Algorithm Library

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2021 年 11 月 23 日

目录

1 多项式

1.1 FFT - tourist

```
/* copy from tourist */
namespace FFT {
    typedef double dbl;

    struct num {
        dbl x, y;
        num() { x = y = 0; }
        num(dbl x, dbl y) : x(x), y(y) { }
    };

    inline num operator+(num a, num b) { return num(a.x + b.x, a.y + b.y); }
    inline num operator-(num a, num b) { return num(a.x - b.x, a.y - b.y); }
    inline num operator*(num a, num b) { return num(a.x * b.x - a.y * b.y, a.x * b.y + a.y * b.x); }
    inline num conj(num a) { return num(a.x, -a.y); }

    int base = 1;
    vector<num> roots = { {0, 0}, {1, 0} };
    vector<int> rev = { 0, 1 };

    const dbl PI = acosl(-1.0);

    void ensure_base(int nbase) {
        if (nbase <= base) {
            return;
        }
        rev.resize(1 << nbase);
        for (int i = 0; i < (1 << nbase); i++) {
            rev[i] = (rev[i >> 1] >> 1) + ((i & 1) << (nbase - 1));
        }
        roots.resize(1 << nbase);
        while (base < nbase) {
            dbl angle = 2 * PI / (1 << (base + 1));
            for (int i = 1 << (base - 1); i < (1 << base); i++) {
                roots[i << 1] = roots[i];
                dbl angle_i = angle * (2 * i + 1 - (1 << base));
                roots[(i << 1) + 1] = num(cos(angle_i), sin(angle_i));
            }
            base++;
        }
```

```cpp
    }

    void fft(vector<num>& a, int n = -1) {
        if (n == -1) {
            n = a.size();
        }
        assert((n & (n - 1)) == 0);
        int zeros = __builtin_ctz(n);
        ensure_base(zeros);
        int shift = base - zeros;
        for (int i = 0; i < n; i++) {
            if (i < (rev[i] >> shift)) {
                swap(a[i], a[rev[i] >> shift]);
            }
        }
        for (int k = 1; k < n; k <<= 1) {
            for (int i = 0; i < n; i += 2 * k) {
                for (int j = 0; j < k; j++) {
                    num z = a[i + j + k] * roots[j + k];
                    a[i + j + k] = a[i + j] - z;
                    a[i + j] = a[i + j] + z;
                }
            }
        }
    }

    vector<num> fa, fb;

    vector<long long> multiply(vector<int>& a, vector<int>& b) {
        int need = a.size() + b.size() - 1;
        int nbase = 1;
        while ((1 << nbase) < need) nbase++;
        ensure_base(nbase);
        int sz = 1 << nbase;
        if (sz > (int)fa.size()) {
            fa.resize(sz);
        }
        for (int i = 0; i < sz; i++) {
            int x = (i < (int)a.size() ? a[i] : 0);
            int y = (i < (int)b.size() ? b[i] : 0);
            fa[i] = num(x, y);
        }
        fft(fa, sz);
```

```
        num r(0, -0.25 / (sz >> 1));
        for (int i = 0; i <= (sz >> 1); i++) {
            int j = (sz - i) & (sz - 1);
            num z = (fa[j] * fa[j] - conj(fa[i] * fa[i])) * r;
            if (i != j) {
                fa[j] = (fa[i] * fa[i] - conj(fa[j] * fa[j])) * r;
            }
            fa[i] = z;
        }
        for (int i = 0; i < (sz >> 1); i++) {
            num A0 = (fa[i] + fa[i + (sz >> 1)]) * num(0.5, 0);
            num A1 = (fa[i] - fa[i + (sz >> 1)]) * num(0.5, 0) * roots[(sz
               >> 1) + i];
            fa[i] = A0 + A1 * num(0, 1);
        }
        fft(fa, sz >> 1);
        vector<long long> res(need);
        for (int i = 0; i < need; i++) {
            if (i % 2 == 0) {
                res[i] = fa[i >> 1].x + 0.5;
            } else {
                res[i] = fa[i >> 1].y + 0.5;
            }
        }
        return res;
    }

    vector<long long> square(const vector<int>& a) {
        int need = a.size() + a.size() - 1;
        int nbase = 1;
        while ((1 << nbase) < need) nbase++;
        ensure_base(nbase);
        int sz = 1 << nbase;
        if ((sz >> 1) > (int)fa.size()) {
            fa.resize(sz >> 1);
        }
        for (int i = 0; i < (sz >> 1); i++) {
            int x = (2 * i < (int)a.size() ? a[2 * i] : 0);
            int y = (2 * i + 1 < (int)a.size() ? a[2 * i + 1] : 0);
            fa[i] = num(x, y);
        }
        fft(fa, sz >> 1);
        num r(1.0 / (sz >> 1), 0.0);
```

```
        for (int i = 0; i <= (sz >> 2); i++) {
            int j = ((sz >> 1) - i) & ((sz >> 1) - 1);
            num fe = (fa[i] + conj(fa[j])) * num(0.5, 0);
            num fo = (fa[i] - conj(fa[j])) * num(0, -0.5);
            num aux = fe * fe + fo * fo * roots[(sz >> 1) + i] * roots[(sz >> 1) + i];
            num tmp = fe * fo;
            fa[i] = r * (conj(aux) + num(0, 2) * conj(tmp));
            fa[j] = r * (aux + num(0, 2) * tmp);
        }
        fft(fa, sz >> 1);
        vector<long long> res(need);
        for (int i = 0; i < need; i++) {
            if (i % 2 == 0) {
                res[i] = fa[i >> 1].x + 0.5;
            } else {
                res[i] = fa[i >> 1].y + 0.5;
            }
        }
        return res;
    }

    vector<int> multiply_mod(vector<int>& a, vector<int>& b, int m, int eq = 0) {
        int need = a.size() + b.size() - 1;
        int nbase = 0;
        while ((1 << nbase) < need) nbase++;
        ensure_base(nbase);
        int sz = 1 << nbase;
        if (sz > (int)fa.size()) {
            fa.resize(sz);
        }
        for (int i = 0; i < (int)a.size(); i++) {
            int x = (a[i] % m + m) % m;
            fa[i] = num(x & ((1 << 15) - 1), x >> 15);
        }
        fill(fa.begin() + a.size(), fa.begin() + sz, num{ 0, 0 });
        fft(fa, sz);
        if (sz > (int) fb.size()) {
            fb.resize(sz);
        }
        if (eq) {
            copy(fa.begin(), fa.begin() + sz, fb.begin());
```

```
        } else {
            for (int i = 0; i < (int)b.size(); i++) {
                int x = (b[i] % m + m) % m;
                fb[i] = num(x & ((1 << 15) - 1), x >> 15);
            }
            fill(fb.begin() + b.size(), fb.begin() + sz, num{ 0, 0 });
            fft(fb, sz);
        }
        dbl ratio = 0.25 / sz;
        num r2(0, -1);
        num r3(ratio, 0);
        num r4(0, -ratio);
        num r5(0, 1);
        for (int i = 0; i <= (sz >> 1); i++) {
            int j = (sz - i) & (sz - 1);
            num a1 = (fa[i] + conj(fa[j]));
            num a2 = (fa[i] - conj(fa[j])) * r2;
            num b1 = (fb[i] + conj(fb[j])) * r3;
            num b2 = (fb[i] - conj(fb[j])) * r4;
            if (i != j) {
                num c1 = (fa[j] + conj(fa[i]));
                num c2 = (fa[j] - conj(fa[i])) * r2;
                num d1 = (fb[j] + conj(fb[i])) * r3;
                num d2 = (fb[j] - conj(fb[i])) * r4;
                fa[i] = c1 * d1 + c2 * d2 * r5;
                fb[i] = c1 * d2 + c2 * d1;
            }
            fa[j] = a1 * b1 + a2 * b2 * r5;
            fb[j] = a1 * b2 + a2 * b1;
        }
        fft(fa, sz);
        fft(fb, sz);
        vector<int> res(need);
        for (int i = 0; i < need; i++) {
            long long aa = fa[i].x + 0.5;
            long long bb = fb[i].x + 0.5;
            long long cc = fa[i].y + 0.5;
            res[i] = (aa + ((bb % m) << 15) + ((cc % m) << 30)) % m;
        }
        return res;
    }

    vector<int> square_mod(vector<int>& a, int m) {
```

```
        return multiply_mod(a, a, m, 1);
    }
};
```

1.2 形式幂级数

```
#define db double
#ifndef ONLINE_JUDGE // 这三个函数是给MSVC用的，G++不需要
inline int __builtin_clz(int v) { // 返回前导0的个数
    return __lzcnt(v);
}
inline int __builtin_ctz(int v) { // 返回末尾0的个数
    if (v == 0) {
        return 0;
    }
    __asm {
        bsf eax, dword ptr[v];
    }
}
inline int __builtin_popcount(int v) { // 返回二进制中1的个数
    return __popcnt(v);
}
#endif
struct Complex {
    db real, imag;
    Complex(db x = 0, db y = 0) :real(x), imag(y) {}
    Complex& operator+=(const Complex& rhs) {
        real += rhs.real; imag += rhs.imag;
        return *this;
    }
    Complex& operator-=(const Complex& rhs) {
        real -= rhs.real; imag -= rhs.imag;
        return *this;
    }
    Complex& operator*=(const Complex& rhs) {
        db t_real = real * rhs.real - imag * rhs.imag;
        imag = real * rhs.imag + imag * rhs.real;
        real = t_real;
        return *this;
    }
    Complex& operator/=(double x) {
        real /= x, imag /= x;
        return *this;
```

```
    }
    friend Complex operator + (const Complex& a, const Complex& b) { return Complex(a) += b; }
    friend Complex operator - (const Complex& a, const Complex& b) { return Complex(a) -= b; }
    friend Complex operator * (const Complex& a, const Complex& b) { return Complex(a) *= b; }
    friend Complex operator / (const Complex& a, const db& b) { return Complex(a) /= b; }
    Complex power(long long p) const {
        assert(p >= 0);
        Complex a = *this, res = { 1,0 };
        while (p > 0) {
            if (p & 1) res = res * a;
            a = a * a;
            p >>= 1;
        }
        return res;
    }
    static long long val(double x) { return x < 0 ? x - 0.5 : x + 0.5; }
    inline long long Real() const { return val(real); }
    inline long long Imag() const { return val(imag); }
    Complex conj()const { return Complex(real, -imag); }
    explicit operator int()const { return Real(); }
    friend ostream& operator<<(ostream& stream, const Complex& m) {
        return stream << complex<db>(m.real, m.imag);
    }
};
constexpr int MOD = 998244353;
constexpr int Phi_MOD = 998244352;
inline int exgcd(int a, int md = MOD) {
    a %= md;
    if (a < 0) a += md;
    int b = md, u = 0, v = 1;
    while (a) {
        int t = b / a;
        b -= t * a; swap(a, b);
        u -= t * v; swap(u, v);
    }
    assert(b == 1);
    if (u < 0) u += md;
    return u;
}
```

```
inline int add(int a, int b) { return a + b >= MOD ? a + b - MOD : a + b; }
inline int sub(int a, int b) { return a - b < 0 ? a - b + MOD : a - b; }
inline int mul(int a, int b) { return 1LL * a * b % MOD; }
inline int powmod(int a, long long b) {
    int res = 1;
    while (b > 0) {
        if (b & 1) res = mul(res, a);
        a = mul(a, a);
        b >>= 1;
    }
    return res;
}

vector<int> inv, fac, ifac;
void prepare_factorials(int maximum) {
    inv.assign(maximum + 1, 1);
    // Make sure MOD is prime, which is necessary for the inverse algorithm
        below.
    for (int p = 2; p * p <= MOD; p++)
        assert(MOD % p != 0);
    for (int i = 2; i <= maximum; i++)
        inv[i] = mul(inv[MOD % i], (MOD - MOD / i));

    fac.resize(maximum + 1);
    ifac.resize(maximum + 1);
    fac[0] = ifac[0] = 1;

    for (int i = 1; i <= maximum; i++) {
        fac[i] = mul(i, fac[i - 1]);
        ifac[i] = mul(inv[i], ifac[i - 1]);
    }
}
namespace FFT {
    vector<Complex> roots = { Complex(0, 0), Complex(1, 0) };
    vector<int> bit_reverse;
    int max_size = 1 << 20;
    const long double pi = acosl(-1.0l);
    constexpr int FFT_CUTOFF = 150;
    inline bool is_power_of_two(int n) { return (n & (n - 1)) == 0; }
    inline int round_up_power_two(int n) {
        assert(n > 0);
        while (n & (n - 1)) {
            n = (n | (n - 1)) + 1;
```

```cpp
        }
        return n;
    }
    // Given n (a power of two), finds k such that n == 1 << k.
    inline int get_length(int n) {
        assert(is_power_of_two(n));
        return __builtin_ctz(n);
    }
    // Rearranges the indices to be sorted by lowest bit first, then second lowest, etc., rather than highest bit first.
    // This makes even-odd div-conquer much easier.
    void bit_reorder(int n, vector<Complex>& values) {
        if ((int)bit_reverse.size() != n) {
            bit_reverse.assign(n, 0);
            int length = get_length(n);
            for (int i = 0; i < n; i++) {
                bit_reverse[i] = (bit_reverse[i >> 1] >> 1) + ((i & 1) << (length - 1));
            }
        }
        for (int i = 0; i < n; i++) {
            if (i < bit_reverse[i]) {
                swap(values[i], values[bit_reverse[i]]);
            }
        }
    }
    void prepare_roots(int n) {
        assert(n <= max_size);
        if ((int)roots.size() >= n)
            return;
        int length = get_length(roots.size());
        roots.resize(n);
        // The roots array is set up such that for a given power of two n >= 2, roots[n / 2] through roots[n - 1] are
        // the first half of the n-th primitive roots of MOD.
        while (1 << length < n) {
            for (int i = 1 << (length - 1); i < 1 << length; i++) {
                roots[2 * i] = roots[i];
                long double angle = pi * (2 * i + 1) / (1 << length);
                roots[2 * i + 1] = Complex(-cos(angle), -sin(angle));
            }
            length++;
        }
```

```cpp
    }
    void fft_iterative(int N, vector<Complex>& values) {
        assert(is_power_of_two(N));
        prepare_roots(N);
        bit_reorder(N, values);
        for (int n = 1; n < N; n *= 2) {
            for (int start = 0; start < N; start += 2 * n) {
                for (int i = 0; i < n; i++) {
                    Complex& even = values[start + i];
                    Complex odd = values[start + n + i] * roots[n + i];
                    values[start + n + i] = even - odd;
                    values[start + i] = even + odd;
                }
            }
        }
    }
    vector<long long> multiply(vector<int> a, vector<int> b) { // 普通FFT
        int n = a.size();
        int m = b.size();
        if (min(n, m) < FFT_CUTOFF) {
            vector<long long> res(n + m - 1);
            for (int i = 0; i < n; i++) {
                for (int j = 0; j < m; j++) {
                    res[i + j] += 1LL * a[i] * b[j];
                }
            }
            return res;
        }
        int N = round_up_power_two(n + m - 1);
        vector<Complex> tmp(N);
        for (int i = 0; i < a.size(); i++) tmp[i].real = a[i];
        for (int i = 0; i < b.size(); i++) tmp[i].imag = b[i];
        fft_iterative(N, tmp);
        for (int i = 0; i < N; i++) tmp[i] = tmp[i] * tmp[i];
        reverse(tmp.begin() + 1, tmp.end());
        fft_iterative(N, tmp);
        vector<long long> res(n + m - 1);
        for (int i = 0; i < res.size(); i++) {
            res[i] = tmp[i].imag / 2 / N + 0.5;
        }
        return res;
    }
    vector<int> mod_multiply(vector<int> a, vector<int> b, int lim =
```

```
        max_size) { // 任意模数FFT
    int n = a.size();
    int m = b.size();
    if (min(n, m) < FFT_CUTOFF) {
        vector<int> res(n + m - 1);
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) {
                res[i + j] += 1LL * a[i] * b[j] % MOD;
                res[i + j] %= MOD;
            }
        }
        return res;
    }
    int N = round_up_power_two(n + m - 1);
    N = min(N, lim);
    vector<Complex> P(N);
    vector<Complex> Q(N);
    for (int i = 0; i < n; i++) {
        P[i] = Complex(a[i] >> 15, a[i] & 0x7fff);
    }
    for (int i = 0; i < m; i++) {
        Q[i] = Complex(b[i] >> 15, b[i] & 0x7fff);
    }
    fft_iterative(N, P);
    fft_iterative(N, Q);
    vector<Complex>A(N), B(N), C(N), D(N);
    for (int i = 0; i < N; i++) {
        Complex P2 = P[(N - i) & (N - 1)].conj();
        A[i] = (P2 + P[i]) * Complex(0.5, 0),
            B[i] = (P2 - P[i]) * Complex(0, 0.5);
        Complex Q2 = Q[(N - i) & (N - 1)].conj();
        C[i] = (Q2 + Q[i]) * Complex(0.5, 0),
            D[i] = (Q2 - Q[i]) * Complex(0, 0.5);
    }
    for (int i = 0; i < N; i++) {
        P[i] = (A[i] * C[i]) + (B[i] * D[i]) * Complex(0, 1),
            Q[i] = (A[i] * D[i]) + (B[i] * C[i]) * Complex(0, 1);
    }
    reverse(P.begin() + 1, P.end());
    reverse(Q.begin() + 1, Q.end());
    fft_iterative(N, P);
    fft_iterative(N, Q);
    for (int i = 0; i < N; i++) {
```

```
            P[i] /= N, Q[i] /= N;
        }
        int size = min(n + m - 1, lim);
        vector<int> res(size);
        for (int i = 0; i < size; i++) {
            long long ac = P[i].Real() % MOD, bd = P[i].Imag() % MOD,
                ad = Q[i].Real() % MOD, bc = Q[i].Imag() % MOD;
            res[i] = ((ac << 30) + bd + ((ad + bc) << 15)) % MOD;
        }
        return res.resize(n + m - 1), res;
    }
    vector<int> mod_inv(vector<int> a) { // 多项式逆
        int n = a.size();
        int N = round_up_power_two(a.size());
        a.resize(N * 2);
        vector<int> res(1);
        res[0] = exgcd(a[0]);
        for (int i = 2; i <= N; i <<= 1) {
            vector<int> tmp(a.begin(), a.begin() + i);
            int n = (i << 1);
            tmp = mod_multiply(tmp, mod_multiply(res, res, n), n);
            res.resize(i);
            for (int j = 0; j < i; j++) {
                res[j] = add(res[j], sub(res[j], tmp[j]));
            }
        }
        res.resize(n);
        return res;
    }
    vector<int> integral(vector<int> a) { // 多项式积分
        assert(a.size() <= inv.size());
        a.push_back(0);
        for (int i = (int)a.size() - 1; i >= 1; i--) {
            a[i] = mul(a[i - 1], inv[i]);
        }
        return a;
    }
    vector<int> differential(vector<int> a) { // 多项式求导
        for (int i = 0; i < (int)a.size() - 1; i++) {
            a[i] = mul(i + 1, a[i + 1]);
        }
        a.pop_back();
        return a;
```

```
    }
    vector<int> ln(vector<int> a) { // 多项式对数函数
        assert((int)a[0] == 1);
        auto b = mod_multiply(differential(a), mod_inv(a));
        b = integral(b);
        b[0] = 0;
        return b;
    }
    vector<int> exp(vector<int> a) { // 多项式指数函数
        int N = round_up_power_two(a.size());
        int n = a.size();
        a.resize(N * 2);
        vector<int> res{ 1 };
        for (int i = 2; i <= N; i <<= 1) {
            auto tmp = res;
            tmp.resize(i);
            tmp = ln(tmp);
            for (int j = 0; j < i; j++) {
                tmp[j] = sub(a[j], tmp[j]);
            }
            tmp[0] = add(tmp[0], 1);
            res.resize(i);
            res = mod_multiply(res, tmp, i << 1);
            fill(res.begin() + i, res.end(), 0);
        }
        res.resize(n);
        return res;
    }
    // Multiplies many polynomials whose total degree is n in O(n log^2 n).
    vector<int> mod_multiply_all(const vector<vector<int>>& polynomials) {
        if (polynomials.empty())
            return { 1 };
        struct compare_size {
            bool operator()(const vector<int>& x, const vector<int>& y) {
                return x.size() > y.size();
            }
        };
        priority_queue<vector<int>, vector<vector<int>>, compare_size> pq(
            polynomials.begin(), polynomials.end());
        while (pq.size() > 1) {
            vector<int> a = pq.top(); pq.pop();
            vector<int> b = pq.top(); pq.pop();
            pq.push(mod_multiply(a, b));
```

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        }
        return pq.top();
    }
    tuple<int, int, bool> power_reduction(string s, int n) { // 多项式快速幂预处理
        int p = 0, q = 0; bool zero = false;
        for (int i = 0; i < s.length(); i++) {
            p = mul(p, 10);
            p = add(p, s[i] - '0');
            q = 1LL * q * 10 % Phi_MOD; // Phi_MOD 是MOD的欧拉函数值
            q = (q + s[i] - '0');
            if (q >= Phi_MOD) q -= Phi_MOD;
            if (q >= (int)n) zero = true;
        }
        return { p,q,zero };
    }
    vector<int> power(vector<int> a, string s) { // 多项式快速幂 a^s O(nlogn)
        int n = a.size();
        auto [p, q, zero] = power_reduction(s, (int)a.size()); // 不需要降幂的话可以省去这部分
        if (a[0] == 1) {
            auto res = ln(a);
            while ((int)res.size() > n) res.pop_back();
            for (auto& i : res) {
                i = mul(p, i);
            }
            res = exp(res);
            return res;
        } else {
            int mn = -1;
            vector<int> copy_a;
            for (int i = 0; i < (int)a.size(); i++) {
                if (a[i]) {
                    mn = i;
                    break;
                }
            }
            if ((mn == -1) || (mn && (zero || (1LL * mn * p > n)))) { // a中所有元素都是0 或 偏移过大
                return vector<int>(n, 0);
            }
            int inverse_amin = exgcd(a[mn]);
```

