Numerical methods for the 3D isentropic dynamical core

1 Notation

- N_i, N_j and N_k are the number of grid points in x-, y- and θ-direction, respectively.
- Δx, Δy and $\Delta\theta$ are the grid spacings in x-, y and θ-direction, respectively, and Δt is the timestep.
- Grid points may be:
  - unstaggered, i.e., $\{(x_i = i\Delta x,\, y_j = j\Delta y, \theta_k = k\Delta\theta)\}_{1\le i\le N_i,\, 1\le j\le N_j,\, 1\le k\le N_k}$;
  - staggered in the x-direction, i.e., $\{(x_{i+1/2} = (i+1/2)\Delta x,\, y_j = j\Delta y, \theta_k = k\Delta\theta)\}_{0\le i\le N_i,\, 1\le j\le N_j,\, 1\le k\le N_k}$;
  - staggered in the y-direction, i.e., $\{(x_i = i\Delta x,\, y_{j+1/2} = (j+1/2)\Delta y, \theta_k = k\Delta\theta)\}_{1\le i\le N_i,\, 0\le j\le N_j,\, 1\le k\le N_k}$;
  - staggered in the θ-direction, i.e., $\{(x_i = i\Delta x,\, y_j = j\Delta y, \theta_{k+1/2} = (k+1/2)\Delta\theta)\}_{1\le i\le N_i,\, 1\le j\le N_j,\, 0\le k\le N_k}$.
- $\sigma^n_{i,j,k}$ is the isentropic density at $(x_i,\, y_j, \theta_k)$ at time $t^n = n\Delta t$.
- $u^n_{i+1/2,j,k}$ is the *staggered* x-velocity at $(x_{i+1/2},\, y_j, \theta_k)$ at time t^n; the *unstaggered* x-velocity at $(x_i,\, y_j, \theta_k)$ at time t^n is then defined as

$$u^n_{i,j,k} = \frac{u^n_{i-1/2,j,k} + u^n_{i+1/2,j,k}}{2}\,.$$

- $U^n_{i,j,k} = \sigma^n_{i,j,k}\, u^n_{i,j,k}$ is the x-momentum at $(x_i,\, y_j, \theta_k)$ at time t^n.
- $v^n_{i,j+1/2,k}$ is the *staggered* y-velocity at $(x_i,\, y_{j+1/2}, \theta_k)$ at time t^n; the *unstaggered* y-velocity at $(x_i,\, y_j, \theta_k)$ at time t^n is then defined as

$$v^n_{i,j,k} = \frac{v^n_{i,j-1/2,k} + u^n_{i,j+1/2,k}}{2}\,.$$

- $V^n_{i,j,k} = \sigma^n_{i,j,k}\, v^n_{i,j,k}$ is the y-momentum at $(x_i,\, y_j, \theta_k)$ at time t^n.
- $(q_v)^n_{i,j,k}$ is the mass fraction of water vapour at $(x_i,\, y_j, \theta_k)$ at time t^n.
- $(Q_v)^n_{i,j,k} = \sigma^n_{i,j,k}\, (q_v)^n_{i,j,k}$.
- $(q_c)^n_{i,j,k}$ is the mass fraction of cloud water at $(x_i,\, y_j, \theta_k)$ at time t^n.
- $(Q_c)^n_{i,j,k} = \sigma^n_{i,j,k}\, (q_c)^n_{i,j,k}$.
- $(q_r)^n_{i,j,k}$ is the mass fraction of rain at $(x_i,\, y_j, \theta_k)$ at time t^n.
- $(Q_r)^n_{i,j,k} = \sigma^n_{i,j,k}\, (q_v)^n_{i,j,k}$.
- $p^n_{i,j,k+1/2}$ is the pressure at $(x_i,\, y_j, \theta_{k+1/2})$ at time t^n.
- $\pi^n_{i,j,k+1/2}$ is the Exner function at $(x_i,\, y_j, \theta_{k+1/2})$ at time t^n.
- $M^n_{i,j,k}$ is the Montgomery potential at $(x_i,\, y_j, \theta_k)$ at time t^n.

