Communications Basics
Lecture 4

Random Signals and Noise

Orga

Textbook:

Rodger E. Ziemer & William H. Tranter, Principles of Communications **… READ!**

Our content comes from chps. 2-7 (according to edition 5)

Chp. 2 (Signals & Systems – separate course) … **polish your Fourier transforms**

Chp. 3 (Modulation + Demodulation)

Chp. 4 (Probability – separate course) … roll a couple of dice ☺ **… we'll bascially need Random variables, (multivariate) Gaussians, expected values, variances, covariances**

Chp. 5 (Random Processes & Noise)

Chp. 6 (Noise in Modulation Systems)

Chp. 7 (Binary Data Transmission)

Main platform: campusnet … course page !!!

Teaching in person … slides will be on campusnet … but make sure to take your own notes!

TA: Yasmine Ammouze … tutorials

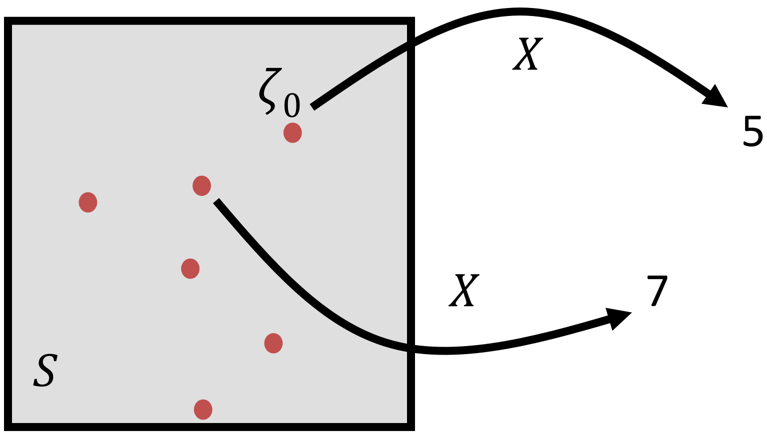
Exam: Written, no cheat sheets (expected end of January), 2 hours, details as announced by the registrar (should show in campusnet)

Random Signals and Noise

Reminder:

A **random variable** is a mapping from
a probability space to the (real) numbers.

$$X\colon\ (S, E, P)\ \rightarrow \mathbb{R}$$



S: Sample space
E: Set of all events = subsets of S with a probability assigned
P: Probability measure with P(S) = 1.

Random Signals and Noise

Random Processes:

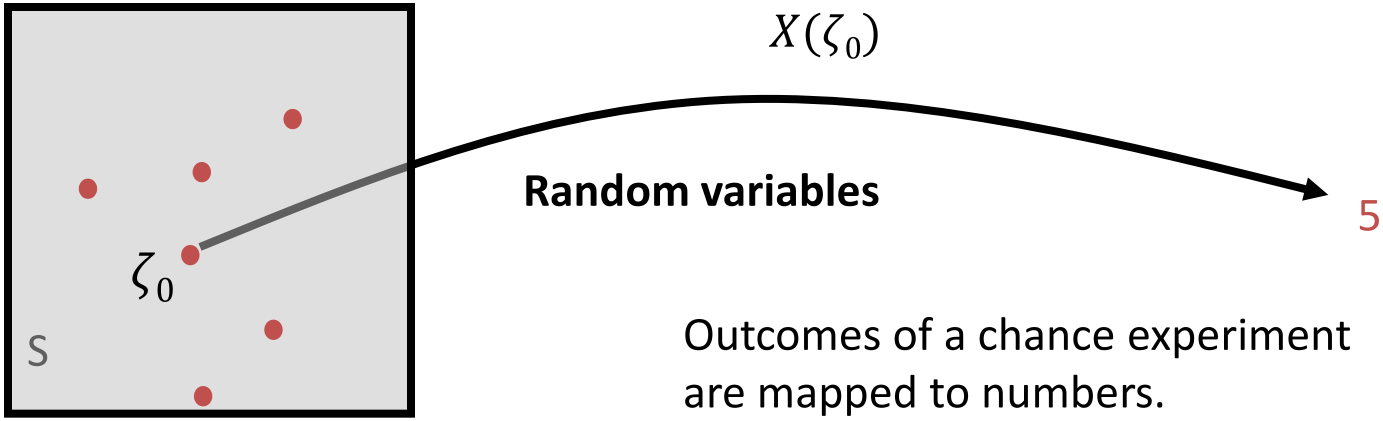
A **random process** is a family of random variables $\{X(t), t \in T\}$, defined on the same probability space indexed by an index t.
Most common: t = time.

For a fixed time t_0, the single $X(t_0)$ is just a normal random variable.

For a fixed outcome ζ_0, the function $X(t, \zeta_0)$ is a fixed (non-random) signal.

