# *MATH223 - Linear Algebra (class notes)*

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## Contents

## List of Theorems

## *January 7th 2019*

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

### *Motivation*

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in $R^3$. Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

*Complex numbers*

**Def:** Let $i$ be a symbol. We declare $i^2 = -1$.

Now, what we'd like to do is take this symbol $i$ and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} 3i \\ i-4 \\ 3i-\pi \\ \sqrt{i}+21 \end{aligned}$$

**Def:** The field of complex numbers $C$ consists of all expressions of the form $a+bi$, where $a, b \in R$.

**Def:** Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

(ii)

$$\begin{aligned} (a+bi)(c+di) &= ac+adi+bci+bdi^2 \\ &= ac+adi+bci-bd \\ &= (ac-bd)+(ad+bc)i \end{aligned}$$

**Notation:**

- $0+bi = bi$
- $a+0i = a$ (a *real* number)
- $0+0i = 0$

**Ex:** If $z_1 = 2-i$, $z_2 = 5i$, then

$$z_1 + z_2 = 2+4i$$

and

$$z_1 z_2 = (2-i)(5i) = 10i - 5i^2 = 5+10i$$

**Def:** Let $z = a+bi \in C$

(i) $\bar{z} = a-bi$, called the *complex conjugate* of $z$

(ii) $|z| = \sqrt{a^2+b^2}$, called the *absolute value* or *modulus*

**Def:** If $z = a + bi \in C$ and $z \neq 0$ (ie $z \neq 0 + 0i$), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i \end{aligned}$$

is called the (multiplicative) inverse of $z$. It has the property $zz^{-1} = 1 = z^{-1}z$.

*Proof.* We have

$$\begin{aligned} zz^{-1} &= (a+bi)(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2+b^2} \\ &= \frac{a^2+0+b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

**Note:** Since $z \neq 0 + 0i$, $a^2 + b^2 \neq 0$ □

**Def:** If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

**Ex:** If $z = 1 + 2i, w = 3 - i$ then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3-i)(\frac{1}{5} - \frac{2}{5}i) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned} \frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3-6i-i+2i^2}{1-2i+2i-4i^2} \\ &= \frac{1-7i}{5} \end{aligned}$$

## *January 9th 2019*

### *Complex numbers as points in $R^2$*

You can view $a + bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in $R^2$, and they will conserve the same properties (addition of complex numbers corresponds to vector addition in $R^2$). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

### *Equations with complex numbers*

**Fact:** Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}(\pm i\sqrt{|a|})^2 &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a && \text{(since } a < 0\text{)}\end{aligned}$$

**Fact:** Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in $C$.
**Ex:** Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.
**Sol:**

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2}\end{aligned}$$

**Note:** If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say $z$, its other root is $\bar{z}$ ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.
**Caution:** $-1$ has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$,

but be careful:

$$\begin{aligned}
-1 &= i^2 \\
&= i \cdot i \\
&= \sqrt{-1} \cdot \sqrt{-1} \\
&= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

**Theorem 1** (Fundamental Theorem of Algebra). *If*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 n^0$$

*is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_0 \in C$, then $p(x)$ factors into linear factors,*

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \ldots \cdot (x - r_n)$$

*for some complex numbers $r_1, r_2, \ldots, r_n$. Some $r_i$'s may be equal.*

**Corollary 1.1.** *Every such polynomial has at least one root, and at most $n$ distinct roots.*

**Note:** *Finding* the roots is, in general, quite difficult.
**Ex:** Factor $2x^3 + 2x$ (over C).
**Sol:**

$$\begin{aligned}
2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\
&= 2(x - 0)(x^2 - i^2) \\
&= 2(x - 0)(x - i)(x + i)
\end{aligned}$$

**Ex:** Solve $x^2 - i = 0$

**Sol:** $x^2 = i$ so $x = \pm\sqrt{i}$. Want $\sqrt{i}$ in format $a + bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a+bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$\begin{aligned} 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} && \text{(so a=b both + or both -)} \\ a^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

### *Vector spaces (Ch 4)*

**Def.** The sets $R$ and $C$ (and also $Q$, rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of $K$" means that $K$ is either $R$ or $C$.

## *January 11th 2019*

**Last time:** *Field* $K$ is $R$ or $C$ (for this class).

### *Geometric vectors ('arrows')*

You can add two vectors (arrows) (see figure 1)

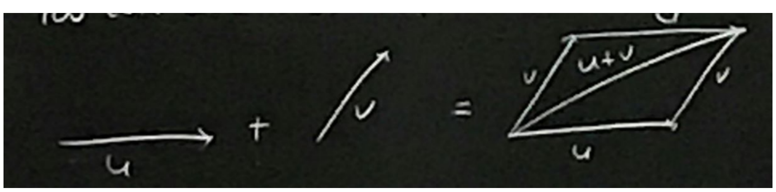



Figure 1: Vector addition

**Observation:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

You can rescale a vector (see figure 2) **Observation:** $a(b\vec{u}) = (ab)\vec{u}$.

Also: $1\vec{u} = \vec{u}$

**Question:** What properties are interesting? What other objects obey the same properties?

**Abstraction:** Focus on properties more than on the objects.

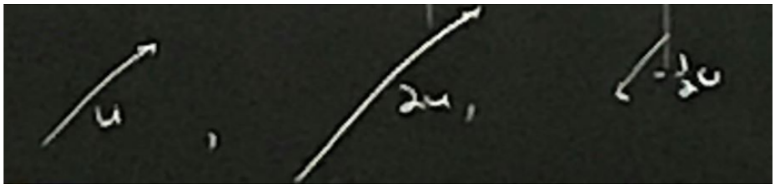



Figure 2: Vector rescaling

*Definition of a vector space*

Let $V$ be a set, called set of "vectors", and let $K$ be a field ($R$ or $C$) (elements of $K$ called *scalars*). Assume that we have already defined two operations:

(1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.

(2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then $V$ is called a *vector space* (over $K$).

(A1) $u + v = v + u$ (commutative laws)

(A2) There exists a vector in $V$, named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$

(A3) For each $u \in V$, there is a vector in $V$, called the (additive) inverse of $u$ and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)

(A4) $(u + v) + w = u + (v + w)$

(SM1) $a(u + v) = au + av$ (distributive laws)

(SM2) $(a + b)u = au + bu$

(SM3) $a(bu) = (ab)u$

(SM4) $1u = u$ ($1 \in R$ or $C$)

These are called the vector space *axioms*.

*Examples of vector spaces*

Some examples:

(1) $K^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

where $c \in K$ (and $K =$ set of scalar).
**Proof that $K^n$ is a vector space**
Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.
*Proof concept:* To prove "for all $x \in A$, something", say "let $x \in A$" (means $x$ is an arbitrary element of $A$, ie you only know $x \in A$). Then, prove something for that $x$.
*Proof:* Let $u, v \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n)$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= (a_1 + b_1, \ldots, a_n + b_n) \quad \text{(definition of addition in } K^n\text{)} \\ &= (b_1 + a_1, \ldots, b_n + a_n) \quad \text{(since } a + b = b + a \text{ for } R \text{ and } C\text{)} \\ &= (b_1, \ldots, b_n) + (a_1, \ldots, a_n) \quad \text{(definition of addition in } K^n\text{)} \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.
Define $0 = (0, 0, \ldots, 0)$ (which *is* in $K^n$). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \ldots, a_n + (0, 0, \ldots, 0) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\ &= (a_1, a_2, \ldots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space $C^2$, $(2 + 3i, 5 - 7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

## *January 14th 2019*

**Problem:** Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that $J$ is not a vector space.
**Solution:** Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned}(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3)\\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3)\\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3)\end{aligned}$$

While

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3)\\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3)\\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3)\end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.
**Actual proof:** Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned}u + (v + w) &= (1, 1) + ((2, 2) + (3, 3))\\ &= (1, 1) + (-2 - 3, 5)\\ &= (1, 1) + (-5, 5)\\ &= (-1 + 5, 6)\\ &= (4, 6)\end{aligned}$$

Whereas,

$$\begin{aligned}(u + v) + w &= ((1, 1) + (2, 2)) + (3, 3)\\ &= (-1 - 2, 3) + (3, 3)\\ &= (-3, 3) + (3, 3)\\ &= (-(-3) - 3, 6)\\ &= (0, 6)\end{aligned}$$

Hence, the axiom does not hold.

*More examples of vector spaces*

(1) $K^n$ (ie $R^n$ or $C^n$). See before

(2) $P(K)$ = polynomials, where coefficients are in $K$. Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}f(x) &= x^2 + 2ix - 4 \quad \in P(C)\\ g(x) &= -x^2 + ix \quad \in P(C) \qquad\qquad \text{(and also in } P(R))\end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned} 2if(x) &= 2ix^2 + 4i^2x - 8i \\ &= 2ix^2 - 4x - 8i \end{aligned}$$

(3) $P_n(K)$ = polynomials of degree $n$ or less, coefficient from $K$. For example,

$$\begin{aligned} x^2 - 2x + 2 &\in P_2(R) \\ x^2 - 2x + 2 &\in P_3(R) \\ x^2 - 2x + 2 &\in P_2(C) \\ x^2 - 2x + 2 &\notin P_1(R) \end{aligned}$$

**Note:** In $P(K), P_n(K)$ the "vectors" are polynomials.

(4) $M_{m\times n}(K) = m \times n$ matrices with entries from $K$. Scalars are $K$, addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2\times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2\times 2}(C) \\ A + B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in $M_{m\times n}(K)$ is the $m \times n$ matrix with all entries 0.

(5) Let $X$ be any set (think $x = R$ or $C$, but not required). Define $F(X,K) = \{f : X \to K\}$ = all functions from $X$ to $K$.
**Ex:** $f(x) = x^2 \in F(R,R)$.
**Ex:** Let $x = \{1,2\}$. Then $g$ defined by

$$\begin{aligned} g(1) &= 3 \\ g(2) &= \sqrt{2} \end{aligned}$$

*Addition* in this space is defined by:

If $f, g \in F(X, K)$ then $f + g$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

*Note that* $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f+g)$ is the addition of vectors $f$ and $g$, while the other $+$ is scalar addition.
*Scalar multiplication* in this space is defined by: if $f \in F(X, K), c \in K$ then $cf$ is the function defined by

$$(cf)(x) = cf(x)$$

*Note that* $cf$ is the name of the function, that "multiplication" is scalar multiplication $F(X, \vDash)$ and $cf(x)$ is the multiplication of two scalars (numbers).
The fact that $F(X, K)$ is a vector space and the axioms are followed is not so obvious.
**Prove (A2) true for $F(X, K)$.** Define $z \in F(X, K)$ by

$$z(x) = 0 \qquad \text{(for all } x \in X)$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f+z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence, $f + z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.
**Exercise:** Try (A3).

## *January 16th 2019*

**Theorem 2** (Cancellation Law). *Suppose $v$ is a vector space over $K$. For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.*

*Note:* To prove "for all" you say let $u \in V$ (means $u$ is an arbitrary vector).

To prove "if $p$ then $q$", denoted $p \to q$, assume $p$ is true and use it to prove $q$.

*Proof.* Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom

A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned}(u+w)+(-w) &= (v+w)+(-w) \\ u+(w+(-w)) &= v+(w+(-w)) && \text{(by A1)} \\ u+\vec{0} &= v+\vec{0} && \text{(by A3)} \\ &= u = v && \text{(by A2)}\end{aligned}$$

□

**Theorem 3.** *Two points:*

1. *The zero vector is unique*
2. *For each $u \in V$, $-u$ is unique*

*Note:* To prove something is unique, suppose you have two of them and show they are the same.

