ES 1 1/2

a) $E[Z] = E[E[Z|X,Y]]$ DIMOSTRARE

→ LEGGE DELLE ASPETTATIVE ITERATE

$$E[Z] = E[E[Z|W]]$$

→ POSSO CONSIDERARE W COME UN VETTORE DI VARIABILI ALEATORIE AD ESEMPIO FORMATO DALLA COMBINAZIONE DI X E Y IN UN UNICO ESPERIMENTO. ESEMPIO, NUMERO COMPLESSO: $W = X + iY$

$$W \triangleq (X, Y)$$

ALTERNATIVAMENTE FACCIO I PASSAGGI:

$$E[Z|X=x, Y=y] = \sum_z z P_{Z|X,Y}(z|x,y)$$ → CONSIDERO UNA PARTICOLARE REALIZZAZIONE

$$E[Z|X,Y] = \sum_z z P_{Z|X,Y}(z|X,Y) = f(X,Y)$$ → FUNZIONE DI DUE V.A.

$$E[E[Z|X,Y]] = E[f(X,Y)] = \sum_x \sum_y \sum_z z \underbrace{P_{Z|X,Y}(z|x,y) P_{X,Y}(x,y)}_{\text{LEGGE COND. } \times \text{ LEGGE DEL COND. } = \text{ LEGGE CONGIUNTA}}$$

$$= \sum_z z \sum_x \sum_y P_{X,Y,Z}(x,y,z)$$

(MARGINALIZZ.) $$= \sum_z z P_Z(z) = E[Z]$$

b) $E[Z|X] = E[E[Z|X,Y]|X]$

$E[Z|X=x] = E[E[Z|X=x,Y]|X=x]$ DIMOSTRARE

CONSIDERO UN SOLO ESP. FISSANDO $X = x$

$$= \sum_z z P_{Z|X}(z|x)$$

(FACCIO COMPARIRE LA V.A. Y PASSAGGIO INVERSO DELLA MARGIN.)

$$= \sum_y \sum_z z P_{Z,Y|X}(z,y|x)$$

ES 1 2/2

REGOLA MOLT

$$E[Z \mid X=x] = \sum_y \underbrace{\sum_z z\, P_{Z|X,Y}(z \mid x, y)}\, P_{Y|X}(y|x) \qquad (Z,Y) \to (Z|Y)\cdot Y$$

$$= \sum_y \underbrace{E[Z \mid X=x, Y=y]}_{g(y)} \cdot P_{Y|X}(y|x)$$

LEGGE DELLE ASP. TOTALI

$$= E\big[\underbrace{E[Z \mid X=x, Y]}_{g(Y)} \mid X=x\big] \to \forall x \text{ QUINDI:}$$

$$E[Z|X] = E\big[E[Z|X,Y] \mid X\big]$$

c) $E[Z] = E\big[E[E[Z|X,Y] \mid X]\big]$

DA b) ABBIAMO DIMOSTRATO:

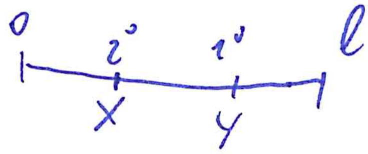
$$E[Z|X] = E\big[E[Z|X,Y] \mid X\big]$$

QUINDI POSSIAMO SOSTITUIRE USANDO LA LEGGE DELLE ASPETTAZIONI ITERATE:

$$\underbrace{E\big[E[Z|X]\big]}_{=E[Z]} = E\big[E[E[Z|X,Y] \mid X]\big]$$

PER LA LEGGE DELLE ASP. IT.

