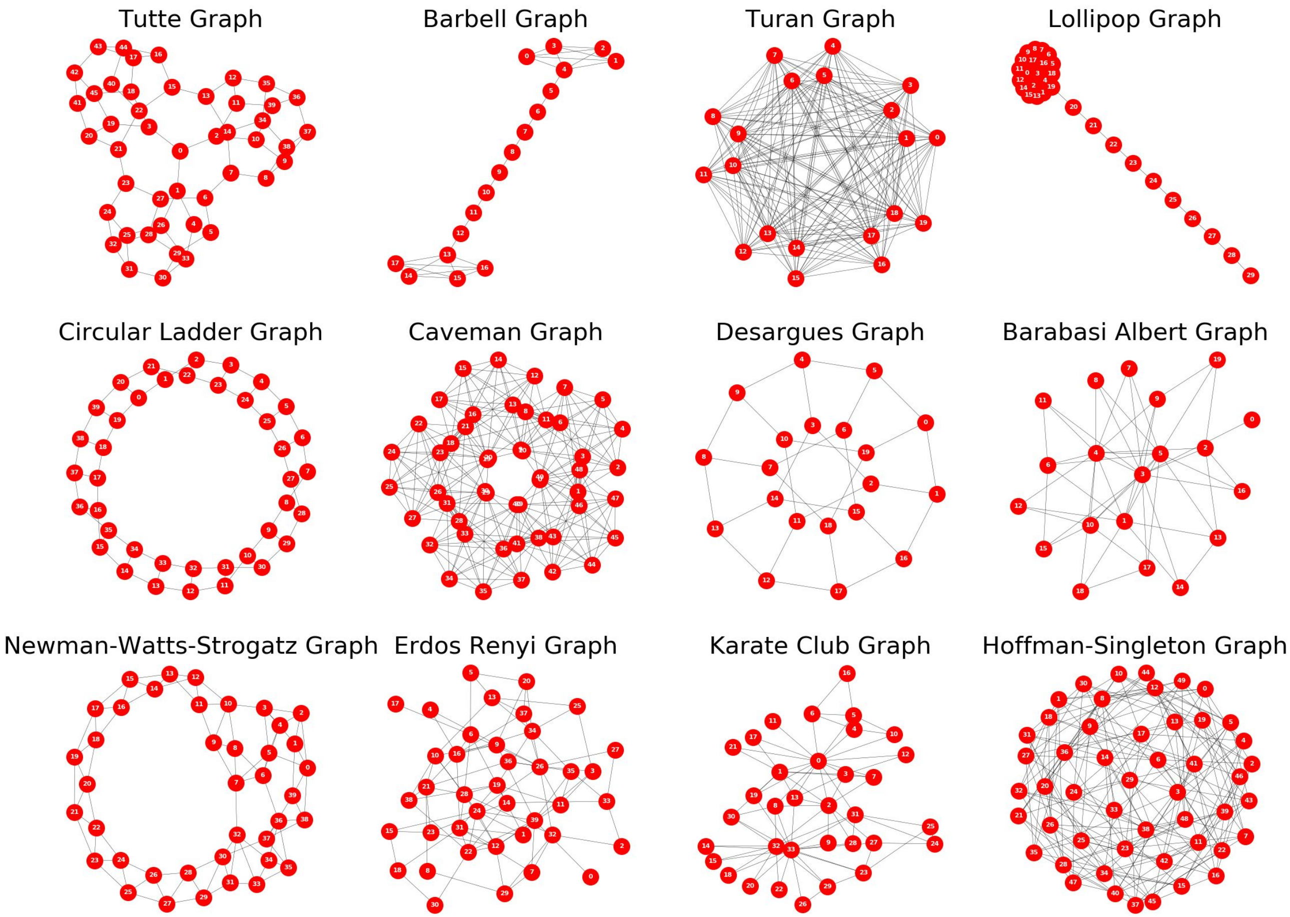
Graph Homomorphisms

Kirill Rodriguez

Graph

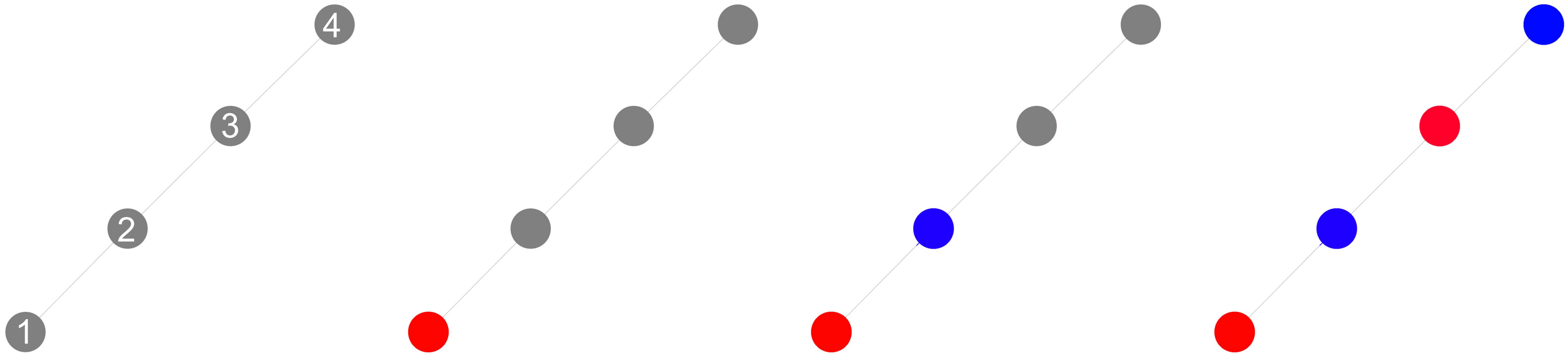
- Nodes and edges
- Complete graphs (K_n)
- Cycles (C_n)
- Paths (P_n)
- V(G) means vertices
- E(G) means edges



Coloring

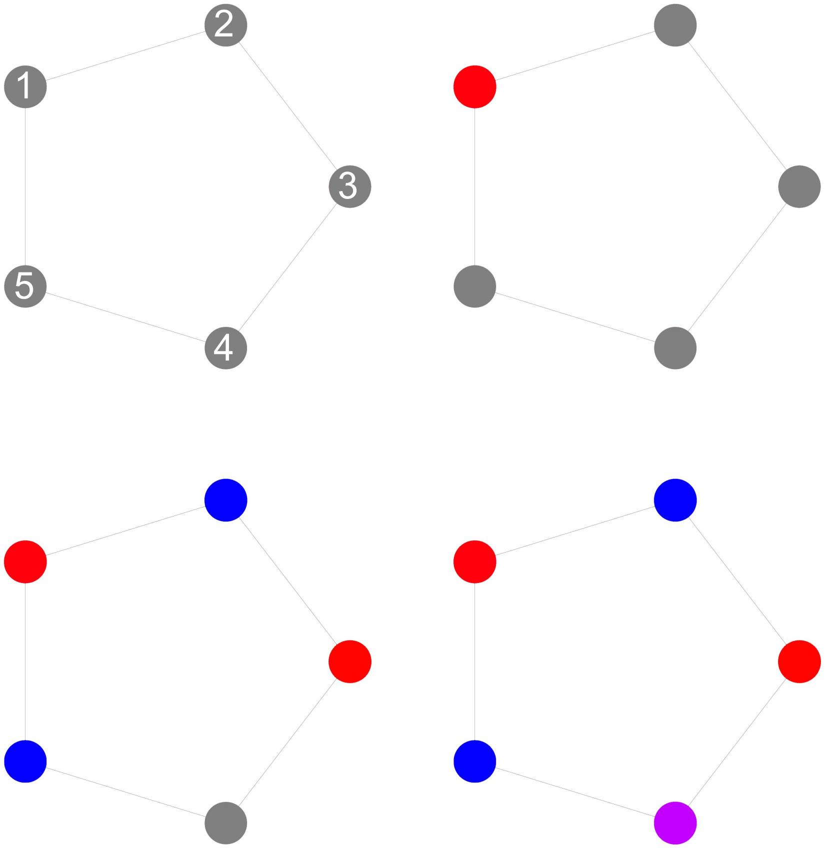
- Vertex coloring
- Can’t color adjacent nodes with same color
- Chromatic number: smallest number of colors

