Robotics

PS07 – Solutions

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Part 7: Localization

Problem 1

Given 4 beacons b_i at 4 known positions (x_i, y_i) in the plane as follows:

i =	x_i	y_i
1	35	40
2	12	23
3	7	18
4	9	9

A robot r at an unknown position $p_r = (x_r, y_r)$ has a sensor to measure the distances to the beacons. Use multilateration to determine p_r when the ranges $D(b_i, r)$ between all beacons and the robot are as follows:

	b_1	b_2	b_3	b_4
$D(b_i, r)$	37	9	2	10

Problem 1

Multilateration as Linear Least Squares: $Ax = b$ with

$$A = \begin{pmatrix} 2(x_1 - x_n) & 2(y_1 - y_n) \\ 2(x_2 - x_n) & 2(y_2 - y_n) \\ & \cdots \\ 2(x_{n-1} - x_n) & 2(y_{n-1} - y_n) \end{pmatrix}$$

$$b = \begin{pmatrix} {x_1}^2 - {x_n}^2 + {y_1}^2 - {y_n}^2 - {d_1}^2 + {d_n}^2 \\ {x_2}^2 - {x_n}^2 + {y_2}^2 - {y_n}^2 - {d_2}^2 + {d_n}^2 \\ \vdots \\ {x_{n-1}}^2 - {x_n}^2 + {y_{n-1}}^2 - {y_n}^2 - {d_{n-1}}^2 + {d_n}^2 \end{pmatrix}$$

compute LLS fit $x^* = (x, y)^T$ with $x^* = (A^T A)^{-1} A^T b = A^+ b$

Problem 1

i =	x_i	y_i
1	35	40
2	12	23
3	7	18
4	9	9

	b_1	b_2	b_3	b_4
$D(b_i, r)$	37	9	2	10

$$A = \begin{pmatrix} 2(35-9) & 2(40-9) \\ 2(12-9) & 2(23-9) \\ 2(7-9) & 2(18-9) \end{pmatrix} = \begin{pmatrix} 52 & 62 \\ 6 & 28 \\ -4 & 18 \end{pmatrix}$$

$$b = \begin{pmatrix} 35^2 - 9^2 + 40^2 - 9^2 - 37^2 + 10^2 \\ 12^2 - 9^2 + 23^2 - 9^2 - 9^2 + 10^2 \\ 7^2 - 9^2 + 18^2 - 9^2 - 2^2 + 10^2 \end{pmatrix} = \begin{pmatrix} 1394 \\ 530 \\ 307 \end{pmatrix}$$

Problem 1

$$A = \begin{pmatrix} 52 & 62 \\ 6 & 28 \\ -4 & 18 \end{pmatrix} = USV^T$$

$$U = \begin{pmatrix} -0.941 & 0.309 \\ -0.306 & -0.610 \\ -0.143 & -0.729 \end{pmatrix}, S = \begin{pmatrix} 85.737 & 0.000 \\ 0.000 & 18.898 \end{pmatrix}, V^T = \begin{pmatrix} -0.586 & -0.811 \\ 0.811 & -0.586 \end{pmatrix}$$

$$A^+ = VS^+U^T = \begin{pmatrix} 0.01968 & -0.02409 & -0.03031 \\ -0.00067 & 0.02181 & 0.02395 \end{pmatrix}$$

Problem 1

$$
\begin{aligned}
x^* &= A^+ b \\
&= \begin{pmatrix} 0.01968 & -0.02409 & -0.03031 \\ -0.00067 & 0.02181 & 0.02395 \end{pmatrix} \begin{pmatrix} 1394 \\ 530 \\ 307 \end{pmatrix} \\
&= \begin{pmatrix} 5.361 \\ 17.978 \end{pmatrix}
\end{aligned}
$$

estimated robot location p_r

Problem 2

Given a sensor network with 4 nodes p_i at 4 positions (x_i, y_i) in the plane. The ranges $D(p_i, p_j)$ between the nodes are given as follows:

$D(p_i, p_j)$	p_1	p_2	p_3	p_4
p_1	0.00	50.00	44.72	31.62
p_2	50.00	0.00	60.83	22.36
p_3	44.72	60.83	0.00	58.31
p_4	31.62	22.36	58.31	0.00

