Robotics

PS10 – Solutions

Andreas Birk

Constructor University

Part 10: Registration

Problem 1

Given two point-sets $A = \{a_i\}$ and $B = \{b_i\}$ where each a_i corresponds to the spatially transformed, i.e., rotated and translated (with noise), point b_i:

	A		B	
i	x	y	x	y
1	0.00	0.00	2.10	2.71
2	1.00	0.00	3.01	3.72
3	0.00	1.00	1.24	3.79
4	1.00	1.00	1.79	4.56
5	2.00	0.50	3.04	4.63

Use Horn's algorithm to determine the underlying rotation R and translation t.

Problem 1

no noise example first
ground truth: rotation by 45 deg, translation by $(2,3)^T$

$$R = \begin{pmatrix} c(45^o) & s(45^o) \\ -s(45^o) & c(45^o) \end{pmatrix}$$
$$= \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$$

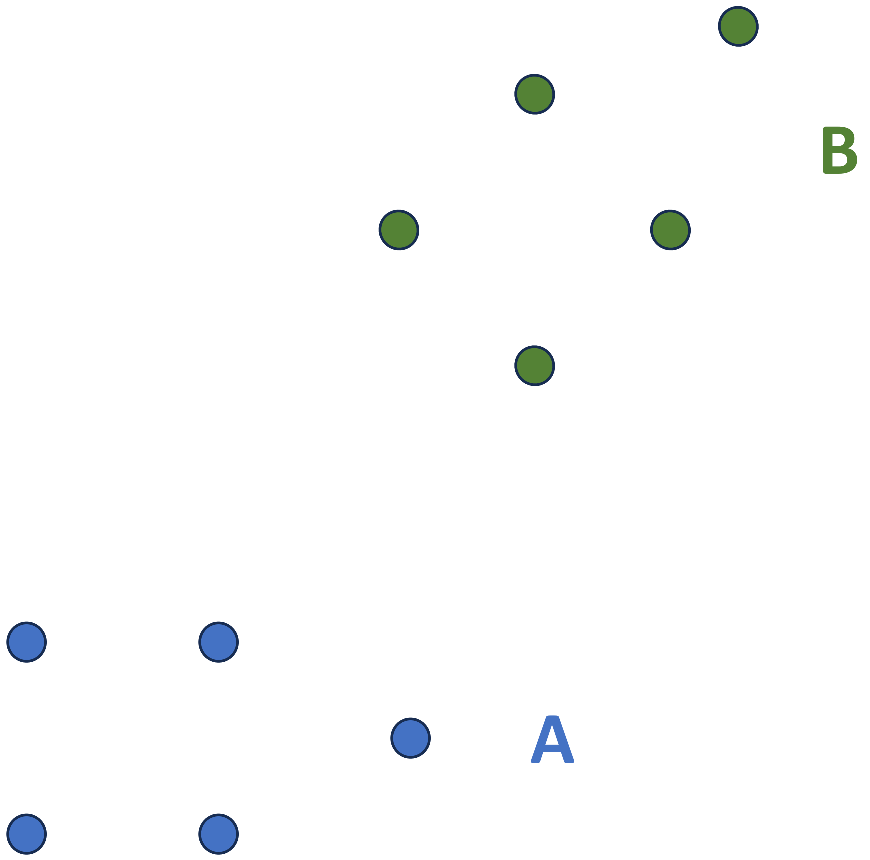
$$t = (2,3)^T$$

	A		B	
i	x	y	x	y
1	0	0	2	3
2	1	0	2.71	3.71
3	0	1	1.29	3.71
4	1	1	2	4.41
5	2	0.5	3.06	4.77

Problem 1

no noise, rotation by 45 deg, translation by $(2,3)^T$

	A		B	
i	x	y	x	y
1	0	0	2	3
2	1	0	2.71	3.71
3	0	1	1.29	3.71
4	1	1	2	4.41
5	2	0.5	3.06	4.77