2 Prognostic equations

The conservative form of the prognostic equations for the isentropic density, the momentums and the air constituents reads:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial u\sigma}{\partial x} + \frac{\partial v\sigma}{\partial y} = 0\,, \tag{2.1a}$$

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial vU}{\partial y} = -\sigma\,\frac{\partial M}{\partial x}\,, \tag{2.1b}$$

$$\frac{\partial V}{\partial t} + \frac{\partial uV}{\partial x} + \frac{\partial vV}{\partial y} = -\sigma\,\frac{\partial M}{\partial y}\,, \tag{2.1c}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial uQ}{\partial x} + \frac{\partial vQ}{\partial y} = 0\,. \tag{2.1d}$$

with $Q = \{Q_v,\, Q_c,\, Q_r$. In the upcoming subsections, we present three numerical techniques for the discretization of (2.1): the leapfrog (Subsection 2.1), upwind (Subsection 2.2) and MacCormack (Subsection 2.3) methods. All the algorithms will be cast in the flux form:

$$\sigma_{i,j}^{n+1} = \sigma_{i,j}^{*} - \Delta t\,\frac{F_{i+1/2,j}^{\sigma,x} - F_{i-1/2,j}^{\sigma,x}}{\Delta x} - \Delta t\,\frac{F_{i,j+1/2}^{\sigma,y} - F_{i,j-1/2}^{\sigma,y}}{\Delta y}\,, \tag{2.2a}$$

$$U_{i,j}^{n+1} = U_{i,j}^{*} - \Delta t\,\frac{F_{i+1/2,j}^{U,x} - F_{i-1/2,j}^{U,x}}{\Delta x} - \Delta t\,\frac{F_{i,j+1/2}^{U,y} - F_{i,j-1/2}^{U,y}}{\Delta y} - c\,\Delta t\,\sigma_{i,j}^{n}\,\frac{M_{i+1,j}^{n} - M_{i-1,j}^{n}}{2\Delta x}\,, \tag{2.2b}$$

$$V_{i,j}^{n+1} = V_{i,j}^{*} - \Delta t\,\frac{F_{i+1/2,j}^{V,x} - F_{i-1/2,j}^{V,x}}{\Delta x} - \Delta t\,\frac{F_{i,j+1/2}^{V,y} - F_{i,j-1/2}^{V,y}}{\Delta y} - c\,\Delta t\,\sigma_{i,j}^{n}\,\frac{M_{i,j+1}^{n} - M_{i,j-1}^{n}}{2\Delta y}\,, \tag{2.2c}$$

$$Q_{i,j}^{n+1} = Q_{i,j}^{*} - \Delta t\,\frac{F_{i+1/2,j}^{Q,x} - Q_{i-1/2,j}^{Q,x}}{\Delta x} - \Delta t\,\frac{F_{i,j+1/2}^{Q,y} - F_{i,j-1/2}^{Q,y}}{\Delta y}\,, \tag{2.2d}$$

where

$$\phi_{i,j}^{*} = \begin{cases} \phi_{i,j}^{n-1} & \text{for the leapfrog scheme}\,, \\ \phi_{i,j}^{n} & \text{otherwise}\,, \end{cases}$$

with $\phi = \{\sigma,\, U,\, V,\, Q_v,\, Q_c,\, Q_r\}$, and

$$c = \begin{cases} 2 & \text{for the leapfrog scheme}\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

The original variables are then recovered by

$$u_{i+1/2,j}^{n+1} = \frac{U_{i,j}^{n+1} + U_{i+1,j}^{n+1}}{\sigma_{i,j}^{n+1} + \sigma_{i,j}^{n+1}}\,, \tag{2.3a}$$

$$v_{i+1/2,j}^{n+1} = \frac{V_{i,j}^{n+1} + V_{i+1,j}^{n+1}}{\sigma_{i,j}^{n+1} + \sigma_{i,j}^{n+1}}\,, \tag{2.3b}$$

$$q_{i,j}^{n+1} = \frac{Q_{i,j}^{n+1}}{\sigma_{i,j}^{n+1}}\,, \tag{2.3c}$$

with $q = \{q_v,\, q_c,\, q_r\}$. Note that each variable considered in this section is defined at the same θ-level. Hence, we can safely drop the subscript refering to the θ-position.