Determine the 4 positions (x_i, y_i) of the nodes via MDS using the step by step algorithm from the lecture, i.e.,:

- compute a suited matrix A from the distance matrix D
- double center A (getting B)
- use SVD on B
- find a good rank approximation B'
- get the locations X

Problem 2

Multidimensional Scaling (MDS) algorithm

1. initialize matrix A $\quad A = -\frac{1}{2} D^{(2)}$
2. compute B with double-centering $\quad B = CAC$
3. compute SVD of B $\quad B = VLV^T$
4. get X $\quad X = VL^{(1/2)}$

Problem 2

$D(p_i, p_j)$	p_1	p_2	p_3	p_4
p_1	0.00	50.00	44.72	31.62
p_2	50.00	0.00	60.83	22.36
p_3	44.72	60.83	0.00	58.31
p_4	31.62	22.36	58.31	0.00

1. initialize A

$$A = -\frac{1}{2} D^{(2)} = -\frac{1}{2} \begin{pmatrix} 0 & 50.0 & 44.72 & 31.62 \\ 50.0 & 0 & 60.83 & 22.36 \\ 44.72 & 60.83 & 0 & 58.31 \\ 31.62 & 22.36 & 58.31 & 0 \end{pmatrix}^{(2)}$$

$$= \begin{pmatrix} 0 & -1250 & -1000 & -500 \\ -1250 & 0 & -1850 & -250 \\ -1000 & -1850 & 0 & -1700 \\ -500 & -250 & -1700 & 0 \end{pmatrix}$$

Problem 2

2. compute $\boldsymbol{B}$ by double-centering $\boldsymbol{A}$

4×4 centering matrix

$$C_{(4)} = I - \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Problem 2

2. compute B by double-centering A

$B = C_{(4)} A C_{(4)}$

$$= \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 0 & -1250 & -1000 & -500 \\ -1250 & 0 & -1850 & -250 \\ -1000 & -1850 & 0 & -1700 \\ -500 & -250 & -1700 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$= \begin{pmatrix} 556.25 & -543.75 & 6.25 & -18.75 \\ -543.75 & 856.25 & -693.75 & 381.25 \\ 6.25 & -693.75 & 1456.25 & -768.75 \\ -18.75 & 381.25 & -768.75 & 406.25 \end{pmatrix}$$

Problem 2

3. compute SVD of $\boldsymbol{B}$

$$B = \begin{pmatrix} 556.25 & -543.75 & 6.25 & -18.75 \\ -543.75 & 856.25 & -693.75 & 381.25 \\ 6.25 & -693.75 & 1456.25 & -768.75 \\ -18.75 & 381.25 & -768.75 & 406.25 \end{pmatrix} = VLV^T$$

$$V = \begin{pmatrix} -0.162 & -0.728 & 0.120 & -0.655 \\ 0.509 & 0.516 & 0.149 & -0.672 \\ -0.746 & 0.407 & -0.402 & -0.342 \\ 0.398 & -0.195 & -0.895 & -0.046 \end{pmatrix}, L = \begin{pmatrix} 2342.061 & 0.000 & 0.000 & 0.000 \\ 0.000 & 932.939 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

here: no noise, 2D => 2 singular values

Problem 2

4. get X

$$V = \begin{pmatrix} -0.162 & -0.728 & 0.120 & -0.655 \\ 0.509 & 0.516 & 0.149 & -0.672 \\ -0.746 & 0.407 & -0.402 & -0.342 \\ 0.398 & -0.195 & -0.895 & -0.046 \end{pmatrix}$$

$$L^{\left(\frac{1}{2}\right)} = \begin{pmatrix} \sqrt{2342.061} & 0.000 & 0.000 & 0.000 \\ 0.000 & \sqrt{932.939} & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$= \begin{pmatrix} 48.39484 & 0.000 & 0.000 & 0.000 \\ 0.000 & 30.54405 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

these parts can be omitted, i.e., reduce V and $L^{\left(\frac{1}{2}\right)}$ to 2 columns

