

Flavour Symmetry Embedded - GLoBES (FaSE-GLoBES)

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Abstract

Neutrino models based on flavour symmetries provide the natural way to explain the origin of tiny neutrino masses. At the dawn of precision measurements of neutrino mixing parameters, it is straightforward to constrain and examine neutrino mass models by on-going and next-generation neutrino experiments. We present a supplemental tool **Flavour Symmetry Embedded (FaSE)** for **General Long Baseline Experiment Simulator (GLoBES)**. It can translate the neutrino mass model inputs to standard neutrino oscillation parameters, and offers prior functions in a user-friendly way. We demonstrate the robustness of **FaSE-GLoBE** with two examples on how the model parameters can be constrained and even whether the model is excluded by an experiment or not. We wish that this toolkit will facilitate the study of new neutrino mass models in an efficient and effective manner.

Keywords: Neutrino Oscillations, Leptonic Flavour Symmetry

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I. INTRODUCTION

The discovery of neutrino oscillations points out the fact that neutrinos have mass, and provides evidence beyond the Standard Model (BSM). This phenomenon is successfully described by a theoretical framework with the help of three neutrino mixing angles (θ_{12} , θ_{13} , θ_{23}), two mass-square splittings (Δm_{21}^2 , Δm_{31}^2), and one Dirac CP phase (δ) [1–4]. Thanks to the great efforts in the past two decades, we almost have a complete understanding of such a neutrino oscillation framework. Nevertheless, more efforts in the neutrino oscillation experiments are needed to determine the sign of Δm_{31}^2 , to measure the value of $\sin \theta_{23}$ more precisely, to discover the potential CP violation in the leptonic sector and even to constrain the size of δ [4]. For these purposes, the on-going long baseline experiments (LBLs), such as the NuMI Off-axis ν_e Appearance experiment (NO ν A) [5] and the Tokai-to-Kamioka experiment (T2K) [6], can answer these questions with the statistical significance $\gtrsim 3\sigma$ in most of the parameter space. Based on the analysis in T2K and NO ν A, the normal mass ordering ($\Delta m_{31}^2 > 0$), the higher θ_{23} octant ($\theta_{23} > 45^\circ$), and $\delta \sim 270^\circ$ are preferred so far [4]. The future LBLs, Deep Underground Neutrino Experiment (DUNE) [7], Tokai to Hyper-Kamiokande (T2HK) [8], and the medium baseline reactor experiment, the Jiangmen Underground Neutrino Observatory (JUNO) [9, 10] will further complete our knowledge of neutrino oscillations.

It is clear that neutrinos carry the peculiar mass smaller than $\mathcal{O}(1)$ eV among the elementary particle, although the absolute value of each neutrino mass is unknown. In addition, one of the fundamental questions with regards to neutrino properties is how neutrino masses are generated naturally. The seesaw mechanism has been used to explain such a tiny mass in the neutrino sector with the help of neutrino flavor mixings. Anyhow new particles have to be introduced in such a mechanism. The more degrees of freedom in a theoretical model, the less predictivity of observables in experiments. Furthermore, flavour symmetries are employed to reduce degrees of freedom in the neutrino mass model. These models can explain the origin of the neutrino mixing, and predict correlations of oscillation parameters (some of recent review articles are [11–17]). Before the discovery of non-zero θ_{13} by Daya Bay reactor neutrino experiment [18], the ‘tri-bi-maximal’ neutrino mixing (TBM) ansatz, which was

proposed in 2002 by Harrison, Perkins, and Scott [19], was in line with experimental data:

$$U_{\text{TBM}} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}.$$

Given $\theta_{13} \approx 8^\circ$ measured by several reactor neutrino experiments, we have to break the symmetry pattern to obtain such non-zero value of θ_{13} . One of popular proposals is to correct the tri-bi-maximal neutrino mixing such that

$$\sin \theta_{12} = (1 + s)/\sqrt{3}, \quad \sin \theta_{13} = r/\sqrt{2}, \quad \text{and} \quad \sin \theta_{23} = (1 + a)/\sqrt{2}.$$

It is relatively easy for model builders to check the validity of the neutrino mass model and constrain model parameters by the public nufit results [Ref:nufit website]. However, there is no such a public toolkit to evaluate model predictions in future neutrino experiments. **General Long Baseline Experiment Simulator (GLOBES)** [25, 26] is a convenient tool to simulate neutrino oscillation experiments via the Abstract Experiment Definition Language (AEDL) as one of the most popular and useful tools in the community of neutrino oscillation physics. Some AEDL files to describe experiments are also available in **GLOBES** website, while the working group in the DUNE experiment also releases their neutrino flux information and detector descriptions in AEDL files [27]. It is to be extended for the purpose of analysing flavour symmetry models in an universal way.

