

MATH223 - Linear Algebra (class notes)

Sandrine Monfourny-Daigneault

McGill University

Contents

January 7th 2019	3
Motivation	3
Complex numbers	4
January 9th 2019	6
Complex numbers as points in \mathbb{R}^2	6
Equations with complex numbers	6
Vector spaces (Ch 4)	8
January 11th 2019	8
Geometric vectors ('arrows')	8
Definition of a vector space	9
Examples of vector spaces	9
January 14th 2019	10
More examples of vector spaces	11
January 16th 2019	13
Linear combinations and spans	15
January 18th 2019	16
Last class	16
Subspaces	17
January 21st 2019	18
A note on logic	18
Subspaces (cont'd)	19
Examples of subspaces and non-subspaces	19
Two special subspaces	21
A refinement on the definition of span	21
January 23rd 2019	21
January 25th 2019	23
Interlude : Symbolic logic (briefly)	23
De Morgan's Laws	24
Quantifiers	24
Negating quantifiers	24

Proof by contradiction	24
Last time	25
Illustration of this theorem	25
Intersection of two subspaces	26
January 28th 2019	26
Linear independence	27
January 30th 2019	28
Last class	28
Some important cases	29
Extending an independent set	30
Basis and dimension	30
February 1st 2019	31
Two important questions	31
Illustration of this thm	32
February 4th 2019	33
Last class	33
Lagrange Interpolation	33
Dimension of subspaces	35
February 6th 2019	36
Intuition	36
General case	37
February 8th 2019	39
Linear transformations - Definition and basic properties . . .	39
February 15th 2019	41
Two special linear transformations	42
Kernel and Image (ch. 5.4)	42

List of Theorems

1	Theorem (Fundamental Theorem of Algebra)	7
1.1	Corollary	7
2	Theorem (Cancellation Law)	13
3	Theorem	14
4	Theorem	14
5	Theorem	16
6	Theorem	18
7	Theorem	19
8	Theorem	19
9	Theorem	21

10	Theorem	21
11	Theorem	22
12	Theorem	22
13	Theorem	23
14	Theorem	25
15	Theorem	26
15.1	Corollary	26
16	Theorem	27
17	Proposition	29
18	Theorem	30
19	Theorem (Bases exist)	32
20	Theorem	32
21	Theorem	32
22	Theorem	33
23	Theorem	33
24	Proposition	34
20	Theorem	35
21	Theorem	38
22	Proposition	40
23	Proposition	41
24	Proposition	41
25	Theorem	43

January 7th 2019

Should know how to solve a linear system and calculate a determinant... things like that.

- Written assignments (5) : 10%
- Webwork assignments (5) : 5%
- Midterm : 20%
- Final : 65%

Textbook: **Schaum's Outline - Linear Algebra.**

Motivation

We have linear systems, with two equations, like such:

$$\begin{aligned} 3x - 2y + z &= 2 \\ x - y + z &= 1 \end{aligned}$$

There is an algebraic way of seeing this, but we can also see this, from the geometric standpoint, as the intersection of the two planes

in R^3 . Linear algebra has to do with things that are "flat", like a plane. As soon as we add in exponents to these equations, we get some curvature, and the techniques to solve these are different.

- Linear equations are the simplest kind, so you *must* understand them. Also, you *can* understand 'everything' about them.
- Theory used to describe solutions, etc.
- Linear equations are often used to approximate or model more complicated equations/situations.
- In applications, linear systems are often quite big (10000 equations/variables)

Complex numbers

Def: Let i be a symbol. We declare $i^2 = -1$.

Now, what we'd like to do is take this symbol i and combine it with the usual real numbers that we are familiar with. We set, for example,

$$\begin{aligned} 3i \\ i - 4 \\ 3i - \pi \\ \sqrt{i} + 21 \end{aligned}$$

Def: The field of complex numbers C consists of all expressions of the form $a + bi$, where $a, b \in R$.

Def: Addition (subtraction) and multiplication of complex numbers is defined by the following rules:

(i)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

(ii)

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Notation:

- $0 + bi = bi$
- $a + 0i = a$ (a *real* number)

- $0 + 0i = 0$

Ex: If $z_1 = 2 - i$, $z_2 = 5i$, then

$$z_1 + z_2 = 2 + 4i$$

and

$$z_1 z_2 = (2 - i)(5i) = 10i - 5i^2 = 5 + 10i$$

Def: Let $z = a + bi \in \mathbb{C}$

- (i) $\bar{z} = a - bi$, called the *complex conjugate* of z
- (ii) $|z| = \sqrt{a^2 + b^2}$, called the *absolute value* or *modulus*

Def: If $z = a + bi \in \mathbb{C}$ and $z \neq 0$ (ie $z \neq 0 + 0i$), then the number

$$\begin{aligned} z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i \end{aligned}$$

is called the (multiplicative) inverse of z . It has the property $zz^{-1} = 1 = z^{-1}z$.

Proof. We have

$$\begin{aligned} zz^{-1} &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i\right) \\ &= \frac{a^2 - abi + abi - b^2i^2}{a^2 + b^2} \\ &= \frac{a^2 + 0 + b^2}{a^2 + b^2} \\ &= 1 \end{aligned}$$

Note: Since $z \neq 0 + 0i$, $a^2 + b^2 \neq 0$

□

Def: If $z, w \in \mathbb{C}$ and $z \neq 0$ then

$$\frac{w}{z} = wz^{-1}$$

Ex: If $z = 1 + 2i$, $w = 3 - i$ then

$$\begin{aligned} \frac{w}{z} &= wz^{-1} \\ &= (3 - i)\left(\frac{1}{5} - \frac{2}{5}i\right) \\ &= \frac{3}{5} - \frac{6}{5}i - \frac{i}{5} + \frac{2}{5}i^2 \\ &= \frac{3}{5} - \frac{2}{5} - \frac{7}{5}i \\ &= \frac{1}{5} - \frac{7}{5}i \end{aligned}$$

Or,

$$\begin{aligned}\frac{3-i}{1+2i} \cdot \frac{(1-2i)}{(1-2i)} &= \frac{3-6i-i+2i^2}{1-2i+2i-4i^2} \\ &= \frac{1-7i}{5}\end{aligned}$$

January 9th 2019

Complex numbers as points in \mathbb{R}^2

You can view $a + bi$ as a point $(a, b) \in \mathbb{R}^2$. The usefulness of this is that we can consider, say, $(3 + 2i)$ and $(3 - i)$ as vectors in \mathbb{R}^2 , and they will conserve the same properties (addition of complex numbers corresponds to vector addition in \mathbb{R}^2). For the interpretation of multiplication to make sense, it's necessary to use polar coordinates.

Equations with complex numbers

Fact: Every real number $a \neq 0$ has two square roots:

- if $a > 0$, roots $\pm\sqrt{a}$
- if $a < 0$, two roots are $\pm i\sqrt{|a|}$, since:

$$\begin{aligned}(\pm i\sqrt{|a|})^2 &= i^2(\sqrt{|a|})^2 \\ &= -1 \cdot |a| \\ &= a\end{aligned}\quad (\text{since } a < 0)$$

Fact: Quadratic equation $ax^2 + bx + c = 0$ has solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which may be in \mathbb{C} .

Ex: Solve $x^2 - 2x + 3 = 0$, and factor $x^2 - 2x + 3$.

Sol:

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 4(1)(3)}}{2} \\ &= \frac{2 \pm \sqrt{-8}}{2} \\ &= \frac{2 \pm i\sqrt{8}}{2} \\ &= \frac{2 \pm i2\sqrt{2}}{2} \\ &= 1 \pm i\sqrt{2}\end{aligned}$$

Note: If $ax^2 + bx + c$ has $a, b, c \in \mathbb{R}$ has a non-real root, say z , its other root is \bar{z} ($z = a + bi$, $\bar{z} = a - bi$). This is not necessarily true if $a, b, c \in \mathbb{C}$.

Back to problem. Factor $x^2 - 2x + 3 = (x - (1 + i\sqrt{2}))(x - (1 - i\sqrt{2}))$.

Caution: -1 has two roots, namely $\pm i$, so you may write $i = \sqrt{-1}$, but be careful:

$$\begin{aligned} -1 &= i^2 \\ &= i \cdot i \\ &= \sqrt{-1} \cdot \sqrt{-1} \\ &= \sqrt{(-1)(-1)} && \text{(this step doesn't quite work)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

Theorem 1 (Fundamental Theorem of Algebra). *If*

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 x^0$$

is a polynomial with $a_n \neq 0$, and $a_n, a_{n-1}, \dots, a_0 \in \mathbb{C}$, then $p(x)$ factors into linear factors,

$$p(x) = a_n \cdot (x - r_1) \cdot (x - r_2) \cdot \dots \cdot (x - r_n)$$

for some complex numbers r_1, r_2, \dots, r_n . Some r_i 's may be equal.

Corollary 1.1. *Every such polynomial has at least one root, and at most n distinct roots.*

Note: Finding the roots is, in general, quite difficult.

Ex: Factor $2x^3 + 2x$ (over \mathbb{C}).

Sol:

$$\begin{aligned} 2(x^3 + x) &= 2(x - 0)(x^2 + 1) \\ &= 2(x - 0)(x^2 - i^2) \\ &= 2(x - 0)(x - i)(x + i) \end{aligned}$$

Ex: Solve $x^2 - i = 0$

Sol: $x^2 = i$ so $x = \pm\sqrt{i}$. Want \sqrt{i} in format $a + bi$, $a, b \in \mathbb{R}$.

$$\begin{aligned}
 \sqrt{i} &= a + bi \\
 i &= (a + bi)^2 \\
 &= a^2 + 2abi + b^2i^2 \\
 0 + i &= (a^2 - b^2) + 2abi \\
 0 &= a^2 - b^2 \\
 1 &= 2ab \\
 a &= \pm b \\
 ab &= \frac{1}{2} & \text{(so } a=b \text{ both } + \text{ or both } -) \\
 a^2 &= \frac{1}{2} \\
 a &= \pm \frac{1}{\sqrt{2}} = b
 \end{aligned}$$

Two solutions, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ and $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Vector spaces (Ch 4)

Def. The sets \mathbb{R} and \mathbb{C} (and also \mathbb{Q} , rational numbers, although we won't go into details of this) are called *fields* (or *fields of scalars*). In this class, "a field of K " means that K is either \mathbb{R} or \mathbb{C} .

January 11th 2019

Last time: Field K is \mathbb{R} or \mathbb{C} (for this class).