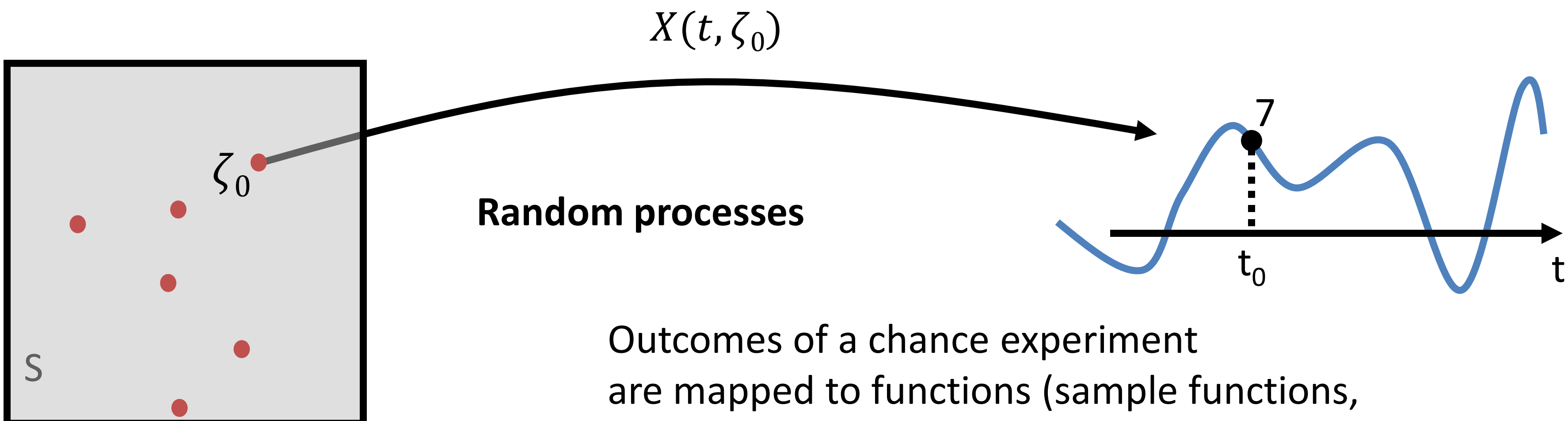
Lecture 4

Random Signals and Noise



Random variables

Outcomes of a chance experiment
are mapped to numbers.



Random processes

Outcomes of a chance experiment
are mapped to functions (sample functions,
also called realizations of the process).

Random Signals and Noise

Random Process

$X(t,\zeta)$ — fixed time → $X(t_0,\zeta)$

random variable

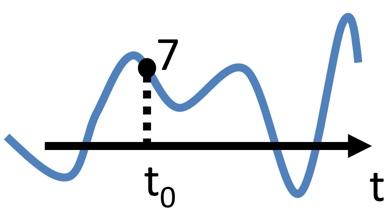
$X(t,\zeta)$ — fixed outcome → $X(t,\zeta_0)$

$X(t_0,\zeta)$ — fixed outcome → $X(t_0,\zeta_0)$

$X(t,\zeta_0)$ — fixed time → $X(t_0,\zeta_0)$

sample function (realization)

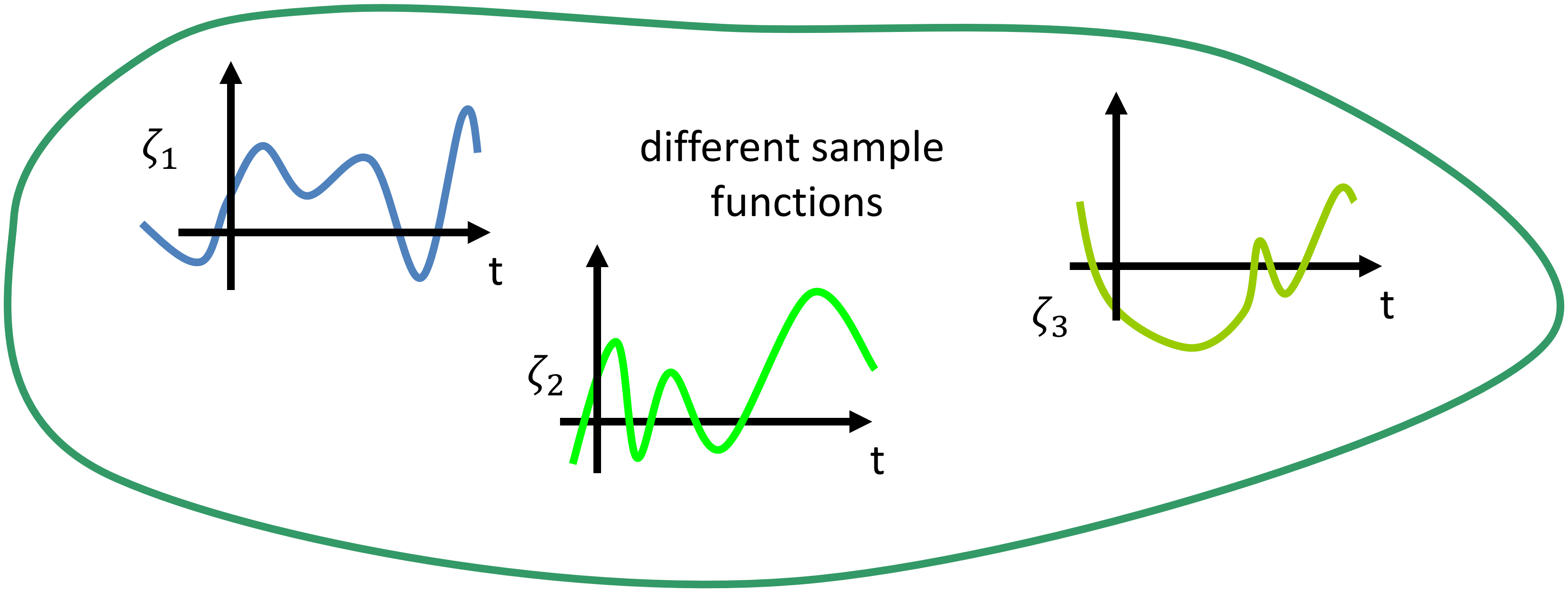
number



Random Signals and Noise

Ensemble

The set of all sample functions (realizations) $\{X(t,\zeta)\ ,\ \zeta \in S\}$ is called the ensemble.



Random Signals and Noise

How can we describe/classify a random process?

Descriptions:

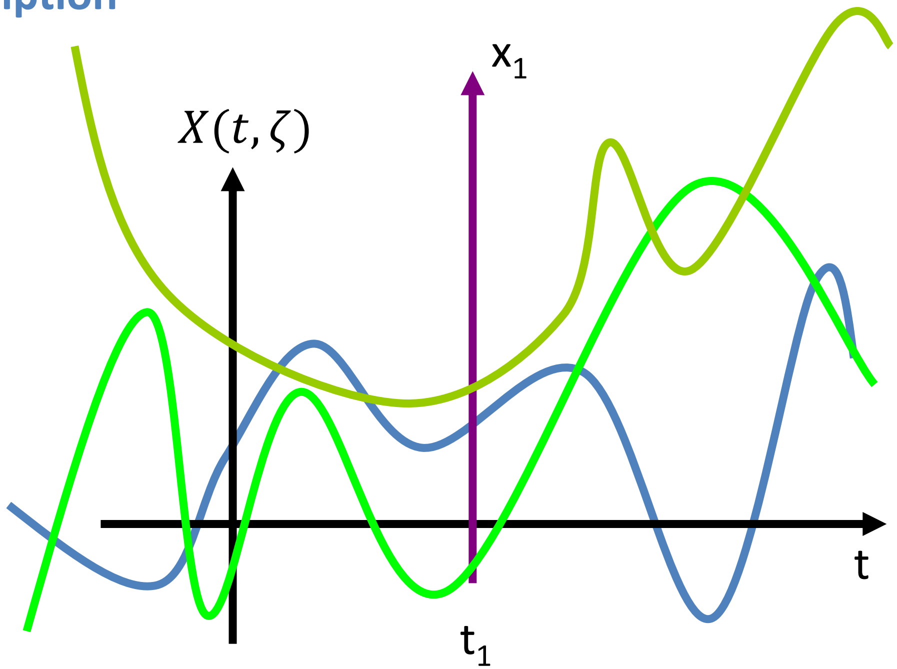
a) probabilistic: pdfs
b) averaged: (low order) moments

Classification:

1) Stationary processes
2) Wide sense stationary processes
3) Ergodic processes

Random Signals and Noise

Probabilistic Description

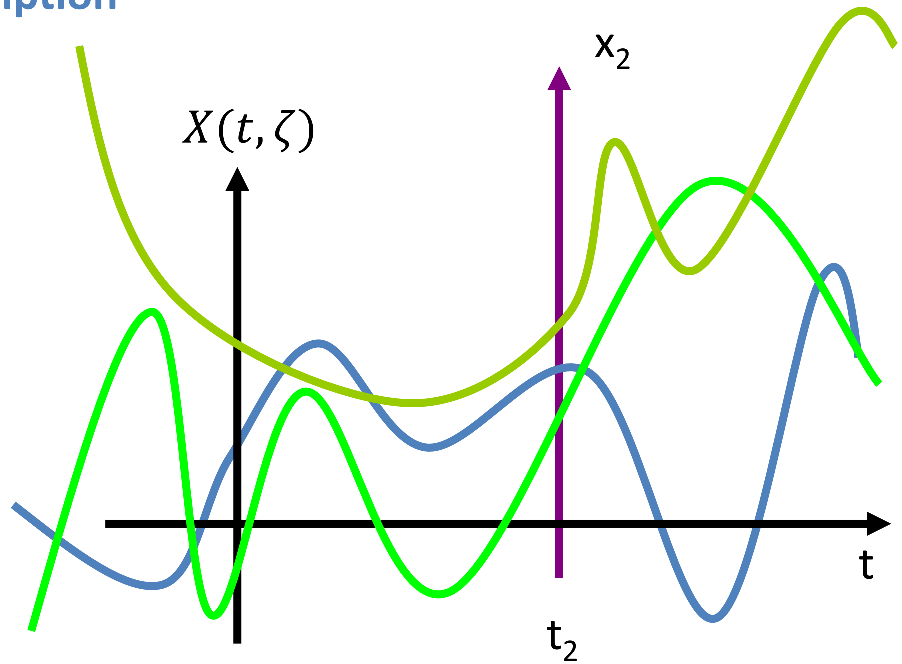


Distribution of values x_1 at t_1 described by pdf

$$f_{X_1}(x_1; t_1)$$

Random Signals and Noise

Probabilistic Description

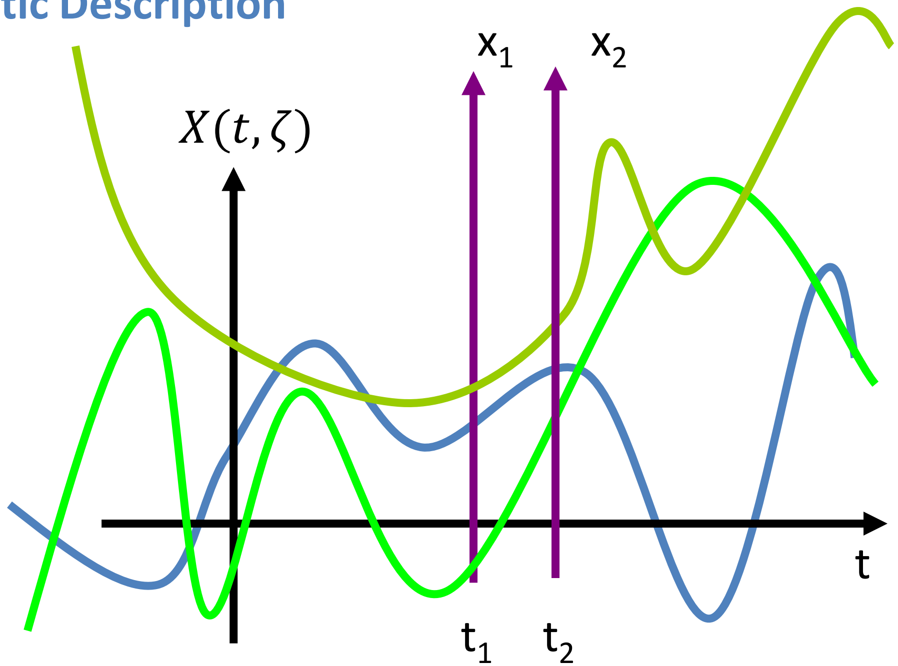


Distribution of values x_2 at t_2 described by pdf

$$f_{X_2}(x_2, t_2)$$

Random Signals and Noise

2^{nd} Order Probabilistic Description

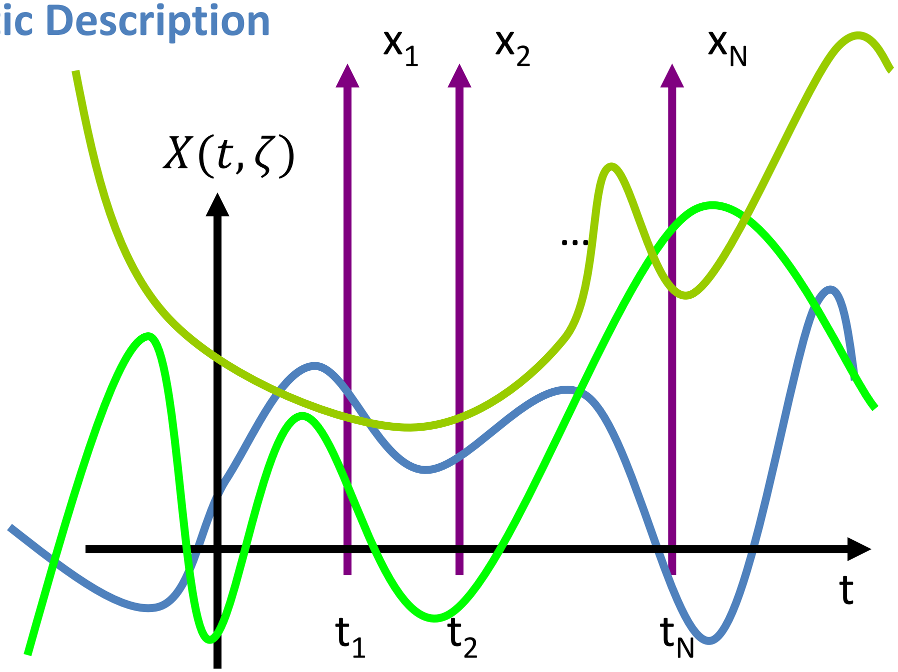


Distribution of values x_1 at t_1 **and** x_2 at t_2 described by the joint pdf

$$f_{X_1X_2}(x_1, x_2\,; t_1, t_2)$$

Random Signals and Noise

N^{th} Order Probabilistic Description



Distribution of values x_1 at t_1, x_2 at t_2 , …, **and** x_N at t_N described by the joint pdf

$$f_{X_1X_2\ldots X_N}(x_1, x_2, \ldots, x_N\,; t_1, t_2, \ldots, t_N)$$

Random Signals and Noise

Probabilistic Description

In order to have a complete description in a probabilistic sense, you need all those pdfs.

In general, that's hopeless!

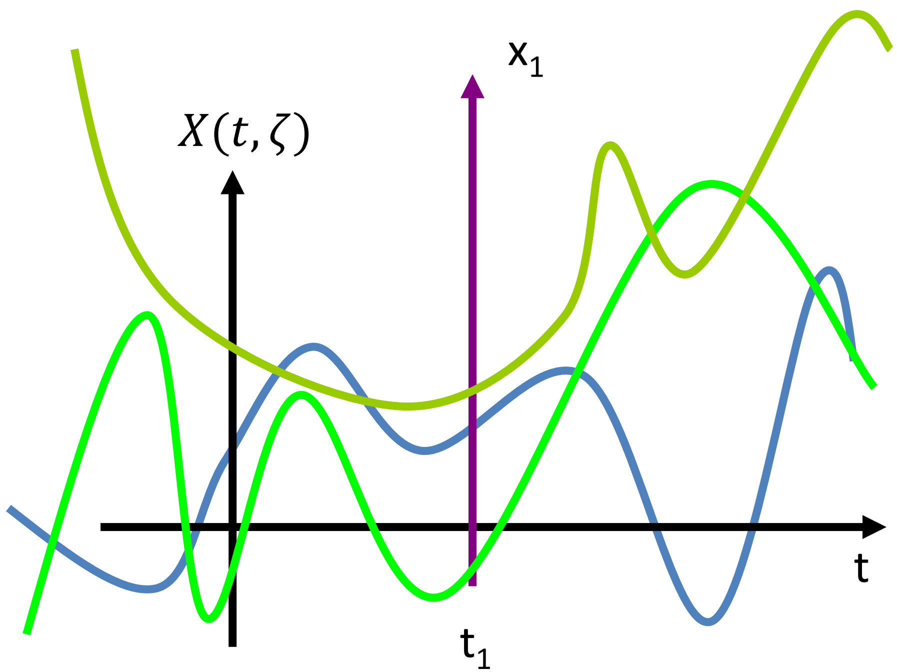
In particular when it comes to estimation.

Try a simpler thing…

Random Signals and Noise

Consider Ensemble Averages

Mean: $$m_X(t_1) = E[X(t_1)] = \overline{X(t_1)} = \overline{X_1} = \int_{-\infty}^{\infty} x_1 f_{X_1}(x_1, t_1) dx_1$$

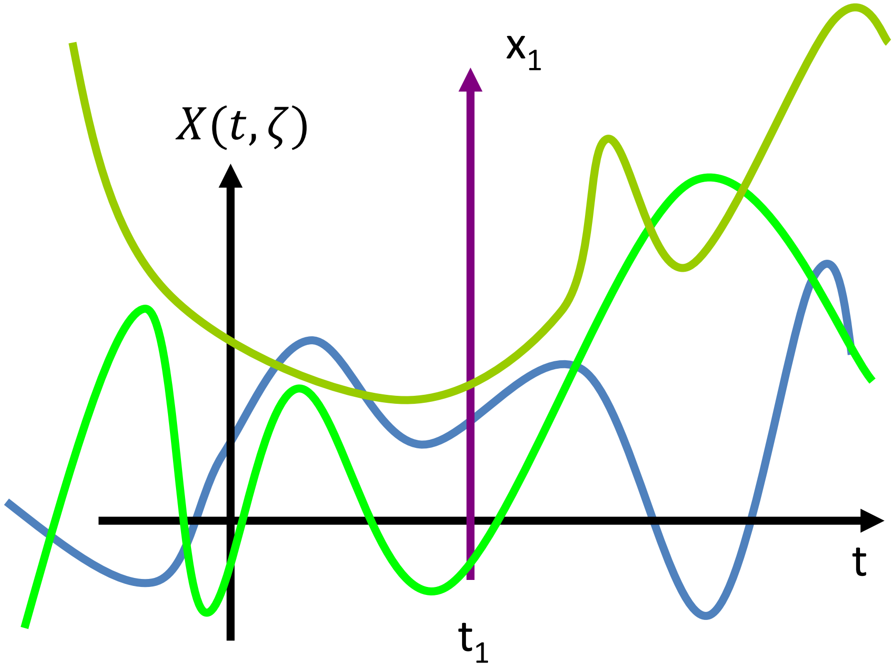


Random Signals and Noise

Ensemble Averages

Variance:

$$\sigma_X^2(t_1) = E\left[\left(X(t_1) - \overline{X(t_1)}\right)^2\right] = \overline{X_1^2} - \overline{X_1}^2$$