```
            for (int i = mn; i < n; i++) {
                copy_a.emplace_back(mul(a[i], inverse_amin));
            }
            copy_a = ln(copy_a);
            while ((int)copy_a.size() > n) copy_a.pop_back();
            for (auto& i : copy_a) {
                i = mul(i, p);
            }
            copy_a = exp(copy_a);
            vector<int> res(n, 0);
            // shift是偏移量 power_k 是a_min^q(q是扩展欧拉定理降出来的幂次)
            int shift = mn * p, power_k = powmod(a[mn], q);
            for (int i = 0; i + shift < n; i++) {
                res[i + shift] = mul(copy_a[i], power_k);
            }
            return res;
        }
    }
    vector<long long> sub_convolution(vector<int> a, vector<int> b) { // 减法卷积 只保留非负次项
        int n = b.size();
        reverse(b.begin(), b.end());
        auto res = multiply(a, b);
        return vector<long long>(res.begin() + n - 1, res.end());
    }
    int bostan_mori(vector<int> p, vector<int> q, long long n) { // [x^n]p(x)/q(x) O(2/3dlog(d)log(n+1)) d是多项式度数
        int i;
        for (; n; n >>= 1) {
            auto r = q;
            for (i = 1; i < r.size(); i += 2) {
                r[i] = MOD - r[i];
            }
            p = mod_multiply(p, r);
            q = mod_multiply(q, r);
            for (i = (n & 1); i < p.size(); i += 2) {
                p[i / 2] = p[i];
            }
            p.resize(i / 2);
            for (i = 0; i < q.size(); i += 2) {
                q[i / 2] = q[i];
            }
            q.resize(i / 2);
```

```
        }
        return p[0];
    }
};
```

1.3 NTT

```
constexpr int MOD = 998244353;
struct Z {
    int val;
    Z(long long v = 0) {
        if (v < 0) v = v % MOD + MOD;
        if (v >= MOD) v %= MOD;
        val = v;
    }
    static int mod_inv(int a, int m = MOD) {
        // https://en.wikipedia.org/wiki/Extended_Euclidean_algorithm#
            Example
        int g = m, r = a, x = 0, y = 1;
        while (r != 0) {
            int q = g / r;
            g %= r; swap(g, r);
            x -= q * y; swap(x, y);
        }
        return x < 0 ? x + m : x;
    }
    explicit operator int() const { return val; }
    explicit operator uint64_t() const { return val; }
    Z& operator+=(const Z& other) {
        val += other.val;
        if (val >= MOD) val -= MOD;
        return *this;
    }
    Z& operator-=(const Z& other) {
        val -= other.val;
        if (val < 0) val += MOD;
        return *this;
    }
    static unsigned fast_mod(uint64_t x, unsigned m = MOD) {
        return x % m;
        /*
        // Optimized mod for Codeforces 32-bit machines.
        // x must be less than 2^32 * m for this to work, so that x / m fits
```

```
            in a 32-bit integer.
        unsigned x_high = x >> 32, x_low = (unsigned) x;
        unsigned quot, rem;
        asm("divl %4\n"
            : "=a" (quot), "=d" (rem)
            : "d" (x_high), "a" (x_low), "r" (m));
        return rem;*/
    }
    Z& operator*=(const Z& other) {
        val = fast_mod((uint64_t)val * other.val);
        return *this;
    }
    Z& operator/=(const Z& other) { return *this *= other.inv(); }
    friend Z operator+(const Z& a, const Z& b) { return Z(a) += b; }
    friend Z operator-(const Z& a, const Z& b) { return Z(a) -= b; }
    friend Z operator*(const Z& a, const Z& b) { return Z(a) *= b; }
    friend Z operator/(const Z& a, const Z& b) { return Z(a) /= b; }
    Z& operator++() {
        val = val == MOD - 1 ? 0 : val + 1;
        return *this;
    }
    Z& operator--() {
        val = val == 0 ? MOD - 1 : val - 1;
        return *this;
    }
    Z operator++(int) { Z before = *this; ++*this; return before; }
    Z operator--(int) { Z before = *this; --*this; return before; }
    Z operator-() const { return val == 0 ? 0 : MOD - val; }
    bool operator==(const Z& other) const { return val == other.val; }
    bool operator!=(const Z& other) const { return val != other.val; }
    Z inv() const { return mod_inv(val); }
    Z pow(long long p) const {
        assert(p >= 0);
        Z a = *this, res = 1;
        while (p > 0) {
            if (p & 1) res *= a;
            a *= a;
            p >>= 1;
        }
        return res;
    }
    friend ostream& operator<<(ostream& stream, const Z& m) {
        return stream << m.val;
```

```cpp
    }
};
vector<Z> inv, fac, ifac;
void prepare_factorials(int maximum) {
    if (maximum + 1 < inv.size()) return;
    inv.assign(maximum + 1, 1);

    // Make sure MOD is prime, which is necessary for the inverse algorithm below.
    for (int p = 2; p * p <= MOD; p++)
        assert(MOD % p != 0);
    for (int i = 2; i <= maximum; i++)
        inv[i] = inv[MOD % i] * (MOD - MOD / i);

    fac.resize(maximum + 1);
    ifac.resize(maximum + 1);
    fac[0] = ifac[0] = 1;

    for (int i = 1; i <= maximum; i++) {
        fac[i] = i * fac[i - 1];
        ifac[i] = inv[i] * ifac[i - 1];
    }
}
inline Z binom(int n, int m) {
    assert(n < fac.size());
    if (n < m || n < 0 || m < 0) return 0;
    return fac[n] * ifac[m] * ifac[n - m];
}
namespace NTT {
    using ull = uint64_t;
    vector<Z> roots = { 0, 1 };
    vector<int> bit_reverse;
    int max_size = -1;
    Z root;

    bool is_power_of_two(int n) { return (n & (n - 1)) == 0; }
    int round_up_power_two(int n) {
        assert(n > 0);
        while (n & (n - 1))
            n = (n | (n - 1)) + 1;
        return n;
    }
    // Given n (a power of two), finds k such that n == 1 << k.
```

```cpp
    int get_length(int n) {
        assert(is_power_of_two(n));
        return __builtin_ctz(n);
    }
    // Rearranges the indices to be sorted by lowest bit first, then second lowest, etc., rather than highest bit first.
    // This makes even-odd div-conquer much easier.
    void bit_reorder(int n, vector<Z>& values) {
        if ((int)bit_reverse.size() != n) {
            bit_reverse.assign(n, 0);
            int length = get_length(n);

            for (int i = 0; i < n; i++)
                bit_reverse[i] = (bit_reverse[i >> 1] >> 1) + ((i & 1) << (length - 1));
        }
        for (int i = 0; i < n; i++)
            if (i < bit_reverse[i])
                swap(values[i], values[bit_reverse[i]]);
    }
    /* 找原根 998244353的原根是31 */
    void find_root() {
        int order = MOD - 1;
        max_size = 1;
        while (order % 2 == 0) {
            order /= 2;
            max_size *= 2;
        }
        root = 2;
        // Find a max_size-th primitive root of MOD.
        while (!((int)root.pow(max_size) == 1 && (int)root.pow(max_size / 2) != 1))
            root = root + 1;
    }

    void prepare_roots(int n) {
        if (max_size < 0) find_root();
        assert(n <= max_size);
        if ((int)roots.size() >= n) return;
        int len = get_length(roots.size());
        roots.resize(n);

        // The roots array is set up such that for a given power of two n >=
```

```
            2, roots[n / 2] through roots[n - 1] are
        // the first half of the n-th primitive roots of MOD.
        while (1 << len < n) {
            // z is a 2^(length + 1)-th primitive root of MOD.
            Z z = root.pow(max_size >> (len + 1));
            for (int i = 1 << (len - 1); i < 1 << len; i++) {
                roots[2 * i] = roots[i];
                roots[2 * i + 1] = roots[i] * z;
            }
            len++;
        }
    }
    void fft_iterative(int N, vector<Z>& values) {
        assert(is_power_of_two(N));
        prepare_roots(N);
        bit_reorder(N, values);
        for (int n = 1; n < N; n *= 2)
            for (int start = 0; start < N; start += 2 * n)
                for (int i = 0; i < n; i++) {
                    Z even = values[start + i];
                    Z odd = values[start + n + i] * roots[n + i];
                    values[start + n + i] = even - odd;
                    values[start + i] = even + odd;
                }
    }

    constexpr int FFT_CUTOFF = 150;
    vector<Z> mod_multiply(vector<Z> left, vector<Z> right) {
        int n = left.size(), m = right.size();
        // Brute force when either n or m is small enough.
        if (min(n, m) < FFT_CUTOFF) {
            const ull ULL_BOUND = numeric_limits<ull>::max() - (ull)MOD * MOD;
            vector<ull> res(n + m - 1);
            for (int i = 0; i < n; i++) {
                for (int j = 0; j < m; j++) {
                    res[i + j] += (ull)((int)left[i]) * ((int)right[j]);
                    if (res[i + j] > ULL_BOUND)
                        res[i + j] %= MOD;
                }
            }
            for (ull& x : res)
                if (x >= MOD)
```

```
                    x %= MOD;
            return vector<Z>(res.begin(), res.end());
        }

        int N = round_up_power_two(n + m - 1);
        left.resize(N);
        right.resize(N);
        fft_iterative(N, left);
        fft_iterative(N, right);
        Z inv_N = Z(N).inv();
        for (int i = 0; i < N; i++)
            left[i] *= right[i] * inv_N;
        reverse(left.begin() + 1, left.end());
        fft_iterative(N, left);
        left.resize(n + m - 1);
        return left;
    }
    vector<Z> sub_conv(vector<Z> a, vector<Z> b) {
        int n = b.size();
        reverse(b.begin(), b.end());
        auto res = mod_multiply(a, b);
        return vector<Z>(res.begin() + n - 1, res.end());
    }
    /*
     * 多项式平移c个单位
     * a(x+c)=\sum x^j 1/j!\sum i!a_i c^{j-i}/(j-i)!
     * O(NlogN)
     */
    vector<Z> shift(vector<Z> a, Z c) {
        int N = round_up_power_two(a.size());
        prepare_factorials(N);
        vector<Z>tmp(a.size());
        Z pc = 1;
        for (int i = 0; i < tmp.size(); i++, pc *= c) {
            tmp[i] = pc * ifac[i];
            a[i] *= fac[i];
        }
        tmp = sub_conv(a, tmp);
        for (int i = 0; i < a.size(); i++) {
            tmp[i] *= ifac[i];
        }
        return tmp;
    }
```

```
    vector<Z> mod_power(const vector<Z>& v, int exponent) {
        assert(exponent >= 0);
        int nn = v.size();
        vector<Z> res = { 1 };
        if (exponent == 0)
            return res;
        for (int k = 31 - __builtin_clz(exponent); k >= 0; k--) {
            res = mod_multiply(res, res);
            if (exponent >> k & 1)
                res = mod_multiply(res, v);
            res.resize(nn);
        }
        return res;
    }
    vector<Z> mod_inv(vector<Z> a) {
        int n = a.size();
        int N = round_up_power_two(a.size());
        a.resize(N * 2);
        vector<Z>res = { a[0].inv() };
        for (int i = 2; i <= N; i <<= 1) {
            vector<Z>tmp(a.begin(), a.begin() + i);
            int n = (i << 1);
            res.resize(n);
            tmp.resize(n);
            fft_iterative(n, tmp);
            fft_iterative(n, res);
            Z inv_n = Z(n).inv();
            for (int j = 0; j < n; j++)
                res[j] = res[j] * (2 - tmp[j] * res[j]) * inv_n;
            reverse(res.begin() + 1, res.end());
            fft_iterative(n, res);
            fill(res.begin() + i, res.end(), 0);
        }
        res.resize(n);
        return res;
    }
    vector<Z> integral(vector<Z> a) {
        assert(a.size() <= inv.size());
        a.push_back(0);
        for (int i = (int)a.size() - 1; i >= 1; i--) {
            a[i] = a[i - 1] * inv[i];
        }
        return a;
```

```
    }
    vector<Z> differential(vector<Z> a) {
        assert(a.size());
        for (int i = 0; i < (int)a.size() - 1; i++) {
            a[i] = a[i + 1] * (i + 1);
        }
        a.pop_back();
        return a;
    }
    vector<Z> ln(vector<Z>a) {
        assert((int)a[0] == 1);
        auto b = mod_multiply(differential(a), mod_inv(a));
        b = integral(b);
        b[0] = 0;
        return b;
    }
    vector<Z> exp(vector<Z>a) {
        int N = round_up_power_two(a.size());
        int n = a.size();
        a.resize(N * 2);
        vector<Z> res{ 1 };
        for (int i = 2; i <= N; i <<= 1) {
            auto tmp = res;
            tmp.resize(i);
            tmp = ln(tmp);
            for (int j = 0; j < i; j++)tmp[j] = a[j] - tmp[j];
            tmp[0] += 1;
            res.resize(i);
            res = mod_multiply(res, tmp);
            fill(res.begin() + i, res.end(), 0);
        }
        res.resize(n);
        return res;
    }
    vector<Z> poly_div(vector<Z> a, vector<Z> b) {
        if (a.size() < b.size()) {
            return { 0 };
        }
        reverse(a.begin(), a.end());
        reverse(b.begin(), b.end());
        b.resize(a.size() - b.size() + 1);
        a.resize(b.size());
        auto d = mod_multiply(mod_inv(b), a);
```

```
        d.resize(b.size());
        reverse(d.begin(), d.end());
        return d;
    }
    /* poly A mod B */
    vector<Z> poly_mod(vector<Z> a, vector<Z> b) {
        auto res = mod_multiply(b, poly_div(a, b));
        a.resize(b.size() - 1);
        res.resize(b.size() - 1);
        for (int i = 0; i < a.size(); i++) {
            a[i] -= res[i];
        }
        return a;
    }
    /* Brute force calc f(k) */
    Z eval(const vector<Z>& f, int k) {
        Z ans;
        for (int i = f.size() - 1; i >= 0; i--)
            ans = (ans * k + f[i]);
        return ans;
    }
    vector<Z> mod_multiply_all(const vector<vector<Z>>& polynomials) {
        if (polynomials.empty())
            return { 1 };
        struct compare_size {
            bool operator()(const vector<Z>& x, const vector<Z>& y) {
                return x.size() > y.size();
            }
        };
        priority_queue<vector<Z>, vector<vector<Z>>, compare_size> pq(
            polynomials.begin(), polynomials.end());
        while (pq.size() > 1) {
            vector<Z> a = pq.top(); pq.pop();
            vector<Z> b = pq.top(); pq.pop();
            pq.push(mod_multiply(a, b));
        }
        return pq.top();
    }
    /*
     * 线性递推求第K项
     * a[n] = \sum seq[i] * a[n-i], calc a[k]
     * O(Nlog(N)log(K))
     */
```

```
    Z linear_seq(const vector<Z>& _init, vector<Z> seq, long long k) {
        reverse(seq.begin(), seq.end());
        for (auto& i : seq)i = -i; seq.push_back(1);
        vector<Z> b = seq;
        reverse(b.begin(), b.end());
        b = mod_inv(b);
        auto poly_mod = [&](vector<Z>a) {
            if (a.size() < seq.size()) {
                a.resize(seq.size() - 1);
                return a;
            }
            vector<Z>tmp = a;
            reverse(a.begin(), a.end());
            b.resize(a.size() - seq.size() + 1);
            a.resize(b.size());
            auto d = mod_multiply(b, a);
            d.resize(b.size());
            reverse(d.begin(), d.end());
            auto res = mod_multiply(seq, d);
            tmp.resize(seq.size() - 1);
            res.resize(seq.size() - 1);
            for (int i = 0; i < tmp.size(); i++) {
                tmp[i] -= res[i];
            }
            return tmp;
        };
        vector<Z> a{ 0,1 };
        vector<Z> res{ 1 };
        for (; k; k >>= 1) {
            if (k & 1)
                res = poly_mod(mod_multiply(res, a));
            a = poly_mod(mod_multiply(a, a));
        }
        Z ans = 0;
        for (int i = 0; i < _init.size(); i++) {
            ans += _init[i] * res[i];
        }
        return ans;
    }
    /*
     * 多点求值
     * 注意如果是1..n的多点求值不要用这个
     * O(Nlog(N)log(M)) 常数偏大
```

```
     */
    vector<Z> multi_eval(vector<Z> F, vector<Z> x) {
        vector<vector<Z>> base;
        function<void(int, int, int)> build = [&](int l, int r, int o) {
            if (r - l == 1) {
                base[o] = { 1,-x[l] };
                return;
            }
            int mid = (l + r) >> 1;
            build(l, mid, o << 1);
            build(mid, r, o << 1 | 1);
            base[o] = (mod_multiply(base[o << 1], base[o << 1 | 1]));
        };
        vector<Z> res(x.size());
        int n = max(x.size(), F.size());
        x.resize(n); F.resize(n + 1);
        base.resize(4 * n);
        build(0, n, 1);
        function<void(vector<Z>, int, int, int)>solve = [&](vector<Z> f, int l, int r, int o) {
            if (r - l == 1) {
                if (l < res.size())res[l] = f[0];
                return;
            }
            int mid = (l + r) >> 1;
            auto L = sub_conv(f, base[o << 1 | 1]);
            auto R = sub_conv(f, base[o << 1]);
            L.resize(mid - l);
            R.resize(r - mid);
            solve(L, l, mid, o << 1);
            solve(R, mid, r, o << 1 | 1);
        };
        solve(sub_conv(F, mod_inv(base[1])), 0, n, 1);
        return res;
    }
    /*
     * 多点插值 给定N个点(xi, yi) 插出一个N-1阶的多项式
     * O(Nlog(N)log(M)) 常数偏大
     */
    vector<Z> multi_inter(const vector<Z>& y, const vector<Z>& x) {
        assert(y.size() == x.size());
        vector<vector<Z>>base;
        function<void(int, int, int)> build = [&](int l, int r, int o) {
```

```cpp
            if (r - l == 1) {
                base[o] = { -x[l],1 };
                return;
            }
            int mid = (l + r) >> 1;
            build(l, mid, o << 1);
            build(mid, r, o << 1 | 1);
            base[o] = (mod_multiply(base[o << 1], base[o << 1 | 1]));
        };
        int n = x.size();
        base.resize(4 * n);
        int s = clock();
        build(0, n, 1);
        vector<Z> pi = differential(base[1]);
        vector<Z> res = multi_eval(pi, x);
        for (int i = 0; i < n; i++)res[i] = y[i] / res[i];
        function<vector<Z>(int, int, int)>solve2 = [&](int l, int r, int o)
          {
            if (r - l == 1) {
                return vector<Z>({ res[l] });
            }
            int mid = (l + r) >> 1;
            vector<Z> L = mod_multiply(solve2(l, mid, o << 1), base[o << 1 |
                1]);
            vector<Z> R = mod_multiply(solve2(mid, r, o << 1 | 1), base[o <<
                1]);
            int n = max(L.size(), R.size());
            L.resize(n); R.resize(n);
            for (int i = 0; i < n; i++) {
                L[i] += R[i];
            }
            return L;
        };
        auto ans = solve2(0, n, 1);
        ans.resize(n);
        return ans;
    }
    vector<Z> mod_power2(const vector<Z>& a, int exponent) {
        int n = a.size();
        auto res = ln(a);
        while ((int)res.size() > n) res.pop_back();
        for (auto& i : res) {
            i = exponent * i;
```

```
        }
        res = exp(res);
        return res;
    }
}
```

1.4 FWT

```
constexpr int MOD = 998244353;
constexpr int MAXN = 1 << 21;
using ll = long long;
namespace FWT {
    int a[MAXN], b[MAXN], ans[MAXN];
    int n;
    ll x, y;
    void print() {
        for (int i = 0; i < n; ++i) cout << a[i] << ' '; cout << "\n";
        for (int i = 0; i < n; ++i) cout << b[i] << ' '; cout << "\n";
        for (int i = 0; i < n; ++i) cout << ans[i] << ' '; cout << "\n";
    }
    void geta(int* t) {
        for (int i = 0; i < n; ++i) a[i] = t[i];
    }
    void getb(int* t) {
        for (int i = 0; i < n; ++i) b[i] = t[i];
    }
    void Or(int* a, int p) {
        for (int i = 1; i < n; i <<= 1) {
            for (int j = 0; j < n; j += (i << 1)) {
                for (int k = 0; k < i; ++k) {
                    x = a[j + k]; y = a[i + j + k];
                    a[j + k] = x;
                    a[i + j + k] = (x * p + y) % MOD;
                }
            }
        }
    }
    void workOr(int* A, int* B, int* Ans, int nn) {
        for (n = 1; n < nn; n <<= 1);
        geta(A); getb(B);
        Or(a, 1);
        Or(b, 1);
        for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;
```

```
        Or(ans, -1);
        for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
    }
    void And(int* a, int p) {
        for (int i = 1; i < n; i <<= 1) {
            for (int j = 0; j < n; j += (i << 1)) {
                for (int k = 0; k < i; ++k) {
                    x = a[j + k]; y = a[i + j + k];
                    a[j + k] = (x + y * p) % MOD;
                    a[i + j + k] = y;
                }
            }
        }
    }
    void workAnd(int* A, int* B, int* Ans, int nn) {
        for (n = 1; n < nn; n <<= 1);
        geta(A); getb(B);
        And(a, 1);
        And(b, 1);
        for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;
        And(ans, -1);
        for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
    }
    void Xor(int* a, int p) {
        for (int i = 1; i < n; i <<= 1) {
            for (int j = 0; j < n; j += (i << 1)) {
                for (int k = 0; k < i; ++k) {
                    x = a[j + k]; y = a[i + j + k];
                    a[j + k] = (x + y) % MOD;
                    a[i + j + k] = (x - y + MOD) % MOD;
                    if (p == -1) {
                        if (a[j + k] & 1) a[j + k] += MOD;
                        if (a[i + j + k] & 1) a[i + j + k] += MOD;
                        a[j + k] >>= 1;
                        a[i + j + k] >>= 1;
                    }
                }
            }
        }
    }
    void workXor(int* A, int* B, int* Ans, int nn) {
        for (n = 1; n < nn; n <<= 1);
        geta(A); getb(B);
```

```cpp
        Xor(a, 1);
        Xor(b, 1);
        for (int i = 0; i < n; ++i) ans[i] = 1LL * a[i] * b[i] % MOD;
        Xor(ans, -1);
        for (int i = 0; i < n; ++i) Ans[i] = (ans[i] + MOD) % MOD;
    }
}
```

1.5 分治 FFT

```cpp
/*
 * given n and g[1], g[2], ..., g[n], f[0] = 1
 * f[i] = sum_{j=1..i}f[i - j]g[j]
 * get f[1], f[2], ..., f[n] in O(nlog^2n)
 */
int n;
vector<int> f, g;
void dfs(int l, int r, int sgn) {
    int mid = (l + r) >> 1;
    if (l != r) {
        dfs(l, mid, 1);
        dfs(mid + 1, r, 0);
    }
    if (sgn) {
        int seg = r - l;
        vector<int> slice(f.begin() + l - 1, f.begin() + r);
        vector<int> cg(g.begin(), g.begin() + min(2 + 2 * seg, n));
        auto h = FFT::mod_multiply(slice, cg);
        for (int i = r; i <= min(r + seg, n - 1); i++) {
            f[i] = add(f[i], h[seg + i - r + 1]);
        }
    }
}
```

2 数论

2.1 简单的防爆模板

2.1.1 类型 1

