MATH223 - Linear Algebra (class notes)

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Contents

List of Theorems

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Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

Motivation

We have linear systems, with two equations, like such:

$$3x - 2y + z = 2$$
$$x - y + z = 1$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in R^3. Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

Complex numbers

Def: Let i be a symbol. We declare $i^2 = -1$.

Now, what we'd like to do is take this symbol i and combine it with the usual real numbers that we are familiar with. We set, for example,

$$3i$$
$$i - 4$$
$$3i - \pi$$
$$\sqrt{i} + 21$$

Def: The field of complex numbers C consists of all expressions of the form $a + bi$, where $a, b \in R$.
Def: Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Notation:

- $0+bi=bi$
- $a+0i=a$ (a *real* number)
- $0+0i=0$

Ex: If $z_1=2-i$, $z_2=5i$, then

$$z_1+z_2=2+4i$$

and

$$z_1z_2=(2-i)(5i)=10i-5i^2=5+10i$$

Def: Let $z=a+bi\in C$

(i) $\bar{z}=a-bi$, called the *complex conjugate* of z

(ii) $|z|=\sqrt{a^2+b^2}$, called the *absolute value* or *modulus*

Def: If $z=a+bi\in C$ and $z\neq 0$ (ie $z\neq 0+0i$), then the number

$$\begin{aligned}z^{-1} &= \frac{\bar{z}}{|z|^2}\\ &= \frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\end{aligned}$$

is called the (multiplicative) inverse of z. It has the property $zz^{-1}=1=z^{-1}z$.

Proof. We have

$$\begin{aligned}zz^{-1} &= (a+bi)(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)\\ &= \frac{a^2-abi+abi-b^2i^2}{a^2+b^2}\\ &= \frac{a^2+0+b^2}{a^2+b^2}\\ &= 1\end{aligned}$$

Note: Since $z\neq 0+0i$, $a^2+b^2\neq 0$ $\square$

Def: If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

Ex: If $z = 1 + 2i, w = 3 - i$ then

$$\begin{aligned}\frac{w}{z} &= wz^{-1} \\ &= (3-i)(\frac{1}{5} - \frac{2}{5}i) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i\end{aligned}$$

Or,

$$\begin{aligned}\frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3-6i-i+2i^2}{1-2i+2i-4i^2} \\ &= \frac{1-7i}{5}\end{aligned}$$

January 9th 2019

Complex numbers as points in R^2

You can view $a + bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in R^2, and they will conserve the same properties (addition of complex numbers corresponds to vector addition in R^2). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

Equations with complex numbers

Fact: Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}(\pm i\sqrt{|a|})^2 &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a & \text{(since } a < 0\text{)}\end{aligned}$$

Fact: Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in C.
Ex: Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.
Sol:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

Note: If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say z, its other root is $\bar{z}$ ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.
Caution: -1 has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Theorem 1 (Fundamental Theorem of Algebra)**.** *If*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 n^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_0 \in \mathrm{C}$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \ldots \cdot (x - r_n)$$

for some complex numbers $r_1, r_2, \ldots, r_n$. Some r_i's may be equal.

Corollary 1.1. *Every such polynomial has at least one root, and at most n distinct roots.*

Note: *Finding* the roots is, in general, quite difficult.
Ex: Factor $2x^3 + 2x$ (over C).

Sol:

$$\begin{aligned} 2(x^3+x) &= 2(x-0)(x^2+1) \\ &= 2(x-0)(x^2-i^2) \\ &= 2(x-0)(x-i)(x+i) \end{aligned}$$

Ex: Solve $x^2 - i = 0$
Sol: $x^2 = i$ so $x = \pm\sqrt{i}$. Want $\sqrt{i}$ in format $a+bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a+bi \\ i &= (a+bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ 0+i &= (a^2-b^2) + 2abi \end{aligned}$$

$$\begin{aligned} 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} && \text{(so a=b both + or both -)} \\ a^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Vector spaces (Ch 4)

Def. The sets R and C (and also Q, rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of K" means that K is either R or C.

January 11th 2019

Last time: *Field* K is R or C (for this class).

Geometric vectors ('arrows')

You can add two vectors (arrows) (see figure 11)
Observation: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
You can rescale a vector (see figure 2) **Observation:** $a(b\vec{u}) = (ab)\vec{u}$.
Also: $1\vec{u} = \vec{u}$
Question: What properties are interesting? What other objects obey the same properties?
Abstraction: Focus on properties more than on the objects.

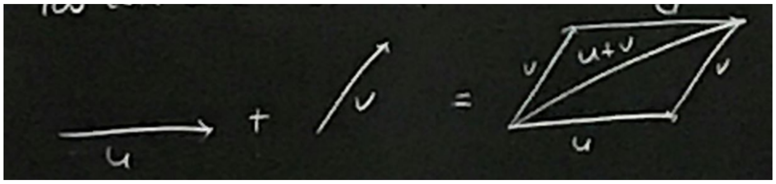


Figure 1: Vector addition

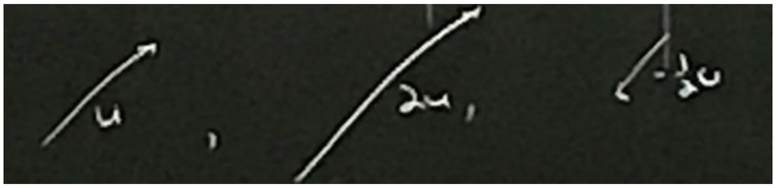


Figure 2: Vector rescaling

Definition of a vector space

Let V be a set, called set of "vectors", and let K be a field (R or C) (elements of K called *scalars*). Assume that we have already defined two operations:

(1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.

(2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then V is called a *vector space* (over K).

(A1) $u + v = v + u$ (commutative laws)

(A2) There exists a vector in V, named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$

(A3) For each $u \in V$, there is a vector in V, called the (additive) inverse of u and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)

(A4) $(u + v) + w = u + (v + w)$

(SM1) $a(u + v) = au + av$ (distributive laws)

(SM2) $(a + b)u = au + bu$

(SM3) $a(bu) = (ab)u$

(SM4) $1u = u$ ($1 \in R$ or C)

These are called the vector space *axioms*.

Examples of vector spaces

Some examples:

(1) $K^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

where $c \in K$ (and $K =$ set of scalar).
Proof that K^n is a vector space
Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.
Proof concept: To prove "for all $x \in A$, something", say "let $x \in A$" (means x is an arbitrary element of A, ie you only know $x \in A$). Then, prove something for that x.
Proof: Let $u, v \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n)$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= (a_1 + b_1, \ldots, a_n + b_n) \qquad \text{(definition of addition in } K^n\text{)} \\ &= (b_1 + a_1, \ldots, b_n + a_n) \quad \text{(since } a + b = b + a \text{ for } R \text{ and } C\text{)} \\ &= (b_1, \ldots, b_n) + (a_1, \ldots, a_n) \quad \text{(definition of addition in } K^n\text{)} \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.
Define $0 = (0, 0, \ldots, 0)$ (which *is* in K^n). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \ldots, a_n + (0, 0, \ldots, 0) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\ &= (a_1, a_2, \ldots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space C^2, $(2 + 3i, 5 - 7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

January 14th 2019

Problem: Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that J is not a vector space.
Solution: Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned}(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3)\\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3)\\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3)\end{aligned}$$

While

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3)\\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3)\\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3)\end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.
Actual proof: Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned}u + (v + w) &= (1, 1) + ((2, 2) + (3, 3))\\ &= (1, 1) + (-2 - 3, 5)\\ &= (1, 1) + (-5, 5)\\ &= (-1 + 5, 6)\\ &= (4, 6)\end{aligned}$$

Whereas,

$$\begin{aligned}(u + v) + w &= ((1, 1) + (2, 2)) + (3, 3)\\ &= (-1 - 2, 3) + (3, 3)\\ &= (-3, 3) + (3, 3)\\ &= (-(-3) - 3, 6)\\ &= (0, 6)\end{aligned}$$

Hence, the axiom does not hold.

More examples of vector spaces

(1) K^n (ie R^n or C^n). See before

(2) $P(K)$ = polynomials, where coefficients are in K. Addition, scalar multiplication are "as expected", ie for multiplication:

$$f(x) = x^2 + 2ix - 4 \quad \in P(C)$$
$$g(x) = -x^2 + ix \quad \in P(C) \qquad \text{(and also in } P(R))$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned} 2if(x) &= 2ix^2 + 4i^2x - 8i \\ &= 2ix^2 - 4x - 8i \end{aligned}$$

(3) $P_n(K)$ = polynomials of degree n or less, coefficient from K. For example,

$$\begin{aligned} x^2 - 2x + 2 &\in P_2(R) \\ x^2 - 2x + 2 &\in P_3(R) \\ x^2 - 2x + 2 &\in P_2(C) \\ x^2 - 2x + 2 &\notin P_1(R) \end{aligned}$$

Note: In $P(K), P_n(K)$ the "vectors" are polynomials.

(4) $M_{m\times n}(K) = m \times n$ matrices with entries from K. Scalars are K, addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2\times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2\times 2}(C) \\ A + B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in $M_{m\times n}(K)$ is the $m \times n$ matrix with all entries 0.

(5) Let X be any set (think $x = R$ or C, but not required). Define $F(X,K) = \{f : X \to K\}$ = all functions from X to K.
Ex: $f(x) = x^2 \in F(R,R)$.
Ex: Let $x = \{1,2\}$. Then g defined by

$$g(1) = 3$$
$$g(2) = \sqrt{2}$$

Addition in this space is defined by:
If $f, g \in F(X,K)$ then $f+g$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

Note that $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f+g)$ is the addition of vectors f and g, while the other $+$ is scalar addition.
Scalar multiplication in this space is defined by: if $f \in F(X,K), c \in K$ then cf is the function defined by

$$(cf)(x) = cf(x)$$

Note that cf is the name of the function, that "multiplication" is scalar multiplication $F(X,\models)$ and $cf(x)$ is the multiplication of two scalars (numbers).
The fact that $F(X,K)$ is a vector space and the axioms are followed is not so obvious.
Prove (A2) true for $F(X,K)$. Define $z \in F(X,K)$ by

$$z(x) = 0 \qquad \text{(for all } x \in X\text{)}$$

Note that 0 here is a scalar. Then if $f \in F(X,K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f+z)(x) &= f(x) + z(x)\\ &= f(x) + 0\\ &= f(x)\end{aligned}$$

Hence, $f+z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.
Exercise: Try (A3).

January 16th 2019

Theorem 2 (Cancellation Law). *Suppose v is a vector space over K. For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.*

Note: To prove "for all" you say let $u \in V$ (means u is an arbitrary vector).

To prove "if p then q", denoted $p \to q$, assume p is true and use it to prove q.

Proof. Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned} (u+w) + (-w) &= (v+w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) && \text{(by A1)} \\ u + \vec{0} &= v + \vec{0} && \text{(by A3)} \\ &= u = v && \text{(by A2)} \end{aligned}$$

□

Theorem 3. *Two points:*

1. *The zero vector is unique*
2. *For each $u \in V$, $-u$ is unique*

Note: To prove something is unique, suppose you have two of them and show they are the same.

Proof. 1) Assume 0 and z both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned} z &= z + 0 && \text{(by *, with } u = z\text{)} \\ &= 0 + z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0\text{)} \end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

Theorem 4. $\forall u \in V, c \in K,$

1) $c\vec{0} = \vec{0}$

2) $0u = \vec{0}$

3) $-(cu) = ((-c)u)$

Proof. Of 2). Let $u \in V$. Then,

$$\begin{aligned}
0u + 0u &= (0+0)u && \text{(By SM2)} \\
0u + 0u &= 0u && \text{(by R addition)} \\
0u + 0u &= 0u + \vec{0} && \text{(by A2)} \\
0u + 0u &= \vec{0} + 0u && \text{(by A4)} \\
0u &= \vec{0} && \text{(by cancellation law)}
\end{aligned}$$

□

Note: $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

Linear combinations and spans

Def: Let $u, v_1, v_2, \ldots, v_n \in V$. If there are scalars $a_1, a_2, \ldots, a_n \in K$ such that $u = a_1v_1, a_2v_2 \ldots a_nv_n$ then u is said to be a linear combination of $v_1, v_2, \ldots, v_n$.
Ex: In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.
Important problem: Given vectors $u, v_1, v_2, \ldots, v_n$, determine if u is a linear combination of $v_1, v_2, \ldots, v_n$ and if so find $a_1, a_2, \ldots, a_n$.
Ex: Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned}
g_1(x) &= x^2 + 2x + 1 \\
g_2(x) &= -2x^2 - 4x - 2 \\
g_3(x) &= 2x^2 - 3
\end{aligned}$$

Sol. Are there a_1, a_2, a_3 s.t.

$$\begin{aligned}
2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\
&= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3)
\end{aligned}$$

Equating coefficients,

$$\begin{aligned}
a_1 - 2a_2 + 2a_3 &= 2 \\
2a_1 - 4a_2 &= 6 \\
a_1 - 2a_2 - 3a_3 &= 8
\end{aligned}$$

Solve the linear system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{(row reduce)}$$

$\therefore$ No solution, because of the last row. f is not a linear combination of g_1, g_2, g_3.
Def: Let $S \subseteq V$ (S is a subset of V) and assume $s \neq 0$. The span of s, denoted $span(s)$ is the set of all linear combinations of vectors from S, ie

$$\begin{aligned} span(s) = \{u \in V | \; \exists \, v_1, v_2, \ldots, v_n \in S \\ \text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\} \end{aligned}$$

January 18th 2019

Last class

$$\begin{aligned} S \subseteq V \\ span(s) = \{u \in V | \; \exists \, v_1, v_2, \ldots, v_n \in S \\ \text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\} \end{aligned}$$

Ex: $S = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Prove $span(S) = R^2$.
Note: $\binom{a}{b}$ means (a, b).
Proof note: To prove two sets A, B are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.
Sol:

(1) Prove $span(S) \subseteq R^2$. Trivial, since any linear combination of vectors in R^2 is still in R^2.

(2) Prove $R^2 \subseteq span(S)$. Let $\binom{a}{b} \in R^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all a, b). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\binom{a}{b} \in span(S)$ so $R^2 \subseteq span(S)$. So by (1), (2), $span(S) = R^2$. □

Note: $Ax = b$, $A_{n \times n}$ if A inv, $x = A^{-1}b$.

Theorem 5. *Let $S \subseteq V$, $S \neq \emptyset$ ($\emptyset$ = empty set). Then,*

(1) If $u, v \in span(S)$ then $u + v \in span(S)$

(2) If $u \in span(S)$ and $c \in K$, then $cu \in span(S)$

(3) $\vec{0} \in span(S)$

Proof. By direct proof.

(1) (Note, "if $u, v \in span(S)$" means for all $u, v \in span(S)$).
Let $u, v \in span(S)$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$
$$v = b_1v_1 + b_2v_2 + \ldots + b_mv_m \text{ where } v_1, \ldots, v_m \in S, b_1, \ldots, b_m \in K$$

Then $u + v = a_1u_1 + \ldots + a_nu_n + b_1v_1 + \ldots b_mv_m$ which is in $span(S)$ since $u_1, \ldots, u_n, v_1, \ldots, v_m \in S$.

(2) Let $u \in span(S), c \in K$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \ldots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \ldots (c_na_n)u_n \end{aligned}$$

Note: If you want to be very formal, you need to write down all of the vector space axioms. Which is in $span(S)$ since it is a linear combination of $a_1, \ldots, a_n$ which are in S.

(3) (Prove $\vec{0} \in span(S)$) Let $u \in S$. **Note**: This is possible only because $S \neq \emptyset$.
Then $u = 1u$, so $u \in span(S)$. Then using $c = 0$ and (2) and fact that $u \in span(S)$,

$$cu = 0u = \vec{0}$$

is also in $span(S)$. **Note:** Since $u = 1u$, $S \subseteq span(S)$.

□

Subspaces

Def. Let V be a vector space and $W \subseteq V$ (subset). If W, using addition and scalar multiplication as defined in V, satisfies the definition of vector space, then W is called a subspace of V, denoted $W \leq V$ (less than equal sign, read as "subspace").
Note: Main issue is that addition and scalar multiplication with vector from W produce vectors which are still in W.

Theorem 6. *Let $W \subseteq V$. Then, if the following three properties hold, $W \leq V$ (subspace).*

- *(SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")*
- *(SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")*
- *(SS3) $\vec{0} \in W$.*

These are the same properties we just proved for spans; in other words, we proved earlier that $span(S)$ is a subspace.

Proof. For W to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

- (A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in V, $u + (v + w) = (u + v) + w$.
- (A2) This is (SS3).
- (A3) This is the one we have to do a bit more work for. Let $w \in W$. Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \qquad \text{(thm from last class)}$$

  is in W, as needed.
- (A4) Still true because it is true in V.
- (SM1-SM4) All hold because they hold in V.

$\square$

January 21st 2019

A note on logic

Let P, Q be statements that are true or false.

(1) "If P then Q", also written symbolically as "$P \Rightarrow Q$" (P *implies* Q) means if P is true, then Q is also true. To *prove* "$P \Rightarrow Q$", assume P and prove Q is true. If you *know* that "$P \Rightarrow Q$" is true, you can *use it*: if you can establish that P is true, you may conclude Q is true.
Ex: Let A be an $n \times n$ matrix:

$$P : dot(A) = 1 \quad Q : \text{"A is invertible"}$$

Thm: $P \Rightarrow Q$

(2) The *converse* of "$P \Rightarrow Q$" is "$Q \Rightarrow P$". This is a (logically) different statement.
Ex: With P and Q as above, "$Q \Rightarrow P$" is not true because $A_{inv} \not\Rightarrow det(A) = 1$.

(3) The *contrapositive* of "$P \Rightarrow Q$" is "$\neg Q \Rightarrow \neg P$" ie "if Q false, then P also false". Logically, this is the same as "$P \Rightarrow Q$".

(4) The *equivalence* "P if and only if Q", written "$P \iff Q$" means "$P \Rightarrow Q$ and also $Q \Rightarrow P$" is true. Also means that either both P and Q are true or both are false.
Ex: $det(A) \neq 0 \iff$ A is invertible.
To prove "$P \iff Q$", need to prove "$P \Rightarrow Q$" and "$Q \Rightarrow P$".
Note: $\neg P \Rightarrow \neg Q$ is the same as $Q \Rightarrow P$.

Subspaces (cont'd)

Thm (last class): Let $W \subseteq V$ (*subset*). If

1. For all $u, v \in W$, $u + v \in W$
2. For all $u \in W$, $c \in K$, $cu \in W$
3. $\vec{0} \in W$

then $W \leq V$ (*subspace*). (ie: $(1), (2), (3)$ are true $\Rightarrow W \leq V$)

Theorem 7. *Let $W \subseteq V$. Then*

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).
Proof. Exercise.

Theorem 8. *Let $W \subseteq V$. Then*

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

Examples of subspaces and non-subspaces

Is each subset a subspace?

(a) $W = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Not a subspace, since the zero vector is not in W. The others are also false, but it's enough to prove that one of the statements does not hold. But $span(W) = R^2$ (so $span(W) \leq R^2$)

(b) $W = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 0\}$. Need to check (1), (2), (3):

(1) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$. Then we know $x + y - z = 0$ and $x' + y' - z' = 0$. Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in W.

(2) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ (means $x + y - z = 0$), let $c \in K$. To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here, $cx + cy - cz = c(x + y - z) = c(0) = 0$. So $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

(3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$, since $0 + 0 - 0 = 0$

Since (1), (2), (3) true, $W \leq R^2$ (subspace)

(c) $W = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 1\}$. This is *not* a subspace. (3) is false.

(d) $W = \{A \in M_{2\times 2} | A_{ij} \geq 0 \forall i, j\}$, where A_{ij} is the entry of A in row i, column j. (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

(2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since, for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ but

$$(-1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

Two special subspaces

Let V be a vector space.

(1) $V \leq V$ is true

(2) $\{\vec{0}\} \leq V$ is true ("zero subspace")

A refinement on the definition of span

Def. If $S = \emptyset$ (emptyset), define $span(S) = \{\vec{0}\}$ (if $S \neq \emptyset$, $span(S)$ defined as before).

Theorem 9. *$span(S) \leq V$.*

Proof Two cases :

1. If $S = \emptyset$, $span(S) = \{\vec{0}\} \leq V$
2. If $S \neq \emptyset$, you already proved $span(S)$ satisfies (1), (2), (3).
   So $span(S) \leq V$.

Theorem 10. *(improved version of subspace conditions) Let $W \subseteq V$. Then*

$$W \leq V \iff W \neq \emptyset \text{ and } \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W$$

Proof We will actually prove $(1), (2), (3) \iff RHS$ (right-hand side). Two parts to proof.

(1) "$(1), (2), (3) \Rightarrow RHS$" or "$\Rightarrow$"

January 23rd 2019

Recap:

(1) If $u, v \in W$ then $u + v \in W$

(2) if $u \in W, c \in K$ then $cu \in W$

(3) $\vec{0} \in W$

Theorem 11. *Let $W \subseteq V$. Then*

$$W \leq V \iff W \neq \emptyset \text{ and } \forall u, v \in W, c \in K \text{ we have } cu + v \in W$$

Proof: Suffices to prove $(1), (2), (3) \iff RHS$.

1. $\Rightarrow$ Assume $(1), (2), (3)$ (prove right-hand side). Two things to prove:

   (1) Since $\vec{0} \in W$ (by (3)), $W \neq \emptyset$

   (2) Let $u, v \in W$ and $c \in K$. Since (2) holds, $cu \in W$. Since (1) holds, $cu \in W$ and $v \in W$, so $cu + v \in W$.

2. $\Leftarrow$ Assume RHS, prove (1), (2), (3).

   (1) Let $u, v \in W$. Apply RHS with $\Leftarrow$ to get

   $$cu + v = 1u + v = u + v \in W$$

   (2) (Prove $\vec{0} \in W$) Since $W \neq \emptyset$, there is a vector $w \in W$. Apply right-hand side with $u = w, v = w, c = -1$. So $cu + v = (-1)w + w = -w + w = \vec{0} \in W$.

   (3) Let $u \in W, c \in K$. Apply RHS $(cu + v \in W)$ with $u = u$, $c = c$, $v = \vec{0}$ (note: $\vec{0} \in W$ by (3) above). Then $cu + v = cu + \vec{0} = cu \in W$ $\square$

Ex: In $F(R, R) = V$ (functions $f : R \to R$), prove that

$$W = \{f \in V | f(3) = 0\}$$

is a subspace. Eg: $f(x) = (x - 3)e^x \in W$.
Solution: (1), (2) together (by last thm). Let $f, g \in W, c \in R$ (prove $cf + g \in W$). We know $f(3) = 0$ and $g(3) = 0$. Then, check $(cf + g)(3) = cf(3) + g(3) = 0 + 0 = 0$. So $cf + g \in W$.

Also, prove $w \neq \emptyset$. $f(x) = x - 3 \in W$, since $f(3) = 0$ (or, $z(3) = 0$ satisfies $z(3) = 0$ so $z \in W$. Note that z is he zero vector of $F(R, R)$).