2.1 Leapfrog method

By discretizing both the space and time derivatives via centered finite difference, we get:

$$F_{i+1/2,j}^{\sigma,x} = U_{i+1,j}^{n} - U_{i,j}^{n}\,, \tag{2.4a}$$

$$F_{i,j+1/2}^{\sigma,y} = V_{i,j+1}^{n} - V_{i,j}^{n}\,, \tag{2.4b}$$

$$F_{i+1/2,j}^{\psi,x} = u_{i+1,j}^{n}\,\psi_{i+1,j}^{n} - u_{i,j}^{n}\,\psi_{i,j}^{n} = \frac{U_{i+1,j}^{n}\,\psi_{i+1,j}^{n}}{\sigma_{i+1,j}^{n}} - \frac{U_{i,j}^{n}\,\psi_{i,j}^{n}}{\sigma_{i,j}^{n}}\,, \tag{2.4c}$$

$$F_{i,j+1/2}^{\psi,y} = v_{i,j+1}^{n}\,\psi_{i,j+1}^{n} - v_{i,j}^{n}\,\psi_{i,j}^{n} = \frac{V_{i,j+1}^{n}\,\psi_{i,j+1}^{n}}{\sigma_{i,j+1}^{n}} - \frac{V_{i,j}^{n}\,\psi_{i,j}^{n}}{\sigma_{i,j}^{n}}\,, \tag{2.4d}$$

with $\psi = \{U, V, Q_v, Q_c, Q_r\}$.

2.2 Upwind method

The upwind method couples the forward Euler scheme in time with a space discretization relying upon the sign of the advective field. In particular:

$$F_{i+1/2,j}^{\phi,x} = \max\left(0,\, u_{i+1/2,j}^{n}\right)\phi_{i,j}^{n} + \min\left(0,\, u_{i+1/2,j}^{n}\right)\phi_{i+1,j}^{n}\,, \tag{2.5a}$$

$$F_{i,j+1/2}^{\phi,y} = \max\left(0,\, v_{i,j+1/2}^{n}\right)\phi_{i,j}^{n} + \min\left(0,\, v_{i,j+1/2}^{n}\right)\phi_{i,j+1}^{n}\,, \tag{2.5b}$$

with $\phi = \{\sigma, U, V, Q_v, Q_c, Q_r\}$. As a result, the scheme is first-order accurate in space and time.

2.3 MacCormack method

As the leapfrog method, the MacCormack method is second-order accurate in space and time. For a generic conserved variable ϕ, it reads:

$$\phi_{i,j}^{n+1} = \phi_{i,j}^{n} + \Delta t \left(\frac{\partial \phi}{\partial t}\right)_{i,j}^{avg}\,, \tag{2.6}$$

where the average derivative is defined as

$$\left(\frac{\partial \phi}{\partial t}\right)_{i,j}^{avg} = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)_{i,j}^{n} + \overline{\left(\frac{\partial \phi}{\partial t}\right)}_{i,j}^{n+1}\right]. \tag{2.7}$$

The average derivatives for the isentropic density, momentums and water components are derived in the following paragraphs.

2.3.1 Continuity equation

1. Approximate the time derivatives of σ, U and V at current time using forward finite differences:

$$\left(\frac{\partial \sigma}{\partial t}\right)_{i,j}^{n} = -\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}\,, \tag{2.8a}$$

$$\left(\frac{\partial U}{\partial t}\right)_{i,j}^{n} = -\frac{u_{i+1,j}^{n}\,U_{i+1,j}^{n} - u_{i,j}^{n}\,U_{i,j}^{n}}{\Delta x} - \frac{v_{i,j+1}^{n}\,U_{i,j+1}^{n} - v_{i,j}^{n}\,U_{i,j}^{n}}{\Delta y} - \sigma_{i,j}^{n}\,\frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta x}\,, \tag{2.8b}$$

$$\left(\frac{\partial V}{\partial t}\right)_{i,j}^{n} = -\frac{u_{i+1,j}^{n}\,V_{i+1,j}^{n} - u_{i,j}^{n}\,V_{i,j}^{n}}{\Delta x} - \frac{v_{i,j+1}^{n}\,V_{i,j+1}^{n} - v_{i,j}^{n}\,V_{i,j}^{n}}{\Delta y} - \sigma_{i,j}^{n}\,\frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta y}\,. \tag{2.8c}$$

