

# MATH223 - Linear Algebra (class notes)

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# 1 January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra.**

## 1.1 Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned}3x - 2y + z &= 2 \\ x - y + z &= 1\end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in  $R^3$ . Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

## 1.2 Complex numbers

**Def:** Let  $i$  be a symbol. We declare  $i^2 = -1$ .

Now, what we'd like to do is take this symbol  $i$  and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned}3i \\ i - 4 \\ 3i - \pi \\ \sqrt{i} + 21\end{aligned}$$

**Def:** The field of complex numbers  $C$  consists of all expressions of the form  $a + bi$ , where  $a, b \in R$ .

**Def:** Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

**Notation:**

- $0 + bi = bi$
- $a + 0i = a$  (a *real* number)
- $0 + 0i = 0$

**Ex:** If  $z_1 = 2 - i$ ,  $z_2 = 5i$ , then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

**Def:** Let  $z = a + bi \in C$

- (i)  $\bar{z} = a - bi$ , called the *complex conjugate* of  $z$
- (ii)  $|z| = \sqrt{a^2 + b^2}$ , called the *absolute value* or *modulus*

**Def:** If  $z = a + bi \in C$  and  $z \neq 0$  (ie  $z \neq 0 + 0i$ ), then the number

$$\begin{aligned}z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\end{aligned}$$

is called the (multiplicative) inverse of  $z$ . It has the property  $zz^{-1} = 1 = z^{-1}z$ .

*Proof.* We have

$$\begin{aligned} zz^{-1} &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

**Note:** Since  $z \neq 0 + 0i$ ,  $a^2 + b^2 \neq 0$

□

**Def:** If  $z, w \in C$  and  $z \neq 0$  then

$$\frac{w}{z} = wz^{-1}$$

**Ex:** If  $z = 1 + 2i, w = 3 - i$  then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3 - i)\left(\frac{1}{5} - \frac{2}{5}i\right) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned} \frac{3 - i}{1 + 2i} \cdot \frac{(1 - 2i)}{(1 - 2i)} &= \frac{3 - 6i - i + 2i^2}{1 - 2i + 2i - 4i^2} \\ &= \frac{1 - 7i}{5} \end{aligned}$$

## 2 January 9th 2019

### 2.1 Complex numbers as points in $R^2$

You can view  $a + bi$  as a point  $(a, b) \in R^2$ . The usefulness of this is that we can consider, say,  $(3 + 2i)$  and  $(3 - i)$  as vectors in  $R^2$ , and they will conserve the same properties (addition of complex numbers corresponds to vector addition in  $R^2$ ). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

## 2.2 Equations with complex numbers

**Fact:** Every real number  $a \neq 0$  has two square roots:

- if  $a > 0$ , roots  $\pm\sqrt{a}$
- if  $a < 0$ , two roots are  $\pm i\sqrt{|a|}$ , since:

$$\begin{aligned}(\pm i\sqrt{|a|}) &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a\end{aligned}\quad (\text{since } a < 0)$$

**Fact:** Quadratic equation  $ax^2 + bx + c = 0$  has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in  $C$ .

**Ex:** Solve  $x^2 - 2x + 3 = 0$ , and factor  $x^2 - 2x + 3$ .

**Sol:**

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2}\end{aligned}$$

**Note:** If  $ax^2 + bx + c$  has  $a, b, c \in R$  has a non-real root, say  $z$ , its other root is  $\bar{z}$  ( $z = a + bi$ ,  $\bar{z} = a - bi$ ). This is not necessarily true if  $a, b, c \in C$ .

Back to problem. Factor  $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$ .

**Caution:**  $-1$  has two roots, namely  $\pm i$ , so you may write  $i = \sqrt{-1}$ , but be careful:

$$\begin{aligned}-1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && (\text{this step doesn't quite work}) \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

**Theorem:** (Fundamental Theorem of Algebra) If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$$

is a polynomial with  $a_n \neq 0$ , and  $a_n, a_{n-1}, \dots, a_0 \in C$ , then  $p(x)$  factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \dots \cdot (x - r_n)$$

for some complex numbers  $r_1, r_2, \dots, r_n$ . Some  $r_i$ 's may be equal.