```cpp
namespace SimpleMod {
    constexpr int MOD = (int)1e9 + 7;
    inline int norm(long long a) { return (a % MOD + MOD) % MOD; }
```

```
    inline int add(int a, int b) { return a + b >= MOD ? a + b - MOD : a + b; }
    inline int sub(int a, int b) { return a - b < 0 ? a - b + MOD : a - b; }
    inline int mul(int a, int b) { return (int)((long long)a * b % MOD); }
    inline int powmod(int a, long long b) {
        int res = 1;
        while (b > 0) {
            if (b & 1) res = mul(res, a);
            a = mul(a, a);
            b >>= 1;
        }
        return res;
    }
    inline int inv(int a) {
        a %= MOD;
        if (a < 0) a += MOD;
        int b = MOD, u = 0, v = 1;
        while (a) {
            int t = b / a;
            b -= t * a; swap(a, b);
            u -= t * v; swap(u, v);
        }
        assert(b == 1);
        if (u < 0) u += MOD;
        return u;
    }
}
```

2.1.2 类型 2

```
template<int MOD> struct Z {
    int x;
    Z(int v = 0) : x(v % MOD) { if (x < 0) x += MOD; }
    Z(long long v = 0) : x(v % MOD) { if (x < 0) x += MOD; }
    Z operator - () const { return x ? MOD - x : 0; }
    Z operator + (const Z& r) { return Z(*this) += r; }
    Z operator - (const Z& r) { return Z(*this) -= r; }
    Z operator * (const Z& r) { return Z(*this) *= r; }
    Z operator / (const Z& r) { return Z(*this) /= r; }
    Z& operator += (const Z& r) {
        x += r.x;
        if (x >= MOD) x -= MOD;
        return *this;
```

```
    }
    Z& operator -= (const Z& r) {
        x -= r.x;
        if (x < 0) x += MOD;
        return *this;
    }
    Z& operator *= (const Z& r) {
        x = 1LL * x * r.x % MOD;
        return *this;
    }
    Z& operator /= (const Z& r) {
        int a = r.x, b = MOD, u = 1, v = 0;
        while (b) {
            long long t = a / b;
            a -= t * b, swap(a, b);
            u -= t * v, swap(u, v);
        }
        x = 1LL * x * u % MOD;
        if (x < 0) x += MOD;
        return *this;
    }
    Z& power(long long k) {
        int a = x; x = 1;
        while (k > 0) {
            if (k & 1) x = 1LL * x * a % MOD;
            a = 1LL * a * a % MOD;
            k >>= 1;
        }
        return *this;
    }
    bool operator == (const Z& r) { return this->x == r.x; }
    bool operator != (const Z& r) { return this->x != r.x; }
    friend constexpr istream& operator >> (istream& is, Z<MOD>& x) noexcept
       {
        is >> x.x;
        x.x %= MOD;
        if (x.x < 0) x.x += MOD;
        return is;
    }
    friend ostream& operator << (ostream& os, const Z<MOD>& x) {
        return os << x.x;
    }
};
```

```cpp
constexpr int MOD = 1e9 + 7;
using mint = Z<MOD>;
```

2.2 约数和

```cpp
long long getSigma(long long n, long long mod = 1e18) {
    if (n == 1) return 1;
    long long ans = 1;
    for (long long i = 2; i * i <= n; ++i) {
        long long cnt = 1;
        while (!(n % i)) {
            cnt = (cnt * i + 1) % mod;
            n /= i;
        }
        ans *= cnt;
    }
    return (n == 1 ? ans : ans * (n + 1) % mod);
}
```

2.3 筛法

2.3.1 线性素数筛

```cpp
vector<bool> isPrime; // true 表示非素数  false 表示是素数
vector<int> prime; // 保存素数
int sieve(int n) {
    isPrime.resize(n + 1, false);
    isPrime[0] = isPrime[1] = true;
    for (int i = 2; i <= n; i++) {
        if (!isPrime[i]) prime.emplace_back(i);
        for (int j = 0; j < (int)prime.size() && prime[j] * i <= n; j++) {
            isPrime[prime[j] * i] = true;
            if (!(i % prime[j])) break;
        }
    }
    return (int)prime.size();
}
```

2.3.2 线性欧拉函数筛

```cpp
bool is_prime[SIZE];
int prime[SIZE], phi[SIZE]; // phi[i] 表示 i 的欧拉函数值
int Phi(int n) { // 线性筛素数的同时线性求欧拉函数
```

```
    phi[1] = 1; is_prime[1] = true;
    int p = 0;
    for (int i = 2; i <= n; i++) {
        if (!is_prime[i]) prime[p++] = i, phi[i] = i - 1;
        for (int j = 0; j < p && prime[j] * i <= n; j++) {
            is_prime[prime[j] * i] = true;
            if (!(i % prime[j])) {
                phi[i * prime[j]] = phi[i] * prime[j];
                break;
            }
            phi[i * prime[j]] = phi[i] * (prime[j] - 1);
        }
    }
    return p;
}
```

2.3.3 线性约数个数函数筛

```
bool is_prime[SIZE];
int prime[SIZE], d[SIZE], num[SIZE]; // d[i] 表示 i 的因子数 num[i] 表示 i 的最小质因子出现次数
int getFactors(int n) { // 线性筛因子数
    d[1] = 1; is_prime[1] = true;
    int p = 0;
    for (int i = 2; i <= n; i++) {
        if (!is_prime[i]) prime[p++] = i, d[i] = 2, num[i] = 1;
        for (int j = 0; j < p && prime[j] * i <= n; j++) {
            is_prime[prime[j] * i] = true;
            if (!(i % prime[j])) {
                num[i * prime[j]] = num[i] + 1;
                d[i * prime[j]] = d[i] / num[i * prime[j]] * (num[i * prime[j]] + 1);
                break;
            }
            num[i * prime[j]] = 1;
            d[i * prime[j]] = d[i] + d[i];
        }
    }
    return p;
}
```

2.3.4 线性素因子个数函数筛

```cpp
bool is_prime[SIZE];
int prime[SIZE], num[SIZE]; // num[i] 表示 i 的质因子数
int getPrimeFactors(int n) { // 线性筛质因子数
    is_prime[1] = true;
    int p = 0;
    for (int i = 2; i <= n; i++) {
        if (!is_prime[i]) prime[p++] = i, num[i] = 1;
        for (int j = 0; j < p && prime[j] * i <= n; j++) {
            is_prime[prime[j] * i] = true;
            if (!(i % prime[j])) {
                num[i * prime[j]] = num[i];
                break;
            }
            num[i * prime[j]] = num[i] + 1;
        }
    }
    return p;
}
```

2.3.5 线性约数和函数筛

```cpp
bool is_prime[SIZE];
int prime[SIZE], f[SIZE], g[SIZE]; // f[i] 表示 i 的约数和
int getSigma(int n) {
    g[1] = f[1] = 1; is_prime[1] = true;
    int p = 0;
    for (int i = 2; i <= n; i++) {
        if (!is_prime[i]) prime[p++] = i, f[i] = g[i] = i + 1;
        for (int j = 0; j < p && prime[j] * i <= n; j++) {
            is_prime[prime[j] * i] = true;
            if (!(i % prime[j])) {
                g[i * prime[j]] = g[i] * prime[j] + 1;
                f[i * prime[j]] = f[i] / g[i] * g[i * prime[j]];
                break;
            }
            f[i * prime[j]] = f[i] * f[prime[j]];
            g[i * prime[j]] = 1 + prime[j];
        }
    }
    return p;
}
```

2.3.6 线性莫比乌斯函数筛

```cpp
bool is_prime[SIZE];
int prime[SIZE], mu[SIZE]; // mu[i] 表示 i 的莫比乌斯函数值
int getMu(int n) { // 线性筛莫比乌斯函数
    mu[1] = 1; is_prime[1] = true;
    int p = 0;
    for (int i = 2; i <= n; i++) {
        if (!is_prime[i]) prime[p++] = i, mu[i] = -1;
        for (int j = 0; j < p && prime[j] * i <= n; j++) {
            is_prime[prime[j] * i] = true;
            if (!(i % prime[j])) {
                mu[i * prime[j]] = 0;
                break;
            }
            mu[i * prime[j]] = -mu[i];
        }
    }
    return p;
}
```

2.4 Pollard-Rho

```cpp
namespace Pollard_Rho {
    typedef long long ll;
    vector<ll> ans; // 存储质因子的数组
    inline ll gcd(ll a, ll b) { ll c; while (b) c = a % b, a = b, b = c; return a; }
    inline ll mulmod(ll x, ll y, const ll z) {
        return (x * y - (ll)(((long double)x * y + 0.5) / (long double)z) * z + z) % z;
    }
    inline ll powmod(ll a, ll b, const ll mo) {
        ll s = 1;
        for (; b; b >>= 1, a = mulmod(a, a, mo)) if (b & 1) s = mulmod(s, a, mo);
        return s;
    }
    bool isPrime(ll p) { // Miller-Rabin O(klog^3(n)) k为素性测试轮数
        const int lena = 10, a[lena] = { 2,3,5,7,11,13,17,19,23,29 };
        if (p == 2) return true;
        if (p == 1 || !(p & 1) || (p == 46856248255981ll)) return false;
        ll D = p - 1;
        while (!(D & 1)) D >>= 1;
```

```
        for (int i = 0; i < lena && a[i] < p; i++) {
            ll d = D, t = powmod(a[i], d, p);
            if (t == 1) continue;
            for (; d != p - 1 && t != p - 1; d <<= 1) t = mulmod(t, t, p);
            if (d == p - 1) return false;
        }
        return true;
    }
    void reportFactor(ll n) { // 得到一个素因子
        ans.emplace_back(n); // 存储素因子
    }
    ll ran() { return rand(); } // 随机数
    void getFactor(ll n) { // Pollard-Rho O(n ^ 1/4)
        if (n == 1) return;
        if (isPrime(n)) { reportFactor(n); return; }
        while (true) {
            ll c = ran() % n, i = 1, x = ran() % n, y = x, k = 2;
            do {
                ll d = gcd(n + y - x, n);
                if (d != 1 && d != n) { getFactor(d); getFactor(n / d); return; }
                if (++i == k) y = x, k <<= 1;
                x = (mulmod(x, x, n) + c) % n;
            } while (y != x);
        }
    }
}
using namespace Pollard_Rho;
```

2.5 扩展欧几里得

2.5.1 线性同余方程最小非负整数解

exgcd 求 $ax + by = c$ 的最小非负整数解详解：

1. 求出 a, b 的最大公约数 $g = \gcd(a, b)$ ，根据裴蜀定理检查是否满足 $c\%g = 0$ ，不满足则无解；
2. 调整系数 a, b, c 为 $a' = \frac{a}{g}, b' = \frac{b}{g}, c' = \frac{c}{g}$ ，这是因为 $ax + by = c$ 和 $a'x + b'y = c'$ 是完全等价的；
3. 实际上 exgcd 求解的方程是 $a'x + b'y = 1$ ，求解前需要注意让系数 $a', b' \geq 0$ （举个例子，如果系数 b' 原本 < 0 ，我们可以翻转 b' 的符号然后令解 (x, y) 为 $(x, -y)$ ，但是求解的时候要把 y 翻回来）;
4. 我们通过 exgcd 求出一组解 (x_0, y_0) ，这组解满足 $a'x_0 + b'y_0 = 1$ ，为了使解合法我们需要令 $x_0 = c'x_0, y_0 = c'y_0$ ，于是有 $a'(c'x_0) + b'(c'y_0) = c''$ ；
5. 考虑到 $a'x_0 + b'y_0 = 1$ 等价于同余方程 $a'x_0 \equiv 1 \pmod{b'}$ ，因此为了求出最小非负整数解，我们最后还需要对 b' 取模；

6. 最后注意特判 $c' = 0$ 的情况，如果要求解 y 且系数 b 发生了翻转，将其翻转回来。

```cpp
long long exgcd(long long a, long long b, long long& x, long long& y) {
    if (!b) {
        x = 1, y = 0;
        return a;
    }
    long long g = exgcd(b, a % b, y, x);
    y -= (a / b) * x;
    return g;
}

ll x, y; // 最小非负整数解
bool solve(ll a, ll b, ll c) { // ax+by=c
    ll g = gcd(a, b);
    if (c % g) return false;
    a /= g, b /= g, c /= g;
    bool flag = false;
    if (b < 0) b = -b, flag = true;
    exgcd(a, b, x, y);
    x = (x * c % b + b) % b;
    if (flag) b = -b;
    y = (c - a * x) / b;
    if (!c) x = y = 0; // ax+by=0
    return true;
}
```

2.5.2 一定范围内线性方程整数解数

exgcd 通解：假设我们通过上方的 exgcd 流程获得了一组解 (x_0, y_0) （没有乘 c ），那么 $a'x + b'y = 1$ 的通解就是 $(x_0 + b't, y_0 - a't)$ ，因此 $a'x + b'y = c'$ 的通解是 $(c'(x_0 + b't), c'(y_0 - a't))$ 。

```cpp
/*
 * 求解 ax+by+c=0 模板CF710D
 * 返回 [xl, xr] x [yl, yr] 的解数
 * 若至少有一组解 则x是一个合法解
 * 可以根据x推出y 但要注意b=0/a=0等特殊情况
 * 特别注意！！！ 设置边界时的取整问题！！！
 * x/y 向上取整时(x+y-1)/y 向下取整时floor(1.0*x/y)
 * */
ll a, b, c, xl, xr, yl, yr;
ll x, y, d;
ll exgcd(ll a, ll b, ll& x, ll& y) {
    if (!b) return x = 1, y = 0, a;
    ll d = exgcd(b, a % b, x, y), t = x;
```

```
    return x = y, y = t - a / b * y, d;
}
ll solve(ll a, ll b, ll c, ll xl, ll xr, ll yl, ll yr) {
    if (xl > xr) return 0;
    if (yl > yr) return 0;
    if (!a && !b) {
        if (c) return 0;
        return (xr - xl + 1) * (yr - yl + 1);
    }
    if (!b) {
        swap(a, b);
        swap(xl, yl);
        swap(xr, yr);
    }
    if (!a) {
        if (c % b) return 0;
        ll y = -c / b;
        if (y < yl || y > yr) return 0;
        return xr - xl + 1;
    }
    d = exgcd((a % abs(b) + abs(b)) % abs(b), abs(b), x, y);
    if (c % d) return 0;
    x = (x % abs(b) + abs(b)) % abs(b) * ((((-c) % abs(b)) + abs(b)) % abs(b
        ) / d) % abs(b / d);
    d = abs(b / d);
    ll kl = (xl - x) / d - 3, kr = (xr - x) / d + 3;
    while (x + kl * d < xl) kl++;
    while (x + kr * d > xr) kr--;
    ll A = (-yl * b - a * x - c) / (a * d), B = (-yr * b - a * x - c) / (a *
        d);
    if (A > B) swap(A, B);
    kl = max(kl, A - 3);
    kr = min(kr, B + 3);
    while (kl <= kr) {
        ll y = (-c - a * x - a * d * kl) / b;
        if (yl <= y && y <= yr) break;
        kl++;
    }
    while (kl <= kr) {
        ll y = (-c - a * x - a * d * kr) / b;
        if (yl <= y && y <= yr) break;
        kr--;
    }
```

```cpp
    if (kl > kr) return 0;
    return kr - kl + 1;
}
```

2.6 类欧几里得

2.6.1 Naive

$$f(a,b,c,n)=\sum_{i=0}^{n}\left\lfloor\frac{ai+b}{c}\right\rfloor$$

原理：

$$\begin{aligned}
f(a,b,c,n)&=\sum_{i=0}^{n}\left\lfloor\frac{ai+b}{c}\right\rfloor\\
&=\sum_{i=0}^{n}\left\lfloor\frac{\left(\left\lfloor\frac{a}{c}\right\rfloor c+a \bmod c\right)i+\left(\left\lfloor\frac{b}{c}\right\rfloor c+b \bmod c\right)}{c}\right\rfloor\\
&=\frac{n(n+1)}{2}\left\lfloor\frac{a}{c}\right\rfloor+(n+1)\left\lfloor\frac{b}{c}\right\rfloor+\sum_{i=0}^{n}\left\lfloor\frac{(a \bmod c)i+(b \bmod c)}{c}\right\rfloor\\
&=\frac{n(n+1)}{2}\left\lfloor\frac{a}{c}\right\rfloor+(n+1)\left\lfloor\frac{b}{c}\right\rfloor+f(a \bmod c,b \bmod c,c,n)
\end{aligned}$$

```cpp
ll f(ll a, ll b, ll c, ll n) { // O(log n)
    if (a == 0)
        return (n + 1) * (b / c);
    if (a >= c || b >= c)
        return (f(a % c, b % c, c, n) + (a / c) * n * (n + 1) / 2 + (b / c)
            * (n + 1));
    ll m = (a * n + b) / c;
    return (n * m - f(c, c - b - 1, a, m - 1));
}
```

2.6.2 General

$$f(a,b,c,n,k1,k2)=\sum_{i=0}^{n}i^{k_1}\left\lfloor\frac{ai+b}{c}\right\rfloor^{k_2}$$

```cpp
// n,a,b,c>=1e9 && k1+k2<=10
// O(k^4log(MAXN))
using ll = long long;
const int K = 10;
const int mod = 1e9 + 7;
const ll lmod = ll(mod) << 32;
const int signs[2] = { 1, mod - 1 };
int invs[K + 2], binom[K + 2][K + 2], B[K + 1], polys[K + 1][K + 2];
```

```cpp

int add(int a, int b) { return (a += b - mod) < 0 ? a + mod : a; }
ll add64(ll a, ll b) { return (a += b - lmod) < 0 ? a + lmod : a;}
int mul(int a, int b) { return 1LL * a * b % mod; }
int power_sum(int e, int x) {
    int ret = 0;
    for (int i = 0; i < e + 2; ++i)
        ret = add(mul(ret, x), polys[e][i]);
    return mul(ret, invs[e + 1]);
}

void init() {
    invs[0] = invs[1] = 1; B[0] = 1;
    for (int i = 2; i <= K + 1; ++i)
        invs[i] = mul(invs[mod % i], mod - mod / i);
    for (int i = 0; i <= K + 1; ++i) {
        binom[i][0] = 1;
        for (int j = 1; j <= i; ++j)
            binom[i][j] = add(binom[i - 1][j - 1], binom[i - 1][j]);
    }
    for (int i = 1; i <= K; ++i) {
        int s = 0;
        for (int j = 0; j < i; ++j)
            s = add(s, mul(binom[i + 1][j], B[j]));
        B[i] = mul(mul(s, invs[i + 1]), signs[1]);
    }
    for (int i = 0; i <= K; ++i) {
        for (int j = 0; j <= i; ++j)
            polys[i][j] = mul(mul(binom[i + 1][j], B[j]), signs[j & 1]);
        polys[i][i + 1] = 0;
    }
    polys[0][1] = 1;
}

int euclidLike(int N, int a, int b, int c, int k1, int k2) {
    assert(N >= 0); assert(a >= 0); assert(b >= 0); assert(c >= 1); assert(
        k1 + k2 <= K);
    using T = tuple<int, int, int, int>;
    stack<T> stac;
    while (1) {
        stac.emplace(N, a, b, c);
        if (N < 0 || a == 0)
            break;
```

```
        if (a >= c) {
            a %= c;
        } else if (b >= c) {
            b %= c;
        } else {
            N = (ll(a) * N + b) / c - 1;
            b = c - 1 - b;
            swap(a, c);
        }
    }

    const int S = k1 + k2;
    static int curr[K + 1][K + 1] = {}, next[K + 1][K + 1] = {};
    while (!stac.empty()) {
        tie(N, a, b, c) = stac.top();
        stac.pop();
        if (N < 0) {
            ;
        } else if (a == 0) {
            int q = b / c;
            for (int e1 = 0; e1 <= S; ++e1) {
                int s = power_sum(e1, N);
                for (int e2 = 0; e2 <= S - e1; ++e2)
                    next[e1][e2] = s, s = mul(s, q);
            }
        } else if (a >= c || b >= c) {
            int q = (a >= c) ? a / c : b / c;
            int d = (a >= c) ? 1 : 0;
            for (int e1 = 0; e1 <= S; ++e1) {
                for (int e2 = 0; e2 <= S - e1; ++e2) {
                    ll s = 0;
                    int p = 1;
                    for (int i2 = 0; i2 <= e2; ++i2) {
                        s = add64(s, ll(p) * mul(binom[e2][i2], curr[e1 + i2 * d][e2 - i2]));
                        p = mul(p, q);
                    }
                    next[e1][e2] = s % mod;
                }
            }
        } else {
            static int cumu[K + 1][K + 1];
            for (int e2 = 0; e2 <= S - 1; ++e2) {
```

```cpp
                for (int e1 = 0; e1 <= S - e2 - 1; ++e1) {
                    ll s = 0;
                    for (int j = 0; j <= e1 + 1; ++j) {
                        s = add64(s, ll(polys[e1][e1 + 1 - j]) * curr[e2][j
                            ]);
                    }
                    cumu[e1][e2] = mul(s % mod, invs[e1 + 1]);
                }
            }
            const int M = (ll(a) * N + b) / c;
            for (int e1 = 0; e1 <= S; ++e1) {
                int p = power_sum(e1, N);
                for (int e2 = 0; e2 <= S - e1; ++e2) {
                    ll t = 0;
                    for (int i2 = 0; i2 < e2; ++i2) {
                        t = add64(t, ll(cumu[e1][i2]) * binom[e2][i2]);
                    }
                    next[e1][e2] = add(p, mod - t % mod);
                    p = mul(p, M);
                }
            }
        }
        swap(curr, next);
    }
    return curr[k1][k2];
}
```

2.7 Wilson 定理

假设 p 是素数，则有：

$$(p-1)! \equiv -1 \pmod p$$

否则除了 $p=4$ 时，$(p-1)! \equiv 0 \pmod p$.

2.8 欧拉定理

$$a^b \equiv \begin{cases} a^{b \bmod \varphi(p)}, & \gcd(a,p)=1 \\ a^b, & \gcd(a,p) \neq 1, b < \varphi(p) \\ a^{b \bmod \varphi(p) + \varphi(p)}, & \gcd(a,p) \neq 1, b \geq \varphi(p) \end{cases} \pmod p$$

2.9 欧拉函数

2.9.1 暴力单点查询