Random Signals and Noise

Ensemble Averages

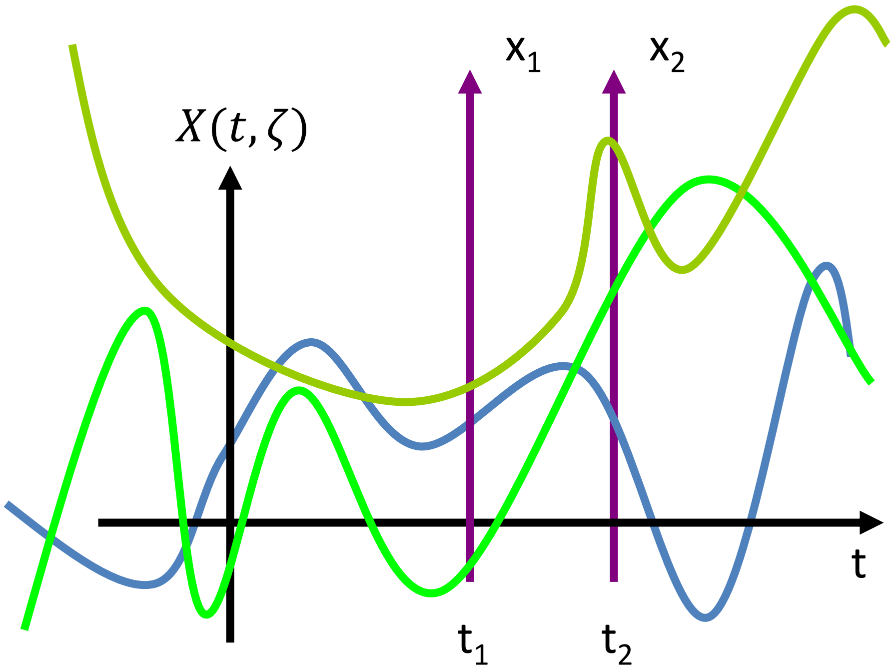
(Statistical) Autocorrelation function:

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \overline{X_1 X_2}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1X_2}(x_1,x_2\,;t_1,t_2)\ dx_1\ dx_2$$

$$R_X(t_1,t_1) = \overline{X_1^2}$$

$$\Rightarrow \sigma_X^2(t_1) = R_X(t_1,t_1) - \overline{X_1}^2$$



Random Signals and Noise

Useful description

Specify:

- Mean and
- Autocorrelation function

(variance is also specified then)

This ignores a lot … but is much more practical!

Random Signals and Noise

Now classify random processes

Random Signals and Noise

Strict – Sense Stationary Processes

If all the pdfs, only depend on time differences,
the random process is called stationary in the strict sense.

2nd order: $$f_{X_1 X_2}(x_1, x_2\,; t_1, t_2) = f_{X_1 X_2}(x_1, x_2\,; t_2 - t_1)$$

Nth order: $$f_{X_1 X_2 \dots X_N}(x_1, x_2, \dots, x_N\,; t_1, t_2, \dots, t_N)$$

$$= f_{X_1 X_2 \dots X_N}(x_1, x_2, \dots, x_N\,; t_2 - t_1, \dots, t_N - t_1)$$

Again: This is hard to check! So, try the ”useful” description…

$\boxed{\text{Compare: LTI systems}}$

Random Signals and Noise

Wide – Sense Stationary (WSS) Processes

A random process is called wide-sense stationary if its mean (and its variance) are independent of time, and its covariance/autocorrelation function depends only on the time difference.

