

# MATH223 - Linear Algebra (class notes)

Sandrine Monfourny-Daigneault

McGill University

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## 1 January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra.**

## 1.1 Motivation

We have linear systems, with two equations, like such:

$$3x - 2y + z = 2$$

$$x - y + z = 1$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in  $R^3$ . Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

## 1.2 Complex numbers

**Def:** Let  $i$  be a symbol. We declare  $i^2 = -1$ .

Now, what we'd like to do is take this symbol  $i$  and combine it with the usual real numbers that we are familiar with. We set, for example,

$$3i$$

$$i - 4$$

$$3i - \pi$$

$$\sqrt{i} + 21$$

**Def:** The field of complex numbers  $C$  consists of all expressions of the form  $a + bi$ , where  $a, b \in R$ .

**Def:** Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

**Notation:**

- $0 + bi = bi$
- $a + 0i = a$  (a *real* number)
- $0 + 0i = 0$

**Ex:** If  $z_1 = 2 - i$ ,  $z_2 = 5i$ , then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

**Def:** Let  $z = a + bi \in C$

- (i)  $\bar{z} = a - bi$ , called the *complex conjugate* of  $z$
- (ii)  $|z| = \sqrt{a^2 + b^2}$ , called the *absolute value* or *modulus*

**Def:** If  $z = a + bi \in C$  and  $z \neq 0$  (ie  $z \neq 0 + 0i$ ), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \end{aligned}$$

is called the (multiplicative) inverse of  $z$ . It has the property  $zz^{-1} = 1 = z^{-1}z$ .

*Proof.* We have

$$\begin{aligned} zz^{-1} &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

**Note:** Since  $z \neq 0 + 0i$ ,  $a^2 + b^2 \neq 0$

□

**Def:** If  $z, w \in C$  and  $z \neq 0$  then

$$\frac{w}{z} = wz^{-1}$$

**Ex:** If  $z = 1 + 2i, w = 3 - i$  then

$$\begin{aligned}\frac{w}{z} &= wz^{-1} \\ &= (3 - i)\left(\frac{1}{5} - \frac{2}{5}i\right) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i\end{aligned}$$

Or,

$$\begin{aligned}\frac{3 - i}{1 + 2i} \cdot \frac{(1 - 2i)}{(1 - 2i)} &= \frac{3 - 6i - i + 2i^2}{1 - 2i + 2i - 4i^2} \\ &= \frac{1 - 7i}{5}\end{aligned}$$

## 2 January 9th 2019

### 2.1 Complex numbers as points in $R^2$

You can view  $a + bi$  as a point  $(a, b) \in R^2$ . The usefulness of this is that we can consider, say,  $(3 + 2i)$  and  $(3 - i)$  as vectors in  $R^2$ , and they will conserve the same properties (addition of complex numbers corresponds to vector addition in  $R^2$ ). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

### 2.2 Equations with complex numbers

**Fact:** Every real number  $a \neq 0$  has two square roots:

- if  $a > 0$ , roots  $\pm\sqrt{a}$
- if  $a < 0$ , two roots are  $\pm i\sqrt{|a|}$ , since:

$$\begin{aligned}(\pm i\sqrt{|a|}) &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a\end{aligned}\quad (\text{since } a < 0)$$

**Fact:** Quadratic equation  $ax^2 + bx + c = 0$  has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in  $C$ .

**Ex:** Solve  $x^2 - 2x + 3 = 0$ , and factor  $x^2 - 2x + 3$ .

**Sol:**

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\&= \frac{2 \pm \sqrt{-8}}{2} \\&= \frac{2 \pm i\sqrt{8}}{2} \\&= \frac{2 \pm i2\sqrt{2}}{2} \\&= 1 \pm i\sqrt{2}\end{aligned}$$

**Note:** If  $ax^2 + bx + c$  has  $a, b, c \in R$  has a non-real root, say  $z$ , its other root is  $\bar{z}$  ( $z = a + bi$ ,  $\bar{z} = a - bi$ ). This is not necessarily true if  $a, b, c \in C$ .

Back to problem. Factor  $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$ .

**Caution:**  $-1$  has two roots, namely  $\pm i$ , so you may write  $i = \sqrt{-1}$ , but be careful:

$$\begin{aligned}-1 &= i^2 \\&= i \cdot i \\&= \sqrt{-1} \cdot \sqrt{-1} \\&= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\&= \sqrt{1} \\&= 1\end{aligned}$$

**Theorem:** (Fundamental Theorem of Algebra) If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

is a polynomial with  $a_n \neq 0$ , and  $a_n, a_{n-1}, \dots, a_0 \in C$ , then  $p(x)$  factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \dots \cdot (x - r_n)$$

for some complex numbers  $r_1, r_2, \dots, r_n$ . Some  $r_i$ 's may be equal.

**Corollary:** Every such polynomial has at least one root, and at most  $n$  distinct roots.

**Note:** *Finding* the roots is, in general, quite difficult.

**Ex:** Factor  $2x^3 + 2x$  (over  $\mathbb{C}$ ).

**Sol:**

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

**Ex:** Solve  $x^2 - i = 0$

**Sol:**  $x^2 = i$  so  $x = \pm\sqrt{i}$ . Want  $\sqrt{i}$  in format  $a + bi$ ,  $a, b \in \mathbb{R}$ .

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$0 = a^2 - b^2$$

$$1 = 2ab$$

$$a = \pm b$$

$$ab = \frac{1}{2} \quad (\text{so } a=b \text{ both } + \text{ or both } -)$$

$$a^2 = \frac{1}{2}$$

$$a = \pm \frac{1}{\sqrt{2}} = b$$

Two solutions,  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$  and  $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ .

