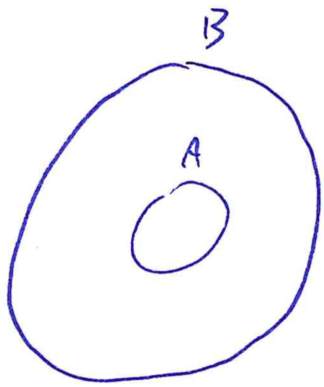
ES 1



$A \subset B$ $\qquad$ $A \perp B$?

DEFINIZIONE DI IND. STATISTICA

$$P(A \cap B) = P(A) \cdot P(B) \Leftrightarrow A \perp B$$

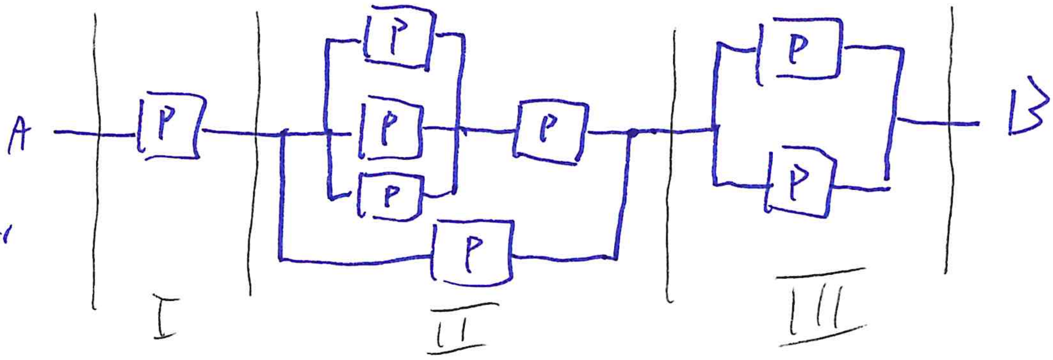
$$P(A \cap B) \overset{A \subset B}{=} P(A)$$

QUINDI

$$P(A) = P(A)\, P(B)$$

- SE $A = \emptyset$ $\quad P(A) = 0 \Rightarrow A \perp B$
- SE $B = \Omega$ $\quad P(B) = 1 \Rightarrow A \perp B$
- ALTRIMENTI $A \not\perp B$

ES 2

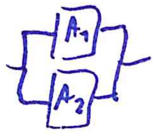


OSSERVAZIONI :

SE HO 2 COMPONENTI IN SERIE ALLORA

$$P(ok) = P(A_1 \cap A_2) \overset{\perp}{=} P(A_1) \cdot P(A_2)$$

MENTRE SE HO DUE COMPONENTI IN PARALLELO HO :



$$P(ok) = 1 - P(\underset{ok^c}{\underset{\nearrow}{ko}}) = 1 - P(A_1^c \cap A_2^c) \overset{\perp}{=} 1 - P(A_1^c) P(A_2^c)$$

$$= 1 - (1 - P(A_1))(1 - P(A_2))$$

ORA POSSO RISOLVERE PER I TRE PEZZI I, II E III

$$P(I) = p$$

$$P(II) = 1 - (1-p)\left(1 - p \cdot \left(1 - (1-p)^3\right)\right)$$

$$P(III) = 1 - (1-p)^2$$

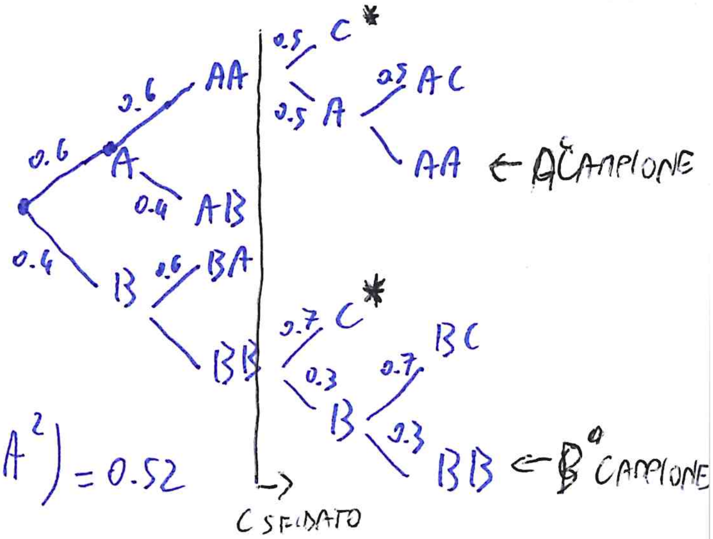
$$P(A \to B) = P(I)\, P(II)\, P(III)$$

ES 3

$P(A \text{ batte } B) = 0.6$

$P(C \text{ batte } A) = 0.5$

$P(C \text{ batte } B) = 0.7$



a) $P(C \text{ sfidato}) = P(A \text{ batte } B)^2 + P(B \text{ batte } A^2) = 0.52$

