Discrete Morse Theory and Applications

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Topology and Geometry Seminar
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Myself

I am a fifth year PhD student at Tulane University.

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Yusu Wang, Computer Science, Ohio State University

MOTIVATION

A reasonably "good" smooth function reveals the Combinatorial Description of a smooth manifold.

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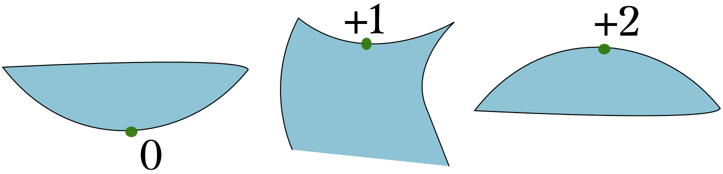

FIGURE: critical points with index

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p has index 0. w has index +2. Others have index +1.

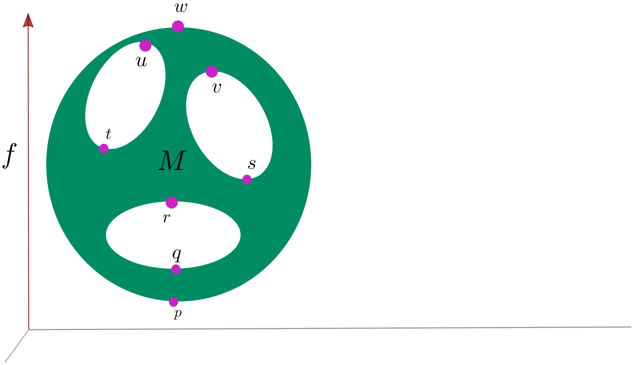


FIGURE: Morse Function

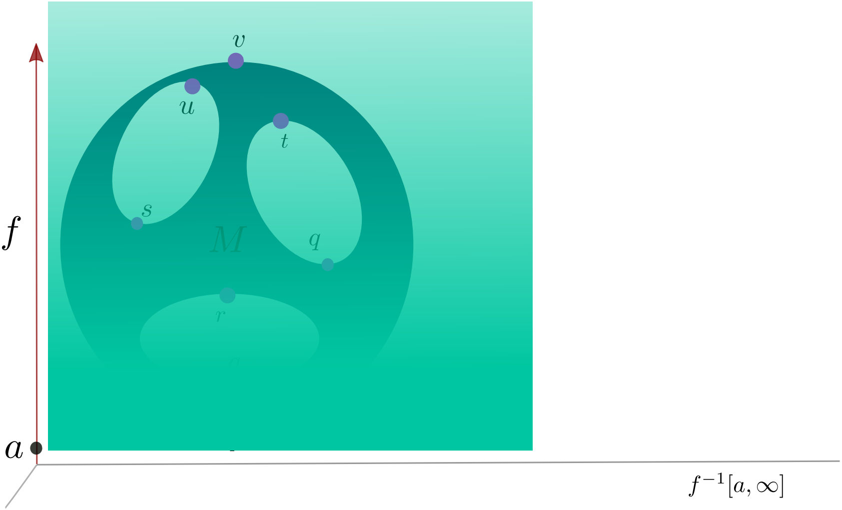
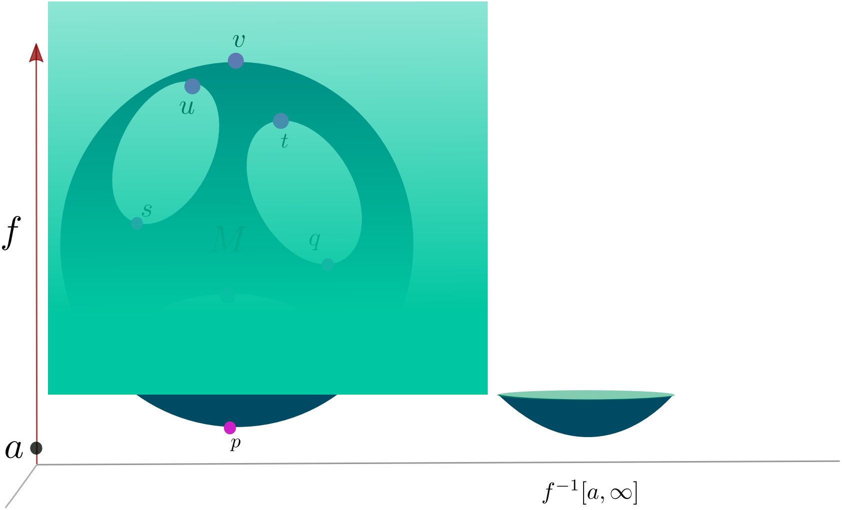


