Robotics

PS06 – Solutions

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Part 6: Locomotion

Problem 1

A differential drive robot has two drive units, each with

- a left respectively right motor with a variable speed s_L, respectively s_R measured in rounds per minute (rpm)
- a planetary gear box with a 1:100 reduction, i.e., the wheel axis turns 100 times slower than the motor axis (but it has 100 times the torque)
- a wheel with a radius $r = 10\ cm$

The distance D between the two wheels is $30\ cm$. The coordinate frame of the robot follows the standards, i.e., it is as follows. The x-axis points from the center of motion of the robot to its front and it is co-aligned with zero degrees; angles are measured counterclockwise.

Suppose the robot drives with constant (motor-)speeds $N_L = 18{,}849\ rpm$, $N_R = 15{,}708\ rpm$ over 40 msec. Suppose its initial pose is $(0,0,0)^T$. Derive its pose after 40 msec once with the (a) approximate and once with the (b) exact arc model.

Problem 1: Approximate Vector Model

$$p_{t+\Delta t} = \begin{pmatrix} x_t + \Delta x \\ y_t + \Delta y \\ \theta_t + \Delta\theta \end{pmatrix} = \begin{pmatrix} x_t + \cos\left(\theta + \dfrac{\Delta\theta}{2}\right)\Delta d \\ y_t + \sin\left(\theta + \dfrac{\Delta\theta}{2}\right)\Delta d \\ \theta_t + \Delta\theta \end{pmatrix}$$

with

- $\Delta d = \frac{d_r + d_l}{2}$, $\Delta\theta = \omega \cdot \Delta \mathrm{t} = \frac{d_\mathrm{r} - d_\mathrm{l}}{D}$
- $d_r = v_r \cdot \Delta t$, $d_l = v_l \cdot \Delta t$

Problem 1

$$d_r = v_r \cdot \Delta t$$
$$d_l = v_l \cdot \Delta t$$

1st, proper angular velocities of the wheel axes in SI units:

- $1\ RPM = 2\pi \frac{1}{60} \frac{rad}{sec}$
- $1\ rad\ = \frac{1\ m}{1\ m}$ („virtual“ unit)

$$\omega_{\mathrm{r}} = GR \cdot N_r = \frac{1}{100} \cdot 15{,}708\ RPM$$

$$= \frac{1}{100} \cdot \frac{15{,}708}{60} \cdot 2\pi \frac{rad}{sec}$$

$$= 16.449 \frac{rad}{sec}$$

$$\omega_{\mathrm{l}} = GR \cdot N_l = \frac{1}{100} \cdot 18{,}849\ RPM$$

$$= \frac{1}{100} \cdot \frac{18{,}849}{60} \cdot 2\pi \frac{rad}{sec}$$

$$= 19.739 \frac{rad}{sec}$$

Problem 1

from angular velocity of wheel axes
to linear velocity of each wheel over ground

- $v_{\mathrm{r}} = \omega_r \cdot \mathrm{r}_{wheel} = 16.449 \, \frac{rad}{sec} \cdot 0.1 \, m = 1.6449 \frac{m}{sec}$
- $v_{\mathrm{l}} = \omega_l \cdot \mathrm{r}_{wheel} = 19.739 \, \frac{rad}{sec} \cdot 0.1 \, m = 1.9739 \frac{m}{sec}$

distances travelled per wheel in the time intervall

- $d_r = v_r \cdot \Delta t = 1.6449 \frac{m}{sec} \cdot 0.04 \, sec = 0.0658 \, m$
- $d_l = v_l \cdot \Delta t = 1.9739 \, \frac{m}{sec} \cdot 0.04 \, \mathrm{sec} = 0.0790 \, m$

Problem 1

length of the line approximation

$$\Delta d = \frac{d_r + d_l}{2} = \frac{0.0658\, m + 0.0790\, m}{2} = 0.072376\, m$$

rotation of the robot around the ICC

$$\Delta\theta = \omega \cdot \Delta \mathrm{t} = \frac{d_{\mathrm{r}} - d_{\mathrm{l}}}{D} = \frac{0.0658\, m + 0.0790\, m}{0.3\, m} = -0.04386\, rad$$