$P(A \text{ sfida } C) = 0.6^2 = 0.36$

$P(C \text{ rimane campione}) = 1 - P(A \text{ campione}) - P(B \text{ campione}) =$

$= 1 - 0.6^2 0.5^2 - 0.4^2 0.3^2 = 0.8956$

b) $P(A \text{ sfidante} \mid C \text{ sfidato}) \overset{\text{BAYES}}{=} \dfrac{P(C \text{ sfidato} \mid A \text{ sfidante}) \cdot P(A \text{ sfidante})}{P(C \text{ sfidato})}$

SE HO A SFIDANTE ALLO C E' SICURAMENTE SFIDATO

$= \dfrac{1 \cdot 0.6^2}{0.52} = 0.6923$

$P(C \text{ campione} \mid C \text{ sfidato}) = 1 - P(A_{cam.} \mid C_{sf.}) - P(B_{cam.} \mid C_{sf}) =$

$= 1 - \dfrac{0.6^2 \cdot 0.5^2 + 0.4^2 \cdot 0.3^2}{0.6^2 + 0.4^2}$

$P(A_{camp} \mid C \text{ sfidato}) \overset{\text{BAYES}}{=} \dfrac{P(C \text{ sfidato} \mid A_{camp.})\, P(A_{camp.})}{P(C \text{ sfidato})} = \dfrac{1 \cdot 0.6^2 \cdot 0.5^2}{0.6^2 + 0.4^2}$

c) $P(A \text{ sfida } C \mid \overbrace{C_{camp}, C \text{ vince subito}}^{*}) = \dfrac{\text{casi favorevoli}}{\text{casi tot.}} = \dfrac{0.6^2 \cdot 0.5}{0.6^2 \cdot 0.5 + 0.4^2 \cdot 0.7}$

$= 0.6164$

ES 4

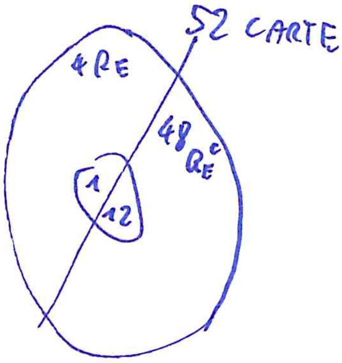
a) $\frac{4}{52} = \frac{1}{13}$ $\left(\frac{\text{CASI FAVOREVOLI}}{\text{CASI TOTALI}} \right.$

b) 1, 2, 3, 4, 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | ...

$\frac{48}{52}$ $\frac{47}{51}$ . . . . . . $\frac{37}{41}$ $\frac{4}{40}$

L'INTERSEZIONE DEGLI EVENTI DA: $\frac{48 \cdot 47 \cdot \ \cdots \ 37 \cdot 4}{52 \cdot 51 \ \cdots \ 41 \cdot 40}$

ALTERNATIVAMENTE SI POSSONO USARE LE ESTRAZIONI SENZA REINSERIMENTO (IPERGEOMETRICHE):



$$P(1\,RE) = \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$$

RICORDANDOMI CHE QUESTA VALE PER QUALSIASI POSIZIONE, QUINDI SE VOGLIO QUELLA DEL RE IN 13MA POSIZIONE HO:

$$P(\text{RE IN } 13^{MA} \text{ POSIZIONE}) = \frac{1}{13} P(1\,RE)$$

ES 5

$X \in \{\pm 3, \pm 2, \pm 1\}$

$$p_X(x) = \begin{cases} \dfrac{x^2}{a} & x \in \{\pm 3, \pm 2, \pm 1\} \\ 0 & \text{ALTR.} \end{cases}$$

a) $\underline{a > 0}$, $\sum_x p_X(x) = \sum_x \dfrac{x^2}{a} \overset{!}{=} 1 \implies a \overset{!}{=} \sum_x x^2 = 28$