*Proof.* 1) Assume $0$ and $z$ both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned} z &= z+0 && \text{(by *, with } u = z) \\ &= 0+z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0) \end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

**Theorem 4.** *$\forall u \in V, c \in K$,*

*1)* $c\vec{0} = \vec{0}$

*2)* $0u = \vec{0}$

*3)* $-(cu) = ((-c)u)$

*Proof.* Of 2). Let $u \in V$. Then,

$$\begin{aligned} 0u+0u &= (0+0)u && \text{(By SM2)} \\ 0u+0u &= 0u && \text{(by R addition)} \\ 0u+0u &= 0u+\vec{0} && \text{(by A2)} \\ 0u+0u &= \vec{0}+0u && \text{(by A4)} \\ 0u &= \vec{0} && \text{(by cancellation law)} \end{aligned}$$

□

*Note:* $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

*Linear combinations and spans*

**Def:** Let $u, v_1, v_2, \ldots, v_n \in V$. If there are scalars $a_1, a_2, \ldots, a_n \in K$ such that $u = a_1 v_1, a_2 v_2 \ldots a_n v_n$ then $u$ is said to be a linear combination of $v_1, v_2, \ldots, v_n$.
**Ex:** In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.
**Important problem:** Given vectors $u, v_1, v_2, \ldots, v_n$, determine if $u$ is a linear combination of $v_1, v_2, \ldots, v_n$ and if so find $a_1, a_2, \ldots, a_n$.
**Ex:** Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

**Sol.** Are there $a_1, a_2, a_3$ s.t.

$$\begin{aligned} 2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\ &= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3) \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a_1 - 2a_2 + 2a_3 &= 2 \\ 2a_1 - 4a_2 &= 6 \\ a_1 - 2a_2 - 3a_3 &= 8 \end{aligned}$$

Solve the linear system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{(row reduce)}$$

$\therefore$ No solution, because of the last row. $f$ is not a linear combination of $g_1, g_2, g_3$.
**Def:** Let $S \subseteq V$ ($S$ is a subset of $V$) and assume $s \neq 0$. The span of $s$, denoted $span(s)$ is the set of all linear combinations of vectors from $S$, ie

$$\begin{aligned} span(s) = \{u \in V | \; &\exists\, v_1, v_2, \ldots, v_n \in S \\ &\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ &u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\} \end{aligned}$$

## *January 18th 2019*

### *Last class*

$$
\begin{aligned}
&S \subseteq V \\
span(s) = \{u \in V| \; &\exists \; v_1, v_2, \ldots, v_n \in S \\
&\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\
&u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\}
\end{aligned}
$$

**Ex:** $S = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Prove $span(S) = R^2$.
**Note:** $\binom{a}{b}$ means $(a, b)$.
**Proof note:** To prove two sets $A, B$ are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.
**Sol:**

(1) Prove $span(S) \subseteq R^2$. Trivial, since any linear combination of vectors in $R^2$ is still in $R^2$.

(2) Prove $R^2 \subseteq span(S)$. Let $\binom{a}{b} \in R^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$
\begin{aligned}
a &= x_1 + 3x_2 \\
b &= 2x_1 + x_2
\end{aligned}
$$

Want to show this has a solution (for all $a, b$). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\binom{a}{b} \in span(S)$ so $R^2 \subseteq span(S)$. So by (1), (2), $span(S) = R^2$. $\square$

**Note:** $Ax = b$, $A_{n \times n}$ if $A$ inv, $x = A^{-1}b$.

**Theorem 5.** *Let $S \subseteq V$, $S \neq \emptyset$ ($\emptyset$ = empty set). Then,*

*(1) If $u, v \in span(S)$ then $u + v \in span(S)$*

*(2) If $u \in span(S)$ and $c \in K$, then $cu \in span(S)$*

*(3) $\vec{0} \in span(S)$*

*Proof.* By direct proof.

(1) (Note, "if $u, v \in span(S)$" means for all $u, v \in span(S)$).
Let $u, v \in span(S)$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$
$$v = b_1v_1 + b_2v_2 + \ldots + b_mv_m \text{ where } v_1, \ldots, v_m \in S, b_1, \ldots, b_m \in K$$

Then $u + v = a_1u_1 + \ldots + a_nu_n + b_1v_1 + \ldots b_mv_m$ which is in $span(S)$ since $u_1, \ldots, u_n, v_1, \ldots, v_m \in S$.

(2) Let $u \in span(S), c \in K$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \ldots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \ldots (c_na_n)u_n \end{aligned}$$

**Note:** If you want to be very formal, you need to write down all of the vector space axioms. Which is in $span(S)$ since it is a linear combination of $a_1, \ldots, a_n$ which are in $S$.

(3) (Prove $\vec{0} \in span(S)$) Let $u \in S$. **Note**: This is possible only because $S \neq \emptyset$.
Then $u = 1u$, so $u \in span(S)$. Then using $c = 0$ and (2) and fact that $u \in span(S)$,

$$cu = 0u = \vec{0}$$

is also in $span(S)$. **Note:** Since $u = 1u$, $S \subseteq span(S)$.

□

*Subspaces*

**Def.** Let $V$ be a vector space and $W \subseteq V$ (subset). If $W$, using addition and scalar multiplication as defined in $V$, satisfies the definition of vector space, then $W$ is called a subspace of $V$, denoted $W \leq V$ (less than equal sign, read as "subspace").
**Note:** Main issue is that addition and scalar multiplication with vector from $W$ produce vectors which are still in $W$.

**Theorem 6.** *Let $W \subseteq V$. Then, if the following three properties hold, $W \leq V$ (subspace).*

*(SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")*

*(SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")*

*(SS3) $\vec{0} \in W$.*

These are the same properties we just proved for spans; in other words, we proved earlier that $span(S)$ is a subspace.

*Proof.* For $W$ to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

(A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in $V$, $u + (v + w) = (u + v) + w$.

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let $w \in W$. Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \qquad \text{(thm from last class)}$$

is in $W$, as needed.

(A4) Still true because it is true in $V$.

(SM1-SM4) All hold because they hold in $V$.

□

## *January 21st 2019*

### *A note on logic*

Let $P, Q$ be statements that are true or false.

(1) "If $P$ then $Q$", also written symbolically as "$P \Rightarrow Q$" ($P$ *implies* $Q$) means if $P$ is true, then $Q$ is also true. To *prove* "$P \Rightarrow Q$", assume $P$ and prove $Q$ is true. If you *know* that "$P \Rightarrow Q$" is true, you can *use it*: if you can establish that $P$ is true, you may conclude $Q$ is true.
**Ex:** Let $A$ be an $n \times n$ matrix:

$$P : dot(A) = 1 \quad Q : \text{"A is invertible"}$$

**Thm:** $P \Rightarrow Q$

(2) The *converse* of "$P \Rightarrow Q$" is "$Q \Rightarrow P$". This is a (logically) different statement.
**Ex:** With $P$ and $Q$ as above, "$Q \Rightarrow P$" is not true because $A_{inv} \not\Rightarrow det(A) = 1$.

(3) The *contrapositive* of "$P \Rightarrow Q$" is "$\neg Q \Rightarrow \neg P$" ie "if $Q$ false, then P also false". Logically, this is the same as "$P \Rightarrow Q$".

(4) The *equivalence* "$P$ if and only if $Q$", written "$P \iff Q$" means "$P \Rightarrow Q$ and also $Q \Rightarrow P$" is true. Also means that either both $P$ and $Q$ are true or both are false.
**Ex:** $det(A) \neq 0 \iff$ A is invertible.
To prove "$P \iff Q$", need to prove "$P \Rightarrow Q$" and "$Q \Rightarrow P$".
**Note:** $\neg P \Rightarrow \neg Q$ is the same as $Q \Rightarrow P$.

*Subspaces (cont'd)*

**Thm (last class):** Let $W \subseteq V$ (*subset*). If

1. For all $u, v \in W$, $u + v \in W$
2. For all $u \in W$, $c \in K$, $cu \in W$
3. $\vec{0} \in W$

then $W \leq V$ (*subspace*). (ie: $(1), (2), (3)$ are true $\Rightarrow W \leq V$)

**Theorem 7.** *Let $W \subseteq V$. Then*

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).
**Proof.** Exercise.

**Theorem 8.** *Let $W \subseteq V$. Then*

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

*Examples of subspaces and non-subspaces*

Is each subset a subspace?

(a) $W = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Not a subspace, since the zero vector is not in $W$. The others are also false, but it's enough to prove that one of the statements does not hold. But $span(W) = R^2$ (so $span(W) \leq R^2$)

(b) $W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 0\}$. Need to check (1), (2), (3):

(1) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$. Then we know $x + y - z = 0$ and $x' + y' - z' = 0$. Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in $W$.

(2) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ (means $x + y - z = 0$), let $c \in K$. To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here, $cx + cy - cz = c(x + y - z) = c(0) = 0$. So $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

(3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$, since $0 + 0 - 0 = 0$

Since (1), (2), (3) true, $W \leq R^2$ (subspace)

(c) $W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 1 \}$. This is *not* a subspace. (3) is false.

(d) $W = \{A \in M_{2\times 2} | A_{ij} \geq 0 \forall i, j\}$, where $A_{ij}$ is the entry of $A$ in row $i$, column $j$. (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

(2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since, for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ but

$$(-1)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

*Two special subspaces*

Let $V$ be a vector space.

(1) $V \leq V$ is true

(2) $\{\vec{0}\} \leq V$ is true ("zero subspace")

*A refinement on the definition of span*

**Def.** If $S = \emptyset$ (emptyset), define $span(S) = \{\vec{0}\}$ (if $S \neq \emptyset$, $span(S)$ defined as before).

**Theorem 9.** *$span(S) \leq V$.*

**Proof** Two cases :

1. If $S = \emptyset$, $span(S) = \{\vec{0}\} \leq V$
2. If $S \neq \emptyset$, you already proved $span(S)$ satisfies (1), (2), (3).
   So $span(S) \leq V$.

**Theorem 10.** *(improved version of subspace conditions) Let $W \subseteq V$. Then*

$$\begin{aligned} W \leq V \iff & W \neq \emptyset \text{ and} \\ & \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W \end{aligned}$$

**Proof** We will actually prove $(1),(2),(3) \iff RHS$ (right-hand side). Two parts to proof.

(1) "$(1),(2),(3) \Rightarrow RHS$" or "$\Rightarrow$"

## January 23rd 2019

**Recap:**

(1) If $u, v \in W$ then $u + v \in W$

(2) if $u \in W, c \in K$ then $cu \in W$

(3) $\vec{0} \in W$

**Theorem 11.** *Let $W \subseteq V$. Then*

$$W \leq V \iff W \neq \emptyset \text{ and } \forall u, v \in W, c \in K \text{ we have } cu + v \in W$$

**Proof:** Suffices to prove $(1), (2), (3) \iff RHS$.

1. $\Rightarrow$ Assume $(1), (2), (3)$ (prove right-hand side). Two things to prove:

   (1) Since $\vec{0} \in W$ (by (3)), $W \neq \emptyset$

   (2) Let $u, v \in W$ and $c \in K$. Since (2) holds, $cu \in W$. Since (1) holds, $cu \in W$ and $v \in W$, so $cu + v \in W$.

2. $\Leftarrow$ Assume RHS, prove (1), (2), (3).

   (1) Let $u, v \in W$. Apply RHS with $\Leftarrow$ to get

   $$cu + v = 1u + v = u + v \in W$$

   (2) (Prove $\vec{0} \in W$) Since $W \neq \emptyset$, there is a vector $w \in W$. Apply right-hand side with $u = w$, $v = w$, $c = -1$. So $cu + v = (-1)w + w = -w + w = \vec{0} \in W$.

   (3) Let $u \in W, c \in K$. Apply RHS $(cu + v \in W)$ with $u = u$, $c = c$, $v = \vec{0}$ (note: $\vec{0} \in W$ by (3) above). Then $cu + v = cu + \vec{0} = cu \in W$ $\square$

**Ex:** In $F(R, R) = V$ (functions $f : R \to R$), prove that

$$W = \{f \in V | f(3) = 0\}$$

is a subspace. Eg: $f(x) = (x - 3)e^x \in W$.
**Solution:** (1), (2) together (by last thm). Let $f, g \in W, c \in R$ (prove $cf + g \in W$). We know $f(3) = 0$ and $g(3) = 0$. Then, check $(cf + g)(3) = cf(3) + g(3) = 0 + 0 = 0$. So $cf + g \in W$.

Also, prove $w \neq \emptyset$. $f(x) = x - 3 \in W$, since $f(3) = 0$ (or, $z(3) = 0$ satisfies $z(3) = 0$ so $z \in W$. Note that $z$ is he zero vector of $F(R, R)$).

**Theorem 12.** *Let $A \in M_{m\times n}(K), b \in K^m$. Define*

$$S = \{x \in K^n | Ax = b\}$$

*ie $S =$ solution set to linear system $Ax = b$. Then,*

$$S \leq K^n \iff b = \vec{0} \text{ (ie system is homogeneous)}$$

**Proof**

(i) $\Rightarrow$ Assume $S \leq K^n$. Then $\vec{0}_n \in S$ (by (3)). So $A\vec{0} = b$ but $A\vec{0}_n = \vec{0}_m$ so $\vec{0} = b$.

(ii) $\Leftarrow$ Assume $b = \vec{0}_m$ (prove $S \leq K^n$). Then $A\vec{0}_n = \vec{0}_m$, so $\vec{0}_n \in S$. Next, let $u, v \in S, c \in K$. So $u, v \in K^n$ and $Au = b, Av = b$. Verify $cu + v$ is a solution.

$$\begin{aligned} A(cu+v) &= A(cu) + Av && \text{(prop of matrix multiplication)} \\ &= c(Au) + Av && \text{(prop of matrix multiplication)} \\ &= cb + b \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \\ &= b \quad \square \end{aligned}$$

**Ex:** Equation $ax + by + cz = d$ describes a plane in $R^3$ (eg $x + y + z = 1$) (and also, every plane can be described this way). That is,

$$\{(x, y, z) \in R^3 | ax + by + z = d\}$$

is a plane.