ES 2



$Y \sim [0, \ell]$

$\{X | Y = y\} \sim U[0, y]$

a) $E[X] = E[E[X|Y]] = E\left[\frac{Y}{2}\right] = \frac{E[Y]}{2} = \frac{\ell}{4}$ ← LEGGE ASP. ITERATE

b) $VAR[X] \overset{\text{LEGGE VARIAZIONE TOTALE}}{=} VAR[E[X|Y]] + E(VAR[X|Y]]$

$VAR[X|Y] = \frac{Y^2}{12}$ PERCHÉ HO $\{X|Y=y\} \sim U[0, y]$

$E[VAR[X|Y]] = \frac{E[Y^2]}{12}$

$E[Y^2] = \int_{-\infty}^{+\infty} y^2 f_Y(y)\, dy = \int_0^{\ell} \frac{y^2}{\ell}\, dy = \frac{\ell^3}{3\ell} = \frac{\ell^2}{3}$

$E[X|Y] = \frac{Y}{2}$

$VAR[E[X|Y]] = VAR\left[\frac{Y}{2}\right] = \frac{VAR(Y)}{4} = \frac{\ell^2}{48}$

$VAR(X) = \frac{\ell^2}{36} + \frac{\ell^2}{48} = \frac{7}{144}\ell^2$

ES 3

N: # CLIENTI

X_i: # ARTICOLI CLIENTE i-ESIMO

T: # TOTALE ARTICOLI

$N \perp X$ $\quad$ $E[N] = E[X] = 10$
$\quad$ iid $\quad$ $VAR[N] = VAR[X] = 16$

$T = \sum_{i=1}^{N} X_i$ $\quad$ $E[T] = ?$ $\quad$ $VAR[T] = ?$

$$E[T] = E\left[\sum_{i=1}^{N} X_i\right] \overset{\text{LEGGE ASP. IT.}}{=} E\left[E\left[\sum_{i=1}^{N} X_i \mid N\right]\right] = E\Big[N \underbrace{E[X_1]}_{iid}\Big] =$$

$$= E[X_1]\, E[N] = 100$$

$$VAR[T] \overset{\text{L. VAR. TOT.}}{=} VAR\big[E[T \mid N]\big] + E\big[VAR[T \mid N]\big]$$

$$E[T \mid N] = E[X_1] \cdot N = 10N$$

$$VAR\big[E[T \mid N]\big] = VAR[10N] = 100\, VAR[N] = 1600$$

$$VAR[T \mid N] = VAR\left[\sum_{i=1}^{N} X_i \mid N\right] \overset{\perp}{=} \sum_{i=1}^{N} VAR[X_i] = 16N$$

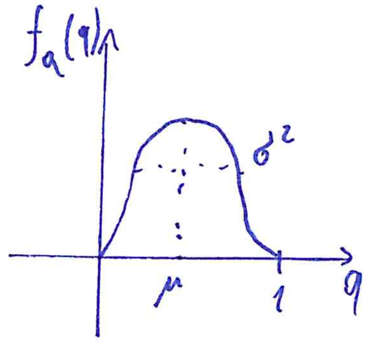
$$E\big[VAR[T \mid N]\big] = 16\, E[N] = 160$$

$$VAR[T] = 1600 + 160 = 1760$$

ES 4 1/2

n : LANCI MONETA

$P(\text{TESTA} \mid Q = q) = q$



$E[Q] = \mu$, $\text{VAR}[Q] = \sigma^2$

$$X_i = \begin{cases} 1 \to \text{TESTA} \\ 0 \to \text{CROCE} \end{cases}$$

$\{X_i \mid Q = q\}$: V.A. iid

$$X = \sum_{i=1}^{n} X_i$$

$\underbrace{\{X_i \mid Q = q\} \sim \text{BERN}(q)}$

a) $E[X_i] = E\left[E(X_i \mid Q)\right] = E[Q] = \mu$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] \overset{iid}{=} n\,E[X_i] = n\mu$$

b) $\text{COV}[X_i, X_j] \underset{\text{SE } i=j}{=} \text{VAR}[X_i] = E[X_i^2] - E(X_i)^2$

$X_i = \{0, 1\} \Rightarrow X_i^2 = X_i \Rightarrow E(X_i^2] = E[X_i] = \mu$

$\text{VAR}[X_i] = \mu - \mu^2 > 0$ PER $\mu > 0$

SE INVECE HO $i \neq j$:

$$\text{COV}[X_i, X_j] \overset{\text{DEF}}{=} E[X_i X_j] - \overbrace{E(X_i]E(X_j)}^{\mu^2}$$

$$E[X_i X_j] = E\left[E[X_i X_j \mid Q]\right]$$

(INDIPENDENZA CONDIZIONATA) $= E\left[E(X_i \mid Q]E[X_j \mid Q]\right] = E[Q^2] =$

$$= \text{VAR}[Q] + E[Q]^2 = \sigma^2 + \mu^2$$

$\text{COV}[X_i, X_j] = \sigma^2 + \mu^2 - \mu^2 = \sigma^2 > 0 \Rightarrow X_i \not\perp X_j$