mean: $$m_X(t) = E[X(t)] = \overline{X(t)} = \text{const}$$

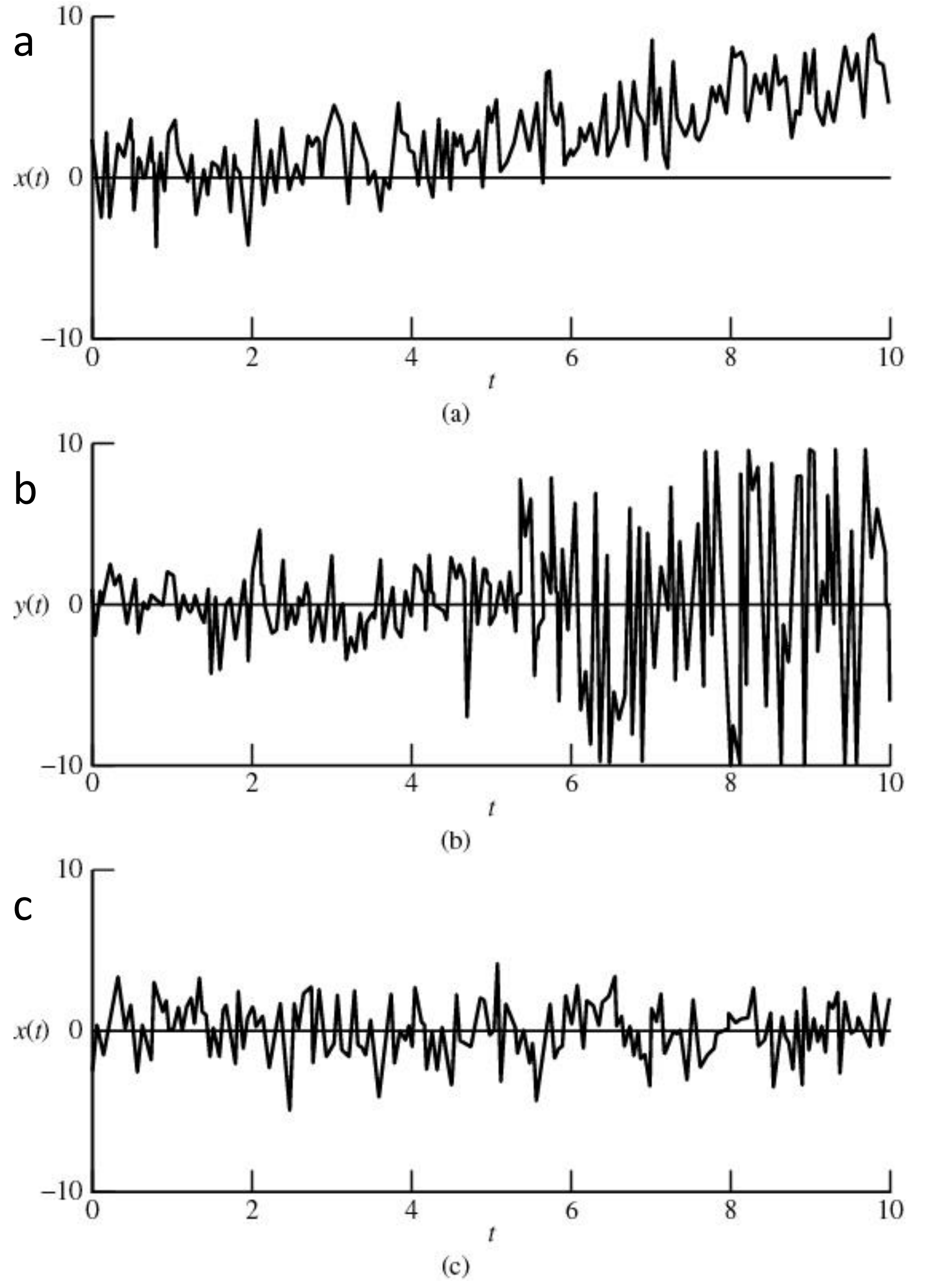
variance: $$\sigma_X^2(t) = E\left[\left(X - \overline{X(t)}\right)^2\right] = \overline{X^2(t)} - \overline{X(t)}^2 = \text{const}$$

covariance: $$Cov_X(t, t+\tau) = E\left[\left(X(t) - \overline{X(t)}\right)\left(X(t+\tau) - \overline{X(t+\tau)}\right)\right]$$
$$= \overline{X(t)X(t+\tau)} - \overline{X(t)} \cdot \overline{X(t+\tau)} = Cov_X(\tau)$$

Mind: Your textbook uses a μ when it comes to covariances.
This may be easily confused with the mean … We use Cov

Def. might also use autocorrelation function instead of covariance

Stationary and Non-Stationary Processes



a) Mean depends on time
b) Variance depends on time
c) Stationay process

[Fig. from Ziemer & Tranter,
Princ. of Comm. 5/e, Wiley 2002]

Lecture 4

Random Signals and Noise

All pdfs

strict-sense stationary

$\Downarrow \qquad \not\Uparrow$

wide-sense stationary

Low order moments

Random Signals and Noise

Ergodic Processes

A random process is called ergodic, if time averages and ensemble averages can be interchanged.

In particular:

ensemble

time

$$m_X = E[X(t)] = \overline{X(t_1)} \stackrel{\downarrow}{=} \langle X(t) \rangle$$

$$\sigma_X^2 = E\left[\left(X(t) - \overline{X(t)}\right)^2\right] \stackrel{\downarrow}{=} \langle (X(t) - \langle X(t) \rangle)^2 \rangle$$

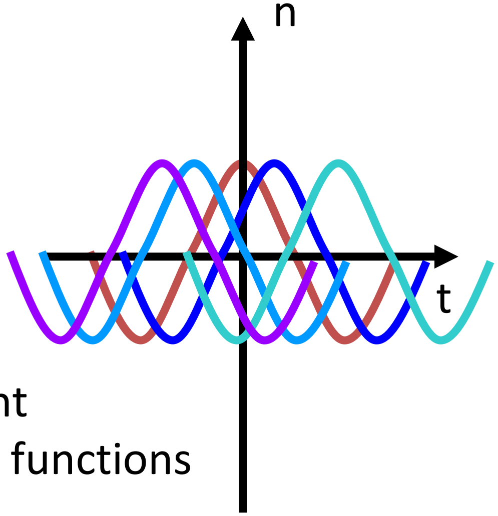
$$R_X(\tau) = E[X(t)X(t+\tau)] \stackrel{\downarrow}{=} \langle X(t)X(t+\tau) \rangle = R(\tau)$$

Random Signals and Noise

Example (Ergodic Process)

$$n(t,\zeta) = A\cos\big(2\pi f_0 t + \Theta(\zeta)\big)$$

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & |\theta| < \pi \\ 0, & \text{otherwise} \end{cases}$$



different
sample functions

Ensemble average:

$$\overline{n(t,\zeta)} = \int_{-\infty}^{\infty} A\cos(2\pi f_0 t + \theta)\, f_\Theta(\theta) d\theta$$

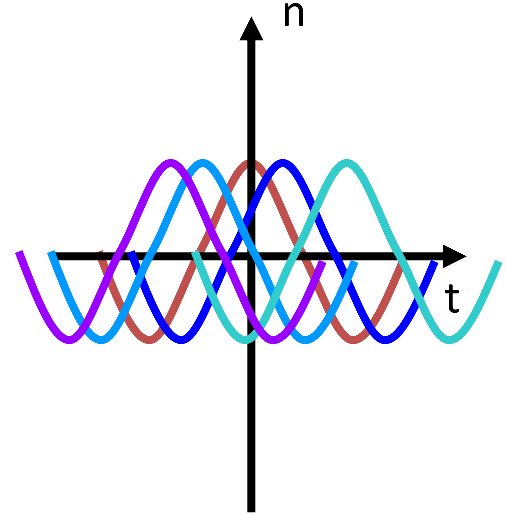
$$= \int_{-\pi}^{\pi} A\cos(2\pi f_0 t + \theta)\frac{1}{2\pi} d\theta = 0$$

Can you see/find the pdf of n
$f_N(n;t)$ **?**

Notice: This average also does NOT depend on time!

