# Robotics

## PS02 – Solutions

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# Part 2: Spatial Transforms

# Problem 1

Given the homogeneous matrix $A$ with

$$A = \begin{pmatrix} 0.866 & 0.433 & -0.250 & 2 \\ 0 & -0.5 & 0.866 & -4 \\ -0.5 & -0.75 & -0.433 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- What is the rotation matrix part $R_A$ of $A$?
- Is it a right- or a left-handed rotation?
- What is the inverse $A^{-1}$ of $A$

# Problem 1

rotation part of A

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{matrix}} & \begin{matrix} 2 \\ -4 \\ 1 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

$$R_A = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}$$

# Problem 1

handedness:

$$\det(R_A) = \det\begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix} = ???$$

use **rule of Sarrus**

# Problem 1

use **rule of Sarrus**

- copy first two columns to the right
- multiply upper-left-to-lower-right diagonals and add them up
- multiply lower-left-to-upper-right diagonals and subtract them

$$\left(\begin{array}{ccc|cc} 0.866 & -0.433 & -0.250 & 0.866 & -0.433 \\ 0 & -0.5 & 0.866 & 0 & -0.5 \\ -0.5 & -0.75 & -0.433 & -0.5 & -0.75 \end{array}\right)$$

$$(\underline{0.866 \cdot -0.5 \cdot -0.433}) + (\underline{-0.433 \cdot 0.866 \cdot -0.5}) + (\underline{-0.25 \cdot 0 \cdot -0.75})$$
$$- (\underline{-0.25 \cdot -0.5 \cdot -0.5}) - (\underline{0.866 \cdot 0.866 \cdot -0.75}) - (\underline{-0.433 \cdot 0 \cdot -0.433}) \approx 1$$

=> **right handed**

# Problem 1

note: right-handed rotation matrices

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_y = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}, R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

$$R_y = \left(\begin{array}{ccc|cc} c\beta & 0 & s\beta & c\beta & 0 \\ 0 & 1 & 0 & 0 & 1 \\ -s\beta & 0 & c\beta & -s\beta & 0 \end{array}\right) \Rightarrow \det(R_y) = c^2\beta + s^2\beta = 1$$

*all diagonals are Zero except those two*

similar for $R_x$ and $R_z$

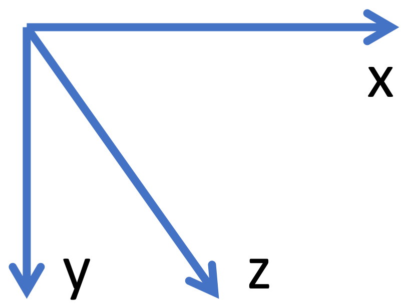
# Problem 1

note:

Computer Graphics and simulation often left-handed

- $z$ points out of monitor
- typically reflection on $y$-axis

$$p^{LH} = A_y^{reflect} p \quad = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}$$



# Problem 1

note: example of a left-handed rotation matrix

$$R_y^{LH} = A_y^{reflect} R_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} = \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & -1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix}$$

$$\left(\begin{array}{ccc|cc} c\beta & 0 & s\beta & c\beta & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -s\beta & 0 & c\beta & -s\beta & 0 \end{array}\right) \Rightarrow \det(R_y^{LH}) = -c^2\beta - s^2\beta = -(c^2\beta + s^2\beta) = -1$$

# Problem 1

rotation part of A : $R_A = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}$

rotation matrix: inverse = transpose

$$R_A{}^{-1} = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}^{-1} = \begin{pmatrix} 0.866 & -0.433 & -0.250 \\ 0 & -0.5 & 0.866 \\ -0.5 & -0.75 & -0.433 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} 0.866 & 0 & -0.5 \\ -0.433 & -0.5 & -0.75 \\ -0.250 & 0.866 & -0.433 \end{pmatrix}$$

# Problem 1

note: inverse rotation = minus angle

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

$$R_x{}^{-1}(\alpha) = R_x{}^{T}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & s\alpha \\ 0 & -s\alpha & c\alpha \end{pmatrix}$$

$$\sin(-\theta) = -\sin(\theta)$$
$$\cos(-\theta) = \cos(\theta)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c(-\alpha) & -s(-\alpha) \\ 0 & s(-\alpha) & c(-\alpha) \end{pmatrix} = R_x(-\alpha)$$

similar for $R_y$ and $R_z$

# Problem 2

Proof that when turning in circles you end up where you started.

