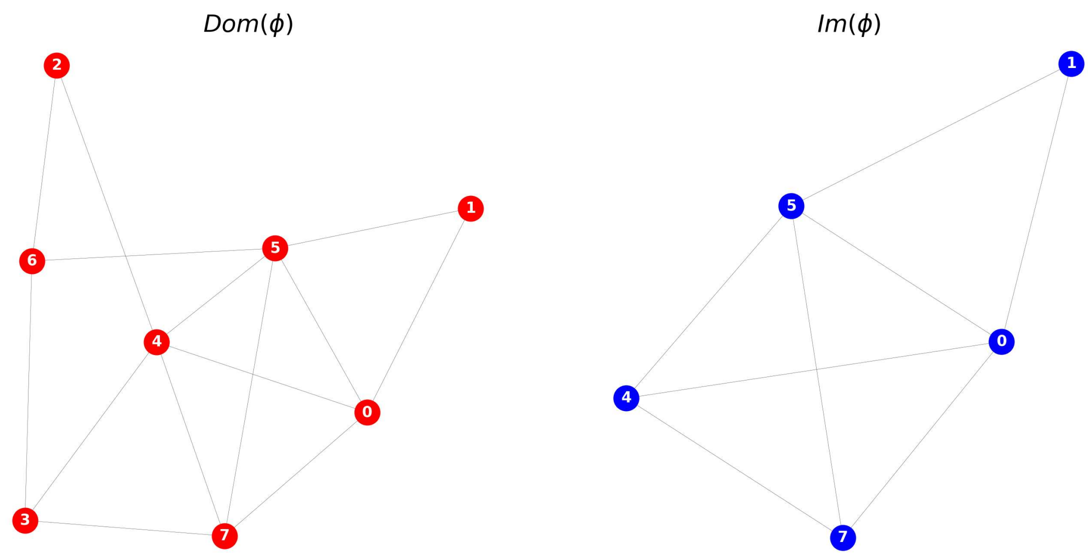
Notes on Graph Homomorphisms

SOURCE: TU Dresden

Homomorphism

Definition:
Graph $G \to H$ (is *homomorphic* to) if $\exists \phi : G \to H : \forall (a,b) \in E(G) \implies (\phi(a), \phi(b)) \in H$.



Definition:
Graph G is called *balanced* if it is homomorphic to a directed path $\vec{P}_n$.

Definition:
Graph G is called *acyclic* if it is homomorphic to a transitive tournament T_n.

Let $C_1 \leq C_2$ be a homomorphic order. Then homomorphism order on digraphs is a *lattice* where every two elements have a *supremum*, called *join*, and an *infimum*, called *meet*.

Definition:
$G \times H = (V(G) \times V(H), \{(a,b) \in E(G)\} \times \{(c,d) \in E(H)\})$ is a direct/cross/categorical product.

Properties:

- $G \times H$ is associative up to isomorphism.
- $G \times H$ is commutative up to isomorphism.

Conjecture *(Hedetniemi)*:
Let G, H be finite graphs. Then $G \times H \to K_n \implies (G \to K_n \text{ or } H \to K_n)$.

Proposition:
Let $(\mathcal{D}, \leq)$ be a homomorphic order of G. Then $\forall C_1, C_2 \in D$:

- $\exists C_1 \cup C_2 \in \mathcal{D} : C_1 \leq (C_1 \cup C_2) \text{ and } C_2 \leq (C_1 \cup C_2) \text{ and } \forall U \in \mathcal{D} : \begin{cases} C_1 \leq U \\ C_2 \leq U \end{cases} \implies C_1 \cup C_2 \leq U$
.
- $\exists C_1 \cap C_2 \in \mathcal{D} : C_1 \leq (C_1 \cap C_2) \text{ and } C_2 \leq (C_1 \cap C_2) \text{ and } \forall U \in \mathcal{D} : \begin{cases} C_1 \leq U \\ C_2 \leq U \end{cases} \implies U \leq C_1 \cap C_2$
.
- $(\mathcal{D}, \leq)$ contains infinite descending chain $C_1 > C_2 > \dots$.
- $(\mathcal{D}, \leq)$ contains infinite *antichains*, sets of pairwise incomparable elements of $\mathcal{D}$ with respect to $\leq$.

Theorem: Let H - finite undirected graph. $\left[H \to K_2 \text{ or } H \text{ contains a loop }\right] \implies H$-coloring can be solved in polynomial time. Otherwise, H-coloring is NP-complete.