As we approach the era of precision measurements in the neutrino oscillations, recent works pay more attentions to how the future neutrino experiments can be used to test these flavour-symmetry neutrino mass models, *e.g.* Ref. [20–24]. In this work, we will present our simulation toolkit **Flavour Symmetry Embedded - GLOBES (FaSE-GLOBES)** in a C-library to facilitate the study in the flavour symmetry neutrino models. **FaSE** is a supplemental tool for GLOBES, written in `c/c++` language, and allows users to assign any flavour symmetry model and analyze how a flavour symmetry model is constrained by the simulated neutrino oscillation experiments.

II. OVERVIEW FASE-GLOBES

FaSE-GLOBES is written in the `c` language, and consists of two source codes `FASE_GLOBES.c` and `model-input.c`. The user defines the correlations between

model inputs and standard neutrino mixing parameters in **model-input.c**, while **FASE_GLoBES.c** is an engine to perform analysis with user-specified experiments in a simulator. Note that we define the standard neutrino mixing/oscillation parameters $\vec{\theta}_{OSC}$ ($\theta_{12}, \theta_{13}, \theta_{23}, \Delta m_{21}^2, \Delta m_{31}^2, \delta$) to separate from model parameters $\vec{\theta}_{Model}$ hereafter.

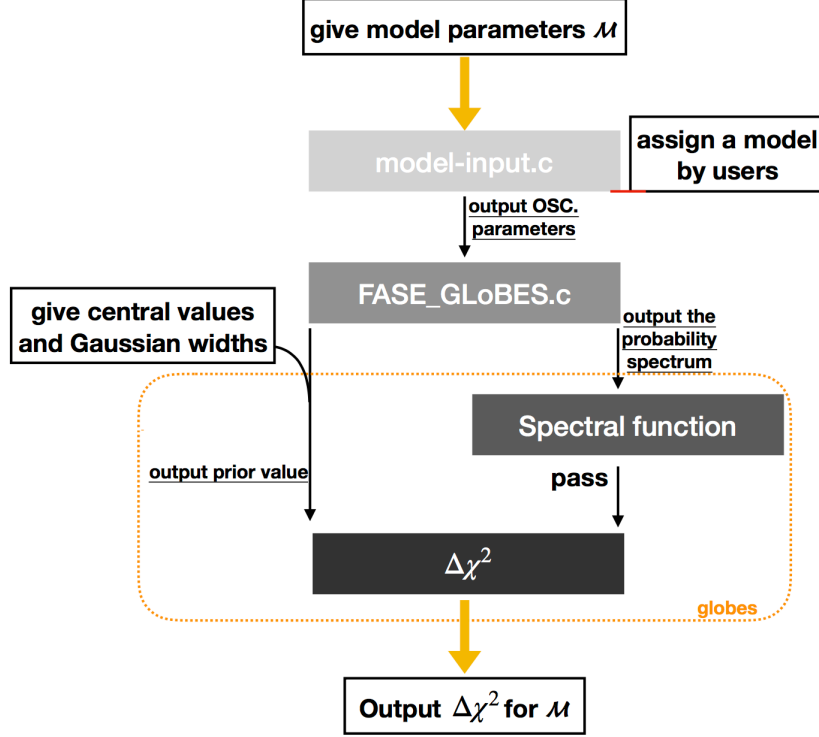


Figure 1: A scheme to correlate the model parameters with standard neutrino oscillation parameters. The error propagation is implemented in the simulation code up to the spectra analysis.