Theorem 12. *Let $A \in M_{m\times n}(K), b \in K^m$. Define*

$$S = \{x \in K^n | Ax = b\}$$

ie $S =$ solution set to linear system $Ax = b$. Then,

$$S \leq K^n \iff b = \vec{0} \text{ (ie system is homogeneous)}$$

Proof

(i) $\Rightarrow$ Assume $S \leq K^n$. Then $\vec{0}_n \in S$ (by (3)). So $A\vec{0} = b$ but $A\vec{0}_n = \vec{0}_m$ so $\vec{0} = b$.

(ii) $\Leftarrow$ Assume $b = \vec{0}_m$ (prove $S \leq K^n$). Then $A\vec{0}_n = \vec{0}_m$, so $\vec{0}_n \in S$. Next, let $u, v \in S, c \in K$. So $u, v \in K^n$ and $Au = b, Av = b$. Verify $cu + v$ is a solution.

$$\begin{aligned} A(cu+v) &= A(cu) + Av && \text{(prop of matrix multiplication)} \\ &= c(Au) + Av && \text{(prop of matrix multiplication)} \\ &= cb + b \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \\ &= b \quad \square \end{aligned}$$

Ex: Equation $ax + by + cz = d$ describes a plane in R^3 (eg $x + y + z = 1$) (and also, every plane can be described this way). That is,

$$\{(x, y, z) \in R^3 | ax + by + z = d\}$$

is a plane.

By last thm,

$$\begin{aligned} P \text{ is a subspace} &\iff ax + by + cz = d \text{ is a homogeneous system} \\ &\iff d = 0 \\ &\iff P \text{ passes through origin } (0, 0, 0) \end{aligned}$$

Theorem 13. *Let $S \subseteq V$. Then,*

(1) $span(S) \leq V$ and $S \subseteq span(S)$

(2) If $S \subseteq W$, and $W \leq V$ (subspace) then $span(S) \subseteq W$ (actually, $span(S) \leq W$, subspace by (1))

Proof:

(1) $\leq$ We know already. Let $u \in S$. Then $u = 1u$, so $u \in span(S)$

(2) Assume $S \subseteq W$, and $W \leq V$. Let $v \in span(S)$. Then $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ for some scalars and vectors $u_1, u_2, \ldots, u_n \in S$.
Since $S \subseteq W$, $u_1, u_2, \ldots, u_n \in W$. But W subspace. So $a_1u_1, a_2u_2, \ldots, a_nu_n \in W$ (by prop (2) subspace) then $a_1u_1 + a_2u_2 \in W$ (by prop (1) of subspaces). So then $(a_1u_1 + a_2u_2) + a_3u_3 \in W$ (etc). So $a_1u_1 + a_2u_2 + \ldots + a_nu_n \in W$.
Note: "etc" here is actually a proof by mathematical induction.
Omit for now.

January 25th 2019

Interlude : Symbolic logic (briefly)

Let P, Q be statements that could be true (T) or false (F). Define:

(1) $\neg P$, "not P", is F when P is T, T when P is F

(2) $P \wedge Q$, "P and Q", is T exactly when P, Q both T

(3) $P \vee Q$, "P or Q" is T when P, Q both F

(4) $P \Rightarrow Q$, "P implies Q", is T *unless* P is T and Q is F. Hence, $P \Rightarrow Q$ is *equivalent to* $\neg P \vee Q$. We will write $P \Rightarrow Q \equiv \neg P \vee Q$.

(5) $P \iff Q$, "P if and only if Q", is T if both T or both F.

De Morgan's Laws

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Quantifiers

- $\forall$ means "for all"
- $\exists$ means "there exists"

Ex. (A4) (commutativity) $\forall u, v \in V \;\; u + v = v + u$.
Ex. 2 (A2) (zero vector) $\exists z \in V \;\; \forall u \in V \;\; (u + z = u) \wedge (z + u = u)$
(textbook version)

Negating quantifiers

- $\neg \forall u \in V P(u) \equiv \exists u \in V \neg P(u)$
- $\neg \exists u \in V P(u) \equiv \forall u \in V \neg P(u)$

Ex.

$$\begin{aligned}\neg(A2) &\equiv \neg\exists z \in V \forall u \in V \quad u+z=u \wedge z+u=u \\ &\equiv \forall z \in V \neg\forall u \in V \quad u+z=u \wedge z+u=u \\ &\equiv \forall z \in V \exists u \in V \quad \neg(u+z=u \wedge z+u=u) \\ &\equiv \forall z \in V \exists u \in V \quad (u+z\neq u \vee z+u\neq u)\end{aligned}$$

Proof by contradiction

You want to prove some statement P. Proof by contradiction works this way:

(1) Assume $\neg P$

(2) Derive a contradiction (hard part)

(3) Conclude P is true

Ex. Outline of how to prove (A2) *does not hold* in some vector space. You want to prove $\neg(A2)$.

$$\begin{aligned}\neg(A2) &\equiv \neg\exists z \in V\ \forall u \in V \quad u+z=u \wedge z+u=u \\ &\equiv \forall z \in V \neg\forall u \in V \quad u+z=u \wedge z+u=u\end{aligned}$$

Let $z \in V$. Prove the right-hand part ($\neg\forall u \in V \quad u+z=u \wedge z+u=u$) by contradiction. Assume (for contradiction) that

$$\forall u \in V \quad u+z=u \wedge z+u=u \tag{1}$$

Use (1) by substituting $u =$ some specific vector (derive a contradiction). Conclude that ($\neg\forall u \in V \quad u+z=u \wedge z+u=u$) is true.

Last time

Theorem 14. *If $S \subseteq W$, $W \leq V$ then $span(S) \subseteq W$.*

Note. This means if you "promote" a subset to a subspace, adding in only what's necessary, what you get is $span(S)$. Or, $span(S)$ is the "smallest" subspace containing S.
Fact. Subspaces are "closed under taking linear combinations". Ie if $W \leq V$, $w_1, \ldots, w_n \in W$ and $a_1, \ldots, a_n \in K$ then

$$a_1w_1 + a_2w_2 + \ldots + a_nw_n \in W$$

Caution. Linear combinations are *finite* sums by definition. So you can't sum up infinitely many vectors.

Illustration of this theorem

Let $S = \{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}\} \subseteq W = \{\begin{pmatrix}x\\y\\0\end{pmatrix} | x, y \in R\}$. Then $span(S) \subseteq W$ ie $span(S)$ is in xy plane. In fact, $span(S) = W$.

Def. If $W = span(S)$, we say that S spans W or is a spanning set for W.

Ex. $S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$, $span(S)$ = xy-plane in R^3. So S spans the xy-plane.

Ex. 2. $S = \{\begin{pmatrix}1\\1\\0\end{pmatrix}\}$, $span(S) = \{\begin{pmatrix}x\\x\\0\end{pmatrix} | x \in R\}$ = line.

Intersection of two subspaces

Theorem 15. *Let $W_1 \leq V, W_2 \leq V$. Then $W_1 \cap W_2 \leq V$ (ie intersection of two subspaces is a subspace).*

Proof. $W_1 \cap W_2 = \{w \in V | w \in W_1 \land w \in W_2\}$.

(1) $\vec{0} \in W_1, \vec{0} \in W_2$ (because subspace). So $\vec{0} \in W_1 \cap W_2$.

(2) Let $u, v \in W_1 \cap W_2, c \in K$. So $u, v \in W_1$ and $W_1 \in V$ so $cu + v \in W_1$ and $u, v \in W_2$ and $W_2 \in V$ so $cu + v \in W_2$. Hence $cu + v \in W_1 \cap W_2$. $\square$

January 28th 2019

Last time: $W_1 \leq V$ and $W_2 \leq V \Rightarrow W_1 \cap W_2 \leq V$.

Corollary 15.1. *The intersection of any number of subspaces is a subspace.*

Problem. Prove that $W = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0 \land f(2) = 0\}$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.

Sol #1: Directly from subspace properties (omit)

Sol #2: We saw an example proving that $\{f : \mathbb{R} \to \mathbb{R} | f(3) = 0\}$ is a subspace. The "3" is not important, so similarly:

$$W_1 = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0\}$$
$$W_2 = \{f : \mathbb{R} \to \mathbb{R} | f(2) = 0\}$$

both subspaces of $F(\mathbb{R}, \mathbb{R})$. Then $W_1 \cap W_2 = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace.
Q: Is union of two subspaces also a subspace?
A: Not in general.
Eg: $W_1 = \text{x-axis} = \{\binom{x}{0} | x \in \mathbb{R}\} \leq \mathbb{R}^2$
$W_2 = \text{y-axis} = \{\binom{0}{y} | y \in \mathbb{R}\} \leq \mathbb{R}^2$
$W_1 \cup W_2 = \text{xy-axis} = \{\binom{x}{y} | x = 0 \vee y = 0\}$, which, importantly, is not $\mathbb{R}^2$. *Not* a subspace, since $\binom{1}{0} \in W_1 \cup W_2$, $\binom{0}{1} \in W_1 \cup W_2$, but $\binom{1}{1} = \binom{1}{0} + \binom{0}{1} \notin W_1 \cup W_2$.
Note: To promote $W_1 \cup W_2$ to a subspace, you form $span(W_1 \cup W_2)$.
Def: Let $W_1 \leq V$m $W_2 \leq V$. The *sum* of W_1 and W_2 is

$$W_1 + W_2 = \{v \in V | \exists w_1 \in W_1, w_2 \in W_2, \text{ such that } v = w_1 + w_2\}$$

Ex:

$$W_1 = \{ax^2 | a \in \mathbb{R}\} \leq P(\mathbb{R})$$
$$W_2 = \{ax | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

We have,

$$W_1 + W_2 = \{ax^2 + bx | a, b \in \mathbb{R}\}$$

Theorem 16. *Let $W_1 \leq V$, $W_2 \leq V$. Then*

(a) $W_1 + W_2 = span(W_1 \cup W_2)$ (hence $W_1 + W_2$ is a subspace)

(b) $W_1 \leq W_1 + W_2$, $W_2 \leq W_1 + W_2$

Proof:

(a)(1) Prove $W_1 + W_2 \subseteq span(W_1 \cup W_2)$. Let $v \in W_1 + W_2$, so $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1, w_2 \in W_1 \cup W_2$ so $v \in span(W_1 \cup W_2)$

(2) "$\supseteq$". Let $v \in span(W_1 \cup W_2)$. Means $v = a_1 u_1 + a_2 u_2 + \ldots a_n u_n, u_1, u_2, \ldots, u_n \in W_1 \cup W_2$ and $a_1, a_2, \ldots, a_n \in K$. Each u_i is in $W_1 \cup W_2$. Separate into two groups and relabel, so that:

- Those in W_1, call these

$$u_1, u_2, \ldots u_l$$

  So $0 \leq l \leq n$, $l = 0$ means *none* in W_1.
- Those in $W_2 \backslash W_1 = \{w \in W_2 | w \notin W_1\}$ ("set difference"), call these

$$u_{l+1}, \ldots, u_n$$

  So $l = 0$ means all in $W_2 \backslash W_1$, $l = n$ means all in W_1.

Then, let $w_1 = a_1u_1 + a_2u_2 + \ldots + a_lu_l$ (or $w_1 = \vec{0}$ if $l = 0$), $w_2 = a_{l+1}u_{l+1} + \ldots + a_nl_n$ (or $w_2 = \vec{0}$ if $l = n$).
Then $w_1 \in W_1$ since W_1 is a subspace, similarly $w_2 \in W_2$. So

$$\begin{aligned} v &= a_1u_1 + \ldots + a_nu_n \\ &= w_1 + w_2 \in W_1 + W_2 \text{ as required} \end{aligned}$$

(b) $W_1 \leq W_1 + W_2$, $W_2 \leq W_1 + W_2$. Follows from (a), since $S \subseteq span(S)$ $\square$.

Linear independence

Def: Vectors $u_1, u_2, \ldots, u_n \in V$ (all distinct) are said to be *linearly dependent* if $\exists$ scalars $a_1, a_2, \ldots, a_n \in K$ *not all 0* such that

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = \vec{0}$$

Above equation called a *dependence relation.*
Note: If $a_1u_1 + a_2u_2 + \ldots + a_nu_n = \vec{0}$ and $a_1 \neq 0$, then you can solve for u_1:

$$u_1 = \frac{-a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \ldots - \frac{a_n}{a_1}u_n$$

ie $u_1 =$ linear combination of others, "depends on" others.
Ex: $\{x^2 + x, 2x^2, \frac{x}{10}\}$ is a dependent set of vectors in $P(\mathbb{R})$ since

$$(x^2 + x) - \frac{1}{2}(2x^2) - 10(\frac{x}{10}) = 0$$

Def: A set of vectors $S \subseteq V$ (possibly infinite) is dependent if $\exists$ a finite subset $\{v_1, v_2, \ldots, v_n\} \subseteq S$ of it which is dependent.
Def: Vectors $v_1, v_2, \ldots, v_n$ are linearly independent if they are *not* dependent. That is,

$$\begin{aligned} \neg\exists a_1, \ldots, a_n \in K \quad & (a_1u_1 + \ldots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0)) \\ \forall a_1, \ldots, a_n \in K \quad & \neg(a_1u_1 + \ldots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0)) \\ \forall a_1, \ldots, a_n \in K \quad & (\neg(a_1u_1 + \ldots + a_nu_n = \vec{0}) \vee (a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0)) \end{aligned}$$

Note that $P \implies Q \equiv \neg P \vee Q$. In other words, $u_1, u_2, \ldots, u_n$ are linearly independent if

$$\forall a_1, \ldots, a_n \in K(a_1u_1 + \ldots + a_nu_n = \vec{0} \implies a_1 = 0 \wedge \ldots \wedge a_n = 0)$$

Which is to say that the only solution to $a_1u_1 + \ldots a_nu_n = \vec{0}$ is the trivial solution $a_1 = 0, a_2 = 0, \ldots, a_n = 0$.

January 30th 2019

Last class

$v_1, v_2, \ldots, v_n$ *independent* if $x_1 v_1 + \ldots + x_n v_n = \vec{0}$ has only trivial solution $x_1 = x_2 = \ldots = x_n = 0$.
Ex: Prove that $\{1+x^2, x+x^2, 1+x+x^2\}$ is independent.
Solution: Consider equation

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0 = 0(1) + 0x + 0x^2$$

Want to show $a = b = c = 0$ is the only solution.
Equation means for all $x \in K$ ($\mathbb{R}$ or $\mathbb{C}$),

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0$$

So, substitute any scalar for x:

$$\begin{aligned} x = 0 &\quad a + c = 0 \\ x = 1 &\quad 2a + 2b + 2c = 0 \\ x = -1 &\quad 2a + 0b + c = 0 \end{aligned}$$

Can translate into linear system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Row-reduce:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Only solution is $a = 0$, $b = 0$, $c = 0$ so vectors are independent. If we obtain infinitely many, then you can find dependent set so dependent.

Some important cases

(i) $S = \emptyset$ is linearly independent since there are no vectors with which to form a dep. relation.

(ii) If $\vec{0} \in S$, then dependent (since $1\vec{0} = \vec{0}$ *is* a dep. relation)

(iii) $\{u\}$ is independent $\iff u \neq \vec{0}$.
Note: $u + (-1)u = \vec{0}$ is *not* a dep. elation, since u is repeated. But, $\{u, -u\}$ is dependent since

$$u + (-u) = \vec{0}$$

is a dep. relation.

Proposition 17. *Let $A, B \subseteq V$ where $A \subseteq B$.*

(i) If A is dependent, B is also dependent

(ii) If B is independent, A is also independent (contrapositive)

Proof:

(i) If A dep, we have a dep relation

$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = \vec{0} \qquad \text{(not all scalars 0, } v_1, \ldots, v_n \in A)$$

which is also a dependence relation in B since $v_1, \ldots, v_n \in B$.

(ii) This is the contrapositive of (i). □

Note: Converse is false, $B\ dep \not\to A\ dep$.

Extending an independent set

Theorem 18. *Let $S \subseteq V$ be linearly independent and suppose $u \notin S$. Then, $S \cup \{u\}$ independent $\iff u \notin span(S)$.*

Proof:

(i) "$\to$" We will prove this as the contrapositive, ie $u \in span(S) \to dep$. Assume $u \in span(S)$. So,

$$u = a_1v_1 + \ldots + a_nv_n \quad \text{where } v_1, v_2, \ldots, v_n \in S$$
$$\vec{0} = (-1)u + a_1v_1 + \ldots + a_nv_n$$

Which is a linear combination of vectors from $S \cup \{u\}$, not all coefficients 0 since first is -1. Also, the vectors $u, v_1, v_2, \ldots, v_n$ are all distinct, since $u \notin S$. So this is a dependence relation on $S \cup \{u\}$, so the set is dependent.

(ii) "$\leftarrow$" Also by contrapositive. Assume $S \cup \{u\}$ dep, want to show that $u \in span(S)$. So there is a dependence relation on $S \cup \{u\}$. Two cases:

- **Case 1:** Dependence relation does not involve u (or, involves u but with coefficient 0), ie we have

  $$a_1 v_1 + \ldots + a_n v_n = \vec{0} \qquad \text{(not all scalars 0, } v_1, \ldots, v_n \in S\text{)}$$

  But this contradicts independence of S, so case 1 does not occur.
- **Case 2:** Dependence relation involves u (with coeff *not* 0), so

  $$au + a_1 v_1 + \ldots + a_n v_n = \vec{0} \quad v_1, \ldots v_n \in S$$

  and $a \neq 0$. Rewrite:

  $$u = \frac{-a_1}{a} v_1 - \frac{a_2}{a} v_2 - \ldots - \frac{a_n}{a} v_n \qquad (a \neq 0)$$

  Hence $u \in span(S)$. $\square$

Note: Conclusion can be restated as

$$S \cup \{u\} \text{ } dependent \iff u \in span(S)$$

Basis and dimension

Fact: If W is subspace, then $span(W) = W$. (Exercise)

So every subspace *is* a span. But thinking of W as $span(W)$ is excessive. Would like to find the *smallest* S such that

$$span(S) = W$$

Def: Let $W \leq V$. A *basis* of W is a set $B \subseteq V$ such that

(i) $span(B) = W$ ("enough vectors to produce W")

(ii) B is linearly independent ("no extra vectors in B")

Examples:

(i) Let $e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow (row\ i)$. Then,

$$\{e_1, e_2, \ldots, e_n\}$$

is a basis for K^n. For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for K^3.

More next class.

February 1st 2019

Recall: B is a basis of W if $span(B) = W$ and B is linearly independent.
Examples:

(1) $P_n(K)$ has basis $\{1, x, x^2, \ldots, x^n\}$

(2) $P(K)$ has basis $\{1, x, x^2, x^3, \ldots\}$ (infinitely many)

(3) $M_{m\times n}(K)$ has basis $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ where $E^{ij} = m \times n$ matrix of 0s except 1 in row i, column j. eg: $M_{2\times 2}(\mathbb{R})$ has basis

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(4) $W = \{\vec{0}\}$ has basis $\emptyset$ since

   (i) span $\emptyset = \{\vec{0}\}$ (by special def)

   (ii) $\emptyset$ is independent

Two important questions

(1) Does W *always* have basis ? (spoiler: yes)

(2) How to *find* a basis?

Theorem 19 (Bases exist)**.** *Let V be vector space and S a* finite *set with $span(S) = V$. Then there is a subset $B \subseteq S$ which is a basis of V.*

Proof. Algorithm to produce B.

(1) If $V = \{\vec{0}\}$, use $B = \emptyset$.

(2) Take one vector, $u_1 \in S(u_1 \neq \vec{0})$. Consider $span\{u_1\}$

(3) If $span\{u_1\} = V$, done. $B = \{u_1\}$ is a basis (set of one non-zero vector is independent)

(4) If $span\{u_1\} \neq V$, there must be a vector $u_2 \in S$ where $u_2 \notin span(\{u_1\})$ (Why? If not, $S \subseteq span(\{u_1\}) \leq V$, then $span(S) \subseteq span\{u_1\}$, but $span(S) = V$ contradicts $V \neq span\{u_1\}$). By previous theorem, since $u_2 \notin span\{u_1\}$, $\{u_1, u_2\}$ is linearly independent.

(5) Consider $\{u_1, u_2\}$. If $span\{u_1, u_2\} = V$, done: $B = \{u_1, u_2\}$. Else, continue as before, finding $u_3 \in S$, $u_3 \notin span\{u_1, u_2\}$ (etc)

Since S is *finite*, this must *stop* and at that point you have basis $B \subseteq S$. $\square$

Illustration of this thm

Find basis of $\mathbb{R}^3$ that is a subset of

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Following this algorithm,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ u_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Theorem 20. *Let V be a vector space, $L \subseteq V$ a linearly indepedent set, and $S \subseteq V$ a spanning set (ie $V = span(S)$). Then $\exists$ a subset $E \subseteq S$ such that $L \cup E$ is a basis of V (ie you can always* extend *it to a basis)*

Proof Omitted.

Theorem 21. *Suppose V has a finite spanning set S. Then V has a basis and all bases have the same size, which is at most $|S|$.*

Proof Omitted.
Def If V has a finite basis B, then the *dimension* of V is

$$dim\ V = |B|$$

If V does not have a finite basis, it is called *infinite dimensional*.
Ex:

(1) $dim\ K^n = n$.

$$\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right\} \right)$$

(2) $dim\ P_n(K) = n + 1$ (basis $\{1, x, x^2, \dots, x^n\}$)

(3) $P(K)$ is infinite dimensional (A#1, proved a finite set of polynomials cannot span $P(K)$)

(4) $dim\ M_{m \times n}(K) = mn$ (see basis E^{ij}, defined above)

Theorem 22. Every *vector space (including the infinite dimensional ones) has a basis.*

Proof Uses Axiom of Choice. Difficult.

Theorem 23. *Suppose dim $V = n$. Let $A \subseteq V$. Then,*

(1) *If span$(A) = V$, then $|A| \geq n$ (or, if $|A| < n$ then A does not span V) and if also $|A| = n$ then A* is *linearly independent, hence basis.*

(2) *If A is linearly independent, then $|A| \leq n$ (or, if $|A| > n$ then A dep) and if also $|A| = n$ then span$(A) = V$ hence A is a basis.*

Proof Omitted.
Note: If you have *correct number* of vectors, you need only check spanning or independent, not both, to check if basis.
Ex: If you have 7 matrices in $M_{3\times 2}(K)$, they *will be* dependent. If you have 5, it's *not* a basis.

February 4th 2019

Last class

Suppose $dim\ V = n$, $S \subseteq V$, $|S| = n$. Then S *span* $V \iff S$ linearly independent (only in case $|S| = dim\ V$).

Lagrange Interpolation

Problem Given "data points" $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ where all a_i are different. Find a polynomial $p(x)$ of degree $n-1$, $p(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1 x + c_0$ whose graph $y = p(x)$ passes through all the points.
Sol #1 Substitute (a_1, b_1) into $y = p(x)$:

$$b_1 = c_{n-1}a_i^{n-1} + \ldots + c_1 a_i + c_0 \qquad \text{(for each } i = 1, \ldots, n\text{)}$$

Which is a system of n linear equations (vars = $c_{n-1}, \ldots, c_0$) in n variables.

