

*MATH223 - Linear Algebra (class notes)*

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*January 7th 2019*

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

### *Motivation*

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in  $R^3$ . Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

### *Complex numbers*

**Def:** Let  $i$  be a symbol. We declare  $i^2 = -1$ .

Now, what we'd like to do is take this symbol  $i$  and combine it with the usual real numbers that we are familiar with. We set, for

example,

$$\begin{array}{c} 3i \\ i - 4 \\ 3i - \pi \\ \sqrt{i} + 21 \end{array}$$

**Def:** The field of complex numbers  $C$  consists of all expressions of the form  $a + bi$ , where  $a, b \in R$ .

**Def:** Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

### Notation:

- $0 + bi = bi$
- $a + 0i = a$  (a *real* number)
- $0 + 0i = 0$

**Ex:** If  $z_1 = 2 - i$ ,  $z_2 = 5i$ , then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

**Def:** Let  $z = a + bi \in C$

- (i)  $\bar{z} = a - bi$ , called the *complex conjugate* of  $z$
- (ii)  $|z| = \sqrt{a^2 + b^2}$ , called the *absolute value* or *modulus*

**Def:** If  $z = a + bi \in C$  and  $z \neq 0$  (ie  $z \neq 0 + 0i$ ), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \end{aligned}$$

is called the (multiplicative) inverse of  $z$ . It has the property  $zz^{-1} = 1 = z^{-1}z$ .

*Proof.* We have

$$\begin{aligned} zz^{-1} &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

**Note:** Since  $z \neq 0 + 0i$ ,  $a^2 + b^2 \neq 0$

□

**Def:** If  $z, w \in C$  and  $z \neq 0$  then

$$\frac{w}{z} = wz^{-1}$$

**Ex:** If  $z = 1 + 2i$ ,  $w = 3 - i$  then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3 - i)\left(\frac{1}{5} - \frac{2}{5}i\right) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5}i - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned} \frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3-6i-i+2i^2}{1-2i+2i-4i^2} \\ &= \frac{1-7i}{5} \end{aligned}$$

*January 9th 2019*

*Complex numbers as points in  $R^2$*

You can view  $a + bi$  as a point  $(a, b) \in R^2$ . The usefulness of this is that we can consider, say,  $(3 + 2i)$  and  $(3 - i)$  as vectors in  $R^2$ , and they will conserve the same properties (addition of complex numbers corresponds to vector addition in  $R^2$ ). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

*Equations with complex numbers*

**Fact:** Every real number  $a \neq 0$  has two square roots:

- if  $a > 0$ , roots  $\pm\sqrt{a}$

- if  $a < 0$ , two roots are  $\pm i\sqrt{|a|}$ , since:

$$\begin{aligned} (\pm i\sqrt{|a|})^2 &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a \quad (\text{since } a < 0) \end{aligned}$$

**Fact:** Quadratic equation  $ax^2 + bx + c = 0$  has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in  $C$ .

**Ex:** Solve  $x^2 - 2x + 3 = 0$ , and factor  $x^2 - 2x + 3$ .

**Sol:**

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2} \end{aligned}$$

**Note:** If  $ax^2 + bx + c$  has  $a, b, c \in R$  has a non-real root, say  $z$ , its other root is  $\bar{z}$  ( $z = a + bi$ ,  $\bar{z} = a - bi$ ). This is not necessarily true if  $a, b, c \in C$ .

Back to problem. Factor  $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$ .

**Caution:**  $-1$  has two roots, namely  $\pm i$ , so you may write  $i = \sqrt{-1}$ , but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} \quad (\text{this step doesn't quite work}) \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

**Theorem 1** (Fundamental Theorem of Algebra). *If*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 n^0$$

*is a polynomial with  $a_n \neq 0$ , and  $a_n, a_{n-1}, \dots, a_0 \in C$ , then  $p(x)$  factors into linear factors,*

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \dots \cdot (x - r_n)$$

for some complex numbers  $r_1, r_2, \dots, r_n$ . Some  $r_i$ 's may be equal.

**Corollary 1.1.** Every such polynomial has at least one root, and at most  $n$  distinct roots.

**Note:** Finding the roots is, in general, quite difficult.

**Ex:** Factor  $2x^3 + 2x$  (over C).

**Sol:**

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

**Ex:** Solve  $x^2 - i = 0$

**Sol:**  $x^2 = i$  so  $x = \pm\sqrt{i}$ . Want  $\sqrt{i}$  in format  $a + bi$ ,  $a, b \in R$ .

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \\ 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} \quad (\text{so } a=b \text{ both + or both -}) \\ a^2 &= \frac{1}{2} \\ a &= \pm \frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions,  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$  and  $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ .

*Vector spaces (Ch 4)*

**Def.** The sets  $R$  and  $C$  (and also  $Q$ , rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of  $K$ " means that  $K$  is either  $R$  or  $C$ .

*January 11th 2019*

**Last time:** Field  $K$  is  $R$  or  $C$  (for this class).

### Geometric vectors ('arrows')

You can add two vectors (arrows) (see figure 1)

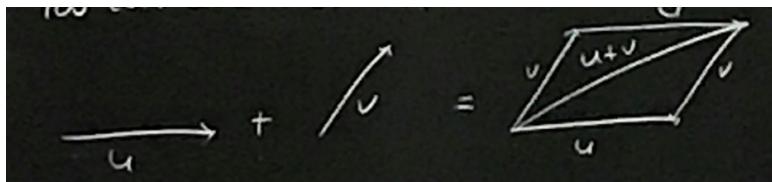


Figure 1: Vector addition

**Observation:**  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .

You can rescale a vector (see figure 2) **Observation:**  $a(b\vec{u}) = (ab)\vec{u}$ .

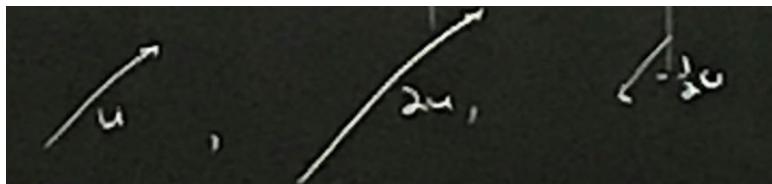


Figure 2: Vector rescaling

Also:  $1\vec{u} = \vec{u}$

**Question:** What properties are interesting? What other objects obey the same properties?

**Abstraction:** Focus on properties more than on the objects.

### Definition of a vector space

Let  $V$  be a set, called set of "vectors", and let  $K$  be a field ( $R$  or  $C$ ) (elements of  $K$  called *scalars*). Assume that we have already defined two operations:

- (1) One called *addition*, which takes two vectors  $\vec{u}, \vec{v} \in V$  and produces another vector denoted  $\vec{u} + \vec{v} \in V$ .
- (2) One called *scalar multiplication* which takes a vector  $\vec{u} \in V$  and a scalar  $a \in K$  and produces another vector denoted  $a\vec{u} \in V$

Then if, for all vectors  $\vec{u}, \vec{v}, \vec{w} \in V$  and all scalars  $a, b \in K$ , the following 8 properties are true, then  $V$  is called a *vector space* (over  $K$ ).

- (A1)  $u + v = v + u$  (commutative laws)
- (A2) There exists a vector in  $V$ , named *zero vector* and denoted  $0$  (or  $\vec{0}$ ) such that for all  $u \in V$ ,  $u + 0 = u$
- (A3) For each  $u \in V$ , there is a vector in  $V$ , called the (additive) inverse of  $u$  and denoted  $-u$ , having the property  $u + (-u) = 0$  (where  $0$  is the zero vector defined in A2)
- (A4)  $(u + v) + w = u + (v + w)$

(SM1)  $a(u + v) = au + av$  (distributive laws)

(SM2)  $(a + b)u = au + bu$

(SM3)  $a(bu) = (ab)u$

(SM4)  $1u = u$  ( $1 \in R$  or  $C$ )

These are called the vector space *axioms*.

*Examples of vector spaces*

Some examples:

(1)  $K^n = \{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in K\}$ , with addition defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$$

where  $c \in K$  (and  $K$  = set of scalar).

#### Proof that $K^n$ is a vector space

Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all  $u, v \in V$ ,  $u + v = v + u$ .

*Proof concept:* To prove "for all  $x \in A$ , something", say "let  $x \in A$ " (means  $x$  is an arbitrary element of  $A$ , ie you only know  $x \in A$ ). Then, prove something for that  $x$ .

*Proof:* Let  $u, v \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ ,  $v = (b_1, b_2, \dots, b_n)$  for some  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in K$ . Then

$$\begin{aligned} u + v &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, \dots, a_n + b_n) \quad (\text{definition of addition in } K^n) \\ &= (b_1 + a_1, \dots, b_n + a_n) \quad (\text{since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \dots, b_n) + (a_1, \dots, a_n) \quad (\text{definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define  $0 = (0, 0, \dots, 0)$  (which is in  $K^n$ ). To prove for all  $u \in K^n$ ,  $u + 0 = u$ , let  $u \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ , so

$$\begin{aligned} u + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) \\ &= u \end{aligned}$$

- (2) In the vector space  $C^2$ ,  $(2 + 3i, 5 - 7i) \in C^2$  is an example of a vector and  $2i \in C$  is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

*January 14th 2019*

**Problem:** Let  $J = \{(x, y) | x \in R, y \in R\}$  but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that  $J$  is not a vector space.

**Solution:** Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned} ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3) \end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.

**Actual proof:** Let  $u = (1, 1)$ ,  $v = (2, 2)$  and  $w = (3, 3)$ . Then,

$$\begin{aligned} u + (v + w) &= (1, 1) + ((2, 2) + (3, 3)) \\ &= (1, 1) + (-2 - 3, 5) \\ &= (1, 1) + (-5, 5) \\ &= (-1 + 5, 6) \\ &= (4, 6) \end{aligned}$$

Whereas,

$$\begin{aligned}
 (u + v) + w &= ((1,1) + (2,2)) + (3,3) \\
 &= (-1 - 2, 3) + (3,3) \\
 &= (-3,3) + (3,3) \\
 &= (-(-3) - 3, 6) \\
 &= (0,6)
 \end{aligned}$$

Hence, the axiom does not hold.

*More examples of vector spaces*

- (1)  $K^n$  (ie  $R^n$  or  $C^n$ ). See before
- (2)  $P(K)$  = polynomials, where coefficients are in  $K$ . Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}
 f(x) &= x^2 + 2ix - 4 \in P(C) \\
 g(x) &= -x^2 + ix \in P(C) \quad (\text{and also in } P(R))
 \end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned}
 2if(x) &= 2ix^2 + 4i^2x - 8i \\
 &= 2ix^2 - 4x - 8i
 \end{aligned}$$

- (3)  $P_n(K)$  = polynomials of degree  $n$  or less, coefficient from  $K$ . For example,

$$\begin{aligned}
 x^2 - 2x + 2 &\in P_2(R) \\
 x^2 - 2x + 2 &\in P_3(R) \\
 x^2 - 2x + 2 &\in P_2(C) \\
 x^2 - 2x + 2 &\notin P_1(R)
 \end{aligned}$$

**Note:** In  $P(K), P_n(K)$  the "vectors" are polynomials.

- (4)  $M_{m \times n}(K)$  =  $m \times n$  matrices with entries from  $K$ . Scalars are  $K$ ,

addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2 \times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2 \times 2}(C) \\ A + B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in  $M_{m \times n}(K)$  is the  $m \times n$  matrix with all entries 0.

- (5) Let  $X$  be any set (think  $x = R$  or  $C$ , but not required). Define  $F(X, K) = \{f : X \rightarrow K\}$  = all functions from  $X$  to  $K$ .  
**Ex:**  $f(x) = x^2 \in F(R, R)$ .  
**Ex:** Let  $x = \{1, 2\}$ . Then  $g$  defined by

$$\begin{aligned} g(1) &= 3 \\ g(2) &= \sqrt{2} \end{aligned}$$

*Addition* in this space is defined by:

If  $f, g \in F(X, K)$  then  $f + g$  is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Note that  $f(x) \in K$  and  $g(x) \in K$ , in other words they are *numbers* (scalars). The  $+$  in  $(f + g)$  is the addition of vectors  $f$  and  $g$ , while the other  $+$  is scalar addition.

*Scalar multiplication* in this space is defined by: if  $f \in F(X, K), c \in K$  then  $cf$  is the function defined by

$$(cf)(x) = cf(x)$$

Note that  $cf$  is the name of the function, that "multiplication" is scalar multiplication  $F(X, \models)$  and  $cf(x)$  is the multiplication of two scalars (numbers).

The fact that  $F(X, K)$  is a vector space and the axioms are followed is not so obvious.

**Prove (A2) true for  $F(X, K)$ .** Define  $z \in F(X, K)$  by

$$z(x) = 0 \quad (\text{for all } x \in X)$$

Note that 0 here is a scalar. Then if  $f \in F(X, K)$  is an arbitrary element, then we need to prove  $f + z = f$ . This is true since for all  $x \in X$ ,

$$\begin{aligned}(f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence,  $f + z, f$  have the same output (namely  $f(x)$ ) for every input. Hence,  $f + z = f$ .

**Exercise:** Try (A3).

January 16th 2019

**Theorem 2** (Cancellation Law). Suppose  $v$  is a vector space over  $K$ . For all vectors  $u, v, w \in V$ , if  $u + w = v + w$  then  $u = v$ .

*Note:* To prove "for all" you say let  $u \in V$  (means  $u$  is an arbitrary vector).

To prove "if  $p$  then  $q$ ", denoted  $p \rightarrow q$ , assume  $p$  is true and use it to prove  $q$ .

*Proof.* Let  $u, v, w \in V$ . Assume  $u + w = v + w$ . By vector space axiom A3, there is a vector  $(-w) \in V$ . Add  $(-w)$  to both sides:

$$\begin{aligned}(u + w) + (-w) &= (v + w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) \quad (\text{by A1}) \\ u + \vec{0} &= v + \vec{0} \quad (\text{by A3}) \\ u &= v \quad (\text{by A2})\end{aligned}$$

□

**Theorem 3.** Two points:

1. The zero vector is unique
2. For each  $u \in V$ ,  $-u$  is unique

*Note:* To prove something is unique, suppose you have two of them and show they are the same.

*Proof.* 1) Assume 0 and  $z$  both satisfy the property (A2:  $\forall u \in V, u + 0 = u$  (\*) and  $u + z = u$  (\*\*)). Goal is to prove  $0 = z$ .

$$\begin{aligned}z &= z + 0 && (\text{by } *, \text{ with } u = z) \\ &= 0 + z && (\text{by A4}) \\ z &= 0 && (\text{by } **, \text{ with } u = 0)\end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

**Theorem 4.**  $\forall u \in V, c \in K,$

$$1) \quad c\vec{0} = \vec{0}$$

$$2) \quad 0u = \vec{0}$$

$$3) \quad -(cu) = ((-c)u)$$

*Proof.* Of 2). Let  $u \in V$ . Then,

$$\begin{aligned} 0u + 0u &= (0 + 0)u && (\text{By SM2}) \\ 0u + 0u &= 0u && (\text{by R addition}) \\ 0u + 0u &= 0u + \vec{0} && (\text{by A2}) \\ 0u + 0u &= \vec{0} + 0u && (\text{by A4}) \\ 0u &= \vec{0} && (\text{by cancellation law}) \end{aligned}$$

□

Note:  $0 + u = u$  is true for all  $u \in V$  (same as  $u + 0 = u$  then apply A4)

### Linear combinations and spans

**Def:** Let  $u, v_1, v_2, \dots, v_n \in V$ . If there are scalars  $a_1, a_2, \dots, a_n \in K$  such that  $u = a_1v_1, a_2v_2 \dots a_nv_n$  then  $u$  is said to be a linear combination of  $v_1, v_2, \dots, v_n$ .

**Ex:** In  $P(R)$ ,  $x^2 + 2x - 4$  is a linear comb of  $x^2, x, 1$ .

**Important problem:** Given vectors  $u, v_1, v_2, \dots, v_n$ , determine if  $u$  is a linear combination of  $v_1, v_2, \dots, v_n$  and if so find  $a_1, a_2, \dots, a_n$ .

**Ex:** Determine if  $f(x) = 2x^2 + 6x + 8$  is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

**Sol.** Are there  $a_1, a_2, a_3$  s.t.

$$\begin{aligned} 2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\ &= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3) \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a_1 - 2a_2 + 2a_3 &= 2 \\ 2a_1 - 4a_2 &= 6 \\ a_1 - 2a_2 - 3a_3 &= 8 \end{aligned}$$

Solve the linear system:

$$\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array} \quad \downarrow \quad \begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \quad (\text{row reduce})$$

$\therefore$  No solution, because of the last row.  $f$  is not a linear combination of  $g_1, g_2, g_3$ .

**Def:** Let  $S \subseteq V$  ( $S$  is a subset of  $V$ ) and assume  $s \neq 0$ . The span of  $s$ , denoted  $\text{span}(s)$  is the set of all linear combinations of vectors from  $S$ , ie

$$\begin{aligned} \text{span}(s) &= \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ &\quad \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ &\quad u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

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Last class

$$\begin{aligned} S &\subseteq V \\ \text{span}(s) &= \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ &\quad \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ &\quad u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

**Ex:**  $S = \{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\} \subseteq \mathbb{R}^2$ . Prove  $\text{span}(S) = \mathbb{R}^2$ .

**Note:**  $\begin{pmatrix} a \\ b \end{pmatrix}$  means  $(a, b)$ .

**Proof note:** To prove two sets  $A, B$  are equal, ie  $A = B$ , you can prove  $A \subseteq B$  and  $B \subseteq A$ .

**Sol:**

- (1) Prove  $\text{span}(S) \subseteq \mathbb{R}^2$ . Trivial, since any linear combination of vectors in  $\mathbb{R}^2$  is still in  $\mathbb{R}^2$ .

- (2) Prove  $R^2 \subseteq \text{span}(S)$ . Let  $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$  (arbitrary). To prove that there exists scalars  $x_1, x_2 \in K$  so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all  $a, b$ ). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution.  $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{span}(S)$  so  $R^2 \subseteq \text{span}(S)$ . So by (1), (2),  $\text{span}(S) = R^2$ .  $\square$

**Note:**  $Ax = b, A_{n \times n}$  if  $A$  inv,  $x = A^{-1}b$ .

**Theorem 5.** Let  $S \subseteq V, S \neq \emptyset$  ( $\emptyset = \text{empty set}$ ). Then,

- (1) If  $u, v \in \text{span}(S)$  then  $u + v \in \text{span}(S)$
- (2) If  $u \in \text{span}(S)$  and  $c \in K$ , then  $cu \in \text{span}(S)$
- (3)  $\vec{0} \in \text{span}(S)$

*Proof.* By direct proof.

- (1) (Note, "if  $u, v \in \text{span}(S)$ " means for all  $u, v \in \text{span}(S)$ ).

Let  $u, v \in \text{span}(S)$ . Then,

$$u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

$$v = b_1 v_1 + b_2 v_2 + \dots + b_m v_m \text{ where } v_1, \dots, v_m \in S, b_1, \dots, b_m \in K$$

Then  $u + v = a_1 u_1 + \dots + a_n u_n + b_1 v_1 + \dots + b_m v_m$  which is in  $\text{span}(S)$  since  $u_1, \dots, u_n, v_1, \dots, v_m \in S$ .

- (2) Let  $u \in \text{span}(S), c \in K$ . Then,

$$u = a_1 u_1 + a_2 u_2 + \dots + a_n u_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \dots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_n)u_n \end{aligned}$$

**Note:** If you want to be very formal, you need to write down all of the vector space axioms. Which is in  $\text{span}(S)$  since it is a linear combination of  $a_1, \dots, a_n$  which are in  $S$ .

- (3) (Prove  $\vec{0} \in \text{span}(S)$ ) Let  $u \in S$ . **Note:** This is possible only because  $S \neq \emptyset$ .

Then  $u = 1u$ , so  $u \in \text{span}(S)$ . Then using  $c = 0$  and (2) and fact that  $u \in \text{span}(S)$ ,

$$cu = 0u = \vec{0}$$

is also in  $\text{span}(S)$ . **Note:** Since  $u = 1u$ ,  $S \subseteq \text{span}(S)$ .

□

### Subspaces

**Def.** Let  $V$  be a vector space and  $W \subseteq V$  (subset). If  $W$ , using addition and scalar multiplication as defined in  $V$ , satisfies the definition of vector space, then  $W$  is called a subspace of  $V$ , denoted  $W \leq V$  (less than equal sign, read as "subspace").

**Note:** Main issue is that addition and scalar multiplication with vector from  $W$  produce vectors which are still in  $W$ .

**Theorem 6.** Let  $W \subseteq V$ . Then, if the following three properties hold,  $W \leq V$  (subspace).

(SS1) For all  $w_1, w_2 \in W$ , we have  $w_1 + w_2 \in W$  ("closure under addition")

(SS2) For all  $w \in W$  and scalars  $c \in K$ , we have  $cw \in W$  ("closure under scalar multiplication")

(SS3)  $\vec{0} \in W$ .

These are the same properties we just proved for spans; in other words, we proved earlier that  $\text{span}(S)$  is a subspace.

*Proof.* For  $W$  to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

- (A1) Let  $u, v, w \in W$ . Then since  $u, v, w \in V$ , and (A1) holds in  $V$ ,
- $$u + (v + w) = (u + v) + w.$$

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let  $w \in W$ .

Want to show  $-w \in W$ . Then, using (SS2) with  $c = -1$  gives

$$-1(w) = -w \quad (\text{thm from last class})$$

is in  $W$ , as needed.

(A4) Still true because it is true in  $V$ .

(SM1-SM4) All hold because they hold in  $V$ .

□

*January 21st 2019*

### *A note on logic*

Let  $P, Q$  be statements that are true or false.

- (1) "If  $P$  then  $Q$ ", also written symbolically as " $P \Rightarrow Q$ " ( $P$  implies  $Q$ ) means if  $P$  is true, then  $Q$  is also true. To prove " $P \Rightarrow Q$ ", assume  $P$  and prove  $Q$  is true. If you know that " $P \Rightarrow Q$ " is true, you can use it: if you can establish that  $P$  is true, you may conclude  $Q$  is true.

**Ex:** Let  $A$  be an  $n \times n$  matrix:

$$P : \det(A) = 1 \quad Q : "A \text{ is invertible}"$$

**Thm:**  $P \Rightarrow Q$

- (2) The *converse* of " $P \Rightarrow Q$ " is " $Q \Rightarrow P$ ". This is a (logically) different statement.

**Ex:** With  $P$  and  $Q$  as above, " $Q \Rightarrow P$ " is not true because  $A_{inv} \neq \det(A) = 1$ .

- (3) The *contrapositive* of " $P \Rightarrow Q$ " is " $\neg Q \Rightarrow \neg P$ " ie "if  $Q$  false, then  $P$  also false". Logically, this is the same as " $P \Rightarrow Q$ ".

- (4) The *equivalence* " $P$  if and only if  $Q$ ", written " $P \iff Q$ " means " $P \Rightarrow Q$  and also  $Q \Rightarrow P$ " is true. Also means that either both  $P$  and  $Q$  are true or both are false.

**Ex:**  $\det(A) \neq 0 \iff A \text{ is invertible}$ .

To prove " $P \iff Q$ ", need to prove " $P \Rightarrow Q$ " and " $Q \Rightarrow P$ ".

**Note:**  $\neg P \Rightarrow \neg Q$  is the same as  $Q \Rightarrow P$ .

*Subspaces (cont'd)*

**Thm (last class):** Let  $W \subseteq V$  (subset). If

1. For all  $u, v \in W$ ,  $u + v \in W$
2. For all  $u \in W$ ,  $c \in K$ ,  $cu \in W$
3.  $\vec{0} \in W$

then  $W \leq V$  (subspace). (ie: (1), (2), (3) are true  $\Rightarrow W \leq V$ )

**Theorem 7.** Let  $W \subseteq V$ . Then

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).

**Proof.** Exercise.

**Theorem 8.** Let  $W \subseteq V$ . Then

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

*Examples of subspaces and non-subspaces*

Is each subset a subspace?

- (a)  $W = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq R^2$ . Not a subspace, since the zero vector is not in  $W$ . The others are also false, but it's enough to prove that one of the statements does not hold. But  $\text{span}(W) = R^2$  (so  $\text{span}(W) \leq R^2$ )

- (b)  $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x + y - z = 0 \right\}$ . Need to check (1), (2), (3):

- (i) Let  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$ . Then we know  $x + y - z = 0$  and  $x' + y' - z' = 0$ . Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in  $W$ .

(2) Let  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$  (means  $x + y - z = 0$ ), let  $c \in K$ . To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here,  $cx + cy - cz = c(x + y - z) = c(0) = 0$ . So  $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

(3)  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$ , since  $0 + 0 - 0 = 0$

Since (1), (2), (3) true,  $W \leq R^3$  (subspace)

(c)  $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x + y - z = 1 \right\}$ . This is *not* a subspace. (3) is false.

(d)  $W = \{A \in M_{2 \times 2} \mid A_{ij} \geq 0 \forall i, j\}$ , where  $A_{ij}$  is the entry of  $A$  in row  $i$ , column  $j$ . (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

(2)  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since,

for example  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$  but

$$(-1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

*Two special subspaces*

Let  $V$  be a vector space.

(1)  $V \leq V$  is true

(2)  $\{\vec{0}\} \leq V$  is true ("zero subspace")

*A refinement on the definition of span*

**Def.** If  $S = \emptyset$  (emptyset), define  $\text{span}(S) = \{\vec{0}\}$  (if  $S \neq \emptyset$ ,  $\text{span}(S)$  defined as before).