The concept of **FaSE-GLoBES** is shown in Fig. 1, in which three parts are shown: 1. **the parameter translation** (the blue box), 2. **giving oscillation-parameter values** (the green box), and 3. **the χ^2 -value calculation** (the orange box). The idea behind this flow chart Fig. 1 is that given a set of model parameters, the corresponding values in standard oscillation parameters are obtained by a translation, which is assigned by the user in **model-input.c**. And then, through **FASE_GLoBES.c** these oscillation-parameter values are passed into **GLoBES** library to simulate the event spectra so that we can perform physics analysis in the newly-defined χ^2 .

Application Programming Interface (API) functions in **FaSE-GLoBES** are listed:

1. `MODEL_init(N_{para})`,
2. `FASE_glb_probability_matrix`,
3. `FASE_glb_set_oscillation_parameters`,
4. `FASE_glb_get_oscillation_parameters`,
5. `FASE_prior_OSC`,
6. `FASE_prior_model`.

The first one is to initialise **FaSE** with the number of input parameters N_{para} . The next three functions need to be included to replace the default **GLOBES** probability engine by the one that can read the output from **model-input.c**, as follows.

```
glbRegisterProbabilityEngine(6,
                            &FASE_glb_probability_matrix,
                            &FASE_glb_set_oscillation_parameters,
                            &FASE_glb_get_oscillation_parameters,
                            NULL);
```

This probability engine can work with oscillation or model parameters. It can be set by the user with the parameter `PARAMETER`. If `PARAMETER=STAN` (`PARAMETER=MODEL`) the probability engine works with oscillation (model) parameters. The final two items on the API list are prior functions. Once the user gives the prior in oscillation (model) parameters, the user needs to call `FASE_prior_OSC` (`FASE_prior_model`) as follows.

```
glbRegisterPriorFunction(FASE_prior_OSC, NULL, NULL, NULL);
```

or

```
glbRegisterPriorFunction(FASE_prior_model, NULL, NULL, NULL);
```

We note that except for setting the probability engine and the prior function, the other parts in the main code should follow with the **GLOBES** manual.

III. MODEL SETTING

The function **MtoS** can do the translation from model parameters $\vec{\theta}_{Model}$ to oscillation parameters $\vec{\theta}_{OSC}$. After the user gives the array $\vec{\theta}_{Model}$ to the function **MtoS**, the output is the corresponding oscillation parameter $\vec{\theta}_{OSC}$, of which components are θ_{12} , θ_{13} , θ_{23} , δ , Δm_{21}^2 , and Δm_{31}^2 . For the first four components, values are given in the unit of **rad**, while the other two are in **eV²**. These values will be passed into **FaSE_GLoBES** to simulate the experimental spectra and compute the prior values.

To transform from $\vec{\theta}_{Model}$ to $\vec{\theta}_{OSC}$, the user can assign the correlation between the standard oscillation and model parameter sets, or define the mass matrix in model parameters, which will be diagonalised by the function **ModelTO** to obtain the corresponding oscillation-parameter values. In the way of directly giving the relation between oscillation and model parameter sets, the user needs to provide $\vec{\theta}_{Model} = \vec{f}(\vec{\theta}_{OSC})$ in the function **MtoS**.

The oscillation parameters can also be obtained in the way based on

$$U^\dagger \mathcal{M} \mathcal{M}^\dagger U = \mathbf{M}^2, \text{ where } \mathbf{M}_{\alpha\beta}^2 = m_\alpha^2 \delta_{\alpha\beta}, \quad (1)$$

where \mathcal{M} (\mathbf{M}) is the neutrino mass matrix in the flavour (mass) state. The matrix \mathcal{M} is given by user with model parameters $\vec{\theta}_{Model}$. The matrix U is the neutrino mixing matrix, and can be used to get mixing angles, mass squared differences and the CP phase. The difference between any two diagonal elements of \mathbf{M} ($\mathbf{M}_{ii} - \mathbf{M}_{jj} = m_i^2 - m_j^2$) is the mass-squared difference (Δm_{ij}^2). This diagonalisation Eq. (1) can be done by the function **ModelTO**, which needs to be called in **MtoS** with outputs of the vector $\vec{\theta}_{OSC}$.

