1 + y_2 w_2 + y_3 w_3$$

You can solve instead 6×6 system:

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = y_1 w_1 + y_2 w_2 + y_3 w_3$$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 - y_1 w_1 - y_2 w_2 - y_3 w_3 = (0, 0, \dots, 0)$$

Coeff matrix: $(v_1 \ v_2 \ v_3 \ -w_1 \ -w_2 \ -w_3)$

Sol gives you $x_1, x_2, x_3, y_1, y_2, y_3$ not z_1, \dots, z_6 . Find $u = (z_1, \dots, z_6)$ from (*) or (**)

Matrix of a linear transformation (ch. 6.2)

Def $T : V \rightarrow W$ linear, $\alpha = \{v_1, \dots, v_n\}$ basis of V , $\beta = \{w_1, \dots, w_n\}$ basis of W . The *standard matrix* of T , relative to α and β , is the $m \times n$ matrix whose i^{th} column is $T(v_i)$, written in β -coordinates, ie $[T(v_i)]_\beta (\in \mathbb{R}^m)$.

It is denoted $[T]_\alpha^\beta$.

Ex Let $T : P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$, $T(f(x)) = f'(x)$. Find $[T]_\alpha^\beta$, $\alpha = \{1, x, x^2\}$, $\beta = \{1, x\}$

Sol Compute T on α

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2x$$

In β -coords,

$$[T(1)]_\alpha^\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (= 0 \ 1 + 0 \ x)$$

$$[T(x)]_\alpha^\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (= 1 \ 1 + 0 \ x)$$

$$[T(x^2)]_\alpha^\beta = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (= 0 \ 1 + 2 \ x)$$

So $[T]_\alpha^\beta$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

So $[T]_\alpha^\beta$ records values of T on α .

Theorem 40. $[T]_\alpha^\beta$ computes T , but in coordinates. That is, for all $v \in V$,

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

Ex $T(f(x)) = f'(x)$. Compute $T(a + bx + cx^2)$ via $[T]_\alpha^\beta$

Sol

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} [T(a + bx + cx^2)]_\beta &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} b \\ 2c \end{pmatrix} \quad (b + 2cx = f(x)) \end{aligned}$$

March 11th 2019

Recall $T : V \rightarrow W$, $\alpha = \{v_1, v_2, \dots, v_n\}$ basis V

$\beta = \{w_1, w_2, \dots, w_n\}$ basis W

Matrix $[T]_{\alpha}^{\beta}$ has i^{th} column being $[T(vi)]_{\beta}$

Theorem 40

$$[T(v)]_{\beta} = [T]_{\alpha}^{\beta}[v]_{\alpha}$$

Proof. Let $A = [T]_{\alpha}^{\beta}$, $v \in V$. Write $v = \sum_{i=1}^n a_i v_i$.

$$\text{So } [v]_{\alpha} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

Then

$$\begin{aligned} A[v]_{\alpha} &= A(a_1 e_1 + \dots + a_n e_n) \\ &= a_1 A e_1 + \dots + a_n A e_n \\ &= a_1 (\text{col } \# 1 \text{ of } A) + \dots + a_n (\text{col } \# n \text{ of } A) \\ &= a_1 [T(v_1)]_{\beta} + \dots + a_n [T(v_n)]_{\beta} \end{aligned}$$

□

Theorem 41. Everything you want to know about T , you can determine from $[T]_{\alpha}^{\beta}$.

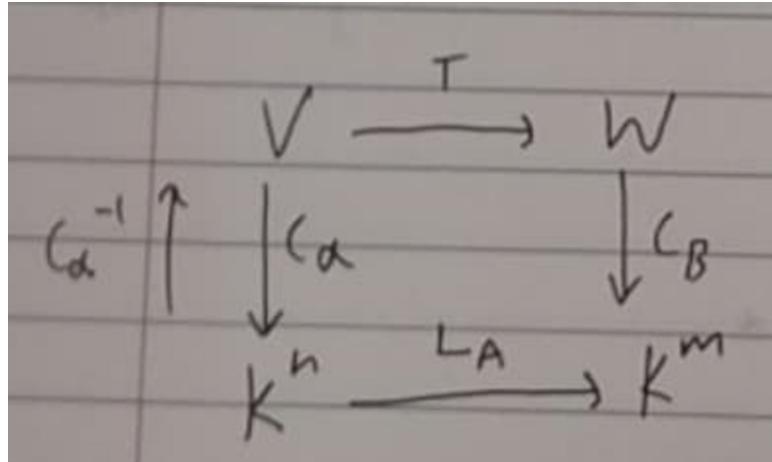
Let $A = [T]_{\alpha}^{\beta}$ ($C_{\alpha} = V \rightarrow \mathbb{R}^n$, $C_{\alpha}(v) = [v]_{\alpha}$). See figure 4.

Then

$$(i) \ Ker(T) = C_{\alpha}^{-1}(Ker(A))$$

$$(ii) \ Im(T) = C_{\beta}^{-1}(Im(A))$$

Figure 4: Theorem 41



Ex $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$ defined by $T(A) = BA$.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Find basis for $\text{Kernel}(T)$, $\text{Image}(T)$ is T inj/surj?

Sol Use basis $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

So we have

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$\text{Ker}(T)$: Solve $[T]x = 0$. Row-reduce

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2s$$

$$x_2 = -2t$$

$$x_3 = s$$

$$x_4 = t$$

$$x = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \text{Ker}([T])$$

So $\left\{ \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} \right\}$ basis for $\text{Ker } T$ so T not injective.

Theorem 42. *The following are true:*

- (i) $T : V \rightarrow W$, linear α basis of V , β basis of W .

$$T \text{ is invertible} \iff [T]_{\alpha}^{\beta} \text{ is invertible}$$

So $\dim(V) = \dim(W)$ must hold, of course.

- (ii) If $S : W \rightarrow U$, γ basis of U , then $[S \cdot T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$

$(S \cdot T : V \rightarrow U)$ is matrix of a composition is product of standard matrices.

Ex $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f(1)$. Prove T is invertible, give formula for $T^{-1}(ax^2 + bx + c)$.

Sol (T is linear, verify)

Use standard basis $\{1, x, x^2\}$.

Calculate T on α

$$T(1) = x(0) + 1 = 1 = 1 + 0x + 0x^2$$

$$T(x) = x(1) + 1 + 1 + 1x + 0x^2$$

$$T(x^2) = x(2x) + 1 = 1 + 0x + 2x^2$$

So $[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $\det([T]) = 2 \neq 0$ so matrix and T are both

invertible.

$$\text{invert}[T] = \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Then

$$\begin{aligned} [T^{-1}(c + bx + ax^2)]_{\alpha} &= \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \\ &= \begin{bmatrix} c - b - \frac{a}{2} \\ b \\ \frac{a}{2} \end{bmatrix} \end{aligned}$$

Formula: $T^{-1}(c + bx + ax^2) = (c - b - \frac{a}{2}) + bx + \frac{a}{2}x^2$.

Check

$$\begin{aligned} T(c - b - \frac{a}{2} + bx + \frac{a}{2}x^2) &= x(b + ax) + c - b - \frac{a}{2} + b + \frac{a}{2} \\ &= c + bx + ax^2 \end{aligned}$$

March 13th 2019

Change of basis (ch 6.3)

Suppose V : vector space, $\alpha = \{u_1, \dots, u_n\}$ and β both bases of V .

How to change from α -coordinates to β -coordinates easily?

Trick: Consider identity lin. transformation I , $I(v) = v$.

$$I : V \rightarrow V$$

Matrix $[I]_{\alpha}^{\beta}$ will change coords, since if $v \in V$,

$$[I]_{\alpha}^{\beta}[v]_{\alpha} = [I(v)]_{\beta} = [v]_{\beta}$$

Def Matrix $Q_{\alpha}^{\beta} = [I]_{\alpha}^{\beta}$ called change-of-basis matrix from α to β . That is Q_{α}^{β} is matrix whose i^{th} column is the i^{th} basis vector of α , written in β -coords ("old basis in new coords, as columns").

Theorem 43. We have

- (i) For all $v \in V$, $Q_{\alpha}^{\beta}[v]_{\alpha} = [v]_{\beta}$ (mult. by Q_{α}^{β} changes coords)
- (ii) $Q_{\beta}^{\alpha} = (Q_{\alpha}^{\beta})^{-1}$ (and Q_{α}^{β} is invertible!)

Proof. (i) Done above.

(ii) $I : V \rightarrow V$ is invertible, so $Q_{\alpha}^{\beta} = [I]_{\alpha}^{\beta}$ also invertible and

$$\begin{aligned} (Q_{\alpha}^{\beta})^{-1} &= ([I]_{\alpha}^{\beta})^{-1} \\ &= [I^{-1}]_{\beta}^{\alpha} \\ &= [I]_{\beta}^{\alpha} \\ &= Q_{\beta}^{\alpha} \end{aligned}$$

□

Ex \mathbb{R}^2 with $\alpha = \{(1, 0), (0, 1)\}$, $\beta = \{(2, 1), (1, 3)\}$. Find Q_{α}^{β} , Q_{β}^{α} , $[(7, 4)]_{\beta}$.

Note In \mathbb{R}^n , $[(a_1)_{\alpha}]_{\alpha} = (a_1)_{\alpha}$ ($\alpha = \{e_1, e_2, \dots, e_n\}$)

Sol Q_{α}^{β} = old basis in α in terms of new basis β = work.

Q_{β}^{α} = easier = β -vectors in terms of α .

$$\begin{aligned} Q_{\beta}^{\alpha} &= \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \\ Q_{\alpha}^{\beta} &= (Q_{\beta}^{\alpha})^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \left[\begin{pmatrix} 7 \\ 4 \end{pmatrix} \right]_{\beta} &= Q_{\beta}^{\alpha} \left[\begin{pmatrix} 7 \\ 4 \end{pmatrix} \right]_{\alpha} \\ &= \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 21 - 4 \\ -28 + 8 \end{pmatrix} \\ &= \begin{pmatrix} \frac{17}{2} \\ -10 \end{pmatrix} \end{aligned}$$

Def $T : V \rightarrow V$ linear transf (some V), called a *linear operator*.

Def Let $A, B \in \mathcal{M}_{n \times n}(K)$. A is *similar* to B if \exists invertible $Q \in \mathcal{M}_{n \times n}(K)$ so that $Q^{-1}AQ = B$

Proposition 44. Note If A similar to B , B similar to A , since

$$\begin{aligned} Q^{-1}AQ &= B \\ QQ^{-1}AQQ^{-1} &= QBQ^{-1} \\ A &= (Q^{-1})^{-1}BQ^{-1} \end{aligned}$$

Theorem 45. Let $T : V \rightarrow V$ linear operator, α, β bases of V . Then,

$$[T]_{\beta}^{\beta} = Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}$$

In particular, $[T]_{\alpha}^{\alpha}$ and $[T]_{\beta}^{\beta}$ are similar since $Q_{\alpha}^{\beta} = (Q_{\beta}^{\alpha})^{-1}$

Proof. Let $v \in V$. Show both compute some linear operator.

$$\text{LHS } [T]_{\beta}^{\beta}[v]_{\beta} = [T(v)]_{\beta}$$

$$\text{RHS } Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}[v]_{\beta}$$

$$\begin{aligned} Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}[v]_{\beta} &= Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha}[v]_{\alpha} \\ &= Q_{\alpha}^{\beta} [T(v)]_{\alpha} \\ &= [T(v)]_{\beta} \end{aligned}$$

So for all $[v]_{\beta}$, mult by LS/RS gives some result, so for std bases

vector e_1, \dots, e_n , LS $e_i = \text{col } i$ of LS, RS $e_i = \text{col } i$ of RS \square

Problem (figure 5) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be reflection in the line $y = mx$.