By last thm,

$$\begin{aligned} P \text{ is a subspace} &\iff ax + by + cz = d \text{ is a homogeneous system} \\ &\iff d = 0 \\ &\iff P \text{ passes through origin } (0, 0, 0) \end{aligned}$$

**Theorem 13.** *Let $S \subseteq V$. Then,*

(1) *$span(S) \leq V$ and $S \subseteq span(S)$*

(2) *If $S \subseteq W$, and $W \leq V$ (subspace) then $span(S) \subseteq W$ (actually, $span(S) \leq W$, subspace by (1))*

**Proof:**

(1) $\leq$ We know already. Let $u \in S$. Then $u = 1u$, so $u \in span(S)$

(2) Assume $S \subseteq W$, and $W \leq V$. Let $v \in span(S)$. Then $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ for some scalars and vectors $u_1, u_2, \ldots, u_n \in S$. Since $S \subseteq W$, $u_1, u_2, \ldots, u_n \in W$. But $W$ subspace. So $a_1u_1, a_2u_2, \ldots, a_nu_n \in W$ (by prop (2) subspace) then $a_1u_1 + a_2u_2 \in W$ (by prop (1) of subspaces). So then $(a_1u_1 + a_2u_2) + a_3u_3 \in W$ (etc). So $a_1u_1 + a_2u_2 + \ldots + a_nu_n \in W$.
**Note:** "etc" here is actually a proof by mathematical induction.
Omit for now.

## *January 25th 2019*

### *Interlude : Symbolic logic (briefly)*

Let $P, Q$ be statements that could be true ($T$) or false ($F$). Define:

(1) $\neg P$, "not P", is $F$ when $P$ is $T$, $T$ when $P$ is $F$

(2) $P \wedge Q$, "P and Q", is $T$ exactly when $P, Q$ both $T$

(3) $P \vee Q$, "P or Q" is $T$ when $P, Q$ both $F$

(4) $P \Rightarrow Q$, "P implies Q", is $T$ *unless* $P$ is $T$ and $Q$ is $F$. Hence, $P \Rightarrow Q$ is *equivalent to* $\neg P \vee Q$. We will write $P \Rightarrow Q \equiv \neg P \vee Q$.

(5) $P \iff Q$, "P if and only if Q", is $T$ if both $T$ or both $F$.

**De Morgan's Laws**

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

**Quantifiers**

- $\forall$ means "for all"
- $\exists$ means "there exists"

**Ex.** (A4) (commutativity) $\forall u, v \in V \ \ u + v = v + u$.
**Ex. 2** (A2) (zero vector) $\exists z \in V \ \ \forall u \in V \ \ (u + z = u) \wedge (z + u = u)$ (textbook version)

**Negating quantifiers**

- $\neg \forall u \in V P(u) \equiv \exists u \in V \neg P(u)$
- $\neg \exists u \in V P(u) \equiv \forall u \in V \neg P(u)$

**Ex.**

$$\begin{aligned}
\neg(A2) &\equiv \neg \exists z \in V \forall u \in V \quad u + z = u \wedge z + u = u \\
&\equiv \forall z \in V \neg \forall u \in V \quad u + z = u \wedge z + u = u \\
&\equiv \forall z \in V \exists u \in V \quad \neg(u + z = u \wedge z + u = u) \\
&\equiv \forall z \in V \exists u \in V \quad (u + z \neq u \vee z + u \neq u)
\end{aligned}$$

**Proof by contradiction**

You want to prove some statement $P$. Proof by contradiction works this way:

(1) Assume $\neg P$

(2) Derive a contradiction (hard part)

(3) Conclude $P$ is true

**Ex.** Outline of how to prove (A2) *does not hold* in some vector space. You want to prove $\neg(A2)$.

$$\begin{aligned}\neg(A2) &\equiv \neg\exists z \in V\ \forall u \in V \quad u + z = u \land z + u = u \\ &\equiv \forall z \in V \neg\forall u \in V \quad u + z = u \land z + u = u\end{aligned}$$

Let $z \in V$. Prove the right-hand part ($\neg\forall u \in V \quad u + z = u \land z + u = u$) by contradiction. Assume (for contradiction) that

$$\forall u \in V \quad u + z = u \land z + u = u \tag{1}$$

Use (1) by substituting $u =$ some specific vector (derive a contradiction). Conclude that ($\neg\forall u \in V \quad u + z = u \land z + u = u$) is true.

*Last time*

**Theorem 14.** *If $S \subseteq W$, $W \leq V$ then $span(S) \subseteq W$.*

**Note.** This means if you "promote" a subset to a subspace, adding in only what's necessary, what you get is $span(S)$. Or, $span(S)$ is the "smallest" subspace containing $S$.
**Fact.** Subspaces are "closed under taking linear combinations". Ie if $W \leq V$, $w_1, \ldots, w_n \in W$ and $a_1, \ldots, a_n \in K$ then

$$a_1 w_1 + a_2 w_2 + \ldots + a_n w_n \in W$$

**Caution.** Linear combinations are *finite* sums by definition. So you can't sum up infinitely many vectors.

*Illustration of this theorem*

Let $S = \{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\3\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}\} \subseteq W = \{\begin{pmatrix}x\\y\\0\end{pmatrix} | x, y \in R\}$. Then $span(S) \subseteq W$ ie $span(S)$ is in $xy$ plane. In fact, $span(S) = W$.
**Def.** If $W = span(S)$, we say that $S$ spans $W$ or is a spanning set for $W$.
**Ex.** $S = \{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\}$, $span(S) =$ xy-plane in $R^3$. So $S$ spans the xy-plane.
**Ex. 2.** $S = \{\begin{pmatrix}1\\1\\0\end{pmatrix}\}$, $span(S) = \{\begin{pmatrix}x\\x\\0\end{pmatrix} | x \in R\} =$ line.

### *Intersection of two subspaces*

**Theorem 15.** *Let $W_1 \leq V, W_2 \leq V$. Then $W_1 \cap W_2 \leq V$ (ie intersection of two subspaces is a subspace).*

**Proof.** $W_1 \cap W_2 = \{w \in V | w \in W_1 \wedge w \in W_2\}$.

(1) $\vec{0} \in W_1, \vec{0} \in W_2$ (because subspace). So $\vec{0} \in W_1 \cap W_2$.

(2) Let $u, v \in W_1 \cap W_2, c \in K$. So $u, v \in W_1$ and $W_1 \in V$ so $cu + v \in W_1$ and $u, v \in W_2$ and $W_2 \in V$ so $cu + v \in W_2$. Hence $cu + v \in W_1 \cap W_2$. □

## *January 28th 2019*

**Last time:** $W_1 \leq V$ and $W_2 \leq V \Rightarrow W_1 \cap W_2 \leq V$.

**Corollary 15.1.** *The intersection of any number of subspaces is a subspace.*

**Problem.** Prove that $W = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.
**Sol #1:** Directly from subspace properties (omit)
**Sol #2:** We saw an example proving that $\{f : \mathbb{R} \to \mathbb{R} | f(3) = 0\}$ is a subspace. The "3" is not important, so similarly:

$$W_1 = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0\}$$
$$W_2 = \{f : \mathbb{R} \to \mathbb{R} | f(2) = 0\}$$

both subspaces of $F(\mathbb{R}, \mathbb{R})$. Then $W_1 \cap W_2 = \{f : \mathbb{R} \to \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace.
**Q:** Is union of two subspaces also a subspace?
**A:** Not in general.
**Eg:** $W_1 = $ x-axis $= \{\binom{x}{0} | x \in \mathbb{R}\} \leq \mathbb{R}^2$
$W_2 = $ y-axis $= \{\binom{0}{y} | y \in \mathbb{R}\} \leq \mathbb{R}^2$
$W_1 \cup W_2 = $ xy-axis $= \{\binom{x}{y} | x = 0 \vee y = 0\}$, which, importantly, is not $\mathbb{R}^2$. *Not* a subspace, since $\binom{1}{0} \in W_1 \cup W_2$, $\binom{0}{1} \in W_1 \cup W_2$, but $\binom{1}{1} = \binom{1}{0} + \binom{0}{1} \notin W_1 \cup W_2$.
**Note:** To promote $W_1 \cup W_2$ to a subspace, you form $span(W_1 \cup W_2)$.
**Def:** Let $W_1 \leq V$m $W_2 \leq V$. The *sum* of $W_1$ and $W_2$ is

$$W_1 + W_2 = \{v \in V | \exists w_1 \in W_1, w_2 \in W_2, \text{ such that } v = w_1 + w_2\}$$

**Ex:**

$$W_1 = \{ax^2 | a \in \mathbb{R}\} \leq P(\mathbb{R})$$
$$W_2 = \{ax | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

We have,

$$W_1 + W_2 = \{ax^2 + bx | a, b \in \mathbb{R}\}$$

**Theorem 16.** *Let $W_1 \leq V$, $W_2 \leq V$. Then*

*(a)* $W_1 + W_2 = span(W_1 \cup W_2)$ *(hence $W_1 + W_2$ is a subspace)*

*(b)* $W_1 \leq W_1 + W_2$, $W_2 \leq W_1 + W_2$

**Proof:**

(a)(1) Prove $W_1 + W_2 \subseteq span(W_1 \cup W_2)$. Let $v \in W_1 + W_2$, so $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1, w_2 \in W_1 \cup W_2$ so $v \in span(W_1 \cup W_2)$

(2) "$\supseteq$". Let $v \in span(W_1 \cup W_2)$. Means $v = a_1u_1 + a_2u_2 + \dots a_nu_n, u_1, u_2, \dots, u_n \in W_1 \cup W_2$ and $a_1, a_2, \dots, a_n \in K$. Each $u_i$ is in $W_1 \cup W_2$. Separate into two groups and relabel, so that:

- Those in $W_1$, call these

$$u_1, u_2, \dots u_l$$

So $0 \leq l \leq n$, $l = 0$ means *none* in $W_1$.

- Those in $W_2 \backslash W_1 = \{w \in W_2 | w \notin W_1\}$ ("set difference"), call these

$$u_{l+1}, \dots, u_n$$

So $l = 0$ means all in $W_2 \backslash W_1$, $l = n$ means all in $W_1$.

Then, let $w_1 = a_1u_1 + a_2u_2 + \dots + a_lu_l$ (or $w_1 = \vec{0}$ if $l = 0$), $w_2 = a_{l+1}u_{l+1} + \dots + a_nl_n$ (or $w_2 = \vec{0}$ if $l = n$).
Then $w_1 \in W_1$ since $W_1$ is a subspace, similarly $w_2 \in W_2$. So

$$\begin{aligned} v &= a_1u_1 + \dots + a_nu_n \\ &= w_1 + w_2 \in W_1 + W_2 \text{ as required} \end{aligned}$$

(b) $W_1 \leq W_1 + W_2$, $W_2 \leq W_1 + W_2$. Follows from (a), since $S \subseteq span(S)$ $\square$.

*Linear independence*

**Def:** Vectors $u_1, u_2, \dots, u_n \in V$ (all distinct) are said to be *linearly dependent* if $\exists$ scalars $a_1, a_2, \dots, a_n \in K$ *not all 0* such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$$

Above equation called a *dependence relation.*

**Note:** If $a_1u_1 + a_2u_2 + \ldots + a_nu_n = \vec{0}$ and $a_1 \neq 0$, then you can solve for $u_1$:

$$u_1 = \frac{-a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \ldots - \frac{a_n}{a_1}u_n$$

ie $u_1 =$ linear combination of others, "depends on" others.
**Ex:** $\{x^2 + x, 2x^2, \frac{x}{10}\}$ is a dependent set of vectors in $P(\mathbb{R})$ since

$$(x^2 + x) - \frac{1}{2}(2x^2) - 10(\frac{x}{10}) = 0$$

**Def:** A set of vectors $S \subseteq V$ (possibly infinite) is dependent if $\exists$ a finite subset $\{v_1, v_2, \ldots, v_n\} \subseteq S$ of it which is dependent.
**Def:** Vectors $v_1, v_2, \ldots, v_n$ are linearly independent if they are *not* dependent. That is,

$$\begin{aligned}
\neg\exists a_1, \ldots, a_n \in K \quad & (a_1u_1 + \ldots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0)) \\
\forall a_1, \ldots, a_n \in K \quad & \neg(a_1u_1 + \ldots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0)) \\
\forall a_1, \ldots, a_n \in K \quad & (\neg(a_1u_1 + \ldots + a_nu_n = \vec{0}) \vee (a_1 = 0 \wedge a_2 = 0 \wedge \ldots \wedge a_n = 0))
\end{aligned}$$

Note that $P \implies Q \equiv \neg P \vee Q$. In other words, $u_1, u_2, \ldots, u_n$ are linearly independent if

$$\forall a_1, \ldots, a_n \in K(a_1u_1 + \ldots + a_nu_n = \vec{0} \implies a_1 = 0 \wedge \ldots \wedge a_n = 0)$$

Which is to say that the only solution to $a_1u_1 + \ldots a_nu_n = \vec{0}$ is the trivial solution $a_1 = 0, a_2 = 0, \ldots, a_n = 0$.