Proposition: Every finite graph H has a core, unique to isomorphism.

Proposition: Let H be a core. Then $CSP(H)$ and the precolored CSP for H are time equivalent.

Proposition: If there is an algorithm that decides $CSP(H)$ in time T, then there is an algorithm that constructs a homomorphism $G \to H$ (if such exists), which runs in time $O(|V(G)|T)$.

Polymorphism

Definition: Homomorphism $H^k \to H \mid k \geq 1$ is called a *(k-ary) polymorphism* of H.

Definition: Digraph H is called *projective* if every *idemponent polymorphism* is a *projection*.

Function clones

Definition: Let $n \geq 1$, $\mathcal{D}$ be a set and $\mathscr{O}_{\mathcal{D}}^{(n)} = \mathcal{D}^{\mathcal{D}^n} := (\mathcal{D}^n \to \mathcal{D})$ of n-ary functions, or *operators*, on $\mathcal{D}$. Let $\mathcal{O}_{\mathcal{D}} := \bigcup\limits_{n \geq 1} \mathscr{O}_{\mathcal{D}}^{(n)}$ be a set of all operators on $\mathcal{D}$. Then a *function clone* (over $\mathcal{D}$) is $\mathscr{C} \subseteq \mathcal{O}_{\mathcal{D}}$ such that:

- $\mathscr{C}$ containts all *projections*: $\forall 1 \leq k \leq n \implies \pi_k^n \in \mathscr{O}_{\mathcal{D}}^{(n)} = \pi_k^n(x_1, \dots, x_n) = x_k \in \mathscr{C}$.
- $\mathscr{C}$ is closed under composition: $\forall f \in \mathscr{C} \cap \mathscr{O}_{\mathcal{D}}^{(n)}; g_1, \dots, g_n \in \mathscr{C} \cap \mathcal{O}_{\mathcal{D}}^{(m)}$:
$\Big[f(g_1, \dots, g_n) \in \mathcal{O}_{\mathcal{D}}^{(m)} = \big[(x_1, \dots, x_m) \to f(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m))\big]\Big] \in \mathscr{C}$.

clone is an abstraction above *function clone*.

SOURCE: Mast Queensu

Basic definitions

Definition:
Graph homomorphism $\phi : G \to H$ is *faithful* if $\Im(G)$ is an induced subgraph of H. It will be called *full* if $(u, v) \in E(G) \iff (\phi(u), \phi(v)) \in E(H)$. That is, when $\phi^{-1}(x) \cup \phi^{-1}(y)$ induces a complete bipartite graph whenever $x, y \in E(H)$.

Definition:
Let G - graph and $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of a $V(G)$. Then the *quotient* $G/\mathcal{P}$ is the graph $(\{V_1, \dots, V_k\}, \{(V_i, V_j) \mid i \neq j, \begin{cases} u_i \in V_i \\ u_j \in V_j \end{cases} \implies (u_i, u_j) \in E(G)\})$. The mapping $\pi_{\mathcal{P}} : V(G) \to V(G\,\mathcal{P})$ defined by $\pi_{\mathcal{P}}(u) = V_i \mid u \in V_i$ is a *natural map* for $\mathcal{P}$.

Lemma: $\phi : G \to H$ - homomorphism $\iff PreIm[\phi^{-1}(I)] \mid I$ - anticlique, $\subset V(H)$ - anticlique.

Proposition: Homomorphism $\phi : G \to H$ is *complete* $\iff \tau : G/\phi \to H$ is an *isomorphism*\invertible.

Definition: Complete homomorphism $\phi : G \to H$ is *elementary* if there is a unique pair of nonadjacent vertices $u, v \in V(G)$ which are identified by ϕ. We call H an *elementary quotient* of G.

Lemma: Let G, H be graphs, and $\phi : G \to H$ be a *homomorphism*. Then there is $k \in \mathbb{N}$ and graphs $G = G_0 \dots G_k$ such that G_{i+1} is an *elementary quotient* of G_i when $i < k$ and $G_k \simeq G\,\phi$.

Definition: *Elementary homomorphism* $\phi : G \to H$ is a *simple fold* if the two vertices which are identified have a common neighbour.

Definition: *Folding* is a homomorphism obtained as a sequence of simple folds. If $\phi : G \to H$ and ϕ is a *folding*, we say G *folds* H.