```
long long phi(long long n) { // O(sqrt(N))
    int m = int(sqrt(n + 0.5));
    long long ans = n;
    for (int i = 2; i <= m; i++) {
        if (n % i == 0) {
            ans = ans / i * (i - 1);
            while (n % i == 0) n /= i;
        }
    }
    if (n > 1) ans = ans / n * (n - 1);
    return ans;
}
```

2.9.2 预处理单点查询

```
vector<int> prime; // 求 n 的欧拉函数需要先把 <= ceil(sqrt(n)) 的素数筛出
long long phi(long long n) { // O(sqrt(N)/log(N))
    long long res = n;
    for (int i = 0; i < (int)prime.size(); i++) {
        long long p = prime[i];
        if (p * p > n) break;
        if (n % p == 0) {
            res = res / p * (p - 1);
            while (n % p == 0) n /= p;
        }
    }
    if (n > 1) res = res / n * (n - 1);
    return res;
}
```

2.10 中国剩余定理

2.10.1 CRT

```
// 求解形如 x = ci (mod mi) 的线性方程组 (mi, mj)必须两两互质
long long CRT(vector<long long>& c, vector<long long>& m) {
    long long M = m[0], ans = 0;
    for (int i = 1; i < (int)m.size(); ++i) M *= m[i];
    for (int i = 0; i < (int)m.size(); ++i) { // Mi * ti * ci
        long long mi = M / m[i];
        long long ti = inv(mi, m[i]); // mi 模 m[i] 的逆元
        ans = (ans + mi * ti % M * c[i] % M) % M;
    }
```

```cpp
    ans = (ans + M) % M; // 返回模 M 意义下的唯一解
    return ans;
}
```

2.10.2 EXCRT

```cpp
long long exgcd(long long a, long long b, long long& x, long long& y) {
    if (!b) {
        x = 1, y = 0;
        return a;
    }
    long long g = exgcd(b, a % b, y, x);
    y -= (a / b) * x;
    return g;
}

long long mulmod(long long x, long long y, const long long z) { // x * y % z 防爆
    return (x * y - (long long)(((long double)x * y + 0.5) / (long double)z) * z + z) % z;
}

// 求解形如 x = ci (mod mi) 的线性方程组
long long EXCRT(vector<long long>& c, vector<long long>& m) {
    long long M = m[0], ans = c[0];
    for (int i = 1; i < (int)m.size(); ++i) { // M * x - mi * y = ci - C
        long long x, y, C = ((c[i] - ans) % m[i] + m[i]) % m[i]; // ci - C
        long long G = exgcd(M, m[i], x, y);
        if (C % G) return -1; // 无解
        long long P = m[i] / G;
        x = mulmod(C / G, x, P); // 防爆求最小正整数解 x
        ans += x * M;
        M *= P; // LCM(M, mi)
        ans = (ans % M + M) % M;
    }
    return ans;
}
```

2.11 BSGS

```cpp
ll bsgs(ll a, ll b, ll m) { // a ^ x = b mod m
    ll n = (ll)sqrt((double)m) + 1, base = 1, val = 1;
    map<int, int> mp; // 可以换 unordered_map
```

```cpp
    b %= m;
    for (int i = 0; i < n; ++i) {
        mp[b * base % m] = i;
        base = (base * a) % m;
    }
    a = base;
    if (!a) return b ? -1 : 1;
    for (int i = 0; i <= n; ++i) {
        int j = (!mp.count(val) ? -1 : mp[val]);
        if (j >= 0 && i * n >= j) return i * n - j;
        val = (val * a) % m;
    }
    return -1; // 无解
}
```

2.12 二次剩余

```cpp
using ll = long long;
inline ll mulmod(ll x, ll y, const ll z) {
    return (x * y - (ll)(((long double)x * y + 0.5) / (long double)z) * z + z) % z;
}
inline ll powmod(ll a, ll b, const ll mo) {
    ll s = 1;
    for (; b; b >>= 1, a = mulmod(a, a, mo)) if (b & 1) s = mulmod(s, a, mo);
    return s;
}
ll tonelliShanks(ll n, ll p) { // O(log p)
    if (n == 0) return 0;
    if (p == 2) return (n & 1) ? 1 : -1;
    if (powmod(n, p >> 1, p) != 1) return -1;
    if (p & 2) return powmod(n, p + 1 >> 2, p);
    int s = __builtin_ctzll(p ^ 1);
    ll q = p >> s, z = 2;
    for (; powmod(z, p >> 1, p) == 1; ++z);
    ll c = powmod(z, q, p);
    ll r = powmod(n, q + 1 >> 1, p);
    ll t = powmod(n, q, p), tmp;
    for (int m = s, i; t != 1;) {
        for (i = 0, tmp = t; tmp != 1; i++) tmp = tmp * tmp % p;
        for (; i < --m;) c = c * c % p;
        r = r * c % p;
```

```
        c = c * c % p;
        t = t * c % p;
    }
    return r;
}
```

2.13 迪利克雷卷积

$$g(1)S(n) = \sum_{i=1}^{n}(f*g)(i) - \sum_{i=2}^{n} g(i)S(\lfloor\frac{n}{i}\rfloor)$$

2.14 杜教筛

$$(f*g)(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{xy=n} f(x)g(y)$$

2.15 Berlekamp Massey

```
namespace Berlekamp_Massey {
    typedef long long ll;
    constexpr ll MOD = 1e9 + 7;
    constexpr int N = 10010;
    ll res[N], base[N], _c[N], _md[N];
    vector<int> Md;
    ll powmod(ll a, ll b) {
        ll res = 1;
        while (b > 0) {
            if (b & 1) res = res * a % MOD;
            a = a * a % MOD;
            b >>= 1;
        }
        return res;
    }
    void mul(ll* a, ll* b, int k) {
        for (int i = 0; i < k + k; i++)
            _c[i] = 0;
        for (int i = 0; i < k; i++) {
            if (!a[i]) continue;
            for (int j = 0; j < k; j++) {
                _c[i + j] = (_c[i + j] + a[i] * b[j]) % MOD;
            }
        }
        for (int i = k + k - 1; i >= k; i--) {
            if (!_c[i]) continue;
```

```cpp
            for (int j = 0; j < Md.size(); j++) {
                _c[i - k + Md[j]] = (_c[i - k + Md[j]] - _c[i] * _md[Md[j]]) % MOD;
            }
        }
        for (int i = 0; i < k; i++)
            a[i] = _c[i];
    }
    int solve(ll n, vector<int> a, vector<int> b) { //a系数 b初值 b[n+1]=a[0]*b[n]+...
        //printf("%d\n", (int)b.size());
        //for (int i = 0; i < b.size(); i++)
        //     printf("b[%d] = %d\n", i, b[i]);
        //printf("%d\n", (int)a.size());
        //printf("b[n]");
        //for (int i = 0; i < a.size(); i++) {
        //     if (!i)putchar('='); else putchar('+');
        //     printf("%d*b[n-%d]", a[i], i + 1);
        //}
        //puts("");
        ll ans = 0, pnt = 0;
        int k = a.size();
        for (int i = 0; i < k; i++) {
            _md[k - 1 - i] = -a[i];
        }
        _md[k] = 1;
        Md.clear();
        for (int i = 0; i < k; i++) {
            if (_md[i]) {
                Md.push_back(i);
            }
            res[i] = base[i] = 0;
        }
        res[0] = 1;
        while ((1LL << pnt) <= n) pnt++;
        for (int p = pnt; p >= 0; p--) {
            mul(res, res, k);
            if ((n >> p) & 1) {
                for (int i = k - 1; i >= 0; i--)
                    res[i + 1] = res[i];
                res[0] = 0;
                for (int j = 0; j < Md.size(); j++) {
                    res[Md[j]] = (res[Md[j]] - res[k] * _md[Md[j]]) % MOD;
```

```
                }
            }
        }
        for (int i = 0; i < k; i++)
            ans = (ans + res[i] * b[i]) % MOD;
        return (ans < 0 ? ans + MOD : ans);
    }
    vector<int> BM(vector<int> s) { // O(n^2)
        vector<int> C(1, 1), B(1, 1);
        int L = 0, m = 1, b = 1;
        for (int n = 0; n < (int)s.size(); n++) {
            ll d = 0;
            for (int i = 0; i <= L; i++)
                d = (d + (ll)C[i] * s[n - i]) % MOD;
            if (!d) {
                ++m;
            } else if (2 * L <= n) {
                auto T = C;
                ll c = MOD - d * powmod(b, MOD - 2) % MOD;
                while (C.size() < B.size() + m)
                    C.push_back(0);
                for (int i = 0; i < B.size(); i++)
                    C[i + m] = (C[i + m] + c * B[i]) % MOD;
                L = n + 1 - L; B = T; b = d; m = 1;
            } else {
                ll c = MOD - d * powmod(b, MOD - 2) % MOD;
                while (C.size() < B.size() + m) C.push_back(0);
                for (int i = 0; i < B.size(); i++) {
                    C[i + m] = (C[i + m] + c * B[i]) % MOD;
                }
                ++m;
            }
        }
        return C;
    }
    int work(vector<int> a, ll n) { // 这里的n不是数组大小 是求数列第n项的值
        vector<int> c = BM(a);          // 求第n项的复杂度为 O(k^2logn) k是递推数列大小
        c.erase(c.begin());
        for (int i = 0; i < c.size(); i++)
            c[i] = (MOD - c[i]) % MOD;
        return solve(n, c, vector<int>(a.begin(), a.begin() + (int)c.size()));
```

```
    }
}
```

3 线性代数

3.1 矩阵

```
template<typename T> struct matrix {
    int n, m;
    vector<vector<T>> a;
    matrix(int n_, int m_, int val = 0) : n(n_), m(m_), a(n_, vector<T>(m_,
        val)) {}
    matrix(vector<vector<T>>& mat) : n(mat.size()), m(mat[0].size()), a(mat)
        {}
    vector<T>& operator [] (int k) { return this->a[k]; }
    matrix operator + (matrix& k) { return matrix(*this) += k; }
    matrix operator - (matrix& k) { return matrix(*this) -= k; }
    matrix operator * (matrix& k) { return matrix(*this) *= k; }
    matrix& operator += (matrix& mat) {
        assert(n == mat.n);
        assert(m == mat.m);
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) {
                a[i][j] += mat[i][j];
            }
        }
        return *this;
    }
    matrix& operator -=(matrix& mat) {
        assert(n == mat.n);
        assert(m == mat.m);
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) {
                a[i][j] -= mat[i][j];
            }
        }
        return *this;
    }
    void input() {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) {
                cin >> a[i][j];
            }
```

```
        }
    }
    void output() {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < m; j++) {
                cout << a[i][j] << " \n"[j == m - 1];
            }
        }
    }
    matrix& operator *= (matrix& mat) {
        assert(m == mat.n);
        int x = n, y = mat.m, z = m;
        matrix<T> c(x, y);
        for (int i = 0; i < x; i++) {
            for (int k = 0; k < z; k++) {
                T r = a[i][k];
                for (int j = 0; j < y; j++) {
                    c[i][j] += mat[k][j] * r;
                }
            }
        }
        return *this = c;
    }
    matrix unit(int n_) {
        matrix res(n_, n_);
        for (int i = 0; i < n_; i++)
            res[i][i] = 1;
        return res;
    }
    matrix power(long long k) {
        assert(n == m);
        auto res = unit(n);
        while (k > 0) {
            if (k & 1) res *= (*this);
            (*this) *= (*this);
            k >>= 1;
        }
        return res;
    }
    matrix inverse() {
        assert(n == m);
        auto b = unit(n);
        for (int i = 0; i < n; i++) {
```

```cpp
            if (a[i][i] == 0) return matrix(0, 0);
            T f = T(1) / a[i][i];
            for (int j = 0; j < n; j++) a[i][j] *= f, b[i][j] *= f;
            for (int j = 0; j < n; j++) {
                if (i == j) continue;
                T g = a[j][i];
                for (int k = 0; k < n; k++) {
                    a[j][k] -= g * a[i][k];
                    b[j][k] -= g * b[i][k];
                }
            }
        }
        return b;
    }
    bool empty() { return (!n && !m); }
};
```

3.2 高斯-约旦消元法

```cpp
/*
 * 高斯-约旦消元法
 * 可以修改为解异或方程组 修改策略为
 * a+b -> a^b
 * a-b -> a^b
 * a*b -> a&b
 * a/b -> a*(b==1)
 * */
constexpr double eps = 1e-7;
double a[SIZE][SIZE], ans[SIZE];
void gauss(int n) {
    vector<bool> vis(n, false);
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            if (vis[j]) continue;
            if (fabs(a[j][i]) > eps) {
                vis[i] = true;
                for (int k = 0; k <= n; k++) swap(a[i][k], a[j][k]);
                break;
            }
        }
        if (fabs(a[i][i]) < eps) continue;
        for (int j = 0; j <= n; j++) {
            if (i != j && fabs(a[j][i]) > eps) {
```

```cpp
                double res = a[j][i] / a[i][i];
                for (int k = 0; k <= n; k++) a[j][k] -= a[i][k] * res;
            }
        }
    }
}

int check(int n) { // 解系检测
    int status = 1;
    for (int i = n - 1; i >= 0; i--) {
        if (fabs(a[i][i]) < eps && fabs(a[i][n]) > eps) return -1; // 无解
        if (fabs(a[i][i]) < eps && fabs(a[i][n]) < eps) status = 0; // 无穷解
        ans[i] = a[i][n] / a[i][i];
        if (fabs(ans[i]) < eps) ans[i] = 0;
    }
    return status; // 唯一解或无穷解
}
```

3.3 高斯消元法-bitset

```cpp
constexpr int SIZE = 1001;
bitset<SIZE> a[SIZE];
int ans[SIZE];
void gauss(int n) { // bitset版高斯消元 用于求解异或线性方程组
    bitset<SIZE> vis;
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            if (vis[j]) continue;
            if (a[j][i]) {
                vis.set(i);
                swap(a[i], a[j]);
                break;
            }
        }
        if (!a[i][i]) continue;
        for (int j = 0; j <= n; j++) {
            if (i != j && (a[j][i] & a[i][i])) {
                a[j] ^= a[i];
            }
        }
    }
}
```

3.4 线性基

```cpp
struct linearBasis {
    /* 线性基性质:
     * 1.若a[i]!=0 (即主元i存在)
     *   则线性基中只有a[i]的第i位是1 (只存在一个主元)
     *   且此时a[i]的最高位就是第i位
     * 2.将数组a插入线性基 假设有|B|个元素成功插入
     *   则数组a中每个不同的子集异或和都出现 2^(n-|B|) 次
     * */
    static const int MAXL = 60;
    long long a[MAXL + 1];
    int id[MAXL + 1];
    int zero;
    /* 0的标志位 =1则表示0可以被线性基表示出来
     * 求第k大元素时 需要注意题意中线性基为空时是否可以表示0
     * 默认不可以表示 有必要时进行修改即可
     * */
    linearBasis() {
        zero = 0;
        fill(a, a + MAXL + 1, 0);
    }
    long long& operator[] (int k) { return a[k]; }
    bool insert(long long x) {
        for (int j = MAXL; ~j; j--) {
            if (!(x & (1LL << j))) { // 如果 x 的第 j 位为 0, 则跳过
                continue;
            }
            if (a[j]) { // 如果 a[j] != 0, 则用 a[j] 消去 x 的第 j 位上的 1
                x ^= a[j];
            } else { // 找到插入位置
                for (int k = 0; k < j; k++) {
                    if (x & (1LL << k)) { // 如果x存在某个低位线性基的主元k则消去
                        x ^= a[k];
                    }
                }
                for (int k = j + 1; k <= MAXL; k++) {
                    if (a[k] & (1LL << j)) { // 如果某个高位线性基存在主元j则消去
                        a[k] ^= x;
                    }
                }
                a[j] = x;
```

```
                return true;
            }
        }
        zero = 1;
        return false;
    }
    long long query_max() { // 最大值
        long long res = 0;
        for (int i = MAXL; ~i; i--) {
            res ^= a[i];
        }
        return res;
    }
    long long query_max(long long x) { // 线性基异或x的最大值
        for (int i = MAXL; ~i; i--) {
            if ((x ^ a[i]) > x) {
                x ^= a[i];
            }
        }
        return x;
    }
    long long query_min() { // 最小值
        for (int i = 0; i < MAXL; i++) {
            if (a[i]) {
                return a[i];
            }
        }
        return -1; // 线性基为空
    }
    long long query_min(long long x) { // 线性基异或x的最小值
        for (int i = MAXL; ~i; i--) {
            if ((x ^ a[i]) < x) {
                x ^= a[i];
            }
        }
        return x;
    }
    int count(long long x) { // 元素 x 能否被线性基内元素表示
        int res = 0;
        vector<long long> b(MAXL + 1);
        for (int i = 0; i <= MAXL; i++) {
            b[i] = a[i];
        }
```

```cpp
        res = this->insert(x);
        for (int i = 0; i <= MAXL; i++) {
            a[i] = b[i];
        }
        return !res; // 成功插入则无法表示 失败则可以表示
    }
    int size() { // 线性基有效元素数量
        int res = 0;
        for (int i = 0; i <= MAXL; i++) {
            if (a[i]) {
                res++;
            }
        }
        return res;
    }
    long long kth_element(long long k) { // 第k大元素
        vector<long long> b;
        for (int i = 0; i <= MAXL; i++) {
            if (a[i]) {
                b.push_back(a[i]);
            }
        }
        if (zero) {
            if (--k == 0) {
                return 0;
            }
        }
        if (k >= (1LL << this->size())) { // k超过了线性基可以表示的最大数量
            return -1;
        }
        long long res = 0;
        for (int i = 0; i <= MAXL; i++) {
            if (k & (1LL << i)) {
                res ^= b[i];
            }
        }
        return res;
    }
    long long rank(long long x) { // 元素x在线性基内的排名（默认不考虑0）
        vector<long long> b;
        for (int i = 0; i <= MAXL; i++) {
            if (a[i]) {
                b.push_back(1LL << i);
```

```
            }
        }
        long long res = 0;
        for (int i = 0; i < (int)b.size(); i++) {
            if (x & b[i]) {
                res |= (1LL << i);
            }
        }
        return res;
    }
    void clear() {
        zero = 0;
        fill(a, a + MAXL + 1, 0);
    }
};
```

3.5 矩阵树定理

```
/*
 * 矩阵树定理
 * 有向图：若 u->v 有一条权值为 w 的边 基尔霍夫矩阵 a[v][v] += w, a[v][u] -= w
 * 生成树数量为除去 根所在行和列 后的n-1阶行列式的值
 * 无向图：若 u->v 有一条权值为 w 的边 基尔霍夫矩阵 a[v][v] += w, a[v][u] -= w, a[u][u] += w, a[u][v] -= w
 * 生成树数量为除去 任意一行和列 后的n-1阶行列式的值
 * 无权图则边权默认为1
 * */
typedef long long ll;
typedef unsigned long long u64;
int a[SIZE][SIZE];
int gauss(int a[][SIZE], int n) { // 任意模数求行列式 O(n^2(n + log(mod)))
    int ans = 1;
    for (int i = 1; i <= n; i++) {
        int* x = 0, * y = 0;
        for (int j = i; j <= n; j++) {
            if (a[j][i] && (x == NULL || a[j][i] < x[i])) {
                x = a[j];
            }
        }
        if (x == 0) {
            return 0;
        }
```

```
        for (int j = i; j <= n; j++) {
            if (a[j] != x && a[j][i]) {
                y = a[j];
                for (;;) {
                    int v = md - y[i] / x[i], k = i;
                    for (; k + 3 <= n; k += 4) {
                        y[k + 0] = (y[k + 0] + u64(x[k + 0]) * v) % md;
                        y[k + 1] = (y[k + 1] + u64(x[k + 1]) * v) % md;
                        y[k + 2] = (y[k + 2] + u64(x[k + 2]) * v) % md;
                        y[k + 3] = (y[k + 3] + u64(x[k + 3]) * v) % md;
                    }
                    for (; k <= n; ++k) {
                        y[k] = (y[k] + u64(x[k]) * v) % md;
                    }
                    if (!y[i]) break;
                    swap(x, y);
                }
            }
        }
        if (x != a[i]) {
            for (int k = i; k <= n; k++) {
                swap(x[k], a[i][k]);
            }
            ans = md - ans;
        }
        ans = 1LL * ans * a[i][i] % md;
    }
    return ans;
}
```

3.6 LGV 引理

一般用于有向无环图不相交路径计数（常见于网格图）。

$$M = \begin{bmatrix} e(A_1,B_1) & e(A_1,B_2) & \cdots & e(A_1,B_n) \\ e(A_2,B_1) & e(A_2,B_2) & \cdots & e(A_2,B_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(A_n,B_1) & e(A_n,B_2) & \cdots & e(A_n,B_n) \end{bmatrix} \det(M) = \sum_{S:A\to B} (-1)^{N(\sigma(S))} \prod_{i=1}^{n} \omega(S_i)$$

4 组合数学

4.1 组合数预处理

```
namespace BinomialCoefficient {
```

```
    vector<int> fac, ifac, iv;
    // 组合数预处理 option=1则还会预处理线性逆元
    void prepareFactorials(int maximum = 1000000, int option = 0) {
        fac.assign(maximum + 1, 0);
        ifac.assign(maximum + 1, 0);
        fac[0] = ifac[0] = 1;
        if (option) { // O(3n)
            iv.assign(maximum + 1, 1);
            for (int p = 2; p * p <= MOD; p++)
                assert(MOD % p != 0);
            for (int i = 2; i <= maximum; i++)
                iv[i] = mul(iv[MOD % i], (MOD - MOD / i));
            for (int i = 1; i <= maximum; i++) {
                fac[i] = mul(i, fac[i - 1]);
                ifac[i] = mul(iv[i], ifac[i - 1]);
            }
        } else { // O(2n + log(MOD))
            for (int i = 1; i <= maximum; i++)
                fac[i] = mul(fac[i - 1], i);
            ifac[maximum] = inv(fac[maximum]);
            for (int i = maximum; i; i--)
                ifac[i - 1] = mul(ifac[i], i);
        }
    }
    inline int binom(int n, int m) {
        if (n < m || n < 0 || m < 0) return 0;
        return mul(fac[n], mul(ifac[m], ifac[n - m]));
    }
}
```

4.2 卢卡斯定理

对于质数 p，有：

$$\binom{n}{m} \bmod p = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \cdot \binom{n \bmod p}{m \bmod p} \bmod p$$