IV. PRIOR SETTING

Given a set of values for model parameters, **FASE_GLoBES.c** will obtain the corresponding oscillation-parameter values from **model-input.c**, and will pass these values to simulate event spectra and to compute the prior values. Two gaussian prior functions are provided in **FaSE**: **FASE_prior_OSC** and **FASE_prior_model**. These two functions are constructed for different purposes. If the user gives the prior in oscillation (model) parameters, the user should register **FASE_prior_OSC** (**FASE_prior_model**) with the **GLoBES** function **glbRegisterPriorFunction**, as we introduced in the previous section. The user also needs

to assign the parameters `PARA=STAN` (`PARA=Model`), when the user prefers to give the prior in oscillation (model) parameters. The Gaussian prior is

$$\chi_{prior}^2 = \sum_i \frac{(\theta_i - \theta_i^c)^2}{\sigma_i^2}, \quad (2)$$

where θ_i is one of parameters constrained by prior, θ_i^c (σ_i) is the central value (Gaussian width) of the prior for θ_i . We note that θ_i OSC can be either model ($\vec{\theta}_{Model}$) or oscillation parameters ($\vec{\theta}_{OSC}$). The values of θ_i^c and σ_i need to be given by the user through three arrays: `Central_prior`, `UPPER_prior`, and `LOWER_prior`, in which there are six components. To treat asymmetry of width for upper ($\theta_i > \theta_i^c$) and lower ($\theta_i < \theta_i^c$) Gaussian widths, we give values in two arrays `UPPER_prior`, and `LOWER_prior`, respectively. If the user gives the prior in model parameters, the order of each component follows with the setup of input of the probability engine. While the user gives the prior in oscillation parameters, the six components in order are θ_{12} , θ_{13} , θ_{23} , δ , Δm_{21}^2 , and Δm_{31}^2 . The first four parameter are in **rad**, and the final two are in **eV²**.

Finally, some restrictions are imposed by the chosen flavour symmetry model. We set up these restrictions in the function `model_restriction` in `model-input.c`. In the function `model_restriction`, the user needs to *return* 1 once the restriction is broken. For example, if the normal ordering is imposed, we give “ `if (DMS31<0) { return 1;}` ” in `model_restriction`, where `DMS31` is the variable for Δm_{31}^2 . Then, when the restriction is broken, `model_restriction` returns the value 1 to the prior function `FASE_prior_OSC` or `FASE_prior_model`. Therefore, if there is no restriction, we simply *return* 0 in `model_restriction` as follows.

```
double model_restriction(double model []){ return 0;};
```

V. THE DEFINITION FOR χ^2 (BASED ON GLOBES)

The user can use **FaSE-GLoBES** to constrain model parameters. The expression χ^2 is used as the default **GLoBES** setting. [JT: What is the meaning of this χ^2 here?] In more detail, the χ^2 function is constructed based on a log-likelihood ratio,

$$\begin{aligned} \chi^2(\vec{\theta}^{hyp}, \xi_s, \xi_b) = & 2 \sum_i \left(\eta_i(\vec{\theta}^{hyp}, \xi_s, \xi_b) - n_i + n_i \ln \frac{n_i}{\eta_i(\vec{\theta}^{hyp}, \xi_s, \xi_b)} \right) \\ & + p(\xi_s, \sigma_s) + p(\xi_b, \sigma_b) + \chi_{prior}^2, \end{aligned} \quad (3)$$

where i runs over the number of bins, $\eta_i(\vec{\theta}, \xi_s, \xi_b)$ is the assumed event rate in the i th bin and E_i is the central value in this energy bin. The vector $\vec{\theta}$ consists of model or oscillation parameters. The parameters ξ_s and ξ_b are introduced to account for the systematic uncertainties in the normalisation for the signal (subscript s) and background (subscript b) components of the event rate, and are allowed to vary in the fit as nuisance parameters. For a given set of parameters $\vec{\theta}$, the event rate in the i th energy bin is calculated as

$$\eta_i(\vec{\theta}^{hyp}, \xi_s, \xi_b) = (1 + \xi_s) \times s_i + (1 + \xi_b) \times b_i, \quad (4)$$

where s_i and b_i are the expected number of signal and background events in i th energy bin, respectively. The nuisance parameters are constrained by the Gaussian prior $p(\xi, \sigma) = \xi^2 / \sigma^2$ with corresponding uncertainties σ_s and σ_b for the signal and background, respectively. Finally, χ^2_{prior} is a set of Gaussian priors for hypothesis, and is expressed as Eq. 2. After doing all minimisations, the user obtain the χ^2 value for a specific hypothesis $\vec{\theta}^{hyp}$, $\chi^2(\vec{\theta}^{hyp})$.