**Corollary:** Every such polynomial has at least one root, and at most  $n$  distinct roots.

**Note:** *Finding* the roots is, in general, quite difficult.

**Ex:** Factor  $2x^3 + 2x$  (over C).

**Sol:**

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

**Ex:** Solve  $x^2 - i = 0$

**Sol:**  $x^2 = i$  so  $x = \pm\sqrt{i}$ . Want  $\sqrt{i}$  in format  $a + bi$ ,  $a, b \in R$ .

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$0 = a^2 - b^2$$

$$1 = 2ab$$

$$a = \pm b$$

$$ab = \frac{1}{2}$$

(so  $a=b$  both + or both -)

$$a^2 = \frac{1}{2}$$

$$a = \pm \frac{1}{\sqrt{2}} = b$$

Two solutions,  $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$  and  $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ .

## 2.3 Vector spaces (Ch 4)

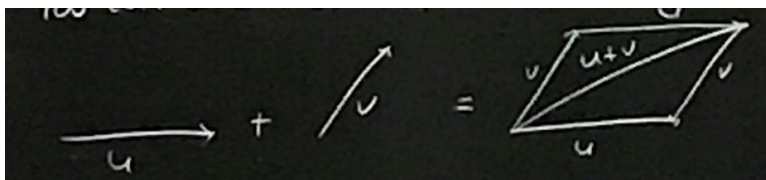
**Def.** The sets  $R$  and  $C$  (and also  $Q$ , rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of  $K$ " means that  $K$  is either  $R$  or  $C$ .

## 3 January 11th 2019

**Last time:** Field  $K$  is  $R$  or  $C$  (for this class).

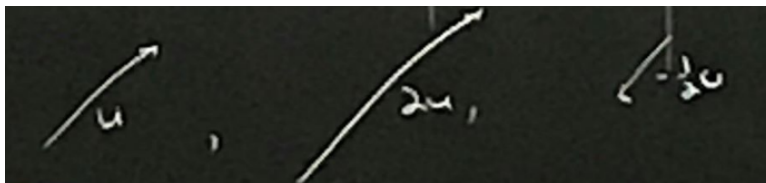
### 3.1 Geometric vectors ('arrows')

You can add two vectors (arrows).



**Observation:**  $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ .

You can rescale a vector:



**Observation:**  $a(b\vec{u}) = (ab)\vec{u}$ .

Also:  $1\vec{u} = \vec{u}$

**Question:** What properties are interesting? What other objects obey the same properties?

**Abstraction:** Focus on properties more than on the objects.

### 3.2 Definition of a vector space

Let  $V$  be a set, called set of "vectors", and let  $K$  be a field ( $R$  or  $C$ ) (elements of  $K$  called *scalars*). Assume that we have already defined two operations:

- (1) One called *addition*, which takes two vectors  $\vec{u}, \vec{v} \in V$  and produces another vector denoted  $\vec{u} + \vec{v} \in V$ .

- (2) One called *scalar multiplication* which takes a vector  $\vec{u} \in V$  and a scalar  $a \in K$  and produces another vector denoted  $a\vec{u} \in V$

Then if, for all vectors  $\vec{u}, \vec{v}, \vec{w} \in V$  and all scalars  $a, b \in K$ , the following 8 properties are true, then  $V$  is called a *vector space* (over  $K$ ).

- (A1)  $u + v = v + u$  (commutative laws)
- (A2) There exists a vector in  $V$ , named *zero vector* and denoted  $0$  (or  $\vec{0}$ ) such that for all  $u \in V$ ,  $u + 0 = u$
- (A3) For each  $u \in V$ , there is a vector in  $V$ , called the (additive) inverse of  $u$  and denoted  $-u$ , having the property  $u + (-u) = 0$  (where  $0$  is the zero vector defined in A2)
- (A4)  $(u + v) + w = u + (v + w)$
- (SM1)  $a(u + v) = au + av$  (distributive laws)
- (SM2)  $(a + b)u = au + bu$
- (SM3)  $a(bu) = (ab)u$
- (SM4)  $1u = u$  ( $1 \in R$  or  $C$ )

These are called the vector space *axioms*.