We'll do something different.
Def For scalars $a_1, a_2, \ldots, a_n$ (all different), define the *Lagrange polynomials* for each $i = 1, 2, \ldots, n$ set

$$\begin{aligned} l_i(x) &= \prod_{k=1, k\neq i}^{n} \frac{(x-a_k)}{(a_i - a_k)} \\ &= \frac{(x-a_1)}{(a_i-a_1)} \cdot \frac{(x-a_2)}{(a_i-a_2)} \cdot \ldots \cdot \frac{(x-a_n)}{a_i - a_n} \qquad \text{(omitting } \tfrac{(x-a_i)}{(a_i-a_i)}\text{)} \end{aligned}$$

Ex For $a_1 = 2, a_2 = 4, a_3 = 6$ we would have

$$l_1(x) = \frac{(x-4)}{2-4} \cdot \frac{(x-6)}{(2-6)}$$
$$l_2(x) = \frac{(x-2)}{4-2} \cdot \frac{(x-6)}{(4-6)}$$
$$l_3(x) = \frac{(x-2)}{6-2} \cdot \frac{(x-4)}{(6-4)}$$

Note: All degree 2, $l_1(4) = 0$, $l_1(6) = 0$, $l_1(2) = 1$.
Fact $l_i(a_j) = 0$ if $i \neq j$ and 1 if $i = j$.
Proof If $i \neq j$, there is a factor $\frac{x-a_j}{a_i-a_j}$, so at $x = a_j$, $\frac{a_j-a_j}{a_i-a_j} = 0$. If $i = j$,

$$l_i(a_i) = \prod_{k=1, k\neq i}^{n} \frac{(a_i - a_k)}{(a_i - a_i)} = 1$$

Proposition 24. *Lagrange polynomials $l_1(x), \ldots, l_n(x)$ form a basis of $P_{n-1}(\mathbb{R})$.*

Proof We have n polunomials (they *are* distinct), *dim* $P_{n-1}(\mathbb{R}) = n - 1 + 1 = n$. So correct number. Suffices to prove *span* or lin independence. We'll prove independence. Suppose

$$d_1 l_1(x) + d_2 l_2(x) + \ldots + d_n l_n(x) = 0 \qquad \text{(note: for } all\ x \in \mathbb{R})$$

Substitute $x = a_1$, $x = a_2$, etc into the above. At $x = a_1$, $l_1(a_1) = 1$ but $l_j(a_1) = 0$ for $j \neq 1$ so

$$d_1 1 + d_2 0 + \ldots + d_n 0 = 0$$

so $d_1 = 0$. Similarly, $d_j = 0$ for all j. More formally, for any $j = 1, 2, \ldots, n$ we have at $x = a_j$

$$\sum_{i=1}^{n} d_i l_i(a_j) = 0$$

but all terms are 0 *except* when $i = j$. Set

$$d_j = d_j(1) = d_j l_j(a_j) = 0$$

Hence Lagrange polynomials for a basis.
Problem Find poly degree $n-1$ through points $(a_1, b_1), \ldots, (a_n, b_n)$.
Sol: Set $p(x) = b_1 l_1(x) + b_2 l_2(x) + \ldots + b_n l_n(x)$ (it has degree $n-1$). Then

$$\begin{aligned} p(a_1) &= b_1 l_1(a_1) + b_2 l_2(a_1) + \ldots + b_n l_n(a_1) \\ &= b_1(1) + 0 + 0 + \ldots + 0 \\ &= b_1 \end{aligned}$$

For each $i = 1, 2, \ldots, n$,

$$\begin{aligned} p(a_i) &= \sum_{j=1}^{n} b_j l_j(a_i) \\ &= 0 + 0 + \ldots + b_i l_i(a_i) + \ldots + 0 \\ &= b_i \end{aligned}$$

Dimension of subspaces

Theorem 20. *Let $W \leq V$, V finite-dimensional. Then*

(i) $dim\ W \leq dim\ V$

(ii) $dim\ W = dim\ V \iff W = V$

Proof

(i) Similar to proof that V has basis. Use W as a spanning set for W. Pick out vectors one at a time (similar to before) to build a basis. You cannot put more than $dim\ V$ vectors into your basis, as this would give an independent set in V of size *more than* $dim\ V$ (impossible). So this process has to stop, and it produces a basis for W.

(ii) "$\rightarrow$" Assume $dim\ W = dim\ V = n$. Take basis B of W. It is a size n linearly independent set inside V, hence B also basis for V, hence,

$$V = span\ B = W$$

"$\leftarrow$" If $W = V$, clearly $dim\ W = dim\ V$. □

Subspaces of $\mathbb{R}^3$ If $W \leq \mathbb{R}^3$, $dim\ W = 0, 1, 2$ or 3.

This allows us to make the following classification: **Problem** Let

$dim\ W$	Classification
0	$\{\vec{0}\}$
1	$span\{u\}$ = line through origin
2	$span\{u, v\}$ = plane through origin
3	$\mathbb{R}^3$

$W = \{A \in M_{n\times n}(\mathbb{R}) | tr(A) = 0\}$, where $tr(A)$ = trace of A = sum of entries on diagonal $= A_{11} + A_{22} + \ldots + A_{nn}$.
Exercise Prove W is a subspace.
Will do next class: Find $dim\ W$ and find a basis of W.

February 6th 2019

Intuition

Solution set W to a homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $K^n (n = \#$ of variables). If no equations, $W = K^n$, $dim\ W = n$. For each equation, *expect* the dimension of W to drop by 1, unless the equation is *redundant*.
Eg: In $\mathbb{R}^3$, one equation

$$\begin{aligned} a_1x + b_1y + c_1z = 0 & \qquad \text{(= plane)} \\ \textbf{add in } a_2x + b_2y + c_2z = 0 & \qquad \text{(intersection of two planes, = line)} \\ \textbf{add in } a_3x + b_3y + c_3z = 0 & \qquad \text{(intersection of three planes, (0,0))} \end{aligned}$$

Problem: $W = \{A \in M_{n\times n}(\mathbb{R}) | tr\ A = 0\}$. Find $dim\ W$, basis of W.
Solution #1: Clever way: "guess" a basis. Note: $tr\ A = A_{11} + A_{22} + \ldots + A_{nn}$ (*one* linear condition). Expecting

$$dim\ W = n^2 - 1$$

Observe that $dim\ W \neq n^2$. This happens only if $W = M_{n\times n}(\mathbb{R})$, and obviously there are matrices which don't have trace 0. Specifically:

$$tr \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ldots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

(In proofs, can choose any example, provided property holds).

Know $dim\ W \leq n^2 - 1$. If you can find independent set of size $n^2 - 1$ in W_1, it *will be* a basis. *Try first* $n = 3$. Looking for $3^2 - 1 = 8$ independent 3×3 matrices, all trace 0.

Want trace $= 0$. Therefore, consider all matrices which have all 0's in the diagonal:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

These 6 are obviously independent. Now, take two more which are independent:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we have 8 independent matrices (check carefully). So for $n = 3$, $dim\ W = 8$, this is a basis.

General case

Two types of basis matrices:

(I) All E^{ij} (1 in (i,j)-pos, 0 elsewhere)) where $i \neq j$. How many are there?

$$\begin{aligned} \text{\# of non-diagonal entries} &= \text{entries - entries on diagonal} \\ &= n^2 - n \end{aligned}$$

Or, $\binom{n}{2}$ ways to choose 2 distinct values from $\{1, 2, \ldots, n\}$, 2 ways to order each pair. Total:

$$\begin{aligned} \binom{n}{2} 2 &= \frac{n!}{2!(n-2)!} 2 \\ &= n(n-1) \\ &= n^2 - n \end{aligned}$$

(II) Looking for $n-1$ more, since $n^2 - n + n - 1 = n^2 - 1$

$$\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \cdots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \cdots & \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \cdots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}, \ldots$$

(n-1 of those)

Formally, let, for $i = 1, 2, \ldots, n-1$, $D_i =$ matrix with 1 in pos (i,i) and -1 in pos $(i+1, i+1)$, 0 elsewhere.
Verifying all matrices E^{ii}, D_i are independent; clear that suffices to check $D_1, D_2, \ldots, D_{n-1}$ independent. Suppose

$$x_1 D_1 + x_2 D_2 + \ldots + x_n D_n = n \times n \text{ zero matrix}$$

The $(1,1)$-entry on left is x_1, so $x_1 = 0$. The $(2,2)$-entry on left is $-x_1 + x_2$,

$$x_1 \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \cdots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \cdots & \\ & & & & 0 \end{pmatrix} + \ldots = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \cdots & \\ & & & & 0 \end{pmatrix}$$

but $x_1 = 0$ so $x_2 = 0$ also, etc. So similarly for all $x_i = 0$, so independent. Formally you'd do a proof by induction, but this is good enough.
Now have $n^2 - 1$ independent vectors in W_1 so $dim\ W \geq n^2 - 1$. Already, know $dim\ W \leq n^2 - 1$. So $dim\ W = n^2 - 1$, have independent set of correct size, so basis.

Solution #2: Let x_{ij} be the (i,j)-entry of A. So have n^2 variables $(x_{ij}, i, j = 1, 2, \ldots, n)$ one equation,

$$x_{11} + x_{22} + \ldots + x_{nn} = 0 \qquad (tr\ A = 0)$$

Solve system. All $x_{ij}, i \neq j$ free variables, so are $x_{22}, \ldots, x_{nn}$.

Theorem 21. *Let U, W be finite dimension subspaces of V. Then,*

$$dim(U+W) = dim\ U + dim\ W - dim\ U \cap W$$

It's like sets, $|A \cup B| = |A| + |B| - |A \cap B|$.
Proof Omitted.
Ex: If W is a plane in $\mathbb{R}^3$ (through $(0,0)$) and L is a line in $\mathbb{R}$ (through $(0,0)$) and L is not in the plane, prove $W + L = \mathbb{R}^3$.
Sol: L not in plane gives $L \cap W = \{\vec{0}\}$. So

$$\begin{aligned} dim(L+W) &= dim\ L + dim\ W - dim\ L \cap W \\ &= 1 + 2 - 0 \\ &= 3 \end{aligned}$$

Hence $L + W = \mathbb{R}^3$.
Problem: Suppose $dim\ V = n$, and U, W subspaces, each of dimension more than $\frac{n}{2}$. Prove that $U \cap W \neq \{\vec{0}\}$.
Proof By contradiction. Suppose $U \cap W = \{\vec{0}\}$. So $dim\ U \cap W = 0$. Then

$$\begin{aligned} dim(U+W) &= dim\ U + dim\ W - dim\ U \cap W \\ > \frac{n}{2} + \frac{n}{2} - 0 = n \end{aligned}$$

Says $U + W$ is a subspace of V of dim more than $dim\ V$. Impossible, so $U \cap W \neq \{\vec{0}\}$.
END OF MIDTERM MATERIAL.

February 8th 2019

Monday: No class, office hours during class time. Tuesday night : Midterm!

Linear transformations - Definition and basic properties

(Chap. 5 in the text) **Def.** Let U, V be vector spaces, both over field K. A funcion $T : U \to V$ is called a *linear transformation* if

(i) $\forall u_1, u_2 \in U \ \ T(u_1 + u_2) = T(u_1) + T(u_2)$. The first '+' is in U, while the second '+' is in V. The vectors spaces need not be related in any way, except that they must be over the same field of scalars.

(ii) $\forall u \in U, c \in K \ \ T(cu) = cT(u)$. Again, the first scalar multiplication happens in U, while the second scalar multiplication happens in V.

Comment: Linear transformations are the functions that are somehow "compatible" with the vector space operations.
Ex: Prove that $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(ax^2 + bx + c) = \begin{pmatrix} a+b \\ b+c \end{pmatrix}$$

Sol:

(i) Let $p_1(x) = a_1x^2 + b_1x + c_1$, $p_2(x) = a_2x^2 + b_2x + c_2$ be in $P_2(x)$. Then,

$$\begin{aligned} T(p_1(x) + p_2(x)) &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2) \\ &= \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ b_1 + b_2 + c_1 + c_2 \end{pmatrix} \\ T(p_1(x)) + T(p_2(x)) &= \begin{pmatrix} a_1 + b_1 \\ b_1 + c_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ b_2 + c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 + a_2 + b_2 \\ b_1 + c_1 + b_2 + c_2 \end{pmatrix} \end{aligned}$$

(ii) Let $p(x) = ax^2 + bx + c \in P_2(\mathbb{R}), d \in K$.

$$\begin{aligned} T(dp(x)) &= T(dax^2 + dbx + dc) \\ &= \begin{pmatrix} da + db \\ db + dc \end{pmatrix} \\ &= d\begin{pmatrix} a+b \\ b+c \end{pmatrix} \\ &= dT(ax^2 + bx + c) \\ &= dT(p(x)) \end{aligned}$$

So T is a linear transformation.
Ex Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x^2, x + y$. Show that T is *not* a linear transformation.
Sol Try $u = (2, 3), v = (3, 4)$.

$$\begin{aligned} T(u + v) &= T(5, 7) \\ &= (25, 12) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(2, 3) + T(3, 4) \\ &= (4, 5) + (9, 7) \\ &= (13, 12) \\ &\neq (25, 12) \end{aligned}$$

So T is *not* linear.
Ex: Define $\frac{d}{dx} : P(\mathbb{R}) \to P(\mathbb{R})$ by

$$\frac{d}{dx}p(x) = p'(x) \qquad \text{(derivative)}$$

Then $\frac{d}{dx}$ is a linear transformation, since we know from calculus that

$$\begin{aligned} \frac{d}{dx}(p(x)+q(x)) &= \frac{d}{dx}p(x) + \frac{d}{dx}q(x) \\ \frac{d}{dx}(cp(x)) &= c\frac{d}{dx}p(x) \qquad (c \in \mathbb{R}) \end{aligned}$$

Proposition 22. *Let $T : U \to V$ be a linear transformation. Then,*

(i) $T(\vec{0}) = \vec{0}$ (where the first $\vec{0}$ is the zero vector of U and the second is the zero vector of V)

(ii) $\forall u_1, u_2, \ldots, u_n \in U$ and $c_1, c_2, \ldots, c_n \in K$,

$$T(c_1u_1 + c_2u_2 + \ldots + c_nu_n) = \\ c_1T(u_1) + c_2T(u_2) + \ldots + c_nT(u_n)$$

Proof. (i)

$$\begin{aligned} T(\vec{0}_U) &= T(\vec{0}_U + \vec{0}_U) \\ T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) && \text{(T linear)} \\ \vec{0}_V + T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) && \text{(A2)} \\ \vec{0}_V &= T(\vec{0}_V) && \text{(cancellation law)} \end{aligned}$$

(ii)

$$\begin{aligned} T(c_1u_1 + (c_2u_2 + \ldots + c_nu_n)) &= T(c_1u_1) + T(c_2u_2 + \ldots + c_nu_n) \\ & \qquad \text{(T linear)} \\ &= c_1T(u_1) + T(c_2u_2 + \ldots + c_nu_n) \\ & \qquad \text{(T linear)} \\ &= \ldots \qquad \text{(proof by induction)} \\ &= c_1T(u_1) + \ldots + c_nT(u_n) \end{aligned}$$

□

Proposition 23. *Let $T : U \to V$ function (U, V vector spaces). Then,*

$$T \text{ is linear transformation} \iff \\ \forall u_1, u_2 \in U\ c \in K, T(cu_1 + u_2) = cT(u_1) + T(u_2)$$

Proof: Exercise. □

February 15th 2019

Def ("matrix defines a linear transformation") Let $A \in M_{m\times n}(K)$. Define a function $L_A : K^n \to K^m$ by

$$L_A(v) = Av \qquad \text{(A an } m \times n \text{ matrix, v } n \times 1\text{)}$$

ie multiply matrix by vector.

Proposition 24. *L_a is a linear transformation.*

Proof. Let $u, v \in K^n, c \in K$. Then

$$\begin{aligned} L_A(cu+v) &= A(cu+v) \\ &= A(cu) + Av && \text{(prop of matrix multiplication)} \\ &= cAu + Av \\ &= cL_A(u) + L_A(v) \end{aligned}$$

□

Ex $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix}$, $L_A : R^3 \to R^2$. Calculate:

$$\begin{aligned} L_A(1,3,-2) &= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2-3-4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -5 \end{pmatrix} \end{aligned}$$

Spoiler: All linear transformations between finite-dim vector spaces can be described in this way, “matrix transformation”.

Two special linear transformations

(1) **Zero transformations:** $0 : V \to W$ defined by $O(v) = \vec{0}$ ($\vec{0}$ of W) for all $v \in V$.

(2) **Identity** transformation, $I : V \to V$ (same vector space) $I(v) = v$ for all $v \in V$

Both are linear transformations (exercise).

Kernel and Image (ch. 5.4)

Def Let $T : V \to W$ be a linear transformation. Define:

(i) **Kernel** or **nullspace** of T,

$$Ker(T) = \{v \in V | T(v) = \vec{0}\}$$

Note: Always one vector which satisfies this.

(ii) **Image** of T is

$$Im(T) = \{w \in W | \exists v \in V \ \ w = T(v)\}$$

Note: $Ker(T) \subseteq V$, $Im(T) \subseteq W$.
Ex Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x,y) = (x,0) \qquad \text{("proj onto x-axis")}$$

Then

$$\begin{aligned} Ker(T) &= \{(x,y) \in \mathbb{R}^2 | T(x,y) = (0,0)\} \\ &= \{(0,y) | y \in \mathbb{R}\} \\ &= "y - axis" \\ Im(T) &= \{(x,y) \in \mathbb{R}^2 | (x,y) = T(x',y') \ \ some \ x', y' \in \mathbb{R}\} \\ &= \{(x,0) | x \in \mathbb{R}\} \\ &= "x - axis" \end{aligned}$$

Ex Define $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ to be derivative, $D(f(x)) = f'(x)$. Find kernel and image of D.
Sol We have

$$\begin{aligned} Ker(D) &= \{f \in P_n(\mathbb{R}) | f'(x) = 0\} \\ &= const.\ polys \\ &= \{a | a \in \mathbb{R}\} \\ &= P_0(\mathbb{R}) \end{aligned}$$

Claim $Im(D) = P_{n-1}(\mathbb{R})$.

Proof. Prove inclusion "$\subseteq$" and "$\supseteq$".

(i) "$\subseteq$" Let $f(x) \in Im(D)$. Then $\exists g(x) \in P_n$ s.t. $f(x) = D(g(x)) = g'(x)$. Since $deg(g) \leq n$, $deg(f) = deg(g') \leq n - 1$ (property of differentiation). So $f(x) \in P_{n-1}$.

(ii) "$\supseteq$" Let $f(x) \in P_{n-1}$. Need to find $g(x) \in P_n$ such that $D(g(x)) = g'(x) = f(x)$. Set $g(x) = \int f(x)dx$. Know from calculus that the degree of g is one higher, ie

$$deg(g(x)) = 1 + deg(f(x))$$

So $deg(g) \leq n$. So $g(x) \in P_n$ and $g'(x) = f(x)$ (calculus).

□

Theorem 25. *Let $T : V \to W$ be linear transformation. Then,*

(i) $Ker(T) \leq V$

(ii) $Im(T) \leq W$

Ie they are subspaces.

Proof. By direct proof.

(i) $T(\vec{0}) = \vec{0}$ always (lin transform) so $\vec{0} \in Ker(T)$. Let $v_1, v_2 \in Ker(T), c \in K$. We know $T(v_1) = \vec{0}, T(v_2) = \vec{0}$. Then

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) && \text{(T linear)} \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

Hence $cv_1 + v_2 \in Ker(T)$. So $Ker(T) \subseteq V$ (we already knew $Ker(T) \subseteq V$)

(ii) $T(\vec{0}) = \vec{0}$, hence $\vec{0}_w = T$ (something), ie $\vec{0}_w \in Im(T)$. Let $w_1, w_2 \in Im(T), c \in K$. We know $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Then

$$\begin{aligned} cw_1 + w_2 &= cT(v_1) + T(v_2) \\ &= T(cv_1 + v_2) && \text{(T linear)} \end{aligned}$$

Hence $cw_1 + w_2 \in Im(T)$. So $Im(T) \leq W$.

□

Def $T : V \to W$ linear. The *nullity* of T is $dim\ Ker(T)$ (dim nullspace). The *rank* of T is $dim\ Im(T)$.
Note: $Ker(T) \leq V$ so $nullity(T) \leq dim\ V$, $Im(T) \leq W$ so $rank(T) \leq dim\ W$.
Ex In $T : \mathbb{R}^2 \to \mathbb{R}^2$, proj onto x-axis,

$$\begin{aligned} Ker(T) &= y - axis && \text{(so } nullity(T) = 1\text{)} \\ Im(T) &= x - axis && \text{(so } rank(T) = 1\text{)} \end{aligned}$$

Ex 2 For $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$, differentiation.

$$\begin{aligned} Ker\ D &= P_0(\mathbb{R}) && \text{(so } nullity(D) = 1\text{)} \\ Im\ D &= P_{n-1} && \text{(so } rank(D) = n\text{)} \end{aligned}$$

February 18th 2019

Notation For set $S = \{v_1, v_2, \dots, v_n\}$, $T : V \to W$ denotes $T(S) = \{T(v_1), T(v_2), \dots, T(v_n)\}$.

Proposition 26. *$T : V \to W$ linear and $V = span(S)$. Then $Im\ T = span(T(S))$. In particular, if B basis of V, $T(B)$ **spans** $Im\ (T)$ (but need not be a basis).*

Proof. By direct proof.

(i) "$\subseteq$". Let $w \in Im(T)$, ie $w = T(v)$, some $v \in V$. Since S spans V, $v = \sum_{i=1}^{n} a_i v_i$, some $v_i \in S$. So

$$\begin{aligned} w = T(v) &= T(\sum_{i=1}^{n} a_i v_i) \\ &= \sum_{i=1}^{n} a_i T(v_i) && (T(v_i) \in T(S), \text{by T linear}) \end{aligned}$$

All of which is $\in span(T(S))$.

(ii) "$\supseteq$" Let $w \in span\ T(S)$. So

$$\begin{aligned} w &= \sum_{i=1}^{n} a_i T(v_i) && (\text{for some vectors } v_i \in S) \\ &= T(\sum_{i=1}^{n} a_i v_i && (\text{T linear}) \\ &= T(something) && (\text{so } w \in Im(T)) \end{aligned}$$

$\square$

Ex Define $T : P_2(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Exercise: T is linear. Find basiss for $Im\ T$.
Sol Take basis $\{1, x, x^2\}$ for P_2. Calculate

$$T(1) = \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$T(x) = \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$T(x^2) = \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$$

So $Im\ T = span\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\}$.

Basis for $Im\ T$ is $\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\}$

(so $Im\ T = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R}\}$

Note: The next theorem is very important!

Theorem 27. *("Dimension theorem") Let $T : V \to W$ linear with V finite-dimensional. Then,*

$$dim\ V = dim\ ker(T) + dim\ Im(T)$$
$$dim\ V = nullity(T) + rank(T)$$

Note *dim W* is *not* involved.