Retracts

Definition: Let G and H be graphs. Then H is called a *retract* of G if there are homomorphisms $\rho : G \to H$ and $\gamma : H \to G$ such that $\rho\gamma = id_H$. The homomorphism ρ is called a *retraction* and γ a *co-retraction*.

Observation: Since there exist homomorphisms in both directions between G and any of its retracts H, it follows that G and H have the same chromatic number, odd girth and clique number. Note also that $\chi(G) = \omega(G) \iff K_{\chi(G)}$ is a retract of G.

Lemma: Let G, H - graphs. Then H is a *retract* of $G \iff \exists[\rho : G \to H, \sigma : H \to G] : \rho\sigma \in Aut(H)$.

Proposition: Any *retraction* of a connected graph is *folding*.

Lemma: Let $\phi : G \rightarrow G$. Then $\exists n \in \mathbb{N} : R = \phi^n(G)$ is a retract of G (and ϕ^n a retraction). Further, $\phi \mid R$ is an *automorphism* of R.

Cores

Definition: Graph G is a *core* if no proper subgraph of G is a retract of G.

Proposition: Let G - graph. Then $\big[G$ - core \$ \big] \iff \forall \phi: G \rightarrow G, \big[\phi\$ - homomorphism $\implies \phi \in Aut(G)\big]$.

Definition: Retract H of G is *core of* G if it's a core.

Proposition: Every finite graph has a core.

Propoition: If H_0 and H_1 are cores of a graph G then they are *isomorphic*.

Homomorphic equivalence

Notation: If there is a homomorphism $G \rightarrow H$, we write $G \rightarrow H$.

Definition: We say $G \rightarrow H$ are homomorphically equivalent if $H \rightarrow G$. In that case we write $G \leftrightarrow H$.

Proposition: Let G, H be graphs such that $G \leftrightarrow H$. Then the cores of G and H are isomorphic.

Corollary: Let G - graph. Then to within isomorphism, $G^{\cdot}$ is a uniue graph of smallest order in $\mathcal{H}(H)$.

Definition: Let G and H be graphs. Then $G \rightarrow H \implies \mathcal{H}(G) \preceq \mathcal{H}(H)$.

Theorem (Welzl 1): Let G and H be graphs s.t. $G \rightarrow h$ and $H \not\rightarrow G$. Then there is a graph K s.t. $G \rightarrow K \rightarrow H$ and $H \not\rightarrow K \not\rightarrow G$.

Products

Definition: Let G and H be graphs. Then the following products G and H are defined on the vertex set $V = V(G) \times V(H)$:

- *Cartesian product*: $G \Box H = \Big[E(G \Box H) = \{[(u,x),(v,y)] : \big[u = v, [x,y] \in E(H) or x = y\big]\}\Big]$.
- *Categorical product*: $G \times H = \Big[E(G \times H) = \{[(u,x),(v,y)] : \big[[u,v] \in E(G), [x,y] \in E(H)\big]\}\Big]$.
- *Strong product*: $G \boxtimes H = \Big[E(G \boxtimes H) = E(G \times H) \cup E(G \Box H)\Big]$.
- *Lexicographic product*: $G[H] = \Big[E(G[H]) = \{[(u,x),(v,y)] : \big[u = v, [x,y] \in E(H) or [umv] \in E(G)\big]\}\Big]$.

Observation: Let G, H be graphs, and let $*$ be any of the *cartesian*, *strong* or *lexicographic* product. Then $G \rightarrow G * H$ and $H \rightarrow G * H$. In fact, in each case G and H are induced subgraphs of $*$.

Lemma: Let G, H - graphs. Then $G \times H \to G$ and $G \times H \to H$.

Theorem: The equivalence classes of graph homomorphisms form a lattice under partial order $\preceq$. The join of $\mathcal{H}(G)$ and $\mathcal{H}(H)$ is the equivalence class containing the disjoint union of G and H, and the meet of $\mathcal{H}(G)$ and $\mathcal{H}(H)$ is $\mathcal{H}(G \times H)$.

Vertex-transitive graphs

Cayley graphs

Theorem (Sabidussi): Any *vertex-transitive* graph is a retract of some *cayley graph*.

Theorem (Multiple version): Let G be a *vertex-transitive* graph. Then some multiple of G is a *cayley graph*.

Theorem (Cayley coset graphs version): Any *vertex-transitive* graph is isomorphic to a *cayley coset graph*.