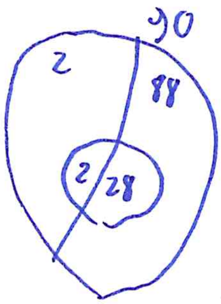
b) $Z = X^2$

X	$Z = X^2$	$p_X(x)$
-3	9	$9/a$
-2	4	$4/a$
-1	1	$1/a$
1	1	$1/a$
2	4	$4/a$
3	9	$9/a$

$\to 1$
$\to 4$
$\to 9$

$$p_Z(z) = \begin{cases} 2/a & z = 1 \\ 8/a & z = 4 \\ 18/a & z = 9 \\ 0 & \text{ALTR.} \end{cases}$$

ES 6



$$P(\text{T e J NELLA CLASSE 1}) = \frac{\binom{2}{2}\binom{88}{28}}{\binom{90}{30}}$$

$$P(\text{T e J NELLA STESSA CLASSE}) = P\left(\bigcup_{i=1}^{3} \{\text{T e J NELLA CLASSE } i\}\right)$$

$$= \sum_{i=1}^{3} P(\{\text{T e J NELLA CLASSE } i\})$$

(SPAZIO PROB. DISCRETO E UNIFORME)

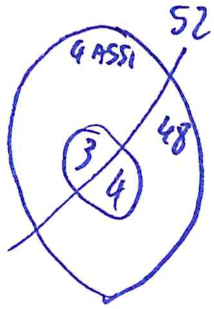
$$= 3\,P(\text{T e J NELLA CLASSE 1})$$

$$= 3 \cdot \frac{\binom{2}{2}\binom{88}{28}}{\binom{90}{30}}$$

ES 7

ESTRAZIONE SENZA REINSERIMENTO

→ PROBABILITÀ IPERGEOMETRICHE



$$P(3 \text{ ASSI}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}$$

ES 8

$X \perp Y \qquad Z = 2X - 3Y$

$$E[Z] = E[2X-3Y] = 2E[X] - 3E[Y]$$

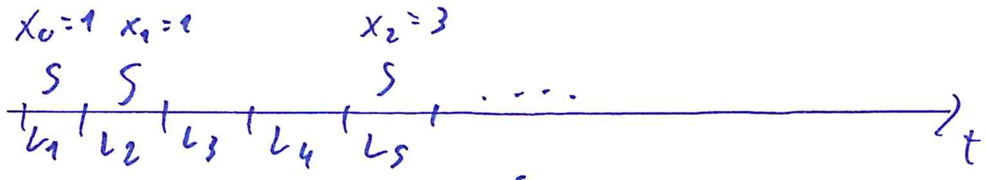
$$\begin{aligned} VAR[Z] = VAR[2X-3Y] &\overset{\perp}{=} VAR[2X] + VAR[-3Y] \\ &= 4\,VAR[X] + 9\,VAR[Y] \end{aligned}$$

ES 9

X: # TOTALE LANCI PER OSSERVARE TUTTE LE FACCE ALMENO UNA VOLTA.

SUCCESSO: OSSERVO UNA FACCIA MAI VISTA

X_i: # LANCI TRA SUCCESSO i E $i+1$



$$X = X_0 + \sum_{i=1}^{5} X_i = 1 + \sum_{i=1}^{5} X_i$$

$$E[X] = E\left[1 + \sum_{i=1}^{5} X_i\right] = 1 + \sum E[X_i]$$

$$X_1 \sim Geom\left(\tfrac{5}{6}\right),\ X_2 \sim Geom\left(\tfrac{4}{6}\right),\ X_3 \sim Geom\left(\tfrac{3}{6}\right) \dots$$

$$X_i \sim Geom(P_i) \rightarrow E[X_i] = \frac{1}{P_i}$$

$$E[X] = 1 + \sum_{i=1}^{5} \frac{6}{6-i} \underset{\uparrow}{=} 1 + \sum_{t=1}^{5} \frac{6}{6-(6-t)} = 1 + \sum_{t=1}^{5} \frac{6}{t} = 14.7$$

