Communications Basics
Lecture 2

Analytic Signals

Orga

Textbook:

Rodger E. Ziemer & William H. Tranter, Principles of Communications **… READ!**

Our content comes from chps. 2-7 (according to edition 5)

Chp. 2 (Signals & Systems – separate course) … **polish your Fourier transforms**

Chp. 3 (Modulation + Demodulation)

Chp. 4 (Probability – separate course) … roll a couple of dice ☺ **… we'll bascially need Random variables, (multivariate) Gaussians, expected values, variances, covariances**

Chp. 5 (Random Processes & Noise)

Chp. 6 (Noise in Modulation Systems)

Chp. 7 (Binary Data Transmission)

Main platform: campusnet … course page !!!

Teaching in person … slides will be on campusnet … but make sure to take your own notes!

TA: Yasmine Ammouze … tutorials

Exam: Written, no cheat sheets (expected end of January), 2 hours, details as announced by the registrar (should show in campusnet)

Lecture 2

Digestion Guide

Did you digest Lecture 1?

Euler‘s formula

Trigonometric identities like $cos^2(\omega t) = \frac{1}{2} + \frac{1}{2}cos(2\omega t)$

How to find those identities?

Calculate:

1) What happens, when we use coherent demodulation for a DSB signal, if the local oscillator‘s frequency is wrong? Consider two cases: just a bit / way off ... how much is ``a bit´´ ?
   How can that happen?
2) What happens, if the local oscillator‘s frequency is right but its phase is wrong? What is a relevant difference? How can this happen?

For coherent demodualtion of DSB signals, would it help to (additionally) transmit the carrier signal alone? How could this be done?

Make sure you understand your Fourier transforms (recap in our textbook, chp. 2)

Start reading chp. 3 of our textbook (DSB + AM)

Lecture 2

Some formulas

Convolution with a delta-peak in time domain:

$$\delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(t-\tau)x(\tau)\,d\tau = x(t)$$

Convolution with a delta-peak in frequency domain:

$$\delta(f) * X(f) = \int_{-\infty}^{\infty} \delta(f-s)X(s)\,ds = X(f)$$

Euler

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}, \qquad \sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi$$

Lecture 2

Euler

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}, \qquad \sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

Use Euler

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi$$

Find trigonometric identity:

$$\cos^2\varphi = \frac{\left(e^{j\varphi} + e^{-j\varphi}\right)^2}{4}$$

$$= \frac{e^{j2\varphi} + 2 + e^{-2j\varphi}}{4}$$

$$= \frac{2}{4} + \frac{e^{j2\varphi} + e^{-2j\varphi}}{4} = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$$

Lecture 2

Euler

$$\cos\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}, \qquad \sin\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}$$

Use Euler

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$e^{-j\varphi} = \cos\varphi - j\sin\varphi$$

Modulation:

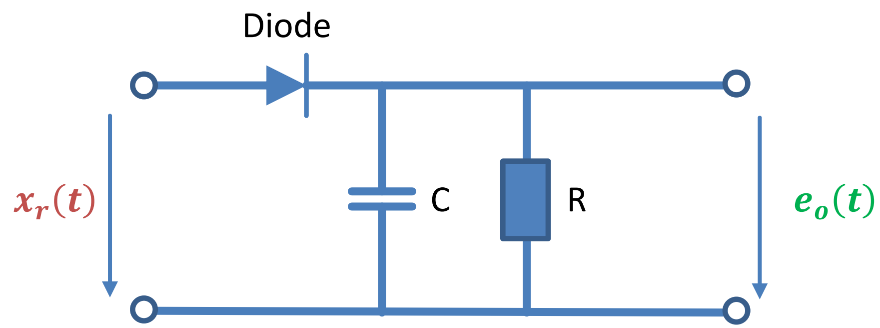
$$\cos\omega_m t \cos\omega_c t = \frac{e^{j\omega_m t} + e^{-j\omega_m}}{2} \cdot \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$= \frac{e^{j(\omega_m+\omega_c)t} + e^{-j(\omega_m+\omega_c)t}}{4} + \frac{e^{j(\omega_m-\omega_c)t} + e^{-j(\omega_m-\omega_c)t}}{4}$$