Geometric vectors ('arrows')

You can add two vectors (arrows) (see figure 1)

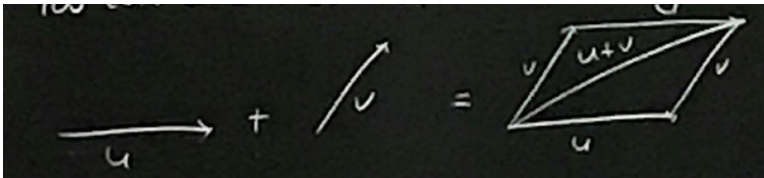


Figure 1: Vector addition

Observation: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

You can rescale a vector (see figure 2) **Observation:** $a(b\vec{u}) = (ab)\vec{u}$.

Also: $1\vec{u} = \vec{u}$

Question: What properties are interesting? What other objects obey the same properties?

Abstraction: Focus on properties more than on the objects.

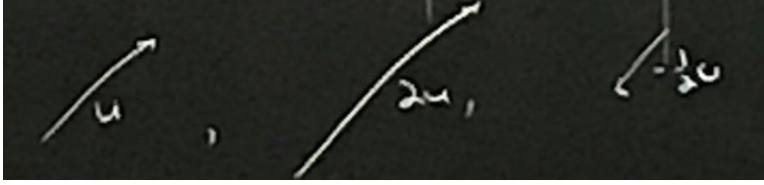


Figure 2: Vector rescaling

Definition of a vector space

Let V be a set, called set of "vectors", and let K be a field (R or C) (elements of K called *scalars*). Assume that we have already defined two operations:

- (1) One called *addition*, which takes two vectors $\vec{u}, \vec{v} \in V$ and produces another vector denoted $\vec{u} + \vec{v} \in V$.
- (2) One called *scalar multiplication* which takes a vector $\vec{u} \in V$ and a scalar $a \in K$ and produces another vector denoted $a\vec{u} \in V$

Then if, for all vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and all scalars $a, b \in K$, the following 8 properties are true, then V is called a *vector space* (over K).

- (A1) $u + v = v + u$ (commutative laws)
- (A2) There exists a vector in V , named *zero vector* and denoted 0 (or $\vec{0}$) such that for all $u \in V$, $u + 0 = u$
- (A3) For each $u \in V$, there is a vector in V , called the (additive) inverse of u and denoted $-u$, having the property $u + (-u) = 0$ (where 0 is the zero vector defined in A2)
- (A4) $(u + v) + w = u + (v + w)$
- (SM1) $a(u + v) = au + av$ (distributive laws)
- (SM2) $(a + b)u = au + bu$
- (SM3) $a(bu) = (ab)u$
- (SM4) $1u = u$ ($1 \in R$ or C)

These are called the vector space *axioms*.

Examples of vector spaces

Some examples:

- (1) $K^n = \{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in K\}$, with addition defined by

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and scalar multiplication by

$$c(a_1, a_2, \dots, a_n) = (ca_1, ca_2, \dots, ca_n)$$

where $c \in K$ (and $K = \text{set of scalar}$).

Proof that K^n is a vector space

Need to prove all 8 properties. We will do 2, the rest are exercises.

(A4) To prove for all $u, v \in V$, $u + v = v + u$.

Proof concept: To prove "for all $x \in A$, something", say "let $x \in A$ " (means x is an arbitrary element of A , ie you only know $x \in A$). Then, prove something for that x .

Proof: Let $u, v \in K^n$. This means $u = (a_1, a_2, \dots, a_n)$, $v = (b_1, b_2, \dots, b_n)$ for some $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in K$. Then

$$\begin{aligned} u + v &= (a_1, \dots, a_n) + (b_1, \dots, b_n) \\ &= (a_1 + b_1, \dots, a_n + b_n) \quad (\text{definition of addition in } K^n) \\ &= (b_1 + a_1, \dots, b_n + a_n) \quad (\text{since } a + b = b + a \text{ for } R \text{ and } C) \\ &= (b_1, \dots, b_n) + (a_1, \dots, a_n) \quad (\text{definition of addition in } K^n) \\ &= v + u \end{aligned}$$

(A2) *Proof concept:* To prove "there exists" something, one method is to describe the thing directly.

Define $0 = (0, 0, \dots, 0)$ (which is in K^n). To prove for all $u \in K^n$, $u + 0 = u$, let $u \in K^n$. This means $u = (a_1, a_2, \dots, a_n)$, so

$$\begin{aligned} u + 0 &= (a_1, a_2, \dots, a_n) + (0, 0, \dots, 0) \\ &= (a_1 + 0, a_2 + 0, \dots, a_n + 0) \\ &= (a_1, a_2, \dots, a_n) \\ &= u \end{aligned}$$

(2) In the vector space C^2 , $(2 + 3i, 5 - 7i) \in C^2$ is an example of a vector and $2i \in C$ is a scalar, so an example of scalar mult is :

$$\begin{aligned} 2i(u) &= 2i(2 + 3i, 5 - 7i) \\ &= (4i + 6i^2, 10i - 14i^2) \\ &= (-6 + 4i, 14 + 10i) \end{aligned}$$

January 14th 2019

Problem: Let $J = \{(x, y) | x \in R, y \in R\}$ but define addition by

$$(x_1, y_1) + (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

and scalar multiplication by

$$c(x, y) = (cx, cy)$$

Show that J is not a vector space.

Solution: Show *one* of the 8 vector space axioms is false. Consider (A1):

$$(x_2, y_2) + (x_1, y_1) = (-x_2 - x_1, y_2 + y_1)$$

This is actually ok! Now consider (A4):

$$\begin{aligned}(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) &= (x_1, y_1) + (-x_2 - x_3, y_2 + y_3) \\ &= (-x_1 - (-x_2 - x_3), y_1 + y_2 + y_3) \\ &= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3)\end{aligned}$$

While

$$\begin{aligned}((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) &= (-x_1 - x_2, y_1 + y_2) + (x_3, y_3) \\ &= (-(-x_1 - x_2) - x_3, y_1 + y_2 + y_3) \\ &= (x_1 + x_2 - x_3, y_1 + y_2 + y_3)\end{aligned}$$

This does not quite yet prove that the axiom is false. To do so, give *specific* case where the equation is false.

Actual proof: Let $u = (1, 1)$, $v = (2, 2)$ and $w = (3, 3)$. Then,

$$\begin{aligned}u + (v + w) &= (1, 1) + ((2, 2) + (3, 3)) \\ &= (1, 1) + (-2 - 3, 5) \\ &= (1, 1) + (-5, 5) \\ &= (-1 + 5, 6) \\ &= (4, 6)\end{aligned}$$

Whereas,

$$\begin{aligned}(u + v) + w &= ((1, 1) + (2, 2)) + (3, 3) \\ &= (-1 - 2, 3) + (3, 3) \\ &= (-3, 3) + (3, 3) \\ &= (-(-3) - 3, 6) \\ &= (0, 6)\end{aligned}$$

Hence, the axiom does not hold.

More examples of vector spaces

- (1) K^n (ie R^n or C^n). See before
- (2) $P(K)$ = polynomials, where coefficients are in K . Addition, scalar multiplication are "as expected", ie for multiplication:

$$\begin{aligned}f(x) &= x^2 + 2ix - 4 \in P(C) \\ g(x) &= -x^2 + ix \in P(C) \quad (\text{and also in } P(R))\end{aligned}$$

For addition,

$$f(x) + g(x) = 3ix - 4$$

And for scalar multiplication,

$$\begin{aligned} 2if(x) &= 2ix^2 + 4i^2x - 8i \\ &= 2ix^2 - 4x - 8i \end{aligned}$$

- (3) $P_n(K)$ = polynomials of degree n or less, coefficient from K . For example,

$$\begin{aligned} x^2 - 2x + 2 &\in P_2(R) \\ x^2 - 2x + 2 &\in P_3(R) \\ x^2 - 2x + 2 &\in P_2(C) \\ x^2 - 2x + 2 &\notin P_1(R) \end{aligned}$$

Note: In $P(K)$, $P_n(K)$ the "vectors" are polynomials.

- (4) $M_{m \times n}(K)$ = $m \times n$ matrices with entries from K . Scalars are K , addition and scalar multiplication as expected.

$$\begin{aligned} A &= \begin{pmatrix} 2 & i \\ 0 & \pi \end{pmatrix} \in M_{2 \times 2}(C) \\ B &= \begin{pmatrix} -2 & 1 \\ 1+i & -\pi \end{pmatrix} \in M_{2 \times 2}(C) \\ A+B &= \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \\ 2iA &= \begin{pmatrix} 4i & 2i^2 \\ 0 & 2i\pi \end{pmatrix} \\ &= \begin{pmatrix} 4i & -2 \\ 0 & 2\pi i \end{pmatrix} \end{aligned}$$

The "zero vector" in $M_{m \times n}(K)$ is the $m \times n$ matrix with all entries 0.

- (5) Let X be any set (think $x = R$ or C , but not required). Define

$F(X, K) = \{f : X \rightarrow K\}$ = all functions from X to K .

Ex: $f(x) = x^2 \in F(R, R)$.

Ex: Let $x = \{1, 2\}$. Then g defined by

$$\begin{aligned} g(1) &= 3 \\ g(2) &= \sqrt{2} \end{aligned}$$

Addition in this space is defined by:

If $f, g \in F(X, K)$ then $f + g$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

Note that $f(x) \in K$ and $g(x) \in K$, in other words they are *numbers* (scalars). The $+$ in $(f + g)$ is the addition of vectors f and g , while the other $+$ is scalar addition.

Scalar multiplication in this space is defined by: if $f \in F(X, K), c \in K$ then cf is the function defined by

$$(cf)(x) = cf(x)$$

Note that cf is the name of the function, that "multiplication" is scalar multiplication $F(X, K)$ and $cf(x)$ is the multiplication of two scalars (numbers).

The fact that $F(X, K)$ is a vector space and the axioms are followed is not so obvious.

Prove (A2) true for $F(X, K)$. Define $z \in F(X, K)$ by

$$z(x) = 0 \quad (\text{for all } x \in X)$$

Note that 0 here is a scalar. Then if $f \in F(X, K)$ is an arbitrary element, then we need to prove $f + z = f$. This is true since for all $x \in X$,

$$\begin{aligned} (f + z)(x) &= f(x) + z(x) \\ &= f(x) + 0 \\ &= f(x) \end{aligned}$$

Hence, $f + z, f$ have the same output (namely $f(x)$) for every input. Hence, $f + z = f$.

Exercise: Try (A3).

January 16th 2019

Theorem 2 (Cancellation Law). Suppose v is a vector space over K . For all vectors $u, v, w \in V$, if $u + w = v + w$ then $u = v$.