**Theorem 9.**  $\text{span}(S) \leq V$ .

**Proof** Two cases :

1. If  $S = \emptyset$ ,  $\text{span}(S) = \{\vec{0}\} \leq V$
2. If  $S \neq \emptyset$ , you already proved  $\text{span}(S)$  satisfies (1), (2), (3).  
So  $\text{span}(S) \leq V$ .

**Theorem 10.** (improved version of subspace conditions) Let  $W \subseteq V$ . Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W$$

**Proof** We will actually prove (1), (2), (3)  $\iff$  RHS (right-hand side). Two parts to proof.

(1) " $(1), (2), (3) \Rightarrow \text{RHS}$ " or " $\Rightarrow$ "

January 23rd 2019

**Recap:**

- (1) If  $u, v \in W$  then  $u + v \in W$
- (2) if  $u \in W, c \in K$  then  $cu \in W$
- (3)  $\vec{0} \in W$

**Theorem 11.** Let  $W \subseteq V$ . Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall u, v \in W, c \in K \text{ we have } cu + v \in W$$

**Proof:** Suffices to prove (1), (2), (3)  $\iff$  RHS.

1.  $\Rightarrow$  Assume (1), (2), (3) (prove right-hand side). Two things to prove:
  - (1) Since  $\vec{0} \in W$  (by (3)),  $W \neq \emptyset$
  - (2) Let  $u, v \in W$  and  $c \in K$ . Since (2) holds,  $cu \in W$ . Since (1) holds,  $cu \in W$  and  $v \in W$ , so  $cu + v \in W$ .
2.  $\Leftarrow$  Assume RHS, prove (1), (2), (3).
  - (1) Let  $u, v \in W$ . Apply RHS with  $\Leftarrow$  to get
 
$$cu + v = 1u + v = u + v \in W$$

(2) (Prove  $\vec{0} \in W$ ) Since  $W \neq \emptyset$ , there is a vector  $w \in W$ . Apply right-hand side with  $u = w, v = w, c = -1$ . So  $cu + v = (-1)w + w = -w + w = \vec{0} \in W$ .

(3) Let  $u \in W, c \in K$ . Apply RHS ( $cu + v \in W$ ) with  $u = u, c = c, v = \vec{0}$  (note:  $\vec{0} \in W$  by (3) above). Then  $cu + v = cu + \vec{0} = cu \in W \quad \square$

**Ex:** In  $F(R, R) = V$  (functions  $f : R \rightarrow R$ ), prove that

$$W = \{f \in V | f(3) = 0\}$$

is a subspace. Eg:  $f(x) = (x - 3)e^x \in W$ .

**Solution:** (1), (2) together (by last thm). Let  $f, g \in W, c \in R$  (prove  $cf + g \in W$ ). We know  $f(3) = 0$  and  $g(3) = 0$ . Then, check  $(cf + g)(3) = cf(3) + g(3) = 0 + 0 = 0$ . So  $cf + g \in W$ .

Also, prove  $w \neq \emptyset$ .  $f(x) = x - 3 \in W$ , since  $f(3) = 0$  (or,  $z(3) = 0$  satisfies  $z(3) = 0$  so  $z \in W$ . Note that  $z$  is the zero vector of  $F(R, R)$ ).

**Theorem 12.** Let  $A \in M_{m \times n}(K), b \in K^m$ . Define

$$S = \{x \in K^n | Ax = b\}$$

i.e  $S$  = solution set to linear system  $Ax = b$ . Then,

$$S \leq K^n \iff b = \vec{0} \text{ (ie system is homogeneous)}$$

### Proof

(i)  $\Rightarrow$  Assume  $S \leq K^n$ . Then  $\vec{0}_n \in S$  (by (3)). So  $A\vec{0} = b$  but  $A\vec{0}_n = \vec{0}_m$  so  $\vec{0} = b$ .

(ii)  $\Leftarrow$  Assume  $b = \vec{0}_m$  (prove  $S \leq K^n$ ). Then  $A\vec{0}_n = \vec{0}_m$ , so  $\vec{0}_n \in S$ .

Next, let  $u, v \in S, c \in K$ . So  $u, v \in K^n$  and  $Au = b, Av = b$ . Verify  $cu + v$  is a solution.

$$\begin{aligned} A(cu + v) &= A(cu) + Av && \text{(prop of matrix multiplication)} \\ &= c(Au) + Av && \text{(prop of matrix multiplication)} \\ &= cb + b \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \\ &= b \quad \square \end{aligned}$$

**Ex:** Equation  $ax + by + cz = d$  describes a plane in  $R^3$  (eg  $x + y + z = 1$ ) (and also, every plane can be described this way). That is,

$$\{(x, y, z) \in R^3 | ax + by + cz = d\}$$

is a plane.

By last thm,

$$\begin{aligned} P \text{ is a subspace} &\iff ax + by + cz = d \text{ is a homogeneous system} \\ &\iff d = 0 \\ &\iff P \text{ passes through origin } (0, 0, 0) \end{aligned}$$

**Theorem 13.** Let  $S \subseteq V$ . Then,

- (1)  $\text{span}(S) \leq V$  and  $S \subseteq \text{span}(S)$
- (2) If  $S \subseteq W$ , and  $W \leq V$  (subspace) then  $\text{span}(S) \subseteq W$  (actually,  $\text{span}(S) \leq W$ , subspace by (1))

**Proof:**

- (1)  $\leq$  We know already. Let  $u \in S$ . Then  $u = 1u$ , so  $u \in \text{span}(S)$
- (2) Assume  $S \subseteq W$ , and  $W \leq V$ . Let  $v \in \text{span}(S)$ . Then  $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$  for some scalars and vectors  $u_1, u_2, \dots, u_n \in S$ . Since  $S \subseteq W$ ,  $u_1, u_2, \dots, u_n \in W$ . But  $W$  subspace. So  $a_1u_1, a_2u_2, \dots, a_nu_n \in W$  (by prop (2) subspace) then  $a_1u_1 + a_2u_2 \in W$  (by prop (1) of subspaces). So then  $(a_1u_1 + a_2u_2) + a_3u_3 \in W$  (etc). So  $a_1u_1 + a_2u_2 + \dots + a_nu_n \in W$ .

**Note:** "etc" here is actually a proof by mathematical induction.

Omit for now.

January 25th 2019

*Interlude : Symbolic logic (briefly)*

Let  $P, Q$  be statements that could be true ( $T$ ) or false ( $F$ ). Define:

- (1)  $\neg P$ , "not  $P$ ", is  $F$  when  $P$  is  $T$ ,  $T$  when  $P$  is  $F$
- (2)  $P \wedge Q$ , " $P$  and  $Q$ ", is  $T$  exactly when  $P, Q$  both  $T$
- (3)  $P \vee Q$ , " $P$  or  $Q$ " is  $T$  when  $P, Q$  both  $F$
- (4)  $P \Rightarrow Q$ , " $P$  implies  $Q$ ", is  $T$  unless  $P$  is  $T$  and  $Q$  is  $F$ . Hence,  $P \Rightarrow Q$  is equivalent to  $\neg P \vee Q$ . We will write  $P \Rightarrow Q \equiv \neg P \vee Q$ .
- (5)  $P \iff Q$ , " $P$  if and only if  $Q$ ", is  $T$  if both  $T$  or both  $F$ .

### De Morgan's Laws

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

## Quantifiers

- $\forall$  means "for all"
- $\exists$  means "there exists"

**Ex. (A4)** (commutativity)  $\forall u, v \in V \ u + v = v + u$ .

**Ex. 2 (A2)** (zero vector)  $\exists z \in V \ \forall u \in V \ (u + z = u) \wedge (z + u = u)$   
(textbook version)

## Negating quantifiers

- $\neg \forall u \in V P(u) \equiv \exists u \in V \neg P(u)$
- $\neg \exists u \in V P(u) \equiv \forall u \in V \neg P(u)$

**Ex.**

$$\begin{aligned} \neg(A2) &\equiv \neg \exists z \in V \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \neg \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \exists u \in V \ \neg(u + z = u \wedge z + u = u) \\ &\equiv \forall z \in V \exists u \in V \ (u + z \neq u \vee z + u \neq u) \end{aligned}$$

## Proof by contradiction

You want to prove some statement  $P$ . Proof by contradiction works this way:

- (1) Assume  $\neg P$
- (2) Derive a contradiction (hard part)
- (3) Conclude  $P$  is true

**Ex.** Outline of how to prove (A2) *does not hold* in some vector space.

You want to prove  $\neg(A2)$ .

$$\begin{aligned} \neg(A2) &\equiv \neg \exists z \in V \forall u \in V \ u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \neg \forall u \in V \ u + z = u \wedge z + u = u \end{aligned}$$

Let  $z \in V$ . Prove the right-hand part ( $\neg \forall u \in V \ u + z = u \wedge z + u = u$ ) by contradiction. Assume (for contradiction) that

$$\forall u \in V \ u + z = u \wedge z + u = u \tag{1}$$

Use (1) by substituting  $u = \text{some specific vector}$  (derive a contradiction). Conclude that ( $\neg \forall u \in V \ u + z = u \wedge z + u = u$ ) is true.

*Last time*

**Theorem 14.** If  $S \subseteq W, W \leq V$  then  $\text{span}(S) \subseteq W$ .

**Note.** This means if you "promote" a subset to a subspace, adding in only what's necessary, what you get is  $\text{span}(S)$ . Or,  $\text{span}(S)$  is the "smallest" subspace containing  $S$ .

**Fact.** Subspaces are "closed under taking linear combinations". Ie if  $W \leq V, w_1, \dots, w_n \in W$  and  $a_1, \dots, a_n \in K$  then

$$a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$$

**Caution.** Linear combinations are *finite* sums by definition. So you can't sum up infinitely many vectors.

*Illustration of this theorem*

Let  $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\} \subseteq W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} | x, y \in R \right\}$ . Then  $\text{span}(S) \subseteq W$  ie  $\text{span}(S)$  is in  $xy$  plane. In fact,  $\text{span}(S) = W$ .

**Def.** If  $W = \text{span}(S)$ , we say that  $S$  spans  $W$  or is a spanning set for  $W$ .

**Ex.**  $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ ,  $\text{span}(S)$  =  $xy$ -plane in  $R^3$ . So  $S$  spans the  $xy$ -plane.

**Ex. 2.**  $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ ,  $\text{span}(S) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} | x \in R \right\}$  = line.

*Intersection of two subspaces*

**Theorem 15.** Let  $W_1 \leq V, W_2 \leq V$ . Then  $W_1 \cap W_2 \leq V$  (ie intersection of two subspaces is a subspace).

**Proof.**  $W_1 \cap W_2 = \{w \in V | w \in W_1 \wedge w \in W_2\}$ .

- (1)  $\vec{0} \in W_1, \vec{0} \in W_2$  (because subspace). So  $\vec{0} \in W_1 \cap W_2$ .
- (2) Let  $u, v \in W_1 \cap W_2, c \in K$ . So  $u, v \in W_1$  and  $W_1 \leq V$  so  $cu + v \in W_1$  and  $u, v \in W_2$  and  $W_2 \leq V$  so  $cu + v \in W_2$ . Hence  $cu + v \in W_1 \cap W_2$ .  $\square$

*January 28th 2019*

**Last time:**  $W_1 \leq V$  and  $W_2 \leq V \Rightarrow W_1 \cap W_2 \leq V$ .

**Corollary 15.1.** *The intersection of any number of subspaces is a subspace.*

**Problem.** Prove that  $W = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$  is a subspace of  $F(\mathbb{R}, \mathbb{R})$ .

**Sol #1:** Directly from subspace properties (omit)

**Sol #2:** We saw an example proving that  $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(3) = 0\}$  is a subspace. The "3" is not important, so similarly:

$$W_1 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0\}$$

$$W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(2) = 0\}$$

both subspaces of  $F(\mathbb{R}, \mathbb{R})$ . Then  $W_1 \cap W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$  is a subspace.

**Q:** Is union of two subspaces also a subspace?

**A:** Not in general.

**Eg:**  $W_1 = \text{x-axis} = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \mathbb{R} \right\} \leq \mathbb{R}^2$

$W_2 = \text{y-axis} = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} | y \in \mathbb{R} \right\} \leq \mathbb{R}^2$

$W_1 \cup W_2 = \text{xy-axis} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | x = 0 \vee y = 0 \right\}$ , which, importantly, is not  $\mathbb{R}^2$ . Not a subspace, since  $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2$ ,  $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2$ , but  $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$ .

**Note:** To promote  $W_1 \cup W_2$  to a subspace, you form  $\text{span}(W_1 \cup W_2)$ .

**Def:** Let  $W_1 \leq V$  and  $W_2 \leq V$ . The sum of  $W_1$  and  $W_2$  is

$$W_1 + W_2 = \{v \in V | \exists w_1 \in W_1, w_2 \in W_2, \text{ such that } v = w_1 + w_2\}$$

**Ex:**

$$W_1 = \{ax^2 | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

$$W_2 = \{ax | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

We have,

$$W_1 + W_2 = \{ax^2 + bx | a, b \in \mathbb{R}\}$$

**Theorem 16.** Let  $W_1 \leq V$ ,  $W_2 \leq V$ . Then

(a)  $W_1 + W_2 = \text{span}(W_1 \cup W_2)$  (hence  $W_1 + W_2$  is a subspace)

(b)  $W_1 \leq W_1 + W_2$ ,  $W_2 \leq W_1 + W_2$

**Proof:**

- (a)(1) Prove  $W_1 + W_2 \subseteq \text{span}(W_1 \cup W_2)$ . Let  $v \in W_1 + W_2$ , so  $v = w_1 + w_2$  where  $w_1 \in W_1$  and  $w_2 \in W_2$ . Then  $w_1, w_2 \in W_1 \cup W_2$  so  $v \in \text{span}(W_1 \cup W_2)$

(2) " $\supseteq$ ". Let  $v \in \text{span}(W_1 \cup W_2)$ . Means  $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$ ,  $u_1, u_2, \dots, u_n \in W_1 \cup W_2$  and  $a_1, a_2, \dots, a_n \in K$ . Each  $u_i$  is in  $W_1 \cup W_2$ . Separate into two groups and relabel, so that:

- Those in  $W_1$ , call these

$$u_1, u_2, \dots, u_l$$

So  $0 \leq l \leq n$ ,  $l = 0$  means none in  $W_1$ .

- Those in  $W_2 \setminus W_1 = \{w \in W_2 | w \notin W_1\}$  ("set difference"), call these

$$u_{l+1}, \dots, u_n$$

So  $l = 0$  means all in  $W_2 \setminus W_1$ ,  $l = n$  means all in  $W_1$ .

Then, let  $w_1 = a_1u_1 + a_2u_2 + \dots + a_lu_l$  (or  $w_1 = \vec{0}$  if  $l = 0$ ),  $w_2 = a_{l+1}u_{l+1} + \dots + a_nu_n$  (or  $w_2 = \vec{0}$  if  $l = n$ ).

Then  $w_1 \in W_1$  since  $W_1$  is a subspace, similarly  $w_2 \in W_2$ . So

$$\begin{aligned} v &= a_1u_1 + \dots + a_nu_n \\ &= w_1 + w_2 \in W_1 + W_2 \text{ as required} \end{aligned}$$

(b)  $W_1 \subseteq W_1 + W_2$ ,  $W_2 \subseteq W_1 + W_2$ . Follows from (a), since  $S \subseteq \text{span}(S)$   $\square$ .

### Linear independence

**Def:** Vectors  $u_1, u_2, \dots, u_n \in V$  (all distinct) are said to be *linearly dependent* if  $\exists$  scalars  $a_1, a_2, \dots, a_n \in K$  not all 0 such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$$

Above equation called a *dependence relation*.

**Note:** If  $a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$  and  $a_1 \neq 0$ , then you can solve for  $u_1$ :

$$u_1 = \frac{-a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \dots - \frac{a_n}{a_1}u_n$$

ie  $u_1$  = linear combination of others, "depends on" others.

**Ex:**  $\{x^2 + x, 2x^2, \frac{x}{10}\}$  is a dependent set of vectors in  $P(\mathbb{R})$  since

$$(x^2 + x) - \frac{1}{2}(2x^2) - 10\left(\frac{x}{10}\right) = 0$$

**Def:** A set of vectors  $S \subseteq V$  (possibly infinite) is dependent if  $\exists$  a finite subset  $\{v_1, v_2, \dots, v_n\} \subseteq S$  of it which is dependent.

**Def:** Vectors  $v_1, v_2, \dots, v_n$  are linearly independent if they are *not* dependent. That is,

$$\neg \exists a_1, \dots, a_n \in K \quad (a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

$$\forall a_1, \dots, a_n \in K \quad \neg(a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

$$\forall a_1, \dots, a_n \in K \quad (\neg(a_1u_1 + \dots + a_nu_n = \vec{0}) \vee (a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0))$$

Note that  $P \implies Q \equiv \neg P \vee Q$ . In other words,  $u_1, u_2, \dots, u_n$  are linearly independent if

$$\forall a_1, \dots, a_n \in K (a_1 u_1 + \dots + a_n u_n = \vec{0} \implies a_1 = 0 \wedge \dots \wedge a_n = 0)$$

Which is to say that the only solution to  $a_1 u_1 + \dots + a_n u_n = \vec{0}$  is the trivial solution  $a_1 = 0, a_2 = 0, \dots, a_n = 0$ .

*January 30th 2019*

*Last class*

$v_1, v_2, \dots, v_n$  independent if  $x_1 v_1 + \dots + x_n v_n = \vec{0}$  has only trivial solution  $x_1 = x_2 = \dots = x_n = 0$ .

**Ex:** Prove that  $\{1+x^2, x+x^2, 1+x+x^2\}$  is independent.

**Solution:** Consider equation

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0 = 0(1) + 0x + 0x^2$$

Want to show  $a = b = c = 0$  is the only solution.

Equation means for all  $x \in K$  ( $\mathbb{R}$  or  $\mathbb{C}$ ),

$$a(1+x^2) + b(x+x^2) + c(1+x+x^2) = 0$$

So, substitute any scalar for  $x$ :

$$\begin{aligned} x = 0 & \quad a + c = 0 \\ x = 1 & \quad 2a + 2b + 2c = 0 \\ x = -1 & \quad 2a + 0b + c = 0 \end{aligned}$$

Can translate into linear system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Row-reduce:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Only solution is  $a = 0, b = 0, c = 0$  so vectors are independent.

If we obtain infinitely many, then you can find dependent set so dependent.

*Some important cases*

- (i)  $S = \emptyset$  is linearly independent since there are no vectors with which to form a dep. relation.
- (ii) If  $\vec{0} \in S$ , then dependent (since  $1\vec{0} = \vec{0}$  is a dep. relation)
- (iii)  $\{u\}$  is independent  $\iff u \neq \vec{0}$ .

**Note:**  $u + (-1)u = \vec{0}$  is not a dep. relation, since  $u$  is repeated. But,  $\{u, -u\}$  is dependent since

$$u + (-u) = \vec{0}$$

is a dep. relation.

**Proposition 17.** Let  $A, B \subseteq V$  where  $A \subseteq B$ .

- (i) If  $A$  is dependent,  $B$  is also dependent
- (ii) If  $B$  is independent,  $A$  is also independent (contrapositive)

**Proof:**

- (i) If  $A$  dep, we have a dep relation

$$a_1v_1 + a_2v_2 + \dots + a_nv_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in A)$$

which is also a dependence relation in  $B$  since  $v_1, \dots, v_n \in B$ .

- (ii) This is the contrapositive of (i).  $\square$

**Note:** Converse is false,  $B$  dep  $\not\rightarrow A$  dep.

*Extending an independent set*

**Theorem 18.** Let  $S \subseteq V$  be linearly independent and suppose  $u \notin S$ . Then,  $S \cup \{u\}$  independent  $\iff u \notin \text{span}(S)$ .

**Proof:**

- (i) " $\rightarrow$ " We will prove this as the contrapositive, ie  $u \in \text{span}(S) \rightarrow$  dep. Assume  $u \in \text{span}(S)$ . So,

$$\begin{aligned} u &= a_1v_1 + \dots + a_nv_n \quad \text{where } v_1, v_2, \dots, v_n \in S \\ \vec{0} &= (-1)u + a_1v_1 + \dots + a_nv_n \end{aligned}$$

Which is a linear combination of vectors from  $S \cup \{u\}$ , not all coefficients 0 since first is  $-1$ . Also, the vectors  $u, v_1, v_2, \dots, v_n$  are all distinct, since  $u \notin S$ . So this is a dependence relation on  $S \cup \{u\}$ , so the set is dependent.

(ii) " $\leftarrow$ " Also by contrapositive. Assume  $S \cup \{u\}$  dep, want to show that  $u \in \text{span}(S)$ . So there is a dependence relation on  $S \cup \{u\}$ .

Two cases:

- **Case 1:** Dependence relation does not involve  $u$  (or, involves  $u$  but with coefficient 0), ie we have

$$a_1v_1 + \dots + a_nv_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in S)$$

But this contradicts independence of  $S$ , so case 1 does not occur.

- **Case 2:** Dependence relation involves  $u$  (with coeff not 0), so

$$au + a_1v_1 + \dots + a_nv_n = \vec{0} \quad v_1, \dots, v_n \in S$$

and  $a \neq 0$ . Rewrite:

$$u = \frac{-a_1}{a}v_1 - \frac{a_2}{a}v_2 - \dots - \frac{a_n}{a}v_n \quad (a \neq 0)$$

Hence  $u \in \text{span}(S)$ .  $\square$

**Note:** Conclusion can be restated as

$$S \cup \{u\} \text{ dependent} \iff u \in \text{span}(S)$$

### Basis and dimension

**Fact:** If  $W$  is subspace, then  $\text{span}(W) = W$ . (Exercise)

So every subspace is a span. But thinking of  $W$  as  $\text{span}(W)$  is excessive. Would like to find the *smallest*  $S$  such that

$$\text{span}(S) = W$$

**Def:** Let  $W \leq V$ . A *basis* of  $W$  is a set  $B \subseteq V$  such that

- (i)  $\text{span}(B) = W$  ("enough vectors to produce  $W$ ")
- (ii)  $B$  is linearly independent ("no extra vectors in  $B$ ")

**Examples:**

- (i) Let  $e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \leftarrow (\text{row } i)$ . Then,

$$\{e_1, e_2, \dots, e_n\}$$

is a basis for  $K^n$ . For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for  $K^3$ .

More next class.

February 1st 2019

**Recall:**  $B$  is a basis of  $W$  if  $\text{span}(B) = W$  and  $B$  is linearly independent.

**Examples:**

- (1)  $P_n(K)$  has basis  $\{1, x, x^2, \dots, x^n\}$
- (2)  $P(K)$  has basis  $\{1, x, x^2, x^3, \dots\}$  (infinitely many)
- (3)  $M_{m \times n}(K)$  has basis  $\{E^{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$  where  $E^{ij} = m \times n$  matrix of 0s except 1 in row  $i$ , column  $j$ . eg:  $M_{2 \times 2}(\mathbb{R})$  has basis

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- (4)  $W = \{\vec{0}\}$  has basis  $\emptyset$  since

- (i)  $\text{span } \emptyset = \{\vec{0}\}$  (by special def)
- (ii)  $\emptyset$  is independent

*Two important questions*

- (1) Does  $W$  always have basis? (spoiler: yes)
- (2) How to find a basis?

**Theorem 19** (Bases exist). Let  $V$  be vector space and  $S$  a finite set with  $\text{span}(S) = V$ . Then there is a subset  $B \subseteq S$  which is a basis of  $V$ .

*Proof.* Algorithm to produce  $B$ .

- (1) If  $V = \{\vec{0}\}$ , use  $B = \emptyset$ .
- (2) Take one vector,  $u_1 \in S (u_1 \neq \vec{0})$ . Consider  $\text{span}\{u_1\}$
- (3) If  $\text{span}\{u_1\} = V$ , done.  $B = \{u_1\}$  is a basis (set of one non-zero vector is independent)
- (4) If  $\text{span}\{u_1\} \neq V$ , there must be a vector  $u_2 \in S$  where  $u_2 \notin \text{span}\{u_1\}$  (Why? If not,  $S \subseteq \text{span}\{u_1\} \leq V$ , then  $\text{span}(S) \subseteq \text{span}\{u_1\}$ , but  $\text{span}(S) = V$  contradicts  $V \neq \text{span}\{u_1\}$ ). By previous theorem, since  $u_2 \notin \text{span}\{u_1\}$ ,  $\{u_1, u_2\}$  is linearly independent.
- (5) Consider  $\{u_1, u_2\}$ . If  $\text{span}\{u_1, u_2\} = V$ , done:  $B = \{u_1, u_2\}$ . Else, continue as before, finding  $u_3 \in S, u_3 \notin \text{span}\{u_1, u_2\}$  (etc)

Since  $S$  is finite, this must stop and at that point you have basis  $B \subseteq S$ . □

*Illustration of this thm*

Find basis of  $\mathbb{R}^3$  that is a subset of

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Following this algorithm,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

**Theorem 20.** Let  $V$  be a vector space,  $L \subseteq V$  a linearly independent set, and  $S \subseteq V$  a spanning set (ie  $V = \text{span}(S)$ ). Then  $\exists$  a subset  $E \subseteq S$  such that  $L \cup E$  is a basis of  $V$  (ie you can always extend it to a basis)

**Proof** Omitted.

**Theorem 21.** Suppose  $V$  has a finite spanning set  $S$ . Then  $V$  has a basis and all bases have the same size, which is at most  $|S|$ .

**Proof** Omitted.

**Def** If  $V$  has a finite basis  $B$ , then the *dimension* of  $V$  is

$$\dim V = |B|$$

If  $V$  does not have a finite basis, it is called *infinite dimensional*.

