ES 14

$Q \sim U[0,1]$ $\quad A_i$: MACCHINA FUNZIONA NEL GIORNO i

$P(A_i \mid Q=q) = q \quad \forall i \qquad \{A_i \mid Q=q\} \perp \{A_j \mid Q=q\} \quad \forall i \neq j$

a) $P(A_i) = \int_0^1 P(A_i \mid Q=q) f_Q(q)\, dq$

$= \int_0^1 q \cdot 1 \, dq = \left[\frac{q^2}{2}\right]_0^1 = \frac{1}{2}$

b) $B = \{$MACCHINA FUNZIONA IN m DI n GIORNI$\}$

$$f_{Q|B}(q) \overset{\text{BAYES}}{=} \frac{P(B \mid Q=q) f_Q(q)}{P(B)}$$

CONDIZIONAMENTO ALL'EVENTO B

$$P(B \mid Q=q) \overset{\text{BINOMIALE}}{=} \begin{cases} \binom{n}{m} q^m (1-q)^{n-m} & 0 \leq m \leq n \\ 0 & \text{ALTR.} \end{cases}$$

$$P(B) = \int_0^1 P(B \mid Q=q) f_Q(q)\, dq = \int_0^1 \binom{n}{m} q^m (1-q)^{n-m} \cdot 1 \, dq$$

USO L'IDENTITÀ FORNITA $= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}$

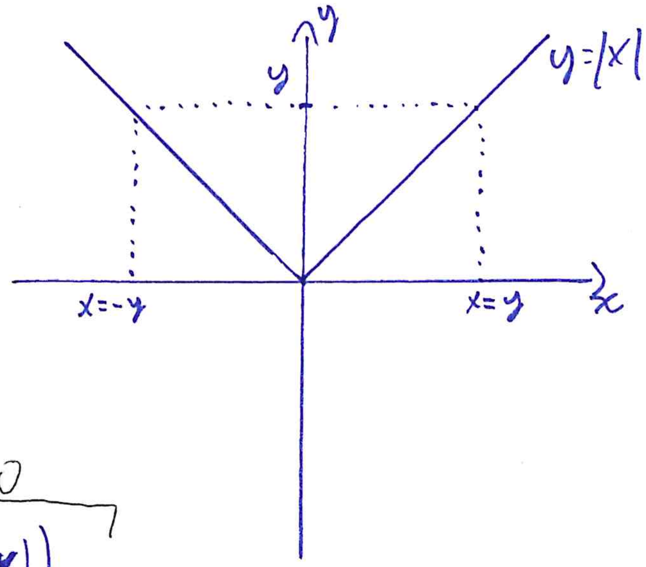
$$f_{Q|B}(q) = \begin{cases} \dfrac{q^m (1-q)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}} & 0 \leq q \leq 1,\ 0 \leq m \leq n \\ 0 & \text{ALTRIMENTI} \end{cases}$$

ES 15 1/2

$X \sim f_X(x)$ $\quad Y = |X|$, $f_Y(y) = ?$

$$y = g(x) = |x| \begin{cases} x>0 : & y = x,\ g'(x) = 1,\ g^{-1}(y) = y \\ x<0 & y = -x,\ g'(x) = -1,\ g^{-1}(y) = -y \end{cases}$$

c) $$f_Y(y) = \begin{cases} 0 & y<0 \\ f_X(-y) + f_X(y) & y>0 \end{cases}$$



METODO DIRETTO:

$$f_Y(y) = \overbrace{\frac{f_X(g^{-1}(y))}{\left|\frac{d}{dx}g(x)\right|_{x=g^{-1}(y)}}}^{X>0} + \overbrace{\frac{f_X(g^{-1}(y))}{\left|\frac{d}{dx}g(x)\right|_{x=g^{-1}(y)}}}^{X<0} = f_X(y) + f_X(-y)$$

