MATH223 - Linear Algebra (class notes)

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Contents

1 January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra**.

1.1 Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes in R^3. Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

1.2 Complex numbers

Def: Let i be a symbol. We declare $i^2 = -1$.
Now, what we'd like to do is take this symbol i and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} 3i \\ i - 4 \\ 3i - \pi \\ \sqrt{i} + 21 \end{aligned}$$

Def: The field of complex numbers C consists of all expressions of the form $a+bi$, where $a, b \in R$.
Def: Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

(ii)

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2 \\ &= ac+adi+bci-bd \\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Notation:

- $0+bi=bi$
- $a+0i=a$ (a *real* number)
- $0+0i=0$

Ex: If $z_1=2-i$, $z_2=5i$, then

$$z_1+z_2=2+4i$$

and

$$z_1z_2=(2-i)(5i)=10i-5i^2=5+10i$$

Def: Let $z=a+bi \in C$

(i) $\bar{z}=a-bi$, called the *complex conjugate* of z

(ii) $|z|=\sqrt{a^2+b^2}$, called the *absolute value* or *modulus*

Def: If $z=a+bi \in C$ and $z \neq 0$ (ie $z \neq 0+0i$), then the number

$$\begin{aligned}z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i\end{aligned}$$

is called the (multiplicative) inverse of z. It has the property $zz^{-1}=1=z^{-1}z$.

Proof. We have

$$\begin{aligned} zz^{-1} &= (a+bi)(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i) \\ &= \frac{a^2 - abi + abi - b^2 i^2}{a^2+b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2+b^2} \\ &= 1 \end{aligned}$$

Note: Since $z \neq 0 + 0i$, $a^2 + b^2 \neq 0$ ☐

Def: If $z, w \in C$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

Ex: If $z = 1 + 2i, w = 3 - i$ then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3-i)(\frac{1}{5} - \frac{2}{5}i) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned} \frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3 - 6i - i + 2i^2}{1 - 2i + 2i - 4i^2} \\ &= \frac{1-7i}{5} \end{aligned}$$

2 January 9th 2019

2.1 Complex numbers as points in R^2

You can view $a+bi$ as a point $(a, b) \in R^2$. The usefulness of this is that we can consider, say, $(3+2i)$ and $(3-i)$ as vectors in R^2, and they will conserve the same properties (addition of complex numbers corresponds to vector addition in R^2). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

2.2 Equations with complex numbers

Fact: Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}(\pm i\sqrt{|a|}) &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a && \text{(since } a < 0\text{)}\end{aligned}$$

Fact: Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in C.

Ex: Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.
Sol:

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2}\end{aligned}$$

Note: If $ax^2 + bx + c$ has $a, b, c \in R$ has a non-real root, say z, its other root is $\bar{z}$ ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in C$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.

Caution: -1 has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned}-1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\ &= \sqrt{1} \\ &= 1\end{aligned}$$

Theorem: (Fundamental Theorem of Algebra) If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 n^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \ldots, a_0 \in C$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \ldots \cdot (x - r_n)$$

for some complex numbers $r_1, r_2, \ldots, r_n$. Some r_i's may be equal.

Corollary: Every such polynomial has at least one root, and at most n distinct roots.

Note: *Finding* the roots is, in general, quite difficult.

Ex: Factor $2x^3 + 2x$ (over C).
Sol:

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

Ex: Solve $x^2 - i = 0$
Sol: $x^2 = i$ so $x = \pm\sqrt{i}$. Want $\sqrt{i}$ in format $a + bi$, $a, b \in R$.

$$\begin{aligned} \sqrt{i} &= a + bi \\ i &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ 0 + i &= (a^2 - b^2) + 2abi \end{aligned}$$

$$\begin{aligned} 0 &= a^2 - b^2 \\ 1 &= 2ab \\ a &= \pm b \\ ab &= \frac{1}{2} && \text{(so a=b both + or both -)} \\ a^2 &= \frac{1}{2} \\ a &= \pm\frac{1}{\sqrt{2}} = b \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

2.3 Vector spaces (Ch 4)

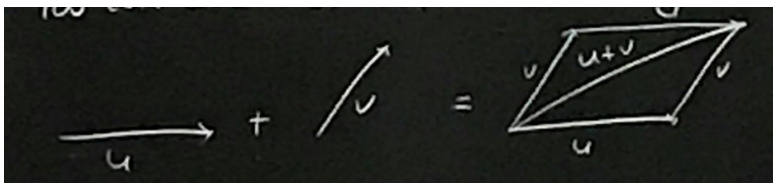
Def. The sets R and C (and also Q, rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of K" means that K is either R or C.