**Ex:**

(1)  $\dim K^n = n$ .

$$\left( \left\{ \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right\} \right)$$

(2)  $\dim P_n(K) = n + 1$  (basis  $\{1, x, x^2, \dots, x^n\}$ )

(3)  $P(K)$  is infinite dimensional (A#1, proved a finite set of polynomials cannot span  $P(K)$ )

(4)  $\dim M_{m \times n}(K) = mn$  (see basis  $E^{ij}$ , defined above)

**Theorem 22.** Every vector space (including the infinite dimensional ones) has a basis.

**Proof** Uses Axiom of Choice. Difficult.

**Theorem 23.** Suppose  $\dim V = n$ . Let  $A \subseteq V$ . Then,

- (1) If  $\text{span}(A) = V$ , then  $|A| \geq n$  (or, if  $|A| < n$  then  $A$  does not span  $V$ ) and if also  $|A| = n$  then  $A$  is linearly independent, hence basis.
- (2) If  $A$  is linearly independent, then  $|A| \leq n$  (or, if  $|A| > n$  then  $A$  dep) and if also  $|A| = n$  then  $\text{span}(A) = V$  hence  $A$  is a basis.

**Proof** Omitted.

**Note:** If you have *correct number* of vectors, you need only check spanning or independent, not both, to check if basis.

**Ex:** If you have 7 matrices in  $M_{3 \times 2}(K)$ , they *will be* dependent. If you have 5, it's *not* a basis.

February 4th 2019

Last class

Suppose  $\dim V = n$ ,  $S \subseteq V$ ,  $|S| = n$ . Then  $S$  spans  $V \iff S$  linearly independent (only in case  $|S| = \dim V$ ).

Lagrange Interpolation

**Problem** Given "data points"  $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$  where all  $a_i$  are different. Find a polynomial  $p(x)$  of degree  $n-1$ ,  $p(x) = c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1x + c_0$  whose graph  $y = p(x)$  passes through all the points.

**Sol #1** Substitute  $(a_1, b_1)$  into  $y = p(x)$ :

$$b_1 = c_{n-1}a_1^{n-1} + \dots + c_1a_1 + c_0 \quad (\text{for each } i = 1, \dots, n)$$

Which is a system of  $n$  linear equations ( $\text{vars} = c_{n-1}, \dots, c_0$ ) in  $n$  variables.

We'll do something different.

**Def** For scalars  $a_1, a_2, \dots, a_n$  (all different), define the *Lagrange polynomials* for each  $i = 1, 2, \dots, n$  set

$$\begin{aligned} l_i(x) &= \prod_{k=1, k \neq i}^n \frac{(x - a_k)}{(a_i - a_k)} \\ &= \frac{(x - a_1)}{(a_i - a_1)} \cdot \frac{(x - a_2)}{(a_i - a_2)} \cdot \dots \cdot \frac{(x - a_n)}{(a_i - a_n)} \quad (\text{omitting } \frac{(x - a_i)}{(a_i - a_i)}) \end{aligned}$$

**Ex** For  $a_1 = 2, a_2 = 4, a_3 = 6$  we would have

$$l_1(x) = \frac{(x-4)}{2-4} \cdot \frac{(x-6)}{(2-6)}$$

$$l_2(x) = \frac{(x-2)}{4-2} \cdot \frac{(x-6)}{(4-6)}$$

$$l_3(x) = \frac{(x-2)}{6-2} \cdot \frac{(x-4)}{(6-4)}$$

**Note:** All degree 2,  $l_1(4) = 0, l_1(6) = 0, l_1(2) = 1$ .

**Fact**  $l_i(a_j) = 0$  if  $i \neq j$  and 1 if  $i = j$ .

**Proof** If  $i \neq j$ , there is a factor  $\frac{x-a_j}{a_i-a_j}$ , so at  $x = a_j, \frac{a_j-a_j}{a_i-a_j} = 0$ . If  $i = j$ ,

$$l_i(a_i) = \prod_{k=1, k \neq i}^n \frac{(a_i - a_k)}{(a_i - a_i)} = 1$$

**Proposition 24.** Lagrange polynomials  $l_1(x), \dots, l_n(x)$  form a basis of  $P_{n-1}(\mathbb{R})$ .

**Proof** We have  $n$  polynomials (they are distinct),  $\dim P_{n-1}(\mathbb{R}) = n - 1 + 1 = n$ . So correct number. Suffices to prove span or lin independence. We'll prove independence. Suppose

$$d_1 l_1(x) + d_2 l_2(x) + \dots + d_n l_n(x) = 0 \quad (\text{note: for all } x \in \mathbb{R})$$

Substitute  $x = a_1, x = a_2$ , etc into the above. At  $x = a_1, l_1(a_1) = 1$  but  $l_j(a_1) = 0$  for  $j \neq 1$  so

$$d_1 1 + d_2 0 + \dots + d_n 0 = 0$$

so  $d_1 = 0$ . Similarly,  $d_j = 0$  for all  $j$ . More formally, for any  $j = 1, 2, \dots, n$  we have at  $x = a_j$

$$\sum_{i=1}^n d_i l_i(a_j) = 0$$

but all terms are 0 except when  $i = j$ . Set

$$d_j = d_j(1) = d_j l_j(a_j) = 0$$

Hence Lagrange polynomials for a basis.

**Problem** Find poly degree  $n - 1$  through points  $(a_1, b_1), \dots, (a_n, b_n)$ .

**Sol:** Set  $p(x) = b_1 l_1(x) + b_2 l_2(x) + \dots + b_n l_n(x)$  (it has degree  $n - 1$ ).

Then

$$\begin{aligned} p(a_1) &= b_1 l_1(a_1) + b_2 l_2(a_1) + \dots + b_n l_n(a_1) \\ &= b_1(1) + 0 + 0 + \dots + 0 \\ &= b_1 \end{aligned}$$

For each  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} p(a_i) &= \sum_{j=1}^n b_j l_j(a_i) \\ &= 0 + 0 + \dots + b_i l_i(a_i) + \dots + 0 \\ &= b_i \end{aligned}$$

### Dimension of subspaces

**Theorem 20.** Let  $W \leq V$ ,  $V$  finite-dimensional. Then

- (i)  $\dim W \leq \dim V$
- (ii)  $\dim W = \dim V \iff W = V$

#### Proof

- (i) Similar to proof that  $V$  has basis. Use  $W$  as a spanning set for  $W$ . Pick out vectors one at a time (similar to before) to build a basis. You cannot put more than  $\dim V$  vectors into your basis, as this would give an independent set in  $V$  of size *more than*  $\dim V$  (impossible). So this process has to stop, and it produces a basis for  $W$ .
- (ii) " $\rightarrow$ " Assume  $\dim W = \dim V = n$ . Take basis  $B$  of  $W$ . It is a size  $n$  linearly independent set inside  $V$ , hence  $B$  also basis for  $V$ , hence,

$$V = \text{span } B = W$$

" $\leftarrow$ " If  $W = V$ , clearly  $\dim W = \dim V$ .  $\square$

**Subspaces of  $\mathbb{R}^3$**  If  $W \leq \mathbb{R}^3$ ,  $\dim W = 0, 1, 2$  or  $3$ .

This allows us to make the following classification: **Problem** Let

$\dim W$	Classification
0	$\{\vec{0}\}$
1	$\text{span}\{u\}$ = line through origin
2	$\text{span}\{u, v\}$ = plane through origin
3	$\mathbb{R}^3$

$W = \{A \in M_{n \times n}(\mathbb{R}) | \text{tr}(A) = 0\}$ , where  $\text{tr}(A)$  = trace of  $A$  = sum of entries on diagonal =  $A_{11} + A_{22} + \dots + A_{nn}$ .

**Exercise** Prove  $W$  is a subspace.

**Will do next class:** Find  $\dim W$  and find a basis of  $W$ .

February 6th 2019

### Intuition

Solution set  $W$  to a homogeneous system  $A\vec{x} = \vec{0}$  is a subspace of  $K^n$  ( $n = \#$  of variables). If no equations,  $W = K^n$ ,  $\dim W = n$ . For each equation, expect the dimension of  $W$  to drop by 1, unless the equation is *redundant*.

Eg: In  $\mathbb{R}^3$ , one equation

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 && (= \text{plane}) \\ \text{add in } a_2x + b_2y + c_2z &= 0 && (\text{intersection of two planes, } = \text{line}) \\ \text{add in } a_3x + b_3y + c_3z &= 0 && (\text{intersection of three planes, } (0,0)) \end{aligned}$$

**Problem:**  $W = \{A \in M_{n \times n}(\mathbb{R}) \mid \text{tr } A = 0\}$ . Find  $\dim W$ , basis of  $W$ .

**Solution #1:** Clever way: "guess" a basis. Note:  $\text{tr } A = A_{11} + A_{22} + \dots + A_{nn}$  (one linear condition). Expecting

$$\dim W = n^2 - 1$$

Observe that  $\dim W \neq n^2$ . This happens only if  $W = M_{n \times n}(\mathbb{R})$ , and obviously there are matrices which don't have trace 0. Specifically:

$$\text{tr} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \dots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

(In proofs, can choose any example, provided property holds).

Know  $\dim W \leq n^2 - 1$ . If you can find independent set of size  $n^2 - 1$  in  $W$ , it will be a basis. Try first  $n = 3$ . Looking for  $3^2 - 1 = 8$  independent  $3 \times 3$  matrices, all trace 0.

Want trace = 0. Therefore, consider all matrices which have all 0's in the diagonal:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

These 6 are obviously independent. Now, take two more which are independent:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we have 8 independent matrices (check carefully). So for  $n = 3$ ,  $\dim W = 8$ , this is a basis.

### General case

Two types of basis matrices:

- (I) All  $E^{ij}$  (1 in  $(i,j)$ -pos, 0 elsewhere)) where  $i \neq j$ . How many are there?

$$\begin{aligned}\# \text{ of non-diagonal entries} &= \text{entries} - \text{entries on diagonal} \\ &= n^2 - n\end{aligned}$$

Or,  $\binom{n}{2}$  ways to choose 2 distinct values from  $\{1, 2, \dots, n\}$ , 2 ways to order each pair. Total:

$$\begin{aligned}\binom{n}{2}2 &= \frac{n!}{2!(n-2)!}2 \\ &= n(n-1) \\ &= n^2 - n\end{aligned}$$

- (II) Looking for  $n-1$  more, since  $n^2 - n + n - 1 = n^2 - 1$

$$\begin{pmatrix} 1 & -1 & & & \\ & \dots & 0 & & \\ & & & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 & & \\ & \dots & 0 & & \\ & & & 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 & & & \\ & \dots & 1 & & \\ & & & -1 & \end{pmatrix}, \dots$$

(n-1 of those)

Formally, let, for  $i = 1, 2, \dots, n-1$ ,  $D_i$  = matrix with 1 in pos  $(i, i)$  and  $-1$  in pos  $(i+1, i+1)$ , 0 elsewhere.

Verifying all matrices  $E^{ii}$ ,  $D_i$  are independent; clear that suffices to check  $D_1, D_2, \dots, D_{n-1}$  independent. Suppose

$$x_1 D_1 + x_2 D_2 + \dots + x_n D_n = n \times n \text{ zero matrix}$$

The  $(1, 1)$ -entry on left is  $x_1$ , so  $x_1 = 0$ . The  $(2, 2)$ -entry on left is  $-x_1 + x_2$ ,

$$x_1 \begin{pmatrix} 1 & -1 & & & \\ & \dots & 0 & & \\ & & & 0 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & -1 & & \\ & \dots & 0 & & \\ & & & 0 & \end{pmatrix} + \dots = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

but  $x_1 = 0$  so  $x_2 = 0$  also, etc. So similarly for all  $x_i = 0$ , so independent. Formally you'd do a proof by induction, but this is good enough.

Now have  $n^2 - 1$  independent vectors in  $W_1$  so  $\dim W \geq n^2 - 1$ . Already know  $\dim W \leq n^2 - 1$ . So  $\dim W = n^2 - 1$ , have independent set of correct size, so basis.

**Solution #2:** Let  $x_{ij}$  be the  $(i, j)$ -entry of  $A$ . So have  $n^2$  variables  $(x_{ij}, i, j = 1, 2, \dots, n)$  one equation,

$$x_{11} + x_{22} + \dots + x_{nn} = 0 \quad (\text{tr } A = 0)$$

Solve system. All  $x_{ij}, i \neq j$  free variables, so are  $x_{22}, \dots, x_{nn}$ .

**Theorem 21.** Let  $U, W$  be finite dimension subspaces of  $V$ . Then,

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W$$

It's like sets,  $|A \cup B| = |A| + |B| - |A \cap B|$ .

**Proof Omitted.**

**Ex:** If  $W$  is a plane in  $\mathbb{R}^3$  (through  $(0, 0)$ ) and  $L$  is a line in  $\mathbb{R}$  (through  $(0, 0)$ ) and  $L$  is not in the plane, prove  $W + L = \mathbb{R}^3$ .

**Sol:**  $L$  not in plane gives  $L \cap W = \{\vec{0}\}$ . So

$$\begin{aligned}\dim(L + W) &= \dim L + \dim W - \dim L \cap W \\ &= 1 + 2 - 0 \\ &= 3\end{aligned}$$

Hence  $L + W = \mathbb{R}^3$ .

**Problem:** Suppose  $\dim V = n$ , and  $U, W$  subspaces, each of dimension more than  $\frac{n}{2}$ . Prove that  $U \cap W \neq \{\vec{0}\}$ .

**Proof** By contradiction. Suppose  $U \cap W = \{\vec{0}\}$ . So  $\dim U \cap W = 0$ .

Then

$$\begin{aligned}\dim(U + W) &= \dim U + \dim W - \dim U \cap W \\ &> \frac{n}{2} + \frac{n}{2} - 0 = n\end{aligned}$$

Says  $U + W$  is a subspace of  $V$  of dim more than  $\dim V$ . Impossible, so  $U \cap W \neq \{\vec{0}\}$ .

**END OF MIDTERM MATERIAL.**

*February 8th 2019*

**Monday:** No class, office hours during class time. Tuesday night : Midterm!

*Linear transformations - Definition and basic properties*

(Chap. 5 in the text) **Def.** Let  $U, V$  be vector spaces, both over field  $K$ . A function  $T : U \rightarrow V$  is called a *linear transformation* if

- (i)  $\forall u_1, u_2 \in U \quad T(u_1 + u_2) = T(u_1) + T(u_2)$ . The first '+' is in  $U$ , while the second '+' is in  $V$ . The vectors spaces need not be related in any way, except that they must be over the same field of scalars.
- (ii)  $\forall u \in U, c \in K \quad T(cu) = cT(u)$ . Again, the first scalar multiplication happens in  $U$ , while the second scalar multiplication happens in  $V$ .

**Comment:** Linear transformations are the functions that are somehow "compatible" with the vector space operations.

**Ex:** Prove that  $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$  defined by

$$T(ax^2 + bx + c) = \begin{pmatrix} a+b \\ b+c \end{pmatrix}$$

**Sol:**

(i) Let  $p_1(x) = a_1x^2 + b_1x + c_1$ ,  $p_2(x) = a_2x^2 + b_2x + c_2$  be in  $P_2(x)$ .

Then,

$$\begin{aligned} T(p_1(x) + p_2(x)) &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2) \\ &= \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ b_1 + b_2 + c_1 + c_2 \end{pmatrix} \\ T(p_1(x)) + T(p_2(x)) &= \begin{pmatrix} a_1 + b_1 \\ b_1 + c_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ b_2 + c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 + a_2 + b_2 \\ b_1 + c_1 + b_2 + c_2 \end{pmatrix} \end{aligned}$$

(ii) Let  $p(x) = ax^2 + bx + c \in P_2(\mathbb{R}), d \in K$ .

$$\begin{aligned} T(dp(x)) &= T(dax^2 + dbx + dc) \\ &= \begin{pmatrix} da + db \\ db + dc \end{pmatrix} \\ &= d \begin{pmatrix} a + b \\ b + c \end{pmatrix} \\ &= dT(ax^2 + bx + c) \\ &= dT(p(x)) \end{aligned}$$

So  $T$  is a linear transformation.

**Ex** Define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(x, y) = (x^2, x + y)$ . Show that  $T$  is not a linear transformation.

**Sol** Try  $u = (2, 3), v = (3, 4)$ .

$$\begin{aligned} T(u + v) &= T(5, 7) \\ &= (25, 12) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(2, 3) + T(3, 4) \\ &= (4, 5) + (9, 7) \\ &= (13, 12) \\ &\neq (25, 12) \end{aligned}$$

So  $T$  is *not* linear.

**Ex:** Define  $\frac{d}{dx} : P(\mathbb{R}) \rightarrow P(\mathbb{R})$  by

$$\frac{d}{dx} p(x) = p'(x) \quad (\text{derivative})$$

Then  $\frac{d}{dx}$  is a linear transformation, since we know from calculus that

$$\begin{aligned} \frac{d}{dx}(p(x) + q(x)) &= \frac{d}{dx}p(x) + \frac{d}{dx}q(x) \\ \frac{d}{dx}(cp(x)) &= c\frac{d}{dx}p(x) \quad (c \in \mathbb{R}) \end{aligned}$$

**Proposition 22.** Let  $T : U \rightarrow V$  be a linear transformation. Then,

(i)  $T(\vec{0}) = \vec{0}$  (where the first  $\vec{0}$  is the zero vector of  $U$  and the second is the zero vector of  $V$ )

(ii)  $\forall u_1, u_2, \dots, u_n \in U$  and  $c_1, c_2, \dots, c_n \in K$ ,

$$\begin{aligned} T(c_1u_1 + c_2u_2 + \dots + c_nu_n) &= \\ c_1T(u_1) + c_2T(u_2) + \dots + c_nT(u_n) \end{aligned}$$

*Proof.* (i)

$$\begin{aligned} T(\vec{0}_U) &= T(\vec{0}_U + \vec{0}_U) \\ T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{T linear}) \\ \vec{0}_V + T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{A2}) \\ \vec{0}_V &= T(\vec{0}_V) \quad (\text{cancellation law}) \end{aligned}$$

(ii)

$$\begin{aligned} T(c_1u_1 + (c_2u_2 + \dots + c_nu_n)) &= T(c_1u_1) + T(c_2u_2 + \dots + c_nu_n) \\ &\quad (\text{T linear}) \\ &= c_1T(u_1) + T(c_2u_2 + \dots + c_nu_n) \\ &\quad (\text{T linear}) \\ &= \dots \quad (\text{proof by induction}) \\ &= c_1T(u_1) + \dots + c_nT(u_n) \end{aligned}$$

□

**Proposition 23.** Let  $T : U \rightarrow V$  function ( $U, V$  vector spaces). Then,

$T$  is linear transformation  $\iff$

$$\forall u_1, u_2 \in U \text{ } c \in K, T(cu_1 + u_2) = cT(u_1) + T(u_2)$$

**Proof:** Exercise. □

February 15th 2019

**Def** ("matrix defines a linear transformation") Let  $A \in M_{m \times n}(K)$ .

Define a function  $L_A : K^n \rightarrow K^m$  by

$$L_A(v) = Av \quad (\text{A an } m \times n \text{ matrix, v } n \times 1)$$

ie multiply matrix by vector.

**Proposition 24.**  $L_A$  is a linear transformation.

*Proof.* Let  $u, v \in K^n, c \in K$ . Then

$$\begin{aligned} L_A(cu + v) &= A(cu + v) \\ &= A(cu) + Av \quad (\text{prop of matrix multiplication}) \\ &= cAu + Av \\ &= cL_A(u) + L_A(v) \end{aligned}$$

□

**Ex**  $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix}, L_A : R^3 \rightarrow R^2$ . Calculate:

$$\begin{aligned} L_A(1, 3, -2) &= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 - 3 - 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -5 \end{pmatrix} \end{aligned}$$

**Spoiler:** All linear transformations between finite-dim vector spaces can be described in this way, "matrix transformation".

*Two special linear transformations*

- (1) **Zero transformations:**  $0 : V \rightarrow W$  defined by  $0(v) = \vec{0}$  ( $\vec{0}$  of  $W$ ) for all  $v \in V$ .
- (2) **Identity transformation,**  $I : V \rightarrow V$  (same vector space)  $I(v) = v$  for all  $v \in V$

Both are linear transformations (exercise).

### Kernel and Image (ch. 5.4)

**Def** Let  $T : V \rightarrow W$  be a linear transformation. Define:

- (i) **Kernel or nullspace** of  $T$ ,

$$\text{Ker}(T) = \{v \in V | T(v) = \vec{0}\}$$

**Note:** Always one vector which satisfies this.

- (ii) **Image** of  $T$  is

$$\text{Im}(T) = \{w \in W | \exists v \in V \ w = T(v)\}$$

**Note:**  $\text{Ker}(T) \subseteq V$ ,  $\text{Im}(T) \subseteq W$ .

**Ex** Define  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T(x, y) = (x, 0) \quad (\text{"proj onto x-axis"})$$

Then

$$\begin{aligned} \text{Ker}(T) &= \{(x, y) \in \mathbb{R}^2 | T(x, y) = (0, 0)\} \\ &= \{(0, y) | y \in \mathbb{R}\} \\ &= \text{"y-axis"} \\ \text{Im}(T) &= \{(x, y) \in \mathbb{R}^2 | (x, y) = T(x', y') \text{ some } x', y' \in \mathbb{R}\} \\ &= \{(x, 0) | x \in \mathbb{R}\} \\ &= \text{"x-axis"} \end{aligned}$$

**Ex** Define  $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$  to be derivative,  $D(f(x)) = f'(x)$ . Find kernel and image of  $D$ .

**Sol** We have

$$\begin{aligned} \text{Ker}(D) &= \{f \in P_n(\mathbb{R}) | f'(x) = 0\} \\ &= \text{const. polys} \\ &= \{a | a \in \mathbb{R}\} \\ &= P_0(\mathbb{R}) \end{aligned}$$

Claim  $\text{Im}(D) = P_{n-1}(\mathbb{R})$ .

*Proof.* Prove inclusion " $\subseteq$ " and " $\supseteq$ ".

- (i) " $\subseteq$ " Let  $f(x) \in \text{Im}(D)$ . Then  $\exists g(x) \in P_n$  s.t.  $f(x) = D(g(x)) = g'(x)$ . Since  $\deg(g) \leq n$ ,  $\deg(f) = \deg(g') \leq n-1$  (property of differentiation). So  $f(x) \in P_{n-1}$ .
- (ii) " $\supseteq$ " Let  $f(x) \in P_{n-1}$ . Need to find  $g(x) \in P_n$  such that  $D(g(x)) = g'(x) = f(x)$ . Set  $g(x) = \int f(x) dx$ . Know from calculus that the degree of  $g$  is one higher, ie

$$\deg(g(x)) = 1 + \deg(f(x))$$

So  $\deg(g) \leq n$ . So  $g(x) \in P_n$  and  $g'(x) = f(x)$  (calculus).

□

**Theorem 25.** Let  $T : V \rightarrow W$  be linear transformation. Then,

$$(i) \ Ker(T) \leq V$$

$$(ii) \ Im(T) \leq W$$

Ie they are subspaces.

*Proof.* By direct proof.

(i)  $T(\vec{0}) = \vec{0}$  always (lin transform) so  $\vec{0} \in Ker(T)$ . Let  $v_1, v_2 \in Ker(T), c \in K$ . We know  $T(v_1) = \vec{0}, T(v_2) = \vec{0}$ . Then

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) && (\text{T linear}) \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

Hence  $cv_1 + v_2 \in Ker(T)$ . So  $Ker(T) \subseteq V$  (we already knew  $Ker(T) \subseteq V$ )

(ii)  $T(\vec{0}) = \vec{0}$ , hence  $\vec{0}_w = T(\text{something})$ , ie  $\vec{0}_w \in Im(T)$ . Let  $w_1, w_2 \in Im(T), c \in K$ . We know  $w_1 = T(v_1), w_2 = T(v_2)$  for some  $v_1, v_2 \in V$ . Then

$$\begin{aligned} cw_1 + w_2 &= cT(v_1) + T(v_2) \\ &= T(cv_1 + v_2) && (\text{T linear}) \end{aligned}$$

Hence  $cw_1 + w_2 \in Im(T)$ . So  $Im(T) \leq W$ .

□

**Def**  $T : V \rightarrow W$  linear. The *nullity* of  $T$  is  $\dim Ker(T)$  ( $\dim$  nullspace). The *rank* of  $T$  is  $\dim Im(T)$ .

**Note:**  $Ker(T) \leq V$  so  $\text{nullity}(T) \leq \dim V$ ,  $Im(T) \leq W$  so  $\text{rank}(T) \leq \dim W$ .

**Ex** In  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , proj onto x-axis,

$$\begin{aligned} Ker(T) &= y-axis && (\text{so } \text{nullity}(T) = 1) \\ Im(T) &= x-axis && (\text{so } \text{rank}(T) = 1) \end{aligned}$$

**Ex 2** For  $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ , differentiation.

$$\begin{aligned} Ker D &= P_0(\mathbb{R}) && (\text{so } \text{nullity}(D) = 1) \\ Im D &= P_{n-1} && (\text{so } \text{rank}(D) = n) \end{aligned}$$

February 18th 2019

**Notation** For set  $S = \{v_1, v_2, \dots, v_n\}$ ,  $T : V \rightarrow W$  denotes  $T(S) = \{T(v_1), T(v_2), \dots, T(v_n)\}$ .

**Proposition 26.**  $T : V \rightarrow W$  linear and  $V = \text{span}(S)$ . Then  $\text{Im } T = \text{span}(T(S))$ . In particular, if  $B$  basis of  $V$ ,  $T(B)$  spans  $\text{Im } (T)$  (but need not be a basis).

*Proof.* By direct proof.