FIGURE: Sub-level Set



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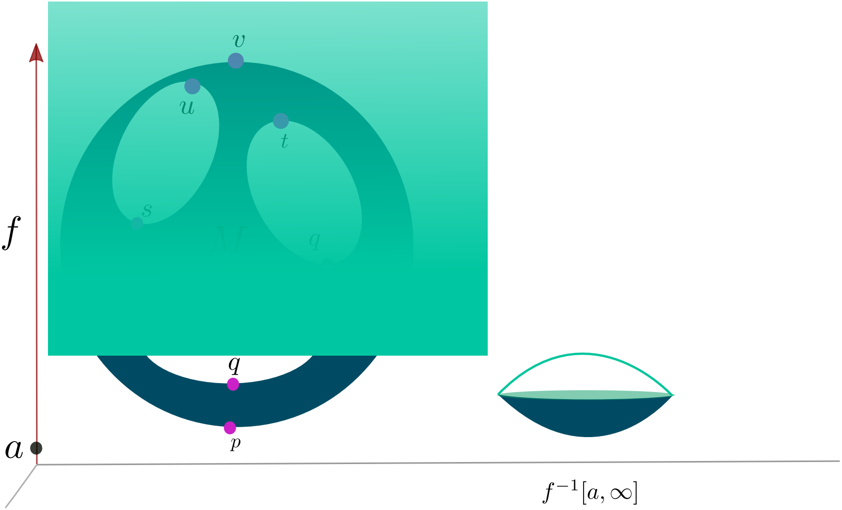

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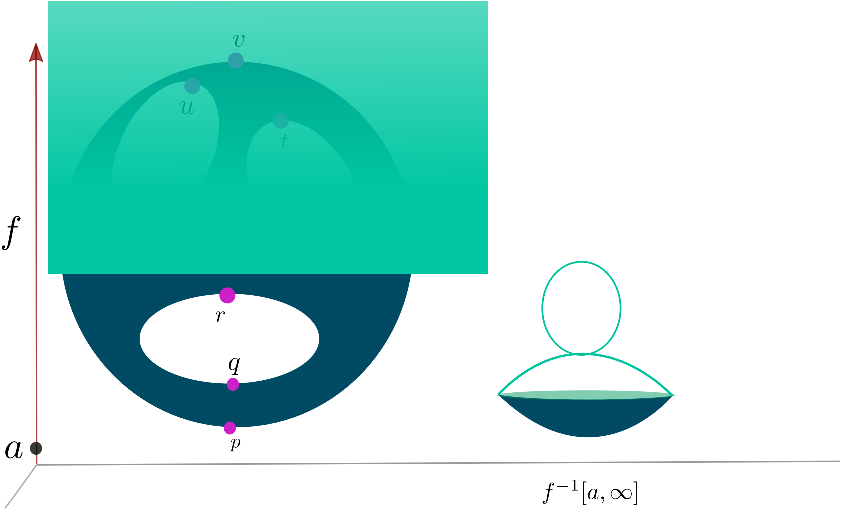