P_4



$\chi(P_4) = \mathbf{2}$

C_5

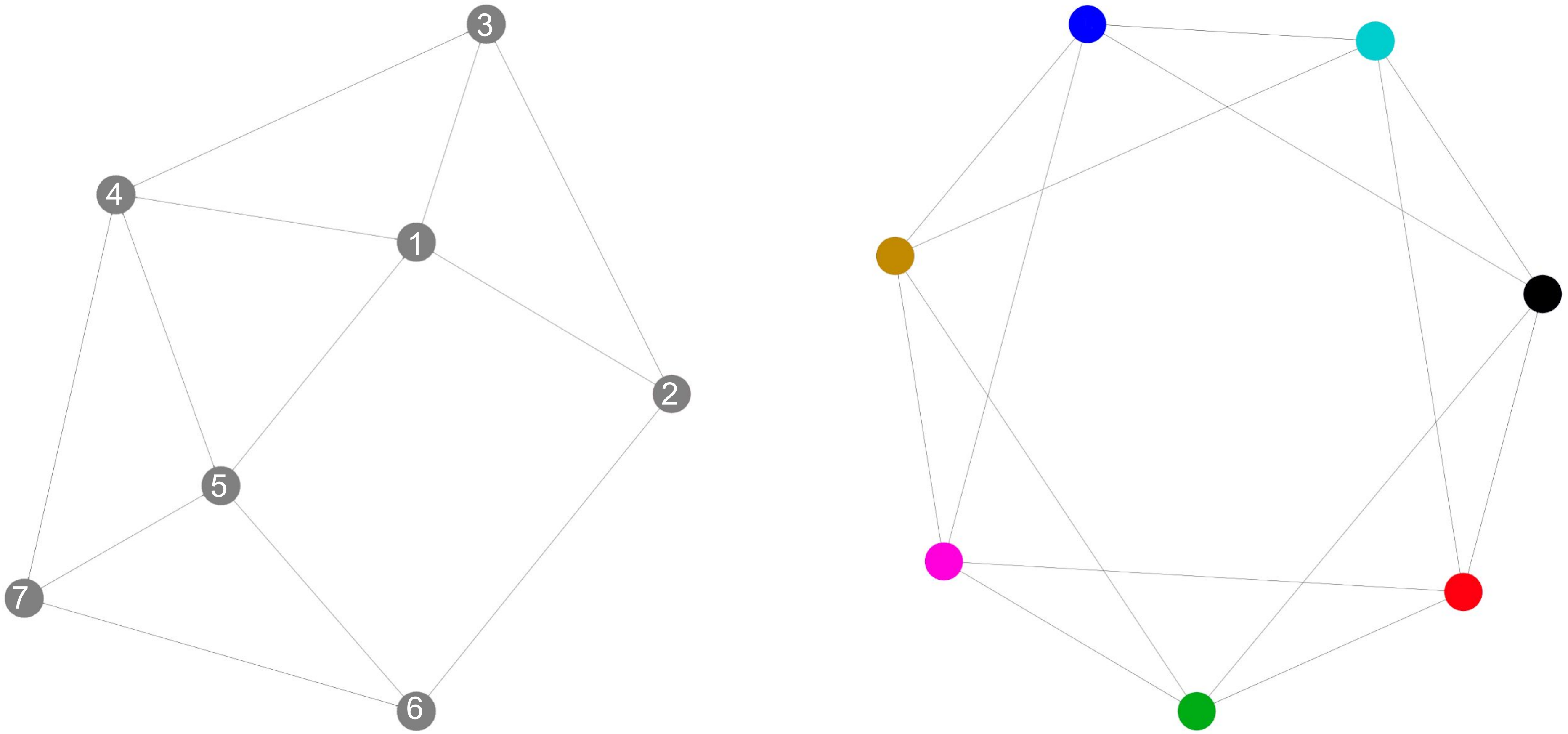


$$\chi(C_5) = 3$$

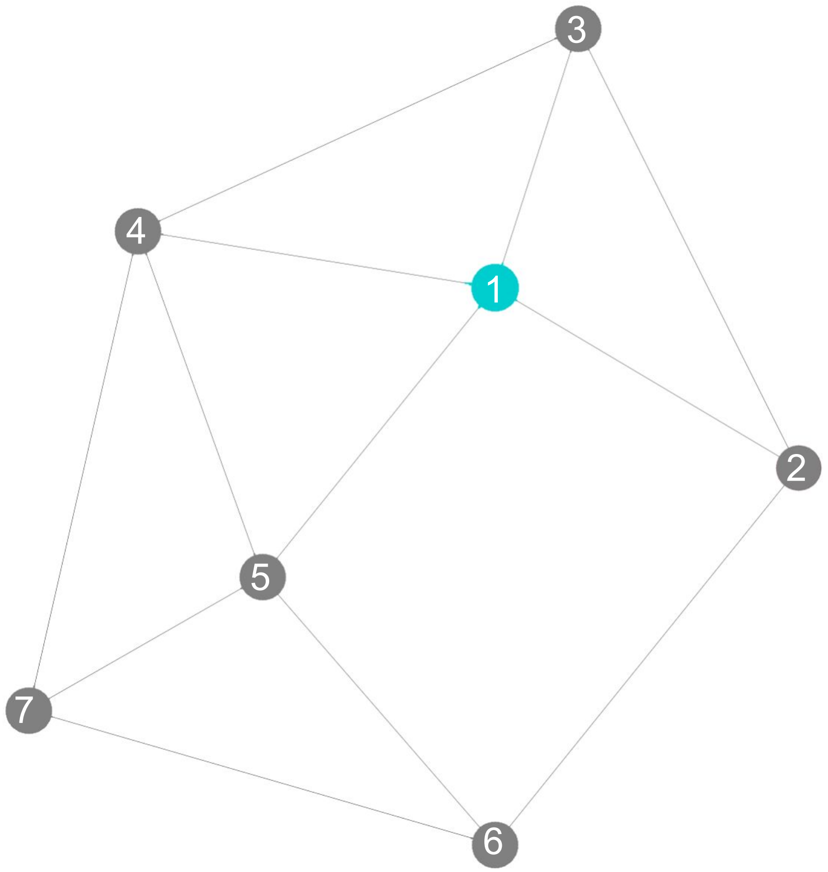
Graph homomorphism

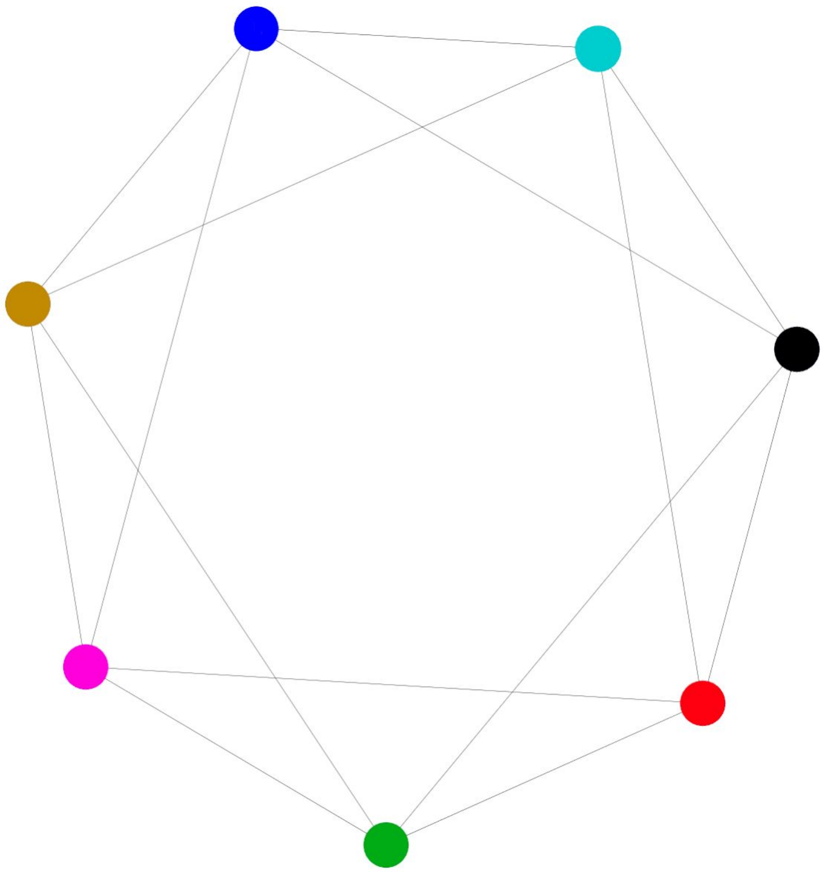
- $G \to H \Leftrightarrow \exists \phi : G \mapsto H : \forall (a, b) \in V(G) : (a, b) \in E(G) \Rightarrow (\phi(a), \phi(b)) \in E(H)$
- Maps edges to edges