### 3.3 Examples of vector spaces

Some examples:

- (1)  $K^n = \{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in K\}$ , with addition defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$$

where  $c \in K$  (and  $K$  = set of scalar).

#### **Proof that $K^n$ is a vector space**

Need to prove all 8 properties. We will do 2, the rest are exercises.

- (A4) To prove for all  $u, v \in V$ ,  $u + v = v + u$ .  
*Proof concept:* To prove "for all  $x \in A$ , something", say "let  $x \in A$ " (means  $x$  is an arbitrary element of  $A$ , ie you only know  $x \in A$ ). Then, prove something for that  $x$ .



*Proof:* Let  $u, v \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ ,  $v = (b_1, b_2, \dots, b_n)$  for some  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in K$ . Then

$$\begin{aligned} u + v &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, \dots, a_n + b_n) \quad (\text{definition of addition in } K^n) \\ &= (b_1 + a_1, \dots, b_n + a_n) \quad (\text{since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \dots, b_n) + (a_1, \dots, a_n) \quad (\text{definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define  $0 = (0, 0, \dots, 0)$  (which is in  $K^n$ ). To prove for all  $u \in K^n$ ,  $u + 0 = u$ , let  $u \in K^n$ . This means  $u = (a_1, a_2, \dots, a_n)$ , so

$$\begin{aligned} u + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space  $C^2$ ,  $(2 + 3i, 5 - 7i) \in C^2$  is an example of a vector and  $2i \in C$  is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

## 4 January 14th 2019

**Problem:** Let  $J = \{(x, y) | x \in R, y \in R\}$  but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that  $J$  is not a vector space.

**Solution:** Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\&= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\&= (x_1 + x_2 - x_3, y_1 + y_2 + y_3)\end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.

**Actual proof:** Let  $u = (1, 1)$ ,  $v = (2, 2)$  and  $w = (3, 3)$ . Then,

$$\begin{aligned}u + (v + w) &= (1, 1) + ((2, 2) + (3, 3)) \\&= (1, 1) + (-2 - 3, 5) \\&= (1, 1) + (-5, 5) \\&= (-1 + 5, 6) \\&= (4, 6)\end{aligned}$$

Whereas,

$$\begin{aligned}(u + v) + w &= ((1, 1) + (2, 2)) + (3, 3) \\&= (-1 - 2, 3) + (3, 3) \\&= (-3, 3) + (3, 3) \\&= -(-3) - 3, 6) \\&= (0, 6)\end{aligned}$$

Hence, the axiom does not hold.

#### 4.1 More examples of vector spaces

- (1)  $K^n$  (ie  $R^n$  or  $C^n$ ). See before
- (2)  $P(K)$  = polynomials, where coefficients are in  $K$ . Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}f(x) &= x^2 + 2ix - 4 \in P(C) \\g(x) &= -x^2 + cx \in P(C) \quad \text{(and also in } P(R))\end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned}2if(x) &= 2ix^2 + 4i^2x - 8i \\&= 2ix^2 - 4x - 8i\end{aligned}$$

- (3)  $P_n(K)$  = polynomials of degree  $n$  or less, coefficient from  $K$ . For example,

$$x^2 - 2x + 2 \in P_2(R)$$

$$x^2 - 2x + 2 \in P_3(R)$$

$$x^2 - 2x + 2 \in P_2(C)$$

$$x^2 - 2x + 2 \notin P_1(R)$$

**Note:** In  $P(K)$ ,  $P_n(K)$  the "vectors" are polynomials.

- (4)  $M_{m \times n}(K)$  =  $m \times n$  matrices with entries from  $K$ . Scalars are  $K$ , addition and scalar multiplication as expected.

$$A = \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2 \times 2}(C)$$

$$B = \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2 \times 2}(C)$$

$$A + B = \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix}$$

$$2iA = \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix}$$

$$= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix}$$

The "zero vector" in  $M_{m \times n}(K)$  is the  $m \times n$  matrix with all entries 0.