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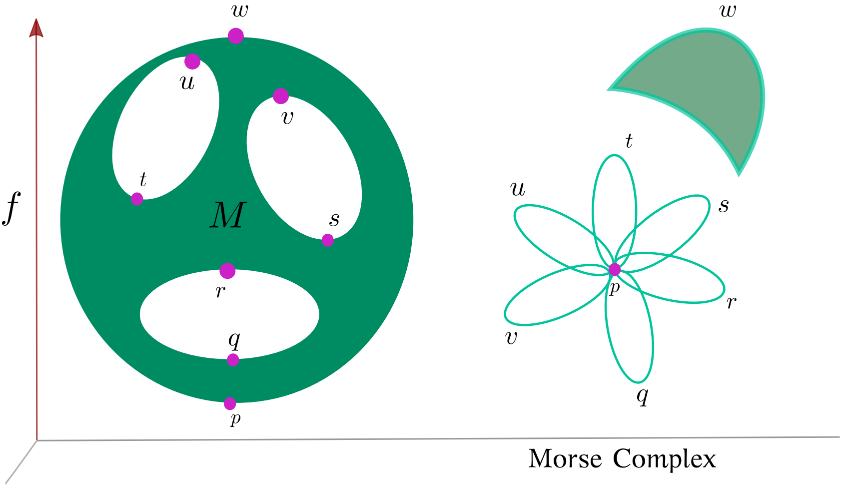

Figure: Morse Complex

MORSE THEOREM

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Morse Inequality

critical d-index critical points $\geq H_d(M)$.

Gradient of Morse Function

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3. no limit cycles.

Problem Statement

Given a (noisy) sample S taken around a (hidden) embedded graph G, how one can "reconstruct" the topology and geometry of G from S.

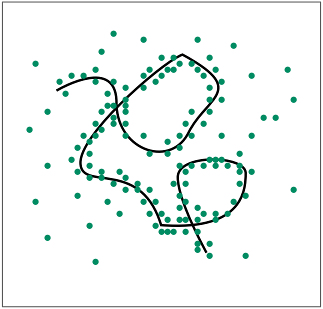


Figure: Sample around an embedded graph

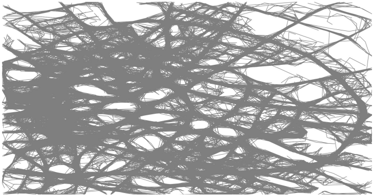
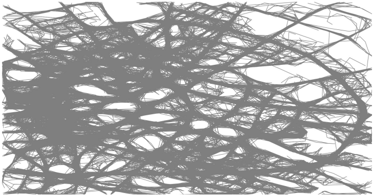


FIGURE: GPS traces of Berlin (mapreconstruction.org)

Map Reconstruction from GPS traces



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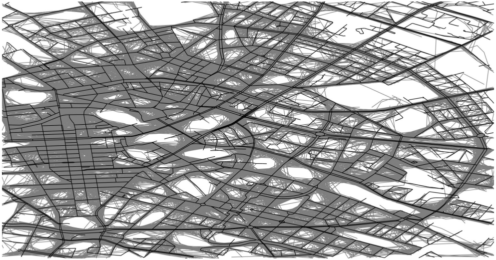

Figure: A reconstruction

Noise Models and Reconstruction

Noise Models

1. Hausdorff noise

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2. Non-Hausdorff noise

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3. heuristic pruning methods are applied this super-level set to approximate G.

RECENT WORKS

1. Choosing thresholds systematically using Persistent Homology. Ahmed, Fasy et al. (AFGW15)

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2. Theoretical guarantees on the topological/geometric correctness is not proved.
3. The output is often a thick region around the hidden graph.

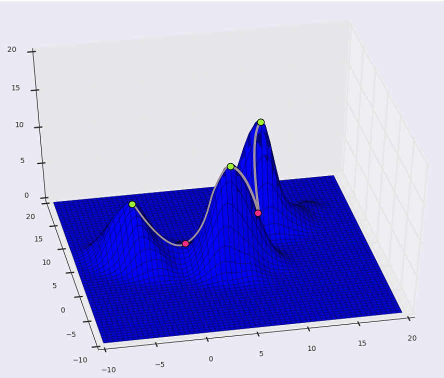

FIGURE: KDE

If the samples are concentrated around a graph, then the mountain ridges on the graph of the density function are expected to capture it.

