# Mac-Cormack's Scheme for Shock Filtering Equation in Image Enhancement

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**Abstract.** Mac-Cormack's scheme is elaborated to approximate the solution of shock filtering equation in image enhancement. This scheme is in second order approximation of spatial and time variables. Here, the comparison results of upwind and Mac-Cormack's scheme are given. The results show that Mac-Cormack's scheme is able to preserve the edge discontinuity. For evaluating the performance of numerical results, the discrete $L^2$ norm error for both numerical schemes is given. From several experiments, along the increasing of image sizes, the error of Mac-Cormack's scheme is observed getting smaller. For instance, using image sizes (64,64) the error is obtained 0.13762, meanwhile using (512,512) the error is observed 0.06640.

**Keywords:** Mac-Cormack's · Shock filtering · Enhancement · Image processing

## 1 Introduction

Image enhancement can be important process in image processing. In some applications, image enhancement is required in pre-processing part to detect the important image features and the details of image (see [7,11] for more detail). For instance in [11], enhancement of medical image is needed to help a surgeon to interpret and diagnosis image more accurately. Since generally, medical images are often degraded by noise and external data acquisition.

Several enhancement methods or algorithms are available in some references [3,4,6,10]. This paper is focus on enhancement method which developed based on partial differential equations (PDE). In PDE-based approach, enhancement image can be done by known as shock filtering equation [2,5]. This equation were introduced by Osher and Rudin [5] which assure the edge preserves discontinuity.

Consider an image $I(x, y, t)$ over domain $\Omega$ and $\mathfrak{L}$ is a nonlinear elliptic PDE operator, then the shock filtering equation is given as

$$I_t = -F\left(\mathfrak{L}(I)\right)|\nabla I|, \quad (x, y) \in \Omega, \quad t > 0 \tag{1}$$


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where $F$ is a Lipschitz function which satisfying the following properties

$$\begin{cases} F(\sigma) = 0, & \sigma = 0, \\ \operatorname{sgn}(\sigma)F(\sigma) > 0, & \sigma \neq 0, \end{cases} \tag{2}$$

with $\operatorname{sgn}(\sigma)$ is signum function which takes the sign of real number,

$$\operatorname{sgn}(\sigma) = \begin{cases} 1, & \sigma > 0, \\ 0, & \sigma = 0, \\ -1, & \sigma < 0. \end{cases} \tag{3}$$

Generally in [2,8], (1) is approximated by a simple finite difference method which depend on the sign of function $F$ (a kind of upwind method). Therefore, this method is first order approximation in space and time. However, in this paper, the second order approximation in space and time approach is proposed. Here, time and spatial derivative are discretized using predictor and corrector step. This numerical scheme is known as Mac-Cormack's scheme.

The organization of this paper is given as follow, in Sect. 2, the numerical schemes, upwind and Mac-Cormack's scheme are given. In Sect. 3 the evaluation and discussion of numerical solution of both schemes are elaborated. Finally, the conclusion is given in Sect. 4.

## 2 Numerical Methods

Before two numerical schemes are elaborated in detail, then the following discrete notations are given. Let the 2-D spatial image domain $\Omega$ is discretized into several points. In this case, the points are described as the coordinate of pixels of image. Given an image with size $(N_x, N_y)$, then a set of spatial discrete points is $\mathcal{M} = \{1, 2, \cdots, N_x\} \times \{1, 2, \cdots, N_y\}$. Moreover, another notations for discrete space and time properties are written as follow,

$$\begin{aligned} &\Delta t = \frac{T_f}{N_t}, \qquad t^n = \Delta t \times n, \qquad n \in \mathcal{T} = \{1, 2, \cdots N_t\}, \quad N_t \in \mathbb{Z}^+, \\ &x_i = i \times \Delta x, \quad y_j = j \times \Delta y, \qquad (i, j) \in \mathcal{M}. \end{aligned}$$

In image processing, the number of grids is similar to the sizes of image, thus spatial steps $\Delta x = \Delta y = 1$. Finally, the notation $I^n_{i,j}$ is used to describe the value of a pixel at spatial point $(x_i, y_j)$ and time $t^n$.