- (5) Let  $X$  be any set (think  $x = R$  or  $C$ , but not required). Define  $F(X, K) = \{f : X \rightarrow K\}$  = all functions from  $X$  to  $K$ .

**Ex:**  $f(x) = x^2 \in F(R, R)$ .

**Ex:** Let  $x = \{1, 2\}$ . Then  $g$  defined by

$$g(1) = 3$$

$$g(2) = \sqrt{2}$$

*Addition* in this space is defined by:

If  $f, g \in F(X, K)$  then  $f + g$  is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

*Note that*  $f(x) \in K$  and  $g(x) \in K$ , in other words they are *numbers* (scalars). The  $+$  in  $(f + g)$  is the addition of vectors  $f$  and  $g$ , while the other  $+$  is scalar addition.

*Scalar multiplication* in this space is defined by: if  $f \in F(X, K)$ ,  $c \in K$  then  $cf$  is the function defined by

$$(cf)(x) = cf(x)$$

*Note that  $cf$  is the name of the function, that "multiplication" is scalar multiplication  $F(X, \mathbb{F})$  and  $cf(x)$  is the multiplication of two scalars (numbers).*

The fact that  $F(X, K)$  is a vector space and the axioms are followed is not so obvious.

**Prove (A2) true for  $F(X, K)$ .** Define  $z \in F(X, K)$  by

$$z(x) = 0 \quad (\text{for all } x \in X)$$

Note that 0 here is a scalar. Then if  $f \in F(X, K)$  is an arbitrary element, then we need to prove  $f + z = f$ . This is true since for all  $x \in X$ ,

$$\begin{aligned} (f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x) \end{aligned}$$

Hence,  $f + z, f$  have the same output (namely  $f(x)$ ) for every input. Hence,  $f + z = f$ .

**Exercise:** Try (A3).

## 5 January 16th 2019

**Theorem:** ("Cancellation Law") Suppose  $v$  is a vector space over  $K$ . For all vectors  $u, v, w \in V$ , if  $u + w = v + w$  then  $u = v$ .

*Note:* To prove "for all" you say let  $u \in V$  (means  $u$  is an arbitrary vector).

To prove "if  $p$  then  $q$ ", denoted  $p \rightarrow q$ , assume  $p$  is true and use it to prove  $q$ .

*Proof.* Let  $u, v, w \in V$ . Assume  $u + w = v + w$ . By vector space axiom A3, there is a vector  $(-w) \in V$ . Add  $(-w)$  to both sides:

$$\begin{aligned} (u + w) + (-w) &= (v + w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) && (\text{by A1}) \\ u + \vec{0} &= v + \vec{0} && (\text{by A3}) \\ &= u = v && (\text{by A2}) \end{aligned}$$

□

**Theorem:**

1. The zero vector is unique
2. For each  $u \in V$ ,  $-u$  is unique

*Note:* To prove something is unique, suppose you have two of them and show they are the same.

*Proof.* 1) Assume 0 and  $z$  both satisfy the property (A2:  $\forall u \in V, u + 0 = u$  (\*) and  $u + z = u$  (\*\*)). Goal is to prove  $0 = z$ .

$$\begin{aligned} z &= z + 0 && \text{(by *, with } u = z) \\ &= 0 + z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0) \end{aligned}$$

So the zero vector is unique.

2) Exercise. □

**Theorem:**  $\forall u \in V, c \in K$ ,

- 1)  $c\vec{0} = \vec{0}$
- 2)  $0u = \vec{0}$
- 3)  $-(cu) = ((-c)u)$

*Proof.* Of 2). Let  $u \in V$ . Then,

$$\begin{aligned} 0u + 0u &= (0 + 0)u && \text{(By SM2)} \\ 0u + 0u &= 0u && \text{(by R addition)} \\ 0u + 0u &= 0u + \vec{0} && \text{(by A2)} \\ 0u + 0u &= \vec{0} + 0u && \text{(by A4)} \\ 0u &= \vec{0} && \text{(by cancellation law)} \end{aligned}$$

□

*Note:*  $0 + u = u$  is true for all  $u \in V$  (same as  $u + 0 = u$  then apply A4)

## 5.1 Linear combinations and spans

**Def:** Let  $u, v_1, v_2, \dots, v_n \in V$ . If there are scalars  $a_1, a_2, \dots, a_n \in K$  such that  $u = a_1v_1 + a_2v_2 + \dots + a_nv_n$  then  $u$  is said to be a linear combination of  $v_1, v_2, \dots, v_n$ .