```
long long p; // C(n, m) % p
long long lucas(long long n, long long m) { // O(p + log(n))
    if (m == 0) return 1;
    return (binom(n % p, m % p) * lucas(n / p, m / p)) % p; // binom(x, y) 在线性预处理组合数之后是 O(1) 的
}
```

4.3 小球盒子模型

设有 n 个球，k 个盒子：

1. 球之间互不相同，盒子之间互不相同，可以空盒

   根据乘法原理，答案就是 k^n 。

2. 球之间互不相同，盒子之间互不相同，每个盒子至多装一个球

   相当于每个球找一个没有被选过的盒子放进去，答案是 $k^{\underline{n}}$ ，即 $k(k-1)\cdots(k-n+1)$ 。

3. 球之间互不相同，盒子之间互不相同，每个盒子至少装一个球

   可以先把盒子视为相同：n 个球放进 k 个相同盒子、不能空盒，这就是第二类斯特林数 S_n^k 的定义。最后由于盒子不同，再乘上一个排列数，因此答案就是 $k!S_n^k$ 。

4. 球之间互不相同，盒子全部相同，可以空盒

   枚举非空盒子数量，相当于第二类斯特林数一行求和：$\sum_{i=1}^{k} S_n^i$ 。

5. 球之间互不相同，盒子全部相同，每个盒子至多装一个球

   因为盒子相同，不论怎么放都是一样的，答案是 $[n \le k]$（这是一个布尔运算式，若 $n \le k$ 成立则取 1 ，否则 0）。

6. 球之间互不相同，盒子全部相同，每个盒子至少装一个球

   就是第二类斯特林数 S_n^k 。

7. 球全部相同，盒子之间互不相同，可以空盒

   隔板法经典应用，$n+k-1$ 个球选 $k-1$ 个板，因此答案是 $\binom{n+k-1}{k-1}$ 。

8. 球全部相同，盒子之间互不相同，每个盒子至多装一个球

   盒子不同，相当于要选出 n 个盒子装球，因此答案是 $\binom{n}{k}$ 。

9. 球全部相同，盒子之间互不相同，每个盒子至少装一个球

   隔板法经典应用，$n-1$ 个空隙选 $k-1$ 个插板（可以看作是情况 7 时每个盒子里都预先加入一个球），因此答案是 $\binom{n-1}{k-1}$ 。

10. 球全部相同，盒子全部相同，可以空盒

    定义划分数 $p_{n,k}$ 表示将自然数 n 拆成 k 份的方案数，那么本例的结论就是 $p_{n,k}$ 。

    这个问题有一个经典递推式：$p(n,k) = p(n,k-1) + p(n-k,k)$ 。意义是将 j 个自然数 +1 或者加入一个 0 。下面给出一个代码实现：

```
p[0][0] = 1;
for (int i = 1; i <= n; i++) {
    p[0][i] = 1;
    for (int j = 1; j <= m; j++) {
        if (i >= j) {
            p[i][j] = add(p[i][j - 1], p[i - j][j]);
        } else {
            p[i][j] = p[i][j - 1];
```

```
        }
    }
}
```

11. 球全部相同，盒子全部相同，每个盒子至多装一个球

    和情况 5 一致，就是 $[n \le k]$ 。

12. 球全部相同，盒子全部相同，每个盒子至少装一个球

    显然也是一个划分数：$p_{n-k,k}$ 。

4.4 斯特林数

4.4.1 第一类斯特林数

第一类斯特林数 $\begin{bmatrix} n \\ k \end{bmatrix}$ 表示将 n 个不同元素划分入 k 个非空圆排列的方案数。

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}$$

边界是 $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ 。

第一类斯特林数三角形，从 s(1, 1) 开始：

1									
1	1								
2	3	1							
6	11	6	1						
24	50	35	10	1					
120	274	225	85	15	1				
720	1764	1624	735	175	21	1			
5040	13068	13132	6769	1960	322	28	1		
40320	109584	118124	67284	22449	4536	546	36	1	
362880	1026576	1172700	723680	269325	63273	9450	870	45	1

4.4.2 第二类斯特林数

第二类斯特林数 $\begin{Bmatrix} n \\ k \end{Bmatrix}$ 表示将 n 个不同元素划分为 k 个非空子集的方案数。

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}$$

边界是 $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 1$ 。

基于容斥原理的递推方法：

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

第二类斯特林数三角形，从 S(1, 1) 开始：

$$
\begin{array}{llllllllll}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
1 & 15 & 25 & 10 & 1 \\
1 & 31 & 90 & 65 & 15 & 1 \\
1 & 63 & 301 & 350 & 140 & 21 & 1 \\
1 & 127 & 966 & 1701 & 1050 & 266 & 28 & 1 \\
1 & 255 & 3025 & 7770 & 6951 & 2646 & 462 & 36 & 1 \\
1 & 511 & 9330 & 34105 & 42525 & 22827 & 5880 & 750 & 45 & 1
\end{array}
$$

5 博弈论

5.1 SG 定理

一个状态的 SG 函数值是它所有后继状态的 MEX，当 MEX 为 0 时该状态后手必胜，反之先手必胜。当某个游戏由多个独立的子游戏组成时，所有子游戏的 SG 函数异或和为 0 时后手必胜，否则先手必胜。

在 Nim 博弈中的简单应用：显然第 i 堆石子的 SG 函数值就是它的石子数量 a_i ，每一堆石子都是一个独立的子游戏，因此 $a_0 \oplus a_1 \oplus \cdots a_{n-1} = 0$ 时后手必胜，否则先手必胜。

5.2 Bash 博弈

一共 N 个石子，先后手轮流取石子，每次最多取 M 个石子，先取完者获胜。

先手必胜：$N \pmod{M+1} \neq 0$ ；必败：$N \pmod{(M+1)} = 0$。

5.3 Nim-K 博弈

有 N 堆石子，先后手轮流取石子，每次最多可以选择 K 堆石子，被选中的每堆石子都可以取任意个，先取完者获胜。

把 N 堆石子的石子数量都用二进制表示，对于二进制意义下的每一位，如果 1 的数量在模 $K+1$ 意义下**全部**为 0 ，则先手必败。

5.4 Anti-Nim 博弈

规则和 Nim 博弈一致，但是获胜条件改为：不能取石子的一方获胜。

满足以下任意条件，则先手必胜：

1. 所有堆的石子数量 ≤ 1 并且异或和 $= 0$ 。

2. 至少存在一堆石子个数 ≥ 2 并且异或和 $\neq 0$ 。

5.5 Anti-SG 博弈

SG 博弈中最先不能行动的一方获胜。满足以下任意条件，则先手必胜：

1. SG 为 0 并且每一个游戏的 SG 都不超过 1 。

2. SG 不为 0 并且至少有一个游戏的 SG 大于 1 。

5.6 阶梯博弈

有 N 个阶梯（下标从 0 开始），每个阶梯上有一定数量的石子，先后手轮流行动，每次可以从一个阶梯上拿走任意个石子放到下一层阶梯上，最先不能操作者失败。

SG 函数为奇数阶梯上的石子的异或和，如果移动偶数层的石子到奇数层，对手一定可以继续移动这些石子到偶数层使得 SG 不变。

5.7 Wythoff 博弈

有两堆石子，石子数量分别为 A, B ，每次可以从一堆或者两堆里拿走**相同数量**的石子，最先不能取石子的人输。

必败态为：$A_k = \lfloor \frac{k(1+\sqrt{5})}{2} \rfloor, B_k = a_k + k$，假设 $A_k < B_k$ 。

5.8 树上删边博弈

给定一棵 N 个点的有根树，两人轮流操作，每次删除树上的一条边，然后删除所有不与根节点联通的部分，最先不能进行删除操作的人失败。

所有叶子节点的 SG 函数值为 0 ，非叶子节点的 SG 值为其所有子节点（SG 值 +1）的异或和。

5.9 无向图删边博弈

和树上删边博弈的规则一样，不过给出的是一个无向图。

可以将图中的任意一个偶环缩成一个新点，任意一个奇环缩成一个新点加一个新边；所有连到原先环上的边全部改为与新点相连。

5.10 二分图博弈

给出一张二分图和起点 S ，先后手轮流操作，每次只能从当前点（最开始是点 S）移动到一个**相邻**的点，且每个点只能被**遍历一次**，无法继续移动的人输。

先手必胜：二分图的最大匹配**一定**包含起点 S ；先手必败：二分图的最大匹配**不一定**包含起点 S 。

6 图论

6.1 并查集

```
struct dsu {
private:
    // number of nodes
    int n;
    // root node: -1 * component size
    // otherwise: parent
    std::vector<int> pa;
public:
    dsu(int n_ = 0) : n(n_), pa(n_, -1) {}
    // find node x's parent
    int find(int x) {
```

```
        return pa[x] < 0 ? x : pa[x] = find(pa[x]);
    }
    // merge node x and node y
    // if x and y had already in the same component, return false, otherwise
        return true
    // Implement (union by size) + (path compression)
    bool unite(int x, int y) {
        int xr = find(x), yr = find(y);
        if (xr != yr) {
            if (-pa[xr] < -pa[yr]) std::swap(xr, yr);
            pa[xr] += pa[yr];
            pa[yr] = xr; // y -> x
            return true;
        }
        return false;
    }
    // size of the connected component that contains the vertex x
    int size(int x) {
        return -pa[find(x)];
    }
};
```

6.2 最短路

```
namespace Dijkstra {
#define ll long long
    static constexpr ll INF = 1e18;
    int n, m; // 点数 边数
    struct edge {
        int to; // 点
        ll val; // 边权
        edge(int to_ = 0, ll val_ = 0) :to(to_), val(val_) {}
        bool operator < (const edge& k) const { return val > k.val; }
    };
    vector<vector<edge>> g;
    void init() { // 建图操作需要根据题意修改
        cin >> n >> m;
        g.resize(n);
        for (int i = 0; i < m; i++) {
            int u, v, w;
            cin >> u >> v >> w;
            --u, --v;
            g[u].push_back(edge(v, w));
```

```
        }
    }
    ll dijkstra(int s, int t) { // 最短路
        vector<ll> dis(n, INF);
        vector<bool> vis(n, false);
        dis[s] = 0;
        priority_queue<edge> pq;
        pq.push(edge(s, 0));
        while (!pq.empty()) {
            auto top = pq.top();
            pq.pop();
            if (!vis[top.to]) {
                vis[top.to] = true;
                for (auto nxt : g[top.to]) {
                    if (!vis[nxt.to] && dis[nxt.to] > nxt.val + dis[top.to]) {
                        dis[nxt.to] = nxt.val + dis[top.to];
                        pq.push(edge(nxt.to, dis[nxt.to]));
                    }
                }
            }
        }
        return dis[t];
    }
#undef ll
}
```

6.3 最小树形图

```
namespace ZL {
    // a 尽量开大, 之后的边都塞在这个里面
    const int N = 100010, M = 100010, inf = 1e9;
    struct edge {
        int u, v, w, use, id;
        edge(int u_ = 0, int v_ = 0, int w_ = 0, int use_ = 0, int id_ = 0)
            : u(u_), v(v_), w(w_), use(use_), id(id_) {}
    }b[M], a[2000100];
    int n, m, ans, pre[N], id[N], vis[N], root, In[N], h[N], len, way[M];
    // 从root 出发能到达每一个点的最小树形图
    // 先调用init 然后把边add 进去, 需要方案就getway, way[i] 为1 表示使用
    // 最小值保存在ans
    void init(int _n, int _root) { // 点数 根节点
        n = _n; m = 0; b[0].w = inf; root = _root;
```

```cpp
    }
    void add(int u, int v, int w) {
        m++;
        b[m] = edge(u, v, w, 0, m);
        a[m] = b[m];
    }
    int work() {
        len = m;
        for (;;) {
            for (int i = 1; i <= n; i++) { pre[i] = 0; In[i] = inf; id[i] =
                0; vis[i] = 0; h[i] = 0; }
            for (int i = 1; i <= m; i++) {
                if (b[i].u != b[i].v && b[i].w < In[b[i].v]) {
                    pre[b[i].v] = b[i].u; In[b[i].v] = b[i].w; h[b[i].v] = b
                        [i].id;
                }
            }
            for (int i = 1; i <= n; i++) if (pre[i] == 0 && i != root)
                return 0;
            int cnt = 0; In[root] = 0;
            for (int i = 1; i <= n; i++) {
                if (i != root) a[h[i]].use++; int now = i; ans += In[i];
                while (vis[now] == 0 && now != root) { vis[now] = i; now =
                    pre[now]; }
                if (now != root && vis[now] == i) {
                    cnt++; int kk = now;
                    while (1) {
                        id[now] = cnt; now = pre[now];
                        if (now == kk) break;
                    }
                }
            }
            if (cnt == 0) return 1;
            for (int i = 1; i <= n; i++) if (id[i] == 0) id[i] = ++cnt;
            // 缩环，每一条接入的边都会茶包原来接入的那条边，所以要调整边权
            // 新加的边是u，茶包的边是v
            for (int i = 1; i <= m; i++) {
                int k1 = In[b[i].v], k2 = b[i].v;
                b[i].u = id[b[i].u];
                b[i].v = id[b[i].v];
                if (b[i].u != b[i].v) {
                    b[i].w -= k1; a[++len].u = b[i].id; a[len].v = h[k2]; b[
                        i].id = len;
```

```cpp
                    }
                }
                n = cnt; root = id[root];
            }
            return 1;
        }
        void getway() {
            for (int i = 1; i <= m; i++) way[i] = 0;
            for (int i = len; i > m; i--) { a[a[i].u].use += a[i].use; a[a[i].v].use -= a[i].use; }
            for (int i = 1; i <= m; i++) way[i] = a[i].use;
        }
}
```

6.4 最近公共祖先

```cpp
constexpr int SIZE = 200010;
constexpr int DEPTH = 21; // 最大深度 2^DEPTH - 1
int pa[SIZE][DEPTH], dep[SIZE];
vector<int> g[SIZE]; //邻接表
void dfs(int rt, int fin) { //预处理深度和祖先
    pa[rt][0] = fin;
    dep[rt] = dep[pa[rt][0]] + 1; //深度
    for (int i = 1; i < DEPTH; i++) { // rt 的 2^i 祖先等价于 rt 的 2^(i-1) 祖先 的 2^(i-1) 祖先
        pa[rt][i] = pa[pa[rt][i - 1]][i - 1];
    }
    int sz = g[rt].size();
    for (int i = 0; i < sz; ++i) {
        if (g[rt][i] == fin) continue;
        dfs(g[rt][i], rt);
    }
}

int LCA(int x, int y) {
    if (dep[x] > dep[y]) swap(x, y);
    int dif = dep[y] - dep[x];
    for (int j = 0; dif; ++j, dif >>= 1) {
        if (dif & 1) {
            y = pa[y][j]; //先跳到同一高度
        }
    }
    if (y == x) return x;
```

```cpp
    for (int j = DEPTH - 1; j >= 0 && y != x; j--) { //从底往上跳
        if (pa[x][j] != pa[y][j]) { //如果当前祖先不相等 我们就需要更新
            x = pa[x][j];
            y = pa[y][j];
        }
    }
    return pa[x][0];
}
```

6.5 欧拉回路

```cpp
// UOJ117 返回欧拉回路的边集（负数代表走了反向边）
#include <bits/stdc++.h>
using namespace std;
// 有向图欧拉回路 任意点的入度=出度
vector<int> directed_euler_circuit(int n, int m, const vector<vector<pair<
    int, int>>>& g) {
    vector<int> d(n);
    for (const auto& A : g) {
        for (auto p : A) {
            d[p.first]++;
        }
    }
    for (int i = 0; i < n; i++) {
        if (g[i].size() != d[i]) {
            return {};
        }
    }
    vector<vector<pair<int, int>>::const_iterator> it(n);
    for (int i = 0; i < n; i++) it[i] = g[i].begin();
    vector<int> vis(m + 1), p;
    function<void(int)> dfs = [&](int u) {
        for (auto& nxt = it[u]; nxt != g[u].end();) {
            if (!vis[nxt->second]) {
                vis[nxt->second] = 1;
                int v = nxt->second;
                dfs(nxt->first);
                p.push_back(v);
            } else {
                nxt = next(nxt);
            }
        }
    };
```

```cpp
    for (int i = 0; i < n; i++) {
        if (!g[i].empty()) {
            dfs(i);
            break;
        }
    }
    if (p.size() < m) return {};
    reverse(p.begin(), p.end());
    return p;
}
// 无向图欧拉回路 任意点的度数为偶数
vector<int> undirected_euler_circuit(int n, int m, const vector<vector<pair<
   int, int>>>& g) {
    for (const auto& A : g) {
        if (A.size() & 1) {
            return {};
        }
    }
    vector<vector<pair<int, int>>::const_iterator> it(n);
    for (int i = 0; i < n; i++) it[i] = g[i].begin();
    vector<int> vis(m + 1), p;
    function<void(int)> dfs = [&](int u) {
        for (auto& nxt = it[u]; nxt != g[u].end();) {
            if (!vis[abs(nxt->second)]) {
                vis[abs(nxt->second)] = 1;
                int v = nxt->second;
                dfs(nxt->first);
                p.push_back(v);
            } else {
                nxt = next(nxt);
            }
        }
    };
    for (int i = 0; i < n; i++) {
        if (!g[i].empty()) {
            dfs(i);
            break;
        }
    }
    if (p.size() < m) return {};
    reverse(p.begin(), p.end());
    return p;
}
```

```cpp

int main() {
    ios::sync_with_stdio(false);
    cin.tie(nullptr);
    cout.tie(nullptr);
    int t, n, m;
    cin >> t >> n >> m; // t=1是无向图 t=-1是有向图
    vector<vector<pair<int, int>>> G(n + 1);
    for (int i = 1, u, v; i <= m; i++) {
        cin >> u >> v;
        if (t == 1) G[v].push_back({ u, -i });
        G[u].push_back({ v, i });
    }
    auto p = t == 1 ? undirected_euler_circuit(n + 1, m, G) : directed_euler_circuit(n + 1, m, G);
    if (p.size() == m) {
        cout << "YES\n";
        for (int x : p)
            cout << x << " \n"[x == p.back()];
    } else {
        cout << "NO\n";
    }
    return 0;
}
```

6.6 强连通分量

```cpp
namespace SCC {
    // Compressed Sparse Row
    template <class E> struct csr {
        std::vector<int> start;
        std::vector<E> elist;
        explicit csr(int n, const std::vector<std::pair<int, E>>& edges)
            : start(n + 1), elist(edges.size()) {
            for (auto e : edges) {
                start[e.first + 1]++;
            }
            for (int i = 1; i <= n; i++) {
                start[i] += start[i - 1];
            }
            auto counter = start;
            for (auto e : edges) {
                elist[counter[e.first]++] = e.second;
```

```
            }
        }
    };

    struct scc_graph {
    public:
        explicit scc_graph(int n) : _n(n) {}

        int num_vertices() { return _n; }

        void add_edge(int from, int to) { edges.push_back({ from, {to} }); }

        // @return pair of (# of scc, scc id)
        std::pair<int, std::vector<int>> scc_ids() {
            auto g = csr<edge>(_n, edges);
            int now_ord = 0, group_num = 0;
            std::vector<int> visited, low(_n), ord(_n, -1), ids(_n);
            visited.reserve(_n);
            auto dfs = [&](auto self, int v) -> void {
                low[v] = ord[v] = now_ord++;
                visited.push_back(v);
                for (int i = g.start[v]; i < g.start[v + 1]; i++) {
                    auto to = g.elist[i].to;
                    if (ord[to] == -1) {
                        self(self, to);
                        low[v] = std::min(low[v], low[to]);
                    } else {
                        low[v] = std::min(low[v], ord[to]);
                    }
                }
                if (low[v] == ord[v]) {
                    while (true) {
                        int u = visited.back();
                        visited.pop_back();
                        ord[u] = _n;
                        ids[u] = group_num;
                        if (u == v) break;
                    }
                    group_num++;
                }
            };
            for (int i = 0; i < _n; i++) {
                if (ord[i] == -1) dfs(dfs, i);
```

```cpp
        }
        for (auto& x : ids) {
            x = group_num - 1 - x;
        }
        return { group_num, ids };
    }

    // O(N + M)
    // It returns the list of the SCC in topological order.
    std::vector<std::vector<int>> scc() {
        auto ids = scc_ids();
        int group_num = ids.first;
        std::vector<int> counts(group_num);
        for (auto x : ids.second) counts[x]++;
        std::vector<std::vector<int>> groups(ids.first);
        for (int i = 0; i < group_num; i++) {
            groups[i].reserve(counts[i]);
        }
        for (int i = 0; i < _n; i++) {
            groups[ids.second[i]].push_back(i);
        }
        return groups;
    }

  private:
    int _n;
    struct edge {
        int to;
    };
    std::vector<std::pair<int, edge>> edges;
  };
}
```

6.7 2-sat

```cpp
struct two_sat {
public:
    two_sat() : _n(0), scc(0) {}
    explicit two_sat(int n) : _n(n), _answer(n), scc(2 * n) {}

    // 加入一个限制：(i=f) or (j=g)
    void add_clause(int i, bool f, int j, bool g) {
        assert(0 <= i && i < _n);
```

```
        assert(0 <= j && j < _n);
        scc.add_edge(2 * i + (f ? 0 : 1), 2 * j + (g ? 1 : 0));
        scc.add_edge(2 * j + (g ? 0 : 1), 2 * i + (f ? 1 : 0));
    }
    // 加入一个限制：i=f => j=g
    void derive(int i, bool f, int j, bool g) {
        add_clause(i, !f, j, g);
    }
    // O(N + M) 如果返回true，则一个方案会保存在answer里
    bool satisfiable() {
        auto id = scc.scc_ids().second;
        for (int i = 0; i < _n; i++) {
            if (id[2 * i] == id[2 * i + 1]) return false;
            _answer[i] = id[2 * i] < id[2 * i + 1];
        }
        return true;
    }
    std::vector<bool> answer() { return _answer; }

private:
    int _n;
    std::vector<bool> _answer;
    SCC::scc_graph scc;
};
```

6.8 最大流

```
template <class T> struct simple_queue {
    std::vector<T> payload;
    int pos = 0;
    void reserve(int n) { payload.reserve(n); }
    int size() const { return int(payload.size()) - pos; }
    bool empty() const { return pos == int(payload.size()); }
    void push(const T& t) { payload.push_back(t); }
    T& front() { return payload[pos]; }
    void clear() {
        payload.clear();
        pos = 0;
    }
    void pop() { pos++; }
};

template <class Cap> struct mf_graph {
```

```cpp
public:
    mf_graph() : _n(0) {}
    mf_graph(int n) : _n(n), g(n) {}

    // returns an integer k such that this is the k-th edge that is added.
    int add_edge(int from, int to, Cap cap) {
        assert(0 <= from && from < _n);
        assert(0 <= to && to < _n);
        assert(0 <= cap);
        int m = int(pos.size());
        pos.push_back({ from, int(g[from].size()) });
        int from_id = int(g[from].size());
        int to_id = int(g[to].size());
        if (from == to) to_id++;
        g[from].push_back(_edge{ to, to_id, cap });
        g[to].push_back(_edge{ from, from_id, 0 });
        return m;
    }

    struct edge {
        int from, to;
        Cap cap, flow;
    };

    edge get_edge(int i) {
        int m = int(pos.size());
        assert(0 <= i && i < m);
        auto _e = g[pos[i].first][pos[i].second];
        auto _re = g[_e.to][_e.rev];
        return edge{ pos[i].first, _e.to, _e.cap + _re.cap, _re.cap };
    }
    std::vector<edge> edges() {
        int m = int(pos.size());
        std::vector<edge> result;
        for (int i = 0; i < m; i++) {
            result.push_back(get_edge(i));
        }
        return result;
    }
    void change_edge(int i, Cap new_cap, Cap new_flow) {
        int m = int(pos.size());
        assert(0 <= i && i < m);
        assert(0 <= new_flow && new_flow <= new_cap);
```

