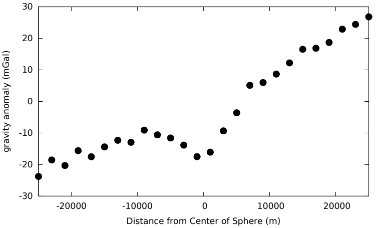
Modeling gravity anomaly with buried sphere approximation

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, a, x)$$



Example application – Katmai volcano gravity anomaly

- Observed gravity data show anomaly consistent with buried sphere, regional trend, and noise
- Estimate trend as polynomial with coefficients $p_1 \ldots p_3$
- Model buried sphere with forward model, assuming depth (z), position (x) and radius (a) are known
- Estimate density contrast, p_4

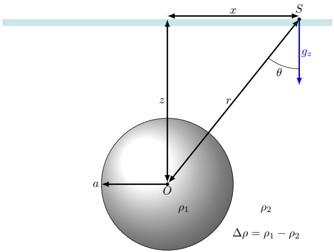
Gravity anomaly due to a buried sphere

It is possible to calculate the exact gravity anomaly due to a wide variety of simple shapes. The simplified geometry of these shapes (spheres, rods, plates) simplifies the analytical solution to find their gravity anomalies (e.g., application of Gauss's law). A surprisingly large number of natural gravity anomalies can be assessed by comparing them to the gravity anomalies of simple shapes. A general form from the vertical component of the gravity anomaly due to a buried mass is:

$$g_z = G \int \frac{dm}{r^2} \cos\theta$$

Note that the vertical component is defined by the Earth's gravity field. It is assumed that the deflection of the equipotential surface (and deflection of the vertical) can be neglected. That is, the anomalous mass is very small compared to the magnitude of the Earth's field. For a sphere with the geometry shown:

$$\begin{aligned} g_z &= g_r \cos\theta \\ &= \frac{GMz}{r^3} \\ &= \frac{4\pi G \Delta\rho a^3}{3} \frac{z}{(x^2+z^2)^{3/2}} \end{aligned}$$



This figure shows the gravity anomaly at point S due to a sphere located entirely below the surface at point O ($z > a$). The Earth's surface is indicated by the thick pale blue line.

Four things are needed for the generalized least squares problem:

- The observed data (e.g., x_i, g_i where $i = 1...n$, where n is the number of gravity observations, at horizontal position x from the center of the sphere with observed gravity value g
- The general linear model (linear in p_j) where $j = 1...m$ and m is the number of unknown model parameters

$$g_{model}(x) = p_1 f_1(x) + p_2 f_2(x) + ... + p_m f_m(x) = \sum_{j=1}^{m} p_j f_j(x)$$

- The m chosen basis functions (a set of linear equations), that describe how the model parameters are thought to relate mathematically to the observations: $f_j(x)$ for $j = 1...m$
- The least-squares misfit criterion (what is the error between the model and the observations that we wish to minimize?)

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (g_{model,i}(x_i) - g_{obs,i}(x_i))^2$$

The gravity observations are arranged in a column vector:

$$\mathbf{b} = \begin{bmatrix} g_1 & g_2 & g_3 & \dots & g_n \end{bmatrix}^T$$

where T indicates the transpose of the matrix
The matrix of basis functions is:

$$\mathbf{A} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \dots & f_{1m} \\ f_{21} & f_{22} & f_{23} & \dots & f_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \dots & f_{nm} \end{bmatrix}$$

The matrix of unknown model parameters is:

$$\mathbf{x} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$$

So, the predicted solution is:

$$\mathbf{g}_{model} = \mathbf{A}\mathbf{x}$$

Knowing $\mathbf{A}$ and $\mathbf{b}$, we need to find the best $\mathbf{x}$, minimizing the difference between $\mathbf{g}_{model}$ and $\mathbf{b}$.
The general least-square solution is

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$$

Which is a linear equation to describe the gravity anomaly due to a buried sphere

One solution to the problem is to implement basis functions in the form:

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, r, x)$$

where p_1, p_2, p_3 are coefficients that attempt to account for the regional variation in gravity using a quadratic equation. The function $g_{sphere}(z, r, x)$ is just like the function implemented to calculate the gravity anomaly due to a sphere, with the important exception that the function returns the estimated gravity anomaly divided by the density. In this case the parameter p_4 is the estimated density contrast. Note this model assumes that the depth of the sphere, its horizontal location, and its radius are known.