**Ex:** In  $P(R)$ ,  $x^2 + 2x - 4$  is a linear comb of  $x^2, x, 1$ .

**Important problem:** Given vectors  $u, v_1, v_2, \dots, v_n$ , determine if  $u$  is a linear combination of  $v_1, v_2, \dots, v_n$  and if so find  $a_1, a_2, \dots, a_n$ .

**Ex:** Determine if  $f(x) = 2x^2 + 6x + 8$  is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

**Sol.** Are there  $a_1, a_2, a_3$  s.t.

$$\begin{aligned} 2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\ &= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3) \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a_1 - 2a_2 + 2a_3 &= 2 \\ 2a_1 - 4a_2 &= 6 \\ a_1 - 2a_2 - 3a_3 &= 8 \end{aligned}$$

Solve the linear system:

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array} \right] \\ \downarrow \\ \left[ \begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{row reduce}) \end{aligned}$$

$\therefore$  No solution, because of the last row.  $f$  is not a linear combination of  $g_1, g_2, g_3$ .

**Def:** Let  $S \subseteq V$  ( $S$  is a subset of  $V$ ) and assume  $s \neq 0$ . The span of  $s$ , denoted  $\text{span}(s)$  is the set of all linear combinations of vectors from  $S$ , ie

$$\begin{aligned} \text{span}(s) &= \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ &\quad \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ &\quad u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

## 6 January 18th 2019

### 6.1 Last class

$$\begin{aligned} S &\subseteq V \\ \text{span}(s) &= \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ &\quad \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ &\quad u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

**Ex:**  $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$ . Prove  $\text{span}(S) = \mathbb{R}^2$ .

**Note:**  $\begin{pmatrix} a \\ b \end{pmatrix}$  means  $(a, b)$ .

**Proof note:** To prove two sets  $A, B$  are equal, ie  $A = B$ , you can prove  $A \subseteq B$  and  $B \subseteq A$ .

**Sol:**

- (1) Prove  $\text{span}(S) \subseteq R^2$ . Trivial, since any linear combination of vectors in  $R^2$  is still in  $R^2$ .
- (2) Prove  $R^2 \subseteq \text{span}(S)$ . Let  $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$  (arbitrary). To prove that there exists scalars  $x_1, x_2 \in K$  so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all  $a, b$ ). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution.  $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{span}(S)$  so  $R^2 \subseteq \text{span}(S)$ . So by (1), (2),  $\text{span}(S) = R^2$ .  $\square$

**Note:**  $Ax = b$ ,  $A_{n \times n}$  if  $A$  inv,  $x = A^{-1}b$ .

**Theorem:** Let  $S \subseteq V$ ,  $S \neq \emptyset$  ( $\emptyset$  = empty set). Then,

- (1) If  $u, v \in \text{span}(S)$  then  $u + v \in \text{span}(S)$
- (2) If  $u \in \text{span}(S)$  and  $c \in K$ , then  $cu \in \text{span}(S)$
- (3)  $\vec{0} \in \text{span}(S)$

*Proof.* By direct proof.

- (1) (Note, "if  $u, v \in \text{span}(S)$ " means for all  $u, v \in \text{span}(S)$ ).  
Let  $u, v \in \text{span}(S)$ . Then,

$$\begin{aligned} u &= a_1u_1 + a_2u_2 + \dots + a_nu_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K \\ v &= b_1v_1 + b_2v_2 + \dots + b_mv_m \text{ where } v_1, \dots, v_m \in S, b_1, \dots, b_m \in K \end{aligned}$$

Then  $u + v = a_1u_1 + \dots + a_nu_n + b_1v_1 + \dots + b_mv_m$  which is in  $\text{span}(S)$  since  $u_1, \dots, u_n, v_1, \dots, v_m \in S$ .