Or more concretely: given the motion $move(\alpha, d)$ (in 2D is sufficient) that turns with angle $\alpha$ and then makes a translation by a distance $d$, proof that the sequence of motions $move(90^0, d), move(90^0, d)$, $move(90^0, d), move(90^0, d)$ executed in pose $p_{start}$ gets you into pose $p_{end}$ with $p_{start} = p_{end}$.

# Problem 2

homogeneous matrix in 2D

- rotation by angle $\alpha$
- and translation $(tx, ty)^T$

$$H = \begin{pmatrix} c\alpha & -s\alpha & tx \\ s\alpha & c\alpha & ty \\ 0 & 0 & 1 \end{pmatrix}$$

# Problem 2

note:

- it is first the rotation,
- then the translation already in the new orientation

rotation angle $\alpha$ , translation vector $(tx, ty)^T$

$$\mathrm{H}(\alpha, \mathrm{t}) = \begin{pmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\alpha & -s\alpha & tx \\ s\alpha & c\alpha & ty \\ 0 & 0 & 1 \end{pmatrix}$$

# Problem 2

wlog, start in the origin $(0,0)^T$ and just chain the motions

$$\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} d \\ d \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & d \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ d \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{pmatrix} \boldsymbol{qed}$$

# Problem 3

Suppose an object, e.g., the earth, has the pose $P_e$ and a 2nd object, e.g., the moon, with pose $P_m$ is rotating around it with angle $\theta$ around the z-axis of $P_e$.

What is the new pose of $P'_m$ for

$$\theta = 90^o, p_e = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, p_m = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 3: Notes – Consider 2D first

homogeneous matrix R’ for

- rotating by α around point $(x_{1,}y_1)^T$
- in frame *F*, i.e., ${}^F(x_{1,}y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

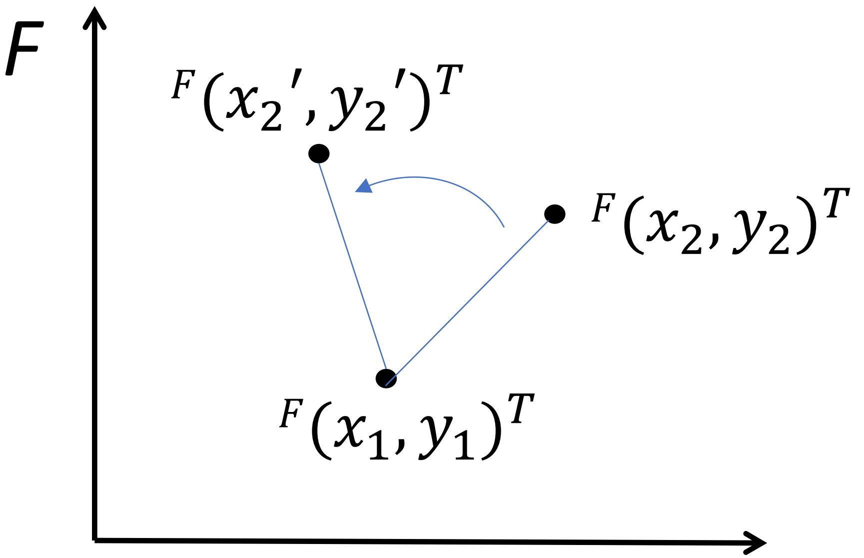
a) shift to origin
b) rotate
c) shift back

# Problem 3: Consider 2D first

rotate ${}^F(x_2, y_2)^T$ by α around point ${}^F(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$p_2' = \begin{pmatrix} x_2{}' \\ y_2{}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$



