# MATH223 - Linear Algebra (class notes)

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## Contents



## 1 January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%

- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

## 1.1 Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in $R^3$. Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

## 1.2 Complex numbers

**Def:** Let $i$ be a symbol. We declare $i^2 = -1$.
Now, what we'd like to do is take this symbol $i$ and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} &3i \\ &i - 4 \\ &3i - \pi \\ &\sqrt{i} + 21 \end{aligned}$$

**Def:** The field of complex numbers $C$ consists of all expressions of the form $a + bi$, where $a, b \in R$.
**Def:** Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

**Notation:**

- $0+bi=bi$
- $a+0i=a$ (a *real* number)
- $0+0i=0$

**Ex:** If $z_1=2-i$, $z_2=5i$, then

$$z_1+z_2=2+4i$$

and

$$z_1z_2=(2-i)(5i)=10i-5i^2=5+10i$$

**Def:** Let $z=a+bi\in C$

(i) $\bar{z}=a-bi$, called the *complex conjugate* of $z$

(ii) $|z|=\sqrt{a^2+b^2}$, called the *absolute value* or *modulus*

**Def:** If $z=a+bi\in C$ and $z\neq 0$ (ie $z\neq 0+0i$), then the number

$$\begin{aligned}z^{-1} &= \frac{\bar{z}}{|z|^2}\\ &= \frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\end{aligned}$$

is called the (multiplicative) inverse of $z$. It has the property $zz^{-1}=1=z^{-1}z$.

*Proof.* We have

$$\begin{aligned}zz^{-1} &= (a+bi)(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i)\\ &= \frac{a^2-abi+abi-b^2i^2}{a^2+b^2}\\ &= \frac{a^2+0+b^2}{a^2+b^2}\\ &= 1\end{aligned}$$

**Note:** Since $z\neq 0+0i$, $a^2+b^2\neq 0$ $\square$

**Def:** If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

**Ex:** If $z = 1 + 2i, w = 3 - i$ then

$$\begin{aligned}
\frac{w}{z} &= wz^{-1} \\
&= (3 - i)(\frac{1}{5} - \frac{2}{5}i) \\
&= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\
&= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\
&= \frac{1}{5} - \frac{7}{5}i
\end{aligned}$$

Or,

$$\begin{aligned}
\frac{3 - i}{1 + 2i} \cdot \frac{(1 - 2i)}{(1 - 2i)} &= \frac{3 - 6i - i + 2i^2}{1 - 2i + 2i - 4i^2} \\
&= \frac{1 - 7i}{5}
\end{aligned}$$

# 2 January 9th 2019

## 2.1 Complex numbers as points in $R^2$

You can view $a + bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in $R^2$, and they will conserve the same properties (addition of complex numbers corresponds to vector addition in $R^2$). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

## 2.2 Equations with complex numbers

**Fact:** Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}
(\pm i\sqrt{|a|}) &= i^2(\sqrt{|a|})^2 \\
&= -1 \cdot |a| \\
&= a && \text{(since } a < 0\text{)}
\end{aligned}$$

**Fact:** Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in $C$.

**Ex:** Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.
**Sol:**

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

**Note:** If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say $z$, its other root is $\bar{z}$ ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.

**Caution:** $-1$ has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

**Theorem:** (Fundamental Theorem of Algebra) If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 n^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_0 \in C$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \ldots \cdot (x - r_n)$$

for some complex numbers $r_1, r_2, \ldots, r_n$. Some $r_i$'s may be equal.

**Corollary:** Every such polynomial has at least one root, and at most $n$ distinct roots.

**Note:** *Finding* the roots is, in general, quite difficult.

**Ex:** Factor $2x^3 + 2x$ (over C).
**Sol:**

$$\begin{aligned} 2(x^3 + x) &= 2(x-0)(x^2+1) \\ &= 2(x-0)(x^2 - i^2) \\ &= 2(x-0)(x-i)(x+i) \end{aligned}$$

**Ex:** Solve $x^2 - i = 0$
**Sol:** $x^2 = i$ so $x = \pm\sqrt{i}$. Want $\sqrt{i}$ in format $a + bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a+bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$\begin{aligned} 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} && \text{(so a=b both + or both -)} \\ a^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

### 2.3 Vector spaces (Ch 4)

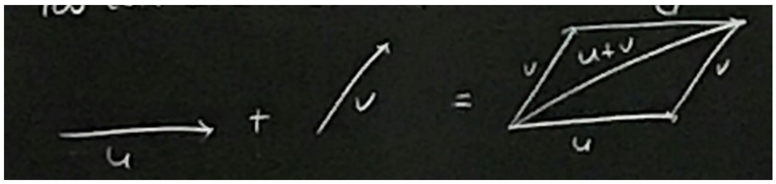
**Def.** The sets $R$ and $C$ (and also $Q$, rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of $K$" means that $K$ is either $R$ or $C$.