Based on the χ^2 value Eq. 3, we can study how model parameters can be constrained and whether a flavour-symmetry neutrino model is excluded by simulated experiments. In the following we will demonstrate with typical examples how it works.

Applications

The user of **FaSE-GLoBES** is able to study how model parameters can be constrained by the simulated experiments. To do so, the user needs to simulate the true event spectrum n_i with a set of model ($\vec{\theta}_{Model}^{true}$) or oscillation parameters ($\vec{\theta}_{OSC}^{true}$), *i.e.* set up $n_i(\vec{\theta}_{Model}^{true})$ or $n_i(\vec{\theta}_{OSC}^{true})$. The hypothesis $\vec{\theta}_{Model}^{hyp}$ predicts the tested event spectrum $\eta_i(\vec{\theta}_{Model}^{hyp}, \xi_s, \xi_b)$. With the default settings for χ^2 function as Eq. (2) in **FaSE-GLoBES**, the user computes the statistical quantity,

$$\chi^2(\vec{\theta}_{Model}^{hyp}), \text{ with } n_i(\vec{\theta}_{Model}^{true}) \text{ or } n_i(\vec{\theta}_{OSC}^{true}). \quad (5)$$

We note that the minimum of χ^2 in the whole parameter space ($\chi^2_{min.}$) may not be 0. Therefore, to get the precision of model parameters, the user should use the value $\Delta\chi^2(\vec{\theta}_{Model}^{hyp}) \equiv \chi^2(\vec{\theta}_{Model}^{hyp}) - \chi^2_{min.}$, instead of χ^2 itself. By varying different hypotheses $\vec{\theta}_{Model}^{hyp}$, we will obtain the allowed region of model parameters with the statistical quantity $\Delta\chi^2(\vec{\theta}_{Model}^{hyp})$.

The user can also study how well a flavour symmetry model explains the computed data, or predict whether the simulated experiment can exclude this model or not. In other words, the user studies the minimum of χ^2 for the flavour symmetry model $\vec{\theta}_{Model}$ as a hypothesis, by assuming different true oscillation values, *i.e.* different $\vec{\theta}_{OSC}^{true}$. To do so, one can compute the same statistical quantity Eq. (5), while the true spectrum is varied with different true values $\vec{\theta}_{OSC}^{true}$. All model parameters are allowed to be varied with the user-defined prior. Finally, the user might adopt Wilk's theorem to interpret results [28]. When we compare nested models, the $\Delta\chi^2$ test statistics is a random variable asymptotically distributed as a χ^2 -distribution with the number of degrees of freedom, which is equal to the difference in the number of free model parameters `dof`.

VI. CONSTRAINT OF MODEL PARAMETERS

FaSE-GLoBES can be used to study how model parameters are constrained by simulated neutrino oscillation experiments. We take *the tri-direct littlest seesaw* (TDLS) [29–31] as an example. In this model, the light left-handed Majorana neutrino mass matrix is given by

$$m_\nu = m_a \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix} + e^{i\eta} m_s \begin{pmatrix} 1 & x & x \\ x & x^2 & x^2 \\ x & x^2 & x^2 \end{pmatrix}, \quad (6)$$

where x , η , m_a , and the ratio $r \equiv m_s/m_a$ are four parameters to be constrained by simulated data. We note that from Eq. (6), $m_1 = 0$ and the normal mass ordering are imposed, and will need to be imposed in **FaSE-GLoBES**. Therefore, the restrictions in this model are $m_a > 0$ and $r > 0$.