ES 4 2/2

c) $\mathrm{VAR}[X] \underset{\substack{\uparrow \\ \text{VAR. TOT.}}}{=} \mathrm{VAR}[E(X|Q)] + E[\mathrm{VAR}[X|Q]] =$

$= \mathrm{VAR}[E(X_1 + X_2 \cdots + X_n | Q)] + E[\mathrm{VAR}[X_1 + \ldots + X_n | Q]]$

$= \mathrm{VAR}[n E[X_1|Q]] + E\left[\sum_{i=1}^{n} \mathrm{VAR}(X_i|Q)\right]$ ← $X_i|Q$ SONO INDIPENDENTI GRAZIE AL COND.

$= n^2 \mathrm{VAR}[Q] + E[n \overbrace{Q(1-Q)}^{\text{VARIANZA BERNOULLI}}]$

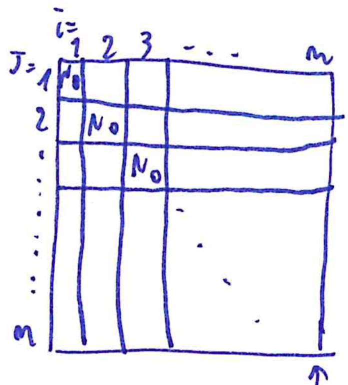
$= n^2 \sigma^2 + n(E[Q] - E[Q^2])$

$= n^2 \sigma^2 + n(\mu - \sigma^2 - \mu^2) = n(\mu - \mu^2) + n(n-1)\sigma^2$

SFRUTTO IL RISULTATO DEL PUNTO b)

$\mathrm{VAR}[X] = \mathrm{VAR}\left[\sum_{i=1}^{n} X_i\right] \underset{\substack{\uparrow \\ \text{FORMULA} \\ \text{GENERALE}}}{=} \sum_{i=1}^{n} \mathrm{VAR}[X_i] + \sum_{i \neq j} \overbrace{\mathrm{COV}[X_i, X_j]}^{= \sigma^2}$

$= n(\mu - \mu^2) + \underbrace{n(n-1)}_{\to\, n^2 - n} \sigma^2$



n^2 ELEM.

A CUI TOLGO n ELEMENTI DELLA DIAGONALE

ES 5

DIMOSTRARE CHE $E[X g(Y) | Y] = g(Y) E[X|Y]$

$$E[X g(Y) | Y = y] = E[X \underbrace{g(y)}_{\text{QUESTO È UN NUMERO}} | Y = y] = g(y) E[X | Y = y]$$

$\forall y \in \mathbb{R} : P(Y=y) > 0$

DATO CHE VALE $\forall y$ POSSO PASSARE ALLA V.A

$$E[X g(Y) | Y] = g(Y) E[X|Y]$$

ES 6

p: PROBABILITÀ DI VITTORIA $\qquad$ x: CAPITALE INIZIALE

n GIOCATE INDIPENDENTI, IDENTICAMENTE DISTRIB. (iid)

$p > \frac{1}{2}$ $\qquad$ X_k : CAPITALE ALLA SCOMMESSA K-ESIMA

$X_0 = x$, $E[X_n] = ?$

OSSERVAZIONE: SE IL CAPITALE È a, LA PUNTATA È $a(2p-1)$, DI CONSEGUENZA IN CASO DI VITTORIA IL CAPITALE DIVENTA $a + a(2p-1) = a2p$ ALTRIMENTI $a - a(2p-1)$ QUINDI MEDIAMENTE DOPO UNA PUNTATA IL CAPITALE DIVENTA:

$$a + p\left(a\cdot(2p-1)\right) - (1-p)\left(a\cdot(2p-1)\right) = a\left(1+(2p-1)^2\right)$$

$$E\left[X_{k+1} \mid X_k = a\right] = \left(1+(2p-1)^2\right)a$$

$$E\left[X_{k+1} \mid X_k\right] = \left(1+(2p-1)^2\right)X_k$$

$$E\left[X_{k+1}\right] \overset{\text{ASP. IT.}}{=} E\left[E\left[X_{k+1} \mid X_k\right]\right] = E\left[\left(1+(2p-1)^2\right)X_k\right] = \left(1+(2p-1)^2\right)E\left[X_k\right]$$

