

Algorithm Library

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1 多项式

1.1 FFT - tourist

```

1  /* copy from tourist */
2  namespace FFT {
3      typedef double dbl;
4
5      struct num {
6          dbl x, y;
7          num() { x = y = 0; }
8          num(dbl x, dbl y) : x(x), y(y) { }
9      };
10
11     inline num operator+(num a, num b) { return num(a.x + b.x, a.y + b.y); }
12     inline num operator-(num a, num b) { return num(a.x - b.x, a.y - b.y); }
13     inline num operator*(num a, num b) { return num(a.x * b.x - a.y * b.y, a
        .x * b.y + a.y * b.x); }
14     inline num conj(num a) { return num(a.x, -a.y); }
15
16     int base = 1;
17     vector<num> roots = { {0, 0}, {1, 0} };
18     vector<int> rev = { 0, 1 };
19
20     const dbl PI = acos(-1.0);
21
22     void ensure_base(int nbase) {
23         if (nbase <= base) {
24             return;
25         }
26         rev.resize(1 << nbase);
27         for (int i = 0; i < (1 << nbase); i++) {
28             rev[i] = (rev[i >> 1] >> 1) + ((i & 1) << (nbase - 1));
29         }
30         roots.resize(1 << nbase);
31         while (base < nbase) {
32             dbl angle = 2 * PI / (1 << (base + 1));
33             for (int i = 1 << (base - 1); i < (1 << base); i++) {
34                 roots[i << 1] = roots[i];
35                 dbl angle_i = angle * (2 * i + 1 - (1 << base));
36                 roots[(i << 1) + 1] = num(cos(angle_i), sin(angle_i));
37             }
38             base++;
39         }

```

```

40     }
41
42     void fft(vector<num>& a, int n = -1) {
43         if (n == -1) {
44             n = a.size();
45         }
46         assert((n & (n - 1)) == 0);
47         int zeros = __builtin_ctz(n);
48         ensure_base(zeros);
49         int shift = base - zeros;
50         for (int i = 0; i < n; i++) {
51             if (i < (rev[i] >> shift)) {
52                 swap(a[i], a[rev[i] >> shift]);
53             }
54         }
55         for (int k = 1; k < n; k <= 1) {
56             for (int i = 0; i < n; i += 2 * k) {
57                 for (int j = 0; j < k; j++) {
58                     num z = a[i + j + k] * roots[j + k];
59                     a[i + j + k] = a[i + j] - z;
60                     a[i + j] = a[i + j] + z;
61                 }
62             }
63         }
64     }
65
66     vector<num> fa, fb;
67
68     vector<long long> multiply(vector<int>& a, vector<int>& b) {
69         int need = a.size() + b.size() - 1;
70         int nbase = 1;
71         while ((1 << nbase) < need) nbase++;
72         ensure_base(nbase);
73         int sz = 1 << nbase;
74         if (sz > (int)fa.size()) {
75             fa.resize(sz);
76         }
77         for (int i = 0; i < sz; i++) {
78             int x = (i < (int)a.size() ? a[i] : 0);
79             int y = (i < (int)b.size() ? b[i] : 0);
80             fa[i] = num(x, y);
81         }
82         fft(fa, sz);

```

```

83     num r(0, -0.25 / (sz >> 1));
84     for (int i = 0; i <= (sz >> 1); i++) {
85         int j = (sz - i) & (sz - 1);
86         num z = (fa[j] * fa[j] - conj(fa[i] * fa[i])) * r;
87         if (i != j) {
88             fa[j] = (fa[i] * fa[i] - conj(fa[j] * fa[j])) * r;
89         }
90         fa[i] = z;
91     }
92     for (int i = 0; i < (sz >> 1); i++) {
93         num A0 = (fa[i] + fa[i + (sz >> 1)]) * num(0.5, 0);
94         num A1 = (fa[i] - fa[i + (sz >> 1)]) * num(0.5, 0) * roots[(sz
95             >> 1) + i];
96         fa[i] = A0 + A1 * num(0, 1);
97     }
98     fft(fa, sz >> 1);
99     vector<long long> res(need);
100    for (int i = 0; i < need; i++) {
101        if (i % 2 == 0) {
102            res[i] = fa[i >> 1].x + 0.5;
103        } else {
104            res[i] = fa[i >> 1].y + 0.5;
105        }
106    }
107    return res;
108 }
109
110 vector<long long> square(const vector<int>& a) {
111     int need = a.size() + a.size() - 1;
112     int nbase = 1;
113     while ((1 << nbase) < need) nbase++;
114     ensure_base(nbase);
115     int sz = 1 << nbase;
116     if ((sz >> 1) > (int)a.size()) {
117         fa.resize(sz >> 1);
118     }
119     for (int i = 0; i < (sz >> 1); i++) {
120         int x = (2 * i < (int)a.size() ? a[2 * i] : 0);
121         int y = (2 * i + 1 < (int)a.size() ? a[2 * i + 1] : 0);
122         fa[i] = num(x, y);
123     }
124     fft(fa, sz >> 1);
125     num r(1.0 / (sz >> 1), 0.0);

```

```

125     for (int i = 0; i <= (sz >> 2); i++) {
126         int j = ((sz >> 1) - i) & ((sz >> 1) - 1);
127         num fe = (fa[i] + conj(fa[j])) * num(0.5, 0);
128         num fo = (fa[i] - conj(fa[j])) * num(0, -0.5);
129         num aux = fe * fe + fo * fo * roots[(sz >> 1) + i] * roots[(sz
            >> 1) + i];
130         num tmp = fe * fo;
131         fa[i] = r * (conj(aux) + num(0, 2) * conj(tmp));
132         fa[j] = r * (aux + num(0, 2) * tmp);
133     }
134     fft(fa, sz >> 1);
135     vector<long long> res(need);
136     for (int i = 0; i < need; i++) {
137         if (i % 2 == 0) {
138             res[i] = fa[i >> 1].x + 0.5;
139         } else {
140             res[i] = fa[i >> 1].y + 0.5;
141         }
142     }
143     return res;
144 }
145
146 vector<int> multiply_mod(vector<int>& a, vector<int>& b, int m, int eq =
    0) {
147     int need = a.size() + b.size() - 1;
148     int nbase = 0;
149     while ((1 << nbase) < need) nbase++;
150     ensure_base(nbase);
151     int sz = 1 << nbase;
152     if (sz > (int)fa.size()) {
153         fa.resize(sz);
154     }
155     for (int i = 0; i < (int)a.size(); i++) {
156         int x = (a[i] % m + m) % m;
157         fa[i] = num(x & ((1 << 15) - 1), x >> 15);
158     }
159     fill(fa.begin() + a.size(), fa.begin() + sz, num{ 0, 0 });
160     fft(fa, sz);
161     if (sz > (int)fb.size()) {
162         fb.resize(sz);
163     }
164     if (eq) {
165         copy(fa.begin(), fa.begin() + sz, fb.begin());

```



```

166     } else {
167         for (int i = 0; i < (int)b.size(); i++) {
168             int x = (b[i] % m + m) % m;
169             fb[i] = num(x & ((1 << 15) - 1), x >> 15);
170         }
171         fill(fb.begin() + b.size(), fb.begin() + sz, num{ 0, 0 });
172         fft(fb, sz);
173     }
174     dbl ratio = 0.25 / sz;
175     num r2(0, -1);
176     num r3(ratio, 0);
177     num r4(0, -ratio);
178     num r5(0, 1);
179     for (int i = 0; i <= (sz >> 1); i++) {
180         int j = (sz - i) & (sz - 1);
181         num a1 = (fa[i] + conj(fa[j]));
182         num a2 = (fa[i] - conj(fa[j])) * r2;
183         num b1 = (fb[i] + conj(fb[j])) * r3;
184         num b2 = (fb[i] - conj(fb[j])) * r4;
185         if (i != j) {
186             num c1 = (fa[j] + conj(fa[i]));
187             num c2 = (fa[j] - conj(fa[i])) * r2;
188             num d1 = (fb[j] + conj(fb[i])) * r3;
189             num d2 = (fb[j] - conj(fb[i])) * r4;
190             fa[i] = c1 * d1 + c2 * d2 * r5;
191             fb[i] = c1 * d2 + c2 * d1;
192         }
193         fa[j] = a1 * b1 + a2 * b2 * r5;
194         fb[j] = a1 * b2 + a2 * b1;
195     }
196     fft(fa, sz);
197     fft(fb, sz);
198     vector<int> res(need);
199     for (int i = 0; i < need; i++) {
200         long long aa = fa[i].x + 0.5;
201         long long bb = fb[i].x + 0.5;
202         long long cc = fa[i].y + 0.5;
203         res[i] = (aa + ((bb % m) << 15) + ((cc % m) << 30)) % m;
204     }
205     return res;
206 }
207
208 vector<int> square_mod(vector<int>& a, int m) {

```

```

209         return multiply_mod(a, a, m, 1);
210     }
211 };

```

1.2 形式幂级数

```

1  #define db double
2  #ifndef ONLINE_JUDGE // 这三个函数是给MSVC用的，G++不需要
3  inline int __builtin_clz(int v) { // 返回前导0的个数
4      return __lzcnt(v);
5  }
6  inline int __builtin_ctz(int v) { // 返回末尾0的个数
7      if (v == 0) {
8          return 0;
9      }
10     __asm {
11         bsf eax, dword ptr[v];
12     }
13 }
14 inline int __builtin_popcount(int v) { // 返回二进制中1的个数
15     return __popcnt(v);
16 }
17 #endif
18 struct Complex {
19     db real, imag;
20     Complex(db x = 0, db y = 0) : real(x), imag(y) {}
21     Complex& operator+=(const Complex& rhs) {
22         real += rhs.real; imag += rhs.imag;
23         return *this;
24     }
25     Complex& operator-=(const Complex& rhs) {
26         real -= rhs.real; imag -= rhs.imag;
27         return *this;
28     }
29     Complex& operator*=(const Complex& rhs) {
30         db t_real = real * rhs.real - imag * rhs.imag;
31         imag = real * rhs.imag + imag * rhs.real;
32         real = t_real;
33         return *this;
34     }
35     Complex& operator/=(double x) {
36         real /= x, imag /= x;
37         return *this;

```

```

38     }
39     friend Complex operator + (const Complex& a, const Complex& b) { return
        Complex(a) += b; }
40     friend Complex operator - (const Complex& a, const Complex& b) { return
        Complex(a) -= b; }
41     friend Complex operator * (const Complex& a, const Complex& b) { return
        Complex(a) *= b; }
42     friend Complex operator / (const Complex& a, const db& b) { return
        Complex(a) /= b; }
43     Complex power(long long p) const {
44         assert(p >= 0);
45         Complex a = *this, res = { 1, 0 };
46         while (p > 0) {
47             if (p & 1) res = res * a;
48             a = a * a;
49             p >>= 1;
50         }
51         return res;
52     }
53     static long long val(double x) { return x < 0 ? x - 0.5 : x + 0.5; }
54     inline long long Real() const { return val(real); }
55     inline long long Imag() const { return val(imag); }
56     Complex conj() const { return Complex(real, -imag); }
57     explicit operator int() const { return Real(); }
58     friend ostream& operator<<(ostream& stream, const Complex& m) {
59         return stream << complex<db>(m.real, m.imag);
60     }
61 };
62 constexpr int MOD = 998244353;
63 constexpr int Phi_MOD = 998244352;
64 inline int exgcd(int a, int md = MOD) {
65     a %= md;
66     if (a < 0) a += md;
67     int b = md, u = 0, v = 1;
68     while (a) {
69         int t = b / a;
70         b -= t * a; swap(a, b);
71         u -= t * v; swap(u, v);
72     }
73     assert(b == 1);
74     if (u < 0) u += md;
75     return u;
76 }

```

```

77 inline int add(int a, int b) { return a + b >= MOD ? a + b - MOD : a + b; }
78 inline int sub(int a, int b) { return a - b < 0 ? a - b + MOD : a - b; }
79 inline int mul(int a, int b) { return 1LL * a * b % MOD; }
80 inline int powmod(int a, long long b) {
81     int res = 1;
82     while (b > 0) {
83         if (b & 1) res = mul(res, a);
84         a = mul(a, a);
85         b >>= 1;
86     }
87     return res;
88 }
89
90 vector<int> inv, fac, ifac;
91 void prepare_factorials(int maximum) {
92     inv.assign(maximum + 1, 1);
93     // Make sure MOD is prime, which is necessary for the inverse algorithm
94     // below.
95     for (int p = 2; p * p <= MOD; p++)
96         assert(MOD % p != 0);
97     for (int i = 2; i <= maximum; i++)
98         inv[i] = mul(inv[MOD % i], (MOD - MOD / i));
99
100     fac.resize(maximum + 1);
101     ifac.resize(maximum + 1);
102     fac[0] = ifac[0] = 1;
103
104     for (int i = 1; i <= maximum; i++) {
105         fac[i] = mul(i, fac[i - 1]);
106         ifac[i] = mul(inv[i], ifac[i - 1]);
107     }
108 }
109 namespace FFT {
110     vector<Complex> roots = { Complex(0, 0), Complex(1, 0) };
111     vector<int> bit_reverse;
112     int max_size = 1 << 20;
113     const long double pi = acos(-1.01);
114     constexpr int FFT_CUTOFF = 150;
115     inline bool is_power_of_two(int n) { return (n & (n - 1)) == 0; }
116     inline int round_up_power_two(int n) {
117         assert(n > 0);
118         while (n & (n - 1)) {
119             n = (n | (n - 1)) + 1;

```

```

119     }
120     return n;
121 }
122 // Given n (a power of two), finds k such that  $n = 1 \ll k$ .
123 inline int get_length(int n) {
124     assert(is_power_of_two(n));
125     return __builtin_ctz(n);
126 }
127 // Rearranges the indices to be sorted by lowest bit first, then second
128 // lowest, etc., rather than highest bit first.
129 // This makes even-odd div-conquer much easier.
130 void bit_reorder(int n, vector<Complex>& values) {
131     if ((int)bit_reverse.size() != n) {
132         bit_reverse.assign(n, 0);
133         int length = get_length(n);
134         for (int i = 0; i < n; i++) {
135             bit_reverse[i] = (bit_reverse[i >> 1] >> 1) + ((i & 1) << (
136                 length - 1));
137         }
138     }
139     for (int i = 0; i < n; i++) {
140         if (i < bit_reverse[i]) {
141             swap(values[i], values[bit_reverse[i]]);
142         }
143     }
144 }
145 void prepare_roots(int n) {
146     assert(n <= max_size);
147     if ((int)roots.size() >= n)
148         return;
149     int length = get_length(roots.size());
150     roots.resize(n);
151     // The roots array is set up such that for a given power of two  $n \geq$ 
152     // 2, roots[n / 2] through roots[n - 1] are
153     // the first half of the n-th primitive roots of MOD.
154     while (1 << length < n) {
155         for (int i = 1 << (length - 1); i < 1 << length; i++) {
156             roots[2 * i] = roots[i];
157             long double angle = pi * (2 * i + 1) / (1 << length);
158             roots[2 * i + 1] = Complex(-cos(angle), -sin(angle));
159         }
160         length++;
161     }

```

```

159     }
160     void fft_iterative(int N, vector<Complex>& values) {
161         assert(is_power_of_two(N));
162         prepare_roots(N);
163         bit_reorder(N, values);
164         for (int n = 1; n < N; n *= 2) {
165             for (int start = 0; start < N; start += 2 * n) {
166                 for (int i = 0; i < n; i++) {
167                     Complex& even = values[start + i];
168                     Complex odd = values[start + n + i] * roots[n + i];
169                     values[start + n + i] = even - odd;
170                     values[start + i] = even + odd;
171                 }
172             }
173         }
174     }
175     vector<long long> multiply(vector<int> a, vector<int> b) { // 普通FFT
176         int n = a.size();
177         int m = b.size();
178         if (min(n, m) < FFT_CUTOFF) {
179             vector<long long> res(n + m - 1);
180             for (int i = 0; i < n; i++) {
181                 for (int j = 0; j < m; j++) {
182                     res[i + j] += 1LL * a[i] * b[j];
183                 }
184             }
185             return res;
186         }
187         int N = round_up_power_two(n + m - 1);
188         vector<Complex> tmp(N);
189         for (int i = 0; i < a.size(); i++) tmp[i].real = a[i];
190         for (int i = 0; i < b.size(); i++) tmp[i].imag = b[i];
191         fft_iterative(N, tmp);
192         for (int i = 0; i < N; i++) tmp[i] = tmp[i] * tmp[i];
193         reverse(tmp.begin() + 1, tmp.end());
194         fft_iterative(N, tmp);
195         vector<long long> res(n + m - 1);
196         for (int i = 0; i < res.size(); i++) {
197             res[i] = tmp[i].imag / 2 / N + 0.5;
198         }
199         return res;
200     }
201     vector<int> mod_multiply(vector<int> a, vector<int> b, int lim =

```

```

max_size) { // 任意模数FFT
202     int n = a.size();
203     int m = b.size();
204     if (min(n, m) < FFT_CUTOFF) {
205         vector<int> res(n + m - 1);
206         for (int i = 0; i < n; i++) {
207             for (int j = 0; j < m; j++) {
208                 res[i + j] += 1LL * a[i] * b[j] % MOD;
209                 res[i + j] %= MOD;
210             }
211         }
212         return res;
213     }
214     int N = round_up_power_two(n + m - 1);
215     N = min(N, lim);
216     vector<Complex> P(N);
217     vector<Complex> Q(N);
218     for (int i = 0; i < n; i++) {
219         P[i] = Complex(a[i] >> 15, a[i] & 0x7fff);
220     }
221     for (int i = 0; i < m; i++) {
222         Q[i] = Complex(b[i] >> 15, b[i] & 0x7fff);
223     }
224     fft_iterative(N, P);
225     fft_iterative(N, Q);
226     vector<Complex> A(N), B(N), C(N), D(N);
227     for (int i = 0; i < N; i++) {
228         Complex P2 = P[(N - i) & (N - 1)].conj();
229         A[i] = (P2 + P[i]) * Complex(0.5, 0),
230         B[i] = (P2 - P[i]) * Complex(0, 0.5);
231         Complex Q2 = Q[(N - i) & (N - 1)].conj();
232         C[i] = (Q2 + Q[i]) * Complex(0.5, 0),
233         D[i] = (Q2 - Q[i]) * Complex(0, 0.5);
234     }
235     for (int i = 0; i < N; i++) {
236         P[i] = (A[i] * C[i]) + (B[i] * D[i]) * Complex(0, 1),
237         Q[i] = (A[i] * D[i]) + (B[i] * C[i]) * Complex(0, 1);
238     }
239     reverse(P.begin() + 1, P.end());
240     reverse(Q.begin() + 1, Q.end());
241     fft_iterative(N, P);
242     fft_iterative(N, Q);
243     for (int i = 0; i < N; i++) {

```

```

244         P[i] /= N, Q[i] /= N;
245     }
246     int size = min(n + m - 1, lim);
247     vector<int> res(size);
248     for (int i = 0; i < size; i++) {
249         long long ac = P[i].Real() % MOD, bd = P[i].Imag() % MOD,
250             ad = Q[i].Real() % MOD, bc = Q[i].Imag() % MOD;
251         res[i] = ((ac << 30) + bd + ((ad + bc) << 15)) % MOD;
252     }
253     return res.resize(n + m - 1), res;
254 }
255 vector<int> mod_inv(vector<int> a) { // 多项式逆
256     int n = a.size();
257     int N = round_up_power_two(a.size());
258     a.resize(N * 2);
259     vector<int> res(1);
260     res[0] = exgcd(a[0]);
261     for (int i = 2; i <= N; i <= 1) {
262         vector<int> tmp(a.begin(), a.begin() + i);
263         int n = (i < 1);
264         tmp = mod_multiply(tmp, mod_multiply(res, res, n), n);
265         res.resize(i);
266         for (int j = 0; j < i; j++) {
267             res[j] = add(res[j], sub(res[j], tmp[j]));
268         }
269     }
270     res.resize(n);
271     return res;
272 }
273 vector<int> integral(vector<int> a) { // 多项式积分
274     assert(a.size() <= inv.size());
275     a.push_back(0);
276     for (int i = (int)a.size() - 1; i >= 1; i--) {
277         a[i] = mul(a[i - 1], inv[i]);
278     }
279     return a;
280 }
281 vector<int> differential(vector<int> a) { // 多项式求导
282     for (int i = 0; i < (int)a.size() - 1; i++) {
283         a[i] = mul(i + 1, a[i + 1]);
284     }
285     a.pop_back();
286     return a;

```



```

287     }
288     vector<int> ln(vector<int> a) { // 多项式对数函数
289         assert((int)a[0] == 1);
290         auto b = mod_multiply(differential(a), mod_inv(a));
291         b = integral(b);
292         b[0] = 0;
293         return b;
294     }
295     vector<int> exp(vector<int> a) { // 多项式指数函数
296         int N = round_up_power_two(a.size());
297         int n = a.size();
298         a.resize(N * 2);
299         vector<int> res{ 1 };
300         for (int i = 2; i <= N; i <= 1) {
301             auto tmp = res;
302             tmp.resize(i);
303             tmp = ln(tmp);
304             for (int j = 0; j < i; j++) {
305                 tmp[j] = sub(a[j], tmp[j]);
306             }
307             tmp[0] = add(tmp[0], 1);
308             res.resize(i);
309             res = mod_multiply(res, tmp, i < 1);
310             fill(res.begin() + i, res.end(), 0);
311         }
312         res.resize(n);
313         return res;
314     }
315     // Multiplies many polynomials whose total degree is n in  $O(n \log^2 n)$ .
316     vector<int> mod_multiply_all(const vector<vector<int>>& polynomials) {
317         if (polynomials.empty())
318             return { 1 };
319         struct compare_size {
320             bool operator()(const vector<int>& x, const vector<int>& y) {
321                 return x.size() > y.size();
322             }
323         };
324         priority_queue<vector<int>, vector<vector<int>>, compare_size> pq(
            polynomials.begin(), polynomials.end());
325         while (pq.size() > 1) {
326             vector<int> a = pq.top(); pq.pop();
327             vector<int> b = pq.top(); pq.pop();
328             pq.push(mod_multiply(a, b));

```

```

329     }
330     return pq.top();
331 }
332 tuple<int, int, bool> power_reduction(string s, int n) { // 多项式快速幂
    预处理
333     int p = 0, q = 0; bool zero = false;
334     for (int i = 0; i < s.length(); i++) {
335         p = mul(p, 10);
336         p = add(p, s[i] - '0');
337         q = 1LL * q * 10 % Phi_MOD; // Phi_MOD 是MOD的欧拉函数值
338         q = (q + s[i] - '0');
339         if (q >= Phi_MOD) q -= Phi_MOD;
340         if (q >= (int)n) zero = true;
341     }
342     return { p, q, zero };
343 }
344 vector<int> power(vector<int> a, string s) { // 多项式快速幂 a^s O(nlogn)
    )
345     int n = a.size();
346     auto [p, q, zero] = power_reduction(s, (int)a.size()); // 不需要降幂
    的话可以省去这部分
347     if (a[0] == 1) {
348         auto res = ln(a);
349         while ((int)res.size() > n) res.pop_back();
350         for (auto& i : res) {
351             i = mul(p, i);
352         }
353         res = exp(res);
354         return res;
355     } else {
356         int mn = -1;
357         vector<int> copy_a;
358         for (int i = 0; i < (int)a.size(); i++) {
359             if (a[i]) {
360                 mn = i;
361                 break;
362             }
363         }
364         if ((mn == -1) || (mn && (zero || (1LL * mn * p > n)))) { // a中
    所有元素都是0 或 偏移过大
365             return vector<int>(n, 0);
366         }
367         int inverse_amin = exgcd(a[mn]);

