Pattern-avoidance in binary fillings of grid shapes

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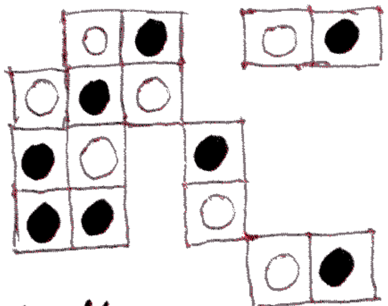
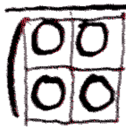
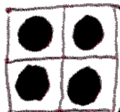
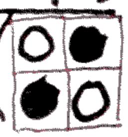
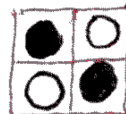
Pattern-Avoidance

Fillings of grid shapes

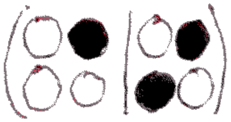


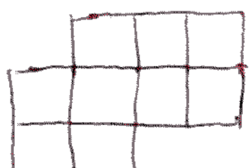
avoids both ( and )

contains one or both of ( and )

(e.g. rows 1,2 ; cols 2,3)

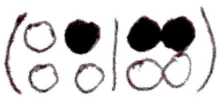
Equivalence

() is equivalent to () in

S = 

means {# fillings of S avoiding ()} =

= 190 = {# fillings of S avoiding ()}

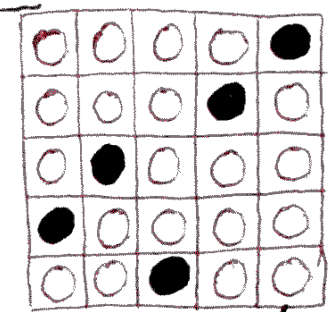
Can also say: () is equivalent to

() in all Young shapes.

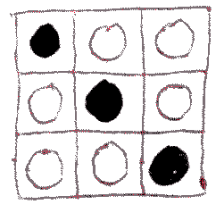
Permutations

○ is 0

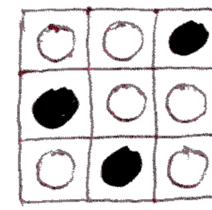
● is 1

54213 ↔ 

avoids

 ↔ 123

contains

 ↔ 312

Wilf-equivalence

123 is Wilf-equivalent to 312 means:

$$\{\# \sigma \in S_n \mid \sigma \text{ avoids } 123\} =$$

$$= C_n =$$

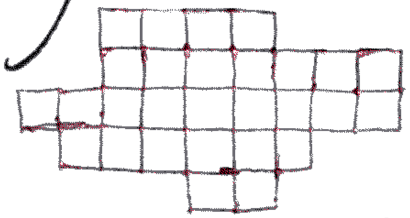
$$\{\# \sigma \in S_n \mid \sigma \text{ avoids } 312\}$$

Preliminaries (Definitions)

<u>Complement</u> Replace ○ by ●, and conversely. E.g.

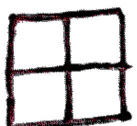
. Complementary pattern pairs are equivalent by the obvious bijection.

<u>Connected (hv) shapes</u> (horizontally vertically) Any row or column of the shape is connected, e.g.

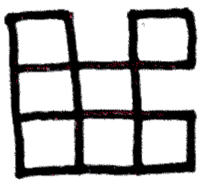


.

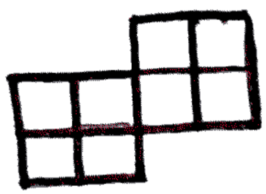
<u>2×2-connected shapes</u> One can walk from any cell to any other, with each step moving between two cells in some



subshape, which contains the current cell. E.g.



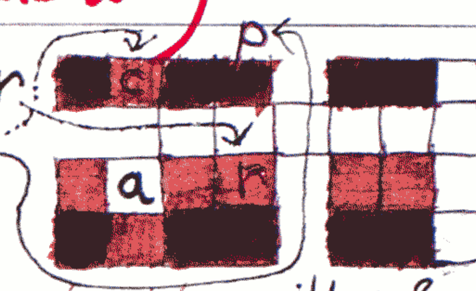
is 2×2-connected, but



is not 2×2-connected.

Complete-rectangle (CR) Erasability

A cell a has a __complete rectangle__ if for any cell r in its row and any c in its column, the cell p in r's column and c's row is in the shape.