$t = 6 - i$ $\quad i = 1 \rightarrow t = 5$

$i = 6 - t$ $\quad i = 5 \rightarrow t = 1$

ES 10

$y=3$	c	c	$2c$
$y=2$	$2c$	0	$4c$
$y=1$	$3c$	c	$6c$
	$x=1$	$x=2$	$x=3$

a) $\sum_{x,y} P_{X,Y}(x,y) \overset{!}{=} 1 \Rightarrow c = \frac{1}{20}$

b) $P(Y=2) = 2c + 4c = \frac{6}{20}$

c) $Z = YX^2 \quad E[Z|Y=2] = ? = E[YX^2|Y=2] = 2E[X^2|Y=2] = \frac{38}{3}$

$E[X^2|Y=2] = \sum_x x^2 \underbrace{P_{X|Y}(x|2)}$

$\Rightarrow P_{X|Y}(x|2) = \frac{P_{X,Y}(x,2)}{P_Y(2)} = \begin{cases} \frac{2c}{6c} & x=1 \\ 0 & x=2 \\ \frac{4c}{6c} & x=3 \end{cases}$

$= 1^2 \cdot \frac{1}{3} + 2^2 \cdot 0 + 3^2 \cdot \frac{2}{3} = \frac{38}{6}$

d) $\{X|X\neq 2\} \perp \{Y|X\neq 2\}$?

$P(X=x, Y=y | X\neq 2) \overset{?}{=} P(X=x|X\neq 2) P(Y=y|X\neq 2) \quad \forall x,y$

POSSO NOTARE CHE:

$P(Y=y|X=1) = P(Y=y|X=3) \quad \forall y$

$\Rightarrow$ QUINDI SI, C'È INDIPENDENZA CONDIZIONATA

e) $Var[Y|X=2] = E[(Y - \underbrace{E[Y|X=2]}_{=2})^2 | X=2]$

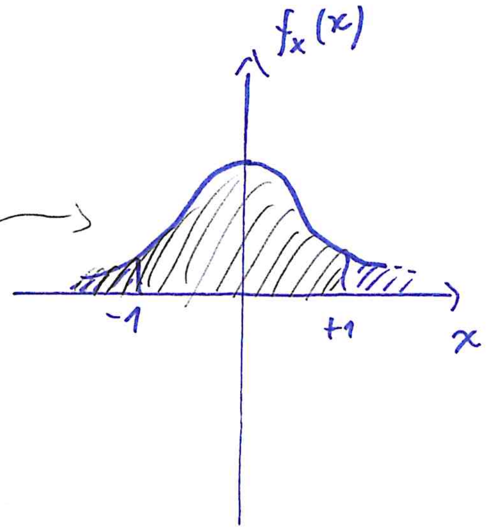
$= 1$

(TUTTE LE DISTANZE VALGONO 1)

$y=3$	c
$y=2$	0
$y=1$	c
	$x=2$

ES 11

$X \sim N(0,1)$, $Y \sim N(1,4)$



a) $P(X \leq 1.5) = \Phi(1.5) = 0.9332$

CUMULATA DELLA GAUSSIANA

$P(X \leq -1) \overset{\text{SIMMETRIA}}{=} P(X \geq 1)$

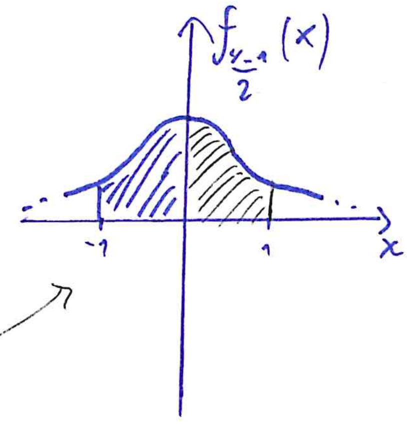
$= 1 - P(X \leq 1)$

$= 1 - \Phi(1) = 0.1587$

b) $\frac{Y-1}{2}$ E' UNA TRASFORMAZIONE LINEARE QUINDI

$$\frac{Y-1}{2} \sim N\left(E\left[\frac{Y-1}{2}\right] = \frac{E[Y]-1}{2} = 0 \ , \ Var\left(\frac{Y-1}{2}\right) = \frac{Var[Y]}{4} = 1\right)$$

↳ STANDARDIZZAZIONE DI Y : $\frac{Y - E[Y]}{\sqrt{Var[Y]}} = \frac{Y-1}{2}$



c) $P(-1 \leq Y \leq 1) = P\left(\frac{-1-1}{2} \leq \frac{Y-1}{2} \leq \frac{1-1}{2}\right)$