```

```

368         for (int i = mn; i < n; i++) {
369             copy_a.emplace_back(mul(a[i], inverse_amin));
370         }
371         copy_a = ln(copy_a);
372         while ((int)copy_a.size() > n) copy_a.pop_back();
373         for (auto& i : copy_a) {
374             i = mul(i, p);
375         }
376         copy_a = exp(copy_a);
377         vector<int> res(n, 0);
378         // shift是偏移量 power_k 是a_min^q(q是扩展欧拉定理降出来的幂次)
379         int shift = mn * p, power_k = powmod(a[mn], q);
380         for (int i = 0; i + shift < n; i++) {
381             res[i + shift] = mul(copy_a[i], power_k);
382         }
383         return res;
384     }
385 }
386 vector<long long> sub_convolution(vector<int> a, vector<int> b) { // 减
    法卷积 只保留非负次项
387     int n = b.size();
388     reverse(b.begin(), b.end());
389     auto res = multiply(a, b);
390     return vector<long long>(res.begin() + n - 1, res.end());
391 }
392 int bostan_mori(vector<int> p, vector<int> q, long long n) { // [x^n]p(x)
    )/q(x) O(2/3dlog(d)log(n+1)) d是多项式度数
393     int i;
394     for (; n; n >>= 1) {
395         auto r = q;
396         for (i = 1; i < r.size(); i += 2) {
397             r[i] = MOD - r[i];
398         }
399         p = mod_multiply(p, r);
400         q = mod_multiply(q, r);
401         for (i = (n & 1); i < p.size(); i += 2) {
402             p[i / 2] = p[i];
403         }
404         p.resize(i / 2);
405         for (i = 0; i < q.size(); i += 2) {
406             q[i / 2] = q[i];
407         }
408         q.resize(i / 2);

```

```

409     }
410     return p[0];
411 }
412 };

```

1.3 NTT

```

1  constexpr int MOD = 998244353;
2  struct Z {
3      int val;
4      Z(long long v = 0) {
5          if (v < 0) v = v % MOD + MOD;
6          if (v >= MOD) v %= MOD;
7          val = v;
8      }
9      static int mod_inv(int a, int m = MOD) {
10         // https://en.wikipedia.org/wiki/Extended\_Euclidean\_algorithm#
11         // Example
12         int g = m, r = a, x = 0, y = 1;
13         while (r != 0) {
14             int q = g / r;
15             g %= r; swap(g, r);
16             x -= q * y; swap(x, y);
17         }
18         return x < 0 ? x + m : x;
19     }
20     explicit operator int() const { return val; }
21     explicit operator uint64_t() const { return val; }
22     Z& operator+=(const Z& other) {
23         val += other.val;
24         if (val >= MOD) val -= MOD;
25         return *this;
26     }
27     Z& operator-=(const Z& other) {
28         val -= other.val;
29         if (val < 0) val += MOD;
30         return *this;
31     }
32     static unsigned fast_mod(uint64_t x, unsigned m = MOD) {
33         return x % m;
34         /*
35         // Optimized mod for Codeforces 32-bit machines.
36         // x must be less than 2^32 * m for this to work, so that x / m fits

```

```

        in a 32-bit integer.
36     unsigned x_high = x >> 32, x_low = (unsigned) x;
37     unsigned quot, rem;
38     asm("divl %4\n"
39         : "=a" (quot), "=d" (rem)
40         : "d" (x_high), "a" (x_low), "r" (m));
41     return rem;*/
42 }
43 Z& operator*=(const Z& other) {
44     val = fast_mod((uint64_t)val * other.val);
45     return *this;
46 }
47 Z& operator/=(const Z& other) { return *this *= other.inv(); }
48 friend Z operator+(const Z& a, const Z& b) { return Z(a) += b; }
49 friend Z operator-(const Z& a, const Z& b) { return Z(a) -= b; }
50 friend Z operator*(const Z& a, const Z& b) { return Z(a) *= b; }
51 friend Z operator/(const Z& a, const Z& b) { return Z(a) /= b; }
52 Z& operator++() {
53     val = val == MOD - 1 ? 0 : val + 1;
54     return *this;
55 }
56 Z& operator--() {
57     val = val == 0 ? MOD - 1 : val - 1;
58     return *this;
59 }
60 Z operator++(int) { Z before = *this; ++*this; return before; }
61 Z operator--(int) { Z before = *this; --*this; return before; }
62 Z operator-() const { return val == 0 ? 0 : MOD - val; }
63 bool operator==(const Z& other) const { return val == other.val; }
64 bool operator!=(const Z& other) const { return val != other.val; }
65 Z inv() const { return mod_inv(val); }
66 Z pow(long long p) const {
67     assert(p >= 0);
68     Z a = *this, res = 1;
69     while (p > 0) {
70         if (p & 1) res *= a;
71         a *= a;
72         p >>= 1;
73     }
74     return res;
75 }
76 friend ostream& operator<<(ostream& stream, const Z& m) {
77     return stream << m.val;

```

```

78     }
79 };
80 vector<Z> inv, fac, ifac;
81 void prepare_factorials(int maximum) {
82     if (maximum + 1 < inv.size()) return;
83     inv.assign(maximum + 1, 1);
84
85     // Make sure MOD is prime, which is necessary for the inverse algorithm
86     // below.
87     for (int p = 2; p * p <= MOD; p++)
88         assert(MOD % p != 0);
89     for (int i = 2; i <= maximum; i++)
90         inv[i] = inv[MOD % i] * (MOD - MOD / i);
91
92     fac.resize(maximum + 1);
93     ifac.resize(maximum + 1);
94     fac[0] = ifac[0] = 1;
95
96     for (int i = 1; i <= maximum; i++) {
97         fac[i] = i * fac[i - 1];
98         ifac[i] = inv[i] * ifac[i - 1];
99     }
100 }
101 inline Z binom(int n, int m) {
102     assert(n < fac.size());
103     if (n < m || n < 0 || m < 0) return 0;
104     return fac[n] * ifac[m] * ifac[n - m];
105 }
106 namespace NTT {
107     using ull = uint64_t;
108     vector<Z> roots = { 0, 1 };
109     vector<int> bit_reverse;
110     int max_size = -1;
111     Z root;
112
113     bool is_power_of_two(int n) { return (n & (n - 1)) == 0; }
114     int round_up_power_two(int n) {
115         assert(n > 0);
116         while (n & (n - 1))
117             n = (n | (n - 1)) + 1;
118         return n;
119     }
120
121     // Given n (a power of two), finds k such that  $n = 1 \ll k$ .

```

```

120  int get_length(int n) {
121      assert(is_power_of_two(n));
122      return __builtin_ctz(n);
123  }
124  // Rearranges the indices to be sorted by lowest bit first, then second
    lowest, etc., rather than highest bit first.
125  // This makes even-odd div-conquer much easier.
126  void bit_reorder(int n, vector<Z>& values) {
127      if ((int)bit_reverse.size() != n) {
128          bit_reverse.assign(n, 0);
129          int length = get_length(n);
130
131          for (int i = 0; i < n; i++)
132              bit_reverse[i] = (bit_reverse[i >> 1] >> 1) + ((i & 1) << (
                  length - 1));
133      }
134      for (int i = 0; i < n; i++)
135          if (i < bit_reverse[i])
136              swap(values[i], values[bit_reverse[i]]);
137  }
138  /* 找原根 998244353的原根是31 */
139  void find_root() {
140      int order = MOD - 1;
141      max_size = 1;
142      while (order % 2 == 0) {
143          order /= 2;
144          max_size *= 2;
145      }
146      root = 2;
147      // Find a max_size-th primitive root of MOD.
148      while (!(int)root.pow(max_size) == 1 && (int)root.pow(max_size / 2)
          != 1))
149          root = root + 1;
150  }
151
152  void prepare_roots(int n) {
153      if (max_size < 0) find_root();
154      assert(n <= max_size);
155      if ((int)roots.size() >= n) return;
156      int len = get_length(roots.size());
157      roots.resize(n);
158
159      // The roots array is set up such that for a given power of two n >=

```

```

160     2, roots[n / 2] through roots[n - 1] are
161     // the first half of the n-th primitive roots of MOD.
162     while (1 << len < n) {
163         // z is a 2^(length + 1)-th primitive root of MOD.
164         Z z = root.pow(max_size >> (len + 1));
165         for (int i = 1 << (len - 1); i < 1 << len; i++) {
166             roots[2 * i] = roots[i];
167             roots[2 * i + 1] = roots[i] * z;
168         }
169         len++;
170     }
171 void fft_iterative(int N, vector<Z>& values) {
172     assert(is_power_of_two(N));
173     prepare_roots(N);
174     bit_reorder(N, values);
175     for (int n = 1; n < N; n *= 2)
176         for (int start = 0; start < N; start += 2 * n)
177             for (int i = 0; i < n; i++) {
178                 Z even = values[start + i];
179                 Z odd = values[start + n + i] * roots[n + i];
180                 values[start + n + i] = even - odd;
181                 values[start + i] = even + odd;
182             }
183 }
184
185 constexpr int FFT_CUTOFF = 150;
186 vector<Z> mod_multiply(vector<Z> left, vector<Z> right) {
187     int n = left.size(), m = right.size();
188     // Brute force when either n or m is small enough.
189     if (min(n, m) < FFT_CUTOFF) {
190         const ull ULL_BOUND = numeric_limits<ull>::max() - (ull)MOD *
191             MOD;
192         vector<ull> res(n + m - 1);
193         for (int i = 0; i < n; i++) {
194             for (int j = 0; j < m; j++) {
195                 res[i + j] += (ull)((int)left[i]) * ((int)right[j]);
196                 if (res[i + j] > ULL_BOUND)
197                     res[i + j] %= MOD;
198             }
199         }
200         for (ull& x : res)
201             if (x >= MOD)

```



```

201         x %= MOD;
202     return vector<Z>(res.begin(), res.end());
203 }
204
205     int N = round_up_power_two(n + m - 1);
206     left.resize(N);
207     right.resize(N);
208     fft_iterative(N, left);
209     fft_iterative(N, right);
210     Z inv_N = Z(N).inv();
211     for (int i = 0; i < N; i++)
212         left[i] *= right[i] * inv_N;
213     reverse(left.begin() + 1, left.end());
214     fft_iterative(N, left);
215     left.resize(n + m - 1);
216     return left;
217 }
218 vector<Z> sub_conv(vector<Z> a, vector<Z> b) {
219     int n = b.size();
220     reverse(b.begin(), b.end());
221     auto res = mod_multiply(a, b);
222     return vector<Z>(res.begin() + n - 1, res.end());
223 }
224 /*
225  * 多项式平移c个单位
226  *  $a(x+c) = \sum x^j \frac{1}{j!} \sum i! a_i c^{j-i} / (j-i)!$ 
227  *  $O(N \log N)$ 
228  */
229 vector<Z> shift(vector<Z> a, Z c) {
230     int N = round_up_power_two(a.size());
231     prepare_factorials(N);
232     vector<Z> tmp(a.size());
233     Z pc = 1;
234     for (int i = 0; i < tmp.size(); i++, pc *= c) {
235         tmp[i] = pc * ifac[i];
236         a[i] *= fac[i];
237     }
238     tmp = sub_conv(a, tmp);
239     for (int i = 0; i < a.size(); i++) {
240         tmp[i] *= ifac[i];
241     }
242     return tmp;
243 }

```

```

244 vector<Z> mod_power(const vector<Z>& v, int exponent) {
245     assert(exponent >= 0);
246     int nn = v.size();
247     vector<Z> res = { 1 };
248     if (exponent == 0)
249         return res;
250     for (int k = 31 - __builtin_clz(exponent); k >= 0; k--) {
251         res = mod_multiply(res, res);
252         if (exponent >> k & 1)
253             res = mod_multiply(res, v);
254         res.resize(nn);
255     }
256     return res;
257 }
258 vector<Z> mod_inv(vector<Z> a) {
259     int n = a.size();
260     int N = round_up_power_two(a.size());
261     a.resize(N * 2);
262     vector<Z> res = { a[0].inv() };
263     for (int i = 2; i <= N; i <= 1) {
264         vector<Z> tmp(a.begin(), a.begin() + i);
265         int n = (i << 1);
266         res.resize(n);
267         tmp.resize(n);
268         fft_iterative(n, tmp);
269         fft_iterative(n, res);
270         Z inv_n = Z(n).inv();
271         for (int j = 0; j < n; j++)
272             res[j] = res[j] * (2 - tmp[j] * res[j]) * inv_n;
273         reverse(res.begin() + 1, res.end());
274         fft_iterative(n, res);
275         fill(res.begin() + i, res.end(), 0);
276     }
277     res.resize(n);
278     return res;
279 }
280 vector<Z> integral(vector<Z> a) {
281     assert(a.size() <= inv.size());
282     a.push_back(0);
283     for (int i = (int)a.size() - 1; i >= 1; i--) {
284         a[i] = a[i - 1] * inv[i];
285     }
286     return a;

```

```

287     }
288     vector<Z> differential(vector<Z> a) {
289         assert(a.size());
290         for (int i = 0; i < (int)a.size() - 1; i++) {
291             a[i] = a[i + 1] * (i + 1);
292         }
293         a.pop_back();
294         return a;
295     }
296     vector<Z> ln(vector<Z>a) {
297         assert((int)a[0] == 1);
298         auto b = mod_multiply(differential(a), mod_inv(a));
299         b = integral(b);
300         b[0] = 0;
301         return b;
302     }
303     vector<Z> exp(vector<Z>a) {
304         int N = round_up_power_two(a.size());
305         int n = a.size();
306         a.resize(N * 2);
307         vector<Z> res{ 1 };
308         for (int i = 2; i <= N; i <<= 1) {
309             auto tmp = res;
310             tmp.resize(i);
311             tmp = ln(tmp);
312             for (int j = 0; j < i; j++) tmp[j] = a[j] - tmp[j];
313             tmp[0] += 1;
314             res.resize(i);
315             res = mod_multiply(res, tmp);
316             fill(res.begin() + i, res.end(), 0);
317         }
318         res.resize(n);
319         return res;
320     }
321     vector<Z> poly_div(vector<Z> a, vector<Z> b) {
322         if (a.size() < b.size()) {
323             return { 0 };
324         }
325         reverse(a.begin(), a.end());
326         reverse(b.begin(), b.end());
327         b.resize(a.size() - b.size() + 1);
328         a.resize(b.size());
329         auto d = mod_multiply(mod_inv(b), a);

```

```

330     d.resize(b.size());
331     reverse(d.begin(), d.end());
332     return d;
333 }
334 /* poly A mod B */
335 vector<Z> poly_mod(vector<Z> a, vector<Z> b) {
336     auto res = mod_multiply(b, poly_div(a, b));
337     a.resize(b.size() - 1);
338     res.resize(b.size() - 1);
339     for (int i = 0; i < a.size(); i++) {
340         a[i] -= res[i];
341     }
342     return a;
343 }
344 /* Brute force calc f(k) */
345 Z eval(const vector<Z>& f, int k) {
346     Z ans;
347     for (int i = f.size() - 1; i >= 0; i--)
348         ans = (ans * k + f[i]);
349     return ans;
350 }
351 vector<Z> mod_multiply_all(const vector<vector<Z>>& polynomials) {
352     if (polynomials.empty())
353         return { 1 };
354     struct compare_size {
355         bool operator()(const vector<Z>& x, const vector<Z>& y) {
356             return x.size() > y.size();
357         }
358     };
359     priority_queue<vector<Z>, vector<vector<Z>>, compare_size> pq(
360         polynomials.begin(), polynomials.end());
361     while (pq.size() > 1) {
362         vector<Z> a = pq.top(); pq.pop();
363         vector<Z> b = pq.top(); pq.pop();
364         pq.push(mod_multiply(a, b));
365     }
366     return pq.top();
367 }
368 /*
369  * 线性递推求第K项
370  * a[n] = \sum seq[i] * a[n-i], calc a[k]
371  * O(Nlog(N)log(K))
372  */

```

```

372 Z linear_seq(const vector<Z>& _init, vector<Z> seq, long long k) {
373     reverse(seq.begin(), seq.end());
374     for (auto& i : seq) i = -i; seq.push_back(1);
375     vector<Z> b = seq;
376     reverse(b.begin(), b.end());
377     b = mod_inv(b);
378     auto poly_mod = [&](vector<Z>a) {
379         if (a.size() < seq.size()) {
380             a.resize(seq.size() - 1);
381             return a;
382         }
383         vector<Z>tmp = a;
384         reverse(a.begin(), a.end());
385         b.resize(a.size() - seq.size() + 1);
386         a.resize(b.size());
387         auto d = mod_multiply(b, a);
388         d.resize(b.size());
389         reverse(d.begin(), d.end());
390         auto res = mod_multiply(seq, d);
391         tmp.resize(seq.size() - 1);
392         res.resize(seq.size() - 1);
393         for (int i = 0; i < tmp.size(); i++) {
394             tmp[i] -= res[i];
395         }
396         return tmp;
397     };
398     vector<Z> a{ 0,1 };
399     vector<Z> res{ 1 };
400     for (; k; k >>= 1) {
401         if (k & 1)
402             res = poly_mod(mod_multiply(res, a));
403         a = poly_mod(mod_multiply(a, a));
404     }
405     Z ans = 0;
406     for (int i = 0; i < _init.size(); i++) {
407         ans += _init[i] * res[i];
408     }
409     return ans;
410 }
411 /*
412 * 多点求值
413 * 注意如果是1..n的多点求值不要用这个
414 *  $O(N \log(N) \log(M))$  常数偏大

```

```

415     */
416 vector<Z> multi_eval(vector<Z> F, vector<Z> x) {
417     vector<vector<Z>> base;
418     function<void(int, int, int)> build = [&](int l, int r, int o) {
419         if (r - l == 1) {
420             base[o] = { 1, -x[l] };
421             return;
422         }
423         int mid = (l + r) >> 1;
424         build(l, mid, o << 1);
425         build(mid, r, o << 1 | 1);
426         base[o] = (mod_multiply(base[o << 1], base[o << 1 | 1]));
427     };
428     vector<Z> res(x.size());
429     int n = max(x.size(), F.size());
430     x.resize(n); F.resize(n + 1);
431     base.resize(4 * n);
432     build(0, n, 1);
433     function<void(vector<Z>, int, int, int)> solve = [&](vector<Z> f, int
434         l, int r, int o) {
435         if (r - l == 1) {
436             if (l < res.size()) res[l] = f[0];
437             return;
438         }
439         int mid = (l + r) >> 1;
440         auto L = sub_conv(f, base[o << 1 | 1]);
441         auto R = sub_conv(f, base[o << 1]);
442         L.resize(mid - 1);
443         R.resize(r - mid);
444         solve(L, l, mid, o << 1);
445         solve(R, mid, r, o << 1 | 1);
446     };
447     solve(sub_conv(F, mod_inv(base[1])), 0, n, 1);
448     return res;
449 }
450 /*
451 * 多点插值 给定N个点(xi, yi) 插出一个N-1阶的多项式
452 * O(Nlog(N)log(M)) 常数偏大
453 */
454 vector<Z> multi_inter(const vector<Z>& y, const vector<Z>& x) {
455     assert(y.size() == x.size());
456     vector<vector<Z>> base;
457     function<void(int, int, int)> build = [&](int l, int r, int o) {

```

```

457         if (r - l == 1) {
458             base[o] = { -x[l], 1 };
459             return;
460         }
461         int mid = (l + r) >> 1;
462         build(l, mid, o << 1);
463         build(mid, r, o << 1 | 1);
464         base[o] = (mod_multiply(base[o << 1], base[o << 1 | 1]));
465     };
466     int n = x.size();
467     base.resize(4 * n);
468     int s = clock();
469     build(0, n, 1);
470     vector<Z> pi = differential(base[1]);
471     vector<Z> res = multi_eval(pi, x);
472     for (int i = 0; i < n; i++) res[i] = y[i] / res[i];
473     function<vector<Z>(<b>int</b>, <b>int</b>, <b>int</b>)> solve2 = [&](int l, int r, int o)
474     {
475         if (r - l == 1) {
476             return vector<Z>({ res[l] });
477         }
478         int mid = (l + r) >> 1;
479         vector<Z> L = mod_multiply(solve2(l, mid, o << 1), base[o << 1 |
480             1]);
481         vector<Z> R = mod_multiply(solve2(mid, r, o << 1 | 1), base[o <<
482             1]);
483         int n = max(L.size(), R.size());
484         L.resize(n); R.resize(n);
485         for (int i = 0; i < n; i++) {
486             L[i] += R[i];
487         }
488         return L;
489     };
490     auto ans = solve2(0, n, 1);
491     ans.resize(n);
492     return ans;
493 }
494 vector<Z> mod_power2(const vector<Z>& a, int exponent) {
495     int n = a.size();
496     auto res = ln(a);
497     while ((int)res.size() > n) res.pop_back();
498     for (auto& i : res) {
499         i = exponent * i;

```

```

497     }
498     res = exp(res);
499     return res;
500 }
501 }

```

1.4 FWT

```

1  constexpr int MOD = 998244353;
2  constexpr int MAXN = 1 << 21;
3  using ll = long long;
4  namespace FWT {
5      int a[MAXN], b[MAXN], ans[MAXN];
6      int n;
7      ll x, y;
8      void print() {
9          for (int i = 0; i < n; ++i) cout << a[i] << ' '; cout << "\n";
10         for (int i = 0; i < n; ++i) cout << b[i] << ' '; cout << "\n";
11         for (int i = 0; i < n; ++i) cout << ans[i] << ' '; cout << "\n";
12     }
13     void geta(int* t) {
14         for (int i = 0; i < n; ++i) a[i] = t[i];
15     }
16     void getb(int* t) {
17         for (int i = 0; i < n; ++i) b[i] = t[i];
18     }
19     void Or(int* a, int p) {
20         for (int i = 1; i < n; i <= 1) {
21             for (int j = 0; j < n; j += (i < 1)) {
22                 for (int k = 0; k < i; ++k) {
23                     x = a[j + k]; y = a[i + j + k];
24                     a[j + k] = x;
25                     a[i + j + k] = (x * p + y) % MOD;
26                 }
27             }
28         }
29     }
30     void workOr(int* A, int* B, int* Ans, int nn) {
31         for (n = 1; n < nn; n <= 1);
32         geta(A); getb(B);
33         Or(a, 1);
34         Or(b, 1);
35         for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;

```



```

36     Or(ans, -1);
37     for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
38 }
39 void And(int* a, int p) {
40     for (int i = 1; i < n; i <= 1) {
41         for (int j = 0; j < n; j += (i < 1)) {
42             for (int k = 0; k < i; ++k) {
43                 x = a[j + k]; y = a[i + j + k];
44                 a[j + k] = (x + y * p) % MOD;
45                 a[i + j + k] = y;
46             }
47         }
48     }
49 }
50 void workAnd(int* A, int* B, int* Ans, int nn) {
51     for (n = 1; n < nn; n <= 1);
52     geta(A); getb(B);
53     And(a, 1);
54     And(b, 1);
55     for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;
56     And(ans, -1);
57     for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
58 }
59 void Xor(int* a, int p) {
60     for (int i = 1; i < n; i <= 1) {
61         for (int j = 0; j < n; j += (i < 1)) {
62             for (int k = 0; k < i; ++k) {
63                 x = a[j + k]; y = a[i + j + k];
64                 a[j + k] = (x + y) % MOD;
65                 a[i + j + k] = (x - y + MOD) % MOD;
66                 if (p == -1) {
67                     if (a[j + k] & 1) a[j + k] += MOD;
68                     if (a[i + j + k] & 1) a[i + j + k] += MOD;
69                     a[j + k] >>= 1;
70                     a[i + j + k] >>= 1;
71                 }
72             }
73         }
74     }
75 }
76 void workXor(int* A, int* B, int* Ans, int nn) {
77     for (n = 1; n < nn; n <= 1);
78     geta(A); getb(B);

```

```

79     Xor(a, 1);
80     Xor(b, 1);
81     for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;
82     Xor(ans, -1);
83     for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
84 }
85 }

```

1.5 分治 FFT

```

1  /*
2  * given n and g[1], g[2], ..., g[n], f[0] = 1
3  * f[i] = sum_{j=1..i} f[i-j]g[j]
4  * get f[1], f[2], ..., f[n] in O(nlog^2n)
5  */
6  int n;
7  vector<int> f, g;
8  void dfs(int l, int r, int sgn) {
9      int mid = (l + r) >> 1;
10     if (l != r) {
11         dfs(l, mid, 1);
12         dfs(mid + 1, r, 0);
13     }
14     if (sgn) {
15         int seg = r - l;
16         vector<int> slice(f.begin() + l - 1, f.begin() + r);
17         vector<int> cg(g.begin(), g.begin() + min(2 * seg, n));
18         auto h = FFT::mod_multiply(slice, cg);
19         for (int i = r; i <= min(r + seg, n - 1); i++) {
20             f[i] = add(f[i], h[seg + i - r + 1]);
21         }
22     }
23 }

