Discrete Morse Theory and Applications in TDA

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Wenk's Team Meeting
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Topological Data Analysis (TDA)

What is TDA?

TDA is a subfield of applied mathematics. Topological tools and techniques are used to Simplify, Analyze and Visualize data.

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What tools are used?

Simplicial Homology, Simple Homotopy Theory, Persistent Homology, Metric Geometry, Morse Theory etc.

data (dictionary.cambridge.org)

noun (U, sing/pl verb)• US /deɪ.tə/

information, especially facts or numbers, collected to be examined and considered and used to help decision-making, or information in an electronic form that can be stored and used by a computer.

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information, especially facts or numbers, collected to be examined and considered and used to help decision-making, or information in an electronic form that can be stored and used by a computer.

Data is an incomplete and discrete projection (sample) of a rather continuous object of interest.

Universal Truth about Data

1. full of noise/error/outliers
2. incomplete
3. high-dimensional

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Still we trust our data!

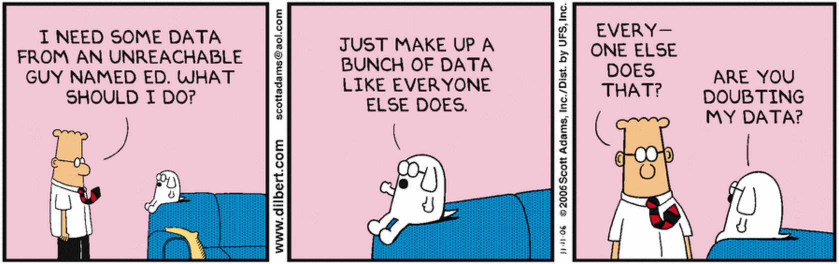

Figure: Joke from bigdata-madesimple.com

Data Analysis

Extract info about the continuous object being sampled.

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1. Statistical Data Analysis (using hyperplanes, hypersurfaces, PDEs etc.)
2. Smooth Interpolation
3. Fractal Interpolation

Why TDA?

Some data are very geometric in nature e.g. medical images, 3D printing.

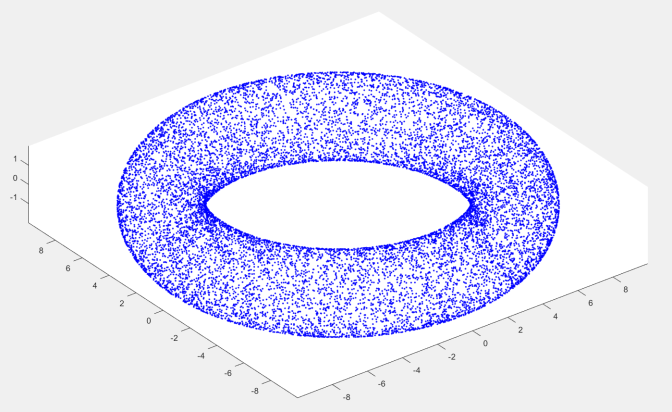

Figure: A sample from an embedded Torus (Matlab®)

Why TDA?

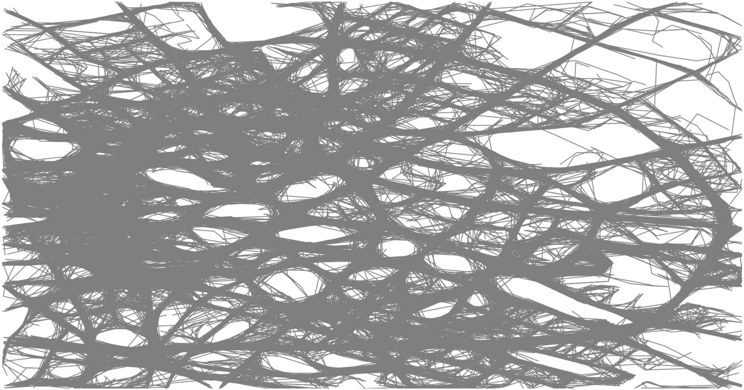

Figure: GPS traces of Berlin, Germany (www.mapconstruction.org)

WHY TDA?

