Autumn Semester 2021
Master's Programs MiQEF, MECON and MBF
University of St. Gallen

Quantitative Risk Management

Assignment

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1 Goals and conditions

The goal of this assignment is the estimation of value-at-risk and expected shortfall (see Chapter 1 for the definition of these risk measures) for the profit and loss distribution of a given portfolio.

The following conditions must be satisfied:

(i) The assignment can be conducted in groups of maximum 3 students. The members of the groups must be announced to Mrs. Margit Albers (email: mathstat@unisg.ch) no later than **Friday 5 November 2021**. No further changes to the composition of the groups will be accepted, unless a member decides to leave her or his group. In this case, the change must announced by the leaving member via email to Mrs. Albers, with all other members of the group in cc. The list of groups will be available on StudyNet (canvas) on 12 November 2021. The members of the group must be clearly indicated in the final solution of the assignment.

(ii) The **due time for the assignment is Monday 29 November 2021 at 6 pm**. The solutions (pdf file in case of a single file or zip file if more than one file) have to be sent via email to Mrs. Albers. She will confirm reception via email. **Solutions delivered after the due time will not be graded**. Please write the subject "7,320 Quantitative Risk Management: Assignment" in the email.

(iii) Present your analysis in a ordered way. Results must be reported (tables and figures) and **commented** in the main document. I will not run codes on my computer in order to obtain the results of your analysis! Formulas which are needed to estimate the parameters of the models or the risk measures must be justified and derived explicitly, also if you finally apply an existing package or function in your software. The presentation of the results will be evaluated!

(iv) You can freely choose the software (MATLAB, S-Plus, R, etc.). Excel is not well-suited.

(v) Codes used for the analysis must be **commented** and attached to the solution.

(vi) The graded assignment will count for the final grade, according to the following rule:

$$\text{final grade G} = 0.75\,F + 0.25\,A$$

where A is the grade of the assignment and F is the grade of the oral exam.

2 Dataset

Please download from StudyNet (canvas) the Excel file `qrm21HSG_assignmentdata.xlsx` with weekly price levels from 1 January 2013 to 15 October 2021 for some of the stocks with the highest market capitalization listed in the S&P 500.

3 Questions

An investor decides to invest 30% of her wealth $W_0 = 10,000,000$ USD in APPLE (AAPL) and 70% in TESLA (TSLA).

The investor wants to compute the portfolio's value-at-risk and expected shortfall over $T = 1$ week. For this purposes, she derives the profit and loss distribution as follows:

$$L_T = W_T - W_0 = W_0\ [\lambda_1\, R_1 + \lambda_2\, R_2]$$

where $\lambda_1 = 0.3$, $\lambda_2 = 0.7$, and R_1 and R_2 are weekly returns of AAPL and TSLA, respectively. Let $\mathbf{R} = (R_1, R_2)'$.

(i) We consider four different models for weekly returns $\boldsymbol{R}$.

M1: $\boldsymbol{R}$ is distributed according to the empirical distribution.

M2: $\boldsymbol{R}$ is bivariate Gaussian distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ (see Subsection 4.1).

M3: R_i is Gaussian distributed with mean μ_i and variance σ_i^2, for $i = 1, 2$. These parameters are the same as for model M2. Moreover, the vector $\left(\Phi\left(\frac{R_1-\mu_1}{\sigma_1}\right), \Phi\left(\frac{R_2-\mu_2}{\sigma_2}\right)\right)'$ possesses a Gumbel-copula with parameter $\theta \geq 1$ (see Subsection 4.3).

M4: $R_i = \mu_i + \sigma_i\, \epsilon_i$, where ϵ_i is standard t-distributed with ν_i degrees of freedom (see Subsection 4.2), for $i = 1, 2$. The parameters μ_i and σ_i, $i = 1, 2$, are the same as for model M2. The vector $(\epsilon_1, \epsilon_2)'$ possesses a Gaussian copula with correlation parameter $\tilde{\rho}$.

Using maximum-likelihood estimation, estimate models M2, M3 and M4 on historical data, using all observations of the given data set.

(ii) Using model M1 for $\boldsymbol{R}$, simulate the portfolio distribution (10,000 simulations of the portfolio return) and estimate 1-day portfolio's value-at-risk and expected shortfall at 90%, 95% and 99% confidence levels.

(iii) Repeat the same exercise as in point (ii), but using models M2 and M4, respectively.

(iv) Analyze the impact of the number of historical observations N on the estimated value-at-risk and expected shortfall. Use the *last* $N = 100, 200, 300$ and 400 observations of 1-week returns, respectively.

(v) Use a rolling window of 100 observations (the first window starts on 1 January 2013) to estimate the 1-week portfolio's value-at-risk at 95% confidence level, under models M2 and M4, respectively. Count the total number of violations of the value-at-risk level using the next out-of-sample portfolio return (a violation of the the value-at-risk level is a profit and loss with $L_T < VaR$). What are your conclusions?