Find formula for $T(a, b)$

Sol First, prove T is linear. (omit)

Option # 1 (figure 6) Compute $T(1, 0), T(0, 1)$, find $[T]_{\alpha}^{\alpha}$, $\alpha = \{(1, 0), (0, 1)\}$

Option # 2 (figure 7) Use better basis, then change basis. Let $v =$

$(1, m)$ so $T(v) = (1, m)$. Let $w = (m, -1)$. Then $T(w) = -w = (-m, 1)$

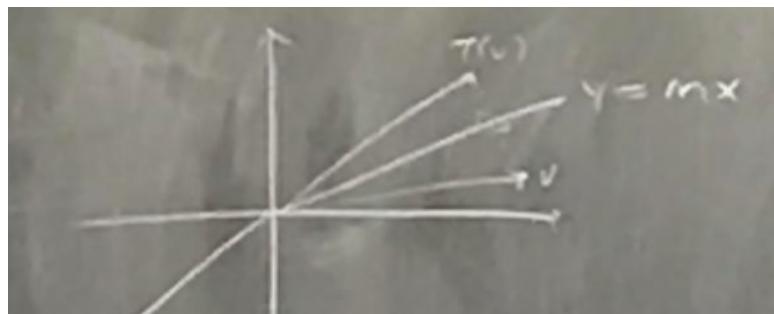


Figure 5: Problem

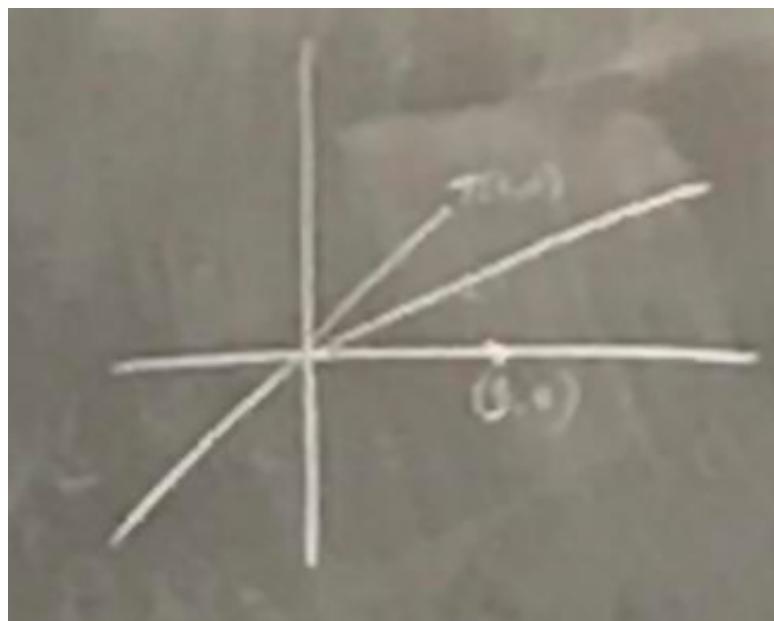


Figure 6: Have to do some geometry! :(

New basis $\beta = \{v, w\}$

$$\begin{aligned}[T]_{\beta}^{\beta} &= ([T(v)]_{\beta}, [T(w)]_{\beta}) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Want $[T]_{\alpha}^{\alpha} = Q_{\beta}^{\alpha} [T]_{\beta}^{\beta} Q_{\alpha}^{\beta}$. Have $Q_{\beta}^{\alpha} = \beta$ in terms of $\alpha = \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}$

$$\begin{aligned}Q_{\alpha}^{\beta} &= (Q_{\beta}^{\alpha})^{-1} \\ &= \frac{1}{-1-m^2} \begin{pmatrix} -1 & -m \\ -m & 1 \end{pmatrix}\end{aligned}$$

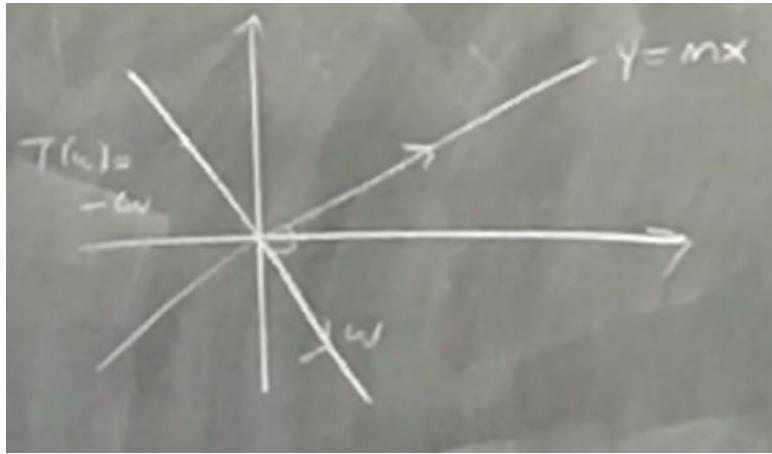


Figure 7: Better option

Compute

$$\begin{aligned}[T]_{\alpha}^{\alpha} &= Q_{\beta}^{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q_{\alpha}^{\beta} && \text{(multiply)} \\ &= \frac{1}{m^2 + 1} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix} \end{aligned}$$

Finally,

$$\begin{aligned} T(a, b) &= \frac{1}{m^2 + 1} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{1}{m^2 + 1} \begin{pmatrix} a - am^2 & 2bm \\ 2am & bm^2 - b \end{pmatrix} \end{aligned}$$

March 15th 2019

Inner Product Spaces (ch. 7 text)

Idea: Dot product on \mathbb{R}^n , $u = (a_1, \dots, a_n)$, $v = (b_1, \dots, b_n)$

$$u \cdot v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

From this,

$$\begin{aligned} \|u\| &= \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = \sqrt{u \cdot u} \\ u \cdot v &= \|u\| \|v\| \cos \theta \end{aligned}$$

Or

$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{u \cdot v}{\|u\| \|v\|} \right) \\ u, v \text{ } \xrightarrow[\text{(}\theta = \frac{\pi}{2}\text{)}} \text{orthogonal} &\iff u \cdot v = 0 \end{aligned}$$

Dot product allows you to *define* lengths, angles, orthogonality. These are geometric ideas.

Def V vector space over K (\mathbb{R} or \mathbb{C}).

An *inner product* on V is a function $\langle u, v \rangle$ which takes two vectors as input and produces a scalar, and satisfies the following:

$$(I1) \quad \forall u, v, w \in V, \forall c \in K$$

$$(i) \quad \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$(ii) \quad \langle cu, w \rangle = c\langle u, w \rangle$$

This is called *linearity in the first component*

$$(I2) \quad \forall u, v \in V$$

$$\langle v, u \rangle = \overline{\langle u, v \rangle}$$

The RHS is the complex conjugate.

This is called *conjugate similarity*.

$$(I3) \quad \forall u \in V, \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \iff u = \vec{0}$$

This is called *positive definite*

Notes:

(1) If $K = \mathbb{R}$, (I2) is $\langle v, u \rangle = \langle u, v \rangle$

(2) If $K = \mathbb{C}$, then by (I2)

$$\langle u, u \rangle = \overline{\langle u, u \rangle}$$

Which means $\langle u, u \rangle \in \mathbb{R}$. So $\langle u, u \rangle \geq 0$ makes sense.

Theorem 46. Properties of inner products

$$(a) \quad \forall u, v, w \in V, \forall c \in K,$$

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$\langle u, cv \rangle = \bar{c}\langle u, v \rangle$$

This is called *conjugate linearity in second component*.

$$(b) \quad \forall u \in V, \langle u, \vec{0} \rangle = 0 \text{ (scalar)}$$

$$(c) \quad \forall u, v, w \in V, \text{ if } \forall w \in V \quad \langle u, w \rangle = \langle v, w \rangle \text{ then } u = v$$

Proof. By direct proof.

(a)

$$\begin{aligned}
\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} && \text{(I2)} \\
&= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} && \text{(I1)} \\
&= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\
&= \langle u, v \rangle + \langle u, w \rangle && \text{(I2)}
\end{aligned}$$

Recall for $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\
\overline{z_1 z_2} &= \overline{z_1} \overline{z_2} \\
z_1 \overline{z_1} &= (a + bi)(a - bi) = a^2 + b^2 = |z_1|^2
\end{aligned}$$

Now, we have:

$$\begin{aligned}
\langle u, cv \rangle &= \overline{\langle cv, u \rangle} && \text{(I2)} \\
&= \overline{c \langle v, u \rangle} && \text{(I1)} \\
&= \overline{c} \overline{\langle v, u \rangle} \\
&= \bar{c} \langle u, v \rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\langle u, \vec{0} \rangle &= \langle u, \vec{0} + \vec{0} \rangle \\
&= \langle u, \vec{0} \rangle + \langle u, \vec{0} \rangle && \text{(by (a))}
\end{aligned}$$

So $0 = \langle u, \vec{0} \rangle$

- (c) Assume $\forall w, \langle u, w \rangle = \langle v, w \rangle$. To show $u = v$, we will show $u - v = \vec{0}$.

Consider

$$\begin{aligned}
\langle u - v, u - v \rangle &= \langle u, u - v \rangle + \langle -v, u - v \rangle && \text{(I1)} \\
&= \langle u, u - v \rangle - \langle v, u - v \rangle && \text{(I1)}
\end{aligned}$$

Using $w = u - v$, $\langle u, u - v \rangle = \langle v, u - v \rangle$. So $\langle u - v, u - v \rangle = 0$ so by (I3). $u - v = \vec{0}$ so $u = v$.

□

March 18th 2019

Standard inner product on K^n :

for $u = \{a_1, \dots, a_n\}, v = \{b_1, \dots, b_n\}$ define

$$\langle u, v \rangle = \sum_{i=1}^n a_i \overline{b_i}$$

So if $K = \mathbb{R}$, $\bar{b}_i = b_i$ so it's the usual dot product.

Ex Compute $\langle u, v \rangle$,

$$u = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

$$v = \begin{pmatrix} i \\ 3-4i \end{pmatrix}$$

Solution

$$\begin{aligned} \langle \begin{pmatrix} 2 \\ 1+i \end{pmatrix}, \begin{pmatrix} i \\ 3-4i \end{pmatrix} \rangle &= 2(\bar{i}) + (1+i)(\overline{3-4i}) \\ &= -2i + (1+i)(3+4i) \\ &= -2i + 3 + 4i + 3i + 4i^2 \\ &= -i + 5i \end{aligned}$$

Proposition 47. Standard inner product in K^n is an inner product

Proof. By direct proof.

(I1) Omit.

(I2)

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{\sum_{i=1}^n b_i \bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i \bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i} \overline{\bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i} a_i \\ &= \sum_{i=1}^n a_i \bar{b}_i \end{aligned}$$

(I3)

$$\begin{aligned} \langle u, u \rangle &= \sum_{i=1}^n a_i \bar{a}_i \\ &= \sum_{i=1}^n |a_i|^2 \end{aligned}$$

Then all $|a_i| \geq 0$ so $\langle u, u \rangle \geq 0$

$$\langle u, u \rangle = 0 \iff |a_i| = 0 \text{ for all } i$$

□

Inner product on $\mathcal{M}_{n \times n}(K)$

For $A, B \in \mathcal{M}_{n \times n}(K)$, define first

- (i) \bar{A} is the matrix obtained by taking the complex conjugate of each entry.
- (ii) $A^* = (\bar{A})^T$, conjugate transpose (adjoint)

Ex:

$$A = \begin{pmatrix} 2+i & 3i \\ 2 & 1+i \end{pmatrix}, \bar{A} = \begin{pmatrix} 2+i & -3i \\ 2 & 1-i \end{pmatrix}, A^* = \begin{pmatrix} 2+i & 2 \\ -3i & 1-i \end{pmatrix}$$

For inner product,

$$\langle A, B \rangle = \text{tr}(B^* A)$$

Ex In $\mathcal{M}_{2 \times 2}(K)$, if

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ \langle A, B \rangle &= \text{tr} \left(\begin{pmatrix} \bar{b}_{11} & \bar{b}_{21} \\ \bar{b}_{12} & \bar{b}_{22} \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} \right) \\ &= (a_{11}\bar{b}_{11} + a_{21}\bar{b}_{21}) + (a_{12}\bar{b}_{12} + a_{22}\bar{b}_{22}) \\ &= \left\langle \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} \right\rangle \quad (\text{Standard inner product on } \mathbb{C}^4) \end{aligned}$$

Proposition 48. $\langle A, B \rangle = \text{tr}(B^* A)$ is an inner product on $\mathcal{M}_{n \times n}(K)$

Proof. Omit. You can prove it directly using matrix properties. \square

Inner product on $P_n(\mathbb{R})$

For $f, g \in P_n(\mathbb{R})$ define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

Ex For $f(x) = x+1$, $g(x) = x$ find $\langle f, g \rangle$

Sol

$$\begin{aligned} \langle x+1, x \rangle &= \int_0^1 (x+1)x dx \\ &= \int_0^1 (x^2 + x) dx \\ &= \frac{x^3}{3} \frac{x^2}{2} \Big|_0^1 \\ &= \frac{1}{3} + \frac{1}{2} \end{aligned}$$

Proposition 49. For any $a < b$

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

defines an inner product on $P_n(\mathbb{R})$ (also on $P(\mathbb{R})$)

Proof. By direct proof.