Proof. Let $B = \{v_1, v_2, \ldots, v_k\}$ be basis $KerT$ (so $k = dim\ Ker\ T$). Let $n = dim\ V$. Note $T(v_i) = 0$, $(i = 1, 2, \ldots, k)$. Let S span V.
Plan: extend B to basis of V, show $T(extra\ vector)$ = basis of Im.
By theorem 20-1, there exists $E \subseteq S$ such that $B \cup E$ is a basis of V.
Denote

$$E = \{v_{k+1}, \ldots, v_n\} \qquad \text{(note } n = dim\ V\text{, |E| = n - k)}$$

Claim $T(E)$ is basis for $Im\ T$.

(i) $T(E)$ spans ImT

(a) "$\subseteq$" is clear since $T(E) \subseteq Im\ T$ by definition. So $span\ T(E) \leq Im\ (T)$ $(Im\ T \leq W)$

(b) "$\supseteq$" Let $w \in Im(T)$, ie $w = T(v)$, some $v \in V$. Since $B \cup E$ is a basis, $v = \sum_{i=1}^{n} a_i v_i$. Then,

$$\begin{aligned} w &= T(\sum_{i=1}^{n} a_i v_i) \\ &= \sum_{i=1}^{n} a_i T(v_i) && \text{(T linear)} \\ &= \sum_{i=k+1}^{n} a_i v_i && \text{(Since } T(v_i) = 0 \text{ for } i = 1, 2, \ldots, k) \end{aligned}$$

Hence $w \in span(T(E))$, since $E = \{v_{k+1}, \ldots, v_n\}$

(ii) $T(E)$ is linearly independent. Suppose

$$\sum_{i=k+1}^{n} b_i T(v_i) = \vec{0} \qquad \text{(linear comb vectors in } T(E))$$

So by linearity of T,

$$T(\sum_{i=k+1}^{n} b_i v_i) = \vec{0}$$

So $\sum_{i=k+1}^{n} b_i v_i \in Ker\ T$, ie is linear comb of B

So $\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} b_i v_i$

ie $\sum_{i=1}^{k}(-b_i)v_i + \sum_{i=k+1}^{n} b_i v_i = \vec{0}$ is linear comb of $v_1, \ldots, v_n$ (ie $B \cup E$) but these independent. So all $b_i = 0$, hence $T(E)$ independent.

Conclude $T(E)$ basis of $Im\ T$. So,

$$dim\ Im\ T = |T(E)| = |E| = n - k$$

So,

$$\begin{aligned} n &= k + n - k \\ dim\ V = |B| + |T(E)| &= dim\ KerT + dim\ Im\ T \end{aligned}$$

□

Why is $|T(E)| = |E|$? True *unless*

$$T(v_i) = T(v_j) \qquad \text{(for some } i, j \geq k+1, i \neq j\text{)}$$

If so,

$$\begin{aligned} T(v_i) - T(v_j) &= 0 \\ T(v_i - v_j) &= 0 \qquad \text{(so } v_i - v_j \in Ker\ T\text{)} \end{aligned}$$

Hence $v_i - v_j = \sum_{l=1}^{n} a_l v_l$, dep relation on $v_1, \ldots, v_n$. Impossible. □

Problem For $T : P_2 \to \mathcal{M}_{2\times 2},$,

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find basis for $Ker\ T$.

Sol Already know $dim\ Im\ T = 2$ (last ex). So

$$\begin{aligned} dimP_2 &= dim\ Ker\ T + dim\ Im\ T \\ 3 &= dim\ Ker\ T + 2 \end{aligned}$$

So $Ker\ T$ is 1-dimensional. Only need to find *one* non-zero $f(x)$ s.t.

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ie need $f(1) = f(2)$ and $f(0) = 0$. For example, $f(x) = x^2 - 3x$ works. So $\{x^2 - 3x\}$ is a basis for $Ker\ T$ (or, $f(x) = ax^2 + bx + c$, $f(1) = a + b + c = f(2) = 4a + 2b + c$, $f(0) = 0 = c$, solve)

February 20th 2019

Comments on dimension theorem

$T : V \to W$, linear.

$$dim\ V = dim\ (Im\ T) + dim\ (Ker\ T)$$

Left-hand part of the sum: Dimensions that are preserved ("saved") by T. Right-hand part: dimensions that are "lost" when you apply T. **Dimension:** Subspaces are *infinite* sets (except $\{\vec{0}\}$). Dimension gives a way to compare the *sizes* of subspaces.

Injective/surjective transformation (ch. 5.5.)

Def Let $f : X \to Y$ be a *function* (X, Y sets).

(i) f is *surjective* ("onto") if

$$\forall y \in Y\ \ \exists x \in X\ \ f(x) = y$$

(equivalently, the image of f is Y)

(ii) f is called *injective* (or "on-to-one") if

$$\forall x_1, x_2 \in X(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

(equivalently, $\forall x_1, x_2 \in X\ \ (f(x_1) = f(x_2) \to x_1 = x_2)$)

Theorem 28. *("How to check if T inj/surj") Let $T : V \to W$. Then,*

(i) T injective $\iff$ $Ker(T) = \{\vec{0}\}$ (nullity $(T) = 0$)

(ii) T surjective $\iff$ $dim\ (Im\ T) = dim\ W$ ($rank(T) = dimW$)

(i) *Proof.* By direct proof.

(1) "$\Rightarrow$" Assume T inj. (know $\{\vec{0}\} \leq Ker\ T$). Let $v \in Ker\ (T)$. So $T(v) = \vec{0}$. But also $T(\vec{0}) = \vec{0}$, so $T(v) = T(\vec{0})$ hence $v = \vec{0}$ since T is injective.

(2) "$\Leftarrow$" Assume $Ker\ T = \{\vec{0}\}$. Let $v_1, v_2 \in V$. Suppose $T(v_1) = T(v_2)$ (prove $v_1 = v_2$).

$$\begin{aligned} T(v_1) - T(v_2) &= \vec{0} \\ T(v_1 - v_2) &= \vec{0} \qquad \text{(linear)} \end{aligned}$$

So $v_1 - v_2 \in Ker\ T = \{\vec{0}\}$. So $v_1 - v_2 = \vec{0}, v_1 = v_2$.

□

(ii) *Proof.* By direct proof.

(1) "$\Rightarrow$" Assume T is surjective, that is $Im\ T = W$. Hence $dim\ Im\ T = dim\ W$.

(2) "$\Leftarrow$" Assume $dim\ Im\ T = dim\ W$. But $Im\ T \leq W$, hence $Im\ T = W$ (by thm 20-2)

$\square$

Problem Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by

$$T(f(x)) = \int_0^1 f(x)dx$$

(Exercise: T is linear). Is T injective? Surjective?
Sol *Dim Thm*:

$$\begin{aligned} dim\ P_2 &= dim\ Im\ T + dim\ Ker\ T \\ 3 &= dim\ Im\ T + dim\ Ker\ T \end{aligned}$$

Hence $ImT \leq \mathbb{R}^1$, so $Im\ T = \{\vec{0}\}$ or $\mathbb{R}$. It is *not* $\{\vec{0}\}$ since $\int_0^1 1dx = 1 \neq 0$, $T(1) \neq 0$. Hence $Im\ T = \mathbb{R}$ so

$$3 = 1 + dim\ Ker\ T$$

So $dim\ Ker\ T = 2$. $Ker\ T \neq \{\vec{0}\}$ *not* injective. $Im\ T = \mathbb{R}$ *is* surjective.

Theorem 29. *("shortcut when dim same") $T : V \to W$ linear, and $dim\ V = dim\ W$. Then,*

$$T\ injective \iff T\ surjective$$

Proof. Dim Thm:

$$dim\ W = dim\ V = dim\ Im\ T + dim\ Ker\ T$$

If T inj, $dim\ Ker\ T = 0$. So

$$dim\ W = dim\ Im\ T + 0$$

So T surjective (thm 28). If T surj, $dim\ Im\ T = dim\ W$ (thm 28), so

$$dim\ W = dim\ W + dim\ Ker\ T$$

So $dim\ Ker\ T = 0$ so $Ker\ T = \{\vec{0}\}$ $\square$

Problem $T : P_2(\mathbb{R}) \to \mathbb{R}^3$, defined by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$$

Is T injective? Surjective?
Sol Same *dim* $(= 3)$. Check only one. Check surjective directly from def surj:
Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Is $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T(f(x))$, some $f(x) \in P_2$?

That is, given $a, b, c \in \mathbb{R}$, is there a degree 2 polynomial such that $f(0) = a, f(1) = b$, $f(2) = c$? By Lagrange Interpolation, $f(x)$ exists (deg $= 1$, less than # of points). So T surj, so also inj.

Isomorphism and coordinates (ch 5.5, 4.11 and 4.12)

Def: (Isomorphism)

(1) If $T : V \to W$ (linear) is injective and surjective, it is called an *isomorphism*.

(2) If V, W vector spaces and *there exists* an isomorphism $T : V \to W$, we say V and W are *isomorphic* and write $V \simeq W$

Note A function that is injective and surjective is called *bijective*.
Ex $T : P_2(\mathbb{R}) \to \mathbb{R}^3$, $T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$ *is* an isomorphism (last ex.)
so $P_2(\mathbb{R}) \simeq \mathbb{R}^3$
Ex Prove that

$$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is isomorphism $P_2 \to \mathbb{R}^3$.
Sol T is linear : let $f(x), g(x) \in P_2(\mathbb{R}), d \in \mathbb{R}$. Then,

$$\begin{aligned} T(df + g) &= T(c(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) \\ &= T((da_1 + a_2)x^2 + (db_1 + b_2) + (dc_1 + c_2)) \\ &= \begin{pmatrix} da_1 + a_2 \\ db_1 + b_2 \\ dc_1 + c_2 \end{pmatrix} \\ &= d\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \\ &= dT(f) + T(g) \end{aligned}$$

So T linear. Same $dim(= 3)$. Check surj. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Then

$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, hence surj., hence inj., hence *isomorphism*.

February 22nd 2019

Notes about functions

(1) If $f : X \to Y$, then f injective and surjective $\iff$ f is invertible, ie $\exists f^{-1} : Y \to X$ such that $\forall x \in X, y \in Y$ $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$

(2) If $g : Y \to Z$, you can compose f and g to get $g \cdot f . X \to Z$, defined by $(g \cdot f)(x) = g(f(x))$ $x \xrightarrow{f} y \xrightarrow{g} z$

Theorem 30. *Let $T : V \to W$ be an isomorphism (ie T linear, inj, surj.). Then T has an inverse $T^{-1} : W \to V$ which is also a* linear *transformation.*

Proof. Fact that T^{-1} *exists* is since T inj and surj. Prove T^{-1} is linear. Let $w_1, w_2 \in W, c \in K$. Since T surjective, $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Also, $T^{-1}(w_1) = T^{-1}(T(v_1)) = v_1$ and $T^{-1}(w_2) = v_2$. Then

$$\begin{aligned} T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\ &= T^{-1}(T(cv_1 + v_2)) && \text{(T linear)} \\ &= cv_1 + v_2 \\ &= cT^{-1}(w_1) + T^{-1}(w_2) \end{aligned}$$

So T^{-1} linear. $\square$

Ex

$$T : P_2(\mathbb{R}) \to \mathbb{R}^3, T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$T^{-1} : \mathbb{R}^3 \to P_2(\mathbb{R}), T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (ax^2 + bx + c)$$

Point Once you know $V \simeq W$ (isomorphic) you can go back and forth between them, do vector space operations in either V or W. That is, V and W have exactly the same *structure* (as far as addition and scalar multiplication are concerned), even though "vectors" look different.

Proposition 31. *If $V \simeq W$, both finite-dimensional, then $dim\ V = dim\ W$*

Proof. $V \simeq W$ so $\exists T : V \to W$, T inj and surj (bijective), linear. So Dim Thm,

$$dim\ V = dim\ Im\ T + dim\ Ker\ T$$

and T inj., so $dim\ Ker\ T = 0$, and T surj., so $Im\ T = W$, so

$$dim\ V = dim\ W + 0$$

$\square$

Theorem 32. *Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V. For any $v \in V$, you can write*

$$v = \sum_{i=1}^{n} a_i v_i$$

Then,

(a) The numbers $(a_1, a_2, \dots, a_n)$ are unique *and are called the* coordinates of v relative to B, *denoted*

$$[v_b] = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(b) The function $C_B : V \to K^n$ defined by

$$C_B(v) = [v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{("find coordinates")}$$

is an isomorphism

Hence, if $dim\ V = n$ then $V \simeq K^n$

Proof. By direct proof.

(a) Assume v can also be written as

$$v = \sum_{i=1}^{n} b_i v_i \qquad \text{(as well as } \textstyle\sum a_1 v_1 = v\text{)}$$

Then

$$\vec{0} = v - v = (\sum_{i=1}^{n} a_i v_i) - (\sum_{i=1}^{n} b_i v_i)$$
$$\vec{0} = \sum_{i=1}^{n} (a_i - b_i) v_i$$

Since $\{v_1, \ldots, v_n\}$ independent (B = basis) all $a_i - b_i = 0$ ($i = 1, 2, \ldots, n$) so $a_1 = b_1$. Hence representation is *unique*.

(b) Let $v = \sum_{i=1}^{n} a_i v_i, u = \sum_{i=1}^{n} b_i v_i$ be in V, $c \in K$. Then,

$$\begin{aligned} C_B(cv+u) &= C_B(\sum_{i=1}^{n}(ca_i + b_i)v_i) \\ &= \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix} \\ &= c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= C_B(v) + C_B(u) \end{aligned}$$

Hence C_B is linear. To check C_B inj. and surj., since $dim\ V = n = dim\ K^n$, need only check on (other will follow). We will prove surj. Let $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in K^n$. Then let $v = \sum_{i=1}^{n} a_i v_i$, so $C_B(v) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

□

Remarks

(1) We need the coords to be *unique* in order for $C_B : V \to K^n$ to be a (well-defined) function.

(2) If you use a different basis, or even same basis but in different order, you get different coords and also different isomorphism.

Always infinitely many isomorphisms

Lemma 33. *Let $T : V \to W$, $S : W \to U$ be a linear transformation. Then*

(a) $S \cdot T : V \to U$ ($Vt-> Ws-> U$) is linear

(b) If T, S both injective (surjective), then $S \cdot T$ is also injective (surjective)

Proof. Exercise. $\square$

Theorem 34. *Let V, W be finite-dimensional vector spaces over field K. Then,*

$$V \simeq W \iff dim\ V = dim\ W$$

That is, as far as vectir space ops go, only the dimension *really matters.*

Proof. By direct proof.

- "$\Rightarrow$" Prop 31.
- "$\Leftarrow$" $dim\ V = dim\ W = n$. By Thm 32, $V \simeq K^n$, $W \simeq K^n$, using $C_{B_1} : V \to K^n$, $C_{B_2} : W \to K^n$. Then $C_{B_2}^{-1} : K^n \to W$ is an isomorphism (Thm 30), so

  $$C_B^{-1} \cdot C_B : V \to W \qquad (V \overset{C_{B_1}}{\to} K^n \overset{C_{B_2}^{-1}}{\to} W)$$

  is linear, injective, surjective by lemma 33 so it is an isomorphism.

$\square$

February 25th 2019

Recall $V \simeq W \iff dim\ V = dim\ W$ (proved for finite-dim vector spaces only).
Note: If $T : V \to W$ isomorphism, $T^{-1} : W \to V$ is also an isomorphism.
Examples of isomorphisms:

- $P_n(K) \simeq K^{n+1}$
- $\mathcal{M}_{m\times n} \simeq K^{mn}$
- $K^n \simeq K^m \iff n = m$

Question If $n = dim\ V$, then $V \simeq K^n$, why bother studying vector spaces other than K^n?
Answer If you only want to know about addition and scalar multiplication, only K^n matters *but* the "vectors" P_n, $\mathcal{M}_{n\times m}$ etc... have other properties not always related to vector space operations.

For example, in $P_2(\mathbb{R})$ we can evaluate polynomials $f(x)$ at say $x = 3$,

$$f(x) = ax^2 + bx + c$$
$$f(3) = 9a + 3b + c$$

If we consider $P_2(\mathbb{R}) \simeq \mathbb{R}^3$, "eval at $x = 3$" is a linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}$$
$$T(a, b, c) = 9a + 3b + c$$

Computations related to linear transformation

Theorem 35 (T is determined by its value on a basis)**.** *Let V, W be vector spaces, $\{v_1, v_2, \ldots, v_n\}$ basis V.*

Let $w_1, w_2, \ldots, w_n \in W$ be any vectors (need not be distinct). Then there is one *linear transformation $T : V \to W$ s.t. $T(v_i) = w_i$*

Idea of proof If you want to calculate $T(v)v \in V$ (arbitrary element), write v uniquely in terms of basis

$$v = \sum_{i=1}^{n} a_i v_i$$

Then since T is supposed to be linear, compute

$$\begin{aligned} T(v) &= T(\sum a_i v_i) \\ &= \sum a_i T(v_i) \\ &= \sum a_i w_i \end{aligned}$$

Problem Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}, \quad T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

Find $T\binom{3}{4}$.

Solution $\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\}$ = basis $\mathbb{R}^2$, should have enough info to know what T is. Need to find

$$\begin{bmatrix}3\\4\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix}$$
$$\begin{bmatrix}1 & 1 & 3\\1 & -1 & 4\end{bmatrix} \to \begin{bmatrix}1 & 0 & \frac{7}{2}\\0 & 1 & -\frac{1}{2}\end{bmatrix}$$

So

$$\begin{aligned} T\begin{pmatrix}3\\4\end{pmatrix} &= T(\frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}) \\ &= \frac{7}{2}T\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}T\begin{pmatrix}1\\-1\end{pmatrix} \\ &= \frac{7}{2}\begin{bmatrix}-2\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix} \end{aligned}$$

Row, column, nullspace of a matrix

Def $A \in \mathcal{M}_{m\times n}(K)$

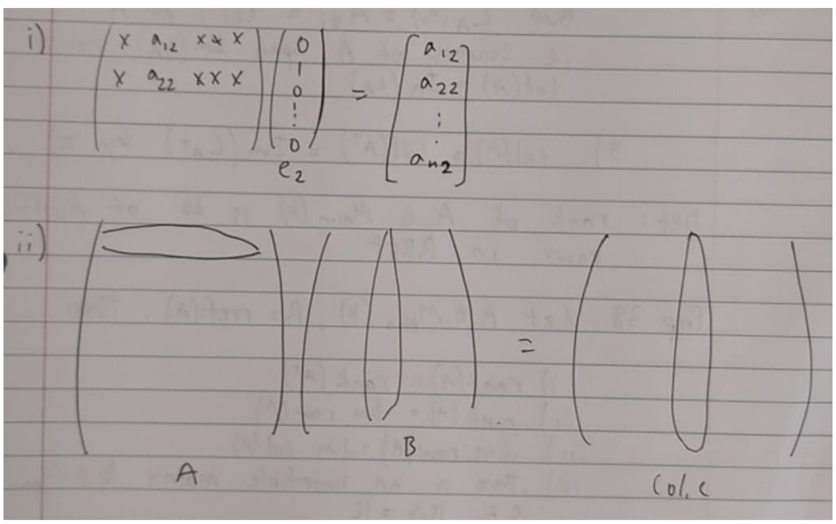
1. The row space, $row(A)$ is the span of the rows of A. Subspace of K^n
2. The column space, $col(A)$ is span of columns. Subspace of K^n
3. $Nullspace(ker)$, is the solution set to the homogeneous system $Ax = \vec{0}$. Subspace of K^n

Proposition 36. *Let* $A \in \mathcal{M}_{m\times n}(K)$*. Then*

(1) A_{ei} = *column* i *of* A

(2) *If* $B \in \mathcal{M}_{n\times p}(K)$ *then column* i *of* AB *is* Ab_i, b_i = *column* i *of* B

Proof. Proof by picture! □



Proposition 37. *Let* $A \in \mathcal{M}_{m\times n}(K)$*, so* $L_A : K^n \to K^m$*.*

(1) $ker(A) = Ker(L_A)$

(2) $col(A) = Im(L_A)$

(3) $row(A) = Im(L_{A^T}$

Proof. By direct proof.

(1)

$$\begin{aligned} Ker(A) &= \{x \in K^n | A_x = \vec{0}\} \\ &= \{x \in K^n | L_A(x) = \vec{0}\} \\ &= Ker(A) \end{aligned}$$

(2) Take basis $\{e_1, e_2, \ldots, e_n\}$ for K^n. Then by prop 26,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots$$

$$L_A(e_1) \ldots L_A(e_n) \text{ spans } Im(L_A)$$

But $L_A(e_1) = A_{ei}$ = column i of A, ie columns of A span $Im(L_A)$ hence $col(A) = Im(L_A)$

(3) $col(A) = col(A^T) = Im(L_{A^T})$ by (2)

□

Def: Rank of $A \in \mathcal{M}_{m \times n}(K)$ is number of non-zero rows in RREF.

Proposition 38. *Let $A \in \mathcal{M}_{m \times n}(K), R = RREF(A)$. Then,*

(i) $rank(A) = rank(A^T)$

(ii) $rank(A) = dim\ row(A)$

(iii) $dim\ row(A) = dim\ col(A)$

(iv) There is an invertible matrix $B \in \mathcal{M}_{m \times n}(K)$ s.t. $BA = R$

Proof. (iii) We have:

$$\begin{aligned} dim\ row(A) &= rank(A) && \text{(by (ii))} \\ &= rank(A^T) && \text{(by (i))} \\ &= dim\ row(A^T) && \text{(by (ii))} \\ &= dim\ col(A) && \text{(by (iii))} \end{aligned}$$

□

February 27th 2019

Theorem 39 (computing bases). *Let $A \in \mathcal{M}_{m\times n}(K)$, let R be the reduced non-echelon form of A. Then,*

(i) The non-zero rows of R form a basis of $row(A)$.

(ii) The columns of A which correspond to the pivot columns (columns containing a leading 1) form a basis of $col(A)$.

(iii) The "basic solutions" obtained when solving $Ax = \vec{0}$ form a basis for nullspace (ker) of A.

Proof. By direct proof.

(i) Elementary row ops do not change the row space so $row(A) = row(R)$. Non-zero rows form basis because of form of R.

(ii) Let $w_1, w_2, \ldots, w_r$ be the columns of R containing leading 1's (pivot columns). Because of form of R, no other non-zero entries above/below a leading 1, so $w_1, w_2, \ldots, w_r$ are standard basis vectors (ie in $\{e_1, e_2, \ldots, e_m\}$). So, $\{w_1, \ldots, w_r\}$ are linearly independent. Let $v_1, v_2, \ldots, v_r$ be corresponding columns.

Note $r = rank(A) = dim\ row(A) = dim\ col(A)$.