2. Compute provisional values at time t^{n+1} exploiting Taylor expansion and (2.8):

$$\overline{\sigma}_{i,j}^{n+1} = \sigma_{i,j}^{n} + \Delta t \left(\frac{\partial \sigma}{\partial t} \right)_{i,j}^{n} = \sigma_{i,j}^{n} - \Delta t \frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \Delta t \frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}, \tag{2.9a}$$

$$\overline{U}_{i,j}^{n+1} = U_{i,j}^{n} + \Delta t \left(\frac{\partial U}{\partial t} \right)_{i,j}^{n} = U_{i,j}^{n} - \Delta t \frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \Delta t \frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y} - \Delta t \, \sigma_{i,j}^{n} \frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta x}, \tag{2.9b}$$

$$\overline{V}_{i,j}^{n+1} = V_{i,j}^{n} + \Delta t \left(\frac{\partial V}{\partial t} \right)_{i,j}^{n} = V_{i,j}^{n} - \Delta t \frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} - \Delta t \frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y} - \Delta t \, \sigma_{i,j}^{n} \frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta y}. \tag{2.9c}$$

3. Approximate the provisional time derivative for σ using backward finite differences:

$$\overline{\left(\frac{\partial \sigma}{\partial t} \right)}_{i,j}^{n+1} = - \frac{\overline{U}_{i,j}^{n+1} - \overline{U}_{i-1,j}^{n+1}}{\Delta x} - \frac{\overline{V}_{i,j}^{n+1} - \overline{V}_{i,j-1}^{n+1}}{\Delta y}. \tag{2.10}$$

4. Step σ using (2.6), (2.8a) and (2.10):

$$\sigma_{i,j}^{n+1} = \sigma_{i,j}^{n} + \frac{\Delta t}{2} \left[- \frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y} - \frac{\overline{U}_{i,j}^{n+1} - \overline{U}_{i-1,j}^{n+1}}{\Delta x} - \frac{\overline{V}_{i,j}^{n+1} - \overline{V}_{i,j-1}^{n+1}}{\Delta y} \right]. \tag{2.11}$$

Inserting (2.9b) and (2.9c) in (2.11) yields:

$$\begin{aligned} \sigma_{i,j}^{n+1} = \sigma_{i,j}^{n} &- \frac{\Delta t}{2\Delta x} \left[U_{i+1,j}^{n} - \Delta t \left(\frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} + \frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y} + \sigma_{i,j}^{n} \frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta x} \right) \right] \\ &+ \frac{\Delta t}{2\Delta x} \left[U_{i-1,j}^{n} - \Delta t \left(\frac{u_{i,j}^{n} U_{i,j}^{n} + u_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta x} + \frac{v_{i-1,j+1}^{n} U_{i-1,j+1}^{n} - v_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta y} + \sigma_{i-1,j}^{n} \frac{M_{i,j}^{n} + M_{i-1,j}^{n}}{\Delta x} \right) \right] \\ &- \frac{\Delta t}{2\Delta y} \left[V_{i,j+1}^{n} - \Delta t \left(\frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} + \frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y} + \sigma_{i,j}^{n} \frac{M_{i,j+1}^{n} - M_{i,j}^{n}}{\Delta y} \right) \right] \\ &- \frac{\Delta t}{2\Delta y} \left[V_{i,j-1}^{n} - \Delta t \left(\frac{u_{i+1,j-1}^{n} V_{i+1,j-1}^{n} - u_{i,j-1}^{n} V_{i,j-1}^{n}}{\Delta x} + \frac{v_{i,j}^{n} V_{i,j}^{n} - v_{i,j-1}^{n} V_{i,j-1}^{n}}{\Delta y} + \sigma_{i,j-1}^{n} \frac{M_{i,j}^{n} - M_{i,j-1}^{n}}{\Delta y} \right) \right]. \end{aligned} \tag{2.12}$$

The fluxes finally follows from (2.12):

$$F_{i+1/2,j}^{\sigma,x} = \frac{U_{i+1,j}^{n} + U_{i,j}^{n}}{2} - \frac{\Delta t}{2} \left(\frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} + \frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y} + \sigma_{i,j}^{n} \frac{M_{i+1,j}^{n} - M_{i,j}^{n}}{\Delta x} \right), \tag{2.13a}$$

$$F_{i,j+1/2}^{\sigma,y} = \frac{U_{i,j+1}^{n} + U_{i,j}^{n}}{2} - \frac{\Delta t}{2} \left(\frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} + \frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y} + \sigma_{i,j}^{n} \frac{M_{i,j+1}^{n} - M_{i,j}^{n}}{\Delta y} \right). \tag{2.13b}$$