Problem 1

- $p_t = (0,0,0)^T$
- $\Delta d = 0.072376\, m$
- $\Delta\theta = -0.04386\, rad$

$$p_{t+\Delta t} = \begin{pmatrix} x_t + \Delta x \\ y_t + \Delta y \\ \theta_t + \Delta\theta \end{pmatrix} = \begin{pmatrix} x_t + \cos\left(\theta + \frac{\Delta\theta}{2}\right)\Delta d \\ y_t + \sin\left(\theta + \frac{\Delta\theta}{2}\right)\Delta d \\ \theta_t + \Delta\theta \end{pmatrix} = \begin{pmatrix} \cos(-0.044/2)\, 0.072 \\ \sin(-0.044/2)\, 0.072 \\ -0.04386 \end{pmatrix}$$

$$= \begin{pmatrix} 0.072358611 \\ \text{-}0.001586957 \\ -0.04386 \end{pmatrix}$$

Problem 1

- $p_t = (0,0,0)^T$
- $\Delta d = 0.072376\, m$
- $\Delta\theta = -0.04386\, rad$

- note that $\Delta\theta$ is relatively large (-2.5128^o)
- due to the very high velocities of the motors (and the comparably low gear ratio)
- a smaller Δt could hence be used to have a better approximation of the arc

$$p_{t+\Delta t} = \begin{pmatrix} x_t + \Delta x \\ y_t + \Delta y \\ \theta_t + \Delta\theta \end{pmatrix} = \begin{pmatrix} x_t + \cos\left(\theta + \frac{\Delta\theta}{2}\right)\Delta d \\ y_t + \sin\left(\theta + \frac{\Delta\theta}{2}\right)\Delta d \\ \theta_t + \Delta\theta \end{pmatrix} = \begin{pmatrix} \cos(-0.044/2)\, 0.072 \\ \sin(-0.044/2)\, 0.072 \\ -0.04386 \end{pmatrix}$$

$$= \begin{pmatrix} 0.072358611 \\ \text{-}0.001586957 \\ -0.04386 \end{pmatrix}$$

Problem 1: Exact Arc Model

$$\begin{pmatrix} x_{t+\Delta t} \\ y_{t+\Delta t} \\ \theta_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} cos(\omega\Delta t) & -sin(\omega\Delta t) & 0 \\ sin(\omega\Delta t) & cos(\omega\Delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_t - x_{ICC} \\ y_t - y_{ICC} \\ \theta_t \end{pmatrix} + \begin{pmatrix} x_{ICC} \\ y_{ICC} \\ \omega\Delta t \end{pmatrix}$$

with

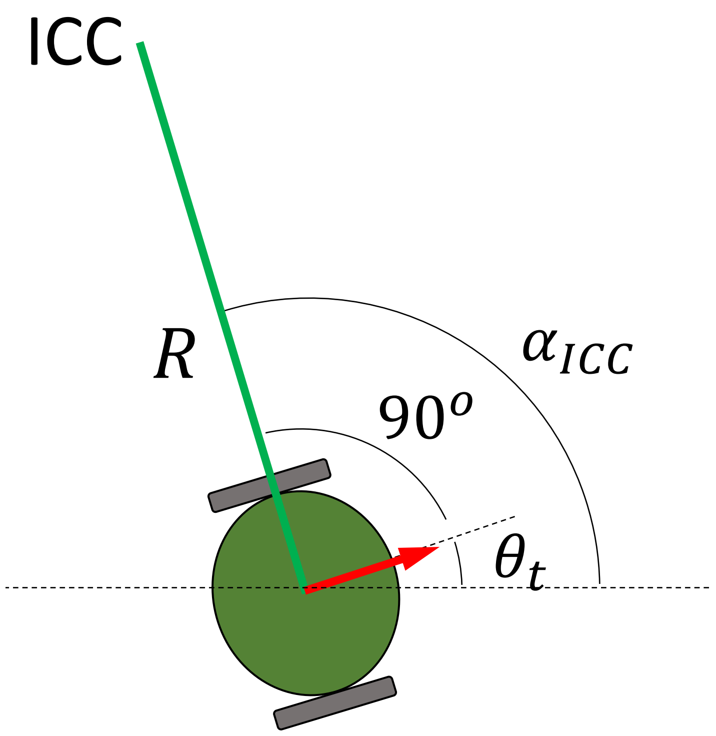
- $ICC = (x_{ICC}, y_{ICC})^T = (x_t - R\sin(\theta_t), y_t + R\cos(\theta_t))^T$
- $R = \frac{D}{2}\frac{v_\mathrm{r}+v_\mathrm{l}}{v_\mathrm{r}-v_\mathrm{l}}$, $\omega = \frac{v_\mathrm{r}-v_\mathrm{l}}{D}$