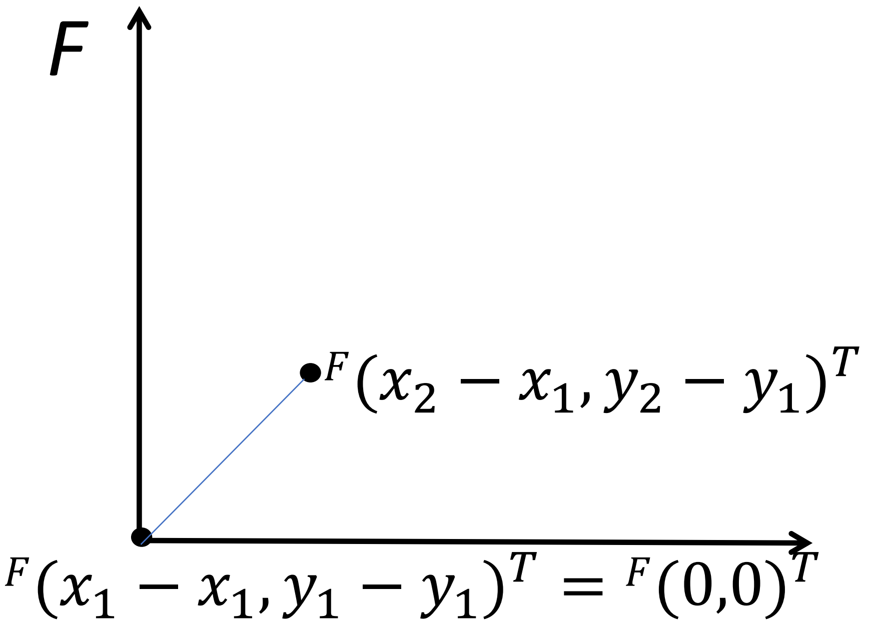
# Problem 3: Consider 2D first

rotate ${}^{F}(x_2, y_2)^T$ by α around point ${}^{F}(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}}$$

a) **shift to origin**
b) rotate
c) shift back

$$p_2' = \begin{pmatrix} {x_2}' \\ {y_2}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$



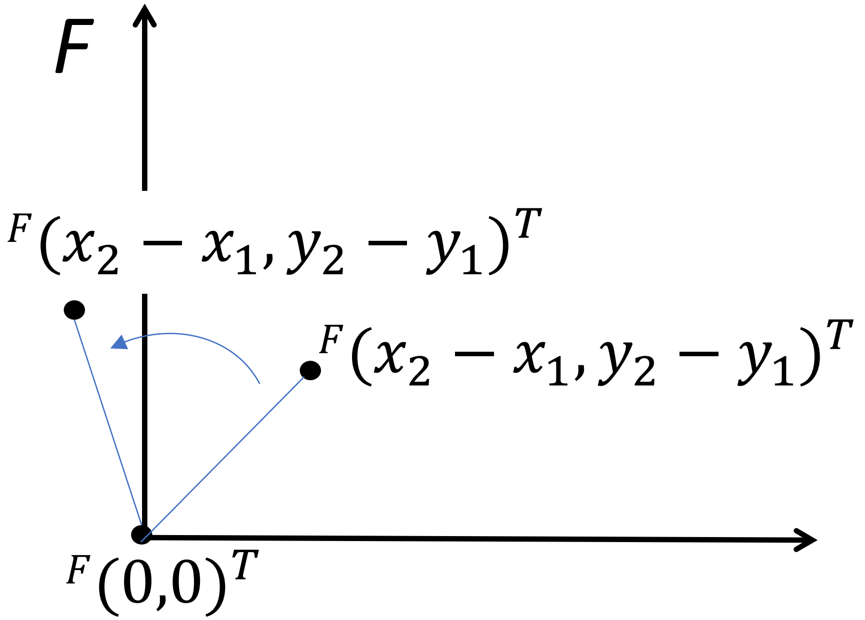
# Problem 3: Consider 2D first

rotate ${}^F(x_2, y_2)^T$ by α around point ${}^F(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) shift to origin
**b) rotate**
c) shift back

$$p_2' = \begin{pmatrix} {x_2}' \\ {y_2}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$



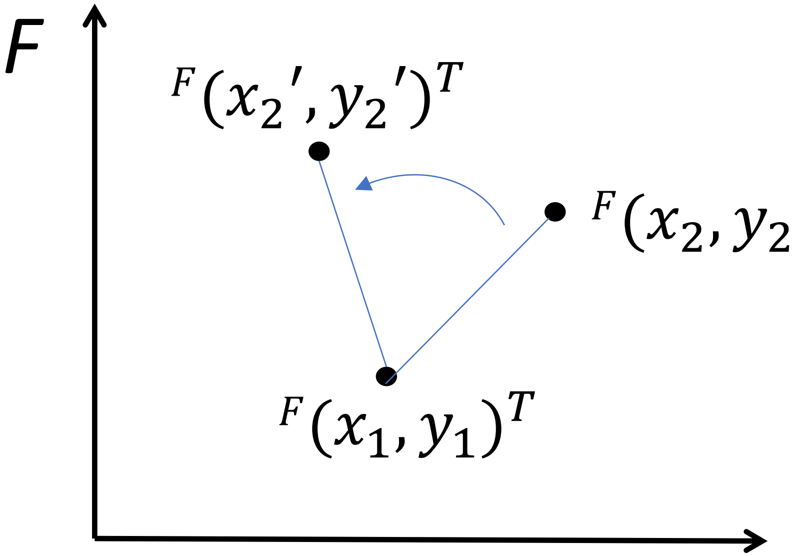
# Problem 3: Consider 2D first

rotate ${}^{F}(x_2, y_2)^T$ by α around point ${}^{F}(x_1, y_1)^T$

$$R' = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{pmatrix}$$

a) shift to origin
b) rotate
c) **shift back**

$$p_2' = \begin{pmatrix} x_2{}' \\ y_2{}' \\ 1 \end{pmatrix} = R' \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}$$