```

2 数论

2.1 简单的防爆模板

2.1.1 类型 1

```

1 namespace SimpleMod {
2     constexpr int MOD = (int)1e9 + 7;
3     inline int norm(long long a) { return (a % MOD + MOD) % MOD; }

```

```

4   inline int add(int a, int b) { return a + b >= MOD ? a + b - MOD : a + b
    ; }
5   inline int sub(int a, int b) { return a - b < 0 ? a - b + MOD : a - b; }
6   inline int mul(int a, int b) { return (int)((long long)a * b % MOD); }
7   inline int powmod(int a, long long b) {
8       int res = 1;
9       while (b > 0) {
10          if (b & 1) res = mul(res, a);
11          a = mul(a, a);
12          b >>= 1;
13      }
14      return res;
15  }
16  inline int inv(int a) {
17      a %= MOD;
18      if (a < 0) a += MOD;
19      int b = MOD, u = 0, v = 1;
20      while (a) {
21          int t = b / a;
22          b -= t * a; swap(a, b);
23          u -= t * v; swap(u, v);
24      }
25      assert(b == 1);
26      if (u < 0) u += MOD;
27      return u;
28  }
29 }

```

2.1.2 类型 2

```

1  template<int MOD> struct Z {
2      int x;
3      Z(int v = 0) : x(v % MOD) { if (x < 0) x += MOD; }
4      Z(long long v = 0) : x(v % MOD) { if (x < 0) x += MOD; }
5      Z operator - () const { return x ? MOD - x : 0; }
6      Z operator + (const Z& r) { return Z(*this) += r; }
7      Z operator - (const Z& r) { return Z(*this) -= r; }
8      Z operator * (const Z& r) { return Z(*this) *= r; }
9      Z operator / (const Z& r) { return Z(*this) /= r; }
10     Z& operator += (const Z& r) {
11         x += r.x;
12         if (x >= MOD) x -= MOD;
13         return *this;

```

```

14     }
15     Z& operator -= (const Z& r) {
16         x -= r.x;
17         if (x < 0) x += MOD;
18         return *this;
19     }
20     Z& operator *= (const Z& r) {
21         x = 1LL * x * r.x % MOD;
22         return *this;
23     }
24     Z& operator /= (const Z& r) {
25         int a = r.x, b = MOD, u = 1, v = 0;
26         while (b) {
27             long long t = a / b;
28             a -= t * b, swap(a, b);
29             u -= t * v, swap(u, v);
30         }
31         x = 1LL * x * u % MOD;
32         if (x < 0) x += MOD;
33         return *this;
34     }
35     Z& power(long long k) {
36         int a = x; x = 1;
37         while (k > 0) {
38             if (k & 1) x = 1LL * x * a % MOD;
39             a = 1LL * a * a % MOD;
40             k >>= 1;
41         }
42         return *this;
43     }
44     bool operator == (const Z& r) { return this->x == r.x; }
45     bool operator != (const Z& r) { return this->x != r.x; }
46     friend constexpr istream& operator >> (istream& is, Z<MOD>& x) noexcept
47     {
48         is >> x.x;
49         x.x %= MOD;
50         if (x.x < 0) x.x += MOD;
51         return is;
52     }
53     friend ostream& operator << (ostream& os, const Z<MOD>& x) {
54         return os << x.x;
55     };

```

```

56 constexpr int MOD = 1e9 + 7;
57 using mint = Z<MOD>;

```

2.2 约数和

```

1 long long getSigma(long long n, long long mod = 1e18) {
2     if (n == 1) return 1;
3     long long ans = 1;
4     for (long long i = 2; i * i <= n; ++i) {
5         long long cnt = 1;
6         while (!(n % i)) {
7             cnt = (cnt * i + 1) % mod;
8             n /= i;
9         }
10        ans *= cnt;
11    }
12    return (n == 1 ? ans : ans * (n + 1) % mod);
13 }

```

2.3 筛法

2.3.1 线性素数筛

```

1 vector<bool> isPrime; // true 表示非素数 false 表示是素数
2 vector<int> prime; // 保存素数
3 int sieve(int n) {
4     isPrime.resize(n + 1, false);
5     isPrime[0] = isPrime[1] = true;
6     for (int i = 2; i <= n; i++) {
7         if (!isPrime[i]) prime.emplace_back(i);
8         for (int j = 0; j < (int)prime.size() && prime[j] * i <= n; j++) {
9             isPrime[prime[j] * i] = true;
10            if (!(i % prime[j])) break;
11        }
12    }
13    return (int)prime.size();
14 }

```

2.3.2 线性欧拉函数筛

```

1 bool is_prime[SIZE];
2 int prime[SIZE], phi[SIZE]; // phi[i] 表示 i 的欧拉函数值
3 int Phi(int n) { // 线性筛素数的同时线性求欧拉函数

```

```

4   phi[1] = 1; is_prime[1] = true;
5   int p = 0;
6   for (int i = 2; i <= n; i++) {
7       if (!is_prime[i]) prime[p++] = i, phi[i] = i - 1;
8       for (int j = 0; j < p && prime[j] * i <= n; j++) {
9           is_prime[prime[j] * i] = true;
10          if (!(i % prime[j])) {
11              phi[i * prime[j]] = phi[i] * prime[j];
12              break;
13          }
14          phi[i * prime[j]] = phi[i] * (prime[j] - 1);
15      }
16  }
17  return p;
18 }

```

2.3.3 线性约数个数函数筛

```

1   bool is_prime[SIZE];
2   int prime[SIZE], d[SIZE], num[SIZE]; // d[i] 表示 i 的因子数 num[i] 表示 i
   的最小质因子出现次数
3   int getFactors(int n) { // 线性筛因子数
4       d[1] = 1; is_prime[1] = true;
5       int p = 0;
6       for (int i = 2; i <= n; i++) {
7           if (!is_prime[i]) prime[p++] = i, d[i] = 2, num[i] = 1;
8           for (int j = 0; j < p && prime[j] * i <= n; j++) {
9               is_prime[prime[j] * i] = true;
10              if (!(i % prime[j])) {
11                  num[i * prime[j]] = num[i] + 1;
12                  d[i * prime[j]] = d[i] / num[i * prime[j]] * (num[i * prime[
13                      j]] + 1);
14                  break;
15              }
16              num[i * prime[j]] = 1;
17              d[i * prime[j]] = d[i] + d[i];
18          }
19      }
20  return p;

```

2.3.4 线性素因子个数函数筛

```

1  bool is_prime[SIZE];
2  int prime[SIZE], num[SIZE]; // num[i] 表示 i 的质因子数
3  int getPrimeFactors(int n) { // 线性筛质因子数
4      is_prime[1] = true;
5      int p = 0;
6      for (int i = 2; i <= n; i++) {
7          if (!is_prime[i]) prime[p++] = i, num[i] = 1;
8          for (int j = 0; j < p && prime[j] * i <= n; j++) {
9              is_prime[prime[j] * i] = true;
10             if (!(i % prime[j])) {
11                 num[i * prime[j]] = num[i];
12                 break;
13             }
14             num[i * prime[j]] = num[i] + 1;
15         }
16     }
17     return p;
18 }

```

2.3.5 线性约数和函数筛

```

1  bool is_prime[SIZE];
2  int prime[SIZE], f[SIZE], g[SIZE]; // f[i] 表示 i 的约数和
3  int getSigma(int n) {
4      g[1] = f[1] = 1; is_prime[1] = true;
5      int p = 0;
6      for (int i = 2; i <= n; i++) {
7          if (!is_prime[i]) prime[p++] = i, f[i] = g[i] = i + 1;
8          for (int j = 0; j < p && prime[j] * i <= n; j++) {
9              is_prime[prime[j] * i] = true;
10             if (!(i % prime[j])) {
11                 g[i * prime[j]] = g[i] * prime[j] + 1;
12                 f[i * prime[j]] = f[i] / g[i] * g[i * prime[j]];
13                 break;
14             }
15             f[i * prime[j]] = f[i] * f[prime[j]];
16             g[i * prime[j]] = 1 + prime[j];
17         }
18     }
19     return p;
20 }

```

2.3.6 线性莫比乌斯函数筛

```

1  bool is_prime[SIZE];
2  int prime[SIZE], mu[SIZE]; // mu[i] 表示 i 的莫比乌斯函数值
3  int getMu(int n) { // 线性筛莫比乌斯函数
4      mu[1] = 1; is_prime[1] = true;
5      int p = 0;
6      for (int i = 2; i <= n; i++) {
7          if (!is_prime[i]) prime[p++] = i, mu[i] = -1;
8          for (int j = 0; j < p && prime[j] * i <= n; j++) {
9              is_prime[prime[j] * i] = true;
10             if (!(i % prime[j])) {
11                 mu[i * prime[j]] = 0;
12                 break;
13             }
14             mu[i * prime[j]] = -mu[i];
15         }
16     }
17     return p;
18 }

```

2.4 Pollard-Rho

```

1  namespace Pollard_Rho {
2      typedef long long ll;
3      vector<ll> ans; // 存储质因子的数组
4      inline ll gcd(ll a, ll b) { ll c; while (b) c = a % b, a = b, b = c;
5          return a; }
6      inline ll mulmod(ll x, ll y, const ll z) {
7          return (x * y - (ll)((long double)x * y + 0.5) / (long double)z *
8              z + z) % z;
9      }
10     inline ll powmod(ll a, ll b, const ll mo) {
11         ll s = 1;
12         for (; b; b >>= 1, a = mulmod(a, a, mo)) if (b & 1) s = mulmod(s, a,
13             mo);
14         return s;
15     }
16     bool isPrime(ll p) { // Miller-Rabin  $O(k \log^3(n))$  k为素性测试轮数
17         const int lena = 10, a[lena] = { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 };
18         if (p == 2) return true;
19         if (p == 1 || !(p & 1) || (p == 4685624825598111)) return false;
20         ll D = p - 1;
21         while (!(D & 1)) D >>= 1;

```



```

19     for (int i = 0; i < lena && a[i] < p; i++) {
20         ll d = D, t = powmod(a[i], d, p);
21         if (t == 1) continue;
22         for (; d != p - 1 && t != p - 1; d <= 1) t = mulmod(t, t, p);
23         if (d == p - 1) return false;
24     }
25     return true;
26 }
27 void reportFactor(ll n) { // 得到一个素因子
28     ans.emplace_back(n); // 存储素因子
29 }
30 ll ran() { return rand(); } // 随机数
31 void getFactor(ll n) { // Pollard-Rho  $O(n^{1/4})$ 
32     if (n == 1) return;
33     if (isPrime(n)) { reportFactor(n); return; }
34     while (true) {
35         ll c = ran() % n, i = 1, x = ran() % n, y = x, k = 2;
36         do {
37             ll d = gcd(n + y - x, n);
38             if (d != 1 && d != n) { getFactor(d); getFactor(n / d);
39                 return; }
40             if (++i == k) y = x, k <= 1;
41             x = (mulmod(x, x, n) + c) % n;
42         } while (y != x);
43     }
44 }
45 using namespace Pollard_Rho;

```

2.5 扩展欧几里得

2.5.1 线性同余方程最小非负整数解

exgcd 求 $ax + by = c$ 的最小非负整数解详解:

1. 求出 a, b 的最大公约数 $g = \gcd(a, b)$ ，根据裴蜀定理检查是否满足 $c \% g = 0$ ，不满足则无解；
2. 调整系数 a, b, c 为 $a' = \frac{a}{g}, b' = \frac{b}{g}, c' = \frac{c}{g}$ ，这是因为 $ax + by = c$ 和 $a'x + b'y = c'$ 是完全等价的；
3. 实际上 exgcd 求解的方程是 $a'x + b'y = 1$ ，求解前需要注意让系数 $a', b' \geq 0$ （举个例子，如果系数 b' 原本 < 0 ，我们可以翻转 b' 的符号然后令解 (x, y) 为 $(x, -y)$ ，但是求解的时候要把 y 翻回来）；
4. 我们通过 exgcd 求出一组解 (x_0, y_0) ，这组解满足 $a'x_0 + b'y_0 = 1$ ，为了使解合法我们需要令 $x_0 = c'x_0, y_0 = c'y_0$ ，于是有 $a'(c'x_0) + b'(c'y_0) = c'$ ；
5. 考虑到 $a'x_0 + b'y_0 = 1$ 等价于同余方程 $a'x_0 \equiv 1 \pmod{b'}$ ，因此为了求出最小非负整数解，我们最后还需要对 b' 取模；

6. 最后注意特判 $c' = 0$ 的情况，如果要求解 y 且系数 b 发生了翻转，将其翻转回来。

```

1 long long exgcd(long long a, long long b, long long& x, long long& y) {
2     if (!b) {
3         x = 1, y = 0;
4         return a;
5     }
6     long long g = exgcd(b, a % b, y, x);
7     y -= (a / b) * x;
8     return g;
9 }
10
11 ll x, y; // 最小非负整数解
12 bool solve(ll a, ll b, ll c) { // ax+by=c
13     ll g = gcd(a, b);
14     if (c % g) return false;
15     a /= g, b /= g, c /= g;
16     bool flag = false;
17     if (b < 0) b = -b, flag = true;
18     exgcd(a, b, x, y);
19     x = (x * c % b + b) % b;
20     if (flag) b = -b;
21     y = (c - a * x) / b;
22     if (!c) x = y = 0; // ax+by=0
23     return true;
24 }

```

2.5.2 一定范围内线性方程整数解数

exgcd 通解：假设我们通过上方的 exgcd 流程获得了一组解 (x_0, y_0) （没有乘 c ），那么 $a'x + b'y = 1$ 的通解就是 $(x_0 + b't, y_0 - a't)$ ，因此 $a'x + b'y = c'$ 的通解是 $(c'(x_0 + b't), c'(y_0 - a't))$ 。

```

1 /*
2  * 求解 ax+by+c=0 模板CF710D
3  * 返回 [xl, xr] x [yl, yr] 的解数
4  * 若至少有一组解 则x是一个合法解
5  * 可以根据x推出y 但要注意b=0/a=0等特殊情况
6  * 特别注意!!! 设置边界时的取整问题!!!
7  * x/y 向上取整时(x+y-1)/y 向下取整时floor(1.0*x/y)
8  * */
9 ll a, b, c, xl, xr, yl, yr;
10 ll x, y, d;
11 ll exgcd(ll a, ll b, ll& x, ll& y) {
12     if (!b) return x = 1, y = 0, a;
13     ll d = exgcd(b, a % b, x, y), t = x;

```

```

14     return x = y, y = t - a / b * y, d;
15 }
16 ll solve(ll a, ll b, ll c, ll xl, ll xr, ll yl, ll yr) {
17     if (xl > xr) return 0;
18     if (yl > yr) return 0;
19     if (!a && !b) {
20         if (c) return 0;
21         return (xr - xl + 1) * (yr - yl + 1);
22     }
23     if (!b) {
24         swap(a, b);
25         swap(xl, yl);
26         swap(xr, yr);
27     }
28     if (!a) {
29         if (c % b) return 0;
30         ll y = -c / b;
31         if (y < yl || y > yr) return 0;
32         return xr - xl + 1;
33     }
34     d = exgcd((a % abs(b) + abs(b)) % abs(b), abs(b), x, y);
35     if (c % d) return 0;
36     x = (x % abs(b) + abs(b)) % abs(b) * (((-c) % abs(b)) + abs(b)) % abs(b)
37         / d % abs(b / d);
38     d = abs(b / d);
39     ll kl = (xl - x) / d - 3, kr = (xr - x) / d + 3;
40     while (x + kl * d < xl) kl++;
41     while (x + kr * d > xr) kr--;
42     ll A = (-yl * b - a * x - c) / (a * d), B = (-yr * b - a * x - c) / (a *
43         d);
44     if (A > B) swap(A, B);
45     kl = max(kl, A - 3);
46     kr = min(kr, B + 3);
47     while (kl <= kr) {
48         ll y = (-c - a * x - a * d * kl) / b;
49         if (yl <= y && y <= yr) break;
50         kl++;
51     }
52     while (kl <= kr) {
53         ll y = (-c - a * x - a * d * kr) / b;
54         if (yl <= y && y <= yr) break;
55         kr--;
56     }
57 }

```

```

55     if (kl > kr) return 0;
56     return kr - kl + 1;
57 }

```

2.6 类欧几里得

2.6.1 Naive

$$f(a, b, c, n) = \sum_{i=0}^n \left\lfloor \frac{ai + b}{c} \right\rfloor$$

原理:

$$\begin{aligned}
 f(a, b, c, n) &= \sum_{i=0}^n \left\lfloor \frac{ai + b}{c} \right\rfloor \\
 &= \sum_{i=0}^n \left\lfloor \frac{(\lfloor \frac{a}{c} \rfloor c + a \bmod c) i + (\lfloor \frac{b}{c} \rfloor c + b \bmod c)}{c} \right\rfloor \\
 &= \frac{n(n+1)}{2} \left\lfloor \frac{a}{c} \right\rfloor + (n+1) \left\lfloor \frac{b}{c} \right\rfloor + \sum_{i=0}^n \left\lfloor \frac{(a \bmod c) i + (b \bmod c)}{c} \right\rfloor \\
 &= \frac{n(n+1)}{2} \left\lfloor \frac{a}{c} \right\rfloor + (n+1) \left\lfloor \frac{b}{c} \right\rfloor + f(a \bmod c, b \bmod c, c, n)
 \end{aligned}$$

```

1  ll f(ll a, ll b, ll c, ll n) { // O(log n)
2      if (a == 0)
3          return (n + 1) * (b / c);
4      if (a >= c || b >= c)
5          return (f(a % c, b % c, c, n) + (a / c) * n * (n + 1) / 2 + (b / c)
6              * (n + 1));
7      ll m = (a * n + b) / c;
8      return (n * m - f(c, c - b - 1, a, m - 1));
9  }

```

2.6.2 General

$$f(a, b, c, n, k1, k2) = \sum_{i=0}^n i^{k1} \left\lfloor \frac{ai + b}{c} \right\rfloor^{k2}$$

```

1  // n, a, b, c >= 1e9 && k1+k2 <= 10
2  // O(k^4 log(MAXN))
3  using ll = long long;
4  const int K = 10;
5  const int mod = 1e9 + 7;
6  const ll lmod = 1ll(mod) << 32;
7  const int signs[2] = { 1, mod - 1 };
8  int invs[K + 2], binom[K + 2][K + 2], B[K + 1], polys[K + 1][K + 2];

```

```

9
10 int add(int a, int b) { return (a += b - mod) < 0 ? a + mod : a; }
11 int add64(int a, int b) { return (a += b - lmod) < 0 ? a + lmod : a;}
12 int mul(int a, int b) { return 1LL * a * b % mod; }
13 int power_sum(int e, int x) {
14     int ret = 0;
15     for (int i = 0; i < e + 2; ++i)
16         ret = add(mul(ret, x), polys[e][i]);
17     return mul(ret, invs[e + 1]);
18 }
19
20 void init() {
21     invs[0] = invs[1] = 1; B[0] = 1;
22     for (int i = 2; i <= K + 1; ++i)
23         invs[i] = mul(invs[mod % i], mod - mod / i);
24     for (int i = 0; i <= K + 1; ++i) {
25         binom[i][0] = 1;
26         for (int j = 1; j <= i; ++j)
27             binom[i][j] = add(binom[i - 1][j - 1], binom[i - 1][j]);
28     }
29     for (int i = 1; i <= K; ++i) {
30         int s = 0;
31         for (int j = 0; j < i; ++j)
32             s = add(s, mul(binom[i + 1][j], B[j]));
33         B[i] = mul(mul(s, invs[i + 1]), signs[1]);
34     }
35     for (int i = 0; i <= K; ++i) {
36         for (int j = 0; j <= i; ++j)
37             polys[i][j] = mul(mul(binom[i + 1][j], B[j]), signs[j & 1]);
38         polys[i][i + 1] = 0;
39     }
40     polys[0][1] = 1;
41 }
42
43 int euclidLike(int N, int a, int b, int c, int k1, int k2) {
44     assert(N >= 0); assert(a >= 0); assert(b >= 0); assert(c >= 1); assert(
45         k1 + k2 <= K);
46     using T = tuple<int, int, int, int>;
47     stack<T> stac;
48     while (1) {
49         stac.emplace(N, a, b, c);
50         if (N < 0 || a == 0)
51             break;

```

```

51     if (a >= c) {
52         a %= c;
53     } else if (b >= c) {
54         b %= c;
55     } else {
56         N = (ll(a) * N + b) / c - 1;
57         b = c - 1 - b;
58         swap(a, c);
59     }
60 }
61
62 const int S = k1 + k2;
63 static int curr[K + 1][K + 1] = {}, next[K + 1][K + 1] = {};
64 while (!stac.empty()) {
65     tie(N, a, b, c) = stac.top();
66     stac.pop();
67     if (N < 0) {
68         ;
69     } else if (a == 0) {
70         int q = b / c;
71         for (int e1 = 0; e1 <= S; ++e1) {
72             int s = power_sum(e1, N);
73             for (int e2 = 0; e2 <= S - e1; ++e2)
74                 next[e1][e2] = s, s = mul(s, q);
75         }
76     } else if (a >= c || b >= c) {
77         int q = (a >= c) ? a / c : b / c;
78         int d = (a >= c) ? 1 : 0;
79         for (int e1 = 0; e1 <= S; ++e1) {
80             for (int e2 = 0; e2 <= S - e1; ++e2) {
81                 ll s = 0;
82                 int p = 1;
83                 for (int i2 = 0; i2 <= e2; ++i2) {
84                     s = add64(s, ll(p) * mul(binom[e2][i2], curr[e1 + i2
85                         * d][e2 - i2]));
86                     p = mul(p, q);
87                 }
88                 next[e1][e2] = s % mod;
89             }
90         } else {
91             static int cumu[K + 1][K + 1];
92             for (int e2 = 0; e2 <= S - 1; ++e2) {

```

```

93         for (int e1 = 0; e1 <= S - e2 - 1; ++e1) {
94             ll s = 0;
95             for (int j = 0; j <= e1 + 1; ++j) {
96                 s = add64(s, ll(polys[e1][e1 + 1 - j]) * curr[e2][j]
97                     );
98             }
99             cumu[e1][e2] = mul(s % mod, invs[e1 + 1]);
100         }
101     const int M = (ll(a) * N + b) / c;
102     for (int e1 = 0; e1 <= S; ++e1) {
103         int p = power_sum(e1, N);
104         for (int e2 = 0; e2 <= S - e1; ++e2) {
105             ll t = 0;
106             for (int i2 = 0; i2 < e2; ++i2) {
107                 t = add64(t, ll(cumu[e1][i2]) * binom[e2][i2]);
108             }
109             next[e1][e2] = add(p, mod - t % mod);
110             p = mul(p, M);
111         }
112     }
113     swap(curr, next);
114 }
115 return curr[k1][k2];
116 }

```

2.7 Wilson 定理

假设 p 是素数，则有：

$$(p-1)! \equiv -1 \pmod{p}$$

否则除了 $p=4$ 时， $(p-1)! \equiv 0 \pmod{p}$ 。

2.8 欧拉定理

$$a^b \equiv \begin{cases} a^{b \bmod \varphi(p)}, & \gcd(a, p) = 1 \\ a^b, & \gcd(a, p) \neq 1, b < \varphi(p) \\ a^{b \bmod \varphi(p) + \varphi(p)}, & \gcd(a, p) \neq 1, b \geq \varphi(p) \end{cases} \pmod{p}$$

2.9 欧拉函数

2.9.1 暴力单点查询

```

1 long long phi(long long n) { // O(sqrt(N))
2     int m = int(sqrt(n + 0.5));
3     long long ans = n;
4     for (int i = 2; i <= m; i++) {
5         if (n % i == 0) {
6             ans = ans / i * (i - 1);
7             while (n % i == 0) n /= i;
8         }
9     }
10    if (n > 1) ans = ans / n * (n - 1);
11    return ans;
12 }