(exact, but computationally more expensive)

Note: Exact Arc Model

ICC is perpendicular to robot forward orientation
i.e., $\alpha_{ICC} = \theta_t + \pi/2$

- $\cos(\alpha) = \cos(\theta + \pi/2) = -\sin(\theta)$
- $\sin(\alpha) = \sin(\theta + \pi/2) = \cos(\theta)$



$$
\begin{aligned} x_{ICC} &= x_t + R \cdot \cos(\alpha_{ICC}) \\ y_{ICC} &= y_t + R \cdot \sin(\alpha_{ICC}) \end{aligned} \Longrightarrow \begin{aligned} x_{ICC} &= x_t - R \cdot \sin(\theta_t) \\ y_{ICC} &= y_t + R \cdot \cos(\theta_t) \end{aligned}
$$

Note: Exact Arc Model

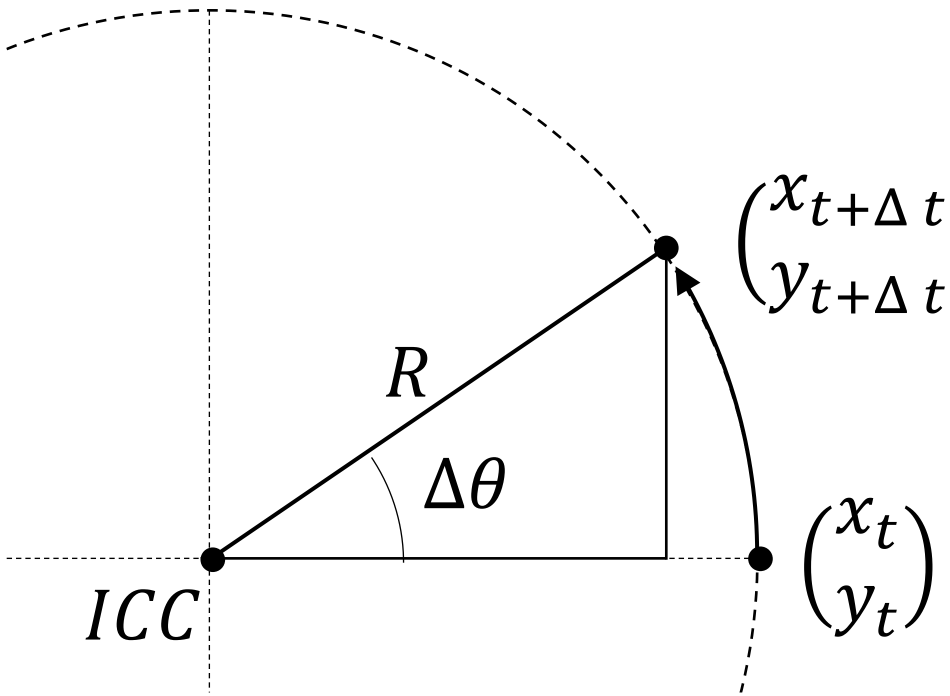
2D rotation matrix around ICC
with $\alpha = \Delta\theta + \frac{\pi}{2}$ and $\Delta\theta = \omega\Delta t$

$$\begin{pmatrix} x_{t+\Delta t} \\ y_{t+\Delta t} \\ \theta_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \cos(\omega\Delta t) & -\sin(\omega\Delta t) & 0 \\ \sin(\omega\Delta t) & \cos(\omega\Delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_t - x_{ICC} \\ y_t - y_{ICC} \\ \theta_t \end{pmatrix} + \begin{pmatrix} x_{ICC} \\ y_{ICC} \\ \omega\Delta t \end{pmatrix}$$

shift to ICC,
rotate there

shift
back

Note: Exact Arc Model



$$\begin{pmatrix} x_{t+\Delta\, t} \\ y_{t+\Delta\, t} \end{pmatrix} = \begin{pmatrix} x_t + R \cdot \cos(\Delta\theta) \\ y_t + R \cdot \sin(\Delta\theta) \end{pmatrix}$$

this model is not an approximation

- i.e., the line Δd is not used but the proper location according to the arc
- the drawback is that it is computationally more expensive