(I1) Let $f, g, h \in P_n(\mathbb{R}), c \in \mathbb{R}$. Then

$$\begin{aligned} \langle f + cg, h \rangle &= \int_a^b (f(x) + cg(x))h(x)dx \\ &= \int_a^b f(x)h(x)dx + c \int_a^b g(x)h(x)dx \\ &= \langle f, h \rangle + c\langle g, h \rangle \quad ((i) \text{ and } (ii) \text{ together}) \end{aligned}$$

(I2)

$$\begin{aligned} \langle f, g \rangle &= \int_a^b f(x)g(x)dx \\ &= \int_a^b g(x)f(x)dx \\ &= \langle g, f \rangle \end{aligned}$$

(I3)

$$\begin{aligned} \langle f, f \rangle &= \int_a^b f(x)f(x)dx \\ &= \int_a^b (f(x))^2 dx \end{aligned}$$

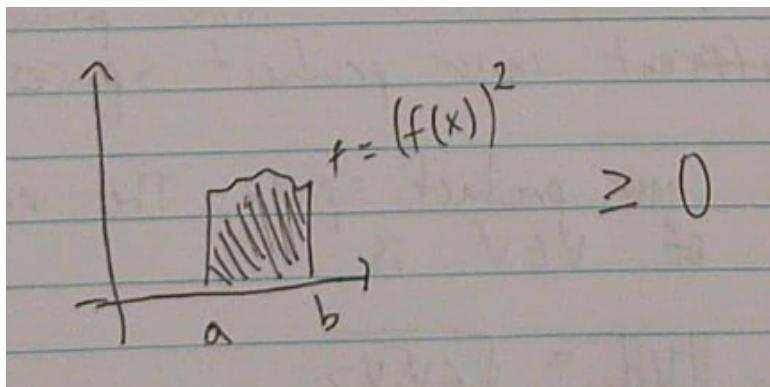


Figure 8: Representation

□

Problem For $P_1(\mathbb{R})$, write formula for

$$\langle a + bx, c + dx \rangle$$

in terms of a, b, c, d

Sol

$$\begin{aligned}\langle a + bx, c + dx \rangle &= \int_0^1 (ac + (ad + bc)x + bdx^2) dx \\ &= acx + \frac{ad + bc}{2}x^2 + \frac{bd}{3}x^3 \Big|_0^1 \\ &= ac + \frac{ad}{2} + \frac{bc}{2} + \frac{bd}{3}\end{aligned}$$

Note $P_1(\mathbb{R}) \simeq \mathbb{R}^2$. Isomorphism,

$$(a + bx) \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

Under this isomorphism, you can compute $\langle a + bx, c + dx \rangle$ using an inner product on \mathbb{R}^2 defined by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = ac + \frac{1}{2}ad + \frac{1}{2}bc + \frac{1}{3}bd$$

The point is, the inner product makes sense in $P_1(\mathbb{R})$.

Def A vector space V with a specified inner product is called an *inner product space*.

Note Some V with different inner products are different inner product spaces.

Def V on inner product space. The norm or length of $v \in V$ is

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Example In $P_1(\mathbb{R})$ with $[0, 1]$,

$$\begin{aligned}\|x + 1\| &= \sqrt{\langle x + 1, x + 1 \rangle} \\ &= \left(\int_0^1 (x + 1)^2 dx \right)^{-\frac{1}{2}} \\ &= \left(\frac{(x + 1)^3}{3} \Big|_0^1 \right)^{-\frac{1}{2}} \\ &= \left(\frac{2^3}{3} - \frac{1}{3} \right)^{-\frac{1}{2}} \\ &= \sqrt{\frac{7}{3}}\end{aligned}$$

March 20th 2019

Last time: Norm (length) is $\|v\| = \sqrt{\langle u, v \rangle}$

Proposition 50. For all $v \in V, c \in K$

$$\|cv\| = |c|\|v\| \quad (\text{note } |c|^2 = c\bar{c} \in \mathbb{C}, |c|^2 = a^2 + b^2)$$

Proof.

$$\begin{aligned}\|cv\| &= \sqrt{\langle cv, cv \rangle} \\ &= \sqrt{c\bar{c}\langle v, v \rangle} \tag{I2} \\ &= |c|\sqrt{\langle v, v \rangle} \\ &= |c| \cdot \|v\|\end{aligned}$$

□

Theorem 51 (Cauchy-Schwarz Inequality). For all $u, v \in V$, (inner product space)

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

So also $\langle u, v \rangle \leq \|u\| \|v\|$ if $K = \mathbb{R}$ or equiv,

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

Further, equality holds $\iff u, v$ are dependent.

Proof. Let $c \in K$ any scalar. Consider

$$0 \leq \langle u - cv, u - cv \rangle \tag{I3}$$

$$= \langle u, u - cv \rangle + \langle -cv, u - cv \rangle \tag{I1}$$

$$\begin{aligned}&= \langle u, u \rangle + \langle u, -cv \rangle + \langle -cv, u \rangle + \langle -cv, -cv \rangle \\&= \|u\|^2 + \overline{(-c)}\langle u, v \rangle + (-c)\langle v, u \rangle + (-c)\overline{(-c)}\langle v, v \rangle \\&0 \leq \|u\|^2 - \bar{c}\langle u, v \rangle - c\langle v, u \rangle + c\bar{c}\|v\|^2\end{aligned}$$

Set $c = \frac{\langle u, v \rangle}{\|v\|^2}$ (unless $\|v\| = 0$, only if $v = \vec{0}$, in which case $\langle u, 0 \rangle = 0 = \|u\|0 = \|u\| \|v\|$) So $c = \frac{1}{\|v\|^2} \langle u, v \rangle$. (LHS $\in \mathbb{R}$, RHS $\in \mathbb{C}$). So

$$\begin{aligned}\bar{c} &= \frac{1}{\|v\|^2} \overline{\langle u, v \rangle} \\&= \frac{\langle v, u \rangle}{\|v\|^2}\end{aligned}$$

So

$$\begin{aligned} 0 &\leq ||u||^2 - \frac{\langle v, u \rangle}{||v||^2} \langle u, v \rangle - \frac{\langle u, v \rangle \langle v, u \rangle}{||v||^2} + \frac{u, v}{||v||^2} \frac{v, u}{||v||^2} ||v||^2 \\ 0 &\leq ||u||^2 - \frac{\langle u, v \rangle \langle v, u \rangle}{||v||^2} \\ \langle u, v \rangle \langle v, u \rangle &\leq ||u||^2 ||v||^2 \\ \langle u, v \rangle \overline{\langle u, v \rangle} &\leq ||u||^2 ||v||^2 \\ |\langle u, v \rangle|^2 &\leq ||u||^2 ||v||^2 \end{aligned}$$

Omit proof about equality. \square

Important cases

(1) \mathbb{R}^n , usual inner product. Let $u = (a_1, \dots, a_n)$, $v = (b_1, \dots, b_n)$. So,

$$\langle u, v \rangle^2 = (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq ||u||^2 ||v||^2$$

So

$$(a_1 b_1 + \dots + a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$$

Ex Prove for all a_1, a_2, \dots, a_n ,

$$(|a_1| + |a_2| + \dots + |a_n|)^2 \leq n(a_1^2 + a_2^2 + \dots + a_n^2)$$

Sol Let

$$\begin{aligned} u &= (|a_1|, |a_2|, \dots, |a_n|) \\ v &= (1, 1, \dots, 1) \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} (|a_1| + |a_2| + \dots + |a_n|)^2 &\leq (a_1^2 + \dots + a_n^2)(1 + 1 + \dots + 1) \\ &= n(a_1^2 + \dots + a_n^2) \end{aligned}$$

(2) $\mathcal{P}(\mathbb{R}), f, g \in \mathcal{P}(\mathbb{R})$

$$\begin{aligned} \langle f, g \rangle^2 &\leq \langle f, f \rangle \langle g, g \rangle \\ (\int_0^1 f(x)g(x)dx)^2 &\leq (\int_0^1 f(x)^2 dx)(\int_0^1 g(x)^2 dx) \end{aligned}$$

Theorem 52. Triangle inequality For all $u, v \in V$,

$$||u + v|| \leq ||u|| + ||v||$$

Proof. Instead of

$$||u + v|| = \sqrt{\langle u + v, u + v \rangle}$$

Look at

$$\begin{aligned}
 \|u + v\|^2 &= \langle u + v, u + v \rangle \\
 &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \quad (\text{I1}) \\
 &= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2
 \end{aligned}$$

For $z = a + bi$, $z + \bar{z} = 2a = 2\operatorname{Re}(z)$ ($\operatorname{Re}(z) = a$, $\operatorname{Im}(z) = b$). Also,

$$a \leq |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

So $\operatorname{Re}(z) \leq |z|$ (*)

Then,

$$\begin{aligned}
 \|u + v\|^2 &= \|u\|^2 + 2\operatorname{Re}(\langle u, v \rangle) + \|v\|^2 \\
 &\leq \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2 \quad (\text{by } (*)) \\
 &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad (\text{Cauchy-Schwarz}) \\
 &= (\|u\| + \|v\|)^2
 \end{aligned}$$

So $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$, take square root. \square

Angles

Since $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$,

$$\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq 1 \text{ or } -1 \leq \frac{\langle u, v \rangle}{\|u\| \|v\|} \quad (K = \mathbb{R})$$

So there is an *angle* θ such that

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

Define the angle between u, v to be θ .

Note When the angle is 0 , $\cos \theta = 1$. When the angle is $\pi/2$, $\cos \theta = 0$. When the angle is π , $\cos \theta = -1$. So $\cos \theta$ measures how "similar" two vectors are in terms of "angle" or "direction".

March 22nd 2019

Application/interpretation

Word counts in textual analysis. Consider \mathbb{R}^n , $n = \#$ of words in the (English) language. Each component corresponds to a word (eg: component 1 is "a", etc). View a text (eg Hamlet) as a vector (v_{hamlet}) , count # times each word occurs.

Norm $\|v_{\text{hamlet}}\| = \sqrt{\sum_{i=1}^n a_i^2}$ (usual dot product)

more words \rightarrow larger norm

Eg $v = (1, 1, \dots, 1)$, $n = 1000$.

$$\begin{aligned} \|v\| &= \sqrt{\sum 1} \\ &= \sqrt{1000} \end{aligned}$$

$w = (1000, 0, \dots, 0)$, $n = 1000$:

$$\begin{aligned} \|v\| &= \sqrt{1000^2} \\ &= 1000 \end{aligned}$$

Angle

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

If u, v have no words in common, $\langle u, v \rangle = 0$, so $\cos \theta = 0$ ($\theta = \frac{\pi}{2}$, "orthogonal"). Suppose you compare "Hamlet" to "2x Hamlet":

$$\begin{aligned} \cos \theta &= \frac{\langle v_{\text{hamlet}}, v_{2x \text{ hamlet}} \rangle}{\|v_{\text{hamlet}}\| \|v_{2x \text{ hamlet}}\|} \\ &= \frac{\langle v_{\text{hamlet}}, 2v_{\text{hamlet}} \rangle}{\|v_{\text{hamlet}}\| \|2v_{\text{hamlet}}\|} \\ &= \frac{2\|v_{\text{hamlet}}\|}{2\|v_{\text{hamlet}}\| \|v_{\text{hamlet}}\|} \\ &= 1 \end{aligned}$$

Ie $\theta = 0$. Texts are "the same".