METODO DELLA CUMULATA

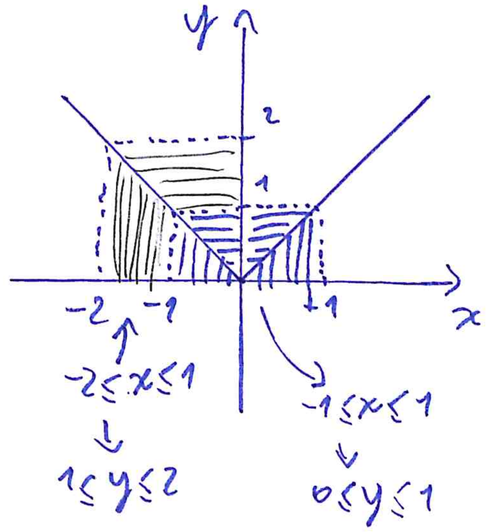
$$P(Y \le y) = \begin{cases} 0 & y<0 \\ P(|X| \le y) = P(-y \le X \le y) = P(X \le y) - P(X \le -y) & \\ \quad = F_X(y) - F_X(-y) & y>0 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(y)}{dy} - \frac{dF_X}{dy}(-y) =$$

$$= f_X(y) \cdot 1 - f_X(-y) \cdot (-1)$$

$$= f_X(y) + f_X(-y) \qquad y>0$$

ES 15 2/2



a)
$$f_X(x)=\begin{cases}1/3 & -2\le x\le 1\\ 0 & \text{ALTR}\end{cases}$$

$$f_Y(y)=f_X(y)+f_X(-y)$$
$$=\begin{cases}2/3 & 0<y<1\\ 1/3 & 1<y<2\\ 0\end{cases}$$

b)
$$f_X(x)=\begin{cases}2e^{-2x} & x>0\\ 0 & \text{ALTR}\end{cases}$$

POSSO SUBITO NOTARE $Y=|X|=X$ PERCHÉ X È SOLO POSITIVA

QUINDI $f_Y(y)=f_X(x)\ \forall y\in\mathbb{R}$

ALTRIMENTI POSSO FARE I PASSAGGI:

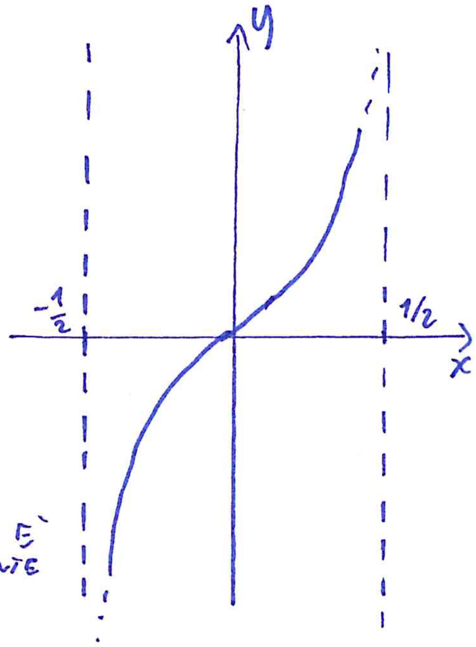
$$f_Y(y)=\begin{cases}f_X(y)+f_X(-y) & y>0\\ 0 & \text{ALTR}\end{cases}$$

$$=\begin{cases}2e^{-2y}+0 & y>0\\ 0 & \text{ALTR}\end{cases}$$

$$=f_X(y)$$

ES 16

$X \sim U\left[-\frac{1}{2}, \frac{1}{2}\right]$



a) $Y = \tan(\pi X) \cdot \quad f_Y(y) = \frac{1}{\pi(1+y^2)}$ ← DIM.

MANCA NEL TESTO

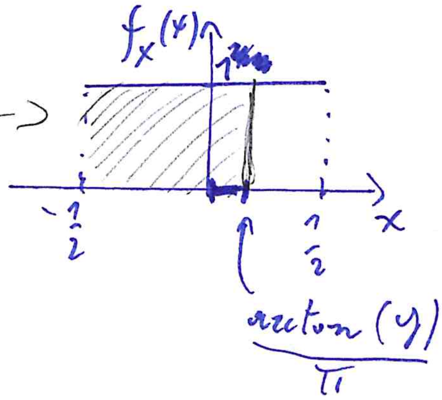
USIAMO LA CUMULATA:

$F_Y(y) = P(Y \leq y)$

POSSO FARLO SENZA CAMBIARE VERSO DI DISEGUAGLIANZA PERCHÉ tan È MONOTONA CRESCENTE

$= P(\tan(\pi X) \leq y) = P(\pi X \leq \mathrm{arctg}(y))$

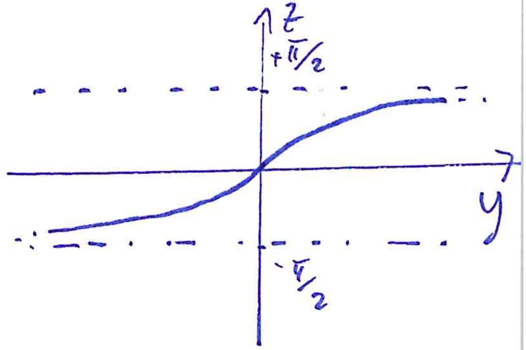
$= \underbrace{F_X\left(\frac{\arctan(y)}{\pi}\right)}_{}$ EVENTO DI INTERESSE