3 January 11th 2019

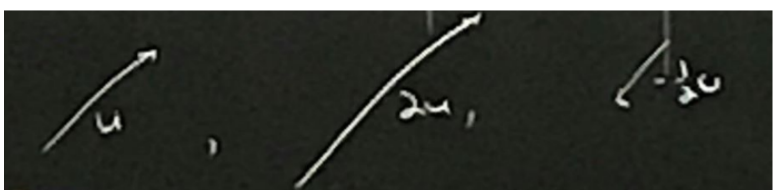
Last time: *Field* K is R or C (for this class).

3.1 Geometric vectors ('arrows')

You can add two vectors (arrows).



Observation: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
You can rescale a vector:



Observation: $a(b\vec{u}) = (ab)\vec{u}$.
Also: $1\vec{u} = \vec{u}$
Question: What properties are interesting? What other objects obey the same properties?
Abstraction: Focus on properties more than on the objects.

3.2 Definition of a vector space

Let V be a set, called set of "vectors", and let K be a field (R or C) (elements of K called *scalars*). Assume that we have already defined two operations:

(1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.

(2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then V is called a *vector space* (over K).

(A1) $u + v = v + u$ (commutative laws)

(A2) There exists a vector in V, named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$

(A3) For each $u \in V$, there is a vector in V, called the (additive) inverse of u and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)

(A4) $(u + v) + w = u + (v + w)$

(SM1) $a(u + v) = au + av$ (distributive laws)

(SM2) $(a + b)u = au + bu$

(SM3) $a(bu) = (ab)u$

(SM4) $1u = u$ ($1 \in R$ or C)

These are called the vector space *axioms*.

3.3 Examples of vector spaces

Some examples:

(1) $K^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2, \ldots, ca_n)$$

where $c \in K$ (and K = set of scalar).

Proof that K^n is a vector space
Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.
Proof concept: To prove "for all $x \in A$, something", say "let $x \in A$" (means x is an arbitrary element of A, ie you only know $x \in A$). Then, prove something for that x.

Proof: Let $u, v \in K^n$. This means $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n)$ for some $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \ldots, a_n) + (b_1, \ldots, b_n) \\ &= (a_1 + b_1, \ldots, a_n + b_n) \qquad \text{(definition of addition in } K^n) \\ &= (b_1 + a_1, \ldots, b_n + a_n) \quad \text{(since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \ldots, b_n) + (a_1, \ldots, a_n) \quad \text{(definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.
Define $0 = (0, 0, \ldots, 0)$ (which *is* in K^n). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \ldots, a_n$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \ldots, a_n + (0, 0, \ldots, 0) \\ &= (a_1 + 0, a_2 + 0, \ldots, a_n + 0) \\ &= (a_1, a_2, \ldots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space C^2, $(2 + 3i, 5 - 7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

4 January 14th 2019

Problem: Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that J is not a vector space.
Solution: Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned} (x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3) \end{aligned}$$

While

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3)\\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3)\\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3)\end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.
Actual proof: Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned}u + (v + w) &= (1, 1) + ((2, 2) + (3, 3))\\ &= (1, 1) + (-2 - 3, 5)\\ &= (1, 1) + (-5, 5)\\ &= (-1 + 5, 6)\\ &= (4, 6)\end{aligned}$$

Whereas,

$$\begin{aligned}(u + v) + w &= ((1, 1) + (2, 2)) + (3, 3)\\ &= (-1 - 2, 3) + (3, 3)\\ &= (-3, 3) + (3, 3)\\ &= (-(-3) - 3, 6)\\ &= (0, 6)\end{aligned}$$

Hence, the axiom does not hold.

4.1 More examples of vector spaces

(1) K^n (ie R^n or C^n). See before

(2) $P(K)$ = polynomials, where coefficients are in K. Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}f(x) &= x^2 + 2ix - 4 \quad \in P(C)\\ g(x) &= -x^2 + cx \quad \in P(C) \qquad \text{(and also in } P(R))\end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned}2if(x) &= 2ix^2 + 4i^2x - 8i\\ &= 2ix^2 - 4x - 8i\end{aligned}$$

(3) $P_n(K)$ = polynomials of degree n or less, coefficient from K. For example,

$$\begin{aligned}
x^2 - 2x + 2 &\in P_2(R) \\
x^2 - 2x + 2 &\in P_3(R) \\
x^2 - 2x + 2 &\in P_2(C) \\
x^2 - 2x + 2 &\notin P_1(R)
\end{aligned}$$

Note: In $P(K), P_n(K)$ the "vectors" are polynomials.