Orthogonality and projections

Def u, v are *orthogonal* if $\langle u, v \rangle = 0$.

Ex In $P_1(\mathbb{R})$, inner product $\int_0^1 fg dx$, find all polynomials (vectors) orthogonal to $1 + x$.

Sol Let $g(x) = a + bx$. Need

$$\begin{aligned} 0 &= \langle 1 + x, a + bx \rangle \\ &= \int_0^1 (a + bx + ax + bx^2) dx \\ &= ax + \frac{b(a)}{2}x^2 + \frac{b}{3}x^3 \Big|_0^1 \\ &= a + \frac{b}{2} + \frac{a}{2} + \frac{b}{3} \\ \frac{-3}{2}a &= \frac{5}{6}b, b = \frac{-3}{2}(\frac{6}{5})a = \frac{-9}{5}a \end{aligned}$$

All vectors $a - \frac{9}{5}ax$, ie $\text{span}\{1 - \frac{9}{5}x\}$.

Def A set S of vectors is

(i) *orthogonal* if $\langle u, v \rangle = 0$ for all $u, v \in S$, $u \neq v$.

(ii) *orthonormal* if orthogonal and $\|u\| = 1$, all $u \in S$

Def A basis α is an *orthognormal basis* (ONB) if it is an orthonormal set.

Ex $\alpha = \{e_1 e_2, \dots, e_n\}$ is ONB.

Notation : Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So if $S = \{v_1, v_2, \dots, v_n\}$ is ONB, $\langle v_i, v_j \rangle = \delta_{ij}$.

Proposition 53. If S is an orthogonal set of non-zero vectors, then S is linearly independent.

Proof. Suppoe

$$\sum_{i=1}^k a_i v_i = 0 \quad (\text{for some } v_1, \dots, v_k \in S, a_1, \dots, a_n \text{ scalars})$$

Trick. Take inner product with each v_j , $j = 1, 2, \dots, k$. So

$$\begin{aligned} 0 &= \langle \vec{0}, v_j \rangle \\ &= \left\langle \sum_{i=1}^k a_i v_i, v_j \right\rangle \\ &= \sum_{i=1}^k \langle a_i v_i, v_j \rangle \quad ((I1)) \\ &= \sum_{i=1}^k a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle \quad (\text{Since } \langle v_i, v_j \rangle = 0, \text{ unless } i = j) \end{aligned}$$

But $v_j \neq \vec{0}$ so $\langle v_j, v_j \rangle \neq 0$. So $a_j = 0$, for all $j = 1, \dots, k$. So all the coefficients are 0, so S is independent. \square

Theorem 54. Let V be inner product space,

$$\alpha = \{v_1, v_2, \dots, v_n\}$$

an orthogonal basis. Then for any $u \in V$,

$$u = \sum_{i=1}^n \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

ie the i^{th} component of coords of U in basis α is $\frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle}$. Further, if α is ONB, then

$$u = \sum_{i=1}^n \langle u, v_i \rangle v_i$$

Proof. We know $u = \sum_{i=1}^n a_i v_i$ for some scalars. Take inner product with each v_j , $j = 1, 2, \dots, n$ in turn. So

$$\begin{aligned}\langle u, v_j \rangle &= \left\langle \sum_{i=1}^n a_i v_i, v_j \right\rangle \\ &= \sum_{i=1}^n a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle \quad (\text{All 0 except when } i = j)\end{aligned}$$

So $a_j = \langle u, v_j \rangle / \langle v_j, v_j \rangle$, α orthog.

□

March 25th 2019

Last time: If $\alpha = \{v_1, v_2, \dots, v_n\}$ orthog. basis then for all $v \in V$

$$v = \sum_{i=1}^n \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

Ex In \mathbb{R}^3 , $\alpha = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is an ONB. Find

coords of $v = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ in α -basis.

Sol Compute its inner products with basis :

$$\begin{aligned}\left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle &= \frac{1}{\sqrt{2}} 3 = \frac{3}{\sqrt{2}} \\ \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle &= \frac{2 - 1 - 3}{\sqrt{3}} = \frac{-2}{\sqrt{3}} \\ \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\rangle &= \frac{-2 + 1 - 6}{\sqrt{6}} = \frac{-7}{\sqrt{6}}\end{aligned}$$

So

$$\left[\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right]_\alpha = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{-2}{\sqrt{3}} \\ \frac{-7}{\sqrt{6}} \end{pmatrix}$$

Def Let $S \subseteq V$. The *orthogonal complement* of S is

$$\begin{aligned}S^\perp &= \{v \in V \mid \forall s \in S, \langle v, s \rangle = 0\} \\ &= \text{all vectors orthogonal to all vectors in } S\end{aligned}$$

S^\perp reads "S perp".

Ex

- (1) $S = xy\text{-plane in } \mathbb{R}^3, S^\perp = z\text{-axis.}$
- (2) $S = z\text{-axis}, S^\perp = xy\text{-plane.}$
- (3) $S = \text{plane through origin}, S^\perp = \text{normal line.}$
- (4) $S = V, S^\perp = \{\vec{0}\}$
- (5) $S = \{\vec{0}\}, S^\perp = V$

Proposition 55. Let $W \leq V$ (subspace). Then

(i) W^\perp is a subspace (true even if W just subset)

(ii) If $\alpha = \{w_1, w_2, \dots, w_k\}$, basis W , then

$$W^\perp = \{v \in V \mid \langle v, w_i \rangle = 0 \text{ for } i = 1, 2, \dots, k\}$$

(ie to compute W^\perp , find all v that are orthogonal to all basis elements)

(iii) $W \cap W^\perp = \{\vec{0}\}$

Proof. By direct proof.

- (i) Let $u, v \in W^\perp, c \in K$. Then, to check if $cu + v \in W^\perp$, calculate for any $w \in W$

$$\begin{aligned} \langle cu + v, w \rangle &= c\langle u, w \rangle + \langle v, w \rangle \\ &= 0 \quad (\text{both parts 0 since } u, v \in W^\perp) \end{aligned}$$

So $cu + v \in W^\perp$. Also, $\vec{0} \in W^\perp$ since $\langle \vec{0}, w \rangle = 0$ for all $w \in W$.

- (ii) Prove two sets are equal:

(a) $LS \subseteq RS$. Let $v \in W^\perp$. Since each $w_i \in W$, $\langle v, w_i \rangle = 0$ since $v \in W^\perp$.

(b) $RS \subseteq LS$. Let $v \in V$ such that $\langle v, w_i \rangle = 0$ all $i = 1, 2, \dots, k$. Let $w \in W$. Write $w = \sum_{i=1}^k a_i w_i$, then

$$\begin{aligned} \langle v, w \rangle &= \langle v, \sum_{i=1}^k a_i w_i \rangle \\ &= \sum_{i=1}^k \langle v, a_i w_i \rangle \\ &= \sum_{i=1}^k \bar{a}_i \langle v, w_i \rangle \\ &= 0 \end{aligned}$$

So $v \in W^\perp = LS$.

- (c) Let $v \in W \cap W^\perp$. Since $v \in W^\perp$, v orthog to all vectors in W , including itself, we have

$$\langle v, v \rangle = 0$$

So $v = \vec{0}$ by (I3).

□

Ex Let $W = \{A \in \mathcal{M}_{2 \times 2}(K) | A^T = A\}$. Find W^\perp .

Sol Find basis W . See A2.

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Find all $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle \\ &= \text{tr}(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\ &= a \end{aligned}$$

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= d \end{aligned}$$

$$\begin{aligned} 0 &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \\ &= b + c \end{aligned}$$

So $a = d = 0$, $c = -b$, general solution: $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$.

$$W^\perp = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

Orthogonal Projection

See figure ???. Decompose v as $v' + w$, $w \in W$, $v' \in W^\perp$.

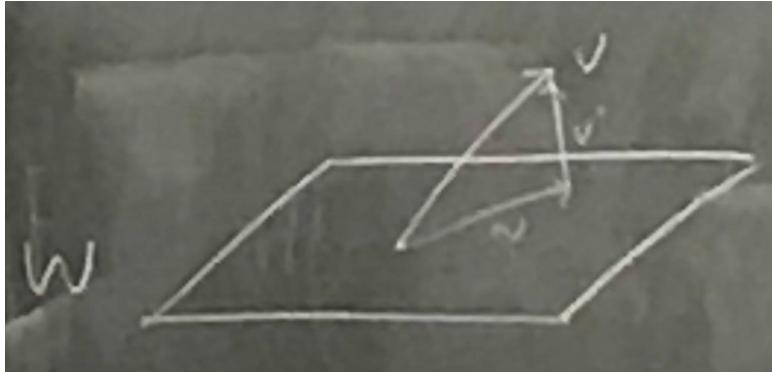


Figure 9: Orthogonal projection

Theorem 56. Let $W \leq V$, $v \in V$. Then \exists unique vectors $w \in W$, $v' \in W^\perp$ such that $v = v' + w$. Vector w called the (orthogonal) projection of v onto W , denoted $\text{proj}_W v = w$. Further, if $\alpha = \{w_1, w_2, \dots, w_k\}$ is an orthogonal basis of W , then

$$w = \text{proj}_W v = \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

Important: α must be orthogonal!

Proof. Set $w = \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$ (so $v = v' + w$). Set $v' = v - w$. So $v' + w = v$, $w \in W$ ($w = \text{comp of } W$ -basis vectors). Need $v' \in W^\perp$. Check if $\langle v', w_j \rangle = 0$, all j .

$$\begin{aligned} \langle v', w_j \rangle &= \langle v - w_i, w_j \rangle \\ &= \langle v - \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \rangle \\ &= \langle v, w_j \rangle - \left\langle \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \right\rangle \\ &= \langle v, w_j \rangle - \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_j \rangle \\ &\quad (\langle w_i, w_j \rangle = 0 \text{ or } \langle w_j, w_j \rangle \text{ since orthog basis}) \\ &= \langle v, w_j \rangle - \frac{\langle v, w_j \rangle}{\langle w_j, w_j \rangle} \langle w_j, w_j \rangle \\ &= 0 \end{aligned}$$

So $\langle v, w_j \rangle = 0$ for all $j = 1, 2, \dots, k$ so $v' \in W^\perp$ \square

March 27th 2019

Last time: Thm 56: $W \leq V$, for all $v \in V$ exists unique $w \in W, v' \in W^\perp$ so that

$$v = v' + w$$

If $\alpha = \{w_1, \dots, w_n\}$ orthog basis W

$$\text{proj}_W v = w = \sum_{i=1}^n \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

Proof. Uniqueness. To prove, suppose $v = \hat{v}' + \hat{w}$, where $\hat{w} \in W, \hat{v}' \in W^\perp$. Then,

$$\begin{aligned} \vec{0} - v - v &= (v' + w) - (\hat{v}' + \hat{w}) \\ \vec{0} &= v' - \hat{v}' + w - \hat{w} \\ \hat{w} - w &= v' - \hat{v}' \end{aligned}$$

LHS in W , RHS in W^\perp , since $v', \hat{v}' \in W^\perp$ and W^\perp subspace and W subspace.

So $\hat{w} - w \in W \cap W^\perp = \{\vec{0}\}$

$$\hat{w} - w = \vec{0}$$

so $\hat{w} = w$. Similarly, $v' - \hat{v}' \in W^\perp \cap W = \{\vec{0}\}$. So $v' = \hat{v}'$ \square

Terminology

If $\alpha = \{w_1, w_2, \dots, w_m\}$ is an orthogonal set of non-zero vectors, for $v \in V$ the scalars $\frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$ are called *Fourier coefficients* of v relative to α .

