Threshold-Based Graph Reconstruction Using Discrete Morse Theory

Brittany Terese Fasy[1,2] **Sushovan Majhi**[3]
Carola Wenk[4]

[1]Department of Computer Science, Montana State University

[2]Department of Mathematics, Montana State University

[3]**Department of Mathematics, Tulane University**

[4]Department of Computer Science, Tulane University

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Problem Statement

Given a (noisy) sample S taken around a (hidden) embedded graph G, how one can "reconstruct" the topology and geometry of G from S.

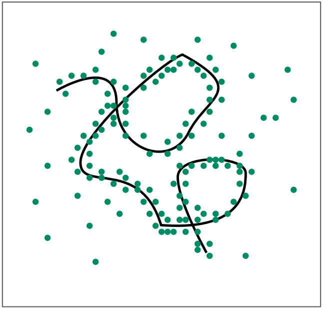


Figure: Sample around an embedded graph

Application: Map Reconstruction from GPS traces

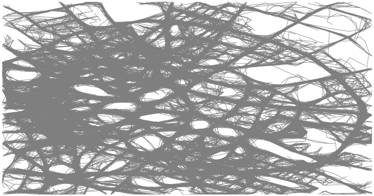
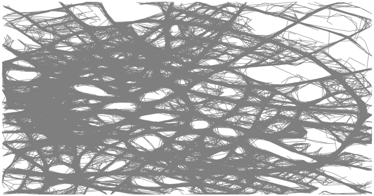


Figure: GPS traces of Berlin (mapreconstruction.org)



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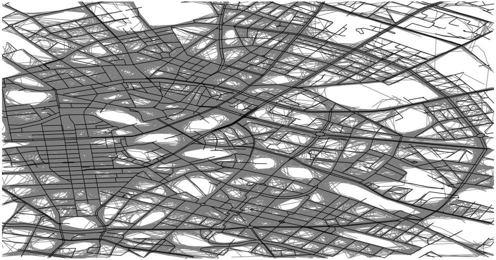

FIGURE: A reconstruction

Noise Models

1. Hausdorff noise

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2. Non-Hausdorff noise

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Generic Density-Based Algorithm

Given a discretized domain $\tilde{D}$ and a sample S around G.

1. Compute density f of S over $\tilde{D}$.

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$$f(x) = \frac{1}{2\pi n \tau^2} \sum_{X_i \in S} K(x, X_i; \tau)$$

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3. heuristic pruning methods are applied this super-level set to approximate G.

Recent Works

1. Choosing thresholds systematically using Persistent Homology. Ahmed, Fasy et al. (AFGW15)

Related Work

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Limitations

1. Thresholds are chosen heuristically.
2. Theoretical guarantees on the topological/geometric correctness is not proved.
3. The output is often a thick region around the hidden graph.

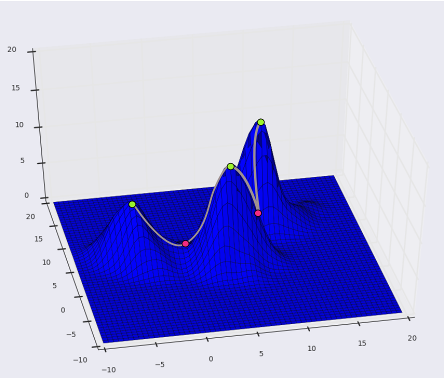


FIGURE: KDE

If the samples are concentrated around a graph, then the mountain ridges on the graph of the density function are expected to capture it.

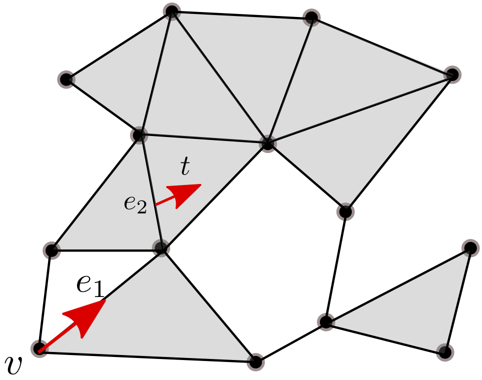
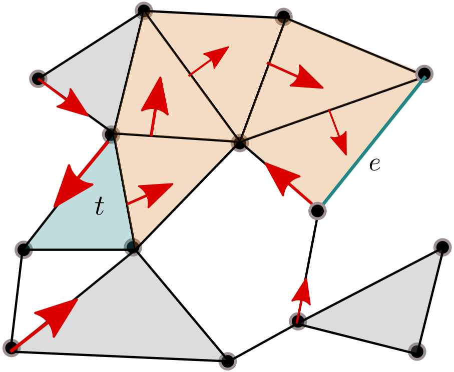


FIGURE: Simplicial Complex K and discrete vector

Discrete Vector Field

(σ, τ) is a discrete vector in K if $\tau < \sigma$.