## 2.3 Vector spaces (Ch 4)

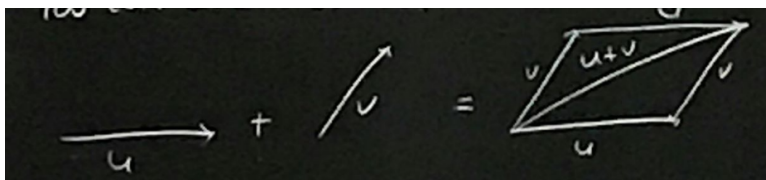
**Def.** The sets  $\mathbb{R}$  and  $\mathbb{C}$  (and also  $\mathbb{Q}$ , rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of  $K$ " means that  $K$  is either  $\mathbb{R}$  or  $\mathbb{C}$ .

## 3 January 11th 2019

**Last time:** *Field*  $K$  is  $\mathbb{R}$  or  $\mathbb{C}$  (for this class).

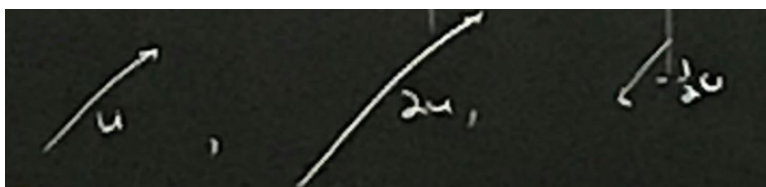
### 3.1 Geometric vectors ('arrows')

You can add two vectors (arrows).



**Observation:**  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .

You can rescale a vector:



**Observation:**  $a(b\vec{u}) = (ab)\vec{u}$ .

Also:  $1\vec{u} = \vec{u}$

**Question:** What properties are interesting? What other objects obey the same properties?

**Abstraction:** Focus on properties more than on the objects.

### 3.2 Definition of a vector space

Let  $V$  be a set, called set of "vectors", and let  $K$  be a field ( $R$  or  $C$ ) (elements of  $K$  called *scalars*). Assume that we have already defined two operations:

- (1) One called *addition*, which takes two vectors  $\vec{u}, \vec{v} \in V$  and produces another vector denoted  $\vec{u} + \vec{v} \in V$ .
- (2) One called *scalar multiplication* which takes a vector  $\vec{u} \in V$  and a scalar  $a \in K$  and produces another vector denoted  $a\vec{u} \in V$

Then if, for all vectors  $\vec{u}, \vec{v}, \vec{w} \in V$  and all scalars  $a, b \in K$ , the following 8 properties are true, then  $V$  is called a *vector space* (over  $K$ ).

- (A1)  $(u + v) + w = u + (v + w)$
- (A2) There exists a vector in  $V$ , named *zero vector* and denoted  $0$  (or  $\vec{0}$ ) such that for all  $u \in V$ ,  $u + 0 = u$
- (A3) For each  $u \in V$ , there is a vector in  $V$ , called the (additive) inverse of  $u$  and denoted  $-u$ , having the property  $u + (-u) = 0$  (where  $0$  is the zero vector defined in A2)
- (A4)  $u + v = v + u$  (commutative laws)
- (S1)  $a(u + v) = au + av$  (distributive laws)

$$(S2) \quad (a + b)u = au + bu$$

$$(S3) \quad a(bu) = (ab)u$$

$$(S4) \quad 1u = u \quad (1 \in R \text{ or } C)$$

These are called the vector space *axioms*.

### 3.3 Examples of vector spaces

Some examples:

- (1)  $K^n = \{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in K\}$ , with addition defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$$

where  $c \in K$  (and  $K$  = set of scalar).

#### **Proof that $K^n$ is a vector space**

Need to prove all 8 properties. We will do 2, the rest are exercises.

- (A4) To prove for all  $u, v \in V$ ,  $u + v = v + u$ .

*Proof concept:* To prove "for all  $x \in A$ , something", say "let  $x \in A$ " (means  $x$  is an arbitrary element of  $A$ , ie you only know  $x \in A$ ).

Then, prove something for that  $x$ .

*Proof:* Let  $u, v \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ ,  $v = (b_1, b_2, \dots, b_n)$  for some  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in K$ . Then

$$\begin{aligned} u + v &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, \dots, a_n + b_n) && \text{(definition of addition in } K^n) \\ &= (b_1 + a_1, \dots, b_n + a_n) && \text{(since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \dots, b_n) + (a_1, \dots, a_n) && \text{(definition of addition in } K^n) \\ &= v + u \end{aligned}$$

- (A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define  $0 = (0, 0, \dots, 0)$  (which is in  $K^n$ ). To prove for all  $u \in K^n$ ,  $u + 0 = u$ , let  $u \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ , so

$$\begin{aligned} u + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) \\ &= u \end{aligned}$$



- (2) In the vector space  $C^2$ ,  $(2 + 3i, 5 - 7i) \in C^2$  is an example of a vector and  $2i \in C$  is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$