# Problem 3: Same Game in 3D

reference frame ${}^{F}F_0$
rotate a frame within ${}^{F}F_0$ by α with R(α)

$${}^{F}_{0}R' = {}^{F}F_0 \cdot {}^{F}R(\alpha) \cdot {}^{F}F_0{}^{-1}$$

i.e.,

- move to origin
- rotate
- move back

# Problem 3: Same Game in 3D

reference frame ${}^{F}F_0$
**“arbitrary” motion** within ${}^{F}F_0$, i.e., **homogeneous transform H**

$$ {}^{F_0}H' = {}^{F}F_0 \cdot H \cdot {}^{F}F_0{}^{-1} $$

i.e.,

- move to origin
- apply the motion
- move back

# Problem 3

$$p'_m = p_e \cdot R_z(90^o) \cdot {p_e}^{-1} \cdot p_m$$

$$p_e = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, p_m = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${p_e}^{-1} = \begin{pmatrix} & R^T & & -R^T t \\ & & & \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 3

$$p'_m = p_e \cdot R_z(90^o) \cdot {p_e}^{-1} \cdot p_m$$

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$, here $\lambda = 90^o, s\lambda = 1, c\lambda = 0$

$$R_z(90^o) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 3

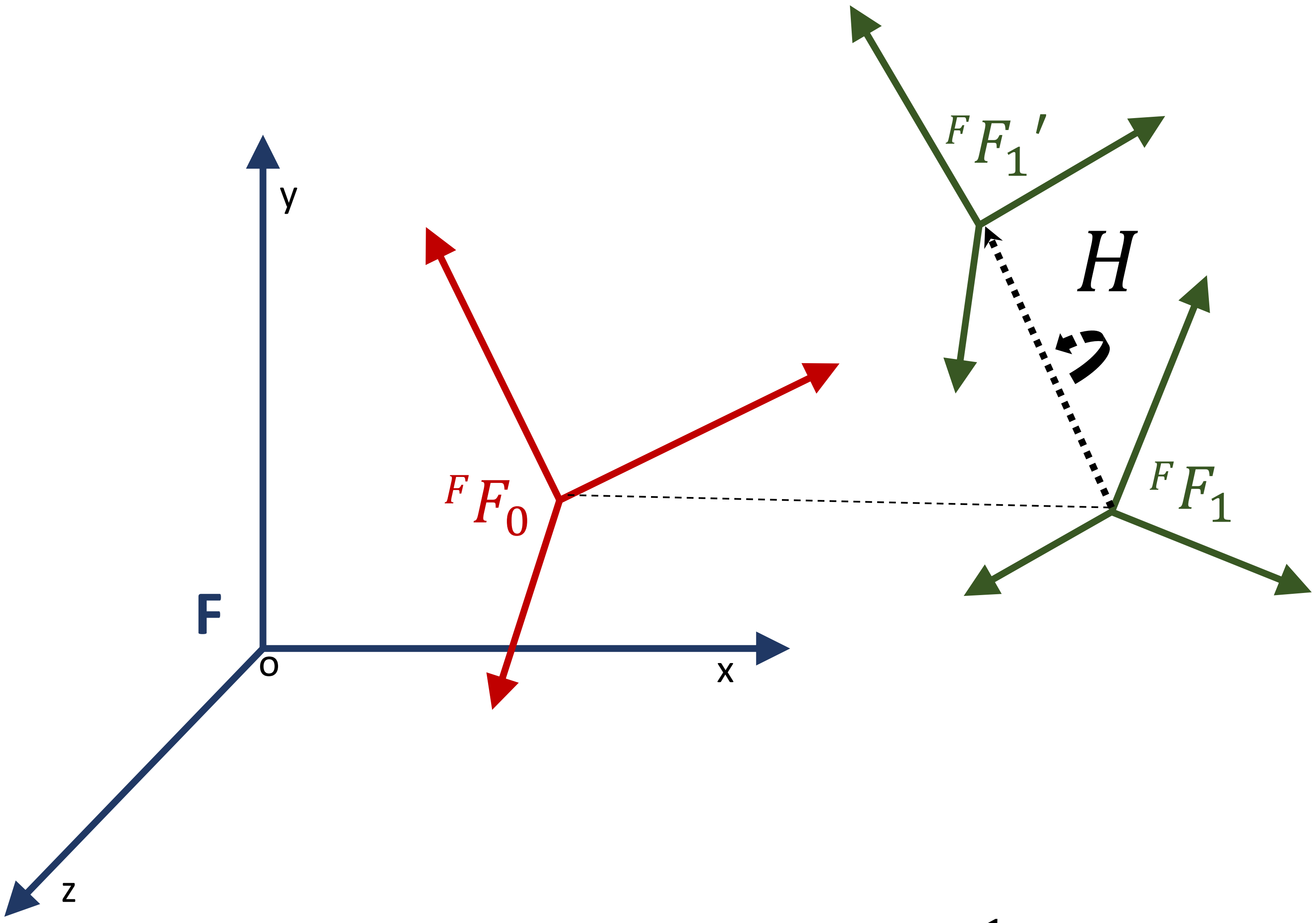
$$p'_m = p_e \cdot R_z(90^o) \cdot {p_e}^{-1} \cdot p_m$$