Note: To prove "for all" you say let $u \in V$ (means u is an arbitrary vector).

To prove "if p then q ", denoted $p \rightarrow q$, assume p is true and use it to prove q .

Proof. Let $u, v, w \in V$. Assume $u + w = v + w$. By vector space axiom

A3, there is a vector $(-w) \in V$. Add $(-w)$ to both sides:

$$\begin{aligned}(u + w) + (-w) &= (v + w) + (-w) \\ u + (w + (-w)) &= v + (w + (-w)) && \text{(by A1)} \\ u + \vec{0} &= v + \vec{0} && \text{(by A3)} \\ &= u = v && \text{(by A2)}\end{aligned}$$

□

Theorem 3. Two points:

1. The zero vector is unique
2. For each $u \in V$, $-u$ is unique

Note: To prove something is unique, suppose you have two of them and show they are the same.

Proof. 1) Assume 0 and z both satisfy the property (A2: $\forall u \in V, u + 0 = u$ (*) and $u + z = u$ (**)). Goal is to prove $0 = z$.

$$\begin{aligned}z &= z + 0 && \text{(by *, with } u = z\text{)} \\ &= 0 + z && \text{(by A4)} \\ z &= 0 && \text{(by **, with } u = 0\text{)}\end{aligned}$$

So the zero vector is unique.

2) Exercise.

□

Theorem 4. $\forall u \in V, c \in K$,

- 1) $c\vec{0} = \vec{0}$
- 2) $0u = \vec{0}$
- 3) $-(cu) = ((-c)u)$

Proof. Of 2). Let $u \in V$. Then,

$$\begin{aligned}0u + 0u &= (0 + 0)u && \text{(By SM2)} \\ 0u + 0u &= 0u && \text{(by R addition)} \\ 0u + 0u &= 0u + \vec{0} && \text{(by A2)} \\ 0u + 0u &= \vec{0} + 0u && \text{(by A4)} \\ 0u &= \vec{0} && \text{(by cancellation law)}\end{aligned}$$

□

Note: $0 + u = u$ is true for all $u \in V$ (same as $u + 0 = u$ then apply A4)

Linear combinations and spans

Def: Let $u, v_1, v_2, \dots, v_n \in V$. If there are scalars $a_1, a_2, \dots, a_n \in K$ such that $u = a_1v_1 + a_2v_2 + \dots + a_nv_n$ then u is said to be a linear combination of v_1, v_2, \dots, v_n .

Ex: In $P(R)$, $x^2 + 2x - 4$ is a linear comb of $x^2, x, 1$.

Important problem: Given vectors u, v_1, v_2, \dots, v_n , determine if u is a linear combination of v_1, v_2, \dots, v_n and if so find a_1, a_2, \dots, a_n .

Ex: Determine if $f(x) = 2x^2 + 6x + 8$ is a linear combination of

$$\begin{aligned} g_1(x) &= x^2 + 2x + 1 \\ g_2(x) &= -2x^2 - 4x - 2 \\ g_3(x) &= 2x^2 - 3 \end{aligned}$$

Sol. Are there a_1, a_2, a_3 s.t.

$$\begin{aligned} 2x^2 + 6x + 8 &= a_1(x^2 + 2x + 1) + a_2(-2x^2 - 4x - 2) + a_3(2x^2 - 3) \\ &= (a_1 - 2a_2 + 2a_3)x^2 + (2a_1 - 4a_2)x + (a_1 - 2a_2 - 3a_3) \end{aligned}$$

Equating coefficients,

$$\begin{aligned} a_1 - 2a_2 + 2a_3 &= 2 \\ 2a_1 - 4a_2 &= 6 \\ a_1 - 2a_2 - 3a_3 &= 8 \end{aligned}$$

Solve the linear system:

$$\begin{aligned} &\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & -4 & 0 & 6 \\ 1 & -2 & -3 & 8 \end{array} \right] \\ &\quad \downarrow \\ &\left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \quad (\text{row reduce}) \end{aligned}$$

\therefore No solution, because of the last row. f is not a linear combination of g_1, g_2, g_3 .

Def: Let $S \subseteq V$ (S is a subset of V) and assume $s \neq 0$. The span of s , denoted $\text{span}(s)$ is the set of all linear combinations of vectors from S , ie

$$\begin{aligned} \text{span}(s) &= \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S \\ &\quad \text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.} \\ &\quad u = a_1v_1 + a_2v_2 + \dots + a_nv_n\} \end{aligned}$$

January 18th 2019

Last class

$$S \subseteq V$$

$$\text{span}(S) = \{u \in V \mid \exists v_1, v_2, \dots, v_n \in S$$

$$\text{and scalars } a_1, a_2, \dots, a_n \text{ s.t.}$$

$$u = a_1 v_1 + a_2 v_2 + \dots + a_n v_n\}$$

Ex: $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$. Prove $\text{span}(S) = \mathbb{R}^2$.

Note: $\begin{pmatrix} a \\ b \end{pmatrix}$ means (a, b) .

Proof note: To prove two sets A, B are equal, ie $A = B$, you can prove $A \subseteq B$ and $B \subseteq A$.

Sol:

- (1) Prove $\text{span}(S) \subseteq \mathbb{R}^2$. Trivial, since any linear combination of vectors in \mathbb{R}^2 is still in \mathbb{R}^2 .
- (2) Prove $\mathbb{R}^2 \subseteq \text{span}(S)$. Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ (arbitrary). To prove that there exists scalars $x_1, x_2 \in K$ so that

$$\begin{pmatrix} a \\ b \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

In other words,

$$a = x_1 + 3x_2$$

$$b = 2x_1 + x_2$$

Want to show this has a solution (for all a, b). System is:

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

But,

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2(3) \neq 0$$

hence the system has (exactly one) solution. $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{span}(S)$ so $\mathbb{R}^2 \subseteq \text{span}(S)$. So by (1), (2), $\text{span}(S) = \mathbb{R}^2$. \square

Note: $Ax = b$, $A_{n \times n}$ if A inv, $x = A^{-1}b$.

Theorem 5. Let $S \subseteq V$, $S \neq \emptyset$ ($\emptyset = \text{empty set}$). Then,

- (1) If $u, v \in \text{span}(S)$ then $u + v \in \text{span}(S)$
- (2) If $u \in \text{span}(S)$ and $c \in K$, then $cu \in \text{span}(S)$
- (3) $\vec{0} \in \text{span}(S)$

Proof. By direct proof.

- (1) (Note, "if $u, v \in \text{span}(S)$ " means for all $u, v \in \text{span}(S)$).

Let $u, v \in \text{span}(S)$. Then,

$$u = a_1u_1 + a_2u_2 + \dots + a_nu_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

$$v = b_1v_1 + b_2v_2 + \dots + b_mv_m \text{ where } v_1, \dots, v_m \in S, b_1, \dots, b_m \in K$$

Then $u + v = a_1u_1 + \dots + a_nu_n + b_1v_1 + \dots + b_mv_m$ which is in $\text{span}(S)$ since $u_1, \dots, u_n, v_1, \dots, v_m \in S$.

- (2) Let $u \in \text{span}(S), c \in K$. Then,

$$u = a_1u_1 + a_2u_2 + \dots + a_nu_n \text{ where } u_1, \dots, u_n \in S, a_1, \dots, a_n \in K$$

So,

$$\begin{aligned} cu &= c(a_1u_1) + c(a_2u_2) + \dots + c(a_nu_n) \\ &= (ca_1)u_1 + (ca_2)u_2 + \dots + (ca_n)u_n \end{aligned}$$

Note: If you want to be very formal, you need to write down all of the vector space axioms. Which is in $\text{span}(S)$ since it is a linear combination of a_1, \dots, a_n which are in S .

- (3) (Prove $\vec{0} \in \text{span}(S)$) Let $u \in S$. **Note:** This is possible only because $S \neq \emptyset$.

Then $u = 1u$, so $u \in \text{span}(S)$. Then using $c = 0$ and (2) and fact that $u \in \text{span}(S)$,

$$cu = 0u = \vec{0}$$

is also in $\text{span}(S)$. **Note:** Since $u = 1u$, $S \subseteq \text{span}(S)$.

□

Subspaces

Def. Let V be a vector space and $W \subseteq V$ (subset). If W , using addition and scalar multiplication as defined in V , satisfies the definition of vector space, then W is called a subspace of V , denoted $W \leq V$ (less than equal sign, read as "subspace").

Note: Main issue is that addition and scalar multiplication with vector from W produce vectors which are still in W .

Theorem 6. Let $W \subseteq V$. Then, if the following three properties hold, $W \leq V$ (subspace).

(SS1) For all $w_1, w_2 \in W$, we have $w_1 + w_2 \in W$ ("closure under addition")

(SS2) For all $w \in W$ and scalars $c \in K$, we have $cw \in W$ ("closure under scalar multiplication")

(SS3) $\vec{0} \in W$.

These are the same properties we just proved for spans; in other words, we proved earlier that $\text{span}(S)$ is a subspace.

Proof. For W to have operations addition, scalar multiplication, just means (SS1) and (SS2) are true. So now, check (A1) - (SM4). Most of them are true because they are true in a larger vector space.

(A1) Let $u, v, w \in W$. Then since $u, v, w \in V$, and (A1) holds in V ,
 $u + (v + w) = (u + v) + w$.

(A2) This is (SS3).

(A3) This is the one we have to do a bit more work for. Let $w \in W$.
 Want to show $-w \in W$. Then, using (SS2) with $c = -1$ gives

$$-1(w) = -w \quad (\text{thm from last class})$$

is in W , as needed.

(A4) Still true because it is true in V .

(SM1-SM4) All hold because they hold in V .

□

January 21st 2019

A note on logic

Let P, Q be statements that are true or false.

- (1) "If P then Q ", also written symbolically as " $P \Rightarrow Q$ " (P implies Q) means if P is true, then Q is also true. To *prove* " $P \Rightarrow Q$ ", assume P and prove Q is true. If you *know* that " $P \Rightarrow Q$ " is true, you can *use it*: if you can establish that P is true, you may conclude Q is true.

Ex: Let A be an $n \times n$ matrix:

$$P : \det(A) = 1 \quad Q : "A \text{ is invertible}"$$

Thm: $P \Rightarrow Q$

- (2) The *converse* of " $P \Rightarrow Q$ " is " $Q \Rightarrow P$ ". This is a (logically) different statement.

Ex: With P and Q as above, " $Q \Rightarrow P$ " is not true because $A_{inv} \nrightarrow \det(A) = 1$.

- (3) The *contrapositive* of " $P \Rightarrow Q$ " is " $\neg Q \Rightarrow \neg P$ " ie "if Q false, then P also false". Logically, this is the same as " $P \Rightarrow Q$ ".