Problem 1

compute centroids $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$ $\bar{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$

	A				B	
	x	y			x	y
	0	0			2	3
	1	0			2.71	3.71
	0	1			1.29	3.71
	1	1			2	4.41
	2	0.5			3.06	4.77
$\bar{a}$	**0.80**	**0.50**		$\bar{b}$	**2.21**	**3.92**

Problem 1

subtract centroids $\bar{a}$ and $\bar{b}$ from A and B, i.e.,

$$A' = \{a_i - \bar{a}\} = \{a_i{}'\} \qquad B' = \{b_i - \bar{b}\} = \{b_i{}'\}$$

	A	
	x	y
	0	0
	1	0
	0	1
	1	1
	2	0.5
$\bar{a}$	**0.80**	**0.50**

	B	
	x	y
	2	3
	2.71	3.71
	1.29	3.71
	2	4.41
	3.06	4.77
$\bar{b}$	**2.21**	**3.92**

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.21	-0.92
2	0.20	-0.50	0.49	-0.21
3	-0.80	0.50	-0.92	-0.21
4	0.20	0.50	-0.21	0.49
5	1.20	0.00	0.85	0.85

Problem 1

compute matrix $W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} b_i' \cdot a_i'^T$

$$W_1 = b_1' \cdot a_1'^T$$

$$= \begin{pmatrix} -0.21 \\ -0.92 \end{pmatrix} \cdot \begin{pmatrix} -0.80 & -0.50 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1697 & 0.1061 \\ 0.7354 & 0.4596 \end{pmatrix}$$

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.21	-0.92
2	0.20	-0.50	0.49	-0.21
3	-0.80	0.50	-0.92	-0.21
4	0.20	0.50	-0.21	0.49
5	1.20	0.00	0.85	0.85

Problem 1

compute matrix $W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} b_i' \cdot a_i'^T$

$$W_2 = b_2' \cdot a_2'^{\,T}$$

$$= \begin{pmatrix} 0.49 \\ -0.21 \end{pmatrix} \cdot \begin{pmatrix} 0.20 & -0.50 \end{pmatrix}$$

$$= \begin{pmatrix} 0.0990 & -0.2475 \\ -0.0424 & 0.1061 \end{pmatrix}$$

and so on for W_3 to W_5…

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.21	-0.92
2	0.20	-0.50	0.49	-0.21
3	-0.80	0.50	-0.92	-0.21
4	0.20	0.50	-0.21	0.49
5	1.20	0.00	0.85	0.85

Problem 1

$$W = W_1 + W_2 + \cdots + W_5$$

$$= \begin{pmatrix} 0.1697 & 0.1061 \\ 0.7354 & 0.4596 \end{pmatrix} + \begin{pmatrix} 0.0990 & -0.2475 \\ -0.0424 & 0.1061 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1.9799 & -0.7071 \\ 1.9799 & 0.7071 \end{pmatrix}$$

compute SVD of $W = USV^T$

Problem 1

$$W = \begin{pmatrix} 1.9799 & -0.7071 \\ 1.9799 & 0.7071 \end{pmatrix} = USV^T$$

$$U = \begin{pmatrix} -0.70711 & -0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}, S = \begin{pmatrix} 2.8 & 0 \\ 0 & 1 \end{pmatrix}, V^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

rotation $R = UV^T$

$$R = \begin{pmatrix} -0.70711 & -0.70711 \\ -0.70711 & 0.70711 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$$

Problem 1

translation $t = \bar{b} - R\bar{a}$

$$t = \begin{pmatrix} 2.212 \\ 3.919 \end{pmatrix} - \begin{pmatrix} -0.70711 & -0.70711 \\ -0.70711 & 0.70711 \end{pmatrix} \begin{pmatrix} 0.80 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

	A		B	
	x	y	x	y
	0	0	2	3
	1	0	2.71	3.71
	0	1	1.29	3.71
	1	1	2	4.41
	2	0.5	3.06	4.77
$\bar{a}$ / $\bar{b}$	**0.80**	**0.50**	**2.21**	**3.92**