$$= \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 11 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 3




$$ {}^{F}F_1{}' = \left({}^{F}F_0 \cdot H \cdot {}^{F}F_0{}^{-1}\right){}^{F}F_1 $$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# Problem 4

Given a world-frame $F_w$ as identity matrix and an object with pose $P_o$ with

$$p_o = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Suppose the object rotates by $90^o$ around the z-axis of $F_w$. What is the new pose $P_o'$ of the object?
- Suppose world frame is an observer/sensor, who/which rotates by $90^o$ around its z-axis. What is the new pose $P_o'$ of the object?

# Problem 4

- object rotates: $p'_o = R_z(90^o) \cdot p_o$
- observer rotates: $p'_o = {R_z}^{-1}(90^o) \cdot p_o$

$$R_z = \begin{pmatrix} c\lambda & -s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ with here } \lambda = 90^o, s\lambda = 1, c\lambda = 0$$

$$R_z(90^o) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, R^{-1}{}_z(90^o) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 4

object rotates:

$$p'_o = R_z(90^o) \cdot p_o$$

$$= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

observer rotates:

$$p'_o = {R_z}^{-1}(90^o) \cdot p_o$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -4 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & -4 \\ 0 & 0 & -1 & -2 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# Problem 5

Given the quaternions

- $q_1 = (1, (2,3,4))$ and
- $q_2 = \ (0.4811480, (\ 0.1984591, 0.7246066, 0.4517253))$

Which of the two represents an orientation? And why?

# Problem 5

representation of orientation requires a **unit** quaternion $\hat{q} = (a, v)$

- quaternion norm: $|q| = \sqrt{q\ \bar{q}} = \sqrt{\bar{q}\ q} = \sqrt{a^2 + b^2 + c^2 + d^2}$
- with conjugate $\bar{q}$ of $q$ : $\bar{q} = (a, -v)$
- $q$ represents orientation $\Longrightarrow |q| = 1$

# Problem 5

$$
\begin{aligned}
|q_1| &= \sqrt{1^2 + 2^2 + 3^2 + 4^2} \\
&= \sqrt{1 + 4 + 9 + 16} \\
&= \sqrt{30} \\
&\approx 5.477
\end{aligned}
$$

$$
\begin{aligned}
|q_2| &= \sqrt{0.4811480^2 + 0.1984591^2 + 0.7246066^2 + 0.4517253^2} \\
&= \sqrt{0.231503 + 0.039386 + 0.525055 + 0.204056} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

# Problem 6

Given point $p = (2,3,4)^T$.