- (4) The *equivalence* " P if and only if Q ", written " $P \iff Q$ " means " $P \Rightarrow Q$ and also $Q \Rightarrow P$ " is true. Also means that either both P and Q are true or both are false.

Ex: $\det(A) \neq 0 \iff A$ is invertible.

To prove " $P \iff Q$ ", need to prove " $P \Rightarrow Q$ " and " $Q \Rightarrow P$ ".

Note: $\neg P \Rightarrow \neg Q$ is the same as $Q \Rightarrow P$.

Subspaces (cont'd)

Thm (last class): Let $W \subseteq V$ (subset). If

1. For all $u, v \in W$, $u + v \in W$
2. For all $u \in W$, $c \in K$, $cu \in W$
3. $\vec{0} \in W$

then $W \leq V$ (subspace). (ie: (1), (2), (3) are true $\Rightarrow W \leq V$)

Theorem 7. Let $W \subseteq V$. Then

$$W \leq V \Rightarrow (1), (2), (3) \text{ are true}$$

(ie the converse of last theorem is true).

Proof. Exercise.

Theorem 8. Let $W \subseteq V$. Then

$$W \leq V \iff (1), (2), (3) \text{ are true}$$

Examples of subspaces and non-subspaces

Is each subset a subspace?

- (a) $W = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^2$. Not a subspace, since the zero vector is not in W . The others are also false, but it's enough to prove that one of the statements does not hold. But $\text{span}(W) = \mathbb{R}^2$ (so $\text{span}(W) \leq \mathbb{R}^2$)

- (b) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + y - z = 0 \right\}$. Need to check (1), (2), (3):

- (1) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in W$. Then we know $x + y - z = 0$ and $x' + y' - z' = 0$. Check:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ z + z' \end{pmatrix}$$

Verify

$$\begin{aligned} (x + x') + (y + y') - (z + z') &= (x + y - z) + (x' + y' - z') \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

So yes, it is in W .

- (2) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$ (means $x + y - z = 0$), let $c \in K$. To prove

$$c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \in W$$

Here, $cx + cy - cz = c(x + y - z) = c(0) = 0$. So $c \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in W$

- (3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in W$, since $0 + 0 - 0 = 0$

Since (1), (2), (3) true, $W \leq R^2$ (subspace)

- (c) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x + y - z = 1 \right\}$. This is *not* a subspace. (3) is false.

- (d) $W = \{A \in M_{2 \times 2} \mid A_{ij} \geq 0 \forall i, j\}$, where A_{ij} is the entry of A in row i , column j . (1) and (3) are true:

- (1) Add two matrices with non-negatives entries, result has non-negative entries.

- (2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$

Note, we wrote these out very informally. Now, (2) is false since,
for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ but

$$(-1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin W$$

Two special subspaces

Let V be a vector space.

- (1) $V \leq V$ is true
- (2) $\{\vec{0}\} \leq V$ is true ("zero subspace")

A refinement on the definition of span

Def. If $S = \emptyset$ (emptyset), define $\text{span}(S) = \{\vec{0}\}$ (if $S \neq \emptyset$, $\text{span}(S)$ defined as before).

Theorem 9. $\text{span}(S) \leq V$.

Proof Two cases :

- 1. If $S = \emptyset$, $\text{span}(S) = \{\vec{0}\} \leq V$
- 2. If $S \neq \emptyset$, you already proved $\text{span}(S)$ satisfies (1), (2), (3).
So $\text{span}(S) \leq V$.

Theorem 10. (improved version of subspace conditions) Let $W \subseteq V$. Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall w_1, w_2 \in W \text{ and } c \in K \text{ we have } cw_1 + w_2 \in W$$

Proof We will actually prove $(1), (2), (3) \iff \text{RHS}$ (right-hand side). Two parts to proof.

- (1) " $(1), (2), (3) \Rightarrow \text{RHS}$ " or " \Rightarrow "

January 23rd 2019

Recap:

- (1) If $u, v \in W$ then $u + v \in W$
- (2) if $u \in W, c \in K$ then $cu \in W$
- (3) $\vec{0} \in W$

Theorem 11. Let $W \subseteq V$. Then

$$W \leq V \iff W \neq \emptyset \text{ and } \forall u, v \in W, c \in K \text{ we have } cu + v \in W$$

Proof: Suffices to prove (1), (2), (3) \iff RHS.

1. \Rightarrow Assume (1), (2), (3) (prove right-hand side). Two things to prove:

(1) Since $\vec{0} \in W$ (by (3)), $W \neq \emptyset$

(2) Let $u, v \in W$ and $c \in K$. Since (2) holds, $cu \in W$. Since (1) holds, $cu \in W$ and $v \in W$, so $cu + v \in W$.

2. \Leftarrow Assume RHS, prove (1), (2), (3).

(1) Let $u, v \in W$. Apply RHS with \Leftarrow to get

$$cu + v = 1u + v = u + v \in W$$

(2) (Prove $\vec{0} \in W$) Since $W \neq \emptyset$, there is a vector $w \in W$. Apply right-hand side with $u = w, v = w, c = -1$. So $cu + v = (-1)w + w = -w + w = \vec{0} \in W$.

(3) Let $u \in W, c \in K$. Apply RHS ($cu + v \in W$) with $u = u, c = c, v = \vec{0}$ (note: $\vec{0} \in W$ by (3) above). Then $cu + v = cu + \vec{0} = cu \in W$ \square

Ex: In $F(R, R) = V$ (functions $f : R \rightarrow R$), prove that

$$W = \{f \in V \mid f(3) = 0\}$$

is a subspace. Eg: $f(x) = (x - 3)e^x \in W$.

Solution: (1), (2) together (by last thm). Let $f, g \in W, c \in R$ (prove $cf + g \in W$). We know $f(3) = 0$ and $g(3) = 0$. Then, check $(cf + g)(3) = cf(3) + g(3) = 0 + 0 = 0$. So $cf + g \in W$.

Also, prove $w \neq \emptyset$. $f(x) = x - 3 \in W$, since $f(3) = 0$ (or, $z(3) = 0$ satisfies $z(3) = 0$ so $z \in W$. Note that z is the zero vector of $F(R, R)$).

Theorem 12. Let $A \in M_{m \times n}(K), b \in K^m$. Define

$$S = \{x \in K^n \mid Ax = b\}$$

ie S = solution set to linear system $Ax = b$. Then,

$$S \leq K^n \iff b = \vec{0} \text{ (ie system is homogeneous)}$$

Proof

(i) \Rightarrow Assume $S \leq K^n$. Then $\vec{0}_n \in S$ (by (3)). So $A\vec{0} = b$ but $A\vec{0}_n = \vec{0}_m$ so $\vec{0} = b$.

- (ii) \Leftarrow Assume $b = \vec{0}_m$ (prove $S \leq K^n$). Then $A\vec{0}_n = \vec{0}_m$, so $\vec{0}_n \in S$.
 Next, let $u, v \in S, c \in K$. So $u, v \in K^n$ and $Au = b, Av = b$. Verify $cu + v$ is a solution.

$$\begin{aligned}
 A(cu + v) &= A(cu) + Av && \text{(prop of matrix multiplication)} \\
 &= c(Au) + Av && \text{(prop of matrix multiplication)} \\
 &= cb + b \\
 &= c\vec{0} + \vec{0} \\
 &= \vec{0} \\
 &= b \quad \square
 \end{aligned}$$

Ex: Equation $ax + by + cz = d$ describes a plane in R^3 (eg $x + y + z = 1$) (and also, every plane can be described this way). That is,

$$\{(x, y, z) \in R^3 | ax + by + cz = d\}$$

is a plane.

By last thm,

$$\begin{aligned}
 P \text{ is a subspace} &\iff ax + by + cz = d \text{ is a homogeneous system} \\
 &\iff d = 0 \\
 &\iff P \text{ passes through origin } (0, 0, 0)
 \end{aligned}$$

Theorem 13. Let $S \subseteq V$. Then,

- (1) $\text{span}(S) \leq V$ and $S \subseteq \text{span}(S)$
- (2) If $S \subseteq W$, and $W \leq V$ (subspace) then $\text{span}(S) \subseteq W$ (actually, $\text{span}(S) \leq W$, subspace by (1))

Proof:

- (1) \leq We know already. Let $u \in S$. Then $u = 1u$, so $u \in \text{span}(S)$
- (2) Assume $S \subseteq W$, and $W \leq V$. Let $v \in \text{span}(S)$. Then $v = a_1u_1 + a_2u_2 + \dots + a_nu_n$ for some scalars and vectors $u_1, u_2, \dots, u_n \in S$. Since $S \subseteq W$, $u_1, u_2, \dots, u_n \in W$. But W subspace. So $a_1u_1, a_2u_2, \dots, a_nu_n \in W$ (by prop (2) subspace) then $a_1u_1 + a_2u_2 \in W$ (by prop (1) of subspaces). So then $(a_1u_1 + a_2u_2) + a_3u_3 \in W$ (etc). So $a_1u_1 + a_2u_2 + \dots + a_nu_n \in W$.

Note: "etc" here is actually a proof by mathematical induction.

Omit for now.

January 25th 2019

Interlude : Symbolic logic (briefly)

Let P, Q be statements that could be true (T) or false (F). Define:

- (1) $\neg P$, "not P", is F when P is T , T when P is F
- (2) $P \wedge Q$, "P and Q", is T exactly when P, Q both T
- (3) $P \vee Q$, "P or Q" is T when P, Q both F
- (4) $P \Rightarrow Q$, "P implies Q", is T *unless* P is T and Q is F . Hence, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$. We will write $P \Rightarrow Q \equiv \neg P \vee Q$.
- (5) $P \iff Q$, "P if and only if Q", is T if both T or both F .

De Morgan's Laws

- $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
- $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Quantifiers

- \forall means "for all"
- \exists means "there exists"

Ex. (A4) (commutativity) $\forall u, v \in V \quad u + v = v + u$.