Independence ratio and the No-Homomorphism Lemma

Definition: The *independence ratio* of a graph G is $i(G) = \alpha(G) \ |V(G)|$

Lemma (No-Homomorphism Lemma, Albertson and Collins): Let G, H - graphs s.t. H is *vertex-transitive* and $G \to H$. Then $i(G) \geq i(H)$.

Lemma: Let G, H - graphs,

$$\begin{cases} H \text{ - vertex-transitive} \\ i(G) = i(H) \\ G \to H \end{cases} \implies \forall \text{ independent sets } I : Card(I) = Card(\alpha(H)) \text{ in } H$$ and $\forall \phi : G \to H$ we have: $|\phi^{-1}(I)| = \alpha(G)$.

Cores of vertex-transitive graphs

Theorem (Welzl 2): Let G - *vertex-transitive*. Then its core G^* is *vertex-transitive*.

Corollary: Let G - *vertex-transitive*. $\alpha(G) \perp \vert G \vert \implies G - $ core.

Theorem: Let G - *vertex-transitive*, $\phi : G \to G^*$. Then $Card(\phi^{-1}(u) \mid u \in V(G^*)) = Card(|G| \ |G^*|) \mid \forall \phi$.

Kneser graphs

Theorem (Lovasz): The chromatic number of $K(r, s)$ is $s - 2r + 2$.

Theorem ((Erdos, Ko, Rado): Let $r, s \in \mathbb{Z} : 1 \leq r < s \ 2$ and $\mathcal{F}$ a family of pairwise intersecting r-subsets of $[s]$. Then $|\mathcal{F}| \leq \binom{s-1}{r-1}$.

Theorem (Hilton, Milner): Let $r, s \in \mathbb{Z} : 1 \le r \le s/2$ and $\mathcal{F}$ a family of pairwise intersecting r-subsets of $[s]$. Then $\bigcap \mathcal{F} = \emptyset \implies |\mathcal{F}| \le \binom{s-1}{r-1} - \binom{s-r-1}{r-1} + 1$.

Proposition: All *Kneser graphs* are *cores*.

Propositio: Let $k, n \in \mathbb{Z} : n \ge 3$. Let $r = k(n-2)$, $s = (2k+1)(n-2)$. Then $K(r, s)$ has *chromatic number* n and odd girth $2k+1$.

Proposition (Stahl 1):
$\left[\left[r, s \in \mathbb{P} : r \perp s, 1 \le r < s/2\right] \implies K(mr, ms) \to K(nr, ns) \mid \forall m, n \in \mathbb{Z}\right] \iff m \div n$.

Proposition (Stahl 2): $r, s \in \mathbb{Z} : 1 \le r < s/2 \implies K(r+1, s+2) \to K(r, s)$.

Corollary: Let r, r', s, s' be integers s.t. $r \ge r', 1 \le r < s/2, 1 \le r' < s/2$. Then $K(r, s) \to K(r', s') \iff s - 2r + 2 \le s' - 2r' + 2$.

Proposition: Let G - graph, r, r', s, s' - integers s.t. $1 \le r < s/2$ and $1 \le r' < s/2$. Then $G \to K(r', s') \implies G \to K(r + r', s + s')$.

Circular graphs

Lemma: Let r, s be positive integers s.t. $r \le s/2$. Then $\alpha(G^r_s) = r$.

Lemma (Bondy, Hell): Let $r, s, k \in \mathbb{Z}$ s.t. $r \le s/2$. Then $G^r_s \leftrightarrow G^{kr}_{ks}$.

Proposition (Bondy, Hell 1): Let $r, s, m, n \in \mathbb{Z}$ s.t. $r \le s/2$, $m \le n/2$. Then $G^r_s \to G^m_n \iff r/s \ge m/n$.

Theorem (Tardif): G, H - *vertex-transitive graphs* s.t. G maps strictly into H, then there is a *vertex-transitive* graph K such that $G \to K \to H$ and no two other graphs are homomorphically equivalent.

Proposition (Bondy, Hell 2): Let $r, s \in \mathbb{Z} : r \perp s, r < s/2$. Then $\exists m, n \in \mathbb{Z} : r/s < m/n$ and $\forall u \in V(G^r_s), G^r_s - u \to G^m_n$.

Graph colourings and variation

Definition: Let $k \in \mathbb{N}$. Then a complete k-colouring of a graph is a complete homomorphism $\phi : G \to K_k$.

Chromatic number:

Lemma: Let G, H - graphs. $G \to H \implies \left[G \text{ has a proper } k\text{-colouring whenever } H \text{ does}\right]$.