## 3 January 11th 2019

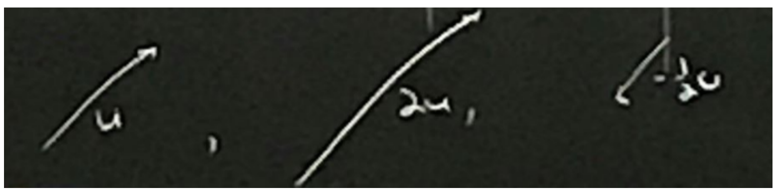
**Last time:** *Field* $K$ is $R$ or $C$ (for this class).

## 3.1 Geometric vectors ('arrows')

You can add two vectors (arrows).




**Observation:** $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
You can rescale a vector:




**Observation:** $a(b\vec{u}) = (ab)\vec{u}$.
Also: $1\vec{u} = \vec{u}$
**Question:** What properties are interesting? What other objects obey the same properties?
**Abstraction:** Focus on properties more than on the objects.

## 3.2 Definition of a vector space

Let $V$ be a set, called set of "vectors", and let $K$ be a field ($R$ or $C$) (elements of $K$ called *scalars*). Assume that we have already defined two operations:

(1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.

(2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then $V$ is called a *vector space* (over $K$).

(A1) $u + v = v + u$ (commutative laws)

(A2) There exists a vector in $V$, named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$

(A3) For each $u \in V$, there is a vector in $V$, called the (additive) inverse of $u$ and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)

(A4) $(u + v) + w = u + (v + w)$

(SM1) $a(u + v) = au + av$ (distributive laws)

(SM2) $(a + b)u = au + bu$

(SM3) $a(bu) = (ab)u$

(SM4) $1u = u$ ($1 \in R$ or $C$)

These are called the vector space *axioms*.

## 3.3 Examples of vector spaces

Some examples:

(1) $K^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

where $c \in K$ (and $K$ = set of scalar).

**Proof that $K^n$ is a vector space**
Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.
*Proof concept:* To prove "for all $x \in A$, something", say "let $x \in A$" (means $x$ is an arbitrary element of $A$, ie you only know $x \in A$). Then, prove something for that $x$.
*Proof:* Let $u, v \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n)$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= (a_1 + b_1, \ldots, a_n + b_n) \qquad \text{(definition of addition in } K^n) \\ &= (b_1 + a_1, \ldots, b_n + a_n) \qquad (\text{since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \ldots, b_n) + (a_1, \ldots, a_n) \qquad \text{(definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define $0 = (0, 0, \ldots, 0)$ (which *is* in $K^n$). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \ldots, a_n$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \ldots, a_n + (0, 0, \ldots, 0) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\ &= (a_1, a_2, \ldots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space $C^2$, $(2+3i, 5-7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2+3i, 5-7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

## 4 January 14th 2019

**Problem:** Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that $J$ is not a vector space.
**Solution:** Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned} ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3) \end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.

**Actual proof:** Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned} u + (v + w) &= (1, 1) + ((2, 2) + (3, 3)) \\ &= (1, 1) + (-2 - 3, 5) \\ &= (1, 1) + (-5, 5) \\ &= (-1 + 5, 6) \\ &= (4, 6) \end{aligned}$$

Whereas,

$$\begin{aligned} (u + v) + w &= ((1, 1) + (2, 2)) + (3, 3) \\ &= (-1 - 2, 3) + (3, 3) \\ &= (-3, 3) + (3, 3) \\ &= (-(-3) - 3, 6) \\ &= (0, 6) \end{aligned}$$

Hence, the axiom does not hold.