(2) Let  $u \in \text{span}(S), c \in K$ . Then,

$$u = a_1u_1 + a_2u_2 + \dots + a_nu_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \dots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_n)u_n \end{aligned}$$

**Note:** If you want to be very formal, you need to write down all of the vector space axioms. Which is in  $\text{span}(S)$  since it is a linear combination of  $a_1, \dots, a_n$  which are in  $S$ .

(3) (Prove  $\vec{0} \in \text{span}(S)$ ) Let  $u \in S$ . **Note:** This is possible only because  $S \neq \emptyset$ .

Then  $u = 1u$ , so  $u \in \text{span}(S)$ . Then using  $c = 0$  and (2) and fact that  $u \in \text{span}(S)$ ,

$$cu = 0u = \vec{0}$$

is also in  $\text{span}(S)$ . **Note:** Since  $u = 1u$ ,  $S \subseteq \text{span}(S)$ .

□

## 6.2 Subspaces

**Def.** Let  $V$  be a vector space and  $W \subseteq V$  (subset). If  $W$ , using addition and scalar multiplication as defined in  $V$ , satisfies the definition of vector space, then  $W$  is called a subspace of  $V$ , denoted  $W \leq V$  (less than equal sign, read as "subspace").

**Note:** Main issue is that addition and scalar multiplication with vector from  $W$  produce vectors which are still in  $W$ .

**Theorem:** Let  $W \subseteq V$ . Then, if the following three properties hold, then  $W \leq V$  (subspace).

(SS1) For all  $w_1, w_2 \in W$ , we have  $w_1 + w_2 \in W$  ("closure under addition")

(SS2) For all  $w \in W$  and scalars  $c \in K$ , we have  $cw \in W$  ("closure under scalar multiplication")

(SS3)  $\vec{0} \in W$ .

These are the same properties we just proved for spans; in other words, we proved earlier that  $\text{span}(S)$  is a subspace.

*Proof.* For  $W$  to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.



(A1) Let  $u, v, w \in W$ . Then since  $u, v, w \in V$ , and (A1) holds in  $V$ ,  $u + (v + w) = (u + v) + w$ .

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let  $w \in W$ . Want to show  $-w \in W$ . Then, using (SS2) with  $c = -1$  gives

$$-1(w) = -w \quad (\text{thm from last class})$$

is in  $W$ , as needed.

(A4) Still true because it is true in  $V$ .

(SM1-SM4) All hold because they hold in  $V$ .

□

## 7 January 21st 2019

### 7.1 A note on logic

Let  $P, Q$  be statements that are true or false.

- (1) “If  $P$  then  $Q$ ”, also written symbolically as “ $P \Rightarrow Q$ ” ( $P$  *implies*  $Q$ ) means if  $P$  is true, then  $Q$  is also true. To *prove* “ $P \Rightarrow Q$ ”, assume  $P$  and prove  $Q$  is true. If you *know* that “ $P \Rightarrow Q$ ” is true, you can *use it*: if you can establish that  $P$  is true, you may conclude  $Q$  is true.

**Ex:** Let  $A$  be an  $n \times n$  matrix:

$$P : \det(A) = 1 \quad Q : \text{“} A \text{ is invertible”}$$

**Thm:**  $P \Rightarrow Q$

- (2) The *converse* of “ $P \Rightarrow Q$ ” is “ $Q \Rightarrow P$ ”. This is a (logically) different statement.

**Ex:** With  $P$  and  $Q$  as above, “ $Q \Rightarrow P$ ” is not true because  $A_{inv} \not\Rightarrow \det(A) = 1$ .

- (3) The *contrapositive* of “ $P \Rightarrow Q$ ” is “ $\neg Q \Rightarrow \neg P$ ” ie “if  $Q$  false, then  $P$  also false”. Logically, this is the same as “ $P \Rightarrow Q$ ”.

- (4) The *equivalence* “ $P$  if and only if  $Q$ ”, written “ $P \iff Q$ ” means “ $P \Rightarrow Q$  and also  $Q \Rightarrow P$ ” is true. Also means that either both  $P$  and  $Q$  are true or both are false.