```
        auto& _e = g[pos[i].first][pos[i].second];
        auto& _re = g[_e.to][_e.rev];
        _e.cap = new_cap - new_flow;
        _re.cap = new_flow;
    }

    // max flow from s to t
    // O(M*N^2) general
    // O(min(M*N^2/3, M^3/2)) if capacities of edges are 1
    Cap flow(int s, int t) {
        return flow(s, t, std::numeric_limits<Cap>::max());
    }
    Cap flow(int s, int t, Cap flow_limit) {
        assert(0 <= s && s < _n);
        assert(0 <= t && t < _n);
        assert(s != t);

        std::vector<int> level(_n), iter(_n);
        simple_queue<int> que;

        auto bfs = [&]() {
            std::fill(level.begin(), level.end(), -1);
            level[s] = 0;
            que.clear();
            que.push(s);
            while (!que.empty()) {
                int v = que.front();
                que.pop();
                for (auto e : g[v]) {
                    if (e.cap == 0 || level[e.to] >= 0) continue;
                    level[e.to] = level[v] + 1;
                    if (e.to == t) return;
                    que.push(e.to);
                }
            }
        };
        auto dfs = [&](auto self, int v, Cap up) {
            if (v == s) return up;
            Cap res = 0;
            int level_v = level[v];
            for (int& i = iter[v]; i < int(g[v].size()); i++) {
                _edge& e = g[v][i];
                if (level_v <= level[e.to] || g[e.to][e.rev].cap == 0)
```

```
                    continue;
                Cap d =
                    self(self, e.to, std::min(up - res, g[e.to][e.rev].cap))
                        ;
                if (d <= 0) continue;
                g[v][i].cap += d;
                g[e.to][e.rev].cap -= d;
                res += d;
                if (res == up) break;
            }
            return res;
        };

        Cap flow = 0;
        while (flow < flow_limit) {
            bfs();
            if (level[t] == -1) break;
            std::fill(iter.begin(), iter.end(), 0);
            while (flow < flow_limit) {
                Cap f = dfs(dfs, t, flow_limit - flow);
                if (!f) break;
                flow += f;
            }
        }
        return flow;
    }

    std::vector<bool> min_cut(int s) {
        std::vector<bool> visited(_n);
        simple_queue<int> que;
        que.push(s);
        while (!que.empty()) {
            int p = que.front();
            que.pop();
            visited[p] = true;
            for (auto e : g[p]) {
                if (e.cap && !visited[e.to]) {
                    visited[e.to] = true;
                    que.push(e.to);
                }
            }
        }
        return visited;
```

```
    }

private:
    int _n;
    struct _edge {
        int to, rev;
        Cap cap;
    };
    std::vector<std::pair<int, int>> pos;
    std::vector<std::vector<_edge>> g;
};
```

6.9 最小费用最大流

```
/*
 * 费用流Cost常用类型的上限：int范围内 0 <= nx <= 2e9 + 1000， long long范围内： 0 <= nx <= 8e18 + 1000
 *
 * min_cost_slope() 函数返回的是一个分段函数F(x)（其中x代表流量上界，F(x)代表当前最大流量的最小费用）
 * 返回的vector是所有F(x)改变的点
 * 时间复杂度 O(f(N + M))log(N + M) f(N + M) 代表图的流量总和
 * */
namespace MCMF {
    template <class T> struct simple_queue {
        std::vector<T> payload;
        int pos = 0;
        void reserve(int n) { payload.reserve(n); }
        int size() const { return int(payload.size()) - pos; }
        bool empty() const { return pos == int(payload.size()); }
        void push(const T& t) { payload.push_back(t); }
        T& front() { return payload[pos]; }
        void clear() {
            payload.clear();
            pos = 0;
        }
        void pop() { pos++; }
    };

    template <class E> struct csr {
        std::vector<int> start;
        std::vector<E> elist;
        explicit csr(int n, const std::vector<std::pair<int, E>>& edges)
```

```cpp
            : start(n + 1), elist(edges.size()) {
            for (auto e : edges) {
                start[e.first + 1]++;
            }
            for (int i = 1; i <= n; i++) {
                start[i] += start[i - 1];
            }
            auto counter = start;
            for (auto e : edges) {
                elist[counter[e.first]++] = e.second;
            }
        }
    };

    template <class Cap, class Cost> struct mcf_graph {
    public:
        mcf_graph() {}
        explicit mcf_graph(int n) : _n(n) {}

        int add_edge(int from, int to, Cap cap, Cost cost) {
            assert(0 <= from && from < _n);
            assert(0 <= to && to < _n);
            assert(0 <= cap);
            assert(0 <= cost);
            int m = int(_edges.size());
            _edges.push_back({ from, to, cap, 0, cost });
            return m;
        }

        struct edge {
            int from, to;
            Cap cap, flow;
            Cost cost;
        };

        edge get_edge(int i) {
            int m = int(_edges.size());
            assert(0 <= i && i < m);
            return _edges[i];
        }
        std::vector<edge> edges() { return _edges; }

        std::pair<Cap, Cost> flow(int s, int t) {
```

```
        return flow(s, t, std::numeric_limits<Cap>::max());
    }
    std::pair<Cap, Cost> flow(int s, int t, Cap flow_limit) {
        return slope(s, t, flow_limit).back();
    }
    std::vector<std::pair<Cap, Cost>> slope(int s, int t) {
        return slope(s, t, std::numeric_limits<Cap>::max());
    }
    std::vector<std::pair<Cap, Cost>> slope(int s, int t, Cap flow_limit
        ) {
        assert(0 <= s && s < _n);
        assert(0 <= t && t < _n);
        assert(s != t);

        int m = int(_edges.size());
        std::vector<int> edge_idx(m);

        auto g = [&]() {
            std::vector<int> degree(_n), redge_idx(m);
            std::vector<std::pair<int, _edge>> elist;
            elist.reserve(2 * m);
            for (int i = 0; i < m; i++) {
                auto e = _edges[i];
                edge_idx[i] = degree[e.from]++;
                redge_idx[i] = degree[e.to]++;
                elist.push_back({ e.from, {e.to, -1, e.cap - e.flow, e.
                    cost} });
                elist.push_back({ e.to, {e.from, -1, e.flow, -e.cost} })
                    ;
            }
            auto _g = csr<_edge>(_n, elist);
            for (int i = 0; i < m; i++) {
                auto e = _edges[i];
                edge_idx[i] += _g.start[e.from];
                redge_idx[i] += _g.start[e.to];
                _g.elist[edge_idx[i]].rev = redge_idx[i];
                _g.elist[redge_idx[i]].rev = edge_idx[i];
            }
            return _g;
        }();

        auto result = slope(g, s, t, flow_limit);

```

```
            for (int i = 0; i < m; i++) {
                auto e = g.elist[edge_idx[i]];
                _edges[i].flow = _edges[i].cap - e.cap;
            }

            return result;
        }

    private:
        int _n;
        std::vector<edge> _edges;

        // inside edge
        struct _edge {
            int to, rev;
            Cap cap;
            Cost cost;
        };

        std::vector<std::pair<Cap, Cost>> slope(csr<_edge>& g,
            int s,
            int t,
            Cap flow_limit) {
            // variants (C = maxcost):
            // -(n-1)C <= dual[s] <= dual[i] <= dual[t] = 0
            // reduced cost (= e.cost + dual[e.from] - dual[e.to]) >= 0 for
               all edge

            // dual_dist[i] = (dual[i], dist[i])
            std::vector<std::pair<Cost, Cost>> dual_dist(_n);
            std::vector<int> prev_e(_n);
            std::vector<bool> vis(_n);
            struct Q {
                Cost key;
                int to;
                bool operator<(Q r) const { return key > r.key; }
            };
            std::vector<int> que_min;
            std::vector<Q> que;
            auto dual_ref = [&]() {
                for (int i = 0; i < _n; i++) {
                    dual_dist[i].second = std::numeric_limits<Cost>::max();
                }
```

```cpp
                std::fill(vis.begin(), vis.end(), false);
                que_min.clear();
                que.clear();

                // que[0..heap_r) was heapified
                size_t heap_r = 0;

                dual_dist[s].second = 0;
                que_min.push_back(s);
                while (!que_min.empty() || !que.empty()) {
                    int v;
                    if (!que_min.empty()) {
                        v = que_min.back();
                        que_min.pop_back();
                    } else {
                        while (heap_r < que.size()) {
                            heap_r++;
                            std::push_heap(que.begin(), que.begin() + heap_r);
                        }
                        v = que.front().to;
                        std::pop_heap(que.begin(), que.end());
                        que.pop_back();
                        heap_r--;
                    }
                    if (vis[v]) continue;
                    vis[v] = true;
                    if (v == t) break;
                    // dist[v] = shortest(s, v) + dual[s] - dual[v]
                    // dist[v] >= 0 (all reduced cost are positive)
                    // dist[v] <= (n-1)C
                    Cost dual_v = dual_dist[v].first, dist_v = dual_dist[v].second;
                    for (int i = g.start[v]; i < g.start[v + 1]; i++) {
                        auto e = g.elist[i];
                        if (!e.cap) continue;
                        // |-dual[e.to] + dual[v]| <= (n-1)C
                        // cost <= C - -(n-1)C + 0 = nC
                        Cost cost = e.cost - dual_dist[e.to].first + dual_v;
                        if (dual_dist[e.to].second - dist_v > cost) {
                            Cost dist_to = dist_v + cost;
                            dual_dist[e.to].second = dist_to;
                            prev_e[e.to] = e.rev;
```

```
                        if (dist_to == dist_v) {
                            que_min.push_back(e.to);
                        } else {
                            que.push_back(Q{ dist_to, e.to });
                        }
                    }
                }
            }
            if (!vis[t]) {
                return false;
            }

            for (int v = 0; v < _n; v++) {
                if (!vis[v]) continue;
                // dual[v] = dual[v] - dist[t] + dist[v]
                //         = dual[v] - (shortest(s, t) + dual[s] - dual[t]) +
                //           (shortest(s, v) + dual[s] - dual[v]) = - shortest(s,
                //           t) + dual[t] + shortest(s, v) = shortest(s, v) -
                //           shortest(s, t) >= 0 - (n-1)C
                dual_dist[v].first -= dual_dist[t].second - dual_dist[v].second;
            }
            return true;
        };
        Cap flow = 0;
        Cost cost = 0, prev_cost_per_flow = -1;
        std::vector<std::pair<Cap, Cost>> result = { {Cap(0), Cost(0)} };
        while (flow < flow_limit) {
            if (!dual_ref()) break;
            Cap c = flow_limit - flow;
            for (int v = t; v != s; v = g.elist[prev_e[v]].to) {
                c = std::min(c, g.elist[g.elist[prev_e[v]].rev].cap);
            }
            for (int v = t; v != s; v = g.elist[prev_e[v]].to) {
                auto& e = g.elist[prev_e[v]];
                e.cap += c;
                g.elist[e.rev].cap -= c;
            }
            Cost d = -dual_dist[s].first;
```

```
                flow += c;
                cost += c * d;
                if (prev_cost_per_flow == d) {
                    result.pop_back();
                }
                result.push_back({ flow, cost });
                prev_cost_per_flow = d;
            }
            return result;
        }
    };
}
```

6.10 上下界网络流

6.10.1 无源汇上下界可行流

给定无源汇流量网络 G。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时每一个点流量平衡。

不妨假设每条边已经流了 $b(u,v)$ 的流量，设其为初始流。同时我们在新图中加入 u 连向 v 的流量为 $c(u,v)-b(u,v)$ 的边。考虑在新图上进行调整。

由于最大流需要满足初始流量平衡条件（最大流可以看成是下界为 0 的上下界最大流），但是构造出来的初始流很有可能不满足初始流量平衡。假设一个点初始流入流量减初始流出流量为 M。

若 $M=0$，此时流量平衡，不需要附加边。

若 $M>0$，此时入流量过大，需要新建附加源点 S'，S' 向其连流量为 M 的附加边。

若 $M<0$，此时出流量过大，需要新建附加汇点 T'，其向 T' 连流量为 $-M$ 的附加边。

如果附加边满流，说明这一个点的流量平衡条件可以满足，否则这个点的流量平衡条件不满足。（因为原图加上附加流之后才会满足原图中的流量平衡。）

在建图完毕之后跑 S' 到 T' 的最大流，若 S' 连出去的边全部满流，则存在可行流，否则不存在。

6.10.2 有源汇上下界可行流

给定有源汇流量网络 G。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。

假设源点为 S，汇点为 T。

则我们可以加入一条 T 到 S 的上界为 ∞，下界为 0 的边转化为无源汇上下界可行流问题。

若有解，则 S 到 T 的可行流流量等于 T 到 S 的附加边的流量。

6.10.3 有源汇上下界最大流

给定有源汇流量网络 G。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。如果存在，询问满足标定的最大流量。

我们找到网络上的任意一个可行流。如果找不到解就可以直接结束。

否则我们考虑删去所有附加边之后的残量网络并且在网络上进行调整。

我们在残量网络上再跑一次 S 到 T 的最大流，将可行流流量和最大流流量相加即为答案。

一个非常易错的问题: S 到 T 的最大流直接在跑完有源汇上下界可行的残量网络上跑。

6.10.4 有源汇上下界最小流

给定有源汇流量网络 G。询问是否存在一种标定每条边流量的方式，使得每条边流量满足上下界同时除了源点和汇点每一个点流量平衡。如果存在，询问满足标定的最小流量。

类似的，我们考虑将残量网络中不需要的流退掉。

我们找到网络上的任意一个可行流。如果找不到解就可以直接结束。

否则我们考虑删去所有附加边之后的残量网络。

我们在残量网络上再跑一次 T 到 S 的最大流，将可行流流量减去最大流流量即为答案。

对于每个点，向 T 连边权 c, 上界 ∞, 下界为 1。

S 点为 1 号节点。

跑一次上下界带源汇最小费用可行流即可。

因为最小费用可行流解法与最小可行流类似，这里不再展开。

6.11 全局最小割

```
constexpr int N = 601;
constexpr int inf = 0x3f3f3f3f;
int edge[N][N]; // 边权存这里
int dis[N], vis[N], bin[N];
int n, m;
int contract(int& s, int& t) {  // Find s,t
    memset(dis, 0, sizeof(dis));
    memset(vis, false, sizeof(vis));
    int i, j, k, mincut, maxc;
    for (i = 1; i <= n; i++) {
        k = -1;
        maxc = -1;
        for (j = 1; j <= n; j++) {
            if (!bin[j] && !vis[j] && dis[j] > maxc) {
                k = j;
                maxc = dis[j];
            }
        }
        if (k == -1) return mincut;
        s = t; t = k;
        mincut = maxc;
        vis[k] = true;
        for (j = 1; j <= n; j++) {
            if (!bin[j] && !vis[j]) {
                dis[j] += edge[k][j];
            }
        }
    }
    return mincut;
```

```
}

int stoerWagner() { // O(NM + N^2logN) <=> O(N^3)
    int mincut, i, j, s, t, ans;
    for (mincut = inf, i = 1; i < n; i++) {
        ans = contract(s, t);
        bin[t] = true;
        if (mincut > ans) mincut = ans;
        if (mincut == 0) return 0;
        for (j = 1; j <= n; j++) {
            if (!bin[j]) {
                edge[s][j] = (edge[j][s] += edge[j][t]);
            }
        }
    }
    return mincut;
}
```

6.12 二分图最大权匹配

```
namespace KM {
    typedef long long ll;
    const int maxn = 510;
    const int inf = 1e9;
    int vx[maxn], vy[maxn], lx[maxn], ly[maxn], slack[maxn];
    int w[maxn][maxn]; // 以上为权值类型
    int pre[maxn], left[maxn], right[maxn], NL, NR, N;
    void match(int& u) {
        for (; u; std::swap(u, right[pre[u]]))
            left[u] = pre[u];
    }
    void bfs(int u) {
        static int q[maxn], front, rear;
        front = 0; vx[q[rear = 1] = u] = true;
        while (true) {
            while (front < rear) {
                int u = q[++front];
                for (int v = 1; v <= N; ++v) {
                    int tmp;
                    if (vy[v] || (tmp = lx[u] + ly[v] - w[u][v]) > slack[v])
                        continue;
                    pre[v] = u;
                    if (!tmp) {
```

```
                        if (!left[v]) return match(v);
                        vy[v] = vx[q[++rear] = left[v]] = true;
                    } else slack[v] = tmp;
                }
            }
            int a = inf;
            for (int i = 1; i <= N; ++i)
                if (!vy[i] && a > slack[i]) a = slack[u = i];
            for (int i = 1; i <= N; ++i) {
                if (vx[i]) lx[i] -= a;
                if (vy[i]) ly[i] += a;
                else slack[i] -= a;
            }
            if (!left[u]) return match(u);
            vy[u] = vx[q[++rear] = left[u]] = true;

        }

    }
    void exec() {
        for (int i = 1; i <= N; ++i) {
            for (int j = 1; j <= N; ++j) {
                slack[j] = inf;
                vx[j] = vy[j] = false;
            }
            bfs(i);
        }
    }
    ll work(int nl, int nr) { // NL , NR 为左右点数，返回最大权匹配的权值和
        NL = nl; NR = nr;
        N = std::max(NL, NR);
        for (int u = 1; u <= N; ++u)
            for (int v = 1; v <= N; ++v)
                lx[u] = std::max(lx[u], w[u][v]);
        exec();
        ll ans = 0;
        for (int i = 1; i <= N; ++i)
            ans += lx[i] + ly[i];
        return ans;
    }
    void output() { // 输出左边点与右边哪个点匹配，没有匹配输出0
        for (int i = 1; i <= NL; ++i)
            printf("%d ", (w[i][right[i]] ? right[i] : 0));
```

```
        printf("\n");
    }
}
```

6.13 一般图最大匹配

```
// UOJ79 copy from jiangly
#include <bits/stdc++.h>
struct Graph {
    int n;
    std::vector<std::vector<int>> e;
    Graph(int n) : n(n), e(n) {}
    void addEdge(int u, int v) {
        e[u].push_back(v);
        e[v].push_back(u);
    }
    std::vector<int> findMatching() {
        std::vector<int> match(n, -1), vis(n), link(n), f(n), dep(n);
        // disjoint set union
        auto find = [&](int u) {
            while (f[u] != u)
                u = f[u] = f[f[u]];
            return u;
        };
        auto lca = [&](int u, int v) {
            u = find(u);
            v = find(v);
            while (u != v) {
                if (dep[u] < dep[v])
                    std::swap(u, v);
                u = find(link[match[u]]);
            }
            return u;
        };

        std::queue<int> que;
        auto blossom = [&](int u, int v, int p) {
            while (find(u) != p) {
                link[u] = v;
                v = match[u];
                if (vis[v] == 0) {
                    vis[v] = 1;
                    que.push(v);
```

```cpp
                }
                f[u] = f[v] = p;
                u = link[v];
            }
        };

        // find an augmenting path starting from u and augment (if exist)
        auto augment = [&](int u) {
            while (!que.empty())
                que.pop();
            std::iota(f.begin(), f.end(), 0);
            // vis = 0 corresponds to inner vertices, vis = 1 corresponds to outer vertices
            std::fill(vis.begin(), vis.end(), -1);
            que.push(u);
            vis[u] = 1, dep[u] = 0;
            while (!que.empty()){
                int u = que.front();
                que.pop();
                for (auto v : e[u]) {
                    if (vis[v] == -1) {
                        vis[v] = 0;
                        link[v] = u;
                        dep[v] = dep[u] + 1;
                        // found an augmenting path
                        if (match[v] == -1) {
                            for (int x = v, y = u, temp; y != -1; x = temp, y = x == -1 ? -1 : link[x]) {
                                temp = match[y];
                                match[x] = y;
                                match[y] = x;
                            }
                            return;
                        }
                        vis[match[v]] = 1;
                        dep[match[v]] = dep[u] + 2;
                        que.push(match[v]);
                    } else if (vis[v] == 1 && find(v) != find(u)) {
                        // found a blossom
                        int p = lca(u, v);
                        blossom(u, v, p);
                        blossom(v, u, p);
                    }
```

```cpp
                }
            }
        };

        // find a maximal matching greedily (decrease constant)
        auto greedy = [&]() {
            for (int u = 0; u < n; ++u) {
                if (match[u] != -1)
                    continue;
                for (auto v : e[u]) {
                    if (match[v] == -1) {
                        match[u] = v;
                        match[v] = u;
                        break;
                    }
                }
            }
        };
        greedy();
        for (int u = 0; u < n; ++u)
            if (match[u] == -1)
                augment(u);
        return match;
    }
};

int main() {
    std::ios::sync_with_stdio(false);
    std::cin.tie(nullptr);
    int n, m;
    std::cin >> n >> m;
    Graph g(n);
    for (int i = 0; i < m; ++i) {
        int u, v;
        std::cin >> u >> v;
        --u, --v;
        g.addEdge(u, v);
    }
    auto match = g.findMatching();
    int ans = 0;
    for (int u = 0; u < n; ++u)
        if (match[u] != -1)
            ++ans;
```

```
    std::cout << ans / 2 << "\n";
    for (int u = 0; u < n; ++u) // 输出每个人匹配的对象，如果没有则输出0
        std::cout << match[u] + 1 << " \n"[u == n - 1];
    return 0;
}
```

6.14 最大团

