Massachusetts Institute of Technology
6.046J/18.410J: Introduction to Algorithms
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Handout 9
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Master Theorem Worksheet

This is a worksheet to help you master solving recurrence relations using the Master Theorem. For each recurrence, either give the asympotic solution using the Master Theorem (state which case), or else state that the Master Theorem doesn't apply. You should be able to go through these **25** recurrences in **10** minutes.

Problem 1-1. $T(n) = 3T(n/2) + n^2$

- $f(n) = n^2 = \Omega(n^{\log_2 3 + \epsilon})$
- $3f(\frac{n}{2}) \leq cf(n)$

$\Rightarrow T(n) = \Theta(f(n))$

$\therefore \boxed{T(n) = \Theta(n^2)}$

(from case 3)

Problem 1-2. $T(n) = 7T(n/2) + n^2$

- $f(n) = n^2 = O(n^{\log_2 7 - \epsilon})$

$\Rightarrow T(n) = \Theta(n^{\log_b a})$

$\therefore \boxed{T(n) = \Theta(n^{\log_2 7})}$

(from case 1)

Problem 1-3. $T(n) = 4T(n/2) + n^2$

- $f(n) = n^2 = \Theta(n^{\log_2 4})$

$\Rightarrow T(n) = \Theta(n^{\log_b a} \lg n)$

$\therefore \boxed{T(n) = \Theta(n^2 \lg n)}$

(from case 2)

Problem 1-4. $T(n) = 3T(n/4) + n \lg n$

- $f(n) = n \lg n = \Omega(n^{\log_4 3 + \epsilon})$
- $3f(\frac{n}{4}) \leq cf(n)$

$\therefore \boxed{T(n) = \Theta(n \lg n)}$

(from case 3)

Problem 1-5. $T(n) = 4T(n/2) + \lg n$

- $f(n) = \lg n = O(n^{\log_2 4 - \epsilon})$

$\therefore \boxed{T(n) = \Theta(n^2)}$

(from case 1)

Problem 1-6. $T(n) = T(n-1) + n$

Does not apply

Problem 1-7. $T(n) = 4T(n/2) + n^2 \lg n$

$\Rightarrow T(n) = \Theta(n \log^{k+1} n)$

$\therefore \boxed{T(n) = \Theta(n^2 \log^2 n)}$

- $n^{\log_b a} \log k = n^2 \log n = f(n)$

(from case 2 special)

Problem 1-8. $T(n) = 5T(n/2) + n^2 \lg n$

$\therefore \boxed{T(n) = \Theta(n^{\log_2 5})}$

- $f(n) = n^2 \lg n = O(n^{\log_2 5 - \epsilon})$

(from case 1)

Problem 1-9. $T(n) = 3T(n/3) + n/\lg n$

Does not apply

Problem 1-10. $T(n) = 2T(n/4) + c$

$\therefore \boxed{T(n) = \Theta(n^{1/2})}$

- $f(n) = c = O(n^{\log_4 2 - \epsilon})$
  $= O(n^{1/2 - \epsilon})$

(case 1)

Problem 1-11. $T(n) = T(n/4) + \lg n$

$\therefore \boxed{T(n) = \Theta(\log^2 n)}$

- $n^{\log_4 1} \log 1 = \lg n$

(case 2 - special)

Problem 1-12. $T(n) = T(n/2) + T(n/4) + n^2$

Does not apply

Problem 1-13. $T(n) = 2T(n/4) + \lg n$

$\therefore \boxed{T(n) = \Theta(n^{1/2})}$

- $f(n) = \lg n = O(n^{\log_4 2 - \epsilon})$
  $= O(n^{\frac{1}{2} - \epsilon})$

(from case 1)

Problem 1-14. $T(n) = 3T(n/3) + n \lg n$

$\Rightarrow T(n) = \Theta(n \log^{k+1} n)$

$\therefore \boxed{T(n) = \Theta(n \log^2 n)}$

- $n \lg n = n^{\log_3 3} \log 1 = \Theta(n)$

(from case 2 - special)

Problem 1-15. $T(n) = 8T((n - \sqrt{n})/4) + n^2$

Does not apply

Problem 1-16. $T(n) = 2T(n/4) + \sqrt{n}$

- $f(n) = n^{1/2} = \Theta(n^{\log_4 2})$

(from case 2)

$\therefore \boxed{T(n) = \Theta(n^{1/2} \lg n)}$

Problem 1-17. $T(n) = 2T(n/4) + n^{0.51}$

- $f(n) = n^{0.51} = \Omega(n^{\log_4 2 + \epsilon})$
  $= \Omega(n^{0.5+\epsilon})$
- $2f(n/4) \leq cf(n)$

(Case 3)

$\therefore \boxed{T(n) = \Theta(n^{0.51})}$

Problem 1-18. $T(n) = 16T(n/4) + n!$

- $f(n) = n! = \Omega(n^{\log_4 (16+\epsilon)})$
  $= \Omega(n^{2+\epsilon})$
- $16 f(n/16) \leq c f(n)$

(case 3)

$\therefore \boxed{T(n) = \Theta(n!)}$

Problem 1-19. $T(n) = 3T(n/2) + n$

- $f(n) = n = O(n^{\log_2 3 - \epsilon})$

(case 1)

$\therefore \boxed{T(n) = \Theta(n^{\log_2 3})}$

Problem 1-20. $T(n) = 4T(n/2) + cn$

- $f(n) = cn = O(n^{\log_2 4 - \epsilon})$
  $= O(n^{2-\epsilon})$

(case 1)

$\therefore \boxed{T(n) = \Theta(n^2)}$

Problem 1-21. $T(n) = 3T(n/3) + n/2$

- $f(n) = \frac{n}{2} = \Theta(n^{\log_3 3})$
  $= \Theta(n)$

(case 2)

$T(n) = \Theta(n^{\log_3 3} \lg n)$

$\therefore \boxed{T(n) = \Theta(n \lg n)}$

Problem 1-22. $T(n) = 4T(n/2) + n/\lg n$

- $f(n) = \frac{n}{\lg n} = O(n^{\log_2 4 - \epsilon})$
  $= O(n^{2-\epsilon})$

(case 1)

$\therefore \boxed{T(n) = \Theta(n^2)}$

Problem 1-23. $T(n) = 7T(n/3) + n^2$

- $f(n) = n^2 = \Omega(n^{\log_3 7 + \epsilon})$
- $7f(n/3) \leq cf(n)$

(case 3)

$\therefore T(n) = \Theta(n^2)$

Problem 1-24. $T(n) = 8T(n/3) + 2^n$

- $f(n) = 2^n = \Omega(n^{\log_3 8 + \epsilon})$
- $8f(n/3) \leq cf(n)$

(case 3)

$\therefore T(n) = \Theta(2^n)$

Problem 1-25. $T(n) = 16T(n/4) + n$

- $f(n) = n = O(n^{\log_4 16 - \epsilon})$
  $= O(n^{2-\epsilon})$

(case 1)

$\therefore T(n) = \Theta(n^2)$