Prove $v_1, v_2, \ldots, v_2$ are linearly independent. Suppose

$$\sum_{i=1}^{n} a_i v_i = \vec{0}$$

By proposition 38, $\exists$ invertible M s.t. $MA = R$. Multiply by M:

$$\begin{aligned} M(\sum_{i=1}^{r} a_i v_i) &= M\vec{0} \\ &= \vec{0} \end{aligned}$$

So $\sum_{i=1}^{r} a_i M v_i = \vec{0}$, but M(column i of A) = col i of MA ie of R (prop 36). So,

$$\sum_{i=1}^{r} a_i w_i = \vec{0}$$

But $\{w_1, \ldots, w_r\}$ are independent. So all $a_i = 0$, so $\{v_1, \ldots, v_r\}$ independent so basis.

(iii) Solve $Ax = 0$, obtain general solution,

$$\begin{aligned} \vec{x} &= x_1 v_1 + x_2 v_2 + \ldots + x_s v_s \\ &= x_1 \begin{pmatrix} 1 \\ \\ \\ \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ \\ \\ \end{pmatrix} + \ldots + x_s \begin{pmatrix} 0 \\ \\ \\ \end{pmatrix} \end{aligned}$$

Where $x_1, x_2, \ldots, x_s$ free variables. Claim is that $v_1, v_2, \ldots, v_s$ form a basis for $ker(A)$. They clearly span. Independent? In the x_1 position, only v_i has a non-zero entry, so they are independent.

Comment The dimension of $Ker(A)$ is therefore the number of *free* variables.

□

Basis-finding problems

Problem Let $W \leq \mathcal{M}_{2\times 2}(\mathbb{R})$, where W consists of all A such that sum of entries in each row and column is the same. Find basis of W.

Solution Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$. So

$$\begin{aligned} a+b &= c+d \\ a+c &= b+d \\ a+b &= a+c && (a+b = b+d \text{ etc are not needed}) \end{aligned}$$

Write as linear system:

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $a = d$, $b = c$, $c = c$ and $d = d$. ie, $\vec{x} = c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

General solution,

$$\begin{aligned} A &= \begin{pmatrix} d & c \\ c & d \end{pmatrix} \\ &= c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Linearly independent by Thm 39 (kernel basis case). So

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \qquad \text{(basis)}$$

Problem: Let

$$\begin{aligned} P_1(x) &= 1 + 2x + 3x^2 - x^3 \\ P_2(x) &= -1 + 3x + x^2 + x^3 \\ P_3(x) &= 3 - 4x + x^2 - 3x^3 \\ P_4(x) &= 1 + 7x + 7x^2 - x^3 \\ P_5(x) &= 2 + 2x - x^2 - x^3 \end{aligned}$$

Let $W = span\{P_1(x), \dots, P_5(x)\} \leq P_3(\mathbb{R})$. Find:

(i) basis of W that is a subset of $\{P_1(x), \dots, P_5(x)\}$

(ii) basis of W consisting of polys of different degree.

Sol Isomorphism $T : P_3 \to \mathbb{R}^4$,

$$T(d + cx + bx^2 + ax^3) = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \qquad \left(\text{or } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right)$$

(i) Put the vectors as columns of a matrix,

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 3 \\ 2 & 3 & -4 & 7 & 2 \\ 3 & 1 & 1 & 7 & -1 \\ -1 & 1 & -3 & -1 & -1 \end{pmatrix}$$

Find basis $col(A)$. Row-reduce to

$$R = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So columns $1, 2$ and 5 of A form a basis for $col(A)$, which corresponds (using isomorphism T) to W, so

$$\{P_1(x), P_2(x), P_5(x)\} \qquad \text{(basis)}$$

(ii) Basis all diff degree. Use row space of a matrix. Put $P_1, \dots, P_5$ as rows. But use isomorphism

$$d + cx + bx^2 + ax^3 \iff \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So

$$A = \begin{pmatrix} -1 & 3 & 2 & 1 \\ 1 & 1 & 3 & -1 \\ -3 & 1 & -4 & 3 \\ -1 & 7 & 7 & 1 \\ -1 & -1 & 2 & 2 \end{pmatrix} \quad \text{(So } W \text{ corresponds to row space.)}$$

$$\rightarrow \quad = \begin{pmatrix} 1 & 0 & 0 & \frac{-27}{20} \\ 0 & 1 & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

First three rows form basis $row(A)$. As polynomials, we get

$$x^3 - \frac{27}{20},\ x^2 - \frac{1}{4},\ x + \frac{1}{5}$$

Which is basis of W, all of different degree. The choice of order was relevant since we knew in advance the general form the reduced form would take.

March 1st 2019

Problem Let

$$\begin{aligned} v_1 &= (1,3,-1,2,0,2) \\ v_2 &= (3,3,5,-4,-7,-5) \\ v_3 &= (2,2,-1,1,2,1) \\ w_1 &= (3,1,-1,0,4,0) \\ w_2 &= (3,3,1,1,1,-1) \\ w_3 &= (1,1,-1,2,3,1) \end{aligned}$$

Let $V = span\{v_1, v_2, v_3\}$, $W = span\{w_1, w_2, w_3\}$. Find bases (and dimensions of) $V + W$, $V \cap W$.

Solution Check that $\{v_1, v_2, v_3\}$, $\{w_1, w_2, w_3\}$ both independent (put into matrix as either rows or columns, verify $rank = 3$)

$V + W = span\{V \cup W\} = span\{v_1, v_2, v_3, w_1, w_2, w_3\}$. For basis, put vectors as rows or columns, solve for row space or col space. I used columns, matrix reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis = cols $1, 2, 3, 5$ of original matrix. So $\{v_1, v_2, v_3, w_2\}$ so $dim\ (V + W) = 4$.
Formula:

$$\begin{aligned} dim\ (V+W) &= dim\ V + dim\ W - dim\ (V \cap W) \\ 4 &= 3 + 3 - dim\ (V \cap W) \end{aligned}$$

So $dim\ (V \cap W) = 2$.
$V \cap W$ is all $u = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{R}^6$ such that $u = x_1 v_1 + x_2 v_2 + x_3 v_3$ (*) (ie $u \in V$) and $u = y_1 w_1 + y_2 w_2 + y_3 w_3$ (**) (ie $u \in W$) for some $x_1, x_2, x_3, y_1, y_2, y_3$. This is linear system. 12 variables, 12 equations (2 for each of 6 components):

$$\begin{aligned} z_1 &= x_1 + 3x_2 + 2x_3 && (z_1-\text{component of (*)}) \\ z_2 &= 3x_1 + 3x_2 + 2x_3 && (z_2-\text{component of (*)}) \\ \dots \end{aligned}$$

And

$$\begin{aligned} z_1 &= 3y_1 + 3y_2 + y_3 && (z_1-\text{component of (**)}) \\ \dots \\ z_6 &= 0y_1 - y_2 + y_3 && (z_6-\text{component of (**)}) \end{aligned}$$

Goal is to solve the system, need only $u = (z_1, \dots, z_6)$. Remember that:

$$\begin{pmatrix} z_1 \\ \dots \\ \\ \\ \\ z_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 3 \\ 5 \\ \dots \\ \\ \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 2 \\ \dots \\ \\ \\ \end{pmatrix}$$

Rewrite as

$$\begin{aligned} z_1 - x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 &= 0 \\ z_2 - 3x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 &= 0 \\ \dots \end{aligned}$$

Coefficient matrix: see fig 12
The form is

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6\times 3} \\ I_6 & 0_{6\times 3} & -w_1 - w_2 - w_3 \end{pmatrix}$$

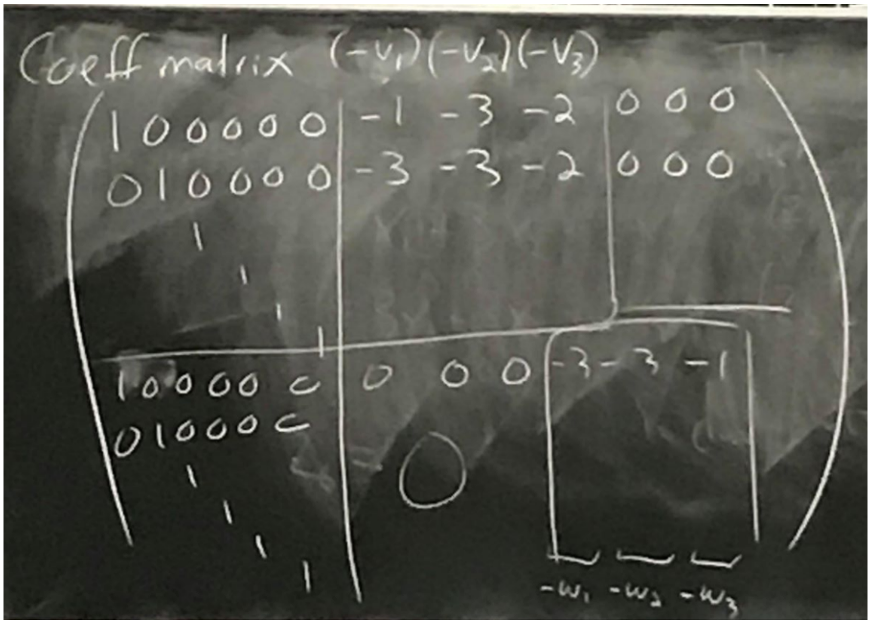


Figure 3: Coefficient matrix

Row-reduce, find basic solutions, each solution is in $\mathbb{R}^{12}$ (12 variables), you only need first 6 components ($(z_1, z_2, \ldots, z_6) = u \in V \cap W$).
Obtain basis

$$\begin{aligned} u_1 &= (3, 1, -1, 0, 4, 0) && (= w_1) \\ u_2 &= (-1, -1, -5/3, 4/3, 7/3, 5/3) && (= \tfrac{-1}{3} v_2) \end{aligned}$$

Shortcut When you row-reduce, after 6 ops, get

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6\times 3} \\ 0_{6\times 6} & v_1 + v_2 + v_3 & -w_1 - w_2 - w_3 \end{pmatrix}$$

Another viewpoint. Had

$$\begin{aligned} u &= x_1 v_1 + x_2 v_2 + x_3 v_3 \\ u &= y_1 w_1 + y_2 w_2 + y_3 w_3 \end{aligned}$$

You can solve instead 6×6 system:

$$\begin{aligned} x_1 v_1 + x_2 v_2 + x_3 v_3 &= y_1 w_1 + y_2 w_2 + y_3 w_3 \\ x_1 v_1 + x_2 v_2 + x_3 v_3 - y_1 w_1 - y_2 w_2 - y_3 w_3 &= (0, 0, \ldots, 0) \end{aligned}$$

Coeff matrix: $\begin{pmatrix} v_1 & v_2 & v_3 & -w_1 & -w_2 & -w_3 \end{pmatrix}$
Sol gives you $x_1, x_2, x_3, y_1, y_2, y_3$ not $z_1, \ldots, z_6$. Find $u = (z_1, \ldots, z_6)$ from (*) or (**)

Matrix of a linear transformation (ch. 6.2)

Def $T : V \to W$ linear, $\alpha = \{v_1, \dots, v_n\}$ basis of V, $\beta = \{w_1, \dots, w_n\}$ basis of W. The *standard matrix of* T, relative to α and β, is the $m \times n$ matrix whose i^{th} column is $T(v_i)$, writeen in $\beta-$coordinates, ie $[T(v_i)]_\beta (\in \mathbb{R}^m)$.
It is denoted $[T]_\alpha^\beta$.
Ex Let $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$, $T(f(x)) = f'(x)$. Find $[T]_\alpha^\beta$, $\alpha = \{1, x, x^2\}$, $\beta = \{1, x\}$
Sol Compute T on α

$$T(1) = 0$$
$$T(x) = 1$$
$$T(x^2) = 2x$$

In $\beta-$coords,

$$[T(1)]_\alpha^\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (= 0\,1 + 0\,x)$$
$$[T(x)]_\alpha^\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (= 1\,1 + 0\,x)$$
$$[T(x^2)]_\alpha^\beta = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad (= 0\,1 + 2\,x)$$

So $[T]_\alpha^\beta$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.
So $[T]_\alpha^\beta$ records values of T on α.

Theorem 40. *$[T]_\alpha^\beta$ computes T, but in coordinates. That is, for all $v \in V$,*

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

Ex $T(f(x)) = f'(x)$. Compute $T(a + bx + cx^2)$ via $[T]_\alpha^\beta$
Sol

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
$$[T(a + bx + cx^2)]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$
$$= \begin{pmatrix} b \\ 2c \end{pmatrix} \qquad (b + 2cx = f(x))$$

March 11th 2019

Recall $T : V \to W$, $\alpha = \{v_1, v_2, \dots, v_n\}$ basis V

$\beta = \{w_1, w_2, \ldots, w_n\}$ basis W
Matrix $[T]_\alpha^\beta$ has i^{th} column being $[T(vi)]_\beta$
Theorem 40

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

Proof. Let $A = [T]_\alpha^\beta, v \in V$. Write $v = \sum_{i=1}^n a_i v_i$.

So $[v]_\alpha = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1e_1 + a_2e_2 + \ldots + a_ne_n$

Then

$$\begin{aligned} A[v]_\alpha &= A(a_1e_1 + \ldots + a_ne_n) \\ &= a_1Ae_1 + \ldots + a_nAe_n \\ &= a_1(\text{col \# 1 of A}) + \ldots + a_n(\text{col \# n of A}) \\ &= a_1[T(v_1)]_\beta + \ldots + a_n[T(v_n)]_\beta \end{aligned}$$

□

Theorem 41. *Everything you want to know about T, you can determine from $[T]_\alpha^\beta$.*
Let $A = [T]_\alpha^\beta$ ($C_\alpha = V \to \mathbb{R}^n$, $C_\alpha(v) = [v]_\alpha$). See figure 4.
Then

(i) $Ker(T) = C_\alpha^{-1}(Ker(A))$

(ii) $Im(T) = C_\beta^{-1}(Im(A))$

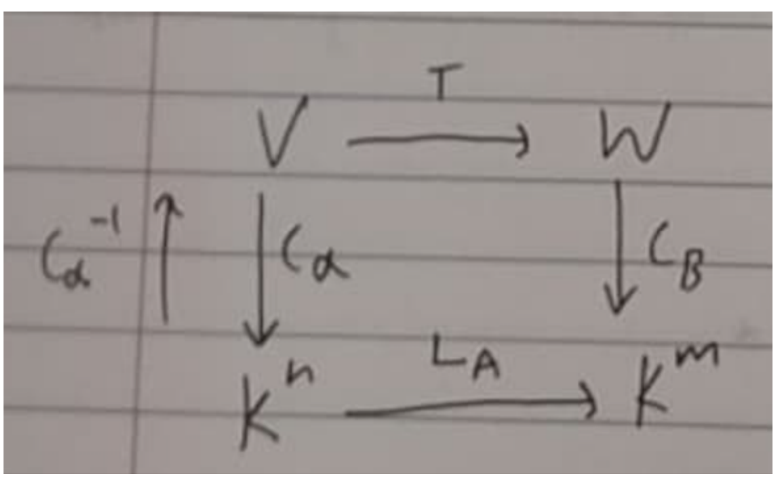


Figure 4: Theorem 41

Ex $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ defined by $T(A) = BA$.

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Find basis for $Kernel(T)$, $Image(T)$ is T inj/surj?

Sol Use basis $\alpha = \{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \}$

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$
$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$
$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$
$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

So we have

$$[T]^{\alpha}_{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$Ker(T)$: Solve $[T]x = 0$. Row-reduce

$$[T]^{\alpha}_{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2s$$
$$x_2 = -2t$$
$$x_3 = s$$
$$x_4 = t$$

$$x = s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis $= \{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \}$ for $Ker([T])$

So $\{ \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} \}$ basis for $Ker\ T$ so T not injective.

Theorem 42. *The following are true:*

(i) $T : V \to W$*, linear* α *basis of* V*,* β *basis of* W*.*

$$T \text{ is invertible } \iff [T]_\alpha^\beta \text{ is invertible}$$

So $dim(V) = dim(W)$ *must hold, of course.*

(ii) *If* $S : W \to U$*,* γ *basis of* U*, then* $[S \cdot T]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$

$(S \cdot T : V \to U)$ *is matrix of a compositiion is product of standard matrices.*

Ex $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f(1)$. Prove T is invertible, give formula for $T^{-1}(ax^2 + bx + c)$.
Sol (T is linear, verify)
Use standard basis $\{1, x, x^2\}$.
Calculate T on α

$$\begin{aligned} T(1) &= x(0) + 1 = 1 = 1 + 0x + 0x^2 \\ T(x) &= x(1) + 1 + 1 + 1x + 0x^2 \\ T(x^2) &= x(2x) + 1 = 1 + 0x + 2x^2 \end{aligned}$$

So $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. $det([T]) = 2 \neq 0$ so matrix and T are both invertible.
$invert[T] = \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$
Then

$$\begin{aligned} [T^{-1}(c + bx + ax^2)]_\alpha &= \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \\ &= \begin{bmatrix} c - b - \frac{a}{2} \\ b \\ \frac{a}{2} \end{bmatrix} \end{aligned}$$

Formula: $T^{-1}(c + bx + ax^2) = (c - b - \frac{a}{2}) + bx + \frac{a}{2}x^2$.
Check

$$\begin{aligned} T(c - b - \frac{a}{2} + bx + \frac{a}{2}x^2) &= x(b + ax) + c - b - \frac{a}{2} + b + \frac{a}{2} \\ &= c + bx + ax^2 \end{aligned}$$

March 13th 2019

Change of basis (ch 6.3)

Suppose V : vector space, $\alpha = \{u_1, \dots, u_n\}$ and β both bases of V.
How to change from α-coordinates to β-coordinates easily?
Trick: Consider identity lin. transformation I, $I(v) = v$.

$$I : V \to V$$

Matrix $[I]_\alpha^\beta$ will change coords, since if $v \in V$,

$$[I]_\alpha^\beta [v]_\alpha = [I(v)]_\beta = [v]_\beta$$

Def Matrix $Q_\alpha^\beta = [I]_\alpha^\beta$ called change-of-basis matrix from α to β. That is Q_α^β is matrix whose i^{th} column is the i^{th} basis vector of α, written in β-coords ("old basis in new coords, as columns").

Theorem 43. *We have*

(i) For all $v \in V$, $Q_\alpha^\beta [v]_\alpha = [v]_\beta$ (mult. by Q_α^β changes coords)

(ii) $Q_\beta^\alpha = (Q_\alpha^\beta)^{-1}$ (and Q_α^β is invertible!)

Proof. (i) Done above.

(ii) $I : V \to V$ is invertible, so $Q_\alpha^\beta = [I]_\alpha^\beta$ also invertible and

$$\begin{aligned}(Q_\alpha^\beta)^{-1} &= ([I]_\alpha^\beta)^{-1} \\ &= [I^{-1}]_\beta^\alpha \\ &= [I]_\beta^\alpha \\ &= Q_\beta^\alpha\end{aligned}$$

□

Ex $\mathbb{R}^2$ with $\alpha = \{\binom{1}{0}, \binom{0}{1}\}$, $\beta = \{\binom{2}{4}, \binom{1}{3}\}$. Find Q_α^β, Q_β^α, $[\binom{7}{4}]_\beta$.
Note In $\mathbb{R}^n$, $[\binom{a_1}{a_n}]_\alpha = \binom{a_1}{a_n}$ ($\alpha = \{e_1, e_2, \dots, e_n\}$)
Sol Q_α^β = old basis in α in terms of new basis β = work.
Q_β^α = easier = β-vectors in terms of α.

$$Q_\beta^\alpha = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$$

$$Q_\alpha^\beta = (Q_\beta^\alpha)^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$$

Then,

$$
\begin{aligned}
[\begin{pmatrix}7\\4\end{pmatrix}]_\beta &= Q_\beta^\alpha[\begin{pmatrix}7\\4\end{pmatrix}]_\alpha \\
&= \frac{1}{2}\begin{pmatrix}3 & -1\\ -4 & 2\end{pmatrix}\begin{pmatrix}7\\4\end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix}21-4\\ -28+8\end{pmatrix} \\
&= \begin{pmatrix}\frac{17}{2}\\ -10\end{pmatrix}
\end{aligned}
$$

Def $T : V \to V$ linear transf (some V), called a *linear operator*.
Def Let $A, B \in \mathcal{M}_{n\times n}(K)$. A is *similar* to B if $\exists$ invertible $Q \in \mathcal{M}_{n\times n}(K)$ so that $Q^{-1}AQ = B$

Proposition 44. ***Note*** *If A similar to B, B similar to A, since*

$$
\begin{aligned}
Q^{-1}AQ &= B \\
QQ^{-1}AQQ^{-1} &= QBQ^{-1} \\
A = (Q^{-1})^{-1}BQ^{-1}
\end{aligned}
$$

Theorem 45. *Let $T : V \to V$ linear operator, α, β bases of V. Then,*

$$[T]_\beta^\beta = Q_\alpha^\beta[T]_\alpha^\alpha Q_\beta^\alpha$$

In particular, $[T]_\alpha^\alpha$ and $[T]_\beta^\beta$ are similar since $Q_\alpha^\beta = (Q_\beta^\alpha)^{-1}$

Proof. Let $v \in V$. Show both compute some linear operator.
LHS $[T]_\beta^\beta[v]_\beta = [T(v)]_\beta$
RHS $Q_\alpha^\beta[T]_\alpha^\alpha Q_\beta^\alpha[v]_\beta$

$$
\begin{aligned}
Q_\alpha^\beta[T]_\alpha^\alpha Q_\beta^\alpha[v]_\beta &= Q_\alpha^\beta[T]_\alpha^\alpha[v]_\alpha \\
&= Q_\alpha^\beta[T(v)]_\alpha \\
&= [T(v)]_\beta
\end{aligned}
$$

So for all $[v]_\beta$, mult by LS/RS gives some result, so for std bases vector $e_1, \ldots, e_n$, LS e_i = col i of LS, RS e_i = col i of RS ☐

Problem (figure 5) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection in the line $y = mx$. Find formula for $T(a, b)$
Sol *First*, prove T is linear. (omit)
Option # 1 (figure 6) Compute $T(1, 0), T(0, 1)$, find $[T]_\alpha^\alpha$, $\alpha = \{\binom{1}{0}, \binom{0}{1}\}$
Option # 2 (figure 7) Use better basis, then change basis. Let $v = (1, m)$ so $T(v) = (1, m)$. Let $w = (m, -1)$. Then $T(w) = -w = (-m, 1)$