```

2.9.2 预处理单点查询

```

1 vector<int> prime; // 求 n 的欧拉函数需要先把 <= ceil(sqrt(n)) 的素数筛出
2 long long phi(long long n) { // O(sqrt(N)/log(N))
3     long long res = n;
4     for (int i = 0; i < (int)prime.size(); i++) {
5         long long p = prime[i];
6         if (p * p > n) break;
7         if (n % p == 0) {
8             res = res / p * (p - 1);
9             while (n % p == 0) n /= p;
10        }
11    }
12    if (n > 1) res = res / n * (n - 1);
13    return res;
14 }

```

2.10 中国剩余定理

2.10.1 CRT

```

1 // 求解形如  $x = c_i \pmod{m_i}$  的线性方程组 ( $m_i, m_j$ ) 必须两两互质
2 long long CRT(vector<long long>& c, vector<long long>& m) {
3     long long M = m[0], ans = 0;
4     for (int i = 1; i < (int)m.size(); ++i) M *= m[i];
5     for (int i = 0; i < (int)m.size(); ++i) { //  $M_i * t_i * c_i$ 
6         long long mi = M / m[i];
7         long long ti = inv(mi, m[i]); //  $m_i$  模  $m[i]$  的逆元
8         ans = (ans + mi * ti % M * c[i] % M) % M;
9     }

```



```

10     ans = (ans + M) % M; // 返回模 M 意义下的唯一解
11     return ans;
12 }

```

2.10.2 EXCRT

```

1  long long exgcd(long long a, long long b, long long& x, long long& y) {
2      if (!b) {
3          x = 1, y = 0;
4          return a;
5      }
6      long long g = exgcd(b, a % b, y, x);
7      y -= (a / b) * x;
8      return g;
9  }
10
11 long long mulmod(long long x, long long y, const long long z) { // x * y % z
12     防爆
13     return (x * y - (long long)(((long double)x * y + 0.5) / (long double)z)
14         * z + z) % z;
15 }
16
17 // 求解形如  $x = c_i \pmod{m_i}$  的线性方程组
18 long long EXCRT(vector<long long>& c, vector<long long>& m) {
19     long long M = m[0], ans = c[0];
20     for (int i = 1; i < (int)m.size(); ++i) { //  $M * x - m_i * y = c_i - C$ 
21         long long x, y, C = ((c[i] - ans) % m[i] + m[i]) % m[i]; //  $c_i - C$ 
22         long long G = exgcd(M, m[i], x, y);
23         if (C % G) return -1; // 无解
24         long long P = m[i] / G;
25         x = mulmod(C / G, x, P); // 防爆求最小正整数解 x
26         ans += x * M;
27         M *= P; // LCM(M, m_i)
28         ans = (ans % M + M) % M;
29     }
30     return ans;
31 }

```

2.11 BSGS

```

1  ll bsgs(ll a, ll b, ll m) { //  $a^x = b \pmod{m}$ 
2      ll n = (ll)sqrt((double)m) + 1, base = 1, val = 1;
3      map<int, int> mp; // 可以换 unordered_map

```

```

4      b %= m;
5      for (int i = 0; i < n; ++i) {
6          mp[b * base % m] = i;
7          base = (base * a) % m;
8      }
9      a = base;
10     if (!a) return b ? -1 : 1;
11     for (int i = 0; i <= n; ++i) {
12         int j = (!mp.count(val) ? -1 : mp[val]);
13         if (j >= 0 && i * n >= j) return i * n - j;
14         val = (val * a) % m;
15     }
16     return -1; // 无解
17 }

```

2.12 二次剩余

```

1  using ll = long long;
2  inline ll mulmod(ll x, ll y, const ll z) {
3      return (x * y - (ll)((long double)x * y + 0.5) / (long double)z * z +
4          z) % z;
5  }
6  inline ll powmod(ll a, ll b, const ll mo) {
7      ll s = 1;
8      for (; b; b >>= 1, a = mulmod(a, a, mo)) if (b & 1) s = mulmod(s, a, mo);
9      return s;
10 }
11 ll tonelliShanks(ll n, ll p) { // O(log p)
12     if (n == 0) return 0;
13     if (p == 2) return (n & 1) ? 1 : -1;
14     if (powmod(n, p >> 1, p) != 1) return -1;
15     if (p & 2) return powmod(n, p + 1 >> 2, p);
16     int s = __builtin_ctzll(p ^ 1);
17     ll q = p >> s, z = 2;
18     for (; powmod(z, p >> 1, p) == 1; ++z);
19     ll c = powmod(z, q, p);
20     ll r = powmod(n, q + 1 >> 1, p);
21     ll t = powmod(n, q, p), tmp;
22     for (int m = s, i; t != 1;) {
23         for (i = 0, tmp = t; tmp != 1; i++) tmp = tmp * tmp % p;
24         for (; i < -m;) c = c * c % p;
25         r = r * c % p;
26     }
27 }

```

```

25     c = c * c % p;
26     t = t * c % p;
27 }
28 return r;
29 }

```

2.13 迪利克雷卷积

$$g(1)S(n) = \sum_{i=1}^n (f * g)(i) - \sum_{i=2}^n g(i)S(\lfloor \frac{n}{i} \rfloor)$$

2.14 杜教筛

$$(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{xy=n} f(x)g(y)$$

2.15 Berlekamp Massey

```

1 namespace Berlekamp_Massey {
2     typedef long long ll;
3     constexpr ll MOD = 1e9 + 7;
4     constexpr int N = 10010;
5     ll res[N], base[N], _c[N], _md[N];
6     vector<int> Md;
7     ll powmod(ll a, ll b) {
8         ll res = 1;
9         while (b > 0) {
10             if (b & 1) res = res * a % MOD;
11             a = a * a % MOD;
12             b >>= 1;
13         }
14         return res;
15     }
16     void mul(ll* a, ll* b, int k) {
17         for (int i = 0; i < k + k; i++)
18             _c[i] = 0;
19         for (int i = 0; i < k; i++) {
20             if (!a[i]) continue;
21             for (int j = 0; j < k; j++) {
22                 _c[i + j] = (_c[i + j] + a[i] * b[j]) % MOD;
23             }
24         }
25         for (int i = k + k - 1; i >= k; i--) {
26             if (!_c[i]) continue;

```

```

27         for (int j = 0; j < Md.size(); j++) {
28             __c[i - k + Md[j]] = (__c[i - k + Md[j]] - __c[i] * __md[Md[j]])
                % MOD;
29         }
30     }
31     for (int i = 0; i < k; i++)
32         a[i] = __c[i];
33 }
34 int solve(ll n, vector<int> a, vector<int> b) { //a系数 b初值 b[n+1]=a
    [0]*b[n]+...
35     //printf("%d\n", (int)b.size());
36     //for (int i = 0; i < b.size(); i++)
37     //    printf("b[%d] = %d\n", i, b[i]);
38     //printf("%d\n", (int)a.size());
39     //printf("b[n]");
40     //for (int i = 0; i < a.size(); i++) {
41     //    if (!i) putchar('='); else putchar('+');
42     //    printf("%d*b[n-%d]", a[i], i + 1);
43     //}
44     //puts("");
45     ll ans = 0, pnt = 0;
46     int k = a.size();
47     for (int i = 0; i < k; i++) {
48         __md[k - 1 - i] = -a[i];
49     }
50     __md[k] = 1;
51     Md.clear();
52     for (int i = 0; i < k; i++) {
53         if (__md[i]) {
54             Md.push_back(i);
55         }
56         res[i] = base[i] = 0;
57     }
58     res[0] = 1;
59     while ((1LL << pnt) <= n) pnt++;
60     for (int p = pnt; p >= 0; p--) {
61         mul(res, res, k);
62         if ((n >> p) & 1) {
63             for (int i = k - 1; i >= 0; i--)
64                 res[i + 1] = res[i];
65             res[0] = 0;
66             for (int j = 0; j < Md.size(); j++) {
67                 res[Md[j]] = (res[Md[j]] - res[k] * __md[Md[j]]) % MOD;

```

```

68         }
69     }
70 }
71 for (int i = 0; i < k; i++)
72     ans = (ans + res[i] * b[i]) % MOD;
73 return (ans < 0 ? ans + MOD : ans);
74 }
75 vector<int> BM(vector<int> s) { // O(n^2)
76     vector<int> C(1, 1), B(1, 1);
77     int L = 0, m = 1, b = 1;
78     for (int n = 0; n < (int)s.size(); n++) {
79         ll d = 0;
80         for (int i = 0; i <= L; i++)
81             d = (d + (ll)C[i] * s[n - i]) % MOD;
82         if (!d) {
83             ++m;
84         } else if (2 * L <= n) {
85             auto T = C;
86             ll c = MOD - d * powmod(b, MOD - 2) % MOD;
87             while (C.size() < B.size() + m)
88                 C.push_back(0);
89             for (int i = 0; i < B.size(); i++)
90                 C[i + m] = (C[i + m] + c * B[i]) % MOD;
91             L = n + 1 - L; B = T; b = d; m = 1;
92         } else {
93             ll c = MOD - d * powmod(b, MOD - 2) % MOD;
94             while (C.size() < B.size() + m) C.push_back(0);
95             for (int i = 0; i < B.size(); i++) {
96                 C[i + m] = (C[i + m] + c * B[i]) % MOD;
97             }
98             ++m;
99         }
100     }
101     return C;
102 }
103 int work(vector<int> a, ll n) { // 这里的n不是数组大小 是求数列第n项的值
104     vector<int> c = BM(a); // 求第n项的复杂度为 O(k^2 log n) k是递推
    数列大小
105     c.erase(c.begin());
106     for (int i = 0; i < c.size(); i++)
107         c[i] = (MOD - c[i]) % MOD;
108     return solve(n, c, vector<int>(a.begin(), a.begin() + (int)c.size()))
    );

```

```

109     }
110 }

```

3 线性代数

3.1 矩阵

```

1  template<typename T> struct matrix {
2      int n, m;
3      vector<vector<T>> a;
4      matrix(int n_, int m_, int val = 0) : n(n_), m(m_), a(n_, vector<T>(m_,
        val)) {}
5      matrix(vector<vector<T>>& mat) : n(mat.size()), m(mat[0].size()), a(mat)
        {}
6      vector<T>& operator [] (int k) { return this->a[k]; }
7      matrix operator + (matrix& k) { return matrix(*this) += k; }
8      matrix operator - (matrix& k) { return matrix(*this) -= k; }
9      matrix operator * (matrix& k) { return matrix(*this) *= k; }
10     matrix& operator += (matrix& mat) {
11         assert(n == mat.n);
12         assert(m == mat.m);
13         for (int i = 0; i < n; i++) {
14             for (int j = 0; j < m; j++) {
15                 a[i][j] += mat[i][j];
16             }
17         }
18         return *this;
19     }
20     matrix& operator -= (matrix& mat) {
21         assert(n == mat.n);
22         assert(m == mat.m);
23         for (int i = 0; i < n; i++) {
24             for (int j = 0; j < m; j++) {
25                 a[i][j] -= mat[i][j];
26             }
27         }
28         return *this;
29     }
30     void input() {
31         for (int i = 0; i < n; i++) {
32             for (int j = 0; j < m; j++) {
33                 cin >> a[i][j];
34             }

```

```

35     }
36 }
37 void output() {
38     for (int i = 0; i < n; i++) {
39         for (int j = 0; j < m; j++) {
40             cout << a[i][j] << " \n"[j == m - 1];
41         }
42     }
43 }
44 matrix& operator *= (matrix& mat) {
45     assert(m == mat.n);
46     int x = n, y = mat.m, z = m;
47     matrix<T> c(x, y);
48     for (int i = 0; i < x; i++) {
49         for (int k = 0; k < z; k++) {
50             T r = a[i][k];
51             for (int j = 0; j < y; j++) {
52                 c[i][j] += mat[k][j] * r;
53             }
54         }
55     }
56     return *this = c;
57 }
58 matrix unit(int n_) {
59     matrix res(n_, n_);
60     for (int i = 0; i < n_; i++)
61         res[i][i] = 1;
62     return res;
63 }
64 matrix power(long long k) {
65     assert(n == m);
66     auto res = unit(n);
67     while (k > 0) {
68         if (k & 1) res *= (*this);
69         (*this) *= (*this);
70         k >>= 1;
71     }
72     return res;
73 }
74 matrix inverse() {
75     assert(n == m);
76     auto b = unit(n);
77     for (int i = 0; i < n; i++) {

```

```

78         if (a[i][i] == 0) return matrix(0, 0);
79         T f = T(1) / a[i][i];
80         for (int j = 0; j < n; j++) a[i][j] *= f, b[i][j] *= f;
81         for (int j = 0; j < n; j++) {
82             if (i == j) continue;
83             T g = a[j][i];
84             for (int k = 0; k < n; k++) {
85                 a[j][k] -= g * a[i][k];
86                 b[j][k] -= g * b[i][k];
87             }
88         }
89     }
90     return b;
91 }
92 bool empty() { return (!n && !m); }
93 };

```

3.2 高斯-约旦消元法

```

1  /*
2   * 高斯-约旦消元法
3   * 可以修改为解异或方程组 修改策略为
4   * a+b -> a^b
5   * a-b -> a^b
6   * a*b -> a&b
7   * a/b -> a*(b==1)
8   * */
9  constexpr double eps = 1e-7;
10 double a[SIZE][SIZE], ans[SIZE];
11 void gauss(int n) {
12     vector<bool> vis(n, false);
13     for (int i = 0; i < n; i++) {
14         for (int j = 0; j < n; j++) {
15             if (vis[j]) continue;
16             if (fabs(a[j][i]) > eps) {
17                 vis[i] = true;
18                 for (int k = 0; k <= n; k++) swap(a[i][k], a[j][k]);
19                 break;
20             }
21         }
22         if (fabs(a[i][i]) < eps) continue;
23         for (int j = 0; j <= n; j++) {
24             if (i != j && fabs(a[j][i]) > eps) {

```



```

25         double res = a[j][i] / a[i][i];
26         for (int k = 0; k <= n; k++) a[j][k] -= a[i][k] * res;
27     }
28 }
29 }
30 }
31
32 int check(int n) { // 解系检测
33     int status = 1;
34     for (int i = n - 1; i >= 0; i--) {
35         if (fabs(a[i][i]) < eps && fabs(a[i][n]) > eps) return -1; // 无解
36         if (fabs(a[i][i]) < eps && fabs(a[i][n]) < eps) status = 0; // 无穷
           解
37         ans[i] = a[i][n] / a[i][i];
38         if (fabs(ans[i]) < eps) ans[i] = 0;
39     }
40     return status; // 唯一解或无穷解
41 }

```

3.3 高斯消元法-bitset

```

1  constexpr int SIZE = 1001;
2  bitset<SIZE> a[SIZE];
3  int ans[SIZE];
4  void gauss(int n) { // bitset版高斯消元 用于求解异或线性方程组
5      bitset<SIZE> vis;
6      for (int i = 0; i < n; i++) {
7          for (int j = 0; j < n; j++) {
8              if (vis[j]) continue;
9              if (a[j][i]) {
10                 vis.set(i);
11                 swap(a[i], a[j]);
12                 break;
13             }
14         }
15         if (!a[i][i]) continue;
16         for (int j = 0; j <= n; j++) {
17             if (i != j && (a[j][i] & a[i][i])) {
18                 a[j] ^= a[i];
19             }
20         }
21     }
22 }

```

3.4 线性基

```

1 struct linearBasis {
2     /* 线性基性质:
3      * 1. 若  $a[i] \neq 0$  (即主元  $i$  存在)
4      *   则线性基中只有  $a[i]$  的第  $i$  位是 1 (只存在一个主元)
5      *   且此时  $a[i]$  的最高位就是第  $i$  位
6      * 2. 将数组  $a$  插入线性基 假设有  $|B|$  个元素成功插入
7      *   则数组  $a$  中每个不同的子集异或和都出现  $2^{(n-|B|)}$  次
8      * */
9     static const int MAXL = 60;
10    long long a[MAXL + 1];
11    int id[MAXL + 1];
12    int zero;
13    /* 0 的标志位 =1 则表示 0 可以被线性基表示出来
14     * 求第  $k$  大元素时 需要注意题意中线性基为空时是否可以表示 0
15     * 默认不可以表示 有必要时进行修改即可
16     * */
17    linearBasis() {
18        zero = 0;
19        fill(a, a + MAXL + 1, 0);
20    }
21    long long& operator[] (int k) { return a[k]; }
22    bool insert(long long x) {
23        for (int j = MAXL; ~j; j--) {
24            if (!(x & (1LL << j))) { // 如果  $x$  的第  $j$  位为 0, 则跳过
25                continue;
26            }
27            if (a[j]) { // 如果  $a[j] \neq 0$ , 则用  $a[j]$  消去  $x$  的第  $j$  位上的 1
28                x ^= a[j];
29            } else { // 找到插入位置
30                for (int k = 0; k < j; k++) {
31                    if (x & (1LL << k)) { // 如果  $x$  存在某个低位线性基的主元  $k$ 
32                        则消去
33                        x ^= a[k];
34                    }
35                }
36                for (int k = j + 1; k <= MAXL; k++) {
37                    if (a[k] & (1LL << j)) { // 如果某个高位线性基存在主元  $j$ 
38                        则消去
39                        a[k] ^= x;
40                    }
41                }
42                a[j] = x;

```

```

41         return true;
42     }
43 }
44 zero = 1;
45 return false;
46 }
47 long long query_max() { // 最大值
48     long long res = 0;
49     for (int i = MAXL; ~i; i--) {
50         res ^= a[i];
51     }
52     return res;
53 }
54 long long query_max(long long x) { // 线性基异或x的最大值
55     for (int i = MAXL; ~i; i--) {
56         if ((x ^ a[i]) > x) {
57             x ^= a[i];
58         }
59     }
60     return x;
61 }
62 long long query_min() { // 最小值
63     for (int i = 0; i < MAXL; i++) {
64         if (a[i]) {
65             return a[i];
66         }
67     }
68     return -1; // 线性基为空
69 }
70 long long query_min(long long x) { // 线性基异或x的最小值
71     for (int i = MAXL; ~i; i--) {
72         if ((x ^ a[i]) < x) {
73             x ^= a[i];
74         }
75     }
76     return x;
77 }
78 int count(long long x) { // 元素 x 能否被线性基内元素表示
79     int res = 0;
80     vector<long long> b(MAXL + 1);
81     for (int i = 0; i <= MAXL; i++) {
82         b[i] = a[i];
83     }

```

```

84         res = this->insert(x);
85         for (int i = 0; i <= MAXL; i++) {
86             a[i] = b[i];
87         }
88         return !res; // 成功插入则无法表示 失败则可以表示
89     }
90     int size() { // 线性基有效元素数量
91         int res = 0;
92         for (int i = 0; i <= MAXL; i++) {
93             if (a[i]) {
94                 res++;
95             }
96         }
97         return res;
98     }
99     long long kth_element(long long k) { // 第k大元素
100         vector<long long> b;
101         for (int i = 0; i <= MAXL; i++) {
102             if (a[i]) {
103                 b.push_back(a[i]);
104             }
105         }
106         if (zero) {
107             if (--k == 0) {
108                 return 0;
109             }
110         }
111         if (k >= (1LL << this->size())) { // k超过了线性基可以表示的最大数量
112             return -1;
113         }
114         long long res = 0;
115         for (int i = 0; i <= MAXL; i++) {
116             if (k & (1LL << i)) {
117                 res ^= b[i];
118             }
119         }
120         return res;
121     }
122     long long rank(long long x) { // 元素x在线性基内的排名（默认不考虑0）
123         vector<long long> b;
124         for (int i = 0; i <= MAXL; i++) {
125             if (a[i]) {
126                 b.push_back(1LL << i);

```

```

127     }
128 }
129 long long res = 0;
130 for (int i = 0; i < (int)b.size(); i++) {
131     if (x & b[i]) {
132         res |= (1LL << i);
133     }
134 }
135 return res;
136 }
137 void clear() {
138     zero = 0;
139     fill(a, a + MAXL + 1, 0);
140 }
141 };

```

3.5 矩阵树定理

```

1  /*
2  * 矩阵树定理
3  * 有向图：若 u->v 有一条权值为 w 的边 基尔霍夫矩阵 a[v][v] += w, a[v][u] -=
      w
4  * 生成树数量为除去 根所在行和列 后的n-1阶行列式的值
5  * 无向图：若 u->v 有一条权值为 w 的边 基尔霍夫矩阵 a[v][v] += w, a[v][u] -=
      w, a[u][u] += w, a[u][v] -= w
6  * 生成树数量为除去 任意一行和列 后的n-1阶行列式的值
7  * 无权图则边权默认为1
8  * */
9  typedef long long ll;
10 typedef unsigned long long u64;
11 int a[SIZE][SIZE];
12 int gauss(int a[][SIZE], int n) { // 任意模数求行列式 O(n^2(n + log(mod)))
13     int ans = 1;
14     for (int i = 1; i <= n; i++) {
15         int* x = 0, * y = 0;
16         for (int j = i; j <= n; j++) {
17             if (a[j][i] && (x == NULL || a[j][i] < x[i])) {
18                 x = a[j];
19             }
20         }
21         if (x == 0) {
22             return 0;
23         }

```

```

24     for (int j = i; j <= n; j++) {
25         if (a[j] != x && a[j][i]) {
26             y = a[j];
27             for (;;) {
28                 int v = md - y[i] / x[i], k = i;
29                 for (; k + 3 <= n; k += 4) {
30                     y[k + 0] = (y[k + 0] + u64(x[k + 0]) * v) % md;
31                     y[k + 1] = (y[k + 1] + u64(x[k + 1]) * v) % md;
32                     y[k + 2] = (y[k + 2] + u64(x[k + 2]) * v) % md;
33                     y[k + 3] = (y[k + 3] + u64(x[k + 3]) * v) % md;
34                 }
35                 for (; k <= n; ++k) {
36                     y[k] = (y[k] + u64(x[k]) * v) % md;
37                 }
38                 if (!y[i]) break;
39                 swap(x, y);
40             }
41         }
42     }
43     if (x != a[i]) {
44         for (int k = i; k <= n; k++) {
45             swap(x[k], a[i][k]);
46         }
47         ans = md - ans;
48     }
49     ans = 1LL * ans * a[i][i] % md;
50 }
51 return ans;
52 }

```

3.6 LGV 引理

一般用于有向无环图不相交路径计数（常见于网格图）。

$$M = \begin{bmatrix} e(A_1, B_1) & e(A_1, B_2) & \cdots & e(A_1, B_n) \\ e(A_2, B_1) & e(A_2, B_2) & \cdots & e(A_2, B_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(A_n, B_1) & e(A_n, B_2) & \cdots & e(A_n, B_n) \end{bmatrix} \det(M) = \sum_{S:A \rightarrow B} (-1)^{N(\sigma(S))} \prod_{i=1}^n \omega(S_i)$$

4 组合数学

4.1 组合数预处理

```

1 namespace BinomialCoefficient {

```

```

2   vector<int> fac, ifac, iv;
3   // 组合数预处理 option=1则还会预处理线性逆元
4   void prepareFactorials(int maximum = 1000000, int option = 0) {
5       fac.assign(maximum + 1, 0);
6       ifac.assign(maximum + 1, 0);
7       fac[0] = ifac[0] = 1;
8       if (option) { // O(3n)
9           iv.assign(maximum + 1, 1);
10          for (int p = 2; p * p <= MOD; p++)
11              assert(MOD % p != 0);
12          for (int i = 2; i <= maximum; i++)
13              iv[i] = mul(iv[MOD % i], (MOD - MOD / i));
14          for (int i = 1; i <= maximum; i++) {
15              fac[i] = mul(i, fac[i - 1]);
16              ifac[i] = mul(iv[i], ifac[i - 1]);
17          }
18      } else { // O(2n + log(MOD))
19          for (int i = 1; i <= maximum; i++)
20              fac[i] = mul(fac[i - 1], i);
21          ifac[maximum] = inv(fac[maximum]);
22          for (int i = maximum; i; i--)
23              ifac[i - 1] = mul(ifac[i], i);
24      }
25  }
26  inline int binom(int n, int m) {
27      if (n < m || n < 0 || m < 0) return 0;
28      return mul(fac[n], mul(ifac[m], ifac[n - m]));
29  }
30 }