In Fig. 2, we study how parameters x and η can be constrained by the MuOn-decay Medium baseline NeuTrino beam experiment (MOMENT) [32]. We input the model by giving model parameters as functions of standard oscillation parameters Eq. (??). Besides, parameters r and ma are varied with the prior that is given in standard oscillation parameters, according to the global-fit result NUFIT4.0. More details are presented in the user manual. The code for this example is provided as `x_eta.c` in the **FaSE** distribution.

To show the generality of **FaSE-GLoBES**, we present the similar result with another model – *the warped flavor symmetry* (WFS) [Ref???]. This model predict further simplified

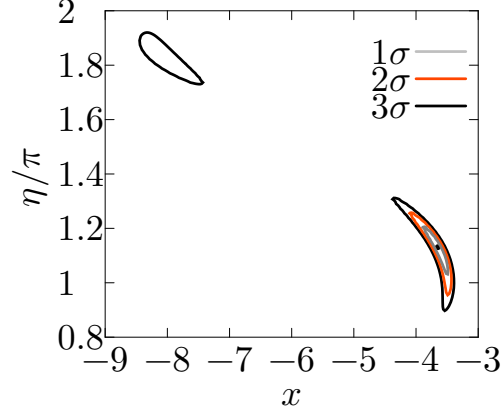


Figure 2: Precision measurements of two model parameters in the framework of three neutrino oscillations taking uncertainties of the current global fit results, for MOMENT, at 1σ (gray), 2σ (orange), 3σ (black). True values for the model parameters are used $(x, \eta, r, M_a) = (-3.65, 1.13\pi, 0.511, 3.71 \text{ meV})$.

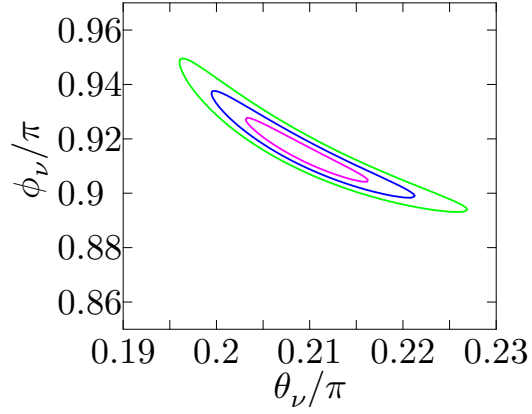


Figure 3: [formate will be further tuned]

correlations that the standard oscillation parameters including mixing angles and the CP

phase are functions of only two model parameters θ_ν and ϕ_ν ,

$$\begin{aligned}\sin^2 \theta_{12} &= \frac{1}{2 - \sin 2\theta_\nu \cos \phi_\nu}, \\ \sin^2 \theta_{13} &= \frac{1}{3}(1 + \sin 2\theta_\nu \cos \phi_\nu), \\ \sin^2 \theta_{23} &= \frac{1 - \sin 2\theta_\nu \sin(\pi/6 - \phi_\nu)}{2 - \sin 2\theta_\nu \cos \phi_\nu}, \\ J_{CP} &= -\frac{1}{6\sqrt{3}} \cos 2\theta_\nu.\end{aligned}\tag{7}$$

The constraint of θ_ν and ϕ_ν is presented in Fig 3, in which we use the best fit of NUFIT 4.0 result as the true values $((\theta_{12}, \theta_{13}, \theta_{23}, \delta, \Delta m_{21}^2, \Delta m_{31}^2) = (33.82^\circ, 8.61^\circ, 49.6^\circ, 215^\circ, 7.39 \times 10^{-5} \text{ eV}^2, 2.525 \times 10^{-3} \text{ eV}^2))$.