## *January 30th 2019*

### *Last class*

$v_1, v_2, \ldots, v_n$ *independent* if $x_1v_1 + \ldots + x_nv_n = \vec{0}$ has only trivial solution $x_1 = x_2 = \ldots = x_n = 0$.
**Ex:** Prove that $\{1 + x^2, x + x^2, 1 + x + x^2\}$ is independent.
**Solution:** Consider equation

$$a(1 + x^2) + b(x + x^2) + c(1 + x + x^2) = 0 = 0(1) + 0x + 0x^2$$

Want to show $a = b = c = 0$ is the only solution.
Equation means for all $x \in K$ ($\mathbb{R}$ or $\mathbb{C}$),

$$a(1 + x^2) + b(x + x^2) + c(1 + x + x^2) = 0$$

So, substitute any scalar for $x$:

$$\begin{aligned}
x = 0 \quad & a + c = 0 \\
x = 1 \quad & 2a + 2b + 2c = 0 \\
x = -1 \quad & 2a + 0b + c = 0
\end{aligned}$$

Can translate into linear system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Row-reduce:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Only solution is $a = 0$, $b = 0$, $c = 0$ so vectors are independent.
If we obtain infinitely many, then you can find dependent set so dependent.

*Some important cases*

(i) $S = \emptyset$ is linearly independent since there are no vectors with which to form a dep. relation.

(ii) If $\vec{0} \in S$, then dependent (since $1\vec{0} = \vec{0}$ *is* a dep. relation)

(iii) $\{u\}$ is independent $\iff u \neq \vec{0}$.
**Note**: $u + (-1)u = \vec{0}$ is *not* a dep. elation, since $u$ is repeated. But, $\{u, -u\}$ is dependent since

$$u + (-u) = \vec{0}$$

*is* a dep. relation.

**Proposition 17.** *Let $A, B \subseteq V$ where $A \subseteq B$.*

*(i) If $A$ is dependent, $B$ is also dependent*

*(ii) If $B$ is independent, $A$ is also independent (contrapositive)*

**Proof:**

(i) If $A$ dep, we have a dep relation

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = \vec{0} \qquad \text{(not all scalars 0, } v_1, \ldots, v_n \in A)$$

which is also a dependence relation in $B$ since $v_1, \ldots, v_n \in B$.

(ii) This is the contrapositive of (i). $\square$

**Note:** Converse is false, $B$ *dep* $\not\to$ $A$ *dep*.

*Extending an independent set*

**Theorem 18.** *Let $S \subseteq V$ be linearly independent and suppose $u \notin S$. Then, $S \cup \{u\}$ independent $\iff$ $u \notin span(S)$.*

**Proof:**

(i) "$\to$" We will prove this as the contrapositive, ie $u \in span(S) \to dep$. Assume $u \in span(S)$. So,

$$u = a_1v_1 + \ldots + a_nv_n \quad \text{where } v_1, v_2, \ldots, v_n \in S$$
$$\vec{0} = (-1)u + a_1v_1 + \ldots + a_nv_n$$

Which is a linear combination of vectors from $S \cup \{u\}$, not all coefficients 0 since first is $-1$. Also, the vectors $u, v_1, v_2, \ldots, v_n$ are all distinct, since $u \notin S$. So this is a dependence relation on $S \cup \{u\}$, so the set is dependent.

(ii) "$\leftarrow$" Also by contrapositive. Assume $S \cup \{u\}$ dep, want to show that $u \in span(S)$. So there is a dependence relation on $S \cup \{u\}$. Two cases:

- **Case 1:** Dependence relation does not involve $u$ (or, involves $u$ but with coefficient 0), ie we have

  $$a_1v_1 + \ldots + a_nv_n = \vec{0} \qquad \text{(not all scalars 0, } v_1, \ldots, v_n \in S)$$

  But this contradicts independence of $S$, so case 1 does not occur.
- **Case 2:** Dependence relation involves $u$ (with coeff *not* 0), so

  $$au + a_1v_1 + \ldots + a_nv_n = \vec{0} \quad v_1, \ldots v_n \in S$$

  and $a \neq 0$. Rewrite:

  $$u = \frac{-a_1}{a}v_1 - \frac{a_2}{a}v_2 - \ldots - \frac{a_n}{a}v_n \qquad (a \neq 0)$$

  Hence $u \in span(S)$. $\square$

**Note:** Conclusion can be restated as

$$S \cup \{u\} \text{ dependent} \iff u \in span(S)$$

*Basis and dimension*

**Fact:** If $W$ is subspace, then $span(W) = W$. (Exercise)

So every subspace *is* a span. But thinking of $W$ as $span(W)$ is excessive. Would like to find the *smallest* $S$ such that

$$span(S) = W$$

**Def:** Let $W \leq V$. A *basis* of $W$ is a set $B \subseteq V$ such that

(i) $span(B) = W$ ("enough vectors to produce $W$")

(ii) $B$ is linearly independent ("no extra vectors in $B$")

**Examples:**

(i) Let $e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow (row\ i)$ . Then,

$$\{e_1, e_2, \ldots, e_n\}$$

is a basis for $K^n$. For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $K^3$.

More next class.

## *February 1st 2019*

**Recall:** $B$ is a basis of $W$ if $span(B) = W$ and $B$ is linearly independent.

**Examples:**

(1) $P_n(K)$ has basis $\{1, x, x^2, \ldots, x^n\}$

(2) $P(K)$ has basis $\{1, x, x^2, x^3, \ldots\}$ (infinitely many)

(3) $M_{m\times n}(K)$ has basis $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ where $E^{ij} = m \times n$ matrix of 0s except 1 in row i, column j. eg: $M_{2\times 2}(\mathbb{R})$ has basis

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(4) $W = \{\vec{0}\}$ has basis $\emptyset$ since

(i) span $\emptyset = \{\vec{0}\}$ (by special def)

(ii) $\emptyset$ is independent

### *Two important questions*

(1) Does $W$ *always* have basis ? (spoiler: yes)

(2) How to *find* a basis?

**Theorem 19** (Bases exist). *Let $V$ be vector space and $S$ a* finite *set with $span(S) = V$. Then there is a subset $B \subseteq S$ which is a basis of $V$.*

*Proof.* Algorithm to produce $B$.

(1) If $V = \{\vec{0}\}$, use $B = \emptyset$.

(2) Take one vector, $u_1 \in S(u_1 \neq \vec{0})$. Consider $span\{u_1\}$

(3) If $span\{u_1\} = V$, done. $B = \{u_1\}$ is a basis (set of one non-zero vector is independent)

(4) If $span\{u_1\} \neq V$, there must be a vector $u_2 \in S$ where $u_2 \notin span(\{u_1\})$ (Why? If not, $S \subseteq span(\{u_1\}) \leq V$, then $span(S) \subseteq span\{u_1\}$, but $span(S) = V$ contradicts $V \neq span\{u_1\}$). By previous theorem, since $u_2 \notin span\{u_1\}$, $\{u_1, u_2\}$ is linearly independent.

(5) Consider $\{u_1, u_2\}$. If $span\{u_1, u_2\} = V$, done: $B = \{u_1, u_2\}$. Else, continue as before, finding $u_3 \in S$, $u_3 \notin span\{u_1, u_2\}$ (etc)

Since $S$ is *finite*, this must *stop* and at that point you have basis $B \subseteq S$. □

*Illustration of this thm*

Find basis of $\mathbb{R}^3$ that is a subset of

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Following this algorithm,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

**Theorem 20.** *Let $V$ be a vector space, $L \subseteq V$ a linearly indepedent set, and $S \subseteq V$ a spanning set (ie $V = span(S)$). Then $\exists$ a subset $E \subseteq S$ such that $L \cup E$ is a basis of $V$ (ie you can always* extend *it to a basis)*

**Proof** Omitted.

**Theorem 21.** *Suppose $V$ has a finite spanning set $S$. Then $V$ has a basis and all bases have the same size, which is at most $|S|$.*

**Proof** Omitted.
**Def** If $V$ has a finite basis $B$, then the *dimension* of $V$ is

$$dim\ V = |B|$$

If $V$ does not have a finite basis, it is called *infinite dimensional*.
**Ex:**

(1) $dim\ K^n = n$.

$$\left(\left\{\begin{pmatrix}1\\0\\\dots\\0\end{pmatrix},\begin{pmatrix}0\\1\\\dots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\0\\\dots\\1\end{pmatrix}\right\}\right)$$

(2) $dim\ P_n(K) = n+1$ (basis $\{1, x, x^2, \dots, x^n\}$)

(3) $P(K)$ is infinite dimensional (A#1, proved a finite set of polynomials cannot span $P(K)$)

(4) $dim\ M_{m\times n}(K) = mn$ (see basis $E^{ij}$, defined above)

**Theorem 22.** Every *vector space (including the infinite dimensional ones) has a basis.*

**Proof** Uses Axiom of Choice. Difficult.

**Theorem 23.** *Suppose $dim\ V = n$. Let $A \subseteq V$. Then,*

*(1) If $span(A) = V$, then $|A| \geq n$ (or, if $|A| < n$ then $A$ does not span $V$) and if also $|A| = n$ then $A$* is *linearly independent, hence basis.*

*(2) If $A$ is linearly independent, then $|A| \leq n$ (or, if $|A| > n$ then $A$ dep) and if also $|A| = n$ then $span(A) = V$ hence $A$ is a basis.*

**Proof** Omitted.
**Note:** If you have *correct number* of vectors, you need only check spanning or independent, not both, to check if basis.
**Ex:** If you have 7 matrices in $M_{3\times 2}(K)$, they *will be* dependent. If you have 5, it's *not* a basis.

## *February 4th 2019*

### *Last class*

Suppose $dim\ V = n$, $S \subseteq V$, $|S| = n$. Then $S\ span\ V \iff S$ linearly independent (only in case $|S| = dim\ V$).

### *Lagrange Interpolation*

**Problem** Given "data points" $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ where all $a_i$ are different. Find a polynomial $p(x)$ of degree $n-1$, $p(x) =$

$c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \ldots + c_1 x + c_0$ whose graph $y = p(x)$ passes through all the points.
**Sol #1** Substitute $(a_1, b_1)$ into $y = p(x)$:

$$b_1 = c_{n-1}a_i^{n-1} + \ldots + c_1 a_i + c_0 \qquad \text{(for each } i = 1, \ldots, n\text{)}$$

Which is a system of $n$ linear equations (vars = $c_{n-1}, \ldots, c_0$) in $n$ variables.

We'll do something different.
**Def** For scalars $a_1, a_2, \ldots, a_n$ (all different), define the *Lagrange polynomials* for each $i = 1, 2, \ldots, n$ set

$$\begin{aligned} l_i(x) &= \prod_{k=1, k\neq i}^{n} \frac{(x - a_k)}{(a_i - a_k)} \\ &= \frac{(x-a_1)}{(a_i - a_1)} \cdot \frac{(x-a_2)}{(a_i-a_2)} \cdot \ldots \cdot \frac{(x-a_n)}{a_i - a_n} \qquad \text{(omitting } \tfrac{(x-a_i)}{(a_i-a_i)}\text{)} \end{aligned}$$

**Ex** For $a_1 = 2, a_2 = 4, a_3 = 6$ we would have

$$\begin{aligned} l_1(x) &= \frac{(x-4)}{2-4} \cdot \frac{(x-6)}{(2-6)} \\ l_2(x) &= \frac{(x-2)}{4-2} \cdot \frac{(x-6)}{(4-6)} \\ l_3(x) &= \frac{(x-2)}{6-2} \cdot \frac{(x-4)}{(6-4)} \end{aligned}$$

**Note:** All degree 2, $l_1(4) = 0$, $l_1(6) = 0$, $l_1(2) = 1$.
**Fact** $l_i(a_j) = 0$ if $i \neq j$ and 1 if $i = j$.
**Proof** If $i \neq j$, there is a factor $\frac{x - a_j}{a_i - a_j}$, so at $x = a_j$, $\frac{a_j - a_j}{a_i - a_j} = 0$. If $i = j$,

$$l_i(a_i) = \prod_{k=1, k\neq i}^{n} \frac{(a_i - a_k)}{(a_i - a_i)} = 1$$

**Proposition 24.** *Lagrange polynomials $l_1(x), \ldots, l_n(x)$ form a basis of $P_{n-1}(\mathbb{R})$.*

**Proof** We have $n$ polunomials (they *are* distinct), *dim* $P_{n-1}(\mathbb{R}) = n - 1 + 1 = n$. So correct number. Suffices to prove *span* or lin independence. We'll prove independence. Suppose

$$d_1 l_1(x) + d_2 l_2(x) + \ldots + d_n l_n(x) = 0 \qquad \text{(note: for } \textit{all } x \in \mathbb{R}\text{)}$$