(i) " $\subseteq$ ". Let  $w \in \text{Im}(T)$ , ie  $w = T(v)$ , some  $v \in V$ . Since  $S$  spans  $V$ ,  $v = \sum_{i=1}^n a_i v_i$ , some  $v_i \in S$ . So

$$\begin{aligned} w &= T(v) = T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \quad (T(v_i) \in T(S), \text{ by T linear}) \end{aligned}$$

All of which is  $\in \text{span}(T(S))$ .

(ii) " $\supseteq$ " Let  $w \in \text{span } T(S)$ . So

$$\begin{aligned} w &= \sum_{i=1}^n a_i T(v_i) \quad (\text{for some vectors } v_i \in S) \\ &= T\left(\sum_{i=1}^n a_i v_i\right) \quad (\text{T linear}) \\ &= T(\text{something}) \quad (\text{so } w \in \text{Im}(T)) \end{aligned}$$

□

**Ex** Define  $T : P_2(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$  by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Exercise: T is linear. Find basiss for  $\text{Im } T$ .

**Sol** Take basis  $\{1, x, x^2\}$  for  $P_2$ . Calculate

$$\begin{aligned} T(1) &= \begin{pmatrix} 1-1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ T(x) &= \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \\ T(x^2) &= \begin{pmatrix} 1-4 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

So  $\text{Im } T = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ .

Basis for  $\text{Im } T$  is  $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

(so  $\text{Im } T = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{R}\right\}$ )

Note: The next theorem is very important!

**Theorem 27.** ("Dimension theorem") Let  $T : V \rightarrow W$  linear with  $V$  finite-dimensional. Then,

$$\dim V = \dim \ker(T) + \dim \text{Im}(T)$$

$$\dim V = \text{nullity}(T) + \text{rank}(T)$$

**Note**  $\dim W$  is not involved.

*Proof.* Let  $B = \{v_1, v_2, \dots, v_k\}$  be basis  $\text{Ker } T$  (so  $k = \dim \text{Ker } T$ ). Let  $n = \dim V$ . Note  $T(v_i) = 0$ , ( $i = 1, 2, \dots, k$ ). Let  $S$  span  $V$ .

Plan: extend  $B$  to basis of  $V$ , show  $T(\text{extra vector}) = \text{basis of Im}$ .

By theorem 20-1, there exists  $E \subseteq S$  such that  $B \cup E$  is a basis of  $V$ .

Denote

$$E = \{v_{k+1}, \dots, v_n\} \quad (\text{note } n = \dim V, |E| = n - k)$$

Claim  $T(E)$  is basis for  $\text{Im } T$ .

(i)  $T(E)$  spans  $\text{Im } T$

(a) " $\subseteq$ " is clear since  $T(E) \subseteq \text{Im } T$  by definition. So  $\text{span } T(E) \leq \text{Im } T$  ( $\text{Im } T \leq W$ )

(b) " $\supseteq$ " Let  $w \in \text{Im}(T)$ , ie  $w = T(v)$ , some  $v \in V$ . Since  $B \cup E$  is a basis,  $v = \sum_{i=1}^n a_i v_i$ . Then,

$$\begin{aligned} w &= T\left(\sum_{i=1}^n a_i v_i\right) \\ &= \sum_{i=1}^n a_i T(v_i) \quad (\text{T linear}) \\ &= \sum_{i=k+1}^n a_i v_i \quad (\text{Since } T(v_i) = 0 \text{ for } i = 1, 2, \dots, k) \end{aligned}$$

Hence  $w \in \text{span}(T(E))$ , since  $E = \{v_{k+1}, \dots, v_n\}$

(ii)  $T(E)$  is linearly independent. Suppose

$$\sum_{i=k+1}^n b_i T(v_i) = \vec{0} \quad (\text{linear comb vectors in } T(E))$$

So by linearity of  $T$ ,

$$T\left(\sum_{i=k+1}^n b_i v_i\right) = \vec{0}$$

So  $\sum_{i=k+1}^n b_i v_i \in \text{Ker } T$ , ie is linear comb of  $B$

$$\text{So } \sum_{i=k+1}^n b_i v_i = \sum_{i=1}^k b_i v_i$$

ie  $\sum_{i=1}^k (-b_i)v_i + \sum_{i=k+1}^n b_i v_i = \vec{0}$  is linear comb of  $v_1, \dots, v_n$  (ie  $B \cup E$ ) but these independent. So all  $b_i = 0$ , hence  $T(E)$  independent.

Conclude  $T(E)$  basis of  $\text{Im } T$ . So,

$$\dim \text{Im } T = |T(E)| = |E| = n - k$$

So,

$$n = k + n - k$$

$$\dim V = |B| + |T(E)| = \dim \text{Ker } T + \dim \text{Im } T$$

□

Why is  $|T(E)| = |E|$ ? True unless

$$T(v_i) = T(v_j) \quad (\text{for some } i, j \geq k+1, i \neq j)$$

If so,

$$\begin{aligned} T(v_i) - T(v_j) &= 0 \\ T(v_i - v_j) &= 0 \quad (\text{so } v_i - v_j \in \text{Ker } T) \end{aligned}$$

Hence  $v_i - v_j = \sum_{l=1}^n a_l v_l$ , dep relation on  $v_1, \dots, v_n$ . Impossible. □

**Problem** For  $T : P_2 \rightarrow \mathcal{M}_{2 \times 2}$ ,

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Find basis for  $\text{Ker } T$ .

**Sol** Already know  $\dim \text{Im } T = 2$  (last ex). So

$$\begin{aligned} \dim P_2 &= \dim \text{Ker } T + \dim \text{Im } T \\ 3 &= \dim \text{Ker } T + 2 \end{aligned}$$

So  $\text{Ker } T$  is 1-dimensional. Only need to find one non-zero  $f(x)$  s.t.

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

ie need  $f(1) = f(2)$  and  $f(0) = 0$ . For example,  $f(x) = x^2 - 3x$  works. So  $\{x^2 - 3x\}$  is a basis for  $\text{Ker } T$  (or,  $f(x) = ax^2 + bx + c$ ,  $f(1) = a + b + c = f(2) = 4a + 2b + c$ ,  $f(0) = 0 = c$ , solve)

February 20th 2019

Comments on dimension theorem

$T : V \rightarrow W$ , linear.

$$\dim V = \dim (\text{Im } T) + \dim (\text{Ker } T)$$

Left-hand part of the sum: Dimensions that are preserved ("saved") by  $T$ . Right-hand part: dimensions that are "lost" when you apply  $T$ .

**Dimension:** Subspaces are *infinite* sets (except  $\{\vec{0}\}$ ). Dimension gives a way to compare the *sizes* of subspaces.

Injective/surjective transformation (ch. 5.5.)

**Def** Let  $f : X \rightarrow Y$  be a *function* ( $X, Y$  sets).

(i)  $f$  is *surjective* ("onto") if

$$\forall y \in Y \ \exists x \in X \ f(x) = y$$

(equivalently, the image of  $f$  is  $Y$ )

(ii)  $f$  is called *injective* (or "on-to-one") if

$$\forall x_1, x_2 \in X (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

(equivalently,  $\forall x_1, x_2 \in X (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$ )

**Theorem 28.** ("How to check if  $T$  inj/surj") Let  $T : V \rightarrow W$ . Then,

(i)  $T$  injective  $\iff \text{Ker}(T) = \{\vec{0}\}$  (nullity  $(T) = 0$ )

(ii)  $T$  surjective  $\iff \dim (\text{Im } T) = \dim W$  (rank( $T$ ) = dim  $W$ )

(i) *Proof.* By direct proof.

- (1) " $\Rightarrow$ " Assume  $T$  inj. (know  $\{0\} \leq \text{Ker } T$ ). Let  $v \in \text{Ker } (T)$ . So  $T(v) = \vec{0}$ . But also  $T(\vec{0}) = \vec{0}$ , so  $T(v) = T(\vec{0})$  hence  $v = \vec{0}$  since  $T$  is injective.
- (2) " $\Leftarrow$ " Assume  $\text{Ker } T = \{\vec{0}\}$ . Let  $v_1, v_2 \in V$ . Suppose  $T(v_1) = T(v_2)$  (prove  $v_1 = v_2$ ).

$$\begin{aligned} T(v_1) - T(v_2) &= \vec{0} \\ T(v_1 - v_2) &= \vec{0} \end{aligned} \quad (\text{linear})$$

So  $v_1 - v_2 \in \text{Ker } T = \{\vec{0}\}$ . So  $v_1 - v_2 = \vec{0}, v_1 = v_2$ .

□

(ii) *Proof.* By direct proof.

- (1) " $\Rightarrow$ " Assume  $T$  is surjective, that is  $\text{Im } T = W$ . Hence  $\dim \text{Im } T = \dim W$ .
- (2) " $\Leftarrow$ " Assume  $\dim \text{Im } T = \dim W$ . But  $\text{Im } T \leq W$ , hence  $\text{Im } T = W$  (by thm 2o-2)

□

**Problem** Define  $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$T(f(x)) = \int_0^1 f(x)dx$$

(Exercise:  $T$  is linear). Is  $T$  injective? Surjective?

**Sol Dim Thm:**

$$\begin{aligned} \dim P_2 &= \dim \text{Im } T + \dim \text{Ker } T \\ 3 &= \dim \text{Im } T + \dim \text{Ker } T \end{aligned}$$

Hence  $\text{Im } T \leq \mathbb{R}^1$ , so  $\text{Im } T = \{\vec{0}\}$  or  $\mathbb{R}$ . It is *not*  $\{\vec{0}\}$  since  $\int_0^1 1dx = 1 \neq 0$ ,  $T(1) \neq 0$ . Hence  $\text{Im } T = \mathbb{R}$  so

$$3 = 1 + \dim \text{Ker } T$$

So  $\dim \text{Ker } T = 2$ .  $\text{Ker } T \neq \{\vec{0}\}$  not injective.  $\text{Im } T = \mathbb{R}$  is surjective.

**Theorem 29.** ("shortcut when dim same")  $T : V \rightarrow W$  linear, and  $\dim V = \dim W$ . Then,

$$T \text{ injective} \iff T \text{ surjective}$$

*Proof.* Dim Thm:

$$\dim W = \dim V = \dim \text{Im } T + \dim \text{Ker } T$$

If  $T$  inj,  $\dim \text{Ker } T = 0$ . So

$$\dim W = \dim \text{Im } T + 0$$

So  $T$  surjective (thm 28). If  $T$  surj,  $\dim \text{Im } T = \dim W$  (thm 28), so

$$\dim W = \dim W + \dim \text{Ker } T$$

So  $\dim \text{Ker } T = 0$  so  $\text{Ker } T = \{\vec{0}\}$

□

**Problem**  $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$ , defined by

$$T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$$

Is  $T$  injective? Surjective?

**Sol** Same  $\dim (= 3)$ . Check only one. Check surjective directly from def surj:

Let  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ . Is  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T(f(x))$ , some  $f(x) \in P_2$ ?

That is, given  $a, b, c \in \mathbb{R}$ , is there a degree 2 polynomial such that  $f(0) = a, f(1) = b, f(2) = c$ ? By Lagrange Interpolation,  $f(x)$  exists ( $\deg = 1$ , less than # of points). So  $T$  surj, so also inj.

*Isomorphism and coordinates (ch 5.5, 4.11 and 4.12)*

**Def:** (Isomorphism)

- (1) If  $T : V \rightarrow W$  (linear) is injective and surjective, it is called an *isomorphism*.
- (2) If  $V, W$  vector spaces and *there exists* an isomorphism  $T : V \rightarrow W$ , we say  $V$  and  $W$  are *isomorphic* and write  $V \simeq W$

**Note** A function that is injective and surjective is called *bijective*.

**Ex**  $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^3$ ,  $T(f(x)) = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$  is an isomorphism (last ex.)

so  $P_2(\mathbb{R}) \simeq \mathbb{R}^3$

**Ex** Prove that

$$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is isomorphism  $P_2 \rightarrow \mathbb{R}^3$ .

**Sol**  $T$  is linear : let  $f(x), g(x) \in P_2(\mathbb{R}), d \in \mathbb{R}$ . Then,

$$\begin{aligned} T(df + g) &= T(c(a_1x^2 + b_1x + c_1) + (a_2x^2 + b_2x + c_2)) \\ &= T((da_1 + a_2)x^2 + (db_1 + b_2) + (dc_1 + c_2)) \\ &= \begin{pmatrix} da_1 + a_2 \\ db_1 + b_2 \\ dc_1 + c_2 \end{pmatrix} \\ &= d \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \\ &= dT(f) + T(g) \end{aligned}$$

So  $T$  linear. Same  $\dim (= 3)$ . Check surj. Let  $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ . Then

$T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ , hence surj., hence inj., hence *isomorphism*.

February 22nd 2019

Notes about functions

- (1) If  $f : X \rightarrow Y$ , then  $f$  injective and surjective  $\iff f$  is invertible,  
ie  $\exists f^{-1} : Y \rightarrow X$  such that  $\forall x \in X, y \in Y$   $f^{-1}(f(x)) = x$  and  
 $f(f^{-1}(y)) = y$
- (2) If  $g : Y \rightarrow Z$ , you can compose  $f$  and  $g$  to get  $g \cdot f : X \rightarrow Z$ , defined  
by  $(g \cdot f)(x) = g(f(x))$   $x \xrightarrow{f} y \xrightarrow{g} z$

**Theorem 30.** Let  $T : V \rightarrow W$  be an isomorphism (ie  $T$  linear, inj, surj.).

Then  $T$  has an inverse  $T^{-1} : W \rightarrow V$  which is also a linear transformation.

*Proof.* Fact that  $T^{-1}$  exists is since  $T$  inj and surj. Prove  $T^{-1}$  is linear.

Let  $w_1, w_2 \in W, c \in K$ . Since  $T$  surjective,  $w_1 = T(v_1), w_2 = T(v_2)$  for some  $v_1, v_2 \in V$ . Also,  $T^{-1}(w_1) = T^{-1}(T(v_1)) = v_1$  and  $T^{-1}(w_2) = v_2$ . Then

$$\begin{aligned} T^{-1}(cw_1 + w_2) &= T^{-1}(cT(v_1) + T(v_2)) \\ &= T^{-1}(T(cv_1 + v_2)) \quad (\text{T linear}) \\ &= cv_1 + v_2 \\ &= cT^{-1}(w_1) + T^{-1}(w_2) \end{aligned}$$

So  $T^{-1}$  linear.  $\square$

Ex

$$\begin{aligned} T : P_2(\mathbb{R}) &\rightarrow \mathbb{R}^3, T(ax^2 + bx + c) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ T^{-1} : \mathbb{R}^3 &\rightarrow P_2(\mathbb{R}), T^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (ax^2 + bx + c) \end{aligned}$$

**Point** Once you know  $V \simeq W$  (isomorphic) you can go back and forth between them, do vector space operations in either  $V$  or  $W$ . That is,  $V$  and  $W$  have exactly the same *structure* (as far as addition and scalar multiplication are concerned), even though "vectors" look different.

**Proposition 31.** If  $V \simeq W$ , both finite-dimensional, then  $\dim V = \dim W$

*Proof.*  $V \simeq W$  so  $\exists T : V \rightarrow W$ ,  $T$  inj and surj (bijective), linear. So Dim Thm,

$$\dim V = \dim \text{Im } T + \dim \text{Ker } T$$

and  $T$  inj., so  $\dim \text{Ker } T = 0$ , and  $T$  surj., so  $\text{Im } T = W$ , so

$$\dim V = \dim W + 0$$

□

**Theorem 32.** Let  $B = \{v_1, v_2, \dots, v_n\}$  be a basis of  $V$ . For any  $v \in V$ , you can write

$$v = \sum_{i=1}^n a_i v_i$$

Then,

- (a) The numbers  $(a_1, a_2, \dots, a_n)$  are unique and are called the coordinates of  $v$  relative to  $B$ , denoted

$$[v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- (b) The function  $C_B : V \rightarrow K^n$  defined by

$$C_B(v) = [v]_B = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad (\text{"find coordinates"})$$

is an isomorphism

Hence, if  $\dim V = n$  then  $V \simeq K^n$

*Proof.* By direct proof.

- (a) Assume  $v$  can also be written as

$$v = \sum_{i=1}^n b_i v_i \quad (\text{as well as } \sum a_i v_i = v)$$

Then

$$\begin{aligned}\vec{0} &= v - v = \left( \sum_{i=1}^n a_i v_i \right) - \left( \sum_{i=1}^n b_i v_i \right) \\ \vec{0} &= \sum_{i=1}^n (a_i - b_i) v_i\end{aligned}$$

Since  $\{v_1, \dots, v_n\}$  independent ( $B$  = basis) all  $a_i - b_i = 0$  ( $i = 1, 2, \dots, n$ ) so  $a_1 = b_1$ . Hence representation is *unique*.

(b) Let  $v = \sum_{i=1}^n a_i v_i, u = \sum_{i=1}^n b_i v_i$  be in  $V, c \in K$ . Then,

$$\begin{aligned}C_B(cv + u) &= C_B\left(\sum_{i=1}^n (ca_i + b_i)v_i\right) \\ &= \begin{pmatrix} ca_1 + b_1 \\ ca_2 + b_2 \\ \vdots \\ ca_n + b_n \end{pmatrix} \\ &= c \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= C_B(v) + C_B(u)\end{aligned}$$

Hence  $C_B$  is linear. To check  $C_B$  inj. and surj., since  $\dim V = n = \dim K^n$ , need only check on (other will follow). We will prove surj.

Let  $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in K^n$ . Then let  $v = \sum_{i=1}^n a_i v_i$ , so  $C_B(v) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

□

### Remarks

- (1) We need the coords to be *unique* in order for  $C_B : V \rightarrow K^n$  to be a (well-defined) function.
- (2) If you use a different basis, or even same basis but in different order, you get different coords and also different isomorphism.

Always infinitely many isomorphisms

**Lemma 33.** Let  $T : V \rightarrow W, S : W \rightarrow U$  be a linear transformation. Then

- (a)  $S \cdot T : V \rightarrow U$  ( $Vt \rightarrow Ws \rightarrow U$ ) is linear
- (b) If  $T, S$  both injective (surjective), then  $S \cdot T$  is also injective (surjective)

*Proof.* Exercise. □

**Theorem 34.** Let  $V, W$  be finite-dimensional vector spaces over field  $K$ .

Then,

$$V \simeq W \iff \dim V = \dim W$$

That is, as far as vector space ops go, only the dimension really matters.

*Proof.* By direct proof.

- " $\Rightarrow$ " Prop 31.
- " $\Leftarrow$ "  $\dim V = \dim W = n$ . By Thm 32,  $V \simeq K^n, W \simeq K^n$ , using  $C_{B_1} : V \rightarrow K^n, C_{B_2} : W \rightarrow K^n$ . Then  $C_{B_2}^{-1} : K^n \rightarrow W$  is an isomorphism (Thm 30), so

$$C_B^{-1} \cdot C_B : V \rightarrow W \quad (V \xrightarrow{C_{B_1}} K^n \xrightarrow{C_{B_2}^{-1}} W)$$

is linear, injective, surjective by lemma 33 so it is an isomorphism.

□

February 25th 2019

Recall  $V \simeq W \iff \dim V = \dim W$  (proved for finite-dim vector spaces only).

**Note:** If  $T : V \rightarrow W$  isomorphism,  $T^{-1} : W \rightarrow V$  is also an isomorphism.

Examples of isomorphisms:

- $P_n(K) \simeq K^{n+1}$
- $\mathcal{M}_{m \times n} \simeq K^{mn}$
- $K^n \simeq K^m \iff n = m$

**Question** If  $n = \dim V$ , then  $V \simeq K^n$ , why bother studying vector spaces other than  $K^n$ ?

**Answer** If you only want to know about addition and scalar multiplication, only  $K^n$  matters but the "vectors"  $P_n, \mathcal{M}_{n \times m}$  etc... have other properties not always related to vector space operations.

For example, in  $P_2(\mathbb{R})$  we can evaluate polynomials  $f(x)$  at say  $x = 3$ ,

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(3) &= 9a + 3b + c \end{aligned}$$

If we consider  $P_2(\mathbb{R}) \simeq \mathbb{R}^3$ , "eval at  $x = 3$ " is a linear transformation:

$$\begin{aligned} T : \mathbb{R}^3 &\rightarrow \mathbb{R} \\ T(a, b, c) &= 9a + 3b + c \end{aligned}$$

*Computations related to linear transformation*

**Theorem 35** (T is determined by its value on a basis). Let  $V, W$  be vector spaces,  $\{v_1, v_2, \dots, v_n\}$  basis  $V$ .

Let  $w_1, w_2, \dots, w_n \in W$  be any vectors (need not be distinct). Then there is one linear transformation  $T : V \rightarrow W$  s.t.  $T(v_i) = w_i$

**Idea of proof** If you want to calculate  $T(v)v \in V$  (arbitrary element), write  $v$  uniquely in terms of basis

$$v = \sum_{i=1}^n a_i v_i$$

Then since  $T$  is supposed to be linear, compute

$$\begin{aligned} T(v) &= T(\sum a_i v_i) \\ &= \sum a_i T(v_i) \\ &= \sum a_i w_i \end{aligned}$$

**Problem** Suppose  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is linear and

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find  $T\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ .

**Solution**  $\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\}$  = basis  $\mathbb{R}^2$ , should have enough info to know what  $T$  is. Need to find

$$\begin{aligned} \begin{bmatrix} 3 \\ 4 \end{bmatrix} &= x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 4 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

So

$$\begin{aligned} T\begin{pmatrix} 3 \\ 4 \end{pmatrix} &= T\left(\frac{7}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\ &= \frac{7}{2}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}T\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \frac{7}{2}\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

*Row, column, nullspace of a matrix*

**Def**  $A \in \mathcal{M}_{m \times n}(K)$

1. The row space,  $\text{row}(A)$  is the span of the rows of  $A$ . Subspace of  $K^n$
2. The column space,  $\text{col}(A)$  is span of columns. Subspace of  $K^n$
3. Nullspace( $\ker$ ), is the solution set to the homogeneous system  $Ax = \vec{0}$ . Subspace of  $K^n$

**Proposition 36.** Let  $A \in \mathcal{M}_{m \times n}(K)$ . Then

- (1)  $A_{ei} = \text{column } i \text{ of } A$
- (2) If  $B \in \mathcal{M}_{n \times p}(K)$  then column  $i$  of  $AB$  is  $Ab_i$ ,  $b_i = \text{column } i \text{ of } B$

*Proof.* Proof by picture!

□

i) 
$$\left( \begin{array}{cc|c} 1 & 2 & \\ 2 & 1 & \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) = \left( \begin{array}{c} 2 \\ 1 \\ \vdots \\ 2 \end{array} \right)$$

ii) 
$$\left( \begin{array}{ccc|c} 1 & 2 & 3 & \\ 2 & 1 & 3 & \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 2 \\ 1 \\ 3 \\ 2 \end{array} \right)$$

**Proposition 37.** Let  $A \in \mathcal{M}_{m \times n}(K)$ , so  $L_A : K^n \rightarrow K^m$ .

- (1)  $\ker(A) = \text{Ker}(L_A)$
- (2)  $\text{col}(A) = \text{Im}(L_A)$
- (3)  $\text{row}(A) = \text{Im}(L_{A^T})$

*Proof.* By direct proof.

(1)

$$\begin{aligned} \text{Ker}(A) &= \{x \in K^n | A_x = \vec{0}\} \\ &= \{x \in K^n | L_A(x) = \vec{0}\} \\ &= \text{Ker}(A) \end{aligned}$$

(2) Take basis  $\{e_1, e_2, \dots, e_n\}$  for  $K^n$ . Then by prop 26,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots$$

$$L_A(e_1) \dots L_A(e_n) \text{ spans } \text{Im}(L_A)$$

But  $L_A(e_1) = A_{ei} = \text{column } i \text{ of } A$ , ie columns of  $A$  span  $\text{Im}(L_A)$   
hence  $\text{col}(A) = \text{Im}(L_A)$

(3)  $\text{col}(A) = \text{col}(A^T) = \text{Im}(L_{A^T})$  by (2)

□

**Def:** Rank of  $A \in \mathcal{M}_{m \times n}(K)$  is number of non-zero rows in RREF.**Proposition 38.** Let  $A \in \mathcal{M}_{m \times n}(K), R = \text{RREF}(A)$ . Then,

- (i)  $\text{rank}(A) = \text{rank}(A^T)$
- (ii)  $\text{rank}(A) = \dim \text{row}(A)$
- (iii)  $\dim \text{row}(A) = \dim \text{col}(A)$
- (iv) There is an invertible matrix  $B \in \mathcal{M}_{m \times n}(K)$  s.t.  $BA = R$

*Proof.* (iii) We have:

$$\begin{aligned} \dim \text{row}(A) &= \text{rank}(A) && \text{(by (ii))} \\ &= \text{rank}(A^T) && \text{(by (i))} \\ &= \dim \text{row}(A^T) && \text{(by (ii))} \\ &= \dim \text{col}(A) && \text{(by (iii))} \end{aligned}$$

□

February 27th 2019

**Theorem 39** (computing bases). Let  $A \in \mathcal{M}_{m \times n}(K)$ , let  $R$  be the reduced non-echelon form of  $A$ . Then,

- (i) The non-zero rows of  $R$  form a basis of  $\text{row}(A)$ .
- (ii) The columns of  $A$  which correspond to the pivot columns (columns containing a leading 1) form a basis of  $\text{col}(A)$ .
- (iii) The “basic solutions” obtained when solving  $Ax = \vec{0}$  form a basis for nullspace ( $\ker$ ) of  $A$ .

*Proof.* By direct proof.