2.3.2 Momentum equations

1. Approximate the time derivatives of σ, U and V at current time using forward finite differences. For U and V, disregard the terms involving the Montgomery potential:

$$\left(\frac{\partial \sigma}{\partial t} \right)_{i,j}^{n} = - \frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}, \tag{2.14a}$$

$$\left(\frac{\partial U}{\partial t} \right)_{i,j}^{n} = - \frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y}, \tag{2.14b}$$

$$\left(\frac{\partial V}{\partial t} \right)_{i,j}^{n} = - \frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} - \frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y}. \tag{2.14c}$$

2. Compute provisional values at time t^{n+1} exploiting Taylor expansion and (2.14):

$$\overline{\sigma}_{i,j}^{n+1} = \sigma_{i,j}^{n} + \Delta t \left(\frac{\partial \sigma}{\partial t}\right)_{i,j}^{n} = \sigma_{i,j}^{n} - \Delta t \frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \Delta t \frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}, \tag{2.15a}$$

$$\overline{U}_{i,j}^{n+1} = U_{i,j}^{n} + \Delta t \left(\frac{\partial U}{\partial t}\right)_{i,j}^{n} = U_{i,j}^{n} - \Delta t \frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \Delta t \frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y}, \tag{2.15b}$$

$$\overline{V}_{i,j}^{n+1} = V_{i,j}^{n} + \Delta t \left(\frac{\partial V}{\partial t}\right)_{i,j}^{n} = V_{i,j}^{n} - \Delta t \frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} - \Delta t \frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y}. \tag{2.15c}$$

3. Approximate the provisional time derivative for U and V using backward finite differences. As in 2.3.2, neglect the terms involving the Montgomery potential:

$$\overline{\left(\frac{\partial U}{\partial t}\right)}_{i,j}^{n+1} = -\frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{U}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{U}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i,j-1}^{n+1}\,\overline{V}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right), \tag{2.16a}$$

$$\overline{\left(\frac{\partial V}{\partial t}\right)}_{i,j}^{n+1} = -\frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{V}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{V}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{V}_{i,j-1}^{n+1}\,\overline{V}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right). \tag{2.16b}$$

4. Step U and V using (2.6), (2.14b), (2.14c) and (2.16):

$$\begin{aligned} U_{i,j}^{n+1} = U_{i,j}^{n} + \frac{\Delta t}{2}\Bigg[&-\frac{1}{\Delta x}\left(u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}\right) - \frac{1}{\Delta y}\left(v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}\right) \\ &- \frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{U}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{U}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i,j-1}^{n+1}\,\overline{V}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right)\Bigg], \end{aligned} \tag{2.17a}$$

$$\begin{aligned} V_{i,j}^{n+1} = V_{i,j}^{n} + \frac{\Delta t}{2}\Bigg[&-\frac{1}{\Delta x}\left(u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}\right) - \frac{1}{\Delta y}\left(v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}\right) \\ &- \frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{V}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{V}_{i,j}^{n+1}\,\overline{V}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{V}_{i,j-1}^{n+1}\,\overline{V}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right)\Bigg]. \end{aligned} \tag{2.17b}$$