## 4.1 More examples of vector spaces

(1) $K^n$ (ie $R^n$ or $C^n$). See before

(2) $P(K)$ = polynomials, where coefficients are in $K$. Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned} f(x) &= x^2 + 2ix - 4 \quad \in P(C) \\ g(x) &= -x^2 + cx \quad \in P(C) \qquad \text{(and also in } P(R)\text{)} \end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned} 2if(x) &= 2ix^2 + 4i^2x - 8i \\ &= 2ix^2 - 4x - 8i \end{aligned}$$

(3) $P_n(K)$ = polynomials of degree $n$ or less, coefficient from $K$. For example,

$$\begin{aligned} x^2 - 2x + 2 &\in P_2(R) \\ x^2 - 2x + 2 &\in P_3(R) \\ x^2 - 2x + 2 &\in P_2(C) \\ x^2 - 2x + 2 &\notin P_1(R) \end{aligned}$$

**Note:** In $P(K), P_n(K)$ the "vectors" are polynomials.

(4) $M_{m\times n}(K) = m \times n$ matrices with entries from $K$. Scalars are $K$, addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2\times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2\times 2}(C) \\ A+B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in $M_{m\times n}(K)$ is the $m \times n$ matrix with all entries 0.

(5) Let $X$ be any set (think $x = R$ or $C$, but not required). Define $F(X,K) = \{f : X \to K\}$ = all functions from $X$ to $K$.
**Ex:** $f(x) = x^2 \in F(R,R)$.
**Ex:** Let $x = \{1,2\}$. Then $g$ defined by

$$\begin{aligned} g(1) &= 3 \\ g(2) &= \sqrt{2} \end{aligned}$$

*Addition* in this space is defined by:
If $f, g \in F(X,K)$ then $f+g$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

*Note that* $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f+g)$ is the addition of vectors $f$ and $g$, while the other $+$ is scalar addition.
*Scalar multiplication* in this space is defined by: if $f \in F(X,K), c \in K$ then $cf$ is the function defined by

$$(cf)(x) = cf(x)$$

*Note that* $cf$ is the name of the function, that "multiplication" is scalar multiplication $F(X,\models)$ and $cf(x)$ is the multiplication of two scalars (numbers).
The fact that $F(X,K)$ is a vector space and the axioms are followed is not so obvious.
**Prove (A2) true for** $F(X,K)$**.** Define $z \in F(X,K)$ by

$$z(x) = 0 \qquad \text{(for all } x \in X)$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f+z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence, $f+z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.
**Exercise:** Try (A3).

# 5 January 16th 2019

**Theorem**: ("Cancellation Law") Suppose $v$ is a vector space over $K$. For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.
*Note:* To prove "for all" you say let $u \in V$ (means $u$ is an arbitrary vector).

To prove "if $p$ then $q$", denoted $p \to q$, assume $p$ is true and use it to prove $q$.

*Proof.* Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned}(u+w) + (-w) &= (v+w) + (-w) && \\ u + (w + (-w)) &= v + (w + (-w)) && \text{(by A1)} \\ u + \vec{0} &= v + \vec{0} && \text{(by A3)} \\ &= u = v && \text{(by A2)}\end{aligned}$$

□

**Theorem**:

1. The zero vector is unique
2. For each $u \in V$, $-u$ is unique

*Note:* To prove something is unique, suppose you have two of them and show they are the same.

*Proof.* 1) Assume 0 and $z$ both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned}z &= z + 0 && \text{(by *, with } u = z) \\ &= 0 + z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0)\end{aligned}$$

So the zero vector is unique.

2) Exercise.

$\square$

**Theorem**: $\forall u \in V, c \in K$,

1) $c\vec{0} = \vec{0}$
2) $0u = \vec{0}$
3) $-(cu) = ((-c)u)$

*Proof.* Of 2). Let $u \in V$. Then,

$$\begin{aligned}
0u + 0u &= (0+0)u && \text{(By SM2)}\\
0u + 0u &= 0u && \text{(by R addition)}\\
0u + 0u &= 0u + \vec{0} && \text{(by A2)}\\
0u + 0u &= \vec{0} + 0u && \text{(by A4)}\\
0u &= \vec{0} && \text{(by cancellation law)}
\end{aligned}$$

$\square$

*Note:* $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

## 5.1 Linear combinations and spans

**Def:** Let $u, v_1, v_2, \dots, v_n \in V$. If there are scalars $a_1, a_2, \dots, a_n \in K$ such that $u = a_1v_1, a_2v_2 \dots a_nv_n$ then $u$ is said to be a linear combination of $v_1, v_2, \dots, v_n$.
**Ex:** In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.

**Important problem:** Given vectors $u, v_1, v_2, \dots, v_n$, determine if $u$ is a linear combination of $v_1, v_2, \dots, v_n$ and if so find $a_1, a_2, \dots, a_n$.