If α is actually basis of V , Fourier coefficients are coords of v relative to α . If α is a basis for a subspace W , Fourier coefficients give the scalars needed to compute $\text{proj}_W v$. If $v \in W$, $\text{proj}_W v = v$, so these coeffs are cords of $v \in W$.

Note To compute proj , need *orthog* basis W . How to find one?

Lemma 57 (Pythagoras' Thm). *If $u, v \in V$ are orthogonal, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$*

Proof. Exercise. \square

Note $\|u - v\|$ = "distance between u and v ". Compare to θ ; two vectors with very different norm can be very far apart yet have a small angle. Similarly, inverting the direction of a vector gives us a large angle but a small distance.

Theorem 58. Let $W \leq V, v \in V, w = \text{proj}_W v$. Then w is the “closest vector in W to v ” in the sense that if $z \in W$ is any vector

$$\|v - w\| \leq \|v - z\|$$

Proof. Recall $\|u\| = \sqrt{\langle u, u \rangle}, \|u\|^2 = \langle u, u \rangle$. Write $v = v' + w$.

$$\begin{aligned} \|v - z\|^2 &= \|v' + w - z\|^2 = \|v' + (w - z)\|^2 \\ &= \|v'\|^2 + \|w - z\|^2 \end{aligned} \quad (\text{Pythagoras})$$

($v' \in W^\perp, w - z \in W$, so $v', w - z$ are orthogonal)

$$\begin{aligned} \|v - z\|^2 &\geq \|v'\|^2 \\ &= \|v - w\|^2 \end{aligned}$$

Take square root. \square

Gram-Schmidt Orthogonalization Process

Or “how to produce an orthogonal basis”. Replace w_2 by $v' = w_2 - \text{proj}_{w_1} w_1$

Let $W \leq V, \alpha = \{w_1, w_2, \dots, w_m\}$ basis of W . Produce a new basis $\beta = \{v_1, v_2, \dots, v_m\}$ for W by

$$\begin{aligned} v_1 &= w_1 \\ v_i &= w_i - \text{proj}_{\beta_{i-1}} w_i \quad (\text{for } i = 2, 3, \dots, m) \end{aligned}$$

Where $\beta_{i-1} = \text{span}\{v_1, v_2, \dots, v_{i-1}\}$. We will see that $\{v_1, v_2, \dots, v_{i-1}\}$ orthogonal basis for β_{i-1} so in fact

$$v_i = w_i - \left(\sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j \right)$$

Theorem 59. For each $i = 1, 2, \dots, m$, $\{v_1, v_2, \dots, v_i\}$ is orthog basis for $\text{span}\{w_1, w_2, \dots, w_i\}$. In particular, $\{v_1, v_2, \dots, v_m\}$ is orthog basis of W (you can make it ONB by normalizing each v_i)

Proof. Omit. Expand some more products. \square

Ex $W = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$. Find ONB of W .

Sol Apply Gram-Schmidt

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = w_2 - \text{proj}_{v_1} w_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{0+2+0+0}{1+4+0+1} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 1/3 \\ 1 \\ -1/3 \end{pmatrix}$$

Replace by $\begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix} = v_2$.

$$v_3 = w_3 - \text{proj}_{\text{span of } \{v_1, v_2\}} w_3 = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \right)$$

$$= \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

Replace by $v_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix}$. Orthonormal basis = $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$

March 29th 2019

Recall: An orthonormal basis $\{v_1, \dots, v_n\}$ is s.t.

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- (1) Say we want an orthonormal basis for $P_2(\mathbb{R})$ with standard inner product.

$$\langle f, g \rangle = \int_0^1 f \cdot g(x) dx$$

Sol Take our standard basis $\{1, x, x^2\}$, apply Gram-Schmidt.

- (i) Consider $v_1 = 1$. check unit length

$$\|v_1\| = 1 = \sqrt{\int_0^1 1 dx}$$

Already normal! Apply G-S process to x . Let $v_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$:

$$\begin{aligned} x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} &= x - \langle x, 1 \rangle \cdot 1 \\ &= x - \int_0^1 x \cdot 1 dx \\ &= x - \frac{1}{2} \end{aligned}$$

Normalize v'_2 .

$$\begin{aligned} \|v'_2\| &= \sqrt{\langle v'_2, v'_2 \rangle} \\ \langle v'_2, v'_2 \rangle &= \int_0^1 (x - \frac{1}{2})^2 dx \\ &= \frac{1}{12} \\ \rightarrow \|v'_2\| &= \frac{1}{\sqrt{12}} \end{aligned}$$

v_2 (normalize v'_2) = $\sqrt{12} - v'_2 = 2\sqrt{3}x = \sqrt{3}$ Consider x^2 . let $v'_3 = x^2 - \text{proj}(x^2) = x^2 - x + \frac{1}{6}$ then normalize v'_3 to get

$$\begin{aligned} \langle v'_3, v'_3 \rangle &= 1/180 \\ \rightarrow \|v'_3\| &= \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}} \\ 1/3 &= 6\sqrt{5} - 6\sqrt{5} + \sqrt{5} \end{aligned}$$

So our orthonormal basis is $\{v_1, v_2, v_3\}$

- (ii) Now, what about finding the proj of x^2 onto span $\{1, x\}$. v_1, v_2 to be basis elements for this subspace. $\{1, x\} \iff \{v_1, v_2\}$.

Let $\text{span}\{1, x\} = W$, with basis $\{v_1, v_2\}$,

$$\begin{aligned} \text{proj}_W(x^2) &= \frac{\langle x^2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle x^2, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\ &= \langle x^2, v_1 \rangle v_1 + \langle x^2, v_2 \rangle v_2 \\ &= \int_0^1 x^2 \cdot 1 dx \cdot 1 + \left(\int_0^1 x^2 (2\sqrt{3}x = \sqrt{3}) dx \right) (2\sqrt{3}x - \sqrt{3}) \\ &= x - \frac{1}{6} \end{aligned}$$

Theorem 60. Let $\alpha_1 = \{v_1, v_2, \dots, v_k\}$ be orthonormal set in V , $n = \dim(V)$, let $W = \text{span}\{v_1, \dots, v_k\} \subseteq V$ be a subspace. Then,

(i) α_1 can be extended to an orthonormal basis of V

$$\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

(ii) $\alpha_2 = \{v_{k+1}, \dots, v_n\}$ is an orthogonal basis of $W^\perp = \{v \in V \mid \langle v, v_i \rangle = 0 \ \forall i = 1, \dots, k\}$

(iii) $\dim(W) + \dim(W^\perp) = \dim(V)$

Proof. (idea)

- (i) Extend $\{v_1, \dots, v_k\}$ to a basis in the usual way, then apply G-S.
Omit.

□

Diagonalization

Eigenvalues + eigenvectors

Def If $T : V \rightarrow V$ is a linear operator, if $\vec{v} \neq 0$ and $T(\vec{v}) = \lambda \vec{v}$ for some $\lambda \in K$, then \vec{v} is called an eigenvector with eigenvalue λ .

Similarly if $A \in M_{n \times n}(K)$ and $A\vec{v} = \lambda \vec{v}$ and $\vec{v} \neq 0$ then \vec{v} is an eigenvector with eigenvalue λ .

Remark $\vec{v} \neq 0!$ $\lambda = 0$ is allowed!

Proposition 61. $\lambda = 0$ is an eigenvalue $\iff T$ is NOT INJECTIVE (in particular, not invertible!)

Proof. Prove both ways.

1. " \Rightarrow " if $\vec{v} \neq 0$ and

$$\begin{aligned} T(\vec{v}) &= \lambda \vec{v} = \vec{0} \\ \Rightarrow \vec{v} &\in \text{Ker}(T), \vec{v} \neq 0 \\ \Rightarrow \text{Ker}(T) &\text{ NOT trivial} \\ \Rightarrow T &\text{ is NOT injective} \end{aligned}$$

2. " \Leftarrow "

$$\begin{aligned} T \text{ not inj} &\Rightarrow \text{Ker}(T) \text{ NOT trivial} \\ &\Rightarrow \exists \vec{v} \neq 0 \text{ s.t. } T(\vec{v}) = 0 \\ &\Rightarrow T(\vec{v}) = \vec{0} = 0 \cdot \vec{v} \\ &\Rightarrow \lambda \text{ is an eigenvalue with eigenvector } \vec{v} \end{aligned}$$

□

Problem Let $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ which gives a rotation by π about the z -axis!
Thinking geometrically, can we find some eigenvalues and vectors?

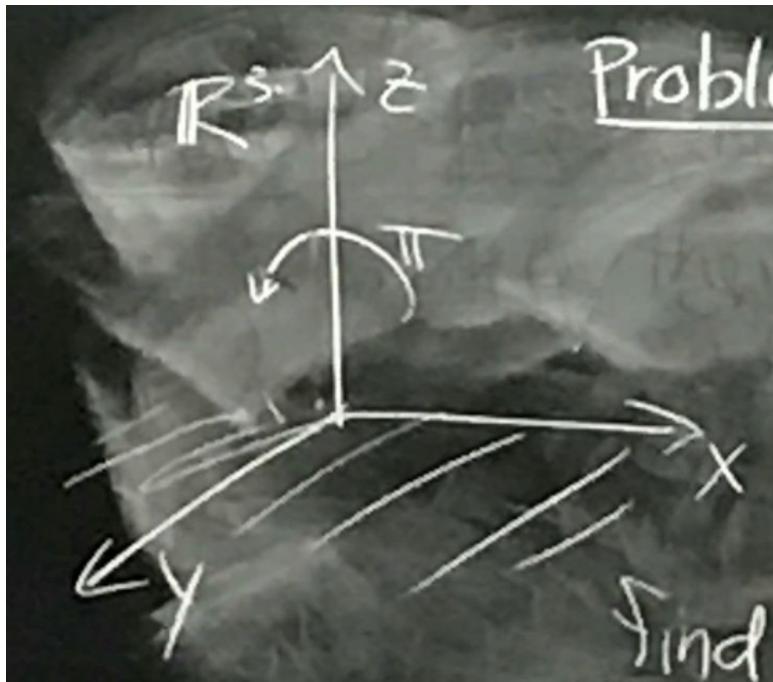


Figure 10: Problem

The z -axis itself $(0, 0, 1) \in \mathbb{R}^3$ is an eigenvector with eigenvalue 1.
 z -axis is fixed by T .

$$\begin{aligned} &\Rightarrow T(\vec{v}_1) = \vec{v}_1 = \vec{v}_1 = 1 \cdot \vec{v}_1 \\ &\Rightarrow \lambda = 1 \text{ e.v.} \end{aligned}$$

Vectors lying in the $x - y$ plane have $\lambda = -1$ as an eigenvalue. Let $v \in x - y$ -plane $\rightarrow v = (x, y, 0)$

$$\begin{aligned} T(\vec{v}) &= (-x, -y, 0) \\ &= -(x, y, 0) \\ &= -1 \cdot \vec{v} \end{aligned}$$

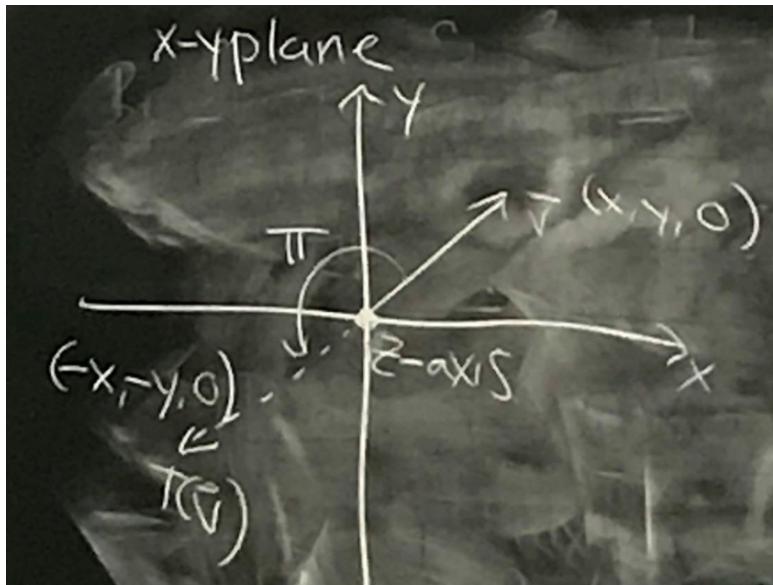


Figure 11: Problem

$\Rightarrow \lambda = -1$ is an eigenvalue!