Inserting (2.16) yields

$$\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} &- \frac{\Delta t}{\Delta x}\boxed{\left[\frac{u_{i+1,j}^{n} U_{i+1,j}^{n}}{2} + \frac{1}{2}\,\frac{\left[U_{i,j}^{n} - \Delta t\frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \Delta t\frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y}\right]\left[U_{i,j}^{n} - \Delta t\frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \Delta t\frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y}\right]}{\sigma_{i,j}^{n} - \Delta t\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \Delta t\frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}}\right]} \quad F_{i+1/2,j}^{U,x} \\
&+ \frac{\Delta t}{\Delta x}\boxed{\left[\frac{u_{i,j}^{n} U_{i,j}^{n}}{2} + \frac{\left[U_{i-1,j}^{n} - \Delta t\frac{u_{i,j}^{n} U_{i,j}^{n} - u_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta x} - \Delta t\frac{v_{i-1,j+1}^{n} U_{i-1,j+1}^{n} - v_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta y}\right]\left[U_{i-1,j}^{n} - \Delta t\frac{u_{i,j}^{n} U_{i,j}^{n} - u_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta x} - \Delta t\frac{v_{i-1,j+1}^{n} U_{i-1,j+1}^{n} - v_{i-1,j}^{n} U_{i-1,j}^{n}}{\Delta y}\right]}{\sigma_{i-1,j}^{n} - \Delta t\frac{U_{i,j}^{n} - U_{i-1,j}^{n}}{\Delta x} - \Delta t\frac{V_{i-1,j+1}^{n} - V_{i-1,j}^{n}}{\Delta y}}\right]} \quad F_{i-1/2,j}^{U,x} \\
&- \frac{\Delta t}{\Delta y}\boxed{\left[\frac{v_{i,j+1}^{n} U_{i,j+1}^{n}}{2} + \frac{\left[U_{i,j}^{n} - \Delta t\frac{u_{i+1,j}^{n} U_{i+1,j}^{n} - u_{i,j}^{n} U_{i,j}^{n}}{\Delta x} - \Delta t\frac{v_{i,j+1}^{n} U_{i,j+1}^{n} - v_{i,j}^{n} U_{i,j}^{n}}{\Delta y}\right]\left[V_{i,j}^{n} - \Delta t\frac{u_{i+1,j}^{n} V_{i+1,j}^{n} - u_{i,j}^{n} V_{i,j}^{n}}{\Delta x} - \Delta t\frac{v_{i,j+1}^{n} V_{i,j+1}^{n} - v_{i,j}^{n} V_{i,j}^{n}}{\Delta y}\right]}{\sigma_{i,j}^{n} - \Delta t\frac{U_{i+1,j}^{n} - U_{i,j}^{n}}{\Delta x} - \Delta t\frac{V_{i,j+1}^{n} - V_{i,j}^{n}}{\Delta y}}\right]} \quad F_{i,j+1/2}^{U,y} \\
&+ \frac{\Delta t}{\Delta y}\boxed{\left[\frac{v_{i,j}^{n} U_{i,j}^{n}}{2} + \frac{\left[U_{i,j-1}^{n} - \Delta t\frac{u_{i+1,j-1}^{n} U_{i+1,j-1}^{n} - u_{i,j-1}^{n} U_{i,j-1}^{n}}{\Delta x} - \Delta t\frac{v_{i,j}^{n} U_{i,j}^{n} - v_{i,j-1}^{n} U_{i,j-1}^{n}}{\Delta y}\right]\left[V_{i,j-1}^{n} - \Delta t\frac{u_{i+1,j-1}^{n} V_{i+1,j-1}^{n} - u_{i,j-1}^{n} V_{i,j-1}^{n}}{\Delta x} - \Delta t\frac{v_{i,j}^{n} V_{i,j}^{n} - v_{i,j-1}^{n} V_{i,j-1}^{n}}{\Delta y}\right]}{\sigma_{i,j-1}^{n} - \Delta t\frac{U_{i+1,j-1}^{n} - U_{i,j-1}^{n}}{\Delta x} - \Delta t\frac{V_{i,j}^{n} - V_{i,j-1}^{n}}{\Delta y}}\right]} \quad F_{i,j-1/2}^{U,y}
\end{aligned}$$