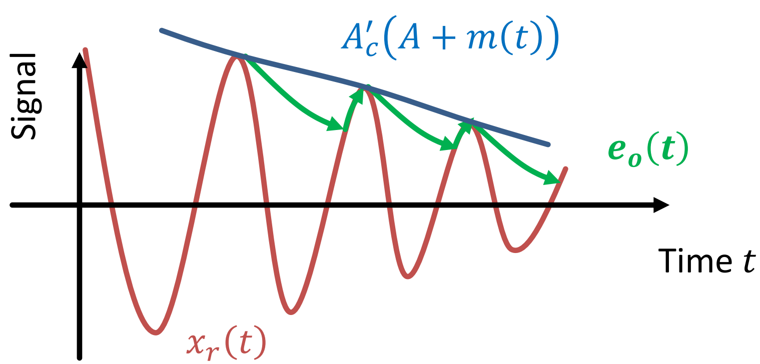
$$= \frac{1}{2}\cos(\omega_m + \omega_c)t + \frac{1}{2}\cos(\omega_m - \omega_c)t = \frac{1}{2}\cos(\omega_c + \omega_m)t + \frac{1}{2}\cos(\omega_c - \omega_m)t$$

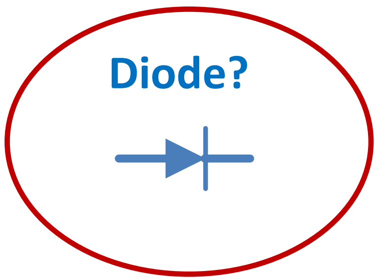
Realize: Sums and differences

Envelopes – Making Signals Analytic



Demodulation
(Envelope detection for AM)





Reminder:

Envelopes are interesting!

For a mathematical access …

… we need a formal concept:

envelope of a signal $x(t)$

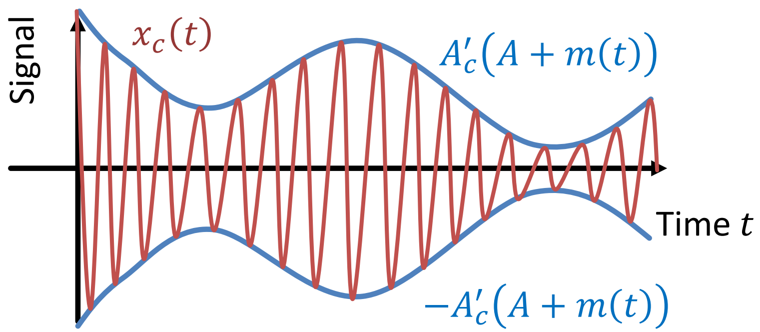
Lecture 2

Envelopes – Making Signals Analytic

Envelope of a sinusoid … and others

$x(t) = A\,cos(\omega_c t)$ ➔ envelope is $|A|$

How to express $A_c'\big(A + m(t)\big)$ as the envelope of $x_c(t)$?



Envelopes – Making Signals Analytic

Envelope of a sinusoid … and others – The trick:

Interpret via complex signals:

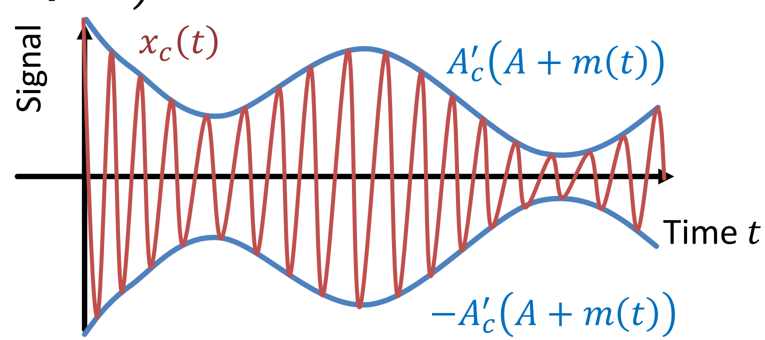
$$x(t) = A\,cos(\omega_c t) = \Re\big(A(cos(\omega_c t) + j\,sin(\omega_c t))\big) = \Re\left(\underbrace{Ae^{j\omega_c t}}_{=x_p(t)}\right)$$

Also, interpret: (real) envelope $|A| = \left|Ae^{j\omega_c t}\right|$

Envision a similar approach for $x_c(t)$: $x_c(t) = \Re\left(x_{c,p}(t)\right)$

Such that we obtain $A_c'\big(A + m(t)\big) = \left|x_{c,p}(t)\right|$

How to find a suitable imaginary signal to complement $x_c(t)$?



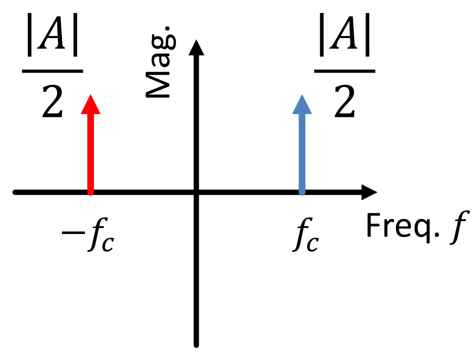
Envelopes – Making Signals Analytic

Envelope of a sinusoid … and others – The trick:

$$x(t) = A\,cos(\omega_c t) = \Re\big(A(cos(\omega_c t) + j\,sin(\omega_c t))\big) = \Re\left(\underbrace{Ae^{j\omega_c t}}_{=x_p(t)}\right)$$

Also, interpret: Envelope $|A| = \left|Ae^{j\omega_c t}\right|$

Another perspective on the same approach for



$$x(t) = A\,cos(\omega_c t) = A\,\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$x_p(t) = Ae^{j\omega_c t} = \text{``}twice\ the\ positive\ frequency\ component, only\text{''}$$

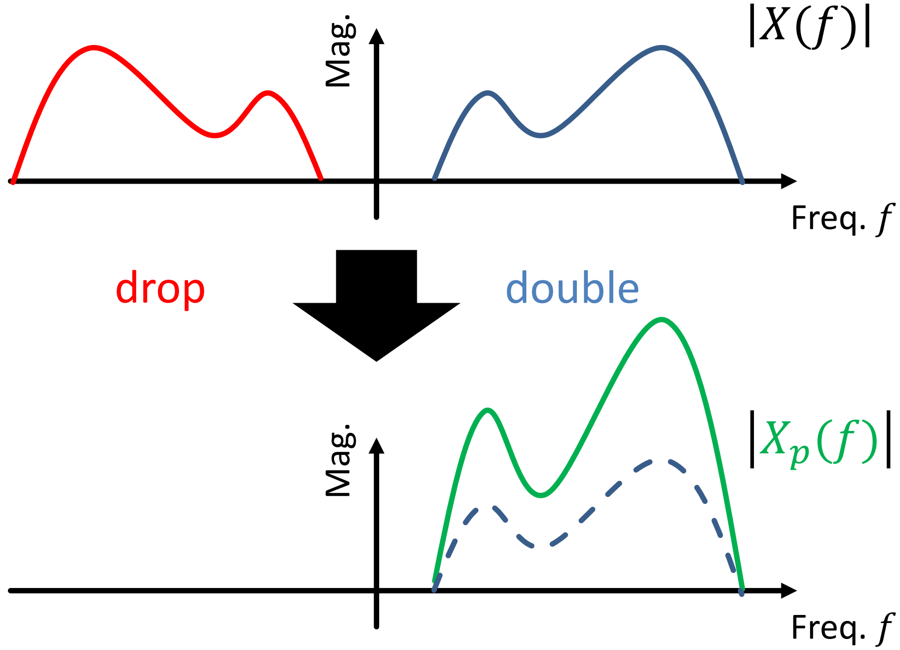
This can be generalized … if we go for a linear transformation ☺