(4) $M_{m\times n}(K)$ = $m \times n$ matrices with entries from K. Scalars are K, addition and scalar multiplication as expected.

$$\begin{aligned}
A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2\times 2}(C) \\
B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2\times 2}(C) \\
A + B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\
2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\
&= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix}
\end{aligned}$$

The "zero vector" in $M_{m\times n}(K)$ is the $m \times n$ matrix with all entries 0.

(5) Let X be any set (think $x = R$ or C, but not required). Define $F(X, K) = \{f : X \to K\}$ = all functions from X to K.
Ex: $f(x) = x^2 \in F(R, R)$.
Ex: Let $x = \{1, 2\}$. Then g defined by

$$\begin{aligned}
g(1) &= 3 \\
g(2) &= \sqrt{2}
\end{aligned}$$

Addition in this space is defined by:
If $f, g \in F(X, K)$ then $f + g$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Note that $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f + g)$ is the addition of vectors f and g, while the other $+$ is scalar addition.
Scalar multiplication in this space is defined by: if $f \in F(X, K), c \in K$ then cf is the function defined by

$$(cf)(x) = cf(x)$$

Note that cf is the name of the function, that "multiplication" is scalar multiplication $F(X, \vDash)$ and $cf(x)$ is the multiplication of two scalars (numbers).
The fact that $F(X, K)$ is a vector space and the axioms are followed is not so obvious.
Prove (A2) true for $F(X, K)$**.** Define $z \in F(X, K)$ by

$$z(x) = 0 \qquad \text{(for all } x \in X\text{)}$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned}(f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x)\end{aligned}$$

Hence, $f+z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.
Exercise: Try (A3).

5 January 16th 2019

Theorem: ("Cancellation Law") Suppose v is a vector space over K. For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.
Note: To prove "for all" you say let $u \in V$ (means u is an arbitrary vector).
To prove "if p then q", denoted $p \to q$, assume p is true and use it to prove q.

Proof. Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned}(u + w) + (-w) &= (v + w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) && \text{(by A1)} \\ u + \vec{0} &= v + \vec{0} && \text{(by A3)} \\ &= u = v && \text{(by A2)}\end{aligned}$$

□

Theorem:

1. The zero vector is unique
2. For each $u \in V$, $-u$ is unique

Note: To prove something is unique, suppose you have two of them and show they are the same.

Proof. 1) Assume 0 and z both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned} z &= z + 0 && \text{(by *, with } u = z\text{)} \\ &= 0 + z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0\text{)} \end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

Theorem: $\forall u \in V, c \in K$,

1) $c\vec{0} = \vec{0}$

2) $0u = \vec{0}$

3) $-(cu) = ((-c)u)$

Proof. Of 2). Let $u \in V$. Then,

$$\begin{aligned} 0u + 0u &= (0+0)u && \text{(By SM2)} \\ 0u + 0u &= 0u && \text{(by R addition)} \\ 0u + 0u &= 0u + \vec{0} && \text{(by A2)} \\ 0u + 0u &= \vec{0} + 0u && \text{(by A4)} \\ 0u &= \vec{0} && \text{(by cancellation law)} \end{aligned}$$

□

Note: $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

5.1 Linear combinations and spans

Def: Let $u, v_1, v_2, \ldots, v_n \in V$. If there are scalars $a_1, a_2, \ldots, a_n \in K$ such that $u = a_1v_1, a_2v_2 \ldots a_nv_n$ then u is said to be a linear combination of $v_1, v_2, \ldots, v_n$.
Ex: In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.

Important problem: Given vectors $u, v_1, v_2, \ldots, v_n$, determine if u is a linear combination of $v_1, v_2, \ldots, v_n$ and if so find $a_1, a_2, \ldots, a_n$.

Ex: Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

Sol. Are there a_1, a_2, a_3 s.t.

$$\begin{aligned} 2x^2+6x+8 &= a_1(x^2+2x+1)+a_2(-2x^2-4x-2)+a_3(2x^2-3) \\ &= (a_1-2a_2+2a_3)x^2+(2a_1-4a_2)x+(a_1-2a_2-3a_3) \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a_1-2a_2+2a_3 &= 2 \\ 2a_1-4a_2 &= 6 \\ a_1-2a_2-3a_3 &= 8 \end{aligned}$$

Solve the linear system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array}\right]$$
$$\downarrow$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \text{(row reduce)}$$

$\therefore$ No solution, because of the last row. f is not a linear combination of g_1, g_2, g_3.