```
/*
 * 最大团 Bron–Kerbosch algorithm
 * 最劣复杂度 O(3^(n/3))
 * 采用位运算形式实现
 * */
namespace Max_clique {
#define ll long long
#define TWOL(x) (1ll<<(x))
    const int N = 60;
    int n, m;       // 点数 边数
    int r = 0;      // 最大团大小
    ll G[N];        // 以二进制形式存图
    ll clique = 0; // 最大团 以二进制形式存储
    void BronK(int S, ll P, ll X, ll R) { // 调用时参数这样设置: 0, TWOL(n)
        -1, 0, 0
        if (P == 0 && X == 0) {
            if (r < S) {
                r = S;
                clique = R;
            }
        }
        if (P == 0) return;
        int u = __builtin_ctzll(P | X);
        ll c = P & ~G[u];
        while (c) {
            int v = __builtin_ctzll(c);
            ll pv = TWOL(v);
            BronK(S + 1, P & G[v], X & G[v], R | pv);
            P ^= pv; X |= pv; c ^= pv;
        }
    }
    void init() {
        cin >> n >> m;
        for (int i = 0; i < m; i++) {
            int u, v;
```

```cpp
            cin >> u >> v;
            --u, --v;
            G[u] |= TWOL(v);
            G[v] |= TWOL(u);
        }
        BronK(0, TWOL(n)-1, 0, 0);
        cout << r << ' ' << clique << '\n';
    }
}
```

7 数据结构

7.1 树状数组

```cpp
template<typename T> struct fenwickTree {
    int n, hbit;
    vector<T> tree;
    fenwickTree(int n_ = 0) : n(n_), tree(n_ + 1), hbit(log2(n_) + 1) {}
    int lowbit(int x) { return x & (-x); }
    int size() { return n; }
    void add(int pos, int x) { // pos位置加上x
        for (; pos <= n; pos += lowbit(pos)) {
            tree[pos] += x;
        }
    }
    T query(int pos) { // 查询pos位置的前缀和 即a[1] + a[2] + ... + a[pos]
        T res = 0;
        for (; pos > 0; pos -= lowbit(pos)) {
            res += tree[pos];
        }
        return res;
    }
    T sum(int l, int r) { // [l, r]区间查询
        return query(r) - query(l - 1);
    }
    int kth(int k) { // 第k大元素
        int ans = 0, cnt = 0;
        for (int i = hbit; i >= 0; i--) {
            ans += (1 << i);
            if (ans > n || cnt + tree[ans] >= k) ans -= (1 << i);
            else cnt += tree[ans];
        }
        return ++ans;
```

```
    }
};
```

7.2 线段树

```
namespace SegmentTree {
    static constexpr int SIZE = 200001;
    int a[SIZE];
    int mx[SIZE << 2], lazy[SIZE << 2];

    void pushup(int rt) {
        mx[rt] = max(mx[rt << 1], mx[rt << 1 | 1]);
    }

    void pushdown(int rt) {
        if (lazy[rt]) {
            lazy[rt << 1] += lazy[rt];
            lazy[rt << 1 | 1] += lazy[rt];
            mx[rt << 1] += lazy[rt];
            mx[rt << 1 | 1] += lazy[rt];
            lazy[rt] = 0;
        }
    }

    void build(int rt, int l, int r) {
        if (l == r) {
            mx[rt] = a[l];
            lazy[rt] = 0;
            return;
        }
        int mid = (l + r) >> 1;
        pushdown(rt);
        build(rt << 1, l, mid), build(rt << 1 | 1, mid + 1, r);
        pushup(rt);
    }

    void rangeAdd(int rt, int l, int r, int ql, int qr, int val) {
        if (ql <= l && qr >= r) {
            mx[rt] += val;
            lazy[rt] += val;
            return;
        }
        int mid = (l + r) >> 1;
```

```cpp
        pushdown(rt);
        if (ql <= mid) rangeAdd(rt << 1, l, mid, ql, qr, val);
        if (qr > mid) rangeAdd(rt << 1 | 1, mid + 1, r, ql, qr, val);
        pushup(rt);
    }

    int rangeMax(int rt, int l, int r, int ql, int qr) {
        if (ql <= l && qr >= r) return mx[rt];
        int mid = (l + r) >> 1;
        int res = 0;
        pushdown(rt);
        if (ql <= mid) res = max(res, rangeMax(rt << 1, l, mid, ql, qr));
        if (qr > mid) res = max(res, rangeMax(rt << 1 | 1, mid + 1, r, ql,
            qr));
        pushup(rt);
        return res;
    }
}
```

8 字符串

8.1 KMP

```cpp
namespace KMP {
    vector<int> getPrefixTable(string s) { // 求前缀表
        int n = s.length();
        vector<int> nxt(n, 0);
        for (int i = 1; i < n; i++) {
            int j = nxt[i - 1];
            while (j > 0 && s[i] != s[j]) {
                j = nxt[j - 1];
            }
            if (s[i] == s[j]) j++;
            nxt[i] = j;
        }
        return nxt;
    }

    vector<int> kmp(string s, string t) { // 返回所有匹配位置的集合
        int n = s.length(), m = t.length();
        vector<int> res;
        vector<int> nxt = getPrefixTable(t);
        for (int i = 0, j = 0; i < n; i++) {
```

```cpp
            while (j > 0 && j < m && s[i] != t[j]) {
                j = nxt[j - 1];
            }
            if (s[i] == t[j]) j++;
            if (j == m) {
                res.push_back(i + 1 - m);
                j = nxt[m - 1];
            }
        }
        return res;
    }
}
```

8.2 Z-Function

```cpp
// O(N) 查询字符串s每一位开始的LCP
vector<int> z_function(string s) {
    int n = (int)s.length();
    vector<int> z(n);
    for (int i = 1, l = 0, r = 0; i < n; ++i) {
        if (i <= r && z[i - l] < r - i + 1) {
            z[i] = z[i - l];
        } else {
            z[i] = max(0, r - i + 1);
            while (i + z[i] < n && s[z[i]] == s[i + z[i]]) ++z[i];
        }
        if (i + z[i] - 1 > r) l = i, r = i + z[i] - 1;
    }
    return z;
}
```

8.3 Manacher

```cpp
namespace Manacher {
    static constexpr int SIZE = 1e5 + 5; // 预设为原串长度
    int len = 1; // manacher 预处理后字符串的长度
    char stk[SIZE << 1]; // manacher预处理字符串 需要2倍空间+1
    void init(string s) { // 初始化stk
        stk[0] = '*'; len = 1;
        for (int i = 0; i < s.length(); ++i) {
            stk[len++] = s[i];
            stk[len++] = '*';
        }
```

```cpp
    }
    int manacher() { // 返回最长回文子串长度
        vector<int> rad(len << 1); // 存储每个点作为对称中心可拓展的最大半径
        int md = 0; // 最远回文串对称中心下标
        for (int i = 1; i < len; ++i) {
            int& r = rad[i] = 0;
            if (i <= md + rad[md]) {
                r = min(rad[2 * md - i], md + rad[md] - i);
            }
            while (i - r - 1 >= 0 && i + r + 1 < len &&
                stk[i - r - 1] == stk[i + r + 1]) ++r;
            if (i + r >= md + rad[md]) md = i;
        }
        int res = 0;
        for (int i = 0; i < len; ++i) {
            if (rad[i] > res) {
                res = rad[i];
            }
        }
        return res;
    }
}
```

8.4 Trie

```cpp
struct trie {
    int cnt;
    vector<vector<int>> nxt;
    vector<bool> vis;
    /* 初始化的时候size需要设置为字符串总长之和 26是字符集大小 */
    trie(int size_ = 0) :cnt(0), vis(size_, false), nxt(size_, vector<int
       >(26, 0)) {}
    void insert(string s) {  // 插入字符串
        int p = 0;
        for (int i = 0; i < (int)s.length(); i++) {
            int c = s[i] - 'a';
            if (!nxt[p][c]) nxt[p][c] = ++cnt;
            p = nxt[p][c];
        }
        vis[p] = true;
    }
    bool find(string s) {  // 查找字符串
        int p = 0;
```

```cpp
        for (int i = 0; i < (int)s.length(); i++) {
            int c = s[i] - 'a';
            if (!nxt[p][c]) return false;
            p = nxt[p][c];
        }
        return vis[p];
    }
};
```

8.5 01-Trie

```cpp
template<typename T> struct xorTrie {
    int HIGHBIT, cnt;
    vector<vector<int>> nxt;
    vector<bool> vis;
    xorTrie(int n_ = 0, int highbit_ = 30) : HIGHBIT(highbit_), cnt(0) {
        int size_ = upperBoundEstimate(n_);
        nxt.resize(size_, vector<int>(2, 0));
        vis.resize(size_, false);
    }
    int upperBoundEstimate(int n) { // 求内存上界
        int hbit = log2(n);
        return n * (HIGHBIT - hbit + 1) + (1 << (hbit + 1)) - 1;
    }
    void insert(T x) { // 插入
        int p = 0;
        for (int i = HIGHBIT; ~i; i--) {
            int s = ((x >> i) & 1);
            if (!nxt[p][s]) nxt[p][s] = ++cnt;
            p = nxt[p][s];
        }
        vis[p] = true;
    }
    bool find(T x) { // 查询
        int p = 0;
        for (int i = HIGHBIT; ~i; i--) {
            int s = ((x >> i) & 1);
            if (!nxt[p][s]) return false;
            p = nxt[p][s];
        }
        return vis[p];
    }
};
```

9 计算几何

```
namespace Geometry {
#define db long double
#define pi acos(-1.0)
    constexpr db eps = 1e-7;
    int sign(db k) {
        if (k > eps) return 1;
        else if (k < -eps) return -1;
        return 0;
    }
    int cmp(db k1, db k2) { // k1 < k2 : -1, k1 == k2 : 0, k1 > k2 : 1
        return sign(k1 - k2);
    }
    int inmid(db k1, db k2, db k3) { // k3 在 [k1, k2] 内
        return sign(k1 - k3) * sign(k2 - k3) <= 0;
    }

    struct point { // 点类
        db x, y;
        point() {}
        point(db x_, db y_) :x(x_), y(y_) {}
        point operator + (const point& k) const { return point(k.x + x, k.y + y); }
        point operator - (const point& k) const { return point(x - k.x, y - k.y); }
        point operator * (db k) const { return point(x * k, y * k); }
        point operator / (db k1) const { return point(x / k1, y / k1); }
        point turn(db k1) { return point(x * cos(k1) - y * sin(k1), x * sin(k1) + y * cos(k1)); } // 逆时针旋转
        point turn90() { return point(-y, x); } // 逆时针方向旋转 90 度
        db len() { return sqrt(x * x + y * y); } // 向量长度
        db len2() { return x * x + y * y; } // 向量长度的平方
        db getPolarAngle() { return atan2(y, x); } // 向量极角
        db dis(point k) { return ((*this) - k).len(); } // 到点k的距离
        point unit() { db d = len(); return point(x / d, y / d); } // 单位向量
        point getdel() { // 将向量的方向调整为指向第一/四象限 包括y轴正方向
            if (sign(x) == -1 || (sign(x) == 0 && sign(y) == -1))
                return (*this) * (-1);
            else return (*this);
        }
        bool operator < (const point& k) const { // 水平序排序 x坐标为第一关键字,y坐标第二关键字
```

```cpp
            return x == k.x ? y < k.y : x < k.x;
        }
        bool operator == (const point& k) const { return cmp(x, k.x) == 0 && cmp(y, k.y) == 0; }
        bool getP() const { // 判断点是否在上半平面 含x负半轴 不含x正半轴及零点
            return sign(y) == 1 || (sign(y) == 0 && sign(x) == -1);
        }
        void input() { cin >> x >> y; }
    };
    db cross(point k1, point k2) { return k1.x * k2.y - k1.y * k2.x; } // 向量 k1,k2 的叉积
    db dot(point k1, point k2) { return k1.x * k2.x + k1.y * k2.y; }   // 向量 k1,k2 的点积
    db rad(point k1, point k2) { // 向量 k1,k2 之间的有向夹角
        return atan2(cross(k1, k2), dot(k1, k2));
    }
    int inmid(point k1, point k2, point k3) { // k1 k2 k3共线时 判断点 k3 是否在线段 k1k2 上
        return inmid(k1.x, k2.x, k3.x) && inmid(k1.y, k2.y, k3.y);
    }
    int compareAngle(point k1, point k2) { // 比较向量 k1,k2 的角度大小 角度按照atan2()函数定义
        // k1 < k2 返回 1, k1 >= k2 返回 0
        return k1.getP() < k2.getP() || (k1.getP() == k2.getP() && sign(cross(k1, k2)) > 0);
    }
    point proj(point k1, point k2, point q) { // q 到直线 k1,k2 的投影
        point k = k2 - k1; return k1 + k * (dot(q - k1, k) / k.len2());
    }
    point reflect(point k1, point k2, point q) { return proj(k1, k2, q) * 2 - q; } // q 关于直线 k1,k2 的对称点
    int counterclockwise(point k1, point k2, point k3) { // k1 k2 k3 逆时针1 顺时针-1 否则0
        return sign(cross(k2 - k1, k3 - k1));
    }
    int checkLL(point k1, point k2, point k3, point k4) { // 判断直线 k1k2 和直线k3k4 是否相交
        // 即判断直线 k1k2 和 k3k4 是否平行 平行返回0 不平行返回1
        return sign(cross(k2 - k1, k4 - k3)) != 0;
    }
    point getLL(point k1, point k2, point k3, point k4) { // 求 k1k2 k3k4 两直线交点
```

```
        db w1 = cross(k1 - k3, k4 - k3), w2 = cross(k4 - k3, k2 - k3);
        return (k1 * w2 + k2 * w1) / (w1 + w2);
    }
    int intersect(db l1, db r1, db l2, db r2) { // 判断 [l1,r1] 和 [l2, r2] 是否相交
        if (l1 > r1) swap(l1, r1);
        if (l2 > r2) swap(l2, r2);
        return cmp(r1, l2) != -1 && cmp(r2, l1) != -1;
    }
    int checkSS(point k1, point k2, point k3, point k4) { // 判断线段 k1k2 和线段 k3k4 是否相交
        return intersect(k1.x, k2.x, k3.x, k4.x) && intersect(k1.y, k2.y, k3.y, k4.y) &&
            sign(cross(k3 - k1, k4 - k1)) * sign(cross(k3 - k2, k4 - k2)) <= 0 &&
            sign(cross(k1 - k3, k2 - k3)) * sign(cross(k1 - k4, k2 - k4)) <= 0;
    }
    db disSP(point k1, point k2, point q) { // 点 q 到线段 k1k2 的最短距离
        point k3 = proj(k1, k2, q);
        if (inmid(k1, k2, k3)) return q.dis(k3);
        else return min(q.dis(k1), q.dis(k2));
    }
    db disLP(point k1, point k2, point q) { // 点 q 到直线 k1k2 的最短距离
        point k3 = proj(k1, k2, q);
        return q.dis(k3);
    }
    db disSS(point k1, point k2, point k3, point k4) { // 线段 k1k2 和线段 k3k4 的最短距离
        if (checkSS(k1, k2, k3, k4)) return 0;
        else return min(min(disSP(k1, k2, k3), disSP(k1, k2, k4)),
            min(disSP(k3, k4, k1), disSP(k3, k4, k2)));
    }
    bool onLine(point k1, point k2, point q) { // 判断点 q 是否在直线 k1k2 上
        return sign(cross(k1 - q, k2 - q)) == 0;
    }
    bool onSegment(point k1, point k2, point q) { // 判断点 q 是否在线段 k1k2 上
        if (!onLine(k1, k2, q)) return false; // 如果确定共线 要删除这个特判
        return inmid(k1, k2, q);
    }
    void polarAngleSort(vector<point>& p, point t) { // p为待排序点集 t为极
```

```
        角排序中心
        sort(p.begin(), p.end(), [&](const point& k1, const point& k2) {
            return compareAngle(k1 - t, k2 - t);
        });
    }

    struct line { // 直线 / 线段类
        point p[2];
        line() {}
        line(point k1, point k2) { p[0] = k1, p[1] = k2; }
        point& operator [] (int k) { return p[k]; }
        point dir() { return p[1] - p[0]; } // 向量 p[0] -> p[1]
        bool include(point k) { // 判断点是否在直线上
            return sign(cross(p[1] - p[0], k - p[0])) > 0;
        }
        bool includeS(point k) { // 判断点是否在线段上
            return onSegment(p[0], p[1], k);
        }
        line push(db len) { // 向外（左手边）平移 len 个单位
            point delta = (p[1] - p[0]).turn90().unit() * len;
            return line(p[0] - delta, p[1] - delta);
        }
    };
    bool parallel(line k1, line k2) { // 判断是否平行
        return sign(cross(k1.dir(), k2.dir())) == 0;
    }
    bool sameLine(line k1, line k2) { // 判断是否共线
        return parallel(k1, k2) && parallel(k1, line(k2.p[0], k1.p[0]));
    }
    bool sameDir(line k1, line k2) { // 判断向量 k1 k2 是否同向
        return parallel(k1, k2) && sign(dot(k1.dir(), k2.dir())) == 1;
    }
    bool operator < (line k1, line k2) {
        if (sameDir(k1, k2)) return k2.include(k1[0]);
        return compareAngle(k1.dir(), k2.dir());
    }
    bool checkLL(line k1, line k2) {
        return checkLL(k1[0], k1[1], k2[0], k2[1]);
    }
    point getLL(line k1, line k2) {  // 求 k1 k2 两直线交点 不要忘了判平行!
        return getLL(k1[0], k1[1], k2[0], k2[1]);
    }
    bool checkpos(line k1, line k2, line k3) {  // 判断是否三线共点
```

```
    return k3.include(getLL(k1, k2));
}

struct circle { // 圆类
    point o;
    double r;
    circle() {}
    circle(point o_, double r_) : o(o_), r(r_) {}
    int inside(point k) {  // 判断点 k 和圆的位置关系
        return cmp(r, o.dis(k)); // 圆外:-1, 圆上:0, 圆内:1
    }
};
int checkposCC(circle k1, circle k2) { // 返回两个圆的公切线数量
    if (cmp(k1.r, k2.r) == -1) swap(k1, k2);
    db dis = k1.o.dis(k2.o);
    int w1 = cmp(dis, k1.r + k2.r), w2 = cmp(dis, k1.r - k2.r);
    if (w1 > 0) return 4; // 外离
    else if (w1 == 0) return 3; // 外切
    else if (w2 > 0) return 2;  // 相交
    else if (w2 == 0) return 1; // 内切
    else return 0; // 内离(包含)
}
vector<point> getCL(circle k1, point k2, point k3) { // 求直线 k2k3 和圆
    k1 的交点
    // 沿着 k2->k3 方向给出 相切给出两个
    point k = proj(k2, k3, k1.o);
    db d = k1.r * k1.r - (k - k1.o).len2();
    if (sign(d) == -1) return {};
    point del = (k3 - k2).unit() * sqrt(max((db)0.0, d));
    return { k - del,k + del };
}
vector<point> getCC(circle k1, circle k2) { // 求圆 k1 和圆 k2 的交点
    // 沿圆 k1 逆时针给出, 相切给出两个
    int pd = checkposCC(k1, k2); if (pd == 0 || pd == 4) return {};
    db a = (k2.o - k1.o).len2(), cosA = (k1.r * k1.r + a -
        k2.r * k2.r) / (2 * k1.r * sqrt(max(a, (db)0.0)));
    db b = k1.r * cosA, c = sqrt(max((db)0.0, k1.r * k1.r - b * b));
    point k = (k2.o - k1.o).unit(), m = k1.o + k * b, del = k.turn90() *
        c;
    return { m - del,m + del };
}
vector<point> tangentCP(circle k1, point k2) { // 点 k2 到圆 k1 的切点
    沿圆 k1 逆时针给出
```

```
        db a = (k2 - k1.o).len(), b = k1.r * k1.r / a, c = sqrt(max((db)0.0,
            k1.r * k1.r - b * b));
        point k = (k2 - k1.o).unit(), m = k1.o + k * b, del = k.turn90() * c
            ;
        return { m - del,m + del };
    }
    vector<line> tangentOutCC(circle k1, circle k2) {
        int pd = checkposCC(k1, k2);
        if (pd == 0) return {};
        if (pd == 1) {
            point k = getCC(k1, k2)[0];
            return { line(k,k) };
        }
        if (cmp(k1.r, k2.r) == 0) {
            point del = (k2.o - k1.o).unit().turn90().getdel();
            return { line(k1.o - del * k1.r,k2.o - del * k2.r),
                line(k1.o + del * k1.r,k2.o + del * k2.r) };
        } else {
            point p = (k2.o * k1.r - k1.o * k2.r) / (k1.r - k2.r);
            vector<point> A = tangentCP(k1, p), B = tangentCP(k2, p);
            vector<line> ans; for (int i = 0; i < A.size(); i++)
                ans.push_back(line(A[i], B[i]));
            return ans;
        }
    }
    vector<line> tangentInCC(circle k1, circle k2) {
        int pd = checkposCC(k1, k2);
        if (pd <= 2) return {};
        if (pd == 3) {
            point k = getCC(k1, k2)[0];
            return { line(k, k) };
        }
        point p = (k2.o * k1.r + k1.o * k2.r) / (k1.r + k2.r);
        vector<point> A = tangentCP(k1, p), B = tangentCP(k2, p);
        vector<line> ans;
        for (int i = 0; i < (int)A.size(); i++) ans.push_back(line(A[i], B[i
            ]));
        return ans;
    }
    vector<line> tangentCC(circle k1, circle k2) { // 求两圆公切线
        int flag = 0;
        if (k1.r < k2.r) swap(k1, k2), flag = 1;
        vector<line> A = tangentOutCC(k1, k2), B = tangentInCC(k1, k2);
```