- (i) Elementary row ops do not change the row space so  $\text{row}(A) = \text{row}(R)$ . Non-zero rows form basis because of form of  $R$ .
- (ii) Let  $w_1, w_2, \dots, w_r$  be the columns of  $R$  containing leading 1’s (pivot columns). Because of form of  $R$ , no other non-zero entries above/below a leading 1, so  $w_1, w_2, \dots, w_r$  are standard basis vectors (ie in  $\{e_1, e_2, \dots, e_m\}$ ). So,  $\{w_1, \dots, w_r\}$  are linearly independent. Let  $v_1, v_2, \dots, v_r$  be corresponding columns.

**Note**  $r = \text{rank}(A) = \dim \text{row}(A) = \dim \text{col}(A)$ .

Prove  $v_1, v_2, \dots, v_r$  are linearly independent. Suppose

$$\sum_{i=1}^n a_i v_i = \vec{0}$$

By proposition 38,  $\exists$  invertible  $M$  s.t.  $MA = R$ . Multiply by  $M$ :

$$\begin{aligned} M\left(\sum_{i=1}^r a_i v_i\right) &= M\vec{0} \\ &= \vec{0} \end{aligned}$$

So  $\sum_{i=1}^r a_i Mv_i = \vec{0}$ , but  $M$  (column  $i$  of  $A$ ) = col  $i$  of  $MA$  ie of  $R$  (prop 36). So,

$$\sum_{i=1}^r a_i w_i = \vec{0}$$

But  $\{w_1, \dots, w_r\}$  are independent. So all  $a_i = 0$ , so  $\{v_1, \dots, v_r\}$  independent so basis.

- (iii) Solve  $Ax = 0$ , obtain general solution,

$$\begin{aligned} \vec{x} &= x_1 v_1 + x_2 v_2 + \dots + x_s v_s \\ &= x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_s \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \end{aligned}$$

Where  $x_1, x_2, \dots, x_s$  free variables. Claim is that  $v_1, v_2, \dots, v_s$  form a basis for  $\ker(A)$ . They clearly span. Independent? In the  $x_1$  position, only  $v_i$  has a non-zero entry, so they are independent.

**Comment** The dimension of  $\ker(A)$  is therefore the number of free variables.

□

### Basis-finding problems

**Problem** Let  $W \subseteq M_{2 \times 2}(\mathbb{R})$ , where  $W$  consists of all  $A$  such that sum of entries in each row and column is the same. Find basis of  $W$ .

**Solution** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$ . So

$$a + b = c + d$$

$$a + c = b + d$$

$$a + b = a + c \quad (a + b = b + d \text{ etc are not needed})$$

Write as linear system:

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{So } a = d, b = c, c = c \text{ and } d = d. \text{ ie, } \vec{x} = c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

General solution,

$$\begin{aligned} A &= \begin{pmatrix} d & c \\ c & d \end{pmatrix} \\ &= c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Linearly independent by Thm 39 (kernel basis case). So

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad (\text{basis})$$

**Problem:** Let

$$\begin{aligned}P_1(x) &= 1 + 2x + 3x^2 - x^3 \\P_2(x) &= -1 + 3x + x^2 + x^3 \\P_3(x) &= 3 - 4x + x^2 - 3x^3 \\P_4(x) &= 1 + 7x + 7x^2 - x^3 \\P_5(x) &= 2 + 2x - x^2 - x^3\end{aligned}$$

Let  $W = \text{span}\{P_1(x), \dots, P_5(x)\} \leq P_3(\mathbb{R})$ . Find:

- (i) basis of  $W$  that is a subset of  $\{P_1(x), \dots, P_5(x)\}$
- (ii) basis of  $W$  consisting of polys of different degree.

**Sol** Isomorphism  $T : P_3 \rightarrow \mathbb{R}^4$ ,

$$T(d + cx + bx^2 + ax^3) = \begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix} \quad (\text{or } \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix})$$

- (i) Put the vectors as columns of a matrix,

$$A = \begin{pmatrix} 1 & -1 & 3 & 1 & 3 \\ 2 & 3 & -4 & 7 & 2 \\ 3 & 1 & 1 & 7 & -1 \\ -1 & 1 & -3 & -1 & -1 \end{pmatrix}$$

Find basis  $\text{col}(A)$ . Row-reduce to

$$R = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So columns 1, 2 and 5 of  $A$  form a basis for  $\text{col}(A)$ , which corresponds (using isomorphism  $T$ ) to  $W$ , so

$$\{P_1(x), P_2(x), P_5(x)\} \quad (\text{basis})$$

- (ii) Basis all diff degree. Use row space of a matrix. Put  $P_1, \dots, P_5$  as rows. But use isomorphism

$$d + cx + bx^2 + ax^3 \iff \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

So

$$A = \begin{pmatrix} -1 & 3 & 2 & 1 \\ 1 & 1 & 3 & -1 \\ -3 & 1 & -4 & 3 \\ -1 & 7 & 7 & 1 \\ -1 & -1 & 2 & 2 \end{pmatrix} \quad (\text{So } W \text{ corresponds to row space.})$$

$$\rightarrow = \begin{pmatrix} 1 & 0 & 0 & \frac{-27}{20} \\ 0 & 1 & 0 & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

First three rows form basis  $\text{row}(A)$ . As polynomials, we get

$$x^3 - \frac{27}{20}, x^2 - \frac{1}{4}, x + \frac{1}{5}$$

Which is basis of  $W$ , all of different degree. The choice of order was relevant since we knew in advance the general form the reduced form would take.

*March 1st 2019*

**Problem** Let

$$\begin{aligned} v_1 &= (1, 3, -1, 2, 0, 2) \\ v_2 &= (3, 3, 5, -4, -7, -5) \\ v_3 &= (2, 2, -1, 1, 2, 1) \\ w_1 &= (3, 1, -1, 0, 4, 0) \\ w_2 &= (3, 3, 1, 1, 1, -1) \\ w_3 &= (1, 1, -1, 2, 3, 1) \end{aligned}$$

Let  $V = \text{span}\{v_1, v_2, v_3\}$ ,  $W = \text{span}\{w_1, w_2, w_3\}$ . Find bases (and dimensions of)  $V + W$ ,  $V \cap W$ .

**Solution** Check that  $\{v_1, v_2, v_3\}$ ,  $\{w_1, w_2, w_3\}$  both independent (put into matrix as either rows or columns, verify  $\text{rank} = 3$ )

$V + W = \text{span}\{V \cup W\} = \text{span}\{v_1, v_2, v_3, w_1, w_2, w_3\}$ . For basis, put vectors as rows or columns, solve for row space or col space. I used columns, matrix reduces to

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis = cols 1, 2, 3, 5 of original matrix. So  $\{v_1, v_2, v_3, w_2\}$  so  $\dim(V + W) = 4$ .

Formula:

$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$$

$$4 = 3 + 3 - \dim(V \cap W)$$

So  $\dim(V \cap W) = 2$ .

$V \cap W$  is all  $u = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{R}^6$  such that  $u = x_1 v_1 + x_2 v_2 + x_3 v_3$  (\*) (ie  $u \in V$ ) and  $u = y_1 w_1 + y_2 w_2 + y_3 w_3$  (\*\*) (ie  $u \in W$ ) for some  $x_1, x_2, x_3, y_1, y_2, y_3$ . This is linear system. 12 variables, 12 equations (2 for each of 6 components):

$$z_1 = x_1 + 3x_2 + 2x_3 \quad (z_1\text{-component of } *)$$

$$z_2 = 3x_1 + 3x_2 + 2x_3 \quad (z_2\text{-component of } *)$$

...

And

$$z_1 = 3y_1 + 3y_2 + y_3 \quad (z_1\text{-component of } **)$$

...

$$z_6 = 0y_1 - y_2 + y_3 \quad (z_6\text{-component of } **)$$

Goal is to solve the system, need only  $u = (z_1, \dots, z_6)$ . Remember that:

$$\begin{pmatrix} z_1 \\ \dots \\ z_6 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ 3 \\ 5 \\ \dots \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 2 \\ \dots \end{pmatrix}$$

Rewrite as

$$z_1 - x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 = 0$$

$$z_2 - 3x_1 - 3x_2 - 2x_3 + 0y_1 + 0y_2 + 0y_3 = 0$$

...

Coefficient matrix: see fig 12

The form is

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6 \times 3} \\ I_6 & 0_{6 \times 3} & -w_1 - w_2 - w_3 \end{pmatrix}$$

(Coeff matrix)  $(-v_1)(-v_2)(-v_3)$

$$\left( \begin{array}{cccccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -3 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -3 & -3 & -2 & 0 & 0 & 0 \\ & & & & & & 1 & 1 & 1 & & & \\ & & & & & & 0 & 0 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \end{array} \right)$$

$\sim$   
 $\sim$   
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 $\sim$   
 $\sim$

$-w_1 \quad -w_2 \quad -w_3$

Figure 3: Coefficient matrix

Row-reduce, find basic solutions, each solution is in  $\mathbb{R}^{12}$  (12 variables), you only need first 6 components  $((z_1, z_2, \dots, z_6)) = u \in V \cap W$ .

Obtain basis

$$\begin{aligned} u_1 &= (3, 1, -1, 0, 4, 0) & (= w_1) \\ u_2 &= (-1, -1, -5/3, 4/3, 7/3, 5/3) & (= \frac{-1}{3} v_2) \end{aligned}$$

**Shortcut** When you row-reduce, after 6 ops, get

$$\begin{pmatrix} I_6 & -v_1 - v_2 - v_3 & 0_{6 \times 3} \\ 0_{6 \times 6} & v_1 + v_2 + v_3 & -w_1 - w_2 - w_3 \end{pmatrix}$$

Another viewpoint. Had

$$\begin{aligned} u &= x_1 v_1 + x_2 v_2 + x_3 v_3 \\ u &= y_1 w_1 + y_2 w_2 + y_3 w_3 \end{aligned}$$

You can solve instead  $6 \times 6$  system:

$$\begin{aligned} x_1 v_1 + x_2 v_2 + x_3 v_3 &= y_1 w_1 + y_2 w_2 + y_3 w_3 \\ x_1 v_1 + x_2 v_2 + x_3 v_3 - y_1 w_1 - y_2 w_2 - y_3 w_3 &= (0, 0, \dots, 0) \end{aligned}$$

Coeff matrix:  $(v_1 \ v_2 \ v_3 \ -w_1 \ -w_2 \ -w_3)$

Sol gives you  $x_1, x_2, x_3, y_1, y_2, y_3$  not  $z_1, \dots, z_6$ . Find  $u = (z_1, \dots, z_6)$  from (\*) or (\*\*)

*Matrix of a linear transformation (ch. 6.2)*

**Def**  $T : V \rightarrow W$  linear,  $\alpha = \{v_1, \dots, v_n\}$  basis of  $V$ ,  $\beta = \{w_1, \dots, w_n\}$  basis of  $W$ . The *standard matrix* of  $T$ , relative to  $\alpha$  and  $\beta$ , is the  $m \times n$  matrix whose  $i^{th}$  column is  $T(v_i)$ , written in  $\beta$ -coordinates, ie  $[T(v_i)]_\beta (\in \mathbb{R}^m)$ .

It is denoted  $[T]_\alpha^\beta$ .

**Ex** Let  $T : P_2(\mathbb{R}) \rightarrow P_1(\mathbb{R})$ ,  $T(f(x)) = f'(x)$ . Find  $[T]_\alpha^\beta$ ,  $\alpha = \{1, x, x^2\}$ ,  $\beta = \{1, x\}$

**Sol** Compute  $T$  on  $\alpha$

$$T(1) = 0$$

$$T(x) = 1$$

$$T(x^2) = 2x$$

In  $\beta$ -coords,

$$[T(1)]_\alpha^\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (= 0 \ 1 + 0 \ x)$$

$$[T(x)]_\alpha^\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (= 1 \ 1 + 0 \ x)$$

$$[T(x^2)]_\alpha^\beta = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad (= 0 \ 1 + 2 \ x)$$

So  $[T]_\alpha^\beta$  is  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ .

So  $[T]_\alpha^\beta$  records values of  $T$  on  $\alpha$ .

**Theorem 40.**  $[T]_\alpha^\beta$  computes  $T$ , but in coordinates. That is, for all  $v \in V$ ,

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$$

**Ex**  $T(f(x)) = f'(x)$ . Compute  $T(a + bx + cx^2)$  via  $[T]_\alpha^\beta$

**Sol**

$$[T]_\alpha^\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} [T(a + bx + cx^2)]_\beta &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} b \\ 2c \end{pmatrix} \quad (b + 2cx = f(x)) \end{aligned}$$

March 11th 2019

Recall  $T : V \rightarrow W$ ,  $\alpha = \{v_1, v_2, \dots, v_n\}$  basis  $V$

$\beta = \{w_1, w_2, \dots, w_n\}$  basis  $W$

Matrix  $[T]_{\alpha}^{\beta}$  has  $i^{th}$  column being  $[T(vi)]_{\beta}$

**Theorem 40**

$$[T(v)]_{\beta} = [T]_{\alpha}^{\beta}[v]_{\alpha}$$

*Proof.* Let  $A = [T]_{\alpha}^{\beta}$ ,  $v \in V$ . Write  $v = \sum_{i=1}^n a_i v_i$ .

$$\text{So } [v]_{\alpha} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

Then

$$\begin{aligned} A[v]_{\alpha} &= A(a_1 e_1 + \dots + a_n e_n) \\ &= a_1 A e_1 + \dots + a_n A e_n \\ &= a_1 (\text{col } \# 1 \text{ of } A) + \dots + a_n (\text{col } \# n \text{ of } A) \\ &= a_1 [T(v_1)]_{\beta} + \dots + a_n [T(v_n)]_{\beta} \end{aligned}$$

□

**Theorem 41.** Everything you want to know about  $T$ , you can determine from  $[T]_{\alpha}^{\beta}$ .

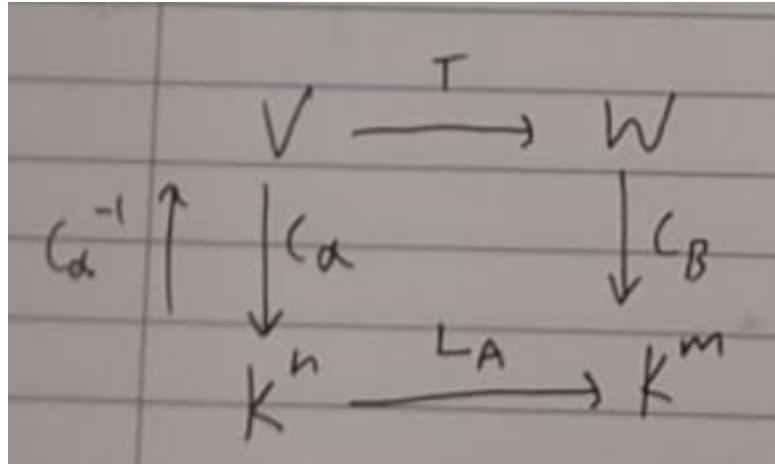
Let  $A = [T]_{\alpha}^{\beta}$  ( $C_{\alpha} = V \rightarrow \mathbb{R}^n$ ,  $C_{\alpha}(v) = [v]_{\alpha}$ ). See figure 4.

Then

$$(i) \ Ker(T) = C_{\alpha}^{-1}(Ker(A))$$

$$(ii) \ Im(T) = C_{\beta}^{-1}(Im(A))$$

Figure 4: Theorem 41



**Ex**  $T : \mathcal{M}_{2 \times 2}(\mathbb{R}) \rightarrow \mathcal{M}_{2 \times 2}(\mathbb{R})$  defined by  $T(A) = BA$ .

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Find basis for  $\text{Kernel}(T)$ ,  $\text{Image}(T)$  is  $T$  inj/surj?

**Sol** Use basis  $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

So we have

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$\text{Ker}(T)$ : Solve  $[T]x = 0$ . Row-reduce

$$[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2s$$

$$x_2 = -2t$$

$$x_3 = s$$

$$x_4 = t$$

$$x = s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \text{Ker}([T])$$

So  $\left\{ \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix} \right\}$  basis for  $\text{Ker } T$  so  $T$  not injective.

**Theorem 42.** The following are true:

- (i)  $T : V \rightarrow W$ , linear  $\alpha$  basis of  $V$ ,  $\beta$  basis of  $W$ .

$$T \text{ is invertible} \iff [T]_{\alpha}^{\beta} \text{ is invertible}$$

So  $\dim(V) = \dim(W)$  must hold, of course.

- (ii) If  $S : W \rightarrow U$ ,  $\gamma$  basis of  $U$ , then  $[S \cdot T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$

$(S \cdot T : V \rightarrow U)$  is matrix of a composition is product of standard matrices.

**Ex**  $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  defined by  $T(f(x)) = xf(x) + f(1)$ . Prove  $T$  is invertible, give formula for  $T^{-1}(ax^2 + bx + c)$ .

**Sol** ( $T$  is linear, verify)

Use standard basis  $\{1, x, x^2\}$ .

Calculate  $T$  on  $\alpha$

$$T(1) = x(0) + 1 = 1 = 1 + 0x + 0x^2$$

$$T(x) = x(1) + 1 + 1 + 1x + 0x^2$$

$$T(x^2) = x(2x) + 1 = 1 + 0x + 2x^2$$

So  $[T]_{\alpha}^{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ .  $\det([T]) = 2 \neq 0$  so matrix and  $T$  are both

invertible.

$$\text{invert}[T] = \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Then

$$\begin{aligned} [T^{-1}(c + bx + ax^2)]_{\alpha} &= \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} \\ &= \begin{bmatrix} c - b - \frac{a}{2} \\ b \\ \frac{a}{2} \end{bmatrix} \end{aligned}$$

Formula:  $T^{-1}(c + bx + ax^2) = (c - b - \frac{a}{2}) + bx + \frac{a}{2}x^2$ .

Check

$$\begin{aligned} T(c - b - \frac{a}{2} + bx + \frac{a}{2}x^2) &= x(b + ax) + c - b - \frac{a}{2} + b + \frac{a}{2} \\ &= c + bx + ax^2 \end{aligned}$$

March 13th 2019

*Change of basis (ch 6.3)*

Suppose  $V$  : vector space,  $\alpha = \{u_1, \dots, u_n\}$  and  $\beta$  both bases of  $V$ .

How to change from  $\alpha$ -coordinates to  $\beta$ -coordinates easily?

**Trick:** Consider identity lin. transformation  $I$ ,  $I(v) = v$ .

$$I : V \rightarrow V$$

Matrix  $[I]_{\alpha}^{\beta}$  will change coords, since if  $v \in V$ ,

$$[I]_{\alpha}^{\beta}[v]_{\alpha} = [I(v)]_{\beta} = [v]_{\beta}$$

**Def** Matrix  $Q_{\alpha}^{\beta} = [I]_{\alpha}^{\beta}$  called change-of-basis matrix from  $\alpha$  to  $\beta$ . That is  $Q_{\alpha}^{\beta}$  is matrix whose  $i^{th}$  column is the  $i^{th}$  basis vector of  $\alpha$ , written in  $\beta$ -coords ("old basis in new coords, as columns").

**Theorem 43.** We have

- (i) For all  $v \in V$ ,  $Q_{\alpha}^{\beta}[v]_{\alpha} = [v]_{\beta}$  (mult. by  $Q_{\alpha}^{\beta}$  changes coords)
- (ii)  $Q_{\beta}^{\alpha} = (Q_{\alpha}^{\beta})^{-1}$  (and  $Q_{\alpha}^{\beta}$  is invertible!)

*Proof.* (i) Done above.

(ii)  $I : V \rightarrow V$  is invertible, so  $Q_{\alpha}^{\beta} = [I]_{\alpha}^{\beta}$  also invertible and

$$\begin{aligned} (Q_{\alpha}^{\beta})^{-1} &= ([I]_{\alpha}^{\beta})^{-1} \\ &= [I^{-1}]_{\beta}^{\alpha} \\ &= [I]_{\beta}^{\alpha} \\ &= Q_{\beta}^{\alpha} \end{aligned}$$

□

**Ex**  $\mathbb{R}^2$  with  $\alpha = \{(1, 0), (0, 1)\}$ ,  $\beta = \{(2, 1), (1, 3)\}$ . Find  $Q_{\alpha}^{\beta}$ ,  $Q_{\beta}^{\alpha}$ ,  $[(7, 4)]_{\beta}$ .

**Note** In  $\mathbb{R}^n$ ,  $[(a_1)_{\alpha}]_{\alpha} = (a_1)_{\alpha}$  ( $\alpha = \{e_1, e_2, \dots, e_n\}$ )

**Sol**  $Q_{\alpha}^{\beta}$  = old basis in  $\alpha$  in terms of new basis  $\beta$  = work.

$Q_{\beta}^{\alpha}$  = easier =  $\beta$ -vectors in terms of  $\alpha$ .

$$\begin{aligned} Q_{\beta}^{\alpha} &= \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \\ Q_{\alpha}^{\beta} &= (Q_{\beta}^{\alpha})^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \end{aligned}$$

Then,

$$\begin{aligned} \left[ \begin{pmatrix} 7 \\ 4 \end{pmatrix} \right]_{\beta} &= Q_{\beta}^{\alpha} \left[ \begin{pmatrix} 7 \\ 4 \end{pmatrix} \right]_{\alpha} \\ &= \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 21 - 4 \\ -28 + 8 \end{pmatrix} \\ &= \begin{pmatrix} \frac{17}{2} \\ -10 \end{pmatrix} \end{aligned}$$

**Def**  $T : V \rightarrow V$  linear transf (some  $V$ ), called a *linear operator*.

**Def** Let  $A, B \in \mathcal{M}_{n \times n}(K)$ .  $A$  is similar to  $B$  if  $\exists$  invertible  $Q \in \mathcal{M}_{n \times n}(K)$  so that  $Q^{-1}AQ = B$

**Proposition 44.** Note If  $A$  similar to  $B$ ,  $B$  similar to  $A$ , since

$$\begin{aligned} Q^{-1}AQ &= B \\ QQ^{-1}AQQ^{-1} &= QBQ^{-1} \\ A &= (Q^{-1})^{-1}BQ^{-1} \end{aligned}$$

**Theorem 45.** Let  $T : V \rightarrow V$  linear operator,  $\alpha, \beta$  bases of  $V$ . Then,

$$[T]_{\beta}^{\beta} = Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}$$

In particular,  $[T]_{\alpha}^{\alpha}$  and  $[T]_{\beta}^{\beta}$  are similar since  $Q_{\alpha}^{\beta} = (Q_{\beta}^{\alpha})^{-1}$

*Proof.* Let  $v \in V$ . Show both compute some linear operator.

$$\text{LHS } [T]_{\beta}^{\beta}[v]_{\beta} = [T(v)]_{\beta}$$

$$\text{RHS } Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}[v]_{\beta}$$

$$\begin{aligned} Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha} Q_{\beta}^{\alpha}[v]_{\beta} &= Q_{\alpha}^{\beta} [T]_{\alpha}^{\alpha}[v]_{\alpha} \\ &= Q_{\alpha}^{\beta} [T(v)]_{\alpha} \\ &= [T(v)]_{\beta} \end{aligned}$$

So for all  $[v]_{\beta}$ , mult by LS/RS gives some result, so for std bases

vector  $e_1, \dots, e_n$ , LS  $e_i = \text{col } i$  of LS, RS  $e_i = \text{col } i$  of RS  $\square$

**Problem** (figure 5) Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be reflection in the line  $y = mx$ .

Find formula for  $T(a, b)$

**Sol** First, prove  $T$  is linear. (omit)

**Option # 1** (figure 6) Compute  $T(1, 0), T(0, 1)$ , find  $[T]_{\alpha}^{\alpha}$ ,  $\alpha = \{(1, 0), (0, 1)\}$

**Option # 2** (figure 7) Use better basis, then change basis. Let  $v =$

$(1, m)$  so  $T(v) = (1, m)$ . Let  $w = (m, -1)$ . Then  $T(w) = -w = (-m, 1)$

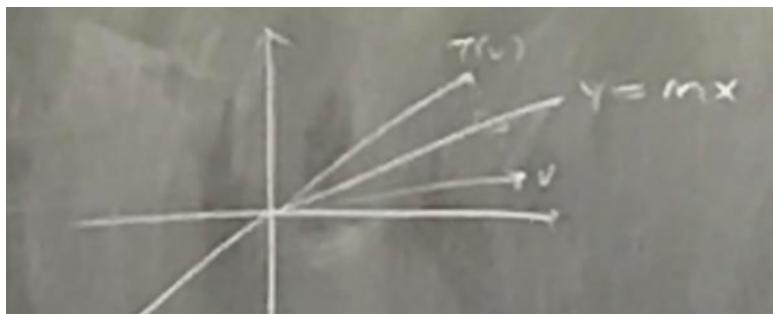


Figure 5: Problem

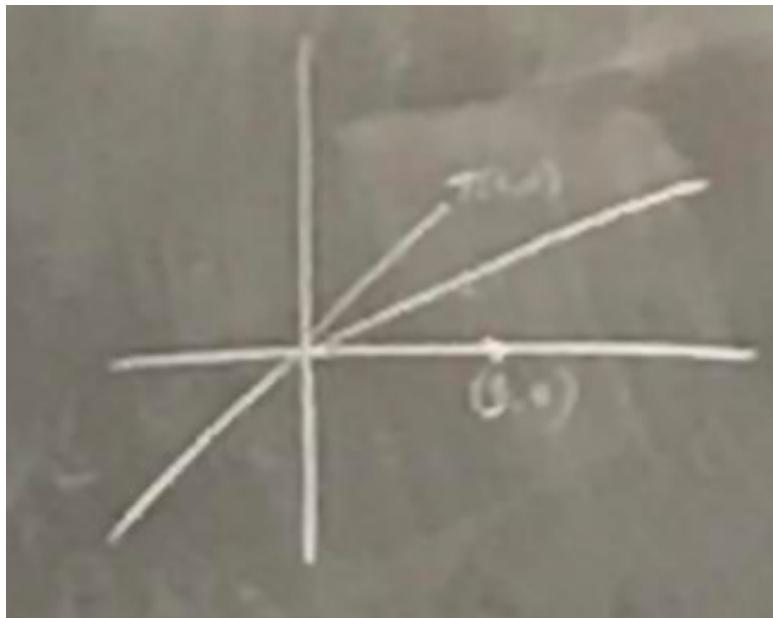


Figure 6: Have to do some geometry! :(

New basis  $\beta = \{v, w\}$

$$\begin{aligned}[T]_{\beta}^{\beta} &= ([T(v)]_{\beta}, [T(w)]_{\beta}) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

Want  $[T]_{\alpha}^{\alpha} = Q_{\beta}^{\alpha} [T]_{\beta}^{\beta} Q_{\alpha}^{\beta}$ . Have  $Q_{\beta}^{\alpha} = \beta$  in terms of  $\alpha = \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}$

$$\begin{aligned}Q_{\alpha}^{\beta} &= (Q_{\beta}^{\alpha})^{-1} \\ &= \frac{1}{-1-m^2} \begin{pmatrix} -1 & -m \\ -m & 1 \end{pmatrix}\end{aligned}$$

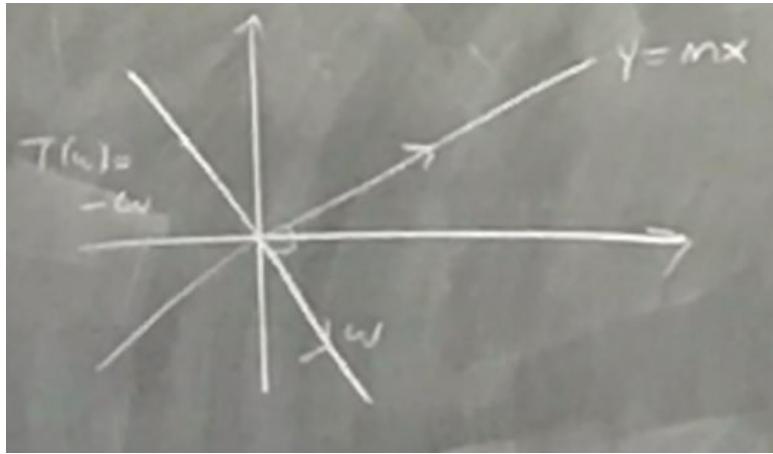


Figure 7: Better option

Compute

$$\begin{aligned}[T]_{\alpha}^{\alpha} &= Q_{\beta}^{\alpha} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Q_{\alpha}^{\beta} && \text{(multiply)} \\ &= \frac{1}{m^2 + 1} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix} \end{aligned}$$

Finally,

$$\begin{aligned} T(a, b) &= \frac{1}{m^2 + 1} \begin{pmatrix} 1 - m^2 & 2m \\ 2m & m^2 - 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{1}{m^2 + 1} \begin{pmatrix} a - am^2 & 2bm \\ 2am & bm^2 - b \end{pmatrix} \end{aligned}$$

March 15th 2019

Inner Product Spaces (ch. 7 text)

**Idea:** Dot product on  $\mathbb{R}^n$ ,  $u = (a_1, \dots, a_n)$ ,  $v = (b_1, \dots, b_n)$

$$u \cdot v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

From this,

$$\begin{aligned} \|u\| &= \sqrt{a_1^2 + a_2^2 + \dots + a_n^2} = \sqrt{u \cdot u} \\ u \cdot v &= \|u\| \|v\| \cos \theta \end{aligned}$$

Or

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{u \cdot v}{\|u\| \|v\|} \right) \\ u, v \text{ } \frac{\text{orthogonal}}{(\theta = \frac{\pi}{2})} &\iff u \cdot v = 0 \end{aligned}$$

Dot product allows you to *define* lengths, angles, orthogonality. These are geometric ideas.