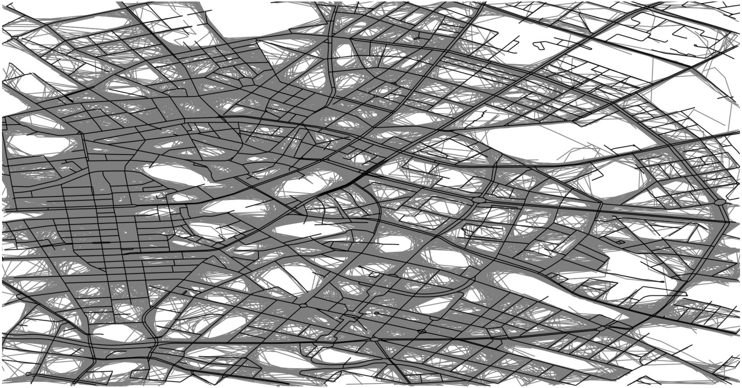

FIGURE: A possible reconstruction of road-network (www.mapconstruction.org)

Problem of Map Reconstruction

Applied math is also Challenging!

A real-world problem does not directly inform us about the math behind it.

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GPS traces $\rightarrow$ road-network

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Our Problem

GPS traces $\longrightarrow$ road-network

Expectations

1. The output is homotopy equivalent to the actual road-network i.e. same homology, homotopy groups.
2. output has a very small Hausdorff distance to the ground truth i.e. close geometric features as well.

Our First Approach

Tools Used

Metric Geometry (geodesic space, Gromov distortion, convexity radius)

Algebraic Topology (Cech and Vietoris-Rips complexes, Simplicial Homology, Simplicial Shadow).

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Theorem (submitted to SOCG'19)

Let X be a compact subset of $\mathbb{R}^N$ with a positive convexity radius ρ and finite distortion δ. And, let $S \subseteq \mathbb{R}^n$ such that $d_H(S, X) < \epsilon < \frac{\rho}{4\delta(2\delta+1)}$. Then, for any non-negative integer k, $H_k(X)$ is isomorphic to the image of the homomorphism induced by the following simplicial inclusion map

$$j : \mathcal{C}_\epsilon(S) \to \mathcal{C}_{(4\delta+1)\epsilon}(S).$$

The Shortcomings

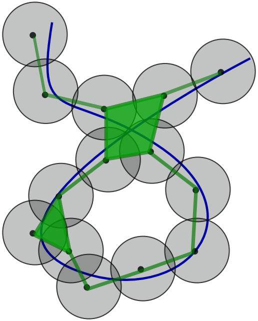

Figure: Reconstruction of a planar graph

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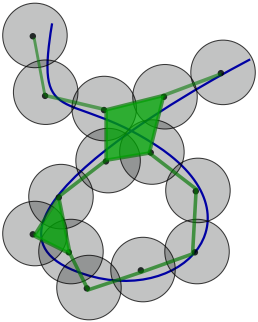

Figure: Reconstruction of a planar graph

1. the output (green) does not look like a graph
2. the method does not like outliers

CLASSICAL MORSE THEORY

MOTIVATION

A reasonably "good" smooth function reveals the Combinatorial Description of a smooth manifold.

Let M^n be a smooth manifold and $f : M \to \mathbb{R}$ be a smooth function.

Critical Point

A point $p \in M$ is called a critical point of f if $Df = 0$.

Classical Morse Theory

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The number of -ve eigen values if called the index of p.

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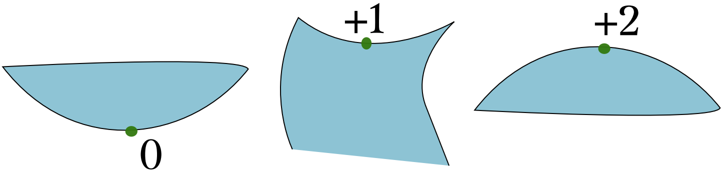

FIGURE: critical points with index

Classical Morse Theory

Morse Function

A smooth function $f : M^n \to \mathbb{R}$ is called a Morse function if all its critical points are non-degenerate.

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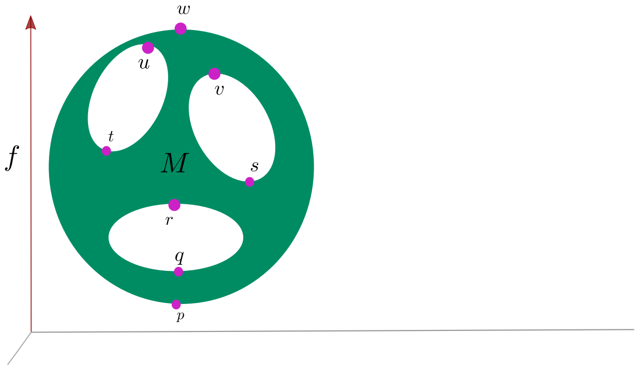
p has index 0. w has index $+2$. Others have index $+1$.