Problem 1

error

$$E(R,t) = \sum_{i=1}^{n} (\|a_i'\|^2 + \|b_i'\|^2) - 2(\sigma_1 + \sigma_2)$$

$$= 7.6 - 2(2.8 + 1)$$

$$= 0$$

with $S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 2.8 & 0 \\ 0 & 1 \end{pmatrix}$

Problem 1

ground truth:

rotation $R = \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$, **translation** $t = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

result of Horn's Algorithm:

rotation $R = \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$, **translation** $t = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

error $E(R, t) = 0$

Problem 1

now, **with noise** on B, i.e., **HW numbers** (and same ground truth)

	A		B	
i	x	y	x	y
1	0.00	0.00	2	3
2	1.00	0.00	2.71	3.71
3	0.00	1.00	1.29	3.71
4	1.00	1.00	2	4.41
5	2.00	0.50	3.06	4.77

B without noise

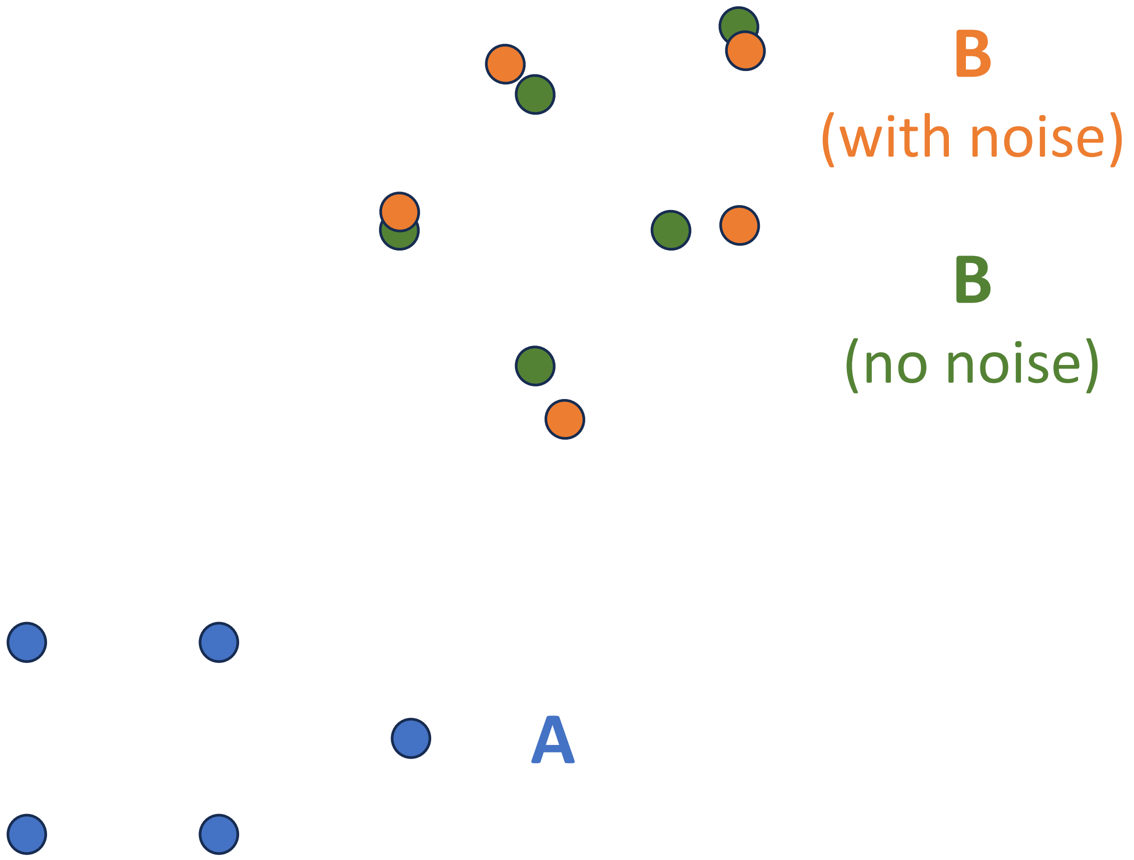
	A		B	
i	x	y	x	y
1	0.00	0.00	2.10	2.71
2	1.00	0.00	3.01	3.72
3	0.00	1.00	1.24	3.79
4	1.00	1.00	1.79	4.56
5	2.00	0.50	3.04	4.63