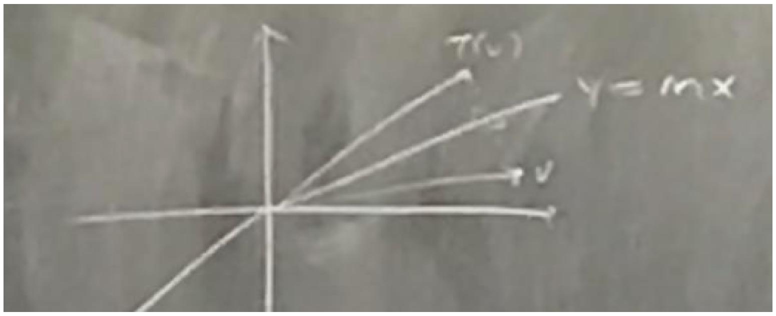


Figure 5: Problem

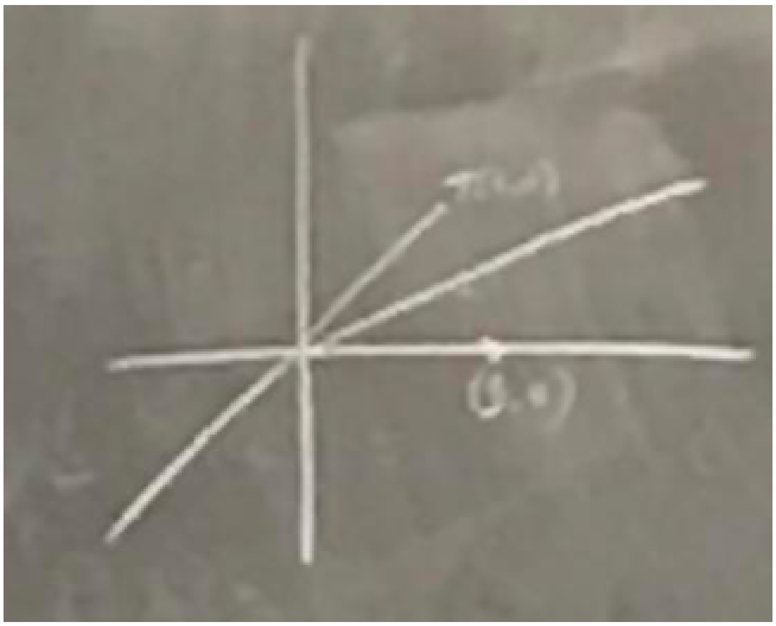


Figure 6: Have to do some geometry! :(

New basis $\beta = \{v, w\}$

$$[T]_\beta^\beta = ([T(v)]_\beta, [T(w)]_\beta)$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Want $[T]_\alpha^\alpha = Q_\beta^\alpha [T]_\beta^\beta Q_\alpha^\beta$. Have $Q_\beta^\alpha = \beta$ in terms of $\alpha = \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}$

$$Q_\alpha^\beta = (Q_\beta^\alpha)^{-1}$$
$$= \frac{1}{-1-m^2} \begin{pmatrix} -1 & -m \\ -m & 1 \end{pmatrix}$$

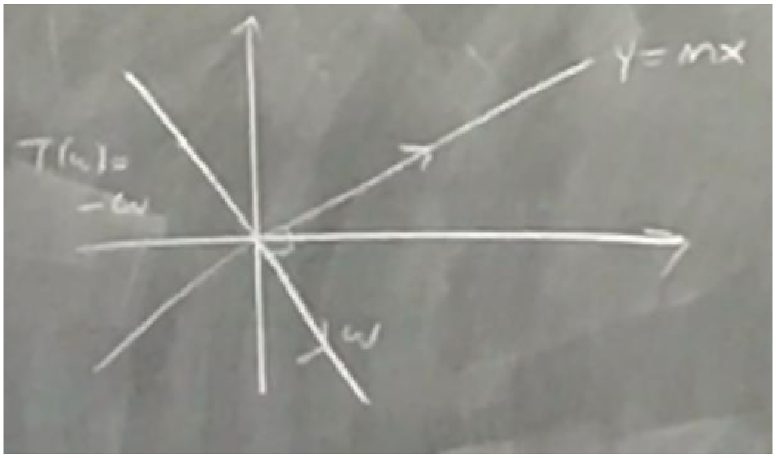


Figure 7: Better option

Compute

$$[T]_\alpha^\alpha = Q_\beta^\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q_\alpha^\beta \qquad \text{(multiply)}$$
$$= \frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

Finally,

$$T(a,b) = \frac{1}{m^2+1} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$= \frac{1}{m^2+1} \begin{pmatrix} a-am^2 & 2bm \\ 2am & bm^2-b \end{pmatrix}$$

March 15th 2019

Inner Product Spaces (ch. 7 text)

Idea: Dot product on $\mathbb{R}^n$, $u = (a_1, \ldots, a_n)$, $v = (b_1, \ldots, b_n)$

$$u \cdot v = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

From this,

$$||u|| = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2} = \sqrt{u \cdot u}$$
$$u \cdot v = ||u||||v||cos\theta$$

Or

$$\theta = cos^{-1}(\frac{u \cdot v}{||u||||v||})$$
$$u, v \ \frac{orthogonal}{(\theta = \frac{\pi}{2})} \iff u \cdot v = 0$$

Dot product allows you to *define* lengths, angles, orthogonality. These are geometric ideas.
Def V vector space over K ($\mathbb{R}$ or $\mathbb{C}$).
An *inner product* on V is a function $\langle u, v\rangle$ which takes two vectors as input and produces a scalar, and satisfies the following:

(I1) $\forall u, v, w \in V, \forall c \in K$

  (i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
  (ii) $\langle cu, w\rangle = c\langle u, w\rangle$

  This is called *linearity in the first component*

(I2) $\forall u, v, \in V$

$$\langle v, u\rangle = \overline{\langle u, v\rangle}$$

  The RHS is the complex conjugate.

  This is called *conjugate similarity*.

(I3) $\forall u \in V$, $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0 \iff u = \vec{0}$
  This is called *positive definite*

Notes:

(1) If $K = \mathbb{R}$, (I2) is $\langle v, u\rangle = \langle u, v\rangle$

(2) If $K = \mathbb{C}$, then by (I2)

$$\langle u, u\rangle = \overline{\langle u, u\rangle}$$

  Which means $\langle u, u\rangle \in \mathbb{R}$. So $\langle u, u\rangle \geq 0$ makes sense.

Theorem 46. *Properties of inner products*

  (a) $\forall u, v, w \in V, \forall c \in K$,

$$\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$$
$$\langle u, cv\rangle = \bar{c}\langle u, v\rangle$$

  This is called ***conjugate linearity in second component***.

  (b) $\forall u \in V, \langle u, \vec{0}\rangle = 0$ *(scalar)*

  (c) $\forall u, v, w \in V$, *if* $\forall w \in V\ \ \langle u, w\rangle = \langle v, w\rangle$ *then* $u = v$

Proof. By direct proof.

(a)

$$\begin{aligned}
\langle u, v+w\rangle &= \overline{\langle v+w, u\rangle} && \text{(I2)}\\
&= \overline{\langle v, u\rangle + \langle w, u\rangle} && \text{(I1)}\\
&= \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} &&\\
&= \langle u, v\rangle + \langle u, w\rangle && \text{(I2)}
\end{aligned}$$

Recall for $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2}\\
\overline{z_1 z_2} &= \overline{z_1}\,\overline{z_2}\\
z_1\overline{z_1} &= (a+bi)(a-bi) = a^2 + b^2 = |z_1|^2
\end{aligned}$$

Now, we have:

$$\begin{aligned}
\langle u, cv\rangle &= \overline{\langle cv, u\rangle} && \text{(I2)}\\
&= \overline{c\langle v, u\rangle} && \text{(I1)}\\
&= \overline{c}\,\overline{\langle v, u\rangle} &&\\
&= \overline{c}\langle u, v\rangle &&
\end{aligned}$$

(b)

$$\begin{aligned}
\langle u, \vec{0}\rangle &= \langle u, \vec{0}+\vec{0}\rangle &&\\
&= \langle u, \vec{0}\rangle + \langle u, \vec{0}\rangle && \text{(by (a))}
\end{aligned}$$

So $0 = \langle u, \vec{0}\rangle$

(c) Assume $\forall w$, $\langle u, w\rangle = \langle v, w\rangle$. To show $u = v$, we will show $u - v = \vec{0}$.

Consider

$$\begin{aligned}
\langle u-v, u-v\rangle &= \langle u, u-v\rangle + \langle -v, u-v\rangle && \text{(I1)}\\
&= \langle u, u-v\rangle - \langle v, u-v\rangle && \text{(I1)}
\end{aligned}$$

Using $w = u - v$, $\langle u, u-v\rangle = \langle v, u-v\rangle$. So $\langle u-v, u-v\rangle = 0$ so by (I3). $u - v = \vec{0}$ so $u = v$.

□

March 18th 2019

Standard inner product on K^n:
for $u = \{a_1, \ldots, a_n\}$, $v = \{b_1, \ldots, b_n\}$ define

$$\langle u, v\rangle = \sum_{i=1}^{n} a_i \overline{b_i}$$

So if $K = \mathbb{R}$, $\overline{b_i} = b_i$ so it's the usual dot product.
Ex Compute $\langle u, v\rangle$,

$$u = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$
$$v = \begin{pmatrix} i \\ 3-4i \end{pmatrix}$$

Solution

$$\begin{aligned}
\langle \begin{pmatrix} 2 \\ 1+i \end{pmatrix}, \begin{pmatrix} i \\ 3-4i \end{pmatrix} \rangle &= 2(\bar{i}) + (1+i)(\overline{3-4i}) \\
&= -2i + (1+i)(3+4i) \\
&= -2i + 3 + 4i + 3i + 4i^2 \\
&= -i + 5i
\end{aligned}$$

Proposition 47. *Standard inner product in K^n is an inner product*

Proof. By direct proof.

(I1) Omit.

(I2)

$$\begin{aligned}
\overline{\langle v, u\rangle} &= \overline{\sum_{i=1}^n b_i\bar{a}_i} \\
&= \sum_{i=1}^n \overline{b_i\bar{a}_i} \\
&= \sum_{i=1}^n \overline{b_i}\overline{\bar{a}_i} \\
&= \sum_{i=1}^n \overline{b_i}a_i \\
&= \sum_{i=1}^n a_i\overline{b_i}
\end{aligned}$$

(I3)

$$\begin{aligned}
\langle u, u\rangle &= \sum_{i=1}^n a_i\overline{a_i} \\
&= \sum_{i=1}^n |a_i|^2
\end{aligned}$$

Then all $|a_i| \geq 0$ so $\langle u, u\rangle \geq 0$

$$\langle u, u\rangle = 0 \iff |a_i| = 0 \text{ for all } i$$

□

Inner product on $\mathcal{M}_{n\times n}(K)$

For $A, B \in \mathcal{M}_{n\times n}(K)$, define first

(i) $\overline{A}$ is the matrix obtained by taking the complex conjugate of each entry.

(ii) $A^* = (\bar{A})^T$, conjugate transpose (adjoint)

Ex:

$$A = \begin{pmatrix} 2+i & 3i \\ 2 & 1+i \end{pmatrix}, \bar{A} = \begin{pmatrix} 2+i & -3i \\ 2 & 1-i \end{pmatrix}, A^* = \begin{pmatrix} 2+i & 2 \\ -3i & 1-i \end{pmatrix}$$

For inner product,

$$\langle A, B \rangle = tr(B^* A)$$

Ex In $\mathcal{M}_{2\times 2}(K)$, if

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{aligned}\langle A, B \rangle &= tr \begin{pmatrix} \overline{b_{11}} & \overline{b_{21}} \\ \overline{b_{12}} & \overline{b_{22}} \end{pmatrix} \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{pmatrix} \\ &= (a_{11}\overline{b_{11}} + a_{21}\overline{b_{21}}) + (a_{12}\overline{b_{12}} + a_{22}\overline{b_{22}}) \\ &= \langle \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} \rangle \qquad \text{(Standard inner product on } \mathbb{C}^4)\end{aligned}$$

Proposition 48. *$\langle A, B \rangle = tr(B^* A)$ is an inner product on $\mathcal{M}_{n\times n}(K)$*

Proof. Omit. You can prove it directly using matrix properties. □

Inner product on $P_n(\mathbb{R})$

For $f, g \in P_n(\mathbb{R})$ define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

Ex For $f(x) = x + 1$, $g(x) = x$ find $\langle f, g \rangle$
Sol

$$\begin{aligned}\langle x+1, x \rangle &= \int_0^1 (x+1)(x)dx \\ &= \int_0^1 (x^2 + x)dx \\ &= \frac{x^3}{3}\frac{x^2}{2}|_0^1 \\ &= \frac{1}{3} + \frac{1}{2}\end{aligned}$$

Proposition 49. *For any $a < b$*

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

defines an inner product on $P_n(\mathbb{R})$ (also on $P(\mathbb{R})$)

Proof. By direct proof.

(I1) Let $f, g, h \in P_n(\mathbb{R}), c \in \mathbb{R}$. Then

$$\begin{aligned} \langle f + cg, h \rangle &= \int_a^b (f(x) + cg(x))h(x)dx \\ &= \int_a^b f(x)h(x)dx + c\int_a^b g(x)h(x)dx \\ &= \langle f, h \rangle + c\langle g, h \rangle \qquad \text{((i) and (ii) together)} \end{aligned}$$

(I2)

$$\begin{aligned} \langle f, g \rangle &= \int_a^b f(x)g(x)dx \\ &= \int_a^b g(x)f(x)dx \\ &= \langle g, f \rangle \end{aligned}$$

(I3)

$$\begin{aligned} \langle f, f \rangle &= \int_a^b f(x)f(x)dx \\ &= \int_a^b (f(x))^2 dx \end{aligned}$$

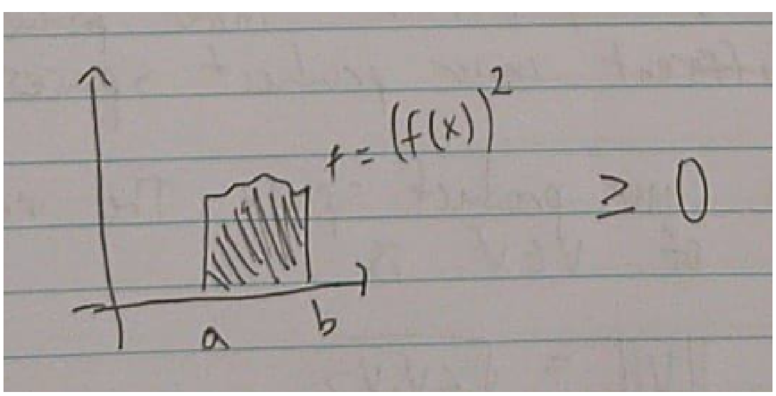


Figure 8: Representation

□

Problem For $P_1(\mathbb{R})$, write formula for

$$\langle a+bx, c+dx \rangle$$

in terms of a, b, c, d
Sol

$$\begin{aligned}\langle a+bx, c+dx \rangle &= \int_0^1 (ac + (ad+bc)x + bdx^2)dx \\ &= acx + \frac{ad+bc}{2}x^2 + \frac{bd}{3}x^3|_0^1 \\ &= ac + \frac{ad}{2} + \frac{bc}{2} + \frac{bd}{3}\end{aligned}$$

Note $P_1(\mathbb{R}) \simeq \mathbb{R}^2$. Isomorphism,

$$(a+bx) \to \begin{pmatrix} a \\ b \end{pmatrix}$$

Under this isomorphism, you can compute $\langle a+bx, c+dx \rangle$ using an inner product on $\mathbb{R}^2$ defined by

$$\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \rangle = ac + \frac{1}{2}ad + \frac{1}{2}bc + \frac{1}{3}bd$$

The point is, the inner product makes sense in $P_1(\mathbb{R})$.
Def A vector space V with a specified inner product is called an *inner product space*.
Note Some V with different inner products are different inner product spaces.
Def V on inner product space. The norm or length of $v \in V$ is

$$||v|| = \sqrt{\langle v, v \rangle}$$

Example In $P_1(\mathbb{R})$ with $[0,1]$,

$$\begin{aligned}||x+1|| &= \sqrt{\langle x+1, x+1 \rangle} \\ &= (\int_0^1 (x+1)^2 dx)^{\frac{-1}{2}} \\ &= (\frac{(x+1)^3}{3}|_0^1)^{\frac{-1}{2}} \\ &= (\frac{2^3}{3} - \frac{1}{3})^{\frac{-1}{2}} \\ &= \sqrt{\frac{7}{3}}\end{aligned}$$

March 20th 2019

Last time: Norm (length) is $||v|| = \sqrt{\langle u, v \rangle}$

Proposition 50. *For all $v \in V, c \in K$*

$$||cv|| = |c|||v|| \qquad \text{(note } |c|^2 = c\bar{c} \in \mathbb{C}, |c|^2 = a^2 + b^2\text{)}$$

Proof.

$$\begin{aligned} ||cv|| &= \sqrt{\langle cv, cv\rangle} \\ &= \sqrt{c\bar{c}\langle v, v\rangle} && \text{(I2)} \\ &= |c|\sqrt{\langle v, v\rangle} \\ &= |c| \cdot ||v|| \end{aligned}$$

□

Theorem 51 (Cauchy-Schwarz Inequality). *For all $u, v \in V$, (inner product space)*

$$|\langle u, v\rangle| \leq ||u||||v||$$

So also $\langle u, v\rangle \leq ||u||||v||$ if $K = \mathbb{R}$ or equiv,

$$|\langle u, v\rangle|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

Further, equality holds $\iff$ u, v are dependent.

Proof. Let $c \in K$ any scalar. Consider

$$\begin{aligned} 0 &\leq \langle u - cv, u - cv\rangle && \text{(I3)} \\ &= \langle u, u - cv\rangle + \langle -cv, u - cv\rangle && \text{(I1)} \\ &= \langle u, u\rangle + \langle u, -cv\rangle + \langle -cv, u\rangle + \langle -cv, -cv\rangle \\ &= ||u||^2 + \overline{(-c)}\langle u, v\rangle + (-c)\langle v, u\rangle + (-c)\overline{(-c)}\langle v, v\rangle \\ 0 &\leq ||u||^2 - \bar{c}\langle u, v\rangle - c\langle v, u\rangle + c\bar{c}||v||^2 \end{aligned}$$

Set $c = \frac{\langle u,v\rangle}{||v||^2}$ (unless $||v|| = 0$, only if $v = \vec{0}$, in which case $\langle u, 0\rangle = 0 = ||u||0 = ||u||||v||$) So $c = \frac{1}{||v||^2}\langle u, v\rangle$. (LHS $\in \mathbb{R}$, RHS $\in \mathbb{C}$). So

$$\begin{aligned} \bar{c} &= \frac{1}{||v||^2}\overline{\langle u, v\rangle} \\ &= \frac{\langle v, u\rangle}{||v||^2} \end{aligned}$$

So

$$
\begin{aligned}
0 &\leq ||u||^2 - \frac{\langle v,u\rangle}{||v||^2}\langle u,v\rangle - \frac{\langle u,v\rangle\langle v,u\rangle}{||v||^2} + \frac{u,v}{||v||^2}\frac{v,u}{||v||^2}||v||^2 \\
0 &\leq ||u||^2 - \frac{\langle u,v\rangle\langle v,u\rangle}{||v||^2} \\
\langle u,v\rangle\langle v,u\rangle &\leq ||u||^2||v||^2 \\
\langle u,v\rangle\overline{\langle u,v\rangle} &\leq ||u||^2||v||^2 \\
|\langle u,v\rangle|^2 &\leq ||u||^2||v||^2
\end{aligned}
$$

Omit proof about equality. □

Important cases

(1) $\mathbb{R}^n$, usual inner product. Let $u = (a_1, \ldots, a_n)$, $v = (b_1, \ldots, b_n)$. So,

$$\langle u,v\rangle^2 = (a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \leq ||u||^2||v||^2$$

So

$$(a_1b_1 + \ldots + a_nb_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots b_n^2)$$

Ex Prove for all $a_1, a_2, \ldots, a_n$,

$$(|a_1| + |a_2| + \ldots + |a_n|)^2 \leq n(a_1^2 + a_2^2 + \ldots + a_n^2)$$

Sol Let

$$
\begin{aligned}
u &= (|a_1|, |a_2|, \ldots, |a_n|) \\
v &= (1, 1, \ldots, 1)
\end{aligned}
$$

By Cauchy-Schwarz inequality,

$$
\begin{aligned}
(|a_1| + |a_2| + \ldots + |a_n|)^2 &\leq (a_1^2 + \ldots + a_n^2)(1 + 1 + \ldots + 1) \\
&= n(a_1^2 + \ldots a_n^2)
\end{aligned}
$$

(2) $\mathcal{P}(\mathbb{R}), f, g \in \mathcal{P}(\mathbb{R})$

$$
\begin{aligned}
\langle f,g\rangle^2 &\leq \langle f,f\rangle\langle g,g\rangle \\
\left(\int_0^1 f(x)g(x)dx\right)^2 &\leq \left(\int_0^1 f(x)^2dx\right)\left(\int_0^1 g(x)^2dx\right)
\end{aligned}
$$

Theorem 52. *Triangle inequality For all $u, v \in V$,*

$$||u+v|| \leq ||u|| + ||v||$$

Proof. Instead of

$$||u+v|| = \sqrt{\langle u+v, u+v\rangle}$$

Look at

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \qquad \text{(I1)} \\ &= ||u||^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + ||v||^2 \end{aligned}$$

For $z = a + bi$, $z + \bar{z} = 2a = 2Re(z)$ $(Re(z) = a, Im(z) = b)$. Also,

$$a \leq |a| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|$$

So $Re(z) \leq |z|$ (*)
Then,

$$\begin{aligned} ||u+v||^2 &= ||u||^2 + 2Re(\langle u,v\rangle) + ||v||^2 \\ &\leq ||u||^2 + 2|\langle u,v\rangle| + ||v||^2 && \text{(by (*))} \\ &\leq ||u||^2 + 2||u||||v|| + ||v||^2 && \text{(Cauchy-Schwarz)} \\ &= (||u|| + ||v||)^2 \end{aligned}$$

So $||u+v||^2 \leq (||u|| + ||v||)^2$, take square root. □

Angles

Since $|\langle u,v\rangle| \leq ||u|| \cdot ||v||$,

$$\frac{|\langle u,v\rangle|}{||u||||v||} \leq 1 \text{ or } -1 \leq \frac{\langle u,v\rangle}{||u||||v||} \qquad (K = \mathbb{R})$$

So there is an *angle* θ such that

$$cos\ \theta = \frac{\langle u,v\rangle}{||u|| \cdot ||v||}$$

Define the angle between u, v to be θ.
Note When the angle is 0, $cos\ \theta = 1$. When the angle is $\pi/2$, $cos\ \theta = 0$. When the angle is π, $cos\ \theta = -1$. So $cos\ \theta$ measures how "similar" two vectors are in terms of "angle" or "direction".