Definition: Let G - graph. The *chromatic number* of G, denoted $\chi(G)$, is the least $n \in \mathbb{N}$ such that $G \to K_n$.

Lemma: Let G - graph, $\chi(G) = n$. Then $\phi - \text{ complete } \forall \phi : G \to K - n$

Corollary: $G \to H \implies \chi(G) \leq \chi(H)$.

Corollary: H - quotient of G by an *elementary homomorphism* $\implies \chi(G) \leq \chi(H) \leq hi(G) + 1$.

Corollary: The chromatic numebr of a graph and its core are the same ($\chi(G) = \chi(G^*)$).

Theorem (X. Zhu 1): For any graph G, $\chi(G) = \inf\{r : g \to I(r)\}$.

Achromatic number

Definition: The *achromatic number* of a graph G, denoted $achr(G)$, is the largest s such that there is a complete homomorphism from $G \to K_n$.

Lemma: $\phi : G \to H$ - complete $\implies achr(G) \geq achr(H)$.

Proposition: If there is an *elementary homomorphism* $\phi : G \to H$ then $achr(G) - 2 \leq achr(H) \leq achr(G)$.

Proposition: Let G - graph. There is a complete homomorphism from G onto K_n for each s s.t. $\chi(G) \leq s \leq achr(G)$.

Theorem: $\forall k \exists K : |V(G)| \leq K \mid$ irreducible graph $G : achr(G) = k$.

Lemma: Let G - graph, $n \in \mathbb{N}$. Then:

1. $\exists$ complete $P_n \to G \iff G$ is an underlying graph for some multigraph G' with $n-1$ edges with an eulerian trail.
2. $\exists$ complete $C_n \to G \iff G$ is an underlying graph of some multigraph G' with n edges and an eulerian tour.

Theorem (Hedrlin, Hell, Ko): For every graph G for which there are complete homomorphisms onto K_n and K_{n+1} there is a graph $\mathcal{K} \in \mathcal{H}_{n+\frac{1}{2}}$ and a complete homomorphism $G \to \mathcal{K}$.

Kneser colourings

Theorem: Let r, s be positive integers, $r < s/2$. A Kneser (r, s)-colouring of a graph G is a homomorphism from G to the Kneser graph $K(r, s)$.

Theorem: Let $r \in \mathbb{N}$. The r-chromatic number $\chi_r(G)$ of a graph G is the least s such that $G \to K(r, s)$.

Circular colourings

Definition: Let $1 \leq r \leq s/2$. A *circular* (r, s)-colouring of a graph G is a homomorphism $\phi : G \to G_s^r$.

Definition: A *circular* (r, s)-colouring of a graph G is a map $\phi : V(G) \to \{0, \ldots, s-1\}$ such that $[u, v] \in E(G) \implies r \leq |\phi(u) - \phi(v)| \leq s - r$.

Theorem (Bondy and Hell): The *circular* chromatic number of a graph G is $\chi_c(G) = min\{s/r : G \to G_s^r \text{ and } s \leq |G|\}$.

Theorem: For any graph G, $\chi_c(G) = \inf\{r : G \to C(R)\}$.

Theorem: If G has a vertex v which is adjacent to every other vertex of G, then $\chi_c(G) = \chi(G) = n$.

Corollary: Suppose G has a chromatic number S and a vertex whose neighbours induce a subgraph of chromatic number $s - 1$. Then $\chi_c(G) = \chi(G)$. In particular, $\chi(G) = s$ and G contains (a copy of) K_s $\implies \chi_c(G) = \chi(G)$.

Theorem: G – graph whose complement is disconnected $\implies \chi_c(G) = \chi(G)$.

Fractional chromatic number

Definition: Let $\mathcal{I}(G)$ denote a set of all independent sets of a graph G. A *fractional colouring* of G is a weight function $\mu : \mathcal{I}(G) \to [0, 1]$ such that the constraints $\sum_{u \in I \in \mathcal{I}(G)} \mu(I) \geq 1$ are satisfied for all $u \in V(G)$.

Proposition: Let G - graph. Then: $\chi_f(G) = \inf\{s\, r : G \to K(r, s)\} = \sup\{|H|\, \alpha(H) : H \to G\}$

Theorem (X Zhu 2): For any graph G, $\chi_f(G) = \inf\{r : G \to M(r)\}$.

Chromatic difference sequence

Definition: Let G be a graph and let $n = |G|$. The *chromatic difference sequence* of G is a sequence $(\beta_1(G)\, n, \ldots, \beta_{\chi(G)}(G)\, n)$.