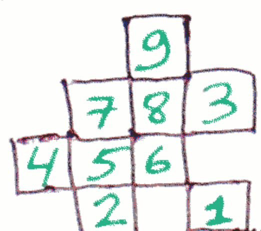
light – possible cs & rs.
dark – possible ps.

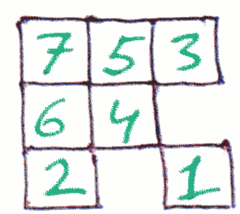
A shape S is __complete-rectangle erasable__ if it is the empty shape __or__ if it has a cell c with a complete rectangle, such that deleting c from S ($S \setminus c$) makes a smaller CR-erasable shape.

S is __(property)-CR-erasable__ — this is just like CR-erasability, with the extra constraints that c has (property), __and__ $S \setminus c$ is (property)-CR-erasable, e.g.

- __bottom-CR-erasable:__ c is bottommost in its column.
- __bottom-right-CR-erasable:__ c is bottommost in its column and rightmost in its row.

Examples, with erasure order:





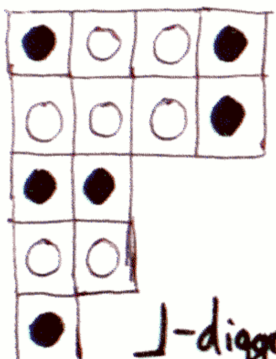
CR-erasable __but not__ bottom-CR-erasable __but not__ bottom-right-CR-erasable

Motivation

Examples — Equinumerous objects — In the language of pattern-avoidance

Totally nonnegative Grassmann cells of Ω_λ, for Young shape λ.

$\updownarrow$ Alex Postnikov [2]



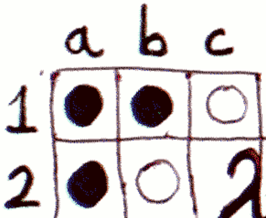
⅃-diagram

⅃("le")-diagrams, invented by Postnikov, q-enumeration by Lauren Williams. $\longleftrightarrow$ filling of λ avoiding

Def: 0/1-filling of λ, which avoids the ⅃ pattern:

[2] Postnikov, [3] key lemma by Williams $\updownarrow$ recurrence

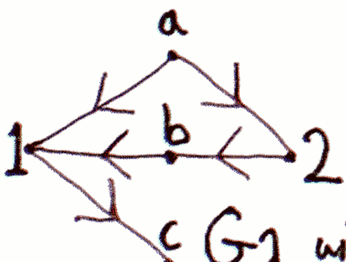
recurrence in bottom-right CR-erasable shapes; bijection by Matthieu Josuat-Vergès for Young shapes [1] $\updownarrow$