Envelopes – Making Signals Analytic

Envelope – The trick:

Consider the Fourier transfrom $X(f)$ of a real signal $x(t)$, and its magnitude $|X(f)|$.

$$X_p(f) = 2 \cdot \text{``}\mathit{positive\ frequency\ components\ of}\ X(f)\text{''}$$



Envelopes – Making Signals Analytic

Envelope – The trick:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$X_p(f) = 2 \cdot \text{``}\textit{positive frequency components of } X(f)\text{''}$$

$$\Rightarrow X_p(f) = X(f)\big(1 + sgn(f)\big) = X(f) + sgn(f)X(f)$$

$$= \begin{cases} 0\,, & for\ neg.\ frequencies \\ 2X(f)\,, & for\ pos.\ frequencies \end{cases}$$

Mind: Components at $f = 0$ would not be changed.

In this course, we usually avoid those, anyway … well, except for AM

Lecture 2

Envelopes – Making Signals Analytic

Envelope – The trick:

$\Re\{X(f)\}$ even ➔ $\Re\{\underbrace{sgn(f)X(f)}\}$ is odd
$\Im\{X(f)\}$ is odd ➔ $\Im\{sgn(f)X(f)\}$ is even

What does that mean? … Where is the complex signal?

Mind: real signal ←→ $\Re\{X(f)\}$ is even, *and* $\Im\{X(f)\}$ is odd

im signal ←→ flipped symmetries

$$X_p(f) = X(f) + \underbrace{sgn(f)X(f)}_{\substack{corresponds\ to \\ an\ imaginary\ signal}} = X(f) + j\left[\underbrace{-j \cdot sgn(f)X(f)}_{\substack{corresponds\ to \\ a\ real\ signal}}\right]$$

Back to time domain … What do we expect?

$$x_p(t) = x(t) + j\hat{x}(t) = real\ part + j \cdot imaginary\ part$$

$$\hat{x}(t) = -j \int_{-\infty}^{\infty} sgn(f)X(f)e^{j2\pi ft}df$$

Envelopes – Making Signals Analytic

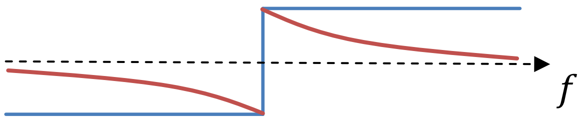
Envelope – The trick:

$$\hat{x}(t) = -j \int_{-\infty}^{\infty} sgn(f) X(f) e^{j2\pi f t} df$$

Product in the frequency domain → Convolution in the time domain …

$$\int_{-\infty}^{\infty} sgn(f) e^{j2\pi f t} df = ??$$

Well, this does not exist in the usual sense.
Technically, this can be remedied via distributions!



Here is a motivation: Consider approximations, $\alpha > 0, \alpha \to 0$: $sgn(f) \approx \begin{cases} +e^{-\alpha f}, & f > 0 \\ -e^{+\alpha f}, & f < 0 \end{cases}$

$$-\int_{-\infty}^{0} e^{\alpha f} e^{j2\pi f t} df + \int_{0}^{\infty} e^{-\alpha f} e^{j2\pi f t} df = -\frac{1}{\alpha + j2\pi t} - \frac{1}{-\alpha + j2\pi t} = \frac{j4\pi t}{\alpha^2 + (2\pi t)^2}$$

Envelopes – Making Signals Analytic

Envelope – The trick:

Hence,

$$\int_{-\infty}^{\infty} sgn(f)e^{j2\pi ft}df = \lim_{\alpha\to 0}\frac{j4\pi t}{\alpha^2+(2\pi t)^2} = \frac{j}{\pi t}$$

That is,

$$-\mathrm{j}\cdot sgn(f) \leftrightarrow \frac{1}{\pi t}$$

And

$$\hat{x}(t) = -j\int_{-\infty}^{\infty} sgn(f)X(f)e^{j2\pi ft}dt = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(t-t')}{t'}dt'$$

$\hat{x}(t) = \mathcal{H}\big(x(t)\big)$ is usually called the Hilbert transform of $x(t)$.