```

4.2 卢卡斯定理

对于质数 p , 有:

$$\binom{n}{m} \bmod p = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \cdot \binom{n \bmod p}{m \bmod p} \bmod p$$

```

1   long long p; // C(n, m) % p
2   long long lucas(long long n, long long m) { // O(p + log(n))
3       if (m == 0) return 1;
4       return (binom(n % p, m % p) * lucas(n / p, m / p)) % p; // binom(x, y)
           在线性预处理组合数之后是 O(1) 的
5   }

```

4.3 小球盒子模型

设有 n 个球， k 个盒子：

1. 球之间互不相同，盒子之间互不相同，可以空盒

根据乘法原理，答案就是 k^n 。

2. 球之间互不相同，盒子之间互不相同，每个盒子至多装一个球

相当于每个球找一个没有被选过的盒子放进去，答案是 k^n ，即 $k(k-1)\cdots(k-n+1)$ 。

3. 球之间互不相同，盒子之间互不相同，每个盒子至少装一个球

可以先把盒子视为相同： n 个球放进 k 个相同盒子、不能空盒，这就是第二类斯特林数 S_n^k 的定义。最后由于盒子不同，再乘上一个排列数，因此答案就是 $k!S_n^k$ 。

4. 球之间互不相同，盒子全部相同，可以空盒

枚举非空盒子数量，相当于第二类斯特林数一行求和： $\sum_{i=1}^k S_n^i$ 。

5. 球之间互不相同，盒子全部相同，每个盒子至多装一个球

因为盒子相同，不论怎么放都是一样的，答案是 $[n \leq k]$ （这是一个布尔运算式，若 $n \leq k$ 成立则取 1，否则 0）。

6. 球之间互不相同，盒子全部相同，每个盒子至少装一个球

就是第二类斯特林数 S_n^k 。

7. 球全部相同，盒子之间互不相同，可以空盒

隔板法经典应用， $n+k-1$ 个球选 $k-1$ 个板，因此答案是 $\binom{n+k-1}{k-1}$ 。

8. 球全部相同，盒子之间互不相同，每个盒子至多装一个球

盒子不同，相当于要选出 n 个盒子装球，因此答案是 $\binom{n}{k}$ 。

9. 球全部相同，盒子之间互不相同，每个盒子至少装一个球

隔板法经典应用， $n-1$ 个空隙选 $k-1$ 个插板（可以看作是情况 7 时每个盒子里都预先加入一个球），因此答案是 $\binom{n-1}{k-1}$ 。

10. 球全部相同，盒子全部相同，可以空盒

定义划分数 $p_{n,k}$ 表示将自然数 n 拆成 k 份的方案数，那么本例的结论就是 $p_{n,k}$ 。

这个问题有一个经典递推式： $p(n, k) = p(n, k-1) + p(n-k, k)$ 。意义是将 j 个自然数 +1 或者加入一个 0。下面给出一个代码实现：

```

1 p[0][0] = 1;
2 for (int i = 1; i <= n; i++) {
3     p[0][i] = 1;
4     for (int j = 1; j <= m; j++) {
5         if (i >= j) {
6             p[i][j] = add(p[i][j-1], p[i-j][j]);
7         } else {
8             p[i][j] = p[i][j-1];

```


9		}
10	}	
11	}	

11. 球全部相同，盒子全部相同，每个盒子至多装一个球

和情况 5 一致，就是 $[n \leq k]$ 。

12. 球全部相同，盒子全部相同，每个盒子至少装一个球

显然也是一个划分数： $p_{n-k,k}$ 。

4.4 斯特林数

4.4.1 第一类斯特林数

第一类斯特林数 $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ 表示将 n 个不同元素划分入 k 个非空圆排列的方案数。

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] + (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right]$$

边界是 $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] = 1$ 。

第一类斯特林数三角形，从 $s(1, 1)$ 开始：

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6769	1960	322	28	1		
40320	109584	118124	67284	22449	4536	546	36	1	
362880	1026576	1172700	723680	269325	63273	9450	870	45	1

4.4.2 第二类斯特林数

第二类斯特林数 $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ 表示将 n 个不同元素划分为 k 个非空子集的方案数。

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\} + k \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$$

边界是 $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ 。

基于容斥原理的递推方法：

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

第二类斯特林数三角形，从 $S(1, 1)$ 开始：

1									
1	1								
1	3	1							
1	7	6	1						
1	15	25	10	1					
1	31	90	65	15	1				
1	63	301	350	140	21	1			
1	127	966	1701	1050	266	28	1		
1	255	3025	7770	6951	2646	462	36	1	
1	511	9330	34105	42525	22827	5880	750	45	1

5 博弈论

5.1 SG 定理

一个状态的 SG 函数值是它所有后继状态的 MEX，当 MEX 为 0 时该状态后手必胜，反之先手必胜。当某个游戏由多个独立的子游戏组成时，所有子游戏的 SG 函数异或和为 0 时后手必胜，否则先手必胜。

在 Nim 博弈中的简单应用：显然第 i 堆石子的 SG 函数值就是它的石子数量 a_i ，每一堆石子都是一个独立的子游戏，因此 $a_0 \oplus a_1 \oplus \cdots \oplus a_{n-1} = 0$ 时后手必胜，否则先手必胜。

5.2 Bash 博弈

一共 N 个石子，先后手轮流取石子，每次最多取 M 个石子，先取完者获胜。

先手必胜： $N \pmod{M+1} \neq 0$ ；必败： $N \pmod{M+1} = 0$ 。

5.3 Nim-K 博弈

有 N 堆石子，先后手轮流取石子，每次最多可以选择 K 堆石子，被选中的每堆石子都可以取任意个，先取完者获胜。

把 N 堆石子的石子数量都用二进制表示，对于二进制意义下的每一位，如果 1 的数量在模 $K+1$ 意义下全部为 0，则先手必败。

5.4 Anti-Nim 博弈

规则和 Nim 博弈一致，但是获胜条件改为：不能取石子的一方获胜。

满足以下任意条件，则先手必胜：

1. 所有堆的石子数量 ≤ 1 并且异或和 $= 0$ 。
2. 至少存在一堆石子个数 ≥ 2 并且异或和 $\neq 0$ 。

5.5 Anti-SG 博弈

SG 博弈中最先不能行动的一方获胜。满足以下任意条件，则先手必胜：

1. SG 为 0 并且每一个游戏的 SG 都不超过 1。
2. SG 不为 0 并且至少有一个游戏的 SG 大于 1。

5.6 阶梯博弈

有 N 个阶梯（下标从 0 开始），每个阶梯上有一定数量的石子，先后手轮流行动，每次可以从一个阶梯上拿走任意个石子放到下一层阶梯上，最先不能操作者失败。

SG 函数为奇数阶梯上的石子的异或和，如果移动偶数层的石子到奇数层，对手一定可以继续移动这些石子到偶数层使得 SG 不变。

5.7 Wythoff 博弈

有两堆石子，石子数量分别为 A, B ，每次可以从一堆或者两堆里拿走相同数量的石子，最先不能取石子的人输。

必败态为： $A_k = \lfloor \frac{k(1+\sqrt{5})}{2} \rfloor, B_k = a_k + k$ ，假设 $A_k < B_k$ 。

5.8 树上删边博弈

给定一棵 N 个点的有根树，两人轮流操作，每次删除树上的一条边，然后删除所有不与根节点联通的部分，最先不能进行删除操作的人失败。

所有叶子节点的 SG 函数值为 0，非叶子节点的 SG 值为其所有子节点（SG 值 +1）的异或和。

5.9 无向图删边博弈

和树上删边博弈的规则一样，不过给出的是一个无向图。

可以将图中的任意一个偶环缩成一个新点，任意一个奇环缩成一个新点加一个新边；所有连到原先环上的边全部改为与新点相连。

5.10 二分图博弈

给出一张二分图和起点 S ，先后手轮流操作，每次只能从当前点（最开始是点 S ）移动到一个相邻的点，且每个点只能被遍历一次，无法继续移动的人输。

先手必胜：二分图的最大匹配一定包含起点 S ；先手必败：二分图的最大匹配不一定包含起点 S 。

6 图论

6.1 并查集

```

1 struct dsu {
2 private:
3     // number of nodes
4     int n;
5     // root node: -1 * component size
6     // otherwise: parent
7     std::vector<int> pa;
8 public:
9     dsu(int n_ = 0) : n(n_), pa(n_, -1) {}
10    // find node x's parent
11    int find(int x) {

```

```

12     return pa[x] < 0 ? x : pa[x] = find(pa[x]);
13 }
14 // merge node x and node y
15 // if x and y had already in the same component, return false, otherwise
    return true
16 // Implement (union by size) + (path compression)
17 bool unite(int x, int y) {
18     int xr = find(x), yr = find(y);
19     if (xr != yr) {
20         if (-pa[xr] < -pa[yr]) std::swap(xr, yr);
21         pa[xr] += pa[yr];
22         pa[yr] = xr; // y -> x
23         return true;
24     }
25     return false;
26 }
27 // size of the connected component that contains the vertex x
28 int size(int x) {
29     return -pa[find(x)];
30 }
31 };

```

6.2 最短路

```

1 namespace Dijkstra {
2 #define ll long long
3     static constexpr ll INF = 1e18;
4     int n, m; // 点数 边数
5     struct edge {
6         int to; // 点
7         ll val; // 边权
8         edge(int to_ = 0, ll val_ = 0) : to(to_), val(val_) {}
9         bool operator < (const edge& k) const { return val > k.val; }
10 };
11     vector<vector<edge>> g;
12     void init() { // 建图操作需要根据题意修改
13         cin >> n >> m;
14         g.resize(n);
15         for (int i = 0; i < m; i++) {
16             int u, v, w;
17             cin >> u >> v >> w;
18             —u, —v;
19             g[u].push_back(edge(v, w));

```

```

20     }
21 }
22 ll dijkstra(int s, int t) { // 最短路
23     vector<ll> dis(n, INF);
24     vector<bool> vis(n, false);
25     dis[s] = 0;
26     priority_queue<edge> pq;
27     pq.push(edge(s, 0));
28     while (!pq.empty()) {
29         auto top = pq.top();
30         pq.pop();
31         if (!vis[top.to]) {
32             vis[top.to] = true;
33             for (auto nxt : g[top.to]) {
34                 if (!vis[nxt.to] && dis[nxt.to] > nxt.val + dis[top.to])
35                     {
36                         dis[nxt.to] = nxt.val + dis[top.to];
37                         pq.push(edge(nxt.to, dis[nxt.to]));
38                     }
39             }
40         }
41         return dis[t];
42     }
43 #undef ll
44 }

```

6.3 Prim 最小生成树

```

1  /*
2   * Prim Algorithm  $O(N^2 + M)$ 
3   * g是一个带边权的邻接表 下标从0开始
4   * 如果图不连通则返回1
5   */
6  long long prim(vector<vector<pair<int, int>>>& g) {
7      int n = g.size();
8      long long res = 0;
9      vector<int> dis(n, INT_MAX), vis(n);
10     dis[0] = 0;
11     for (int i = 0; i < n; i++) {
12         int mn = INT_MAX, p = -1;
13         for (int j = 0; j < n; j++) {
14             if (!vis[j] && dis[j] < mn) {

```

```

15         mn = dis[j];
16         p = j;
17     }
18 }
19 if (p == -1) {
20     return -1;
21 }
22 vis[p] = true;
23 res += dis[p];
24 for (auto& [j, k] : g[p]) {
25     if (!vis[j] && k < dis[j]) {
26         dis[j] = k;
27     }
28 }
29 }
30 return res;
31 }

```

6.4 最小树形图

```

1 namespace ZL {
2     // a 尽量开大，之后的边都塞在这个里面
3     const int N = 100010, M = 100010, inf = 1e9;
4     struct edge {
5         int u, v, w, use, id;
6         edge(int u_ = 0, int v_ = 0, int w_ = 0, int use_ = 0, int id_ = 0)
7             : u(u_), v(v_), w(w_), use(use_), id(id_) {}
8     } b[M], a[2000100];
9     int n, m, ans, pre[N], id[N], vis[N], root, In[N], h[N], len, way[M];
10    // 从root 出发能到达每一个点的最小树形图
11    // 先调用init 然后把边add 进去，需要方案就getway, way[i] 为1 表示使用
12    // 最小值保存在ans
13    void init(int _n, int _root) { // 点数 根节点
14        n = _n; m = 0; b[0].w = inf; root = _root;
15    }
16    void add(int u, int v, int w) {
17        m++;
18        b[m] = edge(u, v, w, 0, m);
19        a[m] = b[m];
20    }
21    int work() {
22        len = m;
23        for (;;) {

```

```

24     for (int i = 1; i <= n; i++) { pre[i] = 0; In[i] = inf; id[i] =
25         0; vis[i] = 0; h[i] = 0; }
26     for (int i = 1; i <= m; i++) {
27         if (b[i].u != b[i].v && b[i].w < In[b[i].v]) {
28             pre[b[i].v] = b[i].u; In[b[i].v] = b[i].w; h[b[i].v] = b
29                 [i].id;
30         }
31     }
32     for (int i = 1; i <= n; i++) if (pre[i] == 0 && i != root)
33         return 0;
34     int cnt = 0; In[root] = 0;
35     for (int i = 1; i <= n; i++) {
36         if (i != root) a[h[i]].use++; int now = i; ans += In[i];
37         while (vis[now] == 0 && now != root) { vis[now] = i; now =
38             pre[now]; }
39         if (now != root && vis[now] == i) {
40             cnt++; int kk = now;
41             while (1) {
42                 id[now] = cnt; now = pre[now];
43                 if (now == kk) break;
44             }
45         }
46     }
47     if (cnt == 0) return 1;
48     for (int i = 1; i <= n; i++) if (id[i] == 0) id[i] = ++cnt;
49     // 缩环，每一条接入的边都会茶包原来接入的那条边，所以要调整边权
50     // 新加的边是u，茶包的边是v
51     for (int i = 1; i <= m; i++) {
52         int k1 = In[b[i].v], k2 = b[i].v;
53         b[i].u = id[b[i].u];
54         b[i].v = id[b[i].v];
55         if (b[i].u != b[i].v) {
56             b[i].w -= k1; a[++len].u = b[i].id; a[len].v = h[k2]; b[
57                 i].id = len;
58         }
59     }
60     n = cnt; root = id[root];
61     return 1;
62 }
63 void getway() {
64     for (int i = 1; i <= m; i++) way[i] = 0;
65     for (int i = len; i > m; i--) { a[a[i].u].use += a[i].use; a[a[i].v

```

```

        ].use -= a[i].use; }
62     for (int i = 1; i <= m; i++) way[i] = a[i].use;
63 }
64 }

```

6.5 最近公共祖先

```

1  constexpr int SIZE = 200010;
2  constexpr int DEPTH = 21; // 最大深度  $2^{\text{DEPTH}} - 1$ 
3  int pa[SIZE][DEPTH], dep[SIZE];
4  vector<int> g[SIZE]; // 邻接表
5  void dfs(int rt, int fin) { // 预处理深度和祖先
6      pa[rt][0] = fin;
7      dep[rt] = dep[pa[rt][0]] + 1; // 深度
8      for (int i = 1; i < DEPTH; i++) { // rt 的  $2^i$  祖先等价于 rt 的  $2^{(i-1)}$ 
          祖先的  $2^{(i-1)}$  祖先
9          pa[rt][i] = pa[pa[rt][i-1]][i-1];
10     }
11     int sz = g[rt].size();
12     for (int i = 0; i < sz; ++i) {
13         if (g[rt][i] == fin) continue;
14         dfs(g[rt][i], rt);
15     }
16 }
17
18 int LCA(int x, int y) {
19     if (dep[x] > dep[y]) swap(x, y);
20     int dif = dep[y] - dep[x];
21     for (int j = 0; dif; ++j, dif >>= 1) {
22         if (dif & 1) {
23             y = pa[y][j]; // 先跳到同一高度
24         }
25     }
26     if (y == x) return x;
27     for (int j = DEPTH - 1; j >= 0 && y != x; j--) { // 从底往上跳
28         if (pa[x][j] != pa[y][j]) { // 如果当前祖先不相等 我们就需要更新
29             x = pa[x][j];
30             y = pa[y][j];
31         }
32     }
33     return pa[x][0];
34 }

```


6.6 欧拉回路

```

1 // UOJ117 返回欧拉回路的边集（负数代表走了反向边）
2 #include <bits/stdc++.h>
3 using namespace std;
4 // 有向图欧拉回路 任意点的入度=出度
5 vector<int> directed_euler_circuit(int n, int m, const vector<vector<pair<
    int, int>>>& g) {
6     vector<int> d(n);
7     for (const auto& A : g) {
8         for (auto p : A) {
9             d[p.first]++;
10        }
11    }
12    for (int i = 0; i < n; i++) {
13        if (g[i].size() != d[i]) {
14            return {};
15        }
16    }
17    vector<vector<pair<int, int>>::const_iterator> it(n);
18    for (int i = 0; i < n; i++) it[i] = g[i].begin();
19    vector<int> vis(m + 1), p;
20    function<void(int)> dfs = [&](int u) {
21        for (auto& nxt = it[u]; nxt != g[u].end(); ) {
22            if (!vis[nxt->second]) {
23                vis[nxt->second] = 1;
24                int v = nxt->second;
25                dfs(nxt->first);
26                p.push_back(v);
27            } else {
28                nxt = next(nxt);
29            }
30        }
31    };
32    for (int i = 0; i < n; i++) {
33        if (!g[i].empty()) {
34            dfs(i);
35            break;
36        }
37    }
38    if (p.size() < m) return {};
39    reverse(p.begin(), p.end());
40    return p;
41 }

```

```

42 // 无向图欧拉回路 任意点的度数为偶数
43 vector<int> undirected_euler_circuit(int n, int m, const vector<vector<pair<
    int, int>>>& g) {
44     for (const auto& A : g) {
45         if (A.size() & 1) {
46             return {};
47         }
48     }
49     vector<vector<pair<int, int>>>::const_iterator> it(n);
50     for (int i = 0; i < n; i++) it[i] = g[i].begin();
51     vector<int> vis(m + 1), p;
52     function<void(int)> dfs = [&](int u) {
53         for (auto& nxt = it[u]; nxt != g[u].end(); ) {
54             if (!vis[abs(nxt->second)]) {
55                 vis[abs(nxt->second)] = 1;
56                 int v = nxt->second;
57                 dfs(nxt->first);
58                 p.push_back(v);
59             } else {
60                 nxt = next(nxt);
61             }
62         }
63     };
64     for (int i = 0; i < n; i++) {
65         if (!g[i].empty()) {
66             dfs(i);
67             break;
68         }
69     }
70     if (p.size() < m) return {};
71     reverse(p.begin(), p.end());
72     return p;
73 }
74
75 int main() {
76     ios::sync_with_stdio(false);
77     cin.tie(nullptr);
78     cout.tie(nullptr);
79     int t, n, m;
80     cin >> t >> n >> m; // t=1是无向图 t=-1是有向图
81     vector<vector<pair<int, int>>> G(n + 1);
82     for (int i = 1, u, v; i <= m; i++) {
83         cin >> u >> v;

```

```

84     if (t == 1) G[v].push_back({ u, -i });
85     G[u].push_back({ v, i });
86 }
87 auto p = t == 1 ? undirected_euler_circuit(n + 1, m, G) :
    directed_euler_circuit(n + 1, m, G);
88 if (p.size() == m) {
89     cout << "YES\n";
90     for (int x : p)
91         cout << x << " \n"[x == p.back()];
92 } else {
93     cout << "NO\n";
94 }
95 return 0;
96 }

```

6.7 强连通分量

```

1 namespace SCC {
2     // Compressed Sparse Row
3     template <class E> struct csr {
4         std::vector<int> start;
5         std::vector<E> elist;
6         explicit csr(int n, const std::vector<std::pair<int, E>>& edges)
7             : start(n + 1), elist(edges.size()) {
8             for (auto e : edges) {
9                 start[e.first + 1]++;
10            }
11            for (int i = 1; i <= n; i++) {
12                start[i] += start[i - 1];
13            }
14            auto counter = start;
15            for (auto e : edges) {
16                elist[counter[e.first]++] = e.second;
17            }
18        }
19    };
20
21    struct scc_graph {
22    public:
23        explicit scc_graph(int n) : _n(n) {}
24
25        int num_vertices() { return _n; }
26    };

```

```

27 void add_edge(int from, int to) { edges.push_back({ from, {to} }); }
28
29 // @return pair of (# of scc, scc id)
30 std::pair<int, std::vector<int>> scc_ids() {
31     auto g = csr<edge>(_n, edges);
32     int now_ord = 0, group_num = 0;
33     std::vector<int> visited, low(_n), ord(_n, -1), ids(_n);
34     visited.reserve(_n);
35     auto dfs = [&](auto self, int v) -> void {
36         low[v] = ord[v] = now_ord++;
37         visited.push_back(v);
38         for (int i = g.start[v]; i < g.start[v + 1]; i++) {
39             auto to = g.elist[i].to;
40             if (ord[to] == -1) {
41                 self(self, to);
42                 low[v] = std::min(low[v], low[to]);
43             } else {
44                 low[v] = std::min(low[v], ord[to]);
45             }
46         }
47         if (low[v] == ord[v]) {
48             while (true) {
49                 int u = visited.back();
50                 visited.pop_back();
51                 ord[u] = _n;
52                 ids[u] = group_num;
53                 if (u == v) break;
54             }
55             group_num++;
56         }
57     };
58     for (int i = 0; i < _n; i++) {
59         if (ord[i] == -1) dfs(dfs, i);
60     }
61     for (auto& x : ids) {
62         x = group_num - 1 - x;
63     }
64     return { group_num, ids };
65 }
66
67 // O(N + M)
68 // It returns the list of the SCC in topological order.
69 std::vector<std::vector<int>> scc() {

```

```

70     auto ids = scc_ids();
71     int group_num = ids.first;
72     std::vector<int> counts(group_num);
73     for (auto x : ids.second) counts[x]++;
74     std::vector<std::vector<int>>> groups(ids.first);
75     for (int i = 0; i < group_num; i++) {
76         groups[i].reserve(counts[i]);
77     }
78     for (int i = 0; i < _n; i++) {
79         groups[ids.second[i]].push_back(i);
80     }
81     return groups;
82 }
83
84 private:
85     int _n;
86     struct edge {
87         int to;
88     };
89     std::vector<std::pair<int, edge>> edges;
90 };
91 }

```

6.8 2-sat

```

1  struct two_sat {
2  public:
3      two_sat() : _n(0), scc(0) {}
4      explicit two_sat(int n) : _n(n), _answer(n), scc(2 * n) {}
5
6      // 加入一个限制: (i=f) or (j=g)
7      void add_clause(int i, bool f, int j, bool g) {
8          assert(0 <= i && i < _n);
9          assert(0 <= j && j < _n);
10         scc.add_edge(2 * i + (f ? 0 : 1), 2 * j + (g ? 1 : 0));
11         scc.add_edge(2 * j + (g ? 0 : 1), 2 * i + (f ? 1 : 0));
12     }
13     // 加入一个限制: i=f ==> j=g
14     void derive(int i, bool f, int j, bool g) {
15         add_clause(i, !f, j, g);
16     }
17     // O(N + M) 如果返回true, 则一个方案会保存在answer里
18     bool satisfiable() {

```

```

19     auto id = scc.scc_ids().second;
20     for (int i = 0; i < _n; i++) {
21         if (id[2 * i] == id[2 * i + 1]) return false;
22         _answer[i] = id[2 * i] < id[2 * i + 1];
23     }
24     return true;
25 }
26 std::vector<bool> answer() { return _answer; }
27
28 private:
29     int _n;
30     std::vector<bool> _answer;
31     SCC::scc_graph scc;
32 };