Coloring left graph with right graph

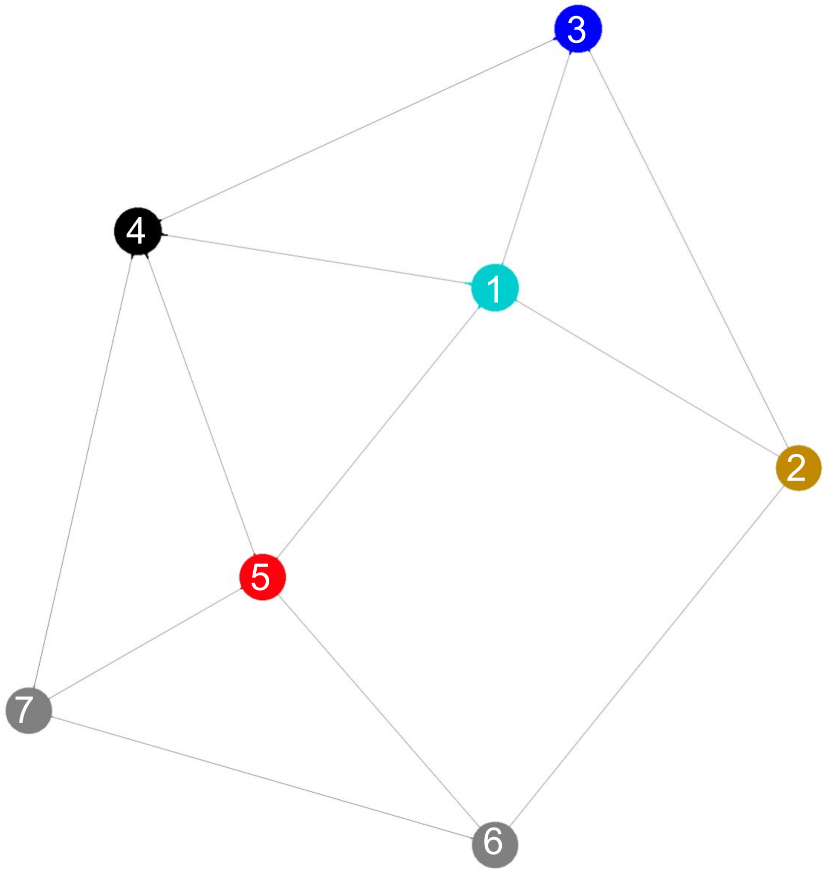


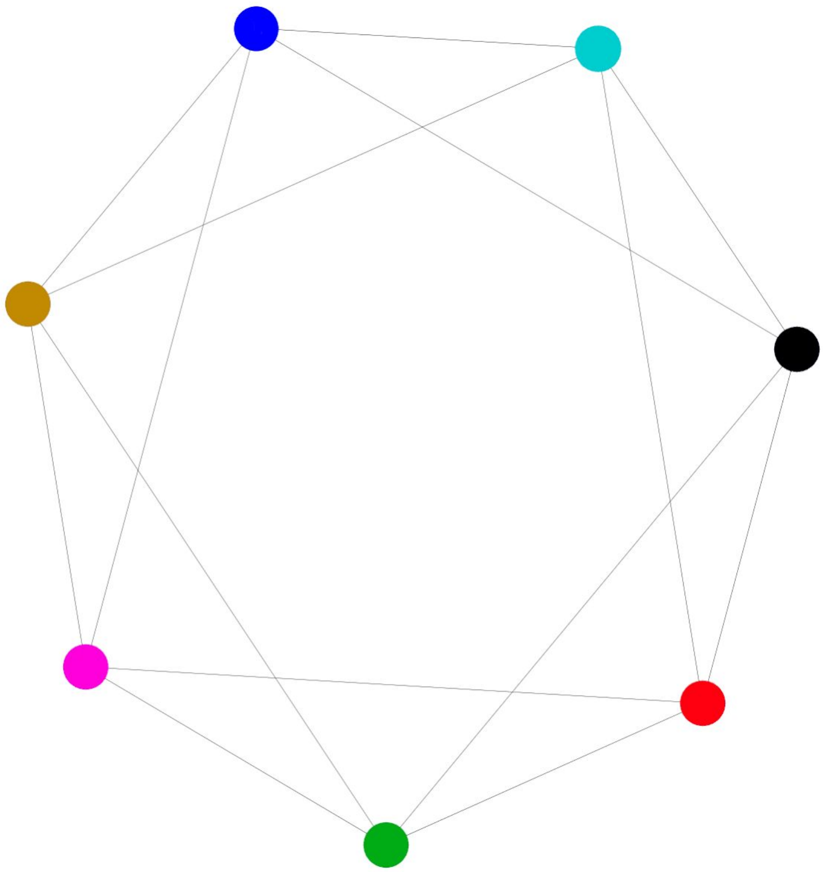
Edge in H means two colors can be adjacent