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DISCRETE VECTOR FIELD

(σ, τ) is a discrete vector in K if $\tau < \sigma$. A discrete vector field is a collection of discrete vectors such that every simplex of K is head/tail of at most one vector.

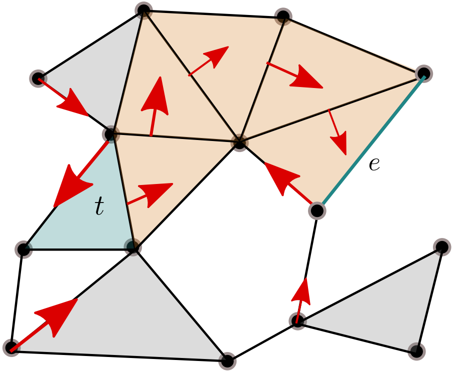


FIGURE: V-path

V-PATH

$$\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{l+1}$$

where (σ_i, τ_i) is a vector and $\tau_i < \sigma_{i+1}$.

Background: Morse Cancellation

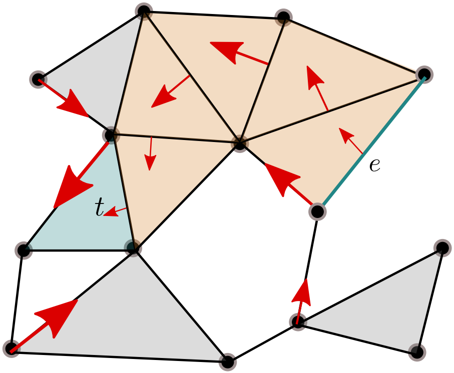
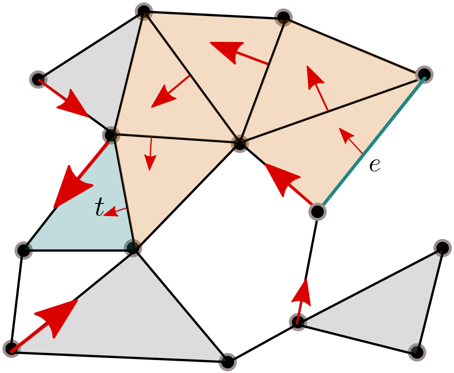


Figure: Morse Cancellation



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STABLE MANIFOLD

For a critical edge e, its stable manifold is the set of all V-paths ending at the boundary of e.

$(\omega, \beta_1, \beta_2, \nu)$-APPROXIMATION OF G

$$f(x) \in \begin{cases} [\beta_1, \beta_1 + \nu], & x \in V^\omega \\ [\beta_2, \beta_2 + \nu], & x \in E^\omega \\ [0, \nu], & \text{otherwise} \end{cases}$$

Input: The discretized domain K, the density function f, the threshold δ
Output: The reconstructed graph $\hat{G}$

1. Initialize V as the trivial vector field on K and $\hat{G} = \emptyset$.
2. Run persistence on the super-level set filtration of f to get the persistence pairs $P(K)$.
3. For each $(\sigma, \tau) \in P(K)$ with $Pers(\sigma, \tau) < \delta$
   Try to perform a Morse cancellation for the pair and update V.
4. For each $(v, e) \in P(K)$ and $(e, t) \in P(K)$ with $Pers(v) \geq \delta$, $\hat{G} = \hat{G} \cup \{$ stable manifold of $e\}$.
5. output $\hat{G}$

Our Result

Theorem

If G is a connected, embedded planar graph in a cubical complex K and f is an $(\omega, \beta_1, \beta_2, \nu)$-approximation then the output $\hat{G}$ has the same homotopy type as G. Moreover, $d_H(G, \hat{G}) < \omega$.

Future Work

1. How to circumvent the heavy persistence computation.

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2. What condition on the density function gives up a small Fréchet distance between the edges of the output and the edges of the graph.
3. Extend the result to higher dimensions.

Thanks

References I

Mahmuda Ahmed, Brittany Terese Fasy, Matt Gibson, and Carola Wenk, *Choosing thresholds for density-based map construction algorithms*, Proceedings of the 23rd SIGSPATIAL International Conference on Advances in Geographic Information Systems (New York, NY, USA), SIGSPATIAL '15, ACM, 2015, pp. 24:1–24:10.

Tamal K. Dey, Jiayuan Wang, and Yusu Wang, *Graph reconstruction by discrete Morse theory*, 34th International Symposium on Computational Geometry, 2018, pp. 31:1–31:15.