B with noise

Problem 1

with noise, rotation by 45 deg, translation by $(2,3)^T$

	A		B	
i	x	y	x	y
1	0.00	0.00	2.10	2.71
2	1.00	0.00	3.01	3.72
3	0.00	1.00	1.24	3.79
4	1.00	1.00	1.79	4.56
5	2.00	0.50	3.04	4.63



Problem 1

compute centroids $\bar{a} = \frac{1}{n}\sum_{i=1}^{n} a_i$ $\bar{b} = \frac{1}{n}\sum_{i=1}^{n} b_i$

	A			B	
	x	y		x	y
	0	0		2.10	2.71
	1	0		3.01	3.72
	0	1		1.24	3.79
	1	1		1.79	4.56
	2	0.5		3.04	4.63
$\bar{a}$	**0.80**	**0.50**	$\bar{b}$	**2.236**	**3.881**

Problem 1

subtract centroids $\bar{a}$ and $\bar{b}$ from A and B, i.e.,

$$A' = \{a_i - \bar{a}\} = \{a_i{}'\} \qquad B' = \{b_i - \bar{b}\} = \{b_i{}'\}$$

	A	
	x	y
	0	0
	1	0
	0	1
	1	1
	2	0.5
$\bar{a}$	**0.80**	**0.50**

	B	
	x	y
	2.10	2.71
	3.01	3.72
	1.24	3.79
	1.79	4.56
	3.04	4.63
$\bar{b}$	**2.236**	**3.881**

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.14	-1.17
2	0.20	-0.50	0.77	-0.16
3	-0.80	0.50	-0.99	-0.09
4	0.20	0.50	-0.45	0.68
5	1.20	0.00	0.80	0.75

Problem 1

compute matrix $W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} b'_i \cdot {a'_i}^T$

$W_1 = b'_1 \cdot {a'_1}^T$

$= \begin{pmatrix} -0.14 \\ -1.17 \end{pmatrix} \cdot \begin{pmatrix} -0.80 & -0.50 \end{pmatrix}$

$= \begin{pmatrix} 0.1089 & 0.0681 \\ 0.9370 & 0.5856 \end{pmatrix}$

and so on for W_2 to W_5…

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.14	-1.17
2	0.20	-0.50	0.77	-0.16
3	-0.80	0.50	-0.99	-0.09
4	0.20	0.50	-0.45	0.68
5	1.20	0.00	0.80	0.75

Problem 1

$$W = W_1 + W_2 + \cdots + W_5 = \begin{pmatrix} 1.9339 & -1.0371 \\ 2.0119 & 0.9621 \end{pmatrix} = USV^T$$

- $U = \begin{pmatrix} -0.697399 & -0.716683 \\ -0.716683 & 0.697399 \end{pmatrix},$
- $S = \begin{pmatrix} 2.7908 & 0 \\ 0 & 1.4144 \end{pmatrix},$
- $V^T = \begin{pmatrix} -0.9999 & 0.0121 \\ 0.0121 & 0.9999 \end{pmatrix}$