λ-filling

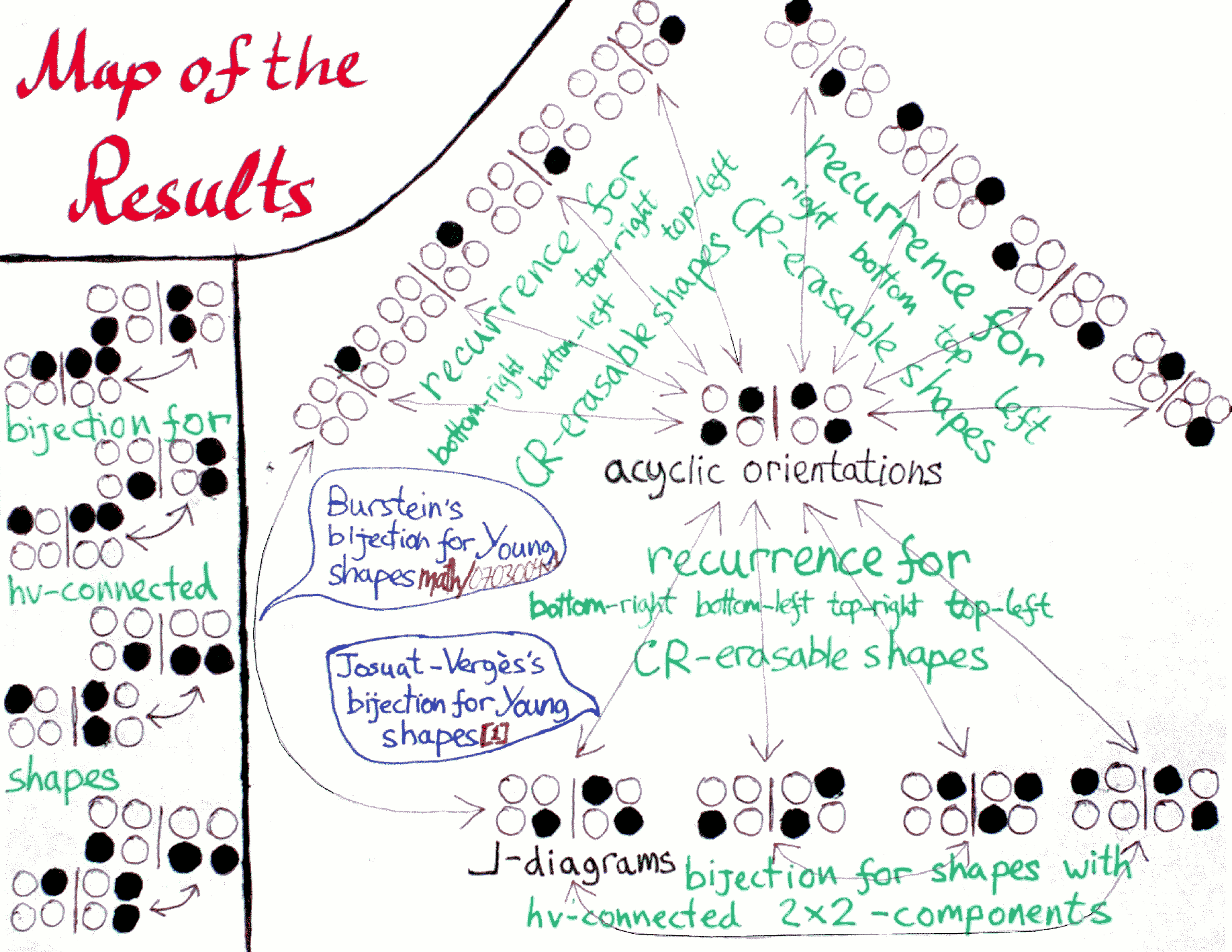
Acyclic orientations of G_λ, a bipartite graph with λ's rows and columns as vertices, and edges corresponding to λ's cells. A filling then naturally orients the edges. $\longleftrightarrow$ filling of λ avoiding

In 2×2-connected, CR-erasable shapes

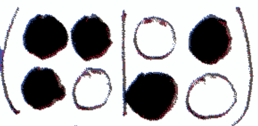
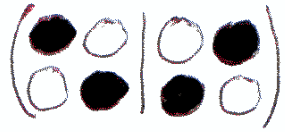


G_λ with orientation

Map of the Results



Goal: Classification

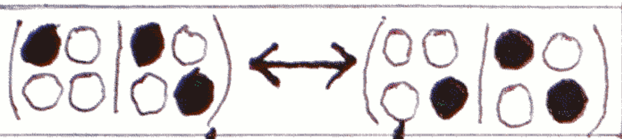
Data: (i)  is equivalent to  in Young shapes. (ii) Computer exploration suggests many more equivalences in various classes of shapes. (iii) Generic shapes have no equivalences.

Want to know: (i) Given a shape, which patterns are equivalent? (ii) Given two patterns, in which shapes are they equivalent? (iii) Given a <u>class</u> of shapes, which patterns are equivalent in <u>each</u> of its shapes?

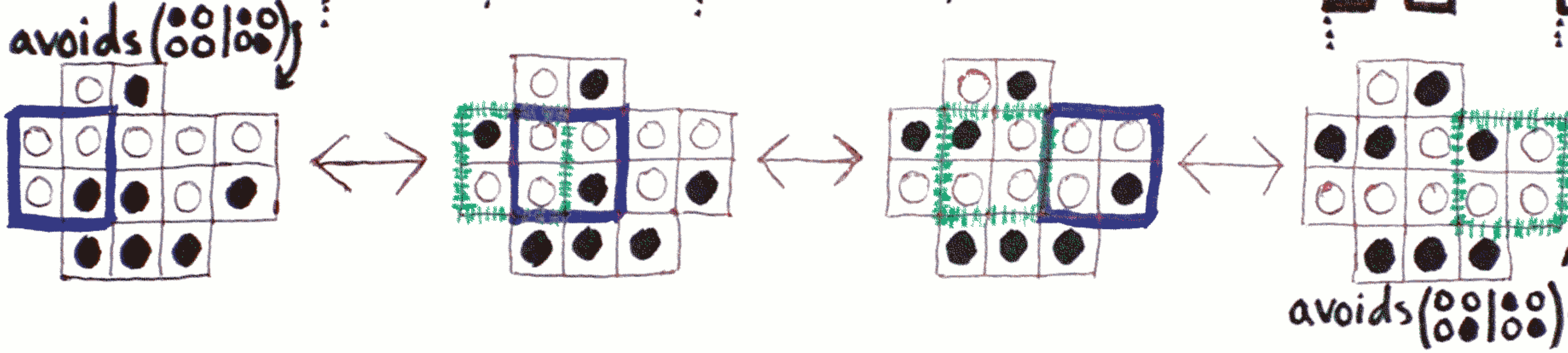
Cannot (?) know: (i) Which patterns are <u>not</u> equivalent, given a shape? (ii) In which shapes are two given patterns <u>not</u> equivalent? (Reason) Pattern-avoiding fillings of two types <u>could</u> just <u>happen</u> to be equinumerous without a systematic cause.

