MATH223 - Linear Algebra (class notes)

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1 January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

1.1 Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in R^3. Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

1.2 Complex numbers

Def: Let i be a symbol. We declare $i^2 = -1$.
Now, what we'd like to do is take this symbol i and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} &3i \\ &i - 4 \\ &3i - \pi \\ &\sqrt{i} + 21 \end{aligned}$$

Def: The field of complex numbers C consists of all expressions of the form $a + bi$, where $a, b \in R$.
Def: Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Notation:

- $0 + bi = bi$
- $a + 0i = a$ (a *real* number)
- $0 + 0i = 0$

Ex: If $z_1 = 2 - i$, $z_2 = 5i$, then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

Def: Let $z = a + bi \in C$

(i) $\bar{z} = a - bi$, called the *complex conjugate* of z

(ii) $|z| = \sqrt{a^2 + b^2}$, called the *absolute value* or *modulus*

Def: If $z = a + bi \in C$ and $z \neq 0$ (ie $z \neq 0 + 0i$), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \end{aligned}$$

is called the (multiplicative) inverse of z. It has the property $zz^{-1} = 1 = z^{-1}z$.

Proof. We have

$$\begin{aligned} zz^{-1} &= (a + bi)(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i) \\ &= \frac{a^2 - abi + abi - b^2 i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

Note: Since $z \neq 0 + 0i$, $a^2 + b^2 \neq 0$ □

Def: If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

Ex: If $z = 1 + 2i, w = 3 - i$ then

$$\begin{aligned}
\frac{w}{z} &= wz^{-1} \\
&= (3 - i)(\frac{1}{5} - \frac{2}{5}i) \\
&= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\
&= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\
&= \frac{1}{5} - \frac{7}{5}i
\end{aligned}$$

Or,

$$\begin{aligned}
\frac{3 - i}{1 + 2i} \cdot \frac{(1 - 2i)}{(1 - 2i)} &= \frac{3 - 6i - i + 2i^2}{1 - 2i + 2i - 4i^2} \\
&= \frac{1 - 7i}{5}
\end{aligned}$$

2 January 9th 2019

2.1 Complex numbers as points in R^2

You can view $a + bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in R^2, and they will conserve the same properties (addition of complex numbers corresponds to vector addition in R^2). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

2.2 Equations with complex numbers

Fact: Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}
(\pm i\sqrt{|a|}) &= i^2(\sqrt{|a|})^2 \\
&= -1 \cdot |a| \\
&= a && \text{(since } a < 0\text{)}
\end{aligned}$$

Fact: Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in C.

Ex: Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.
Sol:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

Note: If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say z, its other root is $\bar{z}$ ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.

Caution: -1 has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Theorem: (Fundamental Theorem of Algebra) If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 n^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_0 \in C$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \ldots \cdot (x - r_n)$$

for some complex numbers $r_1, r_2, \ldots, r_n$. Some r_i's may be equal.

Corollary: Every such polynomial has at least one root, and at most n distinct roots.

Note: *Finding* the roots is, in general, quite difficult.

Ex: Factor $2x^3 + 2x$ (over C).
Sol:

$$\begin{aligned} 2(x^3 + x) &= 2(x-0)(x^2+1) \\ &= 2(x-0)(x^2 - i^2) \\ &= 2(x-0)(x-i)(x+i) \end{aligned}$$

Ex: Solve $x^2 - i = 0$
Sol: $x^2 = i$ so $x = \pm\sqrt{i}$. Want $\sqrt{i}$ in format $a + bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a+bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$\begin{aligned} 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} && \text{(so a=b both + or both -)} \\ a^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

2.3 Vector spaces (Ch 4)

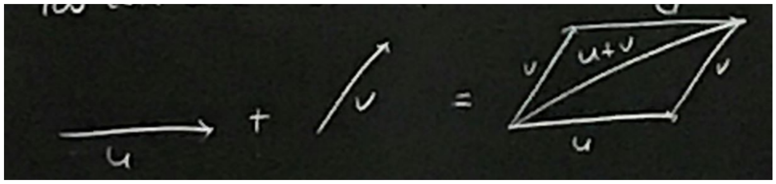
Def. The sets R and C (and also Q, rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of K" means that K is either R or C.

3 January 11th 2019

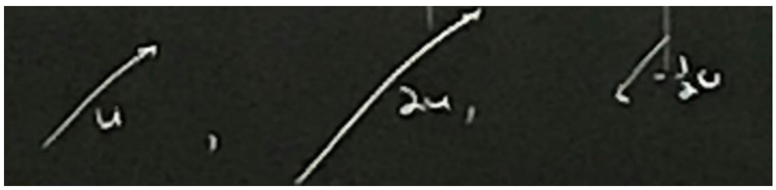
Last time: *Field* K is R or C (for this class).

3.1 Geometric vectors ('arrows')

You can add two vectors (arrows).



Observation: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
You can rescale a vector:



Observation: $a(b\vec{u}) = (ab)\vec{u}$.
Also: $1\vec{u} = \vec{u}$
Question: What properties are interesting? What other objects obey the same properties?
Abstraction: Focus on properties more than on the objects.

3.2 Definition of a vector space

Let V be a set, called set of "vectors", and let K be a field (R or C) (elements of K called *scalars*). Assume that we have already defined two operations:

(1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.