VII. MODEL TESTING

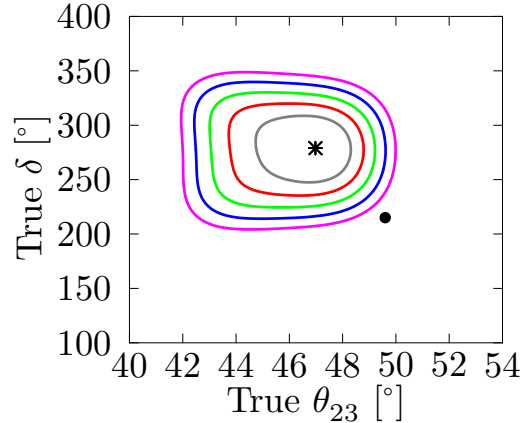


Figure 4: The 2-D exclusion contour for tri-direct littlest seesaw model on the plane of two true standard parameters θ_{23} and δ , from 1σ to 5σ . The range for each parameter is taken according to the 3σ uncertainty in NuFit4.0 results. The black dot denotes the best fit of NuFit4.0 results $((\theta_{12}, \theta_{13}, \theta_{23}, \delta, \Delta m_{21}^2, \Delta m_{31}^2) = (33.82^\circ, 8.61^\circ, 49.6^\circ, 215^\circ, 7.39 \times 10^{-5} \text{ eV}^2, 2.525 \times 10^{-3} \text{ eV}^2))$, while the star is the prediction by the tri-direct littlest seesaw model with NuFit4.0 results $((\theta_{12}, \theta_{13}, \theta_{23}, \delta, \Delta m_{21}^2, \Delta m_{31}^2) \sim (36.25^\circ, 8.63^\circ, 47^\circ, 279^\circ, 7.39 \times 10^{-5} \text{ eV}^2, 2.525 \times 10^{-3} \text{ eV}^2))$.

We can also study on whether a neutrino mass model is excluded or not, assuming different true values for oscillation parameters. In this example, we present testing the TDLS model

in various true values of θ_{23} and δ . We compute the minimal χ^2 value for the model with all four model parameters, varied with the prior Eq. (2) according to NUFIT4.0 results. In addition, the studied statistics function is exactly given by Eq. (3), but the true event rate n_i is predicted by a set of oscillation parameters, which will be varied in the code.

As mentioned in Sec. V, the user needs to consider Wilk's theorem with the different of degrees of freedom between the true theory and the hypothesis, `dof`. [JT: what is the meaning of the true theory?] Here the number `dof` is 2. And that is our output: `gsl_cdf_chisq_Qinv(gsl_cdf_chisq_Q(fabs(res),dof),1)`. [I don't understand this sentence.] We show the result in Fig. 4. More details are presented in the user manual. The code for this example is provided as `th23_delta.c` in the **FaSE** distribution.

We also show the similar result for WFS, and present it in Fig.

VIII. SUMMARY AND CONCLUSIONS

[JT: This section has to be significantly improved.]

We have presented a simulation toolkit **FaSE-GLoBES** to study the leptonic flavour symmetry models with neutrino oscillation experiments in a user-friendly way. **FaSE** contains two c-codes: `model-input.c` and `FASE_GLoBES.c`. Shown in Fig. 1, **FASE_GLoBES**, which calls functions in `model-input.c`, plays a role as a bridge between **FaSE** and **GLoBES** to simulate the event spectrum and compute the prior value. It can be left to be untouched by users. However, all inputs from the user needs to be given in `model-input.c`. The user can provide the relation between model and oscillation parameters, or simply give the mass matrix predicted by the studied model. The prior can be assigned as a Gaussian distribution in model or oscillation parameters. Central values and widths are given by the user in the main code through three arrays. Given a set of model parameters $\vec{\theta}_{Model}$, with **GLoBES**, the output can be the χ^2 value for the hypothesis $\vec{\theta}_{Model}$ in the simulated experiments.

We also present demonstrate examples for **FaSE-GLoBES** with the flavour symmetry model – tri-direct littlest seesaw (TDLS) – and the future neutrino oscillaiton experiment – MOMENT. We demonstrate how to use **FaSE-GLoBES** to obtain the constraint of any two of model parameters, and to study the ability to TDLS by MOMENT experiment.

Finally, **GLoBES** is a popular and powerful simulation tool to analysis the neutrino

oscillation experiments in a simple language (AEDL), without losing too much detail. Considering the success of the flavour symmetry theory to explain the neutrino oscillations, **FaSE-GLoBES** should benefit model builders of leptonic flavour symmetry and phenomenologists for neutrino oscillation physics. We leave the flexibility for the user, and some other improvements and extensions can be expected in the future.

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