Envelopes – Making Signals Analytic

Envelope – The trick:

Combined: In order to make a given real signal $x(t)$ analytic,
complement it by an imaginary part $\hat{x}(t)$ which is its Hilbert transform.

$$x_p(t) = x(t) + j\hat{x}(t)$$

Notice:

By construction, $\hat{x}(t)$ itself, resulting from a real integration, is again a real signal.

Now, the desired (real) envelope of $x(t)$ is $\left|x_p(t)\right| = \sqrt{x^2(t) + \hat{x}^2(t)}$

Envelopes – Making Signals Analytic

Examples:

Consider

1) $x(t) = A\cos(\omega t)$
2) $x(t) = A\sin(\omega t)$

Solve:

1) $x(t) = A\cos(\omega t) = A\frac{e^{j\omega t}+e^{-j\omega t}}{2}$ (use freq. dom.)

If $\omega > 0$

$$\Rightarrow x_p(t) = A\,e^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t)$$

$$\Rightarrow \hat{x}(t) = \Im\left(x_p(t)\right) = \mathrm{A}\sin(\omega t)$$

$$\text{Real envelope} = \left|x_p(t)\right| = \left|A\,e^{j\omega t}\right| = |A|$$

$$= |A|\sqrt{cos^2(\omega t) + sin^2(\omega t)}$$

Envelopes – Making Signals Analytic

Examples:

Consider

1) $\mathrm{x}(t) = A\cos(\omega t)$
2) $\mathrm{x}(t) = A\sin(\omega t)$

Solve:

2) $\mathrm{x}(t) = A\sin(\omega t) = A\frac{e^{j\omega t} - e^{-j\omega t}}{2j}$ (use freq. dom.)

If $\omega > 0$

$$\Rightarrow x_p(t) = -jA\, e^{j\omega t} = A\sin(\omega t) - j\mathrm{A}\cos(\omega t)$$

$$\Rightarrow \hat{x}(t) = \Im\left(x_p(t)\right) = -\mathrm{A}\cos(\omega t)$$

$$\text{Real envelope} = \left|x_p(t)\right| = \left|-jA\, e^{j\omega t}\right| = |A|$$

$$= |A|\sqrt{sin^2(\omega t) + cos^2(\omega t)}$$

Envelopes – Making Signals Analytic

Observations & Insights

For $\omega > 0$, we saw:

$$\mathcal{H}\big(\cos(\omega t)\big) = \widehat{\cos(\omega t)} = \sin(\omega t)\,,\ \ \mathcal{H}\big(\sin(\omega t)\big) = \widehat{\sin(\omega t)} = -\cos(\omega t)$$

In general: What about $\mathcal{H}\Big(\mathcal{H}\big(\mathrm{x}(t)\big)\Big) = \hat{\hat{x}}(t) = ??$
Argue via the frequency domain:

$$\mathrm{x}(t) \leftrightarrow X(f) \Rightarrow \hat{x}(t) \leftrightarrow -j\ \mathrm{sgn}(f)X(f)$$

$$\Rightarrow \hat{\hat{x}}(t) \leftrightarrow \big(-j\ \mathrm{sgn}(f)\big)^2 X(f) = -X(f) \leftrightarrow -\mathrm{x}(t)$$