Problem 1

rotation $R = UV^T$

$$R = \begin{pmatrix} -0.697399 & -0.716683 \\ -0.716683 & 0.697399 \end{pmatrix} \begin{pmatrix} -0.9999 & 0.0121 \\ 0.0121 & 0.9999 \end{pmatrix} = \begin{pmatrix} 0.6887 & -0.7251 \\ 0.7251 & 0.6887 \end{pmatrix}$$

translation $t = \bar{b} - R\bar{a}$

$$t = \begin{pmatrix} 2.236 \\ 3.881 \end{pmatrix} - \begin{pmatrix} 0.6887 & -0.7251 \\ 0.7251 & 0.6887 \end{pmatrix} \begin{pmatrix} 0.80 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 2.02 \\ 2.96 \end{pmatrix}$$

Problem 1

error

$$E(R,t) = \sum_{i=1}^{n} (\|a_i'\|^2 + \|b_i'\|^2) - 2(\sigma_1 + \sigma_2)$$

$$= 8.68 - 2(2.7908 + 1.41435)$$

$$= 0.266815944$$

with $S = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 2.7908 & 0 \\ 0 & 1.41435 \end{pmatrix}$

Problem 1

ground truth:

rotation $R = \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$, **translation** $t = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

result of Horn's Algorithm:

rotation $R = \begin{pmatrix} 0.6887 & -0.7251 \\ 0.7251 & 0.6887 \end{pmatrix}$, **translation** $t = \begin{pmatrix} 2.02 \\ 2.96 \end{pmatrix}$

quite accurate, despite few points and high noise

error $E(R, t) = 0.266815944$

Problem 1

additional example **with scale** (and no noise)
ground truth: rotation by 45 deg, translation by $(2,3)^T$, scale 2.7

$$R = \begin{pmatrix} c(45^o) & s(45^o) \\ -s(45^o) & c(45^o) \end{pmatrix}$$

$$= \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$$

$$t = (2,3)^T$$

$$s = 2.7$$

	A		B	
i	x	y	x	y
1	0	0	2.00	3.00
2	1	0	3.91	4.91
3	0	1	0.09	4.91
4	1	1	2.00	6.82
5	2	0.5	4.86	7.77

Problem 1

compute centroids $\bar{a}$ and $\bar{b}$ and subtract them from A and B

$$A' = \{a_i - \bar{a}\} = \{a_i{}'\} \qquad B' = \{b_i - \bar{b}\} = \{b_i{}'\}$$

	A	
	x	y
	0	0
	1	0
	0	1
	1	1
	2	0.5
$\bar{a}$	**0.80**	**0.50**

	B	
	x	y
	2.00	3.00
	3.91	4.91
	0.09	4.91
	2.00	6.82
	4.86	7.77
$\bar{b}$	**2.573**	**5.482**

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.57	-2.48
2	0.20	-0.50	1.34	-0.57
3	-0.80	0.50	-2.48	-0.57
4	0.20	0.50	-0.57	1.34
5	1.20	0.00	2.29	2.29

Problem 1

compute $W = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} b_i' \cdot a_i'^T = \begin{pmatrix} 5.346 & -1.909 \\ 5.346 & 1.909 \end{pmatrix}$

and its SVD

$$U = \begin{pmatrix} 0.70711 & 0.70711 \\ 0.70711 & -0.70711 \end{pmatrix}, S = \begin{pmatrix} 2.8 & 0 \\ 0 & 1 \end{pmatrix}, V^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

rotation $R = UV^T$

$$R = \begin{pmatrix} 0.70711 & 0.70711 \\ 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 7.56 & 0 \\ 0 & 2.7 \end{pmatrix} = \begin{pmatrix} 0.70711 & 0.70711 \\ -0.70711 & 0.70711 \end{pmatrix}$$

Problem 1

scale factor

$$\boldsymbol{s} = \sqrt{\frac{\sum_{i=1}^{n}\left\|b_i'\right\|^2}{\sum_{i=1}^{n}\left\|a_i'\right\|^2}}$$

$$= \sqrt{\frac{0.89 + 0.29 + 0.89 + 0.29 + 1.44}{6.49 + 2.11 + 6.49 + 2.11 + 10.50}}$$

$$= \sqrt{\frac{3.8}{22.7}} = \mathbf{2.7}$$

	A’		B’	
i	x	y	x	y
1	-0.80	-0.50	-0.57	-2.48
2	0.20	-0.50	1.34	-0.57
3	-0.80	0.50	-2.48	-0.57
4	0.20	0.50	-0.57	1.34
5	1.20	0.00	2.29	2.29

