Communication Complexity

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Born	December 24, 1946 (age 69) Shanghai, China
Residence	Beijing
Citizenship	United States Taiwan
Fields	Computer science
Institutions	Stanford University Princeton University Tsinghua University Chinese University of Hong Kong
Alma mater	National Taiwan University (BS) Harvard University (AM, PhD) University of Illinois at Urbana–Champaign (PhD)
Known for	Yao's Principle
Notable awards	Pólya Prize (SIAM) (1987) Knuth Prize (1996) Turing Award (2000)

Communication Everywhere

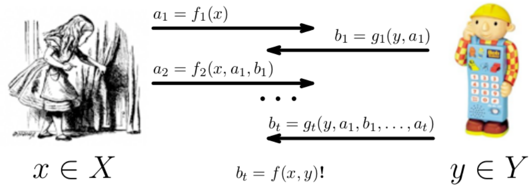
Communcation exists because of the limitation of resources in a single system

Setting Up The Stage

Given a boolean function

$$f : X \times Y \to \{0, 1\}$$

that both Alice and Bob want to compute on an input(x,y).
Let's take $X = Y = \{0, 1\}^n$.



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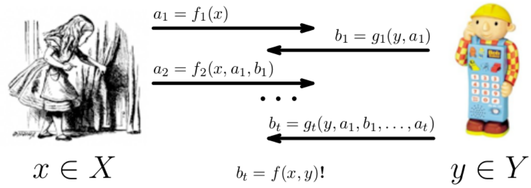

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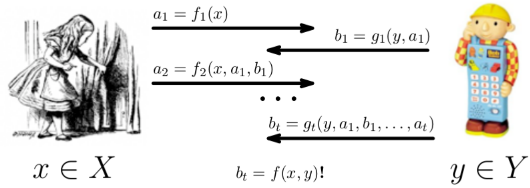

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ii) The communication channel is completely secure and noiseless.

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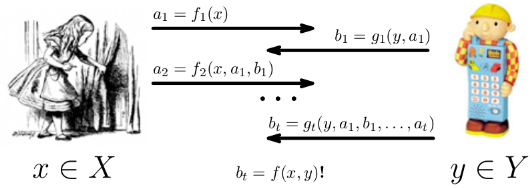

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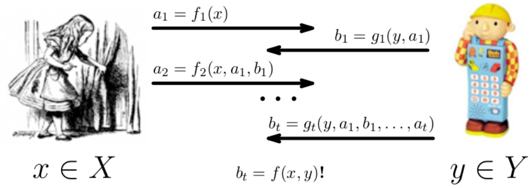

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ii) The communication channel is completely secure and noiseless.
iii) The parties have unbounded/infinte computational power.
iv) The number of rounds or the size of the sets X, Y are not that important to us.

Defining The Communication Complexity

Measuring The Cost

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After having access to x, Bob computes the function and shares the output of f in the second round using a single bit.

ExM:1

Given two integers(in binary) x and y of lenth n
$f(x, y)$ decides whether $x + y$ is the binary representation of an EVEN integer.
Can we have a communication protocol that uses less that $n + 1$ bits?

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Can we have a communication protocol that uses less that $n + 1$ bits?
Think for a moment.......
$C(f) \leq \log(2016) + 1$.
Round one: Alice divids x by 2016 and sends the remainder r to Bob!
Round two: Bob checks divisibility of $(y + r)$ by 2016 and sends it back to Alice!
Hence, $C(f) \in O(1)$!

Fix n.
Let x,$y \in \{0,1\}^n$.

$$H(x,y) = \begin{cases} 1 & \text{if } x = 1^n \text{ and } y \text{ is a Turing machine that halts on the input } x \\ 0 & \text{otherwise} \end{cases}$$

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Round one: Alice confirms whether x is of the form 1^n.
Round two: Bob determines whether the Turing machine halts on x.
Remember: Alice and Bob have unbounded computational power, including the ability to decide the Halting Problem.

Lower Bound

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$C(EQ) \geq n$

Yao proved it.

Fooling Set

We say that a function $f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\}$ has a size M fooling set if there is an M-sized subset $S \subset \{0,1\}^n \times \{0,1\}^n$ and a value $b \in \{0,1\}$ such that

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Disjointness

Input strings x, y can be interpreted as characteristic vectors of subsets of $\{1, 2, ..., n\}$.

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$$S = \left\{ (A, \overline{A}) : A \subset \{1, 2, ..., n\} \right\}$$

is a fooling set of size 2^n.

Theorem

If f has a size-M fooling set then $C(f) \geq \log M$.

Fooling Set Method:Theorem

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Corollary

1) $C(DISJ) \geq n$
2) $C(EQ) \geq n$

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One can also show that
$\log \chi(f) \leq C(f) \leq (\log \chi(f))^2$

Definition

For every function f, $\chi(f) \geq rank(M(f))$.

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Variants

1. Multiparty games

Variants Of The Basic Model And Open Problems

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1. Multiparty games
2. Nondeterministic communication protocols

References

a. "Communication Complexity", Eyal Kushilevitz, Noam Nisan
b. "Computational Complexity", Arora, Barak
c. "1979 Yao",

Thank You!