```

6.9 最大流

```

1  template <class T> struct simple_queue {
2      std::vector<T> payload;
3      int pos = 0;
4      void reserve(int n) { payload.reserve(n); }
5      int size() const { return int(payload.size()) - pos; }
6      bool empty() const { return pos == int(payload.size()); }
7      void push(const T& t) { payload.push_back(t); }
8      T& front() { return payload[pos]; }
9      void clear() {
10         payload.clear();
11         pos = 0;
12     }
13     void pop() { pos++; }
14 };
15
16 template <class Cap> struct mf_graph {
17 public:
18     mf_graph() : _n(0) {}
19     mf_graph(int n) : _n(n), g(n) {}
20
21     // returns an integer k such that this is the k-th edge that is added.
22     int add_edge(int from, int to, Cap cap) {
23         assert(0 <= from && from < _n);
24         assert(0 <= to && to < _n);
25         assert(0 <= cap);
26         int m = int(pos.size());

```

```

27     pos.push_back({ from, int(g[from].size()) });
28     int from_id = int(g[from].size());
29     int to_id = int(g[to].size());
30     if (from == to) to_id++;
31     g[from].push_back(_edge{ to, to_id, cap });
32     g[to].push_back(_edge{ from, from_id, 0 });
33     return m;
34 }
35
36 struct edge {
37     int from, to;
38     Cap cap, flow;
39 };
40
41 edge get_edge(int i) {
42     int m = int(pos.size());
43     assert(0 <= i && i < m);
44     auto _e = g[pos[i].first][pos[i].second];
45     auto _re = g[_e.to][_e.rev];
46     return edge{ pos[i].first, _e.to, _e.cap + _re.cap, _re.cap };
47 }
48 std::vector<edge> edges() {
49     int m = int(pos.size());
50     std::vector<edge> result;
51     for (int i = 0; i < m; i++) {
52         result.push_back(get_edge(i));
53     }
54     return result;
55 }
56 void change_edge(int i, Cap new_cap, Cap new_flow) {
57     int m = int(pos.size());
58     assert(0 <= i && i < m);
59     assert(0 <= new_flow && new_flow <= new_cap);
60     auto& _e = g[pos[i].first][pos[i].second];
61     auto& _re = g[_e.to][_e.rev];
62     _e.cap = new_cap - new_flow;
63     _re.cap = new_flow;
64 }
65
66 // max flow from s to t
67 // O(M*N^2) general
68 // O(min(M*N^2/3, M^3/2)) if capacities of edges are 1
69 Cap flow(int s, int t) {

```

```

70     return flow(s, t, std::numeric_limits<Cap>::max());
71 }
72 Cap flow(int s, int t, Cap flow_limit) {
73     assert(0 <= s && s < _n);
74     assert(0 <= t && t < _n);
75     assert(s != t);
76
77     std::vector<int> level(_n), iter(_n);
78     simple_queue<int> que;
79
80     auto bfs = [&]() {
81         std::fill(level.begin(), level.end(), -1);
82         level[s] = 0;
83         que.clear();
84         que.push(s);
85         while (!que.empty()) {
86             int v = que.front();
87             que.pop();
88             for (auto e : g[v]) {
89                 if (e.cap == 0 || level[e.to] >= 0) continue;
90                 level[e.to] = level[v] + 1;
91                 if (e.to == t) return;
92                 que.push(e.to);
93             }
94         }
95     };
96     auto dfs = [&](auto self, int v, Cap up) {
97         if (v == s) return up;
98         Cap res = 0;
99         int level_v = level[v];
100         for (int& i = iter[v]; i < int(g[v].size()); i++) {
101             _edge& e = g[v][i];
102             if (level_v <= level[e.to] || g[e.to][e.rev].cap == 0)
103                 continue;
104             Cap d =
105                 self(self, e.to, std::min(up - res, g[e.to][e.rev].cap))
106                 ;
107             if (d <= 0) continue;
108             g[v][i].cap += d;
109             g[e.to][e.rev].cap -= d;
110             res += d;
111             if (res == up) break;
112         }

```



```

111         return res;
112     };
113
114     Cap flow = 0;
115     while (flow < flow_limit) {
116         bfs();
117         if (level[t] == -1) break;
118         std::fill(iter.begin(), iter.end(), 0);
119         while (flow < flow_limit) {
120             Cap f = dfs(dfs, t, flow_limit - flow);
121             if (!f) break;
122             flow += f;
123         }
124     }
125     return flow;
126 }
127
128 std::vector<bool> min_cut(int s) {
129     std::vector<bool> visited(_n);
130     simple_queue<int> que;
131     que.push(s);
132     while (!que.empty()) {
133         int p = que.front();
134         que.pop();
135         visited[p] = true;
136         for (auto e : g[p]) {
137             if (e.cap && !visited[e.to]) {
138                 visited[e.to] = true;
139                 que.push(e.to);
140             }
141         }
142     }
143     return visited;
144 }
145
146 private:
147     int _n;
148     struct _edge {
149         int to, rev;
150         Cap cap;
151     };
152     std::vector<std::pair<int, int>> pos;
153     std::vector<std::vector<_edge>> g;

```

```
154 };
```

6.10 最小费用最大流

```

1  /*
2  * 费用流 Cost 常用类型的上限: int 范围内  $0 \leq nx \leq 2e9 + 1000$ , long long 范围
   内:  $0 \leq nx \leq 8e18 + 1000$ 
3  *
4  * min_cost_slope() 函数返回的是一个分段函数  $F(x)$  (其中  $x$  代表流量上界,  $F(x)$  代
   表当前最大流量的最小费用)
5  * 返回的 vector 是所有  $F(x)$  改变的点
6  * 时间复杂度  $O(f(N + M) \log(N + M))$   $f(N + M)$  代表图的流量总和
7  * */
8  namespace MCMF {
9      template <class T> struct simple_queue {
10         std::vector<T> payload;
11         int pos = 0;
12         void reserve(int n) { payload.reserve(n); }
13         int size() const { return int(payload.size()) - pos; }
14         bool empty() const { return pos == int(payload.size()); }
15         void push(const T& t) { payload.push_back(t); }
16         T& front() { return payload[pos]; }
17         void clear() {
18             payload.clear();
19             pos = 0;
20         }
21         void pop() { pos++; }
22     };
23
24     template <class E> struct csr {
25         std::vector<int> start;
26         std::vector<E> elist;
27         explicit csr(int n, const std::vector<std::pair<int, E>>& edges)
28             : start(n + 1), elist(edges.size()) {
29             for (auto e : edges) {
30                 start[e.first + 1]++;
31             }
32             for (int i = 1; i <= n; i++) {
33                 start[i] += start[i - 1];
34             }
35             auto counter = start;
36             for (auto e : edges) {
37                 elist[counter[e.first]++] = e.second;

```

```

38         }
39     }
40 };
41
42 template <class Cap, class Cost> struct mcf_graph {
43 public:
44     mcf_graph() {}
45     explicit mcf_graph(int n) : _n(n) {}
46
47     int add_edge(int from, int to, Cap cap, Cost cost) {
48         assert(0 <= from && from < _n);
49         assert(0 <= to && to < _n);
50         assert(0 <= cap);
51         assert(0 <= cost);
52         int m = int(_edges.size());
53         _edges.push_back({ from, to, cap, 0, cost });
54         return m;
55     }
56
57     struct edge {
58         int from, to;
59         Cap cap, flow;
60         Cost cost;
61     };
62
63     edge get_edge(int i) {
64         int m = int(_edges.size());
65         assert(0 <= i && i < m);
66         return _edges[i];
67     }
68     std::vector<edge> edges() { return _edges; }
69
70     std::pair<Cap, Cost> flow(int s, int t) {
71         return flow(s, t, std::numeric_limits<Cap>::max());
72     }
73     std::pair<Cap, Cost> flow(int s, int t, Cap flow_limit) {
74         return slope(s, t, flow_limit).back();
75     }
76     std::vector<std::pair<Cap, Cost>> slope(int s, int t) {
77         return slope(s, t, std::numeric_limits<Cap>::max());
78     }
79     std::vector<std::pair<Cap, Cost>> slope(int s, int t, Cap flow_limit
80         ) {

```

```

80     assert(0 <= s && s < _n);
81     assert(0 <= t && t < _n);
82     assert(s != t);
83
84     int m = int(_edges.size());
85     std::vector<int> edge_idx(m);
86
87     auto g = [&]() {
88         std::vector<int> degree(_n), redge_idx(m);
89         std::vector<std::pair<int, _edge>> elist;
90         elist.reserve(2 * m);
91         for (int i = 0; i < m; i++) {
92             auto e = _edges[i];
93             edge_idx[i] = degree[e.from]++;
94             redge_idx[i] = degree[e.to]++;
95             elist.push_back({ e.from, {e.to, -1, e.cap - e.flow, e.
96                 cost} });
97             elist.push_back({ e.to, {e.from, -1, e.flow, -e.cost} })
98                 ;
99         }
100         auto _g = csr<_edge>(_n, elist);
101         for (int i = 0; i < m; i++) {
102             auto e = _edges[i];
103             edge_idx[i] += _g.start[e.from];
104             redge_idx[i] += _g.start[e.to];
105             _g.elist[edge_idx[i]].rev = redge_idx[i];
106             _g.elist[redge_idx[i]].rev = edge_idx[i];
107         }
108         return _g;
109     }();
110
111     auto result = slope(g, s, t, flow_limit);
112
113     for (int i = 0; i < m; i++) {
114         auto e = g.elist[edge_idx[i]];
115         _edges[i].flow = _edges[i].cap - e.cap;
116     }
117
118     return result;
119
120 private:
121     int _n;

```

```

121     std::vector<edge> _edges;
122
123     // inside edge
124     struct _edge {
125         int to, rev;
126         Cap cap;
127         Cost cost;
128     };
129
130     std::vector<std::pair<Cap, Cost>> slope(csr<_edge>& g,
131         int s,
132         int t,
133         Cap flow_limit) {
134         // variants (C = maxcost):
135         //  $-(n-1)C \leq \text{dual}[s] \leq \text{dual}[i] \leq \text{dual}[t] = 0$ 
136         // reduced cost  $(= e.\text{cost} + \text{dual}[e.\text{from}] - \text{dual}[e.\text{to}]) \geq 0$  for
            all edge
137
138         //  $\text{dual\_dist}[i] = (\text{dual}[i], \text{dist}[i])$ 
139         std::vector<std::pair<Cost, Cost>> dual_dist(_n);
140         std::vector<int> prev_e(_n);
141         std::vector<bool> vis(_n);
142         struct Q {
143             Cost key;
144             int to;
145             bool operator<(Q r) const { return key > r.key; }
146         };
147         std::vector<int> que_min;
148         std::vector<Q> que;
149         auto dual_ref = [&]() {
150             for (int i = 0; i < _n; i++) {
151                 dual_dist[i].second = std::numeric_limits<Cost>::max();
152             }
153             std::fill(vis.begin(), vis.end(), false);
154             que_min.clear();
155             que.clear();
156
157             // que[0..heap_r) was heapified
158             size_t heap_r = 0;
159
160             dual_dist[s].second = 0;
161             que_min.push_back(s);
162             while (!que_min.empty() || !que.empty()) {

```

```

163         int v;
164         if (!que_min.empty()) {
165             v = que_min.back();
166             que_min.pop_back();
167         } else {
168             while (heap_r < que.size()) {
169                 heap_r++;
170                 std::push_heap(que.begin(), que.begin() + heap_r
171                               );
172             }
173             v = que.front().to;
174             std::pop_heap(que.begin(), que.end());
175             que.pop_back();
176             heap_r--;
177         }
178         if (vis[v]) continue;
179         vis[v] = true;
180         if (v == t) break;
181         // dist[v] = shortest(s, v) + dual[s] - dual[v]
182         // dist[v] >= 0 (all reduced cost are positive)
183         // dist[v] <= (n-1)C
184         Cost dual_v = dual_dist[v].first, dist_v = dual_dist[v].
185             second;
186         for (int i = g.start[v]; i < g.start[v + 1]; i++) {
187             auto e = g.elist[i];
188             if (!e.cap) continue;
189             // |-dual[e.to] + dual[v]| <= (n-1)C
190             // cost <= C - -(n-1)C + 0 = nC
191             Cost cost = e.cost - dual_dist[e.to].first + dual_v;
192             if (dual_dist[e.to].second - dist_v > cost) {
193                 Cost dist_to = dist_v + cost;
194                 dual_dist[e.to].second = dist_to;
195                 prev_e[e.to] = e.rev;
196                 if (dist_to == dist_v) {
197                     que_min.push_back(e.to);
198                 } else {
199                     que.push_back(Q{ dist_to, e.to });
200                 }
201             }
202         }
203         if (!vis[t]) {
204             return false;

```

```

204         }
205
206         for (int v = 0; v < _n; v++) {
207             if (!vis[v]) continue;
208             // dual[v] = dual[v] - dist[t] + dist[v]
209             //          = dual[v] - (shortest(s, t) + dual[s] - dual[
210                 t]) +
211             //          (shortest(s, v) + dual[s] - dual[v]) = -
212                 shortest(s,
213                 //          t) + dual[t] + shortest(s, v) = shortest(s, v
214                 ) -
215             //          shortest(s, t) >= 0 - (n-1)C
216             dual_dist[v].first -= dual_dist[t].second - dual_dist[v
217                 ].second;
218         }
219         return true;
220     };
221     Cap flow = 0;
222     Cost cost = 0, prev_cost_per_flow = -1;
223     std::vector<std::pair<Cap, Cost>> result = { {Cap(0), Cost(0)}
224         };
225     while (flow < flow_limit) {
226         if (!dual_ref()) break;
227         Cap c = flow_limit - flow;
228         for (int v = t; v != s; v = g.elist[prev_e[v]].to) {
229             c = std::min(c, g.elist[g.elist[prev_e[v]].rev].cap);
230         }
231         for (int v = t; v != s; v = g.elist[prev_e[v]].to) {
232             auto& e = g.elist[prev_e[v]];
233             e.cap += c;
234             g.elist[e.rev].cap -= c;
235         }
236         Cost d = -dual_dist[s].first;
237         flow += c;
238         cost += c * d;
239         if (prev_cost_per_flow == d) {
240             result.pop_back();
241         }
242         result.push_back({ flow, cost });
243         prev_cost_per_flow = d;
244     }
245     return result;
246 }

```

```

242     };
243 }

```

6.11 上下界网络流

6.11.1 无源汇上下界可行流

给定无源汇流量网络 G 。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时每一个点流量平衡。

不妨假设每条边已经流了 $b(u, v)$ 的流量，设其为初始流。同时我们在新图中加入 u 连向 v 的流量为 $c(u, v) - b(u, v)$ 的边。考虑在新图上进行调整。

由于最大流需要满足初始流量平衡条件（最大流可以看成是下界为 0 的上下界最大流），但是构造出来的初始流很有可能不满足初始流量平衡。假设一个点初始流入流量减初始流出流量为 M 。

若 $M = 0$ ，此时流量平衡，不需要附加边。

若 $M > 0$ ，此时入流量过大，需要新建附加源点 S' ， S' 向其连流量为 M 的附加边。

若 $M < 0$ ，此时出流量过大，需要新建附加汇点 T' ，其向 T' 连流量为 $-M$ 的附加边。

如果附加边满流，说明这一个点的流量平衡条件可以满足，否则这个点的流量平衡条件不满足。（因为原图加上附加流之后才会满足原图中的流量平衡。）

在建图完毕之后跑 S' 到 T' 的最大流，若 S' 连出去的边全部满流，则存在可行流，否则不存在。

6.11.2 有源汇上下界可行流

给定有源汇流量网络 G 。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。

假设源点为 S ，汇点为 T 。

则我们可以加入一条 T 到 S 的上界为 ∞ ，下界为 0 的边转化为无源汇上下界可行流问题。

若有解，则 S 到 T 的可行流流量等于 T 到 S 的附加边的流量。

6.11.3 有源汇上下界最大流

给定有源汇流量网络 G 。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。如果存在，询问满足标定的最大流量。

我们找到网络上的任意一个可行流。如果找不到解就可以直接结束。

否则我们考虑删去所有附加边之后的残量网络并且在网络上进行调整。

我们在残量网络上再跑一次 S 到 T 的最大流，将可行流流量和最大流流量相加即为答案。

一个非常易错的问题： S 到 T 的最大流直接在跑完有源汇上下界可行的残量网络上跑。

6.11.4 有源汇上下界最小流

给定有源汇流量网络 G 。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。如果存在，询问满足标定的最小流量。

类似的，我们考虑将残量网络中不需要的流退掉。

我们找到网络上的任意一个可行流。如果找不到解就可以直接结束。

否则我们考虑删去所有附加边之后的残量网络。

我们在残量网络上再跑一次 T 到 S 的最大流，将可行流流量减去最大流流量即为答案。

对于每个点，向 T 连边权 c ，上界 ∞ ，下界为 1。

S 点为 1 号节点。

跑一次上下界带源汇最小费用可行流即可。

因为最小费用可行流解法与最小可行流类似，这里不再展开。

6.12 全局最小割

```

1  constexpr int N = 601;
2  constexpr int inf = 0x3f3f3f3f;
3  int edge[N][N]; // 边权存这里
4  int dis[N], vis[N], bin[N];
5  int n, m;
6  int contract(int& s, int& t) { // Find s, t
7      memset(dis, 0, sizeof(dis));
8      memset(vis, false, sizeof(vis));
9      int i, j, k, mincut, maxc;
10     for (i = 1; i <= n; i++) {
11         k = -1;
12         maxc = -1;
13         for (j = 1; j <= n; j++) {
14             if (!bin[j] && !vis[j] && dis[j] > maxc) {
15                 k = j;
16                 maxc = dis[j];
17             }
18         }
19         if (k == -1) return mincut;
20         s = t; t = k;
21         mincut = maxc;
22         vis[k] = true;
23         for (j = 1; j <= n; j++) {
24             if (!bin[j] && !vis[j]) {
25                 dis[j] += edge[k][j];
26             }
27         }
28     }
29     return mincut;
30 }
31
32 int stoerWagner() { // O(NM + N^2logN) <=> O(N^3)
33     int mincut, i, j, s, t, ans;
34     for (mincut = inf, i = 1; i < n; i++) {
35         ans = contract(s, t);
36         bin[t] = true;
37         if (mincut > ans) mincut = ans;
38         if (mincut == 0) return 0;
39     }
39 }

```

```

39     for (j = 1; j <= n; j++) {
40         if (!bin[j]) {
41             edge[s][j] = (edge[j][s] += edge[j][t]);
42         }
43     }
44 }
45 return mincut;
46 }

```

6.13 二分图最大权匹配

```

1 namespace KM {
2     typedef long long ll;
3     const int maxn = 510;
4     const int inf = 1e9;
5     int vx[maxn], vy[maxn], lx[maxn], ly[maxn], slack[maxn];
6     int w[maxn][maxn]; // 以上为权值类型
7     int pre[maxn], left[maxn], right[maxn], NL, NR, N;
8     void match(int& u) {
9         for (; u; std::swap(u, right[pre[u]]))
10             left[u] = pre[u];
11     }
12     void bfs(int u) {
13         static int q[maxn], front, rear;
14         front = 0; vx[q[rear = 1] = u] = true;
15         while (true) {
16             while (front < rear) {
17                 int u = q[++front];
18                 for (int v = 1; v <= N; ++v) {
19                     int tmp;
20                     if (vy[v] || (tmp = lx[u] + ly[v] - w[u][v]) > slack[v])
21                         continue;
22                     pre[v] = u;
23                     if (!tmp) {
24                         if (!left[v]) return match(v);
25                         vy[v] = vx[q[++rear] = left[v]] = true;
26                     } else slack[v] = tmp;
27                 }
28             }
29             int a = inf;
30             for (int i = 1; i <= N; ++i)
31                 if (!vy[i] && a > slack[i]) a = slack[u = i];
32             for (int i = 1; i <= N; ++i) {

```

```

33         if (vx[i]) lx[i] -= a;
34         if (vy[i]) ly[i] += a;
35         else slack[i] -= a;
36     }
37     if (!left[u]) return match(u);
38     vy[u] = vx[q[++rear] = left[u]] = true;
39
40 }
41
42 }
43 void exec() {
44     for (int i = 1; i <= N; ++i) {
45         for (int j = 1; j <= N; ++j) {
46             slack[j] = inf;
47             vx[j] = vy[j] = false;
48         }
49         bfs(i);
50     }
51 }
52 ll work(int nl, int nr) { // NL, NR 为左右点数, 返回最大权匹配的权值和
53     NL = nl; NR = nr;
54     N = std::max(NL, NR);
55     for (int u = 1; u <= N; ++u)
56         for (int v = 1; v <= N; ++v)
57             lx[u] = std::max(lx[u], w[u][v]);
58     exec();
59     ll ans = 0;
60     for (int i = 1; i <= N; ++i)
61         ans += lx[i] + ly[i];
62     return ans;
63 }
64 void output() { // 输出左边点与右边哪个点匹配, 没有匹配输出0
65     for (int i = 1; i <= NL; ++i)
66         printf("%d ", (w[i][right[i]] ? right[i] : 0));
67     printf("\n");
68 }
69 }

```

6.14 一般图最大匹配

```

1 // UOJ79 copy from jiangly
2 #include <bits/stdc++.h>
3 struct Graph {

```

```

4      int n;
5      std::vector<std::vector<int>> e;
6      Graph(int n) : n(n), e(n) {}
7      void addEdge(int u, int v) {
8          e[u].push_back(v);
9          e[v].push_back(u);
10     }
11     std::vector<int> findMatching() {
12         std::vector<int> match(n, -1), vis(n), link(n), f(n), dep(n);
13         // disjoint set union
14         auto find = [&](int u) {
15             while (f[u] != u)
16                 u = f[u] = f[f[u]];
17             return u;
18         };
19         auto lca = [&](int u, int v) {
20             u = find(u);
21             v = find(v);
22             while (u != v) {
23                 if (dep[u] < dep[v])
24                     std::swap(u, v);
25                 u = find(link[match[u]]);
26             }
27             return u;
28         };
29
30         std::queue<int> que;
31         auto blossom = [&](int u, int v, int p) {
32             while (find(u) != p) {
33                 link[u] = v;
34                 v = match[u];
35                 if (vis[v] == 0) {
36                     vis[v] = 1;
37                     que.push(v);
38                 }
39                 f[u] = f[v] = p;
40                 u = link[v];
41             }
42         };
43
44         // find an augmenting path starting from u and augment (if exist)
45         auto augment = [&](int u) {
46             while (!que.empty())

```

```

47         que.pop();
48     std::iota(f.begin(), f.end(), 0);
49     // vis = 0 corresponds to inner vertices, vis = 1 corresponds to
        outer vertices
50     std::fill(vis.begin(), vis.end(), -1);
51     que.push(u);
52     vis[u] = 1, dep[u] = 0;
53     while (!que.empty()) {
54         int u = que.front();
55         que.pop();
56         for (auto v : e[u]) {
57             if (vis[v] == -1) {
58                 vis[v] = 0;
59                 link[v] = u;
60                 dep[v] = dep[u] + 1;
61                 // found an augmenting path
62                 if (match[v] == -1) {
63                     for (int x = v, y = u, temp; y != -1; x = temp,
                        y = x == -1 ? -1 : link[x]) {
64                         temp = match[y];
65                         match[x] = y;
66                         match[y] = x;
67                     }
68                     return;
69                 }
70                 vis[match[v]] = 1;
71                 dep[match[v]] = dep[u] + 2;
72                 que.push(match[v]);
73             } else if (vis[v] == 1 && find(v) != find(u)) {
74                 // found a blossom
75                 int p = lca(u, v);
76                 blossom(u, v, p);
77                 blossom(v, u, p);
78             }
79         }
80     }
81 };
82
83 // find a maximal matching greedily (decrease constant)
84 auto greedy = [&]() {
85     for (int u = 0; u < n; ++u) {
86         if (match[u] != -1)
87             continue;

```

```

88         for (auto v : e[u]) {
89             if (match[v] == -1) {
90                 match[u] = v;
91                 match[v] = u;
92                 break;
93             }
94         }
95     }
96 };
97 greedy();
98 for (int u = 0; u < n; ++u)
99     if (match[u] == -1)
100         augment(u);
101 return match;
102 }
103 };
104
105 int main() {
106     std::ios::sync_with_stdio(false);
107     std::cin.tie(nullptr);
108     int n, m;
109     std::cin >> n >> m;
110     Graph g(n);
111     for (int i = 0; i < m; ++i) {
112         int u, v;
113         std::cin >> u >> v;
114         —u, —v;
115         g.addEdge(u, v);
116     }
117     auto match = g.findMatching();
118     int ans = 0;
119     for (int u = 0; u < n; ++u)
120         if (match[u] != -1)
121             ++ans;
122     std::cout << ans / 2 << "\n";
123     for (int u = 0; u < n; ++u) // 输出每个人匹配的对象, 如果没有则输出0
124         std::cout << match[u] + 1 << " \n"[u == n - 1];
125     return 0;
126 }