Discrete Morse Function

Let K be a simplicial complex. A function $f : K \to \mathbb{R}$ is a discrete Morse function if for every $\alpha^{(p)} \in K$

1. $\#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} \leq 1$,
2. $\#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} \leq 1$.

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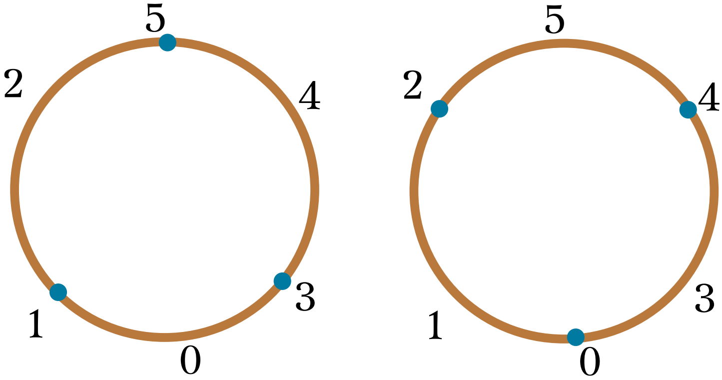

Critical Simplex

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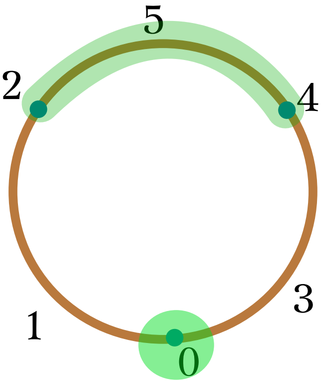

FIGURE: Discrete Morse Function

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K is a simplicial complex with a discrete Morse function. Then K is homotopy equivalent to a CW-complex with exactly one cell of dimension p for each critical simplex of dimension p.

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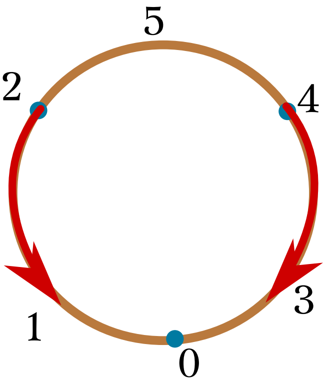

FIGURE: Simplicial Collapse

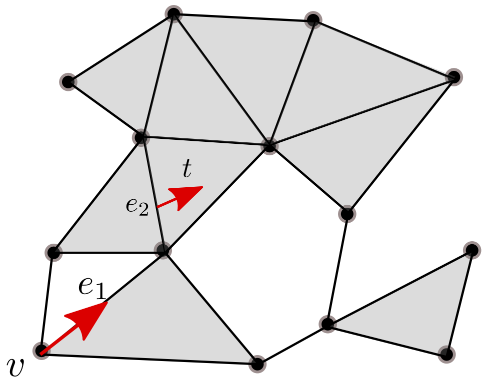

Figure: Simplicial Complex K and discrete vector

Discrete Vector Field

(σ, τ) is a discrete vector in K if $\tau < \sigma$.

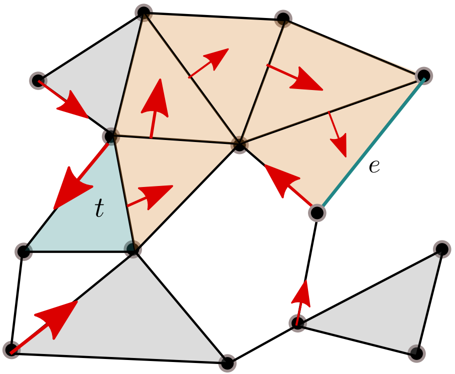

Figure: Simplicial Complex K and discrete vector

Discrete Vector Field

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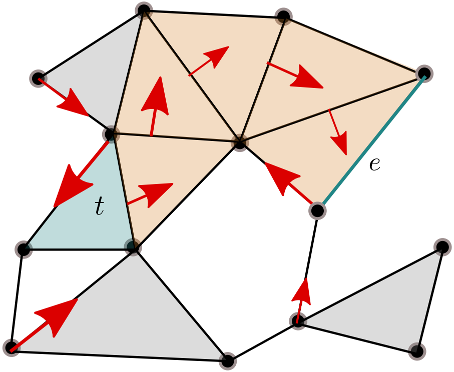