Question How do we find eigenvalues and eigenvectors algebraically?

Def Let $A \in \mathcal{M}_{n \times n}(K)$. The *characteristic polynomial* of A is defined as

$$c_A(t) = \det(A - tI)$$

Example Let $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find $c_A(t)$.

Sol

$$\begin{aligned} c_A(t) &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} \\ &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & t-1 & 1-t \end{vmatrix} \\ &= (t-1) \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= (t-1) \begin{vmatrix} -t & -2 & -1 \\ 1 & 3-t & 1 \\ 0 & 0 & -1 \end{vmatrix} \\ &= (t-1)(-1) \begin{vmatrix} -t & -2 \\ 1 & 3-t \end{vmatrix} \\ &= -(t-1)(t^2 - 3t + 2) \end{aligned}$$

Theorem 62. Let $A \in \mathcal{M}_{n \times n}(K)$. Then

- (i) $c_A(t)$ is a polynomial of degree n
- (ii) λ is an eigenvalue of $A \iff \lambda$ is a root of $c_A(t)$
- (iii) v is an eigenvector $\iff v \in \text{Ker}(A - \lambda I)$ and $v \neq \vec{0}$

April 1st 2019

Last time: Recall theorem 62:

Theorem 62: Let $A \in \mathcal{M}_{n \times n}(K)$. Then

- (i) Characteristic polynomial $c_A(t)$ is poly of degree n
- (ii) λ is eigenvalue of $A \iff \lambda$ is a root of $c_A(t)$
- (iii) $v \in K^n$ is eigenvector of A with eigenvalue $\lambda \iff v \in \text{Ker}(A - \lambda I)$ and $v \neq \vec{0}$

Proof. By direct proof.

(i) Omit.

(ii) We have:

$$\begin{aligned}
 \lambda \text{ is eigenvalue} &\iff \exists \text{ eigenvector } v \text{ with eigenvalue } \lambda \\
 &\iff \text{Ker}(A - \lambda I) \neq \{\vec{0}\} \\
 &\iff \text{lin transformation defined by } A - \lambda I \text{ not injective} \\
 &\iff \text{lin transformation defined by } A - \lambda I \text{ not bijective} \\
 &\iff A - \lambda I \text{ not invertible} \\
 &\iff \det(A - \lambda I) = 0 \quad (\text{ie } c_A(\lambda) = 0)
 \end{aligned}$$

(iii) We have:

$$\begin{aligned}
 v \text{ is eigenvector with eigenvalue } \lambda &\iff Av = \lambda v \\
 &\iff Av - \lambda v = \vec{0} \\
 &\iff Av - \lambda Iv = \vec{0} \\
 &\iff (A - \lambda I)v = \vec{0} \\
 &\iff v \in \text{Ker}(A - \lambda I) \\
 &\quad (\text{and } v \neq \vec{0})
 \end{aligned}$$

□

Def If λ is an eigenvalue of A , the eigenspace for λ is

$$\begin{aligned} E_\lambda &= \text{Ker}(A - \lambda I) \\ &= \{v \in K^n | (A - \lambda I)v = \vec{0}\} \\ &= \{v \in K^n | Av = \lambda v\} \\ &= \text{all eigenvectors for } \lambda \text{ and also } \vec{0} \end{aligned}$$

It is a subspace since Ker is always a subspace.

Ex $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find basis for each eigenspace.

Sol Last class, $c_A(t) = \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} = -(t-1)^2(t-2)$.

Eigenvalues $\lambda = 1, 2$.

Eigenspace E_1 ($\lambda = 1$) Solve $(A - I)\vec{x} = \vec{0}$. Note: Can expect to get at least one free variable in these kind of problems.

$$\begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = -x_2 - x_3$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$\vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So basis $\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$ for E_1 .

Eigenspace E_2 Solve $(A - 2I)\vec{x} = \vec{0}$

$$\begin{pmatrix} -2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = -x_3$$

$$x_2 = x_3$$

$$x_3 = x_3$$

So basis is $\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$

Proposition 63. Let $A, B \in \mathcal{M}_{n \times n}(K)$ be similar matrices, ie $Q^{-1}AQ = B$ for some $Q \in \mathcal{M}_{n \times n}(K)$. Then,

$$(i) \ det(A) = det(B)$$

$$(ii) \ c_A(t) = c_B(t)$$

Proof. (i) Omit. Like (ii) (follows from it)

(ii)

$$\begin{aligned} c_A(t) &= \det(A - tI) \\ &= \det(QBQ^{-1} - tI) \\ &= \det(QBQ^{-1} - tQIQ^{-1}) \\ &= \det(Q(B - tI)Q^{-1}) \\ &= \det(Q)\det(B - tI)\det(Q^{-1}) \\ &= \det(Q)c_B(t)\frac{1}{\det(Q)} \\ &= C_B(t) \end{aligned}$$

□

Def $T : V \rightarrow V$ linear op. The characteristic polynomial of T is

$$c_T(t) = \det([T]_\alpha - tI)$$

where α is *any* basis of V .

Remark α does not matter, since if β is any other basis then

$$[T]_\beta^\beta = Q_\alpha^\beta [T]_\alpha^\alpha Q_\beta^\alpha$$

ie $[T]_\beta[T]_\alpha$ are similar, same characteristic polynomial.

Ex For $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by

$$T(f(x)) = (1+x)f'(x)$$

Find basis for each eigenspace.

Sol Need standard matrix for T , any basis. Use $\alpha = \{1, x, x^2\}$. Calculate

$$T(1) = (1+x)(0) = 0$$

$$T(x) = (1+x)(1) = 1+x$$

$$T(x^2) = (1+x)(2x) = 2x + 2x^2$$

$$[T]_\alpha^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$c_T(t) = \begin{vmatrix} -t & 1 & 0 \\ 0 & 1-t & 2 \\ 0 & 0 & 2-t \end{vmatrix} = t(1-t)(2-t)$$

So $\lambda = 0, 1, 2$ eigenvalues.

For E_0 ($\lambda = 0$) Solve $([T] - 0I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = x_1$$

$$x_2 = 0$$

$$x_3 = 0$$

Basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ie $f(x) = 1$.

Check $T(f(x)) = T(1) = (1+x)0 = 0f(x)$.

For E_1

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = x_2$$

$$x_2 = x_2$$

$$x_3 = 0$$

Basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ie $g(x) = 1+x$.

Check $T(f(x)) = T(1+x) = (1+x)(1) = 1(1+x)$.

For E_2

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = x_3$$

$$x_2 = 2x_3$$

$$x_3 = x_3$$

Basis $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ie $h(x) = 1+2x+x^2$.

Check $T(f(x)) = T(1+2x+x^2) = (1+x)(2+2x) = 2+4x+2x^2 = 2(1+2x+x^2)$.

April 3rd 2019

Last time: $E_\lambda = \{v \in V | T(v) = \lambda v\} = \text{eigenspace}$

Proposition 64. Let $\lambda_1 \neq \lambda_2$ be eigenvalues of T . Then

$$E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$$

Proof. $\{\vec{0}\} \supseteq E_{\lambda_1} \cap E_{\lambda_2}$ since $\vec{0}$ is in both subspaces. For other inclusion, suppose $v \in E_{\lambda_1} \cap E_{\lambda_2}$, so $T(v) = \lambda_1 v$ and $T(v) = \lambda_2 v$ so $\lambda_1 v - \lambda_2 v = \vec{0}$ so

$$(\lambda_1 - \lambda_2)v = \vec{0}$$

If $v \neq \vec{0}$, then $\lambda_1 - \lambda_2 = 0$, contradicts $\lambda_1 \neq \lambda_2$ so $v = \vec{0}$. So $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$. \square

Diagonalization

Idea: Diagonal matrices are very nice. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$$\begin{aligned} AA &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{pmatrix} \end{aligned}$$

In fact, $A^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$. Easy!

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= 3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

In fact, eigenvalues are 2, 3, 4 corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Def

1 $A \in \mathcal{M}_{n \times n}(K)$ is *diagonalizable* if $\exists Q \in \mathcal{M}_{n \times n}(K)$ so that

$$Q^{-1}AQ = D \quad (\text{with } D \text{ diagonal})$$

(ie A is *similar* to a diagonal matrix)

- 2 Linear operator $T : V \rightarrow V$ is *diagonalizable* if \exists basis α of V so that $[T]_\alpha$ is a diagonal matrix.

Note: For any bases α, β of V ,

$$[T]_\alpha = Q^{-1}[T]_\beta Q \quad (Q = Q_\alpha^\beta)$$

ie T diagonalizable $\iff [T]_\beta$ diagonalizable, β any basis.

Theorem 65. Let $T : V \rightarrow V$ be linear operator.

- (1) T diagonalizable $\iff \exists$ basis α composed of eigenvectors of T .
- (2) If $\alpha = \{v_1, v_2, \dots, v_n\}$ is basis of V , composed of eigenvectors of T , then

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Where λ_i is eigenvalue for v_i , $i = 1, 2, \dots, n$.

- (3) If $A \in \mathcal{M}_{n \times n}(K)$ is diagonalizable with

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Then the i -th column q_i of Q is an eigenvector for A with eigenvalue λ_i , $i = 1, 2, \dots, n$.

Also, $\{q_1, q_2, \dots, q_n\}$ is a basis of K^n .

Proof. (1) " \Rightarrow " Assume T diagonalizable, ie \exists basis $\alpha = \{v_1, v_2, \dots, v_n\}$ so that

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (\text{ie is diagonal})$$

Recall column i of $[T]_\alpha$ is $[T(v_i)]_\alpha$. So $[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$, ie

$$T(v_i) = \lambda_i v_i$$

So v_i is eigenvector for λ_i , so α is basis of V composed of eigenvectors.

" \Leftarrow " Assume $\alpha = \{v_1, v_2, \dots, v_n\}$ basis eigenvectors and $T(v_i) = \lambda_i v_i$. Then,

$$[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$$

$$\text{So } [T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}. \text{ So } T \text{ diagonalizable.}$$

(2) Done in proof of (1).

(3) Assume

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let q_i = column i of Q . Write

$$AQ = QD$$

In AQ , column i is Aq_i (proposition 36). On the right side, column i is

$$\begin{aligned} Q(\text{col } i \text{ of } D) &= Q \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix} \\ &= Q(\lambda_i e_i) \\ &= \lambda_i Q e_i \\ &= \lambda_i q_i \end{aligned}$$

So $Aq_i = \lambda_i q_i$. So q_i is eigenvector for eigenvalue λ_i .

□

Ex Diagonalize $A = \begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix}$, ie find Q, D so that $Q^{-1}AQ = D$ = diagonal.

Sol

$$\begin{aligned}
c_A(t) &= \begin{vmatrix} i-t & -3 \\ 1 & -i-t \end{vmatrix} \\
&= (i-t)(-i-t) + 3 \\
&= i-it+it+t^2+3 \\
&= t^2+4 \\
&= t^2-2i \\
&= (t-2i)(t+2i)
\end{aligned}$$

So $\lambda = 2i, -2i$ **Eigenvectors** For $\lambda = 2i$.

$$\begin{aligned}
\begin{pmatrix} -i & -3 & 0 \\ 1 & -3i & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & -3i & 0 \\ -i & -3 & 0 \end{pmatrix} \\
\rightarrow \begin{pmatrix} 1 & -3i & 0 \\ 0 & 0 & 0 \end{pmatrix} & \\
x_1 &= 3ix_2 \\
x_2 &= x_2 \\
\begin{pmatrix} 3i \\ 1 \end{pmatrix} &
\end{aligned}$$

Check

$$\begin{aligned}
\begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 3i \\ 1 \end{pmatrix} &= \begin{pmatrix} -3-3 \\ 3i-i \end{pmatrix} = \begin{pmatrix} -6 \\ 2i \end{pmatrix} \\
&= 2i \begin{pmatrix} 3i \\ 1 \end{pmatrix}
\end{aligned}$$

For $\lambda = -2i$, similar. Get eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. So $\{\begin{pmatrix} 3i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\}$ is basis of \mathbb{C}^2 composed of eigenvectors of A .