4 Formulae

4.1 Bivariate Gaussian distribution

The random vector $\mathbf{X} = (X_1, X_2)'$ possesses a bivariate Gaussian distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\,\sigma_1\,\sigma_2 \\ \rho\,\sigma_1\,\sigma_2 & \sigma_2^2 \end{pmatrix},$$

when its density function is

$$f(x_1, x_2; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{2\pi\,\sigma_1\,\sigma_2\,\sqrt{1-\rho^2}}\; e^{-\frac{1}{2\,(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\,\rho\,(x_1-\mu_1)\,(x_2-\mu_2)}{\sigma_1\,\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}.$$

4.2 Standard t-distribution

A random variable X is (univariate) standard t-distributed with ν degrees of freedom when its density function is

$$f_{t,1}(x; \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\,\pi}\,\Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

The random vector $\mathbf{X} = (X_1, X_2)'$ is bivariate standard t-distributed with ν degrees of freedom and correlation parameter ρ when its density function is

$$f_{t,2}(x_1, x_2; \nu, \rho) = \frac{\Gamma(\frac{\nu+2}{2})}{\Gamma(\frac{\nu}{2})\,\pi\,\nu\,\sqrt{1-\rho^2}}\left(1 + \frac{x_1^2 - 2\,\rho\,x_1\,x_2 + x_2^2}{\nu\,(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}.$$

Γ is the gamma-function, $\Gamma(z) = \int_0^\infty t^{z-1}\,e^{-t}\,dt$.

4.3 Gumbel-copula

The Gumbel-copula is defined by

$$C(u_1, u_2) = \exp\left(-\left((-\ln(u_1))^\theta + (-\ln(u_2))^\theta\right)^{1/\theta}\right),$$

where $\theta \geq 1$.

4.4 Simulations

4.4.1 Univariate distribution

Some softwares (like Excel, see the RAND() function) only allow to simulate a uniform distribution on $[0, 1]$. How do we simulate a general (continuous) distribution?

Let X be a random variable with a continuous cumulative distribution function F and F^{-1} its quantile function. Let U be a uniformly distributed random variable on $[0, 1]$ and define

$$Y = F^{-1}(U).$$

Then Y has the cumulative distribution function F:

$$\mathbb{P}\big[Y \leq y\big] = \mathbb{P}\big[F^{-1}(U) \leq y\big] = \mathbb{P}\big[U \leq F(y)\big] = F(y)$$

since U is uniformly distributed on $[0, 1]$.

4.4.2 Bivariate Gaussian distribution

Some softwares (like Excel) only allow to simulate a uniform distribution on $[0, 1]$. Using the approach explained in Subsection 4.4.1, we can easily simulate any univariate distribution, as long as we know its quantile function. However, how do we simulate bivariate distributions?

Let $\mathbf{X} = (X_1, X_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\boldsymbol{\Sigma}$. Since $\boldsymbol{\Sigma}$ is symmetric and positive definite, we find a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} > 0$ such that

$$\boldsymbol{\Sigma} = \mathbf{A}'\,\mathbf{A}.$$

$\mathbf{A}$ is a upper triangular matrix with strictly positive diagonal entries, and the last equation is called the *Cholesky decomposition* of $\boldsymbol{\Sigma}$.

Let $\mathbf{Z} = (Z_1, Z_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\mathbf{0} = (0, 0)'$ and covariance matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e., Z_1, Z_2 are independent and identically, standard normally distributed. Let $\mathbf{A}$ such that $\mathbf{A}'\,\mathbf{A} = \boldsymbol{\Sigma}$ (Cholesky decomposition of $\boldsymbol{\Sigma}$), then

$$\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}'\,\mathbf{Z}$$

is a bivariate Gaussian distributed random vector with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\boldsymbol{\Sigma}$.

In words: to simulate a bivariate Gaussian distributed random vector $\mathbf{Y}$ with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and covariance matrix $\boldsymbol{\Sigma}$, one can simulate *two independent standard normally distributed random variables* Z_1, Z_2, compute the Cholesky decomposition $\mathbf{A}$ of $\boldsymbol{\Sigma}$, and set

$$\mathbf{Y} = \boldsymbol{\mu} + \mathbf{A}'\,\mathbf{Z},$$

where $\mathbf{Z} = (Z_1, Z_2)'$.

4.4.3 Bivariate t-distribution

Let $\mathbf{Z} = (Z_1, Z_2)'$ be a bivariate Gaussian distributed random vector with mean vector $\mathbf{0} = (0, 0)'$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

Let W be a Chi-squared distributed random variable with ν degrees of freedom random variable, *and* independent from $\mathbf{Z}$.

The random vector

$$\mathbf{X} = \sqrt{\frac{\nu}{W}}\,\mathbf{Z}$$

is standard t-distributed with ν degrees of freedom and correlation parameter ρ.