Use quaternions to rotate it

- by $30^o$ around the y-axis
- by $30^o$ around the axis $(1,-1,3)^T$
- first by $30^o$ the y-axis, then by $90^o$ around the axis $(1,-1,3)^T$

# Problem 6

**3D point** $p = (x, y, z)^T$

represented as quaternion $q = (s, v)$

- with scalar part $s = 0$ and vector part $v = (x, y, z)$
- i.e., $p = (0, v) = (0, (x, y, z))$

note: $p$ is typically *not* a unit quaternion

# Problem 6

rotate point $p$ (represented as quaternion)
by angle $\theta$ around unit axis $v$ to new location $p$'

with

$$p' = q \cdot p \cdot \bar{q}$$

using the rotation quaternion

$$q = \left( cos\left(\frac{\theta}{2}\right), sin\left(\frac{\theta}{2}\right) \cdot v \right)$$

# Problem 6

vector part notation as tupel, i.e., like row vector

$$q_i = (s_i, v_i) = \left(s_i, (v_{i.1}, v_{i.2}, v_{i.2})\right)$$

$$q_1 q_2 = (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

with

- $\mathbf{v}_1 \cdot \mathbf{v}_2 = \sum_{j=1}^{3} v_{1.j} v_{2.j}$
- $\mathbf{v}_1 \times \mathbf{v}_2 = \begin{pmatrix} v_{1.2} \cdot v_{2.3} - v_{1.3} \cdot v_{2.2} \\ v_{1.3} \cdot v_{2.1} - v_{1.1} \cdot v_{2.3} \\ v_{1.1} \cdot v_{2.2} - v_{1.2} \cdot v_{2.1} \end{pmatrix}$

vector part used like
spatial vector, i.e., column vector

# Problem 6: simple example

rotate $x = 2$ by $180^o$ around z-axis

$$p = (0, (2,0,0)^T)$$
$$\theta = 180^o, v = (0,0,1)^T \;\Rightarrow$$
$$q = (cos(\, 90^o), (0,0,1)^T \, sin(\, 90^o)) = (0, (0,0,1)^T)$$

$$p' = q\, p\, \bar{q} = (0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) \cdot (0, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}) \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix})$$

# Problem 6: simple example

$$p' = \boxed{\left(0, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) \cdot \left(0, \begin{pmatrix}2\\0\\0\end{pmatrix}\right)} \cdot \left(0, \begin{pmatrix}0\\0\\-1\end{pmatrix}\right)$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}2\\0\\0\end{pmatrix} = 0 + 0 + 0 = 0$$

$$\boxed{q_1 q_2} = (s_1 s_2 - \boxed{\mathbf{v}_1 \cdot \mathbf{v}_2}, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \boxed{\mathbf{v}_1 \times \mathbf{v}_2})$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}0-0\\2-0\\0-0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$\boxed{\left(0, \begin{pmatrix}0\\0\\1\end{pmatrix}\right) \cdot \left(0, \begin{pmatrix}2\\0\\0\end{pmatrix}\right)} = \left(0 \cdot 0 - \boxed{0}, \left(0 \begin{pmatrix}2\\0\\0\end{pmatrix} + 0 \begin{pmatrix}0\\0\\1\end{pmatrix} + \boxed{\begin{pmatrix}0\\2\\0\end{pmatrix}}\right)\right) = \left(0, \begin{pmatrix}0\\2\\0\end{pmatrix}\right)$$

# Problem 6: simple example

done

$= (0, \begin{pmatrix}0\\2\\0\end{pmatrix})$

next

$$p' = \boxed{(0, \begin{pmatrix}0\\0\\1\end{pmatrix}) \cdot (0, \begin{pmatrix}2\\0\\0\end{pmatrix})} \cdot (0, \begin{pmatrix}0\\0\\-1\end{pmatrix}) = \boxed{(0, \begin{pmatrix}0\\2\\0\end{pmatrix}) \cdot (0, \begin{pmatrix}0\\0\\-1\end{pmatrix})}$$

# Problem 6: simple example

$$p' = \boxed{(0, \begin{pmatrix}0\\2\\0\end{pmatrix}) \cdot (0, \begin{pmatrix}0\\0\\-1\end{pmatrix})}$$

$$\begin{pmatrix}0\\2\\0\end{pmatrix} \cdot \begin{pmatrix}0\\0\\-1\end{pmatrix} = 0$$

$$\begin{pmatrix}0\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}-2\\0\\0\end{pmatrix}$$

$$\boxed{q_1 q_2} = (s_1 s_2 - \boxed{\mathbf{v}_1 \cdot \mathbf{v}_2}, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \boxed{\mathbf{v}_1 \times \mathbf{v}_2})$$

$$\boxed{(0, \begin{pmatrix}0\\0\\1\end{pmatrix}) \cdot (0, \begin{pmatrix}2\\0\\0\end{pmatrix})} = (0 \cdot 0 - \boxed{0}, \left(0 \begin{pmatrix}0\\0\\-1\end{pmatrix} + 0 \begin{pmatrix}0\\2\\0\end{pmatrix} + \boxed{\begin{pmatrix}-2\\0\\0\end{pmatrix}}\right)) = (0, \begin{pmatrix}-2\\0\\0\end{pmatrix})$$