And $Q^{-1}AQ = D$ where

$$D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, Q = \begin{pmatrix} 3i & -i \\ 1 & 1 \end{pmatrix}$$

Or

$$D = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, Q = \begin{pmatrix} -i & 3i \\ 1 & 1 \end{pmatrix}$$

April 5th 2019

Ex Diagonalize $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, if possible.

Sol We already found $\lambda = 1, \lambda = 2$.

$$E_1 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$E_2 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Put bases together, is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$. Basis \mathbb{R}^3 ?

Check

$$\begin{aligned} Q &= \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \det Q &= -1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= -1(-1) - 0 \\ &= 1 \neq 0 \end{aligned}$$

So Q inv, this is basis of eigenvectors.

$$Q^{-1}AQ = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Or, $Q^{-1}AQ = D$, with $Q = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$c_A(t) = (t-1)^2(t-2)$$

Ex Show $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is *not* diagonalizable.

$$\text{Sol } c_A(t) = \begin{vmatrix} 1-t & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)^3. \lambda = 1 \text{ only.}$$

E_1 Solve $(A - I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

On free variable, so $\dim E_1 = 1$. Can't set basis of eigenvector of \mathbb{R}^3 . So *not* diagonalizable.

Ex Is $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ diagonalizable?

Sol

$$\begin{aligned} c_A(t) &= \begin{vmatrix} 1-t & -2 \\ 1 & -1-t \end{vmatrix} = (1-t)(-1-t) + 2 \\ &= -1-t+t+t^2+2=t^2+1 \end{aligned}$$

In \mathbb{R} , no roots, so no eigenvalues, no eigenvectors. A is not diagonalizable as element of $\mathcal{M}_{2 \times 2}(\mathbb{R})$. But in $\mathcal{M}_{2 \times 2}(\mathbb{C})$, have $\lambda = +/-i$ as eigenvalues. Obtain eigenvectors

$$\begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Which are independent, so A is diagonalizable as element of $\mathcal{M}_{2 \times 2}(\mathbb{C})$.

$$\text{So } Q^{-1}AQ = D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Diagonalizability

Def If $T : V \rightarrow V, S : V \rightarrow V$ are linear operators and $a, b \in K$, define
lin op as $aT + bS$ by

$$(aT + bS)(v) = aT(v) + bS(v), v \in V$$

Proposition 66. Let $\lambda_1 \neq \lambda_2$ be eigenvalues of $T : V \rightarrow V$ and v_1 eigenvector for λ_1 , v_2 eigenvector for λ_2 . Then v_1, v_2 are linearly independent.

Proof. Suppose $a_1v_1 + a_2v_2 = \vec{0}$. Consider linear operator $T - \lambda_1 I$
($I : V \rightarrow V, I(v) = v$).

Eval $T - \lambda_1 I$ at $\vec{0} = a_1v_1 + a_2v_2$

$$\begin{aligned} \vec{0} &= (T - \lambda_1 I)(\vec{0}) \\ &= (T - \lambda_1 I)(a_1v_1 + a_2v_2) \\ &= (T - \lambda_1 I)(a_1) + (T - \lambda_1 I)(a_2v_2) \\ &= a_1(T - \lambda_1 I)(v_1) + a_2(T - \lambda_1 I)(v_2) \\ &= a_1(T(v_1) - \lambda_1 I(v_1)) + a_2(T(v_2) - \lambda_1 I(v_2)) \\ &= a_1(\lambda_1 v_1 - \lambda_1 v_1) + a_2(\lambda_2 v_2 - \lambda_1 v_2) \quad (v_1, v_2 \text{ eigenvectors}) \\ \vec{0} &= \vec{0} + a_2(\lambda_2 v_2 - \lambda_1 v_2) \end{aligned}$$

So, $a_2(\lambda_2 - \lambda_1)v_2 = \vec{0}$. But $\lambda_1 - \lambda_2 \neq 0, v_2 \neq \vec{0}$ (v_2 eigenvector).

So $a_2 = 0$. Similarly, using $T - \lambda_2 I$ we get $a_1 = 0$. So v_1, v_2 linearly independent. \square

Theorem 67. Let $T : V \rightarrow V$ lin. op., $\lambda_1, \dots, \lambda_k$ eigenvalues. If β_i is a basis for E_{λ_i} , $i = 1, 2, \dots, k$ then

$$\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$$

is a linearly independent set of size

$$|\beta_1| + |\beta_2| + \dots + |\beta_k|$$

In particular, if $\sum_{i=1}^k \dim E_{\lambda_i} = \dim V$, then T is diagonalizable.

Proof. Similar to prop 66. Omit. \square

Def Let λ be an eigenvalue of T .

- (1) The *geometric multiplicity* of λ is $\dim E_\lambda$
- (2) The *algebraic multiplicity* of λ is the greatest m so that $(t - \lambda)^m$ is a factor of $c_T(t)$

Ex $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for $\lambda = 1$, algebraic mult is 3 since

$$c_A(t) = (1 - t)^3$$

Geometric multiplicity is 1 since $\dim E_1 = 1$ (one free variable, see previous example)

Theorem 68. Let λ be eigenvalue of T . Then,

$$1 \leq \text{geometric mult. of } \lambda \leq \text{algebraic mult. of } \lambda$$

Proof. We have two inequalities to prove.

- (1) Since λ eigenvalue, \exists a (non-zero) eigenvector, so $\dim E_\lambda \geq 1$
- (2) Let $d = \dim E_\lambda$ and let $\{v_1, v_2, \dots, v_d\}$ be a basis for E_λ . Extend this to a basis $\alpha = \{v_1, v_2, \dots, v_d, v_{d+1}, \dots, v_n\}$ of V .

Compute $[T]_\alpha$.

$$T(v_1) = \lambda v_1$$

$$T(v_2) = \lambda v_2$$

...

$$T(v_d) = \lambda v_d$$

$$T(v_{d+1}) = \text{something (doesn't actually matter)}$$

...

$$T(v_n) = \text{something (doesn't actually matter)}$$

$$CT(t) = \det((\lambda - tI)^d \oplus B)$$

Figure 12:

$$CT(t) = (\lambda - t)^d c_C(t)$$

Figure 13:

So we have 12. So we have 13. (with I same size as C) So

$$\begin{aligned} c_T(t) &= (\lambda - t)^d \det(C - tI) \\ &= (\lambda - t)^d c_C(t) \end{aligned}$$

So $(\lambda - t)^d$ is a factor of $c_T(t)$. So the greatest factor of $(\lambda - t)$ in $c_T(t)$ is $(\lambda - t)^m$ where $m \geq d$. Ie, $d \leq m$ ie geometric multiplicity \leq algebraic multiplicity.

□

April 8th 2019

Last time: λ eigenvalue,

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

Theorem 69. $T : V \rightarrow V$ linear operator, $n = \dim V$.

- (1) If $K = \mathbb{R}$ and $c_T(t)$ has a non-real root, then T is not diagonalizable.
- (2) If either $K = \mathbb{C}$, or $K = \mathbb{R}$ and all the roots of $c_T(t)$ are real, then

$$T \text{ diagonalizable} \iff$$

For every eigenvalue λ geom mult of $\lambda =$ algebraic mult of λ

(3) If T diagonalizable for each λ_i eigenvalue, β_{λ_i} is a basis of E_{λ_i} ,
then $\beta = \beta_{\lambda_1} \cup \beta_{\lambda_2} \cup \dots \cup \beta_{\lambda_k}$ is a basis of V (composed of
eigenvectors)

Proof. (briefly) If $\lambda_1, \lambda_2, \dots, \lambda_k$ are the eigenvalues then the largest set of independent eigenvectors has size

$$\begin{aligned}\sum_{i=1}^k \dim E_{\lambda_i} &= \sum_{i=1}^k \text{geom mult of } \lambda_i \\ &\leq \sum_{i=1}^k \text{alg mult of } \lambda_i = n\end{aligned}$$

So if any eigenspace is "short", ie

$$\dim E_{\lambda_i} < \text{alg mult } \lambda_i$$

then can't get n linearly independent eigenvectors. \square

Ex Define $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

(it is linear). Is it diagonalizable?

Sol Find $[T]_\alpha$, $\alpha = \{1, x, x^2\}$

$$\begin{aligned}T(1) &= 1 + 0x + 0x^2 \\ T(x) &= 1 + 1x + (1 + 0)x^2 = 1 + x + x^2 \\ T(x^2) &= 1 + 0x + (0 + 2)x^2 = 1 + 0x + 2x^2 \\ [T]_\alpha &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}c_T(t) &= \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1 & 2-t \end{vmatrix} = (1-t) \begin{vmatrix} 1-t & 0 \\ 1 & 2-t \end{vmatrix} \\ &= (1-t)(2-t)\end{aligned}$$

For $\lambda = 2$. $1 \leq \text{geom mult} \leq \text{alg mult} = 1$. So $\dim E_2 = 1 = \text{alg mult}$.

Ok.

For $\lambda = 1$.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 free variables, so $\dim E_1 = 2 = \text{alg mult}$. Ok.

So T is diagonalizable.

Corollary 70. If $n = \dim V$, and $T : V \rightarrow V$ has n distinct eigenvalues, then T is diagonalizable.

Proof. For each of $\lambda_1, \lambda_2, \dots, \lambda_n$

$$1 = \text{geom mult of } \lambda_i = \text{alg mult of } \lambda_i$$

□

Ex $A = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$. Is A diagonalizable

(i) over $K = \mathbb{R}$

(ii) over $K = \mathbb{C}$

Sol

$$c_A(t) = \begin{vmatrix} 5-t & 3 & 1 \\ 4 & 2-t & 5 \\ 2 & 0 & 1-t \end{vmatrix} = -t^3 + 8t^2 - 3t + 24$$

Possible rational roots are $\frac{\text{factors of const term}}{\text{factors of coeff of } t^n \text{ term}}$. Here, factors of 24:

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

Sub each in $-t^3 + 8t^2 - 3t + 24$, find $t = 8$ is a root. Then do long division.

$$\begin{aligned} c_A(t) &= -(t-8)(t^2+3) \\ &= -(t-8)(t+\sqrt{3}i)(t-\sqrt{3}i) \end{aligned}$$

(i) Over \mathbb{R} , not diagonalizable.

(ii) Over \mathbb{C} , diagonalizable (3 distinct eigenvalues in $\dim 3$)

Application: Matrix Powers

Problem Find formula for A^n , $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$

Sol If $P^{-1}AP = D$, then

$$A = PDP^{-1}$$

Then $A^n = PDP^{-1}PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^nP^{-1}$. Diagonalize

A as

$$\begin{aligned} P &= \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix}, \\ D &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \\ P^{-1} &= \frac{1}{-3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \end{aligned}$$

So

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= P \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{-3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{-3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2^n & -2^{n+1} \end{pmatrix} \\ &= \frac{1}{-3} \begin{pmatrix} -2 - 2^n & 2^{n+1} \\ -1 + 2^n & -2^{n+1} \end{pmatrix} \end{aligned}$$

Symmetric Matrices

Def $A \in \mathcal{M}_{n \times n}(K)$ is *symmetric* if $A^T = A$

Def If $A \in \mathcal{M}_{n \times n}(K)$ is symmetric, it is also called *positive definite* if for all $u \in K^n$, $u \neq \vec{0}$.

$$\underbrace{u^T}_{1 \times n} \underbrace{A}_{n \times n} \underbrace{u}_{n \times 1} > 0$$

Theorem 71. If $A \in \mathcal{M}_{n \times n}(K)$ is symmetric and positive definite, then the formula

$$\langle u, v \rangle_A = u^T A$$

(where $u, v \in K^n$) defines an inner product on K^n

$$\begin{aligned} \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle &= (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= (x_1 \ x_2 \ x_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= \text{usual dot product} \end{aligned}$$

April 10th 2019

Recall: $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ symmetric is *positive definite* if $\forall u \in \mathbb{R}^n, u \neq \vec{0}$

$$u^T A u > 0$$

Fact:

$$(AB)^T = B^T A^T (ABC)^T = C^T (AB)^T = C^T B^T A^T$$

Thm 71 (reminder): If $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ which is symmetric and positive definite, then the formula

$$\langle u, v \rangle = u^T A v$$

defines an inner product on \mathbb{R}^n .