Problem 1

from angular velocity of wheel axes
to linear velocity of each wheel over ground

- $v_{\mathrm{r}} = \omega_r \cdot \mathrm{r}_{wheel} = 16.449 \frac{rad}{sec} \cdot 0.1\, m = 1.6449 \frac{m}{sec}$
- $v_{\mathrm{l}} = \omega_l \cdot \mathrm{r}_{wheel} = 19.739 \frac{rad}{sec} \cdot 0.1\, m = 1.9739 \frac{m}{sec}$

distances travelled per wheel in the time intervall

- $d_r = v_r \cdot \Delta t = 1.6449 \frac{m}{sec} \cdot 0.04\, sec = 0.0658\, m$
- $d_l = v_l \cdot \Delta t = 1.9739 \frac{m}{sec} \cdot 0.04\,\mathrm{sec} = 0.0790\, m$

Problem 1: Exact Arc Model

$$v_{wheel} = r \cdot \omega_{wheel-axis} = r \cdot GR \cdot \omega_{motor-axis}$$

$$v_{\mathrm{l}} = 0.1m \cdot \frac{1}{100} \cdot 18{,}849\ RPM = \frac{2\pi}{1000} m \cdot 18{,}849/60\ \frac{rad}{sec} = 1.6449 \frac{m}{sec}$$

$$v_{\mathrm{r}} = 0.1m \cdot \frac{1}{100} \cdot 15{,}708\ RPM = \frac{2\pi}{1000} m \cdot 15{,}708/60\ \frac{rad}{sec} = 1.9739 \frac{m}{sec}$$

Problem 1: Exact Arc Model

$$\omega = \frac{v_\mathrm{r} - v_\mathrm{l}}{D} = \frac{(1.6449 - 1.9739)\frac{m}{s}}{0.3m} = -1.096415836\,\frac{rad}{s}$$

$$R = \frac{D}{2}\frac{v_\mathrm{r} + v_\mathrm{l}}{v_\mathrm{r} - v_\mathrm{l}} = \frac{0.3m}{2}\frac{(1.6449 + 1.9739)\frac{m}{s}}{(1.6449 - 1.9739)\frac{m}{s}} = -1.65m$$

$$\begin{aligned} p_{ICC} &= (x_{ICC}, y_{ICC})^T \\ &= (x_t - R\sin(\theta_t), y_t + R\cos(\theta_t))^T \\ &= (0 + 1.65\sin(0), 0 - 1.65\cos(0))^T = (0, -1.65)^T \end{aligned}$$

start pose:
$(x_t, y_t, \theta_t)^T = (0,0,0)^T$

Problem 1: Exact Arc Model

$$\omega\Delta t = \Delta\theta = -1.096415836\,\frac{rad}{s}\cdot 0.04\mathrm{s} = -\,0.04386\,rad$$

$$\begin{pmatrix} x_{t+\Delta t} \\ y_{t+\Delta t} \\ \theta_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \cos(\,\omega\Delta t) & -\sin(\,\omega\Delta t) & 0 \\ \sin(\,\omega\Delta t) & \cos(\,\omega\Delta t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_t - x_{ICC} \\ y_t - y_{ICC} \\ \theta_t \end{pmatrix} + \begin{pmatrix} x_{ICC} \\ y_{ICC} \\ \omega\Delta t \end{pmatrix}$$

$$= \begin{pmatrix} \cos(-0.044) & -\sin(-0.044) & 0 \\ \sin(-0.044) & \cos(-0.044) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0-0 \\ 0+1.65 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1.65 \\ -0.04386 \end{pmatrix}$$

$$= \begin{pmatrix} 0.999 & 0.0438 & 0 \\ -0.0438 & 0.999 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1.65 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1.65 \\ -0.04386 \end{pmatrix} = \begin{pmatrix} 0.072352812 \\ -0.00158683 \\ -0.04386 \end{pmatrix}$$