**Def**  $V$  vector space over  $K$  ( $\mathbb{R}$  or  $\mathbb{C}$ ).

An *inner product* on  $V$  is a function  $\langle u, v \rangle$  which takes two vectors as input and produces a scalar, and satisfies the following:

$$(I1) \quad \forall u, v, w \in V, \forall c \in K$$

$$(i) \quad \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$

$$(ii) \quad \langle cu, w \rangle = c\langle u, w \rangle$$

This is called *linearity in the first component*

$$(I2) \quad \forall u, v \in V$$

$$\langle v, u \rangle = \overline{\langle u, v \rangle}$$

The RHS is the complex conjugate.

This is called *conjugate similarity*.

$$(I3) \quad \forall u \in V, \langle u, u \rangle \geq 0 \text{ and } \langle u, u \rangle = 0 \iff u = \vec{0}$$

This is called *positive definite*

### Notes:

(1) If  $K = \mathbb{R}$ , (I2) is  $\langle v, u \rangle = \langle u, v \rangle$

(2) If  $K = \mathbb{C}$ , then by (I2)

$$\langle u, u \rangle = \overline{\langle u, u \rangle}$$

Which means  $\langle u, u \rangle \in \mathbb{R}$ . So  $\langle u, u \rangle \geq 0$  makes sense.

### Theorem 46. Properties of inner products

$$(a) \quad \forall u, v, w \in V, \forall c \in K,$$

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

$$\langle u, cv \rangle = \bar{c}\langle u, v \rangle$$

This is called *conjugate linearity in second component*.

$$(b) \quad \forall u \in V, \langle u, \vec{0} \rangle = 0 \text{ (scalar)}$$

$$(c) \quad \forall u, v, w \in V, \text{ if } \forall w \in V \quad \langle u, w \rangle = \langle v, w \rangle \text{ then } u = v$$

*Proof.* By direct proof.

(a)

$$\begin{aligned}
\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} && \text{(I2)} \\
&= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} && \text{(I1)} \\
&= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\
&= \langle u, v \rangle + \langle u, w \rangle && \text{(I2)}
\end{aligned}$$

Recall for  $z_1, z_2 \in \mathbb{C}$ ,

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{z_1} + \overline{z_2} \\
\overline{z_1 z_2} &= \overline{z_1} \overline{z_2} \\
z_1 \overline{z_1} &= (a + bi)(a - bi) = a^2 + b^2 = |z_1|^2
\end{aligned}$$

Now, we have:

$$\begin{aligned}
\langle u, cv \rangle &= \overline{\langle cv, u \rangle} && \text{(I2)} \\
&= \overline{c \langle v, u \rangle} && \text{(I1)} \\
&= \overline{c} \overline{\langle v, u \rangle} \\
&= \bar{c} \langle u, v \rangle
\end{aligned}$$

(b)

$$\begin{aligned}
\langle u, \vec{0} \rangle &= \langle u, \vec{0} + \vec{0} \rangle \\
&= \langle u, \vec{0} \rangle + \langle u, \vec{0} \rangle && \text{(by (a))}
\end{aligned}$$

So  $0 = \langle u, \vec{0} \rangle$

- (c) Assume  $\forall w, \langle u, w \rangle = \langle v, w \rangle$ . To show  $u = v$ , we will show  $u - v = \vec{0}$ .

Consider

$$\begin{aligned}
\langle u - v, u - v \rangle &= \langle u, u - v \rangle + \langle -v, u - v \rangle && \text{(I1)} \\
&= \langle u, u - v \rangle - \langle v, u - v \rangle && \text{(I1)}
\end{aligned}$$

Using  $w = u - v$ ,  $\langle u, u - v \rangle = \langle v, u - v \rangle$ . So  $\langle u - v, u - v \rangle = 0$  so by (I3).  $u - v = \vec{0}$  so  $u = v$ .

□

*March 18th 2019*

Standard inner product on  $K^n$ :

for  $u = \{a_1, \dots, a_n\}, v = \{b_1, \dots, b_n\}$  define

$$\langle u, v \rangle = \sum_{i=1}^n a_i \overline{b_i}$$

So if  $K = \mathbb{R}$ ,  $\bar{b}_i = b_i$  so it's the usual dot product.

**Ex** Compute  $\langle u, v \rangle$ ,

$$u = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

$$v = \begin{pmatrix} i \\ 3-4i \end{pmatrix}$$

**Solution**

$$\begin{aligned} \langle \begin{pmatrix} 2 \\ 1+i \end{pmatrix}, \begin{pmatrix} i \\ 3-4i \end{pmatrix} \rangle &= 2(\bar{i}) + (1+i)(\overline{3-4i}) \\ &= -2i + (1+i)(3+4i) \\ &= -2i + 3 + 4i + 3i + 4i^2 \\ &= -i + 5i \end{aligned}$$

**Proposition 47.** Standard inner product in  $K^n$  is an inner product

*Proof.* By direct proof.

(I1) Omit.

(I2)

$$\begin{aligned} \overline{\langle v, u \rangle} &= \overline{\sum_{i=1}^n b_i \bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i \bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i} \overline{\bar{a}_i} \\ &= \sum_{i=1}^n \overline{b_i} a_i \\ &= \sum_{i=1}^n a_i \bar{b}_i \end{aligned}$$

(I3)

$$\begin{aligned} \langle u, u \rangle &= \sum_{i=1}^n a_i \bar{a}_i \\ &= \sum_{i=1}^n |a_i|^2 \end{aligned}$$

Then all  $|a_i| \geq 0$  so  $\langle u, u \rangle \geq 0$

$$\langle u, u \rangle = 0 \iff |a_i| = 0 \text{ for all } i$$

□

*Inner product on  $\mathcal{M}_{n \times n}(K)$*

For  $A, B \in \mathcal{M}_{n \times n}(K)$ , define first

- (i)  $\bar{A}$  is the matrix obtained by taking the complex conjugate of each entry.
- (ii)  $A^* = (\bar{A})^T$ , conjugate transpose (adjoint)

**Ex:**

$$A = \begin{pmatrix} 2+i & 3i \\ 2 & 1+i \end{pmatrix}, \bar{A} = \begin{pmatrix} 2+i & -3i \\ 2 & 1-i \end{pmatrix}, A^* = \begin{pmatrix} 2+i & 2 \\ -3i & 1-i \end{pmatrix}$$

For inner product,

$$\langle A, B \rangle = \text{tr}(B^* A)$$

**Ex** In  $\mathcal{M}_{2 \times 2}(K)$ , if

$$\begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ \langle A, B \rangle &= \text{tr} \left( \begin{pmatrix} \bar{b}_{11} & \bar{b}_{21} \\ \bar{b}_{12} & \bar{b}_{22} \end{pmatrix} \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} \right) \\ &= (a_{11}\bar{b}_{11} + a_{21}\bar{b}_{21}) + (a_{12}\bar{b}_{12} + a_{22}\bar{b}_{22}) \\ &= \left\langle \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} \right\rangle \quad (\text{Standard inner product on } \mathbb{C}^4) \end{aligned}$$

**Proposition 48.**  $\langle A, B \rangle = \text{tr}(B^* A)$  is an inner product on  $\mathcal{M}_{n \times n}(K)$

*Proof.* Omit. You can prove it directly using matrix properties.  $\square$

*Inner product on  $P_n(\mathbb{R})$*

For  $f, g \in P_n(\mathbb{R})$  define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

**Ex** For  $f(x) = x+1$ ,  $g(x) = x$  find  $\langle f, g \rangle$

**Sol**

$$\begin{aligned} \langle x+1, x \rangle &= \int_0^1 (x+1)x dx \\ &= \int_0^1 (x^2 + x) dx \\ &= \frac{x^3}{3} \frac{x^2}{2} \Big|_0^1 \\ &= \frac{1}{3} + \frac{1}{2} \end{aligned}$$

**Proposition 49.** For any  $a < b$

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

defines an inner product on  $P_n(\mathbb{R})$  (also on  $P(\mathbb{R})$ )

*Proof.* By direct proof.

(I1) Let  $f, g, h \in P_n(\mathbb{R}), c \in \mathbb{R}$ . Then

$$\begin{aligned} \langle f + cg, h \rangle &= \int_a^b (f(x) + cg(x))h(x)dx \\ &= \int_a^b f(x)h(x)dx + c \int_a^b g(x)h(x)dx \\ &= \langle f, h \rangle + c\langle g, h \rangle \quad ((i) \text{ and } (ii) \text{ together}) \end{aligned}$$

(I2)

$$\begin{aligned} \langle f, g \rangle &= \int_a^b f(x)g(x)dx \\ &= \int_a^b g(x)f(x)dx \\ &= \langle g, f \rangle \end{aligned}$$

(I3)

$$\begin{aligned} \langle f, f \rangle &= \int_a^b f(x)f(x)dx \\ &= \int_a^b (f(x))^2 dx \end{aligned}$$

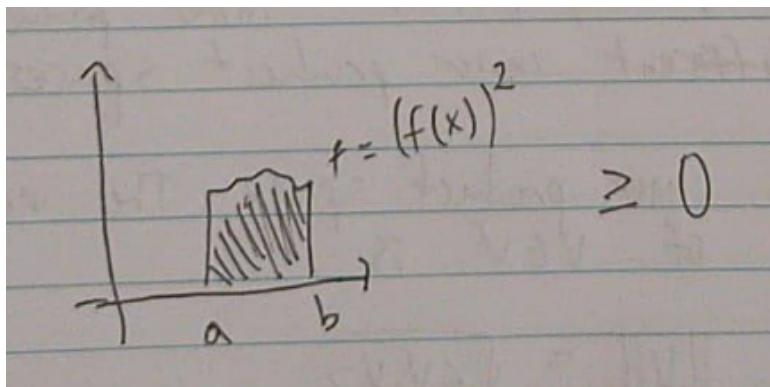


Figure 8: Representation

□

**Problem** For  $P_1(\mathbb{R})$ , write formula for

$$\langle a + bx, c + dx \rangle$$

in terms of  $a, b, c, d$

**Sol**

$$\begin{aligned}\langle a + bx, c + dx \rangle &= \int_0^1 (ac + (ad + bc)x + bdx^2) dx \\ &= acx + \frac{ad + bc}{2}x^2 + \frac{bd}{3}x^3 \Big|_0^1 \\ &= ac + \frac{ad}{2} + \frac{bc}{2} + \frac{bd}{3}\end{aligned}$$

**Note**  $P_1(\mathbb{R}) \simeq \mathbb{R}^2$ . Isomorphism,

$$(a + bx) \rightarrow \begin{pmatrix} a \\ b \end{pmatrix}$$

Under this isomorphism, you can compute  $\langle a + bx, c + dx \rangle$  using an inner product on  $\mathbb{R}^2$  defined by

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle = ac + \frac{1}{2}ad + \frac{1}{2}bc + \frac{1}{3}bd$$

The point is, the inner product makes sense in  $P_1(\mathbb{R})$ .

**Def** A vector space  $V$  with a specified inner product is called an *inner product space*.

**Note** Some  $V$  with different inner products are different inner product spaces.

**Def**  $V$  on inner product space. The norm or length of  $v \in V$  is

$$\|v\| = \sqrt{\langle v, v \rangle}$$

**Example** In  $P_1(\mathbb{R})$  with  $[0, 1]$ ,

$$\begin{aligned}\|x + 1\| &= \sqrt{\langle x + 1, x + 1 \rangle} \\ &= \left( \int_0^1 (x + 1)^2 dx \right)^{-\frac{1}{2}} \\ &= \left( \frac{(x + 1)^3}{3} \Big|_0^1 \right)^{-\frac{1}{2}} \\ &= \left( \frac{2^3}{3} - \frac{1}{3} \right)^{-\frac{1}{2}} \\ &= \sqrt{\frac{7}{3}}\end{aligned}$$

March 20th 2019

**Last time:** Norm (length) is  $\|v\| = \sqrt{\langle u, v \rangle}$

**Proposition 50.** For all  $v \in V, c \in K$

$$\|cv\| = |c|\|v\| \quad (\text{note } |c|^2 = c\bar{c} \in \mathbb{C}, |c|^2 = a^2 + b^2)$$

*Proof.*

$$\begin{aligned}\|cv\| &= \sqrt{\langle cv, cv \rangle} \\ &= \sqrt{c\bar{c}\langle v, v \rangle} \tag{I2} \\ &= |c|\sqrt{\langle v, v \rangle} \\ &= |c| \cdot \|v\|\end{aligned}$$

□

**Theorem 51** (Cauchy-Schwarz Inequality). For all  $u, v \in V$ , (inner product space)

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

So also  $\langle u, v \rangle \leq \|u\| \|v\|$  if  $K = \mathbb{R}$  or equiv,

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

Further, equality holds  $\iff u, v$  are dependent.

*Proof.* Let  $c \in K$  any scalar. Consider

$$0 \leq \langle u - cv, u - cv \rangle \tag{I3}$$

$$= \langle u, u - cv \rangle + \langle -cv, u - cv \rangle \tag{I1}$$

$$\begin{aligned}&= \langle u, u \rangle + \langle u, -cv \rangle + \langle -cv, u \rangle + \langle -cv, -cv \rangle \\&= \|u\|^2 + \overline{(-c)}\langle u, v \rangle + (-c)\langle v, u \rangle + (-c)\overline{(-c)}\langle v, v \rangle \\&0 \leq \|u\|^2 - \bar{c}\langle u, v \rangle - c\langle v, u \rangle + c\bar{c}\|v\|^2\end{aligned}$$

Set  $c = \frac{\langle u, v \rangle}{\|v\|^2}$  (unless  $\|v\| = 0$ , only if  $v = \vec{0}$ , in which case  $\langle u, 0 \rangle = 0 = \|u\|0 = \|u\| \|v\|$ ) So  $c = \frac{1}{\|v\|^2} \langle u, v \rangle$ . (LHS  $\in \mathbb{R}$ , RHS  $\in \mathbb{C}$ ). So

$$\begin{aligned}\bar{c} &= \frac{1}{\|v\|^2} \overline{\langle u, v \rangle} \\&= \frac{\langle v, u \rangle}{\|v\|^2}\end{aligned}$$

So

$$\begin{aligned} 0 &\leq ||u||^2 - \frac{\langle v, u \rangle}{||v||^2} \langle u, v \rangle - \frac{\langle u, v \rangle \langle v, u \rangle}{||v||^2} + \frac{u, v}{||v||^2} \frac{v, u}{||v||^2} ||v||^2 \\ 0 &\leq ||u||^2 - \frac{\langle u, v \rangle \langle v, u \rangle}{||v||^2} \\ \langle u, v \rangle \langle v, u \rangle &\leq ||u||^2 ||v||^2 \\ \langle u, v \rangle \overline{\langle u, v \rangle} &\leq ||u||^2 ||v||^2 \\ |\langle u, v \rangle|^2 &\leq ||u||^2 ||v||^2 \end{aligned}$$

Omit proof about equality.  $\square$

### Important cases

(1)  $\mathbb{R}^n$ , usual inner product. Let  $u = (a_1, \dots, a_n)$ ,  $v = (b_1, \dots, b_n)$ . So,

$$\langle u, v \rangle^2 = (a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq ||u||^2 ||v||^2$$

So

$$(a_1 b_1 + \dots + a_n b_n)^2 \leq (a_1^2 + \dots + a_n^2)(b_1^2 + \dots + b_n^2)$$

**Ex** Prove for all  $a_1, a_2, \dots, a_n$ ,

$$(|a_1| + |a_2| + \dots + |a_n|)^2 \leq n(a_1^2 + a_2^2 + \dots + a_n^2)$$

**Sol** Let

$$\begin{aligned} u &= (|a_1|, |a_2|, \dots, |a_n|) \\ v &= (1, 1, \dots, 1) \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} (|a_1| + |a_2| + \dots + |a_n|)^2 &\leq (a_1^2 + \dots + a_n^2)(1 + 1 + \dots + 1) \\ &= n(a_1^2 + \dots + a_n^2) \end{aligned}$$

(2)  $\mathcal{P}(\mathbb{R}), f, g \in \mathcal{P}(\mathbb{R})$

$$\begin{aligned} \langle f, g \rangle^2 &\leq \langle f, f \rangle \langle g, g \rangle \\ (\int_0^1 f(x)g(x)dx)^2 &\leq (\int_0^1 f(x)^2 dx)(\int_0^1 g(x)^2 dx) \end{aligned}$$

**Theorem 52.** Triangle inequality For all  $u, v \in V$ ,

$$||u + v|| \leq ||u|| + ||v||$$

*Proof.* Instead of

$$||u + v|| = \sqrt{\langle u + v, u + v \rangle}$$

Look at

$$\begin{aligned}
 \|u + v\|^2 &= \langle u + v, u + v \rangle \\
 &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \quad (\text{I1}) \\
 &= \|u\|^2 + \langle u, v \rangle + \overline{\langle u, v \rangle} + \|v\|^2
 \end{aligned}$$

For  $z = a + bi$ ,  $z + \bar{z} = 2a = 2\operatorname{Re}(z)$  ( $\operatorname{Re}(z) = a$ ,  $\operatorname{Im}(z) = b$ ). Also,

$$a \leq |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2} = |z|$$

So  $\operatorname{Re}(z) \leq |z|$  (\*)

Then,

$$\begin{aligned}
 \|u + v\|^2 &= \|u\|^2 + 2\operatorname{Re}(\langle u, v \rangle) + \|v\|^2 \\
 &\leq \|u\|^2 + 2|\langle u, v \rangle| + \|v\|^2 \quad (\text{by } (*)) \\
 &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad (\text{Cauchy-Schwarz}) \\
 &= (\|u\| + \|v\|)^2
 \end{aligned}$$

So  $\|u + v\|^2 \leq (\|u\| + \|v\|)^2$ , take square root.  $\square$

### Angles

Since  $|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$ ,

$$\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq 1 \text{ or } -1 \leq \frac{\langle u, v \rangle}{\|u\| \|v\|} \quad (K = \mathbb{R})$$

So there is an *angle*  $\theta$  such that

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|}$$

Define the angle between  $u, v$  to be  $\theta$ .

**Note** When the angle is  $0$ ,  $\cos \theta = 1$ . When the angle is  $\pi/2$ ,  $\cos \theta = 0$ . When the angle is  $\pi$ ,  $\cos \theta = -1$ . So  $\cos \theta$  measures how "similar" two vectors are in terms of "angle" or "direction".

March 22nd 2019

### Application/interpretation

Word counts in textual analysis. Consider  $\mathbb{R}^n$ ,  $n = \#$  of words in the (English) language. Each component corresponds to a word (eg: component 1 is "a", etc). View a text (eg Hamlet) as a vector  $(v_{\text{hamlet}})$ , count # times each word occurs.

Norm  $\|v_{\text{hamlet}}\| = \sqrt{\sum_{i=1}^n a_i^2}$  (usual dot product)

more words  $\rightarrow$  larger norm

Eg  $v = (1, 1, \dots, 1)$ ,  $n = 1000$ .

$$\begin{aligned} \|v\| &= \sqrt{\sum 1} \\ &= \sqrt{1000} \end{aligned}$$

$w = (1000, 0, \dots, 0)$ ,  $n = 1000$ :

$$\begin{aligned} \|v\| &= \sqrt{1000^2} \\ &= 1000 \end{aligned}$$

Angle

$$\cos \theta = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

If  $u, v$  have no words in common,  $\langle u, v \rangle = 0$ , so  $\cos \theta = 0$  ( $\theta = \frac{\pi}{2}$ , "orthogonal"). Suppose you compare "Hamlet" to "2x Hamlet":

$$\begin{aligned} \cos \theta &= \frac{\langle v_{\text{hamlet}}, v_{2x \text{ hamlet}} \rangle}{\|v_{\text{hamlet}}\| \|v_{2x \text{ hamlet}}\|} \\ &= \frac{\langle v_{\text{hamlet}}, 2v_{\text{hamlet}} \rangle}{\|v_{\text{hamlet}}\| \|2v_{\text{hamlet}}\|} \\ &= \frac{2\|v_{\text{hamlet}}\|}{2\|v_{\text{hamlet}}\| \|v_{\text{hamlet}}\|} \\ &= 1 \end{aligned}$$

Ie  $\theta = 0$ . Texts are "the same".

### Orthogonality and projections

**Def**  $u, v$  are *orthogonal* if  $\langle u, v \rangle = 0$ .

**Ex** In  $P_1(\mathbb{R})$ , inner product  $\int_0^1 fg dx$ , find all polynomials (vectors) orthogonal to  $1 + x$ .

**Sol** Let  $g(x) = a + bx$ . Need

$$\begin{aligned} 0 &= \langle 1 + x, a + bx \rangle \\ &= \int_0^1 (a + bx + ax + bx^2) dx \\ &= ax + \frac{b(a)}{2}x^2 + \frac{b}{3}x^3 \Big|_0^1 \\ &= a + \frac{b}{2} + \frac{a}{2} + \frac{b}{3} \\ \frac{-3}{2}a &= \frac{5}{6}b, b = \frac{-3}{2}(\frac{6}{5})a = \frac{-9}{5}a \end{aligned}$$

All vectors  $a - \frac{9}{5}ax$ , ie  $\text{span}\{1 - \frac{9}{5}x\}$ .

**Def** A set  $S$  of vectors is

(i) *orthogonal* if  $\langle u, v \rangle = 0$  for all  $u, v \in S$ ,  $u \neq v$ .

(ii) *orthonormal* if orthogonal and  $\|u\| = 1$ , all  $u \in S$

**Def** A basis  $\alpha$  is an *orthognormal basis* (ONB) if it is an orthonormal set.

**Ex**  $\alpha = \{e_1 e_2, \dots, e_n\}$  is ONB.

*Notation : Kronecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So if  $S = \{v_1, v_2, \dots, v_n\}$  is ONB,  $\langle v_i, v_j \rangle = \delta_{ij}$ .