Here, the operator $\mathfrak{L}(I)$ for both numerical schemes is defined as

$$\mathfrak{L}(I) := \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{|I_{xx} + I_{yy}|^2}. \tag{4}$$

According to [2,5], this operator is better than ordinary Laplace operator. This operator allows the edges are formed from the zero crossings of second directional derivatives.

## 2.1 Upwind Scheme

In scalar hyperbolic type of PDE, upwind scheme is used to approximate the transport equation with first order approximation in space and time. Here, the discretization of (1) using upwind is given as

$$I_{i,j}^{n+1} = \begin{cases} I_{i,j}^{n} - c\sqrt{\left(\dfrac{I_{i+1,j}^{n} - I_{i,j}^{n}}{\Delta x}\right)^2 + \left(\dfrac{I_{i,j+1}^{n} - I_{i,j}^{n}}{\Delta y}\right)^2}, & \text{if } c < 0, \\ I_{i,j}^{n}, \quad \text{if } c = 0, & \\ I_{i,j}^{n} - c\sqrt{\left(\dfrac{I_{i,j}^{n} - I_{i-1,j}^{n}}{\Delta x}\right)^2 + \left(\dfrac{I_{i,j}^{n} - I_{i,j-1}^{n}}{\Delta y}\right)^2}, & \text{if } c > 0. \end{cases} \tag{5}$$

where $c = F(\mathfrak{L}(I_{i,j}^{n}))$. As shown in (5), this scheme depends on the characteristic wave sign of $(F(\mathfrak{L}(I)))$. If the sign is positive then backward difference scheme is used, meanwhile the forward difference scheme is implemented if the sign is negative.

## 2.2 Mac-Cormack's Scheme

In computational fluid dynamics area, Mac-Cormack's scheme is known as a simple and robust scheme for approximating simple Navier-Stokes equation in [1]. This scheme consists of two steps to obtain second order approximation in space and time. Here, the steps are called prediction and correction step.

**Prediction step:**

In this step, the prediction value is approximated by forward difference scheme in space,

$$\left(\frac{\partial I}{\partial t}\right)_{i,j}^{n} = -F(\mathfrak{L}(I_{i,j}^{n}))\sqrt{\left(\frac{I_{i+1,j}^{n} - I_{i,j}^{n}}{\Delta x}\right)^2 + \left(\frac{I_{i,j+1}^{n} - I_{i,j}^{n}}{\Delta y}\right)^2}, \tag{6}$$

then the prediction value is done by discrete time which is given as follow

$$\overline{I}_{i,j}^{n} = I_{i,j}^{n} + \left(\frac{\partial I}{\partial t}\right)_{i,j}^{n} \Delta t. \tag{7}$$

**Corrector step:**

Next step, the spatial term is discretized using backward scheme using prediction value (7). This scheme is written as

$$\left(\overline{\frac{\partial I}{\partial t}}\right)_{i,j}^{n+1} = -F(\mathfrak{L}(\overline{I}_{i,j}^{n}))\sqrt{\left(\frac{\overline{I}_{i,j}^{n} - \overline{I}_{i-1,j}^{n}}{\Delta x}\right)^2 + \left(\frac{\overline{I}_{i,j}^{n} - \overline{I}_{i,j-1}^{n}}{\Delta y}\right)^2}. \tag{8}$$

Finally, the new value of $I_{i,j}^{n+1}$ is obtained by integrating time evolution using average value of predictor (6) and corrector (8) which given as

$$\left(\frac{\partial I}{\partial t}\right)_{av} = 0.5\left[\left(\frac{\partial I}{\partial t}\right)_{i,j}^{n} + \left(\overline{\frac{\partial I}{\partial t}}\right)_{i,j}^{n+1}\right], \tag{9}$$

$$I_{i,j}^{n+1} = I_{i,j}^{n} + \left(\frac{\partial I}{\partial t}\right)_{av} \Delta t. \tag{10}$$

As shown in [1], Mac-Cormack's scheme is a second order approximation in space and time and analogous to Lax-Wendroff scheme for one-dimensional problem.