# Problem 6: simple example

rotate $x = 2$ by $180^o$ around z-axis

$$p = (0, (2,0,0)^T)$$
$$\theta = 180^o, v = (0,0,1)^T \;\Rightarrow$$
$$q = (cos(\,90^o), (0,0,1)^T\, sin(\,90^o)) = (0, (0,0,1)^T)$$

$$p' = q\; p\; \bar{q} = (0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) \cdot (0, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}) \cdot (0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}) = (0, \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix})$$

# Problem 6: rotate by 30 deg around y

$p = (2,3,4)^T \Leftrightarrow p = (0, (2,3,4)^T)$

$\theta_1 = 30^o, \mathbf{v}_1 = (0,1,0)^T$ :
$q_1 = (\cos(\,15^o), (0,1,0)^T \sin(\,15^o)) = (0.9659, (0,0.2588,0)^T)$

$$
\begin{aligned}
p_1 &= q_1\, p\, \bar{q}_1 \\
&= (0.9659, (0,0.2588,0)^T) \cdot (0, (2,3,4)^T \cdot (0.9659, (0, -0.2588,0)^T)
\end{aligned}
$$

# Problem 6 : rotate by 30 deg around y

1st quaternion multiplication
$q_1 p = (0.9659, (0,0.2588,0)^T) \cdot (0, (2,3,4)^T)$

dot & cross product of vector parts
$(0,0.2588,0)^T \cdot (2,3,4)^T = 0.77645714$
$(0,0.2588,0)^T \times (2,3,4)^T = (1.0352762, 0, -0.5176381)^T$

$$\begin{aligned} q_1 p &= (s_1, \mathbf{v}_1)(s_2, \mathbf{v}_2) \\ &= (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2) \\ &= (0 - 0.77645714, (0.9659 \cdot (2,3,4)^T + \mathbf{0} \\ &= (-0.77645714, (2.9671278, 2.89777748, 3.34606521)^T) \end{aligned}$$

# Problem 6 : rotate by 30 deg around y

2nd quaternion multiplication

$(q_1 p)\bar{q}_1 = (-0.7765, (2.9671, 2.8978, 3.3460) \cdot (0.9659, (0, -0.2588, 0))$

dot & cross product of vector parts

$(2.9671, 2.8978, 3.3460) \cdot (0, -0.2588, 0) = -0.75$

$(2.9671, 2.8978, 3.3460)^T \times (0, -0.2588, 0)^T = (0.8660254, 0, -0.7679492)^T$

$$\begin{aligned}(q_1 p)\bar{q}_1 &= (s_1, \mathbf{v}_1)(s_2, \mathbf{v}_2)\\ &= (s_1 s_2 - \mathbf{v}_1 \cdot \mathbf{v}_2, s_1 \mathbf{v}_2 + s_2 \mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)\\ &= (0, (3.7320508, 3, 2.4641016)^T)\\ &\approx (0, (3.73, 3, 2.46)^T)\end{aligned}$$

# Problem 6 : rotate by 30 deg around (1,-1,3)

$\theta_2 = 30^o, \mathbf{v}_2 = (1, -1,3)^T$

normalize $\mathbf{v}_2$!!!

- $|\mathbf{v}_2| = 3.31662479$
- $\hat{\mathbf{v}}_2 = (0.30151134, -0.301511,0.90453403)^T$

$$q_2 \approx (\cos(15^o), (0.3015, -0.3015,0.9045)^T \sin(15^o))$$
$$\approx (0.9659, (0.0780369, -0.0780369,0.23411063)^T)$$

$$p_2 = q_2\, p\, \bar{q}_2$$

# Problem 6 : rotate by 30 deg around (1,-1,3)

$$
\begin{aligned}
p_2 &= q_2\, p\, \bar{q}_2 \\
&= (0.9659, (0.0780, -0.0780, 0.2341)^T) \cdot (0, (2,3,4)^T) \\
&\quad \cdot \left(0.9659, (-0.0780, 0.0780, -0.2341)^T\right) \\
&= (-0.8584, (0.9173, 3.0538, 4.2539)^T) \\
&\quad \cdot \left(0.9659, (-0.0780, 0.0780, -0.2341)^T\right) \\
&= (0, (-0.093798, 2.76561296, 4.6198038)^T)
\end{aligned}
$$