DAL TESTO SAPPIAMO CHE $E[X_0] = X_0 = x$

$$E\left[X_{k+1}\right] = \left(1+(2p-1)^2\right)\cdot\underbrace{\left(1+(2p-1)^2\right)E\left[X_{k-1}\right]}_{E[X_k]}$$

$\vdots$ (k+1) VOLTE

$\vdots$

$$= \left(1+(2p-1)^2\right)^{k+1} x$$

$$E\left[X_n\right] = \left(1+(2p-1)^2\right)^n x$$

ES 7   1/2

$X_n \sim \text{Bern}(1/n)$, $Y_n = n X_n$ $n = 1, 2, \dots$

a) $E[X_n] = \frac{1}{n}$ $Var[X_n] = \frac{1}{n}\left(1-\frac{1}{n}\right) = \frac{n-1}{n^2}$

$E[Y_n] = E[nX_n] = nE[X_n] = 1$

$Var[Y_n] = Var[nX_n] = n^2 \cdot \frac{n-1}{n^2} = n-1$

b) CHEBYSHEV APPLICATA A X_n:

$$P\left(\left|X_n - \underbrace{\frac{1}{n}}_{\text{PER } n\to\infty,\ \frac{1}{n}\to 0}\right| > \varepsilon\right) \le \frac{Var[X_n]}{\varepsilon^2} = \frac{n-1}{n^2\varepsilon^2} \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon > 0$$

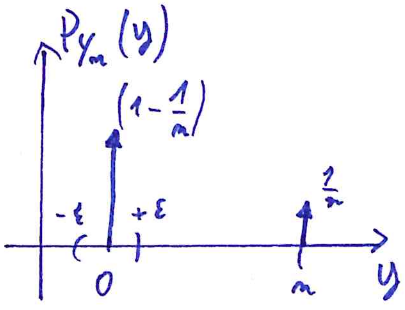
$X_n \xrightarrow{P} 0$

APPLICO A Y_n:

$$P(|Y_n - 1| > \varepsilon) \le \frac{Var[Y_n]}{\varepsilon^2} = \frac{n-1}{\varepsilon^2} \xrightarrow{n\to\infty} +\infty$$

INCONCLUSIVO

c)



$Y_n \xrightarrow{P} 0$

$P(|Y_n - 0| < \varepsilon) = P(-\varepsilon \le Y_n < \varepsilon)$

$= 1 - \frac{1}{n} \xrightarrow{n\to\infty} 1 \quad \forall \varepsilon < n$

$$\boxed{P(|Y_n - 0| > \varepsilon) \le \frac{1}{n} \xrightarrow{n\to\infty} 0 \quad \forall \varepsilon < n}$$

d) NO, AD ESEMPIO HO:

$E[Y_n] = 1$ ANCHE SE $Y_n \xrightarrow{P} 0$ (CONVERGENZA DEBOLE)

ES 7 2/2

CONVERGENZA IN MEDIA QUADRATICA:

$$X_n \xrightarrow{m.q.} c \quad : \quad \lim_{n \to \infty} E\left[(X_n - c)^2\right] = 0$$

$$(X_n - c)^2 \geq 0$$

e) MARKOV:

$$P\left((X_n - c)^2 \geq \alpha^2\right) \leq \frac{E\left[(X_n - c)^2\right]}{\alpha^2} \xrightarrow{n \to \infty} 0 \quad \forall \alpha > 0$$