Also, $\mathcal{H}^3\big(\mathrm{x}(t)\big) = -\mathcal{H}\big(\mathrm{x}(t)\big) = -\hat{x}(t)$, and $\mathcal{H}^4\big(\mathrm{x}(t)\big) = -\mathcal{H}^2\big(\mathrm{x}(t)\big) = \mathrm{x}(t)$

Envelopes – Making Signals Analytic

Properties of the Hilbert transform:

1) Energies of x(t) and $\hat{x}(t)$ are the same. … Use Parseval:

$$\int_{-\infty}^{\infty} \hat{x}^2(t)dt = \int_{-\infty}^{\infty} |-j\mathrm{sgn}(f)X(f)|^2 df = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)dt$$

2) x(t) and $\hat{x}(t)$ are orthogonal.
Again, via the frequency domain (generalized Parseval):

$$\int_{-\infty}^{\infty} x(t)\hat{x}(t)dt = \int_{-\infty}^{\infty} X(f)[-j\mathrm{sgn}(f)X(f)]^* df = j\int_{-\infty}^{\infty} \underbrace{\mathrm{sgn}(f)}_{odd}\,\underbrace{|X(f)|^2}_{even}\,df = 0$$

Envelopes – Making Signals Analytic

Properties of the Hilbert transform:

3) If $m(t)$ is lowpass and $c(t)$ is highpass with nonoverlapping spectra $M(f)$ and $C(f)$, respectively, then

$$\mathcal{H}[m(t)c(t)\,] = m(t)\hat{c}(t)$$

Proof:

$$m(t)c(t) = \int_{-\infty}^{\infty} M(f)e^{j2\pi ft}df \int_{-\infty}^{\infty} C(f')e^{j2\pi f't}df'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')e^{j2\pi(f+f')t}df\,df'$$

This is a (large and double) sum of frequency components. Hence, its Hilbert transform is

$$\mathcal{H}[m(t)c(t)\,] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')[-j\mathrm{sgn}(f+f')]e^{j2\pi(f+f')t}df\,df'$$

Envelopes – Making Signals Analytic

Properties of the Hilbert transform:

$$\mathcal{H}[m(t)c(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')[-j\,\mathrm{sgn}(f+f')]e^{j2\pi(f+f')t}df\,df'$$

Now, $M(f)$ is nonzero only for small f, and $C(f')$ is nonzero only for large f'. Hence, for the integral, we can assume that $|f| < |f'|$.

$$\Rightarrow \mathcal{H}[m(t)c(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(f)C(f')[-j\mathrm{sgn}(f')]e^{j2\pi(f+f')t}df\,df'$$