```

6.15 最大团

```
1  /*
```

```

2  * 最大团 Bron-Kerbosch algorithm
3  * 最劣复杂度  $O(3^{\lfloor n/3 \rfloor})$ 
4  * 采用位运算形式实现
5  * */
6  namespace Max_clique {
7  #define ll long long
8  #define TWOL(x) (1ll << (x))
9      const int N = 60;
10     int n, m;          // 点数 边数
11     int r = 0;          // 最大团大小
12     ll G[N];           // 以二进制形式存图
13     ll clique = 0;      // 最大团 以二进制形式存储
14     void BronK(int S, ll P, ll X, ll R) { // 调用时参数这样设置: 0, TWOL(n)
        -1, 0, 0
15         if (P == 0 && X == 0) {
16             if (r < S) {
17                 r = S;
18                 clique = R;
19             }
20         }
21         if (P == 0) return;
22         int u = __builtin_ctzll(P | X);
23         ll c = P & ~G[u];
24         while (c) {
25             int v = __builtin_ctzll(c);
26             ll pv = TWOL(v);
27             BronK(S + 1, P & G[v], X & G[v], R | pv);
28             P ^= pv; X |= pv; c ^= pv;
29         }
30     }
31     void init() {
32         cin >> n >> m;
33         for (int i = 0; i < m; i++) {
34             int u, v;
35             cin >> u >> v;
36             —u, —v;
37             G[u] |= TWOL(v);
38             G[v] |= TWOL(u);
39         }
40         BronK(0, TWOL(n) - 1, 0, 0);
41         cout << r << ' ' << clique << '\n';
42     }
43 }

```

7 数据结构

7.1 树状数组

```

1  template<typename T> struct fenwickTree {
2      int n, hbit;
3      vector<T> tree;
4      fenwickTree(int n_ = 0) : n(n_), tree(n_ + 1), hbit(log2(n_) + 1) {}
5      int lowbit(int x) { return x & (-x); }
6      int size() { return n; }
7      void add(int pos, int x) { // pos位置加上x
8          for (; pos <= n; pos += lowbit(pos)) {
9              tree[pos] += x;
10         }
11     }
12     T query(int pos) { // 查询pos位置的前缀和 即a[1] + a[2] + ... + a[pos]
13         T res = 0;
14         for (; pos > 0; pos -= lowbit(pos)) {
15             res += tree[pos];
16         }
17         return res;
18     }
19     T sum(int l, int r) { // [l, r]区间查询
20         return query(r) - query(l - 1);
21     }
22     int kth(int k) { // 第k大元素
23         int ans = 0, cnt = 0;
24         for (int i = hbit; i >= 0; i--) {
25             ans += (1 << i);
26             if (ans > n || cnt + tree[ans] >= k) ans -= (1 << i);
27             else cnt += tree[ans];
28         }
29         return ++ans;
30     }
31 };

```

7.2 线段树

```

1  namespace SegmentTree {
2      static constexpr int SIZE = 200001;
3      int a[SIZE];
4      int mx[SIZE << 2], lazy[SIZE << 2];
5
6      void pushup(int rt) {

```



```

7      mx[rt] = max(mx[rt << 1], mx[rt << 1 | 1]);
8  }
9
10 void pushdown(int rt) {
11     if (lazy[rt]) {
12         lazy[rt << 1] += lazy[rt];
13         lazy[rt << 1 | 1] += lazy[rt];
14         mx[rt << 1] += lazy[rt];
15         mx[rt << 1 | 1] += lazy[rt];
16         lazy[rt] = 0;
17     }
18 }
19
20 void build(int rt, int l, int r) {
21     if (l == r) {
22         mx[rt] = a[l];
23         lazy[rt] = 0;
24         return;
25     }
26     int mid = (l + r) >> 1;
27     pushdown(rt);
28     build(rt << 1, l, mid), build(rt << 1 | 1, mid + 1, r);
29     pushup(rt);
30 }
31
32 void rangeAdd(int rt, int l, int r, int ql, int qr, int val) {
33     if (ql <= l && qr >= r) {
34         mx[rt] += val;
35         lazy[rt] += val;
36         return;
37     }
38     int mid = (l + r) >> 1;
39     pushdown(rt);
40     if (ql <= mid) rangeAdd(rt << 1, l, mid, ql, qr, val);
41     if (qr > mid) rangeAdd(rt << 1 | 1, mid + 1, r, ql, qr, val);
42     pushup(rt);
43 }
44
45 int rangeMax(int rt, int l, int r, int ql, int qr) {
46     if (ql <= l && qr >= r) return mx[rt];
47     int mid = (l + r) >> 1;
48     int res = 0;
49     pushdown(rt);

```

```

50         if (ql <= mid) res = max(res, rangeMax(rt << 1, l, mid, ql, qr));
51         if (qr > mid) res = max(res, rangeMax(rt << 1 | 1, mid + 1, r, ql,
52             qr));
53         pushup(rt);
54         return res;
55     }
56 }

```

8 字符串

8.1 KMP

```

1  namespace KMP {
2      vector<int> getPrefixTable(string s) { // 求前缀表
3          int n = s.length();
4          vector<int> nxt(n, 0);
5          for (int i = 1; i < n; i++) {
6              int j = nxt[i - 1];
7              while (j > 0 && s[i] != s[j]) {
8                  j = nxt[j - 1];
9              }
10             if (s[i] == s[j]) j++;
11             nxt[i] = j;
12         }
13         return nxt;
14     }
15
16     vector<int> kmp(string s, string t) { // 返回所有匹配位置的集合
17         int n = s.length(), m = t.length();
18         vector<int> res;
19         vector<int> nxt = getPrefixTable(t);
20         for (int i = 0, j = 0; i < n; i++) {
21             while (j > 0 && j < m && s[i] != t[j]) {
22                 j = nxt[j - 1];
23             }
24             if (s[i] == t[j]) j++;
25             if (j == m) {
26                 res.push_back(i + 1 - m);
27                 j = nxt[m - 1];
28             }
29         }
30         return res;
31     }

```

```
32 }
```

8.2 Z-Function

```
1 // O(N) 查询字符串s每一位开始的LCP
2 vector<int> z_function(string s) {
3     int n = (int)s.length();
4     vector<int> z(n);
5     for (int i = 1, l = 0, r = 0; i < n; ++i) {
6         if (i <= r && z[i - l] < r - i + 1) {
7             z[i] = z[i - l];
8         } else {
9             z[i] = max(0, r - i + 1);
10            while (i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];
11        }
12        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
13    }
14    return z;
15 }
```

8.3 Manacher

```
1 namespace Manacher {
2     static constexpr int SIZE = 1e5 + 5; // 预设为原串长度
3     int len = 1; // manacher 预处理后字符串的长度
4     char stk[SIZE << 1]; // manacher预处理字符串 需要2倍空间+1
5     void init(string s) { // 初始化stk
6         stk[0] = '*'; len = 1;
7         for (int i = 0; i < s.length(); ++i) {
8             stk[len++] = s[i];
9             stk[len++] = '*';
10        }
11    }
12    int manacher() { // 返回最长回文子串长度
13        vector<int> rad(len << 1); // 存储每个点作为对称中心可拓展的最大半径
14        int md = 0; // 最远回文串对称中心下标
15        for (int i = 1; i < len; ++i) {
16            int& r = rad[i] = 0;
17            if (i <= md + rad[md]) {
18                r = min(rad[2 * md - i], md + rad[md] - i);
19            }
20            while (i - r - 1 >= 0 && i + r + 1 < len &&
21                stk[i - r - 1] == stk[i + r + 1]) ++r;
22        }
23    }
24 }
```

```

22         if (i + r >= md + rad[md]) md = i;
23     }
24     int res = 0;
25     for (int i = 0; i < len; ++i) {
26         if (rad[i] > res) {
27             res = rad[i];
28         }
29     }
30     return res;
31 }
32 }

```

8.4 Trie

```

1 struct trie {
2     int cnt;
3     vector<vector<int>> nxt;
4     vector<bool> vis;
5     /* 初始化的时候size需要设置为字符串总长之和 26是字符集大小 */
6     trie(int size_ = 0) : cnt(0), vis(size_, false), nxt(size_, vector<int>
7         >(26, 0)) {}
8     void insert(string s) { // 插入字符串
9         int p = 0;
10        for (int i = 0; i < (int)s.length(); i++) {
11            int c = s[i] - 'a';
12            if (!nxt[p][c]) nxt[p][c] = ++cnt;
13            p = nxt[p][c];
14        }
15        vis[p] = true;
16    }
17    bool find(string s) { // 查找字符串
18        int p = 0;
19        for (int i = 0; i < (int)s.length(); i++) {
20            int c = s[i] - 'a';
21            if (!nxt[p][c]) return false;
22            p = nxt[p][c];
23        }
24        return vis[p];
25    }
26 };

```

8.5 01-Trie

```

1  template<typename T> struct xorTrie {
2      int HIGHBIT, cnt;
3      vector<vector<int>> nxt;
4      vector<bool> vis;
5      xorTrie(int n_ = 0, int highbit_ = 30) : HIGHBIT(highbit_), cnt(0) {
6          int size_ = upperBoundEstimate(n_);
7          nxt.resize(size_, vector<int>(2, 0));
8          vis.resize(size_, false);
9      }
10     int upperBoundEstimate(int n) { // 求内存上界
11         int hbit = log2(n);
12         return n * (HIGHBIT - hbit + 1) + (1 << (hbit + 1)) - 1;
13     }
14     void insert(T x) { // 插入
15         int p = 0;
16         for (int i = HIGHBIT; ~i; i--) {
17             int s = ((x >> i) & 1);
18             if (!nxt[p][s]) nxt[p][s] = ++cnt;
19             p = nxt[p][s];
20         }
21         vis[p] = true;
22     }
23     bool find(T x) { // 查询
24         int p = 0;
25         for (int i = HIGHBIT; ~i; i--) {
26             int s = ((x >> i) & 1);
27             if (!nxt[p][s]) return false;
28             p = nxt[p][s];
29         }
30         return vis[p];
31     }
32 };

```

9 计算几何

```

1  namespace Geometry {
2  #define db long double
3  #define pi acos(-1.0)
4      constexpr db eps = 1e-7;
5      int sign(db k) {
6          if (k > eps) return 1;
7          else if (k < -eps) return -1;

```

```

8         return 0;
9     }
10    int cmp(db k1, db k2) { // k1 < k2 : -1, k1 == k2 : 0, k1 > k2 : 1
11        return sign(k1 - k2);
12    }
13    int inmid(db k1, db k2, db k3) { // k3 在 [k1, k2] 内
14        return sign(k1 - k3) * sign(k2 - k3) <= 0;
15    }
16
17    struct point { // 点类
18        db x, y;
19        point() {}
20        point(db x_, db y_) :x(x_), y(y_) {}
21        point operator + (const point& k) const { return point(k.x + x, k.y
22            + y); }
23        point operator - (const point& k) const { return point(x - k.x, y -
24            k.y); }
25        point operator * (db k) const { return point(x * k, y * k); }
26        point operator / (db k1) const { return point(x / k1, y / k1); }
27        point turn(db k1) { return point(x * cos(k1) - y * sin(k1), x * sin(
28            k1) + y * cos(k1)); } // 逆时针旋转
29        point turn90() { return point(-y, x); } // 逆时针方向旋转 90 度
30        db len() { return sqrt(x * x + y * y); } // 向量长度
31        db len2() { return x * x + y * y; } // 向量长度的平方
32        db getPolarAngle() { return atan2(y, x); } // 向量极角
33        db dis(point k) { return ((*this) - k).len(); } // 到点k的距离
34        point unit() { db d = len(); return point(x / d, y / d); } // 单位向
35            量
36        point getdel() { // 将向量的方向调整为指向第一/四象限 包括y轴正方向
37            if (sign(x) == -1 || (sign(x) == 0 && sign(y) == -1))
38                return (*this) * (-1);
39            else return (*this);
40        }
41        bool operator < (const point& k) const { // 水平序排序 x坐标为第一关
42            键字,y坐标第二关键字
43            return x == k.x ? y < k.y : x < k.x;
44        }
45        bool operator == (const point& k) const { return cmp(x, k.x) == 0 &&
46            cmp(y, k.y) == 0; }
47        bool getP() const { // 判断点是否在上半平面 含x负半轴 不含x正半轴及
48            零点
49            return sign(y) == 1 || (sign(y) == 0 && sign(x) == -1);
50        }

```

```

44     void input() { cin >> x >> y; }
45 };
46 db cross(point k1, point k2) { return k1.x * k2.y - k1.y * k2.x; } // 向
    量 k1,k2 的叉积
47 db dot(point k1, point k2) { return k1.x * k2.x + k1.y * k2.y; } // 向
    量 k1,k2 的点积
48 db rad(point k1, point k2) { // 向量 k1,k2 之间的有向夹角
49     return atan2(cross(k1, k2), dot(k1, k2));
50 }
51 int inmid(point k1, point k2, point k3) { // k1 k2 k3共线时 判断点 k3 是
    否在线段 k1k2 上
52     return inmid(k1.x, k2.x, k3.x) && inmid(k1.y, k2.y, k3.y);
53 }
54 int compareAngle(point k1, point k2) { // 比较向量 k1,k2 的角度大小 角度
    按照atan2()函数定义
55     // k1 < k2 返回 1, k1 >= k2 返回 0
56     return k1.getP() < k2.getP() || (k1.getP() == k2.getP() && sign(
        cross(k1, k2)) > 0);
57 }
58 point proj(point k1, point k2, point q) { // q 到直线 k1,k2 的投影
59     point k = k2 - k1; return k1 + k * (dot(q - k1, k) / k.len2());
60 }
61 point reflect(point k1, point k2, point q) { return proj(k1, k2, q) * 2
    - q; } // q 关于直线 k1,k2 的对称点
62 int counterclockwise(point k1, point k2, point k3) { // k1 k2 k3 逆时针1
    顺时针-1 否则0
63     return sign(cross(k2 - k1, k3 - k1));
64 }
65 int checkLL(point k1, point k2, point k3, point k4) { // 判断直线 k1k2
    和直线k3k4 是否相交
66     // 即判断直线 k1k2 和 k3k4 是否平行 平行返回0 不平行返回1
67     return sign(cross(k2 - k1, k4 - k3)) != 0;
68 }
69 point getLL(point k1, point k2, point k3, point k4) { // 求 k1k2 k3k4 两
    直线交点
70     db w1 = cross(k1 - k3, k4 - k3), w2 = cross(k4 - k3, k2 - k3);
71     return (k1 * w2 + k2 * w1) / (w1 + w2);
72 }
73 int intersect(db l1, db r1, db l2, db r2) { // 判断 [l1, r1] 和 [l2, r2]
    是否相交
74     if (l1 > r1) swap(l1, r1);
75     if (l2 > r2) swap(l2, r2);
76     return cmp(r1, l2) != -1 && cmp(r2, l1) != -1;

```

```

77     }
78     int checkSS(point k1, point k2, point k3, point k4) { // 判断线段 k1k2
        和线段 k3k4 是否相交
79         return intersect(k1.x, k2.x, k3.x, k4.x) && intersect(k1.y, k2.y, k3
            .y, k4.y) &&
80             sign(cross(k3 - k1, k4 - k1)) * sign(cross(k3 - k2, k4 - k2)) <=
                0 &&
81             sign(cross(k1 - k3, k2 - k3)) * sign(cross(k1 - k4, k2 - k4)) <=
                0;
82     }
83     db disSP(point k1, point k2, point q) { // 点 q 到线段 k1k2 的最短距离
84         point k3 = proj(k1, k2, q);
85         if (inmid(k1, k2, k3)) return q.dis(k3);
86         else return min(q.dis(k1), q.dis(k2));
87     }
88     db disLP(point k1, point k2, point q) { // 点 q 到直线 k1k2 的最短距离
89         point k3 = proj(k1, k2, q);
90         return q.dis(k3);
91     }
92     db disSS(point k1, point k2, point k3, point k4) { // 线段 k1k2 和线段
        k3k4 的最短距离
93         if (checkSS(k1, k2, k3, k4)) return 0;
94         else return min(min(disSP(k1, k2, k3), disSP(k1, k2, k4)),
95             min(disSP(k3, k4, k1), disSP(k3, k4, k2)));
96     }
97     bool onLine(point k1, point k2, point q) { // 判断点 q 是否在直线 k1k2
        上
98         return sign(cross(k1 - q, k2 - q)) == 0;
99     }
100    bool onSegment(point k1, point k2, point q) { // 判断点 q 是否在线段
        k1k2 上
101        if (!onLine(k1, k2, q)) return false; // 如果确定共线 要删除这个特判
102        return inmid(k1, k2, q);
103    }
104    void polarAngleSort(vector<point>& p, point t) { // p为待排序点集 t为极
        角排序中心
105        sort(p.begin(), p.end(), [&](const point& k1, const point& k2) {
106            return compareAngle(k1 - t, k2 - t);
107        });
108    }
109
110    struct line { // 直线 / 线段类
111        point p[2];

```



```

112     line() {}
113     line(point k1, point k2) { p[0] = k1, p[1] = k2; }
114     point& operator [] (int k) { return p[k]; }
115     point dir() { return p[1] - p[0]; } // 向量 p[0] -> p[1]
116     bool include(point k) { // 判断点是否在直线上
117         return sign(cross(p[1] - p[0], k - p[0])) > 0;
118     }
119     bool includeS(point k) { // 判断点是否在线段上
120         return onSegment(p[0], p[1], k);
121     }
122     line push(db len) { // 向外（左手边）平移 len 个单位
123         point delta = (p[1] - p[0]).turn90().unit() * len;
124         return line(p[0] - delta, p[1] - delta);
125     }
126 };
127 bool parallel(line k1, line k2) { // 判断是否平行
128     return sign(cross(k1.dir(), k2.dir())) == 0;
129 }
130 bool sameLine(line k1, line k2) { // 判断是否共线
131     return parallel(k1, k2) && parallel(k1, line(k2.p[0], k1.p[0]));
132 }
133 bool sameDir(line k1, line k2) { // 判断向量 k1 k2 是否同向
134     return parallel(k1, k2) && sign(dot(k1.dir(), k2.dir())) == 1;
135 }
136 bool operator < (line k1, line k2) {
137     if (sameDir(k1, k2)) return k2.include(k1[0]);
138     return compareAngle(k1.dir(), k2.dir());
139 }
140 bool checkLL(line k1, line k2) {
141     return checkLL(k1[0], k1[1], k2[0], k2[1]);
142 }
143 point getLL(line k1, line k2) { // 求 k1 k2 两直线交点 不要忘了判平行!
144     return getLL(k1[0], k1[1], k2[0], k2[1]);
145 }
146 bool checkpos(line k1, line k2, line k3) { // 判断是否三线共点
147     return k3.include(getLL(k1, k2));
148 }
149
150 struct circle { // 圆类
151     point o;
152     double r;
153     circle() {}
154     circle(point o_, double r_) : o(o_), r(r_) {}

```

```

155     int inside(point k) { // 判断点 k 和圆的位置关系
156         return cmp(r, o.dis(k)); // 圆外:-1, 圆上:0, 圆内:1
157     }
158 };
159 int checkposCC(circle k1, circle k2) { // 返回两个圆的公切线数量
160     if (cmp(k1.r, k2.r) == -1) swap(k1, k2);
161     db dis = k1.o.dis(k2.o);
162     int w1 = cmp(dis, k1.r + k2.r), w2 = cmp(dis, k1.r - k2.r);
163     if (w1 > 0) return 4; // 外离
164     else if (w1 == 0) return 3; // 外切
165     else if (w2 > 0) return 2; // 相交
166     else if (w2 == 0) return 1; // 内切
167     else return 0; // 内离(包含)
168 }
169 vector<point> getCL(circle k1, point k2, point k3) { // 求直线 k2k3 和圆
    k1 的交点
170     // 沿着 k2->k3 方向给出 相切给出两个
171     point k = proj(k2, k3, k1.o);
172     db d = k1.r * k1.r - (k - k1.o).len2();
173     if (sign(d) == -1) return {};
174     point del = (k3 - k2).unit() * sqrt(max((db)0.0, d));
175     return { k - del, k + del };
176 }
177 vector<point> getCC(circle k1, circle k2) { // 求圆 k1 和圆 k2 的交点
178     // 沿圆 k1 逆时针给出, 相切给出两个
179     int pd = checkposCC(k1, k2); if (pd == 0 || pd == 4) return {};
180     db a = (k2.o - k1.o).len2(), cosA = (k1.r * k1.r + a -
181         k2.r * k2.r) / (2 * k1.r * sqrt(max(a, (db)0.0)));
182     db b = k1.r * cosA, c = sqrt(max((db)0.0, k1.r * k1.r - b * b));
183     point k = (k2.o - k1.o).unit(), m = k1.o + k * b, del = k.turn90() *
        c;
184     return { m - del, m + del };
185 }
186 vector<point> tangentCP(circle k1, point k2) { // 点 k2 到圆 k1 的切点
    沿圆 k1 逆时针给出
187     db a = (k2 - k1.o).len(), b = k1.r * k1.r / a, c = sqrt(max((db)0.0,
        k1.r * k1.r - b * b));
188     point k = (k2 - k1.o).unit(), m = k1.o + k * b, del = k.turn90() * c
        ;
189     return { m - del, m + del };
190 }
191 vector<line> tangentOutCC(circle k1, circle k2) {
192     int pd = checkposCC(k1, k2);

```

```

193     if (pd == 0) return {};
194     if (pd == 1) {
195         point k = getCC(k1, k2)[0];
196         return { line(k, k) };
197     }
198     if (cmp(k1.r, k2.r) == 0) {
199         point del = (k2.o - k1.o).unit().turn90().getdel();
200         return { line(k1.o - del * k1.r, k2.o - del * k2.r),
201                 line(k1.o + del * k1.r, k2.o + del * k2.r) };
202     } else {
203         point p = (k2.o * k1.r - k1.o * k2.r) / (k1.r - k2.r);
204         vector<point> A = tangentCP(k1, p), B = tangentCP(k2, p);
205         vector<line> ans; for (int i = 0; i < A.size(); i++)
206             ans.push_back(line(A[i], B[i]));
207         return ans;
208     }
209 }
210 vector<line> tangentInCC(circle k1, circle k2) {
211     int pd = checkposCC(k1, k2);
212     if (pd <= 2) return {};
213     if (pd == 3) {
214         point k = getCC(k1, k2)[0];
215         return { line(k, k) };
216     }
217     point p = (k2.o * k1.r + k1.o * k2.r) / (k1.r + k2.r);
218     vector<point> A = tangentCP(k1, p), B = tangentCP(k2, p);
219     vector<line> ans;
220     for (int i = 0; i < (int)A.size(); i++) ans.push_back(line(A[i], B[i]
221     ));
222     return ans;
223 }
224 vector<line> tangentCC(circle k1, circle k2) { // 求两圆公切线
225     int flag = 0;
226     if (k1.r < k2.r) swap(k1, k2), flag = 1;
227     vector<line> A = tangentOutCC(k1, k2), B = tangentInCC(k1, k2);
228     for (line k : B) A.push_back(k);
229     if (flag) for (line& k : A) swap(k[0], k[1]);
230     return A;
231 }
232 db getAreaCT(circle k1, point k2, point k3) { // 圆 k1 与三角形 k2k3k1.o
233     的有向面积交
234     point k = k1.o; k1.o = k1.o - k; k2 = k2 - k; k3 = k3 - k;
235     int pd1 = k1.inside(k2), pd2 = k1.inside(k3);