Lemma: $\begin{cases} G \to H \\ H - \text{ vertex-transitive} \end{cases} \implies$ normalized chromatic sequence of G dominates that of H.

Theorem: The normalized chromatic difference sequence of G^k is equal to that of G for any cayley graph G of an abelian group with cartesian product.

Theorem: Let G, H graphs.

1. Normalized chromatic difference sequence of $G \times H$ dominates those of G or H.
2. Normalized chromatic difference sequence of $G \square H$ is dominated by those of G and H and is the normalized chromatic difference sequence of the cartesian power G^k is dominated by that of G.

Ultimate independence ratio

Definition: An independent set cover of a graph G by a graph H is a homomorphism $i : G \to Ind(H)$. When G has an independent set cover by G, we say simply that G has an *independent set cover*.

Lemma: Let G, H - graphs. Then $i(G \square H) \leq min\{i(G), i(H)\}$.

Definition: The *ultimate independence ratio* of a graph G, denoted by $I(G)$, is $I(G) = \lim_{k \to \infty} i(G^k)$.

Theorem: $G^2 \to G \to Ind(G) \implies I(G) = i(G)$.

Theorem: G - cayley graph $Cay(\Gamma, S))$ on an abelian group $\Gamma \implies I(G) = i(G)$.

Corollary: G - *circulant* $\implies I(G) = i(G)$. In particular, $I(C_{2k}) = 1/2$ and $I(C_{2k+1}) = k/(2k+1) \mid k > 0$.

Theorem (Hahn, Hell, Poljak 1): $G \to H \implies I(H) \leq I(G)$. In particular, $[H$ - retract of $G] \implies I(H) = I(G)$.

Theorem (Hahn, Hall, Poljak 2): For any graph $G \frac{1}{\chi(G)} \leq I(G) \leq \frac{1}{\chi_f(G)}$.

Theorem (X Zhu 3): For any graph G, the sequence $(1\ \chi_f(G^k) \mid k > 1)$ is non-increasing and $I(G) = \lim_{k \to \infty} \dfrac{1}{\chi_f(G^k)}$.

Theorem (X Zhi 4): For any graph G, $I(G) \geq \frac{1}{\chi_c(G)}$.

Graph products

Categorical product and Hedietnemi conjecture

Observation: Let G, H be graphs. Then the *odd girth* of $G \times H$ is the maximum of *odd girths* of G and H, and the clique number of $G \times H$ is the minimum of the clique numbers of G and H.

Conjecture (Product Conjecture, Hedietnemi): Let G, H be graphs. Then $\chi(G \times H) = min\{\chi(G), \chi(H)\}$.

Definition: For a graph G and an integer n, we define n*-colouring graph* $\mathcal{C}_n(G)$ of G by putting:

- $V(\mathcal{C}_n(G)) = \{f : V(G) \to |n|\}$.
- $E(\mathcal{C}_n(G) = \{|f, g| : \forall (u, v) \in V(G)^2, [u, v] \in E(G), f(u) \neq g(v)\}$.

Theorem (El-Zabar, Sauer): Let G - graph, $\chi(G) \geq 4$. Then $\chi(\mathcal{C}_3(G)) = 3$.

Corollary: If $\chi(G) \geq 4$ and $\chi(H) \geq 4$, then $\chi(G \times H) \geq 4$.

Lemma (El-Zabar, Sauer): let G - connected graph and $n < \chi(G)$. Then $\mathcal{C}_n(G)$ contains a unique complete subgraph of cardinality n, namely, the subgraph induced by the constant functions.

Theorem (Burr, Erdos, Lovasz): Let $\mathcal{H}$ - graph such that each vertex of H is contained in a complete subgraph of cardinality n and $\chi(H) > n$. Then for any graph G such that $\chi(G) > n$ w have $\chi(G \times H) > n$.

Theorem (Duffus, Sands, Woodrow): Let G, H be two connected graphs, both containing a complete subgraph of cardinality n, and such $\chi(G), \chi(H) > n$. Then $\chi(G \times H) > n$.

Definition: A graph K is called *multiplicative* if for any graphs G and H such that $G \not\to K$ and $H \not\to K$ we have $G \times H \not\to K$.

Conjecture: Let G and H be graphs. Then $\chi_c(G \times H) = min\{\chi_c(G), \chi_c(H)\}$.