# Problem 6: chained rotations

rotate $p$

- first by 30 deg around y
- then by 30 deg around $(1, -1, 3)^T$

$p = (0, (2,3,4)^T)$
$q_1 = (0.9659, (0, 0.2588, 0)^T)$
$q_2 = (0.9659, (0.0780, -0.0780, 0.2341)^T)$

option 1: $p_3 = q_2 \cdot (q_1 \cdot p \cdot \bar{q}_1) \cdot \bar{q}_2$ [4 quat.mult.]
option 2: $q_3 = q_2 \cdot q_1, p_3 = q_3 \cdot p \cdot \bar{q}_3$ [3 quat.mult.] better

# Problem 6: chained rotations

**why does this chaining work?**

$$\overline{\boldsymbol{q_1} \cdot \boldsymbol{q_2}} = \bar{\boldsymbol{q}}_2 \cdot \bar{\boldsymbol{q}}_1$$

- i.e., order of multiplications is swapped in this case
- proof by using definitions of conjugate and quaternion multiplication

but note that in general: $q_1 \cdot q_2 \neq q_2 \cdot q_1$
(quaternion multiplication is not commutative)

# Problem 6: chained rotations

why does this chaining work?

option 1

$$
\begin{aligned}
q_2 \cdot (q_1 \cdot p \cdot \bar{q}_1) \cdot \bar{q}_2 &= q_2 \cdot q_1 \cdot p \cdot \bar{q}_1 \cdot \bar{q}_2 \\
&= q_2 \cdot q_1 \cdot p \cdot \overline{q_2 \cdot q_1} \\
&= q_3 \cdot p \cdot \bar{q}_3 \text{ (with } q_3 = q_2 \cdot q_1\text{)}
\end{aligned}
$$

option 2

# Problem 6: chained rotations

rotate $p$

- first by 30 deg around y
- then by 30 deg around $(1,-1,3)^T$

$p = (0,(2,3,4)^T)$
$q_1 = (0.9659,(0,0.2588,0)^T)$
$q_2 = (0.9659,(0.0780,-0.0780,0.2341)^T)$

option 2: $q_3 = q_2 \cdot q_1, p_3 = q_3 \cdot p \cdot \bar{q}_3$

# Problem 6: chained rotations

rotate $p$: first by 30 deg around y, then by 30 deg around $(1,-1,3)^T$

option 2:

$$\begin{aligned} q_3 &= q_2 \cdot q_1 \\ &= (0.9659, (0.0780, -0.0780, 0.2341)^T) \cdot (0.9659, (0, 0.2588, 0)^T \\ &= (0.9532, (0.0148, 0.1746, 0.2463)^T) \end{aligned}$$

$$\begin{aligned} p_3 &= (0.9532, (0.0148, 0.1746, 0.2463)^T) \cdot (0, (2,3,4)^T \\ &\quad \cdot (0.9532, (-0.0148, -0.1746, -0.2463)^T \\ &= (0, (1.6027, 3.8155, 3.4457)^T) \end{aligned}$$

# Problem 7

Use the Rodrigues formula

- to rotate $p \ = \ (2,3,4)^T$
- by $30^o$ around the axis $(1,-1,3)^T$.

# Problem 7

- rotate $v \;=\; (2,3,4)^T$
- by angle $\theta = \; 30^o$
- around a normalized axis $\boldsymbol{k}$ generated from $\boldsymbol{k}' = (1,-1,3)^T$

Rodrigues formula

$$\mathbf{v}' = \; \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)$$

# Problem 7

normalize: $\boldsymbol{k}' = (1, -1,3)^T$

- $|\boldsymbol{k}'| = 3.31662479$
- $\boldsymbol{k} = {}^{\boldsymbol{k}'}/_{|\boldsymbol{k}'|} = (0.30151134, -0.301511,0.90453403)^T$

# Problem 7

$$v' = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \mathrm{c}\,30^o + \left( \begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) \mathrm{s}\,30^o + \begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \left( \begin{pmatrix} 0.3015 \\ -0.3015 \\ 0.9045 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right) (1 - \mathrm{c}\,30^o)$$

$$= \begin{pmatrix} 1.73205081 \\ 2.5980762 \\ 3.46410162 \end{pmatrix} + \begin{pmatrix} -1.9598237 \\ 0.3015113 \\ 0.75377836 \end{pmatrix} + \begin{pmatrix} 0.1339746 \\ -0.133975 \\ 0.40192379 \end{pmatrix}$$

$$= \begin{pmatrix} -0.0937983 \\ 2.765613 \\ 4.61980377 \end{pmatrix}$$