```

```

234     vector<point> A = getCL(k1, k2, k3);
235     if (pd1 >= 0) {
236         if (pd2 >= 0) return cross(k2, k3) / 2;
237         return k1.r * k1.r * rad(A[1], k3) / 2 + cross(k2, A[1]) / 2;
238     } else if (pd2 >= 0) {
239         return k1.r * k1.r * rad(k2, A[0]) / 2 + cross(A[0], k3) / 2;
240     } else {
241         int pd = cmp(k1.r, disSP(k2, k3, k1.o));
242         if (pd <= 0) return k1.r * k1.r * rad(k2, k3) / 2;
243         return cross(A[0], A[1]) / 2 + k1.r * k1.r * (rad(k2, A[0]) +
                rad(A[1], k3)) / 2;
244     }
245 }
246 db getAreaCC(circle k1, circle k2) { // 两圆面积交
247     db d = k1.o.dis(k2.o);
248     if (cmp(d, k1.r + k2.r) >= 0) return 0; // 两圆相离
249     if (cmp(k1.r, k2.r) == -1) swap(k1, k2);
250     if (cmp(k1.r - k2.r, d) >= 0) return pi * k2.r * k2.r; // 圆k1包含k2
251     db g1 = acos((k1.r * k1.r + d * d - k2.r * k2.r) / (2 * k1.r * d));
252     db g2 = acos((k2.r * k2.r + d * d - k1.r * k1.r) / (2 * k2.r * d));
253     return g1 * k1.r * k1.r + g2 * k2.r * k2.r - k1.r * d * sin(g1);
254 }
255 circle getCircleOut(point k1, point k2, point k3) { // 三角形外接圆
256     db a1 = k2.x - k1.x, b1 = k2.y - k1.y, c1 = (a1 * a1 + b1 * b1) / 2;
257     db a2 = k3.x - k1.x, b2 = k3.y - k1.y, c2 = (a2 * a2 + b2 * b2) / 2;
258     db d = a1 * b2 - a2 * b1;
259     point o((k1.x + (c1 * b2 - c2 * b1) / d, k1.y + (a1 * c2 - a2 * c1) /
                d);
260     return circle(o, k1.dis(o));
261 }
262 circle getCircleIn(point k1, point k2, point k3) { // 三角形内切圆
263     db a = k1.dis(k2), b = k2.dis(k3), c = k3.dis(k1);
264     db len = a + b + c;
265     db r = abs(cross(k1 - k2, k1 - k3)) / len;
266     point o((k1.x * b + k2.x * c + k3.x * a) / len, (k1.y * b + k2.y * c
                + k3.y * a) / len);
267     return circle(o, r);
268 }
269 circle minCircleCovering(vector<point> A) { // 最小圆覆盖 O(n)随机增量法
270     // random_shuffle(A.begin(), A.end()); // <= C++14
271     auto seed = chrono::steady_clock::now().time_since_epoch().count();
272     default_random_engine e(seed);
273     shuffle(A.begin(), A.end(), e); // >= C++11

```

```

274     circle ans = circle(A[0], 0);
275     for (int i = 1; i < A.size(); i++) {
276         if (ans.inside(A[i]) == -1) {
277             ans = circle(A[i], 0);
278             for (int j = 0; j < i; j++) {
279                 if (ans.inside(A[j]) == -1) {
280                     ans.o = (A[i] + A[j]) / 2;
281                     ans.r = ans.o.dis(A[i]);
282                     for (int k = 0; k < j; k++) {
283                         if (ans.inside(A[k]) == -1)
284                             ans = getCircleOut(A[i], A[j], A[k]);
285                     }
286                 }
287             }
288         }
289     }
290     return ans;
291 }
292
293 typedef vector<point> polygon;
294 db area(polygon p) { // 多边形有向面积
295     if (p.size() < 3) return 0;
296     db ans = 0;
297     for (int i = 1; i < p.size() - 1; i++)
298         ans += cross(p[i] - p[0], p[i + 1] - p[0]);
299     return 0.5L * ans;
300 }
301
302 int checkConvexP(polygon p, point a) { // O(logn)判断点是否在凸包内 2内部 1边界 0外部
303     // 必须保证凸多边形是一个水平序凸包且不能退化
304     // 退化情况 比如凸包退化成线段 可使用 onSegment() 函数特判
305     auto check = [&](int x) {
306         int ccw1 = counterclockwise(p[0], a, p[x]),
307             ccw2 = counterclockwise(p[0], a, p[x + 1]);
308         if (ccw1 == -1 && ccw2 == -1) return 2;
309         else if (ccw1 == 1 && ccw2 == 1) return 0;
310         else if (ccw1 == -1 && ccw2 == 1) return 1;
311         else return 1;
312     };
313     if (counterclockwise(p[0], a, p[1]) <= 0 && counterclockwise(p[0], a,
314         p.back()) >= 0) {
315         int l = 1, r = p.size() - 2, mid;

```

```

315         while (l <= r) {
316             mid = (l + r) >> 1;
317             int chk = check(mid);
318             if (chk == 1) l = mid + 1;
319             else if (chk == -1) r = mid;
320             else break;
321         }
322         int res = counterclockwise(p[mid], a, p[mid + 1]);
323         if (res < 0) return 2;
324         else if (res == 0) return 1;
325         else return 0;
326     } else {
327         return 0;
328     }
329 }
330 int checkPolyP(vector<point> p, point q) { // O(n)判断点是否在一般多边形
    内
331     // 必须保证简单多边形的点按逆时针给出 返回 2 内部 1 边界 0 外部
332     int pd = 0, n = p.size();
333     for (int i = 0; i < n; i++) {
334         point u = p[i], v = p[(i + 1) % n];
335         if (onSegment(u, v, q)) return 1;
336         if (cmp(u.y, v.y) > 0) swap(u, v);
337         if (cmp(u.y, q.y) >= 0 || cmp(v.y, q.y) < 0) continue;
338         if (sign(cross(u - v, q - v)) < 0) pd ^= 1;
339     }
340     return pd << 1;
341 }
342 db convexDiameter(polygon p) { // O(n)旋转卡壳求凸包直径 / 平面最远点对
    的平方
343     int n = p.size(); // 请保证多边形是凸包
344     db ans = 0;
345     for (int i = 0, j = n < 2 ? 0 : 1; i < j; i++) {
346         for (; j = (j + 1) % n) {
347             ans = max(ans, (p[i] - p[j]).len2());
348             if (sign(cross(p[i + 1] - p[i], p[(j + 1) % n] - p[j])) <=
                0) break;
349         }
350     }
351     return ans;
352 }
353 polygon convexHull(polygon A, int flag = 1) { // 凸包 flag=0 不严格 flag
    =1 严格

```

```

354     int n = A.size(); polygon ans(n + n);
355     sort(A.begin(), A.end()); int now = -1;
356     for (int i = 0; i < A.size(); i++) {
357         while (now > 0 && sign(cross(ans[now] - ans[now - 1], A[i] - ans
358             [now - 1])) < flag)
359             now--;
360         ans[++now] = A[i];
361     }
362     int pre = now;
363     for (int i = n - 2; i >= 0; i--) {
364         while (now > pre && sign(cross(ans[now] - ans[now - 1], A[i] -
365             ans[now - 1])) < flag)
366             now--;
367         ans[++now] = A[i];
368     }
369     ans.resize(now);
370     return ans;
371 }
372 bool checkConvexHull(polygon p) { // 检测多边形是否是凸包 (可以有三点共
373     线)
374     int sgn, n = p.size(), i = 0; // 如果三点共线不算凸包 去掉ccw=0的情
375     况
376     for (; i++) { // 这一步是为了防止第一步遇到共线的三个点
377         sgn = counterclockwise(p[i], p[(i + 1) % n], p[(i + 2) % n]);
378         if (sgn) break;
379     }
380     for (; i < n; i++) {
381         int ccw = counterclockwise(p[i], p[(i + 1) % n], p[(i + 2) % n])
382         ;
383         if (ccw && ccw != sgn) {
384             return false;
385         }
386     }
387     return true;
388 }
389 polygon convexCut(polygon A, point k1, point k2) { // 半平面 k1k2 切凸包
390     A
391     int n = A.size(); // 保留所有满足 k1 -> p -> k2 为逆时针方向的点
392     A.push_back(A[0]); // 保留的点可能有重点
393     polygon ans;
394     line cut(k1, k2);
395     for (int i = 0; i < n; i++) {
396         int ccw1 = counterclockwise(k1, k2, A[i]);

```

```

391         int ccw2 = counterclockwise(k1, k2, A[i + 1]);
392         if (ccw1 >= 0) ans.push_back(A[i]);
393         if (ccw1 * ccw2 <= 0) {
394             if (sameLine(cut, line(A[i], A[i + 1]))) { // 半平面恰好切到
395                 ans.push_back(A[i]);
396                 ans.push_back(A[i + 1]);
397             } else {
398                 ans.push_back(getLL(k1, k2, A[i], A[i + 1]));
399             }
400         }
401     }
402     return ans;
403 }
404
405 vector<line> getHL(vector<line>& L) { // 求半平面交 逆时针方向存储
406     sort(L.begin(), L.end());
407     deque<line> q;
408     for (int i = 0; i < (int)L.size(); ++i) {
409         if (i && sameDir(L[i], L[i - 1])) continue;
410         while (q.size() > 1 && !checkpos(q[q.size() - 2], q[q.size() - 1], L[i])) q.pop_back();
411         while (q.size() > 1 && !checkpos(q[1], q[0], L[i])) q.pop_front();
412         q.push_back(L[i]);
413     }
414     while (q.size() > 2 && !checkpos(q[q.size() - 2], q[q.size() - 1], q[0])) q.pop_back();
415     while (q.size() > 2 && !checkpos(q[1], q[0], q[q.size() - 1])) q.pop_front();
416     vector<line> ans;
417     for (int i = 0; i < q.size(); ++i) ans.push_back(q[i]);
418     return ans;
419 }
420
421 db closestPoint(vector<point>& A, int l, int r) { // 最近点对, 先要按照
422     x 坐标排序
423     if (r - l <= 5) {
424         db ans = 1e20;
425         for (int i = l; i <= r; ++i)
426             for (int j = i + 1; j <= r; j++)
427                 ans = min(ans, A[i].dis(A[j]));
428         return ans;

```



```

428     }
429     int mid = l + r >> 1;
430     db ans = min(closestPoint(A, l, mid), closestPoint(A, mid + 1, r));
431     vector<point> B;
432     for (int i = l; i <= r; i++)
433         if (abs(A[i].x - A[mid].x) <= ans)
434             B.push_back(A[i]);
435     sort(B.begin(), B.end(), [&](const point& k1, const point& k2) {
436         return k1.y < k2.y;
437     });
438     for (int i = 0; i < B.size(); i++)
439         for (int j = i + 1; j < B.size() && B[j].y - B[i].y < ans; j++)
440             ans = min(ans, B[i].dis(B[j]));
441     return ans;
442 }
443 }
444 using namespace Geometry;

```

10 数学公式及定理

10.1 求导法则

下文中 f, g 代表可微函数，其余字符代表常数。

1. 加法法则: $(af)' = a \cdot f'$
2. 乘法法则: $(fg)' = gf' + fg'$
3. 除法法则: $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$
4. 链式法则: $(f \circ g)' = f'(g(x))g'(x)$

10.2 麦克劳林级数

$$\begin{aligned}\frac{1}{1-x} &= 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \\ \frac{1}{1+x} &= 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \\ e^x &= 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n!} \\ (1+x)^p &= \sum_{k=0}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} x^k\end{aligned}$$

10.3 泰勒公式

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

10.4 微积分

10.5 球缺

以下公式中 H 为球缺的高, R 为大圆半径。

球缺质心: 匀质球缺的质心位于它的中轴线上, 并且与底面的距离为:

$$c = \frac{(4R - H)H}{12R - 4H}$$

球缺的体积:

$$V = \pi H^2 \left(R - \frac{H}{3} \right)$$

球冠的表面积公式:

$$S = 2\pi RH$$

10.6 圆环整点数

$x^2 + y^2 = n$ 整数解数: 将 n 分解为 $2^x \prod p_i^{c_i} \prod q_i^{d_i}$, 其中 p_i 为形如 $4k+1$ 的质因子, q_i 为 $4k+3$ 的质因子, 解的总数为 $4 \prod (c_i + 1)$ 。

10.7 吸收型马尔可夫链

10.7.1 规范型转移矩阵

两个主要定义:

1. 吸收态：从当前状态只能转移到自身的状态，即转移矩阵中 $p_{i,i} = 1$ 。
2. 瞬态：非吸收态的所有状态称为瞬态。

让所有瞬态位于左侧，吸收态位于右侧，即 $S = \{s_1, s_2, \dots, s_k, s'_{k+1}, \dots, a'_r\}$ ，其中 s_i 表示瞬态， s'_j 表示吸收态，假设瞬态集合大小为 k 。此时，转移矩阵就可以写成：

$$\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

这里， \mathbf{I} 是一个单位矩阵， $\mathbf{0}$ 是一个零矩阵， \mathbf{Q} 是一个 $k \times k$ 的非零矩阵， \mathbf{R} 是一个 $k \times (r - k)$ 的非零矩阵。这个转移矩阵被称为规范型转移矩阵，简称规范矩阵（Canonical Matrix）。

10.7.2 基本矩阵

对于一个吸收型马尔可夫链，如果矩阵 $\mathbf{I} - \mathbf{Q}$ 存在一个逆矩阵 \mathbf{N} ，那么矩阵 \mathbf{N} 的第 i, j 项 $n_{i,j}$ 就是从 s_i 移动到 s_j 的期望步数。

10.7.3 吸收时间

令 \mathbf{t} 为一个 $1 \times r$ 的行向量，其中 t_i 表示从状态 s_i 出发被吸收的期望步数，那么有：

$$\mathbf{t} = \mathbf{N}\mathbf{c}$$

其中 \mathbf{c} 是一个 $r \times 1$ 的全 1 列向量。

更具体地，从瞬态 s_i 出发被任意吸收态 s'_j 吸收的期望时间构成一个 $r - k \times r - k$ 的矩阵 \mathbf{E} ，且有：

$$\mathbf{E} = \frac{\mathbf{N}^2\mathbf{R}}{\mathbf{NR}}$$

这里的除法就是两个矩阵之间的对应元素直接相除，而不是矩阵求逆。

10.7.4 吸收概率

令 $b_{i,j}$ 表示从瞬态 s_i 出发在 s_j 被吸收的概率，那么 \mathbf{B} 是一个 $k \times r$ 的矩阵，且有：

$$\mathbf{B} = \mathbf{NR}$$

其中， \mathbf{R} 就是标准型转移矩阵中的 \mathbf{R} 。

11 杂项

11.1 快速 IO

```

1 // fast IO by yosupo
2 struct Scanner {
3     FILE* fp = nullptr;
4     char line[(1 << 15) + 1];
5     size_t st = 0, ed = 0;
6     void reread() {
7         memmove(line, line + st, ed - st);

```

```

8      ed -= st;
9      st = 0;
10     ed += fread(line + ed, 1, (1 << 15) - ed, fp);
11     line[ed] = '\0';
12 }
13 bool succ() {
14     while (true) {
15         if (st == ed) {
16             reread();
17             if (st == ed) return false;
18         }
19         while (st != ed && isspace(line[st])) st++;
20         if (st != ed) break;
21     }
22     if (ed - st <= 50) reread();
23     return true;
24 }
25 template <class T, enable_if_t<is_same<T, string>::value, int> = 0>
26 bool read_single(T& ref) {
27     if (!succ()) return false;
28     while (true) {
29         size_t sz = 0;
30         while (st + sz < ed && !isspace(line[st + sz])) sz++;
31         ref.append(line + st, sz);
32         st += sz;
33         if (!sz || st != ed) break;
34         reread();
35     }
36     return true;
37 }
38 template <class T, enable_if_t<is_integral<T>::value, int> = 0>
39 bool read_single(T& ref) {
40     if (!succ()) return false;
41     bool neg = false;
42     if (line[st] == '-') {
43         neg = true;
44         st++;
45     }
46     ref = T(0);
47     while (isdigit(line[st])) {
48         ref = 10 * ref + (line[st++] - '0');
49     }
50     if (neg) ref = -ref;

```

```

51         return true;
52     }
53     template <class T> bool read_single(vector<T>& ref) {
54         for (auto& d : ref) {
55             if (!read_single(d)) return false;
56         }
57         return true;
58     }
59     void read() {}
60     template <class H, class... T> void read(H& h, T&... t) {
61         bool f = read_single(h);
62         assert(f);
63         read(t...);
64     }
65     Scanner(FILE* _fp) : fp(_fp) {}
66 };
67
68 struct Printer {
69 public:
70     template <bool F = false> void write() {}
71     template <bool F = false, class H, class... T>
72     void write(const H& h, const T&... t) {
73         if (F) write_single(' ');
74         write_single(h);
75         write<true>(t...);
76     }
77     template <class... T> void writeln(const T&... t) {
78         write(t...);
79         write_single('\n');
80     }
81
82     Printer(FILE* _fp) : fp(_fp) {}
83     ~Printer() { flush(); }
84
85 private:
86     static constexpr size_t SIZE = 1 << 15;
87     FILE* fp;
88     char line[SIZE], small[50];
89     size_t pos = 0;
90     void flush() {
91         fwrite(line, 1, pos, fp);
92         pos = 0;
93     }

```

```

94     void write_single(const char& val) {
95         if (pos == SIZE) flush();
96         line[pos++] = val;
97     }
98     template <class T, enable_if_t<is_integral<T>::value, int> = 0>
99     void write_single(T val) {
100         if (pos > (1 << 15) - 50) flush();
101         if (val == 0) {
102             write_single('0');
103             return;
104         }
105         if (val < 0) {
106             write_single('-');
107             val = -val; // todo min
108         }
109         size_t len = 0;
110         while (val) {
111             small[len++] = char('0' + (val % 10));
112             val /= 10;
113         }
114         for (size_t i = 0; i < len; i++) {
115             line[pos + i] = small[len - 1 - i];
116         }
117         pos += len;
118     }
119     void write_single(const string& s) {
120         for (char c : s) write_single(c);
121     }
122     void write_single(const char* s) {
123         size_t len = strlen(s);
124         for (size_t i = 0; i < len; i++) write_single(s[i]);
125     }
126     template <class T> void write_single(const vector<T>& val) {
127         auto n = val.size();
128         for (size_t i = 0; i < n; i++) {
129             if (i) write_single(' ');
130             write_single(val[i]);
131         }
132     }
133     void write_single(long double d) {
134         {
135             long long v = d;
136             write_single(v);

```

```

137         d -= v;
138     }
139     write_single(' ');
140     for (int __ = 0; __ < 8; __++) {
141         d *= 10;
142         long long v = d;
143         write_single(v);
144         d -= v;
145     }
146 }
147 };
148
149 Scanner sc(stdin);
150 Printer pr(stdout);

```

11.2 蔡勒公式

```

1  int zeller(int y, int m, int d) { // 蔡勒公式 返回星期几
2      if (m <= 2) y--, m += 12;
3      int c = y / 100; y %= 100;
4      int w = ((c >> 2) - (c << 1) + y + (y >> 2) +
5              (13 * (m + 1) / 5) + d - 1) % 7;
6      if (w < 0) w += 7;
7      return (w);
8  }
9  int getId(int y, int m, int d) { // 返回到公元1年1月1日的天数
10     if (m < 3) { y--; m += 12; }
11     return 365 * y + y / 4 - y / 100 + y / 400 +
12           (153 * (m - 3) + 2) / 5 + d - 307;
13 }

```

11.3 枚举子集

11.3.1 暴力遍历

```

1  /* O(3^n) 遍历子集 */
2  for (int i = 0; i < (1 << n); i++) { // i是当前需要遍历的全集
3      for (int l = i;; l = i & (l - 1)) { // l是i的子集
4          int r = i - l; // l+r=i
5          if (!l) break;
6      }
7  }

```

11.3.2 遍历大小为 k 的子集

```

1  int n, k;
2  int s = (1 << k) - 1;
3  while (s < (1 << n)) { // O(binom(n, k))
4      // 每次取出s就是一个大小为k的子集
5      int x = s & -s, y = s + x;
6      s = (((s & ~y) / x) >> 1) | y;
7  }
```

11.4 高维前缀和/SoSDP

```

1  /* 高维前缀和/子集前缀和 */
2  for (int i = 0; i < n; i++) { // O(n2^n)
3      for (int j = 0; j < (1 << n); j++) {
4          if (j & (1 << i)) {
5              pre[j] += pre[j ^ (1 << i)];
6          }
7      }
8  }
```

11.5 压位 BFS

给定一个 n 个点的有向图，当所有边权均为 1 时， $O(\frac{n^3}{w})$ 求任意两点之间的最短路。

```

1  #include <bits/stdc++.h>
2  using namespace std;
3  const int SIZE = 1001;
4  bitset<SIZE> g[SIZE], vis, now;
5  int dis[SIZE][SIZE];
6
7  int main() {
8      ios::sync_with_stdio(false);
9      cin.tie(nullptr);
10     cout.tie(nullptr);
11     memset(dis, -1, sizeof(dis));
12     int n;
13     cin >> n;
14     for (int i = 1; i <= n; i++) {
15         for (int j = 1; j <= n; j++) {
16             int x;
17             cin >> x;
18             if (x) g[i].set(j);
19         }
```



```

20     }
21     for (int i = 1; i <= n; i++) {
22         vis.reset(); // 清空已遍历的数组
23         vis.set(i);
24         queue<int> q;
25         q.push(i);
26         dis[i][i] = 0;
27         while (!q.empty()) {
28             auto top = q.front();
29             q.pop();
30             now = g[top] ^ (g[top] & vis); // 去掉已经遍历到的节点
31             // 本方法的关键: O(n/w) 遍历 bitset
32             for (int to = now._Find_first(); to != now.size(); to = now.
                _Find_next(to)) {
33                 dis[i][to] = dis[i][top] + 1;
34                 q.push(to);
35             }
36             vis |= now; // 更新已遍历的节点
37         }
38     }
39     for (int i = 1; i <= n; i++) {
40         for (int j = 1; j <= n; j++) {
41             cout << dis[i][j] << " \n"[j == n];
42         }
43     }
44     return 0;
45 }

```

11.6 数位 DP

```

1  #include <bits/stdc++.h>
2  using namespace std;
3  using ll = long long;
4  ll a[20], dp[20][20];
5  ll dfs(int len, int las, int maxi, int lead) {
6      if (len == 0) return 1;
7      if (!maxi && !lead && dp[len][las] != -1) return dp[len][las];
8      ll sum = 0;
9      int ma = 9;
10     if (maxi) ma = a[len];
11     if (lead) {
12         for (int i = 0; i <= ma; i++) {
13             if (i == 0) sum += dfs(len - 1, i, 0, lead);

```

```

14         else if (i == ma && maxi) sum += dfs(len - 1, i, maxi, 0);
15         else sum += dfs(len - 1, i, 0, 0);
16     }
17 } else {
18     for (int i = 0; i <= ma; i++) {
19         if (abs(i - las) < 2) continue;
20         if (i == ma && maxi) sum += dfs(len - 1, i, maxi, 0);
21         else sum += dfs(len - 1, i, 0, 0);
22     }
23 }
24 if (maxi == 0 && lead == 0) dp[len][las] = sum;
25 return sum;
26 }
27 ll sol(ll x) {
28     int cnt = 0;
29     a[++cnt] = x % 10;
30     x /= 10;
31     while (x) {
32         a[++cnt] = x % 10;
33         x /= 10;
34     }
35     return dfs(cnt, 0, 1, 1);
36 }
37
38 int main() {
39     ll l, r;
40     cin >> l >> r;
41     memset(dp, -1, sizeof(dp));
42     cout << sol(r) - sol(l - 1) << "\n";
43     return 0;
44 }

```

11.7 随机数生成

```

1 int rd(int l, int r) {
2     mt19937_64 gen(chrono::steady_clock::now().time_since_epoch().count());
3     int p = uniform_int_distribution<int>(l, r)(gen);
4     return p;
5 }

```

11.8 简单对拍

```

1 /*

```

```
2  * gen.exe是数据生成器
3  * a.exe 和 std.exe 是对拍程序和标程
4  */
5  while (1) {
6      system("gen.exe > in.txt");
7      system("a.exe < in.txt > a.out");
8      system("std.exe < in.txt > std.out");
9      if (system("fc a.out std.out")) {
10         break;
11     }
12 }
```