$= \left(\frac{1}{2} + \frac{\arctan(y)}{\pi}\right) \cdot 1$

$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{\pi} \cdot \frac{1}{1+y^2} \quad \forall y \in \mathbb{R}$

b) $Z = \arctan(Y)$



$F_Z(z) = P(Z \leq z) = P(\arctan(Y) \leq z) =$

$= P(Y \leq \tan(z)) = \int_{-\infty}^{\tan(z)} \frac{1}{\pi(1+y^2)}\, dy$

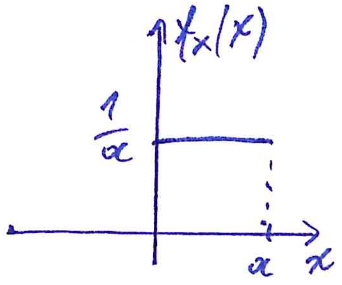
$= \frac{1}{\pi}\left[\arctan(y)\right]_{-\infty}^{\tan(z)} = \begin{cases} \frac{1}{\pi}\left[z - \left(-\frac{\pi}{2}\right)\right] & -\frac{\pi}{2} \leq z \leq \frac{\pi}{2} \\ 0 & z \leq -\frac{\pi}{2} \\ 1 & z \geq \pi/2 \end{cases}$

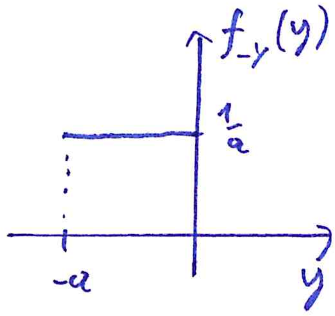
$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} \leq z \leq \frac{\pi}{2} \\ 0 & \text{ALTR} \end{cases} \Rightarrow Z \sim U\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

ES 17

$X \perp Y \quad X \sim Y \sim U[0,a]$

$Z = |X - Y|$ TROVARE f_Z

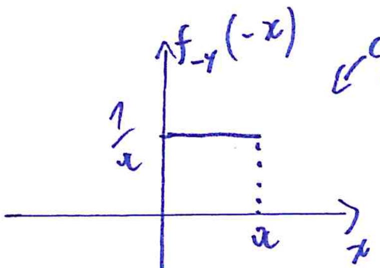




DEFINISCO W

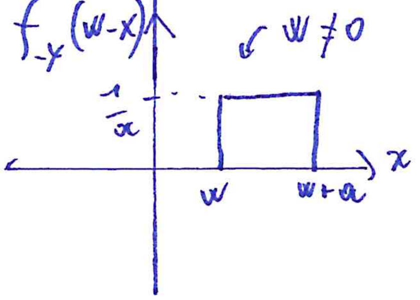
$W = X - Y = X + (-Y)$ LA DDP DELLA SOMMA È LA CONVOLUZIONE QUINDI:

$$f_W(w) = (f_X * f_{-Y})(w) = \int_{-\infty}^{+\infty} f_X(x) f_{-Y}(w-x)\,dx$$

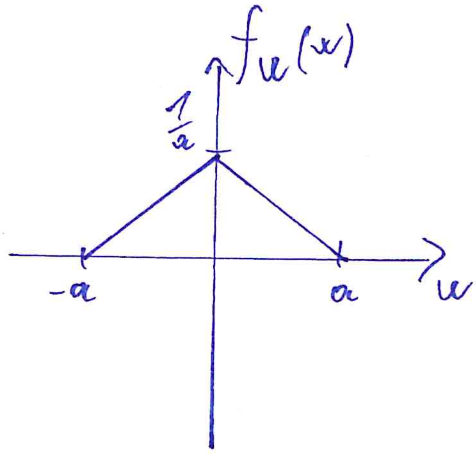


CASO $W=0$

$$f_W(0) = \int_{-\infty}^{\infty} f_X(x) f_{-Y}(-x)\,dx = \int_0^a \frac{1}{a^2}\,dx = \frac{1}{a}$$