Figure: V-path

V-path

$$\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \sigma_{l+1}$$

where (σ_i, τ_i) is a vector and $\tau_i < \sigma_{i+1}$.

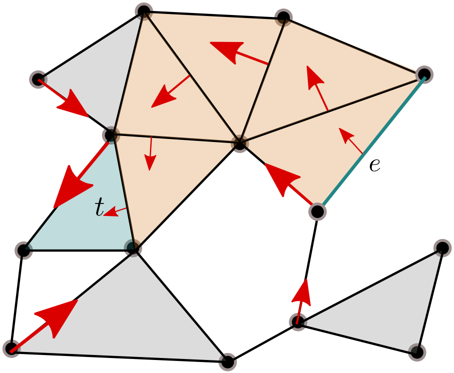

Figure: Morse Cancellation

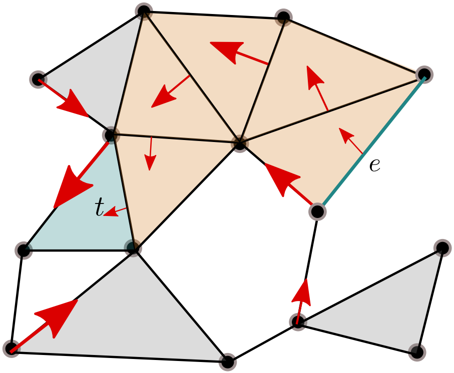

FIGURE: Morse Cancellation

STABLE MANIFOLD

For a critical edge *e*, its stable manifold is the set of all V-paths ending at the boundary of *e*.

Assumption on the Density Function

$(\omega, \beta_1, \beta_2, \nu)$-approximation of G

$$f(x) \in \begin{cases} [\beta_1, \beta_1 + \nu], & x \in V^\omega \\ [\beta_2, \beta_2 + \nu], & x \in E^\omega \\ [0, \nu], & \text{otherwise} \end{cases}$$

Input: The discretized domain K, the density function f, the threshold δ
Output: The reconstructed graph $\hat{G}$

1. Initialize V as the trivial vector field on K and $\hat{G} = \emptyset$.
2. Run persistence on the super-level set filtration of f to get the persistence pairs $P(K)$.
3. For each $(\sigma, \tau) \in P(K)$ with $Pers(\sigma, \tau) < \delta$
   Try to perform a Morse cancellation for the pair and update V.
4. For each $(v, e) \in P(K)$ and $(e, t) \in P(K)$ with $Pers(v) \geq \delta$, $\hat{G} = \hat{G} \cup \{$ stable manifold of $e\}$.
5. output $\hat{G}$

Our Result

Theorem

If G is a connected, embedded planar graph in a cubical complex K and f is an $(\omega, \beta_1, \beta_2, \nu)$-approximation then the output $\widehat{G}$ has the same homotopy type as G. Moreover, $d_H(G, \widehat{G}) < \omega$.

Future Work

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3. Extend the result to higher dimensions.

Thanks

Mahmuda Ahmed, Brittany Terese Fasy, Matt Gibson, and Carola Wenk, *Choosing thresholds for density-based map construction algorithms*, Proceedings of the 23rd SIGSPATIAL International Conference on Advances in Geographic Information Systems (New York, NY, USA), SIGSPATIAL '15, ACM, 2015, pp. 24:1–24:10.

Tamal K. Dey, Jiayuan Wang, and Yusu Wang, *Graph reconstruction by discrete Morse theory*, 34th International Symposium on Computational Geometry, 2018, pp. 31:1–31:15.