Problem 1

translation $t = \bar{b} - sR\bar{a}$

$$\boldsymbol{t} = \begin{pmatrix} 2.573 \\ 5.482 \end{pmatrix} - 2.7 \begin{pmatrix} -0.70711 & -0.70711 \\ -0.70711 & 0.70711 \end{pmatrix} \begin{pmatrix} 0.80 \\ 0.50 \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

	A	
	x	y
	0	0
	1	0
	0	1
	1	1
	2	0.5
$\bar{\boldsymbol{a}}$	**0.80**	**0.50**

	B	
	x	y
	2.00	3.00
	3.91	4.91
	0.09	4.91
	2.00	6.82
	4.86	7.77
$\bar{\boldsymbol{b}}$	**2.573**	**5.482**

Problem 2

Suppose the correspondences between the points in *A* and *B* from the previous problem are not known. What do the nearest neighbor correspondences in a first step of the Iterative Closest Point (ICP) algorithm look like?

Problem 2

ckeck Euclidean distances of all a_i to all b_i

$\|a_i, b_i\|$	b_1	b_2	b_3	b_4	b_5
a_1	3.428	4.781	3.986	4.903	5.537
a_2	2.925	4.224	3.795	4.632	5.058
a_3	2.708	4.053	3.052	3.988	4.734
a_4	2.033	3.378	2.798	3.651	4.162
a_5	2.212	3.371	3.373	4.070	4.257

Problem 2

ckeck Euclidean distances of all a_i to all b_i

$\|a_i, b_i\|$	b_1	b_2	b_3	b_4	b_5
a_1	3.428	4.781	3.986	4.903	5.537
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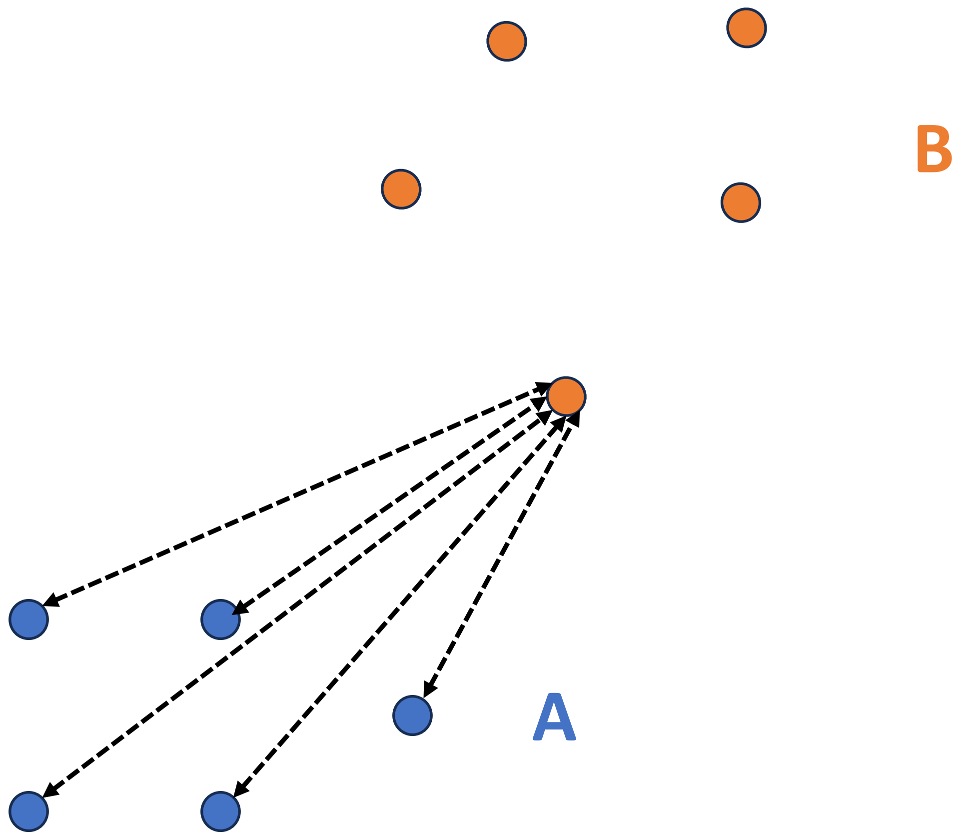
a_4	2.033	3.378	2.798	3.651	4.162
a_5	2.212	3.371	3.373	4.070	4.257