PERL script to implement solution to buried sphere

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, a, x)$$

```
sub gsphere ($$$) {

#subroutine to calculate gravity anomaly due to buried sphere
my $pi = 3.14159265358979;
my $r = shift; #input sphere radius in km
my $z = shift; #input sphere depth in km
my $x = shift; #input offset (location of gravity obs) in km

my $big_G = 6.67e-11;
my $to_mgal = 1e5;
my $R_m = 1000 * $r;
my $z_m = 1000 * $z;
my $x_m = 1000 * $x;

my $V = (4/3)* $pi * $R_m ** 3;

$grav = $to_mgal * $big_G * $V * $z_m* ((($z_m*$z_m + $x_m * $x_m))**(-3/2));

return $grav;
};

```

PERL script to implement linear inversion

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, a, x)$$

```
#build the A mtx
$A = new Math::MatrixReal($n,4);
$grav_data = new Math::MatrixReal($n,1);

for ($i=1;$i<$n+1;$i++) {

  $A->assign($i,1,1);
  $A->assign($i,2,$x[$i]);
  $A->assign($i,3,$x[$i]*$x[$i]);
  $A->assign($i,4,gsphere($r, $z, $x[$i]-2));

  $grav_data->assign($i,1,$gravity[$i]);
  }
$AT = $A->new(4,$n);

$AT->transpose($A);
$AAT = $AT * $A;
$inv_AAT = $AAT->inverse;
$AAAT = $inv_AAT * $AT;
$p = $AAAT*$grav_data;
```

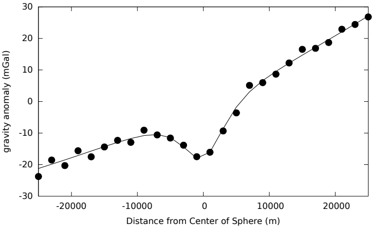
PERL script to implement linear inversion – continued

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, a, x)$$

```
$c1 = $p->element(1,1);
$c2 = $p->element(2,1);
$c3 = $p->element(3,1);
$c4 = $p->element(4,1);

for ($offset=-15.0; $offset<15.0; $offset+=0.1)
{

$x[$i] = $offset;
$mod_gravity[$i] = $c1 + $c2*$offset + $c3*$offset*$offset + $c4*gsphere($r, $z,
$offset-2);

print OUTPUT2 "$x[$i] $mod_gravity[$i] \n";
$i++
};
```

Estimating parameters for gravity anomaly with buried sphere

$$g_{model}(x) = p_1 + p_2 x + p_3 x^2 + p_4 g_{sphere}(z, a, x)$$



- $p_1 = -0.61$ mgal
- $p_2 = 0.0010$ mgal m^{-1}
- $p_3 = 7.3 \times 10^{-4}$ mgal m^{-2}
- $p_4 = -270$ kg m^{-3}

Idea is the x position is constrained by the shape of the anomaly; depth, z, and radius, a, are deduced from seismic tomographic model. Linear problem to solve for density. Great flexibility in modeling the regional trend.

Estimating parameters for gravity anomaly with buried sphere

If the data error is normally distributed with variance σ^2 (e.g., measurement error), it is possible to estimate the parameter error directly:

$$\mathbf{C} = \sigma^2(\mathbf{A}^T\mathbf{A})^{-1}$$

$$\mathbf{p}_{error} = 1.96\sqrt{diag(\mathbf{C})}$$

The number 1.96 indicates that the confidence interval on the parameter estimate is 95 percent.
For example, $p_4 = -270 \pm 25\,\mathrm{kg\,m}^{-3}$

Estimating parameters for gravity anomaly with buried sphere

Steps in assessing the buried sphere model uncertainty:

- Develop a conceptual model of the process
- Collect data and estimate data uncertainty
- Estimate parameters and parameter uncertainty with a mathematical model
- Look back and assess conceptual model (sphere model, r, position)