Lemma: let G, H, K be graphs. Then $G \times H \to K \iff H \to \mathcal{C}_K(G)$.

Theorem (Haagkvist, Hell, Miller, Neumann-Lara): All odd cycles are multiplicative.

The cartesian product and normal cayley graphs

Definition: A graph K is called *hom-idemponent* if $K \square K \leftrightarrow K$.

Definition: A normal Cayley graph is a cayley graph $Cay(\Gamma, S)$ such that $x^{-1}sx \in S \mid \forall x \in \Gamma, s \in S$.

Lemma: A Cayley graph $Cay(\Gamma, S)$ is *normal* $\iff$ both left and right translations of Γ are *automorphisms* of $Cay(\Gamma, S)$.

Corollary (Hahn, Hell, Poljak): Let $Cay(\Gamma, S)$ be *normal* Cayley Graph. Then the map $\phi : Cay(\Gamma, S) \square Cay(\Gamma, S) \to Cay(\Gamma, S)$ defined by putting $\phi(x, y) = xy$ is a homomorphism. Therefore, $Cay(\Gamma, S)$ is *hom-idemponent*.

Definition: Let G, K - graphs. Then we define *homomorphism graph* $Hom(G, K)$ to be:

- $V(Hom(G, K)) = \{\phi : G \to K \mid \phi \text{ is a homomorphism}\}$.
- $E(Hom(G, K)) = \{[\phi, \tau] \mid \forall u \in V(G), [\phi(u), \tau(u)] \in E(K)\}$.

Proposition: Let G, H, K be graphs. Then:

1. Let $\phi : G \square H$ be a homomorphism. Then the map $\tau : G \to Hom(H, K)$ defined by setting $\tau(u) = \phi_u$ is a homomorphism, where $\phi_u : H \to K$ is defined by $\phi_u(v) = \phi(u, v)$.
2. Let $\tau : G \to Hom(H, K)$ be a homomorphism. Then the map $\phi : G \square H \to K$ defined by $\phi(u, v) = \phi_u(v)$ is a homomorphism, where $\phi_u = \tau(u)$.

Corollary: A graph K is hom-independent $\iff K \leftrightarrow Hom(K, K)$.

Proposition: Let K - core. Then $Hom(K, K)$ is a normal Cayley graph.

Theorem (Larose, Laviolette, Tardif 1): A graph is *hom-independent* $\iff$ it is homomorphically equivalent to a normal Cayley graph.

Theorem (Larose, Laviolette, Tardif 2): Let K be a graph and let $m > n$ integers. Then $K^m \to K^n \iff \exists$ normal $Cay(\Gamma_1, S_1), \ldots, Cay(\Gamma_n, S_n)$ such that $Cay(\Gamma_k, S_k) \to K \mid k = 1, \ldots, n$ and $K \to \square_{k=1}^{n} Cay(\Gamma_K, S_K))$.

Definition:

- Let $\phi : G \to \square_{k=1}^{n} G_k$ be a homomorphism. The edge-labeling induced by ϕ is the map $l_\phi : E(G) \to \{1, \ldots, n\}$ where $l_\phi[u, v]$ is the unique index k such that $pr_k \cdot \phi(u) \neq pr_k \cdot \phi(v)$.
- Let $l : E(G) \to \{1, \ldots, n\}$ be a map. Then for $1 \leq k \leq n, k \in \mathbb{N}$, let $G\, l^{-1}(k)$ denote the quotient $G\, \mathcal{Q}_k$ where $\mathcal{Q}_k$ is the partition of $V(G)$ whose cells are the (vertex sets of) connected components of $G - l^{-1}(k)$.

Observation: Let G - graph, n - integer, $l : E(G) \to \{1, \ldots, n\}$ a map. Then the following conditions are equivalent:

- $\exists \phi : G \to \square_{k=1}^{n} G_k$ such that $l_0 = l$.
- The natural map $\pi : G \to \square_{k=1}^{n} G / l^{-1}(k)$ is a homomorphism.
- For any edge $e = [u, v]$ of g and for any path P from u to v, there exists an edge e' of P such that $l(e') = l(e)$.

The strong product and the lexicographic product

Lemma: For any graph G, and integer n, $\chi(G[K_n]) = \chi_n(G)$.

Theorem (Stahl 3): For any graphs G and H, we have $\chi(G[H]) = \chi_n(G)$, where $n = \chi(H)$.

Theorem (Zhu 5): For any graphs G and H, we have $\chi_c(G[H]) = \chi_c(G[K_n])$, where $n = \chi(H)$.