b_1 is always the closest, i.e., start: all a_i correspond to b_1

Problem 2

b_1 is always the closest, i.e., start: all a_i correspond to b_1

$\|a_i, b_i\|$	b_1	b_2	b_3	b_4	b_5
a_1	3.428	4.781	3.986	4.903	5.537
a_2	2.925	4.224	3.795	4.632	5.058
a_3	2.708	4.053	3.052	3.988	4.734
a_4	2.033	3.378	2.798	3.651	4.162
a_5	2.212	3.371	3.373	4.070	4.257



Problem 2

all a_i correspond to b_1, hence B for Horn only consists of b_1

	A		B	
	x	y	x	y
	0	0	2.10	2.71
	1	0	2.10	2.71
	0	1	2.10	2.71
	1	1	2.10	2.71
	2	0.5	2.10	2.71
$\bar{a}$	**0.80**	**0.50**		
$\bar{b}$			**2.10**	**2.71**

Problem 2

therefore, 1st ICP step with Horn finds only a translation as

$$W = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = USV^T \text{ with } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, V^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

hence

- $R = UV^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = I$
- $t = \bar{b} - R\bar{a} = \begin{pmatrix} 2.10 \\ 2.71 \end{pmatrix} - I \cdot \begin{pmatrix} 0.8 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1.30 \\ 2.21 \end{pmatrix}$

Problem 2

apply R and t to B from $k = 1$ (the original B)

$k = 1$

	A		B	
i	x	y	x	y
1	0.00	0.00	2.10	2.71
2	1.00	0.00	3.01	3.72
3	0.00	1.00	1.24	3.79
4	1.00	1.00	1.79	4.56
5	2.00	0.50	3.04	4.63

$k = 2$

	A		B	
i	x	y	x	y
1	0.00	0.00	0.800	0.500
2	1.00	0.00	1.707	1.507
3	0.00	1.00	-0.057	1.577
4	1.00	1.00	0.490	2.354
5	2.00	0.50	1.741	2.418

all points in B moved by $-t = \begin{pmatrix} -1.30 \\ -2.21 \end{pmatrix}$

Problem 2

	A		B	
i	x	y	x	y
1	0.00	0.00	0.800	0.500
2	1.00	0.00	1.707	1.507
3	0.00	1.00	-0.057	1.577
4	1.00	1.00	0.490	2.354
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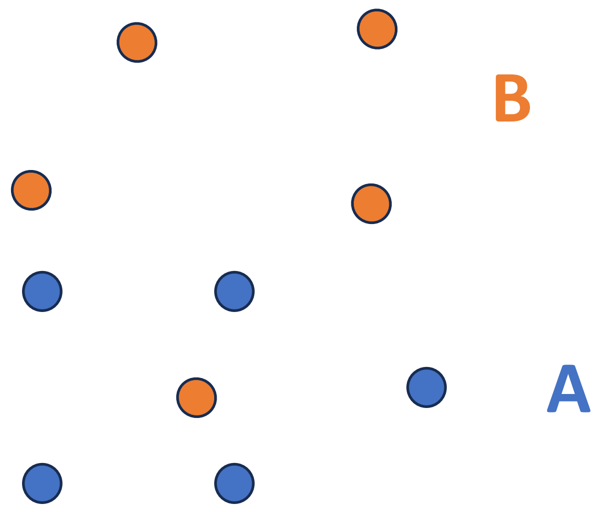
all points in B moved by $-t = \begin{pmatrix} -1.30 \\ -2.21 \end{pmatrix}$