**Proposition 53.** If  $S$  is an orthogonal set of non-zero vectors, then  $S$  is linearly independent.

*Proof.* Suppoe

$$\sum_{i=1}^k a_i v_i = 0 \quad (\text{for some } v_1, \dots, v_k \in S, a_1, \dots, a_n \text{ scalars})$$

Trick. Take inner product with each  $v_j$ ,  $j = 1, 2, \dots, k$ . So

$$\begin{aligned} 0 &= \langle \vec{0}, v_j \rangle \\ &= \left\langle \sum_{i=1}^k a_i v_i, v_j \right\rangle \\ &= \sum_{i=1}^k \langle a_i v_i, v_j \rangle \quad ((I1)) \\ &= \sum_{i=1}^k a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle \quad (\text{Since } \langle v_i, v_j \rangle = 0, \text{ unless } i = j) \end{aligned}$$

But  $v_j \neq \vec{0}$  so  $\langle v_j, v_j \rangle \neq 0$ . So  $a_j = 0$ , for all  $j = 1, \dots, k$ . So all the coefficients are 0, so  $S$  is independent.  $\square$

**Theorem 54.** Let  $V$  be inner product space,

$$\alpha = \{v_1, v_2, \dots, v_n\}$$

an orthogonal basis. Then for any  $u \in V$ ,

$$u = \sum_{i=1}^n \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

ie the  $i^{th}$  component of coords of  $U$  in basis  $\alpha$  is  $\frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle}$ . Further, if  $\alpha$  is ONB, then

$$u = \sum_{i=1}^n \langle u, v_i \rangle v_i$$

*Proof.* We know  $u = \sum_{i=1}^n a_i v_i$  for some scalars. Take inner product with each  $v_j$ ,  $j = 1, 2, \dots, n$  in turn. So

$$\begin{aligned}\langle u, v_j \rangle &= \left\langle \sum_{i=1}^n a_i v_i, v_j \right\rangle \\ &= \sum_{i=1}^n a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle \quad (\text{All 0 except when } i = j)\end{aligned}$$

So  $a_j = \langle u, v_j \rangle / \langle v_j, v_j \rangle$ ,  $\alpha$  orthog.

□

March 25th 2019

**Last time:** If  $\alpha = \{v_1, v_2, \dots, v_n\}$  orthog. basis then for all  $v \in V$

$$v = \sum_{i=1}^n \frac{\langle u, v_i \rangle}{\langle v_i, v_i \rangle} v_i$$

**Ex** In  $\mathbb{R}^3$ ,  $\alpha = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}$  is an ONB. Find

coords of  $v = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$  in  $\alpha$ -basis.

**Sol** Compute its inner products with basis :

$$\begin{aligned}\left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle &= \frac{1}{\sqrt{2}} 3 = \frac{3}{\sqrt{2}} \\ \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle &= \frac{2 - 1 - 3}{\sqrt{3}} = \frac{-2}{\sqrt{3}} \\ \left\langle \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\rangle &= \frac{-2 + 1 - 6}{\sqrt{6}} = \frac{-7}{\sqrt{6}}\end{aligned}$$

So

$$\left[ \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right]_\alpha = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{-2}{\sqrt{3}} \\ \frac{-7}{\sqrt{6}} \end{pmatrix}$$

**Def** Let  $S \subseteq V$ . The *orthogonal complement* of  $S$  is

$$\begin{aligned}S^\perp &= \{v \in V \mid \forall s \in S, \langle v, s \rangle = 0\} \\ &= \text{all vectors orthogonal to all vectors in } S\end{aligned}$$

$S^\perp$  reads "S perp".

**Ex**

- (1)  $S = xy\text{-plane in } \mathbb{R}^3, S^\perp = z\text{-axis.}$
- (2)  $S = z\text{-axis}, S^\perp = xy\text{-plane.}$
- (3)  $S = \text{plane through origin}, S^\perp = \text{normal line.}$
- (4)  $S = V, S^\perp = \{\vec{0}\}$
- (5)  $S = \{\vec{0}\}, S^\perp = V$

**Proposition 55.** Let  $W \leq V$  (subspace). Then

(i)  $W^\perp$  is a subspace (true even if  $W$  just subset)

(ii) If  $\alpha = \{w_1, w_2, \dots, w_k\}$ , basis  $W$ , then

$$W^\perp = \{v \in V \mid \langle v, w_i \rangle = 0 \text{ for } i = 1, 2, \dots, k\}$$

(ie to compute  $W^\perp$ , find all  $v$  that are orthogonal to all basis elements)

(iii)  $W \cap W^\perp = \{\vec{0}\}$

*Proof.* By direct proof.

- (i) Let  $u, v \in W^\perp, c \in K$ . Then, to check if  $cu + v \in W^\perp$ , calculate for any  $w \in W$

$$\begin{aligned} \langle cu + v, w \rangle &= c\langle u, w \rangle + \langle v, w \rangle \\ &= 0 \quad (\text{both parts 0 since } u, v \in W^\perp) \end{aligned}$$

So  $cu + v \in W^\perp$ . Also,  $\vec{0} \in W^\perp$  since  $\langle \vec{0}, w \rangle = 0$  for all  $w \in W$ .

- (ii) Prove two sets are equal:

(a)  $LS \subseteq RS$ . Let  $v \in W^\perp$ . Since each  $w_i \in W$ ,  $\langle v, w_i \rangle = 0$  since  $v \in W^\perp$ .

(b)  $RS \subseteq LS$ . Let  $v \in V$  such that  $\langle v, w_i \rangle = 0$  all  $i = 1, 2, \dots, k$ . Let  $w \in W$ . Write  $w = \sum_{i=1}^k a_i w_i$ , then

$$\begin{aligned} \langle v, w \rangle &= \langle v, \sum_{i=1}^k a_i w_i \rangle \\ &= \sum_{i=1}^k \langle v, a_i w_i \rangle \\ &= \sum_{i=1}^k \bar{a}_i \langle v, w_i \rangle \\ &= 0 \end{aligned}$$

So  $v \in W^\perp = LS$ .

- (c) Let  $v \in W \cap W^\perp$ . Since  $v \in W^\perp$ ,  $v$  orthog to all vectors in  $W$ , including itself, we have

$$\langle v, v \rangle = 0$$

So  $v = \vec{0}$  by (I3).

□

**Ex** Let  $W = \{A \in \mathcal{M}_{2 \times 2}(K) | A^T = A\}$ . Find  $W^\perp$ .

**Sol** Find basis  $W$ . See A2.

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

Find all  $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  so that

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle \\ &= \text{tr}(\overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \\ &= a \end{aligned}$$

$$\begin{aligned} 0 &= \langle B, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle \\ &= d \end{aligned}$$

$$\begin{aligned} 0 &= \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \\ &= b + c \end{aligned}$$

So  $a = d = 0$ ,  $c = -b$ , general solution:  $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ .

$$W^\perp = \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

### Orthogonal Projection

See figure ???. Decompose  $v$  as  $v' + w$ ,  $w \in W$ ,  $v' \in W^\perp$ .

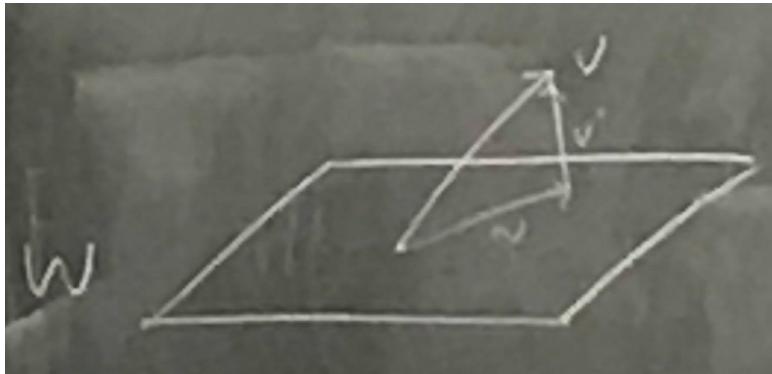


Figure 9: Orthogonal projection

**Theorem 56.** Let  $W \leq V$ ,  $v \in V$ . Then  $\exists$  unique vectors  $w \in W$ ,  $v' \in W^\perp$  such that  $v = v' + w$ . Vector  $w$  called the (orthogonal) projection of  $v$  onto  $W$ , denoted  $\text{proj}_W v = w$ . Further, if  $\alpha = \{w_1, w_2, \dots, w_k\}$  is an orthogonal basis of  $W$ , then

$$w = \text{proj}_W v = \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

**Important:**  $\alpha$  must be orthogonal!

*Proof.* Set  $w = \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$  (so  $v = v' + w$ ). Set  $v' = v - w$ . So  $v' + w = v$ ,  $w \in W$  ( $w = \text{comp of } W$ -basis vectors). Need  $v' \in W^\perp$ . Check if  $\langle v', w_j \rangle = 0$ , all  $j$ .

$$\begin{aligned} \langle v', w_j \rangle &= \langle v - w_i, w_j \rangle \\ &= \langle v - \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \rangle \\ &= \langle v, w_j \rangle - \left\langle \sum \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i, w_j \right\rangle \\ &= \langle v, w_j \rangle - \sum_{i=1}^k \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} \langle w_i, w_j \rangle \\ &\quad (\langle w_i, w_j \rangle = 0 \text{ or } \langle w_j, w_j \rangle \text{ since orthog basis}) \\ &= \langle v, w_j \rangle - \frac{\langle v, w_j \rangle}{\langle w_j, w_j \rangle} \langle w_j, w_j \rangle \\ &= 0 \end{aligned}$$

So  $\langle v, w_j \rangle = 0$  for all  $j = 1, 2, \dots, k$  so  $v' \in W^\perp$   $\square$

March 27th 2019

**Last time:** Thm 56:  $W \leq V$ , for all  $v \in V$  exists unique  $w \in W, v' \in W^\perp$  so that

$$v = v' + w$$

If  $\alpha = \{w_1, \dots, w_n\}$  orthog basis  $W$

$$\text{proj}_W v = w = \sum_{i=1}^n \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

*Proof.* Uniqueness. To prove, suppose  $v = \hat{v}' + \hat{w}$ , where  $\hat{w} \in W, \hat{v}' \in W^\perp$ . Then,

$$\begin{aligned} \vec{0} - v - v &= (v' + w) - (\hat{v}' + \hat{w}) \\ \vec{0} &= v' - \hat{v}' + w - \hat{w} \\ \hat{w} - w &= v' - \hat{v}' \end{aligned}$$

LHS in  $W$ , RHS in  $W^\perp$ , since  $v', \hat{v}' \in W^\perp$  and  $W^\perp$  subspace and  $W$  subspace.

So  $\hat{w} - w \in W \cap W^\perp = \{\vec{0}\}$

$$\hat{w} - w = \vec{0}$$

so  $\hat{w} = w$ . Similarly,  $v' - \hat{v}' \in W^\perp \cap W = \{\vec{0}\}$ . So  $v' = \hat{v}'$  □

### Terminology

If  $\alpha = \{w_1, w_2, \dots, w_m\}$  is an orthogonal set of non-zero vectors, for  $v \in V$  the scalars  $\frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$  are called *Fourier coefficients* of  $v$  relative to  $\alpha$ .

If  $\alpha$  is actually basis of  $V$ , Fourier coefficients are coords of  $v$  relative to  $\alpha$ . If  $\alpha$  is a basis for a subspace  $W$ , Fourier coefficients give the scalars needed to compute  $\text{proj}_W v$ . If  $v \in W$ ,  $\text{proj}_W v = v$ , so these coeffs are cords of  $v \in W$ .

**Note** To compute  $\text{proj}$ , need *orthog* basis  $W$ . How to find one?

**Lemma 57** (Pythagoras' Thm). *If  $u, v \in V$  are orthogonal, then  $\|u + v\|^2 = \|u\|^2 + \|v\|^2$*

*Proof.* Exercise. □

**Note**  $\|u - v\|$  = "distance between  $u$  and  $v$ ". Compare to  $\theta$ ; two vectors with very different norm can be very far apart yet have a small angle. Similarly, inverting the direction of a vector gives us a large angle but a small distance.

**Theorem 58.** Let  $W \leq V, v \in V, w = \text{proj}_W v$ . Then  $w$  is the “closest vector in  $W$  to  $v$ ” in the sense that if  $z \in W$  is any vector

$$\|v - w\| \leq \|v - z\|$$

*Proof.* Recall  $\|u\| = \sqrt{\langle u, u \rangle}, \|u\|^2 = \langle u, u \rangle$ . Write  $v = v' + w$ .

$$\begin{aligned} \|v - z\|^2 &= \|v' + w - z\|^2 = \|v' + (w - z)\|^2 \\ &= \|v'\|^2 + \|w - z\|^2 \end{aligned} \quad (\text{Pythagoras})$$

( $v' \in W^\perp, w - z \in W$ , so  $v', w - z$  are orthogonal)

$$\begin{aligned} \|v - z\|^2 &\geq \|v'\|^2 \\ &= \|v - w\|^2 \end{aligned}$$

Take square root.  $\square$

### Gram-Schmidt Orthogonalization Process

Or “how to produce an orthogonal basis”. Replace  $w_2$  by  $v' = w_2 - \text{proj}_{w_1} w_1$

Let  $W \leq V, \alpha = \{w_1, w_2, \dots, w_m\}$  basis of  $W$ . Produce a new basis  $\beta = \{v_1, v_2, \dots, v_m\}$  for  $W$  by

$$\begin{aligned} v_1 &= w_1 \\ v_i &= w_i - \text{proj}_{\beta_{i-1}} w_i \quad (\text{for } i = 2, 3, \dots, m) \end{aligned}$$

Where  $\beta_{i-1} = \text{span}\{v_1, v_2, \dots, v_{i-1}\}$ . We will see that  $\{v_1, v_2, \dots, v_{i-1}\}$  orthogonal basis for  $\beta_{i-1}$  so in fact

$$v_i = w_i - \left( \sum_{j=1}^{i-1} \frac{\langle w_i, v_j \rangle}{\langle v_j, v_j \rangle} v_j \right)$$

**Theorem 59.** For each  $i = 1, 2, \dots, m$ ,  $\{v_1, v_2, \dots, v_i\}$  is orthog basis for  $\text{span}\{w_1, w_2, \dots, w_i\}$ . In particular,  $\{v_1, v_2, \dots, v_m\}$  is orthog basis of  $W$  (you can make it ONB by normalizing each  $v_i$ )

*Proof.* Omit. Expand some more products.  $\square$

Ex  $W = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$ . Find ONB of  $W$ .

**Sol** Apply Gram-Schmidt

$$\begin{aligned}
 v_1 = w_1 &= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \\
 v_2 = w_2 - \text{proj}_{v_1} w_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \\
 v_2 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{0+2+0+0}{1+4+0+1} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \\
 &= \begin{pmatrix} -1/3 \\ 1/3 \\ 1 \\ -1/3 \end{pmatrix}
 \end{aligned}$$

Replace by  $\begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix} = v_2$ .

$$\begin{aligned}
 v_3 = w_3 - \text{proj}_{\text{span of } \{v_1, v_2\}} w_3 &= w_3 - \left( \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \right) \\
 &= \begin{pmatrix} 3/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}
 \end{aligned}$$

Replace by  $v_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix}$ . Orthonormal basis =  $\left\{ \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} -1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$

March 29th 2019

**Recall:** An orthonormal basis  $\{v_1, \dots, v_n\}$  is s.t.

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- (1) Say we want an orthonormal basis for  $P_2(\mathbb{R})$  with standard inner product.

$$\langle f, g \rangle = \int_0^1 f \cdot g(x) dx$$

**Sol** Take our standard basis  $\{1, x, x^2\}$ , apply Gram-Schmidt.

- (i) Consider  $v_1 = 1$ . check unit length

$$\|v_1\| = 1 = \sqrt{\int_0^1 1 dx}$$

Already normal! Apply G-S process to  $x$ . Let  $v_2 = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} \cdot 1$ :

$$\begin{aligned} x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} &= x - \langle x, 1 \rangle \cdot 1 \\ &= x - \int_0^1 x \cdot 1 dx \\ &= x - \frac{1}{2} \end{aligned}$$

Normalize  $v'_2$ .

$$\begin{aligned} \|v'_2\| &= \sqrt{\langle v'_2, v'_2 \rangle} \\ \langle v'_2, v'_2 \rangle &= \int_0^1 (x - \frac{1}{2})^2 dx \\ &= \frac{1}{12} \\ \rightarrow \|v'_2\| &= \frac{1}{\sqrt{12}} \end{aligned}$$

$v_2$  (normalize  $v'_2$ ) =  $\sqrt{12} - v'_2 = 2\sqrt{3}x = \sqrt{3}$  Consider  $x^2$ . let  $v'_3 = x^2 - \text{proj}(x^2) = x^2 - x + \frac{1}{6}$  then normalize  $v'_3$  to get

$$\begin{aligned} \langle v'_3, v'_3 \rangle &= 1/180 \\ \rightarrow \|v'_3\| &= \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}} \\ 1/3 &= 6\sqrt{5} - 6\sqrt{5} + \sqrt{5} \end{aligned}$$

So our orthonormal basis is  $\{v_1, v_2, v_3\}$

- (ii) Now, what about finding the proj of  $x^2$  onto span  $\{1, x\}$ .  $v_1, v_2$  to be basis elements for this subspace.  $\{1, x\} \iff \{v_1, v_2\}$ .

Let  $\text{span}\{1, x\} = W$ , with basis  $\{v_1, v_2\}$ ,

$$\begin{aligned} \text{proj}_W(x^2) &= \frac{\langle x^2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle x^2, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \\ &= \langle x^2, v_1 \rangle v_1 + \langle x^2, v_2 \rangle v_2 \\ &= \int_0^1 x^2 \cdot 1 dx \cdot 1 + \left( \int_0^1 x^2 (2\sqrt{3}x = \sqrt{3}) dx \right) (2\sqrt{3}x - \sqrt{3}) \\ &= x - \frac{1}{6} \end{aligned}$$

**Theorem 60.** Let  $\alpha_1 = \{v_1, v_2, \dots, v_k\}$  be orthonormal set in  $V$ ,  $n = \dim(V)$ , let  $W = \text{span}\{v_1, \dots, v_k\} \subseteq V$  be a subspace. Then,

(i)  $\alpha_1$  can be extended to an orthonormal basis of  $V$

$$\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$

(ii)  $\alpha_2 = \{v_{k+1}, \dots, v_n\}$  is an orthogonal basis of  $W^\perp = \{v \in V \mid \langle v, v_i \rangle = 0 \ \forall i = 1, \dots, k\}$

(iii)  $\dim(W) + \dim(W^\perp) = \dim(V)$

*Proof.* (idea)

- (i) Extend  $\{v_1, \dots, v_k\}$  to a basis in the usual way, then apply G-S.  
Omit.

□

*Diagonalization*

### Eigenvalues + eigenvectors

**Def** If  $T : V \rightarrow V$  is a linear operator, if  $\vec{v} \neq 0$  and  $T(\vec{v}) = \lambda \vec{v}$  for some  $\lambda \in K$ , then  $\vec{v}$  is called an eigenvector with eigenvalue  $\lambda$ .

Similarly if  $A \in M_{n \times n}(K)$  and  $A\vec{v} = \lambda \vec{v}$  and  $\vec{v} \neq 0$  then  $\vec{v}$  is an eigenvector with eigenvalue  $\lambda$ .

**Remark**  $\vec{v} \neq 0!$   $\lambda = 0$  is allowed!

**Proposition 61.**  $\lambda = 0$  is an eigenvalue  $\iff T$  is NOT INJECTIVE (in particular, not invertible!)

*Proof.* Prove both ways.

1. " $\Rightarrow$ " if  $\vec{v} \neq 0$  and

$$\begin{aligned} T(\vec{v}) &= \lambda \vec{v} = \vec{0} \\ \Rightarrow \vec{v} &\in \text{Ker}(T), \vec{v} \neq 0 \\ \Rightarrow \text{Ker}(T) &\text{ NOT trivial} \\ \Rightarrow T &\text{ is NOT injective} \end{aligned}$$

2. " $\Leftarrow$ "

$$\begin{aligned} T \text{ not inj} &\Rightarrow \text{Ker}(T) \text{ NOT trivial} \\ &\Rightarrow \exists \vec{v} \neq 0 \text{ s.t. } T(\vec{v}) = 0 \\ &\Rightarrow T(\vec{v}) = \vec{0} = 0 \cdot \vec{v} \\ &\Rightarrow \lambda \text{ is an eigenvalue with eigenvector } \vec{v} \end{aligned}$$

□

**Problem** Let  $\mathbb{R}^3 \rightarrow \mathbb{R}^3$  which gives a rotation by  $\pi$  about the  $z$ -axis!  
Thinking geometrically, can we find some eigenvalues and vectors?

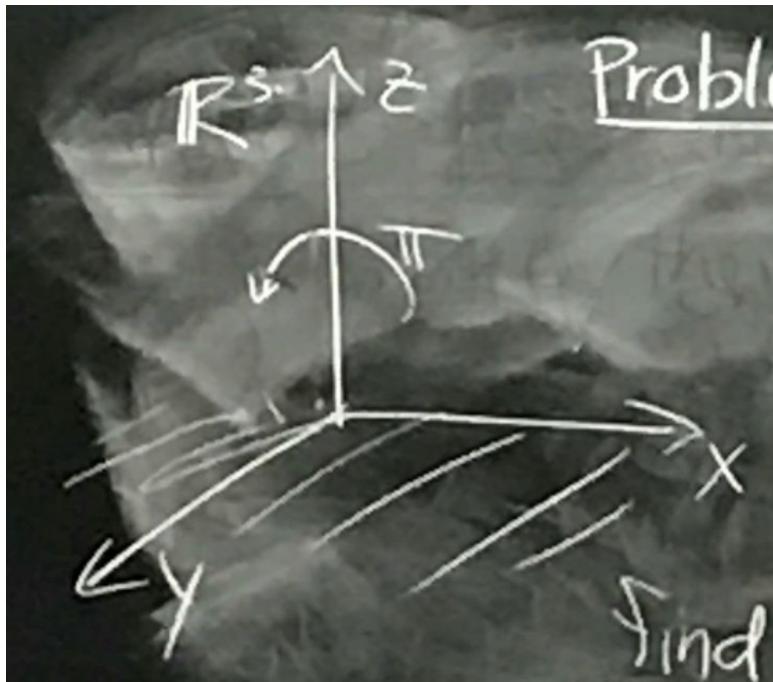


Figure 10: Problem

The  $z$ -axis itself  $(0, 0, 1) \in \mathbb{R}^3$  is an eigenvector with eigenvalue 1.  
 $z$ -axis is fixed by  $T$ .

$$\begin{aligned} &\Rightarrow T(\vec{v}_1) = \vec{v}_1 = \vec{v}_1 = 1 \cdot \vec{v}_1 \\ &\Rightarrow \lambda = 1 \text{ e.v.} \end{aligned}$$

Vectors lying in the  $x - y$  plane have  $\lambda = -1$  as an eigenvalue. Let  $v \in x - y$ -plane  $\rightarrow v = (x, y, 0)$

$$\begin{aligned} T(\vec{v}) &= (-x, -y, 0) \\ &= -(x, y, 0) \\ &= -1 \cdot \vec{v} \end{aligned}$$

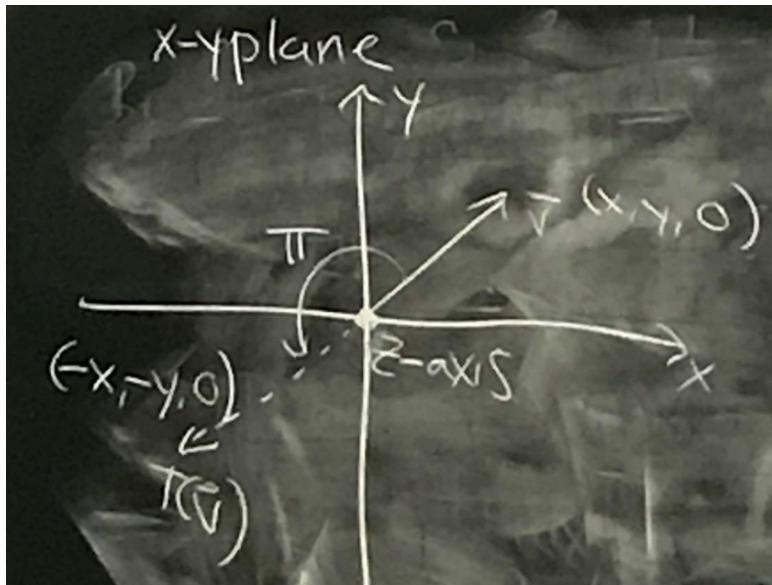


Figure 11: Problem

$\Rightarrow \lambda = -1$  is an eigenvalue!

**Question** How do we find eigenvalues and eigenvectors algebraically?

**Def** Let  $A \in \mathcal{M}_{n \times n}(K)$ . The *characteristic polynomial* of  $A$  is defined as

$$c_A(t) = \det(A - tI)$$

**Example** Let  $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ . Find  $c_A(t)$ .

**Sol**

$$\begin{aligned} c_A(t) &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} \\ &= \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & t-1 & 1-t \end{vmatrix} \\ &= (t-1) \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 0 & 1 & -1 \end{vmatrix} \\ &= (t-1) \begin{vmatrix} -t & -2 & -1 \\ 1 & 3-t & 1 \\ 0 & 0 & -1 \end{vmatrix} \\ &= (t-1)(-1) \begin{vmatrix} -t & -2 \\ 1 & 3-t \end{vmatrix} \\ &= -(t-1)(t^2 - 3t + 2) \end{aligned}$$

**Theorem 62.** Let  $A \in \mathcal{M}_{n \times n}(K)$ . Then

- (i)  $c_A(t)$  is a polynomial of degree  $n$
- (ii)  $\lambda$  is an eigenvalue of  $A \iff \lambda$  is a root of  $c_A(t)$
- (iii)  $v$  is an eigenvector  $\iff v \in \text{Ker}(A - \lambda I)$  and  $v \neq \vec{0}$

April 1st 2019

**Last time:** Recall theorem 62:

**Theorem 62:** Let  $A \in \mathcal{M}_{n \times n}(K)$ . Then

- (i) Characteristic polynomial  $c_A(t)$  is poly of degree  $n$
- (ii)  $\lambda$  is eigenvalue of  $A \iff \lambda$  is a root of  $c_A(t)$
- (iii)  $v \in K^n$  is eigenvector of  $A$  with eigenvalue  $\lambda \iff v \in \text{Ker}(A - \lambda I)$  and  $v \neq \vec{0}$

*Proof.* By direct proof.