```
        for (line k : B) A.push_back(k);
        if (flag) for (line& k : A) swap(k[0], k[1]);
        return A;
    }
    db getAreaCT(circle k1, point k2, point k3) { // 圆 k1 与三角形 k2k3k1.o 的有向面积交
        point k = k1.o; k1.o = k1.o - k; k2 = k2 - k; k3 = k3 - k;
        int pd1 = k1.inside(k2), pd2 = k1.inside(k3);
        vector<point> A = getCL(k1, k2, k3);
        if (pd1 >= 0) {
            if (pd2 >= 0) return cross(k2, k3) / 2;
            return k1.r * k1.r * rad(A[1], k3) / 2 + cross(k2, A[1]) / 2;
        } else if (pd2 >= 0) {
            return k1.r * k1.r * rad(k2, A[0]) / 2 + cross(A[0], k3) / 2;
        } else {
            int pd = cmp(k1.r, disSP(k2, k3, k1.o));
            if (pd <= 0) return k1.r * k1.r * rad(k2, k3) / 2;
            return cross(A[0], A[1]) / 2 + k1.r * k1.r * (rad(k2, A[0]) +
                rad(A[1], k3)) / 2;
        }
    }
    db getAreaCC(circle k1, circle k2) { // 两圆面积交
        db d = k1.o.dis(k2.o);
        if (cmp(d, k1.r + k2.r) >= 0) return 0; // 两圆相离
        if (cmp(k1.r, k2.r) == -1) swap(k1, k2);
        if (cmp(k1.r - k2.r, d) >= 0) return pi * k2.r * k2.r; // 圆k1包含k2
        db g1 = acos((k1.r * k1.r + d * d - k2.r * k2.r) / (2 * k1.r * d));
        db g2 = acos((k2.r * k2.r + d * d - k1.r * k1.r) / (2 * k2.r * d));
        return g1 * k1.r * k1.r + g2 * k2.r * k2.r - k1.r * d * sin(g1);
    }
    circle getCircleOut(point k1, point k2, point k3) { // 三角形外接圆
        db a1 = k2.x - k1.x, b1 = k2.y - k1.y, c1 = (a1 * a1 + b1 * b1) / 2;
        db a2 = k3.x - k1.x, b2 = k3.y - k1.y, c2 = (a2 * a2 + b2 * b2) / 2;
        db d = a1 * b2 - a2 * b1;
        point o(k1.x + (c1 * b2 - c2 * b1) / d, k1.y + (a1 * c2 - a2 * c1) /
            d);
        return circle(o, k1.dis(o));
    }
    circle getCircleIn(point k1, point k2, point k3) {  // 三角形内切圆
        db a = k1.dis(k2), b = k2.dis(k3), c = k3.dis(k1);
        db len = a + b + c;
        db r = abs(cross(k1 - k2, k1 - k3)) / len;
        point o((k1.x * b + k2.x * c + k3.x * a) / len, (k1.y * b + k2.y * c
```

```
            + k3.y * a) / len);
        return circle(o, r);
    }
    circle minCircleCovering(vector<point> A) { // 最小圆覆盖 O(n)随机增量法
        // random_shuffle(A.begin(), A.end()); // <= C++14
        auto seed = chrono::steady_clock::now().time_since_epoch().count();
        default_random_engine e(seed);
        shuffle(A.begin(), A.end(), e); // >= C++11
        circle ans = circle(A[0], 0);
        for (int i = 1; i < A.size(); i++) {
            if (ans.inside(A[i]) == -1) {
                ans = circle(A[i], 0);
                for (int j = 0; j < i; j++) {
                    if (ans.inside(A[j]) == -1) {
                        ans.o = (A[i] + A[j]) / 2;
                        ans.r = ans.o.dis(A[i]);
                        for (int k = 0; k < j; k++) {
                            if (ans.inside(A[k]) == -1)
                                ans = getCircleOut(A[i], A[j], A[k]);
                        }
                    }
                }
            }
        }
        return ans;
    }

    typedef vector<point> polygon;
    db area(polygon p) { // 多边形有向面积
        if (p.size() < 3) return 0;
        db ans = 0;
        for (int i = 1; i < p.size() - 1; i++)
            ans += cross(p[i] - p[0], p[i + 1] - p[0]);
        return 0.5L * ans;
    }

    int checkConvexP(polygon p, point a) { // O(logn)判断点是否在凸包内 2内部 1边界 0外部
        // 必须保证凸多边形是一个水平序凸包且不能退化
        // 退化情况 比如凸包退化成线段 可使用 onSegment() 函数特判
        auto check = [&](int x) {
            int ccw1 = counterclockwise(p[0], a, p[x]),
                ccw2 = counterclockwise(p[0], a, p[x + 1]);
```

```
            if (ccw1 == -1 && ccw2 == -1) return 2;
            else if (ccw1 == 1 && ccw2 == 1) return 0;
            else if (ccw1 == -1 && ccw2 == 1) return 1;
            else return 1;
        };
        if (counterclockwise(p[0], a, p[1]) <= 0 && counterclockwise(p[0], a
            , p.back()) >= 0) {
            int l = 1, r = p.size() - 2, mid;
            while (l <= r) {
                mid = (l + r) >> 1;
                int chk = check(mid);
                if (chk == 1) l = mid + 1;
                else if (chk == -1) r = mid;
                else break;
            }
            int res = counterclockwise(p[mid], a, p[mid + 1]);
            if (res < 0) return 2;
            else if (res == 0) return 1;
            else return 0;
        } else {
            return 0;
        }
    }
    int checkPolyP(vector<point> p, point q) { // O(n)判断点是否在一般多边形
        内
        // 必须保证简单多边形的点按逆时针给出 返回 2 内部 1 边界 0 外部
        int pd = 0, n = p.size();
        for (int i = 0; i < n; i++) {
            point u = p[i], v = p[(i + 1) % n];
            if (onSegment(u, v, q)) return 1;
            if (cmp(u.y, v.y) > 0) swap(u, v);
            if (cmp(u.y, q.y) >= 0 || cmp(v.y, q.y) < 0) continue;
            if (sign(cross(u - v, q - v)) < 0) pd ^= 1;
        }
        return pd << 1;
    }
    db convexDiameter(polygon p) { // 0(n)旋转卡壳求凸包直径 / 平面最远点对
        的平方
        int n = p.size(); // 请保证多边形是凸包
        db ans = 0;
        for (int i = 0, j = n < 2 ? 0 : 1; i < j; i++) {
            for (;; j = (j + 1) % n) {
                ans = max(ans, (p[i] - p[j]).len2());
```

```
            if (sign(cross(p[i + 1] - p[i], p[(j + 1) % n] - p[j])) <=
                0) break;
        }
    }
    return ans;
}
polygon convexHull(polygon A, int flag = 1) { // 凸包 flag=0 不严格 flag
    =1 严格
    int n = A.size(); polygon ans(n + n);
    sort(A.begin(), A.end()); int now = -1;
    for (int i = 0; i < A.size(); i++) {
        while (now > 0 && sign(cross(ans[now] - ans[now - 1], A[i] - ans
            [now - 1])) < flag)
            now--;
        ans[++now] = A[i];
    }
    int pre = now;
    for (int i = n - 2; i >= 0; i--) {
        while (now > pre && sign(cross(ans[now] - ans[now - 1], A[i] -
            ans[now - 1])) < flag)
            now--;
        ans[++now] = A[i];
    }
    ans.resize(now);
    return ans;
}
bool checkConvexHull(polygon p) { // 检测多边形是否是凸包（可以有三点共
    线）
    int sgn, n = p.size(), i = 0; // 如果三点共线不算凸包 去掉ccw=0的情
        况
    for (;; i++) { // 这一步是为了防止第一步遇到共线的三个点
        sgn = counterclockwise(p[i], p[(i + 1) % n], p[(i + 2) % n]);
        if (sgn) break;
    }
    for (; i < n; i++) {
        int ccw = counterclockwise(p[i], p[(i + 1) % n], p[(i + 2) % n])
            ;
        if (ccw && ccw != sgn) {
            return false;
        }
    }
    return true;
}
```

```
    polygon convexCut(polygon A, point k1, point k2) { // 半平面 k1k2 切凸包 A
        int n = A.size();  // 保留所有满足 k1 -> p -> k2 为逆时针方向的点
        A.push_back(A[0]); // 保留的点可能有重点
        polygon ans;
        line cut(k1, k2);
        for (int i = 0; i < n; i++) {
            int ccw1 = counterclockwise(k1, k2, A[i]);
            int ccw2 = counterclockwise(k1, k2, A[i + 1]);
            if (ccw1 >= 0) ans.push_back(A[i]);
            if (ccw1 * ccw2 <= 0) {
                if (sameLine(cut, line(A[i], A[i + 1]))) { // 半平面恰好切到凸包上某条边
                    ans.push_back(A[i]);
                    ans.push_back(A[i + 1]);
                } else {
                    ans.push_back(getLL(k1, k2, A[i], A[i + 1]));
                }
            }
        }
        return ans;
    }

    vector<line> getHL(vector<line>& L) { // 求半平面交 逆时针方向存储
        sort(L.begin(), L.end());
        deque<line> q;
        for (int i = 0; i < (int)L.size(); ++i) {
            if (i && sameDir(L[i], L[i - 1])) continue;
            while (q.size() > 1 && !checkpos(q[q.size() - 2], q[q.size() - 1], L[i])) q.pop_back();
            while (q.size() > 1 && !checkpos(q[1], q[0], L[i])) q.pop_front();
            q.push_back(L[i]);
        }
        while (q.size() > 2 && !checkpos(q[q.size() - 2], q[q.size() - 1], q[0])) q.pop_back();
        while (q.size() > 2 && !checkpos(q[1], q[0], q[q.size() - 1])) q.pop_front();
        vector<line> ans;
        for (int i = 0; i < q.size(); ++i) ans.push_back(q[i]);
        return ans;
    }

```

```cpp
    db closestPoint(vector<point>& A, int l, int r) { // 最近点对, 先要按照 x 坐标排序
        if (r - l <= 5) {
            db ans = 1e20;
            for (int i = l; i <= r; ++i)
                for (int j = i + 1; j <= r; j++)
                    ans = min(ans, A[i].dis(A[j]));
            return ans;
        }
        int mid = l + r >> 1;
        db ans = min(closestPoint(A, l, mid), closestPoint(A, mid + 1, r));
        vector<point> B;
        for (int i = l; i <= r; i++)
            if (abs(A[i].x - A[mid].x) <= ans)
                B.push_back(A[i]);
        sort(B.begin(), B.end(), [&](const point& k1, const point& k2) {
            return k1.y < k2.y;
        });
        for (int i = 0; i < B.size(); i++)
            for (int j = i + 1; j < B.size() && B[j].y - B[i].y < ans; j++)
                ans = min(ans, B[i].dis(B[j]));
        return ans;
    }
}
using namespace Geometry;
```

10 数学公式及定理

10.1 求导法则

下文中 f, g 代表可微函数，其余字符代表常数。

1. 加法法则：$(af)' = a \cdot f'$
2. 乘法法则：$(fg)' = gf' + fg'$
3. 除法法则：$(\frac{f}{g})' = \frac{gf'-fg'}{g^2}$
4. 链式法则：$(f \circ g)' = f'(g(x))g'(x)$

10.2 麦克劳林级数

$$
\begin{aligned}
\frac{1}{1-x} &= 1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n \\
\frac{1}{1+x} &= 1-x+x^2-\cdots=\sum_{n=0}^{\infty}(-1)^n x^n \\
e^x &= 1+x+\frac{x^2}{2!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \\
sin(x) &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} \\
cos(x) &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \\
\ln(1+x) &= x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n!} \\
(1+x)^p &= \sum_{k=0}^{\infty}\frac{p(p-1)\cdots(p-k+1)}{k!}x^k
\end{aligned}
$$

10.3 泰勒公式

$$
f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n
$$

10.4 微积分

10.5 球缺

以下公式中 H 为球缺的高，R 为大圆半径。

球缺质心：匀质球缺的质心位于它的中轴线上，并且与底面的距离为：

$$
c=\frac{(4R-H)H}{12R-4H}
$$

球缺的体积：

$$
V=\pi H^2(R-\frac{H}{3})
$$

球冠的表面积公式：

$$
S=2\pi RH
$$

10.6 圆环整点数

$x^2+y^2=n$ 整数解数：将 n 分解为 $2^x\prod p_i^{c_i}\prod q_i^{d_i}$ ，其中 p_i 为形如 $4k+1$ 的质因子，q_i 为 $4k+3$ 的质因子，解的总数为 $4\prod(c_i+1)$ 。

10.7 吸收型马尔可夫链

10.7.1 规范型转移矩阵

两个主要定义：

1. 吸收态：从当前状态只能转移到自身的状态，即转移矩阵中 $p_{i,i} = 1$ 。
2. 瞬态：非吸收态的所有状态称为瞬态。

让所有瞬态位于左侧，吸收态位于右侧，即 $S = \{s_1, s_2, \ldots, s_k, s'_{k+1}, \ldots, a'_r\}$ ，其中 s_i 表示瞬态，s'_j 表示吸收态，假设瞬态集合大小为 k 。此时，转移矩阵就可以写成：

$$\begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

这里，$\mathbf{I}$ 是一个单位矩阵，$\mathbf{0}$ 是一个零矩阵，$\mathbf{Q}$ 是一个 $k \times k$ 的非零矩阵，$\mathbf{R}$ 是一个 $k \times (r-k)$ 的非零矩阵。这个转移矩阵被称为规范型转移矩阵，简称规范矩阵（Canonical Matrix）。

10.7.2 基本矩阵

对于一个吸收型马尔可夫链，如果矩阵 $\mathbf{I} - \mathbf{Q}$ 存在一个逆矩阵 $\mathbf{N}$ ，那么矩阵 $\mathbf{N}$ 的第 i, j 项 $n_{i,j}$ 就是从 s_i 移动到 s_j 的期望步数。

10.7.3 吸收时间

令 $\mathbf{t}$ 为一个 $1 \times r$ 的行向量，其中 t_i 表示从状态 s_i 出发被吸收的期望步数，那么有：

$$\mathbf{t} = \mathbf{N}\mathbf{c}$$

其中 $\mathbf{c}$ 是一个 $r \times 1$ 的全 1 列向量。

更具体地，从瞬态 s_i 出发被任意吸收态 s'_j 吸收的期望时间构成一个 $r - k \times r - k$ 的矩阵 $\mathbf{E}$ ，且有：

$$\mathbf{E} = \frac{\mathbf{N}^2\mathbf{R}}{\mathbf{N}\mathbf{R}}$$

这里的除法就是两个矩阵之间的对应元素直接相除，而不是矩阵求逆。

10.7.4 吸收概率

令 $b_{i,j}$ 表示从瞬态 s_i 出发在 s_j 被吸收的概率，那么 $\mathbf{B}$ 是一个 $k \times r$ 的矩阵，且有：

$$\mathbf{B} = \mathbf{N}\mathbf{R}$$

其中，$\mathbf{R}$ 就是标准型转移矩阵中的 $\mathbf{R}$ 。

11 杂项

11.1 快速 IO

```
// fast IO by yosupo
struct Scanner {
    FILE* fp = nullptr;
    char line[(1 << 15) + 1];
    size_t st = 0, ed = 0;
    void reread() {
        memmove(line, line + st, ed - st);
```

```
        ed -= st;
        st = 0;
        ed += fread(line + ed, 1, (1 << 15) - ed, fp);
        line[ed] = '\0';
    }
    bool succ() {
        while (true) {
            if (st == ed) {
                reread();
                if (st == ed) return false;
            }
            while (st != ed && isspace(line[st])) st++;
            if (st != ed) break;
        }
        if (ed - st <= 50) reread();
        return true;
    }
    template <class T, enable_if_t<is_same<T, string>::value, int> = 0>
    bool read_single(T& ref) {
        if (!succ()) return false;
        while (true) {
            size_t sz = 0;
            while (st + sz < ed && !isspace(line[st + sz])) sz++;
            ref.append(line + st, sz);
            st += sz;
            if (!sz || st != ed) break;
            reread();
        }
        return true;
    }
    template <class T, enable_if_t<is_integral<T>::value, int> = 0>
    bool read_single(T& ref) {
        if (!succ()) return false;
        bool neg = false;
        if (line[st] == '-') {
            neg = true;
            st++;
        }
        ref = T(0);
        while (isdigit(line[st])) {
            ref = 10 * ref + (line[st++] - '0');
        }
        if (neg) ref = -ref;
```

```
        return true;
    }
    template <class T> bool read_single(vector<T>& ref) {
        for (auto& d : ref) {
            if (!read_single(d)) return false;
        }
        return true;
    }
    void read() {}
    template <class H, class... T> void read(H& h, T&... t) {
        bool f = read_single(h);
        assert(f);
        read(t...);
    }
    Scanner(FILE* _fp) : fp(_fp) {}
};

struct Printer {
public:
    template <bool F = false> void write() {}
    template <bool F = false, class H, class... T>
    void write(const H& h, const T&... t) {
        if (F) write_single(' ');
        write_single(h);
        write<true>(t...);
    }
    template <class... T> void writeln(const T&... t) {
        write(t...);
        write_single('\n');
    }

    Printer(FILE* _fp) : fp(_fp) {}
    ~Printer() { flush(); }

private:
    static constexpr size_t SIZE = 1 << 15;
    FILE* fp;
    char line[SIZE], small[50];
    size_t pos = 0;
    void flush() {
        fwrite(line, 1, pos, fp);
        pos = 0;
    }
```

```
    void write_single(const char& val) {
        if (pos == SIZE) flush();
        line[pos++] = val;
    }
    template <class T, enable_if_t<is_integral<T>::value, int> = 0>
    void write_single(T val) {
        if (pos > (1 << 15) - 50) flush();
        if (val == 0) {
            write_single('0');
            return;
        }
        if (val < 0) {
            write_single('-');
            val = -val; // todo min
        }
        size_t len = 0;
        while (val) {
            small[len++] = char('0' + (val % 10));
            val /= 10;
        }
        for (size_t i = 0; i < len; i++) {
            line[pos + i] = small[len - 1 - i];
        }
        pos += len;
    }
    void write_single(const string& s) {
        for (char c : s) write_single(c);
    }
    void write_single(const char* s) {
        size_t len = strlen(s);
        for (size_t i = 0; i < len; i++) write_single(s[i]);
    }
    template <class T> void write_single(const vector<T>& val) {
        auto n = val.size();
        for (size_t i = 0; i < n; i++) {
            if (i) write_single(' ');
            write_single(val[i]);
        }
    }
    void write_single(long double d) {
        {
            long long v = d;
            write_single(v);
```

```
                d -= v;
            }
            write_single('.');
            for (int _ = 0; _ < 8; _++) {
                d *= 10;
                long long v = d;
                write_single(v);
                d -= v;
            }
        }
};

Scanner sc(stdin);
Printer pr(stdout);
```

11.2 蔡勒公式

```
int zeller(int y, int m, int d) { // 蔡勒公式 返回星期几
        if (m <= 2) y--, m += 12;
        int c = y / 100; y %= 100;
        int w = ((c >> 2) - (c << 1) + y + (y >> 2) +
                (13 * (m + 1) / 5) + d - 1) % 7;
        if (w < 0) w += 7;
        return (w);
}
int getId(int y, int m, int d) { // 返回到公元1年1月1日的天数
        if (m < 3) { y--; m += 12; }
        return 365 * y + y / 4 - y / 100 + y / 400 +
                (153 * (m - 3) + 2) / 5 + d - 307;
}
```

11.3 枚举子集

11.3.1 暴力遍历

```
/* O(3^n)遍历子集 */
for (int i = 0; i < (1 << n); i++) { // i是当前需要遍历的全集
    for (int l = i;; l = i & (l - 1)) { // l是i的子集
        int r = i - l; // l+r=i
        if (!l) break;
    }
}
```

11.3.2 遍历大小为 k 的子集

```cpp
int n, k;
int s = (1 << k) - 1;
while (s < (1 << n)) { // O(binom(n, k))
    // 每次取出s就是一个大小为k的子集
    int x = s & -s, y = s + x;
    s = (((s & ~y) / x) >> 1) | y;
}
```

11.4 高维前缀和/SoSDP

```cpp
/* 高维前缀和/子集前缀和 */
for (int i = 0; i < n; i++) { // O(n2^n)
    for (int j = 0; j < (1 << n); j++) {
        if (j & (1 << i)) {
            pre[j] += pre[j ^ (1 << i)];
        }
    }
}
```

11.5 压位 BFS

给定一个 n 个点的有向图，当所有边权均为 1 时，$O(\frac{n^3}{w})$ 求任意两点之间的最短路。

```cpp
#include <bits/stdc++.h>
using namespace std;
const int SIZE = 1001;
bitset<SIZE> g[SIZE], vis, now;
int dis[SIZE][SIZE];

int main() {
    ios::sync_with_stdio(false);
    cin.tie(nullptr);
    cout.tie(nullptr);
    memset(dis, -1, sizeof(dis));
    int n;
    cin >> n;
    for (int i = 1; i <= n; i++) {
        for (int j = 1; j <= n; j++) {
            int x;
            cin >> x;
            if (x) g[i].set(j);
        }
```

```cpp
    }
    for (int i = 1; i <= n; i++) {
        vis.reset(); // 清空已遍历的数组
        vis.set(i);
        queue<int> q;
        q.push(i);
        dis[i][i] = 0;
        while (!q.empty()) {
            auto top = q.front();
            q.pop();
            now = g[top] ^ (g[top] & vis); // 去掉已经遍历到的节点
             // 本方法的关键: O(n/w) 遍历bitset
            for (int to = now._Find_first(); to != now.size(); to = now.
                _Find_next(to)) {
                dis[i][to] = dis[i][top] + 1;
                q.push(to);
            }
            vis |= now; // 更新已遍历的节点
        }
    }
    for (int i = 1; i <= n; i++) {
        for (int j = 1; j <= n; j++) {
            cout << dis[i][j] << " \n"[j == n];
        }
    }
    return 0;
}
```

11.6 数位 DP

```cpp
#include <bits/stdc++.h>
using namespace std;
using ll = long long;
ll a[20], dp[20][20];
ll dfs(int len, int las, int maxi, int lead) {
    if (len == 0) return 1;
    if (!maxi && !lead && dp[len][las] != -1) return dp[len][las];
    ll sum = 0;
    int ma = 9;
    if (maxi) ma = a[len];
    if (lead) {
        for (int i = 0; i <= ma; i++) {
            if (i == 0) sum += dfs(len - 1, i, 0, lead);
```

```
            else if (i == ma && maxi) sum += dfs(len - 1, i, maxi, 0);
            else sum += dfs(len - 1, i, 0, 0);
        }
    } else {
        for (int i = 0; i <= ma; i++) {
            if (abs(i - las) < 2) continue;
            if (i == ma && maxi) sum += dfs(len - 1, i, maxi, 0);
            else sum += dfs(len - 1, i, 0, 0);
        }
    }
    if (maxi == 0 && lead == 0) dp[len][las] = sum;
    return sum;
}
ll sol(ll x) {
    int cnt = 0;
    a[++cnt] = x % 10;
    x /= 10;
    while (x) {
        a[++cnt] = x % 10;
        x /= 10;
    }
    return dfs(cnt, 0, 1, 1);
}

int main() {
    ll l, r;
    cin >> l >> r;
    memset(dp, -1, sizeof(dp));
    cout << sol(r) - sol(l - 1) << "\n";
    return 0;
}
```

11.7 随机数生成

```
int rd(int l, int r) {
    mt19937_64 gen(chrono::steady_clock::now().time_since_epoch().count());
    int p = uniform_int_distribution<int>(l, r)(gen);
    return p;
}
```

11.8 简单对拍

```
/*
```

```
 * gen.exe是数据生成器
 * a.exe 和 std.exe 是对拍程序和标程
 */
while (1) {
    system("gen.exe > in.txt");
    system("a.exe < in.txt > a.out");
    system("std.exe < in.txt > std.out");
    if (system("fc a.out std.out")) {
        break;
    }
}
```