Matrix Analysis Homework 6

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1.2.P15

Suppose that $\mathbf{A}(t) \in M_n$ is a given continuous matrix-valued function and each of the vector valued functions $x_1(t), \cdots, x_n(t) \in \mathbb{C}^n$ satisfies the system of ordinary differential equations

$$x_j'(t) = \mathbf{A}(t)x_j(t).$$

Let $\mathbf{X}(t) = [x_1(t) \ \cdots \ x_n(t)]$ and let $W(t) = \det \mathbf{X}(t)$. Use (0.8.10) and (0.8.2.11) and provide details for the following argument:

$$\begin{aligned} W'(t) &= \sum_{j=1}^{n} \det\left(\mathbf{X}(t) \underset{j}{\leftarrow} x_j'(t)\right) = \operatorname{tr}\left[\det\left(\mathbf{X}(t) \underset{i}{\leftarrow} x_j'(t)\right)\right]_{i,j=1}^{n} \\ &= \operatorname{tr}\left((\operatorname{adj}\mathbf{X}(t))\mathbf{X}'(t)\right) = \operatorname{tr}((\operatorname{adj}\mathbf{X}(t))\mathbf{A}(t)\mathbf{X}(t)) = W(t)\operatorname{tr}\mathbf{A}(t). \end{aligned}$$

Thus, $W(t)$ satisfies the scalar differential equation $W'(t) = \operatorname{tr}\mathbf{A}(t)W(t)$, whose solution is *Abel's formula* for the *Wronskian*

$$W(t) = W(t_0)\, e^{\int_{t_0}^{t} \operatorname{tr}\mathbf{A}(s)\mathrm{d}s}.$$

Conclude that if the vectors $x_1(t), \cdots, x_n(t)$ are linearly independent for $t = t_0$, then they are linearly independent for all t. How did you use the identity $\operatorname{tr}(\mathbf{BC}) = \operatorname{tr}(\mathbf{CB})$ (1.2.P2)?

Solution.

$$\begin{aligned} W'(t) &= \frac{d}{dt}\det\mathbf{X}(t) \\ &\overset{(0.8.10)}{=} \sum_{j=1}^{n} \det\left(\mathbf{X}(t) \underset{j}{\leftarrow} x_j'(t)\right) \\ &= \operatorname{tr}\left[\det\left(\mathbf{X}(t) \underset{i}{\leftarrow} x_j'(t)\right)\right]_{i,j=1}^{n} \\ &\overset{(0.8.2.11)}{=} \operatorname{tr}\left((\operatorname{adj}\mathbf{X}(t))\mathbf{X}'(t)\right) \\ &\overset{x_j'(t)=\mathbf{A}(t)x_j(t)}{=} \operatorname{tr}((\operatorname{adj}\mathbf{X}(t))\mathbf{A}(t)\mathbf{X}(t)) \\ &\overset{\operatorname{tr}(\mathbf{BC})=\operatorname{tr}(\mathbf{CB})}{=} \operatorname{tr}(\mathbf{X}(t)(\operatorname{adj}\mathbf{X}(t))\mathbf{A}(t)) \\ &= \det\mathbf{X}(t)\operatorname{tr}\mathbf{A}(t) = W(t)\operatorname{tr}\mathbf{A}(t). \end{aligned}$$

If the vectors $x_1(t), \cdots, x_n(t)$ are linearly independent for $t = t_0$, $W(t) = \det\mathbf{X}(t) \neq 0$. Hence $W(t) = W(t_0)\, e^{\int_{t_0}^{t} \operatorname{tr}\mathbf{A}(s)\mathrm{d}s} \neq 0$. Therefore $x_1(t), \cdots, x_n(t)$ are linearly independent for all t.