Random Signals and Noise

Example (Ergodic Process) … contd.



Time average:

$$\langle n(t,\zeta)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(2\pi f_0 t+\theta)\ dt = 0$$

Notice:
In general, the time average is a random variable.

Different sample functions yield different time averages.

Here: The time average does not depend on
the outcome ζ of the chance experiment, and we have

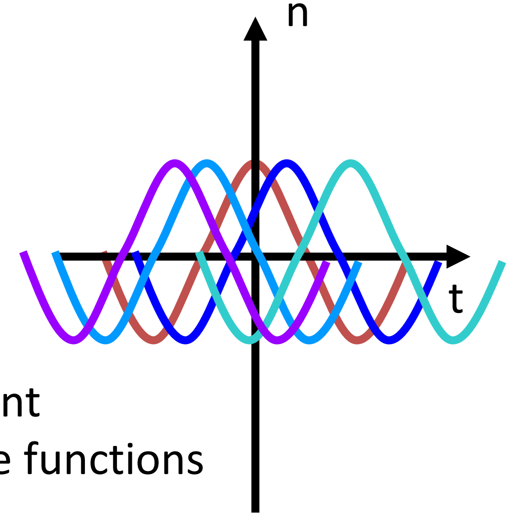
time average $\quad \langle n(t)\rangle = \overline{n(t)} \quad$ ensemble average

Random Signals and Noise

Example (Ergodic Process)

$$n(t) = A\cos(2\pi f_0 t + \Theta)$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & |\theta| < \pi \\ 0, & \text{otherwise} \end{cases}$$



different
sample functions

Ensemble average autocorrelation function:

$$R_n(t_1, t_2) = \overline{n(t_1)n(t_2)} = \int_{-\infty}^{\infty} A^2 \cos(2\pi f_0 t_1 + \theta)\cos(2\pi f_0 t_2 + \theta)\, f_\Theta(\theta) d\theta$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left[\cos(2\pi f_0(t_1 + t_2) + 2\theta) + \cos\big(2\pi f_0(t_2 - t_1)\big)\right] d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi} \cos\big(2\pi f_0(t_2 - t_1)\big)\, d\theta = \frac{A^2}{2}\cos(2\pi f_0 \tau) = R_n(\tau)$$

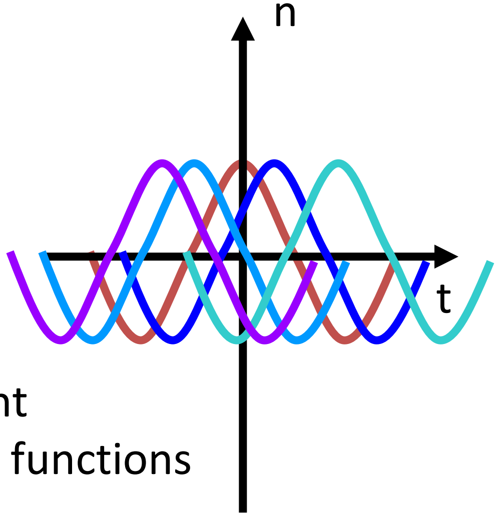
No absolute time t, here!

Random Signals and Noise

Example (Ergodic Process)

$$n(t) = A\cos(2\pi f_0 t + \Theta),\ T_0 = \frac{1}{f_0}$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & |\theta| < \pi \\ 0, & \text{otherwise} \end{cases}$$



different sample functions

Time average autocorrelation function (e.g. average over one or more periods):

$$R(\tau) = \langle n(t)n(t+\tau)\rangle = f_0 \int_{-T_0/2}^{T_0/2} A^2 \cos(2\pi f_0 t + \theta)\cos(2\pi f_0 (t+\tau) + \theta)\,dt$$

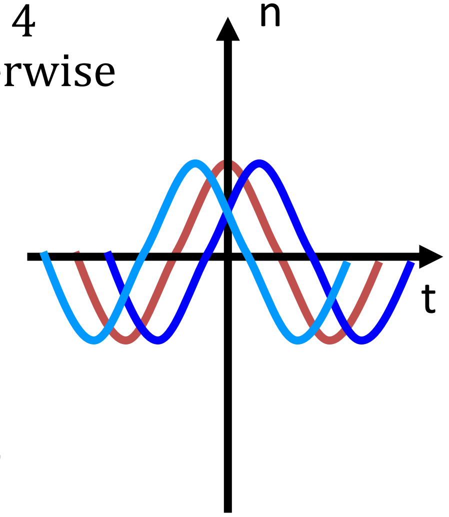
$$= f_0 A^2 \int_{-T_0/2}^{T_0/2} \frac{1}{2}[\cos(2\pi f_0(2t+\tau) + 2\theta) + \cos(2\pi f_0 \tau)]\,dt$$

$$= f_0 A^2 \int_{-T_0/2}^{T_0/2} \frac{1}{2}\cos(2\pi f_0 \tau)\,dt = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$ ✓

Random Signals and Noise

Example II

Just a simple change: $$f_\Theta(\theta) = \begin{cases} \frac{2}{\pi}, & |\theta| < \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$



<u>The time averages are of course the same as before.</u>
<u>But the ensemble average is not:</u>

$$\overline{n(t)} = \int_{-\infty}^{\infty} A\cos(2\pi f_0 t + \theta)\, f_\Theta(\theta) d\theta = \int_{-\pi/4}^{\pi/4} A\cos(2\pi f_0 t + \theta) \frac{2}{\pi} d\theta$$

$$= \frac{2}{\pi} A \sin(2\pi f_0 t + \theta)\Big|_{-\pi/4}^{\pi/4} = \frac{2A}{\pi}\left\{\sin\left(2\pi f_0 t + \pi/4\right) - \sin\left(2\pi f_0 t - \pi/4\right)\right\}$$

$$= \frac{2A}{\pi} 2\cos(2\pi f_0 t) \sin\left(\pi/4\right) = \frac{4A}{\pi}\cos(2\pi f_0 t)\frac{\sqrt{2}}{2} \neq 0$$

$\Rightarrow$ This process is not stationary

→ it is not ergodic.