March 22nd 2019

Application/interpretation

Word counts in textual analysis. Consider $\mathbb{R}^n$, n = # of words in the (English) language. Each component corresponds to a word (eg: component 1 is "a", etc). View a text (eg Hamlet) as a vector (v_{hamlet}), count # times each word occurs.
Norm $||v_{hamlet}|| = \sqrt{\sum_{i=1}^n a_i^2}$ (usual dot product)

more words $\rightarrow$ larger norm

Eg $v = (1, 1, \ldots, 1)$, $n = 1000$.

$$\begin{aligned}||v|| &= \sqrt{\sum 1} \\ &= \sqrt{1000}\end{aligned}$$

$w = (1000, 0, \ldots, 0)$, $n = 1000$:

$$\begin{aligned}||v|| &= \sqrt{1000^2} \\ &= 1000\end{aligned}$$

Angle

$$cos\ \theta = \frac{\langle u, v\rangle}{||u||\ ||v||}$$

If u, v have no words in common, $\langle u, v\rangle = 0$, so $cos\ \theta = 0$ ($\theta = \frac{\pi}{2}$, "orthogonal"). Suppose you compare "Hamlet" to "2x Hamlet" :

$$\begin{aligned}cos\ \theta &= \frac{\langle v_{hamlet}, v_{2x\ hamlet}\rangle}{||v_{hamlet}||\ ||v_{2x\ hamlet}||} \\ &= \frac{\langle v_{hamlet}, 2v_{hamlet}\rangle}{||v_{hamlet}||\ ||2v_{hamlet}||} \\ &= \frac{2||v_{hamlet}||}{2||v_{hamlet}||\ ||v_{hamlet}||} \\ &= 1\end{aligned}$$

Ie $\theta = 0$. Texts are "the same".

Orthogonality and projections

Def u, v are *orthogonal* if $\langle u, v\rangle = 0$.
Ex In $P_1(\mathbb{R})$, inner product $\int_0^1 fg dx$, find all polynomials (vectors) orthogonal to $1 + x$.
Sol Let $g(x) = a + bx$. Need

$$\begin{aligned}0 &= \langle 1 + x, a + bx\rangle \\ &= \int_0^1 (a + bx + ax + bx^2) dx \\ &= ax + \frac{b(a)}{2}x^2 + \frac{b}{3}x^3|_0^1 \\ &= a + \frac{b}{2} + \frac{a}{2} + \frac{b}{3} \\ \frac{-3}{2}a &= \frac{5}{6}b, b = \frac{-3}{2}(\frac{6}{5})a = \frac{-9}{5}a\end{aligned}$$

All vectors $a - \frac{9}{5}ax$, ie $span\{1 - \frac{9}{5}x\}$.
Def A set S of vectors is

(i) *orthogonal* if $\langle u, v\rangle = 0$ for all $u, v \in S$, $u \neq v$.

(ii) *orthonormal* if orthogonal and $||u|| = 1$, all $u \in S$

Def A basis α is an *orthognormal basis* (ONB) if it is an orthonormal set.
Ex $\alpha = \{e_1 e_2, \dots, e_n\}$ is ONB.

Notation : Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So if $S = \{v_1, v_2, \dots, v_n\}$ is ONB, $\langle v_i, v_j \rangle = \delta_{ij}$.

Proposition 53. *If S is an orthogonal set of non-zero vectors, then S is linearly independent.*

Proof. Suppoe

$$\sum_{i=1}^{k} a_i v_i = 0 \qquad \text{(for some } v_1, \dots, v_k \in S,\ a_1, \dots, a_n \text{ scalars)}$$

Trick. Take inner product with each v_j, $j = 1, 2, \dots, k$. So

$$\begin{aligned} 0 &= \langle \vec{0}, v_j \rangle \\ &= \langle \sum_{i=1}^{k} a_i v_i, v_j \rangle \\ &= \sum_{i=1}^{k} \langle a_i v_i, v_j \rangle && \text{((I1))} \\ &= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle && \text{(Since } \langle v_i, v_j \rangle = 0 \text{, unless } i = j) \end{aligned}$$

But $v_j \neq \vec{0}$ so $\langle v_j, v_j \rangle \neq 0$. So $a_j = 0$, for all $j = 1, \dots, k$. So all the coefficients are 0, so S is independent. □

Theorem 54. *Let V be inner product space,*

$$\alpha = \{v_1, v_2, \dots, v_n\}$$

an orthogonal basis. Then for any $u \in V$,

$$u = \sum_{i=1}^{n} \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

ie the i^{th} component of coords of U in basis α is $\frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle}$. Further, if α is ONB, then

$$u = \sum_{i=1}^{n} \langle u, v_i \rangle v_i$$

Proof. We know $u = \sum_{i=1}^{n} a_i v_i$ for some scalars. Take inner product with each v_j, $j = 1, 2, \ldots, n$ in turn. So

$$\begin{aligned}\langle u, v_j\rangle &= \langle \sum_{i=1}^{n} a_i v_i, v_j\rangle \\ &= \sum_{i=1}^{n} a_i \langle v_i, v_j\rangle \\ &= a_j\langle v_j, v_j\rangle \qquad \text{(All 0 except when } i = j)\end{aligned}$$

So $a_j = \langle u, v_j\rangle / \langle v_j, v_j\rangle$, α orthog. $\square$

March 25th 2019

Last time: If $\alpha = \{v_1, v_2, \ldots, v_n\}$ orthog. basis then for all $v \in V$

$$v = \sum_{i=1}^{n} \frac{\langle u, v_i\rangle}{\langle v_i, v_i\rangle} v_i$$

Ex In $\mathbb{R}^3$, $\alpha = \{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\}$ is an ONB. Find coords of $v = \begin{pmatrix}2\\1\\-3\end{pmatrix}$ in $\alpha-$basis.

Sol Compute its inner products with basis :

$$\langle \begin{pmatrix}2\\1\\-3\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}\rangle = \frac{1}{\sqrt{2}}3 = \frac{3}{\sqrt{2}}$$

$$\langle \begin{pmatrix}2\\1\\-3\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\rangle = \frac{2-1-3}{\sqrt{3}} = \frac{-2}{\sqrt{3}}$$

$$\langle \begin{pmatrix}2\\1\\-3\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\1\\2\end{pmatrix}\rangle = \frac{-2+1-6}{\sqrt{6}} = \frac{-7}{\sqrt{6}}$$

So

$$[\begin{pmatrix}2\\1\\-3\end{pmatrix}]_\alpha = \begin{pmatrix}\frac{3}{\sqrt{2}}\\ \frac{-2}{\sqrt{3}}\\ \frac{-7}{\sqrt{6}}\end{pmatrix}$$

Def Let $S \subseteq V$. The *orthogonal complement* of S is

$$\begin{aligned}S^\perp &= \{v \in V | \forall s \in S, \langle v, s\rangle = 0\} \\ &= \text{ all vectors orthogonal to } \textit{all} \text{ vectors in } S\end{aligned}$$

$S^\perp$ reads "S perp".

Ex

(1) $S =$ xy-plane in $\mathbb{R}^3$, $S^\perp =$ z-axis.

(2) $S =$ z-axis, $S^\perp =$ xy-plane.

(3) $S =$ plane through origin, $S^\perp =$ normal line.

(4) $S = V$, $S^\perp = \{\vec{0}\}$

(5) $S = \{\vec{0}\}$, $S^\perp = V$

Proposition 55. *Let $W \leq V$ (subspace). Then*

(i) $W^\perp$ *is a subspace (true even if W just subset)*

(ii) *If $\alpha = \{w_1, w_2, \dots, w_k\}$, basis W, then*

$$W^\perp = \{v \in V | \langle v, w_i \rangle = 0 \text{ for } i = 1, 2, \dots, k\}$$

(ie to compute $W^\perp$, find all v that are orthogonal to all basis *elements)*

(iii) $W \cap W^\perp = \{\vec{0}\}$

Proof. By direct proof.

(i) Let $u, v \in W^\perp, c \in K$. Then, to check if $cu + v \in W^\perp$, calculate for any $w \in W$

$$\begin{aligned}\langle cu + v, w \rangle &= c\langle u, w \rangle + \langle v, w \rangle \\ &= 0 \qquad\qquad \text{(both parts 0 since } u, v \in W^\perp)\end{aligned}$$

So $cu + v \in W^\perp$. Also, $\vec{0} \in W^\perp$ since $\langle \vec{0}, w \rangle = 0$ for all $w \in W$.

(ii) Prove two sets are equal:

(a) $LS \subseteq RS$. Let $v \in W^\perp$. Since each $w_i \in W$, $\langle v, w_i \rangle = 0$ since $v \in W^\perp$.

(b) $RS \subseteq LS$. Let $v \in V$ such that $\langle v, w_i \rangle = 0$ all $i = 1, 2, \dots, k$. Let $w \in W$. Write $w = \sum_{i=1}^k a_i w_i$, then

$$\begin{aligned}\langle v, w \rangle &= \langle v, \sum_{i=1}^k a_i w_i \rangle \\ &= \sum_{i=1}^k \langle v, a_i w_i \rangle \\ &= \sum_{i=1}^k \overline{a_i} \langle v, w_i \rangle \\ &= 0\end{aligned}$$

So $v \in W^\perp = LS$.

(c) Let $v \in W \cap W^{\perp}$. Since $v \in W^{\perp}, v$ orthog to all vectors in W, including itself, we have

$$\langle v, v \rangle = 0$$

So $v = \vec{0}$ by (I3).

$\square$

Ex Let $W = \{A \in \mathcal{M}_{2\times 2}(K) | A^T = A\}$. Find $W^{\perp}$.
Sol Find basis W. See A2.

$$\alpha = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$$

Find all $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle \\ &= tr(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\ &= a \end{aligned}$$

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= d \end{aligned}$$

$$\begin{aligned} 0 &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \\ &= b + c \end{aligned}$$

So $a = d = 0$, $c = -b$, general solution: $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$.
$W^{\perp} = span\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$

Orthogonal Projection

See figure **??**. Decompose v as $v' + w$, $w \in W, v' \in W^{\perp}$.

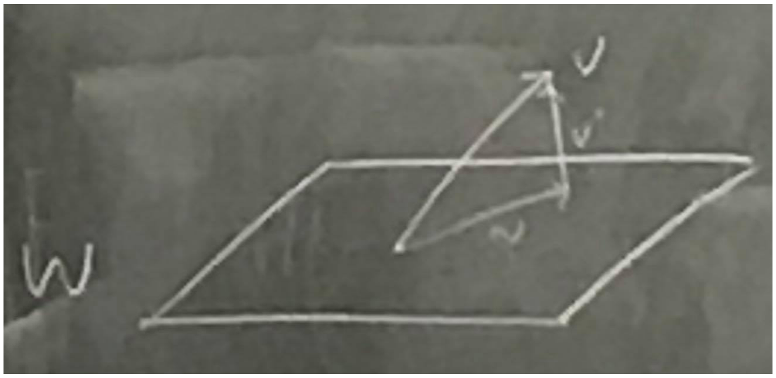


Figure 9: Orthogonal projection

Theorem 56. *Let $W \leq V$, $v \in V$. Then $\exists$ unique vectors $w \in W$, $v' \in W^{\perp}$ such that $v = v' + w$. Vector w called the (orthogonal) projection of v onto W, denoted $proj_W v = w$. Further, if $\alpha = \{w_1, w_2, \ldots, w_k\}$ is an* orthogonal *basis of W, then*

$$w = proj_W v = \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

Important: α must *be orthogonal!*

Proof. Set $w = \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$ (so $v = v' + w$). Set $v' = v - w$. So $v' + w = v, w \in W$ ($w =$ comp of $W-$basis vectors). Need $v' \in W^{\perp}$. Check if $\langle v', w_j \rangle = 0$, all j.

$$\begin{aligned}
\langle v', w_j \rangle &= \langle v - w_i, w_j \rangle \\
&= \langle v - \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \rangle \\
&= \langle v, w_j \rangle - \langle \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \rangle \\
&= \langle v, w_j \rangle - \sum_{i=1}^{k} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_j \rangle \\
&\quad (\langle w_i, w_j \rangle = 0 \text{ or } \langle w_j, w_j \rangle \text{ since orthog basis}) \\
&= \langle v, w_j \rangle - \frac{\langle v, w_j \rangle}{\langle w_j, w_j \rangle} \langle w_j, w_j \rangle \\
&= 0
\end{aligned}$$

So $\langle v, w_j \rangle = 0$ for all $j = 1, 2, \ldots, k$ so $v' \in W^{\perp}$ □

March 27th 2019

Last time: *Thm 56:* $W \leq V$, for all $v \in V$ exists unique $w \in W, v' \in W^{\perp}$ so that

$$v = v' + w$$

If $\alpha = \{w_1, \ldots, w_n\}$ orthog basis W

$$proj_W\ v = w = \sum_{i=1}^{n} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

Proof. Uniqueness. To prove, suppose $v = \hat{v}' + \hat{w}$, where $\hat{w} \in W, \hat{v}' \in W^{\perp}$. Then,

$$\begin{aligned} \vec{0} - v - v &= (v' + w) - (\hat{v}' + w) \\ \vec{0} &= v' - \hat{v}' + w - \hat{w} \\ \hat{w} - w &= v' - \hat{v}' \end{aligned}$$

LHS in W, RHS in $W^{\perp}$, since $v', \hat{v} \in W^{\perp}$ and $W^{\perp}$ subspace and W subspace.

So $\hat{w} - w \in W \cap W^{\perp} = \{\vec{0}\}$

$$\hat{w} - w = \vec{0}$$

so $\hat{w} = w$. Similarly, $v' - \hat{v}' \in W^{\perp} \cap W = \{\vec{0}\}$. So $v' = \vec{v}'$ □

Terminology

If $\alpha = \{w_1, w_2, \ldots, w_m\}$ is an orthogonal set of non-zero vectors, for $v \in V$ the scalars $\frac{\langle v, w_i \rangle}{w_i, w_i}$ are called *Fourier coefficients* of v relative to α.

If α is actually basis of V, Fourier coefficients are coords of v relative to α. If α is a basis for a subspace W, Fourier coefficients give the scalars needed to compute $proj_W\ v$. If $v \in W, proj_W\ v = v$, so these coeffs are cords of $v \in W$.

Note To compute proj, need *orthog* basis W. How to find one?

Lemma 57 (Pythagoras' Thm)**.** *If $u, v \in V$ are orthogonal, then $||u + v||^2 = ||u||^2 + ||v||^2$*

Proof. Exercise. □

Note $||u - v||$ ="distance between u and v". Compare to θ; two vectors with very different norm can be very far apart yet have a small angle. Similarly, inverting the direction of a vector gives us a large angle but a small distance.

Theorem 58. *Let $W \leq V, v \in V, w = proj_W v$. Then w is the "closest vector in W to v" in the sense that if $z \in W$ is any vector*

$$||v - w|| \leq ||v - z||$$

Proof. Recall $||u|| = \sqrt{\langle u, u\rangle}, ||u||^2 = \langle u, u\rangle$. Write $v = v' + w$.

$$\begin{aligned} ||v - z||^2 &= ||v' + w - z||^2 = ||v' + (w - z)||^2 \\ &= ||v||^2 + ||w - z||^2 && \text{(Pythagoras)} \end{aligned}$$

($v' \in W^{\perp}, w - z \in W$, so $v', w - z$ are orthogonal)

$$\begin{aligned} ||v - z||^2 &\geq ||v||^2 \\ &= ||v - w||^2 \end{aligned}$$

Take square root. □

Gram-Schmidt Orthogonalization Process

Or "how to produce an orthogonal basis". Replace w_2 by $v' = w_2 - proj_{w_1} w_1$

Let $W \leq V$, $\alpha = \{w_1, w_2, \ldots, w_m\}$ basis of W. Produce a new basis $\beta = \{v_1, v_2, \ldots, v_m\}$ for W by

$$\begin{aligned} v_1 &= w_1 \\ v_i &= w_i - proj_{\beta_{i-1}} w_i && \text{(for } i = 2, 3, \ldots, m\text{)} \end{aligned}$$

Where $\beta_{i-1} = span\{v_1, v_2, \ldots, v_{i-1}\}$. We will see that $\{v_1, v_2, \ldots, v_{i-1}\}$ orthogonal basis for β_{i-1} so in fact

$$v_i = w_i - (\sum_{i=1}^{i-1} \frac{\langle w_i, v_j\rangle}{\langle v_j, v_j\rangle} v_j)$$

Theorem 59. *For each $i = 1, 2, \ldots, m$, $\{v_1, v_2, \ldots v_i\}$ is orthog basis for $span\{w_1, w_2, \ldots, w_i\}$. In particular, $\{v_1, v_2, \ldots, v_m\}$ is orthog basis of W (you can make it ONB by normalizing each v_i)*

Proof. Omit. Expand some more products. □

Ex $W = span\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}\}$. Find ONB of W.

Sol Apply Gram-Schmidt

$$v_1 = w_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$v_2 = w_2 - proj_{v_i}\, w_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle}$$

$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{0+2+0+0}{1+4+0+1} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 1/3 \\ 1 \\ -1/3 \end{pmatrix}$$

Replace by $\begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix} = v_2$.

$$v_3 = w_3 - proj_{span\ of\ \{v_1,v_2\}}\, w_3 = w_3 - \left(\frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2\right)$$

$$= \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

Replace by $v_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix}$. Orthonormal basis $= \{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix} \}$

March 29th 2019

Recall: An orthonormal basis $\{v_1, \ldots, v_n\}$ is s.t.

$$\langle v_i, v_j \rangle = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

(1) Say we want an orthonormal basis for $P_2(\mathbb{R})$ with standard inner product.

$$\langle f, g \rangle = \int_0^1 f \cdot g(x) dx$$

Sol Take our standard basis $\{1, x, x^2\}$, apply Gram-Schmidt.

(i) Consider $v_1 = 1$. check unit length

$$||v_1|| = 1 = \sqrt{\int_0^1 1 dx}$$

Already normal! Apply G-S process to x. Let $v_2 = x - \frac{\langle x,1 \rangle}{\langle 1,1 \rangle}$:

$$\begin{aligned} x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} &= x - \langle x, 1 \rangle \cdot 1 \\ &= x - \int_0^1 x \cdot 1 dx \\ &= x - \frac{1}{2} \end{aligned}$$

Normalize v_2'.

$$\begin{aligned} ||v_2'|| &= \sqrt{\langle v_2', v_2' \rangle} \\ \langle v_2', v_2' \rangle &= \int_0^1 (x - \frac{1}{2})^2 dx \\ &= \frac{1}{12} \\ &\to ||v_2'|| = \frac{1}{\sqrt{12}} \end{aligned}$$

$v_2($ normalize $v_2') = \sqrt{12} - v_2' = 2\sqrt{3}x = \sqrt{3}$ Consider x^2. let $v_3' = x^2 - proj(x^2) = x^2 - x + \frac{1}{6}$ then normalize v_3' to get

$$\begin{aligned} \langle v_3', v_3' \rangle &= 1/180 \\ \to ||v_3'|| &= \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}} \\ 1/3 = 6\sqrt{5} &- 6\sqrt{5} + \sqrt{5} \end{aligned}$$

So our orthonormal basis is $\{v_1, v_2, v_3\}$

(ii) Now, what about finding the proj of x^2 onto span $\{1, x\}$. v_1, v_2 to be basis elements for this subspace. $\{1, x\} \iff \{v_1, v_2\}$.

Let $span\{1, x\} = W$, with basis $\{v_1, v_2\}$,

$$\begin{aligned} proj_W(x^2) &= \frac{\langle x^2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle x^2, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \\ &= \langle x^2, v_1\rangle v_1 + \langle x^2, v_2\rangle v_2 \\ &= \int_0^1 x^2 \cdot 1dx \cdot 1 + (\int_0^1 x^2(2\sqrt{3}x = \sqrt{3})dx)(2\sqrt{3}x - \sqrt{3}) \\ &= x - \frac{1}{6} \end{aligned}$$

Theorem 60. *Let $\alpha_1 = \{v_1, v_2, \ldots, v_k\}$ be orthonormal set in V, $n = dim(V)$, let $W = span\{v_1, \ldots, v_k\} \subseteq V$ be a subspace. Then,*

(i) α_1 can be extended to an orthonormal basis of V

$$\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

(ii) $\alpha_2 = \{v_{k+1}, \ldots, v_n\}$ is an orthogonal basis of $W^\perp = \{v \in V | \langle v, v_i\rangle = 0 \ \ \forall i = 1, \ldots k\}$

(iii) $dim(W) + dim(W^\perp) = dim(V)$

Proof. (idea)

(i) Extend $\{v_1, \ldots, v_k\}$ to a basis in the usual way, then apply G-S. Omit.

□

Diagonalization

Eigenvalues + eigenvectors

Def If $T : V \to V$ is a linear operator, if $\vec{v} \neq 0$ and $T(\vec{v}) = \lambda\vec{v}$ for some $\lambda \in K$, then $\vec{v}$ is called an eigenvector with eigenvalue λ.

Similarly if $A \in \mathcal{M}_{n\times n}(K)$ and $A\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq 0$ then $\vec{v}$ is an eigenvector with eigenvalue λ.
Remark $\vec{v} \neq 0$! $\lambda = 0$ *is allowed*!

Proposition 61. *$\lambda = 0$ is an eigenvalue $\iff$ T is* NOT INJECTIVE *(in particular,* not invertible*!)*

Proof. Prove both ways.

1. "$\Rightarrow$" if $\vec{v} \neq 0$ and

$$\begin{aligned} T(\vec{v}) &= \lambda\vec{v} = \vec{0} \\ &\Rightarrow \vec{v} \in Ker(T), \vec{v} \neq 0 \\ &\Rightarrow Ker(T) \text{ NOT trivial} \\ &\Rightarrow \text{T is NOT injective} \end{aligned}$$

2. "$\Leftarrow$"

$$\begin{aligned} T \text{ not inj} &\Rightarrow Ker(T) \text{ NOT trivial} \\ &\Rightarrow \exists \vec{v} \neq 0 \text{ s.t. } T(\vec{v}) = 0 \\ &\Rightarrow T(\vec{v}) = \vec{0} = 0 \cdot \vec{v} \\ &\Rightarrow \lambda \text{ is an eigenvalue with eigenvector } \vec{v} \end{aligned}$$

□

Problem Let $\mathbb{R}^3 \to \mathbb{R}^3$ which gives a rotation by π about the z-axis! Thinking geometrically, can we find some eigenvalues and vectors?