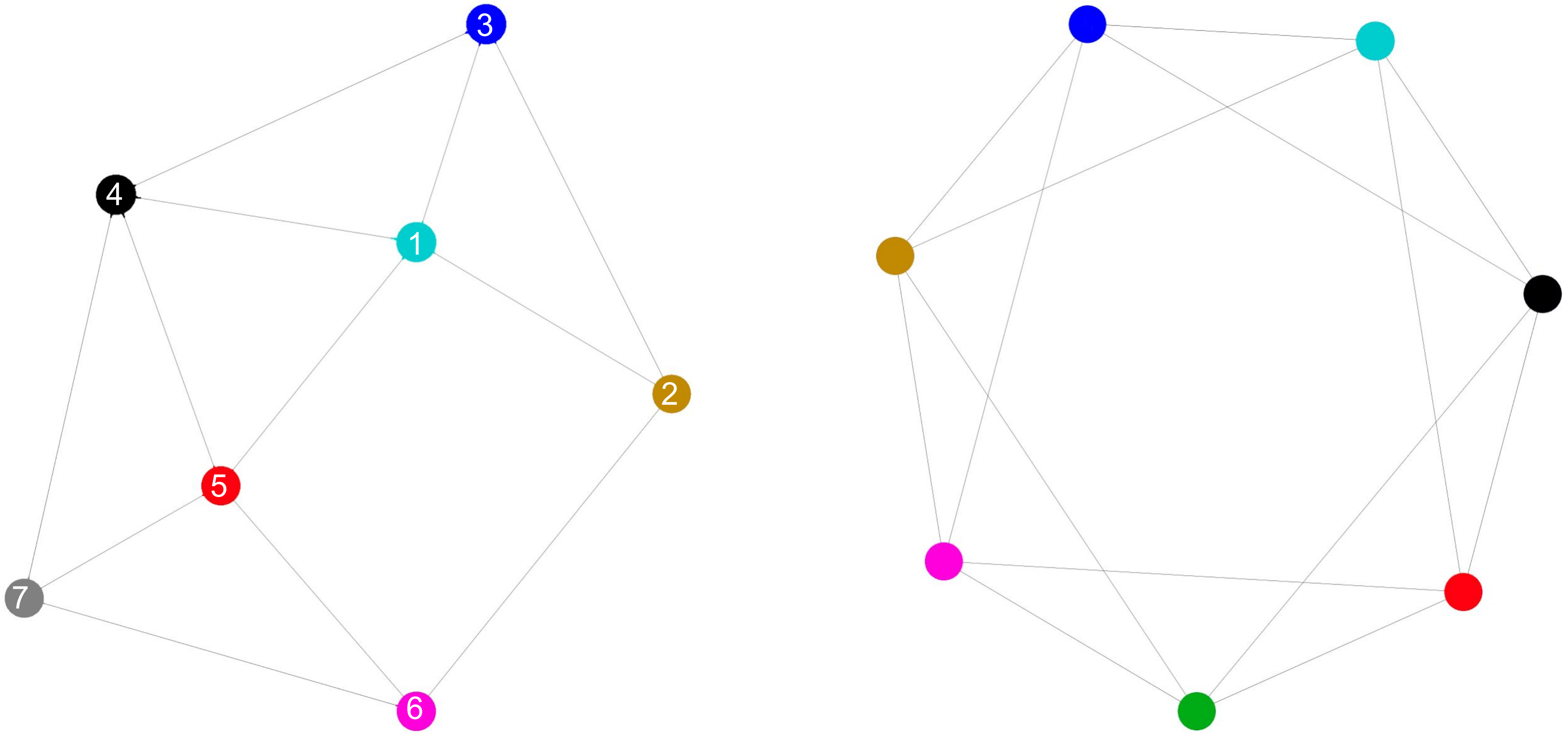


Like coloring, with rules set by H

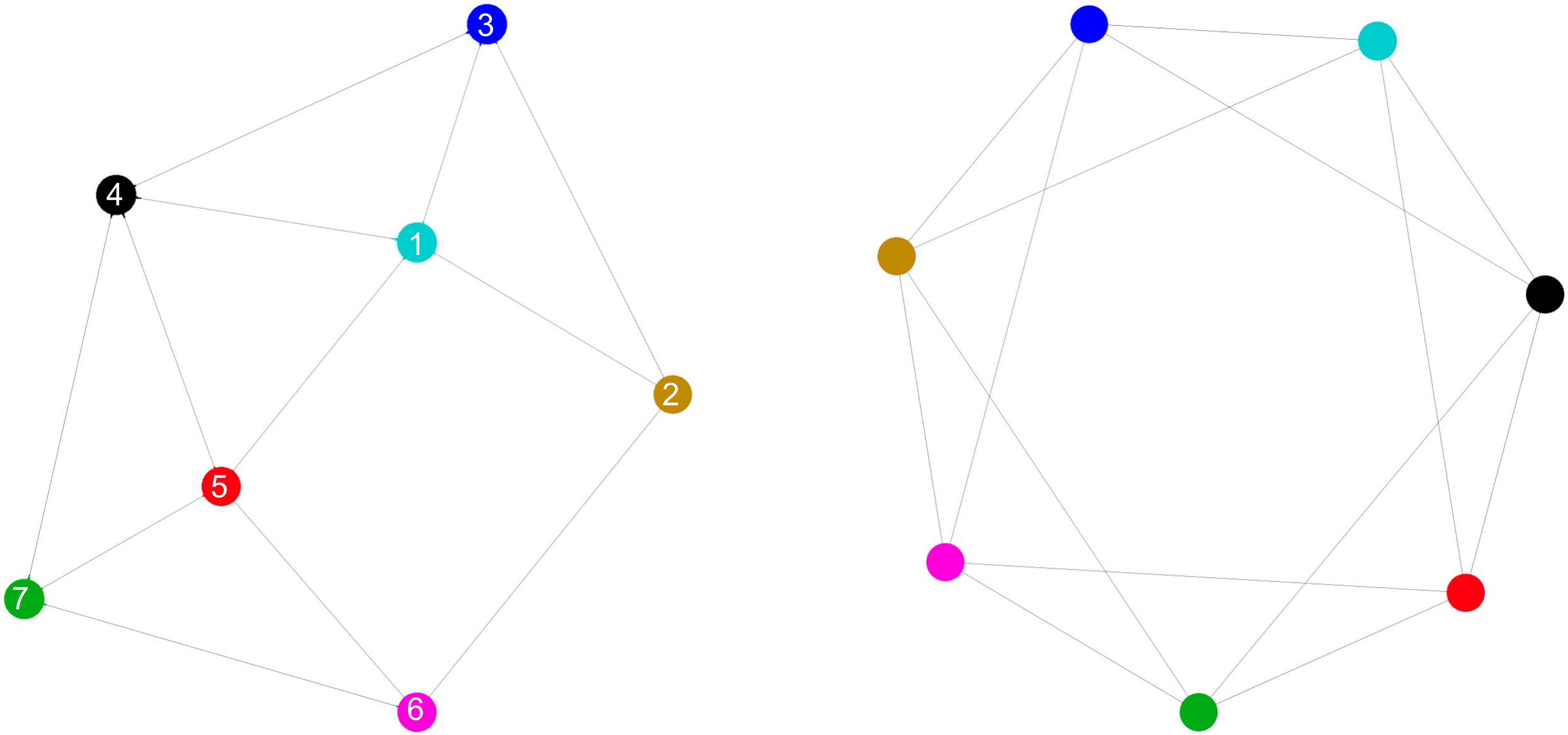




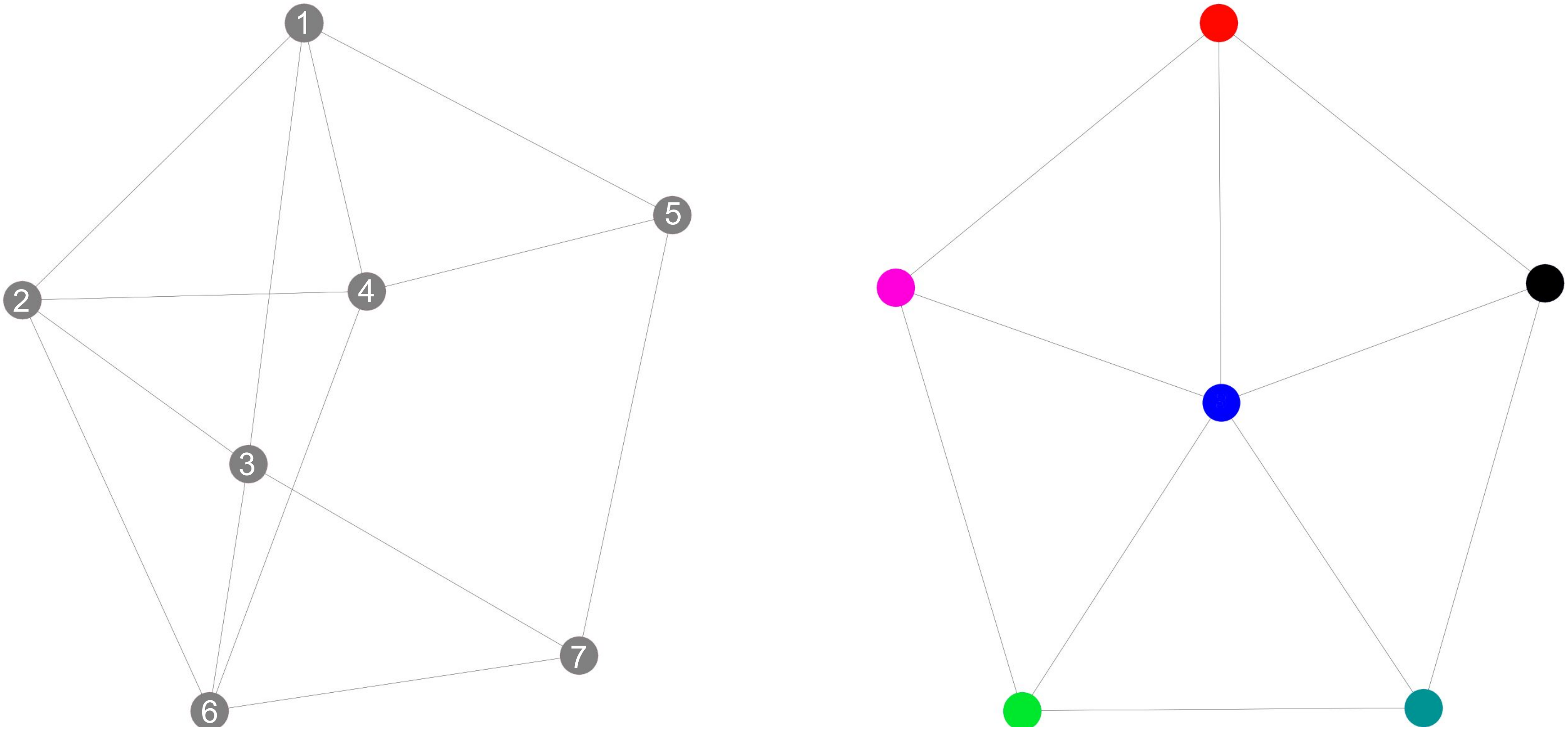
$$G \text{ is } n\text{-colorable} \iff G \to K_n$$