$\Rightarrow$

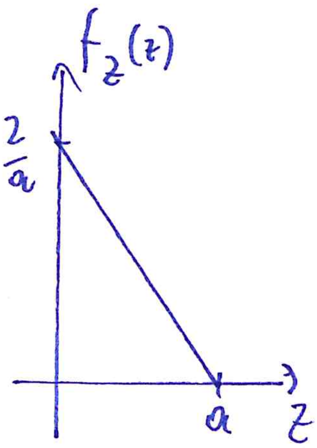


ORA APPLICO IL MODULO

$Z = |W|$

$$f_Z(z) = f_W(z) + f_{-W}(z)$$

$$= \begin{cases} f_W(z) + f_W(-z) & 0 \leq z \leq a \\ 0 & \text{ALTR.} \end{cases}$$



ES 18

$X \sim p_X \qquad Y \sim f_Y \qquad X \perp Y \qquad Z = X + Y$

$$F_Z \overset{DEF}{=} P(Z \leq z) = P(X+Y \leq z)$$

$$\left(\text{Th. PROB. TOTALE}\right) = \sum_x p_X(x) P(X+Y \leq z \mid X = x)$$

$$= \sum_x p_X(x) P(Y \leq z - x \mid X = x)$$

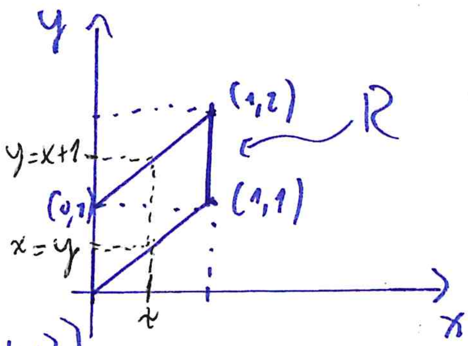
$$(X \perp Y) = \sum_x p_X(x) \underbrace{P(Y \leq z - x)}_{\text{CUMULATA VALUTATA IN } z-x}$$

$$= \sum_x p_X(x) F_Y(z-x)$$

$$\frac{d F_Z(z)}{dz} = f_Z(z) = \sum_x p_X(x) \frac{d F_Y}{dz}(z-x)$$

$$= \sum_x p_X(x) f_Y(z-x)$$

ES 19



$$f_{X,Y}(x,y) = \begin{cases} 1 & (x,y) \in R \\ 0 & (x,y) \notin R \end{cases}$$

VARIAZIONE TOTALE

$$Var[X+Y] \overset{\downarrow}{=} E[Var(X+Y|X)] + Var[E[X+Y|X]]$$

$\hookrightarrow \{Y|X=x\} = U[x, 1+x]$

DIVENTA UNA COST.

$$E[X+Y|X] = X + E[Y|X] = X + \frac{X+X+1}{2} = 2X + \frac{1}{2}$$

X È COME SE FOSSE UNA COST.

$$Var[X+Y|X] \overset{\downarrow}{=} Var[Y|X] = \frac{1^2}{12}$$

VARIANZA DELL'UNIFORME

VARIABILE ALEATORIA DEGENERE

$$Var[X+Y] = E\left[\frac{1}{12}\right] + Var\left[2X + \frac{1}{2}\right] = \frac{1}{12} + 4\,Var[X]$$

$$(X \sim U[0,1]) = \frac{1}{12} + \frac{4}{12} = \frac{5}{12}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{x}^{x+1} dy = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & ALTR \end{cases}$$

MARGINALIZZAZIONE

Es 20

LANCI DI MONETA $X_i \sim Bern\left(\frac{1}{2}\right)$ X_i SONO iid

RISULTATO DEL DADO $N \sim U\{1,2,3,4,5,6\}$

$X = \#$ TOTALE DI TESTE

a) $X = \sum_{i=1}^{N} X_i$

$$E[X] = E\left[\sum_{i=1}^{N} X_i\right] \overset{\text{LEGGE ASP. ITERATE}}{=} E\left[E\left(\sum_{i=1}^{N} X_i \middle| N\right)\right]$$

$$= E[X_1]E[N] = \frac{1}{2} * \frac{7}{2} = \frac{7}{4}$$

$$Var[X] = Var\left[\sum_{i=1}^{N} X_i\right] \overset{\text{VAR. TOT.}}{=} Var[E(X|N)] + E(Var[X|N]) =$$