(i) Omit.

(ii) We have:

$$\begin{aligned} \lambda \text{ is eigenvalue} &\iff \exists \text{ eigenvector } v \text{ with eigenvalue } \lambda \\ &\iff \text{Ker}(A - \lambda I) \neq \{\vec{0}\} \\ &\iff \text{lin transformation defined by } A - \lambda I \text{ not injective} \\ &\iff \text{lin transformation defined by } A - \lambda I \text{ not bijective} \\ &\iff A - \lambda I \text{ not invertible} \\ &\iff \det(A - \lambda I) = 0 \quad (\text{ie } c_A(\lambda) = 0) \end{aligned}$$

(iii) We have:

$$\begin{aligned} v \text{ is eigenvector with eigenvalue } \lambda &\iff Av = \lambda v \\ &\iff Av - \lambda v = \vec{0} \\ &\iff Av - \lambda Iv = \vec{0} \\ &\iff (A - \lambda I)v = \vec{0} \\ &\iff v \in \text{Ker}(A - \lambda I) \\ &\quad (\text{and } v \neq \vec{0}) \end{aligned}$$

□

**Def** If  $\lambda$  is an eigenvalue of  $A$ , the eigenspace for  $\lambda$  is

$$\begin{aligned} E_\lambda &= \text{Ker}(A - \lambda I) \\ &= \{v \in K^n | (A - \lambda I)v = \vec{0}\} \\ &= \{v \in K^n | Av = \lambda v\} \\ &= \text{all eigenvectors for } \lambda \text{ and also } \vec{0} \end{aligned}$$

It is a subspace since  $\text{Ker}$  is always a subspace.

**Ex**  $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ . Find basis for each eigenspace.

**Sol** Last class,  $c_A(t) = \begin{vmatrix} -t & -1 & -1 \\ 1 & 2-t & 1 \\ 1 & 1 & 2-t \end{vmatrix} = -(t-1)^2(t-2)$ .

Eigenvalues  $\lambda = 1, 2$ .

**Eigenspace  $E_1$**  ( $\lambda = 1$ ) Solve  $(A - I)\vec{x} = \vec{0}$ . Note: Can expect to get at least one free variable in these kind of problems.

$$\begin{pmatrix} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = -x_2 - x_3$$

$$x_2 = x_2$$

$$x_3 = x_3$$

$$\vec{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So basis  $\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}$  for  $E_1$ .

**Eigenspace  $E_2$**  Solve  $(A - 2I)\vec{x} = \vec{0}$

$$\begin{pmatrix} -2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = -x_3$$

$$x_2 = x_3$$

$$x_3 = x_3$$

So basis is  $\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$

**Proposition 63.** Let  $A, B \in \mathcal{M}_{n \times n}(K)$  be similar matrices, ie  $Q^{-1}AQ = B$  for some  $Q \in \mathcal{M}_{n \times n}(K)$ . Then,

$$(i) \ det(A) = det(B)$$

$$(ii) \ c_A(t) = c_B(t)$$

*Proof.* (i) Omit. Like (ii) (follows from it)

(ii)

$$\begin{aligned} c_A(t) &= \det(A - tI) \\ &= \det(QBQ^{-1} - tI) \\ &= \det(QBQ^{-1} - tQIQ^{-1}) \\ &= \det(Q(B - tI)Q^{-1}) \\ &= \det(Q)\det(B - tI)\det(Q^{-1}) \\ &= \det(Q)c_B(t)\frac{1}{\det(Q)} \\ &= C_B(t) \end{aligned}$$

□

**Def**  $T : V \rightarrow V$  linear op. The characteristic polynomial of  $T$  is

$$c_T(t) = \det([T]_\alpha - tI)$$

where  $\alpha$  is *any* basis of  $V$ .

**Remark**  $\alpha$  does not matter, since if  $\beta$  is any other basis then

$$[T]_\beta^\beta = Q_\alpha^\beta [T]_\alpha^\alpha Q_\beta^\alpha$$

ie  $[T]_\beta[T]_\alpha$  are similar, same characteristic polynomial.

**Ex** For  $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  defined by

$$T(f(x)) = (1+x)f'(x)$$

Find basis for each eigenspace.

**Sol** Need standard matrix for  $T$ , any basis. Use  $\alpha = \{1, x, x^2\}$ . Calculate

$$T(1) = (1+x)(0) = 0$$

$$T(x) = (1+x)(1) = 1+x$$

$$T(x^2) = (1+x)(2x) = 2x + 2x^2$$

$$[T]_\alpha^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$c_T(t) = \begin{vmatrix} -t & 1 & 0 \\ 0 & 1-t & 2 \\ 0 & 0 & 2-t \end{vmatrix} = t(1-t)(2-t)$$

So  $\lambda = 0, 1, 2$  eigenvalues.

**For**  $E_0$  ( $\lambda = 0$ ) Solve  $([T] - 0I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_1 = x_1$$

$$x_2 = 0$$

$$x_3 = 0$$

Basis  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  ie  $f(x) = 1$ .

**Check**  $T(f(x)) = T(1) = (1+x)0 = 0f(x)$ .

**For**  $E_1$

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = x_2$$

$$x_2 = x_2$$

$$x_3 = 0$$

Basis  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$  ie  $g(x) = 1+x$ .

**Check**  $T(f(x)) = T(1+x) = (1+x)(1) = 1(1+x)$ .

**For**  $E_2$

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 = x_3$$

$$x_2 = 2x_3$$

$$x_3 = x_3$$

Basis  $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$  ie  $h(x) = 1+2x+x^2$ .

**Check**  $T(f(x)) = T(1+2x+x^2) = (1+x)(2+2x) = 2+4x+2x^2 = 2(1+2x+x^2)$ .

April 3rd 2019

**Last time:**  $E_\lambda = \{v \in V | T(v) = \lambda v\} = \text{eigenspace}$

**Proposition 64.** Let  $\lambda_1 \neq \lambda_2$  be eigenvalues of  $T$ . Then

$$E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$$

*Proof.*  $\{\vec{0}\} \supseteq E_{\lambda_1} \cap E_{\lambda_2}$  since  $\vec{0}$  is in both subspaces. For other inclusion, suppose  $v \in E_{\lambda_1} \cap E_{\lambda_2}$ , so  $T(v) = \lambda_1 v$  and  $T(v) = \lambda_2 v$  so  $\lambda_1 v - \lambda_2 v = \vec{0}$  so

$$(\lambda_1 - \lambda_2)v = \vec{0}$$

If  $v \neq \vec{0}$ , then  $\lambda_1 - \lambda_2 = 0$ , contradicts  $\lambda_1 \neq \lambda_2$  so  $v = \vec{0}$ . So  $E_{\lambda_1} \cap E_{\lambda_2} = \{\vec{0}\}$ .  $\square$

*Diagonalization*

**Idea:** Diagonal matrices are very nice. Let  $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ .

$$\begin{aligned} AA &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & 4^2 \end{pmatrix} \end{aligned}$$

In fact,  $A^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}$ . Easy!

$$\begin{aligned} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= 3 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

In fact, eigenvalues are 2, 3, 4 corresponding eigenvectors are  $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ .

**Def**

1  $A \in \mathcal{M}_{n \times n}(K)$  is *diagonalizable* if  $\exists Q \in \mathcal{M}_{n \times n}(K)$  so that

$$Q^{-1}AQ = D \quad (\text{with } D \text{ diagonal})$$

(ie  $A$  is *similar* to a diagonal matrix)

- 2 Linear operator  $T : V \rightarrow V$  is *diagonalizable* if  $\exists$  basis  $\alpha$  of  $V$  so that  $[T]_\alpha$  is a diagonal matrix.

**Note:** For any bases  $\alpha, \beta$  of  $V$ ,

$$[T]_\alpha = Q^{-1}[T]_\beta Q \quad (Q = Q_\alpha^\beta)$$

ie  $T$  diagonalizable  $\iff [T]_\beta$  diagonalizable,  $\beta$  any basis.

**Theorem 65.** Let  $T : V \rightarrow V$  be linear operator.

- (1)  $T$  diagonalizable  $\iff \exists$  basis  $\alpha$  composed of eigenvectors of  $T$ .
- (2) If  $\alpha = \{v_1, v_2, \dots, v_n\}$  is basis of  $V$ , composed of eigenvectors of  $T$ , then

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Where  $\lambda_i$  is eigenvalue for  $v_i$ ,  $i = 1, 2, \dots, n$ .

- (3) If  $A \in \mathcal{M}_{n \times n}(K)$  is diagonalizable with

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Then the  $i$ -th column  $q_i$  of  $Q$  is an eigenvector for  $A$  with eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, n$ .

Also,  $\{q_1, q_2, \dots, q_n\}$  is a basis of  $K^n$ .

*Proof.* (1)  $\Rightarrow$  Assume  $T$  diagonalizable, ie  $\exists$  basis  $\alpha = \{v_1, v_2, \dots, v_n\}$  so that

$$[T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (\text{ie is diagonal})$$

Recall column  $i$  of  $[T]_\alpha$  is  $[T(v_i)]_\alpha$ . So  $[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$ , ie

$$T(v_i) = \lambda_i v_i$$

So  $v_i$  is eigenvector for  $\lambda_i$ , so  $\alpha$  is basis of  $V$  composed of eigenvectors.

" $\Leftarrow$ " Assume  $\alpha = \{v_1, v_2, \dots, v_n\}$  basis eigenvectors and  $T(v_i) = \lambda_i v_i$ . Then,

$$[T(v_i)]_\alpha = \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix}$$

$$\text{So } [T]_\alpha = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}. \text{ So } T \text{ diagonalizable.}$$

(2) Done in proof of (1).

(3) Assume

$$Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Let  $q_i$  = column  $i$  of  $Q$ . Write

$$AQ = QD$$

In  $AQ$ , column  $i$  is  $Aq_i$  (proposition 36). On the right side, column  $i$  is

$$\begin{aligned} Q(\text{col } i \text{ of } D) &= Q \begin{pmatrix} 0 \\ 0 \\ \lambda_i \\ 0 \\ 0 \end{pmatrix} \\ &= Q(\lambda_i e_i) \\ &= \lambda_i Q e_i \\ &= \lambda_i q_i \end{aligned}$$

So  $Aq_i = \lambda_i q_i$ . So  $q_i$  is eigenvector for eigenvalue  $\lambda_i$ .

□

**Ex** Diagonalize  $A = \begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix}$ , ie find  $Q, D$  so that  $Q^{-1}AQ = D$  = diagonal.

**Sol**

$$\begin{aligned}
 c_A(t) &= \begin{vmatrix} i-t & -3 \\ 1 & -i-t \end{vmatrix} \\
 &= (i-t)(-i-t) + 3 \\
 &= i-it+it+t^2+3 \\
 &= t^2+4 \\
 &= t^2-2i \\
 &= (t-2i)(t+2i)
 \end{aligned}$$

So  $\lambda = 2i, -2i$ **Eigenvectors** For  $\lambda = 2i$ .

$$\begin{aligned}
 \begin{pmatrix} -i & -3 & 0 \\ 1 & -3i & 0 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & -3i & 0 \\ -i & -3 & 0 \end{pmatrix} \\
 \rightarrow \begin{pmatrix} 1 & -3i & 0 \\ 0 & 0 & 0 \end{pmatrix} & \\
 x_1 &= 3ix_2 \\
 x_2 &= x_2 \\
 \begin{pmatrix} 3i \\ 1 \end{pmatrix} &
 \end{aligned}$$

Check

$$\begin{aligned}
 \begin{pmatrix} i & -3 \\ 1 & -i \end{pmatrix} \begin{pmatrix} 3i \\ 1 \end{pmatrix} &= \begin{pmatrix} -3-3 \\ 3i-i \end{pmatrix} = \begin{pmatrix} -6 \\ 2i \end{pmatrix} \\
 &= 2i \begin{pmatrix} 3i \\ 1 \end{pmatrix}
 \end{aligned}$$

For  $\lambda = -2i$ , similar. Get eigenvector  $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ . So  $\{\begin{pmatrix} 3i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\}$  is basis of  $\mathbb{C}^2$  composed of eigenvectors of  $A$ .

And  $Q^{-1}AQ = D$  where

$$D = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}, Q = \begin{pmatrix} 3i & -i \\ 1 & 1 \end{pmatrix}$$

Or

$$D = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}, Q = \begin{pmatrix} -i & 3i \\ 1 & 1 \end{pmatrix}$$

April 5th 2019

**Ex** Diagonalize  $A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ , if possible.

**Sol** We already found  $\lambda = 1, \lambda = 2$ .

$$E_1 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$E_2 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

Put bases together, is  $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$ . Basis  $\mathbb{R}^3$ ?

Check

$$\begin{aligned} Q &= \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \det Q &= -1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= -1(-1) - 0 \\ &= 1 \neq 0 \end{aligned}$$

So  $Q$  inv, this is basis of eigenvectors.

$$Q^{-1}AQ = D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Or,  $Q^{-1}AQ = D$ , with  $Q = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ,  $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

$$c_A(t) = (t-1)^2(t-2)$$

**Ex** Show  $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  is *not* diagonalizable.

$$\text{Sol } c_A(t) = \begin{vmatrix} 1-t & 1 & 0 \\ 0 & 1-t & 1 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)^3. \lambda = 1 \text{ only.}$$

$E_1$  Solve  $(A - I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

On free variable, so  $\dim E_1 = 1$ . Can't set basis of eigenvector of  $\mathbb{R}^3$ . So *not* diagonalizable.

**Ex** Is  $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$  diagonalizable?

**Sol**

$$\begin{aligned} c_A(t) &= \begin{vmatrix} 1-t & -2 \\ 1 & -1-t \end{vmatrix} = (1-t)(-1-t) + 2 \\ &= -1-t+t+t^2+2=t^2+1 \end{aligned}$$

In  $\mathbb{R}$ , no roots, so no eigenvalues, no eigenvectors.  $A$  is not diagonalizable as element of  $\mathcal{M}_{2 \times 2}(\mathbb{R})$ . But in  $\mathcal{M}_{2 \times 2}(\mathbb{C})$ , have  $\lambda = +/-i$  as eigenvalues. Obtain eigenvectors

$$\begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

Which are independent, so  $A$  is diagonalizable as element of  $\mathcal{M}_{2 \times 2}(\mathbb{C})$ .

$$\text{So } Q^{-1}AQ = D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

### Diagonalizability

**Def** If  $T : V \rightarrow V, S : V \rightarrow V$  are linear operators and  $a, b \in K$ , define  
lin op as  $aT + bS$  by

$$(aT + bS)(v) = aT(v) + bS(v), v \in V$$

**Proposition 66.** Let  $\lambda_1 \neq \lambda_2$  be eigenvalues of  $T : V \rightarrow V$  and  $v_1$  eigenvector for  $\lambda_1$ ,  $v_2$  eigenvector for  $\lambda_2$ . Then  $v_1, v_2$  are linearly independent.

*Proof.* Suppose  $a_1v_1 + a_2v_2 = \vec{0}$ . Consider linear operator  $T - \lambda_1I$   
( $I : V \rightarrow V, I(v) = v$ ).

Eval  $T - \lambda_1I$  at  $\vec{0} = a_1v_1 + a_2v_2$

$$\begin{aligned} \vec{0} &= (T - \lambda_1I)(\vec{0}) \\ &= (T - \lambda_1I)(a_1v_1 + a_2v_2) \\ &= (T - \lambda_1I)(a_1) + (T - \lambda_1I)(a_2v_2) \\ &= a_1(T - \lambda_1I)(v_1) + a_2(T - \lambda_1I)(v_2) \\ &= a_1(T(v_1) - \lambda_1I(v_1)) + a_2(T(v_2) - \lambda_1I(v_2)) \\ &= a_1(\lambda_1v_1 - \lambda_1v_1) + a_2(\lambda_2v_2 - \lambda_1v_2) \quad (v_1, v_2 \text{ eigenvectors}) \\ \vec{0} &= \vec{0} + a_2(\lambda_2v_2 - \lambda_1v_2) \end{aligned}$$

So,  $a_2(\lambda_2 - \lambda_1)v_2 = \vec{0}$ . But  $\lambda_1 - \lambda_2 \neq 0, v_2 \neq \vec{0}$  ( $v_2$  eigenvector).

So  $a_2 = 0$ . Similarly, using  $T - \lambda_2I$  we get  $a_1 = 0$ . So  $v_1, v_2$  linearly independent.  $\square$

**Theorem 67.** Let  $T : V \rightarrow V$  lin. op.,  $\lambda_1, \dots, \lambda_k$  eigenvalues. If  $\beta_i$  is a basis for  $E_{\lambda_i}$ ,  $i = 1, 2, \dots, k$  then

$$\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$$

is a linearly independent set of size

$$|\beta_1| + |\beta_2| + \dots + |\beta_k|$$

In particular, if  $\sum_{i=1}^k \dim E_{\lambda_i} = \dim V$ , then  $T$  is diagonalizable.

*Proof.* Similar to prop 66. Omit.  $\square$

**Def** Let  $\lambda$  be an eigenvalue of  $T$ .

- (1) The *geometric multiplicity* of  $\lambda$  is  $\dim E_\lambda$
- (2) The *algebraic multiplicity* of  $\lambda$  is the greatest  $m$  so that  $(t - \lambda)^m$  is a factor of  $c_T(t)$

**Ex**  $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$  for  $\lambda = 1$ , algebraic mult is 3 since

$$c_A(t) = (1 - t)^3$$

Geometric multiplicity is 1 since  $\dim E_1 = 1$  (one free variable, see previous example)

**Theorem 68.** Let  $\lambda$  be eigenvalue of  $T$ . Then,

$$1 \leq \text{geometric mult. of } \lambda \leq \text{algebraic mult. of } \lambda$$

*Proof.* We have two inequalities to prove.

- (1) Since  $\lambda$  eigenvalue,  $\exists$  a (non-zero) eigenvector, so  $\dim E_\lambda \geq 1$
- (2) Let  $d = \dim E_\lambda$  and let  $\{v_1, v_2, \dots, v_d\}$  be a basis for  $E_\lambda$ . Extend this to a basis  $\alpha = \{v_1, v_2, \dots, v_d, v_{d+1}, \dots, v_n\}$  of  $V$ .

Compute  $[T]_\alpha$ .

$$T(v_1) = \lambda v_1$$

$$T(v_2) = \lambda v_2$$

...

$$T(v_d) = \lambda v_d$$

$$T(v_{d+1}) = \text{something (doesn't actually matter)}$$

...

$$T(v_n) = \text{something (doesn't actually matter)}$$

Figure 12:

Figure 13:

So we have 12. So we have 13. (with  $I$  same size as  $C$ ) So

$$\begin{aligned} c_T(t) &= (\lambda - t)^d \det(C - tI) \\ &= (\lambda - t)^d c_C(t) \end{aligned}$$

So  $(\lambda - t)^d$  is a factor of  $c_T(t)$ . So the greatest factor of  $(\lambda - t)$  in  $c_T(t)$  is  $(\lambda - t)^m$  where  $m \geq d$ . Ie,  $d \leq m$  ie geometric multiplicity  $\leq$  algebraic multiplicity.

□

April 8th 2019

**Last time:**  $\lambda$  eigenvalue,

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$$

**Theorem 69.**  $T : V \rightarrow V$  linear operator,  $n = \dim V$ .

- (1) If  $K = \mathbb{R}$  and  $c_T(t)$  has a non-real root, then  $T$  is not diagonalizable.
- (2) If either  $K = \mathbb{C}$ , or  $K = \mathbb{R}$  and all the roots of  $c_T(t)$  are real, then

$$T \text{ diagonalizable} \iff$$

For every eigenvalue  $\lambda$  geom mult of  $\lambda =$  algebraic mult of  $\lambda$

(3) If  $T$  diagonalizable for each  $\lambda_i$  eigenvalue,  $\beta_{\lambda_i}$  is a basis of  $E_{\lambda_i}$ ,  
then  $\beta = \beta_{\lambda_1} \cup \beta_{\lambda_2} \cup \dots \cup \beta_{\lambda_k}$  is a basis of  $V$  (composed of  
eigenvectors)

*Proof.* (briefly) If  $\lambda_1, \lambda_2, \dots, \lambda_k$  are the eigenvalues then the largest set of independent eigenvectors has size

$$\begin{aligned}\sum_{i=1}^k \dim E_{\lambda_i} &= \sum_{i=1}^k \text{geom mult of } \lambda_i \\ &\leq \sum_{i=1}^k \text{alg mult of } \lambda_i = n\end{aligned}$$

So if any eigenspace is "short", ie

$$\dim E_{\lambda_i} < \text{alg mult } \lambda_i$$

then can't get  $n$  linearly independent eigenvectors.  $\square$

**Ex** Define  $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$  by

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

(it is linear). Is it diagonalizable?

**Sol** Find  $[T]_\alpha$ ,  $\alpha = \{1, x, x^2\}$

$$\begin{aligned}T(1) &= 1 + 0x + 0x^2 \\ T(x) &= 1 + 1x + (1 + 0)x^2 = 1 + x + x^2 \\ T(x^2) &= 1 + 0x + (0 + 2)x^2 = 1 + 0x + 2x^2 \\ [T]_\alpha &= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}c_T(t) &= \begin{vmatrix} 1-t & 1 & 1 \\ 0 & 1-t & 0 \\ 0 & 1 & 2-t \end{vmatrix} = (1-t) \begin{vmatrix} 1-t & 0 \\ 1 & 2-t \end{vmatrix} \\ &= (1-t)(2-t)\end{aligned}$$

For  $\lambda = 2$ .  $1 \leq \text{geom mult} \leq \text{alg mult} = 1$ . So  $\dim E_2 = 1 = \text{alg mult}$ .

Ok.

For  $\lambda = 1$ .

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 free variables, so  $\dim E_1 = 2 = \text{alg mult}$ . Ok.

So  $T$  is diagonalizable.

**Corollary 70.** If  $n = \dim V$ , and  $T : V \rightarrow V$  has  $n$  distinct eigenvalues, then  $T$  is diagonalizable.

*Proof.* For each of  $\lambda_1, \lambda_2, \dots, \lambda_n$

$$1 = \text{geom mult of } \lambda_i = \text{alg mult of } \lambda_i$$

□

**Ex**  $A = \begin{pmatrix} 5 & 3 & 1 \\ 4 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$ . Is  $A$  diagonalizable

(i) over  $K = \mathbb{R}$

(ii) over  $K = \mathbb{C}$

**Sol**

$$c_A(t) = \begin{vmatrix} 5-t & 3 & 1 \\ 4 & 2-t & 5 \\ 2 & 0 & 1-t \end{vmatrix} = -t^3 + 8t^2 - 3t + 24$$

Possible rational roots are  $\frac{\text{factors of const term}}{\text{factors of coeff of } t^n \text{ term}}$ . Here, factors of 24:

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$$

Sub each in  $-t^3 + 8t^2 - 3t + 24$ , find  $t = 8$  is a root. Then do long division.

$$\begin{aligned} c_A(t) &= -(t-8)(t^2+3) \\ &= -(t-8)(t+\sqrt{3}i)(t-\sqrt{3}i) \end{aligned}$$

(i) Over  $\mathbb{R}$ , not diagonalizable.

(ii) Over  $\mathbb{C}$ , diagonalizable (3 distinct eigenvalues in  $\dim 3$ )

*Application: Matrix Powers*

**Problem** Find formula for  $A^n$ ,  $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$

**Sol** If  $P^{-1}AP = D$ , then

$$A = PDP^{-1}$$

Then  $A^n = PDP^{-1}PDP^{-1}PDP^{-1} \dots PDP^{-1} = PD^nP^{-1}$ . Diagonalize

$A$  as

$$P = \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

$$P^{-1} = \frac{1}{-3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$$

So

$$\begin{aligned} A^n &= PD^nP^{-1} \\ &= P \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{-3} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{-3} \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2^n & -2^{n+1} \end{pmatrix} \\ &= \frac{1}{-3} \begin{pmatrix} -2 - 2^n & 2^{n+1} \\ -1 + 2^n & -2^{n+1} \end{pmatrix} \end{aligned}$$

### Symmetric Matrices

**Def**  $A \in \mathcal{M}_{n \times n}(K)$  is *symmetric* if  $A^T = A$

**Def** If  $A \in \mathcal{M}_{n \times n}(K)$  is symmetric, it is also called *positive definite* if for all  $u \in K^n$ ,  $u \neq \vec{0}$ ,

$$\underbrace{u^T}_{1 \times n} \underbrace{A}_{n \times n} \underbrace{u}_{n \times 1} > 0$$

**Theorem 71.** If  $A \in \mathcal{M}_{n \times n}(K)$  is symmetric and positive definite, then the formula

$$\langle u, v \rangle_A = u^T A$$

(where  $u, v \in K^n$ ) defines an inner product on  $K^n$

$$\begin{aligned} \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle &= (x_1 \ x_2 \ x_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= (x_1 \ x_2 \ x_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \\ &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= \text{usual dot product} \end{aligned}$$