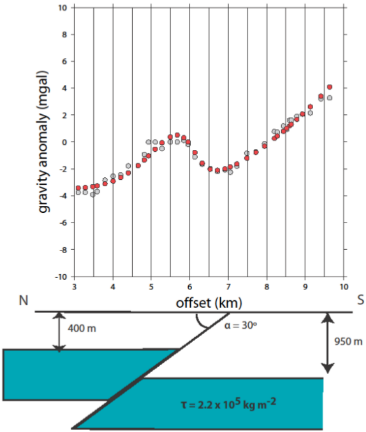
Armenia example – linear inversion of gravity data

Model the gravity anomaly observed near a nuclear power plant. Is it reasonable to explain this gravity anomaly using a thrust fault buried in the sub-surface? The thrust fault might be associated with the spreading of Aragats volcano, or with a regional tectonic fault system.

In this case the inversion method is used to test a hypothesis.

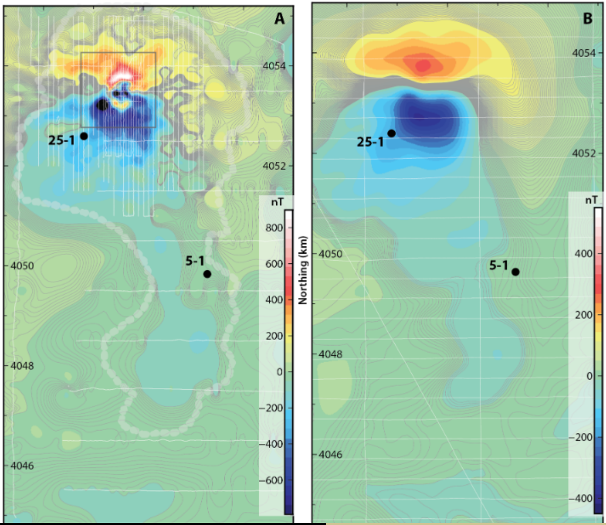
$$\tau = \Delta\rho t$$

where $\Delta\rho$ is the density contrast and t is the thickness.



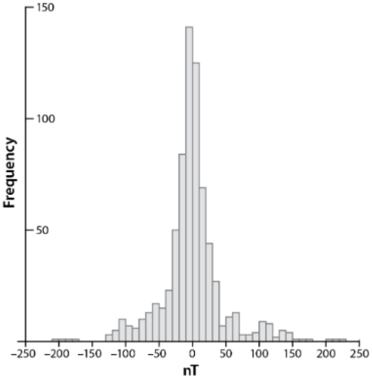
Modeling a complex magnetic anomaly

Magnetic anomalies are observed in the Amargosa Desert (NV). We want to learn about the vent location, depth of burial and lava flow volume. From George et al., 2015, *Statistics in Volcanology*



Data Uncertainty

Differences in magnetic measurements collected along



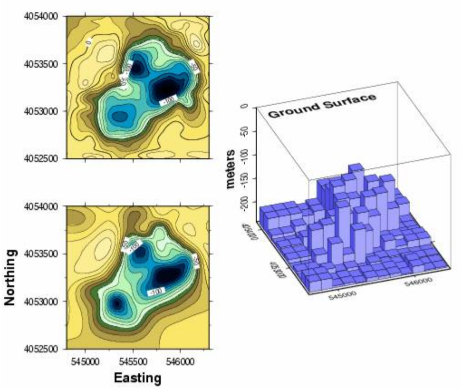
intersecting lines, < 5 m apart:
Standard measurement error - 44 nT

Magnetic anomaly due to prism – complex formulation!