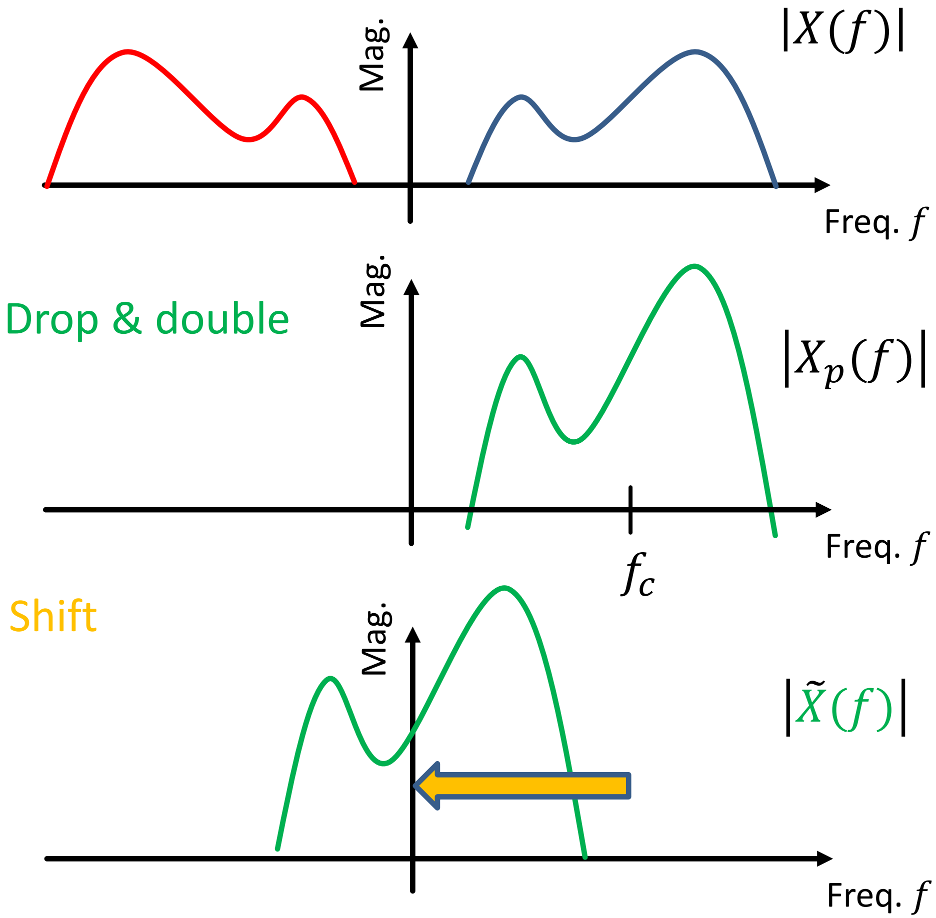
$$= \int_{-\infty}^{\infty} M(f)e^{j2\pi ft}df \int_{-\infty}^{\infty} C(f')[-j\mathrm{sgn}(f')]e^{j2\pi f't}df' = m(t)\hat{c}(t)$$

■

Lecture 2

Envelopes – Making Signals Analytic

Important Application of the Hilbert transform: Bandpass Signals



Original signal: $x(t)$

Analytic signal: $x_p(t) = x(t) + j\hat{x}(t)$

Interpret:

$$x_p(t) = \tilde{x}(t)e^{j2\pi f_0 t}$$

Complex envelope:

$$\tilde{x}(t) = x_p(t)e^{-j2\pi f_0 t}$$

Real envelope:

$$|x_p(t)| = |\tilde{x}(t)|$$

Envelopes – Making Signals Analytic

``Ride the bike´´ :

Consider … make analytic, Hilbert transform, real envelope, …

1) $\mathrm{x}(t) = A\cos(t) + A\sin(t)$
2) $\mathrm{x}(t) = A\cos(99 \cdot t) + A\cos(101 \cdot t)$ … interpret the real envelope!

Reconsider our discussion on $\mathcal{H}^2\big(\mathrm{x}(t)\big)$ etc.
As mentioned before, we assumed that the signal $\mathrm{x}(t)$ has no DC-component.

Just on the side: What happens, if it does ?
Does ``$\mathcal{H}^4 =$ Identity´´ still hold ?

Think: Besides $x_p(t)$, people also define another analytic signal related to a given signal $x(t)$:

$$x_n(t) = x(t) - j\hat{x}(t)$$

What is different, here? … What remains the same?

Envelopes – Making Signals Analytic

``Ride the bike´´ - chance to dig deaper (a long biking tour in a way):

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(t-t')}{t'} dt'$$

Usually, we would not use the time domain approach but go via the frequency domain. However, just for the fun of it, how can this integral be calculated … given the pole at zero?

Such integrals pop up when you try to, systematically, find an adequate imaginary part for a given real function (defined along the real axis) … that is what we motivated, here.

A typical approach is via complex integrals … semi-cricle shaped integration paths … little circles around poles … residues … etc. This leads to the integral above in the sense of the so-called Cauchy principal value:

$$\lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{x(t-t')}{t'} dt' + \int_{\varepsilon}^{\infty} \frac{x(t-t')}{t'} dt' \right]$$

which, due to its symmetry, circumvents the trouble with the pole.

Thank you for your attention!

See you tomorrow …