↳ DALL'IPOTESI $X_n \xrightarrow{m.q.} c$

$$P\left(|X_n - c| \geq \alpha\right) \xrightarrow{n \to \infty} 0 \quad \forall \alpha \geq 0 \quad \Rightarrow X_n \xrightarrow{P} c$$

$$\text{SE } X_n \xrightarrow{m.q.} c \Rightarrow X_n \xrightarrow{P} c$$

E' UNA CONVERGENZA PIU' FORTE

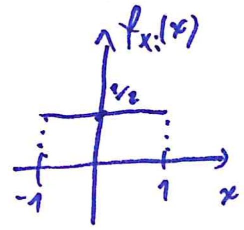
f) ABBIAMO $Y_n \xrightarrow{P} 0$

TUTTAVIA

$$E\left[(Y_n - 0)^2\right] = E\left[Y_n^2\right] = 0^2 \cdot \left(1 - \frac{1}{n}\right) + n^2 \cdot \frac{1}{n} = n \xrightarrow{n \to \infty} \infty$$

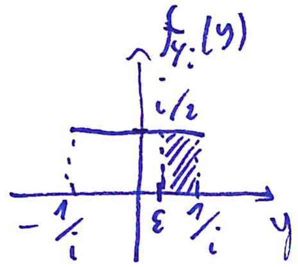
NON C'E' CONVERGENZA IN M.Q. PER Y_n

ES 8 1/2



$X_i \sim U[-1, 1]$ $\quad X_i, \; i = 1, 2 \dots \quad$ i i d

a) $\{X_i\}$ NON CONVERGE A NESSUN VALORE
(NON VARIA CON i)



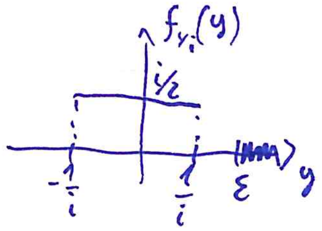
b) $Y_i = \dfrac{X_i}{i}$ TESTO LA CONV. IN PROB. A 0

$$P(|Y_i - 0| > \varepsilon) = P(|Y_i| > \varepsilon) = P(Y_i < -\varepsilon) + P(Y_i > \varepsilon)$$

(PER SIMMETRIA) $= 2P(Y_i > \varepsilon) = 2 \cdot \left(\dfrac{1}{i} - \varepsilon\right) \cdot \dfrac{i}{2}$

$(i \to \infty)$ AL CRESCERE DI i PRIMA O POI L'AREA DIVENTA NULLA $\left(\varepsilon > \frac{1}{i}\right)$ PER $i \to \infty$

CON $\xrightarrow{i}$ SUFFICIENT. GRANDE



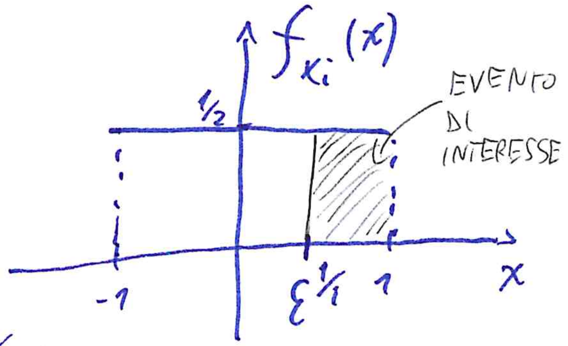
$$Y_i \xrightarrow{P} 0$$

ALTERNATIVAMENTE POSSO ANCHE DIRE

$$2P(Y_i > \varepsilon) = 2P(X_i > \varepsilon i) \xrightarrow{i \to \infty} 0$$

↑ ε È FISSATA, i CRESCE ALL'INFINITO

ES 8 2/2



c) $Z_i = (X_i)^i$

PER i CHE CRESCE, I VALORI TENDONO A "SCHIACCIARSI" ATTORNO A 0

$|x^i| < |X|$ SE $-1 < X < 1$

QUINDI TESTIAMO $Z_i \xrightarrow{P} 0$

SIMMETRIA

$$P(|Z_i - 0| > \varepsilon) = P(|Z_i| > \varepsilon) = 2P(Z_i > \varepsilon) = 2P((X_i)^i > \varepsilon)$$

AREA

$$= 2P(X_i > \varepsilon^{1/i}) = 2 \cdot \frac{1}{2}(1 - \varepsilon^{1/i}) \xrightarrow{i \to \infty} 0 \quad \forall \varepsilon > 0$$

$$Z_i \xrightarrow{P} 0$$