Substitute $x = a_1$, $x = a_2$, etc into the above. At $x = a_1$, $l_1(a_1) = 1$ but $l_j(a_1) = 0$ for $j \neq 1$ so

$$d_1 1 + d_2 0 + \ldots + d_n 0 = 0$$

so $d_1 = 0$. Similarly, $d_j = 0$ for all $j$. More formally, for any $j = 1, 2, \ldots, n$ we have at $x = a_j$

$$\sum_{i=1}^{n} d_i l_i(a_j) = 0$$

but all terms are 0 *except* when $i = j$. Set

$$d_j = d_j(1) = d_j l_j(a_j) = 0$$

Hence Lagrange polynomials for a basis.
**Problem** Find poly degree $n-1$ through points $(a_1, b_1), \ldots, (a_n, b_n)$.
**Sol:** Set $p(x) = b_1 l_1(x) + b_2 l_2(x) + \ldots + b_n l_n(x)$ (it has degree $n-1$). Then

$$\begin{aligned} p(a_1) &= b_1 l_1(a_1) + b_2 l_2(a_1) + \ldots + b_n l_n(a_1) \\ &= b_1(1) + 0 + 0 + \ldots + 0 \\ &= b_1 \end{aligned}$$

For each $i = 1, 2, \ldots, n$,

$$\begin{aligned} p(a_i) &= \sum_{j=1}^{n} b_j l_j(a_i) \\ &= 0 + 0 + \ldots + b_i l_i(a_i) + \ldots + 0 \\ &= b_i \end{aligned}$$

*Dimension of subspaces*

**Theorem 20.** *Let $W \leq V$, $V$ finite-dimensional. Then*

*(i) $\dim W \leq \dim V$*

*(ii) $\dim W = \dim V \iff W = V$*

**Proof**

(i) Similar to proof that $V$ has basis. Use $W$ as a spanning set for $W$. Pick out vectors one at a time (similar to before) to build a basis. You cannot put more than *dim* $V$ vectors into your basis, as this would give an independent set in $V$ of size *more than dim* $V$ (impossible). So this process has to stop, and it produces a basis for $W$.

(ii) "$\rightarrow$" Assume $\dim W = \dim V = n$. Take basis $B$ of $W$. It is a size $n$ linearly independent set inside $V$, hence $B$ also basis for $V$, hence,

$$V = \operatorname{span} B = W$$

"$\leftarrow$" If $W = V$, clearly $\dim W = \dim V$. $\square$

| *dim W* | Classification |
|---|---|
| 0 | $\{\vec{0}\}$ |
| 1 | $span\{u\}$ = line through origin |
| 2 | $span\{u, v\}$ = plane through origin |
| 3 | $\mathbb{R}^3$ |

**Subspaces of** $\mathbb{R}^3$ If $W \leq \mathbb{R}^3$, $dim\ W = 0, 1, 2$ or 3.

This allows us to make the following classification: **Problem** Let $W = \{A \in M_{n\times n}(\mathbb{R}) | tr(A) = 0\}$, where $tr(A)$ = trace of $A$ = sum of entries on diagonal $= A_{11} + A_{22} + \ldots + A_{nn}$.
**Exercise** Prove $W$ is a subspace.
**Will do next class:** Find $dim\ W$ and find a basis of $W$.

## *February 6th 2019*

### *Intuition*

Solution set $W$ to a homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $K^n (n = \#$ of variables ). If no equations, $W = K^n$, $dim\ W = n$. For each equation, *expect* the dimension of $W$ to drop by 1, unless the equation is *redundant*.
**Eg:** In $\mathbb{R}^3$, one equation

$$\begin{aligned} a_1x + b_1y + c_1z = 0 &\qquad \text{(= plane)} \\ \textbf{add in } a_2x + b_2y + c_2z = 0 &\qquad \text{(intersection of two planes, = line)} \\ \textbf{add in } a_3x + b_3y + c_3z = 0 &\qquad \text{(intersection of three planes, (0,0))} \end{aligned}$$

**Problem:** $W = \{A \in M_{n\times n}(\mathbb{R}) | tr\ A = 0\}$. Find $dim\ W$, basis of $W$.
**Solution #1:** Clever way: "guess" a basis. Note: $tr\ A = A_{11} + A_{22} + \ldots + A_{nn}$ (*one* linear condition). Expecting

$$dim\ W = n^2 - 1$$

Observe that $dim\ W \neq n^2$. This happens only if $W = M_{n\times n}(\mathbb{R})$, and obviously there are matrices which don't have trace 0. Specifically:

$$tr \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ldots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

(In proofs, can choose any example, provided property holds).

Know $dim\ W \leq n^2 - 1$. If you can find independent set of size $n^2 - 1$ in $W_1$, it *will be* a basis. *Try first* $n = 3$. Looking for $3^2 - 1 = 8$ independent $3 \times 3$ matrices, all trace 0.

Want trace $= 0$. Therefore, consider all matrices which have all 0's in the diagonal:

$$\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

These 6 are obviously independent. Now, take two more which are independent:

$$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

Now we have 8 independent matrices (check carefully). So for $n = 3$, $dim\ W = 8$, this is a basis.

**General case**

Two types of basis matrices:

(I) All $E^{ij}$ (1 in $(i, j)$-pos, 0 elsewhere)) where $i \neq j$. How many are there?

$$\begin{aligned}\text{\# of non-diagonal entries} &= \text{entries - entries on diagonal}\\ &= n^2 - n\end{aligned}$$

Or, $\binom{n}{2}$ ways to choose 2 distinct values from $\{1, 2, \ldots, n\}$, 2 ways to order each pair. Total:

$$\begin{aligned}\binom{n}{2}2 &= \frac{n!}{2!(n-2)!}2\\ &= n(n-1)\\ &= n^2 - n\end{aligned}$$

(II) Looking for $n - 1$ more, since $n^2 - n + n - 1 = n^2 - 1$

$$\begin{pmatrix}1&&&&\\&-1&&&\\&&\cdots&&\\&&&0&\\&&&&0\end{pmatrix}, \begin{pmatrix}0&&&&\\&1&&&\\&&-1&&\\&&&\cdots&\\&&&&0\end{pmatrix}, \begin{pmatrix}0&&&&\\&0&&&\\&&\cdots&&\\&&&1&\\&&&&-1\end{pmatrix}, \ldots$$

(n-1 of those)

Formally, let, for $i = 1, 2, \ldots, n-1$, $D_i =$ matrix with 1 in pos $(i, i)$ and $-1$ in pos $(i+1, i+1)$, 0 elsewhere.
Verifying all matrices $E^{ii}$, $D_i$ are independent; clear that suffices to check $D_1, D_2, \ldots, D_{n-1}$ independent. Suppose

$$x_1 D_1 + x_2 D_2 + \ldots + x_n D_n = n \times n \text{ zero matrix}$$

The $(1,1)$-entry on left is $x_1$, so $x_1 = 0$. The $(2,2)$-entry on left is $-x_1 + x_2$,

$$x_1 \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \dots & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & & & & \\ & 1 & & & \\ & & -1 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix} + \dots = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix}$$

but $x_1 = 0$ so $x_2 = 0$ also, etc. So similarly for all $x_i = 0$, so independent. Formally you'd do a proof by induction, but this is good enough.
Now have $n^2 - 1$ independent vectors in $W_1$ so $dim\ W \geq n^2 - 1$. Already, know $dim\ W \leq n^2 - 1$. So $dim\ W = n^2 - 1$, have independent set of correct size, so basis.

**Solution #2:** Let $x_{ij}$ be the $(i,j)$-entry of $A$. So have $n^2$ variables $(x_{ij}, i, j = 1, 2, \dots, n)$ one equation,

$$x_{11} + x_{22} + \dots + x_{nn} = 0 \qquad (tr\ A = 0)$$

Solve system. All $x_{ij}, i \neq j$ free variables, so are $x_{22}, \dots, x_{nn}$.

**Theorem 21.** *Let $U, W$ be finite dimension subspaces of $V$. Then,*

$$dim(U + W) = dim\ U + dim\ W - dim\ U \cap W$$

It's like sets, $|A \cup B| = |A| + |B| - |A \cap B|$.
**Proof** Omitted.
**Ex:** If $W$ is a plane in $\mathbb{R}^3$ (through $(0,0)$) and $L$ is a line in $\mathbb{R}$ (through $(0,0)$) and $L$ is not in the plane, prove $W + L = \mathbb{R}^3$.
**Sol:** $L$ not in plane gives $L \cap W = \{\vec{0}\}$. So

$$\begin{aligned} dim(L + W) &= dim\ L + dim\ W - dim\ L \cap W \\ &= 1 + 2 - 0 \\ &= 3 \end{aligned}$$

Hence $L + W = \mathbb{R}^3$.
**Problem:** Suppose $dim\ V = n$, and $U, W$ subspaces, each of dimension more than $\frac{n}{2}$. Prove that $U \cap W \neq \{\vec{0}\}$.
**Proof** By contradiction. Suppose $U \cap W = \{\vec{0}\}$. So $dim\ U \cap W = 0$. Then

$$\begin{aligned} dim(U + W) &= dim\ U + dim\ W - dim\ U \cap W \\ &> \frac{n}{2} + \frac{n}{2} - 0 = n \end{aligned}$$

Says $U + W$ is a subspace of $V$ of $dim$ more than $dim\ V$. Impossible, so $U \cap W \neq \{\vec{0}\}$.
**END OF MIDTERM MATERIAL.**

## *February 8th 2019*

**Monday:** No class, office hours during class time. Tuesday night : Midterm!

### *Linear transformations - Definition and basic properties*

(Chap. 5 in the text) **Def.** Let $U, V$ be vector spaces, both over field $K$. A funcion $T : U \to V$ is called a *linear transformation* if

(i) $\forall u_1, u_2 \in U \;\; T(u_1 + u_2) = T(u_1) + T(u_2)$. The first '+' is in $U$, while the second '+' is in $V$. The vectors spaces need not be related in any way, except that they must be over the same field of scalars.

(ii) $\forall u \in U, c \in K \;\; T(cu) = cT(u)$. Again, the first scalar multiplication happens in $U$, while the second scalar multiplication happens in $V$.

**Comment:** Linear transformations are the functions that are somehow "compatible" with the vector space operations.
**Ex:** Prove that $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(ax^2 + bx + c) = \begin{pmatrix} a+b \\ b+c \end{pmatrix}$$

**Sol:**

(i) Let $p_1(x) = a_1x^2 + b_1x + c_1$, $p_2(x) = a_2x^2 + b_2x + c_2$ be in $P_2(x)$. Then,

$$\begin{aligned} T(p_1(x) + p_2(x)) &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2) \\ &= \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ b_1 + b_2 + c_1 + c_2 \end{pmatrix} \\ T(p_1(x)) + T(p_2(x)) &= \begin{pmatrix} a_1 + b_1 \\ b_1 + c_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ b_2 + c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 + a_2 + b_2 \\ b_1 + c_1 + b_2 + c_2 \end{pmatrix} \end{aligned}$$

(ii) Let $p(x) = ax^2 + bx + c \in P_2(\mathbb{R}), d \in K$.

$$\begin{aligned} T(dp(x)) &= T(dax^2 + dbx + dc) \\ &= \begin{pmatrix} da + db \\ db + dc \end{pmatrix} \\ &= d \begin{pmatrix} a+b \\ b+c \end{pmatrix} \\ &= dT(ax^2 + bx + c) \\ &= dT(p(x)) \end{aligned}$$

So $T$ is a linear transformation.
**Ex** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) = (x^2, x+y$. Show that $T$ is *not* a linear transformation.
**Sol** Try $u = (2,3), v = (3,4)$.

$$\begin{aligned} T(u+v) &= T(5,7) \\ &= (25,12) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u)+T(v) &= T(2,3)+T(3,4) \\ &= (4,5)+(9,7) \\ &= (13,12) \\ &\neq (25,12) \end{aligned}$$

So $T$ is *not* linear.
**Ex:** Define $\frac{d}{dx} : P(\mathbb{R}) \to P(\mathbb{R})$ by

$$\frac{d}{dx}p(x) = p'(x) \qquad \text{(derivative)}$$

Then $\frac{d}{dx}$ is a linear transformation, since we know from calculus that

$$\begin{aligned} \frac{d}{dx}(p(x)+q(x)) &= \frac{d}{dx}p(x) + \frac{d}{dx}q(x) \\ \frac{d}{dx}(cp(x)) &= c\frac{d}{dx}p(x) \qquad (c \in \mathbb{R}) \end{aligned}$$