**Ex:** Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned}
g_1(x) &= x^2 + 2x + 1\\
g_2(x) &= -2x^2 - 4x - 2\\
g_3(x) &= 2x^2 - 3
\end{aligned}$$

**Sol.** Are there $a_1, a_2, a_3$ s.t.

$$\begin{aligned}
2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3)\\
&= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3)
\end{aligned}$$

Equating coefficients,

$$\begin{aligned}
a_1 - 2a_2 + 2a_3 &= 2\\
2a_1 - 4a_2 &= 6\\
a_1 - 2a_2 - 3a_3 &= 8
\end{aligned}$$

Solve the linear system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{(row reduce)}$$

$\therefore$ No solution, because of the last row. $f$ is not a linear combination of $g_1, g_2, g_3$.

**Def:** Let $S \subseteq V$ ($S$ is a subset fof $V$) and assume $s \neq 0$. The span of $s$, denoted $span(s)$ is the set of all linear combinations of vectors from $S$, ie

$$\begin{aligned} span(s) = \{u \in V | \, &\exists \, v_1, v_2, \ldots, v_n \in S \\ &\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ &u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\} \end{aligned}$$

# 6 January 18th 2019

## 6.1 Last class

$$\begin{aligned} S &\subseteq V \\ span(s) = \{u \in V | \, &\exists \, v_1, v_2, \ldots, v_n \in S \\ &\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ &u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n\} \end{aligned}$$

**Ex:** $S = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Prove $span(S) = R^2$.
**Note:** $\binom{a}{b}$ means $(a, b)$.
**Proof note:** To prove two sets $A, B$ are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.
**Sol:**

(1) Prove $span(S) \subseteq R^2$. Trivial, since any linear combination of vectors in $R^2$ is still in $R^2$.

(2) Prove $R^2 \subseteq span(S)$. Let $\binom{a}{b} \in R^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all $a, b$). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\binom{a}{b} \in span(S)$ so $R^2 \subseteq span(S)$. So by (1), (2), $span(S) = R^2$. $\square$

**Note:** $Ax = b$, $A_{n \times n}$ if $A$ inv, $x = A^{-1}b$.
**Theorem:** Let $S \subseteq V$, $S \neq \emptyset$ ($\emptyset =$ empty set). Then,

(1) If $u, v \in span(S)$ then $u + v \in span(S)$

(2) If $u \in span(S)$ and $c \in K$, then $cu \in span(S)$

(3) $\vec{0} \in span(S)$

*Proof.* By direct proof.

(1) (Note, "if $u, v \in span(S)$" means for all $u, v \in span(S)$).
Let $u, v \in span(S)$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$
$$v = b_1v_1 + b_2v_2 + \ldots + b_mv_m \text{ where } v_1, \ldots, v_m \in S, b_1, \ldots, b_m \in K$$

Then $u + v = a_1u_1 + \ldots + a_nu_n + b_1v_1 + \ldots b_mv_m$ which is in $span(S)$ since $u_1, \ldots, u_n, v_1, \ldots, v_m \in S$.

(2) Let $u \in span(S), c \in K$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \ldots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \ldots (c_na_n)u_n \end{aligned}$$

**Note:** If you want to be very formal, you need to write down all of the vector space axioms. Which is in $span(S)$ since it is a linear combination of $a_1, \ldots, a_n$ which are in $S$.

(3) (Prove $\vec{0} \in span(S)$) Let $u \in S$. **Note**: This is possible only because $S \neq \emptyset$.
Then $u = 1u$, so $u \in span(S)$. Then using $c = 0$ and (2) and fact that $u \in span(S)$,

$$cu = 0u = \vec{0}$$

is also in $span(S)$. **Note:** Since $u = 1u$, $S \subseteq span(S)$.

□

## 6.2 Subspaces

**Def.** Let $V$ be a vector space and $W \subseteq V$ (subset). If $W$, using addition and scalar multiplication as defined in $V$, satisfies the definition of vector space, then $W$ is called a subspace of $V$, denoted $W \leq V$ (less than equal sign, read as "subspace").
**Note:** Main issue is that addition and scalar multiplication with vector from $W$ produce vectors which are still in $W$.
**Theorem:** Let $W \subseteq V$. Then, if the following three properties hold, then $W \leq V$ (subspace).

(SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")

(SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")

(SS3) $\vec{0} \in W$.

These are the same properties we just proved for spans; in other words, we proved earlier that $span(S)$ is a subspace.

*Proof.* For $W$ to have operatios addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

(A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in $V$, $u+(v+w) = (u + v) + w$.

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let $w \in W$. Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \qquad \text{(thm from last class)}$$

is in $W$, as needed.

(A4) Still true because it is true in $V$.

(SM1-SM4) All hold because they hold in $V$.

□