(2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then V is called a *vector space* (over K).

(A1) $u + v = v + u$ (commutative laws)

(A2) There exists a vector in V, named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$

(A3) For each $u \in V$, there is a vector in V, called the (additive) inverse of u and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)

(A4) $(u + v) + w = u + (v + w)$

(S1) $a(u + v) = au + av$ (distributive laws)

(S2) $(a+b)u = au + bu$

(S3) $a(bu) = (ab)u$

(S4) $1u = u$ ($1 \in R$ or C)

These are called the vector space *axioms*.

3.3 Examples of vector spaces

Some examples:

(1) $K^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

where $c \in K$ (and K = set of scalar).

Proof that K^n is a vector space
Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.
Proof concept: To prove "for all $x \in A$, something", say "let $x \in A$" (means x is an arbitrary element of A, ie you only know $x \in A$). Then, prove something for that x.
Proof: Let $u, v \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n)$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= (a_1 + b_1, \ldots, a_n + b_n) && \text{(definition of addition in } K^n\text{)} \\ &= (b_1 + a_1, \ldots, b_n + a_n) && \text{(since } a + b = b + a \text{ for } R \text{ and } C\text{)} \\ &= (b_1, \ldots, b_n) + (a_1, \ldots, a_n) && \text{(definition of addition in } K^n\text{)} \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.
Define $0 = (0, 0, \ldots, 0)$ (which *is* in K^n). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \ldots, a_n$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \ldots, a_n + (0, 0, \ldots, 0) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\ &= (a_1, a_2, \ldots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space C^2, $(2+3i, 5-7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2+3i, 5-7i) \\ &= (4i+6i^2, 10i-14i^2) \\ &= (-6+4i, 14+10i) \end{aligned}$$

4 January 14th 2019

Problem: Let $J = \{(x,y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that J is not a vector space.
Solution: Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_2, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned} ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3) \end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.
Actual proof: Let $u = (1,1)$, $v = (2,2)$ and $w = (3,3)$. Then,

$$\begin{aligned} u + (v + w) &= (1,1) + ((2,2) + (3,3)) \\ &= (1,1) + (-2-3, 5) \\ &= (1,1) + (-5, 5) \\ &= (-1+5, 6) \\ &= (4, 6) \end{aligned}$$

Whereas,

$$\begin{aligned}(u+v)+w &= ((1,1)+(2,2))+(3,3)\\ &= (-1-2,3)+(3,3)\\ &= (-3,3)+(3,3)\\ &= (-(-3)-3,6)\\ &= (0,6)\end{aligned}$$

Hence, the axiom does not hold.

4.1 More examples of vector spaces

(1) K^n (ie R^n or C^n). See before

(2) $P(K)$ = polynomials, where coefficients are in K. Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}f(x) &= x^2+2ix-4 \quad \in P(C)\\ g(x) &= -x^2+cx \quad \in P(C) \qquad \text{(and also in } P(R))\end{aligned}$$

For addition,

$$f(x)+g(x) = 3ix-4$$

And for scalar multiplication,

$$\begin{aligned}2if(x) &= 2ix^2+4i^2x-8i\\ &= 2ix^2-4x-8i\end{aligned}$$

(3) $P_n(K)$ = polynomials of degree n or less, coefficient from K. For example,

$$\begin{aligned}x^2-2x+2 &\in P_2(R)\\ x^2-2x+2 &\in P_3(R)\\ x^2-2x+2 &\in P_2(C)\\ x^2-2x+2 &\notin P_1(R)\end{aligned}$$

Note: In $P(K), P_n(K)$ the "vectors" are polynomials.

(4) $M_{m\times n}(K)$ = $m\times n$ matrices with entries from K. Scalars are K, addition

and scalar multiplication as expected.

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2\times 2}(C) \\
B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2\times 2}(C) \\
A+B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\
2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\
&= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix}
\end{aligned}
$$

The "zero vector" in $M_{m\times n}(K)$ is the $m \times n$ matrix with all entries 0.

(5) Let X be any set (think $x = R$ or C, but not required). Define $F(X, K) = \{f : X \to K\}$ = all functions from X to K.
Ex: $f(x) = x^2 \in F(R, R)$.
Ex: Let $x = \{1, 2\}$. Then g defined by

$$
\begin{aligned}
g(1) &= 3 \\
g(2) &= \sqrt{2}
\end{aligned}
$$

Addition in this space is defined by:
If $f, g \in F(X, K)$ then $f + g$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

Note that $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The + in $(f + g)$ is the addition of vectors f and g, while the other + is scalar addition.
Scalar multiplication in this space is defined by: if $f \in F(X, K), c \in K$ then cf is the function defined by

$$(cf)(x) = cf(x)$$

Note that cf is the name of the function, that "multiplication" is scalar multiplication $F(X, \vDash)$ and $cf(x)$ is the multiplication of two scalars (numbers).
The fact that $F(X, K)$ is a vector space and the axioms are followed is not so obvious.
Prove (A2) true for $F(X, K)$**.** Define $z \in F(X, K)$ by

$$z(x) = 0 \qquad \text{(for all } x \in X)$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence, $f+z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.
Exercise: Try (A3).