**Proposition 22.** *Let $T : U \to V$ be a linear transformation. Then,*

*(i) $T(\vec{0}) = \vec{0}$ (where the first $\vec{0}$ is the zero vector of $U$ and the second is the zero vector of $V$)*

*(ii) $\forall u_1, u_2, \ldots, u_n \in U$ and $c_1, c_2, \ldots, c_n \in K$,*

$$\begin{aligned} T(c_1u_1 + c_2u_2 + \ldots + c_nu_n) &= \\ c_1T(u_1) + c_2T(u_2) + \ldots + c_nT(u_n) \end{aligned}$$

*Proof.* (i)

$$\begin{aligned} T(\vec{0}_U) &= T(\vec{0}_U + \vec{0}_U) \\ T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) && \text{(T linear)} \\ \vec{0}_V + T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) && \text{(A2)} \\ \vec{0}_V &= T(\vec{0}_V) && \text{(cancellation law)} \end{aligned}$$

(ii)

$$\begin{aligned}
T(c_1u_1 + (c_2u_2 + \ldots + c_nu_n)) &= T(c_1u_1) + T(c_2u_2 + \ldots + c_nu_n) \\
&\qquad\qquad\qquad\qquad \text{(T linear)} \\
&= c_1T(u_1) + T(c_2u_2 + \ldots + c_nu_n) \\
&\qquad\qquad\qquad\qquad \text{(T linear)} \\
&= \ldots \qquad\qquad \text{(proof by induction)} \\
&= c_1T(u_1) + \ldots + c_nT(u_n)
\end{aligned}$$

$\square$

**Proposition 23.** *Let $T : U \to V$ function ($U, V$ vector spaces). Then,*

$$T \text{ is linear transformation } \iff \forall u_1, u_2 \in U\ c \in K, T(cu_1 + u_2) = cT(u_1) + T(u_2)$$

**Proof:** Exercise. $\square$

## *February 15th 2019*

**Def** ("matrix defines a linear transformation") Let $A \in M_{m\times n}(K)$. Define a function $L_A : K^n \to K^m$ by

$$L_A(v) = Av \qquad \text{(A an } m \times n \text{ matrix, v } n \times 1)$$

ie multiply matrix by vector.

**Proposition 24.** *$L_a$ is a linear transformation.*

*Proof.* Let $u, v \in K^n, c \in K$. Then

$$\begin{aligned}
L_A(cu + v) &= A(cu + v) \\
&= A(cu) + Av \qquad \text{(prop of matrix multiplication)} \\
&= cAu + Av \\
&= cL_A(u) + L_A(v)
\end{aligned}$$

$\square$

**Ex** $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix}$, $L_A : R^3 \to R^2$. Calculate:

$$\begin{aligned} L_A(1,3,-2) &= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2-3-4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -5 \end{pmatrix} \end{aligned}$$

**Spoiler:** All linear transformations between finite-dim vector spaces can be described in this way, "matrix transformation".

*Two special linear transformations*

(1) **Zero transformations:** $0 : V \to W$ defined by $O(v) = \vec{0}$ ($\vec{0}$ of $W$) for all $v \in V$.

(2) **Identity** transformation, $I : V \to V$ (same vector space) $I(v) = v$ for all $v \in V$

Both are linear transformations (exercise).

*Kernel and Image (ch. 5.4)*

**Def** Let $T : V \to W$ be a linear transformation. Define:

(i) **Kernel** or **nullspace** of $T$,

$$Ker(T) = \{v \in V | T(v) = \vec{0}\}$$

**Note:** Always one vector which satisfies this.

(ii) **Image** of $T$ is

$$Im(T) = \{w \in W | \exists v \in V \;\; w = T(v)\}$$

**Note:** $Ker(T) \subseteq V$, $Im(T) \subseteq W$.
**Ex** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x,y) = (x,0) \qquad \text{("proj onto x-axis")}$$

Then

$$\begin{aligned} Ker(T) &= \{(x,y) \in \mathbb{R}^2 | T(x,y) = (0,0)\} \\ &= \{(0,y) | y \in \mathbb{R}\} \\ &= "y-axis" \\ Im(T) &= \{(x,y) \in \mathbb{R}^2 | (x,y) = T(x',y') \ \ some\ x',y' \in \mathbb{R}\} \\ &= \{(x,0) | x \in \mathbb{R}\} \\ &= "x-axis" \end{aligned}$$

**Ex** Define $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ to be derivative, $D(f(x)) = f'(x)$. Find kernel and image of $D$.
**Sol** We have

$$\begin{aligned} Ker(D) &= \{f \in P_n(\mathbb{R}) | f'(x) = 0\} \\ &= const.\ polys \\ &= \{a | a \in \mathbb{R}\} \\ &= P_0(\mathbb{R}) \end{aligned}$$

Claim $Im(D) = P_{n-1}(\mathbb{R})$.

*Proof.* Prove inclusion "$\subseteq$" and "$\supseteq$".

(i) "$\subseteq$" Let $f(x) \in Im(D)$. Then $\exists g(x) \in P_n$ s.t. $f(x) = D(g(x)) = g'(x)$. Since $deg(g) \le n$, $deg(f) = deg(g') \le n-1$ (property of differentiation). So $f(x) \in P_{n-1}$.

(ii) "$\supseteq$" Let $f(x) \in P_{n-1}$. Need to find $g(x) \in P_n$ such that $D(g(x)) = g'(x) = f(x)$. Set $g(x) = \int f(x)dx$. Know from calculus that the degree of $g$ is one higher, ie

$$deg(g(x)) = 1 + deg(f(x))$$

So $deg(g) \le n$. So $g(x) \in P_n$ and $g'(x) = f(x)$ (calculus).

□

**Theorem 25.** *Let $T : V \to W$ be linear transformation. Then,*

*(i) $Ker(T) \le V$*

*(ii) $Im(T) \le W$*

*Ie they are subspaces.*

*Proof.* By direct proof.

(i) $T(\vec{0}) = \vec{0}$ always (lin transform) so $\vec{0} \in Ker(T)$. Let $v_1, v_2 \in Ker(T), c \in K$. We know $T(v_1) = \vec{0}, T(v_2) = \vec{0}$. Then

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) && \text{(T linear)} \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

Hence $cv_1 + v_2 \in Ker(T)$. So $Ker(T) \subseteq V$ (we already knew $Ker(T) \subseteq V$)

(ii) $T(\vec{0}) = \vec{0}$, hence $\vec{0}_w = T$ (something), ie $\vec{0}_w \in Im(T)$. Let $w_1, w_2 \in Im(T), c \in K$. We know $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Then

$$\begin{aligned} cw_1 + w_2 &= cT(v_1) + T(v_2) \\ &= T(cv_1 + v_2) && \text{(T linear)} \end{aligned}$$

Hence $cw_1 + w_2 \in Im(T)$. So $Im(T) \leq W$.

□

**Def** $T : V \to W$ linear. The *nullity* of $T$ is $dim\ Ker(T)$ (dim nullspace). The *rank* of $T$ is $dim\ Im(T)$.
**Note:** $Ker(T) \leq V$ so $nullity(T) \leq dim\ V$, $Im(T) \leq W$ so $rank(T) \leq dim\ W$.
**Ex** In $T : \mathbb{R}^2 \to \mathbb{R}^2$, proj onto x-axis,

$$\begin{aligned} Ker(T) &= y - axis && (\text{so } nullity(T) = 1) \\ Im(T) &= x - axis && (\text{so } rank(T) = 1) \end{aligned}$$

**Ex 2** For $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$, differentiation.

$$\begin{aligned} Ker\ D &= P_0(\mathbb{R}) && (\text{so } nullity(D) = 1) \\ Im\ D &= P_{n-1} && (\text{so } rank(D) = n) \end{aligned}$$

## *February 18th 2019*

**Notation** For set $S = \{v_1, v_2, \ldots, v_n\}$, $T : V \to W$ denotes $T(S) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$.

**Proposition 26.** *$T : V \to W$ linear and $V = span(S)$. Then $Im\ T = span(T(S))$. In particular, if $B$ basis of $V$, $T(B)$* ***spans*** *$Im\ (T)$ (but need not be a basis).*

*Proof.* By direct proof.

(i) "$\subseteq$". Let $w \in Im(T)$, ie $w = T(v)$, some $v \in V$. Since $S$ spans $V$, $v = \sum_{i=1}^{n} a_i v_i$, some $v_i \in S$. So

$$\begin{aligned} w = T(v) &= T(\sum_{i=1}^{n} a_i v_i) \\ &= \sum_{i=1}^{n} a_i T(v_i) && (T(v_i) \in T(S), \text{by T linear}) \end{aligned}$$

All of which is $\in span(T(S))$.

(ii) "$\supseteq$" Let $w \in span\ T(S)$. So

$$\begin{aligned} w &= \sum_{i=1}^{n} a_i T(v_i) && (\text{for some vectors } v_i \in S) \\ &= T(\sum_{i=1}^{n} a_i v_i && (\text{T linear}) \\ &= T(something) && (\text{so } w \in Im(T)) \end{aligned}$$

$\square$

**Ex** Define $T : P_2(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Exercise: T is linear. Find basiss for $Im\ T$.
**Sol** Take basis $\{1, x, x^2\}$ for $P_2$. Calculate

$$\begin{aligned} T(1) &= \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ T(x) &= \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \\ T(x^2) &= \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So $Im\ T = span\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\}$.
Basis for $Im\ T$ is $\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\}$
(so $Im\ T = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R}\}$

Note: The next theorem is very important!

**Theorem 27.** *("Dimension theorem") Let $T : V \to W$ linear with $V$ finite-dimensional. Then,*

$$dim\ V = dim\ ker(T) + dim\ Im(T)$$
$$dim\ V = nullity(T) + rank(T)$$

**Note** $dim\ W$ is *not* involved.

*Proof.* Let $B = \{v_1, v_2, \ldots, v_k\}$ be basis $KerT$ (so $k = dim\ Ker\ T$). Let $n = dim\ V$. Note $T(v_i) = 0$, $(i = 1, 2, \ldots, k)$. Let $S$ span $V$.
Plan: extend $B$ to basis of $V$, show $T(extra\ vector)$ = basis of $Im$.
By theorem 20-1, there exists $E \subseteq S$ such that $B \cup E$ is a basis of $V$.
Denote

$$E = \{v_{k+1}, \ldots, v_n\} \qquad \text{(note } n = dim\ V\text{, |E| = n - k)}$$

Claim $T(E)$ is basis for $Im\ T$.

(i) $T(E)$ spans $ImT$

(a) "$\subseteq$" is clear since $T(E) \subseteq Im\ T$ by definition. So $span\ T(E) \leq Im\ (T)$ ($Im\ T \leq W$)

(b) "$\supseteq$" Let $w \in Im(T)$, ie $w = T(v)$, some $v \in V$. Since $B \cup E$ is a basis, $v = \sum_{i=1}^{n} a_i v_i$. Then,

$$\begin{aligned} w &= T(\sum_{i=1}^{n} a_i v_i) \\ &= \sum_{i=1}^{n} a_i T(v_i) && \text{(T linear)} \\ &= \sum_{i=k+1}^{n} a_i v_i && \text{(Since } T(v_i) = 0 \text{ for } i = 1, 2, \ldots, k) \end{aligned}$$

Hence $w \in span(T(E))$, since $E = \{v_{k+1}, \ldots, v_n\}$

(ii) $T(E)$ is linearly independent. Suppose

$$\sum_{i=k+1}^{n} b_i T(v_i) = \vec{0} \qquad \text{(linear comb vectors in } T(E))$$

So by linearity of $T$,

$$T(\sum_{i=k+1}^{n} b_i v_i) = \vec{0}$$

So $\sum_{i=k+1}^{n} b_i v_i \in Ker\ T$, ie is linear comb of $B$
So $\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} b_i v_i$
ie $\sum_{i=1}^{k} (-b_i) v_i + \sum_{i=k+1}^{n} b_i v_i = \vec{0}$ is linear comb of $v_1, \ldots, v_n$ (ie $B \cup E$) but these independent. So all $b_i = 0$, hence $T(E)$ independent.

Conclude $T(E)$ basis of $Im\ T$. So,

$$dim\ Im\ T = |T(E)| = |E| = n - k$$

So,

$$\begin{aligned} n &= k + n - k \\ dim\ V &= |B| + |T(E)| = dim\ KerT + dim\ Im\ T \end{aligned}$$

$\square$

Why is $|T(E)| = |E|$? True *unless*

$$T(v_i) = T(v_j) \qquad \text{(for some } i, j \geq k+1, i \neq j\text{)}$$

If so,

$$\begin{aligned} T(v_i) - T(v_j) &= 0 \\ T(v_i - v_j) &= 0 \qquad \text{(so } v_i - v_j \in Ker\ T\text{)} \end{aligned}$$

Hence $v_i - v_j = \sum_{l=1}^{n} a_l v_l$, dep relation on $v_1, \ldots, v_n$. Impossible. $\square$
**Problem** For $T : P_2 \to \mathcal{M}_{2\times 2}$,,

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find basis for *Ker T*.
**Sol** Already know $dim\ Im\ T = 2$ (last ex). So

$$\begin{aligned} dimP_2 &= dim\ Ker\ T + dim\ Im\ T \\ 3 &= dim\ Ker\ T + 2 \end{aligned}$$

So *Ker T* is 1-dimensional. Only need to find *one* non-zero $f(x)$ s.t.