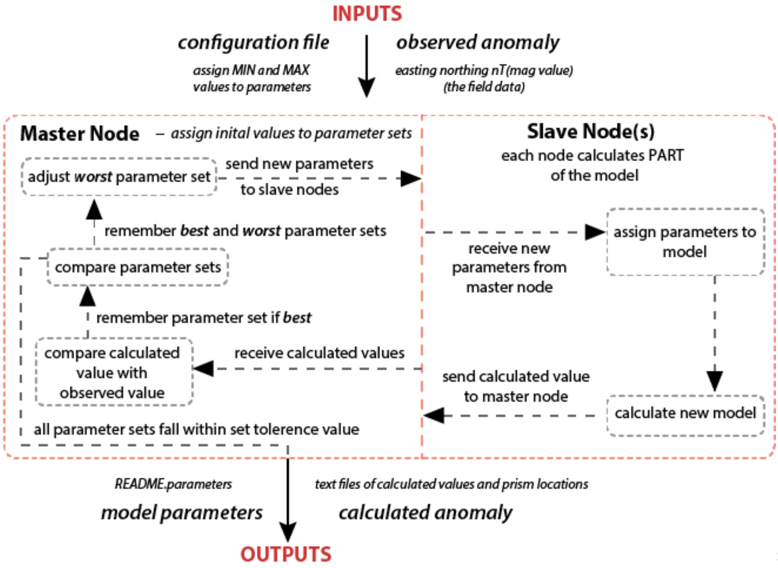
The geology suggests a model framework. Approximately buried volcano with a set of vertically-sided prisms. Magnetic anomalies superimpose (we can add the anomalies due to individual prisms to get the overall anomaly).

Assume the depth to the base of all the prisms is the same (erupted onto a roughly flat surface). Assume the prisms carry uniform remanent magnetization.

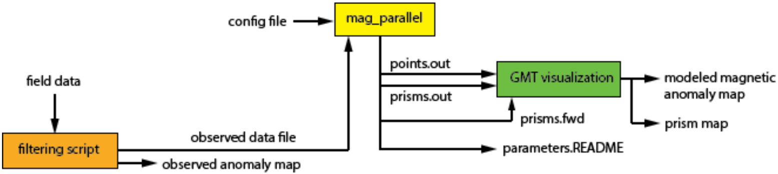
Forward solution due to Rao and Babu (1995). Nonlinear with respect to geometry: depth to top, depth to bottom, orientation of vector of magnetization.



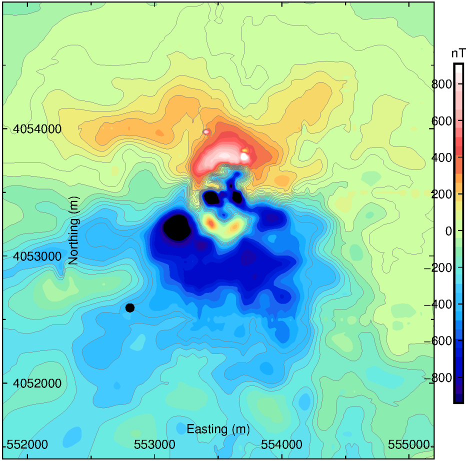
Inversion with many prisms



Execution of code



Magnetic model result

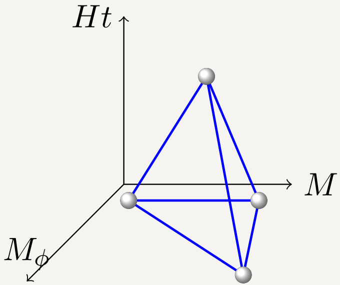


Model results:

- Depth to base of lava flow approximately 150 m
- uniform remanent magnetization – 16 Amp m^{-1}
- area of 25 km^2
- volume 1 ± 0.4 km^3

Understanding the downhill simplex

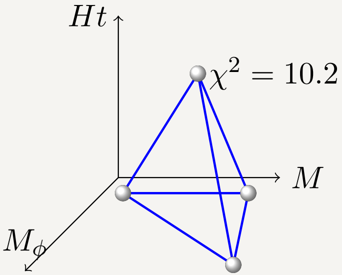
basic simplex



- Hyper-dimensional space (each model parameter is represented by a dimension)
- Here specify three parameters: eruption mass (M), column height (H_t), median grain size (M_ϕ)
- Randomly select $N + 1$ parameter sets – this specifies the simplex. $3 + 1$ parameter sets for this case.