Lemma: For any graphs G and H, $\chi(G \boxtimes H) \geq max\{\chi_{\omega(H)}(G), \chi_{\omega(G)}(H)\}$.

Theorem (Klavzaar, Milutinovic): If G and H are graphs with at least one edge, then $\chi(G \boxtimes H) \geq max\{\chi(G) + 2\omega(H) - 2, \chi(H) + 2\omega(G) - 2\}$.

Theorem (Vesztergombi): For two graphs G and H, $G \boxtimes H$ is m-colourable $\iff H \to \mathcal{C}_m^{\boxtimes}(G)$.

Proposition: For any connected graphs G, H, the core of $G[H]$ is $G'[H^{\cdot}]$, where G' is a subgraph of G which itself is *core*.

Theorem (Imrich, Klavzaar): Let G, H be connceted graphs, R - retract of $G \boxtimes H$. Then $\exists G' \in G, H' \in H : R \simeq G' \boxtimes H'$.

Proposition: Let G, H be connected graphs, such that G does not contain any triangles. Then $G[H]^{\cdot} = G^{\cdot}[H^{\cdot}]$.

Theorem (Imrich, Klavzaar): Let G, H be connected triangle-free graphs and R - retract of $G \boxtimes H$. Then there exist retracts $G' \in G, H' \in H : R \simeq G' \boxtimes H'$. In particular, $(G \boxtimes H)^{\cdot} = G^{\cdot} \boxtimes H^{\cdot}$.

Isometric embeddings and retracts

Definition: Let G, H - connected graphs, $\phi : V(G) \to V(H)$ - map. Then:

1. ϕ is a *contraction* if $d_H(\phi(u), \phi(v)) \leq d_G(u, v)$ for any two vertices u, v in G.
2. ϕ is an *isometric embedding* if $d_H(\phi(u), \phi(v)) = d_G(u, v)$ for any two verties u, v in G.

Lemma: Let R - retract of a connected graph G. Then any *co-retraction* $\gamma : R \to G$ is an *isometric embedding*.

Theorem (Graham, Winkler): Let G - graph, n - integer, $l : E(G) \to \{1, \ldots, n\}$ - map. Then the following conditions are equivalent:

- $\exists$ an *isometric embedding* $\phi : G \to \square_{k=1}^n G_k$ such that $l_0 = l$.
- The *natural map* $\pi : G \to \square_{k=1}^n G\, l^{-1}(k)$ is an *isometric embedding*.
- For all $u, v \in V(G)$, for all $k \in \{1, \ldots, n\}$ and for all uv-geodesic P, we have $|E(P) \cap l^{-1}(k) = min\{|E(Q) \cap l^{-1}(k)| : Q \text{ is a uv-path}\}$.

Theorem (Duffus, Rival): The diameter-preserving retracts of cubes are covering graphs of distributed lattices.

Theorem (Bandelt): The *median graphs* are retracts of cubes.

Proposition: Let $\phi G \to H$ be a contraction, where G and H are bipartite graphs. Then $\exists$ bipartite graph G' and a homomorphism $\tau : G \to H \square G'$ such that $\phi = pr_H \cdot \tau$.

Proposition (Quilliot, Nowakowski, Rival 1): Every connected graph admits an isomorphic embedding into a strong product of paths.

Definition: Let G - connected graph.

1. Let $u \in V(G)$, r - integer. The *ball of center* u and radius r is the set $B(u, r) = \{v \in V(G) \mid d_G(u, v) \leq r)$.
2. G is a *Helly graph* if the family of its *balls* has the *Helly* property, that is, for any family $B(u_1, r_1), \ldots, B(u_n, r_n)$ of pairwise intersecting balls we have $\bigcap_{k=1}^n B(u_k, r_k) \neq \emptyset$.

Proposition (Quilliot, Nowakowski, Rival 2: The fully *Helly graphs* are the weak retracts of strong products of paths.

Theorem (Extension Theorem, Quilliot): Let G - connected graph, H - *Helly graph*. Let $S \subset V(G)$, $\phi_0 : S \to V(H)$ a map such that $d_H(\phi_0(u), \phi_0(v)) \mid \forall u, v \in S$. Then there exists a construction $\phi : G \to H$ whose restriction to S is ϕ_0.

Theroem (Hell): Let T - tree, G - bipartite graph, $\gamma : T \to G$ an isomorphic embedding. Then T is a retract of G
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