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ie need $f(1) = f(2)$ and $f(0) = 0$. For example, $f(x) = x^2 - 3x$ works. So $\{x^2 - 3x\}$ is a basis for *Ker T* (or, $f(x) = ax^2 + bx + c$, $f(1) = a + b + c = f(2) = 4a + 2b + c$, $f(0) = 0 = c$, solve)

## *February 20th 2019*

### *Comments on dimension theorem*

$T : V \to W$, linear.

$$dim\ V = dim\ (Im\ T) + dim\ (Ker\ T)$$

Left-hand part of the sum: Dimensions that are preserved (“saved") by $T$. Right-hand part: dimensions that are “lost" when you apply $T$.
**Dimension:** Subspaces are *infinite* sets (except $\{\vec{0}\}$). Dimension gives a way to compare the *sizes* of subspaces.

*Injective/surjective transformation (ch. 5.5.)*

**Def** Let $f : X \to Y$ be a *function* ($X, Y$ sets).

(i) $f$ is *surjective* (“onto") if

$$\forall y \in Y \ \ \exists x \in X \ \ f(x) = y$$

(equivalently, the image of $f$ is $Y$)

(ii) $f$ is called *injective* (or “on-to-one") if

$$\forall x_1, x_2 \in X(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

(equivalently, $\forall x_1, x_2 \in X \ \ (f(x_1) = f(x_2) \to x_1 = x_2)$)

**Theorem 28.** *("How to check if $T$ inj/surj") Let $T : V \to W$. Then,*

*(i) $T$ injective $\iff$ $Ker(T) = \{\vec{0}\}$ (nullity $(T) = 0$)*

*(ii) $T$ surjective $\iff$ $dim\ (Im\ T) = dim\ W$ ($rank(T) = dimW$)*

(i) *Proof.* By direct proof.

(1) “⇒" Assume $T$ inj. (know $\{\vec{0}\} \leq Ker\ T$). Let $v \in Ker\ (T)$. So $T(v) = \vec{0}$. But also $T(\vec{0}) = \vec{0}$, so $T(v) = T(\vec{0})$ hence $v = \vec{0}$ since $T$ is injective.

(2) “⇐" Assume $Ker\ T = \{\vec{0}\}$. Let $v_1, v_2 \in V$. Suppose $T(v_1) = T(v_2)$ (prove $v_1 = v_2$).

$$\begin{aligned} T(v_1) - T(v_2) &= \vec{0} \\ T(v_1 - v_2) &= \vec{0} && \text{(linear)} \end{aligned}$$

So $v_1 - v_2 \in Ker\ T = \{\vec{0}\}$. So $v_1 - v_2 = \vec{0}, v_1 = v_2$.

□

(ii) *Proof.* By direct proof.

(1) “⇒" Assume $T$ is surjective, that is $Im\ T = W$. Hence $dim\ Im\ T = dim\ W$.

(2) “⇐" Assume $dim\ Im\ T = dim\ W$. But $Im\ T \leq W$, hence $Im\ T = W$ (by thm 20-2)

□

**Problem** Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by

$$T(f(x)) = \int_0^1 f(x)dx$$

(Exercise: $T$ is linear). Is $T$ injective? Surjective?
**Sol** *Dim Thm*:

$$\begin{aligned} dim\ P_2 &= dim\ Im\ T + dim\ Ker\ T \\ 3 &= dim\ Im\ T + dim\ Ker\ T \end{aligned}$$

Hence $ImT \leq \mathbb{R}^1$, so $Im\ T = \{\vec{0}\}$ or $\mathbb{R}$. It is *not* $\{\vec{0}\}$ since $\int_0^1 1dx = 1 \neq 0$, $T(1) \neq 0$. Hence $Im\ T = \mathbb{R}$ so

$$3 = 1 + dim\ Ker\ T$$

So $dim\ Ker\ T = 2$. $Ker\ T \neq \{\vec{0}\}$ *not* injective. $Im\ T = \mathbb{R}$ *is* surjective.

**Theorem 29.** *("shortcut when dim same") $T : V \to W$ linear, and $dim\ V = dim\ W$. Then,*

$$T \text{ injective} \iff T \text{ surjective}$$

*Proof.* Dim Thm:

$$dim\ W = dim\ V = dim\ Im\ T + dim\ Ker\ T$$

If $T$ inj, $dim\ Ker\ T = 0$. So

$$dim\ W = dim\ Im\ T + 0$$

So $T$ surjective (thm 28). If $T$ surj, $dim\ Im\ T = dim\ W$ (thm 28), so

$$dim\ W = dim\ W + dim\ Ker\ T$$

So $dim\ Ker\ T = 0$ so $Ker\ T = \{\vec{0}\}$ ☐

**Problem** $T : P_2(\mathbb{R}) \to \mathbb{R}^3$, defined by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$$

Is $T$ injective? Surjective?
**Sol** Same $dim$ $(= 3)$. Check only one. Check surjective directly from def surj:
Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Is $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T(f(x))$, some $f(x) \in P_2$?

That is, given $a, b, c \in \mathbb{R}$, is there a degree 2 polynomial such that $f(0) = a, f(1) = b$, $f(2) = c$? By Lagrange Interpolation, $f(x)$ exists (deg $= 1$, less than # of points). So $T$ surj, so also inj.

### *Isomorphism and coordinates (ch 5.5, 4.11 and 4.12)*

**Def:** (Isomorphism)

(1) If $T : V \to W$ (linear) is injective and surjective, it is called an *isomorphism*.

(2) If $V, W$ vector spaces and *there exists* an isomorphism $T : V \to W$, we say $V$ and $W$ are *isomorphic* and write $V \simeq W$

**Note** A function that is injective and surjective is called *bijective*.

**Ex** $T : P_2(\mathbb{R}) \to \mathbb{R}^3$, $T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$ *is* an isomorphism (last ex.)
so $P_2(\mathbb{R}) \simeq \mathbb{R}^3$
**Ex** Prove that

$$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is isomorphism $P_2 \to \mathbb{R}^3$.
**Sol** $T$ is linear : let $f(x), g(x) \in P_2(\mathbb{R}), d \in \mathbb{R}$. Then,

$$\begin{aligned} T(df + g) &= T(c(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) \\ &= T((da_1 + a_2)x^2 + (db_1 + b_2) + (dc_1 + c_2)) \\ &= \begin{pmatrix} da_1 + a_2 \\ db_1 + b_2 \\ dc_1 + c_2 \end{pmatrix} \\ &= d \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \\ &= dT(f) + T(g) \end{aligned}$$

So $T$ linear. Same $dim(= 3)$. Check surj. Let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Then

$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, hence surj., hence inj., hence *isomorphism*.

## *February 22nd 2019*

### *Notes about functions*

(1) If $f : X \to Y$, then $f$ injective and surjective $\iff$ $f$ is invertible, ie $\exists f^{-1} : Y \to X$ such that $\forall x \in X, y \in Y$ $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$

(2) If $g : Y \to Z$, you can compose $f$ and $g$ to get $g \cdot f . X \to Z$, defined by $(g \cdot f)(x) = g(f(x))$ $x \xrightarrow{f} y \xrightarrow{g} z$

**Theorem 30.** *Let $T : V \to W$ be an isomorphism (ie T linear, inj, surj.). Then T has an inverse $T^{-1} : W \to V$ which is also a* linear *transformation.*

*Proof.* Fact that $T^{-1}$ *exists* is since $T$ inj and surj. Prove $T^{-1}$ is linear. Let $w_1, w_2 \in W, c \in K$. Since $T$ surjective, $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Also, $T^{-1}(w_1) = T^{-1}(T(v_1)) = v_1$ and $T^{-1}(w_2) = v_2$. Then

$$\begin{aligned} T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\ &= T^{-1}(T(cv_1 + v_2)) && \text{(T linear)} \\ &= cv_1 + v_2 \\ &= cT^{-1}(w_1) + T^{-1}(w_2) \end{aligned}$$

So $T^{-1}$ linear. ☐

**Ex**

$$T : P_2(\mathbb{R}) \to \mathbb{R}^3, T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$T^{-1} : \mathbb{R}^3 \to P_2(\mathbb{R}), T^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (ax^2 + bx + c)$$

**Point** Once you know $V \simeq W$ (isomorphic) you can go back and forth between them, do vector space operations in either $V$ or $W$. That is, $V$ and $W$ have exactly the same *structure* (as far as addition and scalar multiplication are concerned), even though "vectors" look different.

**Proposition 31.** *If $V \simeq W$, both finite-dimensional, then dim $V$ = dim $W$*

*Proof.* $V \simeq W$ so $\exists T : V \to W$, $T$ inj and surj (bijective), linear. So Dim Thm,

$$dim\ V = dim\ Im\ T + dim\ Ker\ T$$

and $T$ inj., so $dim\ Ker\ T = 0$, and $T$ surj., so $Im\ T = W$, so

$$dim\ V = dim\ W + 0$$

☐

**Theorem 32.** *Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis of $V$. For any $v \in V$, you can write*

$$v = \sum_{i=1}^{n} a_i v_i$$

*Then,*

(a) *The numbers $(a_1, a_2, \dots, a_n)$ are* unique *and are called the* coordinates of $v$ relative to $B$, *denoted*

$$[v_b] = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(b) *The function $C_B : V \to K^n$ defined by*

$$C_B(v) = [v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \text{("find coordinates")}$$

*is an isomorphism*

*Hence, if dim $V = n$ then $V \simeq K^n$*

*Proof.* By direct proof.

(a) Assume $v$ can also be written as

$$v = \sum_{i=1}^{n} b_i v_i \qquad \text{(as well as } \textstyle\sum a_1 v_1 = v\text{)}$$

Then

$$\vec{0} = v - v = (\sum_{i=1}^{n} a_i v_i) - (\sum_{i=1}^{n} b_i v_i)$$
$$\vec{0} = \sum_{i=1}^{n} (a_i - b_i) v_i$$

Since $\{v_1, \dots, v_n\}$ independent ($B$ = basis) all $a_i - b_i = 0$ ($i = 1, 2, \dots, n$) so $a_1 = b_1$. Hence representation is *unique*.

(b) Let $v = \sum_{i=1}^{n} a_i v_i, u = \sum_{i=1}^{n} b_i v_i$ be in $V$, $c \in K$. Then,

$$\begin{aligned} C_B(cv+u) &= C_B(\sum_{i=1}^{n}(ca_i+b_i)v_i) \\ &= \begin{pmatrix} ca_1+b_1 \\ ca_2+b_2 \\ \vdots \\ ca_n+b_n \end{pmatrix} \\ &= c\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= C_B(v) + C_B(u) \end{aligned}$$

Hence $C_B$ is linear. To check $C_B$ inj. and surj., since $dim\ V = n = dim\ K^n$, need only check on (other will follow). We will prove surj.

Let $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in K^n$. Then let $v = \sum_{i=1}^{n} a_i v_i$, so $C_B(v) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

□

**Remarks**

(1) We need the coords to be *unique* in order for $C_B : V \to K^n$ to be a (well-defined) function.

(2) If you use a different basis, or even same basis but in different order, you get different coords and also different isomorphism.

Always infinitely many isomorphisms

**Lemma 33.** *Let $T : V \to W$, $S : W \to U$ be a linear transformation. Then*

*(a) $S \cdot T : V \to U$ ($Vt- > Ws- > U$) is linear*

*(b) If $T, S$ both injective (surjective), then $S \cdot T$ is also injective (surjective)*

*Proof.* Exercise. □

**Theorem 34.** *Let $V, W$ be finite-dimensional vector spaces over field $K$. Then,*

$$V \simeq W \iff dim\ V = dim\ W$$

*That is, as far as vectir space ops go, only the* dimension *really matters.*

*Proof.* By direct proof.

- "$\Rightarrow$" Prop 31.
- "$\Leftarrow$" $dim\ V = dim\ W = n$. By Thm 32, $V \simeq K^n$, $W \simeq K^n$, using $C_{B_1} : V \to K^n$, $C_{B_2} : W \to K^n$. Then $C_{B_2}^{-1} : K^n \to W$ is an isomorphism (Thm 30), so

$$C_B^{-1} \cdot C_B : V \to W \qquad (V \xrightarrow{C_{B_1}} K^n \xrightarrow{C_{B_2}^{-1}} W)$$

is linear, injective, surjective by lemma 33 so it is an isomorphism.