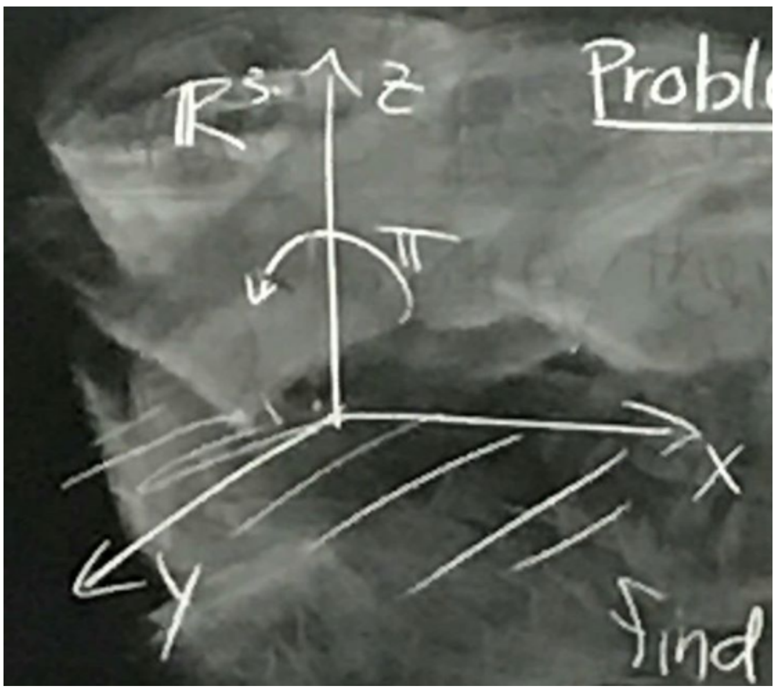


Figure 10: Problem

The z-axis itself $(0,0,1) \in \mathbb{R}^3$ is an eigenvector with eigenvalue 1. z-axis is fixed by T.

$$\begin{aligned} &\Rightarrow T(\vec{v_1}) = \vec{v_1} = \vec{v_1} = 1 \cdot \vec{v_1} \\ &\Rightarrow \lambda = 1 \text{ e.v.} \end{aligned}$$

Vectors lying in the $x - y$ plane have $\lambda = -1$ as an eigenvalue. Let $v \in x - y$-plane $\to v = (x, y, 0)$

$$\begin{aligned} T(\vec{v}) &= (-x, -y, 0) \\ &= -(x, y, 0) \\ &= -1 \cdot \vec{v} \end{aligned}$$

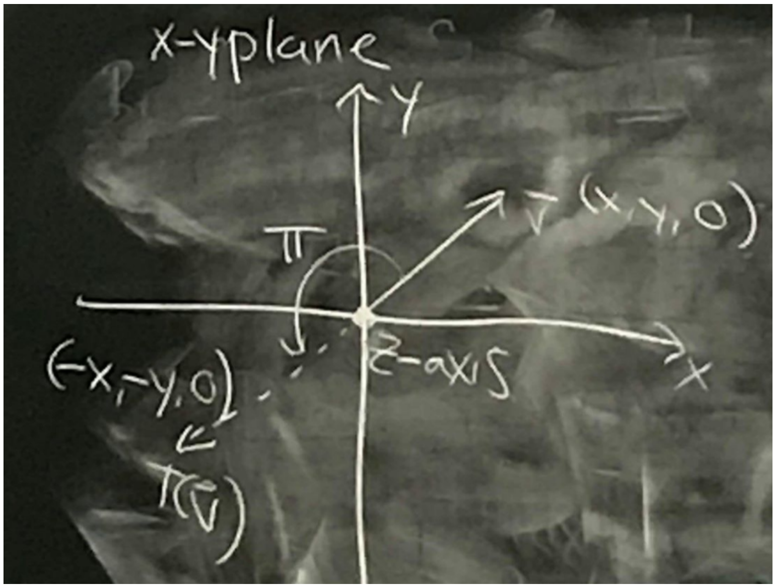


Figure 11: Problem

$\Rightarrow \lambda = -1$ is an eigenvalue!
Question How do we find eigenvalues and eigenvectors algebraically?
Def Let $A \in \mathcal{M}_{n\times n}(K)$. The *characteristic polynomial* of A is defined as

$$c_A(t) = det(A - tI)$$

Example Let $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find $c_A(t)$.

Sol

$$\begin{aligned} c_A(t) &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} \\ &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & t-1 & 1-t \end{vmatrix} \\ &= (t-1)\begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= (t-1)\begin{vmatrix} -t & -2 & -1 \\ 1 & 3-t & 1 \\ 0 & 0 & -1 \end{vmatrix} \\ &= (t-1)(-1)\begin{vmatrix} -t & -2 \\ 1 & 3-t \end{vmatrix} \\ &= -(t-1)(t^2 - 3t + 2) \end{aligned}$$

Theorem 62. *Let $A \in \mathcal{M}_{n\times n}(K)$. Then*

(i) $c_A(t)$ *is a polynomial of degree* n

(ii) λ *is an eigenvalue of* $A \iff \lambda$ *is a root of* $c_A(t)$

(iii) v *is an eigenvector* $\iff v \in Ker(A - \lambda I)$ *and* $v \neq \vec{0}$

April 1st 2019

Last time: Recall theorem 62:
Theorem 62: Let $A \in \mathcal{M}_{n\times n}(K)$. Then

(i) Characteristic polynomial $c_A(t)$ is poly of degree n

(ii) λ is eigenvalue of $A \iff \lambda$ is a root of $c_A(t)$

(iii) $v \in K^n$ is eigenvector of A with eigenvalue $\lambda \iff v \in Ker(A - \lambda I)$ and $v \neq \vec{0}$

Proof. By direct proof.

(i) Omit.

(ii) We have:

$$\begin{aligned}
\lambda \text{ is eigenvalue } &\iff \exists \text{ eigenvector } v \text{ with eigenvalue } \lambda\\
&\iff Ker(A-\lambda I) \neq \{\vec{0}\}\\
&\iff \text{ lin transformation defined by } A-\lambda I \text{ not injective}\\
&\iff \text{ lin transformation defined by } A-\lambda I \text{ not bijective}\\
&\iff A-\lambda I \text{ not invertible}\\
&\iff det(A-\lambda I) = 0 \qquad\qquad (\text{ie } c_A(\lambda)=0)
\end{aligned}$$

(iii) We have:

$$\begin{aligned}
v \text{ is eigenvector with eigenvalue } \lambda &\iff Av = \lambda v\\
&\iff Av - \lambda v = \vec{0}\\
&\iff Av - \lambda I v = \vec{0}\\
&\iff (A-\lambda I)v = \vec{0}\\
&\iff v \in Ker(A-\lambda I)\\
&\qquad\qquad (\text{and } v \neq \vec{0})
\end{aligned}$$

□

Def If λ is an eigenvalue of A, the eigenspace for λ is

$$\begin{aligned} E_\lambda &= Ker(A - \lambda I) \\ &= \{v \in K^n | (A - \lambda I)v = \vec{0}\} \\ &= \{v \in K^n | Av = \lambda v\} \\ &= \text{ all eigenvectors for } \lambda \text{ and also } \vec{0} \end{aligned}$$

It is a subspace since Ker is always a subspace.

Ex $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Find basis for each eigenspace.

Sol Last class, $c_A(t) = \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} = -(t-1)^2(t-2)$.

Eigenvalues $\lambda = 1, 2$.

Eigenspace E_1 ($\lambda = 1$) Solve $(A - I)\vec{x} = \vec{0}$. Note: Can expect to get at least one free variable in these kind of problems.

$$\begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$\vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So basis $\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}$ for E_1.

Eigenspace E_2 Solve $(A - 2I)\vec{x} = \vec{0}$

$$\begin{pmatrix} -2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = -x_3$$

$$\begin{aligned} x_2 &= x_3 \\ x_3 &= x_3 \end{aligned}$$

So basis is $\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \}$

Proposition 63. *Let $A, B \in \mathcal{M}_{n\times n}(K)$ be similar matrices, ie $Q^{-1}AQ = B$ for some $Q \in \mathcal{M}_{n\times n}(K)$. Then,*

(i) $det(A) = det(B)$

(ii) $c_A(t) = c_B(t)$

Proof. (i) Omit. Like (ii) (follows from it)

(ii)

$$\begin{aligned} c_A(t) &= det(A - tI) \\ &= det(QBQ^{-1} - tI) \\ &= det(QBQ^{-1} - tQIQ^{-1}) \\ &= det(Q(B - tI)Q^{-1}) \\ &= det(Q)det(B - \lambda I)det(Q^{-1}) \\ &= det(Q)c_B(t)\frac{1}{det(Q)} \\ &= C_B(t) \end{aligned}$$

□

Def $T : V \to V$ linear op. The characteristic polynomial of T is

$$c_T(t) = det([T]_\alpha - tI)$$

where α is *any* basis of V.
Remark α does not matter, since if β is any other basis then

$$[T]^\beta_\beta = Q^\beta_\alpha [T]^\alpha_\alpha Q^\alpha_\beta$$

ie $[T]_\beta [T]_\alpha$ are similar, same characteristic polynomial.
Ex For $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(f(x)) = (1 + x)f'(x)$$

Find basis for each eigenspace.
Sol Need standard matrix for T, any basis. Use $\alpha = \{1, x, x^2\}$. Calculate

$$\begin{aligned} T(1) &= (1 + x)(0) = 0 \\ T(x) &= (1 + x)(1) = 1 + x \\ T(x^2) &= (1 + x)(2x) && = 2x + 2x^2 \\ [T]^\alpha_\alpha &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \\ c_T(t) &= \begin{vmatrix} -t & 1 & 0 \\ 0 & 1-t & 2 \\ 0 & 0 & 2-t \end{vmatrix} = t(1-t)(2-t) \end{aligned}$$

So $\lambda = 0, 1, 2$ eigenvalues.
For E_0 ($\lambda = 0$) Solve $([T] - 0I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = x_1$$
$$x_2 = 0$$
$$x_3 = 0$$

Basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ie $f(x) = 1$.
Check $T(f(x)) = T(1) = (1+x)0 = 0f(x)$.
For E_1

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$x_1 = x_2$$
$$x_2 = x_2$$
$$x_3 = 0$$

Basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ie $g(x) = 1 + x$.
Check $T(f(x)) = T(1+x) = (1+x)(1) = 1(1+x)$.
For E_2

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$x_1 = x_3$$
$$x_2 = 2x_3$$
$$x_3 = x_3$$

Basis $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ ie $h(x) = 1 + 2x + x^2$.
Check $T(f(x)) = T(1 + 2x + x^2) = (1+x)(2+2x) = 2 + 4x + 2x^2 = 2(1 + 2x + x^2)$.

April 3rd 2019

Last time: $E_\lambda = \{v \in V | T(v) = \lambda v\} =$ eigenspace

Proposition 64. *Let $\lambda_1 \neq \lambda_2$ be eigenvalues of T. Then*

$$E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$$

Proof. $\{\vec{0}\} \supseteq E_{\lambda_1} \cap E_{\lambda_2}$ since $\vec{0}$ in both since both subspaces. For other inclusion, suppose $v \in E_{\lambda_1} \cap E_{\lambda_2}$, so $T(v) = \lambda_1 v$ and $T(v) = \lambda_2 v$ so $\lambda_1 v - \lambda_2 v = \vec{0}$ so

$$(\lambda_1 - \lambda_2)v = \vec{0}$$

If $v \neq \vec{0}$, then $\lambda_1 - \lambda_2 = 0$, contradicts $\lambda_1 \neq \lambda_2$ so $v = \vec{0}$. So $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$. □

Diagonalization

Idea: Diagonal matrices are very nice. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

$$\begin{aligned} AA &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{pmatrix} \end{aligned}$$

In fact, $A^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$. Easy!

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 3 & 0 \end{pmatrix} \\ &= 3 \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

In fact, eigenvalues are $2, 3, 4$ corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Def

1 $A \in \mathcal{M}_{n\times n}(K)$ is *diagonalizable* if $\exists Q \in \mathcal{M}_{n\times n}(K)$ so that

$$Q^{-1}AQ = D \qquad \text{(with } D \text{ diagonal)}$$

(ie A is *similar* to a diagonal matrix)

2 Linear operator $T : V \to V$ is *diagonalizable* if $\exists$ basis α of V so that $[T]_\alpha$ is a diagonal matrix.

Note: For any bases α, β of V,

$$[T]_\alpha = Q^{-1}[T]_\beta Q \qquad (Q = Q_\alpha^\beta)$$

ie T diagonalizable $\iff$ $[T]_\beta$ diagonalizable, β any basis.

Theorem 65. *Let $T : V \to V$ be linear operator.*

(1) T diagonalizable $\iff$ $\exists$ basis α composed of eigenvectors of T.

(2) If $\alpha = \{v_1, v_2, \ldots, v_n\}$ is basis of V, composed of eigenvectors of T, then

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Where λ_i is eigenvalue for v_i, $i = 1, 2, \ldots, n$.

(3) If $A \in \mathcal{M}_{n\times n}(K)$ is diagonalizable with

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Then the i-th column q_i of Q is an eigenvector for A with eigenvalue λ_i, $i = 1, 2, \ldots, n$.

Also, $\{q_1, q_2, \ldots, q_n\}$ is a basis of K^n.

Proof.(1) "$\Rightarrow$" Assume T diagonalizable, ie $\exists$ basis $\alpha = \{v_1, v_2, \ldots, v_n\}$ so that

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \qquad \text{(ie is diagonal)}$$

Recall column i of $[T]_\alpha$ is $[T(v_i)]_\alpha$. So $[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$, ie

$$T(v_i) = \lambda_i v_i$$

So v_i is eigenvector for λ_i, so α is basis of V composed of eigenvectors.

"$\Leftarrow$" Assume $\alpha = \{v_1, v_2, \dots, v_n\}$ basis eigenvectors and $T(v_i) = \lambda_i v_i$. Then,

$$[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$$

So $[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$. So T diagonalizable.

(2) Done in proof of (1).

(3) Assume

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let q_i = column i of Q. Write

$$AQ = QD$$

In AQ, column i is Aq_i (proposition 36). On the right side, column i is

$$\begin{aligned} Q(\text{col } i\text{of } D) &= Q \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix} \\ &= Q(\lambda_i e_i) \\ &= \lambda_i Q e_i \\ &= \lambda_i q_i \end{aligned}$$

So $Aq_i = \lambda_i q_i$. So q_i is eigenvector for eigenvalue λ_i.

□

Ex Diagonalize $A = \begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix}$, ie find Q, D so that $Q^{-1}AQ = D =$ diagonal.

Sol

$$\begin{aligned} c_A(t) &= \begin{vmatrix} i-t & -3 \\ 1 & -i-t \end{vmatrix} \\ &= (i-t)(-i-t)+3 \\ &= i - it + it + t^2 + 3 \\ &= t^2 + 4 \\ &= t^2 - 2i \\ &= (t-2i)(t+2i) \end{aligned}$$

So $\lambda = 2i, -2i$
Eigenvectors For $\lambda = 2i$.

$$\begin{pmatrix} -i & -3 & 0 \\ 1 & -3i & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3i & 0 \\ -i & -3 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -3i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = 3ix_2$$
$$x_2 = x_2$$
$$\begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

Check

$$\begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 3i \\ 1 \end{pmatrix} = \begin{pmatrix} -3-3 \\ 3i-i \end{pmatrix} = \begin{pmatrix} -6 \\ 2i \end{pmatrix}$$
$$= 2i \begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

For $\lambda = -2i$, similar. Get eigenvector $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. So $\{\begin{pmatrix} 3i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\}$ is basis of $\mathbb{C}^2$ composed of eigenvectors of A.

And $Q^{-1}AQ = D$ where

$$D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, Q = \begin{pmatrix} 3i & -i \\ 1 & 1 \end{pmatrix}$$

Or

$$D = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, Q = \begin{pmatrix} -i & 3i \\ 1 & 1 \end{pmatrix}$$

April 5th 2019

Ex Diagonalize $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, if possible.

Sol We already found $\lambda = 1, \lambda = 2$.

$$E_1 = span\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\}$$
$$E_2 = span\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\}$$

Put bases together, is $\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\}$. Basis $\mathbb{R}^3$?

Check

$$Q = \begin{pmatrix}-1 & -1 & -1\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}$$
$$\begin{aligned} det\ Q &= -1\begin{vmatrix}0 & 1\\1 & 1\end{vmatrix} - 1\begin{vmatrix}-1 & -1\\1 & 1\end{vmatrix} \\ &= -1(-1) - 0 \\ &= 1 \neq 0 \end{aligned}$$

So Q inv, this is basis of eigenvectors.

$$Q^{-1}AQ = D = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{pmatrix}$$

Or, $Q^{-1}AQ = D$, with $Q = \begin{pmatrix}-1 & -1 & -1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}, D = \begin{pmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$.
$c_A(t) = (t-1)^2(t-2)$
Ex Show $B = \begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$ is *not* diagonalizable.
Sol $c_A(t) = \begin{vmatrix}1-t & 1 & 0\\0 & 1-t & 1\\0 & 0 & 1-t\end{vmatrix} = (1-t)^3$. $\lambda = 1$ only.
E_1 Solve $(A-I)\vec{x} = \vec{0}$

$$\begin{pmatrix}0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{pmatrix}$$

On free variable, so $dim\ E_1 = 1$. Can't set basis of eigenvector of $\mathbb{R}^3$. So *not* diagonalizable.

Ex Is $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$ diagonalizable?

Sol

$$\begin{aligned} c_A(t) = \begin{vmatrix} 1-t & -2 \\ 1 & -1-t \end{vmatrix} &= (1-t)(-1-t)+2 \\ &= -1-t+t+t^2+2 = t^2+1 \end{aligned}$$

In $\mathbb{R}$, no roots, so no eigenvalues, no eigenvectors. A *not* diagonalizable as element of $\mathcal{M}_{2\times 2}(\mathbb{R})$. But in $\mathcal{M}_{2\times 2}(\mathbb{C})$, have $\lambda = +/-i$ as eigenvalues. Obtain eigenvectors

$$\begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Which are independent, so A *is* diagonalizable as element of $\mathcal{M}_{2\times 2}(\mathbb{C})$.
So $Q^{-1}AQ = D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Diagonalizability

Def If $T: V \to V$, $S: V \to V$ are linear operators and $a, b \in K$, define lin op as $aT + bS$ by

$$(aT+bS)(v) = aT(v) + bS(v), v \in V$$

Proposition 66. *Let $\lambda_1 \neq \lambda_2$ be eigenvalues of $T: V \to V$ and v_1 eigenvector for λ_1, v_2 eigenvector for λ_2. Then v_1, v_2 are linearly independent.*

Proof. Suppose $a_1v_1 + a_2v_2 = \vec{0}$. Consider linear operator $T - \lambda_1 I$
($I: V \to V, I(v) = v$).
Eval $T - \lambda_1 I$ at $\vec{0} = a_1v_1 + a_2v_2$

$$\begin{aligned} \vec{0} &= (T-\lambda_i I)(\vec{0}) \\ &= (T-\lambda_1 I)(a_1v_1 + a_2v_2) \\ &= (T-\lambda_1 I)(a_1) + (T-\lambda_1 I)(a_2v_2) \\ &= a_1(T-\lambda_1 I)(v_1) + a_2(T-\lambda_1 I)(v_2) \\ &= a_1(T(v_1) - \lambda_1 I(v_1)) + a_2(T(v_2) - \lambda_1 I(v_2)) \\ &= a_1(\lambda_1 v_1 - \lambda_1 v_1) + a_2(\lambda_2 v_2 - \lambda_1 v_2) \qquad (v_1, v_2 \text{ eigenvectors}) \\ \vec{0} &= \vec{0} + a_2(\lambda_2 v_2 - \lambda_1 v_2) \end{aligned}$$

So, $a_2(\lambda_2 - \lambda_1)v_2 = \vec{0}$. But $\lambda_1 - \lambda_2 \neq 0, v_2 \neq \vec{0}$ (v_2 eigenvector). So $a_2 = 0$. Similarly, using $T - \lambda_2 I$ we get $a_1 = 0$. So v_1, v_2 linearly independent. □

Theorem 67. *Let $T : V \to V$ lin. op., $\lambda_1, \ldots, \lambda_k$ eigenvalues. If β_i is a basis for E_{λ_i}, $i = 1, 2, \ldots, k$ then*

$$\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$$

is a linearly independent set of size

$$|\beta_1| + |\beta_2| + \ldots + |\beta_k|$$

In particular, if $\sum_{i=1}^{k} dim\ E_{\lambda_i} = dim\ V$, then T is diagonalizable.

Proof. Similar to prop 66. Omit. □

Def Let λ be an eigenvalue of T.

(1) The *geometric multiplicity* of λ is $dim\ E_\lambda$

(2) The *algebraic multiplicity* of λ is the greatest m so that $(t-\lambda)^m$ is a factor of $c_T(t)$

Ex $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ for $\lambda = 1$, algebraic mult is 3 since

$$c_A(t) = (1-t)^3$$

Geometric multiplicity is 1 since $dim\ E_1 = 1$ (one free variable, see previous example)

Theorem 68. *Let λ be eigenvalue of T. Then,*

$$1 \leq \text{ geometric mult. of } \lambda \leq \text{ algebraic mult. of } \lambda$$

Proof. We have two inequalities to prove.

(1) Since λ eigenvalue, $\exists$ a (non-zero) eigenvector, so $dim\ E_\lambda \geq 1$

(2) Let $d = dim\ E_\lambda$ and let $\{v_1, v_2, \ldots, v_d\}$ be a basis for E_λ. Extend this to a basis $\alpha = \{v_1, v_2, \ldots, v_d, v_{d+1}, \ldots, v_n\}$ of V.

Compute $[T]_\alpha$.

$$\begin{aligned} T(v_1) &= \lambda v_1 \\ T(v_2) &= \lambda v_2 \\ &\ldots \\ T(v_d) &= \lambda v_d \\ T(v_{d+1}) &= \text{ something (doesn't actually matter)} \\ &\ldots \\ T(v_n) &= \text{ something (doesn't actually matter)} \end{aligned}$$

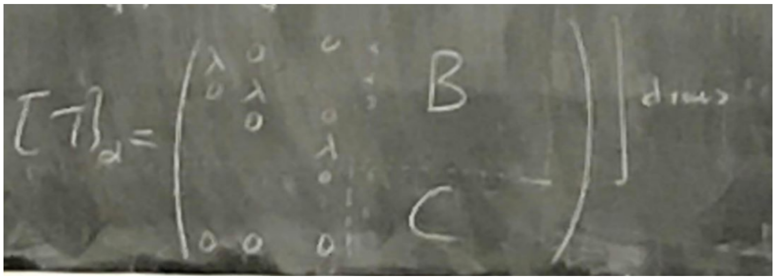


Figure 12:

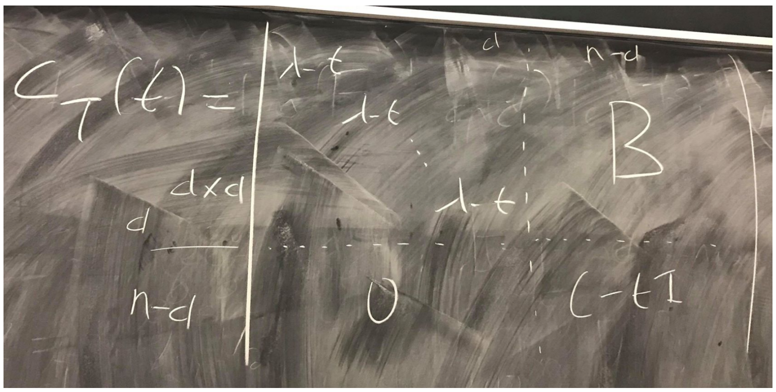


Figure 13:

So we have 12. So we have 13. (with I same size as C) So

$$\begin{aligned} c_T(t) &= (\lambda - t)^d \; det \; (C - tI) \\ &= (\lambda - t)^d \; c_C(t) \end{aligned}$$

So $(\lambda - t)^d$ *is* a factor of $c_T(t)$. So the greatest factor of $(\lambda - t)$ in $c_T(t)$ is $(\lambda - t)^m$ where $m \geq d$. Ie, $d \leq m$ ie geometric multiplicity $\leq$ algebraic multiplicity.

$\square$