Ex. 2 (A2) (zero vector) $\exists z \in V \quad \forall u \in V \quad (u + z = u) \wedge (z + u = u)$
(textbook version)

Negating quantifiers

- $\neg \forall u \in V P(u) \equiv \exists u \in V \neg P(u)$
- $\neg \exists u \in V P(u) \equiv \forall u \in V \neg P(u)$

Ex.

$$\begin{aligned}
 \neg(A2) &\equiv \neg \exists z \in V \forall u \in V \quad u + z = u \wedge z + u = u \\
 &\equiv \forall z \in V \neg \forall u \in V \quad u + z = u \wedge z + u = u \\
 &\equiv \forall z \in V \exists u \in V \quad \neg(u + z = u \wedge z + u = u) \\
 &\equiv \forall z \in V \exists u \in V \quad (u + z \neq u \vee z + u \neq u)
 \end{aligned}$$

Proof by contradiction

You want to prove some statement P . Proof by contradiction works this way:

- (1) Assume $\neg P$
- (2) Derive a contradiction (hard part)
- (3) Conclude P is true

Ex. Outline of how to prove (A2) *does not hold* in some vector space.
You want to prove $\neg(A2)$.

$$\begin{aligned}\neg(A2) &\equiv \neg \exists z \in V \forall u \in V \quad u + z = u \wedge z + u = u \\ &\equiv \forall z \in V \neg \forall u \in V \quad u + z = u \wedge z + u = u\end{aligned}$$

Let $z \in V$. Prove the right-hand part ($\neg \forall u \in V \quad u + z = u \wedge z + u = u$) by contradiction. Assume (for contradiction) that

$$\forall u \in V \quad u + z = u \wedge z + u = u \quad (1)$$

Use (1) by substituting $u =$ some specific vector (derive a contradiction). Conclude that ($\neg \forall u \in V \quad u + z = u \wedge z + u = u$) is true.

Last time

Theorem 14. If $S \subseteq W$, $W \leq V$ then $\text{span}(S) \subseteq W$.

Note. This means if you "promote" a subset to a subspace, adding in only what's necessary, what you get is $\text{span}(S)$. Or, $\text{span}(S)$ is the "smallest" subspace containing S .

Fact. Subspaces are "closed under taking linear combinations". Ie if $W \leq V$, $w_1, \dots, w_n \in W$ and $a_1, \dots, a_n \in K$ then

$$a_1 w_1 + a_2 w_2 + \dots + a_n w_n \in W$$

Caution. Linear combinations are *finite* sums by definition. So you can't sum up infinitely many vectors.

Illustration of this theorem

Let $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\} \subseteq W = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x, y \in R \right\}$. Then

$\text{span}(S) \subseteq W$ ie $\text{span}(S)$ is in xy plan. In fact, $\text{span}(S) = W$.

Def. If $W = \text{span}(S)$, we say that S spans W or is a spanning set for W .

Ex. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\text{span}(S) = xy\text{-plane in } R^3$. So S spans the $xy\text{-plane}$.

Ex. 2. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$, $\text{span}(S) = \left\{ \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} \mid x \in R \right\} = \text{line}$.

Intersection of two subspaces

Theorem 15. Let $W_1 \leq V, W_2 \leq V$. Then $W_1 \cap W_2 \leq V$ (ie intersection of two subspaces is a subspace).

Proof. $W_1 \cap W_2 = \{w \in V | w \in W_1 \wedge w \in W_2\}$.

- (1) $\vec{0} \in W_1, \vec{0} \in W_2$ (because subspace). So $\vec{0} \in W_1 \cap W_2$.
- (2) Let $u, v \in W_1 \cap W_2, c \in K$. So $u, v \in W_1$ and $W_1 \leq V$ so $cu + v \in W_1$ and $u, v \in W_2$ and $W_2 \leq V$ so $cu + v \in W_2$. Hence $cu + v \in W_1 \cap W_2$. \square

January 28th 2019

Last time: $W_1 \leq V$ and $W_2 \leq V \Rightarrow W_1 \cap W_2 \leq V$.

Corollary 15.1. The intersection of any number of subspaces is a subspace.

Problem. Prove that $W = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace of $F(\mathbb{R}, \mathbb{R})$.

Sol #1: Directly from subspace properties (omit)

Sol #2: We saw an example proving that $\{f : \mathbb{R} \rightarrow \mathbb{R} | f(3) = 0\}$ is a subspace. The "3" is not important, so similarly:

$$W_1 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0\}$$

$$W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(2) = 0\}$$

both subspaces of $F(\mathbb{R}, \mathbb{R})$. Then $W_1 \cap W_2 = \{f : \mathbb{R} \rightarrow \mathbb{R} | f(1) = 0 \wedge f(2) = 0\}$ is a subspace.

Q: Is union of two subspaces also a subspace?

A: Not in general.

Eg: $W_1 = x\text{-axis} = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} | x \in \mathbb{R}\right\} \leq \mathbb{R}^2$

$$W_2 = y\text{-axis} = \left\{\begin{pmatrix} 0 \\ y \end{pmatrix} | y \in \mathbb{R}\right\} \leq \mathbb{R}^2$$

$W_1 \cup W_2 = xy\text{-axis} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} | x = 0 \vee y = 0\right\}$, which, importantly, is not \mathbb{R}^2 . Not a subspace, since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in W_1 \cup W_2, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in W_1 \cup W_2$, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin W_1 \cup W_2$.

Note: To promote $W_1 \cup W_2$ to a subspace, you form $\text{span}(W_1 \cup W_2)$.

Def: Let $W_1 \leq V, W_2 \leq V$. The *sum* of W_1 and W_2 is

$$W_1 + W_2 = \{v \in V | \exists w_1 \in W_1, w_2 \in W_2, \text{ such that } v = w_1 + w_2\}$$

Ex:

$$W_1 = \{ax^2 | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

$$W_2 = \{ax | a \in \mathbb{R}\} \leq P(\mathbb{R})$$

We have,

$$W_1 + W_2 = \{ax^2 + bx | a, b \in \mathbb{R}\}$$

Theorem 16. Let $W_1 \leq V, W_2 \leq V$. Then

(a) $W_1 + W_2 = \text{span}(W_1 \cup W_2)$ (hence $W_1 + W_2$ is a subspace)

(b) $W_1 \leq W_1 + W_2, W_2 \leq W_1 + W_2$

Proof:

(a)(1) Prove $W_1 + W_2 \subseteq \text{span}(W_1 \cup W_2)$. Let $v \in W_1 + W_2$, so $v = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. Then $w_1, w_2 \in W_1 \cup W_2$ so $v \in \text{span}(W_1 \cup W_2)$

(2) " \supseteq ". Let $v \in \text{span}(W_1 \cup W_2)$. Means $v = a_1u_1 + a_2u_2 + \dots + a_nu_n, u_1, u_2, \dots, u_n \in W_1 \cup W_2$ and $a_1, a_2, \dots, a_n \in K$. Each u_i is in $W_1 \cup W_2$. Separate into two groups and relabel, so that:

- Those in W_1 , call these

$$u_1, u_2, \dots, u_l$$

So $0 \leq l \leq n, l = 0$ means none in W_1 .

- Those in $W_2 \setminus W_1 = \{w \in W_2 | w \notin W_1\}$ ("set difference"), call these

$$u_{l+1}, \dots, u_n$$

So $l = 0$ means all in $W_2 \setminus W_1, l = n$ means all in W_1 .

Then, let $w_1 = a_1u_1 + a_2u_2 + \dots + a_lu_l$ (or $w_1 = \vec{0}$ if $l = 0$),
 $w_2 = a_{l+1}u_{l+1} + \dots + a_nu_n$ (or $w_2 = \vec{0}$ if $l = n$).

Then $w_1 \in W_1$ since W_1 is a subspace, similarly $w_2 \in W_2$. So

$$\begin{aligned} v &= a_1u_1 + \dots + a_nu_n \\ &= w_1 + w_2 \in W_1 + W_2 \text{ as required} \end{aligned}$$

(b) $W_1 \leq W_1 + W_2, W_2 \leq W_1 + W_2$. Follows from (a), since $S \subseteq \text{span}(S)$ \square .

Linear independence

Def: Vectors $u_1, u_2, \dots, u_n \in V$ (all distinct) are said to be *linearly dependent* if \exists scalars $a_1, a_2, \dots, a_n \in K$ not all 0 such that

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$$

Above equation called a *dependence relation*.

Note: If $a_1u_1 + a_2u_2 + \dots + a_nu_n = \vec{0}$ and $a_1 \neq 0$, then you can solve for u_1 :

$$u_1 = \frac{-a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \dots - \frac{a_n}{a_1}u_n$$

ie u_1 = linear combination of others, "depends on" others.

Ex: $\{x^2 + x, 2x^2, \frac{x}{10}\}$ is a dependent set of vectors in $P(\mathbb{R})$ since

$$(x^2 + x) - \frac{1}{2}(2x^2) - 10(\frac{x}{10}) = 0$$

Def: A set of vectors $S \subseteq V$ (possibly infinite) is dependent if \exists a finite subset $\{v_1, v_2, \dots, v_n\} \subseteq S$ of it which is dependent.

Def: Vectors v_1, v_2, \dots, v_n are linearly independent if they are *not* dependent. That is,

$$\begin{aligned} \neg \exists a_1, \dots, a_n \in K \quad (a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0)) \\ \forall a_1, \dots, a_n \in K \quad \neg(a_1u_1 + \dots + a_nu_n = \vec{0} \wedge \neg(a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0)) \\ \forall a_1, \dots, a_n \in K \quad (\neg(a_1u_1 + \dots + a_nu_n = \vec{0}) \vee (a_1 = 0 \wedge a_2 = 0 \wedge \dots \wedge a_n = 0)) \end{aligned}$$

Note that $P \implies Q \equiv \neg P \vee Q$. In other words, u_1, u_2, \dots, u_n are linearly independent if

$$\forall a_1, \dots, a_n \in K (a_1u_1 + \dots + a_nu_n = \vec{0} \implies a_1 = 0 \wedge \dots \wedge a_n = 0)$$

Which is to say that the only solution to $a_1u_1 + \dots + a_nu_n = \vec{0}$ is the trivial solution $a_1 = 0, a_2 = 0, \dots, a_n = 0$.

January 30th 2019

Last class

v_1, v_2, \dots, v_n independent if $x_1v_1 + \dots + x_nv_n = \vec{0}$ has only trivial solution $x_1 = x_2 = \dots = x_n = 0$.

Ex: Prove that $\{1 + x^2, x + x^2, 1 + x + x^2\}$ is independent.

Solution: Consider equation

$$a(1 + x^2) + b(x + x^2) + c(1 + x + x^2) = 0 = 0(1) + 0x + 0x^2$$

Want to show $a = b = c = 0$ is the only solution.

Equation means for all $x \in K$ (\mathbb{R} or \mathbb{C}),

$$a(1 + x^2) + b(x + x^2) + c(1 + x + x^2) = 0$$

So, substitute any scalar for x :

$$\begin{aligned} x = 0 \quad a + c &= 0 \\ x = 1 \quad 2a + 2b + 2c &= 0 \\ x = -1 \quad 2a + 0b + c &= 0 \end{aligned}$$

Can translate into linear system:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

Row-reduce:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Only solution is $a = 0, b = 0, c = 0$ so vectors are independent.
If we obtain infinitely many, then you can find dependent set so dependent.