**Ex:**  $\det(A) \neq 0 \iff A$  is invertible.

To prove “ $P \iff Q$ ”, need to prove “ $P \Rightarrow Q$ ” and “ $Q \Rightarrow P$ ”.

**Note:**  $\neg P \Rightarrow \neg Q$  is the same as  $Q \Rightarrow P$ .

## 7.2 Subspaces (cont'd)

**Thm (last class):** Let  $W \subseteq V$  (*subset*). If

1. For all  $u, v \in W$ ,  $u + v \in W$
2. For all  $u \in W$ ,  $c \in K$ ,  $cu \in W$
3.  $\vec{0} \in W$

then  $W \leq V$  (*subspace*). (ie: (1), (2), (3) are true  $\Rightarrow W \leq V$ )

**Thm.** Let  $W \subseteq V$ . Then

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).

**Proof.** Exercise.

**Thm.** Let  $W \subseteq V$ . Then

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

## 7.3 Examples of problems

Is each subset a subspace?

(a))  $W = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq R^2$ . Not a subspace, since the zero vector is not in  $W$ . The others are also false, but it's enough to prove that one of the statements does not hold. But  $\text{span}(W) = R^2$  (so  $\text{span}(W) \leq R^2$ )

(b))  $W = \{(x, y, z) \in R^3 \mid x + y - z = 0\}$ . Need to check (1), (2), (3):

(1) Let  $\begin{pmatrix} x & y & z \end{pmatrix}, \begin{pmatrix} x' & y' & z' \end{pmatrix} \in W$ . Then we know  $x + y - z = 0$  and  $x' + y' - z' = 0$ . Check:

$$\begin{pmatrix} x & y & z \end{pmatrix} + \begin{pmatrix} x' & y' & z' \end{pmatrix} = \begin{pmatrix} x + x' & y + y' & z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in  $W$ .

(2) Let  $(x, y, z) \in W$  (means  $x + y - z = 0$ ), let  $c \in K$ . To prove

$$c \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} cx & cy & cz \end{pmatrix} \in W$$

Here,  $cx + cy - cz = c(x + y - z) = c(0) = 0$ . So  $c(x, y, z) \in W$

(3)  $(0, 0, 0) \in W$ , since  $0 + 0 - 0 = 0$

Since (1), (2), (3) true,  $W \leq R^2$  (subspace)

(c))  $W = \{(x, y, z) \in R^3 | x + y - z = 1\}$ . This is *not* a subspace. (3) is false.

(d))  $W = \{A \in M_{2 \times 2} | A_{ij} \geq 0 \forall i, j\}$ , where  $A_{ij}$  is the entry of  $A$  in row  $i$ , column  $j$ . (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

$$(2) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$$

Note, we wrote these out very informally. Now, (2) is false since, for example  $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$  but

$$(-1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

## 7.4 Two special subspaces

Let  $V$  be a vector space.

(1)  $V \leq V$  is true

(2)  $\{\vec{0}\} \leq V$  is true ("zero subspace")

## 7.5 A refinement on the definition of span

**Def.** If  $S = \emptyset$  (emptyset), define  $\text{span}(S) = \{\vec{0}\}$  (if  $S \neq \emptyset$ ,  $\text{span}(S)$  defined as before).

**Thm.**  $\text{span}(S) \leq V$ .

**Proof** Two cases :

1. If  $S = \emptyset$ ,  $\text{span}(S) = \{\vec{0}\} \leq V$

2. If  $S \neq \emptyset$ , you already proved  $\text{span}(S)$  satisfies (1), (2), (3). So  $\text{span}(S) \leq V$

**Thm.** (improved version of subspace conditions) Let  $W \subseteq V$ . Then

$$W \leq V \iff W \neq \emptyset \text{ and}$$

$$\forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W$$

**Proof** We will actually prove (1), (2), (3)  $\iff$  *RHS* (right-hand side). Two parts to proof.

(1) "(1), (2), (3)  $\Rightarrow$  *RHS*" or " $\Rightarrow$ "