and

$$
\begin{aligned}
V^{n+1}_{i,j} = V^{n}_{i,j} &- \underbrace{\boxed{\frac{\Delta t}{\Delta x}\left[\frac{u^n_{i+1,j}\,V^n_{i+1,j}}{2} + \frac{1}{2}\,\frac{\left[U^n_{i,j} - \Delta t\,\frac{u^n_{i+1,j}\,U^n_{i+1,j} - u^n_{i,j}\,U^n_{i,j}}{\Delta x} - \Delta t\,\frac{v^n_{i,j+1}\,U^n_{i,j+1} - v^n_{i,j}\,U^n_{i,j}}{\Delta y}\right]\left[V^n_{i,j} - \Delta t\,\frac{u^n_{i+1,j}\,V^n_{i+1,j} - u^n_{i,j}\,V^n_{i,j}}{\Delta x} - \Delta t\,\frac{v^n_{i,j+1}\,V^n_{i,j+1} - v^n_{i,j}\,V^n_{i,j}}{\Delta y}\right]}{\sigma^n_{i,j} - \Delta t\,\frac{U^n_{i+1,j} - U^n_{i,j}}{\Delta x} - \Delta t\,\frac{V^n_{i,j+1} - V^n_{i,j}}{\Delta y}}\right]}}_{F^{V,x}_{i+1/2,j}} \\
&+ \underbrace{\boxed{\frac{\Delta t}{\Delta x}\left[\frac{u^n_{i,j}\,V^n_{i,j}}{2} + \frac{\left[U^n_{i-1,j} - \Delta t\,\frac{u^n_{i,j}\,U^n_{i,j} - u^n_{i-1,j}\,U^n_{i-1,j}}{\Delta x} - \Delta t\,\frac{v^n_{i-1,j+1}\,U^n_{i-1,j+1} - v^n_{i-1,j}\,U^n_{i-1,j}}{\Delta y}\right]\left[V^n_{i-1,j} - \Delta t\,\frac{u^n_{i,j}\,V^n_{i,j} - u^n_{i-1,j}\,V^n_{i-1,j}}{\Delta x} - \Delta t\,\frac{v^n_{i-1,j+1}\,V^n_{i-1,j+1} - v^n_{i-1,j}\,V^n_{i-1,j}}{\Delta y}\right]}{\sigma^n_{i-1,j} - \Delta t\,\frac{U^n_{i,j} - U^n_{i-1,j}}{\Delta x} - \Delta t\,\frac{V^n_{i-1,j+1} - V^n_{i-1,j}}{\Delta y}}\right]}}_{F^{V,x}_{i-1/2,j}} \\
&- \underbrace{\boxed{\frac{\Delta t}{\Delta y}\left[\frac{v^n_{i,j+1}\,V^n_{i,j+1}}{2} + \frac{\left[V^n_{i,j} - \Delta t\,\frac{u^n_{i+1,j}\,V^n_{i+1,j} - u^n_{i,j}\,V^n_{i,j}}{\Delta x} - \Delta t\,\frac{v^n_{i,j+1}\,V^n_{i,j+1} - v^n_{i,j}\,V^n_{i,j}}{\Delta y}\right]\left[V^n_{i,j} - \Delta t\,\frac{u^n_{i+1,j}\,V^n_{i+1,j} - u^n_{i,j}\,V^n_{i,j}}{\Delta x} - \Delta t\,\frac{v^n_{i,j+1}\,V^n_{i,j+1} - v^n_{i,j}\,V^n_{i,j}}{\Delta y}\right]}{\sigma^n_{i,j} - \Delta t\,\frac{U^n_{i+1,j} - U^n_{i,j}}{\Delta x} - \Delta t\,\frac{V^n_{i,j+1} - V^n_{i,j}}{\Delta y}}\right]}}_{F^{V,y}_{i,j+1/2}} \\
&+ \underbrace{\boxed{\frac{\Delta t}{\Delta y}\left[\frac{v^n_{i,j}\,V^n_{i,j}}{2} + \frac{\left[V^n_{i,j-1} - \Delta t\,\frac{u^n_{i+1,j-1}\,V^n_{i+1,j-1} - u^n_{i,j-1}\,V^n_{i,j-1}}{\Delta x} - \Delta t\,\frac{v^n_{i,j}\,V^n_{i,j} - v^n_{i,j-1}\,V^n_{i,j-1}}{\Delta y}\right]\left[V^n_{i,j-1} - \Delta t\,\frac{u^n_{i+1,j-1}\,V^n_{i+1,j-1} - u^n_{i,j-1}\,V^n_{i,j-1}}{\Delta x} - \Delta t\,\frac{v^n_{i,j}\,V^n_{i,j} - v^n_{i,j-1}\,V^n_{i,j-1}}{\Delta y}\right]}{\sigma^n_{i,j-1} - \Delta t\,\frac{U^n_{i+1,j-1} - U^n_{i,j-1}}{\Delta x} - \Delta t\,\frac{V^n_{i,j} - V^n_{i,j-1}}{\Delta y}}\right]}}_{F^{V,y}_{i,j-1/2}}
\end{aligned}
$$