$(X_i \text{ iid})$ $$= Var[N E[X_1]] + E[N Var(X_1)] =$$

$$= E[X_1]^2 Var[N] + E[N] Var[X_1] =$$

$(E[N^2] - E[N]^2)$ $$= \frac{1}{4} \cdot \underbrace{\frac{35}{12}} + \frac{7}{2} \cdot \frac{1}{4} = \frac{77}{48}$$

b) LANCIO 2 DADI $X + X'$ $X \perp X'$ $X \sim X'$

$$E[X+X'] = E[X] + E[X'] = 2E[X] = \frac{7}{2}$$

$$Var[X+X'] \overset{X \perp X'}{=} Var[X] + Var[X'] = 2Var[X] = \frac{77}{24}$$

ES 21

$\{X_n\}: \quad X_i \in [0, 0.5]$, V.A INDIPENDENTI

a) $E[X_n^2] \to 0$ PER $n \to \infty \overset{?}{\Rightarrow} \{X_n\} \xrightarrow{P} 0$

DEF. CONV. IN PROBABILITÀ:

$$0 \leq \lim_{n\to\infty} P(|X_n - 0| > \varepsilon) = \lim_{n\to\infty} P(|X_n| > \varepsilon) = \lim_{n\to\infty} P(X_n^2 > \varepsilon)$$

PROB. DEVE ESSERE POSITIVA

PER MARKOV

$$\leq \lim_{n\to\infty} \frac{E(X_n^2)}{\varepsilon^2} = 0 \quad \forall \varepsilon > 0$$

CONV. IN M.Q

$\Rightarrow \{X_n\} \xrightarrow{P} 0$

b) $E[X_n] = 0.2 \quad Var[X_n] \to 0 \quad n \to \infty \overset{?}{\Rightarrow} \{X_n\} \xrightarrow{P} 0.2$

$$0 \leq \lim_{n\to\infty} P(|X_n - \underset{E[X_n]}{0,2}| > \varepsilon) \underset{\text{CHEBYSHEV}}{\leq} \lim_{n\to\infty} \frac{Var[X_n]}{\varepsilon^2} \underset{Var(X_n)\to 0}{=} 0 \quad \forall \varepsilon > 0$$

$\Rightarrow \{X_n\} \xrightarrow{P} 0.2$

c) $Z_n = X_1 X_2 \cdots X_n \overset{?}{\Rightarrow} \{Z_n\} \xrightarrow{P} 0$

AL MASSIMO LE X_i VALGONO 0.5

$$0 \leq \lim_{n\to\infty} P(|Z_n - 0| > \varepsilon) = \lim_{n\to\infty} \left(\prod_{i=1}^{n} X_i > \varepsilon\right) \leq \lim_{n\to\infty} P(0.5^n > \varepsilon) = 0$$

$\forall \varepsilon > 0$

ES 22

X_i iid $\quad E[X_i]=0$, $Var[X_i]=2 \quad X_i \perp Y_i$

Y_i iid $\quad E(Y_i)=2$

a) $\frac{X_1+\dots+X_n}{n} \xrightarrow{P} 0$? POSSO NOTARE CHE QUESTA È LA MEDIA CAMPIONARIA

$$M_n = \frac{X_1+\dots+X_n}{n} \xrightarrow{P} E[M_n] \underset{iid}{=} \frac{n\,E(X_1)}{n} = E(X_1) = 0$$

b) $\frac{X_1^2+\dots+X_n^2}{n} \xrightarrow{P} 2$? QUESTA POSSO CONSIDERARLA COME MEDIA CAMPIONARIA DI X_i^2

X_i^2 SONO iid PERCHÈ APPLICO LA STESSA TRASF. A V.A. iid

$$M_n' = \frac{X_1^2+\dots+X_n^2}{n} \xrightarrow{P} E[M_n'] = E[X_1^2] \overset{\text{DEF. VAR}}{=} Var[X_1]+E[X_1]^2 = 2$$

c) $\frac{X_1Y_1+\dots+X_nY_n}{n} \xrightarrow{P} 0$?

STESSO RAGIONAMENTO:

$$M_n'' = \frac{X_1Y_1+\dots+X_nY_n}{n} \xrightarrow{P} E(M_n'') = E[X_1 \cancel{X_1}] \underset{X \perp Y}{=} \underbrace{E[X_1]}_{=0}E[Y_1] =$$

$$= 0$$