Classical Morse Theory

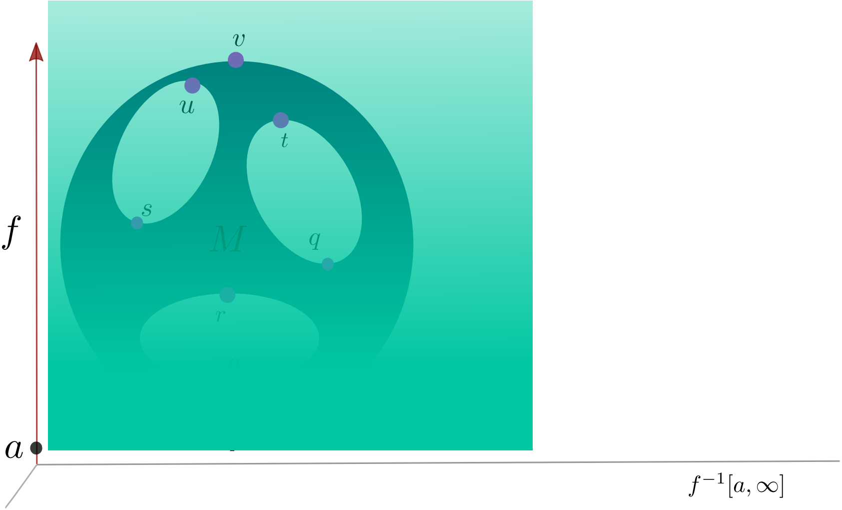
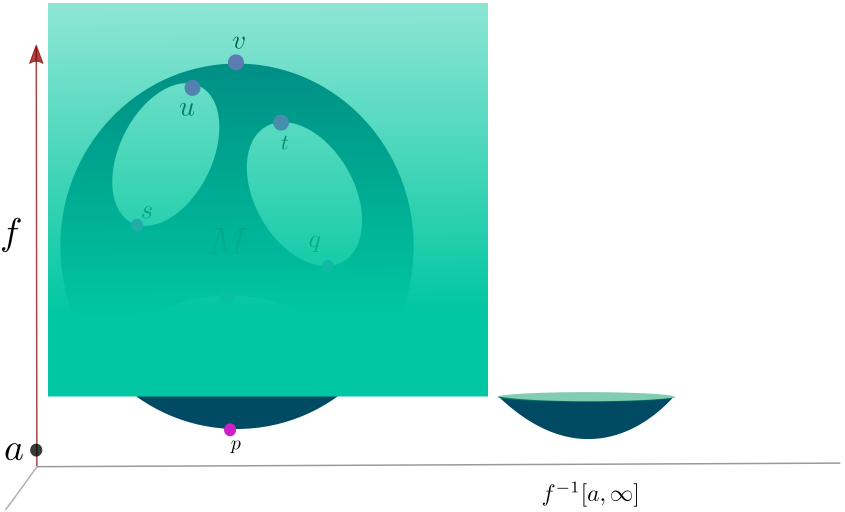


Figure: Sub-level Set

Classical Morse Theory



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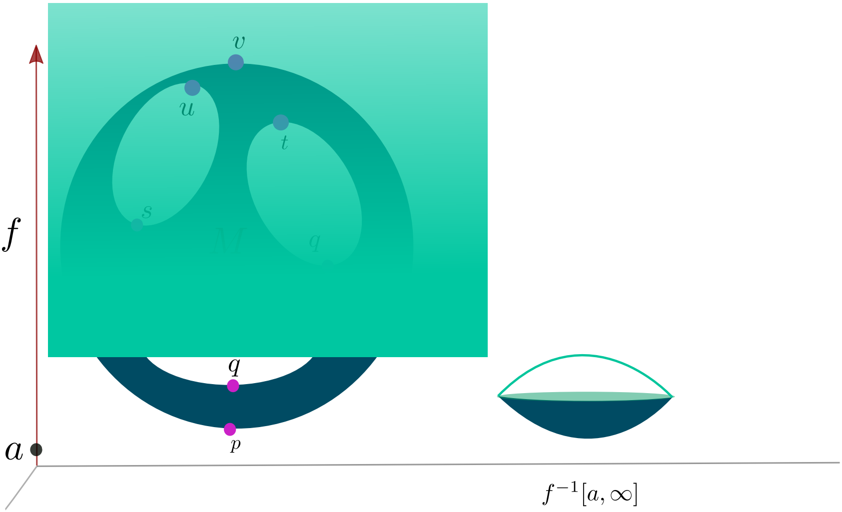