Random Signals and Noise

Think about the Autocorrelation Function ….

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \overline{X_1X_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f_{X_1X_2}(x_1,x_2\,;t_1,t_2)\ dx_1\ dx_2$$

is again an <u>ensemble average</u> with $X_1 = X(t_1)$ and $X_2 = X(t_2)$

If the process is wide-sense stationary

$$\Rightarrow R_X(t_1,t_2) = R_X(t_2-t_1) = R_X(\tau)$$

<u>Compare the time average autocorrelation function:</u>

$$R(\tau) = \langle x(t)x(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\,dt$$

Do we have
$R(\tau) = R_X(\tau)$?

For ergodic processes we do

… continue on your own for the two examples!

Random Signals and Noise

Properties of the WSS Autocorrelation Function $R_X(\tau)$

1. $R_X(-\tau) = R_X(\tau)$

2. $|R_X(\tau)| \leq R_X(0)$

3. $R_X(0) = E[X^2(t)] \geq 0$

(ensemble) average ... power of $X(t,\zeta)$

Can you prove properties 1+2 ?
For 2) consider $\{X(t) + X(t+\tau)\}^2$

Random Signals and Noise

Autocorrelation Function

For deterministic signals, we already know the (time average) autocorrelation function:

$$R(\tau) = \langle x(t)x(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt$$

And we know its Fourier transform,
The power spectral density:

$$S(f) = \int_{-\infty}^{\infty} R(\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$R(\tau) = \int_{-\infty}^{\infty} S(f) \exp(j2\pi f\tau)\, df$$

Mind, the total power is

$$P = \int_{-\infty}^{\infty} S(f)\ df$$

Random Signals and Noise

Autocorrelation Function / Power Spectral Density … understand the FT-relation

Consider windowed versions $x_T(t)$ of the signal $x(t)$ such that $x_T(t) = 0$ for $|t| > T/2$

$$x_T(t) = \int_{-\infty}^{\infty} X_T(f)\exp(j2\pi f t)\,df$$

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_T(t)x_T(t+\tau)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty} \underline{x_T(t)}x_T(t+\tau)\,dt$$

$x_T(t)$ is zero outside the window

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} X_T(f)\exp(j2\pi f t)\,df \int_{-\infty}^{\infty} X_T(f')\exp\big(j2\pi f'(t+\tau)\big)\,df'\right\}dt$$

$x_T(t+\tau)$ is real

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} X_T(f)\exp(j2\pi f t)\,df \int_{-\infty}^{\infty} X_T^*(f')\exp\big(-j2\pi f'(t+\tau)\big)\,df'\right\}dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X_T(f)X_T^*(f')\exp(j2\pi[f-f']t)\exp(-j2\pi f'\tau)\,df\,df'\right\}dt$$

Random Signals and Noise

Autocorrelation Function / Power Spectral Density … understand the FT-relation

So far:

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_T(f) X_T^*(f') \exp(j2\pi[f - f']t) \exp(-j2\pi f'\tau)\, df\, df' \right\} dt$$

Integrate over t → δ

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_T(f) X_T^*(f') \delta[f - f'] \exp(-j2\pi f'\tau)\, df\, df'$$

Integrate over f'

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(f)|^2 \exp(-j2\pi f\tau)\, df$$

$$\lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2 = \lim_{T\to\infty} S_T(f) = S(f)$$

$$= \int_{-\infty}^{\infty} S(f) \exp(-j2\pi f\tau)\, df$$

$S(f)$ is symmetric $\Leftrightarrow$ $R(\tau)$ is real

$$= \int_{-\infty}^{\infty} S(f) \exp(j2\pi f\tau)\, df$$

Random Signals and Noise

… and Power Spectral Density (Deterministic Signals)

Hence:
The power spectral density $S(f)$ for deterministic signals can be easily approximated based on Fourier transforms over finite intervals (of length T):

$$X_T(f) = \int_{-T/2}^{T/2} x(t)\exp(-j2\pi f t)\,dt$$

rectangluar window

Energy:

Energy spectral density $G_T(f)$

$$E_T = \int_{-T/2}^{T/2} |x(t)|^2 \; dt \overset{\text{Parseval}}{=} \int_{-\infty}^{\infty} |X_T(f)|^2 \; df$$

and

Power:

$$P_T = \frac{E_T}{T} = \int_{-\infty}^{\infty} S_T(f) \; df$$

Power spectral density:

$$S_T(f) = \frac{|X_T(f)|^2}{T}$$

and

$$\boxed{S(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}}$$

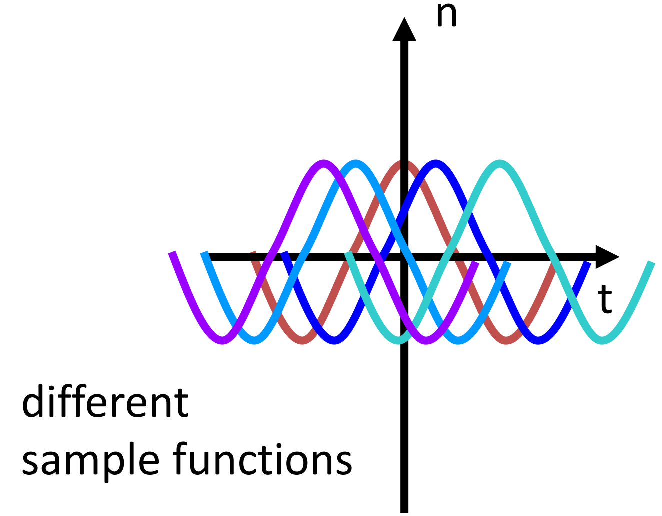
Random Signals and Noise

``Keep riding the bike´´ :

Study random processes … find mean and autocorrelation functions
In particular, for the lecture examples … find the autocorrelation functions yourself
… study the influence of a uniform phase shift …

$$n(t) = A\cos(2\pi f_0 t + \Theta)$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2K}, & |\theta| < K \\ 0, & \text{otherwise} \end{cases}$$



Thank you for your attention!

See you tomorrow …