Goal: Give answers to (i) and (ii) that are as broad as we can get. Answer (iii) <u>precisely</u> for all interesting classes.

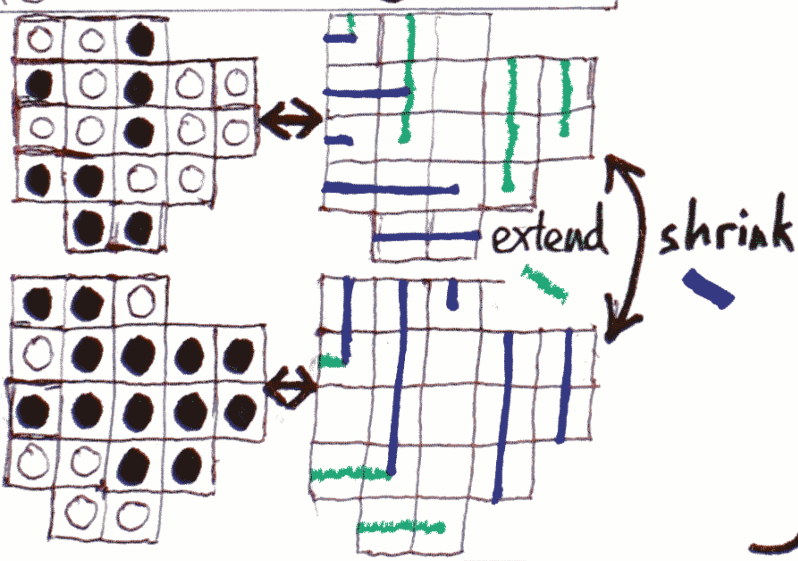
Bijections by Example

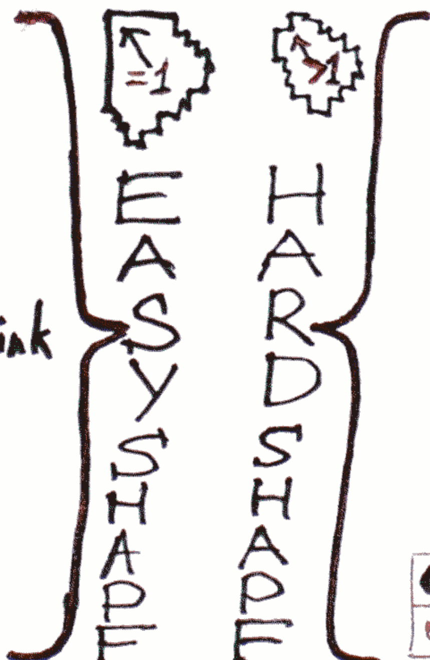


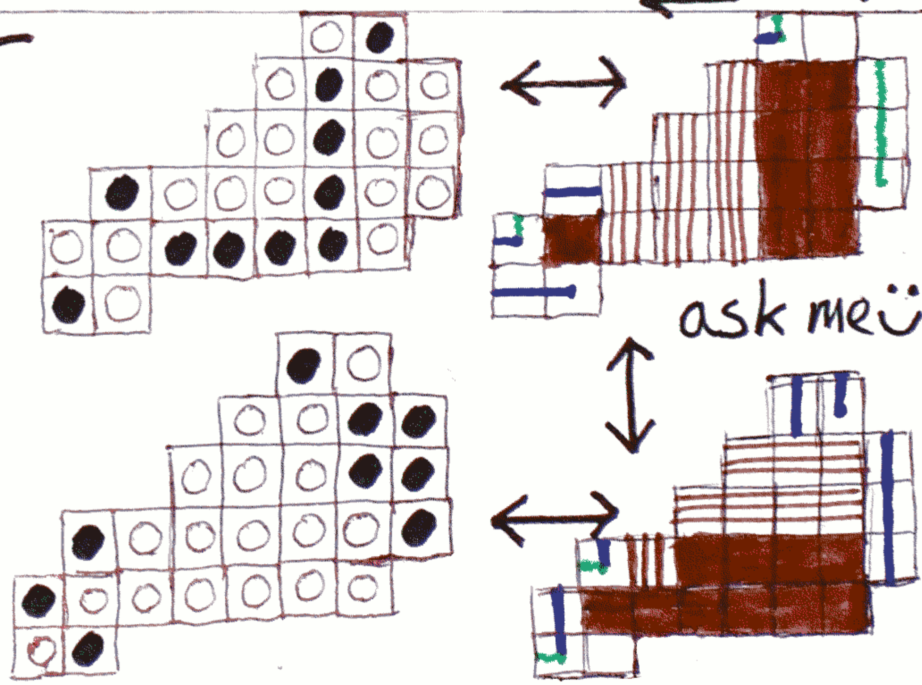
Find forbidden pattern in the following search order, replace it by its ↔ reflection, repeat. The order is read left-to-right for the → map, and right-to-left for ←.



The and boxes mark the first pattern found, for and, resp.

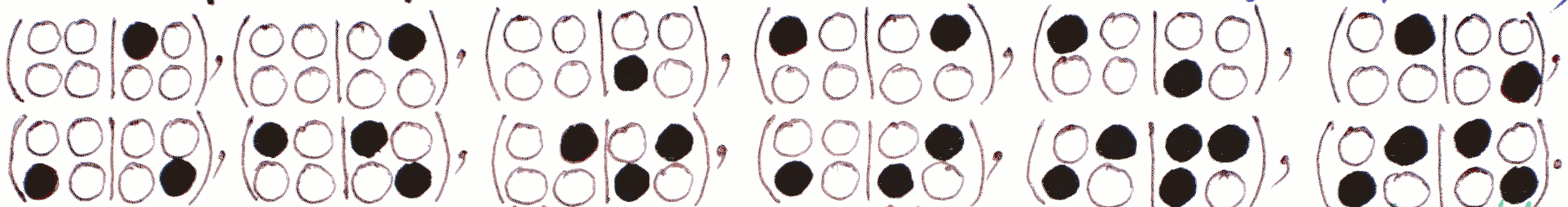






Example Application: All equivalences in Young shapes

Pattern pair equivalence classes: (not listing complements)

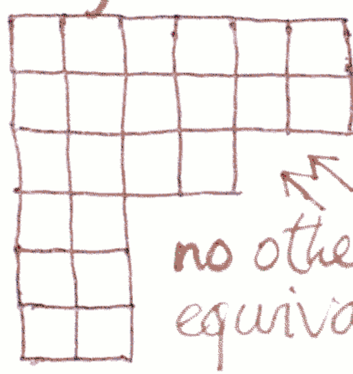
(i) 

↑↑ by recurrences ↑↑ ⅃-diagrams acyclic orientations

(ii) 

(iv) 

↑↑ by bijection ↑↑



no other equivalences.

(iii) 

(v) 

↑↑ by bijection ↑↑

For the above Young shape, computation shows there are no other equivalences. So, the above list of classes is complete.

<u>Goal:</u> Give such classifications for other interesting shape classes.

<u>Note:</u> Several more classifications can likely be given from the current theorems.

Conclusions and Future Work

- Completely described Young shape pattern-avoidance.
- Results partially describe many other, larger shape classes.
- Computer experiments suggest there are other, "less common" equivalences left to prove.

Thanks!

To the FPSAC'08 organisers, to Alex Postnikov, to Matthieu Josuat-Vergès, and to the reviewers. ☺

References

[1] M. Josuat-Vergès, Bijections between pattern-avoiding binary fillings of Young diagrams, February 1, 2008, preprint, on arxiv.

[2] A. Postnikov, Total positivity, Grassmannians and networks, see http://www-math.mit.edu/~apost/papers.html for latest version.

[3] L. Williams, Enumeration of totally positive Grasmann cells, Advances in Math. 190 (2005), no. 2, 319–342, arxiv: math.CO/0307271

Paper, full version: (work in progress) http://www-math.mit.edu/~lesha/papers/pattern-pair-avoidance.pdf