## 3 Results and Discussion

Here, to measure the performance of Mac-Cormack's scheme, a numerical test of deblurring image from the diffusion effect to reduce noise is given. The goal of this test is to see the ability of Mac-Cormack's scheme to improve the quality of image as close as the original image.

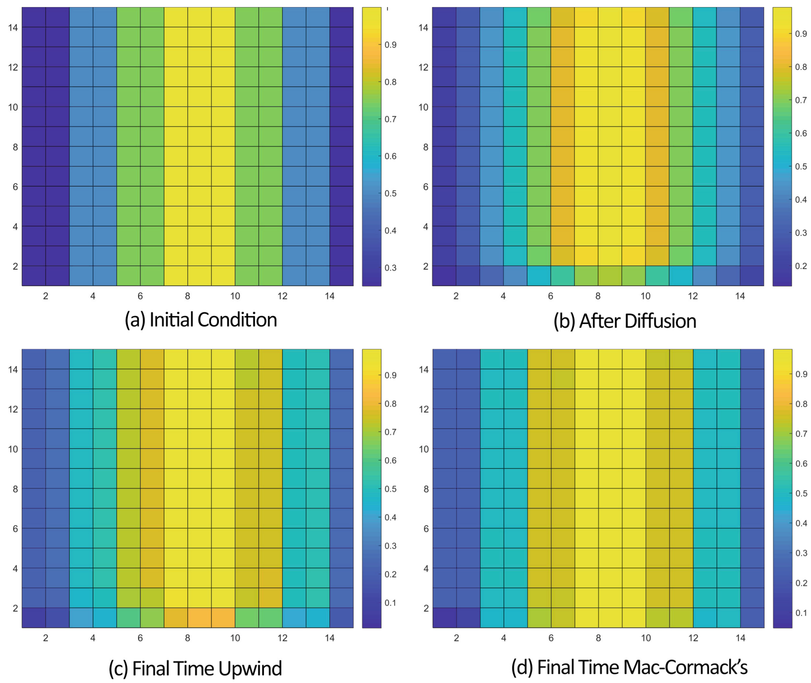


**Fig. 1.** The measurement of numerical scheme in two-dimensional image, (a) the original image, (b) the result of diffusion effect, (c) the result of upwind scheme, and (d) the result of Mac-Cormack's scheme.

Here, the original image with sizes (16,16) is given as in Fig. 1(a), with the value of pixels in this image is in interval [0 : 1]. Further, the original image is blurred using two-dimensional diffusion equation (see [2]) which shown in Fig. 1(b). Finally, in this test, Mac-Cormack's and upwind scheme are used to enhance the image as close as the original.

The results using upwind and Mac-Cormack's numerical scheme are given in Fig. 1(c) and (d) respectively. It is clear that after similar final time of simulation $50t$, with $\Delta t = 10^{-2}$, the result of Mac-Cormack's scheme is shown close to the original image than upwind scheme.

For more clear about this comparison, the observation trough a slice image at $y = 8$ can be seen in Fig. 2. As shown in Fig. 2, the result of Mac-Cormack's scheme is able to approach the original line as close as possible. Meanwhile, the upwind scheme is shown a little bit far from the original line. Moreover, to measure quantitatively, the discrete $L^2$ or $||\{I_{i,j}\}||_{2,\Delta x}$ norm errors (see [9]) for each numerical schemes are given in Table 1.