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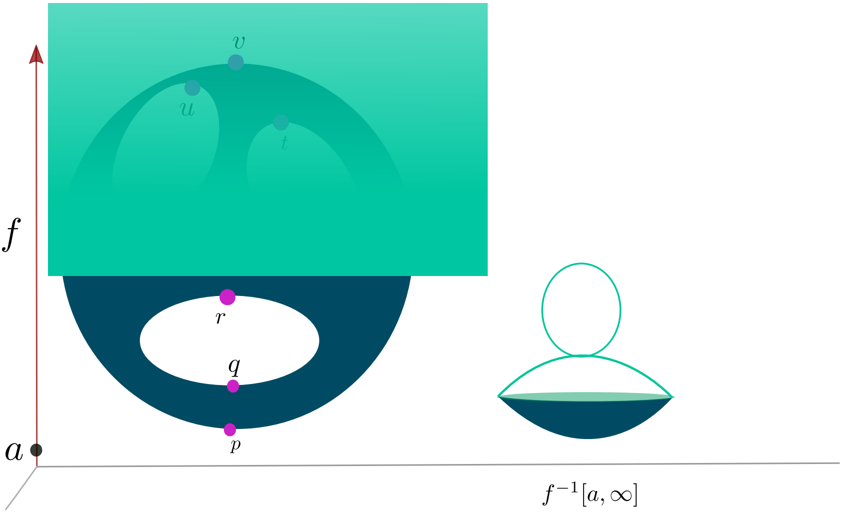

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Classical Morse Theory

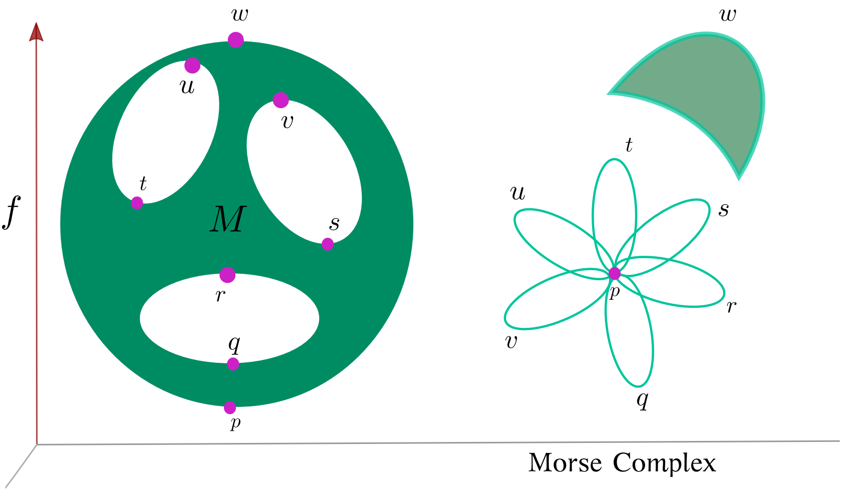

Figure: Morse Complex

Morse Theorem

If f is Morse on M, then M is *homotopy equivalent* to a CW-complex having a d-cell for each critical point of f of index d.

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Morse Inequality

critical d-index critical points $\geq H_d(M)$.

Gradient of Morse Function

The gradient vector field of a smooth function f

$$\langle \nabla f, V \rangle := -Df(V),$$

for any other vector field V on M.

Dynamics of Morse Functions

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Stable Manifold

The stable manifold

$$W_s(p) = \{x \in M \mid \lim_{t \to \infty} \Phi_t(x) = p\}$$

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OBSERVATIONS

For a Morse function f on a compact manifold M,

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Dynamics of Morse Functions

Observations

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1. critical points are the equilibrium points of ∇f.
2. f (strictly) decreases along the flow-lines.
3. no limit cycles.

Back to Our Problem

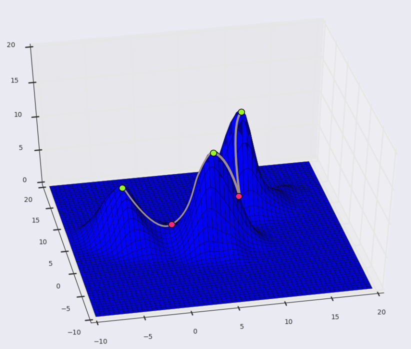

Figure: KDE

If the samples are concentrated around a graph, then the mountain ridges on the graph of the density function are expected to capture it.

Conclusion

1. our project is ongoing.
2. there is a reading group in mathematics department meeting weekly to discuss Morse theory.

Thanks