Problem 2

4. get X

$$X = VL^{\left(\frac{1}{2}\right)} = \begin{pmatrix} -7.840 & -22.236 & \boxed{\begin{matrix} 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \\ 0.000 & 0.000 \end{matrix}} \\ 24.633 & 15.761 & \\ -36.103 & 12.431 & \\ 19.261 & -5.956 & \end{pmatrix}$$

$$X = VL^{\left(\frac{1}{2}\right)} = \begin{pmatrix} -0.162 & -0.728 \\ 0.509 & 0.516 \\ -0.746 & 0.407 \\ 0.398 & -0.195 \end{pmatrix} \begin{pmatrix} 48.39484 & 0.000 \\ 0.000 & 30.54405 \end{pmatrix} = \begin{pmatrix} -7.840 & -22.236 \\ 24.633 & 15.761 \\ -36.103 & 12.431 \\ 19.261 & -5.956 \end{pmatrix}$$

V and $L^{\left(\frac{1}{2}\right)}$ reduced to 2 columns

Problem 2

4. get X

x_i , y_i of the nodes

$$X = VL^{\left(\frac{1}{2}\right)} = \begin{pmatrix} -7.840 & -22.236 \\ 24.633 & 15.761 \\ -36.103 & 12.431 \\ 19.261 & -5.956 \end{pmatrix}$$

node p_i

Problem 2

example with noise

$$A = -\frac{1}{2}D^{(2)} = -\frac{1}{2}\begin{pmatrix} 0.00 & 51.60 & 44.20 & 31.62 \\ 51.60 & 0.00 & 63.83 & 19.36 \\ 44.20 & 63.83 & 0.00 & 59.31 \\ 31.62 & 19.36 & 59.31 & 0.00 \end{pmatrix}^{(2)}$$

$$= \begin{pmatrix} 0 & -1331.28 & -976.82 & -500 \\ -1331.28 & 0 & -2036.98 & -187.418 \\ -976.82 & -2036.98 & 0 & -1758.81 \\ -500 & -187.418 & -1758.81 & 0 \end{pmatrix}$$

Problem 2

example with noise

$$B = C_{(4)} A C_{(4)}$$

$$= \begin{pmatrix} 555.1362 & -589.249 & 69.4443 & -35.3319 \\ -589.249 & 928.9266 & -803.823 & 464.1453 \\ 69.4443 & -803.823 & 1537.392 & -803.013 \\ -35.3319 & 464.1453 & -803.013 & 374.1999 \end{pmatrix}$$

$$= VLV^T$$

Problem 2

example with noise:

- 2D problem => consider only 2 largest singular values in L
- and corresponding columns in V

$$V = \begin{pmatrix} -0.188 & -0.729 & -0.428 & 0.500 \\ 0.534 & 0.485 & -0.478 & 0.500 \\ -0.729 & 0.441 & 0.155 & 0.500 \\ 0.383 & -0.197 & 0.751 & 0.500 \end{pmatrix}, L = \begin{pmatrix} 2566.489 & 0.000 & 0.000 & 0.000 \\ 0.000 & 895.942 & 0.000 & 0.000 \\ 0.000 & 0.000 & 66.776 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix}$$

$$\Longrightarrow V = \begin{pmatrix} -0.188 & -0.729 \\ 0.534 & 0.485 \\ -0.729 & 0.441 \\ 0.383 & -0.197 \end{pmatrix}, L^{\left(\frac{1}{2}\right)} = \begin{pmatrix} 50.661 & 0.000 \\ 0.000 & 29.932 \end{pmatrix}$$

Problem 2

example with noise:

- 2D problem => consider only 2 largest singular values in L
- and corresponding columns in V

$$X = VL^{\left(\frac{1}{2}\right)} = \begin{pmatrix} -0.188 & -0.729 \\ 0.534 & 0.485 \\ -0.729 & 0.441 \\ 0.383 & -0.197 \end{pmatrix} \cdot \begin{pmatrix} 50.661 & 0.000 \\ 0.000 & 29.932 \end{pmatrix}$$

$$= \begin{pmatrix} -9.524 & -21.821 \\ 27.053 & 14.517 \\ -36.932 & 13.200 \\ 19.403 & -5.897 \end{pmatrix}$$