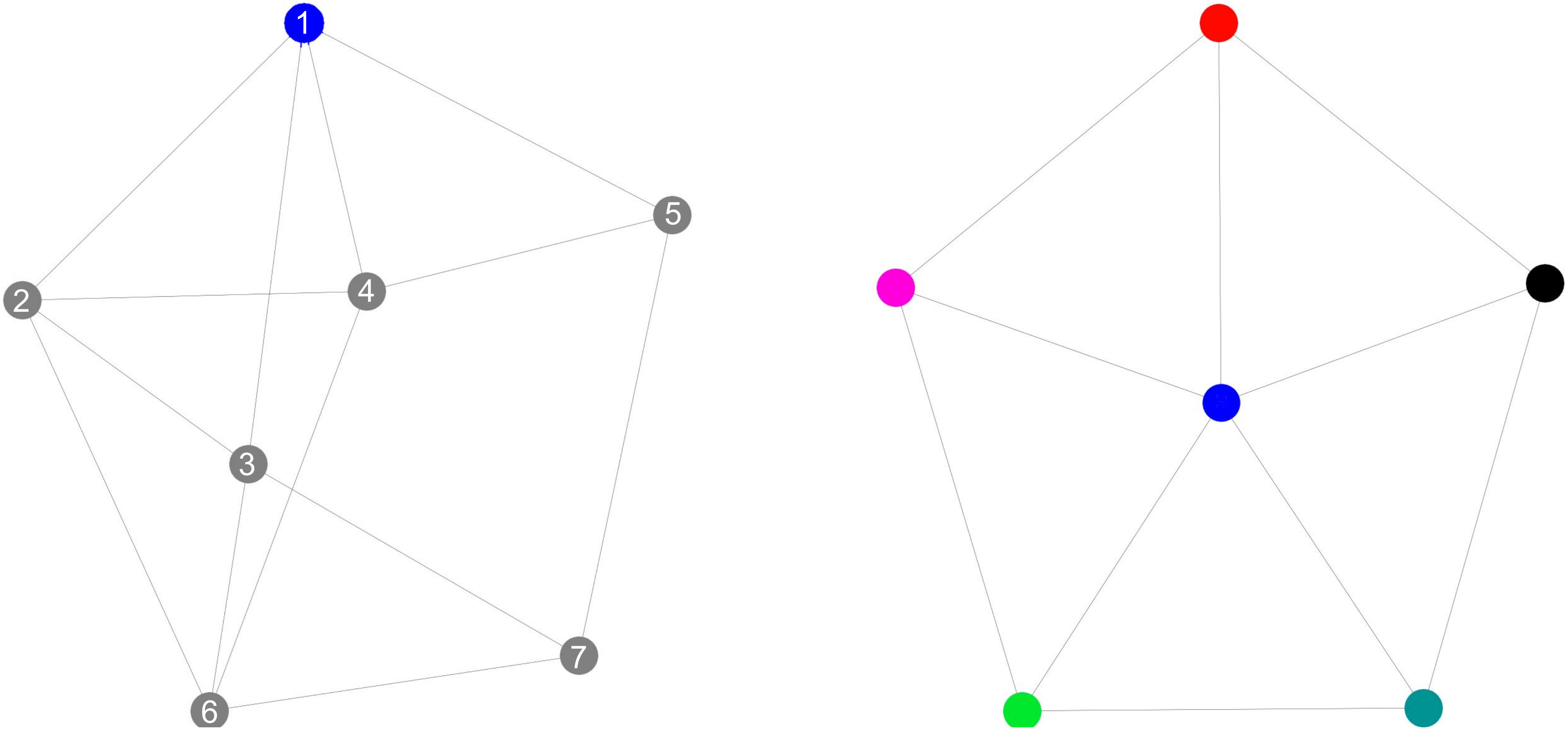
This is a valid homomorphism



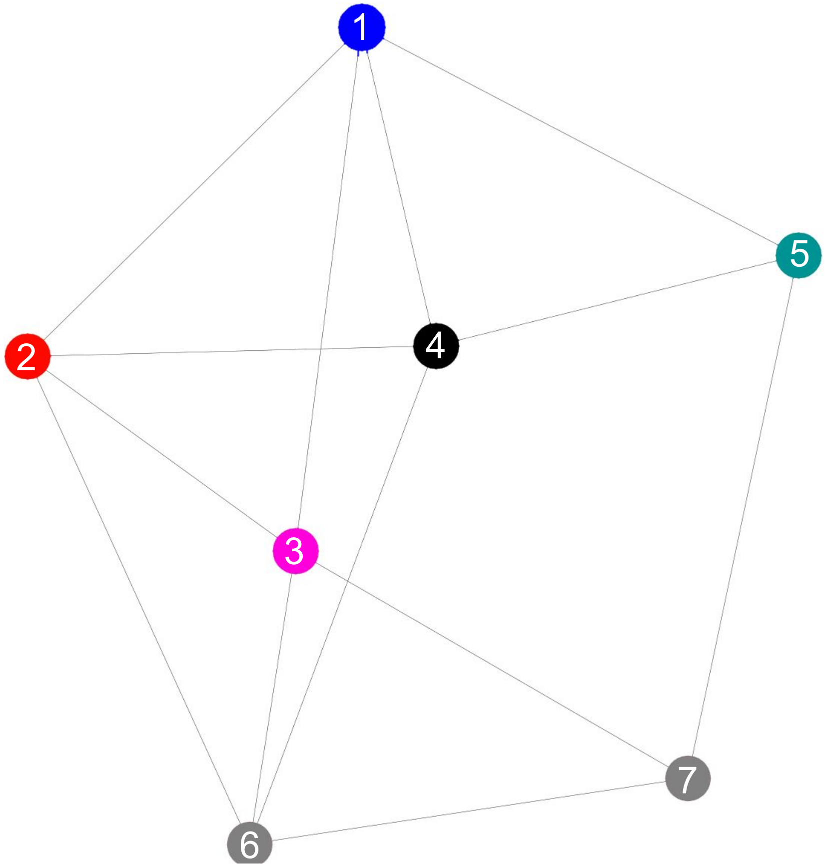
Color bigger graph with smaller graph

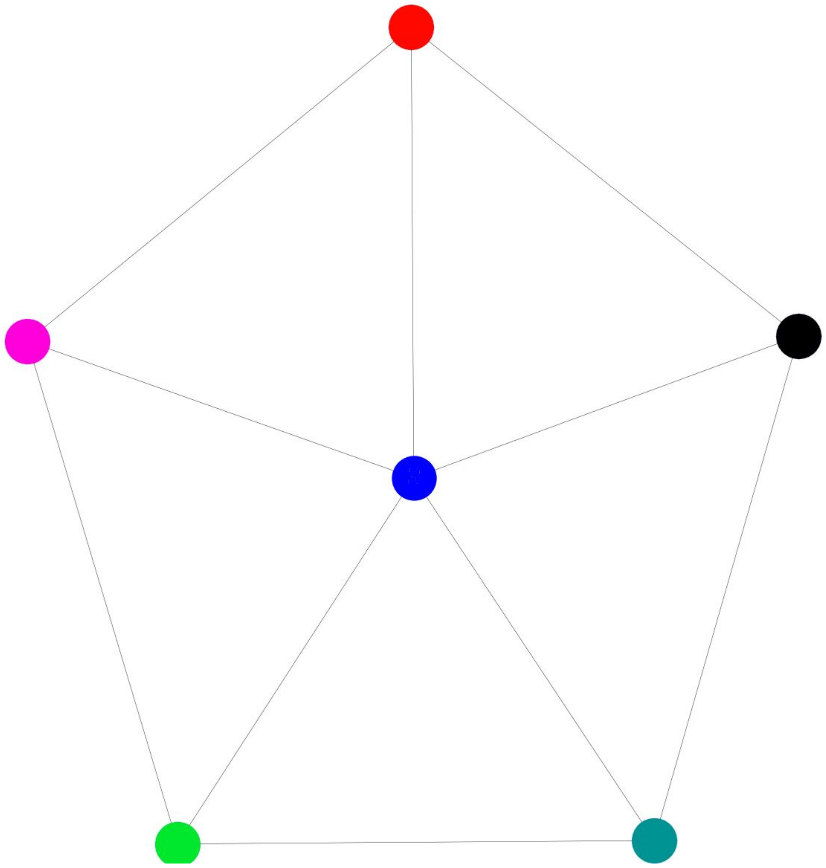


Pick a vertex and some color

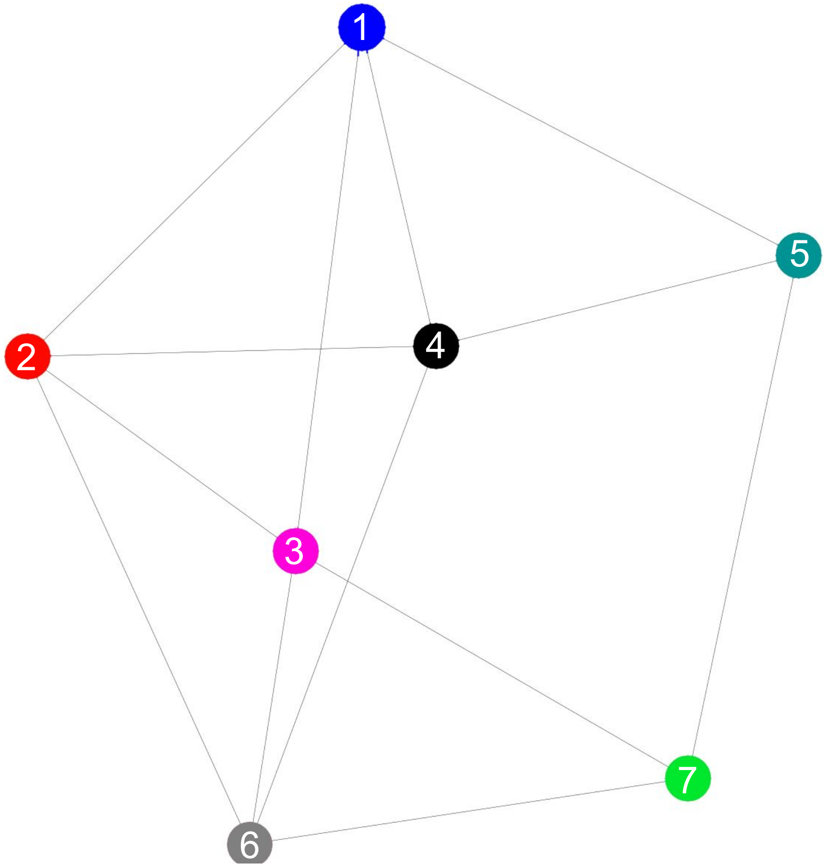


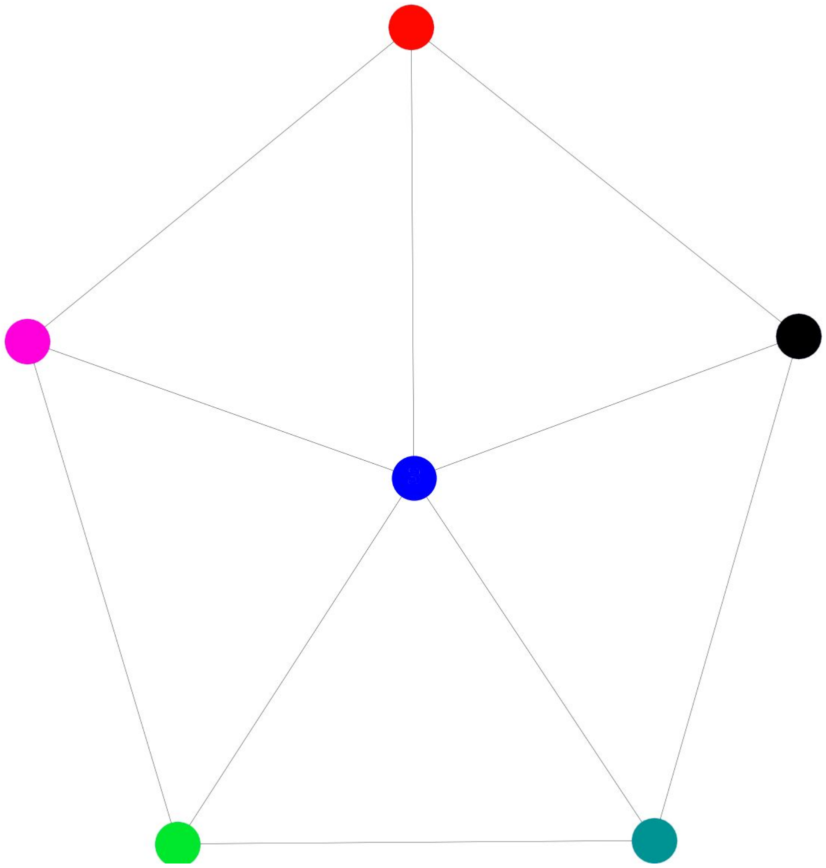
Map edges to edges



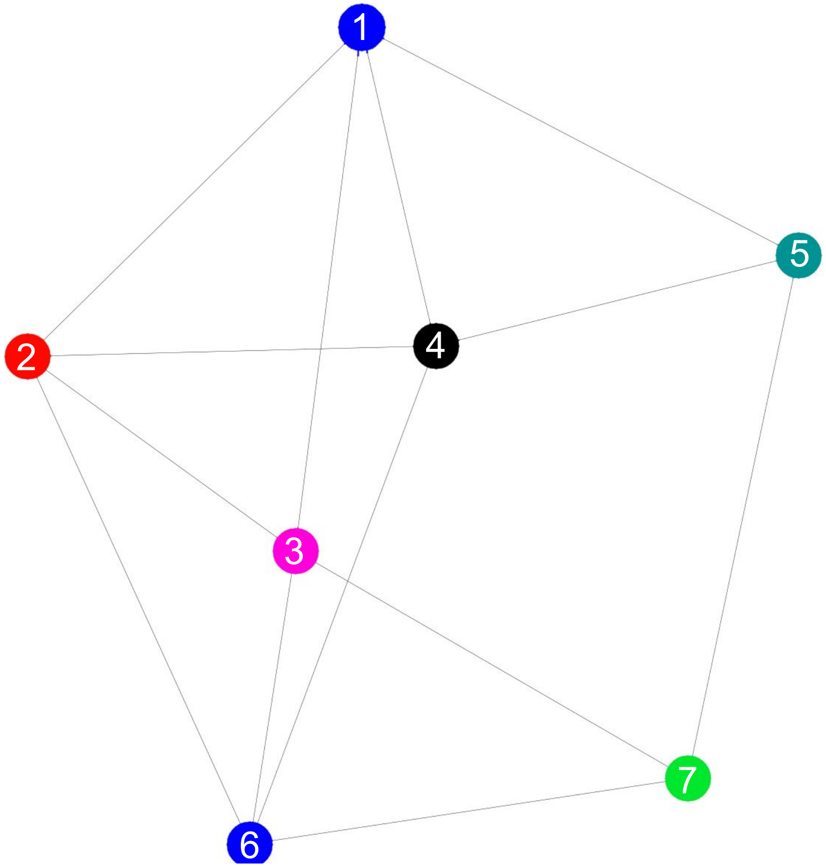


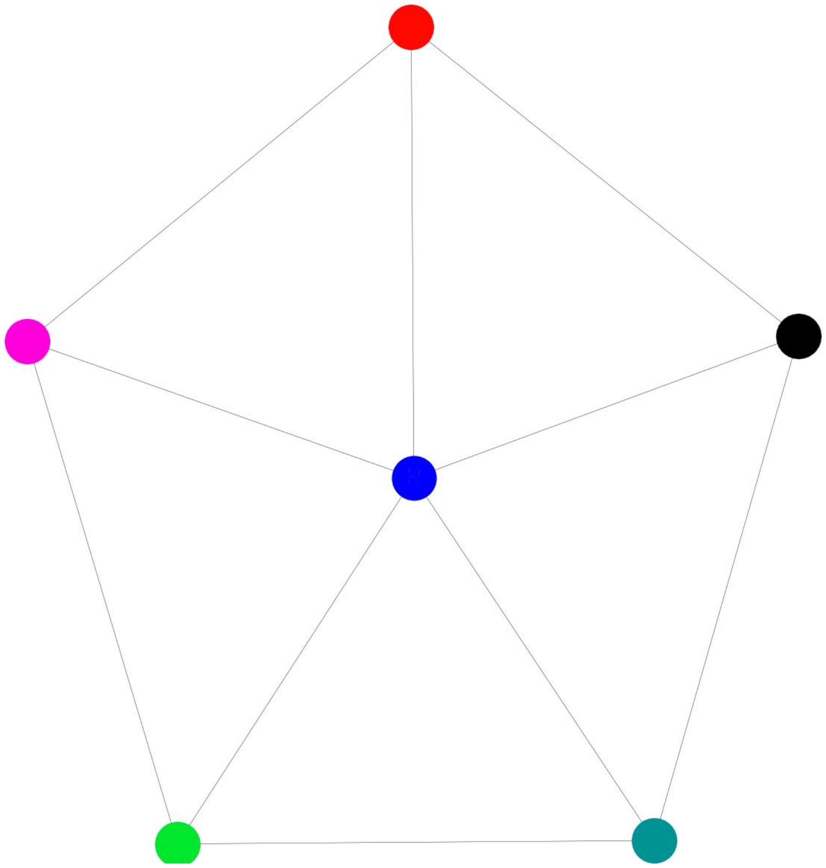
Can use the same color for many nodes





This is a valid homomorphism



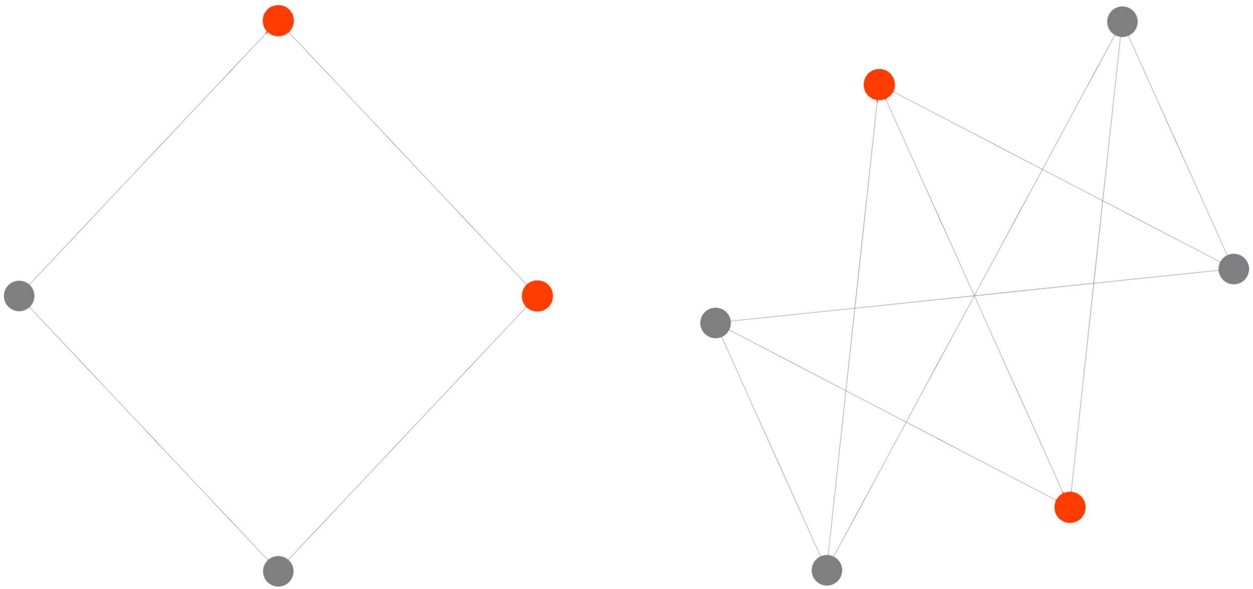