Proof. Prove each property.

(I1) Let $u, v, w \in \mathbb{R}^n, c \in \mathbb{R}$. Then

$$\begin{aligned} \langle cu + v, w \rangle &= (cu + v)^T A w \\ &= (cu^T + v^T) A w \\ &= cu^T A w + v^T A w \\ &= c \langle u, w \rangle + \langle v, w \rangle \end{aligned}$$

(I2)

$$\begin{aligned} \langle v, u \rangle &= v^T A u && \text{(Note: if } B \text{ is } 1 \times 1, B^T = B\text{)} \\ &= (v^T A u)^T && \text{(since it is } 1 \times 1 \text{ matrix)} \\ &= u^T A^T (v^T)^T && \text{(A symmetric)} \\ &= u^T A v = \langle u, v \rangle \end{aligned}$$

(I3) Recall first part of (I3): Let $u \in \mathbb{R}^n, u \neq \vec{0}$.

$$\langle u, u \rangle = u^T A u > 0 \quad (\text{since } A \text{ is positive definite})$$

Also, if $u = \vec{0}$ then

$$\langle u, u \rangle = \vec{0}^T A \vec{0} = 0$$

and if $\langle u, u \rangle = \vec{0}$, then $u = \vec{0}$ since $u \neq \vec{0}$ then $\langle u, u \rangle > 0$

□

Ex: In $P_1(\mathbb{R})$ with $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$, we can represent the inner product as a matrix using isomorphism $a + bx \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix}$. Consider,

$$\begin{aligned}\langle a_1 + b_1x, a_2 + b_2x \rangle &= \int_0^1 (a_1 + b_1x)(a_2 + b_2x)dx \\ &= \int_0^1 (a_1a_2 + (a_1b_2 + a_2b_1)x + b_1b_2x^2)dx \\ &= a_1a_2 + \frac{a_1b_2}{2} + \frac{a_2b_1}{2} + \frac{b_1b_2}{2}\end{aligned}$$

In matrix form,

$$\begin{aligned}\left\langle \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right\rangle &= \begin{pmatrix} a_1 & b_1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & \frac{b_2}{2} \\ \frac{a_2}{2} & \frac{b_2}{2} \end{pmatrix} \\ &= a_1a_2 + \frac{a_1b_2}{2} + \frac{a_2b_1}{2} + \frac{b_1b_2}{2}\end{aligned}$$

Orthogonal Diagonalization

Def $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ is called *orthogonal* if its columns form an orthonormal basis of \mathbb{R}^n .

Eg I is orthonormal, $2I$ is not. (Columns are mutually orthogonal and are unit vectors)

Note Ought to be called "orthonormal matrix" but isn't :(

Lemma 72. $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ then

$$A \text{ orthogonal} \iff A^T = A^{-1}$$

Proof. Let v_1, v_2, \dots, v_n be the columns of A . Then the (i, j) -entry of $A^T A$ is

$$\begin{aligned}&\langle \text{row } i \text{ of } A^T, \text{ column } j \text{ of } A \rangle && \text{(dot product)} \\ &= \langle \text{col } i \text{ of } A, \text{ col } j \text{ of } A \rangle \\ &= \langle v_i, v_j \rangle\end{aligned}$$

Then,

$$\begin{aligned}\{v_1, v_2, \dots, v_n\} \text{ ONB} &\iff \langle v_i, v_j \rangle = \delta_{ij} && \text{(all } i, j\text{)} \\ &\iff (\text{i, j)-entry of } A^T A = I \\ &\iff A^T = A^{-1}\end{aligned}$$

□

Ex

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (\text{Std matrix for a rotation in } \mathbb{R}^2 \text{ by } \theta)$$

is orthogonal since

$$\begin{aligned} R^{-1} &= \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \\ &= R^T \end{aligned}$$

Def $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ is *orthogonally diagonalizable* if there is an orthogonal matrix Q such that $Q^{-1}AQ = D = \text{diagonal}$, ie $Q^T AQ = D$

Equivalently, \mathbb{R}^n has an orthonormal basis composed of eigenvectors of A (the columns of Q).

Note: Basis vectors being unit vectors is not an issue, making the eigenvectors orthogonal to each other is the problem.

Ex Let L be a line through the origin in \mathbb{R}^3 , $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is rotation about L by π .

The eigenspaces are

$$\begin{aligned} E_1 &= L & (\lambda = 1) \\ E_{-1} &= \text{plane orthogonal to } L \text{ through } (0, 0, 0) & (\lambda = -1) \end{aligned}$$

T is orthogonally diagonalizable since

- (1) diagonalizable, since $\dim E_1 + \dim E_{-1} = 1 + 2 = 3$
- (2) the eigenspaces are orthogonal to each other.

For ONB of \mathbb{R}^3 , use

$$\begin{aligned} v_1 &= \text{unit vector along } L \\ v_2, v_3 &= \text{two orthog unit vectors in plane } E_1 \end{aligned}$$

Proposition 73. If $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ is orthogonally diagonalizable, then A is symmetric

Proof. Assume $Q^T AQ = D$. WTS ($Q^T = Q^{-1}$)

$$\begin{aligned} QQ^T AQQ^T &= QDQ^T \\ A &= IAI = QDQ^T \end{aligned}$$

Then

$$\begin{aligned} A^T &= (QDQ^T)^T \\ &= (Q^T)^T D^T Q^T \\ &= QDQ^T = A \end{aligned}$$

□

April 12th 2019

Last time: A orthogonally diagonalizable if $Q^T A Q = D$

Orthogonal matrix $Q^{-1} = Q^T$

A orthogonally diagonalizable $\rightarrow A$ symm

Lemma 74. If $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ is symmetric and $u, v \in \mathbb{R}^n$ then

$$\langle Au, v \rangle = \langle u, Av \rangle \quad (\text{dot product})$$

Proof.

$$\begin{aligned} \langle Au, v \rangle &= (Au)^T Iv = u^T A^T Iv \\ &= u^T A I v = u^T I (Av) \\ &= \langle u, Av \rangle \end{aligned}$$

□

Theorem 75. Let $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ be symmetric. Then,

- 1) All eigenvalues of A are real
- 2) The eigenspaces of A are mutually orthogonal in the sense that if $u \in E_\lambda$ and $v \in E_\lambda$ then $\langle u, v \rangle = 0$

Proof. 1) Let λ be an eigenvalue (possibly $\lambda \in \mathbb{C}$) and $v \in E\lambda$ ($v \neq 0$).

Then,

$$\begin{aligned} \langle Av, v \rangle &= \langle v, Av \rangle && (A \text{ symm, lemma 74}) \\ \langle \lambda v, v \rangle &= \langle v, \lambda v \rangle \\ \lambda \langle v, v \rangle &= \bar{\lambda} \langle v, v \rangle \\ \lambda &= \bar{\lambda} && (\text{since } v \neq \vec{0}, \langle v, v \rangle \neq 0) \end{aligned}$$

So $\lambda \in \mathbb{R}$.

2) Let $u \in E_{\lambda_1}$, $v \in E_{\lambda_2}$, $\lambda_1 \neq \lambda_2$. Then,

$$\begin{aligned} \lambda_1 \langle u, v \rangle &= \langle \lambda_1 u, v \rangle \\ &= \langle Au, v \rangle && (u \in E_{\lambda_1}) \\ &= \langle u, Av \rangle && (\text{lemma 74}) \\ &= \langle u, \lambda_2 v \rangle && (v \in E_{\lambda_2}) \\ &= \lambda_2 \langle u, v \rangle \end{aligned}$$

So

$$\begin{aligned} \lambda_1 \langle u, v \rangle &= \lambda_2 \langle u, v \rangle \\ (\lambda_1 - \lambda_2) \langle u, v \rangle &= 0 && (\lambda_1 - \lambda_2 \neq 0 \text{ since } \lambda_1 \neq \lambda_2) \end{aligned}$$

So $\langle u, v \rangle = 0$.

□

Theorem 76. Let $A \in \mathcal{M}_{n \times n}(\mathbb{R})$. Then A orthogonally diagonalizable
 $\iff A$ symmetric.

Proof. Omit. □

How to orthogonally diagonalize symmetric A ?

- (1) Diagonalize A as usual
- (2) For basis of each E_{λ_i} , apply Gram-Schmidt to get ONB

Ex: Orthogonally diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Sol:

$$\begin{aligned} c_A(t) &= \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{bmatrix} \\ &= \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 0 & t & -t \end{bmatrix} \\ &= t \begin{bmatrix} 1-t & 1 & 1 \\ 1 & 1-t & 1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= t(-1(1-t-1) - ((1-t)^2 - 1)) \\ &= t(t+1 - (1-2t+t^2)) \\ &= t^2(-t+3t) \\ &= t^2(-t+3) \end{aligned}$$

Eigenvectors

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ a+b+c \\ a+b+c \end{bmatrix}$$

For $\lambda = 3$, one eigenvector, $u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

For $\lambda = 0$, $u = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ ($Au = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $Av = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

(Since A diagonalizable, $\dim E_0 = 2 = \text{alg. mult.}$)

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$. Last two not orthog. Apply Gram-

Schmidt to $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$\begin{aligned} v_1 &= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ v_2 &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

Now all orthogonal; need to normalize now.

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \right\}$$

$$\text{And } Q^T A Q = D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Theorem 77. Let A be symmetric, $A \in \mathcal{M}_{n \times n}(\mathbb{R})$. Then

$$A \text{ positive definite} \iff \text{all eigenvalues of } A \text{ are positive}$$

Proof. Prove both directions.

- ' \rightarrow ' Assume A positive definite. Let λ eigenvalue, v eigenvector

for λ . Then

$$\begin{aligned} 0 < v^T A v &= v^T I A v \\ &= \langle v, A v \rangle \\ &= \langle v, \lambda v \rangle \\ &= \lambda \langle v, v \rangle \quad (\lambda \in \mathbb{R}) \end{aligned}$$

So $\lambda \langle v, v \rangle > 0$, so $\lambda > 0$ since $\langle v, v \rangle > 0$.

- ' \leftarrow ' Assume all eigenvalues positive. A symmetric so ONB $\{v_1, v_2, \dots, v_n\}$ of eigenvectors. Let $u \in \mathbb{R}^n, u \neq 0$ (want $u^T A u > 0$). Write

$$u = \sum_{i=1}^n a_i v_i$$

Then

$$\begin{aligned} u^T A u &= u^T I A u \\ &= \langle u, A u \rangle \\ &= \left\langle \sum_{i=1}^n a_i v_i, A \left(\sum_{j=1}^n a_j v_j \right) \right\rangle \\ &= \left\langle \sum_{i=1}^n a_i v_i, \sum_{j=1}^n a_j A v_j \right\rangle \\ &= \left\langle \sum_{i=1}^n a_i v_i, \sum_{j=1}^n a_j \lambda_j v_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \lambda_j \langle v_i, v_j \rangle \\ &= \sum_{i=1}^n a_i^2 \lambda_i > 0 \quad (\langle v_i, v_j \rangle = \delta_{ij} \text{ ONB}) \end{aligned}$$

Since $\lambda_i > 0, a_i^2 \geq 0$, at least one $a_i^2 \neq 0$ since $u \neq \vec{0}$

□