Problem 2

ckeck Euclidean distances of all a_i to all new b_i

$\|a_i, b_i\|$	b_1	b_2	b_3	b_4	b_5
a_1	0.943	2.277	1.578	2.405	2.979
a_2	0.539	1.665	1.899	2.409	2.529
a_3	0.943	1.781	0.580	1.440	2.245
a_4	0.539	0.870	1.204	1.447	1.600
a_5	1.200	1.049	2.322	2.391	1.935

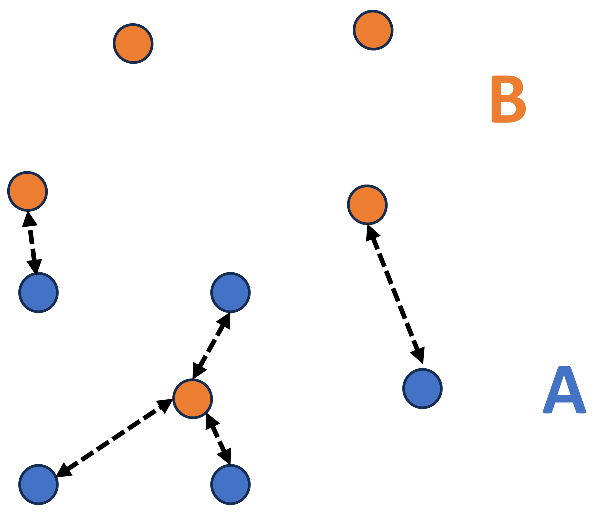
there are now new nearest neighbors and hence new correspondences

$a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_1, a_3 \leftrightarrow b_3, a_4 \leftrightarrow b_1, a_5 \leftrightarrow b_2$

Problem 2

$$a_1 \leftrightarrow b_1, a_2 \leftrightarrow b_1, a_3 \leftrightarrow b_3, a_4 \leftrightarrow b_1, a_5 \leftrightarrow b_2$$

$\|a_i, b_i\|$	b_1	b_2	b_3	b_4	b_5
a_1	0.943	2.277	1.578	2.405	2.979
a_2	0.539	1.665	1.899	2.409	2.529
a_3	0.943	1.781	0.580	1.440	2.245
a_4	0.539	0.870	1.204	1.447	1.600
a_5	1.200	1.049	2.322	2.391	1.935



Problem 2

the B for Horn is based on these correspondences

$k = 2$

	A		B	
i	x	y	x	y
1	0.00	0.00	0.800	0.500
2	1.00	0.00	1.707	1.507
3	0.00	1.00	-0.057	1.577
4	1.00	1.00	0.490	2.354
5	2.00	0.50	1.741	2.418

all points in B moved by $-t = \begin{pmatrix} -1.30 \\ -2.21 \end{pmatrix}$

	A		B	
i	x	y	x	y
1	0.00	0.00	0.800	0.500
2	1.00	0.00	0.800	0.500
3	0.00	1.00	-0.057	1.577
4	1.00	1.00	0.800	0.500
5	2.00	0.50	1.707	1.507

$a_1 \leftrightarrow b_1,$
$a_2 \leftrightarrow b_1,$
$a_3 \leftrightarrow b_3,$
$a_4 \leftrightarrow b_1,$
$a_5 \leftrightarrow b_2$

Problem 2

	A		B	
i	x	y	x	y
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$a_1 \leftrightarrow b_1,$
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$a_4 \leftrightarrow b_1,$
$a_5 \leftrightarrow b_2$

use this as input for Horn's method to get new R and t

iterate until $R \approx I$ and $t \approx (0,0)^{\mathrm{T}}$