Issues

- Comes down to brute-forcing
- NP-complete
- Even “harder” than coloring and isomorphism
- So we will look at the bigger picture

Homomorphic equivalence

- G -> H and H -> G
- Not necessarily the same as isomorphism
- Means G and H have a similar subgraph they are both homomorphic to

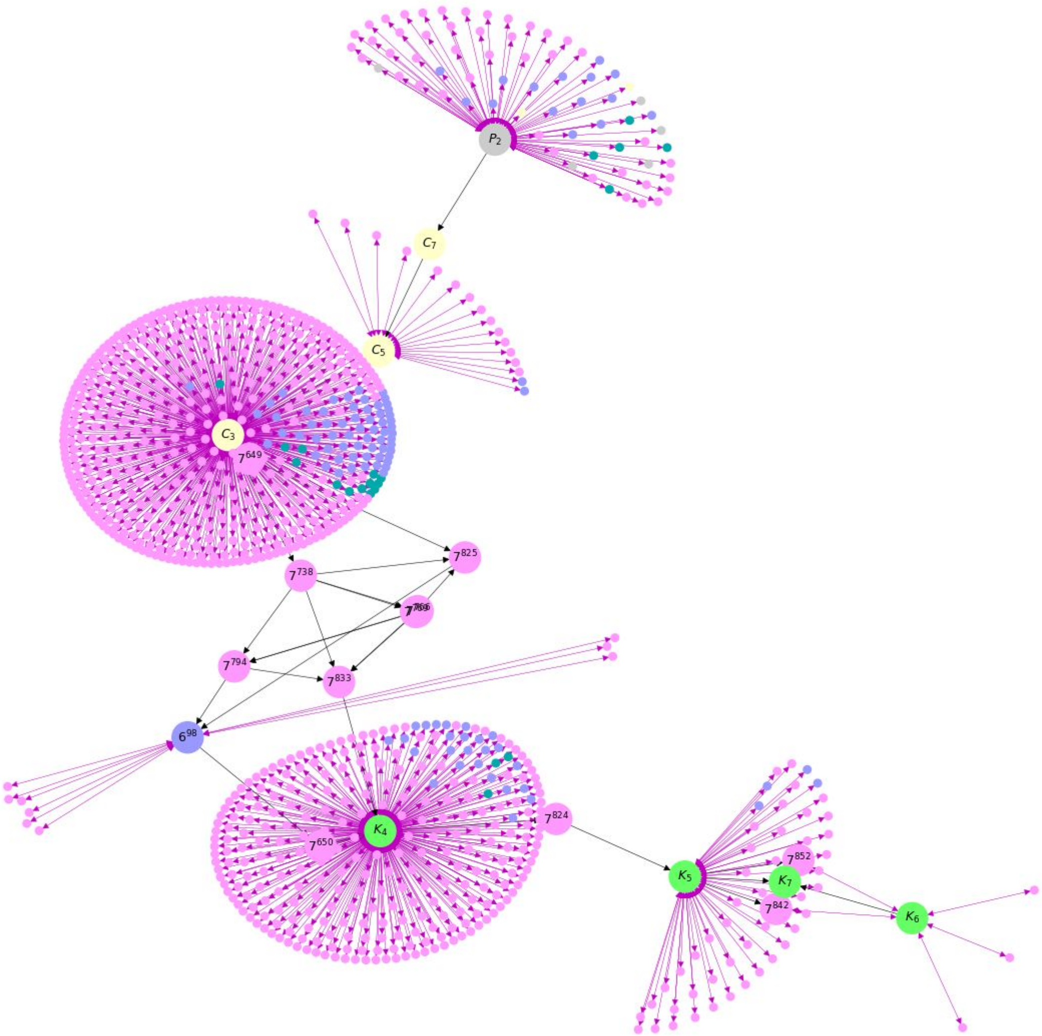


Homomorphism core

- Smallest equivalent subgraph
- Unique up to isomorphism
- Smallest graph in a class

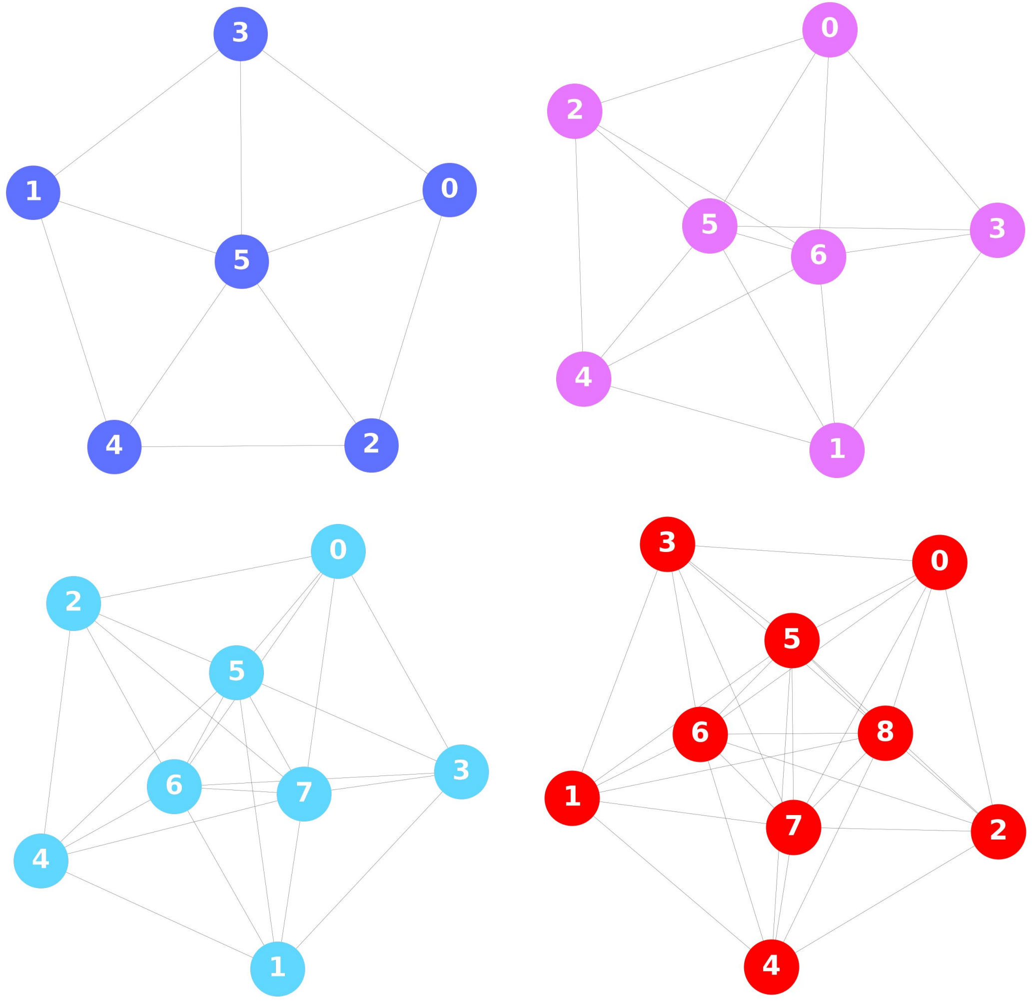
Diagram:

- Grey: path
- Yellow: cycle
- Green: complete
- Blue: 6 vertices
- Pink: 7 vertices
- Black arrows are one-way
- Pink arrows are bidirectional



Examples

- Complete graphs
- Odd cycles
- Kneser graphs
- 5-cycle derivatives (pictures)

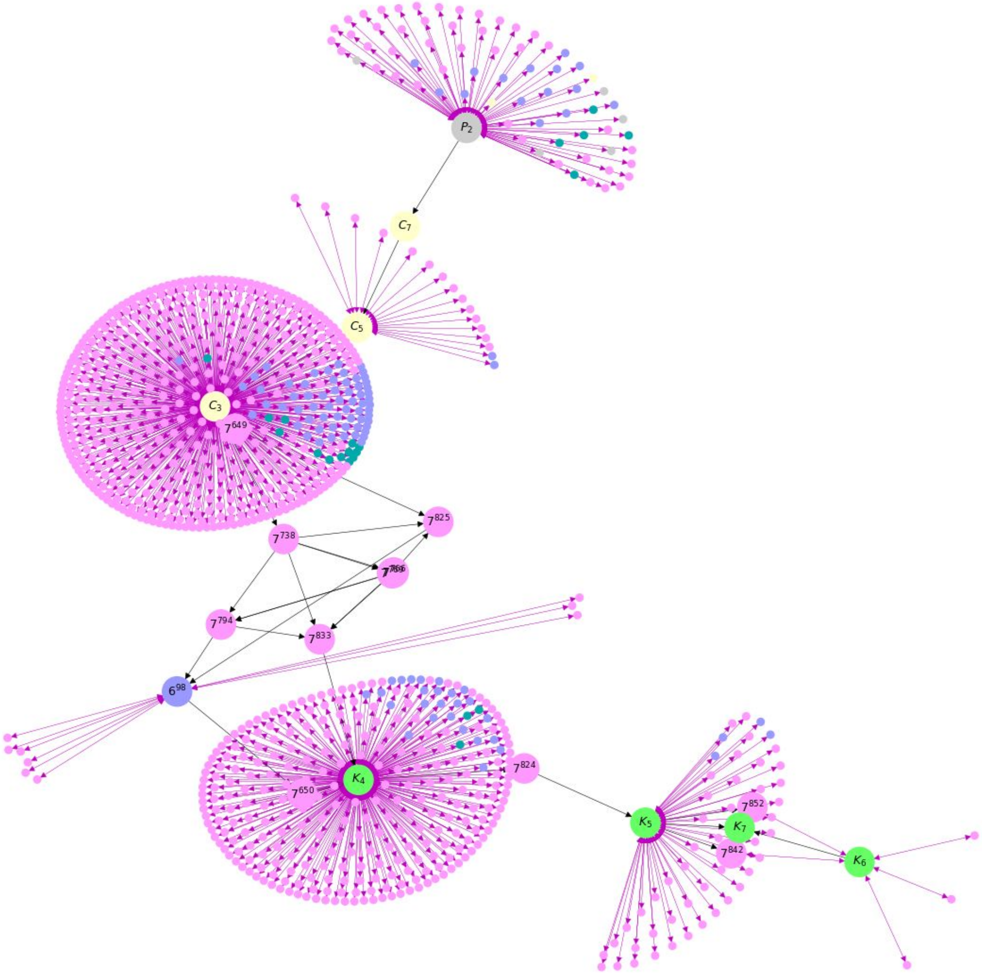


Preorder on graphs ($\leq$)

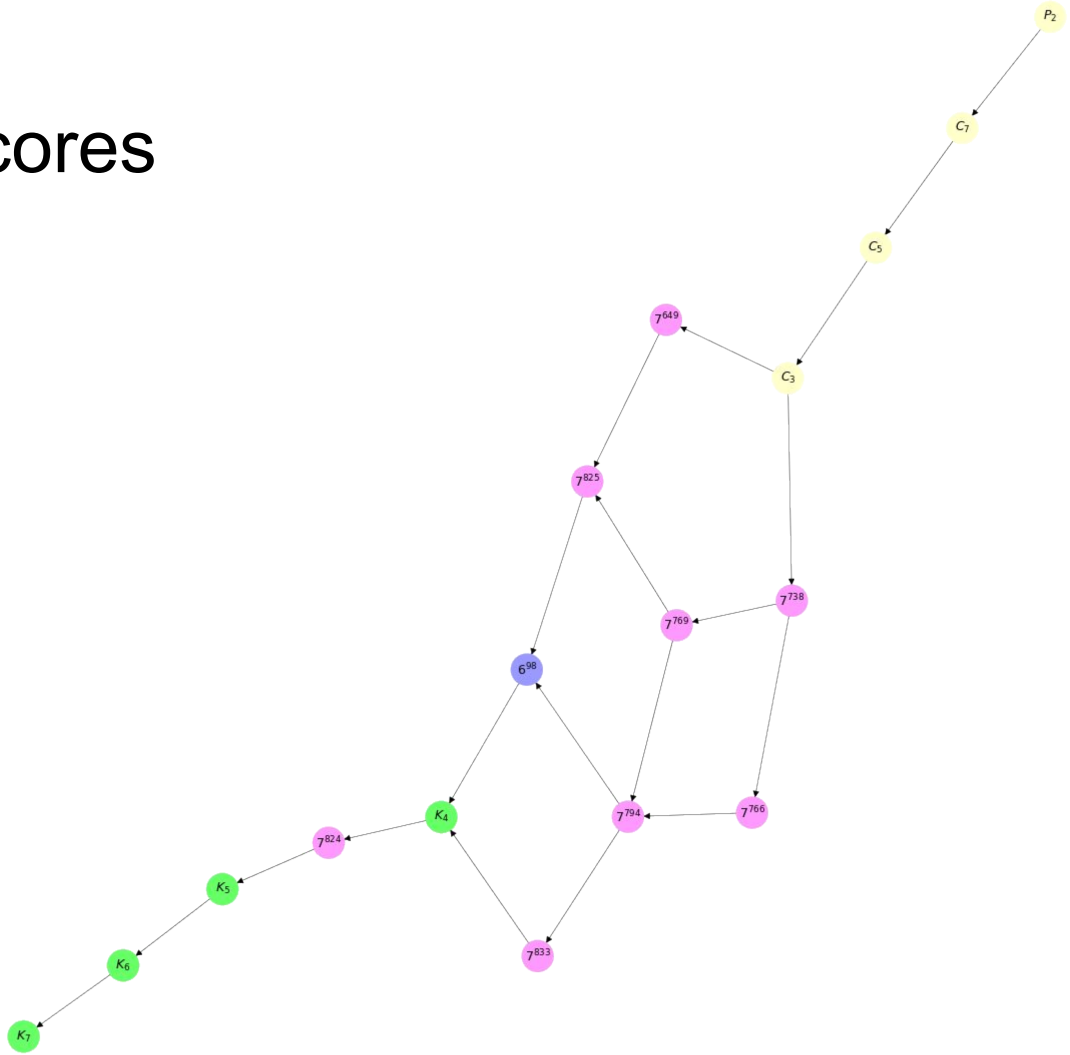
- Reflexive
- Transitive
- Core is the smallest in its class
- Poset on cores

Diagram:

- Grey: path
- Yellow: cycle
- Green: complete
- Blue: 6 vertices
- Pink: 7 vertices
- Black arrows are one-way
- Pink arrows are bidirectional