Def: Let $S \subseteq V$ (S is a subset fof V) and assume $s \neq 0$. The span of s, denoted $span(s)$ is the set of all linear combinations of vectors from S, ie

$$\begin{aligned} span(s) = \{u \in V |\ &\exists\ v_1, v_2, \ldots, v_n \in S \\ &\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ &u = a_1v_1 + a_2v_2 + \ldots + a_nv_n\} \end{aligned}$$

6 January 18th 2019

6.1 Last class

$$\begin{aligned} &S \subseteq V \\ span(s) = \{u \in V |\ &\exists\ v_1, v_2, \ldots, v_n \in S \\ &\text{and scalars } a_1, a_2, \ldots, a_n \text{ s.t.} \\ &u = a_1v_1 + a_2v_2 + \ldots + a_nv_n\} \end{aligned}$$

Ex: $S = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Prove $span(S) = R^2$.
Note: $\binom{a}{b}$ means (a, b).
Proof note: To prove two sets A, B are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.
Sol:

(1) Prove $span(S) \subseteq R^2$. Trivial, since any linear combination of vectors in R^2 is still in R^2.

(2) Prove $R^2 \subseteq span(S)$. Let $\binom{a}{b} \in R^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$\begin{aligned} a &= x_1 + 3x_2 \\ b &= 2x_1 + x_2 \end{aligned}$$

Want to show this has a solution (for all a, b). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\binom{a}{b} \in span(S)$ so $R^2 \subseteq span(S)$. So by (1), (2), $span(S) = R^2$. $\square$

Note: $Ax = b$, $A_{n\times n}$ if A inv, $x = A^{-1}b$.
Theorem: Let $S \subseteq V$, $S \neq \emptyset$ ($\emptyset$ = empty set). Then,

(1) If $u, v \in span(S)$ then $u + v \in span(S)$

(2) If $u \in span(S)$ and $c \in K$, then $cu \in span(S)$

(3) $\vec{0} \in span(S)$

Proof. By direct proof.

(1) (Note, "if $u, v \in span(S)$" means for all $u, v \in span(S)$).
Let $u, v \in span(S)$. Then,

$$\begin{aligned} u &= a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K \\ v &= b_1v_1 + b_2v_2 + \ldots + b_mv_m \text{ where } v_1, \ldots, v_m \in S, b_1, \ldots, b_m \in K \end{aligned}$$

Then $u + v = a_1u_1 + \ldots + a_nu_n + b_1v_1 + \ldots b_mv_m$ which is in $span(S)$ since $u_1, \ldots, u_n, v_1, \ldots, v_m \in S$.

(2) Let $u \in span(S), c \in K$. Then,

$$u = a_1u_1 + a_2u_2 + \ldots + a_nu_n \text{ where } u_1, \ldots, u_n \in S, a_1, \ldots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \ldots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \ldots (c_na_n)u_n \end{aligned}$$

Note: If you want to be very formal, you need to write down all of the vector space axioms. Which is in $span(S)$ since it is a linear combination of $a_1, \ldots, a_n$ which are in S.

(3) (Prove $\vec{0} \in span(S)$) Let $u \in S$. **Note**: This is possible only because $S \neq \emptyset$.
Then $u = 1u$, so $u \in span(S)$. Then using $c = 0$ and (2) and fact that $u \in span(S)$,

$$cu = 0u = \vec{0}$$

is also in $span(S)$. **Note:** Since $u = 1u$, $S \subseteq span(S)$.

□

6.2 Subspaces

Def. Let V be a vector space and $W \subseteq V$ (subset). If W, using addition and scalar multiplication as defined in V, satisfies the definition of vector space, then W is called a subspace of V, denoted $W \leq V$ (less than equal sign, read as "subspace").
Note: Main issue is that addition and scalar multiplication with vector from W produce vectors which are still in W.
Theorem: Let $W \subseteq V$. Then, if the following three properties hold, then $W \leq V$ (subspace).

(SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")

(SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")

(SS3) $\vec{0} \in W$.

These are the same properties we just proved for spans; in other words, we proved earlier that $span(S)$ is a subspace.

Proof. For W to have operatios addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

(A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in V, $u+(v+w) = (u+v)+w$.

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let $w \in W$. Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \qquad \text{(thm from last class)}$$

is in W, as needed.