Problem 1

approximation with vector

$$p_{t+\Delta t} = \begin{pmatrix} 0.072358611 \\ -0.001586957 \\ -0.04386 \end{pmatrix}$$

exact arc

$$\begin{pmatrix} x_{t+\Delta t} \\ y_{t+\Delta t} \\ \theta_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} 0.072352812 \\ -0.00158683 \\ -0.04386 \end{pmatrix}$$

approximation model reasonably accurate
(despite very high velocities of the wheels and relatively large Δt)

Problem 2

Given an omni-drive robot with 4 motors with omni-wheels W_i that are evenly spaced apart at 90^o starting with 0^o, i.e., W_1 is at 0^o, W_2 is at 90^o, and so on. The distance from the center of motion to each wheel is R, the wheel radius is r and the angular velocity of each wheel is ω_i.

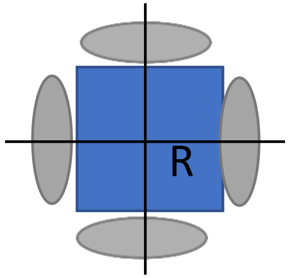
Derive the inverse Kinematics of this robot, i.e., derive the matrix M with

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = M \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$

for the translational velocity $V_t = \left(V_x, V_y\right)^T = (\dot{x}, \dot{y})^T$ and the angular velocity $\omega = \dot{\theta}$ of the robot.

Problem 2

omni-drive with
- 4 wheels
- all with the same distance R to the center of motion



$$\omega_i = \frac{1}{r}\left(-\sin(\alpha_i)\,\dot{x} + \cos(\alpha_i)\,\dot{y} + R\dot{\theta}\right)$$

here with $\alpha_1 = 0^o, \alpha_2 = 90^o, \alpha_3 = 180^o, \alpha_4 = 270^o$

note: variation of notations here
- velocities of the robot as time derivatives
- angular velocities of the wheels denoted in the lecture with φ_i

Problem 2

Omni-Drive Inverse Kinematics

$$\omega_i = \frac{1}{r}\left(-\sin(\alpha_i)\,\dot{x} + \cos(\alpha_i)\,\dot{y} + R\dot{\theta}\right)$$

here with $\alpha_1 = 0^o, \alpha_2 = 90^o, \alpha_3 = 180^o, \alpha_4 = 270^o$

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix} = \frac{1}{r}\begin{pmatrix}-\sin(\alpha_1) & \cos(\alpha_1) & R\\ -\sin(\alpha_2) & \cos(\alpha_2) & R\\ -\sin(\alpha_3) & \cos(\alpha_3) & R\\ -\sin(\alpha_4) & \cos(\alpha_4) & R\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix} = \frac{1}{r}\begin{pmatrix}0 & 1 & R\\ -1 & 0 & R\\ 0 & -1 & R\\ 1 & 0 & R\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix}$$

Problem 2

Omni-Drive Forward Kinematics

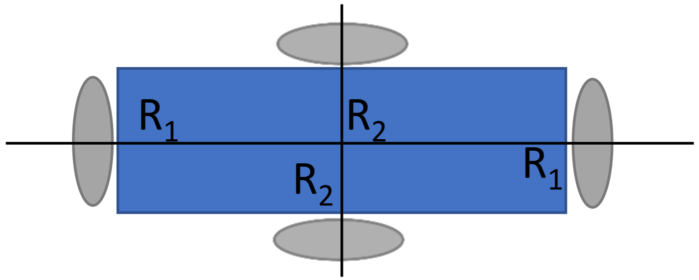
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & R \\ -1 & 0 & R \\ 0 & -1 & R \\ 1 & 0 & R \end{pmatrix}^{+} r \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}$$

note: M not square, hence pseudo-inverse here

Problem 2

note: the distances R_i of the wheels to the center of motion may vary

example:



$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{pmatrix} = \frac{1}{r}\begin{pmatrix}-\sin(\alpha_1) & \cos(\alpha_1) & R_1\\ -\sin(\alpha_2) & \cos(\alpha_2) & R_2\\ -\sin(\alpha_3) & \cos(\alpha_3) & R_1\\ -\sin(\alpha_4) & \cos(\alpha_4) & R_2\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{pmatrix}$$