$= P\left(-1 \leq \frac{Y-1}{2} \leq 0\right)$

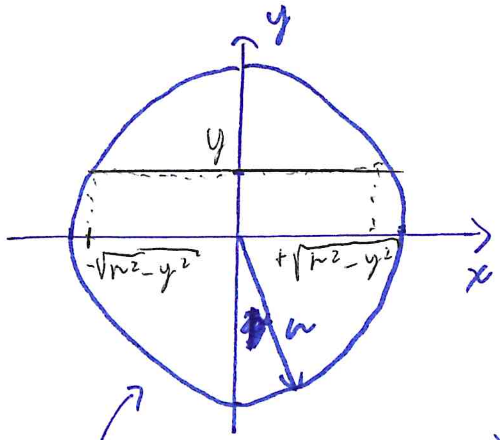
$= P\left(0 \leq \frac{Y-1}{2} \leq 1\right)$ SIMMETRIA

$= P\left(\frac{Y-1}{2} \leq 1\right) - P\left(\frac{Y-1}{2} \leq 0\right)$

$= \Phi(1) - \Phi(0) = 0.8413 - 0.5 = 0.3413$

ES 12

a) SPAZIO DI PROB. UNIFORME



$D_R = \{(x,y) : x^2+y^2 \leq r^2\}$

$$f_{X,Y}(x,y) = \begin{cases} \text{COST.} & \text{PER } (x,y) \in D_R \\ 0 & \text{ALTR.} \end{cases}$$

$$\text{COST.} = \frac{1}{\text{AREA}(D_R)} = \frac{1}{\pi r^2}$$

b) $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx = \begin{cases} \displaystyle\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx & \text{PER } -r \leq y \leq r \\ 0 & \text{ALTR.} \end{cases}$$

$$= \begin{cases} \dfrac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \leq y \leq r \\ 0 & \text{ALTR.} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1/\pi r^2}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}} = \dfrac{1}{2\sqrt{r^2-y^2}} & \begin{array}{l} -r \leq y \leq r \\ -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2} \end{array} \\ \nexists & |y| \geq r \\ 0 & \text{ALTR.} \end{cases}$$

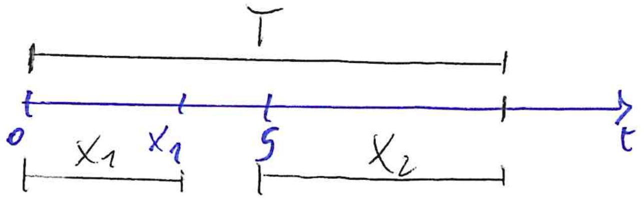
ES 13

$X_{1,2}$: TEMPO COLLOQUIO DEL STUDENTE 1, 2

$X_1 \perp X_2 \qquad X_1 \sim X_2 \sim Exp\left(\frac{1}{30}\right) \qquad E[X_1] = E[X_2] = 30$

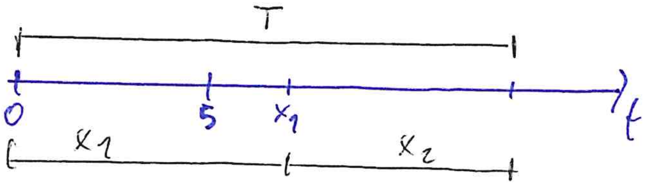
T: TEMPO TOTALE RICEVIMENTO

SCENARIO 1: $\{X_1 < 5\}$



$T = 5 + X_2$

SCENARIO 2: $\{X_1 > 5\}$



$T = X_1 + X_2$

Th. ASPET. TOT.

$$E[T] \overset{\downarrow}{=} E[T \mid X_1 < 5] \cdot P(X_1 < 5) + E[T \mid X_1 > 5] P(X_1 > 5)$$

$$= E[5 + X_2 \mid X_1 < 5] \cdot \underbrace{\left(1 - e^{-5/30}\right)}_{\text{CUMULATA DELL'ESP.}} + E[X_1 + X_2 \mid X_1 > 5] e^{-5/30}$$

$$= \left(5 + E[X_2 \mid X_1 < 5]\right)\left(1 - e^{-5/30}\right) + \left(E[X_1 \mid X_1 > 5] + E[X_2 \mid X_1 > 5]\right) e^{-5/30}$$

$X_1 \perp X_2$

$$= \left(5 + E[X_2]\right)\left(1 - e^{-\frac{5}{30}}\right) + \left(\underbrace{E[X_1 - 5 \mid X_1 > 5]}_{} + 5 + E[X_2]\right) e^{-\frac{5}{30}}$$

PER LA PERDITA DI MEMORIA HO CHE QUESTO È UGUALE A $E[X_1]$

$$= 35\left(1 - e^{-\frac{5}{30}}\right) + 65 e^{-\frac{5}{30}} = 60{,}394$$