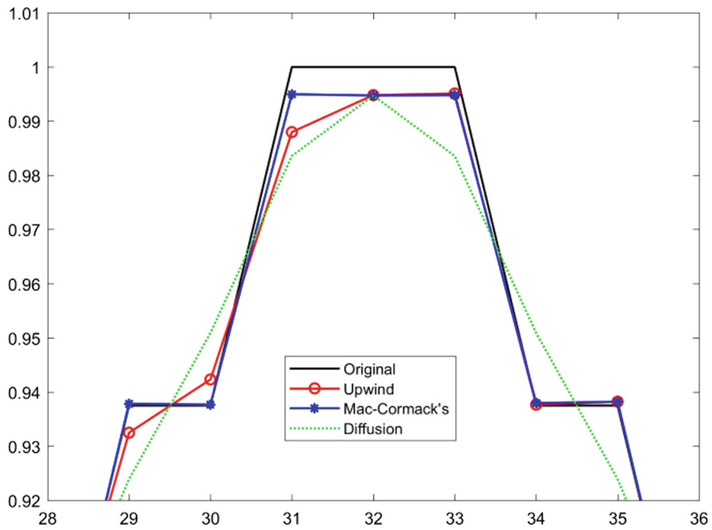


**Fig. 2.** The comparison results of upwind and Mac-Cormack's scheme in slice image at $y = 8$.

**Table 1.** The discrete error $||\{I_{i,j}\}||_{2,\Delta x}$ for upwind and Mac-Cormack's scheme.

| No | Image sizes | Error of upwind | Error of Mac-Cormack |
|---|---|---|---|
| 1 | (16,16) | 0.64085 | 0.37166 |
| 2 | (32,32) | 0.63904 | 0.21817 |
| 3 | (64,64) | 0.79319 | 0.13762 |
| 4 | (128,128) | 1.06730 | 0.09127 |
| 5 | (256,256) | 1.48140 | 0.06626 |
| 6 | (512,512) | 2.07700 | 0.06640 |

In Table 1, several simulations using different image sizes are elaborated. It can be seen clearly that, using Mac-Cormack's scheme, the errors are obtained decreasing along the increasing of image sizes. Meanwhile, the numerical error of upwind scheme is shown slightly increasing, but it is not significantly different. Overall, two schemes are shown in a good agreement as the original image where both schemes can preserve the edge discontinuity very well.

The results using Mac-Cormack's scheme for another test can be seen in Fig. 3. Here at first, original image (Fig. 3(a)) is given a Gaussian noise as shown in Fig. 3(b), then to reduce the noise, the diffusion (blurring) effect is


**Fig. 3.** Results of enhancement using Mac-Cormack's scheme, (a) Original image, (b) image with Gaussian noise, (c) image with diffusion effect to reduce the noise, and (d) the enhanced image using Mac-Cormack's scheme.

implemented (Fig. 3(c)) using diffusion equation. Afterward, the Mac-Cormack's scheme is elaborated to enhance the image as close as the original image (see Fig. 3(d)).

As it can seen in Fig. 3(d), Mac-Cormack's scheme is able to make a blur image becomes sharp image as close to the original image. In this test, the focus is the white area on neck of camera. The results in Fig. 3 show that Mac-Cormack's scheme is capable to sharpen the white area.

## 4 Conclusion

In this paper, the second order approximation of Shock filtering equation in image enhancement is proposed. The numerical scheme consists of two steps, predictor and corrector step, which is called Mac-Cormack's scheme. In order to evaluate the Mac-Cormack's scheme, a numerical test deblurring image from diffusion effect to original image is elaborated. In numerical simulation, this scheme is shown able to preserve the edge discontinuity as close as the original image. The comparison of Mac-Cormack's and first order upwind scheme is also given. Here, the discrete $L^2$ norm error of Mac-Cormack's scheme is observed getting smaller along the increasing of image sizes. By using image sizes (512,512), the discrete $L^2$ norm error of Mac-Cormack's is obtained 0.00640.

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