Understanding the downhill simplex

Measure of model fit



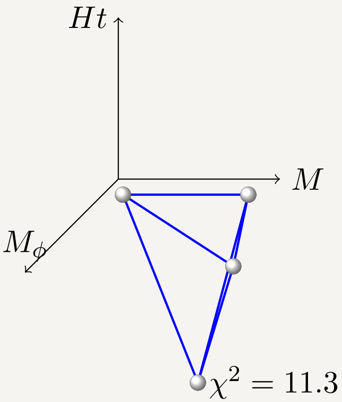
- With each parameter set (4), *calculate* the forward model
- *Compare* model fits using a cost function (misfit function). May be:

$$\chi^2 = \frac{1}{N} \sum \frac{(\hat{t} - t)^2}{\sigma^2}$$

- *Identify* the one parameter set with the worst fit (largest χ^2)

Understanding the downhill simplex

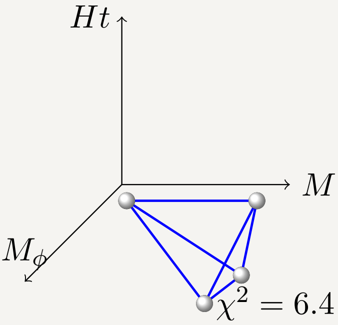
Adjusting the Simplex



- *Change* the parameter set with the worst fit
- A purely geometric operation is applied to the parameter set (in this case – reflection)
- *Calculate* misfit for the new set of parameters
- *Determine* if the misfit improves (in this case it does not improve)

Understanding the downhill simplex

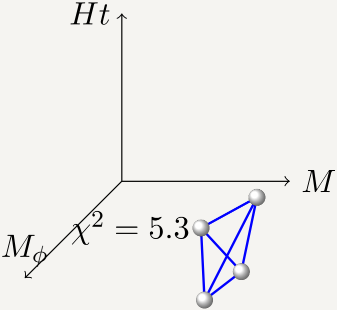
Adjusting the Simplex



- *Change* the parameter set with the worst fit once again using a different operation (in this case contraction)
- *Calculate* misfit for the new set of parameters
- *Determine* if the misfit improves

Understanding the downhill simplex

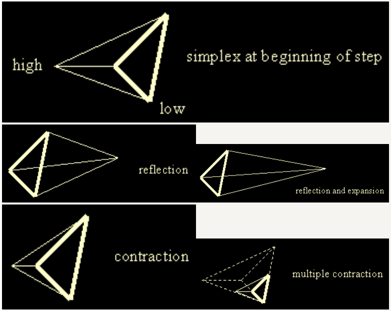
Adjusting the Simplex



- Eventually the geometric move improve the goodness of fit to the point where a different parameter set has the worst fit
- *Repeat* operation until the volume of the simplex is minimized
- Eventually the simplex contracts so that all parameter sets are equal within some tolerance – a global minimum

Understanding the downhill simplex

Range of geometric rules



The downhill simplex method uses three types of operations:

1. reflection
2. contraction
3. expansion

about one of its vertices, to minimize the volume of the specified simplex.

Practical Steps in the Inversion

The basic steps used by the inversion process for finding the optimal set of eruption parameters are:

1. Use field observations of the mass per unit area, $M(x, y)$, of the deposit as input. These input data are compared to the calculated mass per unit area.
2. Define value ranges for the eruption parameters being modeled. For example, the range of possible eruption column heights might be specified as being between 1000 and 20 000 meters. The chosen number of eruption parameters defines an N-dimensional solution space, N being the number of parameters to be optimized by inversion.
3. Randomly initialize $N + 1$ parameter sets. Use Tephra2 as the forward model for calculating initial tephra deposits using each randomly initialized parameter set. Compare the resulting estimates of $M(x, y)$ with field measurements by employing a goodness-of-fit criterion.
4. Choose new parameter values using the downhill simplex method to minimize the difference between calculated values and observed measurements. Iterate through this process of calculating $M(x, y)$ using Tephra2, comparing the calculated values with the field measurements, and adjusting the worst parameter set, until the goodness-of-fit measure, for each parameter set, falls within the tolerance limit set by the user.
5. Explore uncertainty in parameter estimates repeating the process:

- evaluate the goodness of fit of the "optimal solution"
- change the initial parameter sets (random seed)
- change the range of parameter inputs