5. Finally taking the forcing terms into account and slightly differencing ourselves from the general form (2.2), the update values for the momentums are

$$
U^{n+1}_{i,j} = U^n_{i,j} - \Delta t\,\frac{F^{U,x}_{i+1/2,j} - F^{U,x}_{i-1/2,j}}{\Delta x} - \Delta t\,\frac{F^{U,y}_{i,j+1/2} - F^{U,y}_{i,j-1/2}}{\Delta y} - \Delta t\,\sigma^{n+1/2}_{i,j}\,\frac{M^{n+1/2}_{i+1,j} - M^{n+1/2}_{i-1,j}}{2\Delta x}\,, \tag{2.17a}
$$

$$
V^{n+1}_{i,j} = V^n_{i,j} - \Delta t\,\frac{F^{V,x}_{i+1/2,j} - F^{V,x}_{i-1/2,j}}{\Delta x} - \Delta t\,\frac{F^{V,y}_{i,j+1/2} - F^{V,y}_{i,j-1/2}}{\Delta y} - \Delta t\,\sigma^{n+1/2}_{i,j}\,\frac{M^{n+1/2}_{i,j+1} - M^{n+1/2}_{i,j-1}}{2\Delta y}\,, \tag{2.17b}
$$

with

$$
\sigma_{i,j} = \frac{\sigma^n_{i,j} + \sigma^{n+1}_{i,j}}{2}
$$

and $M^{n+1/2}_{i,j}$ diagnosed from $\sigma^{n+1/2}_{i,j}$ (see Subsection **??**).

Note: since the diagnosis of the Montgomery potential requires the vertical axis to be spanned sequentially, (2.17) may be hard to implement in GT4Py.

2.3.3 Water constituents equations

1. Approximate the current time derivative of $Q = \{Q_v, Q_c, Q_r\}$ using forward finite differences:

$$
\left(\frac{\partial Q}{\partial t}\right)^n_{i,j} = -\frac{u^n_{i+1,j}\,Q^n_{i+1,j} - u^n_{i,j}\,Q^n_{i,j}}{\Delta x} - \frac{v^n_{i,j+1}\,Q^n_{i,j+1} - v^n_{i,j}\,Q^n_{i,j}}{\Delta y}\,. \tag{2.18}
$$

2. Compute provisional value for σ, U and V via (2.9);

3. Compute provisional value for Q exploiting Taylor expansion and (2.18):

$$
\overline{Q}^{n+1}_{i,j} = Q^n_{i,j} - \Delta t\,\frac{u^n_{i+1,j}\,Q^n_{i+1,j} - u^n_{i,j}\,Q^n_{i,j}}{\Delta x} - \Delta t\,\frac{v^n_{i,j+1}\,Q^n_{i,j+1} - v^n_{i,j}\,Q^n_{i,j}}{\Delta y}\,. \tag{2.19}
$$

4. Approximate provisional time derivative of Q using backward finite differences:

$$\overline{\left(\frac{\partial Q}{\partial t}\right)}_{i,j}^{n+1} = -\frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{Q}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{V}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{V}_{i,j-1}^{n+1}\,\overline{Q}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right). \tag{2.20}$$

5. Step Q via (2.6):

$$\begin{aligned} Q_{i,j}^{n+1} = Q_{i,j}^{n} + \frac{\Delta t}{2}\Bigg[&- \frac{u_{i+1,j}^{n}\,Q_{i+1,j}^{n} - u_{i,j}^{n}\,Q_{i,j}^{n}}{\Delta x} - \frac{v_{i,j+1}^{n}\,Q_{i,j+1}^{n} - v_{i,j}^{n}\,Q_{i,j}^{n}}{\Delta y} \\ &- \frac{1}{\Delta x}\left(\frac{\overline{U}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{U}_{i-1,j}^{n+1}\,\overline{Q}_{i-1,j}^{n+1}}{\overline{\sigma}_{i-1,j}^{n+1}}\right) - \frac{1}{\Delta y}\left(\frac{\overline{V}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}} - \frac{\overline{V}_{i,j-1}^{n+1}\,\overline{Q}_{i,j-1}^{n+1}}{\overline{\sigma}_{i,j-1}^{n+1}}\right)\Bigg]. \end{aligned} \tag{2.21}$$

Hence:

$$F_{i+1/2,j}^{Q,x} = \frac{1}{2}\left(u_{i+1,j}^{n}\,Q_{i+1,j}^{n} + \frac{\overline{U}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}}\right), \tag{2.22a}$$

$$F_{i,j+1/2}^{Q,y} = \frac{1}{2}\left(v_{i,j+1}^{n}\,Q_{i,j+1}^{n} + \frac{\overline{V}_{i,j}^{n+1}\,\overline{Q}_{i,j}^{n+1}}{\overline{\sigma}_{i,j}^{n+1}}\right). \tag{2.22b}$$