Some important cases

- (i) $S = \emptyset$ is linearly independent since there are no vectors with which to form a dep. relation.
- (ii) If $\vec{0} \in S$, then dependent (since $1\vec{0} = \vec{0}$ is a dep. relation)
- (iii) $\{u\}$ is independent $\iff u \neq \vec{0}$.
Note: $u + (-1)u = \vec{0}$ is *not* a dep. relation, since u is repeated. But, $\{u, -u\}$ is dependent since

$$u + (-u) = \vec{0}$$

is a dep. relation.

Proposition 17. Let $A, B \subseteq V$ where $A \subseteq B$.

- (i) If A is dependent, B is also dependent
- (ii) If B is independent, A is also independent (contrapositive)

Proof:

- (i) If A dep, we have a dep relation

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in A)$$

which is also a dependence relation in B since $v_1, \dots, v_n \in B$.

- (ii) This is the contrapositive of (i). \square

Note: Converse is false, $B \text{ dep} \not\Rightarrow A \text{ dep}$.

Extending an independent set

Theorem 18. Let $S \subseteq V$ be linearly independent and suppose $u \notin S$. Then, $S \cup \{u\}$ independent $\iff u \notin \text{span}(S)$.

Proof:

- (i) " \rightarrow " We will prove this as the contrapositive, ie $u \in \text{span}(S) \rightarrow \text{dep}$. Assume $u \in \text{span}(S)$. So,

$$u = a_1v_1 + \dots + a_nv_n \quad \text{where } v_1, v_2, \dots, v_n \in S$$

$$\vec{0} = (-1)u + a_1v_1 + \dots + a_nv_n$$

Which is a linear combination of vectors from $S \cup \{u\}$, not all coefficients 0 since first is -1 . Also, the vectors u, v_1, v_2, \dots, v_n are all distinct, since $u \notin S$. So this is a dependence relation on $S \cup \{u\}$, so the set is dependent.

- (ii) " \leftarrow " Also by contrapositive. Assume $S \cup \{u\}$ dep, want to show that $u \in \text{span}(S)$. So there is a dependence relation on $S \cup \{u\}$.

Two cases:

- **Case 1:** Dependence relation does not involve u (or, involves u but with coefficient 0), ie we have

$$a_1v_1 + \dots + a_nv_n = \vec{0} \quad (\text{not all scalars } 0, v_1, \dots, v_n \in S)$$

But this contradicts independence of S , so case 1 does not occur.

- **Case 2:** Dependence relation involves u (with coeff *not* 0), so

$$au + a_1v_1 + \dots + a_nv_n = \vec{0} \quad v_1, \dots, v_n \in S$$

and $a \neq 0$. Rewrite:

$$u = \frac{-a_1}{a}v_1 - \frac{a_2}{a}v_2 - \dots - \frac{a_n}{a}v_n \quad (a \neq 0)$$

Hence $u \in \text{span}(S)$. \square

Note: Conclusion can be restated as

$$S \cup \{u\} \text{ dependent} \iff u \in \text{span}(S)$$

Basis and dimension

Fact: If W is subspace, then $\text{span}(W) = W$. (Exercise)

So every subspace is a span. But thinking of W as $\text{span}(W)$ is excessive. Would like to find the *smallest* S such that

$$\text{span}(S) = W$$

Def: Let $W \leq V$. A *basis* of W is a set $B \subseteq V$ such that

- (i) $\text{span}(B) = W$ ("enough vectors to produce W ")
- (ii) B is linearly independent ("no extra vectors in B ")

Examples:

(i) Let $e_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \leftarrow (\text{row } i)$. Then,

$$\{e_1, e_2, \dots, e_n\}$$

is a basis for K^n . For example,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for K^3 .

More next class.

February 1st 2019

Recall: B is a basis of W if $\text{span}(B) = W$ and B is linearly independent.

Examples:

- (1) $P_n(K)$ has basis $\{1, x, x^2, \dots, x^n\}$
- (2) $P(K)$ has basis $\{1, x, x^2, x^3, \dots\}$ (infinitely many)
- (3) $M_{m \times n}(K)$ has basis $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ where $E^{ij} = m \times n$ matrix of 0s except 1 in row i , column j . eg: $M_{2 \times 2}(\mathbb{R})$ has basis

$$E^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- (4) $W = \{\vec{0}\}$ has basis \emptyset since

- (i) $\text{span } \emptyset = \{\vec{0}\}$ (by special def)
- (ii) \emptyset is independent

Two important questions

- (1) Does W *always* have basis? (spoiler: yes)
- (2) How to *find* a basis?

Theorem 19 (Bases exist). *Let V be vector space and S a finite set with $\text{span}(S) = V$. Then there is a subset $B \subseteq S$ which is a basis of V .*

Proof. Algorithm to produce B .

- (1) If $V = \{\vec{0}\}$, use $B = \emptyset$.
- (2) Take one vector, $u_1 \in S (u_1 \neq \vec{0})$. Consider $\text{span}\{u_1\}$
- (3) If $\text{span}\{u_1\} = V$, done. $B = \{u_1\}$ is a basis (set of one non-zero vector is independent)
- (4) If $\text{span}\{u_1\} \neq V$, there must be a vector $u_2 \in S$ where $u_2 \notin \text{span}\{u_1\}$ (Why? If not, $S \subseteq \text{span}\{u_1\} \leq V$, then $\text{span}(S) \subseteq \text{span}\{u_1\}$, but $\text{span}(S) = V$ contradicts $V \neq \text{span}\{u_1\}$). By previous theorem, since $u_2 \notin \text{span}\{u_1\}$, $\{u_1, u_2\}$ is linearly independent.
- (5) Consider $\{u_1, u_2\}$. If $\text{span}\{u_1, u_2\} = V$, done: $B = \{u_1, u_2\}$. Else, continue as before, finding $u_3 \in S, u_3 \notin \text{span}\{u_1, u_2\}$ (etc)

Since S is finite, this must stop and at that point you have basis $B \subseteq S$. □

Illustration of this thm

Find basis of \mathbb{R}^3 that is a subset of

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

Following this algorithm,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Theorem 20. *Let V be a vector space, $L \subseteq V$ a linearly independent set, and $S \subseteq V$ a spanning set (ie $V = \text{span}(S)$). Then \exists a subset $E \subseteq S$ such that $L \cup E$ is a basis of V (ie you can always extend it to a basis)*

Proof Omitted.

Theorem 21. *Suppose V has a finite spanning set S . Then V has a basis and all bases have the same size, which is at most $|S|$.*

Proof Omitted.

Def If V has a finite basis B , then the dimension of V is

$$\dim V = |B|$$

If V does not have a finite basis, it is called *infinite dimensional*.

Ex:

(1) $\dim K^n = n$.

$$\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix} \right\} \right)$$

(2) $\dim P_n(K) = n + 1$ (basis $\{1, x, x^2, \dots, x^n\}$)

(3) $P(K)$ is infinite dimensional (A#1, proved a finite set of polynomials cannot span $P(K)$)

(4) $\dim M_{m \times n}(K) = mn$ (see basis E^{ij} , defined above)

Theorem 22. Every vector space (including the infinite dimensional ones) has a basis.

Proof Uses Axiom of Choice. Difficult.

Theorem 23. Suppose $\dim V = n$. Let $A \subseteq V$. Then,

- (1) If $\text{span}(A) = V$, then $|A| \geq n$ (or, if $|A| < n$ then A does not span V) and if also $|A| = n$ then A is linearly independent, hence basis.
- (2) If A is linearly independent, then $|A| \leq n$ (or, if $|A| > n$ then A dep) and if also $|A| = n$ then $\text{span}(A) = V$ hence A is a basis.

Proof Omitted.

Note: If you have *correct number* of vectors, you need only check spanning or independent, not both, to check if basis.

Ex: If you have 7 matrices in $M_{3 \times 2}(K)$, they *will be* dependent. If you have 5, it's *not* a basis.

February 4th 2019

Last class

Suppose $\dim V = n$, $S \subseteq V$, $|S| = n$. Then $S \text{ span } V \iff S \text{ linearly independent}$ (only in case $|S| = \dim V$).

Lagrange Interpolation

Problem Given "data points" $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$ where all a_i are different. Find a polynomial $p(x)$ of degree $n - 1$, $p(x) =$

$c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_1x + c_0$ whose graph $y = p(x)$ passes through all the points.

Sol #1 Substitute (a_1, b_1) into $y = p(x)$:

$$b_1 = c_{n-1}a_1^{n-1} + \dots + c_1a_1 + c_0 \quad (\text{for each } i = 1, \dots, n)$$

Which is a system of n linear equations (vars = c_{n-1}, \dots, c_0) in n variables.

We'll do something different.

Def For scalars a_1, a_2, \dots, a_n (all different), define the *Lagrange polynomials* for each $i = 1, 2, \dots, n$ set

$$\begin{aligned} l_i(x) &= \prod_{k=1, k \neq i}^n \frac{(x - a_k)}{(a_i - a_k)} \\ &= \frac{(x - a_1)}{(a_i - a_1)} \cdot \frac{(x - a_2)}{(a_i - a_2)} \cdot \dots \cdot \frac{(x - a_n)}{(a_i - a_n)} \quad (\text{omitting } \frac{(x - a_i)}{(a_i - a_i)}) \end{aligned}$$

Ex For $a_1 = 2, a_2 = 4, a_3 = 6$ we would have

$$\begin{aligned} l_1(x) &= \frac{(x - 4)}{2 - 4} \cdot \frac{(x - 6)}{(2 - 6)} \\ l_2(x) &= \frac{(x - 2)}{4 - 2} \cdot \frac{(x - 6)}{(4 - 6)} \\ l_3(x) &= \frac{(x - 2)}{6 - 2} \cdot \frac{(x - 4)}{(6 - 4)} \end{aligned}$$

Note: All degree 2, $l_1(4) = 0, l_1(6) = 0, l_1(2) = 1$.

Fact $l_i(a_j) = 0$ if $i \neq j$ and 1 if $i = j$.