Poset on cores

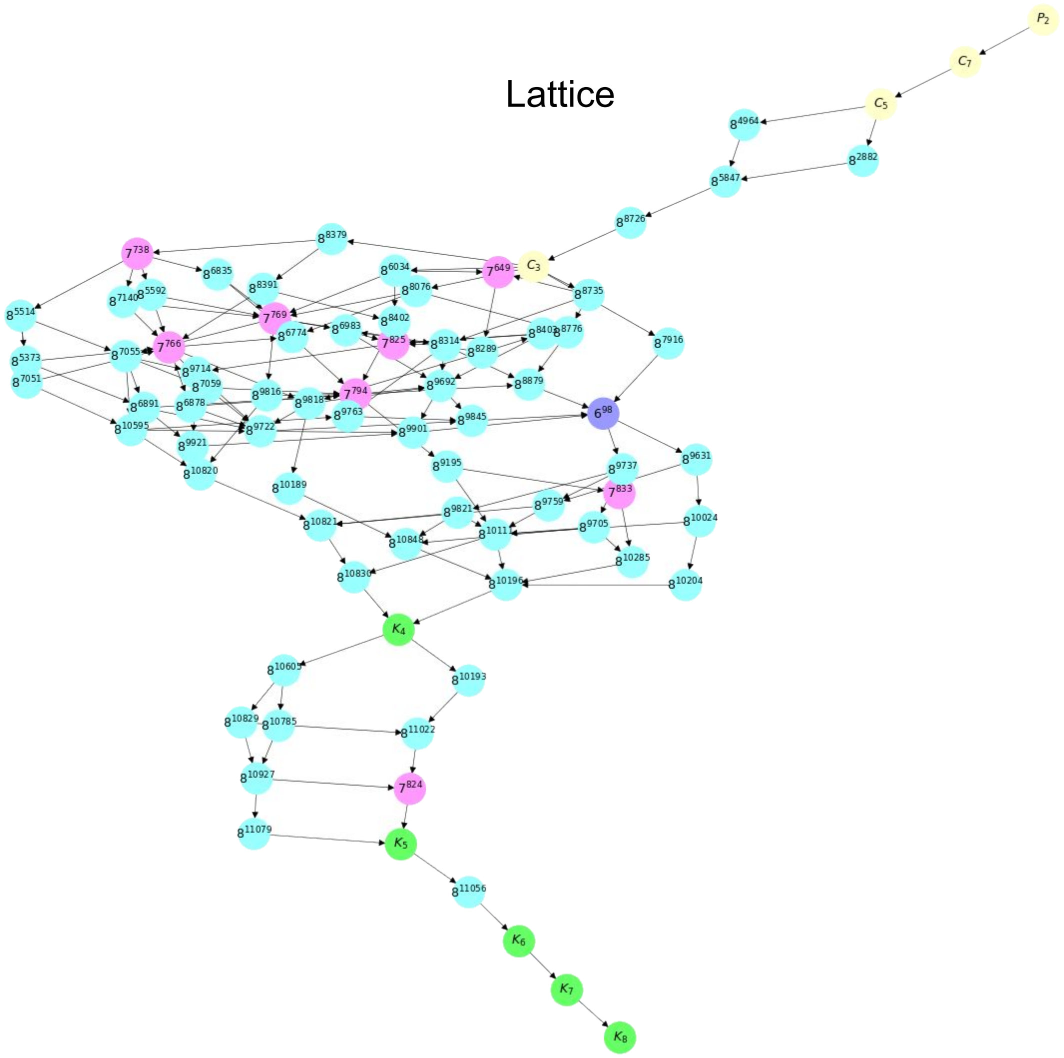


Lattice on cores

- Poset
- Unique common join/meet
- Dense

Diagram:

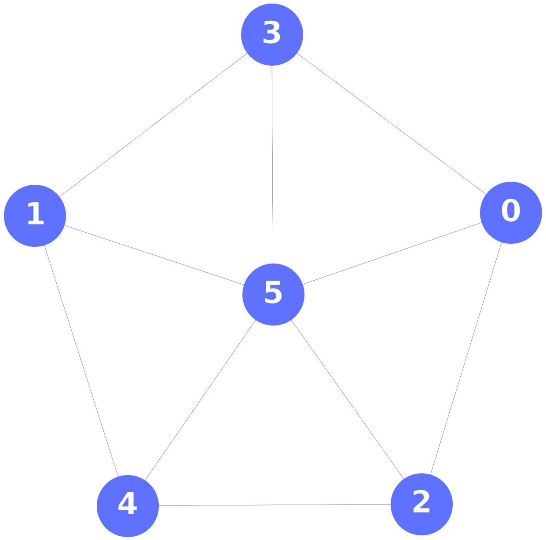
- Yellow: cycle
- Green: complete
- Blue: 6 vertices
- Pink: 7 vertices
- Teal: 8 vertices

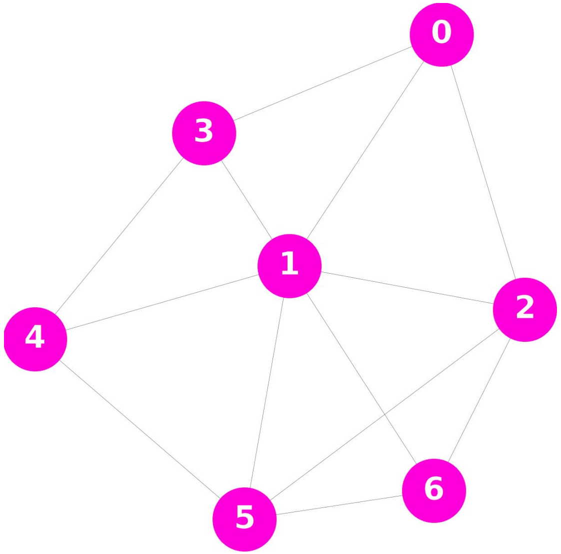


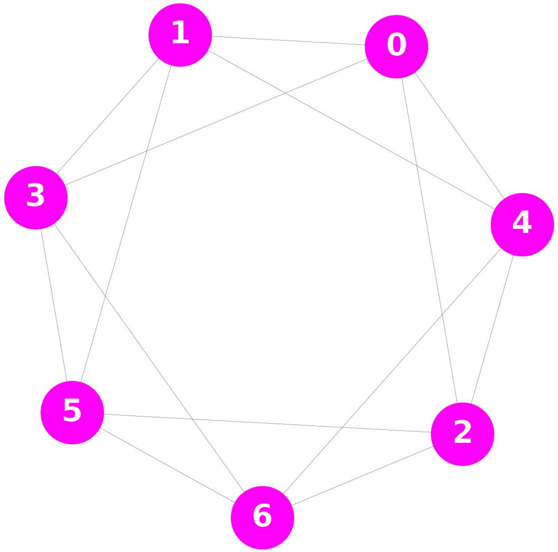
Demonstration

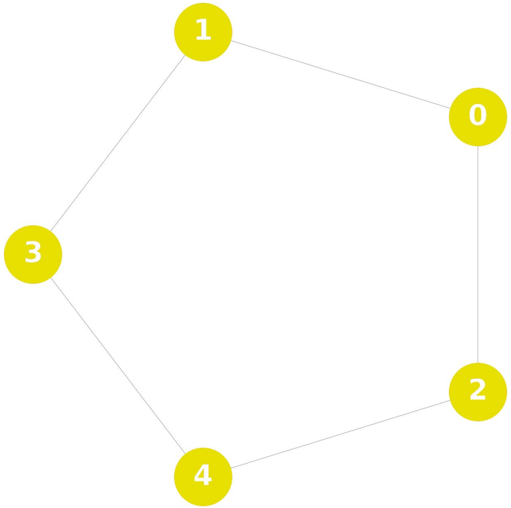
Observation and conjecture

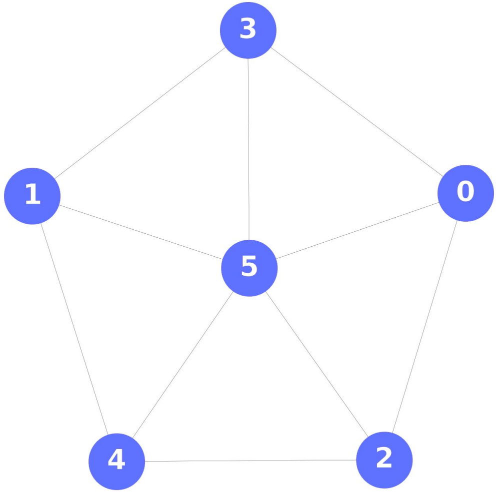
- On n vertices:
  - Exactly 1 core between K_{n-3} and K_{n-2}
  - Exactly 8 cores between K_{n-4} and K_{n-3} (including one of size n - 1)

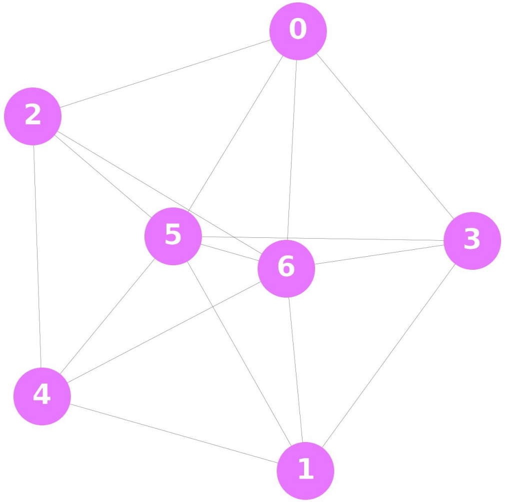


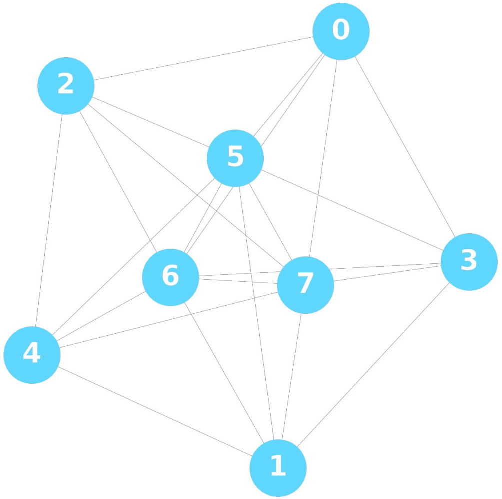


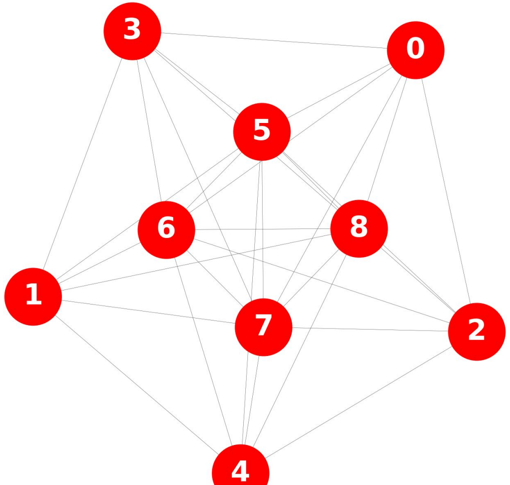



1
0
3
2
4


3
1
0
5
4
2


0
2
5
3
6
4
1


0
2
5
6
7
3
4
1


3
0
5
6
8
1
7
2
4

Thank you for your attention

Q&A