□

## *February 25th 2019*

Recall $V \simeq W \iff dim\ V = dim\ W$ (proved for finite-dim vector spaces only).
**Note:** If $T : V \to W$ isomorphism, $T^{-1} : W \to V$ is also an isomorphism.
Examples of isomorphisms:

- $P_n(K) \simeq K^{n+1}$
- $\mathcal{M}_{m\times n} \simeq K^{mn}$
- $K^n \simeq K^m \iff n = m$

**Question** If $n = dim\ V$, then $V \simeq K^n$, why bother studying vector spaces other than $K^n$?
**Answer** If you only want to know about addition and scalar multiplication, only $K^n$ matters *but* the "vectors" $P_n$, $\mathcal{M}_{n\times m}$ etc... have other properties not always related to vector space operations.

For example, in $P_2(\mathbb{R})$ we can evaluate polynomials $f(x)$ at say $x = 3$,

$$f(x) = ax^2 + bx + c$$
$$f(3) = 9a + 3b + c$$

If we consider $P_2(\mathbb{R}) \simeq \mathbb{R}^3$, "eval at $x = 3$" is a linear transformation:

$$T : \mathbb{R}^3 \to \mathbb{R}$$
$$T(a, b, c) = 9a + 3b + c$$

*Computations related to linear transformation*

**Theorem 35** (T is determined by its value on a basis)**.** *Let $V, W$ be vector spaces, $\{v_1, v_2, \dots, v_n\}$ basis $V$.*

*Let $w_1, w_2, \dots, w_n \in W$ be any vectors (need not be distinct). Then there is* one *linear transformation $T : V \to W$ s.t. $T(v_i) = w_i$*

**Idea of proof** If you want to calculate $T(v) v \in V$ (arbitrary element), write $v$ uniquely in terms of basis

$$v = \sum_{i=1}^{n} a_i v_i$$

Then since $T$ is supposed to be linear, compute

$$\begin{aligned} T(v) &= T(\sum a_i v_i) \\ &= \sum a_i T(v_i) \\ &= \sum a_i w_i \end{aligned}$$

**Problem** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}, \ T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

Find $T\binom{3}{4}$.

**Solution** $\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\}$ = basis $\mathbb{R}^2$, should have enough info to know what $T$ is. Need to find

$$\begin{bmatrix}3\\4\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}1\\-1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1 & 3\\1 & -1 & 4\end{bmatrix} \to \begin{bmatrix}1 & 0 & \frac{7}{2}\\0 & 1 & -\frac{1}{2}\end{bmatrix}$$

So

$$\begin{aligned} T\begin{pmatrix}3\\4\end{pmatrix} &= T(\frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}) \\ &= \frac{7}{2}T\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}T\begin{pmatrix}1\\-1\end{pmatrix} \\ &= \frac{7}{2}\begin{bmatrix}-2\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix} \end{aligned}$$

*Row, column, nullspace of a matrix*

**Def** $A \in \mathcal{M}_{m\times n}(K)$

1. The row space, $row(A)$ is the span of the rows of $A$. Subspace of $K^n$

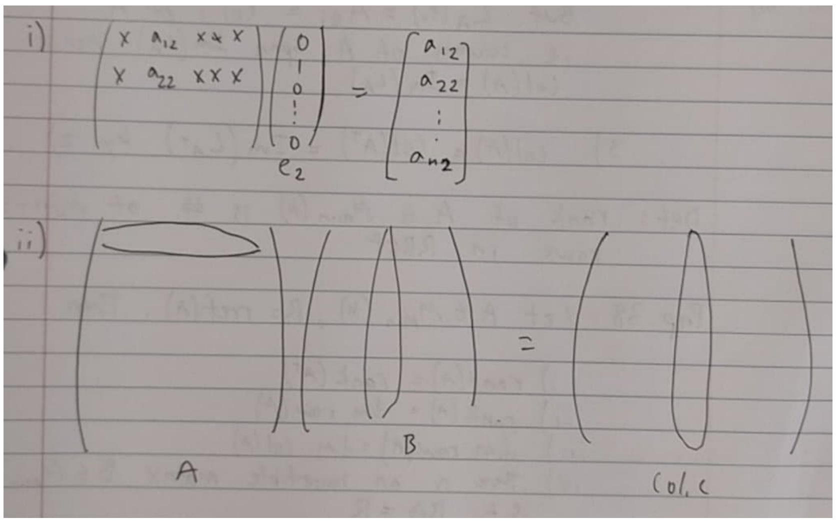
2. The column space, $col(A)$ is span of columns. Subspace of $K^n$
3. $Nullspace(ker)$, is the solution set to the homogeneous system $Ax = \vec{0}$. Subspace of $K^n$

**Proposition 36.** *Let $A \in \mathcal{M}_{m \times n}(K)$. Then*

*(1) $A_{ei}$ = column $i$ of $A$*

*(2) If $B \in \mathcal{M}_{n \times p}(K)$ then column $i$ of $AB$ is $Ab_i$, $b_i$ = column $i$ of $B$*

*Proof.* Proof by picture! □




**Proposition 37.** *Let $A \in \mathcal{M}_{m \times n}(K)$, so $L_A : K^n \to K^m$.*

*(1) $ker(A) = Ker(L_A)$*

*(2) $col(A) = Im(L_A)$*

*(3) $row(A) = Im(L_{A^T}$*

*Proof.* By direct proof.

(1)

$$\begin{aligned} Ker(A) &= \{x \in K^n | A_x = \vec{0}\} \\ &= \{x \in K^n | L_A(x) = \vec{0}\} \\ &= Ker(A) \end{aligned}$$

(2) Take basis $\{e_1, e_2, \ldots, e_n\}$ for $K^n$. Then by prop 26,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots$$

$$L_A(e_1) \ldots L_A(e_n) \text{ spans } Im(L_A)$$

But $L_A(e_1) = A_{ei} =$ column $i$ of $A$, ie columns of $A$ span $Im(L_A)$ hence $col(A) = Im(L_A)$

(3) $col(A) = col(A^T) = Im(L_{A^T})$ by (2)

□

**Def:** Rank of $A \in \mathcal{M}_{m\times n}(K)$ is number of non-zero rows in RREF.

**Proposition 38.** *Let $A \in \mathcal{M}_{m\times n}(K), R = RREF(A)$. Then,*

*(i)* $rank(A) = rank(A^T)$

*(ii)* $rank(A) = dim\ row(A)$

*(iii)* $dim\ row(A) = dim\ col(A)$

*(iv)* *There is an invertible matrix $B \in \mathcal{M}_{m\times n}(K)$ s.t. $BA = R$*

*Proof.* (iii) We have:

$$\begin{aligned} dim\ row(A) &= rank(A) && \text{(by (ii))} \\ &= rank(A^T) && \text{(by (i))} \\ &= dim\ row(A^T) && \text{(by (ii))} \\ &= dim\ col(A) && \text{(by (iii))} \end{aligned}$$

□

## *February 27th 2019*

**Theorem 39** (computing bases)**.** *Let $A \in \mathcal{M}_{m\times n}(K)$, let $R$ be the reduced non-echelon form of $A$. Then,*

*(i) The non-zero rows of $R$ form a basis of $row(A)$.*

*(ii) The columns of $A$ which correspond to the pivot columns (columns containing a leading 1) form a basis of $col(A)$.*

*(iii) The "basic solutions" obtained when solving $Ax = \vec{0}$ form a basis for nullspace (ker) of $A$.*

*Proof.* By direct proof.

(i) Elementary row ops do not change the row space so $row(A) = row(R)$. Non-zero rows form basis because of form of $R$.

(ii) Let $w_1, w_2, \ldots, w_r$ be the columns of $R$ containing leading 1's (pivot columns). Because of form of $R$, no other non-zero entries above/below a leading 1, so $w_1, w_2, \ldots, w_r$ are standard basis vectors (ie in $\{e_1, e_2, \ldots, e_m\}$). So, $\{w_1, \ldots, w_r\}$ are linearly independent. Let $v_1, v_2, \ldots, v_r$ be corresponding columns.

**Note** $r = rank(A) = dim\ row(A) = dim\ col(A)$.

Prove $v_1, v_2, \ldots, v_2$ are linearly independent. Suppose

$$\sum_{i=1}^{n} a_i v_i = \vec{0}$$

By proposition 38, $\exists$ invertible $M$ s.t. $MA = R$. Multiply by $M$:

$$\begin{aligned} M(\sum_{i=1}^{r} a_i v_i) &= M\vec{0} \\ &= \vec{0} \end{aligned}$$

So $\sum_{i=1}^{r} a_i M v_i = \vec{0}$, but $M$( column $i$ of $A$) = col $i$ of $MA$ ie of $R$ (prop 36). So,

$$\sum_{i=1}^{r} a_i w_i = \vec{0}$$

But $\{w_1, \ldots, w_r\}$ are independent. So all $a_i = 0$, so $\{v_1, \ldots, v_r\}$ independent so basis.

(iii) Solve $Ax = 0$, obtain general solution,

$$\begin{aligned} \vec{x} &= x_1 v_1 + x_2 v_2 + \ldots + x_s v_s \\ &= x_1 \begin{pmatrix} 1 \\ \\ \\ \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ \\ \\ \end{pmatrix} + \ldots + x_s \begin{pmatrix} 0 \\ \\ \\ \end{pmatrix} \end{aligned}$$

Where $x_1, x_2, \ldots, x_s$ free variables. Claim is that $v_1, v_2, \ldots, v_s$ form a basis for $ker(A)$. They clearly span. Independent? In the $x_1$ position, only $v_i$ has a non-zero entry, so they are independent.

**Comment** The dimension of $Ker(A)$ is therefore the number of *free* variables.

$\square$

*Basis-finding problems*

**Problem** Let $W \leq \mathcal{M}_{2\times 2}(\mathbb{R})$, where $W$ consists of all $A$ such that sum of entries in each row and column is the same. Find basis of $W$.

**Solution** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$. So

$$\begin{aligned} a+b &= c+d \\ a+c &= b+d \\ a+b &= a+c \qquad\qquad (a+b = b+d \text{ etc are not needed}) \end{aligned}$$

Write as linear system:

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So $a = d$, $b = c$, $c = c$ and $d = d$. ie, $\vec{x} = c\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

General solution,

$$\begin{aligned} A &= \begin{pmatrix} d & c \\ c & d \end{pmatrix} \\ &= c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Linearly independent by Thm 39 (kernel basis case). So

$$\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \qquad\qquad \text{(basis)}$$

**Problem:** Let

$$\begin{aligned} P_1(x) &= 1 + 2x + 3x^2 - x^3 \\ P_2(x) &= -1 + 3x + x^2 + x^3 \\ P_3(x) &= 3 - 4x + x^2 - 3x^3 \\ P_4(x) &= 1 + 7x + 7x^2 - x^3 \\ P_5(x) &= 2 + 2x - x^2 - x^3 \end{aligned}$$

Let $W = span\{P_1(x), \ldots, P_5(x)\} \leq P_3(\mathbb{R})$. Find:

(i) basis of $W$ that is a subset of $\{P_1(x), \ldots, P_5(x)\}$

(ii) basis of $W$ consisting of polys of different degree.

**Sol** Isomorphism $T : P_3 \to \mathbb{R}^4$,

$$T(d + cx + bx^2 + ax^3) = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \qquad \left(\text{or } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}\right)$$

(i) Put the vectors as columns of a matrix,

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 3 \\ 2 & 3 & -4 & 7 & 2 \\ 3 & 1 & 1 & 7 & -1 \\ -1 & 1 & -3 & -1 & -1 \end{pmatrix}$$

Find basis $col(A)$. Row-reduce to

$$R = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So columns 1, 2 and 5 of $A$ form a basis for $col(A)$, which corresponds (using isomorphism $T$) to $W$, so

$$\{P_1(x), P_2(x), P_5(x)\} \qquad \text{(basis)}$$

(ii) Basis all diff degree. Use row space of a matrix. Put $P_1, \ldots, P_5$ as rows. But use isomorphism

$$d + cx + bx^2 + ax^3 \iff \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So

$$A = \begin{pmatrix} -1 & 3 & 2 & 1 \\ 1 & 1 & 3 & -1 \\ -3 & 1 & -4 & 3 \\ -1 & 7 & 7 & 1 \\ -1 & -1 & 2 & 2 \end{pmatrix} \qquad \text{(So } W \text{ corresponds to row space.)}$$

$$\to \qquad = \begin{pmatrix} 1 & 0 & 0 & \frac{-27}{20} \\ 0 & 1 & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

First three rows form basis $row(A)$. As polynomials, we get

$$x^3 - \frac{27}{20},\ x^2 - \frac{1}{4},\ x + \frac{1}{5}$$

Which is basis of $W$, all of different degree. The choice of order was relevant since we knew in advance the general form the reduced form would take.