Proof If $i \neq j$, there is a factor $\frac{x - a_j}{a_i - a_j}$, so at $x = a_j$, $\frac{a_j - a_j}{a_i - a_j} = 0$. If $i = j$,

$$l_i(a_i) = \prod_{k=1, k \neq i}^n \frac{(a_i - a_k)}{(a_i - a_i)} = 1$$

Proposition 24. *Lagrange polynomials $l_1(x), \dots, l_n(x)$ form a basis of $P_{n-1}(\mathbb{R})$.*

Proof We have n polynomials (they are distinct), $\dim P_{n-1}(\mathbb{R}) = n - 1 + 1 = n$. So correct number. Suffices to prove *span* or *lin independence*. We'll prove independence. Suppose

$$d_1l_1(x) + d_2l_2(x) + \dots + d_nl_n(x) = 0 \quad (\text{note: for all } x \in \mathbb{R})$$

Substitute $x = a_1, x = a_2$, etc into the above. At $x = a_1, l_1(a_1) = 1$ but $l_j(a_1) = 0$ for $j \neq 1$ so

$$d_1 \cdot 1 + d_2 \cdot 0 + \dots + d_n \cdot 0 = 0$$

so $d_1 = 0$. Similarly, $d_j = 0$ for all j . More formally, for any $j = 1, 2, \dots, n$ we have at $x = a_j$

$$\sum_{i=1}^n d_i l_i(a_j) = 0$$

but all terms are 0 *except* when $i = j$. Set

$$d_j = d_j(1) = d_j l_j(a_j) = 0$$

Hence Lagrange polynomials for a basis.

Problem Find poly degree $n - 1$ through points $(a_1, b_1), \dots, (a_n, b_n)$.

Sol: Set $p(x) = b_1 l_1(x) + b_2 l_2(x) + \dots + b_n l_n(x)$ (it has degree $n - 1$).

Then

$$\begin{aligned} p(a_1) &= b_1 l_1(a_1) + b_2 l_2(a_1) + \dots + b_n l_n(a_1) \\ &= b_1(1) + 0 + 0 + \dots + 0 \\ &= b_1 \end{aligned}$$

For each $i = 1, 2, \dots, n$,

$$\begin{aligned} p(a_i) &= \sum_{j=1}^n b_j l_j(a_i) \\ &= 0 + 0 + \dots + b_i l_i(a_i) + \dots + 0 \\ &= b_i \end{aligned}$$

Dimension of subspaces

Theorem 20. Let $W \leq V$, V finite-dimensional. Then

$$(i) \dim W \leq \dim V$$

$$(ii) \dim W = \dim V \iff W = V$$

Proof

- (i) Similar to proof that V has basis. Use W as a spanning set for W . Pick out vectors one at a time (similar to before) to build a basis. You cannot put more than $\dim V$ vectors into your basis, as this would give an independent set in V of size *more than* $\dim V$ (impossible). So this process has to stop, and it produces a basis for W .
- (ii) " \rightarrow " Assume $\dim W = \dim V = n$. Take basis B of W . It is a size n linearly independent set inside V , hence B also basis for V , hence,

$$V = \text{span } B = W$$

" \leftarrow " If $W = V$, clearly $\dim W = \dim V$. \square

$\dim W$	Classification
0	$\{\vec{0}\}$
1	$\text{span}\{u\} = \text{line through origin}$
2	$\text{span}\{u, v\} = \text{plane through origin}$
3	\mathbb{R}^3

Subspaces of \mathbb{R}^3 If $W \leq \mathbb{R}^3$, $\dim W = 0, 1, 2$ or 3 .

This allows us to make the following classification: **Problem** Let $W = \{A \in M_{n \times n}(\mathbb{R}) \mid \text{tr}(A) = 0\}$, where $\text{tr}(A) = \text{trace of } A = \text{sum of entries on diagonal} = A_{11} + A_{22} + \dots + A_{nn}$.

Exercise Prove W is a subspace.

Will do next class: Find $\dim W$ and find a basis of W .

February 6th 2019

Intuition

Solution set W to a homogeneous system $A\vec{x} = \vec{0}$ is a subspace of K^n ($n = \#$ of variables). If no equations, $W = K^n$, $\dim W = n$. For each equation, *expect* the dimension of W to drop by 1, unless the equation is *redundant*.

Eg: In \mathbb{R}^3 , one equation

$$a_1x + b_1y + c_1z = 0 \quad (= \text{plane})$$

$$\text{add in } a_2x + b_2y + c_2z = 0 \quad (\text{intersection of two planes, = line})$$

$$\text{add in } a_3x + b_3y + c_3z = 0 \quad (\text{intersection of three planes, (0,0)})$$

Problem: $W = \{A \in M_{n \times n}(\mathbb{R}) \mid \text{tr } A = 0\}$. Find $\dim W$, basis of W .

Solution #1: Clever way: "guess" a basis. Note: $\text{tr } A = A_{11} + A_{22} + \dots + A_{nn}$ (one linear condition). Expecting

$$\dim W = n^2 - 1$$

Observe that $\dim W \neq n^2$. This happens only if $W = M_{n \times n}(\mathbb{R})$, and obviously there are matrices which don't have trace 0. Specifically:

$$\text{tr} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \dots & \\ & & & 1 \\ & & & & 1 \end{pmatrix}$$

(In proofs, can choose any example, provided property holds).

Know $\dim W \leq n^2 - 1$. If you can find independent set of size $n^2 - 1$ in W , it *will be* a basis. Try first $n = 3$. Looking for $3^2 - 1 = 8$ independent 3×3 matrices, all trace 0.

Want trace = 0. Therefore, consider all matrices which have all 0's in the diagonal:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

These 6 are obviously independent. Now, take two more which are independent:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we have 8 independent matrices (check carefully). So for $n = 3$, $\dim W = 8$, this is a basis.

General case

Two types of basis matrices:

- (I) All E^{ij} (1 in (i, j) -pos, 0 elsewhere)) where $i \neq j$. How many are there?

$$\begin{aligned} \# \text{ of non-diagonal entries} &= \text{entries} - \text{entries on diagonal} \\ &= n^2 - n \end{aligned}$$

Or, $\binom{n}{2}$ ways to choose 2 distinct values from $\{1, 2, \dots, n\}$, 2 ways to order each pair. Total:

$$\begin{aligned} \binom{n}{2} 2 &= \frac{n!}{2!(n-2)!} 2 \\ &= n(n-1) \\ &= n^2 - n \end{aligned}$$

- (II) Looking for $n - 1$ more, since $n^2 - n + n - 1 = n^2 - 1$

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \dots & \\ & & & 0 \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & \dots \\ & & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \dots & \\ & & & 1 \\ & & & & -1 \end{pmatrix}, \dots$$

(n-1 of those)

Formally, let, for $i = 1, 2, \dots, n - 1$, D_i = matrix with 1 in pos (i, i) and -1 in pos $(i + 1, i + 1)$, 0 elsewhere.

Verifying all matrices E^{ij} , D_i are independent; clear that suffices to check D_1, D_2, \dots, D_{n-1} independent. Suppose

$$x_1 D_1 + x_2 D_2 + \dots + x_n D_n = n \times n \text{ zero matrix}$$

The $(1,1)$ -entry on left is x_1 , so $x_1 = 0$. The $(2,2)$ -entry on left is $-x_1 + x_2$,

$$x_1 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \dots & \\ & & & 0 \\ & & & & 0 \end{pmatrix} + x_2 \begin{pmatrix} & 1 & & \\ & & -1 & \\ & & & \dots \\ & & & & 0 \end{pmatrix} + \dots = \begin{pmatrix} & & & 0 \\ & & & & 0 \\ & & & & & \dots \\ & & & & & & 0 \end{pmatrix}$$

but $x_1 = 0$ so $x_2 = 0$ also, etc. So similarly for all $x_i = 0$, so independent. Formally you'd do a proof by induction, but this is good enough.

Now have $n^2 - 1$ independent vectors in W_1 so $\dim W \geq n^2 - 1$.

1. Already, know $\dim W \leq n^2 - 1$. So $\dim W = n^2 - 1$, have independent set of correct size, so basis.

Solution #2: Let x_{ij} be the (i,j) -entry of A . So have n^2 variables $(x_{ij}, i, j = 1, 2, \dots, n)$ one equation,

$$x_{11} + x_{22} + \dots + x_{nn} = 0 \quad (\text{tr } A = 0)$$

Solve system. All $x_{ij}, i \neq j$ free variables, so are x_{22}, \dots, x_{nn} .

Theorem 21. Let U, W be finite dimension subspaces of V . Then,

$$\dim(U + W) = \dim U + \dim W - \dim U \cap W$$

It's like sets, $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof Omitted.

Ex: If W is a plane in \mathbb{R}^3 (through $(0,0)$) and L is a line in \mathbb{R}^3 (through $(0,0)$) and L is not in the plane, prove $W + L = \mathbb{R}^3$.

Sol: L not in plane gives $L \cap W = \{\vec{0}\}$. So

$$\begin{aligned} \dim(L + W) &= \dim L + \dim W - \dim L \cap W \\ &= 1 + 2 - 0 \\ &= 3 \end{aligned}$$

Hence $L + W = \mathbb{R}^3$.

Problem: Suppose $\dim V = n$, and U, W subspaces, each of dimension more than $\frac{n}{2}$. Prove that $U \cap W \neq \{\vec{0}\}$.

Proof By contradiction. Suppose $U \cap W = \{\vec{0}\}$. So $\dim U \cap W = 0$. Then

$$\begin{aligned} \dim(U + W) &= \dim U + \dim W - \dim U \cap W \\ &> \frac{n}{2} + \frac{n}{2} - 0 = n \end{aligned}$$

Says $U + W$ is a subspace of V of \dim more than $\dim V$. Impossible, so $U \cap W \neq \{\vec{0}\}$.

END OF MIDTERM MATERIAL.

February 8th 2019

Monday: No class, office hours during class time. Tuesday night :
Midterm!

Linear transformations - Definition and basic properties

(Chap. 5 in the text) **Def.** Let U, V be vector spaces, both over field K .
A function $T : U \rightarrow V$ is called a *linear transformation* if

- (i) $\forall u_1, u_2 \in U \quad T(u_1 + u_2) = T(u_1) + T(u_2)$. The first '+' is in U ,
while the second '+' is in V . The vector spaces need not be related
in any way, except that they must be over the same field of scalars.
- (ii) $\forall u \in U, c \in K \quad T(cu) = cT(u)$. Again, the first scalar multiplication
happens in U , while the second scalar multiplication happens in
 V .

Comment: Linear transformations are the functions that are somehow "compatible" with the vector space operations.