(A4) Still true because it is true in V.

(SM1-SM4) All hold because they hold in V.

□

7 January 21st 2019

7.1 A note on logic

Let P, Q be statements that are true or false.

(1) "If P then Q", also written symbolically as "$P \Rightarrow Q$" (P *implies* Q) means if P is true, then Q is also true. To *prove* "$P \Rightarrow Q$", assume P and prove Q is true. If you *know* that "$P \Rightarrow Q$" is true, you can *use it*: if you can establish that P is true, you may conclude Q is true.
Ex: Let A be an $n \times n$ matrix:

$$P : dot(A) = 1 \quad Q : \text{"A is invertible"}$$

Thm: $P \Rightarrow Q$

(2) The *converse* of "$P \Rightarrow Q$" is "$Q \Rightarrow P$". This is a (logically) different statement.
Ex: With P and Q as above, "$Q \Rightarrow P$" is not true because $A_{inv} \not\Rightarrow det(A) = 1$.

(3) The *contrapositive* of "$P \Rightarrow Q$" is "$\neg Q \Rightarrow \neg P$" ie "if Q false, then P also false". Logically, this is the same as "$P \Rightarrow Q$".

(4) The *equivalence* "P if and only if Q", written "$P \iff Q$" means "$P \Rightarrow Q$ and also $Q \Rightarrow P$" is true. Also means that either both P and Q are true or both are false.
Ex: $det(A) \neq 0 \iff$ A is invertible.
To prove "$P \iff Q$", need to prove "$P \Rightarrow Q$" and "$Q \Rightarrow P$".
Note: $\neg P \Rightarrow \neg Q$ is the same as $Q \Rightarrow P$.

7.2 Subspaces (cont'd)

Thm (last class): Let $W \subseteq V$ (*subset*). If

1. For all $u, v \in W$, $u + v \in W$
2. For all $u \in W$, $c \in K$, $cu \in W$
3. $\vec{0} \in W$

then $W \leq V$ (*subspace*). (ie: $(1), (2), (3)$ are true $\Rightarrow W \leq V$)
Thm. Let $W \subseteq V$. Then

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).
Proof. Exercise.
Thm. Let $W \subseteq V$. Then

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

7.3 Examples of subspaces and non-subspaces

Is each subset a subspace?

(a) $W = \{\binom{1}{2}, \binom{3}{1}\} \subseteq R^2$. Not a subspace, since the zero vector is not in W. The others are also false, but it's enough to prove that one of the statements does not hold. But $span(W) = R^2$ (so $span(W) \leq R^2$)

(b) $W = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 0\}$. Need to check (1), (2), (3):

(1) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$. Then we know $x+y-z = 0$ and $x'+y'-z' = 0$.
Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in W.

(2) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ (means $x + y - z = 0$), let $c \in K$. To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here, $cx + cy - cz = c(x + y - z) = c(0) = 0$. So $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

(3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$, since $0 + 0 - 0 = 0$

Since (1), (2), (3) true, $W \leq R^2$ (subspace)

(c) $W = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 | x + y - z = 1\}$. This is *not* a subspace. (3) is false.

(d) $W = \{A \in M_{2\times2} | A_{ij} \geq 0 \forall i, j\}$, where A_{ij} is the entry of A in row i, column j. (1) and (3) are true:

(1) Add two matrices with non-negatives entries, result has non-negative entries.

(2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since, for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ but

$$(-1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in W$$

7.4 Two special subspaces

Let V be a vector space.

(1) $V \leq V$ is true

(2) $\{\vec{0}\} \leq V$ is true ("zero subspace")

7.5 A refinement on the definition of span

Def. If $S = \emptyset$ (emptyset), define $span(S) = \{\vec{0}\}$ (if $S \neq \emptyset$, $span(S)$ defined as before).
Thm. $span(S) \leq V$.
Proof Two cases :

1. If $S = \emptyset$, $span(S) = \{\vec{0}\} \leq V$
2. If $S \neq \emptyset$, you already proved $span(S)$ satisfies (1), (2), (3).
   So $span(S) \leq V$.

Thm. (improved version of subspace conditions) Let $W \subseteq V$. Then

$$\begin{aligned} W \leq V \iff & W \neq \emptyset \text{ and} \\ & \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W \end{aligned}$$

Proof We will actually prove $(1), (2), (3) \iff RHS$ (right-hand side). Two parts to proof.

(1) "$(1), (2), (3) \Rightarrow RHS$" or "$\Rightarrow$"