Ex: Prove that $T : P_2(\mathbb{R}) \rightarrow \mathbb{R}^2$ defined by

$$T(ax^2 + bx + c) = \begin{pmatrix} a + b \\ b + c \end{pmatrix}$$

Sol:

- (i) Let $p_1(x) = a_1x^2 + b_1x + c_1$, $p_2(x) = a_2x^2 + b_2x + c_2$ be in $P_2(x)$.
Then,

$$\begin{aligned} T(p_1(x) + p_2(x)) &= T((a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2) \\ &= \begin{pmatrix} a_1 + a_2 + b_1 + b_2 \\ b_1 + b_2 + c_1 + c_2 \end{pmatrix} \\ T(p_1(x)) + T(p_2(x)) &= \begin{pmatrix} a_1 + b_1 \\ b_1 + c_1 \end{pmatrix} + \begin{pmatrix} a_2 + b_2 \\ b_2 + c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 + b_1 + a_2 + b_2 \\ b_1 + c_1 + b_2 + c_2 \end{pmatrix} \end{aligned}$$

- (ii) Let $p(x) = ax^2 + bx + c \in P_2(\mathbb{R}), d \in K$.

$$\begin{aligned} T(dp(x)) &= T(dax^2 + dbx + dc) \\ &= \begin{pmatrix} da + db \\ db + dc \end{pmatrix} \\ &= d \begin{pmatrix} a + b \\ b + c \end{pmatrix} \\ &= dT(ax^2 + bx + c) \\ &= dT(p(x)) \end{aligned}$$

So T is a linear transformation.

Ex Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) = (x^2, x + y)$. Show that T is *not* a linear transformation.

Sol Try $u = (2, 3), v = (3, 4)$.

$$\begin{aligned} T(u + v) &= T(5, 7) \\ &= (25, 12) \end{aligned}$$

On the other hand,

$$\begin{aligned} T(u) + T(v) &= T(2, 3) + T(3, 4) \\ &= (4, 5) + (9, 7) \\ &= (13, 12) \\ &\neq (25, 12) \end{aligned}$$

So T is *not* linear.

Ex: Define $\frac{d}{dx} : P(\mathbb{R}) \rightarrow P(\mathbb{R})$ by

$$\frac{d}{dx}p(x) = p'(x) \quad (\text{derivative})$$

Then $\frac{d}{dx}$ is a linear transformation, since we know from calculus that

$$\begin{aligned} \frac{d}{dx}(p(x) + q(x)) &= \frac{d}{dx}p(x) + \frac{d}{dx}q(x) \\ \frac{d}{dx}(cp(x)) &= c \frac{d}{dx}p(x) \quad (c \in \mathbb{R}) \end{aligned}$$

Proposition 22. Let $T : U \rightarrow V$ be a linear transformation. Then,

- (i) $T(\vec{0}) = \vec{0}$ (where the first $\vec{0}$ is the zero vector of U and the second is the zero vector of V)
- (ii) $\forall u_1, u_2, \dots, u_n \in U$ and $c_1, c_2, \dots, c_n \in K$,

$$\begin{aligned} T(c_1u_1 + c_2u_2 + \dots + c_nu_n) &= \\ c_1T(u_1) + c_2T(u_2) + \dots + c_nT(u_n) \end{aligned}$$

Proof. (i)

$$\begin{aligned} T(\vec{0}_U) &= T(\vec{0}_U + \vec{0}_U) \\ T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{T linear}) \\ \vec{0}_V + T(\vec{0}_U) &= T(\vec{0}_U) + T(\vec{0}_U) \quad (\text{A2}) \\ \vec{0}_V &= T(\vec{0}_V) \quad (\text{cancellation law}) \end{aligned}$$

(ii)

$$\begin{aligned}
T(c_1u_1 + (c_2u_2 + \dots + c_nu_n)) &= T(c_1u_1) + T(c_2u_2 + \dots + c_nu_n) \\
&\quad \text{(T linear)} \\
&= c_1T(u_1) + T(c_2u_2 + \dots + c_nu_n) \\
&\quad \text{(T linear)} \\
&= \dots \quad \text{(proof by induction)} \\
&= c_1T(u_1) + \dots + c_nT(u_n)
\end{aligned}$$

□

Proposition 23. Let $T : U \rightarrow V$ function (U, V vector spaces). Then,

$$\begin{aligned}
&T \text{ is linear transformation} \iff \\
&\forall u_1, u_2 \in U \ c \in K, T(cu_1 + u_2) = cT(u_1) + T(u_2)
\end{aligned}$$

Proof: Exercise. □

February 15th 2019

Def ("matrix defines a linear transformation") Let $A \in M_{m \times n}(K)$.

Define a function $L_A : K^n \rightarrow K^m$ by

$$L_A(v) = Av \quad (\text{A an } m \times n \text{ matrix, } v \ n \times 1)$$

ie multiply matrix by vector.

Proposition 24. L_A is a linear transformation.

Proof. Let $u, v \in K^n, c \in K$. Then

$$\begin{aligned}
L_A(cu + v) &= A(cu + v) \\
&= A(cu) + Av \quad \text{(prop of matrix multiplication)} \\
&= cAu + Av \\
&= cL_A(u) + L_A(v)
\end{aligned}$$

□

Ex $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix}$, $L_A : \mathbb{R}^3 \rightarrow \mathbb{R}^2$. Calculate:

$$\begin{aligned} L_A(1, 3, -2) &= \begin{pmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 2-3-4 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 5 \end{pmatrix} \end{aligned}$$

Spoiler: All linear transformations between finite-dim vector spaces can be described in this way, “matrix transformation”.

Two special linear transformations

- (1) **Zero transformations:** $0 : V \rightarrow W$ defined by $O(v) = \vec{0}$ ($\vec{0}$ of W) for all $v \in V$.
- (2) **Identity transformation,** $I : V \rightarrow V$ (same vector space) $I(v) = v$ for all $v \in V$

Both are linear transformations (exercise).

Kernel and Image (ch. 5.4)

Def Let $T : V \rightarrow W$ be a linear transformation. Define:

- (i) **Kernel or nullspace** of T ,

$$\text{Ker}(T) = \{v \in V \mid T(v) = \vec{0}\}$$

Note: Always one vector which satisfies this.

- (ii) **Image** of T is

$$\text{Im}(T) = \{w \in W \mid \exists v \in V \ w = T(v)\}$$

Note: $\text{Ker}(T) \subseteq V$, $\text{Im}(T) \subseteq W$.

Ex Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(x, y) = (x, 0) \quad \text{("proj onto x-axis")}$$

Then

$$\begin{aligned}
 \text{Ker}(T) &= \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (0, 0)\} \\
 &= \{(0, y) \mid y \in \mathbb{R}\} \\
 &= "y - \text{axis}" \\
 \text{Im}(T) &= \{(x, y) \in \mathbb{R}^2 \mid (x, y) = T(x', y') \text{ some } x', y' \in \mathbb{R}\} \\
 &= \{(x, 0) \mid x \in \mathbb{R}\} \\
 &= "x - \text{axis}"
 \end{aligned}$$

Ex Define $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$ to be derivative, $D(f(x)) = f'(x)$. Find kernel and image of D .

Sol We have

$$\begin{aligned}
 \text{Ker}(D) &= \{f \in P_n(\mathbb{R}) \mid f'(x) = 0\} \\
 &= \text{const. polys} \\
 &= \{a \mid a \in \mathbb{R}\} \\
 &= P_0(\mathbb{R})
 \end{aligned}$$

Claim $\text{Im}(D) = P_{n-1}(\mathbb{R})$.

Proof. Prove inclusion " \subseteq " and " \supseteq ".

- (i) " \subseteq " Let $f(x) \in \text{Im}(D)$. Then $\exists g(x) \in P_n$ s.t. $f(x) = D(g(x)) = g'(x)$. Since $\deg(g) \leq n$, $\deg(f) = \deg(g') \leq n - 1$ (property of differentiation). So $f(x) \in P_{n-1}$.
- (ii) " \supseteq " Let $f(x) \in P_{n-1}$. Need to find $g(x) \in P_n$ such that $D(g(x)) = g'(x) = f(x)$. Set $g(x) = \int f(x) dx$. Know from calculus that the degree of g is one higher, ie

$$\deg(g(x)) = 1 + \deg(f(x))$$

So $\deg(g) \leq n$. So $g(x) \in P_n$ and $g'(x) = f(x)$ (calculus).

□

Theorem 25. Let $T : V \rightarrow W$ be linear transformation. Then,

(i) $\text{Ker}(T) \leq V$

(ii) $\text{Im}(T) \leq W$

Ie they are subspaces.

Proof. By direct proof.

- (i) $T(\vec{0}) = \vec{0}$ always (lin transform) so $\vec{0} \in \text{Ker}(T)$. Let $v_1, v_2 \in \text{Ker}(T), c \in K$. We know $T(v_1) = \vec{0}, T(v_2) = \vec{0}$. Then

$$\begin{aligned} T(cv_1 + v_2) &= cT(v_1) + T(v_2) && \text{(T linear)} \\ &= c\vec{0} + \vec{0} \\ &= \vec{0} \end{aligned}$$

Hence $cv_1 + v_2 \in \text{Ker}(T)$. So $\text{Ker}(T) \subseteq V$ (we already knew $\text{Ker}(T) \subseteq V$)

- (ii) $T(\vec{0}) = \vec{0}$, hence $\vec{0}_w = T(\text{something})$, ie $\vec{0}_w \in \text{Im}(T)$. Let $w_1, w_2 \in \text{Im}(T), c \in K$. We know $w_1 = T(v_1), w_2 = T(v_2)$ for some $v_1, v_2 \in V$. Then

$$\begin{aligned} cw_1 + w_2 &= cT(v_1) + T(v_2) \\ &= T(cv_1 + v_2) && \text{(T linear)} \end{aligned}$$

Hence $cw_1 + w_2 \in \text{Im}(T)$. So $\text{Im}(T) \leq W$.

□

Def $T : V \rightarrow W$ linear. The *nullity* of T is $\dim \text{Ker}(T)$ (dim nullspace). The *rank* of T is $\dim \text{Im}(T)$.

Note: $\text{Ker}(T) \leq V$ so $\text{nullity}(T) \leq \dim V$, $\text{Im}(T) \leq W$ so $\text{rank}(T) \leq \dim W$.

Ex In $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, proj onto x-axis,

$$\begin{aligned} \text{Ker}(T) &= y - \text{axis} && \text{(so nullity}(T) = 1) \\ \text{Im}(T) &= x - \text{axis} && \text{(so rank}(T) = 1) \end{aligned}$$

Ex 2 For $D : P_n(\mathbb{R}) \rightarrow P_n(\mathbb{R})$, differentiation.

$$\begin{aligned} \text{Ker } D &= P_0(\mathbb{R}) && \text{(so nullity}(D) = 1) \\ \text{Im } D &= P_{n-1} && \text{(so rank}(D) = n) \end{aligned}$$